Lecture 0: Summary (Part 3)

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1 Normed Linear Space

- **Remark** Note that the definition of a *metric space* is only about the ***topology*** of the space. In the field of functional analysis, we are mostly concerned about ***the vector space***, i.e. a space that equipped with algebraic operations such as vector addition and scalar multiplications. In order to make the ***metric topological structure compatible*** with ***the algebraic structure of vector space***, we need to introduce additional function such as the ***norm***.

- **Definition** (***Normed Linear Space***)
  A ***normed linear space*** is a vector space, V, over $\mathbb{R}$ (or $\mathbb{C}$) and a function, $\|\cdot\| : V \to \mathbb{R}$ which satisfies:

  1. (***Non-Negativity***): $\|v\| \geq 0$ for all v in V;
  2. (***Positive Definiteness***): $\|v\| = 0$ if and only if $v = 0$;
  3. (***Absolute Homogeneity***) $\|\alpha v\| = |\alpha| \, \|v\|$ for all v in V and α in $\mathbb{R}$ (or $\mathbb{C}$)
  4. (***Subadditivity / Triangle Inequality***) $\|v + w\| \leq \|v\| + \|w\|$ for all v and w in V

  We denote the normed linear space as $(V, \|\cdot\|)$.

- **Remark** If the function $p : V \to \mathbb{R}$ only satisfies the condition 1, 3 and 4 (without *positive definiteness*), it is called a ***semi-norm***. The 1. *non-negativity* condition can be derived by the 3. *homogeneity* and 4. *subadditivity* conditions.

- **Remark** *A normed linear space* $(V, \|\cdot\|)$ is a ***metric space*** with *induced metric*

$$d(x, y) = \|x - y\|\,, \quad \text{for all } x, y \in V$$

2 Banach Space

2.1 Definition and Examples

- **Definition** A normed linear space $(V, \|\cdot\|)$ is ***complete*** if it is *complete* as a *metric space* in *the induced metric*.

- **Definition** A ***complete*** *normed linear space* is called a ***Banach space***.

- **Example** ($\mathcal{C}(X)$ and its subspace $\mathcal{C}_{\mathbb{R}}(X)$)
  Let $\mathcal{C}(X)$ be the set of all *complex-valued* **continuous functions** on X and $\mathcal{C}_{\mathbb{R}}(X) \subseteq \mathcal{C}(X)$ be the set of all ***real-valued continuous functions*** on X. Also define $\mathcal{C}^b(X)$ as the set of all *complex-valued* **bounded continuous functions** on X. When X is ***a compact space***, $\mathcal{C}^b(X) = \mathcal{C}(X)$. Define the norm as

$$\|f\|_\infty = \sup_{x \in X} |f(x)|\,.$$

  Then for ***compact Hausdorff space*** X, $\mathcal{C}(X)$ is a *(complex)* ***Banach space*** and $\mathcal{C}(X)$ is ***a (real) Banach space***.

- **Example** ($L^\infty(\mathbb{R})$ ***and its subspace*** $\mathcal{BC}(\mathbb{R})$)
  Let $L^\infty(\mathbb{R})$ be the set of (*equivalence classes of*) *complex-valued measurable functions* on $\mathbb{R}$

such that $|f(x)| \leq M$ a.e. with respect to Lebesgue measure for some $M < \infty$ ($f = g$ means $f(x) = g(x)$ a.e.). Let $\|f\|_\infty$ be ***the smallest such*** M. $L^\infty(\mathbb{R})$ is a ***Banach space*** with norm $\|\cdot\|_\infty$.

The ***bounded continuous functions*** $\mathcal{BC}(\mathbb{R})$ is a ***subspace*** of $L^\infty(\mathbb{R})$ and restricted to $\mathcal{BC}(\mathbb{R})$ the $\|\cdot\|_\infty$-norm is just the usual ***supremum norm*** under which $\mathcal{BC}(\mathbb{R})$ is ***complete*** (since *the uniform limit of continuous functions is continuous*). Thus, $\mathcal{BC}(\mathbb{R})$ ***is a closed subspace of*** $L^\infty(\mathbb{R})$.

Consider the set $\kappa(\mathbb{R})$ of ***continuous functions*** *with* ***compact support***, that is, the continuous functions that *vanish outside of some closed interval.* $\kappa(\mathbb{R})$ is a ***normed linear space*** under $\|\cdot\|_\infty$; but *is* ***not complete***, The ***completion*** of $\kappa(\mathbb{R})$ is ***not all*** *of* $\mathcal{BC}(\mathbb{R})$; for example, if f is the function which is identically equal to one, then I *cannot be approximated by a function in* $\kappa(\mathbb{R})$ since $\|f - g\|_\infty \geq 1$ for all $g \in \kappa(\mathbb{R})$. The ***completion*** of $\kappa(\mathbb{R})$ is just $\mathcal{C}_\infty(\mathbb{R})$, *the continuous functions which* ***approach zero*** *at* ∞.

Some of the most powerful theorems in functional analysis (*Riesz-Markov*, *Stone-Weierstrass*) are generalizations of properties of $\mathcal{BC}(\mathbb{R})$. ∎

- **Example** (L^p ***spaces***)
  Let (X, μ) be a measure space and $p \geq 1$. We denote by $L^p(X, \mu)$ ***the set of equivalence classes*** of measurable functions which satisfy:

$$\|f\|_p := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty$$

Two functions are *equivalent* if they differ only on *a set of measure zero.*

The following theorem collects many of the standard facts about L^p spaces.

Theorem 2.1 *Let* $1 \leq p < \infty$*, then*

1. ***(The Minkowski Inequality)****: If* $f, g \in L^p(X, \mu)$*, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

2. ***(Riesz-Fisher)****:* $L^p(X, \mu)$ *is* ***complete****.*
3. ***(The Hölder Inequality)*** *Let* $p, q,$ *and* r *be positive numbers satisfying* $p, q, r \geq 1$ *and* $p^{-1} + q^{-1} = r^{-1}$*. Suppose* $f \in L^p(X, \mu)$*,* $g \in L^q(X, \mu)$*. Then* $fg \in L^r(X, \mu)$ *and*

$$\|fg\|_r \leq \|f\|_p \, \|g\|_q$$

Remark *The Minkowski inequality* shows that $L^p(X, \mu)$ is a vector space and $\|\cdot\|_p$ satisfies the triangle inequality. This together with *Riesz-Fisher theorem* shows that $L^p(X, \mu)$ ***is a Banach space***.

- **Example** (***Sequence Spaces***)
  There is a nice class of spaces which is easy to describe and which we will often use to illustrate various concepts. In the following definitions,

$$a := (a_n)_{n=1}^\infty$$

always denotes a sequence of complex numbers.

$$\ell^\infty := \left\{a : \|a\|_\infty := \sup_n |a_n| < \infty\right\}$$
$$c_0 := \left\{a : \lim_{n\to\infty} a_n = 0\right\}$$
$$\ell^p := \left\{a : \|a\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty\right\}$$
$$s := \left\{a : \lim_{n\to\infty} n^p a_n = 0 \textit{ for all positive integers } p\right\}$$
$$f := \{a : a_n = 0 \textit{ for all but a finite number of } n\}$$

It is clear that as sets $f \subseteq s \subseteq \ell^p \subseteq c_0 \subseteq \ell^\infty$.

The spaces ℓ^∞ and c_0 are Banach spaces with the $\|\cdot\|_\infty$ norm; ***ℓ^p is a Banach space with the $\|\cdot\|_p$ norm*** (note that $\ell^p = L^p(\mathbb{R}, \mu)$ where μ is the measure with *mass one at each positive integer* and *zero* everywhere else). It will turn out that s ***is a Frechet space***.

One of the reasons that these spaces are easy to handle is that *f is **dense** in ℓ^p* *(in $\|\cdot\|_p$; $p < \infty$* and ***f is dense in c_0*** ***(in the $\|\cdot\|_\infty$ norm)***. Actually, the set of elements of f with only *rational entries* is also **dense** in ℓ^p and c_0. Since this set is ***countable***, ℓ^p and c_0 are ***separable***. ***ℓ^∞ is not separable***.

- **Example** (***Hilbert Space***)
  All ***Hilbert spaces*** $(\mathcal{H}, \langle\cdot\,,\,\cdot\rangle)$ are ***Banach spaces*** with *induced norm* as
  $$\|x\| = (\langle x\,,\,x\rangle)^{\frac{1}{2}}\,.$$

2.2 Isomorphism and Equivalence of Norms

- **Definition** (***Absolutely Summable***)
  A sequence of elements $(x_n)_{n=1}^{\infty}$ in a *normed linear space* X is called ***absolutely summable*** $\sum_{n=1}^{\infty} \|x_n\| < \infty$. It is called ***summable*** if $\sum_{n=1}^{N} x_n$ *converges* as $N \to \infty$ to an $x \in X$.

- **Proposition 2.2** ***(Criterion of Completeness for Normed Linear Space)*** *[Reed and Simon, 1980]*
  A normed linear space is ***complete*** *if and only if every* ***absolutely summable*** *sequence is* ***summable***.

- **Definition** (***Isomorphism between Normed Linear Spaces***)
  A ***bounded linear operator*** from a normed linear space X to a normed linear space Y is called an ***isomorphism*** if it is a ***bijection*** which is ***continuous*** and which has ***a continuous inverse***.

  If it is ***norm preserving***, it is called ***an isometric isomorphism*** (any *norm preserving* map is called an ***isometry***).

- **Definition** (***Norm Equivalence***)
  Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are called ***equivalent*** if there are

positive constants C and C' such that, for all $x \in X$,

$$C \|x\|_2 \leq \|x\|_1 \leq C' \|x\|_2$$

- **Remark** This concept is motivated by the following fact.

  Equivalent norms *on X define* ***the same topology*** *for X.*

- **Proposition 2.3** *The* ***completions*** *of the space in the two norms will be* ***isomorphic*** *if and only if* ***the norms are equivalent****.*

- **Proposition 2.4** *Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are* ***equivalent*** *if and only if* ***the identity map*** *is an* ***isomorphism****.*

- **Remark** An example is provided by *the sequence spaces.* The ***completion*** of f in the $\|\cdot\|_\infty$ norm is c_0 while the completion in the $\|\cdot\|_p$ norm is ℓ^p.

2.3 Subspace of a Banach Space

- **Definition** A ***subspace*** Y of a normed space X is a subspace of X considered as *a vector space*, with the *norm* obtained by ***restricting*** *the norm on X to the subset Y*. This norm on Y is said to be ***induced*** *by the norm on X*.

  If Y is closed in X, then Y is called ***a closed subspace*** of X.

- **Remark** A subspace Y of a ***Banach space*** X is a subspace of X considered as a normed space. Hence *we do not require Y to be complete.*

- **Proposition 2.5** *(****Subspace of a Banach space****). [Kreyszig, 1989]*
  A subspace Y of a Banach space X is ***complete*** *if and only if the set Y is* ***closed*** *in X.*

2.4 Basis and Separability

- **Definition** (***Basis of Normed Space***)
  If a normed space X contains *a sequence* (e_i) with the property that for *every* $x \in X$ there is a ***unique*** *sequence of scalars* (u^i) such that

$$\lim_{n \to \infty} \left\| x - \sum_{i=1}^{n} u^i \, e_i \right\| = 0, \tag{1}$$

  then (e_i) is called *a* ***Schauder basis (or basis)*** *for X*. The series $\sum_{i=1}^{\infty} u^i \, e_i$ which has the sum x is then called *the* ***expansion*** *of x with respect to* (e_i), and we write

$$x = \sum_{i=1}^{\infty} u^i \, e_i$$

- **Example** The (Schauder) basis of ℓ^p is (e_n) and

$$e_n := (\delta_{n,i}) = (0 \, , \ldots , 0, 1, 0, \, \ldots)$$

  where the i-th component is 1 and the others are all zeros.

- **Proposition 2.6** *If a normed space X has a Schauder basis, then X is **separable**.*

- **Theorem 2.7** *(**Completion**). [Kreyszig, 1989]*
  *Let $X = (X, \|\cdot\|)$ be a normed space. Then there is a Banach space X and an isometry A from X onto a subspace W of X which is **dense** in X. The space X is **unique**, except for isometries.*

2.5 Direct Sum of Banach Spaces

- **Definition** (***Direct Sum of Banach Spaces***)
  Let A be an index set (not necessarily countable), and suppose that for each $\alpha \in A$, X_α is a Banach space. Let

$$X := \left\{ (x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha,\ \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha} < \infty \right\}.$$

  Then X with the norm

$$\|(x_\alpha)_{\alpha \in A}\|_X := \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha}$$

  is a Banach space. It is called ***the direct sum*** of the spaces X_α and is often written as $X = \bigoplus_{\alpha \in A} X_\alpha$.

- **Remark** (***Banach Spaces Direct Sum $\neq$ Hilbert Spaces Direct Sum***)
  Note that *the direct sum of Banach spaces* is ***not*** *necesssarily the direct sum of Hilbert spaces.*

  For instance, if we take countable numbers of copies of $\mathbb{C}$, *the Banach space direct sum* is ℓ_1, while *the Hilbert space direct sum* is ℓ_2.

  However, if only ***finite number*** *of Hilbert spaces* are involved, then both *Hilbert space direct sum* and *their Banach space direct sum* are *isomorphic* to each other.

2.6 Finite Dimensional Case

- **Remark** (***Finite Dimensional Normed Space is Simple***)
  We summarizes the ***unique*** simple strcuture of finite dimensional normed space in terms of various concepts we discussed in this chapter:

  1. ***Completeness***: *Every finite dimensional normed vector space* is ***complete*** so it is a ***Banach space***;
  2. ***Norm Equivalence***: ***All norms*** *in a finite dimensional normed space are* ***equivalent***; therefore, ***convergence*** in one norm means convergence in all other norms.
  3. ***Topological Equivalence***: There exists ***only one distinct norm topology*** in a finite dimensional normed space;
  4. ***Compactness***: In a finite dimensional normed space, ***compactness*** is equivalent to ***closedness*** and ***boundedness***.

5. ***Bounded Linear Operator***: ***Every linear operator*** between finite dimensional normed spaces is ***bounded***. Thus in finite dimensional space, every linear operator is ***continuous***.

- **Lemma 2.8** ***(Linear combinations)****. [Kreyszig, 1989]*
  Let $(x_1, \ldots, x_n)$ *be a* ***linearly independent*** *set of vectors in a normed space* X *(of any dimension). Then there is a number* $c > 0$ *such that for every choice of scalars* $\alpha_1, \ldots, \alpha_n$ *we have*

$$\left\| \sum_{i=1}^{n} \alpha_i \, x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \,. \tag{2}$$

- **Theorem 2.9** ***(Completeness)****. [Kreyszig, 1989]*
  Every finite dimensional subspace Y *of a normed space* X *is* ***complete****. In particular, every* ***finite dimensional*** *normed space is* ***complete****.*

- **Remark** In other words, *every finite dimensional normed vector space* is a ***Banach space***.

- **Proposition 2.10** ***(Closedness)****. [Kreyszig, 1989]*
  Every finite dimensional subspace Y *of a normed space* X *is* ***closed*** *in* X*.*

- **Theorem 2.11** ***(Equivalent Norms)****. [Kreyszig, 1989]*
  If a vector space X *is* ***finite dimensional****,* ***all norms are equivalent****.*

- **Remark** This theorem is of considerable practical importance. For instance, it implies that ***convergence*** or *divergence* of a sequence in *a finite dimensional vector space* ***does not depend*** *on the particular* ***choice of a norm*** *on that space.* There is no ambiguity when we say $x_n \to x$ in *finite dimensional space.*

  In fact, ***there exists only one distinct norm topology for finite dimensional space***.

- **Definition** (***Compactness***).
  A metric space X is said to be ***(sequentially) compact*** if *every sequence* in X has a ***convergent subsequence***. A subset M of X is said to be ***compact*** if M is *compact* considered as a subspace of X, that is, if every sequence in M has a convergent subsequence *whose limit is an element of* M*.*

- **Lemma 2.12** ***(Compactness)****.*
  A ***compact*** *subset* M *of a metric space is* ***closed*** *and* ***bounded****.*

- **Remark** *The* ***converse*** *of this lemma is in general* ***false***. But for *finite dimensional space*, the converse is true:

- **Theorem 2.13** ***(Compactness)****. [Kreyszig, 1989]*
  In a ***finite*** *dimensional normed space* X*, any subset* $M \subseteq X$ *is* ***compact if and only if*** M *is* ***closed*** *and* ***bounded****.*

- **Remark** *In finite dimensional space*, the *compact* subsets are precisely *the closed and bounded* subsets, so that this property (***closedness*** and ***boundedness***) can be used for *defining* ***compactness***.

  However, *this can no longer be done* in the case of *an* ***infinite dimensional normed space***.

- **Lemma 2.14** ***(F. Riesz's Lemma)****. [Kreyszig, 1989]*

*Let Y and Z be **subspaces** of a normed space X (of any dimension), and suppose that Y is **closed** and is a **proper subset** of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that*

$$\|z\| = 1, \quad \|z - y\| \geq \theta, \quad \textit{for all } y \in Y.$$

- **Theorem 2.15** *(**Bounded Linear Operator**)*
  *If a normed space X is finite dimensional, then every linear operator on X is **bounded**.*

3 Bounded Linear Operators on Banach Space

3.1 Definitions and Examples

- **Definition** (***Bounded Linear Operator***)
  A ***bounded linear transformation*** (or ***bounded operator***) is a mapping $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ from a normed linear space X to a normed linear space Y that satisfies

  1. (***Linearity***) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$
  2. (***Boundedness***) $\|Tx\|_Y \leq C\, \|x\|_X$ for small $C \geq 0$.

  The smallest such C is called *the **norm** of T*, written $\|T\|$ or $\|T\|_{X,Y}$. Thus

  $$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y$$

- **Remark** *A linear operator T* is a ***homomorphism*** *of a vector space (its domain) into another vector space*, that is, T ***preserves*** *the* ***two operations*** *of vector space.*

- **Proposition 3.1** *[Reed and Simon, 1980, Kreyszig, 1989]*
  *Let T be a linear transformation between two **normed linear spaces**. The following are **equivalent**:*

  1. *T is **continuous** at **one** point.*
  2. *T is **continuous** at **all** points.*
  3. *T is **bounded**.*

- **Definition** (***The Bounded Operators***)
  In above we defined the concept of a *bounded linear transformation* or *bounded operator* from one normed linear space, X, to another Y; we will denote *the set of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$*. We can introduce a *norm* on $\mathcal{L}(X, Y)$ by defining

  $$\|A\| := \sup_{x \neq 0,\, x \in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

  This norm is often called ***the operator norm***.

- We have the following proposition

  Proposition 3.2 *If Y is **complete**, $\mathcal{L}(X, Y)$ is a **Banach space**.*

- **Theorem 3.3** ***(The B.L.T. Theorem)*** *[Reed and Simon, 1980]*
Suppose T is a bounded linear transformation from a normed linear space $(V_1, \|\cdot\|_1)$ to a ***complete*** *normed linear space $(V_2, \|\cdot\|_2)$. Then T can be* ***uniquely extended*** *to a bounded linear transformation (with the same bound), $\widetilde{T}$, from the* ***completion*** *of V_1 to $(V_2, \|\cdot\|_2)$.*

3.2 Dual Space

- **Definition** (***Dual Space***)
The space $\mathcal{L}(X, \mathbb{C})$ of all ***bounded linear functionals*** on *a normed linear space X* is called the ***dual space*** *of X*. This space $\mathcal{L}(X, \mathbb{C})$ is denoted as X^*.

The dual space X^* is a Banach space *if X is a Banach space* (See Proposition 3.2). The ***norm*** of dual space is

$$\|\lambda\| := \sup_{x \neq 0, \, \|x\| \leq 1} |\lambda(x)| \,,$$

for all $\lambda \in X^*$.

- **Remark** (***Cauchy-Schwartz inequality***)
By definition, we have the *dual norm inequality*

$$|\lambda(x)| \leq \|\lambda\|_{X^*} \, \|x\|_{X} \,. \tag{3}$$

In Hilbert space, since $\lambda(x) = \langle y_\lambda \,, x \rangle$ for some y_λ, it becomes ***the Cauchy-Schwartz inequality***.

$$|\langle y_\lambda \,, x\rangle| \leq \|y_\lambda\| \, \|x\|$$

- **Example** (***Hilbert Space***)
Any ***Hilbert space*** $\mathcal{H}$ is ***isomorphic*** to its ***dual*** $\mathcal{H}^*$ according to *the Riesz Representation Theorem.* For instance $L^2(X, \mu) = (L^2(X, \mu))^*$.

- **Example** ($L^p(X, \mu)$ ***Spaces,*** $1 < p < \infty$)
Suppose that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ then. according to *the Hölder inequality*, fg is in $L^1(X, \mu)$. Thus,

$$\int_X f(x)\overline{g(x)} d\mu(x) < \infty$$

makes sense, Let $g \in L^q(X, \mu)$ be fixed and define

$$G(f) := \int_X f \bar{g} d\mu$$

for each $f \in L^p(X, \mu)$. The Hölder inequality shows that $G(f)$ *is a* ***bounded linear functional*** on $L^p(X, \mu)$ with *norm* less than or equal to $\|g\|_q$; actually ***the norm*** $\|G\|$ ***is equal to*** $\|g\|_q$.

The *converse* of this statement is also *true*. That is, ***every bounded linear functional on*** L^p ***is of the form*** $G(f)$ ***for some*** $g \in L^q$. Furthermore, *different* ***functions*** *in L^q give rise to different* ***functionals*** *on L^p*. Thus, the mapping

$$L^q(M, \mu) \to (L^p(X, \mu))^*, \quad g \mapsto G_g(\cdot)$$

is a ***(conjugate linear) isometric isomorphism***.

In this sense, $L^q(M,\mu)$ ***is the dual of*** $L^p(X,\mu)$. Since the roles of p and q in the expression $p^{-1}+q^{-1}=1$ are *symmetric*, it is clear that $L^p(X,\mu)=(L^q(X,\mu))^*=(L^p(X,\mu))^{**}$. That is, the ***dual*** of the ***dual*** of $L^p(X,\mu)$ is again $L^p(X,\mu)$. ∎

- **Remark** Note that $L^\infty(X,\mu)$ space and $L^1(X,\mu)$ space are ***not dual*** spaces to each other. The dual space of $L^\infty(X,\mu)$ space is much larger than $L^1(X,\mu)$ space. In fact, $L^1(X,\mu)$ ***space is not dual to any Banach space***. This is different from ℓ^∞ and ℓ^1.

- **Example** ($\ell^\infty=(\ell^1)^*$, $\ell^1=(c_0)^*$)
Suppose that $(\lambda_k)_{k=1}^\infty\in\ell^1$. Then for each $(a_k)_{k=1}^\infty\in c_0$,
$$\Lambda\left((a_k)_{k=1}^\infty\right)=\sum_{k=1}^\infty \lambda_k\, a_k$$
converges and $\Lambda(\cdot)$ is a ***continuous linear functional*** on c_0 with ***norm*** equal to $\sum_{k=1}^\infty|\lambda_k|$.

- **Remark** We see that $c_0\subseteq(c_0)^{**}=(\ell^1)^*=\ell^\infty$.

- **Definition** (***Double Dual***)
Since the ***dual*** X^* of *a Banach space* is itself *a Banach space*, it also has *a* ***dual*** *space*, denoted by X^{**}. X^{**} is called ***the second dual***, ***the bidual***, or ***the double dual*** *of the space* X.

- **Proposition 3.4** *[Reed and Simon, 1980]*
Let X be a Banach space. For each $x\in X$, let $\widetilde{x}(\cdot)$ be the linear functional on X^* which assigns to each $\lambda\in X^*$ the number $\lambda(x)$. Then the map $J:x\mapsto\widetilde{x}$ is an ***isometric isomorphism*** *of X onto a (possibly proper) subspace of X^{**}.*

- **Remark** From above proposition, we see that there exists an *embedding* from X to a subset of X^{**}
$$X\subseteq X^{**},\quad X\hookrightarrow X^{**}$$

- **Definition** If the map $J:x\mapsto\widetilde{x}$ is ***surjective***, then X is said to be ***reflexive***. In other word, X is reflective if and only if $X=X^{**}$.

- **Example** $L^p(X,\mu)$ spaces are ***reflective*** for $1<p<\infty$. Note that $L^p(X,\mu)=(L^q(X,\mu))^*=(L^p(X,\mu))^{**}$

- **Example** All Hilbert spaces $\mathcal{H}$ are ***reflective***.

- **Example** Since $c_0\subseteq(c_0)^{**}=(\ell^1)^*=\ell^\infty$, c_0 is *not reflective.*

3.3 Dual Space of Compact Supported Continuous Functions

3.3.1 Radon Measure

- **Definition** (***Outer Regularity***) [Folland, 2013]
Let μ be a ***Borel*** measure on X and E a *Borel subset* of X. The measure μ is called

outer regular on E if

$$\mu(E) = \inf \{\mu(U) : U \supseteq E, U \text{ is open}\}$$

- **Definition** (***Inner Regularity***) [Folland, 2013]
  Let μ be a ***Borel*** measure on X and E a *Borel subset* of X. The measure μ is called ***inner regular*** on E if

$$\mu(E) = \sup \{\mu(C) : C \subseteq E, C \text{ is compact}\}$$

- **Definition** If μ is *outer* and *inner regular* on *all Borel sets*, μ is called ***regular***.

- **Remark** ***Baire measure*** is equivalent to a ***regular Borel measure (Randon measure)*** in the context of ***compact space*** X.

- **Definition** (***Radon Measure***) [Folland, 2013]
  A ***Radon measure*** μ on X is a *Borel measure* that is

  1. ***finite*** on *all* ***compact*** *sets*; i.e. for any ***compact subset*** $K \subseteq X$,
  $$\mu(K) < \infty.$$
  2. ***outer regular*** on *all Borel sets*; i.e. for any *Borel set* E
  $$\mu(E) = \inf \{\mu(U) : E \subseteq U, U \textit{ is open}\}.$$
  3. ***inner regular*** on all *open sets*; i.e. for any *open set* E
  $$\mu(E) = \sup \{\mu(C) : C \subseteq E, C \textit{ is compact and Borel}\}.$$

- **Remark** ***Randon measure*** is called ***regular Borel measure***.

3.3.2 The Riesz-Markov Representation Theorem

- **Definition** (***Positive Linear Functional***)
  Let $\mathcal{C}(X)$ be *the space of* **continuous** *functions* on X. A ***positive linear functional*** on $\mathcal{C}(X)$ is a (not necessarily a priori continuous) *linear functiona* I with $I(f) > 0$ for all f with $f(x) \geq 0$ pointwise.

- **Lemma 3.5** ***(Bounded by Unit Ball in Uniform Metric)*** *[Folland, 2013]*
  If I is a positive linear functional on $\mathcal{C}_c(X)$, for each compact $C \subseteq X$ there is a constant κ_C such that $|I(f)| < \kappa_C \|f\|_u$ for all $f \in \mathcal{C}_c(X)$ such that supp$(f) \subset K$.

- **Remark** If μ is a *Borel measure* on X such that $\mu(C) < \infty$ for every compact subset $C \subseteq X$, then $\mathcal{C}_c(X) \subseteq L^1(X, \mu)$. Therefore, $f \mapsto \int f d\mu$ is a ***positive linear functional*** on $\mathcal{C}_c(X)$.

  The following theorem shows that the ***every positive linear functionals*** on $\mathcal{C}_c(X)$ can be ***represented*** as the *integral with respect to* ***some Radon measure*** μ.

- **Theorem 3.6** ***(The Riesz-Markov Representation Theorem).*** *[Folland, 2013]*
Let X be a ***locally compact Hausdorff*** *space, if I is a* ***positive linear functional*** *on $\mathcal{C}_c(X)$, there is a* ***unique Radon measure*** *μ on X such that*

$$I(f) = \int f d\mu$$

for all $f \in \mathcal{C}_c(X)$. Moreover, μ satisfies the following conditions:

1. *for all* ***open*** *sets $U \subseteq X$,*

$$\mu(U) = \sup\left\{I(f) : f \in \mathcal{C}_c(X), supp(f) \subseteq U,\; 0 \le f \le 1\right\}.$$

2. *for all* ***compact*** *sets $K \subseteq X$*

$$\mu(K) = \inf\left\{I(f) : f \in \mathcal{C}_c(X), f \ge \mathbb{1}_K\right\}.$$

- **Remark** Following *the Riesz-Markov Theorem*

$$\mu(X) = \sup\left\{\int_X f d\mu : f \in \mathcal{C}_c(X),\; 0 \le f \le 1\right\}.$$

- The following theorem is another version of *the Riesz representation theorem*:

Theorem 3.7 ***(The Riesz-Markov Theorem)*** *[Reed and Simon, 1980]*
Let X be a ***compact Hausdorff*** *space. For any* ***positive linear functional*** *I on $\mathcal{C}(X)$, there is a* ***unique Baire measure*** *μ on X with*

$$I(f) = \int f d\mu$$

- **Remark** (***Radon Measures*** $\Leftrightarrow$ ***Positive Linear Functionals on*** $\mathcal{C}_c(X)$)
The Riesz-Markov theorem relates ***linear functionals*** on spaces of ***continuous*** *functions on a* ***locally compact*** *space* to ***measures*** in ***measure theory***.

- **Remark** ***Not to be confused*** with *another Riesz representation theorem*, which related *linear functions on Hilbert space* as inner product with some element in Hilbert space

$$I(f) = \langle f\,,\, g_I\rangle$$

for some $g_I \in \mathcal{H}$.

- **Remark** (***Duality between*** $\mathcal{C}_0(X)$ ***and*** $\mathcal{M}(X)$)
The Riesz representation theorem establishes the ***foundation*** of the ***the duality*** between *the space of compactly supported continuous functions* and *the space of all Radon* ***measures*** on X.

In particular, for *locally compact Hausdorff* X,

$$\{\mu : \mu \textit{ is a Radon measure on } X\} \simeq \{I \in \mathcal{C}_0(X)^* : I \text{ is positive}\}$$

3.3.3 Dual Space of $\mathcal{C}_0(X)$

- **Theorem 3.8** *(**Monotone Convergence Theorem for Nets**) [Reed and Simon, 1980]*
  *Let μ be a **regular Borel** measure on a **compact Hausdorff** space X. Let $\{f_\alpha\}_{\alpha \in J}$ be an **increasing net** of continuous functions. Then*

$$f_\alpha \to f \in L^1(X, \mu), \quad a.e.$$

  if and only if $\sup_\alpha \|f_\alpha\|_1 < \infty$ *and in that case*

$$\|f_\alpha - f\|_1 \to 0.$$

- **Lemma 3.9** *[Reed and Simon, 1980]*
  Let $f, g \in \mathcal{C}(X)$ with $f, g \geq 0$. Suppose $h \in \mathcal{C}(X)$ and $0 \leq h \leq f + g$. Then, we can write $h = h_1 + h_2$ with $0 \leq h_1 \leq f$, $0 \leq h_2 \leq g$, $h_1, h_2 \in \mathcal{C}(X)$.

- **Theorem 3.10** *(**Decomposition of Real Linear Functional**) [Reed and Simon, 1980, Folland, 2013]*
  *Let X be a **compact** space, $I \in (\mathcal{C}(X))^*$ be any continuous linear functional on $\mathcal{C}(X)$. Then I can be written*

$$I = I_+ - I_-$$

  *with I_+ and I_- **positive linear functionals**. Moreover,*

$$I_+ + I_- = \|I\|$$

  *and this **uniquely determines** I_+ and I_- .*

- **Definition** (***Complex Radon Measure***)
  A ***signed Radon measure*** is a ***signed Borel measure*** whose ***positive*** and ***negative variations*** are ***Radon***, and a ***complex Radon measure*** is a ***complex Borel measure*** whose *real and imaginary parts* are *signed Radon measures.*

- **Remark** In [Reed and Simon, 1980], one defines ***the complex Baire measure*** as a *finite linear complex combination* of *Baire measures.*

- **Definition** (***Space of Complex Radon Measures***)
  On *locally compact Hausdorff space X*, We denote *the space of complex Radon measures on X* by $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$ we define

$$\|\mu\| = |\mu|(X),$$

  where $|\mu|$ is the ***total variation*** of μ.

- **Proposition 3.11** *($\mathcal{M}(X)$ **is Normed Linear Space**) [Folland, 2013]*
  *If μ is a **complex Borel measure**, then μ is **Radon** if and only if $|\mu|$ is **Radon**. Moreover, $\mathcal{M}(X)$ is a vector space and $\mu \mapsto \|\mu\|$ is a **norm** on it.*

- **Theorem 3.12** *(**The Riesz-Markov Theorem, Locally Compact Version**) [Reed and Simon, 1980, Folland, 2013]*
  *Let X be a **locally compact Hausdorff** space. For any continuous linear functional I*

on $\mathcal{C}_0(X)$, (the space of continuous functions *on X that vanishes at infinity), there is a* ***unique regular countably additive complex Borel measure*** *μ on X such that*

$$I(f) = \int_X f d\mu, \quad \textit{for all } f \in \mathcal{C}_0(X).$$

*The **norm** of I as a linear functional is **the total variation** of μ, that is*

$$\|I\| = |\mu|(X).$$

*Finally, I is **positive** if and only if the measure μ is **non-negative**.*

- **Remark** In other word, the map $\mu \mapsto I_\mu$, is an ***isometric isomorphism*** from $\mathcal{M}(X)$ to $(\mathcal{C}_0(X))^*$, or

$$\mathcal{M}(X) \simeq (\mathcal{C}_0(X))^*.$$

- **Corollary 3.13** *[Reed and Simon, 1980, Folland, 2013]*
*Let X be a **compact Hausdorff** space. Then the **dual space $\mathcal{C}(X)^*$** is **isometric isomorphism** to $\mathcal{M}(X)$.*

- **Definition** Given $\mathcal{M}(X) \simeq (\mathcal{C}_0(X))^*$, we define subspaces of $\mathcal{M}$:

$$\begin{aligned} \mathcal{M}_+(X) &= \{I \in \mathcal{M}(X) : I \textit{ is a positive linear functional}\}, \\ \mathcal{M}_{+,1}(X) &= \{I \in \mathcal{M}(X) : \|I\| = 1\}. \end{aligned}$$

Thus $\mathcal{M}_+(X)$ is *identified* with ***the space of all positive Randon measures on** X.*

- **Remark** (***Isometric Embedding of** $L^1(\mu)$ **into** $M(X)$*)
Let μ be a fixed *positive Radon measure* on X. If $f \in L^1(\mu)$, *the complex measure*

$$d\nu_f = f d\mu$$

is easily seen to be ***Radon***, and $\|\nu\| = \int |f| d\mu = \|f\|_1$. Thus $f \mapsto \nu_f$ is an ***isometric embedding*** of $L^1(\mu)$ into $M(X)$ whose range consists precisely of those $\nu \in \mathcal{M}(X)$ such that $\nu \ll \mu$.

- **Remark** (***Two Perspectives of Measures***)
For *regular Borel measure* μ or in general, *Radon measures* on ***locally compact*** space X, there are two perspectives:

  1. ***Nonegative set function on the σ-algebra $\mathscr{A}$***: as a ***measure of the volume*** of a *subset* in X;
  2. ***Positive linear functional on** $\mathcal{C}_0(X)$*: as a ***integral** of compactly supported continuous functions with respect to **given measure**.*

In some cases, it is important to think of ***measures*** not merely as individual objects but instead as *elements of* $(\mathcal{C}_0(X))^*$, so that we can employ *geometric* ideas.

- **Proposition 3.14** ***(Criterion for Weak* (Vague) Convergence on** $\mathcal{M}(X)$) [Folland, 2013]*
Suppose $\mu_1, \mu_2, \ldots \in \mathcal{M}(\mathbb{R})$, and let $F_n(x) = \mu_n((-\infty, x])$ and $F(x) = \mu((-\infty, x])$.

1. *If* $\sup_n \|\mu_n\| < \infty$ *and* $F_n(x) \to F(x)$ *for* ***every x*** *at which* F *is* ***continuous****, then* $\mu_n \to \mu$ ***vaguely****.*

2. *If* $\mu_n \to \mu$ ***vaguely****, then* $\sup_n \|\mu_n\| < \infty$*. If, in addition, the* μ_n*s are* ***positive****, then* $F_n(x) \to F(x)$ *at* ***every*** x *at which* F *is* ***continuous****.*

- Finally, we tends to the geometrical properties of subspace of $\mathcal{M}(X)$

  Definition (***Convex Cone***)
  A set A in a vector space Y is called ***convex*** if x and $y \in A$ and $0 \le t \le 1$ implies $tx + (1-t)y \in A$. Thus A is *convex* if the ***line segment*** between x and y is in A whenever x and y are in A. A is called a ***cone*** if $x \in A$ implies $tx \in A$ for all $t > 0$. If A is *convex* and a *cone*, it is called a ***convex cone***.

- **Proposition 3.15** *(**Geometry of** $\mathcal{M}_+(X)$ **and** $\mathcal{M}_{+,1}(X)$) [Reed and Simon, 1980]*
  Let X *be a* ***compact Hausdorff*** *space. Then* $\mathcal{M}_{+,1}(X)$ *is* ***convex*** *and* $\mathcal{M}_+(X)$ *is a* ***convex cone****.*

3.4 Adjoints of Bounded Operator

- **Definition** (***Banach Space Adjoint***)
  Let X and Y be *Banach spaces*, T *a* ***bounded linear operator*** from X to Y. The ***Banach space adjoint of*** T, denoted by $T^{'}$, is *the* ***bounded linear operator*** *from* Y^* *to* X^* defined by

$$\left(T' f\right)(x) = f\,(Tx)$$

  for all $f \in Y^*$, $x \in X$.

- **Example** (***Adjoint of Right Shift Operator***)
  Let $X = \ell^1 = Y$ and let be ***the right shift operator***

$$T\,(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

  Then $T' : \ell^\infty \to \ell^\infty$ is ***the left shift operator***

$$T'\,(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)\,.$$

- **Proposition 3.16** *(**Isomorphism between Bounded Operator and its Adjoint**). [Reed and Simon, 1980]*
  Let X *and* Y *be* ***Banach spaces****. The map* $T \to T'$ *is an* ***isometric isomorphism*** *of* $\mathcal{L}(X, Y)$ *into* $\mathcal{L}(Y^*, X^*)$*.*

- **Remark** (***Hilbert Space Adjoint***)
  Let $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ be *the space of bounded linear operators* on $\mathcal{H}$. *The Banach space adjoint* of T^* then a mapping of $\mathcal{H}^*$ to $\mathcal{H}^*$. Let $C : \mathcal{H} \to \mathcal{H}^*$ be the map which assigns to each $y \in \mathcal{H}$, the bounded linear functional $\langle y\,, \cdot\rangle$ in $\mathcal{H}^*$. C is a ***conjugate linear isometry*** which is ***surjective*** by *the Riesz Representation theorem* (so it is ***unitary***). Now define a map $T^* : \mathcal{H} \to \mathcal{H}$ by

$$T^* = C^{-1} T' C$$

Then T^* satisfies

$$\langle x\,,\, Ty\rangle = (Cx)(Ty) = (T'Cx)(y) = \left\langle C^{-1}T'Cx\,,\, y\right\rangle = \langle T^*x\,,\, y\rangle\,,$$

T^* is called ***the Hilbert space adjoint of*** T, but usually we will just call it the adjoint and let the T^* distinguish it from T'. Notice that the map $T \to T^*$ is ***conjugate linear***, that is, $\alpha T \to \bar{\alpha}T^*$. This is because C is conjugate linear.

- **Proposition 3.17** *[Reed and Simon, 1980]*
  The map $T \to T^*$ is always ***continuous*** *in the* ***weak*** *and* ***uniform operator topologies*** *but is only continuous in the* ***strong operator topology*** *if $\mathcal{H}$ is* ***finite dimensional****.*

4 Compactness in Banach Space

Remark (***Compactness in Function Space***)
The importance of ***compactness*** in analysis is well-known, and the fact tha *closed bounded sets* are *compact* in *finite dimensional spaces* lies at the heart of much of the analysis on these spaces. ***Unfortunately***, as we have seen, this is *not true* in *infinite dimensional spaces.*

There are ***two main compactness results*** in *function space*:

1. The ***Ascoli's theorem***: Let X be a *compact Hausdorff space*; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding ***uniform topology***. *A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is* ***compact*** *if and only if it is* ***closed, bounded*** *under the* ***sup metric*** *ρ, and* ***equicontinuous*** *under d.*

2. The ***Banach-Alaoglu theorem***: Let X be a *Banach space. The* ***unit ball*** *in X^*, $\{f \in X^* : \|f\| \le 1\}$ is* ***compact*** *in the* ***weak* topology****.*

   In this section we will show that a *partial analogue* of this result can be obtained in ***infinite dimensions*** if we adopt a *weaker definition of the convergence* of a sequence than the usual definition.

4.1 Strong and Weak Convergence

- **Definition** (***Strong Convergence***). [Kreyszig, 1989]
  A sequence (x_n) in a normed space X is said to be ***strongly convergent*** (or ***convergent in the norm***) if there is an $x \in X$ such that

  $$\lim_{n\to\infty} \|x_n - x\| = 0.$$

  This is written $\lim\limits_{n\to\infty} x_n = x$ or simply $x_n \to x$ is called *the* ***strong limit*** *of* (x_n), and we say that (x_n) *converges* ***strongly*** *to x.*

- **Definition** (***Weak Convergence***). [Kreyszig, 1989]
  A sequence (x_n) in a normed space X is said to be ***weakly convergent*** if there is an $x \in X$ such that *for* ***every*** $f \in X^*$,

  $$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written $x_n \overset{w}{\to} x$ or $x_n \rightharpoonup x$. The element x is called ***the weak limit*** *of* (x_n), and we say that (x_n) ***converges weakly*** *to* x.

- **Remark** For weak convergence, we see it as convergence of *real numbers* $s_n = f(x_n)$ in $\mathbb{R}$.

- **Remark** (***Weak Convergence Analysis is Common***)
Weak convergence has various applications throughout analysis (for instance, in the *calculus of variations*, *the general theory of differential equations* and *probability theory*).

  The concept illustrates ***a basic principle of functional analysis***, namely, the fact that ***the investigation of spaces is often related to that of their dual spaces***, i.e. *probing a variable by using a test functional.*

- **Remark** In *Hilbert space* $\mathcal{H}$, we say $x_n \overset{w}{\to} x$ if there exists an $x \in \mathcal{H}$ such that for all $y \in \mathcal{H}$

$$\lim_{n\to\infty} \langle x_n\,,\, y\rangle = \langle x\,,\, y\rangle\,.$$

  Note that given a set of orthonormal basis (e_n), we have $f(e_n) := \langle e_n\,,\, y\rangle$ and from Bessel inequality

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle e_n\,,\, y\rangle|^2 \le \|y\|^2 < \infty \\ \Rightarrow \lim_{n\to\infty} |\langle e_n\,,\, y\rangle| \to 0 \\ \Rightarrow e_n \overset{w}{\to} 0.\end{aligned}$$

  But $\|e_n - e_m\| \not\to 0$, (e_n) does not converge in norm (strongly).

- **Lemma 4.1** *(**Weak Convergence**).*
*Let (x_n) be a **weakly convergent** sequence in a normed space X, say, $x_n \overset{w}{\to} x$. Then:*

  1. *The weak limit x of (x_n) is **unique**.*
  2. *Every **subsequence** of (x_n) converges weakly to x.*
  3. *The sequence $(\|x_n\|)$ is **bounded**.*

- **Proposition 4.2** *(**Strong and Weak Convergence**). [Kreyszig, 1989]*
Let (x_n) be a sequence in a normed space X. Then:

  1. ***Strong convergence** implies **weak convergence** with the same limit.*
  2. *The converse of (1) is **not** generally true.*
  3. *If dim $X < \infty$, then **weak convergence** implies **strong convergence**.*

- **Remark** From above, we see that in ***finite dimensional normed spaces*** the distinction between ***strong*** and ***weak convergence*** disappears completely.

4.2 Weak Topology

- **Remark** The weak convergence, $x_n \overset{w}{\to} x$, can be considered as *convergence of net* $\{x_n\}_{n=1}^{\infty}$ *in the **weak topology**.*

- **Definition** (***Weak Topology on a Set*** S) [Reed and Simon, 1980]
Let $\mathcal{F}$ be a family of functions from a set S to a topological vector space $(X, \mathscr{T})$. The ***$\mathcal{F}$-weak*** *(or simply* ***weak****)* ***topology*** on S is the weakest topology for which ***all the functions*** $f \in \mathcal{F}$ *are* ***continuous***.

- **Remark** (***Construction of Weak Topology***) [Reed and Simon, 1980]
To construct a ***$\mathcal{F}$-weak topology*** on S, we take the family of *all* ***finite intersections*** *of sets* of the form $f^{-1}(U)$ where $f \in \mathcal{F}$ and $U \in \mathscr{T}$. The collections of these finite intersections of sets *form* ***a basis*** *of* ***the $\mathcal{F}$-weak topology***.

  In other word, ***the subbasis*** for the *$\mathcal{F}$-weak topology* on S is of form

$$\mathscr{S} = \left\{ f^{-1}(U) : f \in \mathscr{F}, \text{ and } U \in \mathscr{T} \right\}$$

  And the basis of $\mathscr{T}$

$$\mathscr{B} = \left\{ f_1^{-1}(U_1) \cap \ldots \cap f_k^{-1}(U_k) : f_1, \ldots, f_k \in \mathscr{F}, \;\; U_1, \ldots, U_k \in \mathscr{T}, \; 1 \le k < \infty \right\}$$

$$\begin{aligned} B \in \mathscr{B} \Rightarrow B &= \{x : f_1(x) \in U_1, \ldots, f_k(x) \in U_k\}, \; 1 \le k < \infty \\ &= \{x : (f_1(x), \ldots, f_k(x)) \in U\}. \end{aligned}$$

  The basis element is called a ***k-dimensional cylinder set***.

- **Remark** Given a topology on Y and a family of functions in $Y^X = \{f : X \to Y\}$, $\mathscr{F}$-weak topolgy is ***a natural topology*** on X without additional information.

  A product topology on Y^ω can be seen as a $\mathscr{F}$-weak topology when $\mathscr{F} = \{\pi_\alpha : \prod_i Y_i \to Y_\alpha\}$.

- **Remark** A set S equipped with *$\mathcal{F}$-weak topology* ***has little knowledge on itself besides the output of functions*** $f \in \mathcal{F}$ from a family $\mathcal{F}$. The induced topology through a family of functions thus does not tell much besides the behavior of its output.

  For instance, S is the space of hidden states, $\mathcal{F} = \{f_1, \ldots, f_n\} \subset 2^S$ is a series of binary statistical tests, the weak topology on S *partition the domain according to the output of each test*.

- **Remark** By construction, the ***neighborhood base*** of each point $x \in S$ under the $\mathcal{F}$-weak topology is contained in the pre-images $\{f_n^{-1}(U_n)\}$ for ***finitely many*** of $(f_n) \in \mathcal{F}$.

- **Definition** (***Weak Topology on Banach Space***)
Let X be a ***Banach space*** with dual space X^*. The ***weak topology*** on X is *the weakest topology* on X so that $f(x)$ *is* ***continuous for all*** $f \in X^*$.

- **Remark** For infinite dimensional Banach spaces, ***the weak topology does not arise from a metric***. This is one of the main reasons we have introduced topological spaces.

- **Remark** Thus a ***neighborhood*** *base at zero* for ***the weak topology*** is given by the sets of the form

$$N(f_1, \ldots, f_n; \epsilon) = \{x : |f_j(x)| < \epsilon; \; j = 1, \ldots, n\}$$

  that is, neighborhoods of zero contain ***cylinders*** *with* ***finite-dimensional*** *open bases*. A net $\{x_\alpha\}$ converges *weakly* to x, written $x_\alpha \xrightarrow{w} x$, if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

- **Proposition 4.3** *[Reed and Simon, 1980]*

1. *The weak topology is **weaker** than **the norm topology**, that is, every weakly open set is norm open.*

2. *Every **weakly convergent** sequence is **norm bounded**.*

3. *The weak topology is a **Hausdorff** topology.*

- **Proposition 4.4** *(**Weak Topology on Hilbert Space**) [Reed and Simon, 1980]*
*Let $\mathcal{H}$ be a **Hilbert space**. Let $\{\varphi_\alpha\}_{\alpha\in I}$ be an **orthonormal basis** for $\mathcal{H}$. Given a sequence $\psi_n \in \mathcal{H}$, let*

$$\psi_n^{(\alpha)} = \langle \psi_n \, , \, \varphi_\alpha \rangle$$

*be the coordinates of ψ_n. Then $\psi_n \to \psi$ in the **weak topology** (or $\psi_n \stackrel{w}{\to} \psi$) **if and only if***

1. *$\psi_n^{(\alpha)} \to \psi^{(\alpha)}$ for each α; and*

2. *$\|\psi_n\|$ is **bounded**.*

Proof: Suppose $\psi_n \stackrel{w}{\to} \psi$; then (1) follows by definition and (2) comes from the fact that *every weakly convergent sequence is norm bounded.*

On the other hand, let (1) and (2) hold and let $\mathcal{F} \subset \mathcal{H}$ be the subspace of *finite linear combinations* of the φ_α. By (1), $\langle \psi_n \, , \, \varphi_\alpha \rangle \to \langle \psi \, , \, \varphi_\alpha \rangle$ if $\varphi \in \mathcal{F}$. Using the fact that $\mathcal{F}$ is dense, (2), and an $\epsilon/3$ argument, the weak convergence follows. ∎

- **Proposition 4.5** *(**Weak Topology of $\mathcal{C}(X)$ on Compact Hausdorff Space**) [Reed and Simon, 1980]*
*Let X be a **compact Hausdorff** space and consider the **weak topology on** $\mathcal{C}(X)$ (i.e. $\mathcal{C}(X, \mathbb{R})$). Let $\{f_n\}$ be a sequence in $\mathcal{C}(X)$. Then $f_n \to f$ in the **weak topology** (or $f_n \stackrel{w}{\to} f$) **if and only if***

1. *$f_n(x) \to f(x)$ for each $x \in X$; and*

2. *$\|f_n\|$ is **bounded**.*

Proof: For if $f_n \stackrel{w}{\to} f$, then (1) holds since $f \to f(x)$ is an element of $\mathcal{C}(X)^*$ and (2) comes from the fact that *every weakly convergent sequence is norm bounded.*

On the other hand, if (1) and (2) hold, then

$$|f_n(x)| \leq \sup_n \|f_n\|_\infty$$

which is L^1 with respect to any *Baire measure* μ. Thus, by the *dominated convergence theorem*, for any $\mu \in \mathcal{M}_+(X)$, $\int f_n d\mu \to \int f d\mu$. Since any $\lambda \in \mathcal{M}(X) = \mathcal{C}(X)^*$ is a *finite linear combination* of measures in $\mathcal{M}_+(X)$, we conclude that $f_n \to f$ weakly. ∎

- **Proposition 4.6** *(**Banach Space Weak Continuity = Norm Continuity**) [Reed and Simon, 1980]*
*A linear functional f on a **Banach space** is **weakly continuous if and only if** it is **norm continuous**.*

4.3 Weak* Topology

- **Definition** (***Weak* Topology on Banach Space***)
Let X be a *normed vector space* and X^* be its dual space. The ***weak* topology*** on X^* is *the weakest topology* on X^* so that $f(x)$ *is* ***continuous for all*** $x \in X$.

- **Remark** ***The weak* topology*** can be considered as a topology induced by $x \in X$ on dual space X^*, i.e. a topology on functional space on X induced by point in X.

  In fact, ***the weak* topology*** is *the* ***topology*** *of* ***pointwise convergence***:

$$f_\alpha \to f \quad \Leftrightarrow \quad f_\alpha(x) \to f(x) \text{ for all } x \in X.$$

  Moreover, ***the weak* topology*** is *the* **product topology** *on product space* $\mathbb{R}^X$.

- **Definition** (Y***-Weak Topology*** $\sigma(X, Y)$)
Let X be a *vector space* and let Y be a *family of* ***linear functionals*** on X which ***separates points*** of X. That is, for any $x_1 \neq x_2$ in X, there exists a $f \in Y$ so that $f(x_1) \neq f(x_2)$. Then ***the*** Y***-weak topology on*** X, written $\sigma(X, Y)$, is *the weakest topology* on X for which all the *functionals* in Y are *continuous*.

- **Remark** Y*-weak topology* $\sigma(X, Y)$ is *the* $\mathscr{F}$*-weak topology* when *domain* of $\mathscr{F}$ is a *vector space* and $\mathscr{F}$ is a family of *linear functionals*.

- **Remark** Because Y is assumed to *separate points*, $\sigma(X, Y)$ is *a* ***Hausdorff topology*** on X. Note that

  1. *the weak topology* on X is *the* $\sigma(X, X^*)$ *topology*
  2. *the weak* topology* on X^* is *the* $\sigma(X^*, X)$ *topology*

  The $\sigma(X, Y)$ *topology depends* only on ***the vector space generated by*** Y so we henceforth suppose that Y *is a vector space*.

- **Remark** Notice that ***the weak* topology*** *is even* ***weaker*** *than* ***the weak topology***.

$$\textbf{\textit{the norm topology}} \subset \textbf{\textit{the weak topology}} \subseteq \textbf{\textit{the weak* topology}}$$

- **Example** (***Weak* Topology on*** $\mathcal{M}(X)$)
The weak topology* on $\mathcal{M}(X)$, X a ***compact Hausdorff*** space, is often called ***the vague topology***. Note that $\mu_n \xrightarrow{w^*} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in \mathcal{C}_0(X)$.

  It can be shown that *the linear combinations of point masses* are ***weak* dense*** in $\mathcal{M}(X)$. That is, for given $\mu \in \mathcal{M}(X)$, $f_1, \ldots, f_n \in \mathcal{C}(X)$ and $\epsilon > 0$, that we can find $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and $x_1, \ldots, x_m \in X$ so that

$$\left|\mu(f_i) - \sum_{j=1}^{m} \alpha_j f_i(x_j)\right| < \epsilon, \quad \forall\, i = 1, \ldots, n,$$

  i.e. $\sum_{j=1}^{m} \alpha_j \delta_{x_j} \to \mu$ where $\delta_x(f) = f(x)$ is the ***evaluation map*** and $\delta_x(\cdot) \mapsto \delta_x$ is identified with the ***point mass***.

- **Remark** As one might expect, X is ***reflexive*** if and only if the ***weak*** and ***weak*** * ***topologies coincide***, and many *characterizations* of *reflexivity* depend on relations involving the *weak* and *weak** *topologies*.

- **Proposition 4.7** *($\sigma(X, Y)$ **Topology** = **Pointwise Convergence Topology on** X) [Reed and Simon, 1980]*
*The $\sigma(X, Y)$-**continuous** linear functionals on X are **precisely** Y, in particular the only **weak* continuous functionals** on X^* are the **elements** of X.*

- **Theorem 4.8** *(**The Banach-Alaoglu Theorem**) [Reed and Simon, 1980]*
*Let X^* be the dual of some Banach space, X. Then **the unit ball** in X^*, $\{f \in X^* : \|f\| \leq 1\}$ is **compact** in the **weak* topology**.*

- **Corollary 4.9** *(**The Banach-Alaoglu Theorem, Sequential Version**) [Rynne and Youngson, 2007]*
*If X is **separable** and $\{f_n\}$ is a **bounded** sequence in X^*, then $\{f_n\}$ has a **weak* convergent subsequence**.*

- **Theorem 4.10** *(**Kakutani's Theorem**) [Rynne and Youngson, 2007]*
*X is **reflexive** Banach space **if and only if the unit ball** in X, $\{x \in X : \|x\| \leq 1\}$ is **compact** in the **weak topology**.*

- **Corollary 4.11** *[Rynne and Youngson, 2007]*
*If X is **reflexive** Banach space and $\{x_n\}$ is a **bounded** sequence in X, then $\{x_n\}$ has a **weakly convergent subsequence**.*

- **Corollary 4.12** *[Rynne and Youngson, 2007]*
*If X is **reflexive** Banach space and $M \subseteq X$ is **bounded, closed** and **convex**, then any sequence in M has a **subsequence** which is **weakly convergent** to an element of M.*

- **Exercise 4.13** *[Rynne and Youngson, 2007]*
*Suppose that X is **reflexive** Banach space, M is a **closed, convex** subset of X, and $y \in X \setminus M$. Show that there is a point $y_M \in M$ such that*

$$y - y_M = \inf \{y - x : x \in M\}.$$

*Show that this result is **not true** if the assumption that M is **convex** is omitted.*

- **Example** (***Convergence in Distribution***)
Convergence in distribution is also called ***weak convergence*** in probability theory [Folland, 2013]. In functional analysis, however, ***weak convergence*** is actually reserved for a different mode of convergence, while ***the convergence in distribution*** is ***the weak*** * ***convergence*** on $\mathcal{M}(X)$.

In general, it is actually ***not*** *a mode of* ***convergence of functions*** f_n ***itself*** but instead is *the* ***convergence*** *of* ***bounded linear functionals*** $\int f d\mu_n$. Equivalently, it is *the* ***convergence of measures*** F_n *on* $\mathscr{B}(\mathbb{R})$.

$$\begin{aligned} &\text{weak convergence} && \int f_n d\mu \to \int f d\mu, \quad \forall \mu \in \mathcal{M}(X), \\ &\text{convergence in distribution} && \int f d\mu_n \to \int f d\mu, \quad \forall f \in \mathcal{C}_0(X) \end{aligned}$$

Definition (***Cumulative Distribution Function***) [Van der Vaart, 2000]
Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. Given any real-valued measurable function $\xi : \Omega \to \mathbb{R}$, we define the ***cumulative distribution function*** $F : \mathbb{R} \to [0, \infty]$ of ξ to be the function

$$F_{\xi}(\lambda) := \mu\left(\{x \in X : \xi(x) \leq \lambda\}\right) = \int_X \mathbb{1}\left\{\xi(x) \leq \lambda\right\} d\mu(x).$$

Definition (***Converge in Distribution***) [Van der Vaart, 2000]
Let $\xi_n : \Omega \to \mathbb{R}$ be a sequence of real-valued *measurable functions*, and $\xi : \Omega \to \mathbb{R}$ be another measurable function. We say that ξ_n ***converges in distribution*** to ξ if *the cumulative distribution function* $F_n(\lambda)$ of ξ_n ***converges pointwise*** to the cumulative distribution function $F(\lambda)$ of ξ at all $\lambda \in \mathbb{R}$ for which F is continuous. Denoted as $\xi_n \xrightarrow{F} \xi$ or $\xi_n \xrightarrow{d} \xi$ or $\xi_n \rightsquigarrow \xi$.

$$\xi_n \xrightarrow{d} \xi \;\Leftrightarrow\; F_n(\lambda) \to F(\lambda), \text{ for all } \lambda \in \mathbb{R}$$

Theorem 4.14 ***(The Portmanteau Theorem)****. [Van der Vaart, 2000]*
The following statements are equivalent.

1. $X_n \rightsquigarrow X$.
2. $\mathbb{E}\left[h(X_n)\right] \to \mathbb{E}\left[h(X)\right]$ *for all* ***continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *that are non-zero only on a* ***closed*** *and* ***bounded*** *set.*
3. $\mathbb{E}\left[h(X_n)\right] \to \mathbb{E}\left[h(X)\right]$ *for all* ***bounded continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$.
4. $\mathbb{E}\left[h(X_n)\right] \to \mathbb{E}\left[h(X)\right]$ *for all* ***bounded measurable functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *for which* $\mathbb{P}(X \in \{x : h \text{ is continuous at } x\}) = 1$.

We can reformulate the definition of *convergence in distribution* as below:

Definition [Wellner et al., 2013]
Let $(\mathcal{X}, d)$ be a *metric space*, and $(\mathcal{X}, \mathscr{B})$ be *a measurable space*, where $\mathscr{B}$ is ***the Borel*** σ***-field on*** $\mathcal{X}$, the smallest σ-field containing *all the open balls* (as the basis of *metric topology* on $\mathcal{X}$). Let $\{\mathcal{P}_n\}$ and $\mathcal{P}$ be ***Borel probability measures*** on $(\mathcal{X}, \mathscr{B})$.

Then the sequence $\mathcal{P}_n$ ***converges in distribution*** to $\mathcal{P}$, which we write as $\mathcal{P}_n \rightsquigarrow \mathcal{P}$, if and only if

$$\int_{\Omega} f d\mathcal{P}_n \to \int_{\Omega} f d\mathcal{P}, \qquad \text{for all } f \in \mathcal{C}_b(\mathcal{X}).$$

Here $\mathcal{C}_b(\mathcal{X})$ denotes the set of *all* ***bounded, continuous,*** *real functions on* $\mathcal{X}$.

We can see that ***the convergence in distribution*** is actually ***a weak**** ***convergence***. That is, it is ***the weak convergence*** of ***bounded linear functionals*** $I_{\mathcal{P}_n} \xrightarrow{w^*} I_{\mathcal{P}}$ on *the space of all probability measures* $\mathcal{P}(\mathcal{X}) \simeq (\mathcal{C}_b(\mathcal{X}))^*$ on $(\mathcal{X}, \mathscr{B})$ where

$$I_{\mathcal{P}} : f \mapsto \int_{\Omega} f d\mathcal{P}.$$

Note that the $I_{\mathcal{P}_n} \xrightarrow{w^*} I_{\mathcal{P}}$ is equivalent to $I_{\mathcal{P}_n}(f) \to I_{\mathcal{P}}(f)$ *for all* $f \in \mathcal{C}_b(\mathcal{X})$.

5 Fundamental Theorems

5.1 The Hahn-Banach Theorem

5.1.1 Extension Form of The Hahn-Banach Theorem

- **Remark** In dealing with Banach spaces, one often needs to *construct* ***linear functionals*** with ***certain properties***. This is usually done in two steps:

  1. one ***defines the linear functional*** on *a* ***subspace*** *of the Banach space* where it is easy to verify the desired properties;
  2. one appeals to (or proves) a general theorem which says that *any such functional* can be ***extended to the whole space*** while ***retaining the desired properties***.

  One of the basic tools of the second step is *the Hahn-Banach theorem*.

- **Definition** (***Sublinear Functional***)
  If X is a vector space, a ***sublinear functional*** on X is a map $p : X \to \mathbb{R}$ such that

  1. (***Homogeneity***): $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ and $x \in X$;
  2. (***Sublinearity***): $p(x + y) \leq p(x) + p(y)$,

- **Example** Every ***semi-norm*** is a *sublinear functional*. If p is a semi-norm, then the condition $f \leq p$ is equivalent to $|f| \leq p$.

- **Theorem 5.1** *(**The Hahn-Banach Theorem, Extension Form**) [Kreyszig, 1989, Reed and Simon, 1980, Luenberger, 1997, Folland, 2013]*
  *Let X be a real normed linear space and p a **sublinear functional** on X. Let f be a **linear functional** defined on a **subspace** M of X satisfying $f(x) \leq p(x)$ for all $x \in M$. Then there exists a **linear functional** F on X such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_M = f$. (F is called an **extension** of f.)*

- **Theorem 5.2** *(**The Complex Hahn-Banach Theorem, Extension Form**) [Kreyszig, 1989, Reed and Simon, 1980, Luenberger, 1997, Folland, 2013]*
  *Let X be a complex normed linear space and p a **semi-norm** on X. Let f be a **complex linear functional** defined on a **subspace** M of X satisfying $|f(x)| \leq |p(x)|$ for all $x \in M$. Then there exists a **complex linear functional** F on X such that $|F(x)| \leq |p(x)|$ for all $x \in X$ and $F|_M = f$. (F is called an **extension** of f.)*

- **Corollary 5.3** *(**The Existance of Minimum Norm Extension**)*
  *Let $f \in M^*$ be a bounded linear functional defined on a **subspace** M of a real normed vector space X. Then there is a bounded linear functional $F \in X^*$ defined on X which is an **extension** of f satisfying $\|F\|_{X^*} = \|f\|_{M^*}$.*

  Note let $p(x) = \|f\|_{M^*} \|x\|$.

- **Corollary 5.4** *Let y be an element of a normed linear space X. Then there is a nonzero $F \in X^*$ such that $F(y) = \|F\|_{X^*} \|y\|_X$.*

- **Corollary 5.5** *(**The Existance of Distance Functional**)*
  Let Z be a subspace of a normed linear space X and suppose that y is an element of X whose

distance *from Z is $d = \inf_{z \in Z} \|y - z\|$. Then there exists a $F \in X^*$ so that $\|F\| \leq 1$, $F(y) = d$, and $F(z) = 0$* ***for all*** *z* ***in*** *Z.*

- **Remark** *The Hahn-Banach theorem*, particularly *Corollary 3.3*, is perhaps most profitably viewed as ***an existence theorem for a minimization problem***. Given an f on a subspace M of a normed space, it is not difficult to extend f to the whole space. ***An arbitrary extension***, however, will in general be ***unbounded*** or *have norm greater than the norm of f on M*. We therefore pose the problem of *selecting the extension of minimum norm*. The Hahn-Banach theorem both guarantees ***the existence of a minimum norm extension*** and tells us ***the norm of the best extension***.

- **Proposition 5.6** *Let X be a Banach space. If X^* is* ***separable****, then X is* ***separable****.*

5.1.2 Geometric Form of The Hahn-Banach Theorem

- **Definition** The ***translation*** of a subspace is called *a* ***linear variety***. It is written as $x + M$ where $x \in X$ is a fixed point and $M \subseteq X$ is a subspace of X.

- **Remark** *A linear variety* is also called *an* ***affine subspace***.

- **Definition** A ***hyperplane*** H in a linear vector space X is a ***maximal*** *proper linear variety*, that is, a linear variety H such that $H \neq X$, and if V is any linear variety containing H, then either $V = X$ or $V = H$.

- **Remark** A *hyperplane* $H = x + M$ where M has ***codimension*** 1 in X, i.e.

$$X = \text{span}\{x, \text{basis of } M\}.$$

- **Proposition 5.7** *[Luenberger, 1997]*
Let H be a **hyperplane** *in a linear vector space X. Then there is a* **linear functional** *f on X and a constant c such that $H = \{x : f(x) = c\}$.* ***Conversely****, f is a nonzero linear functional on X, the set $\{x : f(x) = c\}$ is a hyperplane in X.*

- There exists an ***one-to-one correspondence*** between linear functional and hyperplane that does not passes the origin.

  Proposition 5.8 ***(Unique Linear Functional for Hyperplane)*** *[Luenberger, 1997]*
  Let H be a hyperplane in a linear vector space X. If H **does not contain the origin***, there is a* **unique** *linear functional f on X such that $H = \{x : f(x) = 1\}$.*

- **Proposition 5.9** *Let f be a nonzero linear functional on a normed space X. Then the hyperplane $H = \{x : f(x) = c\}$ is* **closed** *for every c if and only if f is* **continuous***.*

- **Remark** If f is a nonzero linear functional on a linear vector space X, we associate with the hyperplane $H = \{x : f(x) = c\}$ *the four sets*

$$\{x : f(x) \leq c\}, \quad \{x : f(x) < c\}, \quad \{x : f(x) \geq c\}, \quad \{x : f(x) > c\}$$

  called ***half-spaces determined by H***. The first two of these are referred to as ***negative half-spaces*** *determined by f* and the second two as ***positive half-spaces***.

  If f is *continuous*, *the first and the third* half-spaces are ***closed*** and *the second and fourth* are ***open***.

- **Definition** (***The Minkowski Functional***) [Luenberger, 1997]
  Let K be ***a convex set*** in a *normed linear vector space* X and suppose 0 is *an* ***interior point*** *of* K. Then ***the Minkowski functional (or gauge)*** p of K is defined on X by

$$p(x) := \inf\left\{r : \frac{x}{r} \in K, r > 0\right\} = [\sup\{t : t\,x \in X, t > 0\}]^{-1}.$$

  We note that for K *equal to the unit sphere in* X, *the Minkowski functional is* $\|x\|$. In the general case, $p(x)$ defines a kind of ***distance*** from *the origin to* x measured *with respect to* K; it is the *factor* by which K must be expanded so as to *include* x.

- **Lemma 5.10** *Let K be a convex set containing* 0 *as an interior point. Then* ***the Minkowski functional*** *p of K satisfies:*
  1. $0 \le p(x) < \infty$ *for all* $x \in X$;
  2. *(**Homogeneity**):* $p(\lambda x) = \lambda p(x)$ *for all* $\lambda \ge 0$ *and* $x \in X$;
  3. *(**Sublinearity**):* $p(x + y) \le p(x) + p(y)$,
  4. *p is **continuous**;*
  5. $\overline{K} = \{x : p(x) \le 1\}$ *and* $\mathring{K} = \{x : p(x) < 1\}$.

  That is, *the Minkowski functional* is *a sublinear functional.*

- **Theorem 5.11** *(**Mazur's Theorem, Geometric Hahn-Banach Theorem**) [Luenberger, 1997]*
  *Let K be a **convex set** having a **nonempty interior** in a real normed linear vector space X. Suppose V is a **linear variety** in X containing no interior points of K. Then there is a **closed hyperplane** in X containing V but **containing no interior points** of K; i.e., there is an element $f \in X^*$ and a constant c such that $f(v) = c$ for all $v \in V$ and $f(k) < c$ for all $k \in K$.*

- **Remark** (***Geometric Interpretation of the Hahn-Banach theorem***)
  *The **geometric form** of the **Hahn-Banach theorem***, in simplest form, says that given *a* ***convex set*** K containing ***an interior point***, and given a point x_0 *not in* $\mathring{K}$, there is a ***closed hyperplane containing*** x_0 but ***disjoint*** from $\mathring{K}$.

- **Definition** (***Supporting Hyperplane***)
  A ***closed*** *hyperplane* H in a normed space X is said to be ***a supporting hyperplane*** (or a ***support***) for the ***convex set*** K if K is *contained* in *one of the* ***closed half-spaces*** determined by H and H contains a point of $\overline{K}$.

- **Remark** Suppose $K \subseteq \mathbb{R}^n$, and x_0 is a point in its boundary ∂K, i.e.,

$$x_0 \in \partial K = \overline{K} \setminus \mathring{K}.$$

  If $a \neq 0$ satisfies $\langle a\,,\,x\rangle \le \langle a\,,\,x_0\rangle$ for all $x \in K$, then the hyperplane $\{x : \langle a\,,\,x\rangle = \langle a\,,\,x_0\rangle\}$ is called ***a supporting hyperplane*** to K at the point x_0.

- **Theorem 5.12** *(**Supporting Hyperplane Theorem**) [Luenberger, 1997, Rockafellar, 1970]*
  *If x is **not an interior point** of a convex set K which contains interior points, there is a **closed hyperplane** H containing x such that K lies on one side of H.*

- As a consequence of the above theorem, it follows that, for a convex set K with interior points, ***a supporting hyperplane*** *can be constructed containing* ***any boundary point of*** $\overline{K}$.

  Theorem 5.13 *(**Eidelheit's Separation Theorem**) [Luenberger, 1997, Rockafellar, 1970]* *Let K_1 and K_2 be* ***convex sets*** *in X such that K_1 has interior points and K_2* ***contains no interior point of*** *K_1. Then there is a* ***closed hyperplane*** *H* ***separating*** *K_1 and K_2; i.e., there exists $f \in X^*$ such that*

  $$\sup_{x \in K_1} f(x) \le \inf_{x \in K_2} f(x) \tag{4}$$

  In other words, K_1 and K_2 lie in ***opposite half-spaces*** *determined by H.*

  Proof: Let $K = K_1 - K_2 = \{x_1 - x_2 : x_1 \in K_1, x_2 \in K_2\}$; then K contains an interior point and 0 not one of them. Also K is a convex set. By *The Supporting Hyperplane Theorem*, there is an $f \in X^*$, $f \neq 0$, such that $f(x) \le 0$ for $x \in K$. Thus for $x_1 \in K_l$, $x_2 \in K_2$, $f(x_1) \le f(x_2)$. Consequently, there is a real number c such that $\sup_{x \in K_1} f(x) \le c \le \inf_{x \in K_2} f(x)$. The desired hyperplane is $H = \{x : f(x) = c\}$. ∎

- **Corollary 5.14** *If K is a* ***closed convex*** *set and $x \notin K$, there is a* ***closed halfspace*** *that contains K but does not contain x.*

- **Theorem 5.15** *(**Dual Representation of Convex Set**)[Luenberger, 1997, Rockafellar, 1970]*
  If K is a ***closed convex*** *set in a normed space, then K is equal to the* ***intersection*** *of all the* ***closed half-spaces*** *that contain it.*

- **Remark** (***Duality for Convex Set***)
  Theorem above is often regarded as ***the geometric foundation*** *of* ***duality theory*** *for* ***convex sets***. By associating *closed hyperplanes* (or *half-spaces*) with elements of X^*, the theorem expresses ***a convex set in X as a collection of elements in X^****. See more in [Rockafellar, 1970].

- **Definition** Let K be a convex set in a real normed vector space X. The functional

  $$h(f) := \sup_{x \in K} f(x)$$

  defined on X^* is called ***the support functional*** of K. $h \in X^{**}$.

- **Remark** The ***support functional*** of a *convex set K completely specifies the set (to within* ***closure****)*

  $$\overline{K} = \bigcap_{f \in X^*} \{x : f(x) \le h(f)\}.$$

5.2 Baire Category Theorem

- **Remark** (***Empty Interior = Complement is Dense***)
  Recall that if A is a subset of a space X, the ***interior*** of A is defined as *the union of all open sets of X that are contained in A.*

To say that A has ***empty interior*** is to say then that A ***contains no open set of*** X *other than the empty set.* ***Equivalently***, A has ***empty interior*** if every point of A is *a* ***limit point*** *of the* ***complement*** *of* A, that is, if *the* ***complement*** *of* A *is* ***dense*** *in* X.

$$\mathring{A} = \emptyset \quad \Leftrightarrow \quad A^c \text{ is dense in } X$$

In [Reed and Simon, 1980], if a subset $\overline{A}$ of X has *empty interior*, A is said to be ***nowhere dense*** in X.

- **Example** Some examples:
  1. The set $\mathbb{Q}$ of *rationals* has ***empty interior*** as a subset of $\mathbb{R}$
  2. The *interval* $[0, 1]$ has ***nonempty interior***.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Space***)
  A space X is said to be a ***Baire space*** if the following condition holds: Given ***any countable*** collection $\{A_n\}$ of ***closed*** sets of X each of which has ***empty interior*** in X, their ***union*** $\bigcup_{n=1}^{\infty} A_n$ also has ***empty interior*** in X.

- **Example** Some examples:
  1. The space $\mathbb{Q}$ of *rationals* is ***not a Baire space***. For each one-point set in $\mathbb{Q}$ is *closed and has empty interior* ***in*** $\mathbb{Q}$; and $\mathbb{Q}$ is *the countable union of its one-point subsets.*
  2. The space $\mathbb{Z}_+$, on the other hand, does form *a* ***Baire space***. Every subset of $\mathbb{Z}_+$ is *open*, so that there exist *no subsets* of $\mathbb{Z}_+$ having *empty interior*, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Category***)
  A subset A of a space X was said to be of ***the first category in*** X if it ***was contained*** *in the* ***union*** *of a* ***countable*** *collection of* ***closed*** *sets of* X *having* ***empty interiors*** *in* X; ***otherwise***, it was said to be of ***the second category in*** X.

- **Remark** *A space* X *is a* ***Baire space*** *if and only if every* ***nonempty open*** *set in* X *is of* ***the second category***.

- **Lemma 5.16** ***(Open Set Definition of Baire Space)*** *[Munkres, 2000]*
  X is a ***Baire space if and only if*** *given any* ***countable*** *collection* $\{U_n\}$ *of* ***open*** *sets in* X*, each of which is* ***dense*** *in* X*, their* ***intersection*** $\bigcap_{n=1}^{\infty} U_n$ *is also* ***dense*** *in* X.

- **Theorem 5.17** ***(Baire Category Theorem)****. [Munkres, 2000]*
  If X *is a* ***compact Hausdorff*** *space or a* ***complete metric space****, then* X *is a* ***Baire space***.

- **Remark** In other word, neither ***compact Hausdorff*** space or a ***complete metric space*** is a *countable union of closed subsets with empty interior (that are nowhere dense).*

- **Lemma 5.18** *[Munkres, 2000]*

Let $C_1 \supset C_2 \supset \ldots$ be a ***nested*** *sequence of* ***nonempty closed sets*** *in the* ***complete metric space*** *X. If diam $C_n \to 0$, then $\bigcap_n C_n = \emptyset$.*

- **Lemma 5.19** *[Munkres, 2000]*
  Any ***open*** *subspace Y of a Baire space X is itself a Baire space.*

- **Theorem 5.20** ***(Discontinuity Point of Pointwise Convergence Function)*** *[Munkres, 2000]*
  Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a ***Baire space****, the set of points at which f is* ***continuous*** *is* ***dense*** *in X.*

- **Remark** (***Use Baire Category Theorem as Proof by Contradition***)
  The Baire category theorem is used to prove a certain subset C is ***dense*** in X by stating that X is a Baire space and C is countable intersection of dense open subsets in X (*C is a G_δ sets*).

  On the other hand, if $M = \bigcup_{n=1}^{\infty} A_n$ has ***nonempty interior***, then ***some*** of the sets $\bar{A}_n$ ***must have nonempty interior***. Otherwise, it contradicts with the Baire space definition.

5.3 Uniform Boundedness Theorem

- **Proposition 5.21** *[Reed and Simon, 1980]*
  Let X and Y be normed linear spaces. Then a linear map $: X \to Y$ is ***bounded*** *if and only if*

$$T^{-1}\left\{y : \|y\|_Y \le 1\right\}$$

  has a ***nonempty interior****.*

- **Theorem 5.22** ***(The Uniform Boundedness Theorem)****. [Reed and Simon, 1980]*
  Let X be a ***Banach space****. Let $\mathscr{F}$ be a family of* ***bounded*** *linear transformations from X to some* ***normed linear space*** *Y. Suppose that for each $x \in X$, $\{\|Tx\|_Y : T \in \mathscr{F}\}$ is* ***bounded****, i.e.*

$$\sup_{T \in \mathscr{F}} \|Tx\|_Y < \infty.$$

  Then $\{\|T\| : T \in \mathscr{F}\}$ is ***bounded****, i.e.*

$$\sup_{T \in \mathscr{F}} \|T\| < \infty.$$

- **Corollary 5.23** ***(Separately Continuity of Bilinear Form on Banach Space = Joint Continuity)*** *[Reed and Simon, 1980]*
  Let X and Y be Banach spaces and let $B(\cdot,\cdot)$ be a ***separately continuous bilinear mapping*** *from $X \times Y$ to $\mathbb{C}$, that is, it is a* ***bounded*** *linear transformation if one of the two arguments is fixed. Then $B(\cdot,\cdot)$ is* ***jointly continuous****, that is, if $x_n \to 0$ and $y_n \to 0$ then $B(x_n, y_n) \to 0$.*

5.4 Open Mapping Theorem

- **Theorem 5.24** ***(Open Mapping Theorem)*** *[Reed and Simon, 1980]*
  Let $T : X \to Y$ be a ***surjective bounded linear transformation*** *of one* ***Banach*** *space*

onto *another* ***Banach*** *space* Y. *Then if* M *is an* ***open*** *set in* X, $T(M)$ *is* ***open*** *in* Y.

- **Corollary 5.25** ***(Inverse Mapping Theorem)*** *[Reed and Simon, 1980]*
  A ***continuous bijection*** *of one Banach space onto another has a* ***continuous inverse***.

- **Remark** Note T is an open map and $A = T^{-1}(T(A))$ for surjective map, then T^{-1} is *continuous.*

- **Theorem 5.26** ***(Banach-Schauder Theorem)*** *[Reed and Simon, 1980]*
  Let T *be a* ***continuous*** *linear map,* $T : E \to F$, *where* E *and* F *are Banach spaces. Then either* $T(A)$ *is* ***open*** *in* $T(E)$ *for* ***each open*** $A \subseteq E$, *or* $T(E)$ *is of* ***first category*** *in* $\overline{T(E)}$.

5.5 Closed Graph Theorem

- **Definition** (***Graph of Function***)
  Let be a mapping of a normed linear space X into a normed linear space Y. The ***graph of*** T, denoted by $\Gamma(T)$, is defined as

$$\Gamma(T) := \{(x, y) \in X \times Y : y = Tx\}\,.$$

- **Theorem 5.27** ***(Closed Graph Theorem)****[Reed and Simon, 1980]*
  Let X *and* Y *be Banach spaces and* T *a linear map of* X *into* Y. *Then* T *is* ***bounded*** *if and only if the* ***graph*** *of is* ***closed***.

- **Remark** To avoid future confusion, we emphasize that the T in this theorem is implicitly assumed to be ***defined on all of*** X.

- **Remark** Consider the following statements:

  1. x_n converges to some element x;
  2. Tx_n converges to some element y;
  3. $Tx_n = y$.

  Usually to prove T is continuous, one need to show that given statement 1, the statement 2 and 3 are true. That is, we need to ***prove convergence*** of Tx_n and need to show ***identification*** of Tx and the limit of Tx_n.

  With ***close graph theorem***, we just need to show that given statement 1 ***and*** 2, statement 3 is true; that is, we just need to prove the identification part.

- **Corollary 5.28** ***(The Hellinger-Toeplitz Theorem)*** *[Reed and Simon, 1980]*
  Let A *be an* ***everywhere defined*** *linear operator on a* ***Hilbert space*** $\mathcal{H}$ *with*

$$\langle x\,,\, Ay\rangle = \langle Ax\,,\, y\rangle$$

  for all $x, y \in \mathcal{H}$; *that is* A *is* ***self-adjoint***. *Then* A *is* ***bounded***.

6 Spectrum of Bounded Linear Operator in Banach Space

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- **Definition** (***Spectral Radius of Linear Operator***)
  Let

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

  $r(T)$ is called ***the spectral radius of*** T.

- **Proposition 6.1** *(**Spectral Radius Calculation**) [Reed and Simon, 1980]*
  *Let X be a **Banach space**, $T \in \mathcal{L}(X)$. Then*

$$\lim_{n\to\infty} \|T^n\|^{1/n}$$

  exists and is equal to $r(T)$.

- **Theorem 6.2** *(**Spectrum and Resolvent of Adjoint**) (**Phillips**) [Reed and Simon, 1980]*
  *Let X be a **Banach space**, $T \in \mathcal{L}(X)$. Then*

$$\sigma(T) = \sigma(T') \;\; \textit{and} \;\; R_\lambda(T') = (R_\lambda(T))'.$$

- **Proposition 6.3** *(**Spectrum of Adjoint**) [Reed and Simon, 1980]*
  Let X be a Banach space and $T \in \mathcal{L}(X)$. Then,

  1. *If λ is in the **residual spectrum** of T, then λ is in the **point spectrum** of T'.*
  2. *If λ is in the **point spectrum** of T, then λ is in **either** the **point** or the **residual spectrum** of T'.*

References

Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

Erwin Kreyszig. *Introductory functional analysis with applications*, volume 81. wiley New York, 1989.

David G Luenberger. *Optimization by vector space methods.* John Wiley & Sons, 1997.

James R Munkres. *Topology, 2nd.* Prentice Hall, 2000.

Michael Reed and Barry Simon. *Methods of modern mathematical physics: Functional analysis*, volume 1. Gulf Professional Publishing, 1980.

R Tyrrell Rockafellar. *Convex analysis*, volume 18. Princeton university press, 1970.

Bryan Rynne and Martin A Youngson. *Linear functional analysis.* Springer Science & Business Media, 2007.

Aad W Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 2000.

Jon Wellner et al. *Weak convergence and empirical processes: with applications to statistics.* Springer Science & Business Media, 2013.