# Lecture 8: Spectral Theorem

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## Contents

## 1 Spectral Theorem in Finite Dimensional Space

- **Definition** (***Similarity***) [Horn and Johnson, 2012]
  Let $A, B \in M_n$ be given $n \times n$ matrices. We say that $B$ ***is similar to*** $A$ if there exists a ***nonsingular*** $S \in M_n$ such that
  $$B = S^{-1}AS$$
  The transformation $A \to S^{-1}AS$ is called a ***similarity transformation*** by the *similarity matrix* $S$.

- **Definition** (***Normal Matrix***) [Horn and Johnson, 2012]
  A matrix $A \in M_n$ is ***normal*** if
  $$AA^* = A^*A,$$
  that is, if $A$ ***commutes*** with its ***conjugate transpose (adjoint)***.

- **Definition** (***Diagonalizable***) [Horn and Johnson, 2012]
  If $A \in M_n$ is *similar* to a *diagonal matrix*, then $A$ is said to be ***diagonalizable***.

- **Definition** (***Unitary Similarity***) [Horn and Johnson, 2012]
  Let $A, B \in M_n$ be given. We say that $A$ is ***unitarily similar*** to $B$ if there is a ***unitary*** $U \in M_n$ such that
  $$A = UBU^*$$
  We say that $A$ is ***unitarily diagonalizable*** if it is ***unitarily similar*** to a diagonal matrix.

  We say that $A$ is ***orthogonally similar*** to $B$ if there is a ***unitary (real orthorgonal)*** $U \in M_n(\mathbb{R})$ such that
  $$A = UBU^T$$
  We say that $A$ is ***orthogonally diagonalizable*** if it is ***orthogonally similar*** to a diagonal matrix.

- **Theorem 1.1** *(**Spectral Theorem of Normal Matrix**) [Horn and Johnson, 2012]*
  *Let $A = [a_{i,j}] \in M_n$ have **eigenvalues** $\lambda_1, \ldots, \lambda_n$. The following statements are **equivalent**:*

  1. *$A$ is **normal**.*
  2. *$A$ is **unitarily diagonalizable**, i.e. there exists unitary matrix $U \in M_n$ such that*
     $$A = U\Lambda U^*$$
     *where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$.*
  3. $\sum_{i,j=1}^{n} |a_{i,j}|^2 = \sum_{i=1}^{n} \lambda_i^2$
  4. *$A$ has $n$ **orthonormal eigenvectors***

- **Definition** (***Spectral Decomposition***)
  A *representation* of a ***normal matrix*** $A \in M_n$ as $A = U\Lambda U^*$, in which $U$ is ***unitary*** and $\Lambda$ is ***diagonal***, is called a ***spectral decomposition of*** $A$.

- The Hermitian matrix is normal matrix, so the following theorem is a special case of the spectral theorem for normal matrix.

  **Theorem 1.2** *(**Spectral Theorem for Hermitian Matrices**) [Horn and Johnson, 2012] Let $A \in M_n$ be **Hermitian** and have eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. Then*

  1. *$\lambda_1, \ldots, \lambda_n$ are **real** numbers.*
  2. *$A$ is **unitarily diagonalizable***
  3. *There is a **unitary** $U \in M_n$ such that*

  $$A = U\Lambda U^*$$

- **Remark** This is equivalent to say that for ***self-adjoint bounded linear operator*** $A$ on finite dimensional space $V$, there exists ***unitary operator*** $U : \mathbb{C}^n \to V$ such that

  $$[U^{-1}AUf]_k = \lambda_k f_k$$

  for any $f = (f_k)_{k=1}^n \in \mathbb{C}^n$.

## 2 The Continuous Functional Calculus

- **Remark** (***Spectral Theorem for Self-Adjoint Bounde Linear Operator in Hilbert Space***)
  Given a *bounded self-adjoint operator* $A \in \mathcal{L}(\mathcal{H})$ on *Hilbert space* $\mathcal{H}$, we can find a ***measure*** $\mu$ on a *measure space* $\mathcal{M}$ and a ***unitary operator*** $U : L^2(\mathcal{M}, \mu) \to \mathcal{H}$ so that

  $$[U^{-1}AUf](x) = F(x)f(x)$$

  for some ***bounded real-valued measurable function*** $F$ on $\mathcal{M}$.

  In practice, $\mathcal{M}$ will be a *union of copies of* $\mathbb{R}$ and $F$ will be $x$, so the ***core*** of the proof of the theorem will be ***the construction of certain measures*** $\mu$.

- **Remark** (***Functional Calculus***) [Borthwick, 2020]
  In operator theory, the term "***functional calculus***" refers to the ability to *apply a function to an operator*.

  For $A \in \mathcal{L}(\mathcal{H})$, one need to make sense of $f(A)$ for some continuous function $f$. For instance, If $f(x) = \sum_{j=0}^{n} a_j x^j$ is a *polynomial*, we want

  $$f(A) = \sum_{j=0}^{n} a_j A^j.$$

  Similarly, suppose that $f(x) = \sum_{j=0}^{\infty} c_j x^j$ is a *power series* with *radius of convergence* $R$. If $\|A\| < R$, then $\sum_{j=0}^{\infty} c_j A^j$ *converges* in $\mathcal{H}$ so it is natural to set

  $$f(A) = \sum_{j=0}^{\infty} a_j A^j.$$

- In particular, we have

  **Lemma 2.1** ***(Spectrum of Polynomial of Operators)*** *[Reed and Simon, 1980]*
  *Let* $P(x) = \sum_{n=0}^{N} a_n x^n$ *and* $P(A) = \sum_{n=0}^{N} a_n A^n$*. Then*

$$\sigma\left(P(A)\right) = \{P(\lambda) : \lambda \in \sigma(A)\}$$

- **Lemma 2.2** ***(Norm of Polynomial of Bounded Self-Adjoint Operators)*** *[Reed and Simon, 1980]*
  *Let $A$ be a* ***bounded self-adjoint*** *operator. Then*

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$$

- **Theorem 2.3** ***(Continuous Functional Calculus)*** *[Reed and Simon, 1980]*
  *Let $A$ be a* ***self-adjoint*** *operator on a* ***Hilbert space*** *$\mathcal{H}$. Then there is a* ***unique*** *map $\phi : \mathcal{C}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:*

  1. *$\phi$ is an* ***algebraic $*$-homomorphism****, that is,*
     - ***(Preserve Operator Product)*** $\phi(fg) = \phi(f)\phi(g)$
     - ***(Preserve Scalar Product)*** $\phi(\lambda f) = \lambda\phi(f)$
     - ***(Preserve Identity)*** $\phi(1) = I$
     - ***(Preserve Adjoint/Conjugacy)*** $\phi(\bar{f}) = \phi(f)^*$
  2. *$\phi$ is* ***continuous****, that is,*

$$\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq C\,\|f\|_\infty\,.$$

  3. *Let $f$ be the function $f(x) = x$; then $\phi(f) = A$. Moreover, $\phi$ has the* ***additional*** *properties:*
  4. *If $A\psi = \lambda\psi$, then*

$$\phi(f)\psi = f(\lambda)\psi \tag{1}$$

  5. ***(Spectral Mapping Theorem)***

$$\sigma(\phi(f)) = \{f(\lambda) : \lambda \in \sigma(A)\} \tag{2}$$

  6. ***(Preserve Positivity)*** *If $f \geq 0$, then $\phi(f) \succeq 0$.*
  7. ***(Preserve Norm)*** *(This strengthens the (2)).*

$$\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_\infty \tag{3}$$

  *We sometimes write $f(A)$ or $\phi_A(f)$ for $\phi(f)$ to emphasize the dependency on $A$.*

**Proof:** Sketch of the proof. Let $\phi(P) = P(A)$ for polynomial $P$. Then, by previous Lemma, we have

$$\|\phi(P)\|_{\mathcal{L}(\mathcal{H})} = \|P\|_{\mathcal{C}(\sigma(A))}$$

so $\phi$ has a *unique linear extension* to the closure of polynomials $\overline{P(\sigma(A))} \subset \mathcal{C}(\sigma(A))$. Note that $\overline{P(\sigma(A))} = \{P(x) : x \in \sigma(A),$ *all polynomials* $P(x)\}$ forms an *algebra* (with respect to *function multiplication*) that contains 1, and complex conjugates. Moreover, $\overline{P(\sigma(A))}$ *separate points*, i.e. for any $x, y \in \sigma(A)$, we can find $P \in \overline{P(\sigma(A))}$ so that $P(x) \neq P(y)$. By, *Stone-Weierstrass theorem*, $\overline{P(\sigma(A))} = \mathcal{C}(\sigma(A))$. In other word, the domain of $\phi$ can be extended to $\mathcal{C}(\sigma(A))$.

Property (1), (2), (3), (7) is directly from extension of $\phi$ in polynomial function space. If a map $\widetilde{\phi}$ obeys (1), (2), (3) it agrees on $\phi$ on polynomials and thus by *continuity* (since $\overline{P(\sigma(A))} = \mathcal{C}(\sigma(A))$), $\widetilde{\phi} = \phi$ on $\mathcal{C}(\sigma(A))$.

To show (4), note that $\phi(f)\psi = f(A)\psi = f(\lambda)\psi$ for any $f \in \overline{P(\sigma(A))}$, then (4) is proved by continuity. To prove (6), note that if $f \geq 0$, $f = g^2$ with $g$ real and $g \in \mathcal{C}(\sigma(A))$. Thus $\phi(f) = \phi(g)^2$ with $\phi(g)$ being self-adjoint, so $\phi(f) \geq 0$. (5) comes from extension of results in Lemma 2.1. ∎

- **Remark** Note that the continuous function $f$ in defining $f(A)$ is defined on $\sigma(A)$, i.e. ***the spectrum of operator*** $A$, so $f$ ***is a spectral domain transformation function***. In the map,

$$\phi : f \mapsto \phi(f) := f(A) : \mathcal{H} \to \mathcal{H}.$$

  1. So in equation

$$\begin{aligned}
\phi(fg) = \phi(f)\phi(g) &\Leftrightarrow (fg)(A) = f(A)g(A)\\
\phi(\lambda f) = \lambda\phi(f) &\Leftrightarrow (\lambda f)(A) = \lambda\, f(A)\\
\phi(1) = I &\Leftrightarrow 1(A) = I\\
\phi(\bar{f}) = \phi(f)^* &\Leftrightarrow (\bar{f})(A) = (f(A))^*\\
\phi(\mathrm{Id}) = \mathrm{Id} &\Leftrightarrow (\mathrm{id})(A) = A
\end{aligned}$$

     the LHS of first equation is an operator corresponding to the ***product of two functions***, while the RHS of first equation is ***the product of two operators***, each corresponding to one function.

  2. The equation (1) makes sure that *the spectral decomposition* of $f(A)$ and that of $A$ ***shares the same set of eigenfunctions***.

  3. The spectral mapping theorem in (2) actually defines $f(A)$ as the operator whose spectrum is transformed by $f$. In other words, $f(A)$ ***is the operator obtained by spectral domain transformation via*** $f$.

     In signal processing, $f(A)$ corresponds to ***the spectral filtering*** of $A$.

- **Remark** There are some more remarks:

  1. $\phi(f) \succeq 0$ ***if and only if*** $f \geq 0$.

2. (***Abelian*** $C^*$***-Algebra***)
   Since $fg = gf$ for all $f, g$,

   $$\{f(A) : f \in \mathcal{C}(\sigma(A))\}$$

   forms an ***abelian algebra*** closed under ***adjoints***. Since $\|\phi(f)\| = \|f\|_\infty$ and $\mathcal{C}(\sigma(A))$ is ***complete***, $\{f(A) : f \in \mathcal{C}(\sigma(A))\}$ is ***norm-closed***. It is thus an ***abelian*** $C^*$***-algebra*** of *operators*.

3. ($C^*$***-Algebra Generated by*** $A$)
   The image of $\phi$, i.e. $\{f(A) : f \in \mathcal{C}(\sigma(A))\}$ is actually the $C^*$***-algebra generated by*** $A$, that is, the ***smallest*** $C^*$*-algebra containing* $A$.

4. This result shows that *the space of* ***continuous function on spectrum of*** $A$, $\mathcal{C}(\sigma(A))$ and *the* $C^*$*-algebra generated by* $A$ are ***isometrically isomorphic***.

   $$\mathcal{C}(\sigma(A)) \simeq \text{Ran } \phi = \{f(A) : f \in \mathcal{C}(\sigma(A))\}.$$

5. The property (1) and (3) ***uniquely*** *determines* the mapping $\phi$.

- **Example** (***Existence of Square Root for Positive Operator***)
  For $A \succeq 0$, $\sigma(A) \geq 0$ and $\sigma(A) \subset \mathbb{R}$, so let $f(x) = \sqrt{x}$, then

  $$A = (f(A))^2.$$

- **Example** For $f(x) = (\lambda - x)^{-1}$,

  $$\left\|(A - \lambda I)^{-1}\right\| = \sup_{x \in \sigma(A)} |x - \lambda|^{-1} = \frac{1}{\text{dist }(\lambda, \sigma(A))}$$

  for $A$ bounded and $\lambda \notin \sigma(A)$.

# 3 Spectral Theorem for Bounded Self-Adjoint Operator

## 3.1 Spectral Measure

- **Remark** (***Positive Linear Functional on*** $\mathcal{C}(\sigma(A))$)
  For each $\psi \in \mathcal{H}$, the following *quadratic form* defines a *bounded linear functional* on $\mathcal{L}(\mathcal{H})$

  $$\widetilde{I}_\psi : A \mapsto \langle \psi\,,\, A\psi \rangle_{\mathcal{H}}\,.$$

  Then by continuous functional calculus, we can define a map $I_\psi = \widetilde{I}_\psi \circ \phi : \mathcal{C}(\sigma(A)) \to \mathbb{R}$, which is seen as a ***positive linear functional*** (*not positive operator*) on $\mathcal{C}(\sigma(A))$, i.e. $\forall \psi \in \mathcal{H}$,

  $$I_\psi(f) := \langle \psi\,,\, f(A)\psi \rangle \geq 0 \text{ whenever } f \geq 0.$$

  For *a* ***bounded self-adjoint operator*** $A$, the *spectrum* $\sigma(A) \subset \mathbb{R}$ is a ***closed bounded subset*** of $\mathbb{R}$ so it is ***compact***. Thus $\mathcal{C}(\sigma(A))$ is a space of continuous functions on compact domain, which, by Riesz-Markov theorem, has ***dual space*** that is isomorphic to *the space of*

***complex signed Radon measures*** *on* $\sigma(A)$. In other word, for each $\psi \in \mathcal{H}$, there ***exists a positive <u>Radon measure on spectral domain</u>*** $\mu_\psi \in \mathcal{M}(\sigma(A)) \simeq (\mathcal{C}(\sigma(A)))^*$ so that

$$I_\psi(f) := \langle \psi \,,\, f(A)\psi \rangle = \int_{\sigma(A)} f d\mu_\psi. \tag{4}$$

Here let $f = \bar{g}g$, the equation (4) becomes

$$\begin{aligned}
\|g(A)\psi\|_{\mathcal{H}}^2 &= \langle g(A)\psi \,,\, g(A)\psi \rangle_{\mathcal{H}} = \langle \psi \,,\, \bar{g}g(A)\psi \rangle_{\mathcal{H}} \\
&= \int_{\sigma(A)} \bar{g}g d\mu_\psi = \int_{\sigma(A)} |g(\lambda)|^2 \, d\mu_\psi(\lambda) \\
\Rightarrow \|g(A)\psi\|_{\mathcal{H}}^2 &= \int_{\sigma(A)} |g(\lambda)|^2 \, d\mu_\psi(\lambda),
\end{aligned} \tag{5}$$

which confirms that ***the energy in time-domain should match the energy in spectral domain***.

- **Definition** (***Spectral Measure***)
  For each $\psi \in \mathcal{H}$, the measure $\mu_\psi \in \mathcal{M}(\sigma(A))$ defined in (4) is called *the* ***<u>spectral measure</u> associated with the vector*** $\psi$.

## 3.2 Spectral Theorem in Functional Calculus Form

- **Remark** (***Extension to Bounded Borel Functions on*** $\mathbb{R}$) [Reed and Simon, 1980]
  The first and simplest application of the $\mu_\psi$ is to allow us to ***extend*** *the functional calculus to* $B(\mathbb{R})$, *the* ***<u>bounded Borel measurable functions</u>*** *on* $\mathbb{R}$.

  1. Note that *the double dual of* $\mathcal{C}(X)$ on *compact* metric space $X$ is *the space of bounded Borel measurable function* $B(X) = L^\infty(X, \mu)$ [Lax, 2002].

     $$B(X) \simeq (\mathcal{C}(X))^{**}$$

     In other word, for fixed bounded self-adjoint operator $A$ and $\psi \in \mathcal{H}$, the map

     $$I_\psi : g \mapsto \int_{\sigma(A)} g d\mu_\psi$$

     is well-defined for $g \in B(\sigma(A))$. Extending to $B(\mathbb{R})$ is natural since $\mathbb{R}$ is *locally compact.*

  2. Use *the polarization identity*, and the fact that for self-adjoint operator $A$, $I_\psi$ is real-valued

     $$\langle x \,,\, y \rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2),$$

     we can construct *the sesquilinear form* for any $\psi, \varphi \in \mathcal{H}$

     $$F(\psi, \varphi) = \frac{1}{2}(I_{(\psi+\varphi)}(g) - I_{(\psi)}(g) - I_{(\varphi)}(g))$$

  3. By *Riesz representation theorem*, there exists a unique linear operator $\widetilde{A}_g$ on $\mathcal{H}$ so that

     $$F(\psi, \varphi) = \left\langle \psi \,,\, \widetilde{A}_g \varphi \right\rangle = \frac{1}{2}(I_{(\psi+\varphi)}(g) - I_{(\psi)}(g) - I_{(\varphi)}(g))$$

Note that Thus we identifies $g(A) \equiv \widetilde{A}_g$ for any $g \in B(\mathbb{R})$ so that

$$\langle \psi\, ,\, g(A)\psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}} g d\mu_{\psi}.$$

This shows that ***the functional calculus can be extended to all bounded Borel functions***.

- **Theorem 3.1** ***(Spectral Theorem, Functional Calculus Form)*** *[Reed and Simon, 1980]* *Let $A$ be a **bounded self-adjoint** operator on $\mathcal{H}$. There is a **unique map** $\widehat{\phi} : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ so that*

  1. *$\widehat{\phi}$ is an **algebraic $*$-homomorphism**.*
  2. *$\widehat{\phi}$ is **norm continuous**:*
  $$\|\widehat{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \le C\, \|f\|_{\infty}\,.$$
  3. *Let $f$ be the function $f(x) = x$; then $\widehat{\phi}(f) = A$.*
  4. ***(Pointwise Convergence $\Rightarrow$ Strong Convergence)*** *Suppose $f_n(x) \to f(x)$ for each $x$ and $\|f_n\|_{\infty}$ is bounded. Then $\widehat{\phi}(f_n) \to \widehat{\phi}(f)$ **strongly**. Moreover $\widehat{\phi}$ has the properties :*
  5. *If $A\psi = \lambda\psi$, then*
  $$\widehat{\phi}(f)\psi = f(\lambda)\psi \tag{6}$$
  6. ***(Preserve Positivity)*** *If $f \ge 0$, then $\widehat{\phi}(f) \succeq 0$.*
  7. ***(Preserve Commutative)*** *If $BA = AB$, then $B\widehat{\phi}(f) = \widehat{\phi}(f)B$.*

- **Remark** The proof of (4) is via dominated convergence theorem.

- **Remark** ***The norm equality*** of *the continuous functional calculus* carries over if we define $\|f\|'_{\infty}$ to be *the $L^{\infty}$-norm with respect to a suitable notion* of "***almost everywhere***." Namely, pick *an orthonormal basis* $\{\varphi_n\}$ and say that a property is true a.e. if it is true a.e. *with respect to each* $\mu_{\varphi_n}$. Then $\|\widehat{\phi}(f)\|_{L^2(\mathcal{H})} = \|f\|'_{\infty}$.

## 3.3 Spectral Theorem in Multiplication Operator Form

- **Definition** (***Cyclic Vector***)
  A vector $\psi \in \mathcal{H}$ is called a ***cyclic vector** for $A$* if *finite linear combinations* of the elements $\{A^n\psi\}_{n=0}^{\infty}$ are ***dense*** in $\mathcal{H}$.

- **Remark** Not all operators have cyclic vectors.

- Recall the following theorem for normed vector space

  **Theorem 3.2** ***(Bounded Linear Transformation Theorem)*** *[Reed and Simon, 1980]* *Suppose $T$ is a **bounded** linear transformation from a **normed vector space** $(V_1, \|\|_1)$ to a **complete normed vector space** $(V_2, \|\|_2)$. Then $T$ can be **uniquely extended** to a bounded linear transformation (with the same bound), $\widetilde{T}$, from the **completion** of $V_1$ to $(V_2, \|\|_2)$*

- **Lemma 3.3** ***(Spectral Theorem for Bounded Self-Adjoint Operator with Cyclic Vector)*** *[Reed and Simon, 1980]*
  *Let $A$ be a* ***bounded self-adjoint operator*** *with* ***cyclic vector*** *$\psi$. Then, there is a* ***unitary operator*** *$U : L^2(\sigma(A), \mu_\psi) \to \mathcal{H}$ with*

$$[U^{-1}AUf](\lambda) = \lambda f(\lambda)$$

  *Equality is in the sense of elements of $L^2(\sigma(A), \mu_\psi)$.*

  **Proof:** Define $U : \mathcal{C}(\sigma(A)) \to \mathcal{H}$ by

$$Uf = \phi(f)\psi, \tag{7}$$

  where $f$ is *continuous*. We see that $U$ is essentially the map $\phi : \mathcal{C}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ in the *continuous functional calculus theorem.* To show that $U$ is well-defined, we see that

$$\begin{aligned}\|Uf\|_{\mathcal{H}}^2 &= \|\phi(f)\psi\|_{\mathcal{H}}^2 \\ &= \langle \phi(f)\psi\, ,\, \phi(f)\psi \rangle \\ &= \langle \psi\, ,\, (\phi(f)^*\phi(f))\psi \rangle \\ &= \left\langle \psi\, ,\, (\phi(\bar{f}f)\psi \right\rangle \\ &= \int_{\mathcal{C}(\sigma(A))} |f(\lambda)|^2\, d\mu_\psi(\lambda) = \|f\|_{L^2(\mu_\psi)}^2 .\end{aligned}$$

  Therefore if $f = g$ a.e. with respect to $\mu_\psi$, then $\phi(f)\psi = \phi(g)\psi$ (i.e. $Uf = Ug$, so $U$ is *injective*). Thus $U$ is well defined on $\{\phi(f)\psi : f \in \mathcal{C}(\sigma(A))\}$ and is *norm preserving.* By the bounded linear transformation theorem, $U$ can be *extended uniquely* to an ***isometric map*** $L^2(\sigma(A), \mu_\psi) \to \mathcal{H}$, since $L^2(\sigma(A), \mu_\psi)$ is the *completion* of $\mathcal{C}(\sigma(A))$ in $\|\cdot\|_{L^2(\mu_\psi)}^2$ norm.

  Finally, if $f \in \mathcal{C}(\sigma(A))$,

$$\begin{aligned}[U^{-1}AUf](\lambda) &= [U^{-1}A\phi(f)\psi](\lambda) \\ &= [U^{-1}\phi(xf)\psi](\lambda) \\ &= (xf)(\lambda) = \lambda f(\lambda).\end{aligned}$$

  By continuity and denseness of power series of cyclic vectors, this extends from $f \in \mathcal{C}(\sigma(A))$ to $f \in L^2(\sigma(A), \mu_\psi)$. ∎

- **Lemma 3.4** ***(Direct Sum Decomposition of Hilbert Space via Invariant Subspaces)*** *[Reed and Simon, 1980]*
  *Let $A$ be a* ***self-adjoint*** *operator on a* ***separable Hilbert space*** *$\mathcal{H}$. Then there is a* ***direct sum decomposition***

$$\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$$

  *with $N = 1, 2, , \ldots ,$ or $\infty$ so that:*

  1. *$\mathcal{H}_n$ is* ***<u>invariant</u>*** *under operator $A$; that is, for any $\psi \in \mathcal{H}_n$, $A\psi \in \mathcal{H}_n$.*
  2. *For each $n$, there exists a $\psi_n \in \mathcal{H}_n$ that is* ***cyclic*** *for $A|_{\mathcal{H}_n}$, i.e.*

$$\mathcal{H}_n = \overline{\{f(A)\psi_n : f \in \mathcal{C}(\sigma(A))\}}.$$

- **Theorem 3.5** *(**Spectral theorem, Multiplication Operator Form**) [Reed and Simon, 1980]*
  *Let $A$ be a **bounded self-adjoint** operator on $\mathcal{H}$, a **separable Hilbert space**. Then, there exist **measures** $\{\mu_{\psi_n}\}_{n=1}^N$ ($N = 1, 2, \ldots,$ or $\infty$) on $\sigma(A)$ and a **unitary** operator*

$$U : \bigoplus_{n=1}^{N} L^2(\mathbb{R}, \mu_{\psi_n}) \to \mathcal{H}$$

  *so that*

$$[U^{-1}AU\psi]_n(\lambda) = \lambda\psi_n(\lambda) \tag{8}$$

  *where we write an element $\psi \in \bigoplus_{n=1}^N L^2(\sigma(A), \mu_{\psi_n})$ as an $N$-tuple $(\psi_1(\lambda), \ldots, \psi_N(\lambda))$. This realization of $A$ is called a **spectral representation**.*

- **Remark** (***Self-Adjoint Bounded Operator = Mulitplication Operator in Spectral Domain***)
  This theorem tells us that ***every bounded self-adjoint operator is a multiplication operator on a suitable measure space***; what changes as *the operator changes* are *the underlying measures.*

- **Remark** (***Multiplication Operator***)
  Define ***the multiplication operator*** $M_f : v \mapsto fv$ on $L^2$ for $f \in L^2$, so (8) becomes

$$U^{-1}AU = M_\alpha \tag{9}$$

  where $\alpha(x) = x$.

- **Corollary 3.6** *(**Spectral theorem, Single Spectral Measure**) [Reed and Simon, 1980]*
  *Let $A$ be a **bounded self-adjoint** operator on a **separable Hilbert space** $\mathcal{H}$. Then there exists a **finite measure space** $(M, \mu)$, a **bounded function** $F$ on $M$, and a **unitary map**, $U : L^2(M, \mu) \to \mathcal{H}$, so that*

$$[U^{-1}AUf]_n(m) = F(m)f(m)$$

  **Proof:** Choose *the cyclic vectors* $\psi_n$ so that $\|\psi_n\| = 2^{-n}$. Let $M = \bigcup_{n=1}^N \mathbb{R}$, i.e. *the **union** of copies of* $\mathbb{R}$. Define $\mu$ by requiring that its restriction to the $n$-th copy of $\mathbb{R}$ be $\mu_{\psi_n}$. Since $\mu(M) = \sum_{n=1}^N \mu_{\psi_n}(\mathbb{R}) < \sum_{n=1}^N 2^{-n} < \infty$, $\mu$ is ***finite***. ∎

- **Example** (***Self-Adjoint Operator on Finite Dimensional Space***)
  Let $A$ be an $n \times n$ ***self-adjoint (Hermitian)*** *matrix.* The ***finite dimensional spectral theorem*** says that $A$ has a *complete orthonormal set* of ***eigenvectors***, $\psi_1, \ldots, \psi_n$, with

$$A\psi_i = \lambda_i\psi_i.$$

  Suppose first that *the eigenvalues are **distinct***. The *spectral measure* is just *the sum of **Dirac measures***,

$$\mu = \sum_{i=1}^{n} \delta_{\lambda_i}, \tag{10}$$

and $L^2(\mathbb{R}, \mu)$ is just $\mathbb{C}^n$ since $f \in L^2$ is ***determined*** by

$$(f(\lambda_1), \ldots, f(\lambda_n)).$$

Clearly, the function $\lambda f$ corresponds to the $n$-tuple $(\lambda_1 f(\lambda_1), \ldots, \lambda_n f(\lambda_n))$, so $A$ is ***multiplication*** by $\lambda$ on $L^2(\mathbb{R}, \mu)$.

If we take

$$\bar{\mu} = \sum_{i=1}^{n} a_i \delta_{\lambda_i},$$

with $a_1, \ldots, a_n > 0$, *$A$ can also be represented as* ***multiplication*** *by $\lambda$ on $L^2(\mathbb{R}, \bar{\mu})$.* Thus, we explicitly see *the* ***nonuniqueness*** *of the* ***measure*** in this case.

We can also see when ***more than one measure is needed***: *one can represent a finite-dimensional self-adjoint operator as multiplication on $L^2(\mathbb{R}, \mu)$ with* ***only one measure if and only if*** *$A$ has* ***no repeated eigenvalues***. ∎

- **Example** (***Self-Adjoint Compact Operator***)
Let $A$ be ***compact*** and ***self-adjoint***. *The Hilbert-Schmidt theorem* tells us there is a *complete orthonormal set of* ***eigenvectors*** $\{\psi_n\}_{n=1}^{\infty}$, with

$$A\psi_n = \lambda_n \psi_n.$$

If there is *no repeated eigenvalue*,

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{\lambda_n} \tag{11}$$

works as a ***spectral measure***. ∎

- **Example** (***Fourier Transform***)
Note that for $f \in L^2(\mathbb{R}, dx)$, the Fourier transform of $f$ is written as

$$\mathcal{F}f(\lambda) := F(\lambda) = \frac{1}{(2\pi)^{-1}} \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx$$
$$f(x) = \int_{\mathbb{R}} F(\lambda) e^{i\lambda x} d\lambda$$

The Fourier transform $\mathcal{F}$ can be seen as a unitary map $\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, \mu(d\lambda))$, which is the inverse of $U$ where $e^{i\lambda x} d\lambda = \mu(d\lambda)$.

Consider $A = \frac{1}{i}\frac{d}{dx}$ on $L^2(\mathbb{R}, dx)$, which is *self-adjoint* but ***unbounded***. The Fourier transform of $A$ gives

$$\mathcal{F}\left(\frac{1}{i}\frac{d}{dx} f\right)(\lambda) = \lambda\, \mathcal{F}f(\lambda)$$
$$\Leftrightarrow (U^{-1}AUF)(\lambda) = \lambda\, F(\lambda)$$

where the unitary map $U : L^2(\mathbb{R}, \mu(d\lambda)) \to L^2(\mathbb{R}, dx)$ is ***the inverse Fourier transform***

$$(UF)(x) = f(x) = \int_{\mathbb{R}} F(\lambda) e^{i\lambda x} d\lambda.$$

And the spectral measure acts on $f$ is

$$\mu f = \frac{1}{(2\pi)^{-1}} \int_{\sigma(A)} \left[ \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx \right] e^{i\lambda x} d\lambda. \qquad \blacksquare$$

- **Definition** (***Essential Range***)
Let $F$ be a real-valued function on a measure space $(X, \mu)$. We say $\lambda$ is in ***the essential range of*** $F$ *if and only if* for all $\epsilon > 0$,

$$\mu \{x : F(x) \in (\lambda - \epsilon, \lambda + \epsilon)\} = \mu \circ F^{-1}(B(\lambda, \epsilon)) > 0.$$

- **Proposition 3.7** *(**Spectrum of Multiplication Operator via Essential Range**) [Reed and Simon, 1980]*
*Let $F$ be a **bounded real-valued** function on a measure space $(X, \mu)$. Let $M_F$ be the multiplication operator on $L^2(X, \mu)$ given by*

$$(M_F g)(x) = F(x) g(x)$$

*Then $\sigma(M_F)$ is **the essential range** of $F$.*

### 3.4 Decompose of Spectral Measure

- **Definition** (***Support of a Family of Measures***)
If $\{\mu_n\}_{n=1}^N$ is a *family of measures*, ***the support of*** $\{\mu_n\}_{n=1}^N$ is the *complement* of *the largest open set* with $\mu_n(B) = 0$ for all $n$; so

$$supp(\{\mu_n\}_{n=1}^N) = \overline{\bigcup_{n=1}^N supp(\mu_n)}$$

- **Proposition 3.8** *(**Support of All Spectral Measures = the Spectrum**) [Reed and Simon, 1980]*
*Let $A$ be a **self-adjoint operator** and $\{\mu_n\}_{n=1}^N$ a family of **spectral measures**. Then*

$$\sigma(A) = \text{supp}(\{\mu_n\}_{n=1}^N).$$

- **Definition** (***Pure Point of Measure***)
Given measure space $(X, \mu)$, a collection of ***closed one-point sets*** with *nonzero measure* is called ***the pure point set of measure*** $\mu$. That is,

$$P := \{x \in X : \mu(\{x\}) > 0\}.$$

For $X = \mathbb{R}$ and $\mu$ is Borel measure, the pure point set is ***countable***.

- **Definition** (***Pure Point Measure and Continuous Measure***)
***The pure point measure*** is defined as *the restriction of $\mu$ on the pure point set $P$ of that measure*. For Borel measure $\mu$ on $\mathbb{R}$, and any ***Borel set*** $S \in \mathcal{B}(\mathbb{R})$,

$$\mu_{pp}(S) = \mu(S \cap P) = \sum_{x \in S \cap P} \mu(\{x\}).$$

A measure $\mu = \mu_{cont}$ is ***continuous*** if it has ***no pure point***, i.e. $\mu(\{x\}) = 0$ for any $\{x\} \in \mathcal{B}(\mathbb{R})$.

By definition, the following decomposition of measure $\mu$ holds:

$$\mu = \mu_{pp} + \mu_{cont}, \quad \mu_{pp} \perp \mu_{cont}$$

- **Remark** (***Decomposition of Borel Measure with respect to Lebesgue Measure***)
  Recall from Lebesgue decomposition theorem, given $\lambda$ as the Lebesgue measure on $\mathbb{R}$, any measure $\mu$ on $\mathbb{R}$ can be decomposed into two mutually singular parts:

  $$\mu = \mu_{ac} + \mu_{sing}, \quad \mu_{ac} \perp \mu_{sing}$$

  where $\mu_{ac} \ll \lambda$ and $\mu_{sing} \perp \lambda$. Combining with decomposition of pure point measure and continuous measure, we have the decomposition of any measure on $\mathbb{R}$ with respect to Lebesgue measure on $\mathbb{R}$,

  $$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing} \tag{12}$$

  where $\mu_{pp}$ is ***the pure point measure***, $\mu_{ac}$ is the part of ***continuous*** *measure* that is ***absolutely continuous*** *with respect to Lebesgue measure*, and $\mu_{sing}$ is the part of ***continuous*** *measure* that is ***singular*** *with respect to Lebesgue measure.*

- **Remark** (***Decomposition of Invariant Subspace***)
  We apply above decomposition to spectral measure $\mu$. Since these parts are mutually singular to each other, we have

  $$L^2(\mathbb{R}, \mu) = L^2(\mathbb{R}, \mu_{pp}) \oplus L^2(\mathbb{R}, \mu_{ac}) \oplus L^2(\mathbb{R}, \mu_{sing}). \tag{13}$$

  We can verify that any $\psi \in L^2(\mathbb{R}, \mu)$ has an ***absolutely continuous spectral measure*** $\mu_{ac}$ *with respect to Lebesgue measure* ***if and only if***

  $$\psi \in L^2(\mathbb{R}, \mu_{ac}) \Leftrightarrow \int_{\mathbb{R}} |\psi|^2 \, d\mu_{ac} = \int_{\mathbb{R}} |\psi|^2 \, p \, d\lambda < \infty$$

  where $p = d\mu_{ac}/d\lambda$ a.e.. Similarly for *pure point* and *singular measures.*

- **Definition** Let $A$ be a ***bounded self-adjoint*** operator on $\mathcal{H}$. Let

  1. $\mathcal{H}_{pp} := \{\psi \in \mathcal{H} : \mu_\psi$ ***is a pure point measure***$\}$
  2. $\mathcal{H}_{ac} := \{\psi \in \mathcal{H} : \mu_\psi$ ***has no pure point and*** $\mu_\psi \ll \lambda$ ***Lebesgue measure***$\}$
  3. $\mathcal{H}_{sing} := \{\psi \in \mathcal{H} : \mu_\psi$ ***has no pure point and*** $\mu_\psi \perp \lambda$ ***Lebesgue measure***$\}$

- **Proposition 3.9** ***(Direct Sum Decomposition of Hilbert Space via Spectral Measure Decomposition)*** *[Reed and Simon, 1980]*
  *Let $A$ be a* ***bounded self-adjoint*** *operator on separable Hilbert space $\mathcal{H}$. For any $\psi \in \mathcal{H}$, $\mu_\psi$ is the spectral measure on $\sigma(A)$ corresponding to $\psi$. Then the following direct sum decompositon holds*

  $$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$$

  *Moreover,*

1. *Each of these subspaces is* ***invariant*** *under $A$, i.e. for any $\psi$ in these subspaces, $A\psi$ is in the same subspace.*
2. *$A|_{\mathcal{H}_{pp}}$ has a* ***complete set of eigenvectors;***
3. *$A|_{\mathcal{H}_{ac}}$ has* ***only absolutely continuous spectral measures***
4. *$A|_{\mathcal{H}_{sing}}$ has* ***only continuous singular spectral measures****.*

- **Definition** (***Partition of Spectrum***)
  We define the following *subsets of spectrum* $\sigma(A)$:
  1. ***Pure Point Spectrum***: $\sigma_{pp}(A) := \{\lambda \in \sigma(A) : \lambda$ ***is an eigenvalue of*** $A\}$
  2. ***Absolutely Continuous Spectrum***: $\sigma_{ac}(A) := \sigma\left(A|_{\mathcal{H}_{ac}}\right)$
  3. ***(Continuous) Singular Spectrum***: $\sigma_{sing}(A) := \sigma\left(A|_{\mathcal{H}_{sing}}\right)$

  We can also defines ***the continuous spectrum*** as $\sigma_{cont}(A) := \sigma\left(A|_{\mathcal{H}_{ac}\oplus\mathcal{H}_{sing}}\right)$.

- **Remark** These *spectrums* are ***spectrum*** of *the linear operator $A$* ***restricted*** *in* ***each invariant subspace***. They are also the ***support*** of corresponding ***spectral measure***.

- **Remark** Unlike pure point spectrum, the singular spectrum $\sigma_{sing}(A)$ may contains spectral measure that is singular to Lebesgue measure but still without pure point.

- **Proposition 3.10** *[Reed and Simon, 1980]*

$$
\begin{aligned}
\sigma(A) &= \overline{\sigma_{pp}(A)} \cup \sigma_{ac}(A) \cup \sigma_{sing}(A) \\
&= \overline{\sigma_{pp}(A)} \cup \sigma_{cont}(A)
\end{aligned}
$$

- **Remark** The sets ***need not be disjoint***, however. The reader should be warned that $\sigma_{sing}(A)$ may have nonzero Lebesgue measure.

## 3.5 Spectral Theorem in Spectral Projection Form

- **Definition** (***Spectral Projection***)
  Let $A$ be a ***bounded self-adjoint*** operator and $S$ a ***Borel set*** of $\mathbb{R}$.

$$
P_S := \mathbb{1}_S(A) = \widehat{\phi}(\mathbb{1}\{\lambda \in \sigma(A) \cap S\})
$$

  is called a ***spectral projection of*** $A$. It is result of applying the ***characteristic function of set*** $R$, $\mathbb{1}_S(x)$, on operator $A$ via *functional calculus.*

- **Remark** (***Spectral Projection is Orthorgonal Projection***)
  $P_S$ is an ***orthogonal projection*** since for each $x$

$$
\mathbb{1}_S^2(x) = \mathbb{1}_S(x) = \bar{\mathbb{1}}_S(x).
$$

  It is equivalent to a 0-1 ***test*** *to check if each point of spectrum of $A$ is in $S$.*

- **Proposition 3.11** ***(Properties of Spectral Projection)*** *[Reed and Simon, 1980]*
  *The family $\{P_S\}$ of* ***spectral projections*** *of a* ***bounded self-adjoint*** *operator, $A$, has the following properties:*

1. *Each $P_S$ is an* ***orthogonal projection****.*
2. $P_\emptyset = 0$*;* $P_{(-a,a)} = 1$ *for* ***some*** $a$*.*
3. *(****Countable Disjoint Union****) If* $S = \bigcup_{n=1}^{\infty} S_n$ *with* $S_n \cap S_m = \emptyset$ *for all* $n \neq m$*, then in norm topology*

$$P_S = \sum_{n=1}^{\infty} P_{S_n}.$$

4. $P_{S_1} P_{S_2} = P_{S_1 \cap S_2}$

- **Definition** (***Projection-Valued Measure***)
  A family of ***projections*** obeying (1)-(3) is called a ***(bounded) projection-valued measure (p.v.m.)***.

- **Remark** For a family of projections $\{P_S : S \in \mathcal{B}(\mathbb{R})\}$, we have this mapping

$$P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}).$$

  $P$ as a set function is finite i.e. $P(\mathbb{R}) = 1$ and $P(\emptyset) = 0$ and countably additive, therefor $P$ is a ***vector-valued*** *Borel measure on spectral domain* $\mathcal{B}(\mathbb{R})$.

- **Remark** We can obtain a *spectral measure* $\mu_{\psi,S}$ from $P_S$ via

$$\langle \psi \,,\, P_S \psi \rangle = \int_{\sigma(A)} \mathbb{1}_S d\mu_\psi = \mu_\psi(S \cap \sigma(A)) = \int_{\sigma(A)} d\mu_{\psi,S}$$

  for any $\psi \in \mathcal{H}$. We will use the ***symbol*** $d \langle P_S \psi \,,\, \psi \rangle$ to mean ***integration*** with respect to this measure $d\mu_{\psi,S} = \mathbb{1}_S d\mu_\psi$.

  By *standard Riesz representation theorem* methods, there is a ***unique*** operator with

$$\langle \psi \,,\, B\psi \rangle = \int f(\lambda) \, d \langle \psi \,,\, P_S \psi \rangle$$

- **Proposition 3.12** *(****Linear Operator Corresponding to Projection-Value Measure****) [Reed and Simon, 1980]*
  *If $P_S$ is a* ***projection-valued measure*** *and $f$ a* ***bounded*** *Borel function on supp($P_S$), then there is a* ***unique*** *operator $B$ such that*

$$\langle \psi \,,\, B\psi \rangle = \int f(\lambda) \, d \langle \psi \,,\, P_S \psi \rangle \,.$$

  *We denote*

$$B := \int f(\lambda) dP_S(\lambda).$$

$$\Rightarrow \left\langle \psi \,,\, \left( \int f(\lambda) dP_S(\lambda) \right) \psi \right\rangle = \int f(\lambda) \, d \langle \psi \,,\, P_S \psi \rangle$$

- **Theorem 3.13** *(****Spectral Theorem, Projection-Valued Measure Form****) [Reed and Simon, 1980]*
  *There is a* ***one-one correspondence*** *between* ***(bounded) self-adjoint operators*** *$A$ and* ***(bounded) projection valued measures*** *$\{P_S\}$. In particular:*

1. *Given $A$, each projection-valued measure $P_S$ can be obtained as*

$$P_S := \mathbb{1}_S(A) = \widehat{\phi}(\mathbb{1}_S)$$

2. *Given $\{P_S : S \subset \mathbb{R},\ \text{Borel set}\}$, the operator $A$ can be obtained as*

$$A = \int_{\mathbb{R}} \lambda \, dP_\lambda \tag{14}$$

*and*

$$f(A) = \int_{\mathbb{R}} f(\lambda) \, dP_\lambda. \tag{15}$$

- **Remark** (***Understand Integration w.r.t. Projection-Valued Measure***)
As always, we can develop the integration with respect to projection-valued measure from simple function $f \in \mathcal{L}^2(\sigma(A), \mu_\psi)$:

$$f(\lambda) = \sum_{n=1}^{N} c_n \mathbb{1}_{S_n}(\lambda)$$

where $S_n := f^{-1}(\{c_n\})$, $\sigma(A) = \bigcup_{n=1}^{N} S_n$ and $S_n \cap S_m = \emptyset$. Using $\widehat{\phi} : \mathcal{L}^2(\sigma(A), \mu_\psi) \to \mathcal{L}(\mathcal{H})$, we can apply *functional calculus* on $A$ to have

$$f(A) = \sum_{n=1}^{N} c_n \mathbb{1}_{S_n}(A) := \sum_{n=1}^{N} c_n P_{S_n} = \widehat{\phi}\left(\sum_{n=1}^{N} c_n \mathbb{1}_{S_n}\right).$$

Recall that when we define integration of simple function we have

$$\text{simp} \int f(\lambda) d\lambda = \sum_{n=1}^{N} c_n \mu_\psi(S_n) = \sum_{n=1}^{N} c_n \left\langle \psi\,,\, P_{S_n}\psi \right\rangle.$$

Equivalently, we can have integration of simple function with respect to the projection-valued measure $\{P_{S_n}\}$

$$\text{simp} \int f(\lambda) dP_\lambda = \sum_{n=1}^{N} c_n P(S_n) = \sum_{n=1}^{N} c_n P_{S_n} = f(A).$$

Then for unsigned function $f \geq 0$,

$$\underline{\int} f(\lambda) dP_\lambda = \sup_{g \text{ simple},\, 0 \leq g \leq f} \text{simp} \int g(\lambda) dP_\lambda$$

and for any absolutely integrable function $f = f_+ - f_-$,

$$\int f(\lambda) dP_\lambda = \underline{\int} f_+(\lambda) dP_\lambda - \underline{\int} f_-(\lambda) dP_\lambda.$$

Finally we see that $P_{B(\lambda,\epsilon)} = 0$ if $\lambda \notin \sigma(A)$ so this integral is well-defined all over $\mathbb{R}$.

- **Remark** (***Bounded Real-Valued Measurable Function*** $\Leftrightarrow$ ***Bounded Self-Adjoint Operator***) [Halmos, 2017]
  *The* ***essence*** *of spectral theorem* (in *functional calculus form* and in *spectral projection form*):

  The ***analogs*** of ***bounded, real-valued, measurable*** *function* in Hilbert space thoery are ***bounded, self-adjoint linear operators***. Since a function is the *characteristic function of a set if and only if* it is ***idempotent***, it is clear on the algebraic gounds that the analogs of ***characteristic functions*** are ***projections***. The ***approximability*** of functions by ***simple functions*** corresponds in the analogy to the *approximability* of self-adjoint operators by ***real, finite linear combinations of projections***.

- **Remark** (***Comparison of Spectral Projection***)
  Consider the spectral theorem in projection form

$$
\begin{aligned}
A &= \int_{\mathbb{R}} \lambda dP_{\lambda} && \textbf{\textit{general}}\ \textit{self-adjoint}\\
A &= \sum_{i=1}^{n} \lambda_i \varphi_i \varphi_i^T = \sum_{i=1}^{n} \lambda_i P_{\mathcal{H}_i} && \textbf{\textit{finite dimensional}}\\
A &= \sum_{i=1}^{\infty} \lambda_i P_{\mathcal{H}_i} && \textbf{\textit{compact}}\ \textit{self-adjoint}
\end{aligned}
$$

  where $\mathcal{H}_i = \mathrm{Ker}\,(\lambda_i I - A) = \mathrm{span}\,\{A^n \varphi_i : n = 0, 1, \ldots\}$ is ***the invariant subspace***, $\varphi_i$ is ***cyclic vector*** *as the eigenvectors / eigenfunctions* corresponding to $\lambda_i$. For finite dimensional and compact operator case, $\mathcal{H}_i$ is *finite dimensional*.

  The decomposition of spectrum tells us that for general bounded self-adjoint operator

$$
A = \int_{\mathbb{R}} \lambda dP_{\lambda} = \sum_{\{i:\lambda_i \in \sigma_{disc}(A)\}} \lambda_i P_{\mathcal{H}_i} + \int_{\sigma_{ess}(A)} \lambda dP_{\lambda} \tag{16}
$$

  where $\mathcal{H}_i = \mathrm{Ker}\,(\lambda_i I - A)$ is ***the invariant subspace (eigenspace)*** and $\mathcal{H}_i$ is ***finite dimensional***.

### 3.6 Understanding Spectrum via Spectral Projection

- **Proposition 3.14** *(**Criterion for Spectrum**) [Reed and Simon, 1980]*
  $\lambda \in \sigma(A)$ ***if and only if***

$$
P_{B(\lambda,\epsilon)}(A) = P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \neq 0
$$

  *for any* $\epsilon > 0$.

- **Definition** (***Essential Spectrum and Discrete Spectrum***)
  1. We say $\lambda \in \sigma_{ess}(A)$, ***the essential spectrum of*** $A$, *if and only if*

$$
P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \ \textbf{\textit{is infinite dimensional}}
$$

     ***for all*** $\epsilon > 0$. $P$ is *infinite dimensional* means $\overline{\mathrm{Ran}(P)}$ is *infinite dimensional.*

2. If $\lambda \in \sigma(A)$, but

$$P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \textit{ \textbf{is finite dimensional}}$$

***for some*** $\epsilon > 0$, we say $\lambda \in \sigma_{disc}(A)$, <u>***the discrete spectrum of***</u> .

- **Proposition 3.15** *[Reed and Simon, 1980]*
  $\sigma_{ess}(A)$ *is always* ***closed***.

- **Proposition 3.16** *[Reed and Simon, 1980]*
  $\lambda \in \sigma_{disc}(A)$ ***if and only if*** <u>***both***</u> *the following hold:*

  1. $\lambda$ *is an* ***isolated*** *point of* $\sigma(A)$, *that is, for some* $\epsilon$, $(\lambda-\epsilon, \lambda+\epsilon) \cap \sigma(A) = \{\lambda\}$.
  2. $\lambda$ *is an* ***eigenvalue*** *of* ***finite multiplicity***, *i.e.,*

  $$dim\{\varphi : A\varphi = \lambda\varphi\} = dim\ Ker\{A - \lambda I\} < \infty.$$

- **Proposition 3.17** $\lambda \in \sigma_{ess}(A)$ ***if and only if*** <u>***at least one***</u> *of the following holds:*

  1. $\lambda \in \sigma_{cont}(A) = \sigma_{ac}(A) \cup \sigma_{sing}(A)$.
  2. $\lambda$ *is a* ***limit point*** *of* $\sigma_{pp}(A)$.
  3. $\lambda$ *is an* ***eigenvalue*** *of* ***infinite multiplicity***.

- **Remark** (***Multiple Ways to Decompose the Spectrum***)
  The recall the ***partition*** of spectrum by ***point spectrum***, ***continuous spectrum*** *and* ***residual spectrum***. We see that

  1. $$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

     This is related to the ***resolvent*** $R_\lambda(A) = (\lambda I - A)^{-1}$: its ***existence***, its ***range*** (***dense*** or not) and its ***boundedness***. These subsets are *disjoint*. Importantly, this decomposition is ***general*** and it applies to ***all linear operator***.

  2. $$\sigma(A) = \overline{\sigma_{pp}(A)} \cup \sigma_{ac}(A) \cup \sigma_{sing}(A).$$

     This is related to the ***decompose*** of ***spectral measure*** $\mu_\psi$ *with respect to Lebesgue measure* and the ***pure point set***. These sets *may not be disjoint*. Both this and the one below are related to ***spectral theorem*** of ***self-adjoint operator***.

  3. $$\sigma(A) = \sigma_{disc}(A) \cup \sigma_{ess}(A).$$

     This is related to the ***dimensionality of image set*** of ***spectral projection*** $P_{B(\lambda,\epsilon)}$ on any open intervals around $\lambda$. It is related to the multiplicity of the kernel Ker $\{A - \lambda I\}$. These sets *are disjoint*.

- **Theorem 3.18** *(**Weyl's Criterion**) [Reed and Simon, 1980]*
  *Let $A$ be a **bounded self-adjoint** operator. Then $\lambda \in \sigma(A)$ **if and only if** there exists $\{\psi_n\}_{n=1}^{\infty}$ so that $\|\psi_n\| = 1$ and*

  $$\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0.$$

  *$\lambda \in \sigma_{ess}(A)$ **if and only if** the above $\{\psi_n\}_{n=1}^{\infty}$ can be chosen to be **orthogonal**.*

- **Remark** *The essential spectrum **cannot be removed*** by ***essentially finite dimensional perturbations***.

  A general implies that $\sigma_{ess}(A) = \sigma_{ess}(B)$ if $A - B$ is ***compact***.

- **Remark** Finally, we discuss one useful formula relating the resolvent and spectral projections. It is a matter of computation to see that the box on $[a, b]$

  $$f_\epsilon(x) = \begin{cases} 0 & x \notin [a,b] \\ \frac{1}{2} & x = a \text{ or } x = b \\ 1 & x \in (0,1) \end{cases}$$

  We can find

  $$f_\epsilon(x) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_a^b \left( \frac{1}{x-\lambda-i\epsilon} - \frac{1}{x-\lambda+i\epsilon} \right) d\lambda$$

  Moreover, $|f_\epsilon(x)|$ is ***bounded uniformly*** in $\epsilon$. Applying the functional calculus on $A$, we have

  **Theorem 3.19** *(**Stone's formula**) [Reed and Simon, 1980]*
  *Let $A$ be a **bounded self-adjoint** operator. Then*

  $$\begin{aligned} \frac{1}{2}\left(P_{[a,b]} + P_{(a,b)}\right) &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_a^b \left[ (A-\lambda-i\epsilon)^{-1} - (A-\lambda+i\epsilon)^{-1} \right] d\lambda \qquad (17) \\ &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_a^b \left[ R_{\lambda+i\epsilon}(A) - R_{\lambda-i\epsilon}(A) \right] d\lambda \end{aligned}$$

  *for $R_\lambda(A) = (A-\lambda)^{-1}$, the **resolvent** of $A$.*

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