Lecture 10: Product Measure

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Contents

1 Product σ-Algebra

- **Definition** (***Product Topology***)
  Let X and Y be topological spaces. ***The product topology*** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

  Definition Let $\pi_X : X \times Y \to X$ be defined by the equation

  $$\pi_X(x, y) = x;$$

  $\pi_Y : X \times Y \to Y$ he defined by the equation

  $$\pi_Y(x, y) = y.$$

  The maps π_X and π_Y are called *the **projections** of $X \times Y$ **onto** its **first** and **second** factors*, respectively.

- **Definition** (***Product σ-Algebra***)
  Suppose that $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ are measurable spaces. We can form the ***pullback σ-algebras***

  $$\begin{aligned}\pi_X^*(\mathscr{B}_X) &:= \left\{\pi_X^{-1}(E) : E \in \mathscr{B}_X\right\} = \{E \times Y : E \in \mathscr{B}_X\}\\ \pi_Y^*(\mathscr{B}_Y) &:= \left\{\pi_Y^{-1}(F) : F \in \mathscr{B}_Y\right\} = \{X \times F : F \in \mathscr{B}_Y\}\end{aligned}$$

  We then define ***the product σ-algebra $\mathscr{B}_X \times \mathscr{B}_Y$*** to be the ***$\sigma$-algebra generated by the union of these two pull-back σ-algebras***:

  $$\mathscr{B}_X \times \mathscr{B}_Y := \langle \pi_X^*(\mathscr{B}_X) \cup \pi_Y^*(\mathscr{B}_Y) \rangle.$$

  This definition has several equivalent formulations:

- **Proposition 1.1** *(**Equivalent Definition of Product σ-Algebra**) [Tao, 2011]*
  Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be measurable spaces.

  1. *$\mathscr{B}_X \times \mathscr{B}_Y$ is the **σ-algebra generated by the sets** $E \times F$ **with** $E \in \mathscr{B}_X$**,** $Y \in \mathscr{B}_Y$. In other words, $\mathscr{B}_X \times \mathscr{B}_Y$ is the **coarsest σ-algebra** on $X \times Y$ with the property that **the product** of a $\mathscr{B}_X$-measurable set and a $\mathscr{B}_Y$-measurable set is always $\mathscr{B}_X \times \mathscr{B}_Y$ measurable.*

  2. *$\mathscr{B}_X \times \mathscr{B}_Y$ is the **coarsest σ-algebra** on $X \times Y$ that makes the **projection maps** π_X, π_Y both measurable.*

- **Proposition 1.2** *(**Property of Product σ-Algebra**) [Tao, 2011]*
  Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be measurable spaces.

  1. *(**Slices of Meaurable Set**) If $E \in \mathscr{B}_X \times \mathscr{B}_Y$, then sets*

     $$E_x := \{y \in Y : (x, y) \in E\} \in \mathscr{B}_Y$$

     for every $x \in X$, and similarly that the sets

     $$E^y := \{x \in X : (x, y) \in E\} \in \mathscr{B}_X$$

     for every $y \in Y$.

2. ***(Slices of Meaurable Function)*** *If $f : X \times Y \to [0, +\infty]$ is measurable (with respect to $\mathscr{B}_X \times \mathscr{B}_Y$), then the function*

$$f_x : y \to f(x, y)$$

is $\mathscr{B}_Y$-measurable for every $x \in X$, and similarly that the function

$$f^y : x \to f(x, y)$$

is $\mathscr{B}_X$-measurable for every $y \in Y$.

3. *The product of two **trivial σ-algebras** (on two different spaces X, Y) is again **trivial**.*

4. *The product of two **atomic σ-algebras** is again **atomic**.*

5. *The product of two **finite σ-algebras** is again **finite**.*

6. *The product of two **Borel σ-algebras** (on two Euclidean spaces $\mathbb{R}^d$, $R^{d'}$ with $d, d' \geq 1$) is again the **Borel σ-algebra** (on $\mathbb{R}^d \times \mathbb{R}^{d'} \equiv \mathbb{R}^{d+d'}$).*

7. *The product of two **Lebesgue σ-algebras** (on two Euclidean spaces $\mathbb{R}^d$, $R^{d'}$ with $d, d' \geq 1$) is **not** the **Lebesgue σ-algebra**. (Hint: argue by **contradiction** and use slices of measurable set as above proposition.)*

8. *However, the Lebesgue σ-algebra on $\mathbb{R}^{d+d'}$ is the **completion** of the product of the Lebesgue σ-algebras of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ with respect to $(d+d')$-dimensional Lebesgue measure.*

- **Exercise 1.3** *[Tao, 2011]*
  *If $E \in \mathscr{B}_X \times \mathscr{B}_Y$, show that the slices $E_x := \{y \in Y : (x, y) \in E\}$ lie in a countably generated σ-algebra. In other words, show that there exists an at most countable collection $\mathscr{A} = \mathscr{A}_E$ of sets (which can depend on E) such that s $\{E_x : x \in X\} \subseteq \langle \mathscr{A} \rangle$. Conclude in particular that the number of **distinct** slices E_x is at most c, the **cardinality** of the continuum.*

- **Exercise 1.4** *[Tao, 2011]*
  *Give an example to show that the product of two **discrete σ-algebras** is **not** necessarily **discrete**.*

  *On the other hand, show that the product of two **discrete σ-algebras** 2^X, 2^Y is again a **discrete σ-algebra** if at least one of the domains X, Y is **at most countably infinite**.*

2 Product Measure

- **Definition** (***σ-Finite***).
  A measure space $(X, \mathscr{B}, \mu)$ is $\overline{\textit{$\sigma$-finite}}$ if X can be expressed as the **countable union** of sets of **finite** measure, i.e. $X = \bigcup_n X_n$, $\mu(X_n) < \infty$ for all n.

- **Example** ($\mathbb{R}^d$)
  $\mathbb{R}^d$ with ***Lebesgue measure*** is σ-finite, as $\mathbb{R}^d$ can be expressed as the union of (for instance) the balls $B(0, n)$ for $n = 1, 2, 3, \ldots$, each of which has finite measure.

  On the other hand, $\mathbb{R}^d$ with ***counting measure is not σ-finite***.

- **Proposition 2.1** ***(Existence and Uniqueness of Product Measure)*** *[Tao, 2011]*
  *Let $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$ be **σ-finite measure spaces**. Then there exists a **unique***

measure $\mu_X \times \mu_Y : \mathscr{B}_X \times \mathscr{B}_Y \to [0, \infty]$ on product σ-algebra $\mathscr{B}_X \times \mathscr{B}_Y$ that obeys

$$\mu_X \times \mu_Y(E \times F) = \mu_X(E)\, \mu_Y(F)$$

whenever $E \in \mathscr{B}_X$ and $F \in \mathscr{B}_Y$.

- **Remark** When X, Y are **not both** σ-finite, then one can still construct **at least one product measure**, but it will, in general, **not be unique**.

- **Remark** (***Product Measure of Lebesgue Measures***)
The ***product*** $m^d \times m^{d'}$ of the ***Lebesgue measures*** $m^d, m^{d'}$ on $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d])$ and $(\mathbb{R}^{d'}, \mathcal{L}[\mathbb{R}^{d'}])$ respectively will ***agree*** with Lebesgue measure $m^{d+d'}$ on the product space $\mathcal{L}[\mathbb{R}^d] \times \mathcal{L}[\mathbb{R}^{d'}]$, which is a ***subalgebra*** of $\mathcal{L}[\mathbb{R}^{d+d'}]$. After taking the ***completion*** $\overline{m^d \times m^{d'}}$ of this product measure, one obtains the *full Lebesgue measure* $m^{d+d'}$.

- **Proposition 2.2** *[Tao, 2011]*
Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be measurable spaces.

    1. *The product of two **Dirac measures** on $(X, \mathscr{B}_X)$, $(Y, \mathscr{B}_Y)$ is a **Dirac measure** on $(X \times Y, \mathscr{B}_X \times \mathscr{B}_Y)$.*

    2. *If X, Y are **at most countable**, the product of the two **counting measures** on $(X, \mathscr{B}_X)$, $(Y, \mathscr{B}_Y)$ is the **counting measure** on $(X \times Y, \mathscr{B}_X \times \mathscr{B}_Y)$.*

- **Proposition 2.3** ***(Associativity of Product)****. [Tao, 2011]*
*Let $(X, \mathscr{B}_X, \mu_X)$, $(Y, \mathscr{B}_Y, \mu_Y)$, $(Z, \mathscr{B}_Z, \mu_Z)$ be σ-**finite sets**. We may identify the Cartesian products $(X \times Y) \times Z$ and $X \times (Y \times Z)$ with each other in the obvious manner. If we do so, then*

$$(\mathscr{B}_X \times \mathscr{B}_Y) \times \mathscr{B}_Z = \mathscr{B}_X \times (\mathscr{B}_Y \times \mathscr{B}_Z)$$

and

$$(\mu_X \times \mu_Y) \times \mu_Z = \mu_X \times (\mu_Y \times \mu_Z).$$

3 Integration in Product Space

3.1 Tonelli's Theorem

- **Definition** (***Monotone Class***)
Define a ***monotone class*** *in* X is a collection $\mathscr{B}$ of subsets of X with the following two **closure properties**:

    1. If $E_1 \subset E_2 \subset \ldots$ are a ***countable increasing*** sequence of sets in $\mathscr{B}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathscr{B}$.

    2. If $E_1 \supset E_2 \supset \ldots$ are a ***countable decreasing*** sequence of sets in $\mathscr{B}$, then $\bigcap_{n=1}^{\infty} E_n \in \mathscr{B}$.

- **Lemma 3.1** ***(Monotone Class Lemma)****. [Tao, 2011]*
*Let $\mathscr{A}$ be a **Boolean algebra** on X. Then $\langle \mathscr{A} \rangle$ is the **smallest monotone class** that contains $\mathscr{A}$.*

- **Theorem 3.2** ***(Tonelli's Theorem, Incomplete Version)****. [Tao, 2011]*
*Let $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$ be σ-**finite measure spaces**, and let $f : X \times Y \to [0, +\infty]$ be measurable with respect to $\mathscr{B}_X \times \mathscr{B}_Y$. Then:*

1. *The functions*

$$x \to \int_Y f(x,y) d\mu_Y(y)$$
$$y \to \int_X f(x,y) d\mu_X(x)$$

(which are well-defined) are ***measurable*** *with respect to* $\mathscr{B}_X$ *and* $\mathscr{B}_Y$ *respectively.*

2. *We have*

$$\begin{aligned}\int_{X\times Y} f(x,y)\, d\left(\mu_X \times \mu_Y\right)(x,y) &= \int_X \left(\int_Y f(x,y) d\mu_Y(y)\right) d\mu_X(x) \\ &= \int_Y \left(\int_X f(x,y) d\mu_X(x)\right) d\mu_Y(y) \end{aligned} \tag{1}$$

- **Corollary 3.3** ***(Slice of Null Set)****[Tao, 2011]*
Let $(X, \mathscr{B}_X, \mu_X)$ *and* $(Y, \mathscr{B}_Y, \mu_Y)$ *be* σ***-finite measure spaces****, and let* $E \in \mathscr{B}_X \times \mathscr{B}_Y$ *be a* ***null*** *set with respect to* $\mu_X \times \mu_Y$*. Then for* μ_X*-almost every* $x \in X$*, the set* $E_x := \{y \in Y : (x,y) \in E\}$ *is a* μ_Y***-null set****; and similarly, for* μ_Y*-almost every* $y \in Y$*, the set* $E^y := \{x \in X : (x,y) \in E\}$ *is a* μ_X***-null set****.*

  With this corollary, we can extend *Tonelli's theorem* to the completion $(X\times Y, \overline{\mathscr{B}_X \times \mathscr{B}_Y}, \overline{\mu_X \times \mu_Y})$ of the product space $(X \times Y, \mathscr{B}_X \times \mathscr{B}_Y, \mu_X \times \mu_Y)$.

- **Theorem 3.4** ***(Tonelli's Theorem, Complete Version)****. [Tao, 2011]*
Let $(X, \mathscr{B}_X, \mu_X)$ *and* $(Y, \mathscr{B}_Y, \mu_Y)$ *be* ***complete*** σ***-finite measure spaces****, and let* $f : X \times Y \to [0, +\infty]$ *be measurable with respect to* $\mathscr{B}_X \times \mathscr{B}_Y$*. Then:*

  1. *For* μ_X*-almost every* $x \in X$*, the function*

  $$y \to f(x,y)$$

  is $\mathscr{B}_Y$***-measurable*** *and in particular,* $\int_Y f(x,y)d\mu_Y(y)$ *exists. Furthermore, the (*μ_X*-almost everywhere defined) map*

  $$x \to \int_Y f(x,y) d\mu_Y(y)$$

  is $\mathscr{B}_X$***-measurable****.*

  2. *For* μ_Y*-almost every* $y \in Y$*, the function*

  $$x \to f(x,y)$$

  is $\mathscr{B}_X$***-measurable*** *and in particular,* $\int_X f(x,y)d\mu_X(x)$ *exists. Furthermore, the (*μ_Y*-almost everywhere defined) map*

  $$y \to \int_X f(x,y) d\mu_X(x)$$

  is $\mathscr{B}_Y$***-measurable****.*

3. *We have*

$$\int_{X\times Y} f(x,y)\, d\left(\overline{\mu_X \times \mu_Y}\right)(x,y) = \int_X \left(\int_Y f(x,y) d\mu_Y(y)\right) d\mu_X(x)$$
$$= \int_Y \left(\int_X f(x,y) d\mu_X(x)\right) d\mu_Y(y) \qquad (2)$$

- Specialising to the case when f is an indicator function $f = \mathbb{1}_E$, we conclude

  Corollary 3.5 *(**Tonellis Theorem for Sets**). [Tao, 2011]*
  *Let $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$ be **complete σ-finite measure spaces**, and let $E \in \overline{\mathscr{B}_X \times \mathscr{B}_Y}$. Then:*

  1. *For μ_X-almost every $x \in X$, the set*

     $$E_x := \{y \in Y : (x,y) \in E\} \in \mathscr{B}_Y$$

     and the (μ_X-almost everywhere defined) map

     $$x \to \mu_Y(E_x)$$

     *is $\mathscr{B}_X$**-measurable.***

  2. *For μ_Y-almost every $y \in Y$, the set*

     $$E^y := \{x \in X : (x,y) \in E\} \in \mathscr{B}_X$$

     and the (μ_Y-almost everywhere defined) map

     $$y \to \mu_X(E^y)$$

     *is $\mathscr{B}_Y$**-measurable.***

  3. *We have*

     $$\overline{\mu_X \times \mu_Y}(E) = \int_X \mu_Y(E_x)\, d\mu_X(x)$$
     $$= \int_Y \mu_X(E^y)\, d\mu_Y(y) \qquad (3)$$

- **Remark** Tonellis theorem can ***fail*** if ***the σ-finite hypothesis*** *is* ***removed***, and also that *product measure need not be unique.*

3.2 Fubini's Theorem

- **Remark** *Tonelli's theorem* is for *the **unsigned integral***, but it leads to an important analogue for *the absolutely integral*, known as *Fubini's theorem*:

- **Theorem 3.6** *(**Fubinis Theorem**). [Tao, 2011]*
  *Let $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$ be **complete σ-finite measure spaces**, and let $f : X \times Y \to \mathbb{C}$ be **absolutely integrable with respect to** $\overline{\mathscr{B}_X \times \mathscr{B}_Y}$. Then:*

1. *For μ_X-almost every $x \in X$, the function*

$$y \to f(x, y)$$

is ***absolutely integrable with respect to*** *μ_Y and in particular, $\int_Y f(x, y) d\mu_Y(y)$ exists. Furthermore, the (μ_X-almost everywhere defined) map*

$$x \to \int_Y f(x, y) d\mu_Y(y)$$

is ***absolutely integrable with respect to*** *μ_X.*

2. *For μ_Y-almost every $y \in Y$, the function*

$$x \to f(x, y)$$

is ***absolutely integrable with respect to*** *μ_X and in particular, $\int_X f(x, y) d\mu_X(x)$ exists. Furthermore, the (μ_Y-almost everywhere defined) map*

$$y \to \int_X f(x, y) d\mu_X(x)$$

is ***absolutely integrable with respect to*** *μ_Y.*

3. *We have*

$$\begin{aligned} \int_{X \times Y} f(x, y)\, d\left(\overline{\mu_X \times \mu_Y}\right)(x, y) &= \int_X \left(\int_Y f(x, y) d\mu_Y(y) \right) d\mu_X(x) \\ &= \int_Y \left(\int_X f(x, y) d\mu_X(x) \right) d\mu_Y(y) \end{aligned} \tag{4}$$

- **Remark** Fubini's theorem fails when one drops the hypothesis that f is *absolutely integrable* ***with respect to the product space***.
- **Remark** Despite the failure of Tonelli's theorem in the σ-finite setting, it is possible to *(carefully) extend Fubini's theorem to* ***the non-σ-finite setting***, as *the absolute integrability hypotheses*, when combined with *Markov's inequality*, can provide a substitute for *the σ-finite property*.
- **Remark** Informally, *Fubini's theorem* allows one to *always* ***interchange the order of two integrals***, as long as the *integrand is absolutely integrable* ***in the product space*** *(or its completion)*. In particular, specialising to Lebesgue measure, we have

$$\int_{\mathbb{R}^{d+d'}} f(x, y) d(x, y) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{d'}} f(x, y) dy \right) dx = \int_{\mathbb{R}^{d'}} \left(\int_{\mathbb{R}^d} f(x, y) dx \right) dy$$

  whenever $f : \mathbb{R}^{d+d'} \to \mathbb{C}$ is *absolutely integrable*. In view of this, we often write $dxdy$ (or $dydx$) for $d(x, y)$.
- By combining *Fubini's theorem* with *Tonelli's theorem*, we can recast the *absolute integrability hypothesis*:

Corollary 3.7 *(**Fubini-Tonelli Theorem**). [Tao, 2011]*
*Let $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$ be **complete σ-finite measure spaces**, and let $f : X \times Y \to \mathbb{C}$ be **measurable with respect to $\overline{\mathscr{B}_X \times \mathscr{B}_Y}$**. If*

$$\int_X \left(\int_Y |f(x,y)| \, d\mu_Y(y) \right) d\mu_X(x) < \infty$$

*then f is **absolutely integrable** with respect to $\overline{\mathscr{B}_X \times \mathscr{B}_Y}$, and in particular the conclusions of **Fubini's theorem** hold.*

Similarly if we use $\int_Y \left(\int_X |f(x,y)| \, d\mu_X(x) \right) d\mu_Y(y)$ instead of $\int_X \left(\int_Y |f(x,y)| \, d\mu_Y(y) \right) d\mu_X(x)$.

- **Proposition 3.8** *(**Area Interpretation of Integral**). [Tao, 2011]*
  *Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, and let $\mathbb{R}$ be equipped with Lebesgue measure m and the Borel σ-algebra $\mathcal{B}[\mathbb{R}]$. Then $f : X \to [0, +\infty]$ is **measurable if and only if** its **subgraph***

  $$\{(x,t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}$$

  *is **measurable** in $\mathscr{B} \times \mathcal{B}[\mathbb{R}]$, in which case we have*

  $$(\mu \times m)\{(x,t) \in X \times \mathbb{R} : 0 \le t \le f(x)\} = \int_X f(x) d\mu(x).$$

  Similarly if we replace $\{(x,t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}$ by $\{(x,t) \in X \times \mathbb{R} : 0 \le t < f(x)\}$.

- **Proposition 3.9** *(**Distribution Formula**). [Tao, 2011]*
  *Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, and let $f : X \to [0, +\infty]$ be **measurable**. Then*

  $$\int_X f(x) d\mu(x) = \int_{[0,\infty]} \mu\{x \in X : f(x) \ge \lambda\} \, d\lambda \tag{5}$$

  (Note that the integrand on the right-hand side is monotone and thus Lebesgue measurable.) Similarly if we replace $\{x \in X : f(x) \ge \lambda\}$ by $\{x \in X : f(x) > \lambda\}$.

References

Terence Tao. *An introduction to measure theory*, volume 126. American Mathematical Soc., 2011.