Lecture 2: Introduction to Markov Chain

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Contents

1 Markov Chain

1.1 Basic Concepts

- *Markov Chain* $(X_t)_t$ is a *probabilistic graphical model* over a chain graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_C)$, where each random variable X_t only has exactly one children X_{t+1} and one parent X_{t-1}. Denote the index of variable t as the time. Markov chain $(X_t)_t$ is also a *stochastic process*.

- By Markov property,

$$P(X_{t+1}|X_t, X_{t-1}, \ldots, X_1) = P(X_{t+1}|X_t).$$

It is seen that the transition probability does not depend on the time t, i.e. Markov chain is **time-invariant**.

- We can see that the joint distribution on $\boldsymbol{X}_{0:t} = [X_0, \ldots, X_t]$ can be factorized by transition probabilities

$$P(\boldsymbol{X}_{1:t}) = P(X_0) \prod_{s=1}^{t} P(X_s|X_{s-1})$$

by Markov property. Denote $\pi_0(i) := P(X_0 = i)$ as the *initial probability*.

- Define the *transition kernel* of Markov Chain as the *time-invariant* *transition probability*

$$K(x, y) = p(x, y) := P(X_{t+1} = y|X_t = x). \tag{1}$$

Then the m-*step transition probability* is defined as

$$K^m(x, y) = P(X_{t+m} = y|X_t = x). \tag{2}$$

- In *general setting* [Robert and Casella, 1999], $K : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to \mathbb{R}$ is a function so that $K(x, \cdot)$ is a *probability measure* for all $x \in \mathcal{X}$ and $K(\cdot, A)$ is a *measurable function* for all $A \in \mathcal{B}(\mathcal{X})$. K can also be considered as a *functional* such that

$$K\left(h(x)\right) = \int h(y)K(x, dy), \quad h \in L_1(\lambda).$$

where λ is the dominated measure.

- For $X_t \in \mathcal{X} := \{1, \ldots, n\}$ as discrete random variable with $|\mathcal{X}| = n$, we can define the *transition matrix*

$$\boldsymbol{K} = [K(i, j)]_{n \times n} \tag{3}$$

- We can see that $\boldsymbol{X}_{(t+1):(t+m-1)} := [X_{t+1}, \ldots, X_{t+m-1}]$, the m-step transition can be computed using transition kernel

$$\begin{aligned}
P(X_{t+m}|X_t) &= \sum_{\boldsymbol{X}_{(t+1):(t+m-1)}} P(X_{t+m}|\boldsymbol{X}_{(t+1):(t+m-1)}, X_t)P(\boldsymbol{X}_{(t+1):(t+m-1)}) \\
&= \sum_{\boldsymbol{X}_{(t+1):(t+m-1)}} P(X_{t+m}|X_{t+m-1})P(\boldsymbol{X}_{(t+1):(t+m-1)})
\end{aligned}$$

2

$$= \dots$$

$$= \sum_{\boldsymbol{X}_{(t+1):(t+m-1)}} \prod_{i=0}^{m-1} P(X_{t+i+1}|X_{t+i}) \tag{4}$$

$$\Rightarrow [K^m(i,j)] = [K(l,m)]^m = \boldsymbol{K}^m \tag{5}$$

- We have the **Chapman-Kolmogorov equation** [Ross, 2014]:

$$K^{m+n}(x,y) = \sum_z K^m(x,z)K^n(z,y), \quad \forall x, y \in \mathcal{X} \tag{6}$$

$$\Rightarrow \boldsymbol{K}^{m+n} = \boldsymbol{K}^m \boldsymbol{K}^n$$

That is, we split the $(m+n)$-step path from $x \to y$ into all possible combination of a m-step path from $x \to z$ and a n-step path from $z \to y$ for some intermediate state z.

- And the marginal distribution on state X_t can be computed as

$$P(X_t) = \sum_{X_{t-1} \in \mathcal{X}} P(X_t|X_{t-1})P(X_{t-1})$$

$$\Rightarrow \boldsymbol{\pi}_t = \boldsymbol{K}\boldsymbol{\pi}_{t-1} \tag{7}$$

where $\boldsymbol{\pi}_t := [P(X_t = i)]$

1.2 Hitting time

- **Definition** Define $T_j = \min\{t \geq 1 : X_t = j\}$ as the time steps for Markov Chain $(X_t)_t$ to *hit* state j for the **first time**. T_j is called the state j's **first hitting time**.

  Denote $f_{i,j}$ be the **probability of ever hitting state j (within finite time) starting from state** i. That is

$$f_{i,j} := P(T_j < \infty|X_0 = i) \tag{8}$$

  Denote $f_{i,j}^{(m)}$ be the *probability of hitting at state j at time m* starting from state i

$$f_{i,j}^{(m)} := P(T_j = m|X_0 = i) \tag{9}$$

  We can generalize the hitting time for a set of states $T_A := \min\{t \geq 1 : X_t \in A\}$.

- We can the following the **kernel recursion formula**

$$K^m(x,y) = \sum_{n=1}^{m} P(T_y = n|X_0 = x)K^{m-n}(y,y) \tag{10}$$

$$:= \sum_{n=1}^{m} f_{x,y}^{(n)} K^{m-n}(y,y).$$

That is, we *categorize* all possible m-step path from $x \to y$ according to the first time the path visiting y. (This is called the **First-Step analysis**)

3

- Similarly, we have the **hitting time recursion formula**:

$$f_{x,y}^{(m)} := P(T_y = m | X_0 = x) = \sum_{z \neq i} P(T_y = m - 1 | X_0 = z) \, K(x, z). \tag{11}$$

$$:= \sum_{z \neq i} f_{z,y}^{(m-1)} \, K(x, z).$$

This formula break down the m-step path from $x \to y$ into two parts: a path from $x \to z$ and a $(m-1)$-step path from intermediate state $z \to y$ (This is also the *First-Step analysis*).

- Define $N(y) := \sum_{t=0}^{\infty} \mathbb{1} \{X_t = y\}$ is the **total number of times hitting the state** y.

$$P(N(y) \geq 1 | X_0 = x) = P(T_y < \infty | X_0 = x) = f_{x,y} \tag{12}$$

$$P(N(y) \geq m | X_0 = x) = P(N(y) \geq 1 | X_0 = x) P^{m-1}(N(y) \geq 1 | X_0 = y) \tag{13}$$

$$= f_{x,y} \, f_{y,y}^{m-1}$$

Note that in order to visit y at least m times, we need to visit y first time and stating from y recurrently visit y $(m-1)$ times.

The random variable $N(x) | X_0 = x$ follows a **geometric distribution** with mean $1/(1 - f_{x,x})$.

$$P(N(x) = m | X_0 = x) = (1 - f_{x,x}) f_{x,x}^{m-1} \tag{14}$$

- Define $G(x, y) := \mathbb{E}\left[N(y) | X_0 = x\right]$ as the **expected number of total visits** of y starting from x.

$$G(x, y) := \mathbb{E}\left[N(y) | X_0 = x\right]$$

$$= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbb{1} \{X_t = y\} | X_0 = x\right]$$

$$= \sum_{t=0}^{\infty} \mathbb{E}\left[\mathbb{1} \{X_t = y\} | X_0 = x\right] = \sum_{t=0}^{\infty} K^t(x, y) \tag{15}$$

Note that $\mathbb{E}\left[\sum_{t=0}^{\infty} \mathbb{1} \{X_t = y\} | X_0 = x\right] = \sum_{t=0}^{\infty} \mathbb{E}\left[\mathbb{1} \{X_t = y\} | X_0 = x\right]$ is true since $Z_t := \mathbb{1} \{X_t = y\}$ is non-negative random variable.

Since $N(y)$ is geometric distributed, we can compute $G(x, y)$ via

$$G(x, y) = \frac{f_{x,y}}{1 - f_{y,y}} \tag{16}$$

- Define $G(x, x) = \mathbb{E}\left[N(x) | X_0 = x\right]$ as the *expected number of **total returns*** starting from state x.

2 Classification of States

2.1 Equivalence class by communication

- **Definition** For any pair $x, y \in \mathcal{X}$, if there exists $n \in \mathbb{N}_+$ so that $K^n(x, y) > 0$, then the state y is **accessible** from state x. This is equivalent to say that the probability of hitting time of y being finite starting from x is above zero, i.e. $\underline{f_{x,y} > 0}$.

4

- If x is accessible from y, and y is accessible from x, then we say that x and y ***coummunicate***, $x \leftrightarrow y$. It is easy to check that this is an ***equivalence relation***:

    1. $x \leftrightarrow x$;

    2. If $x \leftrightarrow y$, then $y \leftrightarrow x$;

    3. If $x \leftrightarrow z$ and $z \leftrightarrow y$, then $x \leftrightarrow y$

- Thus we can partition the state space $\mathcal{X}$ into several ***equivalence classes*** $\mathcal{X} = \bigcup_k \mathcal{X}^k$ and within each class, all states communicate to each other.

    Equivalently, it means that the kernel $\boldsymbol{K}$ can be *rearranged* into a ***block-diagonal matrix***.

- **Definition** A Markov Chain is ***irreducible*** if it has **only one equivalence class**, i.e. all states in $\mathcal{X}$ communicate to each other.

- Based on hitting time, we can categorize states into two groups:

    - **Definition** A state i is ***recurrent*** if and only if $f_{i,i} = P(T_i < \infty | X_0 = i) = 1$, i.e. the Markov Chain will definitely revisit the state i after stating from i.

      Note that it follows from (15) that

      Proposition 2.1 *(Characterization of recurrence via n-**step return probabilities**) A state i is recurrent if and only if $\sum_{t=0}^{\infty} K^t(i,i) = \infty$.*

    - **Definition** A recurrent state i is ***positive recurrent*** if the *expected returning time* $\mathbb{E}[T_i | X_0 = i] < \infty$; otherwise we say it is ***null recurrent***.

    - **Definition** A state i is called ***transient*** if $f_{i,i} < 1$.

- **Proposition 2.2** *The following conditions are* **equivalent***:*

    1. *state i is recurrent state;*

    2. *The ever returning probability $f_{i,i} = 1$;*

    3. *The probability of total number of visiting is $P(N(i) = \infty | X_0 = j) = f_{j,i}$ and $P(N(i) = \infty | X_0 = i) = 1$;*

    4. *The expected total number of returning is infinite $G(i,i) = \infty$;*

    5. *The sum of all n-step return probabilities $\sum_{t=0}^{\infty} K^t(i,i) = \infty$.*

- **Proposition 2.3** *If i is recurrent, and $i \to j$, then also $j \to i$.*

- **Proposition 2.4** *If i is positive recurrent, and $i \leftrightarrow j$, then j is also positive recurrent.*

- **Proposition 2.5** *If i is recurrent, and $i \to j$, then j is also recurrent. Therefore, in any* ***equivalent class***, ***either all states are recurrent*** *or* ***all are transient***. *In particular, if the chain is* **irreducible***, then either all states are recurrent or all are transient.*

    Based above proposition, we can classify ***each class***, and ***an irreducible Markov Chain*** *as recurrent or transient.*

- **Proposition 2.6** *If a closed subset $S_0 \subset \mathcal{X}$ only has finitely many states, then there must be at least one recurrent state. In particular any finite Markov chain must contain at least one*

positive recurrent state.

Proposition 2.7 *An **irreducible finite state Markov chain** must be **positive recurrent**.*

- **Proposition 2.8** *Any recurrent class is a **closed** subset of states.*

- Let S_T be a set of **transient** states and C be a closed set of **irreducible**, **recurrent** state, the **absorption probability** is defined as

$$p_C(x) = P(T_C < \infty | X_0 = x), \quad \forall x \in S_T. \tag{17}$$

It is the probability of hitting recurrent state set starting from a transient state.

- **Theorem 2.9** *Suppose S_T is a set of **transient** states and C is a closed irreducible set of **recurrent** state, then the following system of equations has **unique** solution,*

$$f(x) = \sum_{y \in C} K(x, y) + \sum_{y \in S_T} K(x, y) f(y), \quad \forall x \in S_T \tag{18}$$

and the unique solution is the absorption probability $f(x) = p_C(x)$.

- The recurrence definition ("with *infinite number of visits*") can be generalized as the **Harris recurrence** in general theory [Robert and Casella, 1999].

Definition A set A is **Harris recurrent** if $P(N_A = \infty | X_0 = x) = 1$ for all $x \in A$, where $N_A := \sum_{t=0}^{\infty} \mathbb{1}\{X_t \in A\}$. The chain $(X_t)_t$ is **Harris recurrent** if there exists a measure p such that $(X_t)_t$ is *p-irreducible* and for every set A with $p(A) > 0$, A is *Harris recurrent*.

2.2 Foster's theorem and Poke's lemma

- **Theorem 2.10** *(**Foster's theorem**)*
  Consider an irreducible Markov chain $(X_t)_t$ with state space $\mathcal{X} = \{0, 1, \dots\}$ and transition matrix $\boldsymbol{K}$ and suppose there exists a function $h : \mathcal{X} \to \mathbb{R}$ such that

  (1) $\inf_{x \in \mathcal{X}} h(x) > -\infty$

  (2) $\sum_{y \in \mathcal{X}} K(x, y) h(y) < \infty \quad \forall x \in \mathcal{S}$

  (3) $\sum_{y \in \mathcal{X}} K(x, y) h(y) < h(x) - \epsilon \quad \forall x \notin \mathcal{S}$

  *for some finite set $\mathcal{S} \subset \mathcal{X}$ and some $\epsilon > 0$, then the Markov chain $(X_t)_t$ is **positive recurrent**.*

- **Lemma 2.11** *(**Poke's lemma**)*
  *Consider an irreducible Markov chain $(X_t)_t$ with state space $\mathcal{X} = \{0, 1, \dots\}$ and transition matrix $\boldsymbol{K}$. Assume that for all $x \in \mathcal{X}$ and all $t \geq 0$, $\mathbb{E}[X_{t+1} | X_t = x] < \infty$ and $\lim_{i \to \infty} \sup_{j \geq i} \mathbb{E}[X_{t+1} - X_t | X_t = j] < 0$. Then the Markov chain $(X_t)_t$ is **positive recurrent**.*

3 Limiting and stationary distribution

3.1 Property of limiting distributions

- **Definition** The probability of states $\{\pi(x), \forall x \in \mathcal{X}\}$ is a **_stationary distribution_** if and only if

$$\pi(y) = \sum_{x \in X} K(x, y)\pi(x), \forall y \in \mathcal{X} \tag{19}$$

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{K} \tag{20}$$

That is, $\boldsymbol{\pi}$ is the eigenvector of stochastic matrix $\boldsymbol{K}$ corresponding to eigenvalue $\lambda_0 = 1$.

- A stationary distribution is also called an **_invariant distribution_** [Robert and Casella, 1999, Liu, 2001], **_steady-state distribution_** [Ross, 2014] or **_equilibrium distribution_** [Brooks et al., 2011, Ross, 2014]. This is due to its _time invariant property_ or _the global balance equation_ (23).

- Let the initial distribution be the stationary distribution $P(X_0 = x) = \pi(x)$. Note that

$$\pi_1(y) = P(X_1 = y) = \sum_x K(x, y)\pi(x) = \pi(y), \forall y \in \mathcal{X}. \tag{21}$$

In other word, **_the stationary distribution does not change over time_**.

In measure theory, the invariant measure π satisfies:

$$\pi(B) = \int_{\mathcal{X}} K(x, B)\pi(dx), \quad \forall B \in \mathcal{B}(\mathcal{X}).$$

- **Proposition 3.1** _Suppose that the **limiting distribution** $\lim_{t \to \infty} P(X_t = y)$ exists, and_

$$\lim_{t \to \infty} K^t(x, y) = \pi(y), \quad \forall x, y \in \mathcal{X}$$

which is independent of where it starts from, then the Markov Chain has a **unique stationary distribution** and

$$\lim_{t \to \infty} P(X_t = y) = \pi(y), \quad \forall y \in \mathcal{X} \tag{22}$$

i.e. the limit distribution is stationary distribution.

Note that $P(X_t = y) = \sum_{x \in \mathcal{X}} K^t(x, y)\pi_0(x)$.

- **Proposition 3.2** _(**Global Balance Equation**)_
The stationary distribution $\{\pi(x), \forall x \in \mathcal{X}\}$ satisfies the following **global balance equation**:

$$\sum_{j \in \mathcal{X}} \pi(i)K(i, j) = \sum_{j \in \mathcal{X}} \pi(j)K(j, i). \tag{23}$$

This means the total flow out of i (LHS) is equal to the total flow into i (RHS) in steady state.

- **Proposition 3.3** *(Detailed Balance Equation)*
  *For distribution $\{\pi(x), \forall x \in \mathcal{X}\}$, if the following **detailed balance equation** is satisfied*

$$\pi(i)K(i,j) = \pi(j)K(j,i), \quad \forall i, j \in \mathcal{X} \tag{24}$$

  then $\{\pi(x), \forall x \in \mathcal{X}\}$ is a stationary distribution.

- **Definition** Define $\mu_i := \mathbb{E}\left[T_i | X_0 = i\right]$ as the ***expected first return time***, i.e. the number of transition that it takes for Markov chain when starting from state i and returning to that state.

- Let $G^{(n)}(x, y) = \mathbb{E}\left[N^{(n)}(y) | X_0 = x\right]$ where $N^{(n)}(y) = \sum_{t=0}^{n} \mathbb{1}\{X_t = y\}$. $N^{(n)}(y)$ is the total amount of time staying at state y within n transitions. Then

  - **Theorem 3.4** *For **transient state** y*

$$\lim_{n \to \infty} N^{(n)} < \infty, \quad (w.p.1)$$

$$\Rightarrow \lim_{n \to \infty} \frac{N^{(n)}}{n} = 0, \quad (w.p.1)$$

$$\lim_{n \to \infty} G^{(n)}(x, y) = \frac{f_{x,y}}{1 - f_{y,y}} < \infty$$

$$\Rightarrow \lim_{n \to \infty} \frac{G^{(n)}(x, y)}{n} = 0, \quad \forall x \in \mathcal{X}$$

    That is, the frequency of visiting transient state y goes to 0 as $n \to \infty$.

  - **Theorem 3.5** *For **recurrent state** y*

$$\lim_{n \to \infty} \frac{N^{(n)}}{n} = \frac{\mathbb{1}\{T_y < \infty\}}{\mu_y}, \quad (w.p.1)$$

$$\lim_{n \to \infty} \frac{G^{(n)}(x, y)}{n} = \frac{f_{x,y}}{\mu_y}, \quad \forall x \in \mathcal{X}$$

    *where $\mu_y := \mathbb{E}\left[T_y | X_0 = y\right]$ is the **expected first return time** of state y. That is, the frequency of visiting **positive recurrent** state y converge to $\frac{1}{\mu_y}$ as $n \to \infty$; otherwise for **null recurrent state** y, it converges to zero.*

- **Theorem 3.6** *(Stationary distribution for transient and null recurrent states)*
  *Let $\{\pi(x), \forall x \in \mathcal{X}\}$ be stationary distribution. If $x \in \mathcal{X}$ is **transient** or **null recurrent** state, then*

$$\pi(x) = 0.$$

- **Theorem 3.7** *(**Kac's Theorem**)[Ross, 2014]*
  *An **irreducible recurrent** Markov Chain has a **unique stationary distribution** $\{\pi(x)\}$, given*

$$\pi(x) = \frac{1}{\mu_x}, \quad \forall x \in \mathcal{X} \tag{25}$$

  *where $\mu_x := \mathbb{E}\left[T_x | X_0 = x\right]$ is the **expected first return time** of state x.*

  It implies that as $n \to \infty$, for any state $x \in \mathcal{X}$, the fraction of time that Markov Chain stays at x is unchanged and is the reciprocal of the expected first return time.

8

3.2 Ergodicity

- Under what condition we have $\forall y \in \mathcal{X}$,

$$\lim_{t \to \infty} P(X_t = y) = \pi(y)? \tag{26}$$

This is the question that ergodicity property tries to answer.

- **Definition** The ***periodicity*** of a state $x \in \mathcal{X}$ is defined as

$$d(x) = \text{g.c.d.} \left\{ t \geq 0 : K^t(x, x) > 0 \right\} \tag{27}$$

where g.c.d. is the **greatest common divisor**.

- **Theorem 3.8** *If $x \leftrightarrow y$ (i.e. $f_{x,y} > 0$ and $f_{y,x} > 0$), then $d(x) = d(y)$.*

- **Definition** If $d(x) \geq 2$, then state x is **periodic**. If $d(x) = 1$, then state x is ***aperiodic***

Based on above theorem, the periodicity property is *closed* under the equivalence class C.

- **Definition** A Markov Chain is ***irreducible***, ***positive recurrent*** and ***aperiodic***, then it is called ***ergodic***.

- **Theorem 3.9** *A Markov Chain is **irreducible and positive recurrent** having stationary distribution π.*

  - *If the Markov Chain is also **aperiodic**, then*

  $$\lim_{t \to \infty} K^t(x, y) = \pi(y), \quad \forall x, y \in \mathcal{X} \tag{28}$$

  - *If the Markov chain is **periodic** with period d, then there exists $r \in \mathbb{N}_+$, $0 \leq r \leq d - 1$ such that*

  $$K^t(x, y) = 0, \quad \forall x, y \in \mathcal{X} \tag{29}$$

  unless *$t = m\,d + r$ for some $m \in \mathbb{N}_+$ and*

  $$\lim_{m \to \infty} K^{m\,d+r}(x, y) = d\,\pi(y), \quad \forall x, y \in \mathcal{X} \tag{30}$$

  Note that periodicity only appears on discrete time Markov chain.

Based on the Theorem 3.9 and Proposition 3.1, when a Markov chain is ergodic, its marginal state distribution will converge to the stationary distribution.

3.3 Mean hitting time formula

- **Definition** Let $(X_t)_t$ be a stochastic process and let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of σ-field.

A random variable $T : (\Omega, \mathcal{F}) \to (\mathbb{N}_+ \cup \{+\infty\}, 2^{\mathbb{N}_+ \cup \{+\infty\}})$ is called a ***stopping time*** with respect to $\{\mathcal{F}_t, t \geq 0\}$, if $\forall k \geq 0$, $\mathbb{1}\{T = k\}$ is $\mathcal{F}_t$-**measurable** (i.e. $\{T = k\} \in \mathcal{F}_t$ and $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$.)

For Markov chain $(X_t)_t$, the ***first hitting time*** is defined as $T_A^{(1)} = \min\{t > 0 : X_t \in A\}$. Also we can define n-th hitting time as $T_A^{(n)} = \min\left\{t > T_A^{(n-1)} : X_t \in A\right\}$. All of these $\left\{T_A^{(1)}, \ldots, T_A^{(n)}, \ldots\right\}$ are all ***stopping time***.

- **Theorem 3.10** (***Strong Markov property***) *[Robert and Casella, 1999]*
  For every initial distribution $\boldsymbol{\pi}$ and every stopping time τ which is almost surely finite,

$$\mathbb{E}\left[h(X_{\tau+1}, X_{\tau+2}, \ldots) \mid x_1 \ldots, x_\tau\right] = \mathbb{E}\left[h(X_1, X_2, \ldots)\right], \tag{31}$$

  provided the expectations exist.

  We can thus condition on a random number of instants while keeping the fundamental properties of a Markov chain. We can proof that for the intervals $\tau_1 = T_x^{(1)}$, $\tau_i := \tau_x^{(i)} - \tau_x^{(i-1)}$, $i = 2, \ldots,$, then $\{\tau_1, \ldots, \tau_n, \ldots\}$ are i.i.d.

- **Theorem 3.11** *Let $(X_t)_t$ be a **positive recurrent** Markov chain with state space $\mathcal{X}$ and stationary distribution $\boldsymbol{\pi}$. Let T be any **stopping time** of $(X_t)_t$ such that for arbitrary $x \in \mathcal{X}$, $X_T = x$. Then for all $y \in \mathcal{X}$,*

$$\mathbb{E}_T\left[\sum_{t=0}^{T-1} K^t(x, y) \mid x\right] = \mathbb{E}\left[\sum_{t=0}^{T-1} \mathbb{1}\{X_t = y\} \mid X_0 = x\right] = \pi(y)\, \mathbb{E}\left[T | X_0 = x\right]. \tag{32}$$

- **Theorem 3.12** *Let $i \neq j$ and T be the first time returning i after visiting j, $T = \min\{t > \tau_j, X_t = i | X_0 = i\}$, $\tau_j = \min\{t > 0 : X_t = j\}$ and $\tau_i = \min\{t > 0 : X_t = i\}$. Then*

  (1) $\mathbb{E}\left[T | X_0 = i\right] = \mathbb{E}\left[\tau_i | X_0 = i\right] + \mathbb{E}\left[\tau_j | X_0 = i\right]$

  (2) $\mathbb{E}\left[\sum_{t=0}^{T-1} \mathbb{1}\{X_t = j\} \mid X_0 = i\right] = \mathbb{E}\left[\sum_{t=\tau_j}^{T-1} \mathbb{1}\{X_t = j\} \mid X_0 = i\right] + \mathbb{E}\left[\sum_{t=0}^{\tau_i-1} \mathbb{1}\{X_t = j\} \mid X_0 = j\right]$

  (3) $\mathbb{E}\left[\sum_{t=0}^{\tau_i-1} \mathbb{1}\{X_t = j\} \mid X_0 = j\right] = \pi(j)\left(\mathbb{E}\left[\tau_j \mid X_0 = i\right] + \mathbb{E}\left[\tau_i \mid X_0 = j\right]\right)$

  The "*number of visits to j before first returning to i*" is geometric distributed with mean $p := P(\tau_j > \tau_i | X_0 = i)$, thus (3) can be computed as

$$\mathbb{E}\left[\sum_{t=0}^{\tau_i-1} \mathbb{1}\{X_t = j\} \mid X_0 = j\right] = \frac{1}{P(\tau_j > \tau_i | X_0 = i)}.$$

- **Definition** For finite state, ergodic Markov chain $(X_t)_t$ with stationary distribution $\boldsymbol{\pi}$, define the ***fundamental matrix*** as

$$\boldsymbol{Z} := \left(\boldsymbol{I} - \left(\boldsymbol{K} - \boldsymbol{1}\boldsymbol{\pi}^T\right)\right)^{-1} \tag{33}$$

$$= \boldsymbol{I} + \sum_{t \geq 0} \left(\boldsymbol{K}^t - \boldsymbol{1}\boldsymbol{\pi}^T\right)$$

  and its (i, j) element is

$$Z_{i,j} = \sum_{t=0}^{\infty} (K^t(i, j) - \pi_j) \tag{34}$$

  Note that $\boldsymbol{Z} = (\boldsymbol{I} - \boldsymbol{Q})^{-1} = \sum_{t=0}^{\infty} \boldsymbol{Q}^t$, where $\boldsymbol{Q} := \left(\boldsymbol{K} - \boldsymbol{1}\boldsymbol{\pi}^T\right)$.

- **Theorem 3.13** *(Mean hitting time formula)*
  For finite state, ergodic Markov chain $(X_t)_t$ with stationary distribution $\boldsymbol{\pi}$, and $Z_{i,j}$ is defined as (34), then

$$Z_{i,i} = \pi(i)\, \mathbb{E}\left[\tau_i | X_0 \sim \boldsymbol{\pi}\right], \quad i \in \mathcal{X}, \tag{35}$$

$$Z_{j,j} - Z_{i,j} = \pi(j)\, \mathbb{E}\left[\tau_j | X_0 = i\right], \quad i, j \in \mathcal{X}, \tag{36}$$

  where $\tau_i := \min\left\{t \geq 0 : X_t = i\right\}$ is the stopping time/first hitting time. Thus

$$\mathbb{E}\left[\tau_i | X_0 \sim \boldsymbol{\pi}\right] = \frac{Z_{i,i}}{\pi(i)}$$

$$\mathbb{E}\left[\tau_j | X_0 = i\right] = \frac{(Z_{j,j} - Z_{i,j})}{\pi(j)}$$

4 Time-reversible Markov Chain

- **Definition** A Markov chain $(X_t)_t$ is called **_time-reversible_**, if it has stationary distribution $\boldsymbol{\pi}$ and the detailed balance equation is satisfied:

$$\pi(i)K(i,j) = \pi(j)K(j,i), \quad \forall i, j \in \mathcal{X}. \tag{37}$$

- From this definition, we can see that reversibility implies that the stationary distribution exists, but not *vice versa*.

- The reversed process $(Y_k)_k := (X_{t-k})_k$ is a Markov chain and its transition probability

$$Q(i,j) = \frac{\pi(j)K(j,i)}{\pi(i)} \tag{38}$$

  Note that $(Y_k)_k$ and $(X_t)_t$ are statistically equivalent since $Q(i,j) = K(i,j)$.

- **Theorem 4.1** *An ergodic Markov chain $(X_t)_t$ for which $K(i,j) = 0$ whenever $K(j,i) = 0$ is **time-reversible** if and only if starting from any state i, any path back to i has the **same probability** as its reverse path. That is, for path $i \to i_1 \to i_2 \to \ldots \to i_k \to i$ and its reverse path $i \leftarrow i_1 \leftarrow i_2 \leftarrow \ldots \leftarrow i_k \leftarrow i$*

$$K(i,i_1)\,K(i_1,i_2)\,\ldots\,K(i_k,i) = K(i,i_k)\,\ldots\,K(i_2,i_1)\,K(i_1,i), \quad \forall i, i_1, \ldots, i_k \in \mathcal{X} \tag{39}$$

- **Theorem 4.2** *(Reversal Test)*
  Let $\boldsymbol{K}$ be a stochastic matrix indexed by a countable set $\mathcal{X}$ and let $\boldsymbol{\pi}$ be a probability distribution on $\mathcal{X}$. Let $\boldsymbol{Q}$ be a stochastic matrix indexed by $\mathcal{X}$ such that

$$\pi(i)Q(i,j) = \pi(j)K(j,i), \quad \forall i, j \in \mathcal{X}. \tag{40}$$

  Then $\boldsymbol{\pi}$ is a stationary distribution of $\boldsymbol{K}$

- **Proposition 4.3** *For finite state, ergodic Markov chain $(X_t)_t$ with stationary distribution $\boldsymbol{\pi}$, and $Z_{i,j}$ is defined as (34), then $(X_t)_t$ is time-reversible if and only if*

$$\pi(i)Z_{i,j} = \pi(j)Z_{j,i}, \quad \forall i, j \in \mathcal{X}. \tag{41}$$

  Note that $\pi(i)\mathbb{E}\left[\tau_j | X_0 = i\right] \neq \pi(j)\mathbb{E}\left[\tau_i | X_0 = j\right]$.

- **Theorem 4.4** *(Cycle-tour property)*
  For states $(i_0, i_1, \ldots, i_m) \subset \mathcal{X}$ of a time-reversible Markov chain,

$$\mathbb{E}\left[\tau_{i_1}|X_0 = i_0\right] + \mathbb{E}\left[\tau_{i_2}|X_0 = i_1\right] + \ldots + \mathbb{E}\left[\tau_{i_0}|X_0 = i_m\right]$$
$$= \mathbb{E}\left[\tau_{i_m}|X_0 = i_0\right] + \mathbb{E}\left[\tau_{i_{m-1}}|X_0 = i_m\right] + \ldots + \mathbb{E}\left[\tau_{i_0}|X_0 = i_1\right] \tag{42}$$

5 Ergodic Theorem and Central Limit Theorem

- Consider the empirical mean of samples generated by Markov Chain

$$S_T(h) = \frac{1}{T}\sum_{t=1}^{T} h(X_t). \tag{43}$$

  We are considering the limit behavior of (43).

- **Theorem 5.1** *(**Ergodic Theorem**) [Robert and Casella, 1999]*
  If $(X_t)_t$ is Harris recurrent with a σ-finite invariant measure π, then for any $f, g \in L_1(\pi)$ with $\mathbb{E}_\pi[g] \neq 0$,

$$\lim_{T \to \infty} \frac{S_T(f)}{S_T(g)} = \frac{\mathbb{E}_\pi[f]}{\mathbb{E}_\pi[g]} = \frac{\int f(x)d\pi(x)}{\int g(x)d\pi(x)} \tag{44}$$

  It can be shown that if $(X_t)_t$ is **Harris positive** with **stationary distribution** π and if $S_T(h)$ *converges* μ_0-almost surely (μ_0 a.s.) to $\mathbb{E}_\pi[h]$, for an initial distribution μ_0, this convergence occurs for **every initial distribution** μ.

  Corollary 5.2 *[Liu, 2001]*
  *If a **finite state-space** Markov chain $(X_t)_t$ is irreducible and aperiodic with stationary distribution π, then $S_T(h)$ converges to $\mathbb{E}_\pi[h]$ **almost surely** for any initial distribution μ.*

- **Theorem 5.3** *(**Central Limit Theorem for discrete atoms**)*
  If $(X_t)_t$ is Harris positive recurrent with an atom α such that

$$\mathbb{E}\left[T_\alpha^2|X_0 \in \alpha\right] < \infty, \qquad \mathbb{E}\left[\left(\sum_{t=1}^{T_\alpha} |h(X_t)|\right)^2 \middle| X_0 \in \alpha\right] < \infty,$$

$$\text{and the variance } \sigma_h^2 := \pi(\alpha)\,\mathbb{E}\left[\left(\sum_{t=1}^{T_\alpha} \{h(X_t) - \mathbb{E}_{\boldsymbol{\pi}}[h(X)]\}\right)^2 \middle| X_0 \in \alpha\right] > 0,$$

  *then the **Central Limit Theorem** applies:*

$$\sqrt{T}\left(S_T(h) - \mathbb{E}_\pi[h]\right) \overset{\mathcal{L}}{\sim} \mathcal{N}(0, \sigma_h^2) \tag{45}$$

  Corollary 5.4 *[Liu, 2001]*
  *For **finite state-space**, irreducible and aperiodic Markov chain $(X_t)_t$, the Central Limit Theorem holds, i.e. $\sqrt{T}\left(S_T(h) - \mathbb{E}_\pi[h]\right) \overset{\mathcal{L}}{\sim} \mathcal{N}(0, \sigma_h^2)$ for any initial distribution μ.*

- **Theorem 5.5** *(**Central Limit Theorem for reversible chains**)*
  *If $(X_t)_t$ is **aperiodic, irreducible, and reversible** with stationary distribution $\boldsymbol{\pi}$, the **Central Limit Theorem** applies when*

$$0 < \sigma_h^2 = \mathbb{E}_{\boldsymbol{\pi}}\left[h^2(X_t)\right] + 2\sum_{s=1}^{\infty} \mathbb{E}_{\boldsymbol{\pi}}\left[h(X_t)\,h(X_{t+s})\right] < \infty. \tag{46}$$

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