Lecture 0: Summary (part 1)

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Contents

1 Topology

1.1 Topological Space

- **Definition** Let X be a set. *A **topology*** on X is *a collection* $\mathscr{T}$ of *subsets* of X, called ***open subsets***, satisfying

  1. X and $\emptyset$ are *open*.
  2. The ***union*** of ***any family*** of open subsets is open.
  3. The ***intersection*** of *any **finite** family* of open subsets is open.

  A pair $(X, \mathscr{T})$ consisting of a set X together with a topology $\mathscr{T}$ on X is called ***a topological space***.

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Definition** A ***continuous bijective*** map $F : X \to Y$ with ***continuous inverse*** is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Definition** A map $F : X \to Y$ (continuous or not) is said to be ***an open map*** if for every *open* subset $U \subseteq X$, the image set $F(U)$ is *open* in Y, and ***a closed map*** if for every *closed* subset $K \subseteq X$, the image $F(K)$ is *closed* in Y .

- **Definition** A topological space X is said to be a ***Hausdorff space*** if for every pair of ***distinct*** points $p, q \in X$, there exist ***disjoint open subsets*** $U, V \subseteq X$ such that $p \in U$ and $q \in V$.

- **Definition** Suppose X is a topological space. A collection $\mathscr{B}$ of open subsets of X is said to be ***a basis*** for *the topology of* X (plural: ***bases***) if every open subset of X is the *union of some collection of elements* of $\mathscr{B}$.

  More generally, suppose X is merely a set, and $\mathscr{B}$ is a collection of *subsets* of X satisfying the following conditions:

  1. $X = \bigcup_{B \in \mathscr{B}} B$.
  2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B2$.

  Then *the collection of **all unions** of elements of* $\mathscr{B}$ is a *topology* on X, called ***the topology generated by*** $\mathscr{B}$, and $\mathscr{B}$ is a ***basis*** for this *topology*.

- **Definition** See the following definitions

  1. A set is said to be ***countably infinite*** if it admits a *bijection* with the set of *positive integers*, and
  2. ***countable*** if it is *finite* or *countably infinite*.
  3. A topological space X is said to be ***first-countable*** if there is a ***countable neighborhood basis*** at each point, and
  4. ***second-countable*** if there is ***a countable basis*** for its topology.

1.2 Subspaces and Quotients

- **Definition** If X is a topological space and $S \subseteq X$ is an arbitrary subset, we define ***the subspace topology*** on S (sometimes called *the* ***relative topology***) by declaring a subset $U \subseteq S$ to be *open* in S *if and only* if there exists an open subset $V \subseteq X$ such that $U = V \cap S$.

  Any subset of X endowed with the subspace topology is said to be ***a subspace of*** X.

- **Definition** If X and Y are topological spaces, a continuous injective map $F : X \to Y$ is called a ***topological embedding*** if it is a ***homeomorphism*** onto its image $F(X) \subseteq Y$ in the subspace topology.

- **Definition** If X is a *topological space*, Y is a set, and $\pi : X \to Y$ is a **surjective** map, ***the quotient topology*** on Y determined by π is defined by declaring a subset $U \subseteq Y$ to be *open if and only* if $\pi^{-1}(U)$ is *open* in X.

  If X and Y are topological spaces, a map $\pi : X \to Y$ is called ***a quotient map*** if it is ***surjective*** and ***continuous*** and Y has the quotient topology determined by π.

- **Definition** The following construction is the most common way of producing quotient maps. *A* ***relation*** on a set X is called ***an equivalence relation*** if it is

  1. ***reflexive***: $x \sim x$ for all $x \in X$,
  2. ***symmetric***: $x \sim y$ implies $y \sim x$,
  3. ***transitive***: $x \sim y$ and $y \sim z$ imply $x \sim z$.

  If $R \subseteq X \times X$ is any ***relation*** on X, then ***the intersection of all equivalence relations*** on X ***containing*** R is *an equivalence relation*, called ***the equivalence relation generated by*** R.

  Remark If is an equivalence relation on X, then for each $x \in X$, ***the equivalence class*** *of* x, denoted by $[x]$, is the *set of all* $y \in X$ *such that* $y \sim x$. The set of *all equivalence classes* is a ***partition*** of X: a collection of disjoint nonempty subsets whose union is X.

- **Definition** Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $X/\sim$ denote ***the set of equivalence classes*** in X, and let $\pi : X \to X/\sim$ be the ***natural projection*** sending each *point* to its *equivalence class*. Endowed with ***the quotient topology*** determined by π, the space $X/\sim$ is called ***the quotient space*** (or *identification space*) of X determined by π.

- **Definition** If $\pi : X \to Y$ is a map, a subset $U \subseteq X$ is said to be ***saturated*** with respect to π if U is the ***entire preimage*** of its ***image***: $U = \pi^{-1}(\pi(U))$.

  Given $y \in Y$, the ***fiber*** of π over y is the set $\pi^{-1}(y)$.

1.3 Connectedness and Compactness

- **Definition** A topological space X is said to be ***disconnected*** if it has two ***disjoint nonempty open subsets*** whose union is X, and it is ***connected*** otherwise. Equivalently, X is connected if and only if the only subsets of X that are ***both open and closed*** are $\emptyset$ and X itself.

- **Definition** Recall that a topological space X is
  - ***connected*** if there do not exist two *disjoint*, *nonempty*, *open* subsets of X whose union is X;
  - ***path-connected*** if every pair of points in X can be ***joined by a path*** in X, and
  - ***locally path-connected*** if X has a ***basis*** of *path-connected open subsets.*
- **Definition** A ***maximal connected subset*** of X (i.e., a connected subset that is not properly contained in any larger connected subset) is called a ***component*** (or ***connected component***) of X.
- **Definition** A topological space X is said to be ***compact*** if every open cover of X has a ***finite*** *subcover.* A ***compact subset*** of a topological space is one that is a compact space in the subspace topology.
- **Definition** If X and Y are topological spaces, a map $F : X \to Y$ (continuous or not) is said to be ***proper*** if for every ***compact*** set $K \subseteq Y$, the ***preimage*** $F^{-1}(K)$ is ***compact***.
- **Definition** A topological space X is said to be ***locally compact*** if every point has a *neighborhood* contained in a ***compact subset*** of X.

  A subset of X is said to be ***precompact*** in X if its ***closure*** in X is *compact.*
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- For a ***Hausdorff space*** X, the following are equivalent:
  1. X is ***locally compact***.
  2. Each point of X has a ***precompact*** neighborhood.
  3. X has a basis of ***precompact*** open subsets.

2 Smooth Manifolds and Smooth Maps

2.1 From Topological Manifolds to Smooth Manifolds

- **Definition** Suppose M is a ***topological space***. We say that M is a ***topological manifold*** of *dimension* n or a ***topological*** n***-manifold*** if it has the following properties:

  1. M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.

  2. M is ***second-countable***: there exists a ***countable basis*** for the topology of M.

  3. M is ***locally Euclidean of dimension*** n: each point of M has a neighborhood that is ***homeomorphic*** to *an open subset of* $\mathbb{R}^n$.

- The third property means, more specifically, that for each $p \in M$ we can find

  – an open subset $U \subseteq M$ containing p,

  – an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and

  – a *homeomorphism* $\varphi : U \to \widehat{U}$.

- **Proposition 2.1** ***(Manifolds Are Locally Compact).***
  Every topological manifold is ***locally compact****.*

- **Definition** Let M be a *topological n-manifold*. A ***coordinate chart*** (or just a *chart*) on M is a ***pair*** (U, φ), where U is an open subset of M and $\varphi : U \to \widehat{U}$ is a ***homeomorphism*** from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

- **Definition** Given a chart (U, φ), we call the set U a ***coordinate domain***, or a ***coordinate neighborhood*** of each of its points. The map φ is called a ***(local) coordinate map***, and the ***component functions*** $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p))$, are called ***local coordinates*** on U.

- **Remark** We sometimes write things such as "(U, φ) is a chart containing p" as shorthand for "(U, φ) is a chart whose domain U contains p." If we wish to emphasize the coordinate function $(x^1, \ldots, x^n)$ instead of coordinate map φ, we sometimes denote the chart by $(U, (x^1, \ldots, x^n))$ or $(U, (x^i))$.

- **Definition** If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is said to be ***smooth*** (or $\mathcal{C}^\infty$, or ***infinitely differentiable***) if each of its component functions has continuous partial derivatives of ***all orders***.

  If in addition F is ***bijective*** and has a ***smooth inverse map***, it is called a ***diffeomorphism***. A *diffeomorphism* is, in particular, a *homeomorphism*.

- **Definition** Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map*** from φ to ψ. It is a *homeomorphism*.

  Two charts (U, φ) and (V, ψ) are said to be ***smoothly compatible*** if either $U \cap V = \emptyset$ or *the transition map $\psi \circ \varphi^{-1}$ is a* ***diffeomorphism****.*

- **Definition** We define an ***atlas*** $\mathcal{A}$ for M to be a collection of charts whose domains ***cover***

M, i.e. $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that $M = \bigcup_{\alpha \in A} U_\alpha$. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are ***smoothly compatible*** with each other.

- **Definition** A *smooth atlas* $\mathcal{A}$ on M is ***maximal*** if it is **not** properly contained in ***any larger*** *smooth atlas*. This just means that any chart that is *smoothly compatible* with *every chart* in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also said to be ***complete***.)

- **Definition** If M is a topological manifold, a ***smooth structure*** on M is a ***maximal smooth atlas***.

  A ***smooth manifold*** is a pair $(M, \mathcal{A})$, where M is a ***topological manifold*** and $\mathcal{A}$ is a *smooth structure* on M.

- **Remark** When the smooth structure is understood, we usually omit mention of it and just say "*M is a smooth manifold.*" Smooth structures are also called ***differentiable structures*** or $\mathcal{C}^\infty$ ***structures*** by some authors. We also use the term ***smooth manifold structure*** to mean a manifold topology together with a smooth structure.

- **Remark** When defining smooth manifold, we ask for any two coordinate charts (U, φ), (V, ψ), ***the transition map*** $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. This is **stronger** than *the coordinate map* φ itself being a diffeomorphism.

- **Remark** In practice, instead of specifying the maximal smooth atlas, it is sufficient to specify *some* smooth atlas to verify if M is smooth manifold

  Proposition 2.2 *Let M be a topological manifold.*

  1. *Every smooth atlas $\mathcal{A}$ for M is contained in a **unique** maximal smooth atlas, called **the smooth structure determined by $\mathcal{A}$**.*

  2. *Two smooth atlases for M determine the same smooth structure if and only if **their union is a smooth atlas**.*

- **Definition** If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***.

  It is useful also to introduce the terms ***smooth coordinate domain*** or ***smooth coordinate neighborhood*** for the domain of a smooth coordinate chart.

- The following result is a generalization of *second-countable* definition for the general topological manifold.

  Proposition 2.3 *Every smooth manifold has a **countable** basis of **regular coordinate balls**.*

- **Remark** Each coordinate map φ maps a smooth neighborhood $U \subseteq M$ to a neighborhood in Euclidean space $\widetilde{U} \subseteq \mathbb{R}^n$. Under this map, we can ***(locally) represent*** a point $p \in U$ by its ***coordinates*** $(x^1, \ldots, x^n) = \varphi(p)$, and think of this n-tuple as *being* the point p. We typically express this by saying "$(x^1, \ldots, x^n)$ *is **the (local) coordinate representation** for p*" or "$p = (x^1, \ldots, x^n)$ ***in local coordinates.***"

- **Remark** As we see, a smooth manifold M does not comes with any predetermined ***choice of coordinates***. Thus any objects we wish to define globally on a manifold ***shall not dependent on a particular choice of coordinates***.

There are generally two ways of doing this:

- either by writing down a ***coordinate-dependent definition*** and then proving that the ***definition gives the same results in any coordinate chart***,
- or by writing down *a definition that is* ***manifestly coordinate-independent*** (often called an ***invariant definition***).

2.2 Manifolds with boundary

- **Definition** The ***closed n-dimensional upper half-space*** $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined as

$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\right\}.$$

We will use the notations $\operatorname{Int}\mathbb{H}^n$ and $\partial\,\mathbb{H}^n$ to denote the ***interior*** and ***boundary*** of $\mathbb{H}^n$, respectively, as a subset of $\mathbb{R}^n$. When $n > 0$, this means

$$\operatorname{Int}\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n > 0\right\},$$
$$\partial\,\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n = 0\right\}.$$

- **Definition** An ***n-dimensional topological manifold with boundary*** is a *second-countable Hausdorff space* M in which every point has a neighborhood ***homeomorphic*** either to *an open subset* of $\mathbb{R}^n$ or to a (*relatively*) *open subset* of $\mathbb{H}^n$.

- **Proposition 2.4** *Let M be a topological n-manifold with boundary.*

  1. *$\operatorname{Int} M$ is an* ***open subset*** *of M and a topological n-manifold* ***without boundary****.*
  2. *$\partial\, M$ is a* ***closed subset*** *of M and a* ***topological*** *$(n-1)$****-manifold without boundary****.*
  3. *M is a topological manifold if and only if $\partial\, M = \emptyset$.*
  4. *If $n = 0$,then partial $M = \emptyset$ and M is a 0-manifold.*

- **Definition** Now let M be a topological manifold ***with boundary***. As in the manifold case, a ***smooth structure*** for M is defined to be a *maximal smooth atlas* – a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described. With such a structure, M is called a ***smooth manifold with boundary***.

- **Remark** Note that, despite their name, ***manifolds with boundary are not in general manifolds***, because boundary points do not have locally Euclidean neighborhoods. Moreover, a manifold with boundary might have ***empty boundary*** – there is nothing in the definition that requires the boundary to be a nonempty set.

  On the other hand, ***a manifold is also a manifold with boundary***, whose boundary is empty. Thus, every manifold is a manifold with boundary, but a manifold with boundary is a manifold if and only if its boundary is empty.

- **Remark** We will often use redundant phrases such as ***manifold without boundary*** if we wish to emphasize that we are talking about a manifold in the original sense, and ***manifold with or without boundary*** to refer to a manifold with boundary if we wish emphasize that the boundary might be empty.

2.3 Smooth Maps on Manifolds

- **Definition** Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a ***smooth chart*** (U, φ) for M whose domain contains p and such that the ***composite function*** $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

  If M is a smooth manifold ***with boundary***, the definition is exactly the same, except that $\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, and in the latter case we interpret smoothness of $f \circ \varphi^{-1}$ to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ ***extends to a smooth function*** in the ordinary sense.

- **Definition** Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the ***(local) coordinate representation*** of f.

- **Remark** With the help of coordinate chart (U, φ), we can generalize a lot of concepts from Euclidean space to Manifolds. The process of applying φ^{-1} is called ***(local) parameterization*** and $f \circ \varphi^{-1}$ is the local coordinate representation of the function f, which is ***parametric***.

  Even though many objects in this course is ***independent*** *of the choice of coordinates*, in practice, one need to use *the coordinate representation* of these objects to **compute** associated quanitites.

- The definition of smooth functions generalizes easily to maps *between* manifolds.

  Definition Let M, N be *smooth manifolds*, and let $F : M \to N$ be any map. We say that F is a ***smooth map*** if for every $p \in M$, there exist *smooth charts* (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and ***the composite map*** $\psi \circ F \circ \varphi^{-1}$ is ***smooth*** from $\varphi(U)$ to $\psi(V)$.

  If M and N are smooth manifolds ***with boundary***, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

- **Definition** If $F : M \to N$ is a *smooth map*, and (U, φ) and (V, ψ) are any smooth charts for M and N, respectively, we call $\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ the ***coordinate representation*** of F with respect to the given coordinates.

- **Remark** Use $F^{-1}(V) \cap U$ is a safer way to make sure the neighborhood V in coordindate chart in N is covered, as compared to using $F(U) \cap V$.

- **Remark** In practice, $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is the function we used in computation involving F.

- **Proposition 2.5** ***(Equivalent Characterizations of Smoothness)*** *[Lee, 2003.]*
  Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is ***smooth*** *if and only if either of the following conditions is satisfied:*

  1. *For every $p \in M$, there exist* ***smooth charts*** *(U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is* ***open*** *in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is* ***smooth*** *from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.*

  2. *F is* ***continuous*** *and there exist* ***smooth atlases*** *$\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for* ***each*** *α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from*

$\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ *to* $\psi_\beta(V_\beta)$.

- **Definition** If M and N are smooth manifolds with or without boundary, a ***diffeomorphism*** from M to N is a ***smooth bijective map*** $F : M \to N$ that has a ***smooth inverse***. We say that M and N are ***diffeomorphic*** if there exists a *diffeomorphism* between them. Sometimes this is symbolized by $M \approx N$.

2.4 Einstein summation convention

- **Remark** We interpret any such expression according to the *following rule*, called the ***Einstein summation convention***:
  - if the ***same index name*** (such as i in the expression above) appears exactly ***twice*** in any ***monomial term***, *once* as ***an upper index*** and once as ***a lower index***, that term is understood to be summed over *all possible values of that index*, generally from 1 to the dimension of the space in question.
  - ***Basis (vector, functions)*** (such as E_i) uses ***lower indices***. The coordinate vector of tangent space ($\frac{\partial}{\partial x^i}$) are considered as lower indices.
  - ***Cooefficients*** of a linear combination, ***coordinate or component (functions)*** of a vector with respect to a basis (such as x^i) use ***upper indices***
  - ***Basis of covectors (cotangent vectors)*** (such as ϵ^i, dx^i) use ***upper indices***, while the component of covector with respect to a basis (such as ω_i) use ***lower indices***. That is, vector and covector notation rules are switched.
  - Any index that is implicitly summed over is a "***dummy index***," meaning that the value of such an expression is *unchanged* if *a different name is substituted for each dummy index*. For example, $x^i E_i$ and $x^j E_j$ mean exactly the same thing.

3 Submersions, Immersions and Embeddings

- **Remark** The following two chapters concerns about the local properties of smooth function F on manifolds by analyzing its differentials dF_p. "***The differential of*** dF_p ***of a function*** F ***is its best linear approximation in the neighborhood of*** p ***.***" This statement can be generalized from $\mathbb{R}^n$ to n-dimensional smooth manifold. These two chapters discuss results that centered around this idea.
  1. The first theorem is ***the Inverse Function Theorem for Manifolds***. This is a direct generalization of the existing results from Euclidean space, since this theorem only concerns the property of the function ***locally*** and the manifolds is diffeomorphic to Euclidean space locally. The result of this theorm confirms that for a smooth map with invertible differential at a point p, in the neighborhood of this point, this map is ***an open map*** and ***has smooth inverse***.
  2. The second theorm concerns about the smooth map ***with constant rank***. Note that the rank of a smooth map is ***a local property*** since it is about the rank of dF_p at point p. But when we enforce the rank to be constant all over the space, we generalize the

local properties to *global*.

3. ***The Rank Thoerem*** for smooth map *with constant rank* reflects a ***stronger*** result than *the Inverse Function Theorem*. It states that we *only need to know the* ***rank*** of the differential dF_p to determine ***a local representation of the smooth map*** F, ***regardless*** of *the form of function itself*. In fact, all smooth maps ***with constant rank*** can be *locally* represented as ***coordinate projection with zero padding***. The ***rank*** of map determines ***the number of coordinate maintained*** and the others are all zero-padded.

4. *The Rank Theorem* confirms that the smooth function F with constant rank is ***locally linear***, thus is best represented by the differential dF_p.

- **Remark** Another important topic that is primiarily discussed in the following two chapters is two special type of smooth map with constant ranks: the ***smooth submersion*** and ***smooth immersion***. Both of these properties characterize ***the local differential properties*** of a smooth function F ***with full rank*** (the *shape* of the differential matrix dF_p is determined by the dimension of the manifolds in domain and codomain).

  1. The ***smooth submersion*** corresponds to F with ***surjective differential*** $dF_p \in \mathbb{R}^{m \times n}$ with $m \geq n$, i.e. the "*fat*" differential matrix. dF_p has *full column rank*.

     A smooth submersion can be *locally* represented as ***a natural coordinate projection***. Therefore it is critical when studying ***projections*** and its ***section*** between manifolds. It is also closely associated with ***quotient map*** and ***the quotient manifolds***. The *pre-image* of a smooth submersion is a submanifold itself, making it important to define new subspace structure.

  2. Similarly, ***a smooth immersion*** corresponds to F with ***injective differential*** $dF_p \in \mathbb{R}^{m \times n}$ with $m \leq n$, i.e. the differential matrix is a "*tall*" matrix with *full row rank*.

     A smooth immersion can be locally represented as ***a natural coordinate inclusion***, or a zero-padding function. It is critical when we try to *put a manifold inside another manifold* and makes it a submanifold of the latter.

     – A sub-class of smooth immersion is ***smooth embedding***. In addition being smooth immersion, a smooth embedding is also ***locally homemorphic*** *to its image*. In other word, it allows one to build an ***equivalence relationship*** between two *local regions* in *topological sense*.

     – Like smooth immersion, a smooth embedding locally is an ***inclusion*** but the embedding map globally is an ***injective*** map with ***smooth inverse*** *locally*.

     – *Smooth embedding map* is commonly used to generate k-dimensional ***embedded submanifolds***, which can be seen as a subset equipped with the subspace topology and ***locally homemorphic to a*** k***-dimensional subspace in*** $\mathbb{R}^n$.

3.1 Definitions

- **Definition** Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$, we define ***the rank of*** F ***at*** p to be ***the rank of the linear map*** $dF_p : T_pM \to T_{F(p)}N$; it is ***the rank of the Jacobian matrix*** of F in any smooth chart, or ***the dimension of*** $\text{Im } dF_p \subseteq T_{F(p)}N$. If F has the same rank r *at every point*, we say that it has constant rank, and write $\text{rank } F = r$.

- **Definition** Note that rank $dF_p \leq \min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, we say that F ***has full rank at*** p, and if F has full rank everywhere, we say F ***has full rank***.

- **Definition** The most important *constant-rank maps* are those *of full rank*. A smooth map $F : M \to N$ is called ***a smooth submersion*** if its differential is ***surjective*** *at each point* (or ***equivalently***, if $\text{rank } F = \dim N$).

  It is called ***a smooth immersion*** if its differential is ***injective*** at each point (***equivalently***, $\text{rank } F = \dim M$).

- **Proposition 3.1** *Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is **surjective**, then p has a neighborhood U such that $F|_U$ is a **submersion**. If dF_p is **injective**, then p has a neighborhood U such that $F|_U$ is an **immersion**.*

- **Remark** F is a ***surjective/injective*** is different from F is a smooth ***submersion/immersion***. The latter is the property of the differential map dF_p at each p not the property of the map itself. But F is ***a smooth embedding*** $\Rightarrow$ F is ***an injective smooth immersion***. The converse is not true since F also need to *have* ***continous inverse*** from its image to its domain.

3.2 Local Diffeomorphisms

- **Definition** If M and N are smooth manifolds with or without boundary, a map $F : M \to N$ is called ***a local diffeomorphism*** if every point $p \in M$ has a neighborhood U such that $F(U)$ is ***open*** in N and the restriction $F|_U : U \to F(U)$ is a ***diffeomorphism***.

- The next theorem is the key to the most important properties of local diffeomorphisms.

  Theorem 3.2 ***(Inverse Function Theorem for Manifolds)***. *[Lee, 2003.]*
  *Suppose M and N are smooth manifolds, and $F : M \to N$ is a **smooth map**. If $p \in M$ is a point such that dF_p is **invertible**, then there are **connected** neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a **diffeomorphism**.*

3.3 The Rank Theorem

- **Theorem 3.3** ***(Rank Theorem)***. *[Lee, 2003.]*
  *Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map **with constant rank** r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has*

a ***coordinate representation*** *of the form*

$$\widehat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0). \tag{1}$$

In particular, if F is a ***smooth submersion****, this becomes*

$$\widehat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n). \tag{2}$$

and if F is a ***smooth immersion****, it is*

$$\widehat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0). \tag{3}$$

- **Corollary 3.4** *Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are* ***equivalent:***

  1. *For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which* ***the coordinate representation of F is linear****.*

  2. *F has* ***constant rank****.*

- The rank theorem is a purely ***local statement.*** However, it has the following powerful ***global*** *consequence.*

  Theorem 3.5 ***(Global Rank Theorem)****. [Lee, 2003.]*
  Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of ***constant rank****.*

  1. *If F is* ***surjective****, then it is a* ***smooth submersion****.*

  2. *If F is* ***injective****, then it is a* ***smooth immersion****.*

  3. *If F is* ***bijective****, then it is a* ***diffeomorphism****.*

3.4 Embeddings

- One special kind of ***immersion*** is particularly important.

  Definition If M and N are smooth manifolds with or without boundary, a ***smooth embedding*** of M into N is a ***smooth immersion*** $F : M \to N$ that is ***also a topological embedding***, i.e., *a* ***homeomorphism*** *onto its image* $F(M) \subseteq N$ in the *subspace topology*.

- **Remark** ***A smooth embedding*** is a map that is ***both*** a ***topological embedding*** and a ***smooth immersion***, not just a topological embedding that happens to be smooth. Also the map need to be ***injective*** and its inverse from $F(U)$ to U needs to be continuous.

- **Proposition 3.6** *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is an* ***injective smooth immersion****. If* ***any*** *of the following holds, then F is a* ***smooth embedding****.*

  1. *F is an* ***open*** *or* ***closed*** *map. (i.e. it maps an open/closed set to an open/closed set)*

  2. *F is a* ***proper map****. (i.e. the preimage of every compact set is compact)*

  3. *M is* ***compact****.*

4. M has empty boundary and dim M = dim N

- **Theorem 3.7** ***(Local Embedding Theorem).***
  Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Then F is a ***smooth immersion*** *if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a* ***smooth embedding****.*

3.5 Submersions

- **Definition** If $\pi : M \to N$ is any continuous map, a ***section*** of π is a ***continuous right inverse*** for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$:

$$
\begin{tikzcd} M \arrow[r, "\pi"] & N \arrow[l, bend left, "\sigma"] \end{tikzcd}
$$

- **Definition** A ***local section*** of π is a continuous map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$

- **Theorem 3.8** ***(Local Section Theorem).*** *[Lee, 2003.]*
  Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then π is a ***smooth submersion*** *if and only if every point of M is in the* ***image*** *of a* ***smooth local section*** *of π.*

- **Proposition 3.9** ***(Properties of Smooth Submersions).***
  Let M and N be smooth manifolds, and suppose $\pi : M \to N$ is a smooth submersion. Then π is ***an open map****, and if it is* ***surjective*** *it is a* ***quotient map****.*

- The next three theorems provide important tools that we will use frequently when studying submersions.

  Theorem 3.10 ***(Characteristic Property of Surjective Smooth Submersions).***
  Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a ***surjective smooth submersion****. For any smooth manifold P with or without boundary, a map $F : N \to P$ is* ***smooth*** *if and only if $F \circ \pi$ is* ***smooth****:*

$$
\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F\circ\pi"] & \\ N \arrow[r, "F"'] & P. \end{tikzcd}
$$

  Theorem 3.11 ***(Passing Smoothly to the Quotient).***
  Suppose M and N are smooth manifolds and $\pi : M \to N$ is a ***surjective smooth submersion****. If P is a smooth manifold with or without boundary and $F : M \to P$ is a smooth map that is* ***constant*** *on* ***the fibers of*** *π, then there exists a* ***unique smooth map*** *$\widetilde{F} : N \to P$ such that $\widetilde{F} \circ \pi = F$:*

$$
\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\widetilde{F}"'] & P. \end{tikzcd}
$$

  Theorem 3.12 ***(Uniqueness of Smooth Quotients).***
  Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are

surjective smooth submersions *that are* **constant** *on* **each other's fibers**. *Then there exists a* **unique diffeomorphism** $F : N_1 \to N_2$ *such that* $F \circ \pi_1 = \pi_2$*:*

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2. \end{tikzcd}$$

4 Submanifolds

- **Remark** A great number of manifolds of interest can be considered as a submanifold of some other (simpler) manifolds. This chaper mainly concerns about ***the embedded submanifolds*** and then extends to ***immersed submanifolds***.

- **Remark** The definition of ***embedded submanifold*** S of M consists of three parts:

  1. $S \subseteq M$ has ***the subspace topology*** (inherited from the topology of the ambient manifold M);

  2. S is endowed with ***a smooth structure***;

  3. Under this smooth structure, ***the inclusion map*** $S \hookrightarrow M$ is a ***smooth embedding***, i.e. it has injective differential everywhere and is *locally homemorphic* to its image.

- **Remark** Besides the definition, there are two other ways to identify embedded submanifold:

  1. ***Local Slice Criterion***: If a subset S under local representation of M is *homemorphic* to an open subset of $\mathbb{R}^k \subseteq \mathbb{R}^n$. *The Local Slice Criterion* relies on *the* ***topology*** *and* ***the smooth structure*** *of* ***the ambient manifold*** M to determine both the topology and the smooth structure of submanifold S.

  2. ***Level Set Criterion***: If every point of subset S has a neighborhood in M so that $S \cap U$ is a level set of some ***smooth submersion***, then this set is a k-dimensional embedded submanifold. In particular, *every embedded submanifold admits a local defining function in a neighborhood of each of its points.* This result provides ***a constructive way*** to build embedded submanifold by defining map.

- **Remark** An important result of *smooth map with constant rank* is that each of its ***level set*** is a *properly embedded submanifold* with **codimension** equal to the ***rank*** of smooth map. In particular,

  1. If the map is a ***submersion***, then the level set is ***a regular level set***, which is a properly embedded submanifold. Moreover, the ***dimension*** of the submanifold is determined by the difference of dimension between domain and codomain.

  2. The ***image of immersion*** is also a submanifold, ***the immersed submanifold***. An immersed submanifold may not be an embedded submanifold. But ***locally*** it is *an embedded submanifold.*

- **Remark** The final important remark on embedded submanifold is that it has a ***unique topology and smooth structure***, which is ***the subspace topology*** and the coordinate map that makes the $S \cap U$ *is a* ***k-slice*** *of* U.

4.1 Embedded Submanifolds

4.1.1 Definitions

- **Definition** Suppose M is a smooth manifold with or without boundary. ***An embedded submanifold*** of M is a subset $S \subseteq M$ that is a *manifold* (without boundary) in the ***subspace topology***, endowed with a ***smooth structure*** with respect to which the ***inclusion map*** $S \hookrightarrow M$ is a ***smooth embedding***. Embedded submanifolds are also called ***regular submanifolds***.

- **Definition** If S is an *embedded submanifold* of M, the ***difference*** $\dim M - \dim S$ is called ***the codimension*** of S in M, and the ***containing manifold*** M is called ***the ambient manifold*** for S.

  An embedded ***hypersurface*** is an embedded submanifold of codimension 1. The *empty set* is an embedded submanifold of *any dimension*.

- **Proposition 4.1** ***(Open Submanifolds)**. [Lee, 2003.]*
  Suppose M is a smooth manifold. The embedded submanifolds of ***codimension 0*** *in M are exactly the* ***open submanifolds****.*

- There are several other ways to create submanifolds:

  Proposition 4.2 ***(Images of Embeddings as Submanifolds)**. [Lee, 2003.]*
  Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is a ***smooth embedding****. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a* ***unique smooth structure*** *making it into an* ***embedded submanifold*** *of M with the property that F is a* ***diffeomorphism*** *onto its image.*

- **Proposition 4.3** ***(Slices of Product Manifolds)**. [Lee, 2003.]*
  Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$ (called ***a slice of the product manifold****) is an* ***embedded submanifold*** *of $M \times N$ diffeomorphic to M.*

- **Proposition 4.4** ***(Graphs as Submanifolds)**. [Lee, 2003.]*
  Suppose M is a smooth m-manifold (without boundary), N is a smooth n-manifold with or without boundary, $U \subseteq M$ is open, and $f : U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote ***the graph of*** *f:*

  $$\Gamma(f) = \{(x, y) \in M \times N : x \in U, y = f(x)\}.$$

  Then $\Gamma(f)$ is an ***embedded m-dimensional*** *submanifold of $M \times N$*

- **Definition** An embedded submanifold $S \subseteq M$ is said to be ***properly embedded*** if the inclusion $S \hookrightarrow M$ is a ***proper map***.

- **Proposition 4.5** *Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedded submanifold. Then S is* ***properly*** *embedded if and only if it is a* ***closed*** *subset of M.*

- **Corollary 4.6** *Every* ***compact*** *embedded submanifold is* ***properly*** *embedded.*

- **Proposition 4.7** ***(Global Graphs Are Properly Embedded)**. [Lee, 2003.]*
  Suppose M is a smooth manifold, N is a smooth manifold with or without boundary, and $f : M \to N$ is a ***smooth*** *map. With the smooth manifold structure as above, the graph of f $\Gamma(f)$ is properly embedded in $M \times N$.*

4.1.2 Slice Charts for Embedded Submanifolds

- **Definition** if U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$, a ***k-dimensional slice*** of U (or simply a k-slice) is any subset of the form
$$S = \left\{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n\right\}$$
for some constants $c^{k+1}, \ldots, c^n$. (When $k = n$, this just means $S = U$.) Clearly, ***every k-slice is homeomorphic to an open subset of*** $\mathbb{R}^k$.

- **Definition** Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S ***is a k-slice of*** U.

- **Definition** Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S ***satisfies the local k-slice condition*** if *each point* of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ *is a **single k-slice** in* U. Any such chart is called ***a slice chart for S in M*** , and the corresponding coordinates $(x^1, \ldots, x^n)$ are called ***slice coordinates***.

- **Remark** The key to understand the ***the local k-slice condition*** for $S \subseteq M$:

  1. It is a condition on the ***subset*** S only; it does ***not presuppose*** any particular ***topology*** or ***smooth structure*** on S. All it needs is the topology and smooth structure from the ambient manifold M.

  2. The *local neighborhood* $U \subseteq M$ is a ***neighborhood** of p in the **ambient manifold** M* not a neigborhood in S (since we do not define such topology);

  3. The k-slice representation is for the ***intersection*** $S \cap U$ under ***the smooth chart*** (U, φ) of ***the ambient manifold*** M.

- **Theorem 4.8** ***(Local Slice Criterion for Embedded Submanifolds)*** *[Lee, 2003.].*
Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. ***Conversely,*** *if $S \subseteq M$ is a subset that* ***satisfies the local k-slice condition****, then with the subspace topology, S is a topological manifold of dimension k, and it* ***has a smooth structure*** *making it into a k-dimensional embedded submanifold of M .*

- **Theorem 4.9** *If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly* ***embedded submanifold*** *of M.*

4.1.3 Level Sets

- **Remark** In practice, embedded submanifolds are most often presented as ***solution sets*** of *equations or systems of equations.*

- **Definition** If $\Phi : M \to N$ is any map and c is any point of N, we call the set $\Phi^{-1}.(c)$ ***a level set of*** Φ (Fig. **??**). (In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called ***the zero set of*** Φ.)

- **Remark** It is easy to find *level sets of smooth functions* that are *not smooth submanifolds.*
$$\Theta(x, y) = x^2 - y, \quad \Phi(x, y) = x^2 - y^2, \quad \Psi(x, y) = x^2 - y^3.$$

(Note that the zero set $\Theta^{-1}(0)$ is an embedded submanifolds in $\mathbb{R}^2$ but not for others.) In fact, ***every closed subset of*** M can be expressed as ***the zero set*** of some smooth real-valued function.

- **Theorem 4.10** ***(Constant-Rank Level Set Theorem).*** *[Lee, 2003.]*
Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map ***with constant rank*** *r.* ***Each level set*** *of Φ is a properly embedded submanifold of* ***codimension*** *r in M.*

- **Corollary 4.11** ***(Submersion Level Set Theorem).*** *[Lee, 2003.]*
If M and N are smooth manifolds and $\Phi : M \to N$ is a ***smooth submersion****, then each level set of Φ is a* ***properly embedded*** *submanifold whose* ***codimension*** *is equal to the* ***dimension of*** *N.*

- **Remark** This result should be compared to the corresponding result in linear algebra: if $L : \mathbb{R}^m \to \mathbb{R}^r$ is a surjective linear map, then the kernel of L is a linear subspace of codimension r by ***the rank-nullity law***. The vector equation $Lx = 0$ is equivalent to r linearly independent scalar equations, each of which can be thought of as cutting down one of the degrees of freedom in $\mathbb{R}^m$, leaving a subspace of codimension r.

  In the context of smooth manifolds, the analogue of *a surjective linear map* is ***a smooth submersion***, ***each*** of whose ***(local) component functions cuts down the dimension by one***.

- **Definition** If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be ***a regular point*** of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is ***surjective***; it is ***a critical point*** of Φ otherwise.

  This means, in particular, that ***every point*** of M is ***critical*** if $\dim M < \dim N$, and every point is ***regular*** if and only if Φ is a ***submersion***.

- **Definition** A point $c \in N$ is said to be ***a regular value*** of Φ if ***every point*** *of the level set* $\Phi^{-1}(c)$ *is a regular point*, and ***a critical value*** otherwise. In particular, if $\Phi^{-1}(c) = \emptyset$, then c is *a regular value*. Finally, a level set $\Phi^{-1}(c)$ is called ***a regular level set*** if c is a regular value of Φ; in other words, a regular level set is a level set consisting ***entirely*** *of regular points* of Φ (points p such that $d\Phi_p$ is surjective).

- **Remark** *Every properly embedded submanifold $M = \Phi^{-1}(c)$ is a regular level set.* The following theorem shows that the converse is true as well.

- **Theorem 4.12** ***(Regular Level Set Theorem).*** *[Lee, 2003.]*
Every regular level set *of a smooth map between smooth manifolds is a* ***properly embedded*** *submanifold whose codimension is equal to the dimension of the codomain.*

- **Proposition 4.13** ***(Local Level Set Criterion for Smooth Embedded Submanifolds)***
Let S be a subset of a smooth m-manifold M. Then S is an ***embedded k-submanifold*** *of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a* ***level set*** *of a* ***smooth submersion*** *$\Phi : U \to \mathbb{R}^{m-k}$.*

- **Proposition 4.14** ***(Local Level Set Criterion for Smooth Embedded Submanifolds)***
Let S be a subset of a smooth m-manifold M. Then S is an ***embedded k-submanifold*** *of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a* ***level set*** *of a* ***smooth submersion*** *$\Phi : U \to \mathbb{R}^{m-k}$.*

- **Definition** If $S \subseteq M$ is an embedded submanifold, *a smooth map* $\Phi : M \to N$ such that S is

a regular level set of Φ is called ***a defining map for S***. In the special case $N = \mathbb{R}^{m-k}$ (so that Φ is a real-valued or vector-valued function), it is usually called ***a defining function***.

More generally, if U is an open subset of M and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called ***a local defining map (or local defining function) for*** S.

4.2 Immersed Submanifolds

4.2.1 Definitions and Examples

- **Definition** Let M be a smooth manifold with or without boundary. ***An immersed submanifold*** of M is a subset $S \subseteq M$ endowed with a *topology* (*not necessarily* ***the subspace topology***) with respect to which it is ***a topological manifold*** (without boundary), and *a smooth structure* with respect to which ***the inclusion map*** $S \hookrightarrow M$ is ***a smooth immersion***.

  As for embedded submanifolds, we define the ***codimension*** of S in M to be $\dim M - \dim S$.

- **Remark** Immersed submanifolds *do not require* ***the submanifold topology*** *to be* ***the subspace topology*** which is more general than embedded submanifold.

- The immersed submanifolds arise in natural way:

  Proposition 4.15 ***(Images of Immersions as Submanifolds)***. *[Lee, 2003.]*
  Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is an ***injective smooth immersion***. *Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a* ***smooth submanifold*** *of M and such that $F : N \to S$ is a* ***diffeomorphism*** *onto its image.*

- **Example** (***Immersed Submanifold but Not an Embedded Submanifold***)
  Both examples of ***The Figure-Eight*** and ***the Dense Curve on the Torus*** are *images of injective smooth immersions*, they are ***immersed submanifolds*** when given appropriate topologies and smooth structures. As smooth manifolds, they are ***diffeomorphic*** to $\mathbb{R}$. *They are* ***not embedded submanifolds***, because ***neither*** one has ***the subspace topology***. In fact, their image sets cannot be made into embedded submanifolds even if we are allowed to change their topologies and smooth structures. ∎

- **Remark** Suppose M is a smooth manifold and $S \subseteq M$ is ***an immersed submanifold***. It can be shown that every subset of S that is ***open*** in the ***subspace topology*** is also ***open*** in its given ***submanifold topology***; and the **converse** is true if and only if S is ***embedded***.

- **Proposition 4.16** ***(Immersed Submanifolds Are Locally Embedded)***. *[Lee, 2003.]*
  If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an ***immersed submanifold***, *then for each $p \in S$ there exists a neighborhood U of p in S that is an* ***embedded submanifold*** *of M.*

  Note that a smooth immersion is locally a smooth embedding.

- **Remark** It is important to be clear about what this proposition does and does not say: given an immersed submanifold $S \subseteq M$ and a point $p \in S$, it is possible to find *a neighborhood U*

of p ***in** S* such that $\underline{U}$ is *embedded*; but it may not be possible to find *a neighborhood* V *of* p ***in** M* such that $\underline{V \cap S}$ is embedded.

- **Definition** Suppose $S \subseteq M$ is an immersed k-dimensional submanifold. ***A local parametrization*** of S is a continuous map $X : U \to M$ whose domain is an ***open subset*** $U \subseteq \mathbb{R}^k$, whose image is an ***open subset*** of S, and which, considered as a map into S, is a ***homeomorphism*** *onto its image*. It is called a ***smooth local parametrization*** if it is a ***diffeomorphism*** onto its image (with respect to Ss smooth manifold structure). If the image of X is all of S, it is called ***a global parametrization***.

- **Remark** For *a smooth chart* (U, φ) of M, $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ is a *diffeomorphism*, *its inverse* $\varphi^{-1} : \widehat{U} \to U \subseteq M$ is ***a smooth local parameterization*** (in fact $X = \mathrm{Id}_M \circ \varphi^{-1}$).

- **Proposition 4.17** *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold, $\iota : S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X : U \to M$ is a **smooth local parametrization** of S **if and only if** there is a smooth coordinate chart (V, φ) for S such that $\underline{X = \iota \circ \varphi^{-1}}$. Therefore, every point of S is in the image of some local parametrization.*

4.3 Restricting Maps to Submanifolds

4.3.1 Restricting Domains and Codomains

- **Remark** Given a smooth map $F : M \to N$, it is important to know whether F is still smooth when its ***domain*** or ***codomain*** is restricted to a submanifold. See Fig. **??**.

  1. ***Restricting Domains***: The answer is ***yes***.

     Theorem 4.18 *(**Restricting the Domain of a Smooth Map**). [Lee, 2003.]*
     *If M and N are smooth manifolds with or without boundary, $F : M \to N$ is a smooth map, and $S \subseteq M$ is an **immersed or embedded submanifold**, then $F|_S : S \to N$ is smooth.*

  2. ***Restricting Codomains***: We provides ***sufficient conditions***: *the **image** of the smooth map should be **contained in the submanifold**.*

     - ***Immersed Submanifolds***:

       Theorem 4.19 *(**Restricting the Codomain of a Smooth Map**). [Lee, 2003.]*
       *Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an immersed submanifold, and $F : N \to M$ is a smooth map whose **image is contained in** S. If F is **continuous** as a map from N to S, then $F : N \to S$ is smooth.*

     - ***Embedded Submanifolds***:

       Corollary 4.20 *(**Embedded Case**).*
       *Let M be a smooth manifold and $S \subseteq M$ be an embedded submanifold. Then every smooth map $F : N \to M$ whose **image is contained in** S is also **smooth** as a map from N to S.*

  3. We can generalize the corollary above as the definition of weakly embedded submanifold.

Definition If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be ***weakly embedded*** in M if every smooth map $F : N \to M$ ***whose image lies in*** S is ***smooth*** as a map from N to S. (*Weakly embedded submanifolds* are called ***initial submanifolds*** by some authors.)

4.3.2 Uniqueness of Smooth Structures on Submanifolds

- **Theorem 4.21** *Suppose M is a smooth manifold and $S \subseteq M$ is an* ***embedded submanifold****. The subspace topology on S and the smooth structure from the local k-slice condition are* ***the only topology and smooth structure*** *with respect to which S is an embedded or immersed submanifold.*

- **Remark** Thanks to this uniqueness result, we now know that a *subset $S \subseteq M$* is an *embedded submanifold* ***if and only if*** it satisfies *the local slice condition*, and if so, its topology and smooth structure are ***uniquely determined***.

  Because the local slice condition is ***a local condition***, if every point $p \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is an embedded k-submanifold *of U*, then S is an embedded k-submanifold of M.

- **Theorem 4.22** *Suppose M is a smooth manifold and $S \subseteq M$ is an* ***immersed submanifold****. For the* ***given topology*** *on S, there is* ***only one smooth structure*** *making S into an immersed submanifold.*

- **Theorem 4.23** *If M is a smooth manifold and $S \subseteq M$ is a* ***weakly embedded submanifold****, then S has* ***only one topology and smooth structure*** *with respect to which it is an immersed submanifold.*

4.3.3 Extending Functions from Submanifolds

- **Remark** Complementary to the restriction problem is the problem of extending smooth functions from a submanifold to the ambient manifold. Here we say $f \in \mathcal{C}^{\infty}(S)$ for submanifold $S \subseteq M$, when f is considered as a function on the manifold S.

- **Lemma 4.24** ***(Extension Lemma for Functions on Submanifolds)****.*
  Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in \mathcal{C}^{\infty}(S)$.

  1. *If S is* ***embedded****, then there exist a* ***neighborhood*** *U of S in M and a smooth function $\widetilde{f} \in \mathcal{C}^{\infty}(U)$ such that $\widetilde{f}|_S = f$.*

  2. *If S is* ***properly embedded****, then the neighborhood U above can be taken to be* ***all*** *of M.*

4.4 The Tangent Space to a Submanifold

- **Remark** The ***tangent space to a smooth submanifold*** of an abstract smooth manifold can be viewed as *a* ***subspace*** of ***the tangent space to the ambient manifold***, once we make appropriate identifications. The following proof is based on the ***differential*** of ***the inclusion map*** as a smooth immersion.

Proof: Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota : S \hookrightarrow M$ is a ***smooth immersion***, at each point $p \in S$ we have an *injective linear map* $d\iota_p : T_pS \to T_pM$. In terms of ***derivations***, this injection works in the following way: for any vector $v \in T_pS$, the image vector $\widetilde{v} = d\iota_p(v) \in T_pM$ acts on smooth functions on M by

$$\widetilde{v}f = d\iota_p(v)f = v\,(f \circ \iota) = v\,(f|_S)\,.$$

We adopt the convention of ***identifying*** T_pS with ***its image*** *under this map*, thereby thinking of T_pS as a certain linear subspace of T_pM. This identification makes sense regardless of whether S is *embedded or immersed*. ∎

- There are several *alternative* ways to *characterize* the tangent space of a submanifold

  1. ***Smooth curve*** *on* ***submanifold***.

     Proposition 4.25 *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose* ***image is contained in*** *S, and which is also* ***smooth*** *as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.*

  2. ***Derivations*** *on functions whose* ***restriction on submanifold are constant zero***.

     Proposition 4.26 *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by*

     $$T_pS = \{v \in T_pM : vf = 0 \textbf{ whenever } f \in \mathcal{C}^{\infty}(M) \textbf{ and } f|_S = 0\}\,.$$

  3. ***Kernel subspace*** *of* ***differential of local defining map***.

     Proposition 4.27 *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any* ***local defining map*** *for S, then $T_pS =$* ***Ker*** *$(d\Phi_p) : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.*

     Note that $S \cap U = (\Phi \circ \iota)^{-1}(c)$ is the level set of $\Phi \circ \iota$ thus it is constant for $\Phi \circ \iota$. So $d\Phi_p \circ d\iota_p = 0$.

     Corollary 4.28 *Suppose $S \subseteq M$ is a* ***level set*** *of a* ***smooth submersion*** *$\Phi = (\Phi^1, \ldots, \Phi^k) : M \to \mathbb{R}^k$. A vector $v \in T_pM$ is tangent to S if and only if $v\Phi^1 = \ldots = v\Phi^k = 0$.*

- **Remark** If M is a smooth manifold ***with boundary*** and $p \in \partial M$, the vectors in T_pM can be separated into ***three classes***:

  1. those ***tangent to the boundary***;

  2. those pointing ***inward***;

  3. those pointing ***outward***.

  Definition If $p \in \partial M$, a vector $v \in T_pM \setminus T_p\partial M$ is said to be ***inward-pointing*** if for some $\epsilon > 0$ there exists a smooth curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is ***outward-pointing*** if there exists such a curve whose domain is $(-\epsilon, 0]$.

  Proposition 4.29 ***(Characterization of Tangent Vectors on Boundary using Component Functions)***

Suppose M is a smooth n-dimensional manifold with boundary, $p \in \partial M$, and (x^i) are any smooth boundary coordinates defined on a neighborhood of p. The ***inward-pointing vectors*** *in T_pM are precisely those with* ***positive x^n-component****, the* ***outward-pointing*** *ones are those with* ***negative x^n-component****, and the ones* ***tangent to ∂M*** *are those with* ***zero x^n-component****. Thus, T_pM is the* ***disjoint union*** *of $T_p\partial M$, the set of inward-pointing vectors, and the set of outwardpointing vectors, and $v \in T_pM$ is inward-pointing if and only if $-v$ is outward-pointing.*

4.5 Vector Fields and Submanifolds

- **Remark** If $S \subseteq M$ is an immersed or embedded submanifold (with or without boundary), *a vector field X on M* does ***not necessarily*** *restrict to a vector field on S*, because X_p may not lie in the *subspace* $T_pS \subseteq T_pM$ at a point $p \in S$.

- **Definition** Given a point $p \in S$, a vector field X on M is said to ***be tangent to*** S at p if $X_p \in T_pS \subseteq T_pM$. It ***is tangent to*** S if it is tangent to S at every point of S.

- **Proposition 4.30** *Let M be a smooth manifold, $S \subseteq M$ be an* ***embedded submanifold*** *with or without boundary, and X be a smooth vector field on M. Then X is* ***tangent*** *to S if and only if $(Xf)|_S = 0$ for* ***every*** *$f \in \mathcal{C}^{\infty}(M)$* ***such that*** *$f|_S \equiv 0$.*

- **Remark** Suppose $S \subseteq M$ is an ***immersed submanifold*** with or without boundary, and Y is a smooth vector field on M. If there is a vector field $X \in \mathfrak{X}(S)$ that is ***ι-related to Y***, where $\iota : S \hookrightarrow M$ is the inclusion map, then clearly Y ***is tangent to*** S, because $Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$.

  The converse is true as well.

  Proposition 4.31 ***(Restricting Vector Fields to Submanifolds****). [Lee, 2003.]*
  Let M be a smooth manifold, let $S \subseteq M$ be an ***immersed submanifold*** *with or without boundary, and let $\iota : S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathfrak{X}(M)$ is* ***tangent to*** *S, then there is a* ***unique smooth vector field*** *on S, denoted by $Y|_S$, that is* ***ι-related to*** *Y.*

4.6 Restricting Covector Fields to Submanifolds

- **Remark** Compare to restricting vector fields to submanifolds, the restriction of covector fields to submanifolds is much simpler.

- **Remark** (***The Pullback of Covector Field by the Inclusion Map is a Covector Field on Submanifold***)
  Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an ***immersed submanifold*** with or without boundary, and $\iota : S \hookrightarrow M$ is *the inclusion map*. If ω is any smooth covector field on M, ***the pullback by ι yields a smooth covector field $\iota^*\omega$ on S.***

  To see what this means, let $v \in T_pS$ be arbitrary, and compute

  $$(\iota^*\omega)_p(v) = \omega_p\left(d\iota_p(v)\right) = \omega_p(v).$$

  since $d\iota_p : T_pS \to T_pM$ is just the inclusion map, under our usual identification of T_pS with a subspace of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called ***the restriction of ω to S***.

Be warned, however, that $\iota^*\omega$ might equal ***zero*** at a given point of S, even though ***considered as a covector field on*** M, ω ***might not vanish there***.

- **Example** ($\omega \neq 0$ but $\iota^*\omega = 0$)
  Let $\omega = dy$ on $\mathbb{R}^2$, and let S be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is ***nonzero*** everywhere, because one of its component functions is ***always*** 1. However, the restriction $\iota^*\omega$ is ***identically zero***, because y vanishes identically on S:

  $$\iota^*\omega = \iota^*dy = d(y \circ \iota) = 0.$$

- **Remark** One usually says that "ω ***vanishes along*** S" or "ω ***vanishes at points of*** S" if $\omega_p = 0$ for every point $p \in S$.

  The ***weaker condition*** that $\iota^*\omega = 0$ is expressed by saying that "***the restriction of*** ω ***to*** S ***vanishes***", or "***the pullback of*** ω ***to*** S ***vanishes***".

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.