Lecture 1: Development of Measures

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1 Elementary Measure and Jordan Measure

1.1 Measure and its motivations

- **Remark** The basic **motivation**: an extension of *measure* $m(E)$ in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ as *length, area and volume* of a geometric body E.

  [Tao, 2011] A set of intuitive axioms for a measure function m defined on power set $2^{\mathbb{R}}$:

  1. The ***unit length*** of interval: $E = (0, 1]$, then $m((0, 1]) = 1$;
  2. If E is ***congruent*** to F: (There exists a proper translation, rotation or reflection from E to F), then $m(E) = m(F)$;
  3. The ***countably additive***: for a countable union of disjoint sets, $\bigcup_{k=1}^{\infty} E_k$, the measure

     $$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} m\left(E_k\right)$$

     Note that for ***uncountable union***, the intuition falls. For example, $f : \boldsymbol{x} \mapsto 2\boldsymbol{x}$ is *one-to-one correspondence*, but when it applies to measure it means that a unit length interval can be dissembled and reassembled as a length of any $k \geq 1$.

     It is seen that even with finite partition, the dissemble-reassemble procedure could generates bizarre results: see "Banach-Tarski paradox". [Tao, 2011] The idea is that the pieces used there is quite "uncommon" as the interval or cubes used in general work.

- **Remark** Unfortunately, ***these three axioms are inconsistent***: ***no proper definition of measure function*** m ***could satisfies all these three axioms for any subset in*** $\mathbb{R}$. The measure theory should be built on a collection of "ordinary" subsets, which motivates the introduction of σ***-algebra***.

- The measure for elementary sets: just as $E = (a, b), [a, b], (a, b], [a, b)$, boxes E^n, or the set E is partitioned as the *finite* union *disjoint* boxes $E_1, \cdots, E_k$. Here $m(E) = (b - a)^n$ for a box E^n, and $m(\bigcup_{i=1}^{k} E_i) = \sum_{i=1}^{k} m(E_i)$.

1.2 Elementary Measure

- **Definition** The ***elementary measure*** m is an ***algebra*** $\mathscr{A}_0$ on set $\mathbb{R}^n$ (,which is *closed* under ***finite union*** and ***complements***). Here $\mathscr{A}_0$ is the ***minimal algebra*** generated by a collection of all ***boxes*** $\bigotimes_{i=1}^{n}(a_i, b_i] \subset \mathbb{R}^n$. Define $m : \mathscr{A}_0 \to \mathbb{R}_+$ should satisfy

  1. ***Non-negative***: $m(E) \geq 0$, for all $E \in \mathscr{A}_0$;
  2. $m(\emptyset) = 0$;
  3. $m((0, 1]^n) = 1$;
  4. ***Translation-invariant***: $m\left(\boldsymbol{x} + E\right) = m(E)$ for any $\boldsymbol{x} \in \mathbb{R}^n$;

5. ***Finitely additive***: For *a finite collection* of *disjoint* sets $\{E_i : 1 \le i \le k\} \subset \mathscr{A}_0$,

$$m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i)$$

.

- **Remark** From the property above, the following properties hold

  1. ***Monotonicity property***: If $E \subseteq F$, then

  $$m(E) \le m(F),$$

  2. ***Finitely sub-additive***: For a finite collection of sets $\{E_i : 1 \le i \le k\} \subset \mathscr{A}_0$,

  $$m\left(\bigcup_{i=1}^{k} E_i\right) \le \sum_{i=1}^{k} m(E_i).$$

- **Remark** A ***box*** in $\mathbb{R}^d$ is a Cartesian product $B := I_1 \times \ldots \times I_d$ of d intervals $I_1, \ldots, I_d$ (not necessarily of the same length), thus for instance *an* ***interval*** is a one-dimensional box. The ***volume*** $|B|$ of such a box B is defined as $|B| := |I_1| \times \ldots \times |I_d|$. *An* ***elementary set*** is any *subset* of $\mathbb{R}^d$ which is the *union* of a *finite number of boxes*.

- **Remark** The collection of all *elementary sets* forms a ***boolean algebra***. That is, if $E, F \subset \mathbb{R}^d$ are elementary sets, then the *union* $E \cup F$, the *intersection* $E \cap F$, and *the set theoretic difference* $E \setminus F := \{x \in E : x \notin F\}$, and *the symmetric difference* $E \Delta F := (E \setminus F) \cup (F \setminus E)$ are also elementary.

- **Exercise 1.1** ***(Uniqueness of elementary measure)****. [Tao, 2011]*
  Let $d \le 1$. Let $\widetilde{m} : E(\mathbb{R}^d) \to \mathbb{R}_+$ be a map from the collection $E(\mathbb{R}^d)$ of elementary subsets of $\mathbb{R}^d$ to the nonnegative reals that obeys the non-negativity, finite additivity, *and* translation invariance properties. *Show that there exists a constant $c \in \mathbb{R}_+$ such that $\widetilde{m}(E) = c\, m(E)$ for all elementary sets E. In particular, if we impose the additional normalisation $\widetilde{m}(E)([0;1)^d) = 1$, then $\widetilde{m} \equiv m$. (Set $c := \widetilde{m}([0,1)^d)$, and then compute $\widetilde{m}(E)([0, \frac{1}{n})^d)$ for any positive integer n.)*

1.3 Jordan Measure

- **Definition** A generalized measure of set E can be induced by *elementary measure* of subset that ***inscribed*** F or ***circumscribed*** G of it: $F \subseteq E \subseteq G$.

  – The ***<u>outer Jordan measure</u>*** is defined as

  $$m^{*,J}(E) = \inf_{G \in \mathscr{A}_0, G \supseteq E} m(G)$$

  – The ***<u>inner Jordan measure</u>*** is defined as

  $$m_{*,J}(E) = \sup_{F \in \mathscr{A}_0, F \subseteq E} m(F)$$

- If $m^{*,J}(E) = m_{*,J}(E)$, then E is ***Jordan measureable*** and denote $m(E) \equiv m^{*,J}(E) = m_{*,J}(E)$.

- **Remark** Jordan measurable sets are those sets which are "*almost elementary*" with respect to Jordan outer measure.

- The Jordan measure has following properties:

  1. *Non-negative*: $m(E) \geq 0$, for all $E \subset \mathbb{R}^n$, E is Jordan measureable;

  2. *Translation-invariant*: $m\left(\boldsymbol{x} + E\right) = m(E)$ for any $\boldsymbol{x} \in \mathbb{R}^n$;

  3. *Finitely additive*: For a finite collection of *disjoint* sets $\{E_i : 1 \leq i \leq k\} \subset \mathbb{R}^n$ and Jordan measureable,
$$m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i).$$

  4. *Finitely sub-additive*: For a finite collection of Jordan measureable sets $\{E_i : 1 \leq i \leq k\}$,
$$m\left(\bigcup_{i=1}^{k} E_i\right) \leq \sum_{i=1}^{k} m(E_i).$$

  5. *Monotonicity*: If $E \subseteq F$, then $m(E) \leq m(F)$.

  6. *Boolean closure*: if $E, F \subset \mathbb{R}^d$ are Jordan measurable sets, then the *union* $E \cup F$, the *intersection* $E \cap F$, and *the set theoretic difference* $E \setminus F := \{x \in E : x \notin F\}$, and *the symmetric difference* $E \Delta F := (E \setminus F) \cup (F \setminus E)$ are also Jordan measurable.

- **Proposition 1.2** ***(Characterisation of Jordan measurability)****. [Tao, 2011] Let $E \subseteq \mathbb{R}^d$ be* ***bounded****. Show that the following are* ***equivalent****:*

  1. E is Jordan measurable.

  2. For every $\epsilon > 0$, there exist elementary sets $A \subseteq E \subseteq B$ such that $m(B \Delta A) \leq \epsilon$.

  3. For every $\epsilon > 0$, there exists an elementary set A such that $m^{*,J}(A \Delta E) \leq \epsilon$.

- Example of Jordan measureable set:

  - Every ***elementary set*** E is *Jordan measurable*.

  - Every ***compact convex polytope*** in $\mathbb{R}^d$ is *Jordan measurable*.

  - All ***open and closed Euclidean balls*** $B(x;r) := \{y \in \mathbb{R}^d : \|y - x\|_2 < r\}$, $\overline{B(x;r)} := \{y \in \mathbb{R}^d : \|y - x\|_2 \leq r\}$ in $\mathbb{R}^d$ are *Jordan measurable*, with Jordan measure $c_d r^d$ for some constant $c_d > 0$ depending only on d.

  - The ***graph of continuous function*** $f : B \to \mathbb{R}$ for B compact in $\mathbb{R}^n$, $G = \{(\boldsymbol{x}, f(\boldsymbol{x})), \boldsymbol{x} \in B\} \subset \mathbb{R}^{n+1}$ is *Jordan measurable*, with $m(G) = 0$.

  - The ***epigraph of continuous function*** $f : B \to \mathbb{R}$ as defined above is the set $\{(\boldsymbol{x}, t) : 0 \leq t \leq f(\boldsymbol{x}), \boldsymbol{x} \in B\} \subset \mathbb{R}^{n+1}$ is *Jordan measurable*.

- **Exercise 1.3** ***(Uniqueness of Jordan measure)****. [Tao, 2011]*
  Let $d \leq 1$. Let $\widetilde{m} : \mathcal{J}(\mathbb{R}^d) \to \mathbb{R}_+$ be a map from the collection $\mathcal{J}(\mathbb{R}^d)$ of elementary subsets of

$\mathbb{R}^d$ *to the nonnegative reals that obeys the* non-negativity, finite additivity, *and* translation invariance properties*. Show that there exists a constant* $c \in \mathbb{R}_+$ *such that* $\widetilde{m}(E) = c\, m(E)$ *for all elementary sets* E*. In particular, if we impose the additional normalisation* $\widetilde{m}(E)([0;1)^d) = 1$*, then* $\widetilde{m} \equiv m$*.*

- **Exercise 1.4** *Show that* ***the bullet-riddled square*** $[0,1]^2 \cap \mathbb{Q}^2$*, and set of* ***bullets*** $[0,1]^2 \setminus \mathbb{Q}^2$*, both have* Jordan inner measure zero *and* Jordan outer measure one*. In particular,* ***both sets are not Jordan measurable****.*

- **Remark** Informally, any set with a lot of "*holes*", or a very "*fractal*" boundary, is unlikely to be Jordan measurable. In order to measure such sets we will need to develop *Lebesgue measure.*

- **Exercise 1.5** *(**Area interpretation of the Riemann integral**). [Tao, 2011]*
  Let $[a,b]$ *be an interval, and let* $f : [a,b] \to \mathbb{R}$ *be a* ***bounded*** *function. Show that* f *is* ***Riemann integrable if and only if*** *the sets* $E_+ := \{(x,t) : x \in [a,b]; 0 \le t \le f(x)\}$ *and* $E_- := \{(x;t) : x \in [a,b]; f(x) \le t \le 0\}$ *are both* ***Jordan measurable*** *in* $\mathbb{R}^2$*, in which case one has*

$$\int_a^b f(x)dx = m^2(E_+) - m^2(E_-)$$

  where m^2 *denotes two-dimensional* ***Jordan measure****.*

2 Lebesgue Measure

2.1 Lebesgue outer measure

- **Remark** ***The countable union*** of *disjoint Jordan measureable sets* may not be *Jordan measureable.*

  Exercise 2.1 *Show that the countable union* $\bigcup_{n=1}^{\infty} E_n$ *or countable intersection* $\bigcap_{n=1}^{\infty} E_n$ *of Jordan measurable sets* $E_1, E_2, \ldots \subset \mathbb{R}$ *need not be Jordan measurable, even when bounded.*

  Also, for $E = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset \mathbb{R}^n$, the Jordan outer measure $m^{*,J}$ could be very large. For example, $m^{*,J}(\mathbb{Q} \cap [-R,R]) = 2\,R$ as $[-R,R]$ is the closure of them.

- **Definition** Define the ***Lebesgue outer measure*** [Tao, 2011]

$$m^*(E) = \inf_{\substack{E \subseteq \cup_{k=1}^{\infty} G_k, \\ \forall G_k \in \mathscr{A}_0}} \sum_{k=1}^{\infty} m(G_k) \tag{1}$$

  That is, if E has a ***a countable covering of elementary sets*** $\{G_k\} \subset \mathscr{A}_0$, then *the Lebesgue outer measure* is the ***infimum of the countable sum*** of *the elementary measures* of these sets. Here ***the countable sum*** is defined as the supremum over $k \ge 1$ of the k-summation

$$\sum_{n=1}^{\infty} a_n = \sup_{k \ge 1} \sum_{n=1}^{k} a_n$$

- **Remark** Compare to *the Lebesgue outer measure* with *the Jordan outer measure* below,

$$m^{*,J}(E) = \inf_{\substack{E \subseteq \cup_{k=1}^{n} G_k, \\ \forall G_k \in \mathscr{A}_0}} \sum_{k=1}^{n} m(G_k),$$

  we see that the Jordan outer measure is *the infimal cost* required to *cover* E by a ***finite union of boxes***, while the Lebesgue outer measure is that for ***a countable infinite union of boxes***. When the countable sum is infinite, the Lebesgue outer measure is also infinite.

  Moreover, we can show that $m^*(E) \leq m^{*,J}(E)$. This is because we can always ***pad out*** a finite union of boxes into *an infinite union* by adding *an infinite number of* ***empty boxes***.

- **Remark** Note that the similar defined "***Lebesgue inner measure***" *does not improve over the Jordan inner measure*, due to the *subadditivity* of the measure.

- **Proposition 2.2** *The Lebesgue outer measure $m^* : 2^{\mathbb{R}^n} \to \mathbb{R}_+$ satisfies the following three properties:*

  1. ***Empty-set:*** $m^*(\emptyset) = 0$*;*
  2. ***Monotonicity:*** *If $E \subset F$, then $m^*(E) \leq m^*(F)$;*
  3. ***Countably subadditivity:*** *For any countable union of sets $\{E_i\}_{i \geq 1}$ in $\mathscr{A}$*

  $$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

  Conversely, any set function $m^* : \mathscr{A} \to \mathbb{R}$ on the σ-algebra $\mathscr{A}$ on X that satisfies the three axioms above is called ***an outer measure****. [Rudin, 1987, Royden and Fitzpatrick, 1988, Folland, 2013]*

- **Lemma 2.3** *(**Finite additivity for separated sets**).*
  Let $E; F \subset \mathbb{R}^d$ be such that dist$(E; F) > 0$, where

  $$\mathit{dist}(E; F) = \inf\left\{\|\boldsymbol{x} - \boldsymbol{y}\|_2 \;\mid \boldsymbol{x} \in E, \boldsymbol{y} \in F\right\}$$

  is the distance between two sets E, F. Then $m^*(E \cup F) = m^*(E) + m^*(F)$.

  Proof: It suffice to prove that $m^*(E \cup F) \geq m^*(E) + m^*(F)$ and the other direction is the subadditivity. Suppose $m^*(E \cup F) < \infty$. (It is trivial to have infinite outer measure.)

  For any $\epsilon > 0$, we can cover the $E \cup F$ by countably infinite boxes $B_1, \cdots$, such that

  $$\sum_{n=1}^{\infty} |B_n| \leq m^*(E \cup F) + \epsilon.$$

  Suppose that each of these boxes intersects at most one of E and F. Note that for those boxes that intersect both E and F, we can partition them into smaller pieces with diameter $r < \text{dist}(E; F)$. This guarantee that each box only intersect one set.

  We divide these boxes into two parts $B_1', \cdots$, and $B_1'', \cdots$,, which only intersects E and F, respectively. Clearly, the first subfamily covers E and the second covers F.

By definition of Lebesgue outer measure,

$$m^*(E) \le \sum_{n=1}^{\infty} |B'_n|$$

and

$$m^*(F) \le \sum_{n=1}^{\infty} |B''_n|$$

Summing up these two terms, we have

$$m^*(E) + m^*(F) \le \sum_{n=1}^{\infty} |B_n| \le m^*(E \cup F) + \epsilon$$

so

$$m^*(E) + m^*(F) \le m^*(E \cup F),$$

which completes the proof. ∎

- **Lemma 2.4** ***(Outer measure of elementary sets)***
  *Let E be an **elementary set**. Then the Lebesgue outer measure $m^*(E)$ of E is equal to the elementary measure $m(E)$ of E: $m^*(E) = m(E)$.*

  Proof: Note that $m^*(E) \le m^{*,J}(E) = m(E)$, so it suffice to prove $m(E) \le m^*(E)$.

  1. If E is closed, since elementary set E is bounded, then by Heine-Borel theorem, E is compact. Then for any $\epsilon > 0$, a countable family of boxes $B_1, \cdots$, will cover E and

  $$E \subset \bigcup_{k=1}^{\infty} B_k$$
  $$\sum_{k=1}^{\infty} |B_k| \le m^*(E) + \epsilon.$$

  This family of boxes need not to be open, while we can add one more $\epsilon > 0$ so that the above inequality holds for the family of open boxes $B'_1, \cdots$, where $B_k \subset B'_1$ and $|B'_k| \le |B_k| + \epsilon/2^n$, so that

  $$E \subset \bigcup_{k=1}^{\infty} B'_k$$
  $$\sum_{k=1}^{\infty} |B'_k| \le \sum_{k=1}^{\infty} |B_k| + \sum_{k=1}^{\infty} \frac{\epsilon}{2^n} \le m^*(E) + 2\epsilon.$$

  Then by compactness, there are finite subcover $\{B'_1, \dots, B'_n\}$ of E; i.e.

  $$E \subset \bigcup_{k=1}^{n} B'_k$$
  $$m(E) \le \sum_{k=1}^{n} |B'_k|,$$

where the last inequality holds due to the subadditivity elementary measure. Note that

$$\sum_{k=1}^{n} \left|B_k'\right| \leq \sum_{k=1}^{\infty} \left|B_k'\right| \leq m^*(E) + 2\epsilon,$$

so

$$m(E) \leq m^*(E) + 2\epsilon,$$

for all $\epsilon > 0$. It suffice to show that $m(E) \leq m^*(E)$.

2. If E is not closed, we can partition E as a finite collection of disjoint boxes $Q_1 \cup \cdots Q_m$, which need not to be closed. Then for any $\epsilon > 0$, for any $1 \leq j \leq m$, so that there exists closed box $Q_j' \subset Q_j$ such that $\left|Q_j'\right| \geq |Q_j| - \epsilon/m$.

   Then E contains a finite collection of closed boxes $Q_1', \cdots, Q_m'$, so

$$\begin{aligned} m(\bigcup_{j=1}^{m} Q_j') = \sum_{j=1}^{m} |Q_j'| &\geq \sum_{j=1}^{m} |Q_j| - \sum_{j=1}^{m} \frac{\epsilon}{m} \\ &= m(E) - \epsilon, \end{aligned}$$

   for $\forall \epsilon > 0$.

   By monotonicity of outer measure, we see that

$$\begin{aligned} m^*(E) &\geq m^*(\bigcup_{j=1}^{m} Q_j') \\ &\geq m(\bigcup_{j=1}^{m} Q_j') \\ &\geq m(E) - \epsilon, \ \forall \epsilon > 0, \end{aligned}$$

   so $m^*(E) \geq m(E)$.

This completes the whole proof. ∎.

- **Proposition 2.5** ***(Lebesgue outer measure vs. Jordan outer / inner measure)*** *For any subset $E \subseteq \mathbb{R}^n$, we have the following relation between the Lebesgue outer measure and the Jordan outer and inner measure.*

$$m_{*,J}(E) \leq m^*(E) \leq m^{*,J}(E)$$

Proof: Note that we have already shown the upper bound before. Suppose that for some elementary set $F \subseteq E$, $m(F)$ attained the Jordan inner measure of E, i.e. $m_{*,J}(E) = m(F)$. Lemma 2.4 shows that the outer measure of all elementary sets are the elmentary measures. So $m(F) = m^*(F) = m_{*,J}(E)$. By monotonicity, $m^*(F) \leq m^*(E)$ thus proved the lower bound. ∎

- **Lemma 2.6** *A collection of sets are* ***almost disjoint****, if their* ***interiors*** *are disjoint. Show that for $E = \bigcup_{k=1}^{\infty} B_k$, where $B_1, \cdots$ are countable collection of almost disjoint boxes, then*

$$m^*(E) = \sum_{k=1}^{\infty} |B_k|$$

Proof: Due to the subadditivity, $m^*(E) \le \sum_{k=1}^{\infty} |B_k|$. We need to show that $m^*(E) \ge \sum_{k=1}^{\infty} |B_k|$.

Note that since $B_1, \cdots$ are almost disjoint, their volumes does not change by taking the interior, so $m(\bigcup_{k=1}^{n} B_k) = \sum_{k=1}^{n} |B_k|$ for any n.

It is seen that finite union of subcollections $\bigcup_{k=1}^{n} B_k \subset E$ for any $n \ge 1$, it follows by monotonicity that

$$m^*(E) \ge \sum_{k=1}^{n} |B_k| \, .$$

Take both size for $n \to \infty$, we have the desired result. ∎.

- **Lemma 2.7** *Let $E \subset \mathbb{R}^d$ be an **open set**. Then E can be expressed as **the countable union of almost disjoint boxes** (and, in fact, as the countable union of almost disjoint closed cubes).*

  Proof: we use the *dyadic mesh*, which is a discretized structure in $\mathbb{R}^d$. Define a *closed dyadic cube* to be a cube Q of the form

  $$Q_n = \left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \cdots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right],$$

  for $n \in \mathbb{N}, i_1, \cdots, i_d \in \mathbb{Z}$. It has side length $\frac{1}{2^n}$. These cubes for $i_1, \cdots, i_d \in \mathbb{Z}$ are almost disjoint and covers $\mathbb{R}^d$.

  Also given Q_n, $\exists Q_{n-1}$, such that $Q_n \subset Q_{n-1}$ and Q_{n-1} can be partitioned into 2^d Q_n's. As a consequence of these facts, we also obtain the important *dyadic nesting property*: given any two closed dyadic cubes (possibly of different side-length), either they are *almost disjoint*, or one of them is *contained in the other* (, since the grid points are integers.).

  For any E open, $\boldsymbol{x} \in E$, then there exists an open neighborhood $U \ni \boldsymbol{x}$ and $U \subset E$. Note that there also exists a closed dyadic cube Q_n such that $\boldsymbol{x} \in Q_n \subset U \subset E$ for some n. Let $\mathcal{Q}$ be the collection of all closed dyadic cubes that are contained in E, so $\bigcup_{Q \in \mathcal{Q}} Q \subseteq E$. It is also clear that $\bigcup_{Q \in \mathcal{Q}} Q \supseteq E$, since $\mathcal{Q}$ is a closed cover of E, thus $\bigcup_{Q \in \mathcal{Q}} Q = E$.

  Note that $\mathcal{Q}$ should be countable as the collection of all Q is countable. To make sure they are almost disjoint, we use the nested property. Note that $\mathcal{Q}$ is endowed with the partial order relation as proper inclusion. Since any simply-ordered subcollection of $\mathcal{Q}$ has an upper bound, then $\mathcal{Q}$ has maximal elements. Let $\mathcal{Q}'$ denote those cubes in $\mathcal{Q}$ that are maximal. By definition of maximal and the dyadic nested property, the elements in $\mathcal{Q}'$ are almost disjoint and $\bigcup_{Q \in \mathcal{Q}'} Q = E$. As $\mathcal{Q}'$ is at most countable, we have proved the claim. ∎

- **Lemma 2.8** ***(Outer regularity)*** *[Tao, 2011]*
  Let $E \subset \mathbb{R}^d$ be arbitrary set. Then one has

  $$m^*(E) = \inf_{E \subset U,\, U\, open} m^*(U).$$

  Proof: For monotonicity, $m^*(E) \le \inf_{E \subset U,\, U \text{open}} m^*(U)$, so we only need to show $m^*(E) \ge \inf_{E \subset U,\, U \text{open}} m^*(U)$. Assume that $m^*(E) < \infty$.

By definition of outer measure of E, there exists a countable collection of boxes $B_1, \cdots$ that covers E and

$$\sum_{k=1}^{\infty} |B_k| \leq m^*(E) + \epsilon$$

for any $\epsilon > 0$. Note that we can enlarge these boxes by open boxes $B_1', \cdots$ such that $B_k \subseteq B_k'$ and $|B_k'| \leq |B_k| + \epsilon/2^k$. Note that $\bigcup_{k=1}^{\infty} B_k' \supset E$ and it is open cover, but

$$\begin{aligned} m^*\left(\bigcup_{k=1}^{\infty} B_k'\right) &\leq \sum_{k=1}^{\infty} |B_k'| \\ &\leq m^*(E) + 2\epsilon \end{aligned}$$

Thus

$$\inf_{E \subset U,\, U \text{open}} m^*(U) \leq m^*(E) + 2\epsilon$$

for any $\epsilon > 0$, which proves the claim. ∎

- **Remark** Lemma 2.8 shows that under ***the Euclidean topology*** of $\mathbb{R}^d$, *the Lebesgue outer measure* is ***regular***; i.e., let $E \subset \mathbb{R}^d$ be arbitrary set. Then one has

  1. ***outer regular***

     $$m^*(E) = \inf_{E \subset U,\, U \textbf{ open}} m^*(U). \tag{2}$$

     and

  2. ***inner regular***

     $$m^*(E) = \sup_{E \supset C,\, C \textbf{ compact}} m^*(C). \tag{3}$$

- **Exercise 2.9** *Give an example to show that the reverse statement*

  $$m^*(E) = \sup_{E \supset U,\, U \textit{ open}} m^*(U)$$

  *is **false**.*

  Note see the ***example*** section.

2.2 Lebesgue measure

- **Definition** A set $E \subset \mathbb{R}^d$ is ***Lebesgue measureable*** if and only if for any $\epsilon > 0$, there exists **open set** U that **contains** E such that $m^*(U \setminus E) < \epsilon$.

  We refer $m(E) = m^*(E)$ as the ***Lebesgue measure*** of E, for E is ***Lebesgue measureable***.

- **Lemma 2.10** *(**Existence of Lebesgue measurable sets**).*
  *The following sets are **Lebesgue measureable sets***

1. *Any* ***open sets*** *or* ***closed set*** *in* $\mathbb{R}^d$*; The empty set* $\emptyset$ *is both open and closed, so it is Lebesgue measurable.*

2. *Any sets with* ***Lebesgue outer measure zero****; (called* ***null set****)*

3. *If* $E \subset \mathbb{R}^d$ *is Lebesgue measureable, then the* ***complement*** E^c *is also Lebesgue measureable.*

4. *Any* ***countable union*** *of Lebesgue measureable sets,* $\bigcup_{n=1}^{\infty} E_n$ *is Lebesgue measureable.*

5. *Any* ***countable intersection*** *of Lebesgue measureable sets,* $\bigcap_{n=1}^{\infty} E_n$ *is Lebesgue measureable.*

Proof: – *The proof of 1. the open set being Lebesgue measurable* follows the definition where $U = E$. *2. The null set is also Lebesgue measurable* by definition.

– We prove that *any closed set is Lebesgue measureable*. For any closed set E, we need to show that, for any ϵ, there exists open $U \supset E$ such that

$$m^*(U \setminus E) \leq \epsilon.$$

By outer regularity, for any ϵ, there exists open $U \supset E$ such that

$$m^*(U) \leq m^*(E) + \epsilon.$$

Note that $U \setminus E$ is open, so it can be decomposed into a countable collection of closed dyadic cubes $Q_1, \cdots$, that are *almost disjoint*. Thus

$$U = E \cup \left(\bigcup_{k=1}^{\infty} Q_k \right);$$

$$\text{and } m^*(U \setminus E) \leq \sum_{k=1}^{\infty} |Q_k|$$

Since E and Q_k's are almost disjoint,

$$\begin{aligned} m^*(U) &= m^*(E) + \sum_{k=1}^{\infty} |Q_k| \\ &\leq m^*(E) + \epsilon \quad \text{(by construction of } U\text{)}, \end{aligned}$$

and $m^*(E) \leq \infty$, so $\sum_{k=1}^{\infty} |Q_k| \leq \epsilon \Rightarrow m^*(U \setminus E) \leq \epsilon$, which complete our proof.

– We prove *4. any countable union of Lebesgue measureable sets is Lebesgue measureable.* Let $\epsilon > 0$ be arbitrary. By hypothesis, each E_n is contained in an open set U_n whose difference $U_n \setminus E_n$ has Lebesgue outer measure at most $\epsilon/2^n$. By *countable subadditivity*, this implies that $\bigcup_{n=1}^{\infty} E_n$ is contained in $\bigcup_{n=1}^{\infty} U_n$, and the difference $(\bigcup_{n=1}^{\infty} U_n) \setminus (\bigcup_{n=1}^{\infty} E_n)$ has Lebesgue outer measure at most ϵ. The set $\bigcup_{n=1}^{\infty} U_n$, being *a union of open sets*, is itself *open*, and the claim follows.

– Now we prove *3. the complement of a Lebesgue measureable set is Lebesgue measureable.* If E is Lebesgue measurable, then for every n we can find an open set U_n containing E such that

$$m^*(U_n \setminus E) \leq \frac{1}{n}.$$

Letting F_n be the *complement* of U_n, we conclude that the complement $\mathbb{R}^d \setminus E$ of E contains all of the F_n, and that $m^*((\mathbb{R}^d \setminus E) \setminus F_n) \leq \frac{1}{n}$. If we let $F := \bigcup_{n=1}^{\infty} F_n$, then $\mathbb{R}^d \setminus E$ contains F, and from *monotonicity* $m^*((\mathbb{R}^d \setminus E) \setminus F) = 0$, thus $\mathbb{R}^d \setminus E$ is the union of F and a set of *Lebesgue outer measure zero*. But F is in turn *the union of countably many closed sets* F_n. The claim now follows from statement 1., 2., 4.

  - The statement 5. is the result of statement 3, 4. and *de Morgans laws.* ∎

- **Remark** Based on above Lemma, the *collection of all Lebesgue measureable set* in $\mathbb{R}^d$ form a ***σ-algebra***, called ***Borel σ-algebra*** $\mathcal{B}^d$.

- **Remark** (***Lebesgue Measure vs. Jordan Measureable***)
Now we look at the Lebesgue measure $m(E)$ of a *Lebesgue measurable set* E, which is defined to equal its *Lebesgue outer measure* $m^*(E)$. If E is *Jordan measurable*, we see from last section that *the Lebesgue measure and the Jordan measure of E coincide*, thus ***Lebesgue measure extends Jordan measure***. This justifies the use of the notation $m(E)$ to denote both *Lebesgue measure* of *a Lebesgue measurable set*, and *Jordan measure* of *a Jordan measurable set* (as well as *elementary measure* of *an elementary set*)

- **Remark** Note that by outer regularity 2, there always exists U open containing E such that $m^*(U) \leq m^*(E) + \epsilon$. However, ***outer measure does not preserve the set difference***, i.e., $m^*(U \setminus E) \geq m^*(U) - m^*(E)$.

- **Lemma 2.11** *The **Lebesgue measure** $m : \mathscr{A} \to \mathbb{R}_+$, where $\mathscr{A}$ is σ-algebra **containing** Borel sets, satisfies the following properties:*

  1. *$m(\emptyset) = 0$;*

  2. ***Countably additivity**: For any countable union of disjoint sets $\{E_i\}_{i\geq 1}$ in $\mathscr{A}$*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

  *This also infers **the Finitely-additivity**.*

Proof: We just need to show number 2.

  - Case 1: *all of E_i are **compact***: Note that for two sets are disjoint, one is closed and one is compact, then *their distance* must above zero. Thus the outer measure is finitely additiveable. $m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m\left(E_i\right)$.

    By monotonicity, $m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m\left(E_i\right)$. Then we take $n \to \infty$ on both sides, $m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m\left(E_i\right)$.

    And by countable subadditivity, $m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m\left(E_i\right)$. Then it completes the proof.

  - Case 2: *E_i are **bounded***: Since E_i are measureable set, by inner regularity, we see that for every E_n, any $\epsilon > 0$, there exists a closed (bounded as a subset, thus ***compact***) set $K_n \subset E_n$ and

$$m(E_n) \leq m(K_n) + \frac{\epsilon}{2^n}.$$

Hence

$$m(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} m(E_n) \le \sum_{n=1}^{\infty} m(K_n) + \epsilon$$

and for compact K_n, we have

$$\sum_{n=1}^{\infty} m(K_n) = m\left(\bigcup_{n=1}^{\infty} K_n\right).$$

By monotonicity,

$$m\left(\bigcup_{n=1}^{\infty} K_n\right) \le m\left(\bigcup_{n=1}^{\infty} E_n\right),$$

so

$$\sum_{n=1}^{\infty} m(E_n) \le m\left(\bigcup_{n=1}^{\infty} E_n\right),$$

which complete our proof.

– Case 3: *general measureable E_i*: The basic idea is to ***decompose*** E_i as ***countable disjoint union*** of ***bounded*** measureable sets.

  First, decompose $\mathbb{R}^d$ as the countable disjoint union $\mathbb{R}^d = \bigcup_{m=1}^{\infty} A_m$ of bounded measurable sets A_m. This is due to the separable property of $\mathbb{R}^d$. Therefore each E_n can be decomposed as a countable disjoint union of bounded measureable sets $E_n \cap A_m, m = 1, \cdots,$ and

$$m(E_n) = \sum_{m=1}^{\infty} m(E_n \cap A_m).$$

  and also $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_n \cap A_m$ with countable disjoint union of bounded measureable sets. Therefore,

$$\begin{aligned} m\left(\bigcup_{n=1}^{\infty} E_n\right) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(E_n \cap A_m) \\ &= \sum_{n=1}^{\infty} m(E_n), \end{aligned}$$

  which completes our proof. ∎

- **Remark** Due to *the countably additivity property*, ***Lebesgue measure*** obeys significantly ***better*** properties than ***Lebesgue outer measure***.

- **Proposition 2.12** *(**Criteria for measurability**) [Tao, 2011]*
  The followings are equivalent:

  1. *E is Lebesgue measureable.*
  2. *(**Outer approximation** by **open**) For every $\epsilon > 0$, one can contain E in an open set U with $m^*(U \setminus E) \le \epsilon$.*

3. ***(Almost open)*** *For every $\epsilon > 0$, one can find an open set U such that $m^*(U\Delta E) \leq \epsilon$, where $U\Delta E = (U \setminus E) \cup (E \setminus U) = (U \cup E) \setminus (U \cap E)$ is the symmetric difference. (In other words, E differs from an open set by a set of outer measure at most ϵ.)*

4. ***(Inner approximation** by **closed)*** *For every $\epsilon > 0$, one can find a closed set F contained in E with $m^*(E \setminus F) \leq \epsilon$.*

5. ***(Almost closed)*** *For every $\epsilon > 0$, one can find a closed set F such that $m^*(E\Delta F) \leq \epsilon$. (In other words, E differs from a closed set by a set of outer measure at most ϵ.)*

6. ***(Almost measurable)*** *For every $\epsilon > 0$, one can find a Lebesgue measurable set E_ϵ such that $m^*(E\Delta E_\epsilon) \leq \epsilon$. (In other words, E differs from a measurable set E_ϵ by a set of outer measure at most ϵ.)*

Proof: – (1) $\Rightarrow$ (2) is the definition;

– (2) $\Rightarrow$ (3): given that every $\epsilon > 0$, one can contain E in an open set U with $m^*(U \setminus E) \leq \epsilon/2$, we want to show that E is almost open as cited above.

  Note that $U \supset E$, so $E \setminus U = \emptyset$ and since $E \setminus U$ and $U \setminus E$ are disjoint, $m^*(U\Delta E) = m^*((U \setminus E) \cup (U \setminus V)) = m^*(E \setminus U) + m^*(U \setminus E) = m^*(U \setminus E) \leq \epsilon$, which completes the proof.

– (3) $\Rightarrow$ (4): For every $\epsilon > 0$, one can find an open set U such that $m^*(U\Delta E) \leq \epsilon/2$. We need to show that for every $\epsilon > 0$, one can find a closed set F contained in E with $m^*(E - F) \leq \epsilon$.

  Note that $m^*(U\Delta E) = m^*(E \setminus U) + m^*(U \setminus E) \leq \epsilon$, where U is open. Decompose $E = (E \cap U) \cup (E \setminus U)$ and $m^*(E) = m^*(E \cap U) + m^*(E \setminus U)$. If $m^*(E \cap U) = 0$, then $m^*(E) = m^*(E \setminus U) \leq \epsilon$, then $F = \emptyset$ and $m^*(E \setminus F) = m^*(E) \leq \epsilon$.

  Suppose $m^*(E \cap U) > 0$ and $m^*(E \setminus U) \leq \epsilon/2$ with $\epsilon/2 \leq m^*(U \setminus E) \leq \epsilon$. The open set U can be decomposed by a countable collection of almost disjoint closed dyadic cubes $Q_1, \cdots$, as $U = \bigcup_{k=1}^{\infty} Q_k$. Choose a subcollection of $Q'_1, \cdots$ that intersects E and $Q''_1, \cdots$ are the rest of cubes that included in $U - E$, which result in

$$U = \bigcup_{k=1}^{\infty} Q_k = \bigcup_{k=1}^{\infty} Q'_k + \bigcup_{k=1}^{\infty} Q''_k$$

  where $E \cap U \subset \bigcup_{k=1}^{\infty} Q'_k$ and $\bigcup_{k=1}^{\infty} Q''_k \subset U - E$.

  Note that

$$m^*(E \cap U) \leq \sum_{k=1}^{\infty} |Q'_k|$$

$$\epsilon/2 \geq m^*(E \setminus U) \geq m^*(E) - \sum_{k=1}^{\infty} |Q'_k|$$

$$m^*(E) \geq \sum_{k=1}^{\infty} |Q'_k| \geq m^*(E) - \epsilon/2.$$

  For $m^*(E) < \infty$, we see that $\sum_{k=1}^{\infty} |Q'_k| < \infty$, so for given $\epsilon > 0$, there exists $N \in \mathbb{N}$

such that for all $m \geq N$

$$\sum_{k=m}^{\infty} |Q'_k| < \epsilon/2;$$

So we can choose a collection of $Q'_1, \cdots, Q'_m$, $m \geq N$ such that $\bigcup_{k=1}^{m} Q'_k \subset E \cap U$. Note that it is possible since $m^*(E) \leq m^*(E \cap U) + \epsilon$ and $m^*(E) \geq \sum_{k=1}^{\infty} |Q'_k| \geq m^*(E) - \epsilon/2$, thus $m^*(E \cap U) \geq m^*(\bigcup_{k=1}^{m} Q'_k)$ for large m.

We define $F \equiv \bigcup_{k=1}^{m} Q'_k$, and it is a closed set. Also $\bigcup_{k=1}^{m} Q'_k = F \subset E \cap U \subset \bigcup_{k=1}^{\infty} Q'_k$. Then

$$\begin{aligned}
E - F &\subseteq E - \bigcup_{k=1}^{m} Q'_k \\
&= (E \cap U) \cup (E \setminus U) - \bigcup_{k=1}^{m} Q'_k \\
&= \left(E \cap U - \bigcup_{k=1}^{m} Q'_k \right) \cup (E \setminus U) \\
&\subseteq \left(\bigcup_{k=m}^{\infty} Q'_k \right) \cup (E \setminus U) \\
m^*(E - F) &\leq \sum_{k=m}^{\infty} |Q'_k| + m^*(E \setminus U) \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

which completes the proof.

- (4) $\Rightarrow$ (5) is similar to (2) $\Rightarrow$ (3): We just see that $F \subset E$, so $F \setminus E = \emptyset$. So $m^*(E \Delta F) = m^*((E \setminus F) \cup (F \setminus E)) = m^*(E \setminus F) + m^*(F \setminus E) = m^*(E \setminus F) \leq \epsilon$, which completes the proof.
- (5) $\Rightarrow$ (6). It is trivial since any closed set $F = E_\epsilon$ is mesureable as required.
- (6) $\Rightarrow$ (1): Given every $\epsilon > 0$, one can find a Lebesgue measurable set E_ϵ such that $m^*(E \Delta E_\epsilon) = m^*(E - E_\epsilon) + m^*(E_\epsilon \setminus E) \leq \epsilon/3$. First, since E_ϵ is Lebesgue measurable set, there exists open set $U_\epsilon \supset E_\epsilon$ such that $m^*(U_\epsilon \setminus E_\epsilon) \leq \epsilon/3$. We need to modify U_ϵ to be $U \supset E$ and $m^*(U \setminus E) \leq \epsilon$.

  Second, consider for any open set $U \supset E$, we decompose $U \setminus E$ as

$$\begin{aligned}
U \setminus E &= ((U \setminus E) \cap U_\epsilon) \cup ((U \setminus E) \cap (U \setminus U_\epsilon)) \\
&= ((U \setminus E) \cap (U_\epsilon \setminus E_\epsilon)) \cup ((U \setminus E) \cap E_\epsilon) \cup ((U \setminus E) \cap (U \setminus U_\epsilon)) \\
m^*(U \setminus E) &\leq m^*((U \setminus E) \cap (U_\epsilon \setminus E_\epsilon)) + m^*((U \setminus E) \cap E_\epsilon) + m^*((U \setminus E) \cap (U \setminus U_\epsilon))
\end{aligned} \tag{4}$$

  Note that

$$\begin{aligned}
m^*((U \setminus E) \cap (U_\epsilon \setminus E_\epsilon)) &\leq m^*(U_\epsilon \setminus E_\epsilon) \leq \epsilon/3 \\
m^*((U \setminus E) \cap (U \setminus U_\epsilon)) &\leq m^*(U \setminus U_\epsilon) \\
m^*((U \setminus E) \cap E_\epsilon) &\leq m^*(E_\epsilon \setminus E) \leq \epsilon/3
\end{aligned} \tag{5}$$

Then our goal is to find $U \supset E$ such that $m^*(U \setminus U_\epsilon) < \epsilon/3$.

We can decompose U_ϵ into countable union of almost disjoint cubes $Q_1, \cdots$, as $U_\epsilon = \bigcup_{k=1}^\infty Q_k$ and let $Q'_1, \cdots$, are those cubes that meet E, so $E_\epsilon \subset \bigcup_{k=1}^\infty Q'_k$. We can enlarge each Q'_k as open set B_k so that $m^*(B_k \setminus Q'_k) \leq \frac{1}{6}\epsilon/2^k$. Then $E_\epsilon \subset \bigcup_{k=1}^\infty B_k$ and $E \subset (\bigcup_{k=1}^\infty B_k) \cup (E \setminus E_\epsilon) \subset (\bigcup_{k=1}^\infty B_k) \cup V$, where open set $V \supset (E \setminus E_\epsilon)$ with $m^*(V) \leq \epsilon$.

Finally, let $U = (\bigcup_{k=1}^\infty B_k) \cup V \supset E$ be the open set we need. Hence

$$\begin{aligned}
U \setminus U_\epsilon &= \left(\bigcup_{k=1}^\infty B_k\right) \cup V \setminus \left(\bigcup_{k=1}^\infty Q'_k\right) \cup \left(U_\epsilon \setminus \bigcup_{k=1}^\infty Q'_k\right) \\
&\subset \left[\left(\bigcup_{k=1}^\infty B_k\right) \setminus \left(\bigcup_{k=1}^\infty Q'_k\right)\right] \cup V^* \\
&\subset \left(\bigcup_{k=1}^\infty \left[B_k \setminus Q'_k\right]\right) \cup V^*; \\
m^*(U \setminus U_\epsilon) &\leq \sum_{k=1}^\infty m^*(B_k \setminus Q'_k) + m^*(V^*) \\
&\leq \sum_{k=1}^\infty \frac{1}{6}\frac{\epsilon}{2^k} + \frac{1}{6}\epsilon \\
&= \frac{1}{3}\epsilon
\end{aligned} \tag{6}$$

where $V^* = V \cup (U_\epsilon \setminus \bigcup_{k=1}^\infty Q'_k)$ is a null set with outer measure

$$\begin{aligned}
m^*(V^*) &\leq m^*(V) + m^*(U_\epsilon \setminus E_\epsilon) \\
&\leq \frac{1}{12}\epsilon + \frac{1}{12}\epsilon = \frac{1}{6}\epsilon
\end{aligned}$$

Note that $(U_\epsilon \setminus \bigcup_{k=1}^\infty Q'_k) \subset U_\epsilon \setminus E_\epsilon$ and $m^*(U_\epsilon \setminus E_\epsilon) \leq \epsilon/12$, so $(U_\epsilon \setminus \bigcup_{k=1}^\infty Q'_k)$ is a null set.

Substituting (6), and (5) into (4), we have $m^*(U \setminus E) \leq \epsilon$, which completes our proof. ∎

- **Proposition 2.13** *The Lebesgue measure satisfying the following property*

  1. *(**Upward monotone convergence**) Let $E_1 \subseteq E_2 \cdots$ be countable **non-decreasing** nested sets, we have $m\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{k\to\infty} m\left(E_k\right)$.*

  2. *(**Downward monotone convergence**) Let $E_1 \supseteq E_2 \cdots$ be countable **non-increasing** nested sets, and **if at least one E_k has finite measure** $m(E_k) < \infty$, we have $m\left(\bigcap_{k=1}^\infty E_k\right) = \lim_{k\to\infty} m\left(E_k\right)$.*

- **Proposition 2.14** *(**Carathéodory criterion**): [Tao, 2011]*
  Let $E \subset \mathbb{R}^d$, the followings are equivalent:

  1. *E is **Lebesgue measurable**;*

2. *For every* ***elementary set*** $A \subset \mathbb{R}^d$*, one has* $m(A) = m^*(A \setminus E) + m^*(A \cap E)$.

3. *For every* ***box*** B*, one has* $|B| = m^*(B \setminus E) + m^*(B \cap E)$.

Proof: (1) $\Rightarrow$ (2). We see that both A and E are Lebesgue measureable, so does $A \setminus E$ and $A \cap E$. Then since $A = (A \setminus E) \cup (A \cap E)$ for two disjoint set, then by countable additivity,

$$m(A) = m\left((A \setminus E) \cup (A \cap E)\right) = m(A \setminus E) + m(A \cap E) = m^*(A \setminus E) + m^*(A \cap E).$$

(2) $\Rightarrow$ (3). Trivial, as the box B is an elementary set.

(2) $\Rightarrow$ (1). To show E is measureable, we see to show that for any $\epsilon > 0$, there exists an open subset $U \supset E$ such that $m^*(U \setminus E) \leq \epsilon$. Suppose $m^*(E) < \infty$. By definition of outer measure, for any $\epsilon > 0$, there exists a countable collection of elementary sets $A_1, \cdots$ so that $E \subset \bigcup_{k=1}^{\infty} A_k$ and $\sum_{k=1}^{\infty} m(A_k) \leq m^*(E) + \epsilon/2$. Then since elementary set are measurable, there exists a countable collection of open sets $U_1, \cdots$ so that $A_k \subset U_k$, $m^*(U_k \setminus A_k) \leq \epsilon/2^{k+1}$.

Let $U = \bigcup_{k=1}^{\infty} U_k$ open and $E \subset \bigcup_{k=1}^{\infty} A_k \subset U$. Consider $U \setminus E \supset \bigcup_{k=1}^{\infty} A_k \setminus E$, as

$$\begin{aligned}
U \setminus E &= \left(U \setminus \bigcup_{k=1}^{\infty} A_k\right) \cup \left(\bigcup_{k=1}^{\infty} A_k \setminus E\right) \\
&= \left(\bigcup_{k=1}^{\infty} U_k \cap \bigcap_{k=1}^{\infty} A_k^c\right) \cup \left(\bigcup_{k=1}^{\infty} (A_k \cap E^c)\right) \\
&\subset \bigcup_{k=1}^{\infty} (U_k \cap A_k^c) \cup \left(\bigcup_{k=1}^{\infty} (A_k \cap E^c)\right) \\
m^*(U \setminus E) &\leq \sum_{k=1}^{\infty} m^*(U_k \setminus A_k) + \sum_{k=1}^{\infty} m^*(A_k \setminus E) \\
&\leq \epsilon/2 \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{k=1}^{\infty} m^*(A_k \setminus E) \\
&= \epsilon/2 + \sum_{k=1}^{\infty} m^*(A_k \setminus E) \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

The last inequality comes from

$$\begin{aligned}
\sum_{k=1}^{\infty} m^*(A_k \setminus E) &= \sum_{k=1}^{\infty} m(A_k) \setminus \sum_{k=1}^{\infty} m^*(A_k \cap E) \quad \text{(by additivity assumption)} \\
&\leq m^*(E) + \epsilon/2 \setminus \sum_{k=1}^{\infty} m^*(A_k \cap E) \\
&\leq m^*(E) + \epsilon/2 \setminus m^*(\bigcup_{k=1}^{\infty} A_k \cap E) \\
&= m^*(E) + \epsilon/2 \setminus m^*(E) = \epsilon/2 \quad (\text{since } E \subset \bigcup_{k=1}^{\infty} A_k)
\end{aligned}$$

(3) $\Rightarrow$ (1) Trivial, as the box B is an elementary set. ∎

- **Proposition 2.15** *(**Inner regularity**).*
  Let $E \subset \mathbb{R}^d$ be Lebesgue measurable, then

$$m(E) = \sup_{K \subset E,\, K \text{ is compact}} m(K)$$

  Proof: For E is Lebesgue measureable, for any $\epsilon > 0$, we can find a ***closed*** subset $K \subset E$, such that $m^*(E \setminus K) \leq \epsilon$. If $m(E) = \infty$, it is then clear that $m(K) = \infty$.

  Suppose $m(E) < \infty$, we only need to show that K is bounded, then for any $\epsilon > 0$, there exists compact (i.e. closed and bounded set) K such that $m^*(E \setminus K) \leq \epsilon$, so $m(E) \leq m^*(K) + \epsilon$. Then $m(E) = \sup_{K \subset E,\, K \text{ is compact}} m(K)$.

  Clearly if E is bounded, K is bounded. If else, $E = E' \cup S$, where E' is bounded with S unbounded but $m(S) = 0$. Then choose $K \subset E'$, then K is bounded. That is, if E is finite measureable, then K is as required. This completes our proof. ∎

- **Remark** The ***inner*** and ***outer regularity*** properties of measure can be used to define the concept of a ***Radon measure***.

- **Proposition 2.16** *[Tao, 2011]*
  Any Lebesgue measureable set E can be seen as

  1. *$G \setminus N$, where G is a G_δ set (i.e. $\cap_{n=1}^{\infty} U_n$ for **open sets** U_n) and N is a null set; or*

  2. *$F \cup N$ where F is a F_σ set (i.e. $\cup_{n=1}^{\infty} F_n$ for **closed sets** F_n) and N is a null set.*

- **Proposition 2.17** *(**Translation invariance**).*
  If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, show that $E + x$ is Lebesgue measurable for any $x \in \mathbb{R}^d$, and that $m(E + x) = m(E)$.

- **Remark** (***Change of Variables***).
  If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a *linear transformation*, $T(E)$ is *Lebesgue measurable*, and that $m(T(E)) = |\det T|\, m(E)$. We caution that if $T : \mathbb{R}^d \to \mathbb{R}^{d'}$ is a linear map to a space $\mathbb{R}^{d'}$ of strictly ***smaller dimension*** than $\mathbb{R}^d$, then $T(E)$ *need not be Lebesgue measurable*;

- **Proposition 2.18** *(**Uniqueness of Lebesgue measure**).*
  *Lebesgue measure $E \mapsto m(E)$ is **the only map** from Lebesgue measurable sets to $[0, +\infty]$ that obeys the following **axioms:***

  1. *(**Empty set**) $m(\emptyset) = 0$.*

  2. ***Countably additivity**: For any countable union of disjoint Lebesgue measurable sets $\{E_i\}_{i \geq 1}$ in $\mathscr{A}$*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

  3. *(**Translation invariance**) If E is Lebesgue measurable and $x \in \mathbb{R}^d$, then $m(E + x) = m(E)$.*

  4. *(**Normalisation**) $m([0, 1]^d) = 1$.*

- **Exercise 2.19** Lebesgue measure *can be viewed as a* ***metric completion*** *of* ***elementary measure****. [Tao, 2011];*

  1. *Let $2^A := \{E : E \subseteq A\}$ be the power set of A. We say that two sets $E, F \in 2^A$ are equivalent if $E \Delta F$ is a null set. Show that this is a* ***equivalence relation****.*

  2. *Let $2^A/\sim$ be the set of equivalence classes $[E] := \left\{F \in 2^A : E \sim F\right\}$ of 2^A with respect to the above equivalence relation. Define a* ***distance*** *$d : 2^A/\sim \times 2^A/\sim \to \mathbb{R}_+$ between two equivalence classes $[E], [E']$ by defining $d([E],[E']) := m^*(E\Delta E')$.*

     Show that this distance is well-defined (in the sense that $m(E\Delta E') = m(F\Delta F')$ whenever $[E] = [F]$ and $[E'] = [F']$) and gives $2^A/\sim$ the structure of ***a complete metric space****.*

  3. *Let $\mathcal{E} \subset 2^A$ be the* ***elementary subsets*** *of A, and let $\mathcal{L} \subset 2^A$ be the* ***Lebesgue measurable subsets*** *of A. Show that $\mathcal{L}/\sim$ is the* ***closure*** *of $E/\sim$ with respect to the* ***metric*** *defined above.*

     In particular, $\mathcal{L}/\sim$ is a ***complete metric space*** *that contains $E/\sim$ as a* ***dense*** *subset; in other words, $\mathcal{L}/\sim$ is a* ***metric completion*** *of $E/\sim$.*

- **Remark** (***Lebesgue Measurable Sets Do Not Cover All Subsets***)
  There exists a subset $E \subset [0,1]$ which is not Lebesgue measurable. (by axiom of choice). Thus the Lebesgue measure is not defined on the whole power set of $[0,1]$.

  Consider ***the quotient group*** $\mathbb{R}/\mathbb{Q} = \{x + \mathbb{Q} : x \in \mathbb{R}\}$ and let $E := \{x_C \in C \cap [0,1] : C \in \mathbb{R}/\mathbb{Q}\}$ be the collection of all the coset representatives $x_C \in C \cap [0,1]$. Note that each coset C of $\mathbb{R}/\mathbb{Q}$ is ***dense*** in $\mathbb{R}$ so $C \cap [0,1] \neq \emptyset$. We can show that E ***is not Lebesgue measurable***.

  On the other hand, one can construct ***finitely additive translation invariant extensions*** of Lebesgue measure to the power set of $\mathbb{R}$ by the Hahn-Banach theorem.

- **Example** ***Projections of measurable sets need not be measurable***:
  Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate projection $\pi(x; y) := x$. Then there exists a measurable subset E of R^2 such that $m(E)$ is not measurable.

- **Remark** Recall from the beginning that there is no hope to have *countably additivity*, *translation invariance* and *normalisation* for all subsets in $\mathbb{R}^d$. But we can see that there is a large collection of Lebesgue measurable sets that fit all three desireable properties. We will see that the rest are all with ***zero measures***.

3 The Development of Measure Theory

3.1 Definition Summary

1. Begin with the ***minimal algebra*** $\mathscr{A}_0$ generated by *a collection of all boxes* $\bigotimes_{i=1}^{n}(a_i, b_i] \subset \mathbb{R}^n$, ***the elementary measure*** is *a generalization of volumes* in $\mathbb{R}^n$:
   (a) ***Non-negative***: $m(E) \geq 0$, for all $E \in \mathscr{A}_0$;
   (b) $m(\emptyset) = 0$;
   (c) $m((0,1]^n) = 1$;
   (d) ***Translation-invariant***: $m(\boldsymbol{x} + E) = m(E)$ for any $\boldsymbol{x} \in \mathbb{R}^n$;
   (e) ***Finitely additive***: For a finite collection of *disjoint* sets $\{E_i : 1 \leq i \leq k\} \subset \mathscr{A}_0$,
   $$m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i).$$

   From the property above, the following properties hold
   (a) ***Monotonicity property***: If $E \subseteq F$, then
   $$m(E) \leq m(F),$$
   (b) ***Finitely sub-additive***: For a finite collection of sets $\{E_i : 1 \leq i \leq k\} \subset \mathscr{A}_0$,
   $$m\left(\bigcup_{i=1}^{k} E_i\right) \leq \sum_{i=1}^{k} m(E_i).$$

   Remark The elementary set has a lot of desireable properties but it is ***very limited***. For instance, open balls, and triangles are not counted as elementary set. Similar for many convex polytopes etc.

2. For ***arbitrary bounded subset*** $E \subset \mathbb{R}^n$, it is possible that $E \notin \mathscr{A}_0$. ***The Jordan measure*** is proposed to *generalize* the elementary measure m on E,
   - The ***outer Jordan measure*** is defined as
   $$m^{*,J}(E) = \inf_{G \in \mathscr{A}_0, G \supset E} m(G)$$
   - The ***inner Jordan measure*** is defined as
   $$m_{*,J}(E) = \sup_{F \in \mathscr{A}_0, F \subset E} m(G)$$
   - If $m^{*,J}(E) = m_{*,J}(E)$, then E is ***Jordan measureable*** and denote $m(E) \equiv m^{*,J}(E) = m_{*,J}(E)$.

   The collection of Jordan measurable sets form an ***algebra*** $\mathscr{A}_1 \supset \mathscr{A}_0$ on $\mathbb{R}^n$ and ***the measure function*** extended on $\mathscr{A}_1$ preserve the property as above:

(a) *Non-negative*: $m(E) \geq 0$, for all $E \subset \mathbb{R}^n$, E is Jordan measureable;

(b) *Translation-invariant*: $m(\boldsymbol{x} + E) = m(E)$ for any $\boldsymbol{x} \in \mathbb{R}^n$;

(c) *Finitely additive*: For a finite collection of *disjoint* sets $\{E_i : 1 \leq i \leq k\} \subset \mathbb{R}^n$ and Jordan measureable,

$$m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i).$$

(d) *Finitely sub-additive*: For a finite collection of Jordan measureable sets $\{E_i : 1 \leq i \leq k\}$,

$$m\left(\bigcup_{i=1}^{k} E_i\right) \leq \sum_{i=1}^{k} m(E_i).$$

(e) *Monotonicity*: If $E \subseteq F$, then $m(E) \leq m(F)$.

Remark The algebra $\mathscr{A}_1$ formed by all Jordan measureable sets are much larger than that of elementary set $\mathscr{A}_0$. It can be shown that the Jordan measures are closely related to ***the Riemann integral***. However, like the Riemann integral, the definition of Jordan measure sets still has a lot of ***limitations***, esp. when dealing with sets that are ***countable infinite union*** of Jordan measureable sets. In general, when the set has a lot of "holes" or very "fractal", the set is not likely Jordan measurable. Thus, we need to generalize the definition of Jordan measure to cover the limit of sets.

3. ***Lebesgue outer measure*** is defined as

$$m^*(E) = \inf\left\{\sum_{i=1}^{\infty} m(B_i) \middle| \, E \subset \bigcup_{i=1}^{\infty} B_i, B_i \in \mathscr{A}_0\right\}.$$

The ***Lesbuge outer measure*** $m^* : 2^X = \mathscr{P}(\mathbb{R}^n) \to \mathbb{R}_+$ satisfies the following three properties:

(a) $m^*(\emptyset) = 0$;

(b) *Monotonicity*: If $E \subset F$, then $m^*(E) \leq m^*(F)$;

(c) ***Countably subadditivity***: For any *countable union of subsets* $\{E_i\}_{i \geq 1}$ in $\mathbb{R}^n$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

Note that outer measure does not need to be defined on σ-algebra.

4. A set $E \subset \mathbb{R}^d$ is ***Lebesgue measureable*** if and only if for any $\epsilon > 0$, there exists ***open set*** U that contains E such that $m^*(U - E) < \epsilon$. If E is *Lebesgue measureable*, the outer measure of E is called ***Lebesgue measure***, $m(E) = m^*(E)$.

In other word, ***the Lebesgue measure*** $m : \mathscr{A} \to \mathbb{R}_+$, where $\mathscr{A}$ is ***σ-algebra*** containing *Borel sets*, satisfies the following properties:

(a) $m(\emptyset) = 0$;

(b) ***Countably subadditivity***: For any countable union of disjoint sets $\{E_i\}_{i\geq 1}$ in $\mathscr{A}$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(c) ***Finitely-additivity*** *(derived from above)*: For any finite union of disjoint sets $\{E_i\}_{1\leq i\leq k}$ in $\mathscr{A}$

$$m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i).$$

The *collection of* ***all Lebesgue measureable set*** form a ***σ-algebra*** $\mathscr{A} \supset \mathscr{A}_1$ (Borel sets in $\mathbb{R}^d$).

5. Finally, we see that *the collection of all Lebesgue measureable sets* is ***the only one*** that contains all desired properties of *measure*:

(a) (***Empty set***) $m(\emptyset) = 0$.

(b) ***Countably additivity***: For any countable union of disjoint Lebesgue measurable sets $\{E_i\}_{i\geq 1}$ in $\mathscr{A}$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

(c) (***Translation invariance***) If E is Lebesgue measurable and $x \in \mathbb{R}^d$, then $m(E + x) = m(E)$.

(d) (***Normalisation***) $m([0, 1]^d) = 1$.

3.2 Table Summary

Table 1: Comparison between different measures in measure theory

	Elementary measure	***Jordan measure***	***Lebesgue outer measure***	***Lebesgue measure***
compatibility		$\Leftarrow \checkmark$	$\Leftarrow \checkmark$	$\Leftarrow \checkmark$
non-negative	✓	✓	✓	✓
$m(\emptyset) = 0$	✓	✓	✓	✓
$m([0,1]^d) = 1$	✓	✓	✓	✓
translation-invariant	✓	✓	✓	✓
finitely additive	✓	✓	✓	✓
monotonicity	✓	✓	✓	✓
finitely subadditive	✓	✓	✓	✓
outer regularity			✓	✓
inner regularity			✓	✓
countably subadditivity			✓	✓
countably additivity				✓
measurable set	box $I_1 \times \ldots \times I_d$	All *elementary sets*; any ***compact convex*** *polytope*; any ***open sets*** and ***closed sets***; ***finite union*** of measurable sets; ***graph/epigraph*** *of* ***continous function***;	All *Jordan measurable sets*; ***countable union*** of measurable sets, e.g. G_δ and F_σ	forms a σ-***algebra*** that includes ***all Borel sets***; sets with Lebesgue outer measure zero (***null sets***).
non-measurable set	*any subsets* other than box	*countable union* of Jordan measurable sets; ***bullet-riddled square*** and ***sets of bullets***; subsets with a lot of "*holes*" or "*fractal*"	same as right	$E = \mathbb{R}/\mathbb{Q} \cap [0,1]$
algebra *for collection of measurable sets*	***boolean algebra*** $\mathscr{A}_0$	*boolean algebra* $\mathscr{A}_1 \supsetneq \mathscr{A}_0$		σ-***algebra*** $\mathscr{A}_2 \supsetneq \mathscr{A}_1$
relation to ***integration***		***Riemann integration***		***Lebesgue integration***

4 Counterexamples

- **Example** For the countable set $\mathbb{Q} \cap [0,1]$, it has countable open covers

$$U \equiv \bigcup_{k=1}^{\infty} (q_k - \epsilon/2^{n+1}, q_k + \epsilon/2^{n+1}), \quad q_k \in \mathbb{Q} \cap [0,1],$$

  for any $\epsilon > 0$.

  The by countable subadditivity,

$$\begin{aligned} m^*(U) &\le \sum_{k=1}^{\infty} m((q_k - \epsilon/2^n, q_k + \epsilon/2^n)) \\ &= \sum_{k=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \end{aligned}$$

  Also it is seen that since U is dense in $[0,1]$, i.e., $\overline{U} \supseteq [0,1]$, therefore

$$m^{*,J}(U) = m^{*,J}(\overline{U}) \ge m^{*,J}([0,1]) = 1.$$

  It shows that ***the Jordan outer measure is different from the Lebesgue outer measure***.

  Also, it is seen that ***bounded open set*** U is ***not Jordan measureable***, but it is Lebesgue measureable. ∎

- **Example** Give an example that satisfies the following

$$m^*(E) > \sup_{E \supset U,\, U \text{ open}} m^*(U).$$

  There are ***Cantor sets*** C that is ***nowhere dense*** with positive measure. That is $m^*(C) > 0$ but C contains ***no interval*** so it is $\sup_{E \supset U,\, U \text{ open}} m^*(U) = 0$.

  The set of irrational numbers $[0,1] - \mathbb{Q} \cap [0,1]$ has outer measures 1 but contains no interval, so $\sup_{E \supset U,\, U \text{ open}} m^*(U) = 0$. ∎

- **Example** (***Non-Lebesgue-Measurable Set in*** $[0,1]$)
  Consider ***the quotient group*** $\mathbb{R}/\mathbb{Q} = \{x + \mathbb{Q} : x \in \mathbb{R}\}$ and let $E := \{x_C \in C \cap [0,1] : C \in \mathbb{R}/\mathbb{Q}\}$ be *the collection of all the coset representatives* $x_C \in C \cap [0,1]$. Note that each coset C of $\mathbb{R}/\mathbb{Q}$ is ***dense*** in $\mathbb{R}$ so $C \cap [0,1] \neq \emptyset$. We can show that E ***is not Lebesgue measurable***.

  Let y be any element of $[0,1]$. Then it must lie in some coset C of $\mathbb{R}/\mathbb{Q}$, and thus differs from x_C by some *rational number* in $[-1,1]$. In other words, we have

$$[0,1] \subseteq \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (E + q). \tag{7}$$

  On the other hand, we clearly have

$$\bigcup_{q \in \mathbb{Q} \cap [-1,1]} (E + q) \subseteq [-1,2]. \tag{8}$$

Also, the different translates $E + q$ are ***disjoint***, because E contains only one element from each *coset* of $\mathbb{Q}$.

To see why E is not Lebesgue measurable, suppose *for* ***contradiction*** that E was Lebesgue measurable. Then the translates $E + q$ would also be Lebesgue measurable. By countable additivity, we thus have

$$m\left(\bigcup_{q\in\mathbb{Q}\cap[-1,1]}(E+q)\right) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m\,(E+q)$$

and thus by translation invariance and (7), (8) we have

$$1 = m([0,1]) \leq m\left(\bigcup_{q\in\mathbb{Q}\cap[-1,1]}(E)\right) \leq m([-1,2]) = 2-(-1) = 3$$

On the other hand, the sum $\sum_{q\in\mathbb{Q}\cap[-1,1]} m\,(E)$ is either ***zero*** (if $m(E) = 0$) or ***infinite*** (if $m(E) > 0$), leading to the desired ***contradiction***. ∎

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