# Lecture 0: Summary of Topology (Part 3)

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Nov. 23th., 2022

## Contents

# 1 Metrization Theorems and Paracompactness

## 1.1 Paracompactness

### 1.1.1 Local Finiteness

- **Definition** (***Local Finiteness***)
  Let $X$ be a topological space. A *collection* $\mathscr{A}$ of *subsets* of $X$ is said to be ***locally finite in*** $X$ if every point of $X$ has a *neighborhood* that ***intersects*** *only* ***finitely many elements*** of $\mathscr{A}$.

- **Remark** (***Understanding Locally Finite***)
  *A locally finite* collection of subsets in a topological space is ***evenly spread across the space***. In other word, there exists ***no cluster point*** $x \in X$ for these subsets so that *every neighborhood of $x$ will intersect with infinitely many subsets in the collection.*

  *Local finiteness* describe *the* ***distribution*** *of the given collection of subsets* in $X$. We can think of $\mathscr{A}$ as the result of "***uniform sampling***" of subsets across the space.

- **Example** (***Locally Finite Collections in*** $\mathbb{R}$)
  The collection of intervals

  $$\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}$$

  is *locally finite* in the topological space $\mathbb{R}$.

  On the other hand, the collection

  $$\mathscr{B} = \{(0, 1/n) : n \in \mathbb{Z}\}$$

  has a cluster point $0 \in \mathbb{R}$ so it is not locally finite in $\mathbb{R}$. However, it is locally finite for $(0, 1)$.

- **Lemma 1.1** *(**Properties of Locally Finiteness**) [Munkres, 2000]*
  *Let $\mathscr{A}$ be a locally finite collection of subsets of $X$. Then:*

  1. *Any **subcollection** of $\mathscr{A}$ is locally finite.*
  2. *The collection $\mathscr{B} = \{\bar{A}\}_{A \in \mathscr{A}}$ of the **closures** of the elements of $A$ is locally finite.*
  3. $\overline{\bigcup_{A \in \mathscr{A}} A} = \bigcup_{A \in \mathscr{A}} \bar{A}$.

- **Remark** There exists collection of sets that is ***not locally finite*** but the collection of *their* ***closures*** is ***locally finite***.

  For instance, consider $X = (0, 1)$, and let $A_N = \bigcup_{n=1}^{N} (\frac{n-1}{N}, \frac{n}{N})$ and $\mathscr{A} = \{A_N\}_{N=1}^{\infty}$. For each point of $x \in (0, 1)$, every neighborhood of $x$ intersects with infinite many $A_N$. But the closure of $A_N = (0, 1)$ itself for every $N$. Thus $\mathscr{B} = \left\{\bar{A} : A \in \mathscr{A}\right\} = \{(0, 1)\}$, which is finite thus locally finite.

- **Definition** (***Locally Finite Indexed Family***)
  The indexed family $\{A_\alpha\}_{\alpha \in J}$ is said to be ***locally finite indexed family in*** $X$ if every $x \in X$ has a neighborhood that *intersects* $A_\alpha$ for only ***finitely many values*** of $\alpha$.

- **Remark** $\{A_\alpha\}_{\alpha \in J}$ is a ***locally finite indexed family*** if and only if it is ***locally finite*** as *a collection of sets* and each *nonempty* subset $A$ of $X$ equals $A_\alpha$ for *at most finitely many values* of $\alpha$.

- **Definition** (***Countably Local Finiteness***)
  A collection $\mathscr{B}$ of subsets of $X$ is said to be ***countably locally finite*** if $\mathscr{B}$ can be written as ***the countable union*** of collections $\mathscr{B}_n$, each of which is ***locally finite***.

$$\mathscr{B} = \bigcup_{n \in \mathbb{Z}_+} \mathscr{B}_n$$

  Countably locally finite is also called ***$\sigma$-locally finite***.

- **Remark** Note that both *a **countable** collection* and *a **locally finite** collection* are ***countably locally finite***.

- **Remark** We can consider a ***countably locally finite*** collection as the result of ***superposition*** of ***countable layers*** of *uniform sampling* of subsets in a topological space.

- **Definition** (***Refinement of Collection***)
  Let $\mathscr{A}$ be a collection of subsets of the space $X$. A collection $\mathscr{B}$ of subsets of $X$ is said to be a ***refinement of $\mathscr{A}$*** (or is said to ***refine*** $\mathscr{A}$) if for each element $B$ of $\mathscr{B}$, there is an element $A$ of $\mathscr{A}$ ***containing*** $B$.

  If the elements of $\mathscr{B}$ are ***open sets***, we call $\mathscr{B}$ an ***open refinement of $\mathscr{A}$***; if they are ***closed sets***, we call $\mathscr{B}$ a ***closed refinement***.

- **Remark** (***Finer $\Rightarrow$ Smaller Subsets***)
  $\mathscr{B}$ is a ***refinement*** of $\mathscr{A} \Rightarrow \forall B \in \mathscr{B}$, $B$ is a subset of some element in $\mathscr{A}$.

  Note that there may exists some $A \in \mathscr{A}$ does not intersect with any $B \in \mathscr{B}$.

- **Theorem 1.2** *[Munkres, 2000]*
  *Let $X$ be a **metrizable** space. If $\mathscr{A}$ is an open covering of $X$, then there is an **open covering** $\mathscr{E}$ of $X$ **refining $\mathscr{A}$** that is **countably locally finite**.*

- **Remark** For ***metrizable*** space $X$, *every **open covering** has a **countable locally finite refinement** that also **covers** $X$.*

### 1.1.2 Paracompactness

- **Definition** (***Compactness in terms of Refinement***)
  A space $X$ is ***compact*** if every *open covering* $\mathscr{A}$ of $X$ has ***a finite open refinement*** $\mathscr{B}$ that ***covers $X$***.

- We generalize the definition of compactness by relaxing the finiteness to locally finiteness

  **Definition** (***Paracompactness***)
  A space $X$ is ***paracompact*** if *every open covering* $\mathscr{A}$ of $X$ has a ***locally finite open refinement*** $\mathscr{B}$ that ***covers $X$***.

- **Remark** (***Compactness** vs. **Paracompactness***)
  *Paracompactness is a **generalization** of compactness*, i.e. *all compact space is paracompact.*

  Both compactness and paracompactness assert *the **existence** of an **open subcovering** with **some structure***. But ***the constraint on the structure*** is different:

  1. *Compactness* ***controls the cardinality*** of *subcovering*, i.e. to be ***finite***.

2. *Paracompactness* ***controls the distribution*** of *subcovering*, i.e. to be ***evenly distributed*** *across space without cluster point* or to be ***locally finite***.

- **Example** ($\mathbb{R}^n$)
The space $\mathbb{R}^n$ is ***paracompact***. Let $X = \mathbb{R}^n$. Let $\mathscr{A}$ be an *open covering* of $X$. Let $B_0 = \emptyset$, and for each *positive integer* $m$, let $B_m = B(0, m)$ denote *the open ball of* ***radius*** $m$ ***centered at the origin***. Note that $B_m \subseteq B_{m+1}$ for all $m$ and its closure $\bar{B}_m$ is a *compact subset* of $\mathbb{R}^n$.

  Given $m$, choose ***finitely many elements*** of $\mathscr{A}$ that ***cover*** $\bar{B}_m$ (since $\bar{B}_m$ is compact) and ***intersect*** *each one* with ***the open set*** $X \setminus \bar{B}_{m-1}$; let this *finite collection* of open sets be denoted $\mathscr{C}_m$. That is $\mathscr{C}_m = \{A_i \cap (X \setminus \bar{B}_{m-1}) : A_i \in \mathscr{A}, \bar{B}_m \subseteq \bigcup_i^k A_i, 1 \le i \le k\}$. Then the collection $\mathscr{C} = \bigcup_m \mathscr{C}_m$ is a *refinement* of $\mathscr{A}$.

  It is clearly *locally finite*, for the open set $B_m$ intersects only *finitely many elements* of $\mathscr{C}$, namely those elements belonging to the *collection* $\mathscr{C}_1 \cup \ldots \cup \mathscr{C}_m$. Finally, $\mathscr{C}$ covers $X$. For, given $x$, let $m$ be the *smallest integer* such that $x \in \bar{B}_m$. Then $x$ belongs to an element of $\mathscr{C}_m$, by definition. ∎

- **Example** ($k$***-Dimensional Topological Manifold***)
*Every $k$-dimensional topological manifold is* ***paracompact****.*

- **Theorem 1.3** *[Munkres, 2000]*
*Every* ***paracompact Hausdorff*** *space $X$ is* ***normal****.*

- **Proposition 1.4** *(**Paracompactness by Closed Subspace**) [Munkres, 2000]*
*Every* ***closed*** *subspace of a paracompact space is paracompact*

- **Remark** ***A paracompact subspace*** of a *Hausdorff space* $X$ ***need not be closed*** in $X$.

  Indeed, the open interval $(0, 1)$ is *paracompact*, being *homeomorphic* to $\mathbb{R}$, but it is *not closed* in $\mathbb{R}$.

- **Remark** ***The product of two paracompact spaces need not be paracompact***.

  The space $\mathbb{R}_\ell$ is *paracompact*, for it is *regular and Lindelöf*. However, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is *not paracompact*, for it is *Hausdorff* but ***not normal***.

- **Remark** ***A subspace of a paracompact space need not be paracompact***.

  The space $\bar{S}_\Omega \times \bar{S}_\Omega$ is *compact* and, therefore, ***paracompact***. But the *subspace* $S_\Omega \times \bar{S}_\Omega$ is ***not paracompact***, for it is *Hausdorff* but *not normal*.

- **Lemma 1.5** *[Munkres, 2000]*
*Let $X$ be* ***regular****. Then the following conditions on $X$ are* ***equivalent****: Every open covering of $X$ has a* ***refinement*** *that is:*

  1. *An* ***open*** *covering of $X$ and* ***countably locally finite****.*

  2. *A* ***covering*** *of $X$ and* ***locally finite****.*

  3. *A* ***closed*** *covering of $X$ and* ***locally finite****.*

  4. *An* ***open*** *covering of $X$ and* ***locally finite****.*

- **Remark** Given *regularity* (*$T_3$ axioms of separation*), "*open subcovering that is countably* *locally finite*" =“*open subcovering that is locally finite*”

- **Theorem 1.6** *[Munkres, 2000]*
  *Every **metrizable** space is paracompact.*

- **Proposition 1.7** *[Munkres, 2000]*
  *Every **regular Lindelöf space** is paracompact.*

- **Example** ($\mathbb{R}^\omega$ *with **Product** and **Uniform Topologies***)
  The space $\mathbb{R}^\omega$ is ***paracompact*** in both *the **product** and **uniform** topologies.* This result follows from the fact that $\mathbb{R}^\omega$ is ***metrizable*** in these topologies.

  It is *not known* whether $\mathbb{R}^\omega$ is *paracompact* in *the box topology.*

- **Example** ($\mathbb{R}^J$ *for **Uncountable Product is Not Paracompact***)
  For $\mathbb{R}^J$ is *Hausdorff* but ***not normal***.

### 1.1.3 Partition of Unity

- **Remark** *One of **the most useful properties*** that a *paracompact space* $X$ possesses has to do with *the **existence** of **partitions of unity*** on $X$.

## 1.2 Metrization Theorems

### 1.2.1 The Nagata-Smirnov Metrization Theorem

- **Theorem 1.8** ***(Nagata-Smirnov Metrization Theorem)**. [Munkres, 2000]*
  *A space $X$ is **metrizable** **if and only if** $X$ is **regular** and has a **basis** that is **countably locally finite**.*

### 1.2.2 The Smirnov Metrization Theorem

- **Definition** (***Locally Metrizable***)
  A space $X$ is ***locally metrizable*** if every point $x$ of $X$ has a ***neighborhood*** $U$ that is ***metrizable*** in *the subspace topology.*

- **Theorem 1.9** ***(Smirnov Metrization Theorem)**. [Munkres, 2000]*
  *A space $X$ is **metrizable** **if and only if** it is a **paracompact Hausdorff** space that is **locally metrizable**.*

- **Remark** (***Sufficient and Necessary Conditions for Metrization***)

$$\begin{tikzcd}
\textbf{\textit{metrizable}}\ \textit{space} \arrow[rr, bend left] \arrow[drr, leftrightarrow, "\textit{countable locally finite basis}"] \arrow[ddr, leftrightarrow, "\textit{locally metrizable}"'] & & \textbf{\textit{Hausdorff}}\ \textit{space} \\
& & \textbf{\textit{regular}}\ \textit{space} \arrow[u] \\
& \textbf{\textit{paracompact Hausdorff}}\ \textit{space} \arrow[r] & \textbf{\textit{normal}}\ \textit{space} \arrow[u] \\
& \textbf{\textit{compact Hausdorff}}\ \textit{space} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

- **Example** (***Locally Convex Space is Metrizable***)

  **Definition** (***Locally Convex Space***)
  A topological vector space $(X, \mathscr{T})$ is called ***locally convex space*** if its topology $\mathscr{T}$ is *the weakest topology* for which all ***semi-norms*** $\{q_\theta, \theta \in \Theta\}$ are *continuous*. $\mathscr{T}$ is generated by ***the convex basis*** $U_{\boldsymbol{x},r,\theta} = \{\boldsymbol{y} \in X \,|\, q_\theta(\boldsymbol{y} - \boldsymbol{x}) \le r\} \in \mathscr{B}, \boldsymbol{x} \in X, r > 0$.

  From the *Smirnov Metrization Theorem*, we see that ***the locally convex space is metrizable***.

# 2 Complete Metric Spaces and Function Spaces

## 2.1 Complete Metric Space

- **Definition** (***Cauchy Net in Topological Vector Space***)
  A *net* $\{x_\alpha\}_{\alpha \in I}$ in ***toplogocial vector space*** $X$ is called ***Cauchy*** if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta) \in I \times I}$ ***converges*** *to zero*. (Here $I \times I$ is ***directed*** in the usual way: $(\alpha, \beta) \prec (\alpha', \beta')$ if and only if $\alpha \prec \alpha'$ and $\beta \prec \beta'$.)

- **Definition** (***Completeness***)
  A toplogocial vector space $X$ is ***complete*** if every Cauchy net converges.

- **Proposition 2.1** *(**Complete First Countable Topological Vector Space**)*
  *If $X$ is a **first-countable topological vector space** and every **Cauchy sequence** in $X$ converges, then every **Cauchy net** in $X$ converges.*

- **Proposition 2.2** *(**Completeness of Euclidean Space**) [Munkres, 2000]*
  *Euclidean space $\mathbb{R}^k$ is **complete** in either of its usual **metrics**, the **euclidean metric** $d$ or the **square metric** $\rho$.*

- **Lemma 2.3** *(**Convergence in Product Space is Weak Convergence**) [Munkres, 2000]*
  *Let $X$ be the product space $X = \prod_\alpha X_\alpha$; let $x_n$ be a sequence of points of $X$. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each $\alpha$.*

- **Proposition 2.4** ***(Completeness of Countable Product Space)*** *[Munkres, 2000]*
  *There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is* ***complete****.*

- **Definition** (***Uniform Metric in Function Space***)
  Let $(Y, d)$ be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the ***standard bounded metric*** on $Y$ derived from $d$. If $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ are points of the cartesian product $Y^J$, let

  $$\bar{\rho}(x, y) = \sup \left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\}.$$

  It is easy to check that $\bar{\rho}$ is a metric; it is called ***the uniform metric*** on $Y^J$ corresponding to the metric $d$ on $Y$.

  Note that ***the space of all functions*** $f : J \to Y$, ***denoted*** as $Y^J$, is a subset of the product space $J \times Y$. We can define uniform metric in the function space: if $f$, $g : J \to Y$, then

  $$\bar{\rho}(f, g) = \sup \left\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\right\}.$$

- **Proposition 2.5** ***(Completeness of Function Space Under Uniform Metric)*** *[Munkres, 2000]*
  *If the space $Y$ is* ***complete*** *in the metric $d$, then the space $Y^J$ is* ***complete*** *in the* ***uniform metric*** *$\bar{\rho}$ corresponding to $d$.*

- **Definition** (***Space of Continuous Functions and Bounded Functions***)
  Let $Y^X$ be *the space of all functions* $f : X \to Y$, where $X$ is a *topological space* and $Y$ is a *metric space with metric $d$*. Denote the ***subspace*** of $Y^X$ consisting of all ***continuous functions*** $f$ as $\mathcal{C}(X, Y)$.

  Also denote *the set of all* ***bounded functions*** $f : X \to Y$ as $\mathcal{B}(X, Y)$. (A function $f$ is said to be ***bounded*** if its image $f(X)$ is a ***bounded subset*** of *the metric space* $(Y, d)$.)

- **Proposition 2.6** ***(Completeness of $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ Under Uniform Metric)*** *[Munkres, 2000]*
  *Let $X$ be a topological space and let $(Y, d)$ be a metric space. The set $\mathcal{C}(X, Y)$ of* ***continuous*** *functions is* ***closed*** *in $Y^X$ under the* ***uniform metric****. So is the set $\mathcal{B}(X, Y)$ of* ***bounded functions****. Therefore, if $Y$ is* ***complete****, these spaces are* ***complete*** *in the* ***uniform metric****.*

- **Definition** (***Sup Metric on Bounded Functions***)
  If $(Y, d)$ is a metric space, one can define another metric *on the set $\mathcal{B}(X, Y)$ of* ***bounded functions*** from $X$ to $Y$ by the equation

  $$\rho(x, y) = \sup \{d(f(x), g(x)) : x \in X\}.$$

  It is easy to see that $\rho$ is well-defined, for the set $f(X) \cup g(X)$ is ***bounded*** if both $f(X)$ and $g(X)$ are. The metric $\rho$ is called ***the sup metric***.

- **Theorem 2.7** ***(Existence of Completion)*** *[Munkres, 2000]*
  *Let $(X, d)$ be a metric space. There is an* ***isometric embedding*** *of $X$ into a* ***complete*** *metric space.*

- **Definition** (***Completion***)
  Let $X$ be a *metric space.* If $h : X \to Y$ is an ***isometric embedding*** of $X$ into a ***complete*** *metric space* $Y$, then the ***subspace*** $h(X)$ of $Y$ is a *complete metric space.* It is called ***the completion of $X$***.

- **Definition** (***Topological Complete***)
  A space $X$ is said to be ***topologically complete*** if there *exists* a metric for the *topology* of $X$ relative to which $X$ is *complete*.

- **Proposition 2.8** *(**Properties of Topological Complete**) [Munkres, 2000]*
  *The followings are properties of topological completeness:*

  1. *A **closed** subspace of a topologically complete space is topologically complete.*
  2. *A **countable product** of topologically complete spaces is topologically complete (in the **product topology**).*
  3. *An **open** subspace of a topologically complete space is topologically complete.*
  4. *A $G_\delta$ **set** in a topologically complete space is topologically complete.*

## 2.2 Compactness in Metric Spaces

### 2.2.1 Total Boundedness and Equicontinuous

- **Remark** (*Relate **Compactness** to **Completeness***)
  How is ***compactness*** of a metric space $X$ related to ***completeness*** of $X$?

  The followings is from *the sequential compactness* and definition of *completeness*:

  **Proposition 2.9** *Every **compact** metric space is **complete**.*

  The *converse* does not hold – ***a complete metric space need not be compact***. It is reasonable to ask what ***extra condition*** one needs to impose on a complete space to be assured of its compactness. Such a condition is the one called *total boundedness*.

- **Definition** (***Total Boundedness***)
  A metric space $(X, d)$ is said to be ***totally bounded*** if for every $\epsilon > 0$, there is a ***finite covering** of $X$ by $\epsilon$**-balls***.

- **Theorem 2.10** *(**Total Boundedness** + **Completeness** = **Compactness**)[Munkres, 2000]*
  *A metric space $(X, d)$ is **compact if and only if** it is **complete** and **totally bounded**.*

- **Remark** We now apply this result to find ***the compact subspaces*** of the space $\mathcal{C}(X, \mathbb{R}^n)$, *in the **uniform topology***. We know that a subspace of $\mathbb{R}^n$ is compact if and only if it is ***closed*** and ***bounded***.

  One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. ***But*** it does not, even if $X$ is *compact*. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies *an **additional condition***, called ***equicontinuity***.

- **Definition** (***Equicontinuity***) [Reed and Simon, 1980, Munkres, 2000]
  Let $(Y, d)$ be a *metric space*. Let $\mathscr{F}$ be a *subset* of the function space $\mathcal{C}(X, Y)$ (i.e. $f \in \mathscr{F}$ is continuous). If $x_0 \in X$, *the set $\mathscr{F}$ of functions* is said to be ***equicontinuous at** $x_0$* if given $\epsilon > 0$, there is a neighborhood $U$ of $x_0$ such that *for all $x \in U$* and ***all** $f \in \mathscr{F}$*,

  $$d(f(x), f(x_0)) < \epsilon.$$

  If the set $\mathscr{F}$ is *equicontinuous* at $x_0$ for each $x_0 \in X$, it is said simply to be ***equicontinuous*** or $\mathscr{F}$ is an ***equicontinuous family***.

We say $\mathscr{F}$ is a ***uniformly equicontinuous family*** if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ whenever $p(x, x') < \delta$ for all $x, x' \in X$ and ***every*** $f \in \mathscr{F}$.

- **Remark** An *equicontinuous family* of functions is *a family of continuous functions.*

- **Remark** ***Continuity*** of the function $f$ at $x_0$ means that ***given*** $f$ and given $\epsilon > 0$, there exists a neighborhood $U$ of $x_0$ such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$. ***Equicontinuity* of $\mathscr{F}$ means that *a single neighborhood* $U$ can be chosen that will work for all the functions** $f$ in the collection $\mathscr{F}$.

- **Lemma 2.11** *(**Total Boundedness $\Rightarrow$ Equicontinuous**) [Munkres, 2000]*
*Let $X$ be a **space**; let $(Y, d)$ be a **metric** space. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is **totally bounded** under the **uniform metric** corresponding to $d$, then $\mathscr{F}$ is **equicontinuous** under $d$.*

- **Lemma 2.12** *(**Equicontinuous + Compactness $\Rightarrow$ Total Boundedness**) [Munkres, 2000]*
*Let $X$ be a space; let $(Y, d)$ be a metric space; assume $X$ and $Y$ are **compact**. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is **equicontinuous** under $d$, then $\mathscr{F}$ is **totally bounded** under the **uniform** and **sup** metrics corresponding to $d$.*

- **Definition** (***Pointwise Bounded***)
If $(Y, d)$ is a *metric space*, a *subset* $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is said to be ***pointwise bounded*** under $d$ if for each $x \in X$, the subset

$$F_a = \{f(a) : f \in \mathscr{F}\}$$

of $Y$ is ***bounded*** under $d$.

- **Theorem 2.13** *(**Ascoli's Theorem, Classical Version**). [Munkres, 2000]*
*Let $X$ be a **compact** space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding **uniform topology**. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has **compact closure** **if and only if** $\mathscr{F}$ is **equicontinuous** and **pointwise bounded** under $d$.*

- **Corollary 2.14** *Let $X$ be **compact**; let $d$ denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding **uniform topology**. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is **compact** **if and only if** it is **closed, bounded** under the **sup metric** $\rho$, and **equicontinuous** under $d$.*

- **Remark** (***Ascoli's Theorem, Sequence Version***) [Reed and Simon, 1980]
*Let $\{f_n\}$ be a family of **uniformly bounded equicontinuous functions** on $[0, 1]$. Then **some subsequence** $\{f_{n,m}\}$ converges **uniformly** on $[0, 1]$.*

### 2.2.2 Pointwise and Compact Convergence

- **Definition** (***Topology of Pointwise Convergence / Point-Open Topology***)
Given a point $x$ of the set $X$ and an open set $U$ of the space $Y$, let

$$S(x, U) = \left\{f : f \in Y^X \text{ and } f(x) \in U\right\}.$$

The sets $S(x, U)$ are a ***subbasis*** for *topology* on $Y^X$, which is called ***the topology*** of ***pointwise convergence*** (or ***the point-open topology***)

- **Remark** (***Basis of Point-Open Topology***)
The general *basis element* for this topology is a *finite intersection* of subbasis elements $S(x, U)$. Thus a typical ***basis element*** about the function $f$ *consists of all functions $g$ that are **"close"** to $f$ **at finitely many points***.

- **Remark** ***The topology of pointwise convergence on*** $Y^X$ ***is the product topology***.

If we replace $X$ by $J$ and denote the general element of $J$ by $\alpha$ to make it look more familiar, then the set $S(\alpha, U)$ of all functions $x : J \to Y$ such that $x(\alpha) \in U$ is just the subset $\pi_\alpha^{-1}(U)$ of $Y^J$, which is *the standard subbasis element* for the product topology.

- **Proposition 2.15** *(**Pointwise Convergence Topology**)[Munkres, 2000]*
*A sequence $f_n$ of functions **converges** to the function $f$ in the **topology of pointwise convergence if and only if** for **each** $x$ in $X$, the sequence $f_n(x)$ of **points of** $Y$ converges to the point $f(x)$.*

- **Remark** Compare the *subbasis* of *the point-open topology* on function space $Y^X$ and *the weak topology* on space $X$

$$
\begin{aligned}
S(x, U) &= \left\{f : f \in Y^X \text{ and } f(x) \in U\right\} && \textit{point-open topology}. \\
B(f, U) &= \{x : x \in X \text{ and } f(x) \in U\} && \textit{weak topology}.
\end{aligned}
$$

- **Example** (**Pointwise Convergence Does Not Preserve Continuity**)
Consider the space $\mathbb{R}^I$, where $I = [0, 1]$. The sequence $(f_n)$ of continuous functions given by $f_n(x) = x^n$ *converges* in the ***topology of pointwise convergence*** to the function $f$ defined by

$$
f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases},
$$

This example shows that the subspace $\mathcal{C}(I, \mathbb{R})$ of continuous functions is ***not closed*** in $\mathbb{R}^I$ *in the topology of pointwise convergence.* Note that $\mathcal{C}(I, \mathbb{R})$ is ***closed*** *in* $\mathbb{R}^I$ *under* ***uniform topology*** due to *Uniform Limit theorem.*

- **Definition** (***Topology of Compact Convergence***)
Let $(Y, d)$ be a *metric space*; let $X$ be a *topological space*. Given an element $f$ of $Y^X$, a ***compact subspace*** $C$ of $X$, and a number $\epsilon > 0$, let $B_C(f, \epsilon)$ denote the set of all those elements $g$ of $Y^X$ for which

$$
\sup\{d(f(x), g(x)) : x \in C\} < \epsilon.
$$

The sets $B_C(f, \epsilon)$ form a ***basis*** for a topology on $Y^X$. It is called the ***topology of compact convergence*** (or sometimes the "***topology of uniform convergence on compact sets***").

- **Proposition 2.16** *(**Topology of Uniform Convergence in Compact Sets**) [Munkres, 2000]*
*A sequence $f_n : X \to Y$ of functions converges to the function $f$ in the **topology of compact convergence** if and only if for **each compact subspace** $C$ of $X$, the sequence $f_n|_C$ converges **uniformly** to $f|_C$.*

- **Definition** A space $X$ is said to be ***compactly generated*** if it satisfies the following condition: A set $A$ is ***open*** in $X$ if $A \cap C$ is ***open*** in $C$ for each **compact subspace** $C$ of $X$.

- **Lemma 2.17** *[Munkres, 2000]*
  *If $X$ is* ***locally compact****, or if $X$ satisfies* ***the first countability axiom****, then $X$ is* ***compactly generated****.*

- The crucial fact about compactly generated spaces is the following:

  **Lemma 2.18** *(**Continuous Extension on Compact Generated Space**) [Munkres, 2000]*
  *If $X$ is compactly generated, then a function $f : X \to Y$ is* ***continuous*** *if for each* ***compact subspace*** *$C$ of $X$, the restricted function $f|_C$ is* ***continuous****.*

- **Theorem 2.19** *($\mathcal{C}(X, Y)$ **on Compact Generated Space**) [Munkres, 2000]*
  *Let $X$ be a* ***compactly generated space****: let $(Y, d)$ be a metric space. Then $\mathcal{C}(X, Y)$ is* ***closed*** *in $Y^X$ in the* ***topology of compact convergence****.*

- **Remark** (***Useful Topologies on*** $Y^X$)
  1. ***Uniform Topology***: generated by the ***basis***
  $$B_U(f, \epsilon) = \left\{g \in Y^X \text{ and } \bar{\rho}(f, g) < \epsilon\right\}$$
  where $\bar{\rho}(f, g) = \sup\left\{\bar{d}(f(x), g(x)) : x \in X\right\}$ is *the uniform metric*. It corresponds to ***the uniform convergence*** of $f_n$ to $f$ in $Y^X$.
  2. ***Topology of Pointwise Convergence***: generated by the ***subbasis***
  $$S(x, U) = \left\{f : f \in Y^X \text{ and } f(x) \in U\right\}.$$
  It corresponds to ***the pointwise convergence*** of $f_n$ to $f$ in $Y^X$.
  3. ***Topology of Compact Convergence***: generated by the ***basis***
  $$B_C(f, \epsilon) = \left\{g \in Y^X \text{ and } \sup_{x \in C} d(f(x), g(x)) < \epsilon\right\}.$$
  It corresponds to ***the uniform convergence*** of $f_n$ to $f$ in $Y^X$ for $x \in C$.

- **Remark** Note that both *uniform topology* and *topology of compact convergence* rmade specific use of the metric $d$ for the space $Y$, i.e. it can only be defined when the image of function $Y$ is a metric space.

  But ***the topology of pointwise convergence*** does *not use the definition of metric $d$* in $Y$. In fact, ***it is defined for any image space*** $Y$.

- **Definition** (***Compact-Open Topology on Continuous Function Space***)
  Let $X$ and $Y$ be topological spaces. If $C$ is a ***compact subspace*** of $X$ and $U$ is an *open* subset of $Y$, define
  $$S(C, U) = \{f \in \mathcal{C}(X, Y) : f(C) \subseteq U\}.$$
  The sets $S(C, U)$ form a ***subbasis*** for a *topology* on $\mathcal{C}(X, Y)$ that is called ***the compact-open topology***.

- **Proposition 2.20** *(**Compact-Open on** $\mathcal{C}(X, Y)$ = **Compact Convergence**) [Munkres, 2000]*
  *Let $X$ be a space and let $(Y, d)$ be a metric space. On the set $\mathcal{C}(X, Y)$, the* ***compact-open topology*** *and the* ***topology of compact convergence coincide****.*

- **Corollary 2.21** ***(Compact Convergence on*** $\mathcal{C}(X,Y)$ ***Need Not*** $d$*) [Munkres, 2000]*
  *Let $Y$ be a metric space. The **compact convergence topology** on $\mathcal{C}(X,Y)$ does **not** depend on the **metric** of $Y$. Therefore if $X$ is **compact**, the **uniform topology** on $\mathcal{C}(X,Y)$ does not depend on the metric of $Y$.*

- **Remark** The fact that the definition of ***the compact-open topology*** does not involve a ***metric*** is just one of its useful features.

  Another is the fact that it satisfies the requirement of "***joint continuity***. Roughly speaking, this means that the expression $f(x)$ is *continuous* not only in the *single*"variable $x$, but is ***continuous jointly*** *in* ***both*** *the $x$ and $f$ .*

- **Theorem 2.22** ***(Compact-Open Topology*** $\Rightarrow$ ***Joint Continuity for*** $x$ ***and*** $f$*)*
  *Let $X$ be **locally compact Hausdorff**; let $\mathcal{C}(X,Y)$ have the **compact-open topology**. Then the map*

$$e : X \times \mathcal{C}(X,Y) \to Y$$

  *defined by the equation*

$$e(x,f) = f(x)$$

  *is **continuous**. The map $e$ is called **the evaluation map**.*

- **Definition** Given a function $f : X \times Z \to Y$, there is a corresponding function $F : Z \to \mathcal{C}(X,Y)$, defined by the equation

$$(F(z))(x) = f(x,z).$$

  Conversely, given $F : Z \to \mathcal{C}(X,Y)$, this equation defines a corresponding function $f : X \times Z \to Y$. We say that $F$ is the map of $Z$ into $\mathcal{C}(X,Y)$ that is induced by $f$.

- **Proposition 2.23** *Let $X$ and $Y$ be spaces; give $\mathcal{C}(X,Y)$ the **compact-open topology**. If $f : X \times Z \to Y$ is **continuous**, then **so is** the induced function $F : Z \to \mathcal{C}(X,Y)$. The **converse** holds if $X$ is **locally compact Hausdorff**.*

### 2.2.3 Ascoli's Theorem

- **Theorem 2.24** ***(Ascoli's Theorem, General Version)****. [Munkres, 2000]*
  *Let $X$ be a space and let $(Y,d)$ be a **metric** space. Give $\mathcal{C}(X,Y)$ the **topology of compact convergence**; let $\mathcal{F}$ be a subset of $\mathcal{C}(X,Y)$.*

  1. *If $\mathcal{F}$ is **equicontinuous** under $d$ and the set*

  $$F_a = \{f(a) : f \in \mathcal{F}\}$$

     *has **compact closure** for each $a \in X$, then $\mathcal{F}$ is **contained** in a **compact subspace** of $\mathcal{C}(X,Y)$.*

  2. *The **converse** holds if $X$ is **locally compact Hausdorff**.*

- **Remark** Compare with classical version, we see generalizations:

1. $X$ need not to be ***compact***; $\Rightarrow$ does not even need $X$ to be topological. $\Leftarrow$ holds when $X$ is ***locally compact Hausdorff***.

2. $\mathcal{C}(X, Y)$ is under ***compact-open topology*** which is ***weaker*** than ***uniform topology***, i.e. we does not require convergence of sequence *uniformly* but only *uniformly in a compact subset.*

3. $\mathcal{F}$ does not need to be ***pointwise bounded*** under $d$. In other word, the set

$$F_a = \{f(a) : f \in \mathcal{F}\}$$

need not to be ***bounded*** but need to have ***compact closure*** for each $a \in X$. Note that for metric space $Y$, if $Y$ is finite dimensional, it is the same requirement as boundness. But compact closure is stronger than bounded.

- **Proposition 2.25** *(**Equicontinuity** + **Pointwise Convergence** $\Rightarrow$ **Compact Convergence**) [Munkres, 2000]*
*Let $(Y, d)$ be a metric space; let $f_n : X \to Y$ be a sequence of **continuous** functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose $f_n$ converges to $f$ in the **topology of pointwise convergence**. If $\{f_n\}$ is **equicontinuous**, then $f$ is **continuous** and $f_n$ converges to $f$ in the **topology of compact convergence**.*

# 3 Baire Spaces

- **Remark** (***Empty Interior*** $=$ ***Complement is Dense***)
Recall that if $A$ is a subset of a space $X$, the ***interior*** of $A$ is defined as *the union of all open sets of $X$ that are contained in $A$.*

To say that $A$ has ***empty interior*** is to say then that $A$ ***contains no open set of*** $X$ *other than the empty set.* ***Equivalently***, $A$ has ***empty interior*** if every point of $A$ is *a* ***limit point*** *of the* ***complement*** *of* $A$, that is, if *the* ***complement*** *of $A$ is* ***dense*** *in $X$*.

$$\mathring{A} = \emptyset \quad \Leftrightarrow \quad A^c \text{ is dense in } X$$

In [Reed and Simon, 1980], if a subset $\overline{A}$ of $X$ has *empty interior*, $A$ is said to be ***nowhere dense*** in $X$.

- **Example** Some examples:

1. The set $\mathbb{Q}$ of *rationals* has ***empty interior*** as a subset of $\mathbb{R}$

2. The *interval* $[0, 1]$ has ***nonempty interior***.

3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Space***)
A space $X$ is said to be a ***Baire space*** if the following condition holds: Given ***any countable*** collection $\{A_n\}$ of ***closed*** sets of $X$ each of which has ***empty interior*** in $X$, their ***union*** $\bigcup_{n=1}^{\infty} A_n$ also has ***empty interior*** in $X$.

- **Example** Some examples:

1. The space $\mathbb{Q}$ of *rationals* is ***not** a **Baire space***. For each one-point set in $\mathbb{Q}$ is *closed and has empty interior* ***in*** $\mathbb{Q}$; and $\mathbb{Q}$ is *the countable union of its one-point subsets.*

2. The space $\mathbb{Z}_+$, on the other hand, does form *a **Baire space***. Every subset of $\mathbb{Z}_+$ is *open*, so that there exist *no subsets* of $\mathbb{Z}_+$ having *empty interior*, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.

3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Category***)
  A subset $A$ of a space $X$ was said to be of ***the first category in** $X$* if it ***was contained*** *in the* ***union*** *of a* ***countable*** *collection of* ***closed*** *sets of $X$ having* ***empty interiors*** *in $X$*; ***otherwise***, it was said to be of ***the second category in** $X$*.

- **Remark** *A space $X$ is a* ***Baire space*** *if and only if every* ***nonempty open*** *set in $X$ is of* ***the second category***.

- **Lemma 3.1** *(**Open Set Definition of Baire Space**) [Munkres, 2000]*
  *$X$ is a* ***Baire space if and only if*** *given any* ***countable*** *collection $\{U_n\}$ of* ***open*** *sets in $X$, each of which is* ***dense*** *in $X$, their* ***intersection*** *$\bigcap_{n=1}^{\infty} U_n$ is also* ***dense*** *in $X$.*

- **Theorem 3.2** *(**Baire Category Theorem**). [Munkres, 2000]*
  *If $X$ is a* ***compact Hausdorff*** *space or a* ***complete metric space****, then $X$ is a* ***Baire space***.

- **Remark** In other word, neither ***compact Hausdorff*** space or a ***complete metric space*** is a *countable union of closed subsets with empty interior (that are nowhere dense).*

- **Lemma 3.3** *[Munkres, 2000]*
  *Let $C_1 \supset C_2 \supset \ldots$ be a* ***nested*** *sequence of* ***nonempty closed sets*** *in the* ***complete metric space*** *$X$. If diam $C_n \to 0$, then $\bigcap_n C_n = \emptyset$.*

- **Lemma 3.4** *[Munkres, 2000]*
  *Any* ***open*** *subspace $Y$ of a Baire space $X$ is itself a Baire space.*

- **Theorem 3.5** *(**Discontinuity Point of Pointwise Convergence Function**) [Munkres, 2000]*
  *Let $X$ be a space; let $(Y, d)$ be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If $X$ is a* ***Baire space****, the set of points at which $f$ is* ***continuous*** *is* ***dense*** *in $X$.*

- **Remark** (***Use Baire Category Theorem as Proof by Contradition***)
  ***The Baire category theorem*** is used to prove a certain subset $C$ is ***dense*** in $X$ by stating that $X$ is a Baire space and $C$ is countable intersection of dense open subsets in $X$ (*$C$ is a $G_\delta$ sets*).

  On the other hand, if $M = \bigcup_{n=1}^{\infty} A_n$ has ***nonempty interior***, then ***some*** of the sets $\bar{A}_n$ ***must have nonempty interior***. Otherwise, it contradicts with the Baire space definition.

# 4 The Fundamental Group

## References

James R Munkres. *Topology, 2nd*. Prentice Hall, 2000.

Michael Reed and Barry Simon. *Methods of modern mathematical physics: Functional analysis*, volume 1. Gulf Professional Publishing, 1980.