Lecture 13: Riemannian Metrics

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Oct. 26th., 2022

Contents

1 Riemannian Metrics

1.1 Definitions

- **Remark** The most important examples of symmetric tensors on a vector space are ***inner products***. Any inner product allows us to define ***lengths*** of vectors and ***angles*** between them, and thus to do Euclidean geometry.

- **Definition** Let M be a smooth manifold with or without boundary. ***A Riemannian metric*** on M is a smooth ***symmetric covariant*** 2***-tensor field*** on M that is ***positive definite*** at each point.

  A Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M. One sometimes simply says "*M is a Riemannian manifold*" if M is understood to be endowed with *a specific Riemannian metric*. *A Riemannian manifold* ***with boundary*** is defined similarly.

- **Remark** If g is a Riemannian metric on M, then for each $p \in M$, *the* 2*-tensor* g_p is an ***inner product*** on T_pM. Because of this, we often use the notation $\langle v\,,\,w\rangle_g$ to denote the real number $g_p(v, w)$ for $v, w \in T_pM$.

- **Remark** (***Coordinate Representation of Riemannian Metric***)
  In any smooth local coordinates (x^i), ***a Riemannian metric*** can be written
  $$g = g_{i,j}\, dx^i \otimes dx^j, \tag{1}$$
  where $(g_{i,j})$ is ***a symmetric positive definite matrix*** of smooth functions.

- **Remark** (***Alternative Coordinate Representation of Riemannian Metric***)
  The ***symmetry*** of g allows us to write g also in terms of ***symmetric products*** as follows:
  $$\begin{aligned} g &= g_{i,j}\, dx^i \otimes dx^j, \\ &\text{(since a symmetric tensor is equal to its symmetrization)} \\ &= \frac{1}{2}\left(g_{i,j} dx^i \otimes dx^j + g_{j,i} dx^j \otimes dx^i\right) \\ &\text{(since } g_{i,j} = g_{j,i}) \\ &= \frac{1}{2} g_{i,j}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right) \\ &\text{(by definition of symmetric product)} \\ &= \frac{1}{2} g_{i,j} dx^i\, dx^j \end{aligned} \tag{2}$$

- **Example** (***The Euclidean Metric***).
  The simplest example of a Riemannian metric is ***the Euclidean metric*** $\bar{g}$ on $\mathbb{R}^n$, given in standard coordinates by
  $$\bar{g} = \delta_{i,j} dx^i dx^j,$$
  where $\delta_{i,j}$ is *the Kronecker delta*. It is common to abbreviate ***the symmetric product*** *of a tensor* α ***with itself*** by α^2, so the Euclidean metric can also be written
  $$\bar{g} = \left(dx^1\right)^2 + \ldots + (dx^n)^2.$$

Applied to vectors $v, w \in T_p\mathbb{R}^n$, this yields

$$\bar{g}_p(v, w) = \delta_{i,j} v^i w^j = \sum_i v^i w^i = \langle \boldsymbol{v}\,,\, \boldsymbol{w} \rangle$$

In other words, $\bar{g}$ is the 2-tensor field whose value at each point is ***the Euclidean dot product***. We denote the value of this 2-tensor field as $g(v, w) := \langle v\,,\, w \rangle_g$.

- **Example** (***Product Metrics***).
  If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, we can define a Riemannian metric $\hat{g} = g \oplus \tilde{g}$ on the product manifold $M \times \widetilde{M}$, called ***the product metric***, as follows:

  $$\hat{g}((v, \widetilde{v}), (w, \widetilde{w})) = g(v, w) + \widetilde{g}(\widetilde{v}, \widetilde{w}) \tag{3}$$

  for any $(v, \widetilde{v}), (w, \widetilde{w}) \in T_pM \times T_q\widetilde{M} \simeq T_{(p,q)}(M \times \widetilde{M})$. Given any local coordinates $x^1, \ldots, x^n$ for M and $y^1, \ldots, y^m$ for $\widetilde{M}$, we obtain local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ for $M \times \widetilde{M}$, and you can check that the product metric is represented locally by the block diagonal matrix

  $$\hat{g}_{i,j} = \begin{bmatrix} g_{i,j} & 0 \\ 0 & \widetilde{g}_{i,j} \end{bmatrix}.$$

  For example, it is easy to verify that the Euclidean metric on $\mathbb{R}^{n+m}$ is the same as the product metric determined by the Euclidean metrics on $\mathbb{R}^n$ and $\mathbb{R}^m$. (Note that the product metrics is the sum of tensors ***not tensor product*** of Riemannian metrics, which would increase the rank of the metric.)

- **Proposition 1.1** *(**Existence of Riemannian Metrics**). [Lee, 2003., 2018]*
  ***Every smooth manifold** with or without boundary admits a **Riemannian metric**.*

  Proof: (A sketch of the proof). Let M be a smooth manifold with or without boundary, and choose a covering of M by smooth coordinate charts $(U_\alpha, \varphi_\alpha)$. In each coordinate domain, there is a Riemannian metric $g_\alpha = \varphi^* \bar{g}$ via *pullback of Euclidean metric* $\bar{g}$ by φ, whose coordinate expression is $\delta_{i,j} dx^i dx^j$. Let $\{\Psi_\alpha\}$ be a smooth partition of unity subordinate to the cover U_α, and define

  $$g = \sum_\alpha \Psi_\alpha g_\alpha,$$

  with each term interpreted to be zero outside supp Ψ_α. By local finiteness, there are *only finitely many nonzero terms* in a neighborhood of each point, so this expression defines *a smooth tensor field*. It is obviously *symmetric*. We can proof this term $g(v, v)$ is postive for each nonzero $v \in T_pM$. ∎

- **Definition** The ***length*** or ***norm*** of a tangent vector $v \in T_pM$ is defined to be

  $$|v|_g = \sqrt{g_p(v, v)} := \sqrt{\langle v\,,\, v \rangle_g}$$

- **Definition** The ***angle*** between two nonzero tangent vectors $v, w \in T_pM$ is the unique $\theta \in [0, \pi]$ satisfying:

  $$\theta = \frac{\langle v\,,\, w \rangle_g}{|v|_g\; |w|_g}.$$

- **Definition** Tangent vectors $v, w \in T_pM$ are said to be ***orthogonal*** if $\langle v\,,\,w\rangle_g = 0$. This means either one or both vectors are zero, or the angle between them is $\pi/2$.

- **Definition** Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. *A local frame* $(E_1\,,\dots,E_n)$ for M on an open subset $U \subseteq M$ is an ***orthonormal frame*** if the vectors $(E_1|_p\,,\dots,E_n|_p)$ form an ***orthonormal basis*** for T_pM at each point $p \in U$, or equivalently if $\langle E_i\,,\,E_j\rangle_g = \delta_{i,j}$.

- **Proposition 1.2** *Suppose (M, g) is a Riemannian manifold with or without boundary, and (X_j) is a smooth local frame for M over an open subset $U \subseteq M$. Then there is a smooth **orthonormal frame** (E_j) over U such that span$\{E_1|_p\,,\dots,E_n|_p\}$ = span$\{X_1|_p\,,\dots,X_n|_p\}$ for each $j = 1\,,\dots,n$ and each $p \in U$.*

- **Corollary 1.3** *(**Existence of Local Orthonormal Frames**).*
  Let (M, g) be a Riemannian manifold with or without boundary. For each $p \in M$, there is a smooth orthonormal frame on a neighborhood of p.

- **Definition** For a Riemannian manifold (M, g) with or without boundary, we define the ***unit tangent bundle*** to be the subset $UTM \subseteq TM$ consisting of unit vectors:

$$UTM = \left\{(p, v) \in TM : |v|_g = 1\right\}.$$

- **Proposition 1.4** *(**Properties of the Unit Tangent Bundle**). [Lee, 2018]*
  *If (M, g) is a Riemannian manifold with or without boundary, its unit tangent bundle UTM is a **smooth**, **properly embedded codimension-**1 submanifold with boundary in TM, with $\partial(UTM) = \pi^{-1}(\partial M)$ (where $\pi : UTM \to M$ is the canonical projection). The unit tangent bundle is **connected** if and only if M is **connected**, and **compact** if and only if M is **compact**.*

1.2 Pullback Metrics

- **Definition** Suppose M, N are smooth manifolds with or without boundary, g is a Riemannian metric on N, and $F : M \to N$ is smooth. The ***pullback*** F^*g is a smooth 2-tensor field on M. If it is ***positive definite***, it is a Riemannian metric on M, called ***the pullback metric*** determined by F.

- **Proposition 1.5** *(**Pullback Metric Criterion**). [Lee, 2003.]*
  *Suppose $F : M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a **Riemannian metric** on M if and only if F is a **smooth immersion**.*

- **Definition** If (M, g) and $(\widetilde{M}, \widetilde{g})$ are both Riemannian manifolds, a smooth map $F : M \to \widetilde{M}$ is called a ***(Riemannian) isometry*** if it is a ***diffeomorphism*** that satisfies $F^*\widetilde{g} = g$. More generally, F is called ***a local isometry*** if every point $p \in M$ has a neighborhood U such that $F|_U$ is an *isometry* of U onto an open subset of $\widetilde{M}$; or equivalently, if F is a ***local diffeomorphism*** satisfying $F^*\widetilde{g} = g$.

  If there exists a *Riemannian isometry* between (M, g) and $(\widetilde{M}, \widetilde{g})$, we say that they are ***isometric*** as Riemannian manifolds. If each point of M has a *neighborhood* that is *isometric* to an *open subset* of $(\widetilde{M}, \widetilde{g})$, then we say that (M, g) is ***locally isometric*** to $(\widetilde{M}, \widetilde{g})$.

- **Definition** The study of properties of Riemannian manifolds that are ***invariant under (local or global) isometries*** is called ***Riemannian geometry***.
- **Definition** A Riemannian n-manifold (M, g) is said to be a ***flat Riemannian manifold***, and g is a ***flat metric***, if .(M, g) is **locally isometric** to $(\mathbb{R}^n, \bar{g})$.
- **Theorem 1.6** *For a Riemannian manifold (M, g), the following are equivalent:*
  1. *g is flat.*
  2. *Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{i,j} dx^i\, dx^j$.*
  3. *Each point of M is contained in the domain of a smooth coordinate chart in which* ***the coordinate frame*** *is* ***orthonormal.***
  4. *Each point of M is contained in the domain of a* ***commuting orthonormal frame****.*

2 Methods for Constructing Riemannian Metrics

2.1 Riemannian Submanifolds

2.2 Riemannian Submersions

2.3 Riemannian Coverings

3 Basic Constructions on Riemannian Manifolds

3.1 Raising and Lowering Indices

- **Definition** Given a Riemannian metric g on M, we define a ***bundle homomorphism*** $\widehat{g} : TM \to T^*M$ by setting

  $$\widehat{g}(v)(w) = g_p(v, w)$$

  for all $p \in M$ and $v, w \in T_pM$.
- **Remark** If X and Y are smooth vector fields on M, this yields

  $$\widehat{g}(X)(Y) = g(X, Y).$$

  $\widehat{g}(X)(Y)$ is ***linear*** over $\mathcal{C}^\infty(M)$ in Y and thus $\widehat{g}(X)$ is a ***smooth covector field*** by the tensor characterization lemma. On the other hand, the covector field $\widehat{g}(X)$ is ***linear*** over $\mathcal{C}^\infty(M)$ as a function of X, and thus $\widehat{g}$ is a ***smooth bundle homomorphism***. As usual, we use the same symbol for both the *pointwise bundle homomorphism* $\widehat{g} : TM \to T^*M$ and *the* ***linear map*** *on* ***sections*** $\widehat{g} : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.
- **Definition** Given a smooth local frame (E_i) and its dual coframe (ϵ^i), let $g = g_{i,j}\epsilon^i\epsilon^j$ be the ***local expression*** for g. If $X = X^i\, E_i$ is a smooth vector field, the *covector field* $\widehat{g}(X)$ has the ***coordinate expression***:

  $$\widehat{g}(X) = \left(g_{i,j} X^i\right) \epsilon^j := X_j\, \epsilon^j,$$

where the ***components*** of ***the covector field*** $\widehat{g}(X)$ is denoted by

$$X_j = g_{i,j} X^i. \tag{4}$$

We say that $\widehat{g}(X)$ *is obtained from* X ***by lowering an index***. And ***the covector field*** $\widehat{g}(X)$ is denoted by $X^\flat$ and called ***X flat***, borrowing from the musical notation for lowering a tone.

- **Remark** Because the matrix $(g_{i,j})$ is nonsingular at each point, the map $\widehat{g}$ is ***invertible***, and the matrix of $\widehat{g}^{-1}$ is just ***the inverse matrix of*** $(g_{i,j})$. We denote ***this inverse matrix*** by $(g^{i,j})$, so that $g^{i,j} g_{j,k} = g_{k,j} g^{j,i} = \delta^i_k$. The ***symmetry*** of $(g_{i,j})$ easily implies that $(g^{i,j})$ is also ***symmetric*** in i and j.

- **Definition** Given $\omega = \omega_j\, \epsilon^j$, the inverse map $\widehat{g}^{-1}$ is given by

$$\widehat{g}^{-1}(\omega) = \omega^i\, E_i$$

where

$$\omega^i = g^{i,j}\, \omega_j \tag{5}$$

If ω is *a covector field*, the ***vector field*** $\widehat{g}^{-1}(\omega)$ is called ***ω sharp*** and denoted by $\omega^\sharp$, and we say that *it is obtained from ω by* ***raising an index***.

The *two* ***inverse isomorphisms*** $\flat$ and $\sharp$ are known as ***the musical isomorphisms***.

- **Definition** If g is a Riemannian metric on M and $f : M \to \mathbb{R}$ is a smooth function, the ***gradient*** of f is ***the vector field***

$$\text{grad } f = (df)^\sharp := \widehat{g}^{-1}(df)$$

obtained from df by ***raising an index***. It is also denoted as ∇f.

- **Remark** Unwinding the definition we have

$$\begin{aligned} \langle \text{grad } f \,,\, X \rangle_g &= \widehat{g}\,(\text{grad } f)\,(X) \\ &= \widehat{g}\left(\widehat{g}^{-1}(df)\right)(X) \\ &= df(X) = Xf \end{aligned}$$

We see that grad f is ***characterized*** by the fact that

$$df(X) = \langle \text{grad } f \,,\, X \rangle_g \qquad \forall X \in \mathfrak{X}(M), \tag{6}$$

and has the ***local basis expression***

$$\text{grad } f = \left(g^{i,j} E_i f\right) E_j. \tag{7}$$

Thus if (E_i) is an ***orthonormal frame***, then grad f is the *vector field* whose ***components are the same as the components of*** df; but *in other frames, this will not be the case.*

- **Remark** In smooth coordinates $(\partial/\partial x^i)$, we have

$$\text{grad } f = g^{i,j} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{8}$$

- **Definition** If f is a smooth real-valued function on a smooth manifold M, recall that a point $p \in M$ is called ***a regular point*** of f if $df_p \neq 0$, and ***a critical point*** of f otherwise; and a level set $f^{-1}(c)$ is called ***a regular level set*** if every point of $f^{-1}(c)$ is a regular point of f

- **Proposition 3.1** *Suppose (M, g) is a Riemannian manifold, $f \in \mathcal{C}^{\infty}(M)$, and $R \subseteq M$ is the set of regular points of f. For each $c \in R$, the set $M_c = f^{-1}(c) \cap R$, if nonempty, is an **embedded smooth hypersurface** in M, and grad f is everywhere **normal** to M_c.*

- **Remark** If h is any covariant k-tensor field on a Riemannian manifold with $k \geq 2$, we can ***raise*** one of its indices (say the last one for definiteness) and obtain a $(1, k-1)$-tensor $h^{\sharp}$. The ***trace of*** $h^{\sharp}$ is thus a well-defined ***covariant*** $(k-2)$***-tensor field***.

  We define ***the trace of** h **with respect to** g* as

$$\mathrm{tr}_g(h) = \mathrm{tr}(h^{\sharp}).$$

  The most important case is that of a *covariant* 2*-tensor field*. In this case, $h^{\sharp}$ is a $(1,1)$-tensor field, which can equivalently be regarded as an ***endomorphism field***, and $\mathrm{tr}_g h$ is just ***the ordinary trace*** *of this endomorphism field*. In terms of a basis, this is

$$\mathrm{tr}_g(h) = h_i^{\,i} = g^{i,j}\, h_{i,j}.$$

  In particular, ***in an orthonormal frame*** this is ***the ordinary trace*** *of the matrix* $[h_{i,j}]$ *(the sum of its diagonal entries)*; but if the frame is not orthonormal, then this trace is different from the ordinary trace.

3.2 Inner Products of Tensors

- **Definition** Suppose g is a Riemannian metric on M, and $x \in M$. We can define an ***inner product*** on ***the cotangent space*** T_x^*M by

$$\langle\, \omega \,,\, \eta \,\rangle_g = \langle\, \omega^{\sharp} \,,\, \eta^{\sharp} \,\rangle_g.$$

- **Remark** (***Coordinate Representation of Inner Product on Covectors***)
  We see that under the formula for sharp operator

$$\begin{aligned}\langle\, \omega \,,\, \eta \,\rangle_g &= \langle\, \omega^{\sharp} \,,\, \eta^{\sharp} \,\rangle_g \\ &= g_{k,l} \left(g^{k,i}\, \omega_i\right) \left(g^{l,j}\, \eta_j\right) \\ &= \delta_l^i \omega_i \left(g^{l,j}\, \eta_j\right) \\ &= g^{i,j} \omega_i\, \eta_j.\end{aligned}$$

  In other words, ***the inner product on covectors*** *is represented by* ***the inverse matrix*** $g^{i,j}$. Using our conventions for raising and lowering indices, this can also be written

$$\langle\, \omega \,,\, \eta \,\rangle_g = \omega_i\, \eta^i = \omega^j\, \eta_j$$

  where $\eta^i = g^{i,j} \eta_j$ and $\omega^j = g^{i,j} \omega_i$.

- **Definition** If $E \to M$ is a smooth vector bundle, ***a smooth fiber metric*** on E is an ***inner product*** *on each fiber* E_p that varies ***smoothly***, in the sense that for any (local) smooth sections σ, τ of E, the inner product $\langle \sigma \, , \, \tau \rangle$ is a ***smooth*** *function.*

- **Proposition 3.2** ***(Inner Products of Tensors)****. [Lee, 2018]*
Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. There is a ***unique smooth fiber metric*** *on each tensor bundle $T^{(k,l)}TM$ with the property that if $\alpha_1 \, , \ldots, \alpha_{k+l}, \beta_1 \, , \ldots, \beta_{k+l}$ are vector or covector fields as appropriate, then*

$$\langle \alpha_1 \otimes \ldots \otimes \alpha_{k+l} \, , \, \beta_1 \otimes \ldots \otimes \beta_{k+l} \rangle = \langle \alpha_1 \, , \, \beta_1 \rangle \cdot \ldots \cdot \langle \alpha_{k+l} \, , \, \beta_{k+l} \rangle \tag{9}$$

With this inner product, if $(E_1 \, , \ldots, E_n)$ is a ***local orthonormal frame*** *for TM and $(\epsilon^1 \, , \ldots, \epsilon^n)$ is the corresponding dual* ***coframe****, then the collection of tensor fields $E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_l}$ as all the indices range from 1 to n* ***forms a local orthonormal frame*** *for $T^{(k,l)}(T_pM)$. In terms of any (not necessarily orthonormal) frame, this* ***fiber metric*** *satisfies*

$$\langle F \, , \, G \rangle = g_{i_1, r_1} \, \ldots \, g_{i_k, r_k} \, g^{j_1, s_1} \, \ldots \, g^{j_l, s_l} \, F^{i_1 \,, \ldots, \, i_k}_{j_i \,, \ldots, \, j_l} \, G^{r_1 \,, \ldots, \, r_k}_{s_1 \,, \ldots, \, s_l} \tag{10}$$

If F and G are both covariant, this can be written

$$\langle F \, , \, G \rangle = F_{j_1 \,, \ldots, \, j_l} G^{j_1 \,, \ldots, \, j_l}.$$

where the last factor on the right represents the components of G with ***all of its indices raised:***

$$G^{j_1 \,, \ldots, \, j_l} = g^{j_1, s_1} \, \ldots \, g^{j_l, s_l} G_{s_1 \,, \ldots, \, s_l}.$$

3.3 The Volume Form and Integration

3.4 The Divergence and the Laplacian

4 Length and Distance

4.1 The Riemannian Distance Function

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