Lecture 0: Summary of Topology (Part 1)

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Nov. 6th., 2022

Contents

1 Set Theory

1.1 Set Operations and Logics

- **Definition** Given a set X, the collection of all subsets of X, denoted as 2^X, is defined as

$$2^X := \{E : E \subseteq X\}$$

- **Remark** The followings are basic operation on 2^X: For $A, B \in 2^X$,

  1. ***Inclusion***: $A \subseteq B$ if and only if $\forall x \in A$, $x \in B$.
  2. ***Union***: $A \cup B = \{x : x \in A \vee x \in B\}$.
  3. ***Intersection***: $A \cap B = \{x : x \in A \wedge x \in B\}$.
  4. ***Difference***: $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
  5. ***Complement***: $A^c = X \setminus A = \{x : x \in X \wedge x \notin A\}$.
  6. ***Symmetric Difference***: $A \Delta B = (A \setminus B) \cup (B \setminus A) = \{x \in X : x \notin A \vee x \notin B\}$.

  We have ***deMorgan's laws***:

$$\left(\bigcup_{a \in A} U_a\right)^c = \bigcap_{a \in A} U_a^c, \quad \left(\bigcap_{a \in A} U_a\right)^c = \bigcup_{a \in A} U_a^c$$

- **Remark** Note that the following equality is useful:

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

- The forms of logic statement using "if ... then":

  1. Orignal statement: "*If P then Q*", or "*Q holds **if** P holds*";

$$P \Rightarrow Q$$

  2. ***<u>Converse statement</u>***: "*If Q then P*", or "*Q holds **only if** P holds*";

$$Q \Rightarrow P$$

  3. ***<u>Contrapositive statement</u>***: "*If not Q then not P*", or "*P not holds **if** Q not holds*";

$$\neg Q \Rightarrow \neg P$$

  The contrapositive and the original statements are ***logically equivalent***.

  If it should happen that both the statement $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are *true*, we express this fact by the notation

$$P \Leftrightarrow Q$$

  "*P holds **if and only if** Q holds*""

1.2 Functions

- **Definition** A ***rule of assignment*** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate ***at most one ordered pair belonging to*** r. Thus, a subset r of $C \times D$ is *a rule of assignment* if

$$[(c, d) \in r \text{ and } (c, d') \in r] \Rightarrow [d = d'].$$

  Given a rule of assignment r, ***the domain*** of r is defined to be the *subset* of C consisting of *all first coordinates of elements* of r, and ***the image*** set of r is defined as the *subset* of D consisting of *all second coordinates of elements* of r.

  A ***function*** f is *a rule of assignment* r, together with a set B that *contains the image set* of r.

- **Definition** $f : X \to Y$ is a ***function*** if for each $x \in X$, there exists a unique $y = f(x) \in Y$. X is called the ***domain*** of f and Y is called the ***codomain or image*** of f. $f(X) = \{y \in Y : y = f(x)\}$ is called the ***range*** of f

  The ***pre-image*** of f is defined as

$$f^{-1}(E) = \{x \in X : f(x) \in E\}\,.$$

- **Definition** If $f : A \to B$ and if A_0 is a subset of A, we define the ***restriction*** of f to A_0 to be the function mapping A_0 into B whose rule is

$$\{(a, f(a)) : \; a \in A_0\}\,.$$

  It is denoted by $f|_{A_0}$, which is read ""*f restricted to A_0*."

- **Remark** The pre-image operation ***commutes*** with ***all basic set operations***:

$$\begin{aligned} A \subseteq B &\Rightarrow f^{-1}(A) \subseteq f^{-1}(B) \\ f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) &= \bigcup_{\alpha \in A} f^{-1}(E_\alpha) \\ f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) &= \bigcap_{\alpha \in A} f^{-1}(E_\alpha) \\ f^{-1}(A \setminus B) &= f^{-1}(A) \setminus f^{-1}(B) \\ f^{-1}(E^c) &= \left(f^{-1}(E)\right)^c \end{aligned}$$

- **Remark** The image operation ***commutes*** with only ***inclusion*** *and* ***union*** *operations*:

$$\begin{aligned} A \subseteq B &\Rightarrow f(A) \subseteq f(B) \\ f\left(\bigcup_{\alpha \in A} E_\alpha\right) &= \bigcup_{\alpha \in A} f(E_\alpha) \end{aligned}$$

  For the other operations:

$$\begin{aligned} f\left(\bigcap_{\alpha \in A} E_\alpha\right) &\subseteq \bigcap_{\alpha \in A} f(E_\alpha) \\ f(A \setminus B) &\supseteq f(A) \setminus f(B) \end{aligned}$$

- **Definition** A map $f : X \to Y$ is ***surjective, or, onto***, if for every $y \in Y$, there exists a $x \in X$ such that $y = f(x)$. In set theory notation:

$$f : X \to Y \text{ is surjective } \Leftrightarrow f^{-1}(Y) \subseteq X.$$

  A map $f : X \to Y$ is ***injective, or one-to-one***, if for every $x_1 \neq x_2 \in X$, their map $f(x_1) \neq f(x_2)$, or equivalently, $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

  If a map $f : X \to Y$ is both *surjective* and *injective*, we say f is a ***bijective***, or there exists an ***one-to-one correspondence*** between X and Y. Thus $Y = f(X)$.

- **Remark**

$$\begin{aligned}
f^{-1}(f(B)) &\supseteq B, \quad \forall B \subseteq X \\
f(f^{-1}(E)) &\subseteq E, \quad \forall E \subseteq Y \\
f : X \to Y \text{ is surjective } &\Leftrightarrow f^{-1}(Y) \subseteq X. \\
&\Rightarrow f(f^{-1}(E)) = E. \\
f : X \to Y \text{ is injective } &\Rightarrow f^{-1}(f(B)) = B \\
&\Rightarrow f\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f(E_\alpha) \\
&\Rightarrow f(A \setminus B) = f(A) \setminus f(B)
\end{aligned}$$

- **Proposition 1.1** *The following statements for composite functions are true:*

  1. *If f, g are both injective, then $g \circ f$ is injective.*
  2. *If f, g are both surjective, then $g \circ f$ is surjective.*
  3. *Every **injective** map $f : X \to Y$ can be writen as $f = \iota \circ f_R$ where $f_R : X \to f(X)$ is a **bijective** map and ι is the **inclusion map**.*
  4. *Every **surjective** map $f : X \to Y$ can be writen as $f = f_p \circ \pi$ where $\pi : X \to (X/\sim)$ is **a quotient map** (projection $x \mapsto [x]$) for the equivalent relation $x \sim y \Leftrightarrow f(x) = f(y)$ and $f_p : (X/\sim) \to Y$ is defined as $f_p([x]) = f(x)$ **constant** in each coset $[x]$.*
  5. *If $g \circ f$ is **injective**, then f is **injective**.*
  6. *If $g \circ f$ is **surjective**, then g is **surjective**.*

1.3 Equivalence Relation

- **Definition** A ***relation*** on a set A is a *subset* R of *the cartesian product* $A \times A$.

  If R is a relation on A, we use the notation xRy to mean the same thing as $(x, y) \in R$. We read it "x is in the relation R to y."

- **Remark** ***A rule of assignment*** r for a function $f : A \to A$ is also a *subset* of $A \times A$. But it is a subset of a *very **special** kind*: namely, one such that ***each element*** of A appears as the ***first coordinate*** of an element of r ***exactly once***. ***Any subset*** *of $A \times A$ is a relation on A.*

- **Definition** ***An equivalence relation*** on X is a relation R on X such that

  1. (***Reflexivity***): xRx for all $x \in X$;
  2. (***Symmetry***): xRy if and only if yRx for all $x, y \in X$;
  3. (***Transitivity***): xRy and yRz then xRz for all $x, y, z \in X$.

  We usually denote the equivalence relation R as $\sim$.

- **Definition** (***Equivalence Class***)
  The equivalence class of an element x is denoted as $[x] := \{y \in X : xRy\}$.

- **Lemma 1.2** *[Munkres, 2000]*
  Two equivalence classes E and E' are either ***disjoint*** *or* ***equal****.*

- **Definition** A ***partition*** of a set A is a collection of ***disjoint*** nonempty subsets of A whose ***union*** is all of A.

- **Remark** The set of equivalence classes provides ***a partition of the set*** X in that every $z \in X$ can must belong to *only one equivalence class* $[x]$. That is $[x] \cap [y] = \emptyset$ if $x \not\sim y$ and $X = \bigcup_{x \in X} [x]$.

- **Definition** *The set of all equivalence classes* of X by $\sim$, denoted $X/\sim \, := \{[x] : x \in X\}$, is ***the quotient set*** of X by $\sim$. $X = \bigcup_{C \in X/\sim} C$.

- **Remark** Since $x \sim y \Rightarrow y \in [x]$, we see that if $[x] \neq [y]$, then $x \not\sim y$, i.e. representative of different equivalence classes are not in the given relationship.

1.4 Order Relation

- **Definition** A relation C on a set A is called ***an order relation*** (or ***a simple order***, or ***a linear order***) if it has the following properties:

  1. (***Comparability***) For every x and y in A for which $x \neq y$, either xCy or yCx.
  2. (***Nonreflexivity***) For no x in A does the relation xCx hold.
  3. (***Transitivity***) If xCy and yCz, then xCz.

  We denote order relation as $>$ or $<$. We shall use the notation $x \leq y$ to stand for the statement "either $x < y$ or $x = y$"; and we shall use the notation $y > x$ to stand for the statement "$x < y$." We write $x < y < z$ to mean "$x < y$ and $y < z$"

- **Remark** If $x \neq y$, then $x < y$ and $y < x$ cannot hold simultaneously.

- **Remark** *The Comparability condition* means ***every two elements are comparable under simple order***. Without this condition, we have partial order $x \prec y$. Consider *the simple ordering* as along ***a chain graph***, while *the partial ordering* is along ***the general graphs***.

- **Definition** (***Order Type***)
  Suppose that A and B are two sets with order relations $<_A$, and $<_B$ respectively. We say that A and B have *the same* ***order type*** if there is a ***bijective*** correspondence between them

that ***preserves order***; that is, if there exists a bijective function $f : A \to B$ such that

$$x <_A y \Rightarrow f(x) <_B f(y)$$

- **Definition** (***Dictionary Order Relation***)
  Suppose that A and B are two sets with order relations $\prec_A$ and $\prec_B$ respectively. Define an order relation $<$ on $A \times B$ by defining

  $$a_1 \times b_1 < a_2 \times b_2$$

  if $a_1 <_A a_2$, ***or*** if $a_1 = a_2$ and $b_1 <_B b_2$. It is called ***the dictionary order relation*** on $A \times B$.

- **Definition** Suppose that A is a set ordered by the relation $<$. Let A_0 be a subset of A. We say that the element b is *the* ***largest element*** *of* A_0 if $b \in A_0$ and $x \leq b$ for every $x \in A_0$.

  Similarly, we say that a is *the* ***smallest element*** of A_0 if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$.

- **Remark** It is easy to see that a set has ***at most one*** largest element and *at most one* smallest element.

- **Definition** (***The Upper Bound and The Supremum of Subset***)
  We say that *the subset* A_0 *of* A *is* ***bounded above*** if there is *an element* b *of* A such that $x \leq b$ for every $x \in A_0$; the element $b \in A$ is called ***an upper bound for*** A_0.

  If *the set of all upper bounds* for A_0 has *a* ***smallest element***, that element is called ***the least upper bound***, or ***the supremum***, of A_0. It is denoted by $\sup A_0$, it may or may not belong to A_0. If it *does*, it is ***the largest element*** of A_0.

- **Definition** (***The Lower Bound and The Infimum of Subset***)
  Similarlly, we say that *the subset* A_0 *of* A *is* ***bounded below*** if there is *an element* a *of* A such that $a \leq x$ for every $x \in A_0$; the element $a \in A$ is called ***a lower bound for*** A_0.

  If *the set of all lower bounds* for A_0 has *a* ***largest element***, that element is called ***the greatest lower bound***, or ***the infimum***, of A_0. It is denoted by $\inf A_0$, it may or may not belong to A_0. If it *does*, it is ***the smallest element*** of A_0.

- **Definition** (***The Least Upper Bound Property*** *and* ***The Greatest Lower Bound Property***)
  An ordered set A is said to have ***the least upper bound property*** if *every nonempty* subset A_0 of A that is *bounded above* has *a least upper bound.*

  Analogously, the set A is said to have ***the greatest lower bound property*** if *every nonempty* subset A_0 of A that is *bounded below* has *a greatest lower bound.*

1.5 Cartesian Products

- **Definition** (***Indexed Family of Sets***)
  Let $\mathcal{A}$ be a nonempty collection of sets. ***An indexing function*** for $\mathcal{A}$ is a ***surjective*** function f from some set J, called ***the index set***, to $\mathcal{A}$. The *collection* $\mathcal{A}$, together with *the*

indexing function f, is called ***an indexed family of sets***. Given $\alpha \in J$, we shall denote the set $f(\alpha)$ by the symbol A_α. And we shall denote the indexed family itself by the symbol

$$\{A_\alpha\}_{\alpha \in J},$$

which is read "*the family of all A_α, as a ranges over J.*" Sometimes we write merely $\{A_\alpha\}$, if it is clear what the index set is.

- **Definition** (***Cartesian Product of Indexed Family of Sets***)
  Let m be a positive integer. Given a set X, we define an ***m-tuple of elements*** of X to be a function

  $$x : \{1, \ldots, m\} \to X.$$

  If X is an m-tuple, we often denote the value of x at i by *the symbol* x_i; rather than $x(i)$ and call it ***the i-th coordinate of*** x. And we often denote the function x itself by the symbol

  $$(x_1, \ldots, x_m).$$

  Now let $\{A_1, \ldots, A_m\}$ be *a family of sets indexed with the set* $\{1, \ldots, m\}$. Let $X = A_1 \cup \ldots \cup A_m$. We define ***the cartesian product*** *of this indexed family*, denoted by

  $$\prod_{i=1}^{m} A_i \quad \text{or} \quad A_1 \times \ldots \times A_m$$

  to be *the set of all m-tuples* $(x_1, \ldots, x_m)$ of elements of X such that $x_i \in A_i$ for each i.

- **Definition** (***Countable Cartesian Product of Indexed Family of Sets***)
  Given a set X, we define an ***ω-tuple of elements*** of X to be a function

  $$x : \mathbb{Z}_+ \to X;$$

  we also call such a function *a **sequence***, or *an **infinite sequence***, of elements of X. If x is *an ω-**tuple***, we often denote the value of x at i by x_i rather than $x(i)$, and call it ***the i-th coordinate*** of x. We denote x itself by the symbol

  $$(x_1, x_2, \ldots) \quad \text{or} \quad (x_n)_{n \in \mathbb{Z}_+}$$

  Now let $\{A_1, A_2, \ldots\}$ be a family of sets, *indexed with the positive integers*; let X be the union of the sets in this family. ***The cartesian product*** *of this indexed family of sets*, denoted by

  $$\prod_{i \in \mathbb{Z}_+} A_i \quad \text{or} \quad A_1 \times A_2 \times \ldots$$

  is defined to be the set of *all ω-tuples* $(x_1, x_2, \ldots)$ of elements of X such that $x_i \in A_i$ for each i.

1.6 Infinite Set and the Principle of Recursive Definition

- **Definition** See the following definitions

1. A set is said to be ***countably infinite*** if it admits a ***bijection*** with the set of *positive integers* $f : A \to \mathbb{Z}_+$, and

2. A set is said to be ***countable*** if it is *finite* or *countably infinite*.

3. A set that is not countable is said to be ***uncountable***.

- **Proposition 1.3** *Let B be a nonempty set. Then the following are equivalent:*

  1. *B is **countable**.*

  2. *There is a **surjective** function $f : \mathbb{Z}_+ \to B$.*

  3. *There is an **injective** function $g : B \to \mathbb{Z}_+$.*

- **Lemma 1.4** *If C is an infinite subset of $\mathbb{Z}_+$, then C is countably infinite.*

- **Principle 1.5** *(**Principle of Recursive Definition**). [Munkres, 2000]*
  *Let A be a set. Given a **formula** that defines $h(1)$ as a **unique** element of A, and for $i > 1$ defines $h(i)$ **uniquely** as an element of A in terms of the values of h **for positive integers less than** i, this formula determines **a unique function** $h : \mathbb{Z}_+ \to A$.*

- **Theorem 1.6** *(**Principle of Recursive Definition**). [Munkres, 2000]*
  *Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to **each function** f mapping a **nonempty section** of **the positive integers** into A, an eiement of A. Then there exists a **unique function***

$$h : \mathbb{Z}_+ \to A$$

  such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \ldots, (i-1)\}) \text{ for all } i > 1. \end{aligned} \tag{1}$$

  *The formula (2) is called a **recursion formula** for h. It specifies $h(1)$, and it expresses the value of h at $i > 1$ in terms of the values of h for positive integers less than i. A definition given by such a formula is called a **recursive definition**.*

- **Remark** (*Recusive Definition*)
  Given the *infinite subset* C of $\mathbb{Z}_+$, there is a *unique function* $h : \mathbb{Z}_+ \to C$ satisfying the formula:

$$\begin{aligned} h(1) &= \text{ smallest element of } C, \\ h(i) &= \text{ smallest element of } [C - h(\{1, \ldots, (i-1)\})] \text{ for all } i > 1. \end{aligned} \tag{2}$$

  The formula (2) is called a *recursion formula* for h; it defines the function h *in terms of itself*. A definition given by such a formula is called *a recursive definition*.

- **Corollary 1.7** *A subset of a countable set is countable.*

- **Corollary 1.8** *The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.*

- **Proposition 1.9** *A countable union of countable sets is countable.*

- **Proposition 1.10** *A finite product of countable sets is countable.*

- It is very tempting to assert that *countable products of countable sets should be countable*; but this assertion is in fact ***not true***:

  Theorem 1.11 *Let X denote the two element set $\{0, 1\}$. Then the set X^ω is uncountable.*

- **Theorem 1.12** *Let A be a set. There is* ***no injective map*** *$f : 2^A \to A$, and there is* ***no surjective map*** *$g : A \to 2^A$.*

- **Proposition 1.13** *Let A be a set. The following statements about A are equivalent:*

  1. *There exists an* ***injective*** *function $f : \mathbb{Z}_+ \to A$.*
  2. *There exists a* ***bijection*** *of A with a proper subset of itself.*
  3. *A is infinite.*

1.7 The Axioms of Choice

- **Principle 1.14** *(**Axiom of Choice**). [Munkres, 2000]*
  Given a collection $\mathscr{A}$ of ***disjoint*** *nonempty sets, there exists a set C consisting of* ***exactly one element from each element of $\mathscr{A}$****; that is, a set C such that C is contained in the* ***union of the elements*** *of $\mathcal{A}$, and for each $A \in \mathscr{A}$, the set $C \cap A$ contains* ***a single element****.*

- **Lemma 1.15** *(**Existence of a Choice Function**). [Munkres, 2000]*
  Given a collection $\mathscr{B}$ of nonempty sets (not necessarily disjoint), there exists a function
  $$c : \mathscr{B} \to \bigcup_{B \in \mathscr{B}} B$$
  such that $c(B)$ is an element of B, for each $B \in \mathscr{B}$.

  Remark The function c is called ***a choice function*** for the collection $\mathscr{B}$. The difference between this lemma and the axiom of choice is that in this lemma the sets of the collection $\mathscr{B}$ are not required to be disjoint.

- **Remark** The axiom of choice is used when someone construct *an infinite set* using *infinite number of arbitrary choices.*

- **Corollary 1.16** *If $\{A_\alpha\}_{\alpha \in J}$ is a* ***disjoint*** *collection of nonempty sets, there is a set $C \subset \bigcup_{\alpha \in J} A_\alpha$ such that $C \cap A_\alpha$ contains* ***precisely one element*** *for each $\alpha \in J$.*

1.8 Well-Ordering Theorem and Zorn's Lemma

- **Definition** (***Well-Ordered Set***)
  A set A with an order relation $<$ is said to be ***well-ordered*** if *every nonempty subset* of A has a ***smallest element***.

- **Proposition 1.17** *(**Finite Ordered Set is Well-Ordered**) [Munkres, 2000]*
  Every nonempty ***finite*** *ordered set has the order type of a section $\{1, \ldots, n\}$ of $\mathbb{Z}_+$, so it is* ***well-ordered****.*

- **Theorem 1.18** *(**Well-Ordering Theorem**). [Munkres, 2000]*
  If A is a set, there ***exists*** *an order relation on A that is a well-ordering.*

- **Remark** The proof of Well-Ordering Theorem is based on a construction involving *an infinite number of arbitrary choices*, that is, a construction involvingthe choice axiom.

- **Corollary 1.19** *There exists an uncountable well-ordered set.*

- **Definition** (***Strict Partial Order***)
  Given a set A, a relation $\prec$ on A is called a ***strict partial order*** on A if it has the following two properties;

  1. (***Nonreflexivity***) The relation $a \prec a$ never holds.

  2. (***Transitivity***) If $a \prec b$ and $b \prec c$, then $a \prec c$.

  Moreover, suppose that we define $a \preceq b$ either $a \prec b$ or $a = b$. Then the relation $\preceq$ is called ***a partial order*** on A.

- **Theorem 1.20** ***(The Maximum Principle).***
  Let A be a set; let $\prec$ be a strict partial order on A. Then there exists a ***maximal simply ordered subset*** *B of A.*

- **Definition** (***Upper Bound and Maximal Element for Strict Partial Order***)
  Let A be a set and let $\prec$ be a *strict partial order* on A. If B is a subset of A, ***an upper bound*** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

  A maximal element of A is an element m of A such that for ***no element*** *a of A does the relation $m \prec a$ hold.*

- **Remark** ***The upper bound*** of a set A is *not necessarily in A*, but ***the maximal element*** of A is in A.

- **Theorem 1.21** ***(Zorn's Lemma).*** *[Munkres, 2000]*
  Let A be a set that is ***strictly partially ordered****. If every* ***simply ordered subset*** *of A has an* ***upper bound in*** *A, then A has a* ***maximal element.***

1.9 Principles in Set Theory

- **Remark** We summarizes the main principles in the set theory chapter:

  1. ***The Axioms of Choice***: Given a collection $\mathscr{A}$ of ***disjoint*** nonempty sets, there exists a set C consisting of ***exactly one*** *element from each element of $\mathscr{A}$*. $\Rightarrow$ one can construct a infinite set with infinite number of arbitrary choices

  2. ***Well-Ordering Theorem***: Every set has a ***well-ordering relation*** so that every non-empty subset has a *smallest* element.

  3. ***The Maximum Principle***: Any ***strict partial ordered*** *set* has a ***maximal simply ordered subset***.

     Zorn's Lemma: For a *strictly partially ordered set A*, if ***every simply ordered subset*** has an ***upper bound*** in A, then A has ***a maximal element***.

  4. ***Principle of Recursive Definition***: To determine a ***unique*** function $h : \mathbb{Z}_{+} \to A$, one first define $h(1)$ ***uniquely*** as an element of A. Then for $i > 1$, one define h ***uniquely*** on $[1 : i]$ in terms of value of h on ***all positive integers less than*** i.

2 Topology

2.1 Topological Space

- **Definition** [Munkres, 2000]
Let X be a set. ***A topology*** on X is *a collection* $\mathscr{T}$ of *subsets* of X, called ***open subsets***, satisfying

  1. X and $\emptyset$ are *open*.
  2. The ***union*** of ***any family*** of open subsets is open.
  3. The ***intersection*** of *any* ***finite*** *family* of open subsets is open.

  A pair $(X, \mathscr{T})$ consisting of a set X together with a topology $\mathscr{T}$ on X is called ***a topological space***.

- **Example** (***Discrete and Trivial Topology***)
If X is any set, *the collection of* ***all subsets*** of X is a *topology* on X; it is called ***the discrete topology***.

  The collection consisting of X *and* 0 *only* is also a *topology* on X; we shall call it *the* ***indiscrete topology***, or ***the trivial topology***.

- **Example** (***The Finite Complement Topology***)
Let X be a set; let $\mathscr{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ *either is* ***finite*** *or is* ***all of*** X. Then $\mathscr{T}_f$ is a topology on X, called ***the finite complement topology***.

- **Definition** (***Comparable Topologies on the Same Set***)
Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set X. If $\mathscr{T}' \supseteq \mathscr{T}$, we say that $\mathscr{T}'$ is ***finer*** (or ***stronger***) than $\mathscr{T}$; if $\mathscr{T}'$ ***properly*** *contains* $\mathscr{T}$, we say that $\mathscr{T}'$ is ***strictly finer*** than $\mathscr{T}$.

  We also say that $\mathscr{T}$ is ***coarser*** (or ***weaker***) than $\mathscr{T}'$, or ***strictly coarser***, in these two respective situations. We say $\mathscr{T}$ is ***comparable*** with $\mathscr{T}'$ if either $\mathscr{T}' \subseteq \mathscr{T}$ or $\mathscr{T} \subseteq \mathscr{T}'$.

- **Definition** Suppose X is a topological space. A collection $\mathscr{B}$ of open subsets of X is said to be ***a basis*** for *the topology of* X (plural: ***bases***) if every open subset of X is the *union of some collection of elements* of $\mathscr{B}$.

  More generally, suppose X is merely a set, and $\mathscr{B}$ is a collection of *subsets* of X satisfying the following conditions:

  1. $X = \bigcup_{B \in \mathscr{B}} B$.
  2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B2$.

  Then *the collection of* ***all unions*** *of elements of* $\mathscr{B}$ is a *topology* $\mathscr{T}$ on X, called ***the topology*** $\mathscr{T}$ ***generated by*** $\mathscr{B}$, and $\mathscr{B}$ is a ***basis*** for this *topology*.

- **Remark** (***Basis Element in Each Neighborhood***)
By definition, a subset U of X is said to be ***open*** in X (that is, to be an element of $\mathscr{T}$) if for each $x \in U$, there exists a ***basis element*** $B \in \mathscr{B}$ such that $x \in B \subset U$. Note that each basis element is itself an element of $\mathscr{T}$.

- **Lemma 2.1** *Let X be a set; let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. Then $\mathscr{T}$ equals the* *collection of **all unions** of elements of $\mathscr{B}$.*

- **Remark** This lemma states that every open set U in X can be expressed as a *union* of *basis elements*. This expression for U is ***not***, however, ***unique***.

- **Lemma 2.2** *(**Obtaining Basis from Given Topology**). [Munkres, 2000]*
  Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then C is a basis for the topology of X.

- **Lemma 2.3** *(**Topology Comparison via Bases**). [Munkres, 2000]*
  Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively, on X. Then the following are equivalent:

  1. *$\mathscr{T}'$ is **finer** than $\mathscr{T}$.*
  2. *For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.*

- **Remark** The ***basis element*** of ***finer*** *topology* are always ***smaller*** than *the basis element* of *coaser topology* so the finer basis element should be included in coaser basis element.

- **Example** (***Topology*** *in* $\mathbb{R}$)
  If $\mathscr{B}$ is the collection of all ***open intervals*** in the real line,

  $$(a, b) = \{x : a < x < b\},$$

  the topology generated by $\mathscr{B}$ is called ***the standard topology*** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

  If $\mathscr{B}'$ is the collection of all *half-open* intervals of the form

  $$[a, b) = \{x : a \leq x < b\},$$

  where $a < b$, the topology generated by $\mathscr{B}'$ is called ***the lower limit topology*** *on* $\mathbb{R}$. When $\mathbb{R}$ is given *the lower limit topology*, we denote it by $\mathbb{R}_\ell$.

  Finally let K denote *the set of all numbers of the form* $1/n$, for $n \in \mathbb{Z}_+$,and let $\mathscr{B}''$ be the collection of *all open intervals* (a, b), along with *all sets of the form* $(a, b) \setminus K$. The topology generated by $\mathscr{B}$" will be called ***the K-topology*** *on* $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote $\mathbb{R}_K$

  Lemma 2.4 *The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are **strictly finer** than **the standard topology** on $\mathbb{R}$, but are not comparable with one another.*

- **Definition** (***Subbasis***)
  A subbasis $\mathscr{S}$ for a *topology* on X is a collection of subsets of X whose union equals X. The topology generated by the *subbasis* $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of ***all unions*** *of **finite intersections** of elements of* $\mathscr{S}$.

- **Remark** (***Basis from Subbasis***)
  For a *subbasis* $\mathscr{S}$, the collection $\mathscr{B}$ of ***all finite intersections*** of elements of $\mathscr{S}$ is a ***basis***,

- **Remark** ***Topology*** of a set X defines ***all local information*** we know regarding a set. For each point $x \in X$, it specifies what do we mean by a "***neighborhood***" U of x. Thus properties that relies on the ***local characteristic*** of the space likely depend on the topology of the space. Examples include *the continuity* of function, *the convergence properties* of sequence and *differential properties* of function.

2.2 The Order Topology

- **Example** (***Order Topology***)
  If X is ***a simply ordered set***, there is a *standard topology* for X, defined using the order relation. It is called ***the order topology***. The order topology is generated by *intervals*.

- **Definition** (***Intervals based on Simple Order Relation***)
  Suppose that X is a set having *a simple order relation* $<$. Given elements a and b of X such that $a < b$, there are *four subsets* of X that are called ***the intervals*** determined by a and b. They are the following :

$$\begin{aligned}(a, b) &= \{x : a < x < b\},\\ (a, b] &= \{x : a < x \le b\},\\ [a, b) &= \{x : a \le x < b\},\\ [a, b] &= \{x : a \le x \le b\}.\end{aligned}$$

  A set of the *first* type is called ***an open interval*** in X, a set of the *last* type is called ***a closed interval*** in X, and sets of *the second and third* types are called ***half-open intervals***.

- **Definition** Let X be a set with a ***simple order relation***; assume X has more than one element. Let $\mathscr{B}$ be the collection of all sets of the following types:

  1. ***All open intervals*** (a, b) in X.
  2. ***All intervals of the form*** $[a_0, b)$, where a_0 is the ***smallest element*** (if any) of X.
  3. ***All intervals of the form*** $(a, b_0]$, where b_0 is the ***largest element*** (if any) of X.

  The collection $\mathscr{B}$ is a basis for a topology on X, which is called ***the order topology***.

- **Definition** (***Rays***)
  If X is an ordered set, and a is an element of X, there are four subsets of X that are called ***the rays*** *determined by* a. They are the following:

$$\begin{aligned}(a, +\infty) &= \{x : x > a\},\\ (-\infty, a) &= \{x : x < a\},\\ [a, +\infty) &= \{x : x \ge a\},\\ (-\infty, a] &= \{x : x \le a\}.\end{aligned}$$

  Sets of the first two types are called ***open rays***, and sets of the last two types are called ***closed rays***.

- **Remark** The ***open rays*** in X are *open sets* in ***the order topology***. In fact, ***the open rays*** *form a* ***subbasis*** *for* ***the order topology*** *on* X.

2.3 The Product Topology

- We now generalize to topology of *arbitrary Cartesian products*.

  Definition (***J*-tuples**)
  Let J be an index set. Given a set X, we define a ***J*-tuple** *of elements* of X to be a function $x : J \to X$. If α is an element of J, we often denote ***the value of** X **at** α* by X_α rather than $x(\alpha)$; we call it ***the α-th coordinate*** of x. And we often *denote the function x itself* by the symbol

$$(x_\alpha)_{\alpha \in J}$$

  which is as close as we can come to a "*tuple notation*" for an arbitrary index set J. We denote ***the set of all J-tuples*** of elements of X by X^J.

- **Remark** Compare with the m-tuple $(x_n)_{n=1}^m$ and ω-tuple $(x_n)_{n \in \mathbb{Z}_+}$, the J-tuple $(x_\alpha)_{\alpha \in J}$ is not necessarily countable.

- **Definition** (***Arbitrary Cartesian Products***)
  Let $\{A_\alpha\}_{\alpha \in J}$ be an *indexed* family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. *The **cartesian product** of this indexed family*, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

  is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

  such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

- **Remark** The existance of just construction is due to *the Axioms of Choice* since J is an arbitrary set.

- **Remark** If $A_\alpha = X$ for all $\alpha \in J$, then we use the notation X^J to represent the cartestian product $\prod_{\alpha \in J} X$

- **Definition** (***Box Topology***)
  Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of ***topological spaces***. Let us take as a ***basis*** for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

  the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

  where U_α is ***open*** in X_α, for each $\alpha \in J$. The topology generated by this basis is called ***the box topology***.

- **Definition** (***Projection Mapping or Coordinate Projection***)
Let
$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$
be the function assigning to each element of the product space its β-th coordinate,
$$\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta;$$
it is called ***the projection mapping*** *associated with the index* β.

- **Definition** (***Product Topology***)
Let $\mathscr{S}_\beta$ denote the collection
$$\mathscr{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\right\},$$
and let $\mathscr{S}$ denote *the union of these collections*,
$$\mathscr{S} = \bigcup_{\beta\in J} \mathscr{S}_\beta.$$
The topology generated by *the* ***subbasis*** S is called ***the product topology***. In this topology $\prod_{\alpha\in J} X_\alpha$ is called ***a product space***.

- **Remark** (***Product Topology = Weak Topology by Coordinate Projections***)
The product topology on $\prod_{\alpha\in J} X_\alpha$ is ***the weak topology*** generated by *a family of projection mappings* $(\pi_\beta)_{\beta\in J}$. It is ***the coarest (weakest) topology*** *such that* $(\pi_\beta)_{\beta\in J}$ *are* ***continuous***.

  A typical element of the basis from *the product topology* is ***the finite intersection*** *of subbasis* where the *index is different*:
$$\pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \ldots \cap \pi_{\beta_n}^{-1}(V_{\beta_n})$$
Thus a ***neighborhood*** of x in ***the product topology*** is
$$N(x) = \{(x_\alpha)_{\alpha\in J} : x_{\beta_1} \in V_{\beta_1}, \ldots, x_{\beta_n} \in V_{\beta_n}\}$$
where there is ***no restriction*** for $\alpha \in \{\beta_1, \ldots, \beta_n\}$.

  Note that for ***the box topology***, a *neighborhood* of x is
$$N_b(x) = \{(x_\alpha)_{\alpha\in J} : x_\alpha \in U_\alpha,\ \forall \alpha \in J\} \subset N(x)$$
Thus ***the box topology*** is ***finer*** than ***the product topology***. Moreover, *for* ***finite product*** $\prod_{\alpha=1}^{n} X_\alpha$, the box topology and the product topology is the ***same***.

- **Proposition 2.5** *(**Comparison of the Box and Product Topologies**). [Munkres, 2000]* *The box topology on* $\prod_{\alpha\in J} X_\alpha$ *has as basis all sets of the form* $\prod_{\alpha\in J} U_\alpha$, *where* U_α *is* ***open*** *in* X_α ***for each*** α. *The product topology on* $\prod_{\alpha\in J} X_\alpha$ *has as basis all sets of the form* $\prod_{\alpha\in J} U_\alpha$, *where* U_α *is* ***open*** *in* X_α *for each* α *and* U_α ***equals*** X_α ***except for finitely many values*** *of a.*

- **Remark** Whenever we consider the product $\prod_{\alpha\in J} X_\alpha$, we shall ***assume*** it is given ***the product topology*** unless we specifically state otherwise.

- **Proposition 2.6** ***(Basis for Box and Product Topology)***
  Suppose the topology on each space X_α is given by a basis $\mathscr{B}_\alpha$. The collection of all sets of the form

$$\prod_\alpha B_\alpha$$

  where $B_\alpha \in \mathscr{B}_\alpha$ for ***each*** *α, will serve as a* ***basis*** *for* ***the box topology*** *on $\prod_{\alpha\in J} X_\alpha$.*

  The collection of all sets of the same form, where $B_\alpha \in \mathscr{B}_\alpha$ for ***finitely many indices*** *α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a* ***basis*** *for* ***the product topoiogy*** *$\prod_{\alpha\in J} X_\alpha$.*

- **Proposition 2.7** *Let A_α be a* ***subspace*** *of X_α, for each $\alpha \in J$. Then $\prod_\alpha A_\alpha$ is a* ***subspace*** *of $\prod_\alpha X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

- **Proposition 2.8** *If each space X_α is a* ***Hausdorff space****, then $\prod_\alpha X_\alpha$ is a* ***Hausdorff*** *space in both the box and product topologies.*

- **Proposition 2.9** *Let (X_α) be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod_\alpha X_\alpha$ is given either the product or the box topology, then*

$$\prod_\alpha \bar{A}_\alpha = \overline{\prod_\alpha A_\alpha}$$

- **Proposition 2.10** ***(Maps into Aribitrary Products)****. [Munkres, 2000]*
  Let $f : A \to \prod_\alpha X_\alpha$ is given by the equation

$$f(x) = (f_\alpha(x))_{\alpha\in J}$$

  where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_\alpha X_\alpha$ be the ***product topology****. Then the function f is* ***continuous*** *if and only if each function f_α is* ***continuous****.*

- **Remark** *The above proposition does* ***not hold*** *for* ***the box topology****.* See example in [Munkres, 2000].

2.4 The Subspace Topology

- **Definition** If $(X, \mathscr{T})$ is a topological space and $S \subseteq X$ is an arbitrary subset, we define ***the*** ***subspace topology*** on S (sometimes called *the* ***relative topology***) as

$$\mathscr{T}_S = \{S \cap U : U \in \mathscr{T}\}$$

  That is, a subset $U \subseteq S$ to be *open* in S *if and only* if there exists an *open* subset $V \subseteq X$ such that $U = V \cap S$. Any subset of X endowed with *the subspace topology* is said to be ***a subspace of*** X.

- **Lemma 2.11** ***(Basis of Subspace Topology)***
  If $\mathscr{B}$ is a basis for the topology of X then the collection

$$\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$$

  is a ***basis*** *for the subspace topology on $S \subset X$.*

- **Remark** (***Relative Openness***)
  When dealing with a space X and a *subspace* Y, one needs to be careful when one uses the term "*open set*". Does one mean *an element of* ***the topology of*** Y or *an element of* ***the topology of*** X ? We make the following definition : If Y is a subspace of X, we say that ***a set*** U ***is open in*** Y (or *open* ***relative*** *to* Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U ***is open in*** X if it belongs to the topology of X.

- **Lemma 2.12** *(Open Subspace)*
  Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

- **Proposition 2.13** *(**Product of Subspace** = **Subspace of Product**) [Munkres, 2000]*
  *If A is a subspace of X and B is a subspace of Y, then **the product topology** on $A \times B$ is the same as the topology $A \times B$ inherits as a **subspace** of $X \times Y$.*

- **Remark** (***Subspace Topology*** $\neq$ ***Order Topology on Subspace***)
  Now let X be an ordered set in *the order topology*, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. *However, the resulting order topology on Y* ***need not be the same*** *as the topology that Y inherits as a subspace of X.*

  Let $I = [0, 1]$. The *dictionary order* on $I \times I$ is just *the restriction to $I \times I$ of the dictionary order* on the plane $\mathbb{R} \times \mathbb{R}$. However, ***the dictionary order topology*** on $I \times I$ is ***not the same*** as ***the subspace topology*** on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

  The set $I \times I$ in *the dictionary order topology* will be called ***the ordered square***, and denoted by I_o^2.

- **Definition** Given an ordered set X, let us say that a subset Y of X is ***convex*** in X if for each pair of points $a < b$ of Y, ***the entire interval*** (a, b) of points of X *lies in* Y. Note that *intervals* and *rays* in X are *convex* in X.

- **Proposition 2.14** *(**Convex Subspace Preserve Order Topology**)[Munkres, 2000]*
  *Let X be an ordered set in the order topology; let Y be a subset of X that is **convex** in X. Then **the order topology on** Y is the same as the topology Y inherits as a **subspace** of X.*

2.5 Closure of Set and Limit Point

- **Definition** A subset A of a topological space X is said to be ***closed*** if the set $X \setminus A$ is *open*.

- **Proposition 2.15** *Let X be a topological space. Then the following conditions hold:*

  1. *$\emptyset$ and X are **closed**.*
  2. ***Arbitrary intersections*** *of* ***closed*** *sets are* ***closed***.
  3. ***Finite unions*** *of* ***closed*** *sets are* ***closed***.

- **Remark** When dealing with ***subspaces***, one needs to be careful in using the term "***closed set***." If Y is a subspace of X, we say that a set A is ***closed in*** Y if A is a subset of Y and if A is ***closed*** in ***the subspace topology*** of Y (that is, if $Y \setminus A$ is *open* in Y).

Proposition 2.16 ***(Closed Set in Subspace Topology)***
Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

- **Remark** A set A that is ***closed in*** the subspace Y may or may not ***not be closed in*** the larger space X.

  Proposition 2.17 *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

- **Definition** Given a subset A of a topological space X, ***the interior of*** A is defined as *the union of all open sets* ***contained*** in A, and ***the closure of*** A is defined as *the intersection of all closed sets* ***containing*** A.

  The interior of A is denoted by Int A or by $\mathring{A}$ and ***the closure of*** A is denoted by CI A or by $\bar{A}$. Obviously $\mathring{A}$ is *an open set* and $\bar{A}$ is *a closed set*; furthermore,

  $$\mathring{A} \subseteq A \subseteq \bar{A}.$$

  If A is ***open***, $A = \mathring{A}$; while if A is ***closed***, $A = \bar{A}$.

- **Proposition 2.18** ***(Closure in Subspace Topology)***
  Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

- **Proposition 2.19** ***(Characterization of Closure in terms of Basis)*** *[Munkres, 2000]*
  Let A be a subset of the topological space X.

  1. *Then $x \in \bar{A}$ if and only if every* ***open*** *set U* ***containing*** *x* ***intersects*** *A.*
  2. *Supposing the topology of X is given by a* ***basis****, then $x \in \bar{A}$ if and only if every basis element B* ***containing*** *x* ***intersects*** *A.*

- **Remark** We can say "*U is a* ***neighborhood*** *of x*" if "*U is an open set containing x*".

- **Definition** (***Limit Point***)
  If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point*** (or "***cluster point***," or "***point of accumulation***") of A if ***every neighborhood*** *of x* ***intersects*** *A in some point* ***other than*** *x itself.*

  Said differently, x is ***a limit point*** of A if it belongs to ***the closure of*** $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.

- **Theorem 2.20** ***(Decomposition of Closure)***
  Let A be a subset of the topological space X; let A' be the set of ***all limit points*** *of A. Then*

  $$\bar{A} = A \cup A'.$$

- **Corollary 2.21** *A subset of a topological space is* ***closed*** *if and only if it contains all its* ***limit points****.*

- **Definition** A topological space is called ***Hausdorff*** (or T_2) if and only if for all all x and y, $x \neq y$, there are ***open sets*** U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

- **Proposition 2.22** *Every* ***finite*** *point set in a* ***Hausdorff*** *space X is* ***closed****.*

- **Proposition 2.23** *(**Limit Point in T_1 Axiom**). [Munkres, 2000]*
  *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is **a limit point** of A if and only if every **neighborhood** of x contains **infinitely many points** of A.*

- **Proposition 2.24** *(**Limit Point is Unique in Hausdorff Space**). [Munkres, 2000]*
  *If X is a **Hausdorff space**, then a sequence of points of X **converges to at most one point** of X.*

2.6 Continuous Function

2.6.1 Definitions

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Remark** ***Continuity of a function*** depends *not only upon* ***the function** f **itself***, but also *on the **topologies** specified for **its domain** and **range***. If we wish to emphasize this fact, we can say that f *is **continuous relative to** specific topologies on X and Y*.

- **Remark** (***Prove Continuity via Basis***)
  If the topology of ***the range space*** Y is given by a ***basis*** $\mathscr{B}$, then to prove ***continuity of*** f it suffices to show that *the **inverse image** of every **basis element** is **open***: The arbitrary open set V of Y can be written as *a union of basis elements*
  $$V = \bigcup_{\alpha \in J} B_\alpha$$
  $$\Rightarrow f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$$

- **Remark** (***Prove Continuity via Subbasis***)
  If the topology on Y is given by ***a subbasis*** $\mathscr{S}$, to prove continuity of f it will even suffice to show that ***the inverse image** of each **subbasis** element is **open***: The arbitrary basis element B for Y can be written as ***a finite intersection*** $S_1 \cap \ldots \cap S_n$ of subbasis elements; it follows from the equation
  $$f^{-1}(B) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n)$$
  that the inverse image of every basis element is *open*.

- **Example** (***$\mathscr{F}$-Weak Topology using Continuity Only***)
  One can ***define a topology just*** based on ***the notion of continuity*** from a family of functions. Let $\mathscr{F}$ be a family of functions from a set S to a topological space $(X, \mathscr{T})$. The ***$\mathscr{F}$-weak** (or simply **weak**) **topology*** on S is *the **coarest topology*** for which ***all the functions** $f \in \mathscr{F}$ are **continuous***.

  The ***$\mathscr{F}$-weak** topology $\mathscr{T}$* is generated by ***subbasis*** $\mathscr{S}$ of the preimage sets $S = f^{-1}(U)$ where $f \in \mathscr{F}$ and $U \in \mathscr{T}$. And the basis of $\mathscr{T}$ is *the collection* of ***all finite intersections*** of preimages $f^{-1}(U)$ for $f \in \mathscr{F}$ and $U \in \mathscr{T}$.

- **Proposition 2.25** *(**Equivalent Definition of Continuity**) [Munkres, 2000]*
  Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

1. *f is **continuous**.*

2. *For every subset A of X, one has $f(\bar{A}) \subseteq \overline{f(A)}$.*

3. *For every **closed** set B of Y, the set $f^{-1}(B)$ is **closed** in X.*

4. *For **each** $x \in X$ and each **neighborhood** V of $f(x)$, there is a **neighborhood** U of X such that $f(U) \subseteq V$.*

*If the condition in (4) holds for the point x of X, we say that f **is continuous at the point** x.*

2.6.2 Homemorphism

- **Definition** (***Homemorphism***)
  A ***continuous bijective*** map $f : X \to Y$ with ***continuous inverse***

  $$f^{-1} : Y \to X$$

  is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Remark** (***Homemorphism is Topological Equivalence (Isomorphism)***)
  A ***homeomorphism*** $f : X \to Y$ gives us a *bijective correspondence* not only between X and Y but between *the collections of open sets of X and of Y*. As a result, *any **property** of X that is **entirely expressed in terms of the topology** of X* (that is, in terms of *the open sets* of X) ***yields***, via the correspondence f, *the **corresponding property** for the space Y*.

  Such a *property* of X is called *a **topological property** of X*. A *homemorphism* is an ***isomorphism*** between topological space, i.e. it ***preserves the topological structure*** during the transformation.

- **Definition** (***Topological Embedding***)
  If X and Y are topological spaces, a ***continuous injective*** map $f : X \to Y$ is called a ***topological embedding*** if it is a ***homeomorphism*** *onto* its image $f(X) \subseteq Y$ *in the subspace topology* (i.e. $f^{-1}|_{f(X)} : f(X) \to X$ is *continuous in* Y).

- **Remark** (***Smooth Embedding***)
  If X and Y are smooth manifoolds, ***a smooth embedding*** $f : X \to Y$ when it is a ***topological embedding***, and it is *smooth map* with *injective differential* df_x for all $x \in X$ (called a ***smooth immersion***).

2.6.3 Constructing Continuous Functions

- **Proposition 2.26** ***(Rules for Constructing Continuous Functions)**. [Munkres, 2000] Let X, Y, and Z be topological spaces.*

  1. ***(Constant Function)** If $f : X \to Y$ maps all of X into the **single point** y_0 of Y, then f is **continuous**.*

  2. ***(Inclusion)** If A is a subspace of X, the **inclusion function** $\iota : A \overset{X}{\hookrightarrow}$ is **continuous**.*

3. ***(Composites)*** *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous*

4. ***(Restricting the Domain)*** *If $f : X \to Y$ is* ***continuous****, and if A is a subspace of X, then* ***the restricted function*** *$f|_A : A \to Y$ is continuous.*

5. ***(Restricting or Expanding the Range)*** *Let $f : X \to Y$ be* ***continuous****. If Z is a* ***subspace*** *of Y containing the* ***image*** *set $f(X)$, then the function $g : X \to Z$ obtained by* ***restricting the range*** *of f is* ***continuous****. If Z is a space having Y as a* ***subspace****, then the function $h : X \to Z$ obtained by* ***expanding the range*** *of f is* ***continuous****.*

6. ***(Local Formulation of Continuity)*** *The map $f : X \to Y$ is* ***continuous*** *if X can be written as the* ***union of open sets*** *U_α such that $f|_{U_\alpha}$ is* ***continuous*** *for each α.*

- **Theorem 2.27** ***(The Pasting Lemma / Gluing Lemma)****. [Munkres, 2000]*
  Let $X = A \cup B$, where A and B are ***closed*** *in X. Let $f : A \to Y$ and $g : B \to Y$ be* ***continuous****. If $f(x) = g(x)$ for* ***every*** *$x \in A \cap B$, then f and g combine to give a* ***continuous function*** *$h : X \to Y$, defined by setting $h|_A = f$, and $h|_B = g$.*

- **Remark** The set A and B can be open sets, and the gluing lemma comes "***Local Formulation of Continuity***".

- **Remark** Notice the condition for *the gluing lemma*:
  1. The domain X is a union of two ***closed sets (or open sets)*** A and B
  2. The two functions f and g are ***continuous*** each of closed domain sets, respectively
  3. f and g ***agree on the intersection*** of two sets $A \cap B$.

- **Theorem 2.28** ***(Maps into Products)****. [Munkres, 2000]*
  Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

  Then f is ***continuous*** *if and only if the functions*

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

  are ***continuous****. The maps f_1 and f_2 are called* ***the coordinate functions*** *of f.*

- **Remark** There is no useful criterion for the *continuity* of a map $f : A \times B \to X$ whose ***domain is a product space***. One might conjecture that f *is continuous if it is continuous* "*in each variable separately,*" but ***this conjecture is not true***.

2.7 Metric Topology

2.7.1 Metric Topology and Metrizability

- **Definition** (***Metric Space***)
  A ***metric space*** is a set M and a real-valued function $d(\cdot, \cdot) : M \times M \to \mathbb{R}$ which satisfies:
  1. (***Non-Negativity***) $d(x, y) \geq 0$

2. (***Definiteness***) $d(x, y) = 0$ if and only if $x = y$
3. (***Symmetric***) $d(x, y) = d(y, x)$
4. (***Triangle Inequality***) $d(x, z) \leq d(x, y) + d(y, z)$

The function d is called a ***metric*** on M. The metric space M equipped with metric d is denoted as (M, d).

- **Definition** (***ϵ-Ball***)
  Given a metric d on X, the number $d(x, y)$ is often called *the **distance** between x and y in the metric d.* Given $\epsilon > 0$, consider the set

  $$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

  of all points y whose distance from x is less than ϵ. It is called ***the ϵ-ball centered at x***. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

- **Definition** (***Metric Topology***)
  If d is a *metric* on the set X, then *the collection of all ϵ-balls $B_d(x, \epsilon)$*, for $x \in X$ and $\epsilon > 0$, is a ***basis*** for a *topology* on X, called ***the metric topology*** *induced by d.*

- **Remark** (***The Triangle Inequality is Necessary for Basis***)
  The triangle inequality conditon is a necessary condition for the ϵ-balls to form a *basis*. It guarantees that for any $y \in B(x, \epsilon)$, there exists a neighborhood of y, $B(y, \delta)$ such that $B(y, \delta) \subset B(x, \epsilon)$.

  Definition (***Open Set in Metric Topology***)
  A set U is ***open*** *in the metric topology* induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

- **Remark** (***Metric Topology is Quantitative***)
  A ***metric*** provides a measurement on the ***closeness*** between two points. ***The metric topology*** generated by open balls thus provides a ***quantitative descripition of the neighborhood*** and it answers the question "*how close the neighborhood of x is* ?" On the other hand, ***the general topology*** answer this question using ***qualitative description*** via ***comparison*** with other neighborhoods via the *inclusion* operation $\subset$. Note that inclusion $\subset$ is ***partially ordered***, while the metric maps onto the real line where $<$ is ***simply ordered***.

  The study of **topology** is to acknowledge that *in many areas of research,* ***there might not exist a properly defined metric*** *in the set of interest.* On the other hand, *the study of* ***analysis*** mainly focus on *the space equipped with metric topology.*

- **Definition** (***Metrizability***)
  If X is a topological space, X is said to be ***metrizable*** if *there exists a metric d* on the set X that *induces the topology* of X. ***A metric space*** is *a metrizable space* X together with a specific metric d that *gives the topology of X*.

- **Remark** (***Metrizability as Inverse Problem***)
  Given a *metric d* on X, we can generate *a metric topology* using ϵ-balls as basis. ***Conversely, given a topology $\mathscr{T}$ on X, is $\mathscr{T}$ a metric topology for some unknown metric d ?***

  This is the question that ***the metrization theory*** is trying to answer.

- **Remark** (***Metrizability is Valuable***)
  Many of the spaces important for mathematics are metrizable, but some are not. ***Metrizability*** is always a highly desirable attribute for a space to possess, for the existence of a *metric* gives one *a valuable tool* for *proving theorems* about the space.

- **Definition** Let X be a metric space with metric d. A subset A of X is said to be ***bounded*** if there is some number M such that

$$d(a_1, a_2) \le M$$

  for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the ***diameter*** of A is defined to be the number

$$\text{diam } A = \sup\left\{d(a_1, a_2) : \forall a_1, a_2 \in A\right\}.$$

- **Remark** The boundedness property depends on specific metric topology, thus it is not a topological property.

  For instance, the following metric guarantee that every open set is bounded.

  Definition (***Standard Bounded Metric***)
  Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

  Then $\bar{d}$ is a *metric* that induces ***the same topology*** as d.

  The metric $\bar{d}$ is called ***the standard bounded metric*** *corresponding to d.*

- **Definition** (***Euclidean Metric and Square Metric***)
  Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the ***norm*** of x by the equation

$$\|x\|_2 = \left(x_1^2 + \ldots + x_n^2\right)^{1/2};$$

  and we define ***the euclidean metric*** d on $\mathbb{R}^n$ by the equation

$$d(x, y) = \|x - y\|_2 = \left((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right)^{1/2}.$$

  We define ***the square metric*** ρ by the equation

$$\rho(x, y) = \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\}.$$

- **Lemma 2.29** *Let d and d' be two metrics on the set X; let $\mathscr{T}$ and $\mathscr{T}'$ be the topologies they induce, respectively. Then $\mathscr{T}'$ is **finer** than $\mathscr{T}$ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon).$$

- **Proposition 2.30** ***(Product Topology = Euclidean Metric Topology in $\mathbb{R}^n$)***
  *The topologies on $\mathbb{R}^n$ induced by **the euclidean metric** d and **the square metric** ρ are the **same** as the **product topology** on $\mathbb{R}^n$.*

- **Remark** (***Finite Dimensional* Vector Space has Only One Meaningful Topology**)
  In ***finite dimensional*** vector space, ***all norms are equivalent***, and *all norm-induced metric topologies* are *the same*. For infinite dimenisonal space, these topologies are different.

- **Definition** (***Uniform Metric on Infinite Dimensional Space***)
  Given an index set J, and given points $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha\in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(x, y) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\},$$

  where $\bar{d}$ is ***the standard bounded metric*** on $\mathbb{R}$, It is easy to check that $\bar{\rho}$ is indeed a metric; it is called ***the uniform metric*** *on* $\mathbb{R}^J$, and the topology it induces is called ***the uniform topology***.

- **Proposition 2.31** *The **uniform topology** on $\mathbb{R}^J$ is **finer** than the product topology and **coarser** than the box topology; these three topologies are all different if J is infinite.*

$$\mathscr{T}_{\boldsymbol{product}} \subset \mathscr{T}_{\boldsymbol{uniform}} \subset \mathscr{T}_{\boldsymbol{box}}$$

- **Theorem 2.32** ***(Countable Product Space with Product Topology is Metrizable).*** *[Munkres, 2000]*
  *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the **standard bounded metric** on $\mathbb{R}$. If x and y are two points of W", define*

$$D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

  *Then D is a metric that induces **the product topology** on $\mathbb{R}^\omega$.*

2.7.2 Constructing Continuous Functions on Metric Space

- The followings are some important facts about the metric topology:
  1. **Proposition 2.33** *Every **metric space** (X, d) is **Hausdorff**.*
  2. **Proposition 2.34** *Every **subspace** of metric space (X, d) is a metric space. That is, if A is a subspace of the topological space X and d is a metric for X, then the restriction of d on $A \times A$ is a metric for the topology of A.*

- **Theorem 2.35** *(**ϵ-δ Definition of Continuous Function in Metric Space**). [Munkres, 2000]*
  *Lei $f : X \to Y$; let X and Y be **metrizable** with metrics d_x and d_y, respectively. Then **continuity** of f is **equivalent** to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) < \epsilon.$$

- **Remark** To use ϵ-δ definition, *both **domain** and **codomain*** need to be ***metrizable***.

- **Lemma 2.36** ***(The Sequence Lemma).*** *[Munkres, 2000]*
  *Let X be a topologicaJ space; let $A \subseteq X$. If there is a sequence of points of A **converging** to x, then $x \in \bar{A}$; the **converse** holds if X is **metrizable**.*

- **Proposition 2.37** *Let $f : X \to Y$. If the function f is **continuous**, then for every **convergent** sequence $x_n \to x$ in X, the sequence $f(x_n)$ **converges** to $f(x)$. The **converse** holds if X is **metrizable**.*

- **Remark** To show the converse part, i.e. "*if $x_n \to x \Rightarrow f(x_n) \to f(x)$ then f is continuous*", we just need the space X to be ***first countable***. That is, at each point x, there is ***a countable collection*** $(U_n)_{n\in\mathbb{Z}_+}$ of ***neighborhoods*** of x such that any neighborhood U of x *contains* at least one of the sets U_n.

- **Proposition 2.38** *(**Arithmetic Operations of Continuous Functions**).*
  If X is a topological space, and if $f, g : X \to Y$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

- **Definition** (***Uniform Convergence***)
  Let $f_n : X \to Y$ be a sequence of functions from the ***set*** X to ***the metric space*** Y. Let d be the metric for Y. We say that the sequence (f_n) ***converges uniformly*** to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

  $$d(f_n(x), f(x)) < \epsilon$$

  for all $n > N$ and ***all** x **in** X*.

- **Theorem 2.39** *(**Uniform Limit Theorem**). [Munkres, 2000]*
  *Let $f_n : X \to Y$ be a sequence of **continuous** functions from the **topological** space X to the **metric space** Y. If (f_n) converges **uniformly** to f, then f is **continuous**.*

- **Remark** (***Uniform Convergence = Convergence of Functions in Uniform Metric***)
  A sequence of functions $f_n : X \to \mathbb{R}$ ***converges uniformly*** to $f : X \to \mathbb{R}$ ***if and only if*** the sequence (f_n) converges to f when they are considered as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$, where $\mathbb{R}^X$ is the space of all real-valued functions on X and $\bar{\rho}$ is ***the unform metric*** defined before.

- **Example** The space of all countable infinite sequences $\mathbb{R}^\omega$ in the ***box topology*** is ***not metrizable***. (on the other hand, it is metrizable *in product topology*).

- **Example** The ***countable product space*** $\mathbb{R}^\omega$ in the ***box topology*** is ***not metrizable***. (on the other hand, it is metrizable *in product topology*).

- **Example** An ***uncountable product*** of $\mathbb{R}$ with itself is ***not metrizable***.

2.8 The Quotient Topology

2.8.1 Definitions and Properties

- **Remark** (***Quotient Topology as "Cut-and-Paste"***)
  One motivation of ***quotient topology*** comes from geometry, where one often has occasion to use "***cut-and-paste***" techniques to construct such geometric objects as surfaces.:

  1. The ***torus*** (surface of a doughnut), for example, can be constructed by taking a ***rectangle*** and "*pasting*" its edges together appropriately

  2. The ***sphere*** (surface of a ball) can be constructed by taking a ***disc*** and *collapsing* its entire boundary to a single point;

- **Definition** (***Quotient Map***)
  Let X and Y be topological spaces; let $\pi : X \to Y$ be a ***surjective map***. The map π is said to be ***a quotient map*** provided a subset U of Y is ***open*** in Y ***if and only if*** $\pi^{-1}(U)$ is ***open*** in X.

- **Remark** (***Quotient Map = Strong Continuity***)
  The condition of quotient map is ***stronger*** than continuity (it is called ***strong continuity*** in some literature).

$$\begin{aligned} \text{continuity}: &\quad U \text{ is open in } Y \Rightarrow \pi^{-1}(U) \text{ is open in } X \\ \text{open map}: &\quad \pi(V) \text{ is open in } Y \Leftarrow V \text{ is open in } X \\ \text{quotient map}: &\quad U \text{ is open in } Y \Leftrightarrow \pi^{-1}(U) \text{ is open in } X \end{aligned}$$

  An equivalent condition is to require that a subset A of K be ***closed*** in Y *if and only if* $\pi^{-1}(A)$ is ***closed*** in X. Equivalence of the two conditions follows from equation

$$\pi^{-1}(Y \setminus B) = X \setminus \pi^{-1}(B).$$

- **Definition** (***Saturated Set and Fiber***)
  If $\pi : X \to Y$ is a ***surjective*** *map*, a subset $U \subseteq X$ is said to be ***saturated*** with respect to π if U contains every set $\pi^{-1}(\{y\})$ that it ***intersects***. Thus U is ***saturated*** if it equals to the ***entire preimage*** of its ***image***: $U = \pi^{-1}(\pi(U))$.

  Given $y \in Y$, the ***fiber*** of π over y is the set $\pi^{-1}(\{y\})$.

- **Definition** (***Quotient Map via Saturated Set***)
  A surjective map $\pi : X \to Y$ is a ***quotient map*** if π is ***continuous*** and π maps ***saturated open sets*** of X to ***open sets*** of Y (or *saturated closed sets* of X to *closed sets* of Y).

- **Definition** (***Open Map*** *and* ***Closed Map***)
  A map $f : X \to Y$ (continuous or not) is said to be an ***open map*** if for every *open* subset $U \subseteq X$, the image set $f(U)$ is *open* in Y, and a ***closed map*** if for every *closed* subset $K \subseteq X$, the image $f(K)$ is *closed* in Y .

- **Proposition 2.40** *If $\pi : X \to Y$ is a* ***surjective continuous*** *map that is* ***either open*** *or* ***closed****, then π is a* ***quotient*** *map.*

  Remark There are *quotient maps* that are ***neither*** *open* nor *closed*. See Exercise in [Munkres, 2000].

- **Example** (***Coordinate Projection as Quotient Map***)
  Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be ***projection onto the first coordinate***; then π_1 is *continuous* and *surjective*. Furthermore, π_1 is an ***open map***. For if $U \times V$ is a nonempty *basis element* for $\mathbb{R} \times \mathbb{R}$, then $\pi_1(U \times V) = U$ is *open* in $\mathbb{R}$; it follows that π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. That is, π_1 is a ***quotient map***.

  However, π_1 is ***not*** *a* ***closed map***. The subset

$$C = \{(x, y) : x \cdot y = 1\}$$

  of $\mathbb{R} \times \mathbb{R}$ is *closed*, but $\pi_1(C) = \mathbb{R} \setminus \{0\}$, which is *not closed* in $\mathbb{R}$.

- **Definition** (***Quotient Topology***)
  If X is a space and A is a set and if $\pi : X \to A$ is a **surjective** map, then there exists **exactly one topology** $\mathscr{T}$ on A relative to which π is a quotient map; it is called ***the quotient topology*** induced by π.

- **Definition** (***Quotient Space***)
  Suppose X is a topological space and $\sim$ is *an equivalence relation* on X. Let $X/\sim$ denote ***the set of equivalence classes*** in X, and let $\pi : X \to X/\sim$ be the ***natural projection*** sending each *point* to its *equivalence class*. Endowed with ***the quotient topology*** determined by π, the space $X/\sim$ is called ***the quotient space*** (or *identification space*) of X determined by π.

  Definition [Munkres, 2000]
  Let X be a topological space, and let X^* be a ***partition*** of X into *disjoint subsets whose union is X*. Let $\pi : X \to X^*$ be the ***surjective*** map that carries each point of X to the element of X^* *containing* it. In ***the quotient topology*** induced by π, the space X^* is called a ***quotient space*** *of X*.

- **Remark** (***Understanding Topology of Quotient Space***)
  We can describe the topology of $X/\sim$ in another way. A *subset U* of $X/\sim$ is ***a collection of equivalence classes***, and the set $\pi^{-1}(U)$ is just ***the union*** *of* ***the equivalence classes belonging to*** *U*.

  Thus the typical ***open set*** of $X/\sim$ is ***a collection*** *of* ***equivalence classes*** whose ***union*** is ***an open set*** of X.

  $$V \text{ open in } X/\sim \quad \Leftrightarrow \quad U := \pi^{-1}(V) = \bigcup_{[y]\in V} [y] \text{ open in } X$$

- **Remark** (***Geometrical Understanding of Quotient Space***)
  A set of points in X in the ***same equivalence class*** *$[y]$ is considered as* ***one point*** *in quotient space $X/\sim$.* Geometrically, it is seen as ***collapsing a set of points*** *into* ***one*** if this set of points are in a connected neighborhood, or, it is seen as ***cut-and paste*** *a set of points in boundary* with another set of points in boundary.

  In general, if a property is considered as irrelevant for the problem of concern, we can ***identify*** a set of instances that share this property as one instance, which forms the quotient space.

- **Proposition 2.41** ***(Restricting Quotient Map to Subspace)***. *[Munkres, 2000]*
  Let $\pi : X \to Y$ be a ***quotient map;*** *let A be a subspace of X that is* ***saturated*** *with respect to π; let $q : A \to \pi(A)$ be the map obtained by restricting π.*

  1. *If A is either* ***open*** *or* ***closed*** *in X, then q is a* ***quotient map.***
  2. *If π is either an* ***open map*** *or a* ***closed map****, then q is a* ***quotient map****.*

- **Remark** (***Composite of Quotient Maps is Quotient Map***).
  Composites of maps behave nicely; it is easy to check that the *composite of two quotient maps is a quotient map*; this fact follows from the equation

  $$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U).$$

- **Remark** (***Product of Quotient Maps Need Not to be Quotient Map***).

On the other hand, products of maps do not behave well; *the cartesian product of two quotient maps* ***need not*** *be a quotient map.*

One needs further conditions on either the maps or the spaces in order for this statement to be true.

1. One such, a condition on the spaces, is called ***local compactness***; we shall study it later.
2. Another, a condition on the *maps*, is the condition that ***both*** *the maps p and q be* ***open maps***. In that case, it is easy to see that $p \times q$ is also ***an open map***, so it is a quotient map.

- **Remark** (***Quotient Space of Haudorff Space Need Not to be Hausdorff***)
The Hausdorff condition does not behave well; *even if X is Hausdorff, there is no reason that the quotient space $X/\sim$ needs to be Hausdorff.* There is a simple condition for $X/\sim$ to satisfy the T_1 axiom; one simply requires that ***each element*** *of the partition $X/\sim$ be a* ***closed subset*** *of X*. Conditions that will ensure $X/\sim$ is *Hausdorff* are harder to find.

2.8.2 Constructing Continuous Function on Quotient Space

- We want to know if $f : (X/\sim) \to Z$ is *continuous function.*
- **Theorem 2.42** ***(Passing Continuity to the Quotient).*** *[Munkres, 2000]*
Let $\pi : X \to Y$ be a ***quotient map****. Let Z be a space and let $g : X \to Z$ be a map that is* ***constant*** *on* ***each fiber*** *$\pi^{-1}(\{y\})$, for $y \in Y$. Then g* ***induces*** *a map $f : Y \to Z$ such that $f \circ \pi = g$. The induced map f is* ***continuous*** *if and only if g is* ***continuous****: f is a* ***quotient map*** *if and only if g is a* ***quotient map****.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z. \end{tikzcd}$$

- **Corollary 2.43** *Let $g : X \to Z$ be a* ***surjective continuous*** *map. Let $X/\sim$ be the following collection of subsets of X:*

$$X/\sim := \left\{ g^{-1}(\{z\}) : z \in Z \right\},$$

Given $X/\sim$ the ***quotient topology****,*

1. *The map g induces a* ***bijective continuous map*** *$f : (X/\sim) \to Z$, which is a* ***homeomorphism if and only if*** *g is a* ***quotient map****.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ (X/\sim) \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

2. *If Z is* ***Hausdorff****, so is $X/\sim$.*

2.9 Topological Groups

- **Definition** (***Topological Group***)
  A ***topological group*** G is a ***group*** that is also a ***topological space*** satisfying ***the*** T_1 ***axiom***, such that ***the multiplication map*** $m : G \times G \to G$ and ***inversion map*** $i : G \to G$, given by
  $$m(x, y) = xy, \quad i(x) = x^{-1}.$$
  are both ***continuous maps***. Here, $G \times G$ is viewed as a *topological space* by using *the product topology.*

- **Example** (***Common Topological Groups***)
  The following are topological groups:

  1. $(\mathbb{Z}, +)$
  2. $(\mathbb{R}, +)$
  3. $(\mathbb{R}_+, \cdot)$
  4. $(\mathbb{S}^1, \cdot)$, where we take $\mathbb{S}^1$ to be *the space of* ***all complex numbers*** z for which $|z| = 1$

- **Example** (***Lie Groups***)

  Definition (***Lie Group***) [Lee, 2003.]
  A ***Lie group*** is a ***smooth manifold*** $\mathcal{G}$ (without boundary) that is also *a* ***group*** in the *algebraic sense*, with the property that *the multiplication map* $m : G \times G \to G$ and *inversion map* $i : G \to G$, given by
  $$m(g, h) = gh, \quad i(g) = g^{-1}.$$
  are both ***smooth***.

  A Lie group is a topological group. The followings are all ***Lie groups***:

  1. ***The general linear group*** $GL(n, \mathbb{R})$ is the set of ***invertible*** $n \times n$ *matrices* with real entries.
     $$GL(n, \mathbb{R}) \equiv \left\{ \boldsymbol{A} \in \mathbb{R}^{n \times n} : \; \det(\boldsymbol{A}) \neq 0 \right\}.$$
     It is a *group* under ***matrix multiplication***, and it is an *open submanifold* of the vector space $M(n, \mathbb{R}) \simeq \mathbb{R}^{n \times n}$. *Multiplication is smooth* because *the matrix entries* of a product matrix AB are *polynomials* in the entries of A and B. Inversion is *smooth* by *Cramer's rule.*

  2. Let $GL_+(n, \mathbb{R})$ denote the subset of $GL(n, \mathbb{R})$ consisting of matrices with ***positive determinant***. Because $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = (\det(A))^{-1}$, it is a subgroup of $GL(n, \mathbb{R})$; and because it is the *preimage* of $(0, +\infty)$ under *the continuous determinant function*, it is an open subset of $GL(n, \mathbb{R})$ and therefore an n^2-dimensional manifold. The *group operations* are the restrictions of those of $GL(n, \mathbb{R})$, so they are smooth. Thus $GL_+(n, \mathbb{R})$ is a *Lie group.*

  3. The ***special linear group*** $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ consisting of matrices with a ***determinant of*** 1.
     $$SL(n, \mathbb{R}) \equiv \left\{ \boldsymbol{A} \in \mathbb{R}^{n \times n} : \; \det(\boldsymbol{A}) = 1 \right\}.$$

It is a *Lie group* with dimension $\dim SL(n,\mathbb{R}) = n^2 - 1$.

4. The ***orthogonal group*** *of dimension* n, denoted $\mathcal{O}(n)$, is the group of ***distance-preserving transformations*** of a Euclidean space of dimension n that preserve a fixed point, where the group operation is given by composing transformations. Also, $(\mathcal{O}(n), \cdot)$ is the group of $n \times n$ ***orthogonal matrices***, where the group operation $(\cdot)$ is given by matrix multiplication, and an orthogonal matrix is a real matrix whose inverse equals its transpose. *The orthogonal group is a Lie group* with *dimension* $n(n-1)/2$.

$$\mathcal{O}(n) \equiv \left\{ \boldsymbol{Q} \in GL(n,\mathbb{R}) : \; \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}_n \right\}.$$

5. The ***special orthogonal group*** $\mathcal{SO}(n)$ is the group of *the* ***orthogonal matrices*** *of* ***determinant*** 1. This group is also called the ***rotation group***

$$\mathcal{SO}(n) \equiv \left\{ \boldsymbol{Q} \in \mathcal{O}(n) : \; \det(\boldsymbol{Q}) = 1 \right\}.$$

   It is an open subgroup of $\mathcal{O}(n)$, which is a *Lie group* of dimension $\dim \mathcal{SO}(n) = \dim \mathcal{O}(n) = n(n-1)/2$.

6. ***The complex general linear group*** $GL(n,\mathbb{C})$ is the group of *invertible complex* $n \times n$ *matrices* under matrix multiplication. It is an open submanifold of $M(n,\mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold, and it is a *Lie group* because *matrix products and inverses* are *smooth functions* of the real and imaginary parts of the matrix entries.

7. If V is any *real or complex vector space*, $GL(V)$ denotes *the set of* ***invertible linear maps*** from V to itself. It is a *group under composition*. If V has ***finite dimension*** n, any basis for V determines an *isomorphism* of $GL(V)$ with $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$, so $GL(V)$ is a *Lie group*.

8. $(\mathbb{Z}, +)$

9. $(\mathbb{R}, +)$

10. The set $\mathbb{R}^*$ of *nonzero real numbers* is a 1-***dimensional Lie group*** *under multiplication*. (In fact, it is exactly $GL(1,\mathbb{R})$ if we identify a 1×1 matrix with the corresponding real number.) The subset $\mathbb{R}_+$ of ***positive real numbers*** is an *open subgroup*, and is thus itself a 1-*dimensional Lie group*.

11. The set $\mathbb{C}^*$ of ***nonzero complex numbers*** is a 2-*dimensional Lie group* under complex multiplication, which can be identified with $GL(1,\mathbb{C})$.

12. The ***circle*** $\mathbb{S}^1 \subset \mathbb{C}^*$ is a smooth manifold and a group under complex multiplication. With appropriate ***angle functions*** as *local coordinates* on open subsets of $\mathbb{S}^1$, *multiplication and inversion* have the *smooth coordinate expressions* $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ and $\theta \mapsto -theta$, and therefore $\mathbb{S}^1$ is a Lie group, called ***the circle group***.

13. The ***n-torus*** $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is an n-***dimensional abelian Lie group***.

- **Example** (***Discrete Group***)
  Any ***group*** with ***the discrete topology*** is a *topological group*, called a ***discrete group***. If in addition the group is *finite* or *countably infinite*, then it is a ***zero-dimensional Lie group***, called a ***discrete Lie group***.

- **Definition** (***Homogeneous Space***)
  A topological space G is a <u>***homogeneous space***</u> if for every pair $x, y \in G$, there exists a *homemorphism* $f : G \to G$ such that $f(x) = y$.

- **Proposition 2.44** *(**Topological Groups Are Homogeneous**)*
  *Every **topological group** is a **homogeneous** space; in particular, define map $h_\alpha : G \to G$ as $h_\alpha(x) = \alpha \cdot x$ and $g_\alpha : G \to G$ as $g_\alpha(x) = x \cdot \alpha$, for $\alpha \in G$. Then h_α, g_α are **homemorphisms**.*

- **Proposition 2.45** *(**Subgroup of Topological Group**)*
  *Let H be a **subspace** of topological group G. If H is also a **subgroup** of G, then both H and its closure $\bar{H}$ are **topological groups**.*

- **Definition** (***Left Coset*** *and* ***Right Coset***)
  For $H \subset G$ as the *subgroup* of G, define the <u>***left coset***</u> as $xH = \{x \cdot h : h \in H\}$. Similarly, define <u>***the right coset***</u> as $Hx = \{h \cdot x : h \in H\}$

- **Definition** (***Quotient Group***)
  The collection of *left cosets* defines a <u>***quotient group***</u> $G/H = \{xH \,|\, x \in G\}$ with *the group operation $xH \cdot yH = (x \cdot y)H$.*

- **Proposition 2.46** *Let G be a topological group.*

  1. *If $\alpha \in G$, the map $f_\alpha : x \mapsto \alpha \cdot x$ induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$. Thus G/H is a **homogeneous space**.*

  2. *If H is a **closed** set in the topology of G, then **one-point sets** are **closed** in G/H.*

  3. *The **quotient map** $\pi : G \to G/H$ is **open**.*

  4. *If H is **closed** in the topology of G and is a **normal subgroup** of G, then the (left) quotient group G/H under quotient topology is a **topological group**.*

  5. *If H is **compact subgroup** of G and $\pi : G \to G/H$ is closed, then G/H is **compact**.*

- **Example** ($GL(n, \mathbb{R})/SL(n, \mathbb{R}) \simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\}$).
  Given the *generalized linear group $GL(n, \mathbb{R})$, the special linear group $SL(n, \mathbb{R})$* is a subgroup of $GL(n, \mathbb{R})$. The *quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$* $\simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

- **Example** ($\mathcal{O}(n)/\mathcal{SO}(n) \simeq \mathbb{Z}_2 = \{-1, 1\} = \mathbb{Z}/2\mathbb{Z}$).
  The quotient group of orthogonal group $\mathcal{O}(n)$ over the special orthorognal group $\mathcal{SO}(n) = \{\boldsymbol{Q} \in \mathcal{O}(n) : \det(\boldsymbol{Q}) = 1\}$ is homemorphic to $\mathbb{Z}_2 = \{-1, 1\}$.

- **Example** ($\mathcal{O}(n)/\mathcal{O}(n-1) \simeq \mathbb{S}^{n-1}$).
  The quotient group of n-dimensional orthogonal group $\mathcal{O}(n)$ over $(n-1)$-dimensional orthogonal group $\mathcal{O}(n-1)$ is homemorphic to $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$.

- **Definition** (***Topological Group Action***)
  An <u>***action***</u> of a ***topological group*** G on a ***topological space*** X is a ***continuous map*** $\phi : G \times X \to X$ such that for $g(x) := \phi(g, x)$,

  $$\begin{aligned} (g_1 \cdot g_2)(x) &= g_1(g_2(x)), && \forall g_1, g_2 \in G, x \in X \\ 1_G(x) &= x, && \forall x \in X \end{aligned}$$

  where 1_G is the unit element of group G. Together with the group action, X is called a <u>***G-space***</u>.

- **Remark** The map $x \mapsto g(x)$ is a ***continuous map*** on X for each $g \in G$. This map has ***inverse map*** $x \mapsto g^{-1}(x)$ which is continuous as well. Thus the map $x \mapsto g(x)$ is a ***homemorphism***.

- **Example** *The topological group* $\mathcal{O}(n)$ *acts* on $\mathbb{R}^n$ is the rotation transformation of vectors in $\mathbb{R}^n$. Similarly, $\mathcal{O}(n)$ *acts* on $\mathbb{S}^1$ is the rotation of circle $\mathbb{S}^1$.

- **Definition** (***Orbit under Topological Group Actions***)
  If *the topological group* G *acts* on topological space X, and $x \in X$, then ***the orbit*** *of* x is defined as
  $$G(x) = \{g(x) : g \in G\}$$

- **Definition** The ***stablizer*** of x under group actions G is defined as
  $$G_x = \{g \in G : g(x) = x\}$$

- **Definition** (***Orbit Space*** X/G)
  Let G be a topological group and X be a *G-space* so that G acts on X. ***The orbit space*** is the set of *all orbits of action* with *quotient topology*. The quotient map $\pi : x \mapsto G(x)$ maps x to its orbit. *The orbit space* is often called ***the quotient of*** X ***by group actions*** G, i.e.
  $$X/G = \{G(x) : x \in X\}.$$

- **Proposition 2.47** *(**Orbit Space by Compact Group**)*
  *Let G be a **compact** topological group and X be a topological space so that G acts on X. Let X/G be the **orbit space**, i.e. the quotient space of X by group actions G. Then*
  1. *X/G is **Hausdorff** if X is **Hausdorff**;*
  2. *X/G is **regular** if X is **regular**;*
  3. *X/G is **normal** if X is **normal**;*
  4. *X/G is **locally compact** if X is **locally compact**;*
  5. *X/G is **second countable** if X is **second countable**;*

- **Example** (***Global Flow on Smooth Manifold***) [Lee, 2003.]

  Definition A ***global flow on*** M (also called ***a one-parameter group action***) is defined as a ***continuous left*** $\mathbb{R}$***-action on*** M; that is, a *continuous map* $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:
  $$\theta_{t+s}(p) = \theta_t \circ \theta_s(p),$$
  $$\theta_0(p) = p$$
  where $\theta_t = \theta(t, \cdot) : M \to M$ is a *continuous map* and $\theta_0 = \mathrm{Id}_M$.

  As we can see that, ***the global flow is topological group action*** *of* $(\mathbb{R}, +)$ *on the smooth manifold* M *(a topological space).*

  Definition For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by
  $$\theta^{(p)}(t) = \theta(t, p).$$
  The image of this curve is ***the orbit of*** p ***under the group action***.

References

John Marshall Lee. *Introduction to smooth manifolds*. Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.

James R Munkres. *Topology, 2nd*. Prentice Hall, 2000.