Lecture 4: Gaussian measure

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1 Definitions

1.1 Probability measure on infinite dimensional function space

- Denote $\mathcal{B}^1 \equiv \mathcal{B}(\mathbb{R})$ as the Borel σ-algebra on $\mathbb{R}$ and $\mathcal{B}^n \equiv \mathcal{B}(\mathbb{R}^n)$ as the Borel σ-algebra on $\mathbb{R}^n, n \geq 1$. Consider the sample space Ω as a locally convex Hausdorff topological space, where the algebra of cylinders $\mathscr{C}_0$ and cylindrical σ-algebra $\mathscr{C}$ is defined. Note that $\mathscr{C}$ is the σ-algebra generated by $\mathscr{C}_0$ and $\mathscr{C} \subset \mathscr{B} \equiv \mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra on Ω. Let Ω^* be the dual space of continuous linear functionals on Ω consists of the random variable $\xi : \Omega \to \mathbb{R}$, which is $(\mathscr{B}, \mathcal{B}^1)$ measureable. Here

  Note that the cylinder set

  $$C_\xi[A; t_1, \dots t_n] \equiv \{\omega \mid (\xi_{t_1}(\omega), \cdots, \xi_{t_n}(\omega)) \in A\} \in \mathscr{C}; \quad A \in \mathcal{B}^n, \forall n \geq 1$$

- Here consider the random function as $\xi_\cdot : T \times \Omega \to \mathbb{R}$, and for each subset $N = \{t_1, \dots t_n\}$, $\xi_N : (\Omega, \mathscr{B}) \to (\mathbb{R}^n, \mathcal{B}^n)$, $n \geq 1$. On the other hand, for fixed ω, the whole sample-path $\xi_t(\omega)$ is seen as a function in $\mathbb{R}^T$ (usually smooth, or integrable functions s.t. structure of T). Therefore, the random function $\xi_\cdot$ is a mapping $(\Omega, \mathscr{B}) \to (\mathbb{R}^T, \mathcal{B}^T)$. Here $\mathcal{B}^T \equiv \mathcal{B}(\mathbb{R}^T)$ is the Borel σ-algebra on function space $\mathbb{R}^T$, with respect to which each coordinate functional (evaluation functional) $\pi_t : \mathbb{R}^T \to \mathbb{R}$, $\pi_t(x) = x(t)$, are $(\mathcal{B}^T, \mathcal{B}^1)$ measureable.

- We can define the probability measure $\mathcal{P}$ on the cylindrical σ-algebra $\mathscr{C}$ and for locally convex Hausdorff topological space Ω, $\mathcal{P}$ can be uniquely extended to the Borel σ-algebra $\mathscr{B}$. Therefore, the probability measure $\mathbb{P}$ on Borel set of $\mathbb{R}$ can be induced from $\mathcal{P}$ via the measureable function $\xi \in \Omega^*$; i.e.,

  $$\mathbb{P}(A) \equiv \mathcal{P}\{\omega \in \Omega \mid \xi(\omega) \in A\} = \mathcal{P} \circ \xi^{-1}(A), \quad A \in \mathcal{B}^1, \tag{1}$$

- Similarly, the probability measure $\mathbb{P}$ on function space $\mathbb{R}^T$ can be induced from the measureable random function $\xi_\cdot : (\Omega, \mathscr{B}) \to (\mathbb{R}^T, \mathcal{B}^T)$; i.e.,

  $$\mathbb{P}(A) \equiv \mathcal{P}\{\omega \in \Omega \mid \xi_\cdot \equiv \xi(\cdot, \omega) \in A\} = \mathcal{P} \circ \xi_\cdot^{-1}(A), \quad A \in \mathcal{B}^T.$$

  The above $\mathbb{P}$ is said to be the *distribution of random function* $\xi_\cdot \equiv \xi(\cdot, \omega)$.

- In particular, for any $n \geq 1$, any $A \in \mathcal{B}^n \subset \mathcal{B}^T$, $\mathbb{P}$ can be defined via each n-dimensional cylinder set $C_\xi[A; t_1, \dots t_n] \in \mathscr{C}$.

  $$\begin{aligned} \mathbb{P}(A) &\equiv \mathcal{P}\left(C_\xi[A_n; t_1, \dots t_n]\right) \\ &= \mathcal{P}\left\{\omega \in \Omega \mid (\xi_{t_1}(\omega), \cdots, \xi_{t_n}(\omega)) \in A_n\right\} \\ &= \mathcal{P} \circ \xi_{T_n}^{-1}(A), \quad A_n \in \mathcal{B}^n \\ &= \mathcal{P}\left[(\xi_{t_1}, \cdots, \xi_{t_n}) \in A_n\right], \end{aligned} \tag{2}$$

  where $\xi_{T_n} = \xi_\cdot \circ \pi_{T_n}$, $\xi_\cdot : (\Omega, \mathscr{B}) \to (\mathbb{R}^T, \mathcal{B}^T)$ and $\pi_N : (\mathbb{R}^T, \mathcal{B}^T) \to (\mathbb{R}^n, \mathcal{B}^n)$ as $\pi_{T_n}(f) = (f_{t_1}, \cdots, f_{t_n})$, $T_n = \{t_1, \dots t_n\} \subset T$, $n \geq 1$ is the evaluation map. The above formula holds for all $n \geq 1$, $(t_1, \dots t_n) \subset T$.

  This is called a *finite n-dimensional joint distribution* of random function $\xi_\cdot$.

Note that each cylinder set can be obtained from the evaluation map . Then

$$C_\xi[A;N] = (\hat{\pi}_{T_n}(\xi.))^{-1}(A); A \in \mathcal{B}^n$$

where $\hat{\pi}_t : \mathbb{R}^{T\times\Omega} \to \mathbb{R}^\Omega : \xi. \mapsto \xi_t$ is considered as the coordinate operator (evaluation operator).

- For a nondecreasing sequence of sets $A_n \uparrow A$, $A_n = \pi_{T_n} A_{n+1} \subset A_{n+1}$, $n \geq 1$ and $T_n = [t_1, \ldots, t_n] \uparrow T$, where $T_{n+1} = T_n \cup \{t_{n+1}\}$, the cylinder sets $C_\xi[A_n;T_n] \downarrow C_\xi[A;T] \equiv \bigcap_{n=1}^\infty C_\xi[A_n;T_n]$.

  Therefore the *distribution of random function* $\xi.$, $\mathbb{P}$, is completely determined by all its finite n-dimensional joint distribution; i.e.,

$$\begin{aligned}
\mathbb{P}(A) &= \mathcal{P}\left(C_\xi[A;T]\right) \\
&= \mathcal{P}\left(\lim_{n\to\infty} C_\xi[A_n;T_n]\right) \\
&= \sum_{n\to\infty} \mathcal{P}\left(C_\xi[A_n;T_n]\right) \\
&= \sum_{n\to\infty} \mathbb{P}(A_n),
\end{aligned}$$

  where $A_n = \pi_{T_n} A$ for any $n \geq 1$, any $T_n \subset T$, any $A \in \mathcal{B}^T$.

1.2 Important functional, operators

- Define the *barycenter* $\omega_a \in \Omega$ of a measure $\mathcal{P}$ if for any continuous linear functional (random variable) $\xi \in \Omega^*$,

$$\xi(\omega_a) = \int_\Omega \xi(\omega)\mathcal{P}(d\omega) \equiv m(\xi), \tag{3}$$

  where $m \in \Omega^{**}$ is a linear functional on random variable $\xi \in \Omega^*$, called *mean* functional.

- The linear operator $K : \Omega^* \to \Omega$ is called the *covariance operator* of a measure $\mathcal{P}$ if for any $\xi, \eta \in \Omega^*$, the following equality holds,

$$\begin{aligned}
\xi\left(K(\eta)\right) &= \int_\Omega \xi(\omega - \omega_a)\eta(\omega - \omega_a)\mathcal{P}(d\omega) \\
&= \int_\Omega \left(\xi(\omega) - \xi(\omega_a)\right)\left(\eta(\omega) - \eta(\omega_a)\right)\mathcal{P}(d\omega) \\
&= \int_\Omega \left(\xi(\omega) - m(\xi)\right)\left(\eta(\omega) - m(\eta)\right)\mathcal{P}(d\omega)
\end{aligned} \tag{4}$$

- For ξ and $\eta \in \Omega^*$, so $\xi(\omega) = \langle \omega\,,\, \xi\rangle$ and $\eta(\omega) = \langle \omega\,,\, \eta\rangle$, where $\langle \cdot\,,\, \cdot\rangle : \Omega \times \Omega^* \to \mathbb{R}$ is the *duality bilinear products*, so the covariance operator K corresponds to

$$\begin{aligned}
\widehat{K}(\xi,\eta) &= \int_\Omega \xi(\omega - \omega_a)\eta(\omega - \omega_a)\mathcal{P}(d\omega) \\
&= \int_\Omega \langle \omega - \omega_a\,,\, \xi\rangle \langle \omega - \omega_a\,,\, \eta\rangle \mathcal{P}(d\omega) \\
&= \langle K\eta\,,\, \xi\rangle \equiv \xi\left(K(\eta)\right)
\end{aligned} \tag{5}$$

where $\widehat{K} : \Omega^* \times \Omega^* \to \mathbb{R}$ is a functional on $\Omega^* \times \Omega^*$.

- Note that K is self-adjoint, i.e. $\xi\,(K\eta) = \eta\,(K\xi)$, or $\langle K\eta\,,\,\xi\rangle = \langle K\xi\,,\,\eta\rangle$.
- Define the *characteristic functional* of measure $\mathcal{P}$ as a complex-valued functional on Ω^* given by the formula,

$$\phi_{\mathcal{P}}(\xi) = \int_{\Omega} \exp\left(j\ \xi(\omega)\right) \mathcal{P}(d\omega). \tag{6}$$

1.3 Gaussian measure

- A measure $\mathcal{P}$ defined on some algebra that contains $\mathscr{C}$ is called *Gaussian* if the distribution for any (continuous) *linear* functional $\xi \in X^*$ with respect to the measure $\mathcal{P}$ is a Gaussian distribution in $\mathbb{R}$; i.e.,

$$\mathcal{P} \circ \xi^{-1} = \mathcal{N}(\omega_a, \sigma^2) \tag{7}$$

  for some m, σ.
- For any *finite-dimensional joint distribution*, we see that

$$\mathcal{P} \circ \xi_T^{-1} = \mathcal{N}(m, K), \tag{8}$$

  where m is the mean functional and K is covariance operator.
- Denote the class of all Radon Gaussian measures on the Borel σ-algebra $\mathscr{B}$ of Ω as $\mathcal{G}(\Omega)$, and the subclass of all centered Radon Gaussian measures as $\mathcal{G}_0(\Omega)$.

2 Theorem

3 Examples