Lecture 4: Submersions, Immersions, and Embeddings

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Oct. 17th., 2022

Contents

1 Maps of Constant Rank

1.1 Submersions and Immersions

- The key linear-algebraic property of *a linear map* is its ***rank***. In fact, the rank is the ***only property*** that distinguishes different linear maps if we are free to choose bases independently for the domain and codomain.

- **Definition** Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$, we define *the **rank** of F at p* to be ***the rank of the linear map*** $dF_p : T_pM \to T_{F(p)}N$; it is ***the rank of the Jacobian matrix*** of F in any smooth chart, or ***the dimension of*** $\text{Im}\, dF_p \subseteq T_{F(p)}N$. If F has the same rank r *at every point*, we say that it has constant rank, and write $\text{rank}\, F = r$.

- **Definition** Note that rank $dF_p \leq \min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, we say that F ***has full rank at*** p, and if F has full rank everywhere, we say F ***has full rank***.

- **Definition** The most important *constant-rank maps* are those *of full rank*. A smooth map $F : M \to N$ is called ***a smooth submersion*** if its differential is ***surjective*** *at each point* (or ***equivalently***, if $\text{rank}\ F = \dim N$).

  It is called ***a smooth immersion*** if its differential is ***injective*** at each point (***equivalently***, $\text{rank}\ F = \dim M$).

- **Remark** (***Submersion vs. Surjective and Immersion vs. Injective***)
  A map $F : M \to N$ is *surjective* if the preimage $F^{-1}(N)$ covers its domain M. It is submersion if ***its differential*** dF_p at p is *surjective* for all p. Similarly, F is injective, if $F(a) \neq F(b)$ when $a \neq b$. It is immersion if ***its differential*** dF_p at p is *injective* for all p.

  The concept of ***submersion/immersion*** is about ***the local differential property*** of F while ***the surjective/injective*** is about ***the global property*** of F. Local property can be determined by the global property but not vice versa. Thus *a map can be submersion but may not be surjective. A map can be immerison but may not be injective.*

- **Proposition 1.1** *Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is **surjective**, then p has a neighborhood U such that $F|_U$ is a **submersion**. If dF_p is **injective**, then p has a neighborhood U such that $F|_U$ is an **immersion**.*

- **Remark** As we will see in this chapter, *smooth submersions* and *immersions* behave *locally* like surjective and injective *linear maps*, respectively.

- **Example** (***Submersions and Immersions***).

  - Suppose $M_1, \ldots, M_k$ are smooth manifolds. Then each of ***the projection maps*** $\pi_i : M_1 \times \ldots \times M_k \to M_i$ is ***a smooth submersion***. In particular, *the projection* $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ *onto* the first n coordinates is *a smooth submersion*.

  - If $\gamma : J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is ***a smooth immersion*** if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

  - If M is a smooth manifold and its tangent bundle TM is given the smooth manifold structure described in Chapter 3, ***the projection*** $\pi : TM \to M$ is ***a smooth submer-***

sion.

To verify this, just note that with respect to any smooth local coordinates (x^i) on an open subset $U \subseteq M$ and the corresponding natural coordinates (x^i, v_i) on $\pi^{-1}(U) \subseteq TM$, the coordinate representation of π is $\widehat{\pi}(x, v) = x$.

– The smooth map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = ((2 + \cos(2\pi u)) \cos(2\pi v),\ (2 + \cos(2\pi u)) \sin(2\pi v),\ \sin(2\pi u))$$

is a *smooth immersion* of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis .

1.2 Local Diffeomorphisms

- **Definition** If M and N are smooth manifolds with or without boundary, a map $F : M \to N$ is called ***a local diffeomorphism*** if every point $p \in M$ has a neighborhood U such that $F(U)$ is ***open*** in N and the restriction $F|_U : U \to F(U)$ is a ***diffeomorphism***.

- The next theorem is the key to the most important properties of local diffeomorphisms.

  Theorem 1.2 *(**Inverse Function Theorem for Manifolds**). [Lee, 2003.]*
  *Suppose M and N are smooth manifolds, and $F : M \to N$ is a **smooth map**. If $p \in M$ is a point such that dF_p is **invertible**, then there are **connected** neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a **diffeomorphism**.*

- **Remark** It is important to notice that we have stated Theorem above ***only for manifolds without boundary***. In fact, it can fail for a map whose domain has nonempty boundary.

- **Proposition 1.3** *(**Elementary Properties of Local Diffeomorphisms**).*

  1. *Every **composition** of local diffeomorphisms is a local diffeomorphism.*
  2. *Every **finite product** of local diffeomorphisms between smooth manifolds is a local diffeomorphism.*
  3. *Every local diffeomorphism is a **local homeomorphism** and **an open map**.*
  4. *The **restriction** of a local diffeomorphism to an **open submanifold** with or without boundary is a local diffeomorphism.*
  5. *Every **diffeomorphism** is a local diffeomorphism.*
  6. *Every **bijective** local diffeomorphism is a diffeomorphism.*
  7. *A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a **coordinate representation** that is a **local diffeomorphism**.*

- **Proposition 1.4** *Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is **a map**.*

  1. *F is a local diffeomorphism if and only if it is **both** a **smooth immersion** and a **smooth submersion**.*

2. *If dim M = dim N and F is* ***either*** *a* ***smooth immersion*** *or a* ***smooth submersion****, then it is a local diffeomorphism.*

1.3 The Rank Theorem

- The most important fact about constant-rank maps is the following consequence of the inverse function theorem, which says that a constant-rank smooth map can be placed locally into a particularly simple canonical form by a change of coordinates.

- **Theorem 1.5** ***(Rank Theorem)****. [Lee, 2003.]*
Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map ***with constant rank*** *r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a* ***coordinate representation*** *of the form*
$$\widehat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0). \tag{1}$$
In particular, if F is a ***smooth submersion****, this becomes*
$$\widehat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n). \tag{2}$$
and if F is a ***smooth immersion****, it is*
$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0). \tag{3}$$

- **Corollary 1.6** *Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are* ***equivalent****:*

  1. *For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which* ***the coordinate representation of F is linear****.*

  2. *F has* ***constant rank****.*

- **Remark** The canonical representation for a *smooth submersion* in (2) is called the ***canonical surjection*** or ***the projection map***, denoted as π; Similarly, the canonical representation for a *smooth immersion* in (3) is called the ***canonical injection*** or ***the inclusion map***, denoted as ι.

  The Rank Theorem states that ***every smooth immersion is an inclusion map locally*** and ***every smooth submersion is a projection map locally***, both regardless of the form of the map itself.

- The rank theorem is a purely ***local statement***. However, it has the following powerful ***global*** *consequence.*

  Theorem 1.7 ***(Global Rank Theorem)****. [Lee, 2003.]*
  Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of ***constant rank****.*

  1. *If F is* ***surjective****, then it is a* ***smooth submersion****.*

  2. *If F is* ***injective****, then it is a* ***smooth immersion****.*

  3. *If F is* ***bijective****, then it is a* ***diffeomorphism****.*

1.4 The Rank Theorem for Manifolds with Boundary

- **Remark** In the context of manifolds with boundary, we need the rank theorem only in one special case: that of a smooth immersion whose domain is a smooth manifold with boundary.

- **Theorem 1.8** ***(Local Immersion Theorem for Manifolds with Boundary).*** *[Lee, 2003.]*
  Suppose M is a smooth m-manifold with boundary, N is a smooth n-manifold, and $F : M \to N$ is a smooth immersion. For any $p \in \partial M$; there exist a smooth boundary chart (U, φ) for M centered at p and a smooth coordinate chart (V, ψ) for N centered at $F(p)$ with $F(U) \subseteq V$, in which F has the coordinate representation

$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0) \tag{4}$$

2 Embeddings

2.1 Definitions

- One special kind of ***immersion*** is particularly important.

  Definition If M and N are smooth manifolds with or without boundary, a ***smooth embedding*** of M into N is a ***smooth immersion*** $F : M \to N$ that is ***also*** ***a topological embedding***, i.e., *a* ***homeomorphism*** *onto its image* $F(M) \subseteq N$ in the *subspace topology*.

- **Remark** ***A smooth embedding*** is a map that is ***both*** a ***topological embedding*** and a ***smooth immersion***, not just a topological embedding that happens to be smooth.

  Also a map is a ***smooth embedding*** $\Rightarrow$ ***the map is an injective smooth immersion***. The reverse is **not** necessarily **true** since this map also need to have ***continuous inverse*** from $F(M)$ to domain M.

- **Example** (***Smooth Embeddings***).

  1. If M is a smooth manifold with or without boundary and $U \subseteq M$ is an ***open submanifold***, the ***inclusion map*** $U \hookrightarrow M$ is a smooth embedding.

  2. If $M_1, \ldots, M_k$ are smooth manifolds and $p_i \in M_i$ are arbitrarily chosen points, each of the maps $\iota_j : M_j \to M_1 \times \ldots \times M_k$ given by
  $$\iota_j(q) = (p_1, \ldots, p_{j-1}, q, p_{j+1}, \ldots, p_k)$$
  is ***a smooth embedding***. In particular, the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ given by sending $(x^1, \ldots, x^n)$ to $(x^1, \ldots, x^n, 0, \ldots, 0)$ is *a smooth embedding*.

  3. The smooth map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by
  $$X(u, v) = ((2 + \cos(2\pi u)) \cos(2\pi v),\ (2 + \cos(2\pi u)) \sin(2\pi v),\ \sin(2\pi u))$$
  descends to ***a smooth embedding*** of the ***torus*** $\mathbb{S}^1 \times \mathbb{S}^1$ into $\mathbb{R}^3$.

- **Remark** To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of ***injective*** *smooth maps that are* ***not smooth embeddings***.

1. **Example** The map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, 0)$ is a smooth map and ***a topological embedding***, but it is ***not a smooth embedding*** because $\gamma'(0) = 0$. (i.e. it is ***not a smooth immerision***.)

2. **Example** (***A Dense Curve on the Torus***). Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let α be any ***irrational number***. The map $\gamma : \mathbb{R} \to \mathbb{T}^2$ given by

   $$\gamma(t) = \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right)$$

   is a *smooth immersion* because $\gamma'(t)$ never vanishes. It is also *injective*, because $\gamma(t_1) = \gamma(t_2)$ implies that both $t_1 - t_2$ and $\alpha t_1 - \alpha t_2$ are integers, which is impossible unless $t_1 = t_2$.

   Consider the set $\gamma(\mathbb{Z}) = \{\gamma(n) : n \in \mathbb{Z}\}$. It follows from Dirichlets approximation theorem that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$. But this means that γ is ***not a homeomorphism*** onto its image, because $\mathbb{Z}$ has no limit point in $\mathbb{R}$.

3. **Example** (***The Figure-Eight Curve***).
   Consider the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ defined by

   $$\beta(t) = (\sin 2t, \sin t).$$

   Its image is a set that looks like a figure-eight in the plane, sometimes called a ***lemniscate***. (It is the locus of points (x, y) where $x^2 = 4y^2(1 - y^2)$, as you can check.) It is easy to see that β is an ***injective smooth immersion*** because $\beta'(t)$ never vanishes; but it is ***not a topological embedding***, because its image is compact in the subspace topology, while its domain is not.

- **Proposition 2.1** *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is an* ***injective smooth immersion****. If* **any** *of the following holds, then F is a* ***smooth embedding****.*

  1. *F is an* **open** *or* **closed** *map. (i.e. it maps an open/closed set to an open/closed set)*
  2. *F is a* ***proper map****. (i.e. the preimage of every compact set is compact)*
  3. *M is* ***compact****.*
  4. *M has empty boundary and dim M = dim N*

- **Theorem 2.2** *(**Local Embedding Theorem**). [Lee, 2003.]*
  Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Then F is a ***smooth immersion*** *if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a* ***smooth embedding****.*

- **Definition** If X and Y are topological spaces, a continuous map $F : M \to N$ is called ***a topological immersion*** if every point of X has a neighborhood U such that $F|_U$ is a **topological embedding**.

- **Remark** Thus, **every smooth immersion is a topological immersion**; but, just as with embeddings, a topological immersion that happens to be smooth need not be a smooth immersion.

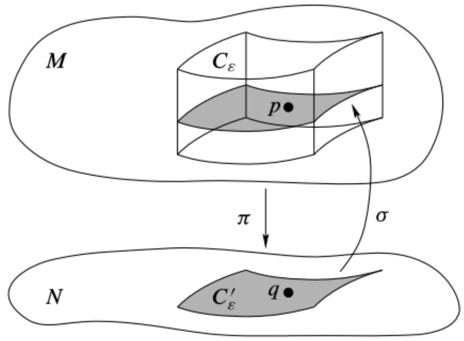

Figure 1: The local section of a submersion. [Lee, 2003.]

3 Submersions

- **Definition** If $\pi : M \to N$ is any continuous map, a ***section*** of π is a ***continuous right inverse*** for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$:

$$\begin{tikzcd} M \arrow[r, "\pi"] & N \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

- **Definition** A ***local section*** of π is a continuous map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$

- **Theorem 3.1** *(**Local Section Theorem**). [Lee, 2003.]*
*Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then π is a **smooth submersion** if and only if **every point** of M is in the **image** of a **smooth local section** of π.*

- **Definition** If X and Y are topological spaces, a continuous map $\pi : M \to N$ is called ***a topological submersion*** if every point of X is in the **image** of a **(continuous) local section** of π.

- **Proposition 3.2** *(**Properties of Smooth Submersions**).*
*Let M and N be smooth manifolds, and suppose $\pi : M \to N$ is a smooth submersion. Then π is **an open map**, and if it is **surjective** it is a **quotient map**.*

- The next three theorems provide important tools that we will use frequently when studying submersions.

  Theorem 3.3 *(**Characteristic Property of Surjective Smooth Submersions**).*
  *Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a **surjective smooth submersion**. For any smooth manifold P with or without boundary, a map $F : N \to P$ is **smooth** if and only if $F \circ \pi$ is **smooth**:*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F\circ\pi"] & \\ N \arrow[r, "F"'] & P. \end{tikzcd}$$

  Theorem 3.4 *(**Passing Smoothly to the Quotient**).*
  *Suppose M and N are smooth manifolds and $\pi : M \to N$ is a **surjective smooth submer-***

sion. *If P is a smooth manifold with or without boundary and $F : M \to P$ is a smooth map that is* ***constant*** *on* ***the fibers of*** *π, then there exists a* ***unique smooth map*** *$\widetilde{F} : N \to P$ such that $\widetilde{F} \circ \pi = F$:*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\widetilde{F}"'] & P. \end{tikzcd}$$

Theorem 3.5 *(**Uniqueness of Smooth Quotients**).*
Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are ***surjective smooth submersions*** *that are* ***constant*** *on* ***each other's fibers***. *Then there exists a* ***unique diffeomorphism*** *$F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:*

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2. \end{tikzcd}$$

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.