Lecture 3: Intrinsic Geometry of Surface

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1 Isometries and Conformal Maps

- This chapter focus on the ***geometry*** of *the first fundamental form*. Many important local properties of a surface can be expressed only in terms of the first fundamental form. The study of such properties is called the ***intrinsic geometry*** of the surface.

- **Definition** For two regular surfaces $\mathcal{S}$ and $\mathcal{S}'$, a *diffeomorphism* $\varphi : \mathcal{S} \to \bar{\mathcal{S}}$ is an ***isometry*** if for all $p \in \mathcal{S}$ and all pairs $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$ we have

$$\langle \boldsymbol{w}_1 \, , \, \boldsymbol{w}_2 \rangle_p = \langle d\varphi_p(\boldsymbol{w}_1) \, , \, d\varphi_p(\boldsymbol{w}_2) \rangle_{\varphi(p)} \, .$$

  The surface $\mathcal{S}$ and $\bar{\mathcal{S}}$ are said to be ***isometric***.

- The diffeomorphism φ is an isometry if the differential $d\varphi$ it **preserves the inner product**. It follows that the first fundamental form

$$I_p(\boldsymbol{w}) = \langle w \, , \, w \rangle_p = \langle d\varphi_p(\boldsymbol{w}) \, , \, d\varphi_p(\boldsymbol{w}) \rangle_{\varphi(p)} = I_{\varphi(p)}(d\varphi_p(\boldsymbol{w})), \quad \forall \, \boldsymbol{w} \in T_pS. \tag{1}$$

  Conversely, if the differential of a *diffeomorphism* preserves the first fundamental form, it is an *isometry*.

- **Definition** A map $\varphi : V \to \bar{\mathcal{S}}$ of a neighborhood V of $p \in \mathcal{S}$ is a ***local isometry*** at p if there exists a neighborhood $\bar{V}$ of $\varphi(p) \in \bar{\mathcal{S}}$ such that $\varphi : V \to \bar{V}$ is an *isometry*.

  If there exists a local isometry into $\bar{\mathcal{S}}$ at every $p \in \mathcal{S}$, the surface $\mathcal{S}$ is said to be ***locally isometric*** to $\bar{\mathcal{S}}$. Then $\mathcal{S}$ and $\bar{\mathcal{S}}$ are *locally isometric* if $\mathcal{S}$ is locally isometric to $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}$ is locally isometric to $\mathcal{S}$.

  Note that for a diffeomorphism φ that is a *local isometry* for *every* $p \in \mathcal{S}$, then φ is a *(global) isometry*.

  It is possible that two surfaces are locally isometric but are not *globally isometric*, e.g. the plane and the cylinder.

- **Proposition 1.1** *Assume the existence of parameterization $\boldsymbol{x} : U \to \mathcal{S}$ and $\bar{\boldsymbol{x}} : U \to \bar{\mathcal{S}}$ such that $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ in U. Then the map $\varphi = \bar{\boldsymbol{x}} \circ \boldsymbol{x}^{-1} : \boldsymbol{x}(U) \to \bar{\mathcal{S}}$ is a local isometry.*

  Proof: Let $p \in \boldsymbol{x}(U)$, and $\boldsymbol{w} \in T_pS$. Then $\boldsymbol{w}$ is tangent to a curve $\boldsymbol{x}(\beta(t))$ at $t = 0$, where $\beta(t) = (u(t), v(t))$ is a curve in U; thus, $\boldsymbol{w}$ can be written as (at $t = 0$)

$$\boldsymbol{w} = \boldsymbol{x}_u u' + \boldsymbol{x}_v v'$$

  for $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ basis in T_pS.

  By definition, the vector $d\varphi_p(\boldsymbol{w})$ is the tangent vector to the curve $\varphi \circ \boldsymbol{x} \circ \beta(t) = \bar{\boldsymbol{x}} \circ \beta(t) = \bar{\boldsymbol{x}}(\beta(t))$, i.e.

$$d\varphi_p(\boldsymbol{w}) = \bar{\boldsymbol{x}}_u u' + \bar{\boldsymbol{x}}_v v'$$

  Since

$$\begin{aligned} I_p(\boldsymbol{w}) &= E(u')^2 + 2F(u'v') + G(v')^2 \\ I_{\varphi(p)}(d\varphi_p(\boldsymbol{w})) &= \bar{E}(u')^2 + 2\bar{F}(u'v') + \bar{G}(v')^2, \end{aligned}$$

  we conclude that $I_p(\boldsymbol{w}) = I_{\varphi(p)}(d\varphi_p(\boldsymbol{w}))$ for all $p \in \boldsymbol{x}(U)$ and for all $\boldsymbol{w} \in T_pS$; hence, φ is an isometry. ■

- Given the first fundamental form, the ***intrinsic distance*** between two points on the surface can be defined as the ***infimum*** of the arc length between these points. ***This distance is invariant under isometry***, i.e. $\varphi : \mathcal{S} \to \bar{\mathcal{S}}$ is an isometry, then $d(p,q) = d(\varphi(p), \varphi(q)),\ p, q \in \mathcal{S}$.

- The notion of ***isometry*** is a natural concept of equivalence for the ***metric*** properties of regular surface. Similarly, the notion of ***diffeomorphism*** is an equivalence relationship form the point of view of ***differentiability***.

- **Definition** A diffeomorphism $\varphi : \mathcal{S} \to \bar{\mathcal{S}}$ is called a ***conformal map*** if for all $p \in \mathcal{S}$ and all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in T_p\mathcal{S}$ we have:

$$\langle d\varphi_p(\boldsymbol{v}_1)\,,\, d\varphi_p(\boldsymbol{v}_2)\rangle = \lambda^2(p)\,\langle \boldsymbol{v}_1\,,\, \boldsymbol{v}_2\rangle_p\,, \tag{2}$$

  where λ^2 is a **nowhere-zero differentiable** function on $\mathcal{S}$; the surfaces $\mathcal{S}$ and $\bar{\mathcal{S}}$ are then said to be ***conformal***.

  A map $\varphi : V \to \bar{\mathcal{S}}$ of a neighborhood V of $p \in \mathcal{S}$ into $\bar{\mathcal{S}}$ is a ***local conformal map*** at p if there exists a neighborhood $\bar{V}$ of $\varphi(p)$ such that $\varphi : V \to \bar{V}$ is a *conformal map*. If for each $p \in \mathcal{S}$, there exists a local conformal map at p, the surface $\mathcal{S}$ is said to be ***locally conformal*** to $\bar{\mathcal{S}}$.

- **Proposition 1.2** *Let $\boldsymbol{x} : U \to \mathcal{S}$ and $\bar{\boldsymbol{x}} : U \to \bar{\mathcal{S}}$ be parametrizations such that $E = \lambda^2\,\bar{E}$, $F = \lambda^2\,\bar{F}$, $G = \lambda^2\,\bar{G}$ in U, where λ^2 is a* ***nowhere-zero differentiable*** *function on U. Then the map $\varphi = \bar{\boldsymbol{x}} \circ \boldsymbol{x}^{-1} : \boldsymbol{x}(U) \to \bar{\mathcal{S}}$ is a local conformal map.*

- **Theorem 1.3** *Any two regular surfaces are locally conformal.*

2 The Gauss Theorem and the Equations of Compatibility

2.1 The fundamental theorem of the local theory of surfaces

- Given a parameterization $\boldsymbol{x} : U \to \mathcal{S}$ in the orientation of a regular surface $\mathcal{S}$, it is possible to assign a ***natural trihedron*** $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$ at each point $p \in \boldsymbol{x}(U)$.

- (*The representation of* ***partial derivatives*** *of basis under basis*)
  Note that given parameterization $\boldsymbol{x} : U \to \mathcal{S}$ and a point $p \in \mathcal{S}$, the trihedron $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$ at p form a basis in ambient space. In terms of this, the partial derivatives of these basis vector in this space can be linearly represented by this basis, i.e.

$$\begin{aligned}
\frac{\partial \boldsymbol{x}_u}{\partial u} &= \boldsymbol{x}_{uu} = \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + e\,N \\
\frac{\partial \boldsymbol{x}_u}{\partial v} &= \boldsymbol{x}_{uv} = \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + f\,N \\
\frac{\partial \boldsymbol{x}_v}{\partial u} &= \boldsymbol{x}_{vu} = \Gamma_{21}^1 \boldsymbol{x}_u + \Gamma_{21}^2 \boldsymbol{x}_v + f\,N \\
\frac{\partial \boldsymbol{x}_v}{\partial v} &= \boldsymbol{x}_{vv} = \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + g\,N \\
\frac{\partial N}{\partial u} &= N_u = a_{11} \boldsymbol{x}_u + a_{21} \boldsymbol{x}_v \\
\frac{\partial N}{\partial v} &= N_v = a_{12} \boldsymbol{x}_u + a_{22} \boldsymbol{x}_v
\end{aligned} \tag{3}$$

The coefficients $\Gamma^k_{i,j}$ for $i, j, k = 1, 2$ are called ***Christoffel symbols*** of $\mathcal{S}$ in paramterization. It is a function of intrinsic parameters. From (3), it is seen that the Christoffel symbols are linear coefficients of the projection of $\boldsymbol{x}_{uu}, \boldsymbol{x}_{uv}, \boldsymbol{x}_{vv}$ onto the tangent plane of the surface, whereas their normal complements are represented via e, f, g, the coefficients of second fundamental form. The coefficients $[a_{i,j}]$ determines the differential of Gauss map dN_p, which is a function of first fundamental form E, F, G.

Like Frenet formula, the above formula (3) is ***the fundamental theorem of the local theory of surfaces***.

- The linear coefficients of the ***second partial derivatives*** of the parameterization $(\boldsymbol{x}_{uu}, \boldsymbol{x}_{uv}, \boldsymbol{x}_{vv})$ under the basis vectors $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ at p is referred as the ***Christoffel symbol***, $\Gamma^k_{i,j}$, where the upper index $k = 1, 2$ is related to the basis vector $(\boldsymbol{x}_u, \boldsymbol{x}_v)$, and the lower index $(i, j) \in \{1, 2\} \times \{1, 2\}$ is related to the intrinsic parameter (u, v) under second order partial derivatives.

  Note that $(\boldsymbol{x}_{uu}, \boldsymbol{x}_{uv}, \boldsymbol{x}_{vv})$ is seen also as the partial derivative of the basis vector $(\boldsymbol{x}_u, \boldsymbol{x}_v)$. Thus the Christoffel symbol is the linear coefficient in representing the partial derivative of the basis vector $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ under these basis vectors itself.

- (***Christoffel symbols*** *via coefficients of* ***first fundamental form***)
  The Christoffel symbols can be determined by taking the inner product of the first four equations in (3) with $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, i.e.

$$\begin{cases} \Gamma^1_{11}E + \Gamma^2_{11}F &= \langle \boldsymbol{x}_{uu}\,,\, \boldsymbol{x}_u \rangle &= \frac{1}{2}E_u \\ \Gamma^1_{11}F + \Gamma^2_{11}G &= \langle \boldsymbol{x}_{uu}\,,\, \boldsymbol{x}_v \rangle &= F_u - \frac{1}{2}E_v \end{cases}$$
$$\begin{cases} \Gamma^1_{12}E + \Gamma^2_{12}F &= \langle \boldsymbol{x}_{uv}\,,\, \boldsymbol{x}_u \rangle &= \frac{1}{2}E_v \\ \Gamma^1_{12}F + \Gamma^2_{12}G &= \langle \boldsymbol{x}_{uv}\,,\, \boldsymbol{x}_v \rangle &= \frac{1}{2}G_u \end{cases}$$
$$\begin{cases} \Gamma^1_{22}E + \Gamma^2_{22}F &= \langle \boldsymbol{x}_{vv}\,,\, \boldsymbol{x}_u \rangle &= F_v - \frac{1}{2}G_u \\ \Gamma^1_{22}F + \Gamma^2_{22}G &= \langle \boldsymbol{x}_{vv}\,,\, \boldsymbol{x}_v \rangle &= \frac{1}{2}G_v \end{cases} \tag{4}$$

  There are three pairs of equations and each pair uniquely determines a pair of Christoffel symbol $(\Gamma^1_{i,j}, \Gamma^2_{i,j}), i, j = 1, 2$. *This system of equations in* (4) *determines the* ***Christoffel symbol*** *only in terms of the coefficients of* ***first fundamental form*** (E, F, G).

  Note that $\Gamma^k_{i,j} = \Gamma^k_{j,i}$, i.e. the Chrisoffel symbol is *symmetric* w.r.t. the lower indices.

  In particular, for orthogonal parameterization, $F = 0$, the Christoffel symbol can be computed as

$$\Gamma^1_{11} = \frac{1}{2}\frac{E_u}{E}; \qquad \Gamma^2_{11} = -\frac{1}{2}\frac{E_v}{G};$$
$$\Gamma^1_{12} = \frac{1}{2}\frac{E_v}{E}; \qquad \Gamma^2_{12} = \frac{1}{2}\frac{G_u}{G};$$
$$\Gamma^1_{22} = -\frac{1}{2}\frac{G_u}{E}; \qquad \Gamma^2_{22} = \frac{1}{2}\frac{G_v}{G}.$$

- The Christoffel symbols $\Gamma^k_{i,j},\ i, j, k = 1, 2$ are **uniquely determined** via the coefficients of first fundamental form (E, F, G).

  All geometric concepts and properties expressed in terms of Christoffel symbols are invariant under isometries.

2.2 THEOREMA EGREGIUM

- **Theorem 2.1** *(***THEOREMA EGREGIUM***) [Gauss]*
  The Gaussian curvature K of a surface is invariant by local isometries.

  Proof: Given parameterization $\boldsymbol{x} : U \to \mathcal{S}$ and a point $p \in \mathcal{S}$, the trihedron $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$ at p form a basis in ambient space. We consider the expression,

$$(\boldsymbol{x}_{uu})_v - (\boldsymbol{x}_{uv})_u = 0. \tag{5}$$

By fact that $\boldsymbol{x}_{uu}, \boldsymbol{x}_{uv}$ lies in the space spanned by $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$ at p, using the Christoffel symbol, we have the following equations

$$\begin{aligned}
\boldsymbol{x}_{uu} &= \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + e\boldsymbol{N} \\
\boldsymbol{x}_{uv} &= \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + f\boldsymbol{N} \\
\boldsymbol{x}_{vv} &= \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + g\boldsymbol{N} \\
\boldsymbol{N}_u &= a_{11} \boldsymbol{x}_u + a_{21} \boldsymbol{x}_v \\
\boldsymbol{N}_v &= a_{12} \boldsymbol{x}_u + a_{22} \boldsymbol{x}_v
\end{aligned} \tag{6}$$

and substitute the above equations into (5), we obtain

$$\begin{aligned}
\Gamma_{11}^1 \boldsymbol{x}_{uv} + \Gamma_{11}^2 \boldsymbol{x}_{vv} + e\boldsymbol{N}_v + \left(\Gamma_{11}^1\right)_v \boldsymbol{x}_u + \left(\Gamma_{11}^2\right)_v \boldsymbol{x}_v + e_v \boldsymbol{N} \\
&= \Gamma_{12}^1 \boldsymbol{x}_{uu} + \Gamma_{12}^2 \boldsymbol{x}_{uv} + f\boldsymbol{N}_u \\
&\quad + \left(\Gamma_{12}^1\right)_u \boldsymbol{x}_u + \left(\Gamma_{12}^2\right)_u \boldsymbol{x}_v + f_u \boldsymbol{N} \\
\Leftrightarrow \left(\Gamma_{11}^1 \boldsymbol{x}_{uv} + \Gamma_{11}^2 \boldsymbol{x}_{vv} - \Gamma_{12}^1 \boldsymbol{x}_{uu} - \Gamma_{12}^2 \boldsymbol{x}_{uv}\right) &= \left(\left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v\right) \boldsymbol{x}_u \\
&\quad + \left(\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v\right) \boldsymbol{x}_v + \left(f\boldsymbol{N}_u - e\boldsymbol{N}_v\right) + \left(f_u - e_v\right) \boldsymbol{N}
\end{aligned}$$

Substitute (4) into above equations, and the LHS is

$$\begin{aligned}
&\Gamma_{11}^1 \left(\Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + f\boldsymbol{N}\right) + \Gamma_{11}^2 \left(\Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + g\boldsymbol{N}\right) \\
&- \Gamma_{12}^1 \left(\Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + e\boldsymbol{N}\right) - \Gamma_{12}^2 \left(\Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + f\boldsymbol{N}\right) \\
&= \left(\Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{11}^2 \Gamma_{22}^1 - \Gamma_{12}^1 \Gamma_{11}^1 - \Gamma_{12}^2 \Gamma_{12}^1\right) \boldsymbol{x}_u + \left(\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - \left(\Gamma_{12}^2\right)^2\right) \boldsymbol{x}_v \\
&+ \left(\Gamma_{11}^1 f + \Gamma_{11}^2 g - \Gamma_{12}^1 e - \Gamma_{12}^2 f\right) \boldsymbol{N}
\end{aligned}$$

And the RHS

$$\begin{aligned}
&\left(\left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v\right) \boldsymbol{x}_u + \left(\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v\right) \boldsymbol{x}_v + \left(f\boldsymbol{N}_u - e\boldsymbol{N}_v\right) + \left(f_u - e_v\right) \boldsymbol{N} \\
&= \left(\left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v + a_{11} f - a_{12} e\right) \boldsymbol{x}_u + \left(\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + a_{21} f - a_{22} e\right) \boldsymbol{x}_v + \left(f_u - e_v\right) \boldsymbol{N}
\end{aligned}$$

Thus we have the equation as

$$A_1 \boldsymbol{x}_u + B_1 \boldsymbol{x}_v + C_1 \boldsymbol{N} = 0$$

where

$$\begin{aligned}
A_1 &= -\left(\Gamma_{12}^1\right)_u + \left(\Gamma_{11}^1\right)_v + \Gamma_{11}^2 \Gamma_{22}^1 - \Gamma_{12}^2 \Gamma_{12}^1 - a_{11} f + a_{12} e \\
B_1 &= -\left(\Gamma_{12}^2\right)_u + \left(\Gamma_{11}^2\right)_v - a_{21} f + a_{22} e + \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - \left(\Gamma_{12}^2\right)^2
\end{aligned}$$

$$C_1 = -f_u + e_v + \Gamma_{11}^1 f + \Gamma_{11}^2 g - \Gamma_{12}^1 e - \Gamma_{12}^2 f$$

By independence of $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$ at p, $A_1 = 0, B_1 = 0, C_1 = 0$, and by **the equations of Weingarten**, we have for $B_1 = 0$

$$\begin{aligned}
\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 + \Gamma_{12}^1\Gamma_{11}^2 + \left(\Gamma_{12}^2\right)^2 &= -a_{21}f + a_{22}e \\
&= -\frac{eg - f^2}{EG - F^2}E \\
&= -\boldsymbol{K}E
\end{aligned} \tag{7}$$

Similarly for $A_1 = 0$

$$\begin{aligned}
\left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v - \Gamma_{11}^2\Gamma_{22}^1 + \Gamma_{12}^2\Gamma_{12}^1 &= -a_{11}f + a_{12}e \\
&= F\frac{eg - f^2}{EG - F^2} \\
&= \boldsymbol{K}F
\end{aligned}$$

Note that by (7), the Gaussian curvature $\boldsymbol{K}$ only on the coefficient of first fundamental form E, and the Christoffel symbols $\Gamma_{11}^1, \Gamma_{11}^2, \Gamma_{12}^1, \Gamma_{12}^2, \Gamma_{22}^2$ and their derivatives $\left(\Gamma_{12}^2\right)_u, \left(\Gamma_{11}^2\right)_v$, which is invariant under local isometries. ■

- It is noted that in essence, the definition of the Gaussian curvature make use of the ***position*** of the surface in the space. However, the *Gaussian theorem* shows that *it only depends on the* ***metric structure*** (i.e. the first fundamental form) of the surface not on the position of the surface in the ambient space.

- (***The linear relationship between coefficients of first and second fundamental forms***)
The relationship btw coefficients of first and second fundamental forms can be computed via the following equations

$$\begin{aligned}
(\boldsymbol{x}_{uu})_v - (\boldsymbol{x}_{uv})_u &= 0 \\
(\boldsymbol{x}_{vv})_u - (\boldsymbol{x}_{uv})_v &= 0 \\
N_{uv} - N_{vu} &= 0
\end{aligned} \tag{8}$$

By substituting (3), it equals to

$$\begin{aligned}
A_1\boldsymbol{x}_u + B_1\boldsymbol{x}_v + C_1 N &= 0 \\
A_2\boldsymbol{x}_u + B_2\boldsymbol{x}_v + C_2 N &= 0 \\
A_3\boldsymbol{x}_u + B_3\boldsymbol{x}_v + C_3 N &= 0
\end{aligned} \tag{9}$$

where $A_i, B_i, C_i, i = 1, 2, 3$ are functions of e, f, g, E, F, G and of their derivatives. By linearly independence of $(\boldsymbol{x}_u, \boldsymbol{x}_v, N)$, it yields nine equations

$$A_i = 0; \quad B_i = 0; \quad C_i = 0 \quad i = 1, 2, 3, \tag{10}$$

This system of equations are related to the ***compatibility equations*** of the theory of surfaces.

- By solving the equations (10), one obtain the following equations

$$\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 + \Gamma_{12}^1\Gamma_{11}^2 + \left(\Gamma_{12}^2\right)^2 = -\boldsymbol{K}E \tag{11}$$

$$\left(\Gamma_{12}^1\right)_u - \left(\Gamma_{11}^1\right)_v - \Gamma_{11}^2\Gamma_{22}^1 + \Gamma_{12}^2\Gamma_{12}^1 = \boldsymbol{K}F \tag{12}$$

$$e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 = e_v - f_u \tag{13}$$

$$e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2 = f_v - g_u, \tag{14}$$

  where $\boldsymbol{K}$ is the ***Gaussian curvature*** shown in Gaussian theorem. The first two equations are called the ***Gauss formula*** and the last two equations are called the ***Mainardi-Codazzi equations***. These four equations are known as the *compatibility equations of the theory of surfaces.*

- **Theorem 2.2** ***(the completeness of the equations of compatibility)****[Bonnet].*
  Let E, F, G, e, f, g be differentiable functions defines in an open set $V \subset \mathbb{R}^2$, with $E > 0, G > 0$. Assume that the given functions satisfies formally the Gauss and Mainardi-Codazzi equations and that $EG - F^2 > 0$. Then, for every $q \in V$, there exits a neighborhood $U \subset V$ of q and a diffeomorphism $\boldsymbol{x} : U \to \boldsymbol{x}(U) \subset \mathbb{R}^3$ such that the regular surface $\boldsymbol{x}(U) \subset \mathbb{R}^3$ has E, F, G, e, f, g as a coefficient of the first and second fundamental forms, respectively. Furthermore, if U is connected and if $\hat{\boldsymbol{x}} : U \to \hat{\boldsymbol{x}}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exits a proper linear orthogonal transformation ρ and translation T so that $\hat{\boldsymbol{x}} = T \circ \rho \circ \boldsymbol{x}$.

- The ***compatibility equations*** (i.e. the ***Gauss formula*** (11) and (12) and *Mainardi-Codazzi equations* (13), (14)) is a system of *differential equations* for the coefficients of the ***first and the second fundamental forms*** (E, F, G, e, f, g) and also there is **no further relations** btw these coefficients.

- In Bonnet theorem 2.2, it shows that the coefficients of the first and the second fundamental forms (E, F, G, e, f, g) ***uniquely determines*** the **parameterization** of the surface locally up to a *rigid transformation.* That is, these coefficients are **sufficient** to determine the local structure of a surface.

3 Summary of first and second fundamental form

1. The *first fundamental form* [do Carmo Valero, 1976] of a regular surface $\mathcal{S} \subset \mathbb{R}^3$ at $p \in \mathcal{S}$ is defined as a quadratic form, $I_p : T_pS \to \mathbb{R}$ given by

$$I_p(\boldsymbol{w}) = \langle w\,,\,w\rangle_p = \|w\|_2^2 \geq 0 \quad \boldsymbol{w} \in T_pS.$$

2. The quadratic form Π_p defined in T_pS by $\Pi_p(\boldsymbol{v}) = -\langle dN_p(\boldsymbol{v})\,,\,\boldsymbol{v}\rangle$ is called the *second fundamental form* of $\mathcal{S}$ at p, where dN_p is the differential of Gauss map at p, referred as the shape operator [O'neill, 2006].

3. The coefficients for the first and second fundamental form

$$E(u,v) = \langle \boldsymbol{x}_u\,,\,\boldsymbol{x}_u\rangle$$
$$F(u,v) = \langle \boldsymbol{x}_u\,,\,\boldsymbol{x}_v\rangle$$
$$G(u,v) = \langle \boldsymbol{x}_v\,,\,\boldsymbol{x}_v\rangle$$

$$
\begin{aligned}
e(u,v) &= -\langle N_u \,,\, \boldsymbol{x}_u\rangle = \langle N \,,\, \boldsymbol{x}_{uu}\rangle \\
f(u,v) &= -\langle N_u \,,\, \boldsymbol{x}_v\rangle = \langle N \,,\, \boldsymbol{x}_{vu}\rangle = \langle N \,,\, \boldsymbol{x}_{uv}\rangle = -\langle N_v \,,\, \boldsymbol{x}_u\rangle \\
g(u,v) &= -\langle N_v \,,\, \boldsymbol{x}_v\rangle = \langle N \,,\, \boldsymbol{x}_{vv}\rangle
\end{aligned} \tag{15}
$$

4. See that E, G are *squared length of tangent vector along the coordinate curve* $\alpha(u, v_0)$, with $\alpha'_u \equiv \boldsymbol{x}_u$ and $\alpha(u_0, v)$, with $\alpha'_v \equiv \boldsymbol{x}_v$.

   Also, e, g are seen as the *normal curvature of the coordinate curve* $\alpha(u, v_0)$, with $\alpha'_u \equiv \boldsymbol{x}_u$ and $\alpha(u_0, v)$, with $\alpha'_v \equiv \boldsymbol{x}_v$, (i.e. the projection of second-order derivatives along $\boldsymbol{N}$) or curvature of the normal section of the surface along the direction $\boldsymbol{x}_u, \boldsymbol{x}_v$.

   The quantity F measures the orthogonality between two coordinate curves (i.e. the angles). $F = 0$ means that two coordinate curves are orthogonal to each other and $F = 0 \Rightarrow f = 0$. The quantity f measures the projection of the rate of the change of vector field $\boldsymbol{x}_u$ w.r.t. the other coordinate curve $\alpha(u_0, v)$, with $\alpha'_v \equiv \boldsymbol{x}_v$ along $\boldsymbol{N}$.

5.
   - E, F, G are quantities related to the *first-order derivatives* of the coordinate curve (metric term in *unit* velocity field);
   - The Christoffel symbols $\Gamma^k_{i,j}$ determines the projection of the second-order derivatives of the coordinate curve, or the derivative of the tangent vector field along each basis of the tangent space; that is, they determine the *tangential component of the second-order derivatives* of the coordinate curve. It is a function of E, F, G and its first derivatives.
   - e, f, g determines the *normal component of the second-order derivatives* of the coordinate curve along $\boldsymbol{N}$;
   - The Gaussian curvature by Gaussian formula is related to the third-order derivatives of the coordinate curve (i.e. the differential of the Christoffel symbol).

6. The Christoffel symbols $\Gamma^k_{i,j}$ only depends on the coefficients of the first fundamental form E, F, G and its first-order derivatives.

4 Homework and Examples

1. **Example** Show that if $\boldsymbol{x}$ is an orthogonal parameterization, i.e. $F = 0$, then

$$
\boldsymbol{K} = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.
$$

2. **Example** Show that if $\boldsymbol{x}$ is an isothermal parameterization, i.e. $E = G = \lambda(u, v)$, then

$$
\boldsymbol{K} = -\frac{1}{2\lambda}\Delta\left(\log\lambda\right).
$$

   where $\Delta\phi$ denotes the Laplacian $(\partial^2\phi/\partial u^2 + \partial^2\phi/\partial v^2)$ of the function ϕ. Conclude that when $E = G = (u^2 + v^2 + c)^{-2}$ and $F = 0$, then $\boldsymbol{K} = const. = 4c$.

References

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