Lecture 2: Connectedness and Compactness

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Contents

1 Connected Spaces

1.1 Definitions

- **Definition** (***Separation*** *and* ***Connectedness***)
  Let X be a topological space. A ***separation*** of X is a pair U, V of ***disjoint*** *nonempty* ***open*** *subset*s of X whose union is X.

  The space X is said to be <u>***connected***</u> if there *does not exist a separation* of X.

- **Definition** (***Connected: Equivalent Definition***)
  Equivalently, X is ***connected*** if and only if the only subsets of X that are ***both open and closed*** are $\emptyset$ and X itself.

- **Remark** (***Proof of Connectedness***)
  As the definition suggests, the proof of connectedness is done ***by contradition***. One first assume that the set X has a ***seperation***; it can be separated into two ***disjoint nonempty open*** sets such that $X = A \cup B$. Then we proof by contradiction using ***existing connectedness conditions*** and the ***property of open subsets (basis, continuity etc.)***.

- **Remark** ***Connectedness*** is obviously a ***topological property***, since it is formulated entirely in terms of *the collection of open sets* of X.

  Said differently, if X is ***connected***, so is any space ***homeomorphic*** to X.

- **Lemma 1.1** *(**Separation and Connected Subspace**) [Munkres, 2000]*
  *If Y is a **subspace** of X, a **separation** of Y is a pair of disjoint nonempty sets A and B whose union is Y, **neither** of which contains a **limit point** of the other. The space Y is connected if there exists no separation of Y.*

- **Example** (***Indiscrete Topology is Connected***)
  Let X denote a two-point space in ***the indiscrete topology***. Obviously there is *no separation* of X, so X is *connected*.

- **Example** ($\mathbb{Q}$ ***is Not Connected***)
  The *rationals* $\mathbb{Q}$ are ***not connected***. Indeed, *the only connected subspaces* of $\mathbb{Q}$ are the *one-point sets*: If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose *an irrational number* a lying between p and q, and write Y as the union of the open sets

  $$Y \cap (-\infty, a) \text{ and } Y \cap (a, +\infty).$$

- **Lemma 1.2** *If the sets C and D form a **separation** of X, and if Y is a **connected** subspace of X, then Y lies **entirely within** either C or D.*

- **Proposition 1.3** *(**Connectedness by Union**) [Munkres, 2000]*
  *The **union** of a collection of connected subspaces of X that **have a point in common** is connected.*

- **Proposition 1.4** *(**Connectedness by Closure**)[Munkres, 2000]*
  Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$, then B is also connected.

- **Remark** If B is formed by *adjoining* to the ***connected*** *subspace A some or all of its* ***limit points***, then B is connected.

- **Proposition 1.5** ***(Connectedness by Continuity)*** *[Munkres, 2000]*
  The ***image*** *of a connected space under a* ***continuous*** *map is connected.*

- **Proposition 1.6** ***(Connectedness by Finite Product)*** *[Munkres, 2000]*
  A ***finite*** *cartesian product of connected spaces is connected.*

- **Remark** Countable infinite product of connected spaces ***may not be connected***. It depends on the ***topology*** of the product space.

- **Example** ($\mathbb{R}^\omega$ ***is Not Connected under Box Topology***)
  Consider the cartesian product $\mathbb{R}^\omega$ in ***the box topology***. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of *all* ***bounded*** *sequences of real numbers*, and the set B of *all* ***unbounded*** *sequences*. These sets are ***disjoint***, and each is ***open*** in the box topology.

  For if $a = (a_1, a_2, \dots)$ is a point of $\mathbb{R}^\omega$, the open set

  $$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$$

  consists ***entirely*** of ***bounded*** sequences if a is ***bounded***, and of *unbounded sequences* if a is *unbounded*. Thus, even though $\mathbb{R}$ is *connected* (as we shall prove in the next section), $\mathbb{R}^\omega$ is *not connected in the box topology.* ∎

- **Example** ($\mathbb{R}^\omega$ ***is Connected under Product Topology***)
  Consider the cartesian product $\mathbb{R}^\omega$ in ***the product topology***. Let $\widetilde{\mathbb{R}}^n$ denote the ***subspace*** of $\mathbb{R}^\omega$ consisting of all sequences $x = (x_1, x_2, \dots)$ such that $x_i = 0$ for $i > n$. The space $\widetilde{\mathbb{R}}^n$ is clearly ***homeomorphic*** to $\mathbb{R}^n$, so that it is ***connected***. It follows that the space $\mathbb{R}^\infty$ that is the ***union*** of the spaces $\widetilde{\mathbb{R}}^n$ is ***connected***, for these spaces have the point $0 = (0, 0, \dots)$ in common. We show that the ***closure*** of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is ***connected*** as well.

  Let $a = (a_1, a_2, \dots)$ be a point of $\mathbb{R}^\omega$. Let $U = \prod_i U_i$ be a ***basis*** element for the product topology that contains a. We show that U ***intersects*** $\mathbb{R}^\infty$. There is an integer N such that $U_i = \mathbb{R}$ for $i > N$. Then the point

  $$x = (a_1, \dots, a_n, 0, 0, \dots)$$

  of $\mathbb{R}^\omega$ belongs to U, since $a_i \in U_i$ for all i, and $0 \in U_i$ for $i > N$. ∎

1.2 Connected Subspaces of the Real Line

- **Definition** (***Linear Continuum***)
  A ***simply ordered set*** L having *more than one element* is called a ***<u>linear continuum</u>*** if the following hold:

  1. L has the ***least upper bound property***.
  2. If $x < y$, there exists z such that $x < z < y$.

- **Proposition 1.7** ***(Linear Continuum is Connected)*** *[Munkres, 2000]*
  If L is a ***linear continuum*** *in the* ***order topology****, then L is* ***connected****, and so are* ***intervals*** *and* ***rays*** *in L.*

Proof: Recall that a subspace Y of L is said to be ***convex*** if for every pair of points a, b of Y with $a < b$, the entire interval $[a, b]$ of points of L lies in Y. We prove that if Y is a ***convex subspace*** of L, then Y is ***connected*** (L itself is *convex*).

So suppose that Y is the union of the disjoint nonempty sets A and B, each of which is ***open*** in Y. Choose $a \in A$ and $b \in B$; suppose for convenience that $a < b$. The interval $[a, b]$ of points of L is contained in Y. Hence $[a, b]$ is the union of the disjoint sets

$$A_0 = A \cap [a, b] \text{ and } B_0 = B \cap [a, b],$$

each of which is *open* in $[a, b]$ in the *subspace topology*, which is the same as the *order topology*. The sets A_0 and B_0 are *nonempty* because $a \in A_0$ and $b \in B_0$. Thus, A_0 and B_0 constitute a ***separation*** of $[a, b]$.

Let $c = \sup A_0$ is ***the least upper bound*** of A_0. We show that c belongs ***neither*** to A_0 nor to B_0, which contradicts the fact that $[a, b]$ is the *union* of A_0 and B_0.

1. Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is ***open*** in $[a, b]$ that there is *some interval* of the form $(d, c]$ *contained* in B_0. If $c = b$, we have a contradiction at once, for d is a ***smaller upper bound*** on A_0 *than* c. (To prove $d > x$ for any $x \in A_0$, we assume that there exists some $x_0 \in A_0$ such that $d < x_0$. However, the interval $(d, x_0) \subset (d, b]$ belongs to B_0, contradiction.)

   If $c < b$, we note that $(c, b]$ does not intersect A_0 (because c is an upper bound on A_0). Then

   $$(d, b] = (d, c] \cup (c, b]$$

   does not intersect A_0. Again, d is a ***smaller upper bound*** on A_0 than c, contrary to construction.

2. Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. Because of *order property (2)* of ***the linear continuum*** L, we can choose a point z of L such that $c < z < e$. Then $z \in A_0$, contrary to the fact that c is *an upper bound* for A_0. ∎

- **Corollary 1.8** *($\mathbb{R}$ **is Connected**)*
  *The real line $\mathbb{R}$ is **connected** and so are **intervals** and **rays** in $\mathbb{R}$.*

- **Theorem 1.9** ***(Intermediate Value Theorem)**. [Munkres, 2000]*
  *Let $f : X \to Y$ be a **continuous** map, where X is a **connected** space and Y is an ordered set in the **order topology**. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there **exists** a point c of X such that $f(c) = r$.*

  Proof: Assume the hypotheses of the theorem. The sets

  $$A = f(X) \cap (-\infty, r) \text{ and } B = f(X) \cap (r, +\infty)$$

  are *disjoint*, and they are *nonempty* because one contains $f(a)$ and the other contains $f(b)$. Each is *open* in $f(X)$, being the *intersection* of an open ray in Y with $f(X)$.

  If there were no point c of X such that $f(c) = r$, then $f(X)$ would be the *union* of the sets A and B. Then A and B would constitute a separation of $f(X)$, contradicting the fact that *the image of a connected space under a continuous map* is *connected*. ∎

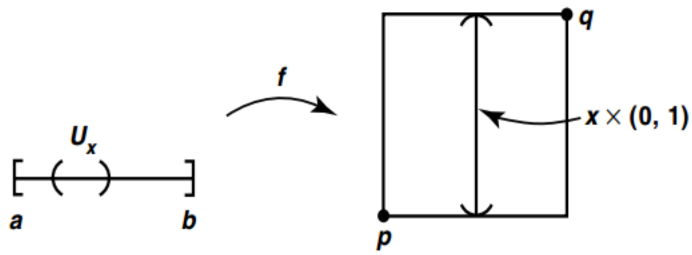

Figure 1: The proof that ordered square is not path connected. [Munkres, 2000]

- **Definition** (***Path Connectedness***)
  Given points x and y of the space X, a ***path*** in X from x to y is a *continuous map* $f : [a, b] \to X$ of some ***closed interval*** in the real line into X, such that $f(a) = x$ and $f(b) = y$.

  A space X is said to be ***path connected*** if ***every pair*** of points of X can be ***joined by a path*** in X.

- **Remark** It is easy to see that ***a path-connected space*** X ***is connected*** since $X = f([a, b])$ is the image of connected space under continuous function f. The *converse* is *not true*, i.e. connected $\not\Rightarrow$ path-connected.

- **Example** (***Punctured Euclidean Space*** $\mathbb{R}^n \setminus \{0\}$ ***is Path Connected***)
  Define ***punctured euclidean space*** to be the space $\mathbb{R}^n \setminus \{0\}$, where 0 is the origin in $\mathbb{R}^n$. If $n > 1$, this space is ***path connected***: Given x and y different from 0, we can join x and y by the *straight-line path* between them if that path does not go through the origin. Otherwise, we can choose a point z *not on the line joining* x *and* y, and take the *broken-line path* from x to z, and then from z to y.

- **Example** (*Common Path-Connected Spaces*)
  The following spaces are ***path-connected***:

  1. ***The unit ball*** $\mathbb{B}^n = \{x : \|x\| \le 1\}$ is *path-connected*;

  2. ***The unit sphere*** $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ by the equation $\mathbb{S}^{n-1} = \{x : \|x\| = 1\}$ is *path connected*. For the map $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ defined by $g(x) = x/\|x\|$ is *continuous* and *surjective*; and the continuous image of path connected space is path connected.

- **Example** ***The ordered square*** I_o^2 is ***connected*** but ***not path connected***.

  Proof: Being a linear continuum, *the ordered square* is ***connected***. Let $p = (0, 0)$ and $q = (1, 1)$. We suppose there is a path $f : [a, b] \to I_o^2$ joining p and q and derive a contradiction.

  The image set $f([a, b])$ must contain every point (x, y) of I_o^2, by *the intermediate value theorem*. Therefore, for each $x \in I$, the set

  $$U_x = f^{-1}(x \times (0, 1))$$

  is a *nonempty subset* of $[a, b]$; by *continuity*, it is ***open*** in $[a, b]$. See Figure 1. Choose, for each $x \in I$, a ***rational number*** q_x belonging to U_x. Since the sets U_x are *disjoint*, the map $x \mapsto q_x$ is an ***injective*** mapping of I into $\mathbb{Q}$. This contradicts the fact that the interval I is ***uncountable*** (which we shall prove later) ∎

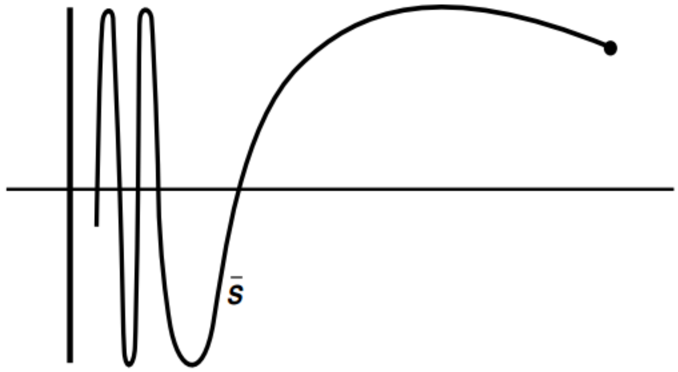

Figure 2: The topologist's sine curve is connected but not path connected. [Munkres, 2000]

- **Example** ***The topologist's sine curve*** is defined as the ***closure*** $\bar{S}$ of the set

  $$S = \{(x, \sin(1/x)) : 0 < x \le 1\}.$$

  S is ***connected*** but ***not path-connected***.

  Note that $f(t) = (x(t), y(t))$ where $x(t) = t$ and $y(t) = \sin(1/x(t))$ are both *continuous*, so $f(t)$ is *continuous*. S is the image of the connected set $(0, 1]$ under a continuous map f, so S is ***connected***. Therefore, ***its closure*** $\bar{S}$ in $\mathbb{R}^2$ is also *connected*. From Figure 2, we see that $\bar{S}$ equals the union of S and the vertical interval $0 \times [-1, 1]$. We show that $\bar{S}$ is *not path connected*.

  Suppose there is a path $f : [a, c] \to \bar{S}$ beginning at the origin and ending at a point of S. The set of those t for which $f(t) \in 0 \times [-1, 1]$ is ***closed***, so it has a ***largest element*** b. Then $f : [b, c] \to \bar{S}$ is a *path* that maps b into the vertical interval $0 \times [-1, 1]$ and maps the other points of $[b, c]$ to points of S.

  Replace $[b, c]$ by $[0, 1]$ for convenience; let $f(t) = (x(t), y(t))$. Then $x(0) = 0$, while $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. We show there is a sequence of points $t_n \to 0$ such that $y(t_n) = (-1)^n$. Then the sequence $y(t_n)$ ***does not converge***, contradicting ***continuity*** of f.

  To find t_n, we proceed as follows: Given n, choose u with $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then use *the intermediate value theorem* to find t_n with $0 < t_n < 1/n$ such that $x(t_n) = u$. ∎

1.3 Components and Local Connectedness

- Given an arbitrary space X, there is a natural way to ***break*** *it up into pieces* that are connected (or path connected).

  Definition (***Connected Component*** *as* ***Equivalence Class***)
  Given X, define an *equivalence relation* on X by setting $x \sim y$ if there is *a* ***connected subspace*** of X containing *both* x and y. The *equivalence classes* are called ***the components*** (or the ***connected components***) of X.

- **Proposition 1.10** ***(Characterization of Connected Components)***
The components of X are ***connected disjoint subspaces*** *of X whose union is X, such that each nonempty* ***connected*** *subspace of X* ***intersects only one*** *of them.*

- **Definition** (***Path Component***)
We define another *equivalence relation* on the space X by defining $x \sim y$ if there is a *path* in X from x to y. The *equivalence classes* are called ***the path components*** *of X*.

- **Proposition 1.11** ***(Characterization of Path Components)***
The path components of X are ***path-connected disjoint subspaces*** *of X whose union is X, such that each nonempty* ***path-connected*** *subspace of X* ***intersects only one*** *of them.*

- **Example** Each connected component of $\mathbb{Q}$ in $\mathbb{R}$ consists of *a single point*. ***None*** of the components of $\mathbb{Q}$ are ***open*** in $\mathbb{Q}$.

- **Example** The "***topologists sine curve*** $\bar{S}$ of the preceding section is a space that has ***a single component*** (since it is *connected*) and ***two path components***. One path component is the curve S and the other is *the vertical interval* $V = 0 \times [-1, 1]$. Note that S is ***open*** in $\bar{S}$ but ***not closed***, while V is ***closed*** but ***not open***.

  If one forms a space from $\bar{S}$ by ***deleting*** all points of V having ***rational second coordinate***, one obtains a space that has ***only one component*** but ***uncountably many path components***.

- **Remark** From the example of topologist's sine curve, we see that the *connectedness does not imply the path-connectedness* since ***neither*** *of* ***two path components*** *are* ***both*** *open and closed*. Note that the vertical line is the set of ***limit points*** of the curve $\sin(1/x)$ but not every sequence approaches to the vertical curve is convergent.

- **Definition** (***Locally Connected*** *and* ***Locally Path-Connected***)
A space X is said to be ***locally connected at*** x if for every *neighborhood* U of x, there is a ***connected neighborhood*** V of x *contained* in U. If X is *locally connected at each of its points*, it is said simply to be ***locally connected***.

  Similarly, a space X is said to be ***locally path connected at*** x if for every *neighborhood* U of x, there is a ***path-connected neighborhood*** V of x contained in$U$. If X is *locally path connected at each of its points*, then it is said to be ***locally path connected***.

- **Example** See some of examples below:

  1. The intervals and rays in $\mathbb{R}$ are ***both connected and locally connected***.
  2. The subspace $[1, 0) \cup (0, 1]$ of $\mathbb{R}$ is ***not connected***, but it is ***locally connected***.
  3. The rationals $\mathbb{Q}$ are ***neither connected nor locally connected***.
  4. *The topologists sine curve* is ***connected*** but ***not locally connected***.

- **Proposition 1.12** ***(Characterization of Locally Connectedness)*** *[Munkres, 2000]*
A space X is locally connected ***if and only if*** *for every* ***open*** *set U of X, each* ***component*** *of U is* ***open*** *in X.*

- **Proposition 1.13** ***(Characterization of Locally Path-Connectedness)*** *[Munkres, 2000]*
A space X is locally path connected ***if and only if*** *for every* ***open*** *set U of X, each* ***path component*** *of U is* ***open*** *in X.*

- **Proposition 1.14** ***(Relationship between Components and Path Components)***
  If X is a topological space, each ***path component*** *of X lies in a* ***component*** *of X. If X is* ***locally path connected****, then the* ***components*** *and* ***the path components*** *of X are the* ***same.***

2 Compact Spaces

Remark (***Metric Space*** *and* ***Compact Hausdorff Space***)
Two of the most well-behaved classes of spaces to deal with in mathematics are ***the metrizable spaces*** and ***the compact Hausdorff spaces***.

1. ***Metrizable space*** (X, d):
   - ***subspace*** of *metrizable* space is *meterizable*;
   - ***compact subspace*** of *metric* space is ***bounded*** in that metric and is ***closed***;
   - *every metrizable space* is ***normal*** (T_4);
   - ***compactness*** = ***sequential compactness*** = ***limit point compactness***;
   - ***sequence lemma***: for $A \subset X$, $x \in \bar{A}$ if and only if there exists a squence of points in A that converges to x. ($\Rightarrow$ need X being metric space);
   - f is ***continuous*** at x if and only if $x_n \to x$ leads to $f(x_n) \to f(x)$ ($\Leftarrow$ part holds for metric space)
   - ***unform limit theorem***: If the *range* of f_n is a *metric space* and f_n are *continuous*, then $f_n \to f$ *uniformly* means that f is a *continuous* function.
   - ***unform continuity theorem***: if f is a *countinous* map between two *metric spaces*, and the domain is ***compact***, then f is ***uniformly continuous***.
2. ***Compact Hausdorff Space***:
   - ***subspace*** of *compact Hausdorff space* is *compact Hausdorff* if and only if it is ***closed.***
   - ***closed subspace*** of *compact* space is ***compact***;
   - ***compact subspace*** of *Hausdorff* space is ***closed***;
   - *compact Hausdorff space* X is ***normal*** (T_4), thus it is ***completely regular***;
   - ***arbitrary product*** of *compact (Hausdorff)* space is *compact (Hausdorff)*;
   - ***compactness*** $\Rightarrow$ ***sequential compactness***;
   - ***compactness*** = ***net compactness***, i.e. every *net* has a convergence *subnet*;
   - ***image*** of *compact* space under continuous map f is *compact*;
   - ***continuous bijection*** between two *compact Hausdorff* spaces is a ***homemorphism*** (and is a ***closed map***);
   - ***closed graph theorem***: f is ***continuous*** if and only if its ***graph*** is ***closed***;

- ***uncountability***: for *compact Hausdorff space*, if the space has *no isolated points*, then it is *uncountable*;

2.1 Definitions

- **Definition** (***Covering of Set*** *and* ***Open Covering of Topological Set***)
A collection $\mathscr{A}$ of subsets of a space X is said to ***cover*** X, or to be *a* ***covering*** *of* X, if the union of the elements of $\mathscr{A}$ is equal to X.

  It is called an ***open covering of*** X if its elements are *open subsets* of X.

- **Definition** (***Compactness***)
A topological space X is said to be ***compact*** if *every open covering* $\mathscr{A}$ of X contains a ***finite*** *subcollection* that also *covers* X.

- **Example** (*Compactness is a strong condition*)
Consider the following examples that are *connected* by *not compact*:

  1. The ***real line*** $\mathbb{R}$ is ***not compact*** since the open covering $\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ has no finite sub-covering.

  2. The ***half interval*** $(0, 1]$ is ***not compact*** since the open covering $\mathscr{A} = \{(1/n, 1] : n \in \mathbb{Z}_+\}$ has no finite sub-covering.

- **Example** (***Finite Set is Compact***)
Any space X containing only ***finitely*** *many points* is necessarily ***compact***, because in this case *every open covering of* X *is finite.*

- **Example** The following *subspace* of $\mathbb{R}$ is ***compact***:

$$X = \{0\} \cup \{1/n : n \in \mathbb{Z}_+\}.$$

  (It is *not connected*.)

  Given an open covering $\mathscr{A}$ of X, there is ***an element*** U of $\mathscr{A}$ containing 0. The set U contains ***all but finitely many*** *of the points* $1/n$; choose, for each point of X ***not in*** U, an element of $\mathscr{A}$ containing it. The collection consisting of these elements of $\mathscr{A}$, along with the element U, is *a finite subcollection* of $\mathscr{A}$ that covers X.

- **Definition** If Y is a subspace of X, a collection $\mathscr{A}$ of *subsets of* X is said to ***cover*** Y if the *union* of its elements *contains* Y.

- **Lemma 2.1** *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets* open *in X contains* a finite subcollection covering Y.

- **Remark** A ***compact subset*** of a topological space is one that is a compact space in the ***subspace topology***.

- **Proposition 2.2** *(****Compactness by Closed Subspace****) [Munkres, 2000]*
Every ***closed subspace*** *of a compact space is compact.*

  Proof: Let Y be a ***closed*** subspace of the compact space X. Given a ***covering*** $\mathscr{A}$ of Y by sets ***open in*** X, let us form an open covering $\mathscr{B}$ of X by adjoining to $\mathscr{A}$ the single open set

$X \setminus Y$, that is,

$$\mathscr{B} = \mathscr{A} \cup (X \setminus Y).$$

Some ***finite*** subcollection of $\mathscr{B}$ covers X. If this subcollection contains the set $X \setminus Y$, discard $X \setminus Y$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of $\mathscr{A}$ that covers Y. ∎

- **Proposition 2.3** *(**Compact Subspace** + **Hausdorff** ⇒ **Closedness**) [Munkres, 2000]*
  *Every **compact** subspace of a **Hausdorff** space is **closed**.*

  Proof: Let Y be a *compact* subspace of the *Hausdorff* space X. We shall prove that $X \setminus Y$ is *open*, so that Y is *closed*.

  Let x_0 be a point of $X \setminus Y$. We show there is a ***neighborhood*** of x_0 that is ***disjoint*** from Y. For each point y of Y, let us choose *disjoint neighborhoods* U_y and V_y of the points x_0 and y, respectively (using the *Hausdorff condition*). The collection $\{V_y : y \in Y\}$ is a *covering of Y by sets open in X*; therefore, *finitely many of them* $V_{y_1}, \ldots, V_{y_n}$ cover Y. The open set

  $$V = V_{y_1} \cup \ldots \cup V_{y_n}$$

  contains Y, and it is ***disjoint*** from the open set

  $$U = U_{y_1} \cap \ldots \cap U_{y_n}$$

  formed by taking *the **intersection** of the corresponding neighborhoods of x_0*. For if z is a point of V, then $z \in V_{y_i}$ for some i, hence $z \notin U_{y_i}$ and so $z \notin U$. Then U is a neighborhood of x_0 disjoint from Y, as desired. ∎

- **Proposition 2.4** *If Y is a **compact subspace** of the **Hausdorff** space X and x_0 is not in Y, then there exist **disjoint open** sets U and V of X containing x_0 and Y, respectively.*

- **Remark** To prove *the compact subspace* is *closed*, one need the *Hausdorff condition*.

- **Exercise 2.5** *(**Compact Subspace in Metric Space**)*
  *Show that every **compact subspace** of **a metric space** is **bounded** in that **metric** and is **closed**. Find a metric space in which not every closed bounded subspace is compact.*

- **Proposition 2.6** *(**Compactness by Continuity**) [Munkres, 2000]*
  *The **image** of a **compact** space under a **continuous** map is compact.*

- **Exercise 2.7** *Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a **closed map** (that is, f carries closed sets to closed sets)*

- **Exercise 2.8** *Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

- **Theorem 2.9** *(**Closed Graph Theorem**) [Reed and Simon, 1980, Munkres, 2000]*
  *Let $f : X \to Y$; let Y be **compact Hausdorff**. Then f is **continuous if and only if** the **graph** of f,*

  $$G(f) = \{(x, f(x)) : x \in X\},$$

  *is **closed** in $X \times Y$.*

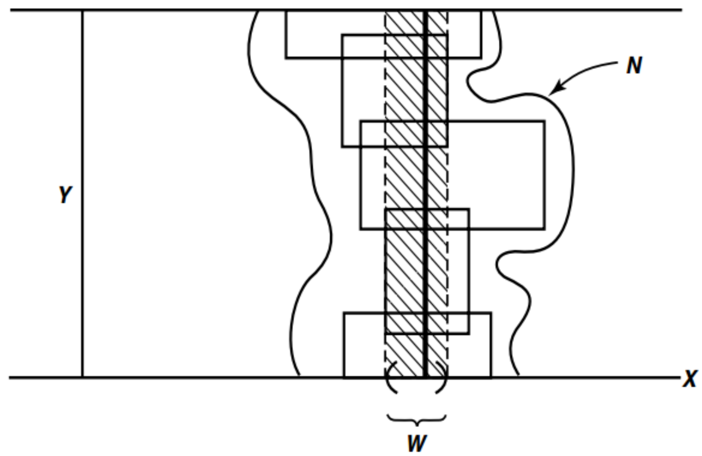

Figure 3: The tube of a slice $x_0 \times Y$ in neighborhood N of product space. [Munkres, 2000]

- **Theorem 2.10** ***(Homemorphism by Compactness and Hausdorff)****[Munkres, 2000]*
  Let $f : X \to Y$ be a ***bijective continuous*** *function. If X is* ***compact*** *and Y is* ***Hausdorff****, then f is a* ***homeomorphism****.*

  Proof: We shall prove that ***images*** of ***closed sets*** of X under f are *closed* in Y (i.e. f is a ***closed map***); this will prove ***continuity*** *of the inverse map* f^{-1}. If A is a *closed subspace* in X, then A is ***compact***. Therefore, by the proposition above, $f(A)$ is ***compact***. Since Y is ***Hausdorff***, *the compact subspace* $f(A)$ is ***closed*** in Y. ∎

- **Proposition 2.11** ***(Compactness by Finite Product)*** *[Munkres, 2000]*
  The product of ***finitely*** *many compact spaces is compact.*

  Lemma 2.12 ***(The Tube Lemma)****.[Munkres, 2000]*
  Consider the product space $X \times Y$, where Y is ***compact****. If N is an open set of $X \times Y$ containing the* ***slice*** *$x_0 \times Y$ of $X \times Y$, then N contains some* ***tube*** *$W \times Y$ about $x_0 \times Y$, where W is a* ***neighborhood*** *of x_0 in X.*

- **Remark** (***Compactness by Infinite Product***)
  Unlike *the connectedness property*, which *may not hold* for infinite product space, *the infinite product of compact space is indeed compact*. This is called ***the Tychonoff theorem***,

- To prove *compactness*, the following property is useful:

  Definition (***Finite Intersection Property***)
  A collection $\mathscr{C}$ of subsets of X is said to have ***the finite intersection property*** if for *every finite subcollection*

$$\{C_1, \ldots, C_n\}$$

  of $\mathscr{C}$, the ***intersection*** $C_1 \cap \ldots \cap C_n$ is ***nonempty***.

- **Proposition 2.13** ***(Equivalent Definition of Compactness)*** *[Munkres, 2000]*
  Let X be a topological space. Then X is ***compact if and only if*** *for every collection $\mathscr{C}$ of* ***closed*** *sets in X having* ***the finite intersection property****, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is* ***nonempty****.*

Proof: Given a collection $\mathscr{A}$ of subsets of X, let

$$\mathscr{C} = \{X \setminus A : A \in \mathscr{A}\}$$

be the collection of their *complements*. Then the following statements hold:

1. $\mathscr{A}$ is a collection of open sets if and only if $\mathscr{C}$ is a collection of closed sets.
2. The collection $\mathscr{A}$ covers X if and only if the *intersection* $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is ***empty***.
3. The ***finite subcollection*** $\{A_1, \ldots, A_n\}$ of $\mathscr{A}$ covers X if and only if the ***intersection*** of the corresponding elements $C_i = X \setminus A_i$ of $\mathscr{C}$ is ***empty***.

The proof of the theorem now proceeds in two easy steps: taking the ***contrapositive*** (of the theorem), and then the ***complement*** (of the sets)!

There are two equivalent statements regarding the compactness of set:

1. "*Given any collection $\mathscr{A}$ of open subsets of X, if $\mathscr{A}$ covers X, then some finite subcollection of $\mathscr{A}$ covers X.*"
2. "*Given any collection $\mathscr{A}$ of open sets, if* ***no finite subcollection*** *of $\mathscr{A}$ covers X, then A* ***does not cover*** *X.*"
3. $\Rightarrow$ "*Given any collection $\mathscr{C}$ of* ***closed sets****, if* ***every finite intersection*** *of elements of $\mathscr{C}$ is* ***nonempty****, then the* ***intersection of all the elements*** *of $\mathscr{C}$ is* ***nonempty***"
$\blacksquare$

- **Remark** (***Nested Sequence of Closed Sets in Compact Space***)
A special case of this proposition occurs when we have a ***nested*** *sequence* $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ *of* ***closed sets*** in *a* ***compact space*** X.

If each of the sets C_n is nonempty, then the collection $\mathscr{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ automatically has ***the finite intersection property***. Then the intersection

$$\bigcap_{n \in \mathbb{Z}_+} C_n$$

is nonempty.

2.2 Compact Subspaces of the Real Line

- **Theorem 2.14** *[Munkres, 2000]*
Let X be a ***simply ordered*** *set having the* ***least upper bound property****. In the order topology, each* ***closed interval*** *in X is* ***compact****.*

- **Corollary 2.15** ***(Closed Interval in Real Line is Compact)****[Munkres, 2000]*
Every ***closed interval*** *in $\mathbb{R}$ is* ***compact****.*

- **Proposition 2.16** ***(Closed and Bounded in Euclidean Metric = Compact)****[Munkres, 2000]*
A subspace A of $\mathbb{R}^n$ is ***compact*** *if and only if it is* ***closed*** *and is* ***bounded*** *in the* ***euclidean metric*** *d or the* ***square metric*** *ρ*

- **Theorem 2.17** ***(Extreme Value Theorem)***. *[Munkres, 2000]*
  Let $f : X \to Y$ be ***continuous****, where Y is an* ***ordered set*** *in the order topology. If X is* ***compact****, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

- **Definition** (***Distance to Subset***)
  Let (X, d) be a ***metric space***; let A be a nonempty subset of X. For each $x \in X$, we define ***the distance*** *from x to A* by the equation

$$d(x, A) = \inf \{d(x, a) : a \in A\}.$$

- **Remark** The distance to subset $d(x : A)$ is a ***continuous*** function with respect to the first argument.

- **Remark** Recall that the ***diameter*** of a *bounded subset A* of a *metric space* (X, d) is the number

$$\sup \{d(a_1, a_2) : a_1, a_2 \in A\}.$$

- **Lemma 2.18** ***(The Lebesgue Number Lemma)***. *[Munkres, 2000]*
  Let $\mathscr{A}$ be an ***open covering*** *of the* ***metric space*** *(X, d). If X is* ***compact****, there is a $\delta > 0$ such that for each subset of X having* ***diameter less than*** *δ, there exists an element of $\mathscr{A}$ containing it.*

  The number δ is called a ***Lebesgue number*** *for the covering $\mathscr{A}$.*

- **Remark** ***The Lebesgue number*** is a ***threshold*** *on* ***diameter of subset*** so that all of subsets with diameter less than this threshold is fully contained in one of the open sets in the covering of X. The *existance* of this number relies on the ***compactness*** of domain X.

  This number is used in ϵ-δ ***condition*** to prove *the uniform continuity.*

- **Definition** (***Uniform Continuity***)
  A function $f : (X, d_X) \to (Y, d_Y)$ is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \quad \Rightarrow \quad d_Y(f(x_0), f(x_1)) < \epsilon.$$

- **Theorem 2.19** ***(Uniform Continuity Theorem)***. *[Munkres, 2000]*
  Let $f : X \to Y$ be a ***continuous*** *map of the* ***compact*** *metric space (X, d_X) to the metric space (Y, d_Y). Then f is* ***uniformly continuous****.*

  Proof: Given $\epsilon > 0$, take ***the open covering*** of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathscr{A}$ be ***the open covering*** of X by ***the inverse images*** *of these balls under f*. Choose δ to be a ***Lebesgue number*** for the covering $\mathscr{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has *diameter less than* δ, so that its image $\{f(x_1), f(x_2)\}$ *lies in some ball* $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired. ∎

- **Remark**

$$f \text{ continous } + \text{ compact domain } \Rightarrow f \text{ uniformly continous}$$

- **Definition** If X is a space, a point x of X is said to be ***an isolated point*** of X if *the one-point set* $\{x\}$ is ***open*** in X.

- **Theorem 2.20** *(**Uncountability in Compact Hausdorff Space**) [Munkres, 2000]*
  *Let X be a nonempty **compact Hausdorff** space. If X has **no isolated points**, then X is **uncountable**.*

- **Corollary 2.21** *[Munkres, 2000]*
  *Every **closed interval** in $\mathbb{R}$ is **uncountable**.*

- **Exercise 2.22** *(**Cantor Set**) [Munkres, 2000]*
  *Let A_0 be the **closed interval** $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by **deleting** its "**middle third** $(1/3, 2/3)$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, define A_n by the equation*

  $$A_n = A_{n-1} \setminus \left(\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^k}, \frac{2+3k}{3^k} \right) \right).$$

  *The **intersection***

  $$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

  *is called **the Cantor set**; it is a **subspace** of $[0, 1]$.*

  1. *Show that C is **totally disconnected**.*
  2. *Show that C is **compact**.*
  3. *Show that each set A_n is a **union** of **finitely** many disjoint **closed intervals** of length $1/3^n$; and show that the **end points** of these intervals lie in C.*
  4. *Show that C has **no isolated points**.*
  5. *Conclude that C is **uncountable**.*

2.3 Limit Point Compactness

- **Definition** (***Limit Point Compactness***)
  A space X is said to be ***limit point compact*** if every infinite subset of X has a ***limit point***.

- **Proposition 2.23** *(**Compactness $\Rightarrow$ Limit Point Compactness**) [Munkres, 2000]*
  ***Compactness** implies **limit point compactness**, but **not conversely**.*

- **Example** (***Limit Point Compactness $\not\Rightarrow$ Compactness***)
  Let Y consist of ***two points***; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is ***limit point compact***, for *every nonempty subset of X has a* ***limit point***. It is ***not compact***, for the *covering* of X by the open sets $U_n = \{n\} \times Y$ has *no finite subcollection covering X.* ∎

- **Definition** (***Sequential Compactness***)
  Let X be a topological space. If (x_n) is a *sequence* of points of X, and if

  $$n_1 < n_2 < \ldots < n_i < \ldots$$

  is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n).

The space X is said to be ***sequentially compact*** if *every sequence of points of X has a **convergent subsequence**.*

- **Theorem 2.24** *(**Equivalent Definitions of Compactness in Metric Space**) [Munkres, 2000]*
  *Let X be a **metrizable space**. Then the following are **equivalent**:*

  1. *X is **compact**.*
  2. *X is **limit point compact**.*
  3. *X is **sequentially compact**.*

2.4 Local Compactness

- **Definition** (***Local Compactness***)
  A space X is said to be ***locally compact at x*** if there is some ***compact subspace*** C of X that contains a ***neighborhood*** of x.

  If X is *locally compact* at *each of its points*, X is said simply to be ***locally compact***.

- **Example** For the one-dimensional space:

  1. The real line $\mathbb{R}$ is ***locally compact***. The point x lies in some interval (a, b), which in turn is *contained* in *the compact subspace* $[a, b]$.
  2. The subspace $\mathbb{Q}$ of rational numbers is ***not locally compact***.

- **Example** For product space of $\mathbb{R}$:

  1. The ***finite dimensional space*** $\mathbb{R}^n$ is ***locally compact***; the point x lies in some basis element $(a_1, b_1) \times \ldots \times (a_n, b_n)$, which in turn lies in *the compact subspace* $[a_1, b_1] \times \ldots \times [a_n, b_n]$.
  2. The ***countable infinite dimensional space*** $\mathbb{R}^\omega$ is ***not locally compact***; *none of its basis elements are contained in compact subspaces.* For if

  $$B = (a_1, b_1) \times \ldots \times (a_n, b_n) \times \mathbb{R} \times \ldots \times \mathbb{R} \times \ldots$$

  were contained in a *compact subspace*, then its ***closure***

  $$\bar{B} = [a_1, b_1] \times \ldots \times [a_n, b_n] \times \mathbb{R} \times \ldots \times \mathbb{R} \times \ldots$$

  would be ***compact***, which it is not.

- **Example** (***Simply Ordered Set with Least Upper Bound Property***)
  Every ***simply ordered*** *set* X *having* ***the least upper bound property*** is ***locally compact***: Given a basis element for X, it is contained in a *closed interval* in X, which is compact.

- **Example** (***Manifold***) [Lee, 2018]
  Every ***topological manifold*** is ***locally compact Hausdorff***.

  Thus ***every smooth manifold*** is ***locally compact Hausdorff***.

- **Definition** (***Precompactness***)
  A subset of X is said to be ***precompact*** in X if its ***closure*** in X is ***compact***.

- If X is *not a compact Hausdorff space*, then *under what conditions* is X *homeomorphic* with *a* ***subspace*** *of a compact Hausdorff space* ?

  Theorem 2.25 ***(Unique One-Point Compactification)*** *[Munkres, 2000]*
  Let X be a space. Then X is ***locally compact Hausdorff*** *if and only if there exists a space Y satisfying the following conditions:*

  1. *X is a subspace of Y.*
  2. *The set $Y \setminus X$ consists of* ***a single point*** *(which is the limit point of X).*
  3. *Y is a* ***compact Hausdorff*** *space.*

  If Y and Y' are two spaces satisfying these conditions, then there is a ***homeomorphism*** *of Y with Y' that equals* ***the identity map*** *on X.*

- **Definition** (***One-Point Compactification***)
  If Y is a ***compact Hausdorff*** space and X is a proper *subspace* of Y whose ***closure*** equals Y, then Y is said to be a ***compactification*** of X.

  If $Y \setminus X$ equals *a single point*, then Y is called ***the one-point compactification*** of X.

- **Remark** (***Locally Compact Hausdorff*** = *Existance of Unique One-Point Compactification*)
  X has a ***one-point compactification*** Y if and only if X is a ***locally compact Hausdorff space*** that is *not itself compact*.

  We speak of Y as "***the***" *one-point compactification* because Y is ***uniquely*** determined up to a *homeomorphism*.

- **Example** ***The one-point compactification*** of the real line $\mathbb{R}$ is ***homeomorphic*** with the ***circle*** $\mathbb{S}^1$.

  Similarly, ***the one-point compactification*** of $\mathbb{R}^2$ is ***homeomorphic*** to the ***sphere*** $\mathbb{S}^2$.

- **Proposition 2.26** ***(Locally Compact Hausdorff = Precompact Basis)*** *[Munkres, 2000]*
  Let X be a ***Hausdorff*** *space. Then X is* ***locally compact if and only if*** *given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is* ***compact*** *and $\bar{V} \subseteq U$.*

- **Corollary 2.27** ***(Closed or Open Subspace)*** *[Munkres, 2000]*
  Let X be locally compact Hausdorff; let A be a subspace of X. If A is ***closed*** *in X or* ***open*** *in X, then A is locally compact.*

- **Corollary 2.28** *[Munkres, 2000]*
  A space X is ***homeomorphic*** *to an* ***open*** *subspace of a* ***compact Hausdorff*** *space* ***if and only if*** *X is* ***locally compact Hausdorff***.

- **Remark** *Locally Compact Hausdorff = Open Subspace of Compact Hausorff*

- **Theorem 2.29** *[Treves, 2016]*
  Every locally compact Hausdorff topological vector space is ***finite-dimensional***.

- **Remark** (***Equivalent Definition of Locally Compact Hausdorff Space***)
  For a ***Hausdorff space*** X, the following are ***equivalent***:

1. X is ***locally compact***.

2. Each point of X has a ***precompact*** neighborhood.

3. X has a basis of ***precompact*** open subsets.

3 Nets and Convergence in Topological Space

- **Definition** (***Directed System of Index Set***)
  A ***directed system*** is *an index set* I together with an ***ordering*** $\prec$ which satisfies:

  1. If $\alpha, \beta \in l$ then there exists $\gamma \in I$ so that $\gamma \succ \alpha$ and $\gamma \succ \beta$.

  2. $\prec$ is a ***partial ordering***.

- **Definition** A subset K of I is said to be ***cofinal*** in I if for each $\alpha \in I$, there exists $\beta \in K$ such that $\alpha \preceq \beta$.

- **Proposition 3.1** *If I is a directed system, and K is cofinal in I, then K is a directed system.*

- **Definition** (***Net***)
  A ***net*** in a topological space X is a mapping from a ***directed system*** I to X; we denote it by $\{x_\alpha\}_{\alpha \in I}$

- **Remark** (***Net*** *vs.* ***Sequence***)
  Net is a generalization and abstraction of ***sequence***. The directed system I is ***not necessarily countable***. So $\{x_\alpha\}_{\alpha \in I}$ may not be a countable sequence. *A sequence is a net with countable index set* $I \subseteq \mathbb{N}$. The directed system can be any set e.g. a graph.

- **Definition** If $P(\alpha)$ is a ***proposition*** depending on an ***index*** α in a *directed set* I we say $P(\alpha)$ ***is eventually true*** if there is a β in I with $P(\alpha)$ *true* if *for all* $\alpha \succ \beta$.

  We say $P(\alpha)$ ***is frequently true*** if it is ***not eventually false***, that is, if for any β *there exists* an $\alpha \succ \beta$ with $P(\alpha)$ *true.*

- **Definition** (***Convergence***)
  A ***net*** $\{x_\alpha\}_{\alpha \in I}$ in a topological space X is said to ***converge*** to a point $x \in X$ (written $x_\alpha \to x$) if for ***any neighborhood*** N of x, ***there exists*** a $\beta \in l$ so that $x_\alpha \in N$ if $\alpha \succeq \beta$. The point x that being converged to is called ***the limit point*** of x_α.

  Note that if $x_\alpha \to x$, then x_α is ***eventually in all neighborhoods of*** x. If x_α is ***frequently in any neighborhood of*** x, we say that x is a ***cluster point*** of x_α.

- **Remark** This definition *generalizes* the ϵ-δ language for convergence in metric space. Notice that the notions of *limit* and *cluster point* generalize the same notions for sequences in a metric space..

- **Proposition 3.2** *(**Net Lemma**) [Reed and Simon, 1980]*
  *Let A be a set in a topological space X. Then, a point $x \in \bar{A}$, the **closure** of A **if and only if** there is a net $\{x_\alpha\}_{\alpha \in I}$ with $x_\alpha \in A$, So that $x_\alpha \to x$.*

- **Proposition 3.3** *[Munkres, 2000]*

  1. *(**Continuous Function**): A function f from a topological space X to a topological*

*space Y is **continuous** if and only if for **every convergent net** $\{x_\alpha\}_{\alpha\in I}$ **in** X, with $x_\alpha \to x$, the net $\{f(x_\alpha)\}_{\alpha\in I}$ **converges in** Y to $f(x)$.*

2. *(**Uniqueness of Limit Point for Hausdorff Space**): Let X be a **Hausdorff space**. Then a net $\{x_\alpha\}_{\alpha\in I}$ in X can have **at most one limit**; that is, if $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.*

- **Definition** A net $\{x_\alpha\}_{\alpha\in I}$ is a ***subnet*** of a net $\{y_\beta\}_{\beta\in J}$ if and only if there is a function $F : I \to J$ such that

  1. $x_\alpha = y_{F(\alpha)}$ for each $\alpha \in I$.
  2. For all $\beta' \in J$, there is an $\alpha' \in I$ such that $\alpha \succ \alpha'$ implies $F(\alpha) \succ \beta'$ (that is, $F(\alpha)$ is ***eventually larger*** *than* ***any fixed*** $\beta \in J \Rightarrow$ $F(I)$ is ***cofinal*** in J).

- **Proposition 3.4** *A point x in a topological space X is a **cluster point** of a **net** $\{x_\alpha\}_{\alpha\in I}$ if and only if **some subnet** of $\{x_\alpha\}_{\alpha\in I}$ **converges** to x.*

- **Theorem 3.5** *(**The Bolzano-Weierstrass Theorem**) [Reed and Simon, 1980, Munkres, 2000]*
  *A space X is **compact if and only if every net** in X **has a convergent subnet**.*

  Proof: To prove the implication $\Rightarrow$, let $B_\alpha = \{x_\beta : \alpha \preceq \beta\}$ and show that $\{B_\alpha\}$ has ***the finite intersection property***.

  To prove $\Leftarrow$, let $\mathscr{A}$ be a collection of ***closed sets*** having *the finite intersection property*, and let $\mathscr{B}$ be the collection of *all finite intersections* of elements of $\mathscr{A}$, ***partially ordered*** by *reverse inclusion.*

- **Remark** (***Compactness*** *via* ***Generalized Sequential Compactness***)
  By *generalization* of ***squences*** $\Rightarrow$ ***nets***, we obtain a *generalization* of the result of ***sequential compactnesss*** *in* ***metric space*** to ***compactness*** in *general topological space.*

  With ***first countable property***, we can use subsequence and sequence in place of subnet and net.

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