Lecture 16: Geodesics

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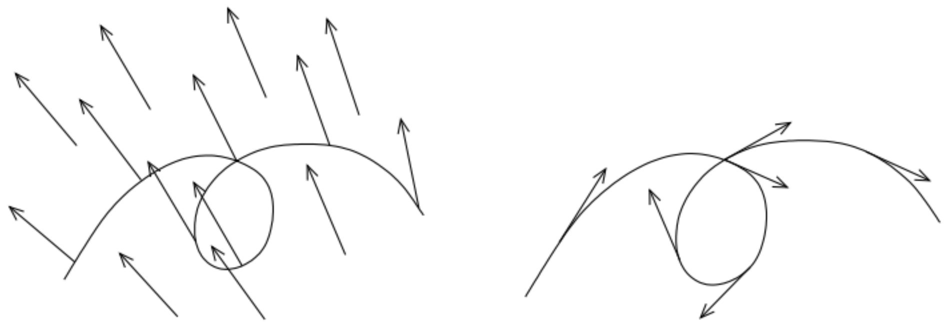

Figure 1: An extendible vector field (Left) vs a non-extendible vector field [Lee, 2018]

1 Vector and Tensor Fields Along Curves

1.1 Definition

- **Definition** Let M be a smooth manifold with or without boundary. Given a smooth curve $\gamma : I \to M$, ***a vector field along*** γ is a *continuous* map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$; it is ***a smooth vector field along*** γ if it is *smooth* as a map from I to TM.

  We let $\mathfrak{X}(\gamma)$ denote ***the set of all smooth vector fields along*** γ. It is a *real vector space* under pointwise vector addition and multiplication by constants, and it is a module over $\mathcal{C}^{\infty}(I)$ with multiplication defined pointwise:

$$(fX)(t) = f(t)X(t).$$

- **Example** (***The Velocity Vector Field***)
  The most obvious example of *a vector field along a smooth curve* γ is the curve's ***velocity***: $\gamma'(t) \in T_{\gamma(t)}M$ for each t, and its coordinate expression

$$\gamma'(t) = \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}$$

  shows that it is ***smooth***.

- **Example** (***The Normal Vector Field***)
  If γ is a curve in $\mathbb{R}^2$, let $N(t) = R\gamma'(t)$, where R is ***counterclockwise rotation*** by $\pi/2$, so $N(t)$ is ***normal*** to $\gamma'(t)$. In standard coordinates,

$$N(t) = -\dot{\gamma}^2(t)\frac{\partial}{\partial x^1} + \dot{\gamma}^1(t)\frac{\partial}{\partial x^2},$$

  so N ***is a smooth vector field along*** γ.

- **Remark** (***Construction of A Smooth Vector Field Along the Curve***)
  Suppose $\gamma : I \to M$ is a smooth curve and $\widetilde{V} \in \mathfrak{X}(M)$ is a smooth vector field on an open subset of M containing the image of γ. The smooth vector field along the curve γ, $V = \widetilde{V} \circ \gamma$:

$$V(t) = \widetilde{V}_{\gamma(t)} \in T_{\gamma(t)}M.$$

A smooth vector field along γ is said to be ***extendible*** if there *exists* a smooth vector field $\widetilde{V}$ on a neighborhood of the image of γ that is related to V in this way.

Not every vector field along a curve need be extendible; for example, if $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$ (Fig. 1), then γ' is not extendible.

- **Definition** More generally, ***a tensor field along*** γ is a *continuous* map σ from I to some tensor bundle $T^{(k,l)}TM$ such that $\sigma(t) \in T^{(k,l)}T_{\gamma(t)}M$ for each $t \in I$.

  It is a ***smooth tensor field along*** γ if it is *smooth* as a map from I to $T^{(k,l)}TM$, and it is ***extendible*** if there is a smooth tensor field $\widetilde{\sigma}$ on a neighborhood of $\gamma(I)$ such that $\widetilde{\sigma} = \sigma \circ \gamma$.

1.2 Covariant Derivatives Along Curves

- **Theorem 1.1** ***(Covariant Derivative Along a Curve).***
  Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For each smooth curve $\gamma : I \to M$, the **connection** *determines* **a unique operator**

  $$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

  called ***the covariant derivative along*** γ*, satisfying the following properties:*

  1. *(**Linearity over** $\mathbb{R}$):*

  $$D_t\,(a\,V + b\,W) = a\,D_t(V) + b\,D_t(W), \quad \text{for } a, b \in \mathbb{R}.$$

  2. *(**Product Rule**):*

  $$D_t(f\,V) = f'\,V + f\,D_t(V), \quad \text{for } f \in \mathcal{C}^\infty(I).$$

  3. *If $V \in \mathfrak{X}(\gamma)$ is* **extendible***, then for every extension $\widetilde{V}$ of V,*

  $$D_t(V(t)) = \nabla_{\gamma'(t)}\widetilde{V}.$$

  There is **an analogous operator** *on the space of* **smooth tensor fields** *of any type along γ.*

- **Remark** (***Coordinate Representation for Covariant Derivatives Along a Curve***)
  Choose smooth coordinates (x^i) for M in a neighborhood of $\gamma(t_0)$, and write

  $$V(t) = V^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

  for t near t_0, where $V^1, \ldots, V^n$ are *smooth real-valued functions* defined on some neighborhood of t_0 in I. By the properties of D_t, since each $\frac{\partial}{\partial x^i}$ is extendible,

  $$\begin{aligned} D_t\,(V_t) &= \dot{V}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + V^i(t)\,\nabla_{\gamma'(t)}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \\ &= \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\,\Gamma^k_{i,j}(\gamma(t))\right)\frac{\partial}{\partial x^k}\Big|_{\gamma(t)} \end{aligned} \tag{1}$$

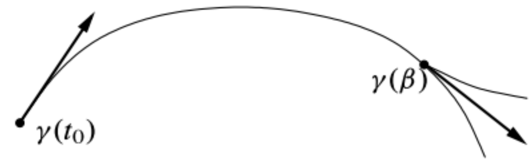

Figure 2: The uniqueness of a geodesic [Lee, 2018]

- **Proposition 1.2** *Let M be a smooth manifold with or without boundary, let ∇ be a connection in TM, and let $p \in M$ and $v \in T_pM$. Suppose Y and $\widetilde{Y}$ are two smooth vector fields that* ***agree*** *at points in the image of some smooth curve $\gamma : I \to M$ such that $\gamma(t_0) = p$ and $\gamma'(t_0) = v$. Then $\nabla_v Y = \nabla_v \widetilde{Y}$.*

2 Geodesics

- **Definition** Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth curve $\gamma : I \to M$, we define the ***acceleration*** of γ to be ***the vector field** $D_t(\gamma')$ **along** γ*.

- **Definition** A smooth curve γ is called a ***geodesic*** (***with respect to*** ∇) if *its acceleration is zero*: $D_t(\gamma'(t)) = 0$.

- **Remark** ***Geodesic*** is the curve whose ***tangential acceleration*** is ***zero***. From *the connection ∇ point of view*, it specify both the directional vector field and the target vector field equal to $\gamma'(t)$. That is, the *tangential acceleration along a curve* γ is

$$\nabla_{\gamma'(t)}\gamma'(t).$$

- **Remark** (***The Ordinary Differential Equations for the Geodesic***)
  In terms of smooth coordinates (x^i), if we write the component functions of γ as $\gamma(t) = (x^1(t), \ldots, x^n(t))$. From (1) and $D_t(\gamma'(t))$, we have a set of ordinary differential equations called ***the geodesic equations***:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\,\Gamma^k_{i,j}(x(t)) = 0, \quad k = 1, \ldots, n. \tag{2}$$

  where $x(t) := (x^1(t), \ldots, x^n(t))$. A (parameterized) curve γ is a geodesic ***if and only if*** its component functions satiesfy *the geodesic equations*. Note that (2) is ***a set of second-order nonlinear ODEs***.

- **Theorem 2.1** *(**Existence and Uniqueness of Geodesics**). [Lee, 2018]*
  Let M be a smooth manifold and ∇ a connection in TM. For every $p \in M$, $w \in T_pM$, and $t_0 \in \mathbb{R}$, there ***exist*** *an open interval $I \subseteq \mathbb{R}$ containing t_0 and a* ***geodesic*** *$\gamma : I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = w$. Any two such geodesics* ***agree*** *on their common domain.*

- **Remark** From the geodesic equation, we see that ***the only parameters*** *of the ODE that determines the geodesic is* ***the cooefficients of the connection*** $\{\Gamma^k_{i,j}\}$. That is, *the geodesic* is solely determined by *the connection* ∇. Thus we also call it a ∇***-geodesic***.

- **Remark** The ***geodesic equation*** *under the initial boundary condition* can be written in the following form:

$$\dot{x}^k(t) = v^k(t) \tag{3}$$
$$\dot{v}^k(t) = -v^i(t)v^j(t)\,\Gamma_{i,j}^k(x(t)) \tag{4}$$

Treating $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ as coordinates on $U \times \mathbb{R}^n$, we can recognize (4) as the equations for the ***flow*** of ***the vector field*** $G \in \mathfrak{X}(U \times \mathbb{R}^n)$ given by

$$G_{(x,v)} = v^k \frac{\partial}{\partial x^k}\Big|_{(x,v)} - v^i v^j\,\Gamma_{i,j}^k(x)\frac{\partial}{\partial v^k}\Big|_{(x,v)}. \tag{5}$$

The importance of G stems from the fact that it actually defines ***a global vector field on the total space of*** TM, called ***the geodesic vector field***. It can be verified that the components of G under a change of coordinates *take the same form* in *every coordinate chart*.

Note that G acts on a function $f \in \mathcal{C}^\infty(U \times \mathbb{R}^n)$ as

$$Gf(p, v) = \frac{d}{dt}\Big|_{t=0} f(\gamma_v(t), \gamma_v'(t)). \tag{6}$$

- **Definition** A geodesic $\gamma : I \to M$ is said to be ***maximal*** if it *cannot be extended* to a geodesic on a *larger interval*, that is, if there does not exist a geodesic $\widetilde{\gamma} : \widetilde{I} \to M$ defined on an interval $\widetilde{I}$ properly containing I and satisfying $\widetilde{\gamma}|_I = \gamma$.

  A ***geodesic segment*** is a geodesic whose domain is a ***compact interval***.

- **Corollary 2.2** *Let M be a smooth manifold and let ∇ be a connection in TM. For each $p \in M$ and $v \in T_pM$, there is a* ***unique maximal geodesic*** *$\gamma : I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing* 0.

- **Definition** The ***unique maximal geodesic*** γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is often called simply ***the geodesic with initial point*** p ***and initial velocity*** v, and is denoted by γ_v. (Note that we can always find $p = \pi(v)$ where $\pi : TM \to M$ is the natural projection.)

3 Parallel Transport

- **Definition** Let M be a smooth manifold and let ∇ be a connection in TM. *A smooth vector or tensor field V along a smooth curve γ* is said to be ***parallel along*** γ *(with respect to ∇)* if $D_t(V) \equiv 0$.

- **Remark** A *geodesic* can be characterized as a curve whose ***velocity vector field is parallel along the curve***.

- **Remark** (***Coordinate Representation of Vector Field Parallel Along a Curve***) Given a smooth curve γ with a local coordinate representation $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, formula (1) shows that a vector field V is parallel along γ if and only if

$$\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\,\Gamma_{i,j}^k(\gamma(t)) = 0, \quad k = 1, \ldots, n \tag{7}$$

This is a set of ***linear ordinary differential equations*** with respect to $(V^1(t), \ldots, V^n(t))$.

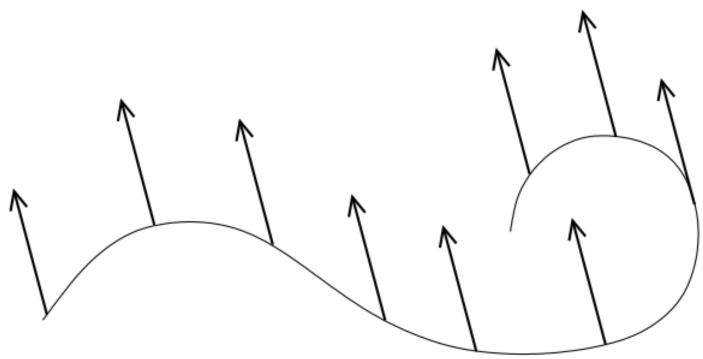

Figure 3: The parallel transport of a vector field along a curve [Lee, 2018]

- For linear ODEs, we have stronger results:

  Theorem 3.1 ***(Existence, Uniqueness, and Smoothness for Linear ODEs).*** *[Lee, 2018]*
  Let $I \subseteq R$ be an open interval, and for $1 \leq j, k \leq n$, let $A^k_j : I \to \mathbb{R}$ be smooth functions. For all $t_0 \in I$ and every initial vector $(c^1, \ldots, c^n) \in \mathbb{R}^n$, the ***linear initial value problem***

$$\dot{V}^k(t) = A^k_j(t)\, V^j(t), \qquad (8)$$
$$V^k(t_0) = c^k,$$

  has a ***unique smooth solution*** *on* ***all*** *of I, and the solution depends* ***smoothly*** *on $(t, c) \in I \times \mathbb{R}^n$.*

- **Theorem 3.2** ***(Existence and Uniqueness of Parallel Transport).***
  Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given a smooth curve $\gamma : I \to M$, $t_0 \in I$, and a vector $v \in T_{\gamma(t_0)}M$ or tensor $v \in T^{(k,l)}T_{\gamma(t_0)}M$, there exists a ***unique parallel vector or tensor field*** *V along γ such that $V(t_0) = v$.*

- **Remark** Compare to results for geodesic, there is no need for definition similar to *the maximal geodesic* since ***the solution for parallel transport is global on all*** *I.*

- **Remark** *The vector or tensor field* whose existence and uniqueness are proved in Theorem aboive is called ***the parallel transport of*** *v* ***along*** *γ.*

- **Definition** For each $t_0, t_1 \in I$, we define a map

$$P^{\gamma}_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M, \qquad (9)$$

  called ***the parallel transport map***, by setting

$$P^{\gamma}_{t_0,t_1}(v) = V(t_1), \quad \forall v \in T_{\gamma(t_0)}M$$

  where V *is the* ***parallel transport*** *of v along γ.*

  This map is ***linear***, because *the equation of parallelism is linear.* It is in fact an ***isomorphism***, because $P^{\gamma}_{t_1,t_0}$ is an ***inverse*** for it.

- **Definition** Given an ***admissible curve*** $\gamma : [a, b] \to M$, a map $V : [a, b] \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for each t is called ***a piecewise smooth vector field along*** γ if V

is continuous and there is an *admissible partition* $(a_0, \ldots, a_k)$ for γ such that V is smooth on each subinterval $[a_{i-1}, a_i]$. We will call any such partition ***an admissible partition for** V. A piecewise smooth vector field V along γ* is said to be ***parallel*** along γ if $D_t(V) = 0$ wherever V is smooth.

- **Corollary 3.3** ***(Parallel Transport Along Piecewise Smooth Curves).***
  Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given an admissible curve $\gamma : [a,b] \to M$ and a vector $v \in T_{\gamma(t_0)}M$ or tensor $v \in T^{(k,l)}T_{\gamma(t_0)}M$, there exists a ***unique piecewise smooth parallel*** *vector or tensor field V along γ such that $V(a) = v$, and V is smooth wherever γ is.*

- **Remark** (***Parallel Frames Along a Curve***)
  Given any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)}M$, we can ***parallel transport the vectors*** b_i ***along*** γ, thus obtaining an n-tuple of *parallel vector fields* $(E_1, \ldots, E_n)$ along γ. Because each parallel transport map is an *isomorphism*, ***the vectors*** $(E_i(t))$ ***form a basis for*** $T_{\gamma(t)}M$ ***at each point*** $\gamma(t)$. Such an n-tuple of vector fields along γ is called ***a parallel frame along*** γ.

  Every smooth (or piecewise smooth) vector field along γ can be expressed in terms of such a frame as
  $$V(t) = V^i(t)\, E_i(t),$$
  and then the properties of covariant derivatives along curves, together with the fact that the E_i's are parallel, imply
  $$D_t(V_t) = \dot{V}^i(t)\, E_i(t) \tag{10}$$
  wherever V and γ are smooth. This means that *a vector field is **parallel** along γ if and only* if ***its component functions with respect to the frame*** (E_i) ***are constants***.

- **Theorem 3.4** ***(Parallel Transport Determines Covariant Differentiation).*** *[Lee, 2018]*
  Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose $\gamma : I \to M$ is a smooth curve and V is a smooth vector field along γ. For each $t_0 \in I$,
  $$D_t V(t_0) = \lim_{\Delta t \to 0} \frac{P^{\gamma}_{(t_0+\Delta t), t_0}(V(t_0 + \Delta t)) - V(t_0)}{\Delta t} \tag{11}$$

- **Corollary 3.5** ***(Parallel Transport Determines the Connection).*** *[Lee, 2018]*
  Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose X and Y are smooth vector fields on M. For every $p \in M$,
  $$\nabla_X Y\big|_p = \lim_{t \to 0} \frac{P^{\gamma}_{t,0}(Y_{\gamma(t)}) - Y_p}{t}, \tag{12}$$
  where $\gamma : I \to M$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

- **Remark** See similarity between (12) and the definition of Lie derivatives:
  $$(\mathscr{L}_X Y)_p = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(Y_{\theta_t(p)}\right) - Y_p}{t},$$
  where θ is the ***flow of*** X in the neighborhood of p such that $\theta_0(p) = p$, $(\theta^{(p)})'(0) = X_p$.

- **Remark** A smooth vector or tensor field on M is said to be ***parallel*** *(with respect to ∇)* if it is *parallel along* ***every smooth curve*** *in M.*

- **Proposition 3.6** *Suppose M is a smooth manifold with or without boundary, ∇ is a connection in TM, and A is a* ***smooth vector or tensor field*** *on M. Then A is parallel on M if and only if $\nabla A \equiv 0$.*

- **Remark** It is always possible to extend a vector at a point to a parallel vector field along any given curve. However, it may not be possible in general to extend it to a ***parallel vector field*** on an open subset of the manifold. The impossibility of finding such extensions is intimately connected with the phenomenon of ***curvature***.

4 Pullback Connections

- **Remark** Like vector fields, connections in the tangent bundle **cannot** be either pushed forward or pulled back by arbitrary smooth maps.

- **Lemma 4.1** *(**Pullback Connections**). [Lee, 2018]*
Suppose M and $\widetilde{M}$ are smooth manifolds with or without boundary. If $\widetilde{\nabla}$ is a connection in $T\widetilde{M}$ and $\varphi : M \to \widetilde{M}$ is a ***diffeomorphism****, then the map $\varphi^*\widetilde{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by*

$$(\varphi^*\widetilde{\nabla})_X Y = (\varphi^{-1})_* \left(\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y)\right) \tag{13}$$

is a connection in TM, called ***the pullback of $\widetilde{\nabla}$ by φ****. Here $\varphi_* X, \varphi_* Y$ are pushforward of X and Y by φ. $(\varphi^{-1})_*(Z)$ is the pushforward of Z by φ^{-1}.*

- The next proposition shows that various important concepts defined in terms of connections – covariant derivatives along curves, parallel transport, and geodesics all behave as expected with respect to pullback connections.

  Proposition 4.2 *(**Properties of Pullback Connections**).*
  Suppose M and $\widetilde{M}$ are smooth manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a diffeomorphism. Let $\widetilde{\nabla}$ be a connection in $T\widetilde{M}$ and let $\nabla = \varphi^*\widetilde{\nabla}$ be the ***pullback connection*** *in TM. Suppose $\gamma : I \to M$ is a smooth curve.*

  1. *φ takes* ***covariant derivatives along curves to covariant derivatives along curves****: if V is a smooth vector field along γ, then*

     $$d\varphi \circ D_t(V) = \widetilde{D}_t(d\varphi \circ V),$$

     where D_t is covariant differentiation along γ with respect to ∇, and $\widetilde{D}_t$ is covariant differentiation along $\varphi \circ \gamma$ with respect to $\widetilde{\nabla}$.

  2. *φ takes* ***geodesics to geodesics****: if γ is a ∇-geodesic in M, then $\varphi \circ \gamma$ is a $\widetilde{\nabla}$-geodesic in $\widetilde{M}$.*

  3. *φ takes* ***parallel transport to parallel transport****: for every $t_0, t_1 \in I$,*

     $$d\varphi_{\gamma(t_1)} \circ P^{\gamma}_{t_0,t_1} = P^{\varphi\circ\gamma}_{t_0,t_1} \circ d\varphi_{\gamma(t_0)}.$$

References

John M Lee. *Introduction to Riemannian manifolds*, volume 176. Springer, 2018.