

Lecture 5: Abstract Integrations

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Contents

1	Recall	2
2	Measurable Functions, and Integration on a Measure Space	7
2.1	Measurable Functions	7
2.2	Simple Integral of Simple Functions	8
2.3	Unsigned Integral	10
2.4	Absolutely Convergent Integral	12
2.5	The Convergence Theorems	13

1 Recall

- **Definition** Let \mathcal{B} be a *Boolean algebra* on a space X . An (unsigned) **finitely additive measure** μ on \mathcal{B} is a map $\mu : \mathcal{B} \rightarrow [0, +\infty]$ that obeys the following axioms

1. $\mu(\emptyset) = 0$;
2. **Finite union**: for any *disjoint sets* $A, B \in \mathcal{B}$,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

- **Proposition 1.1 (Properties of Finitely Additive Measure)** [Tao, 2011]
Let $\mu : \mathcal{B} \rightarrow [0, +\infty]$ be a finitely additive measure on a Boolean σ -algebra \mathcal{B} .

1. (**Monotonicity**) If E, F are \mathcal{B} -measurable and $E \subseteq F$, then

$$\mu(E) \leq \mu(F).$$

2. (**Finite additivity**) If k is a natural number, and E_1, \dots, E_k are \mathcal{B} -measurable and *disjoint*, then

$$\mu(E_1 \cup \dots \cup E_k) = \mu(E_1) + \dots + \mu(E_k).$$

3. (**Finite subadditivity**) If k is a natural number, and E_1, \dots, E_k are \mathcal{B} -measurable, then

$$\mu(E_1 \cup \dots \cup E_k) \leq \mu(E_1) + \dots + \mu(E_k).$$

4. (**Inclusion-exclusion for two sets**) If E, F are \mathcal{B} -measurable, then

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F).$$

(Caution: remember that the cancellation law $a + c = b + c \Rightarrow a = b$ does not hold in $[0; +1]$ if c is infinite, and so the use of cancellation (or subtraction) should be avoided if possible.)

- **Example (Dirac measure).**

Let $x \in X$ and \mathcal{B} be an arbitrary Boolean algebra on X . Then **the Dirac measure** δ_x at x , defined by setting $\delta_x(E) := \mathbb{1}\{x \in E\}$, is **finitely additive**.

- **Example (Zero measure).**

The **zero measure** $0 : E \mapsto 0$ is a *finitely additive measure* on any Boolean algebra.

- **Example (Linear combinations of measures).**

If \mathcal{B} is a Boolean algebra on X , and $\mu, \nu : \mathcal{B} \rightarrow [0, +\infty]$ are *finitely additive measures* on \mathcal{B} , then $\mu + \nu : E \mapsto \mu(E) + \nu(E)$ is also a **finitely additive measure**, as is $c\mu : E \mapsto c \times \mu(E)$ for any $c \in [0, +\infty]$. Thus, for instance, the sum of Lebesgue measure and a Dirac measure is also a finitely additive measure on the Lebesgue algebra (or on any of its sub-algebras).

In other word, **the space of all finitely additive measures on \mathcal{B} is a vector space.**

- **Example (*Restriction of a measure*).**

If \mathcal{B} is a Boolean algebra on X , $\mu : \mathcal{B} \rightarrow [0, +\infty]$ is a *finitely additive measure*, and Y is a \mathcal{B} -measurable subset of X , then **the restriction** $\mu|_Y : \mathcal{B}|_Y \rightarrow [0, +\infty]$ of \mathcal{B} to Y , defined by setting $\mu|_Y(E) := \mu(E)$ whenever $E \in \mathcal{B}|_Y$ (i.e. if $E \in \mathcal{B}$ and $E \subseteq Y$), is also a **finitely additive measure**.

- **Example (*Counting measure*).**

If \mathcal{B} is a Boolean algebra on X , then the function $\# : \mathcal{B} \rightarrow [0, +\infty]$ defined by setting $\#(E)$ to be the **cardinality** of E if E is *finite*, and $\#(E) := +\infty$ if E is infinite, is a **finitely additive measure**, known as counting measure.

- **Proposition 1.2 (*Finitely Additive Measures on Atomic Algebra*)**

Let \mathcal{B} be a **finite** Boolean algebra, generated by a finite family A_1, \dots, A_k of non-empty atoms. For every **finitely additive measure** μ on \mathcal{B} there exists $c_1, \dots, c_k \in [0, +\infty]$ such that

$$\mu(E) = \sum_{1 \leq j \leq k: A_j \subseteq E} c_j.$$

Equivalently, if x_j is a point in A_j for each $1 \leq j \leq k$, then

$$\mu = \sum_{j=1}^k c_j \delta_{x_j}.$$

where c_1, \dots, c_k are **uniquely** determined by μ .

- **Definition** Let (X, \mathcal{B}) be a measurable space. An (*unsigned*) **countably additive measure** μ on \mathcal{B} , or **measure** for short, is a map $\mu : \mathcal{B} \rightarrow [0, +\infty]$ that obeys the following axioms:

1. (**Empty set**) $\mu(\emptyset) = 0$.
2. (**Countable additivity**) Whenever $E_1, E_2, \dots \in \mathcal{B}$ are a **countable sequence** of **disjoint** measurable sets, then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n).$$

A triplet (X, \mathcal{B}, μ) , where (X, \mathcal{B}) is a **measurable space** and $\mu : \mathcal{B} \rightarrow [0, +\infty]$ is a **countably additive measure**, is known as a measure space.

- **Remark** Note the distinction between a **measure space** and a **measurable space**. The latter has the **capability** to be equipped with a *measure*, but the former is **actually** equipped with a *measure*.

- **Definition** [Folland, 2013]

Let (X, \mathcal{B}, μ) be a measure space.

- If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \mathcal{B}$), then μ is called **finite**.
- If $X = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{B}$ and $\mu(E_j) < \infty$, then μ is called **σ -finite**. More generally, if $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathcal{B}$ and $\mu(E_j) < \infty$, then E is said to be **σ -finite** for μ .
- If for each $E \in \mathcal{B}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{B}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, then μ is called **semi-finite**.

- **Proposition 1.3** Let (X, \mathcal{B}, μ) be a **measure space**.

1. (**Countable subadditivity**) If E_1, E_2, \dots are \mathcal{B} -measurable, then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

2. (**Upwards monotone convergence**) If $E_1 \subseteq E_2 \subseteq \dots$ are \mathcal{B} -measurable, then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \rightarrow \infty} \mu(E_n) = \sup_n \mu(E_n). \quad (1)$$

3. (**Downwards monotone convergence**) If $E_1 \supseteq E_2 \supseteq \dots$ are \mathcal{B} -measurable, and $\mu(E_n) < \infty$ for **at least one** n , then

$$\mu \left(\bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \rightarrow \infty} \mu(E_n) = \inf_n \mu(E_n). \quad (2)$$

- **Proposition 1.4** (**Dominated convergence for sets**). [Tao, 2011]

Let (X, \mathcal{B}, μ) be a measure space. Let E_1, E_2, \dots be a sequence of \mathcal{B} -measurable sets that **converge** to another set E , in the sense that $\mathbb{1}_{E_n}$ converges **pointwise** to $\mathbb{1}_E$. Then

1. E is also \mathcal{B} -measurable.
2. If there exists a \mathcal{B} -measurable set F of **finite measure** (i.e. $\mu(F) < \infty$) that **contains all of the** E_n , then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(E).$$

(Hint: Apply downward monotonicity to the sets $\bigcup_{n>N} (E_n \Delta E)$.)

3. The previous part of this proposition can **fail** if the hypothesis that all the E_n are contained in a set of finite measure is **omitted**.

- **Exercise 1.5** (**Countably Additive Measures on Countable Set with Discrete σ -Algebra**)

Let X be an at most **countable** set with **the discrete σ -algebra**. Show that every measure μ on this measurable space can be uniquely represented in the form

$$\mu = \sum_{x \in X} c_x \delta_x$$

for some $c_x \in [0, +\infty]$, thus

$$\mu(E) = \sum_{x \in E} c_x$$

for all $E \subseteq X$. (This claim fails in the **uncountable** case, although showing this is slightly tricky.)

- **Definition (Completeness).** [Tao, 2011]

A **null set** of a measure space (X, \mathcal{B}, μ) is defined to be a \mathcal{B} -measurable set of **measure zero**. A **sub-null set** is any subset of a null set.

A measure space is said to be **complete** if every sub-null set is a null set.

- **Theorem 1.6** The **Lebesgue measure space** $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$ is **complete**, but the **Borel measure space** $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ is **not**.
- Completion is a convenient property to have in some cases, particularly when dealing with properties that hold almost everywhere. Fortunately, it is fairly easy to modify any measure space to be complete:

Proposition 1.7 (Completion).

Let (X, \mathcal{B}, μ) be a measure space. There exists a **unique refinement** $(X, \overline{\mathcal{B}}, \overline{\mu})$, known as **the completion** of (X, \mathcal{B}, μ) , which is the **coarsest** refinement of (X, \mathcal{B}, μ) that is **complete**. Furthermore, $\overline{\mathcal{B}}$ consists precisely of those sets that differ from a \mathcal{B} -measurable set by a \mathcal{B} -subnull set.

- **Definition (Abstract outer measure).** [Tao, 2011]

Let X be a set. An **abstract outer measure** (or **outer measure** for short) is a map $\mu^* : 2^X \rightarrow [0, +\infty]$ that assigns an **unsigned extended real number** $\mu^*(E) \in [0, +\infty]$ to every set $E \subseteq X$ which obeys the following axioms:

1. (**Empty set**) $\mu^*(\emptyset) = 0$.
2. (**Monotonicity**) If $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$.
3. (**Countable subadditivity**) If $E_1, E_2, \dots \subseteq X$ is a countable sequence of subsets of X , then

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Outer measures are also known as **exterior measures**.

- **Definition (Carathéodory measurability).**

Let μ^* be an outer measure on a set X . A set $E \subseteq X$ is said to be **Carathéodory measurable with respect to μ^*** (or, **μ^* -measurable**) if one has

$$\mu^*(A) = \mu^*(A \setminus E) + \mu^*(A \cap E)$$

for every set $A \subseteq X$.

- **Example (Null sets are Carathéodory measurable).**
Suppose that E is a **null set** for an outer measure μ^* (i.e. $\mu^*(E) = 0$). Then that E is Carathéodory measurable with respect to μ^* .
- **Example (Compatibility with Lebesgue measurability).** A set $E \subseteq \mathbb{R}^d$ is Carathéodory measurable with respect to Lebesgue outer measure if and only if it is Lebesgue measurable.
- **Theorem 1.8 (Carathéodory extension theorem).** [Tao, 2011]
Let $\mu^* : 2^X \rightarrow [0, +\infty]$ be an outer measure on a set X , let \mathcal{B} be the collection of all subsets of X that are **Carathéodory measurable with respect to μ^*** , and let $\mu : \mathcal{B} \rightarrow [0, +\infty]$ be the

restriction of μ^ to \mathcal{B} (thus $\mu(E) := \mu^*(E)$ whenever $E \in \mathcal{B}$). Then \mathcal{B} is a σ -algebra, and μ is a measure.*

- **Definition (Pre-measure).**

A pre-measure on a **Boolean algebra** \mathcal{B}_0 is a function $\mu_0 : \mathcal{B}_0 \rightarrow [0, +\infty]$ that satisfies the conditions:

1. (**Empty Set**): $\mu_0(\emptyset) = 0$
2. (**Countably Additivity**): IF $E_1, E_2, \dots \in \mathcal{B}_0$ are *disjoint sets* such that $\bigcup_{n=1}^{\infty} E_n$ is in \mathcal{B}_0 ,

$$\mu_0 \left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

- **Remark** A pre-measure μ_0 is a **finitely additive measure** that **already** is *countably additive within* a Boolean algebra \mathcal{B}_0 .
- **Remark** The countably additivity condition for pre-measure can be relaxed to be the *countably subadditivity* $\mu_0(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu_0(E_n)$ without affecting the definition of a pre-measure.
- **Proposition 1.9** Let $\mathcal{B} \subset 2^X$ and $\mu_0 : \mathcal{B} \rightarrow [0, +\infty]$ be such that $\emptyset, X \in \mathcal{B}$, and $\mu_0(\emptyset) = 0$. For any $A \subseteq X$, define

$$\mu^*(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : E_j \in \mathcal{B}, \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

Then μ^* is an outer measure.

- **Theorem 1.10 (Hahn-Kolmogorov Theorem).**

Every **pre-measure** $\mu_0 : \mathcal{B}_0 \rightarrow [0, +\infty]$ on a Boolean algebra \mathcal{B}_0 in X can be **extended** to a **countably additive measure** $\mu : \mathcal{B} \rightarrow [0, +\infty]$.

- **Remark** We can construct an *outer measure* μ^* according to Proposition 1.9. Let \mathcal{B} be the *collection* of all sets $E \subseteq X$ that are *Carathéodory measurable with respect to μ^** (μ^* -measurable), and let μ be the *restriction* of μ^* to \mathcal{B} . The tuple (X, \mathcal{B}, μ) is what we want in *Hahn-Kolmogorov theorem*.

The measure μ constructed in this way is called the Hahn-Kolmogorov extension of the pre-measure μ_0 .

- **Proposition 1.11 (Uniqueness of the Hahn-Kolmogorov Extension)**

Let $\mu_0 : \mathcal{B}_0 \rightarrow [0, +\infty]$ be a **pre-measure**, let $\mu : \mathcal{B} \rightarrow [0, +\infty]$ be the **Hahn-Kolmogorov extension** of μ_0 , and let $\mu' : \mathcal{B}' \rightarrow [0, +\infty]$ be **another** countably additive extension of μ_0 . Suppose also that μ_0 is **σ -finite**, which means that one can express the whole space X as the countable union of sets $E_1, E_2, \dots \in \mathcal{B}_0$ for which $\mu_0(E_n) < \infty$ for all n . Then μ and μ' agree on their common domain of definition. In other words, show that $\mu(E) = \mu'(E)$ for all $E \in \mathcal{B} \cap \mathcal{B}'$.

2 Measurable Functions, and Integration on a Measure Space

2.1 Measurable Functions

- **Definition** Let (X, \mathcal{B}) be a measurable space, and let $f : X \rightarrow [0, +\infty]$ or $f : X \rightarrow \mathbb{C}$ be an *unsigned* or *complex-valued function*. We say that f is measurable if $f^{-1}(U)$ is \mathcal{B} -measurable for every *open subset* U of $[0, +\infty]$ or \mathbb{C} .

- **Remark** The inverse image of a Lebesgue measurable set by a *measurable function* need not remain Lebesgue measurable. This is due to the definition of above measurable function. The pre-image of E is Lebesgue measurable, if E has a slightly stronger measurability property than Lebesgue measurability, namely **Borel measurability**.

- In general, we have the following

Definition For $f : X \rightarrow Y$, and $X \equiv (X, \mathcal{F})$, $Y \equiv (Y, \mathcal{B})$ are measurable spaces, then f is called $(\mathcal{F}, \mathcal{B})$ measurable (or $(\mathcal{F}/\mathcal{B})$ measurable or, simply, *measurable*), if $f^{-1}(E) \in \mathcal{F}$ for every $E \in \mathcal{B}$.

- **Definition** Note that if $\{(Y_\alpha, \mathcal{B}_\alpha)\}$ is a family of measurable spaces, and $\{f_\alpha\}$ for $f_\alpha : X \rightarrow Y_\alpha$, then there is a **unique smallest** σ -algebra on X so that $\{f_\alpha\}$ are all measurable. It is generated by $f_\alpha^{-1}(E_\alpha)$, $E_\alpha \in \mathcal{B}_\alpha$. It is called the **σ -algebra generated by $\{f_\alpha\}$** .

In particular, $X = \prod_\alpha Y_\alpha$ has **product σ -algebra** that is *generated by coordinate functions* $\{\pi_\alpha\}$.

- **Proposition 2.1** Let (X, \mathcal{B}) be a measurable space.

1. $f : X \rightarrow [0, +\infty]$ is **measurable** if and only if the **level sets** $\{x \in X : f(x) > \lambda\}$ are \mathcal{B} -measurable.
2. The **indicator function** $\mathbb{1}_E$ of a set $E \subseteq X$ is **measurable** if and only if E itself is \mathcal{B} -measurable.
3. $f : X \rightarrow [0, +\infty]$ or $f : X \rightarrow \mathbb{C}$ is measurable if and only if $f^{-1}(E)$ is \mathcal{B} -measurable for every **Borel-measurable** subset E of $[0, +\infty]$ or \mathbb{C} .
4. $f : X \rightarrow \mathbb{C}$ is measurable if and only if its real and imaginary parts are measurable.
5. $f : X \rightarrow \mathbb{R}$ is measurable if and only if the **magnitudes** $f_+ := \max\{f, 0\}$, $f_- := \max\{-f, 0\}$ of its **positive** and **negative** parts are **measurable**.
6. If $f_n : X \rightarrow [0, +\infty]$ are a sequence of **measurable** functions that converge **pointwise** to a limit $f : X \rightarrow [0, +\infty]$, then f is also **measurable**. The same claim holds if $[0, +\infty]$ is replaced by \mathbb{C} .
7. If $f : X \rightarrow [0, +\infty]$ is measurable and $\varphi : [0, +\infty] \rightarrow [0, +\infty]$ is **continuous**, the composite $\varphi \circ f$ is measurable. The same claim holds if $[0, +\infty]$ is replaced by \mathbb{C} .
8. The **sum** or **product** of two **measurable** functions in $[0, +\infty]$ or \mathbb{C} is still measurable.

- **Definition** A function $f : (X, \mathcal{F}) \rightarrow (Y, \mathcal{B})$ is simple if it only takes **finitely many** different values $s_1, \dots, s_k \in Y$.

Then the σ -algebra $f^{-1}(\mathcal{B})$ reduce to $\sigma\left(\{f^{-1}(\{s_\alpha\})\}_{\alpha=1}^k\right)$, the **finite σ -algebra** generated by *atomic algebra* with atoms $E_\alpha \equiv f^{-1}(\{s_\alpha\})$. The **canonical representation** of f is

$$f = \sum_{\alpha=1}^k s_\alpha \mathbb{1}_{\{E_\alpha\}},$$

which is determined up to a reordering.

- **Proposition 2.2 (Measurable Function with respect to Atomic Algebra is Simple)**
Let (X, \mathcal{B}) be a measurable space that is **atomic**, thus $\mathcal{B} = \mathcal{A}((A_\alpha)_{\alpha \in I})$ for some partition $X = \bigcup_{\alpha \in I} A_\alpha$ of X into disjoint non-empty atoms. A function $f : X \rightarrow [0, +\infty]$ or $f : X \rightarrow \mathbb{C}$ is measurable if and only if it is **constant** on each atom, or equivalently if one has a **representation of the form**

$$f(x) = \sum_{\alpha \in I} c_\alpha \mathbb{1}_{\{x \in A_\alpha\}},$$

for some constants $c_\alpha \in [0; +\infty]$ or in \mathbb{C} as appropriate. Furthermore, the c_α are uniquely determined by f .

- **Theorem 2.3 (Egorov's theorem).** [Tao, 2011]
Let (X, \mathcal{B}, μ) be a **finite measure space** (so $\mu(X) < \infty$), and let $f_n : X \rightarrow \mathbb{C}$ be a sequence of measurable functions that **converge pointwise almost everywhere** to a limit $f : X \rightarrow \mathbb{C}$. For $\epsilon > 0$, there exists a measurable set E of measure **at most** ϵ such that f_n converges **uniformly** to f outside of E .
- **Remark** Give an example to show that the claim can **fail** when the measure μ is not finite.

2.2 Simple Integral of Simple Functions

- **Definition (Simple integral).**
Let (X, \mathcal{B}, μ) be a measure space with \mathcal{B} **finite** (i.e., its cardinality is finite and there are only finitely many measurable sets). X can then be partitioned into a finite number of atoms A_1, \dots, A_n . If $f : X \rightarrow [0, +\infty]$ is measurable, it has a **unique representation** of the form

$$f(x) = \sum_{\alpha \in I} c_\alpha \mathbb{1}_{\{x \in A_\alpha\}},$$

for some constants $c_\alpha \in [0; +\infty]$. We then define the **simple integral** $\text{simp} \int_X f d\mu$ of f by the formula

$$\text{simp} \int_X f d\mu \equiv \sum_{\alpha \in I} c_\alpha \mu(A_\alpha)$$

- **Remark** Note that the precise decomposition into atoms *does not affect* the definition of the simple integral.

Proposition 2.4 (Simple integral unaffected by refinements). [Tao, 2011]
Let (X, \mathcal{B}, μ) be a measure space, and let (X, \mathcal{B}', μ') be a **refinement** of (X, \mathcal{B}, μ) , which

means that \mathcal{B}' contains \mathcal{B} and $\mu' : \mathcal{B}' \rightarrow [0, +\infty]$ **agrees** with $\mu : \mathcal{B} \rightarrow [0, +\infty]$ on \mathcal{B} . Suppose that both $\mathcal{B}, \mathcal{B}'$ are **finite**, and let $f : \mathcal{B} \rightarrow [0, +\infty]$ be measurable. We have

$$\text{simp} \int_X f d\mu = \text{simp} \int_X f d\mu'.$$

Proof: Since taking simple integrals both w.r.t. \mathcal{B} and \mathcal{B}' implies that f is both \mathcal{B} -measurable and \mathcal{B}' -measurable, we see that for the finite values a_1, \dots, a_k of f we have $f^{-1}(a_i) \in \mathcal{B}$.

$$\text{simp} \int_X f d\mu' = \sum_{i=1}^k a_i \mu'(f^{-1}(a_i)) = \sum_{i=1}^k a_i \mu'|_{\mathcal{B}}(f^{-1}(a_i)) = \sum_{i=1}^k a_i \mu(f^{-1}(a_i)) = \text{simp} \int_X f d\mu. \quad \blacksquare$$

- The above proposition allows one to extend the *simple integral* to *simple functions*:

Definition (Integral of simple functions).

An **(unsigned) simple function** $f : X \rightarrow [0, +\infty]$ on a measurable space (X, \mathcal{B}) is a measurable function that takes on **finitely many values** a_1, \dots, a_k . Note that such a function is then automatically *measurable* with respect to *at least one finite sub- σ -algebra* \mathcal{B}' of \mathcal{B} , namely the σ -algebra \mathcal{B}' **generated by the preimages** $f^{-1}\{a_1\}, \dots, f^{-1}\{a_k\}$ of a_1, \dots, a_k .

We then define the **simple integral** $\text{simp} \int_X f d\mu$ by the formula

$$\begin{aligned} \text{simp} \int_X f d\mu &\equiv \text{simp} \int_X f d\mu|_{\mathcal{B}'} \\ &= \sum_{i=1}^k a_i \mu(f^{-1}\{a_i\}) \end{aligned}$$

where $\mu|_{\mathcal{B}'} : \mathcal{B}' \rightarrow [0, +\infty]$ is the **restriction** of $\mu : \mathcal{B} \rightarrow [0, +\infty]$ to \mathcal{B}' .

- **Remark** Note that there could be **multiple finite σ -algebras** with respect to which f is *measurable*, but all such extensions will give the same simple integral. Indeed, if f were measurable with respect to two separate finite sub- σ -algebras \mathcal{B}' and \mathcal{B}'' of \mathcal{B} , then it would also be *measurable* with respect to their **common refinement** $\mathcal{B}' \vee \mathcal{B}'' := (\mathcal{B}' \cup \mathcal{B}'')$, which is also *finite* and then by Proposition 2.4, $\int_X f d\mu|_{\mathcal{B}'}$ and $\int_X f d\mu|_{\mathcal{B}''}$ are both equal to $\int_X f d\mu|_{\mathcal{B}' \vee \mathcal{B}''}$, and hence equal to each other.
- **Remark** As with the Lebesgue theory, we say that a property $P(x)$ of an element $x \in X$ of a measure space (X, \mathcal{B}, μ) **holds μ -almost everywhere** if it **holds outside** of a **sub-null set**, i.e. $\mu(\{P(x) \text{ does not hold}\}) = 0$.
- **Proposition 2.5** Let (X, \mathcal{B}, μ) be a measure space, and let $f, g : X \rightarrow [0, +\infty]$ be simple unsigned functions.

1. (**Monotonicity**) If $f \leq g$ then $\text{simp} \int_X f d\mu \leq \text{simp} \int_X g d\mu$.
2. (**Compatibility with measure**) For every \mathcal{B} -measurable set E , we have $\text{simp} \int_X \mathbf{1}_E d\mu = \mu(E)$.
3. (**Homogeneity**) For every $c \in [0, +\infty]$, one has $\text{simp} \int_X (cf) d\mu = c \times \text{simp} \int_X f d\mu$.

4. (**Finite additivity**) We have $\text{simp} \int_X (f + g) d\mu = \text{simp} \int_X f d\mu + \text{simp} \int_X g d\mu$.
5. (**Insensitivity to refinement**) Let (X, \mathcal{B}, μ) be a measure space, and let (X, \mathcal{B}', μ') be its refinement, which means that \mathcal{B}' contains \mathcal{B} and $\mu' : \mathcal{B}' \rightarrow [0, +\infty]$ **agrees** with $\mu : \mathcal{B} \rightarrow [0, +\infty]$ on \mathcal{B} . Suppose that both $\mathcal{B}, \mathcal{B}'$ are **finite**, and let $f : \mathcal{B} \rightarrow [0, +\infty]$ be measurable. We have

$$\text{simp} \int_X f d\mu = \text{simp} \int_X f d\mu'.$$

6. (**Almost everywhere equivalence**) If μ -almost everywhere $f = g$, then $\text{simp} \int_X f d\mu = \text{simp} \int_X g d\mu$.
7. (**Finiteness**) $\text{simp} \int_X f d\mu < \infty$ if and only if f is **finite** μ -almost everywhere and is **supported** on a set of **finite measure**.
8. (**Vanishing**) $\text{simp} \int_X f d\mu = 0$ if and only if $f = 0$ μ -almost everywhere.

• **Exercise 2.6 (Inclusion-exclusion principle).**

Let (X, \mathcal{B}, μ) be a measure space, and let A_1, \dots, A_n be \mathcal{B} -measurable sets of **finite measure**. Show that

$$\mu \left(\bigcup_{i=1}^n A_i \right) = \sum_{J \subseteq [1:n], J \neq \emptyset} (-1)^{|J|-1} \mu \left(\bigcap_{i \in J} A_i \right)$$

(Hint: Compute $\text{simp} \int_X (1 - \prod_{i=1}^n (1 - \mathbb{1}_{A_i})) d\mu$ in two different ways.)

- **Remark** The simple integral could also be defined on *finitely additive measure spaces*, rather than *countably additive ones*, and all the above properties would still apply. However, on a finitely additive measure space one would have difficulty extending the integral beyond simple functions.

2.3 Unsigned Integral

- **Definition** Let (X, \mathcal{B}, μ) be a measure space, and let $f : X \rightarrow [0, +\infty]$ be (*unsigned*) measurable. Then we define the **unsigned integral** $\int_X f d\mu$ of f by the formula

$$\int_X f d\mu \equiv \sup_{\substack{0 \leq g \leq f, \\ g \text{ simple}}} \text{simp} \int_X g d\mu$$

- **Proposition 2.7 (Properties of the unsigned integral).**

Let (X, \mathcal{B}, μ) be a measure space, and let $f, g : X \rightarrow [0, +\infty]$ be measurable.

1. (**Almost everywhere equivalence**) If $f = g$ μ -almost everywhere, then $\int_X f d\mu = \int_X g d\mu$.
2. (**Monotonicity**) If $f \leq g$ μ -almost everywhere, then $\int_X f d\mu \leq \int_X g d\mu$.
3. (**Homogeneity**) We have $\int_X (cf) d\mu = c \int_X f d\mu$ for every $c \in [0, +\infty]$.
4. (**Superadditivity**) We have $\int_X (f + g) d\mu \geq \int_X f d\mu + \int_X g d\mu$.

5. (**Compatibility with the simple integral**) If f is **simple**, then $\int_X f d\mu = \text{simp} \int_X f d\mu$.
6. (**Markov's inequality**) For any $0 < \lambda < 1$, one has

$$\mu(\{x \in X : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f d\mu$$

In particular, if $\int_X f d\mu < \infty$, then the sets $\{x \in X : f(x) \geq \lambda\}$ have finite measure for each $\lambda > 0$.

7. (**Finiteness**) If $\int_X f d\mu < \infty$, then $f(x)$ is **finite** for μ -**almost every** x .
8. (**Vanishing**) If $\int_X f d\mu = 0$, then $f(x)$ is zero for μ -almost every x .
9. (**Vertical truncation**) We have

$$\lim_{n \rightarrow \infty} \int_X \min\{f, n\} d\mu = \int_X f d\mu$$

10. (**Horizontal truncation**) If $E_1 \subseteq E_2 \subseteq \dots$ is an **increasing sequence** of \mathcal{B} -measurable sets, then

$$\lim_{n \rightarrow \infty} \int_X f \mathbf{1}_{E_n} d\mu = \int_X f \mathbf{1}_{\bigcup_{n=1}^{\infty} E_n} d\mu.$$

11. (**Restriction**) If Y is a measurable subset of X , then

$$\int_X f \mathbf{1}_Y d\mu = \int_Y f|_Y d\mu|_Y,$$

where $f|_Y : Y \rightarrow [0, +\infty]$ is the **restriction** of $f : X \rightarrow [0, +\infty]$ to Y , and $\mu|_Y$ is the restriction μ on Y . We will often abbreviate $\int_Y f|_Y d\mu|_Y$ (by slight abuse of notation) as $\int_Y f d\mu$.

- **Theorem 2.8** Let (X, \mathcal{B}, μ) be a measure space, and let $f, g : X \rightarrow [0, +\infty]$ be measurable. Then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

- **Proposition 2.9 (Linearity in μ).**

Let (X, \mathcal{B}, μ) be a measure space, and let $f : X \rightarrow [0, +\infty]$ be measurable.

1. $\int_X f d(c\mu) = c \times \int_X f d\mu$ for every $c \in [0, +\infty]$.
2. If μ_1, μ_2, \dots are a sequence of measures on \mathcal{B} ,

$$\int_X f d\left(\sum_{n=1}^{\infty} \mu_n\right) = \sum_{n=1}^{\infty} \int_X f d\mu_n.$$

- **Proposition 2.10 (Pushforward Measure).**

Let (X, \mathcal{B}, μ) be a measure space, and let $\varphi : X \rightarrow Y$ be $(\mathcal{B}, \mathcal{C})$ measurable from (X, \mathcal{B}) to another measurable space (Y, \mathcal{C}) . Define the **pushforward** $\varphi_*\mu : \mathcal{C} \rightarrow [0, +\infty]$ of μ **by** φ by the formula

$$\varphi_*\mu(E) := \mu(\varphi^{-1}(E)).$$

1. $\varphi_*\mu$ is a **measure** on \mathcal{C} , so that $(Y, \mathcal{C}, \phi_*\mu)$ is a measure space.
2. (**Change of variables formula**). If $f : Y \rightarrow [0, +\infty]$ is \mathcal{C} -measurable, then

$$\int_Y f d(\phi_*\mu) = \int_X (f \circ \phi) d\mu.$$

- **Corollary 2.11** Let $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an invertible linear transformation, and let m be Lebesgue measure on \mathbb{R}^d . Then $T_*m = \frac{1}{|\det T|}m$, where T_*m is **the pushforward of m** .
- **Example (Sums as integrals)**. Let X be an arbitrary set (with the **discrete σ -algebra**), let $\#$ be **counting measure**, and let $f : X \rightarrow [0, +\infty]$ be an arbitrary unsigned function. Then f is **measurable** with

$$\int_X f d\# = \sum_{x \in X} f(x).$$

2.4 Absolutely Convergent Integral

- **Definition (Absolutely convergent integral)**.

Let (X, \mathcal{F}, μ) be a measure space. A measurable function $f : X \rightarrow \mathbb{C}$ is said to be **absolutely integrable** if the *unsigned integral*

$$\|f\|_{L^1(X)} \equiv \int_X |f| d\mu$$

is **finite**. We refer to this quantity $\|f\|_{L^1(X)}$ as **the $L^1(X)$ norm of f** , and use $L^1(X)$ or $L^1(X, \mathcal{F}, \mu)$ or $L^1(\mu)$ to denote the space of absolutely integrable functions. If f is *real-valued* and absolutely integrable, we define **the Lebesgue integral** $\int_X f d\mu$ by the formula

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu$$

where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ are the magnitudes of the positive and negative components of f . (note that the two unsigned integrals on the right-hand side are finite, as f_+, f_- are pointwise dominated by $|f|$). If f is *complex-valued* and absolutely integrable, we define **the Lebesgue integral** $\int_X f(x) d\mu$ by the formula

$$\int_X f d\mu = \int_X \Re(f) d\mu + i \int_X \Im(f) d\mu,$$

where the two integrals on the right are interpreted as real-valued absolutely integrable Lebesgue integrals.

- **Remark** Sometimes $\int_X f d\mu$ is also denoted as $\int_X f(x) \mu(dx)$ or $\int_X f(x) d\mu(x)$, where $X \subseteq \mathbb{R}^d$ and $\mu(E) = \int_E \mu dx$.
- **Proposition 2.12 (Properties of absolutely convergent integral)**
Let (X, \mathcal{B}, μ) be a measure space.

1. $L^1(X, \mathcal{B}, \mu)$ is a **complex vector space**.

2. The integration map $f \mapsto \int_X f d\mu$ is a **complex linear map** from $L^1(X, \mathcal{B}, \mu)$ to \mathbb{C} .
3. The **triangle inequality**

$$\|f + g\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)} + \|g\|_{L^1(\mu)}$$

and the **homogeneity property**

$$\|c f\|_{L^1(\mu)} = |c| \|f\|_{L^1(\mu)}$$

hold for all $f, g \in L^1(X, \mathcal{B}, \mu)$ and $c \in \mathbb{C}$.

4. If $f, g \in L^1(X, \mathcal{B}, \mu)$ are such that $f(x) = g(x)$ for μ -almost every $x \in X$, then $\int_X f d\mu = \int_X g d\mu$.
5. If $f \in L^1(X, \mathcal{B}, \mu)$, and (X, \mathcal{B}', μ') is a **refinement** of (X, \mathcal{B}, μ) , then $f \in L^1(X, \mathcal{B}', \mu')$, and

$$\int_X f d\mu' = \int_X f d\mu.$$

(Hint: it is easy to get one inequality. To get the other inequality, first work in the case when f is both bounded and has finite measure support (i.e. is both vertically and horizontally truncated).)

6. If $f \in L^1(X, \mathcal{B}, \mu)$, then $\|f\|_{L^1(\mu)} = 0$ if and only if f is zero μ -almost everywhere.
7. If $Y \subseteq X$ is \mathcal{B} -measurable and $f \in L^1(X, \mathcal{B}, \mu)$, then $f|_Y \in L^1(Y, \mathcal{B}|_Y, \mu|_Y)$ and

$$\int_Y f|_Y d\mu|_Y = \int_X f \mathbb{1}_Y d\mu.$$

As before, by abuse of notation we write $\int_Y f d\mu$ for $\int_Y f|_Y d\mu|_Y$.

2.5 The Convergence Theorems

- **Proposition 2.13 (Uniform Convergence on a Finite Measure Space).** [Tao, 2011] Suppose that (X, \mathcal{B}, μ) is a **finite measure space** (so $\mu(X) < \infty$), and $f_n : X \rightarrow [0, +\infty]$ (resp. $f_n : X \rightarrow \mathbb{C}$) are a sequence of unsigned measurable functions (resp. absolutely integrable functions) that **converge uniformly** to a limit f . Then $\int_X f_n d\mu$ **converges** to $\int_X f d\mu$.

Proof: Since $f_n \rightarrow f$ uniformly, we have for all $\epsilon > 0$, $\exists N$, for $n \geq N$, $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon$, thus

$$\sup_{x \in X} (|f_n(x)| - |f(x)|) \leq \sup_{x \in X} (|f_n(x) - f(x)|) < \epsilon$$

So $|f_n| \rightarrow |f|$ and f is absolutely integrable if f_n is absolutely integrable

$$\int_X |f| d\mu = \int_X |f - f_n + f_n| d\mu \leq \int_X |f - f_n| d\mu + \int_X |f_n| d\mu \leq \epsilon \mu(X) + \int_X |f_n| d\mu < \infty$$

To prove convergence, see that we can choose N so that $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon/\mu(X)$ since $\mu(X) < \infty$, then

$$\left| \int_X f_n d\mu - \int_X f d\mu \right| = \left| \int_X (f_n - f) d\mu \right| \leq \int_X |f_n - f| d\mu \leq \epsilon/\mu(X) \int_X d\mu = \epsilon. \quad \blacksquare$$

• **Theorem 2.14 (Monotone Convergence Theorem).** [Tao, 2011]

Let (X, \mathcal{B}, μ) be a measure space, and let $0 \leq f_1 \leq f_2 \leq \dots$ be a **monotone non-decreasing** sequence of **unsigned** measurable functions on X . Then we have

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X \left(\lim_{n \rightarrow \infty} f_n \right) d\mu$$

Proof: Let $f \equiv \lim_{n \rightarrow \infty} f_n = \sup_n f_n$, then $f : X \rightarrow [0, +\infty]$ is measurable. Since the f_n are non-decreasing to f , we see from monotonicity that $\int_X f_n d\mu$ are non-decreasing and bounded above $\int_X f d\mu$, which gives the bound

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \leq \int_X f d\mu.$$

It remains to establish the reverse inequality

$$\int_X f d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

By definition, it is equivalent to show that

$$\int_X g d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

for any $0 \leq g \leq f$ simple function. By horizontal truncation we may assume without loss of generality that g is also finite everywhere. Thus by canonical representation

$$g = \sum_{i=1}^m c_i \mathbb{1}_{\{E_i\}}$$

for some $m, c_1, \dots, c_m \in (0, \infty)$ and E_1, \dots, E_m being \mathcal{F} -measurable. The integral

$$\int_X g d\mu = \sum_{i=1}^m c_i \mu \{E_i\}.$$

Let $0 < \epsilon < 1$ be arbitrary. Then we have $f(x) = \sup_n \{f_n(x)\} > (1 - \epsilon)c_i$ for all $x \in E_i$. Thus, if we define the sets

$$E_{i,n} = \{x \in E_i : f_n(x) > (1 - \epsilon)c_i\}$$

then the $E_{i,n}$ increase to E_i and are measurable. By upwards monotonicity of measure, we conclude that

$$\lim_{n \rightarrow \infty} \mu(E_{i,n}) = \mu(E_i), \quad 1 \leq i \leq m.$$

On the other hand, observe the pointwise bound

$$f_n(x) \geq (1 - \epsilon) \sum_{i=1}^m c_i \mathbb{1}_{\{E_{i,n}\}}$$

hold for any n . Integrate both sides,

$$\int_X f_n d\mu \geq (1 - \epsilon) \sum_{i=1}^m c_i \mu(E_{i,n}).$$

and take the limits $n \rightarrow \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X f_n d\mu &\geq (1 - \epsilon) \sum_{i=1}^m c_i \lim_{n \rightarrow \infty} \mu(E_{i,n}) \\ &= (1 - \epsilon) \sum_{i=1}^m c_i \mu(E_i). \end{aligned}$$

Finally, send $\epsilon \rightarrow 0$, we have the required formula. ■

- **Remark** Note that in the special case when each f_n is an indicator function $f_n = \mathbb{1}\{E_n\}$, this theorem collapses to the upwards monotone convergence property. Conversely, the upwards monotone convergence property will play a key role in the proof of this theorem.
- **Remark** Note that the result still holds if the monotonicity $f_n \leq f_{n+1}$ only holds almost everywhere rather than everywhere.
- **Corollary 2.15 (Tonelli's Theorem for Sums and Integrals)**
Let (X, \mathcal{B}, μ) be a measure space, and let f_1, f_2, \dots be a sequence of **unsigned** measurable functions on X . Then

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu$$

- **Exercise 2.16** Give an example to show that this corollary can fail if the f_n are assumed to be absolutely integrable rather than unsigned measurable, even if the sum $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for each x . (Hint: think about the three escapes to infinity.)
- **Lemma 2.17 (Borel-Cantelli Lemma).** [Tao, 2011, Resnick, 2013]
Let (X, \mathcal{B}, μ) be a measure space, and let E_1, E_2, \dots be a sequence of \mathcal{B} -measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then

$$\mu \left\{ \limsup_{n \rightarrow \infty} E_n \right\} = 0.$$

That is, almost every $x \in X$ is contained in **at most finitely many** of the E_n (i.e. $\{n \in \mathbb{N} : x \in E_n\}$ is finite for almost every $x \in X$).

(Hint: Apply Tonelli's theorem to the indicator functions $\mathbb{1}_{E_n}$.)

Proof: Consider the indicator function $f_n = \mathbb{1}\{x \in E_n\}$, which is unsigned \mathcal{B} -measurable

functions since E_n is \mathcal{B} -measurable. By Tonelli's theorem,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(E_n) &= \sum_{n=1}^{\infty} \int_X \mathbb{1}_{\{x \in E_n\}} d\mu \\
&= \int_X \left(\sum_{n=1}^{\infty} \mathbb{1}_{\{x \in E_n\}} \right) d\mu \\
&= \int_X \left(\mathbb{1}_{\left\{x \in \bigcup_{n=1}^{\infty} E_n\right\}} \right) d\mu \\
&= \mu \left(\bigcup_{n=1}^{\infty} E_n \right)
\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\mu(\bigcup_{n=1}^{\infty} E_n) < \infty$. Then by *the downwards monotone convergence* property

$$\begin{aligned}
\mu \left(\limsup_{n \rightarrow \infty} E_n \right) &= \mu \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \right) \\
&= \lim_{N \rightarrow \infty} \mu \left(\bigcup_{n=N}^{\infty} E_n \right) \\
&= \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \mu(E_n) \\
&\leq \limsup_{N \rightarrow \infty} \sum_{n=N}^{\infty} \mu(E_n) = 0,
\end{aligned}$$

since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\sum_{n=N}^{\infty} \mu(E_n) \rightarrow 0$ as $N \rightarrow \infty$. \blacksquare

- When one *does not have monotonicity*, one can at least obtain an important inequality, known as *Fatou's lemma*:

Corollary 2.18 (Fatou's Lemma).

Let (X, \mathcal{B}, μ) be a measure space, and let $f_1, f_2, \dots : X \rightarrow [0, \infty]$ be a sequence of unsigned measurable functions. Then

$$\int_X \left(\liminf_{n \rightarrow \infty} f_n \right) d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu$$

Proof: Write $F_N \equiv \inf_{n \geq N} f_n$ for each N . Then the F_N are measurable and non-decreasing, and hence by the monotone convergence theorem

$$\begin{aligned}
\int_X \left(\lim_{N \rightarrow \infty} F_N \right) d\mu &= \lim_{N \rightarrow \infty} \int_X F_N d\mu \\
\Rightarrow \int_X \left(\liminf_{n \rightarrow \infty} f_n \right) d\mu &= \lim_{N \rightarrow \infty} \int_X \inf_{n \geq N} f_n d\mu \\
&\leq \lim_{N \rightarrow \infty} \inf_{n \geq N} \int_X f_n d\mu \\
&= \liminf_{n \rightarrow \infty} \int_X f_n d\mu
\end{aligned}$$

The second last inequality holds since

$$\begin{aligned} \int_X \inf_{n \geq N} f_n d\mu &\leq \int_X f_n d\mu, \quad \forall n \geq N \\ \Rightarrow \int_X \inf_{n \geq N} f_n d\mu &\leq \inf_{n \geq N} \int_X f_n d\mu, \end{aligned}$$

which completes the proof. \blacksquare

Remark Informally, *Fatou's lemma* tells us that when taking **the pointwise limit** of **unsigned functions** f_n , that mass $\int_X f_n d\mu$ can be **destroyed in the limit** (as was the case in the three key moving bump examples), **but it cannot be created in the limit**. Of course the unsigned hypothesis is necessary here.

While this lemma was stated only for pointwise limits, the same general **principle** (*that mass can be destroyed, but not created, by the process of taking limits*) tends to hold for other “weak” notions of convergence.

- Finally, we give the other major way to shut down loss of mass via *escape to infinity*, which is to *dominate* all of the functions involved by an *absolutely convergent one*. This result is known as *the dominated convergence theorem*:

Theorem 2.19 (Dominated Convergence Theorem).

Let (X, \mathcal{B}, μ) be a measure space, and let $f_1, f_2, \dots : X \rightarrow \mathbb{C}$ be a sequence of measurable functions that converge **pointwise μ -almost everywhere** to a measurable limit $f : X \rightarrow \mathbb{C}$. Suppose that there is an **unsigned absolutely integrable** function $G : X \rightarrow [0, +\infty]$ such that $|f_n|$ are pointwise μ -almost everywhere **bounded** by G for each n . Then we have

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof: By modifying f_n, f on a null set, we may assume without loss of generality that the f_n converge to f *pointwise everywhere* rather than μ -almost everywhere, and similarly we can assume that $|f_n|$ are *bounded* by G *pointwise everywhere* rather than μ -almost everywhere.

By taking real and imaginary parts we may assume without loss of generality that f_n, f are *real*, thus $-G \leq f_n \leq G$ *pointwise*. Of course, this implies that $-G \leq f \leq G$ *pointwise* also.

If we apply Fatou's lemma to the unsigned functions $f_n + G$, we see that

$$\begin{aligned} \int_X \left(\liminf_{n \rightarrow \infty} f_n + G \right) d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (f_n + G) dx \\ \Rightarrow \int_X (f + G) d\mu &\leq \liminf_{n \rightarrow \infty} \int_X (f_n + G) dx \\ \int_X f d\mu &\leq \liminf_{n \rightarrow \infty} \int_X f_n dx \quad \left(\text{since } \int_X G dx < \infty \right) \end{aligned}$$

Similarly, if we apply that lemma to the unsigned functions $G - f_n$, we obtain

$$\begin{aligned} - \int_X f d\mu &\leq \liminf_{n \rightarrow \infty} - \int_X f_n dx \quad \left(\text{since } \int_X G dx < \infty \right) \\ \Rightarrow \int_X f d\mu &\geq \limsup_{n \rightarrow \infty} \int_X f_n dx \end{aligned}$$

It concludes that $\limsup_{n \rightarrow \infty} \int_X f_n dx = \liminf_{n \rightarrow \infty} \int_X f_n dx = \lim_{n \rightarrow \infty} \int_X f_n dx = \int_X f d\mu$. \blacksquare

- **Remark** From the moving bump examples we see that this statement *fails* if there is no *absolutely integrable dominating function* G .

- **Remark** Note also that when each of the f_n is an indicator function $f_n = \mathbb{1}_{E_n}$, the dominated convergence theorem collapses to *dominated convergence for sets* in previous chapter.

- **Proposition 2.20 (Almost dominated convergence).**

Let (X, \mathcal{B}, μ) be a measure space, and let $f_1, f_2, \dots : X \rightarrow \mathbb{C}$ be a sequence of measurable functions that converge pointwise μ -almost everywhere to a measurable limit $f : X \rightarrow \mathbb{C}$. Suppose that there is an unsigned absolutely integrable functions $G, g_1, g_2, \dots : X \rightarrow [0, +\infty]$ such that the $|f_n|$ are pointwise μ -almost everywhere bounded by $G + g_n$, and that $\int_X g_n d\mu \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

- **Exercise 2.21 (Defect version of Fatou's lemma).**

Let (X, \mathcal{B}, μ) be a measure space, and let $f_1, f_2, \dots : X \rightarrow [0, +\infty]$ be a sequence of **unsigned absolutely integrable functions** that converges **pointwise** to an absolutely integrable limit f . Show that

$$\int_X f_n d\mu - \int_X f d\mu - \|f - f_n\|_{L^1(\mu)} \rightarrow 0$$

as $n \rightarrow \infty$. (Hint: Apply the dominated convergence theorem to $\min(f_n, f)$.) Informally, this result tells us that the gap between the left and right hand sides of Fatou's lemma can be measured by the quantity $\|f - f_n\|_{L^1(\mu)}$.

- **Proposition 2.22** Let (X, \mathcal{B}, μ) be a measure space, and let $g : X \rightarrow [0, +\infty]$ be measurable. Then the function $\mu_g : \mathcal{B} \rightarrow [0, +\infty]$ defined by the formula

$$\mu_g(E) := \int_X g \mathbb{1}_E d\mu = \int_E g d\mu$$

is a measure.

- The monotone convergence theorem is, in some sense, a **defining property** of the unsigned integral:

Proposition 2.23 (Characterisation of the unsigned integral).

Let (X, \mathcal{B}) be a measurable space. $I : f \mapsto I(f)$ be a map from the space $U(X, \mathcal{B})$ of **unsigned measurable functions** $f : X \rightarrow [0, +\infty]$ to $[0, +\infty]$ that obeys the following axioms:

1. (**Homogeneity**) For every $f \in U(X, \mathcal{B})$ and $c \in [0, +\infty]$, one has $I(cf) = cI(f)$.
2. (**Finite additivity**) For every $f, g \in U(X, \mathcal{B})$, one has $I(f + g) = I(f) + I(g)$.
3. (**Monotone convergence**) If $0 \leq f_1 \leq f_2 \leq \dots$ are a **nondecreasing** sequence of unsigned measurable functions, then $I(\lim_{n \rightarrow \infty} f_n) = \lim_{n \rightarrow \infty} I(f_n)$.

Then there exists a **unique measure** μ on (X, \mathcal{B}) such that

$$I(f) = \int_X f d\mu, \quad \text{for all } f \in U(X, \mathcal{B}).$$

Furthermore, μ is given by the formula $\mu(E) := I(\mathbb{1}_E)$ for all \mathcal{B} -measurable sets E .

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