Lecture 0: Notations, Expressions and Formulas

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Contents

1 Notations and Symbols

1.1 Tangent Space and Differential at p

- (U,φ): ***a smooth (coordinate) chart*** for M. $U \subseteq M$ is ***coordinate domain***, $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ is ***coordinate map***. $\varphi(p) = (x^1(p), \ldots, x^n(p))$ is the ***coordinate representation*** of $p \in M$.
- $x^i : U \to \mathbb{R}$ is ***the i-th coordinate function***. It is also simplied as the coordinate value itself.
- $$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM, \quad i = 1, \ldots, n$$
  is ***the partial derivative operation*** $\mathcal{C}^\infty(M) \to \mathbb{R}$ with respect to the i-th coordinate. It is *one of* ***the basis vector*** in T_pM. It is the derivation operation at p along i-th basis vector in T_pM.
- $vf \equiv v(f) \in \mathbb{R}$ for $v \in T_pM$: This is the ***derivation*** of f at p ***along direction of*** v. Since $v \in T_pM$ is also a *derivation operator at p* on $\mathcal{C}^\infty(M) \to \mathbb{R}$, it can ***acts on*** $f \in \mathcal{C}^\infty(M)$.
- T_pM: ***the tangent space*** of M at p. It is also *the vector space* of *all* ***derivations operations*** *on $\mathcal{C}^\infty(M)$ at p*. This is a ***n-dimensional space***.
- dF_p: *the* ***differential*** *of F at p*. It is ***a linear map*** from the tangent spaces T_pM to $T_{F(p)}N$, for $F : M \to N$. It is also called *the* ***pointwise pushforward*** *by F*.
- $dF_p(v) \in T_{F(p)}N$: This is ***the tangent vector*** on ***codomain*** N at $F(p)$.
- $dF_p(v)\, g \in \mathbb{R}$: This is *the tangent vector on N* at $F(p)$ acts on g, which produce the directional derivatives of g along $dF_p(v)$ at $F(p)$.
- $\gamma'(t)$: For a curve $\gamma : J \to M$, it is *the* ***differential*** *of γ at t*. It is also ***the tangent direction*** of γ at t. It is ***velocity*** of the curve at t.
- $\gamma'(t)f$: The ***directional derivatives*** of a function f along ***the tangent direction of curve***. It is ***the rate of change*** of f along the curve γ.

1.2 Cotangent Space

- $dx^i|_p$: a ***linear functional*** $T_pM \to \mathbb{R}$. The set (dx^i) is also ***the dual basis*** in T_p^*M corresponds to $(\partial/\partial x^i)$. We can also see it as ***the differential of the coordinate function*** x^i at p, i.e. $dx^i|_p = dx^i_p$.
- $\omega \in T_p^*M$: a ***linear functional*** on T_pM, i.e. $T_pM \to \mathbb{R}$. It is called ***(tangent) covector, or cotangent vector***
- $\omega(v) \in \mathbb{R}$: when *a linear functional ω taking value at* given *tangent vector v*, it returns a real value.
- df_p: for *real-valued function* $f \in \mathcal{C}^\infty(M)$. df_p can be thought as ***the differential*** *of f at p*, which is a linear map between T_pM to $T_{f(p)}\mathbb{R}$. It can also be thought as ***the linear***

functional on T_pM, i.e. the linear map $T_pM \to \mathbb{R}$. Thus it is a ***covector***. It is also ***the differential*** 1***-form*** evaluated at p.

- $df_p(v)$: for $v \in T_pM$, this is a ***real number*** since df_p is a linear functional on T_pM and $df_p(v) = vf$. But it is also a linear operator on function on $\mathbb{R}$. This is also equal to the directional derivative of f along v, by definition of differentials.

- $df_p(v)g$: when df_p treated as linear operator, this is the derivation v act on the composite $g \circ f$, i.e. $v(g \circ f)$.

- T_p^*M: ***the tangent covector (cotangent) space***. It is the vector space of *all linear functionals on* T_pM. It is the dual space of T_pM, i.e. $T_p^*M = (T_pM)^*$. This is a n-***dimensional space***.

- F^*: ***the pullback operator***: $T_{F(p)}^*N \to T_p^*M$ for $F : M \to N$. It maps a covector on $T_{F(p)}N$ to a covector on T_pM.

1.3 Tangent Bundle and Vector Field

- TM: The ***tangent bundle*** on M. It is the ***union*** of *all tangent spaces* for all $p \in M$. *Tangent bundle* itself is a $2n$***-manifold***.

$$TM = \bigsqcup_{p \in M} T_pM$$

- π: The ***natural projection*** $\pi : TM \to M$ onto the manifold M. $\pi(p, v) = p$. It is a ***smooth surjective submersion***. Each tangent space is ***the level set*** of π, i.e. $T_pM = \pi^{-1}(p)$.

- X: A ***vector field***. It is a ***section*** of π, i.e. a continuous map $X : M \to TM$ so that $\pi \circ X = \mathrm{Id}_M$. That is, $\pi(X(p)) = p$. The value of X at p is denoted as $X_p := X(p) \in T_pM$. X_p is *a tangent vector* at p. A vector field X also defines ***a derivation operation*** on $\mathcal{C}^\infty(M)$, i.e. $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

- $\mathfrak{X}(M) := \Gamma(TM)$: The ***vector space*** of all vector fields on M. $\Gamma(TM) =$ the vector space of all sections on tangent bundle TM. This is a n-***dimensional space***.

- $Xf \in \mathcal{C}^\infty(M)$: This is a ***smooth function*** since the derivation of a smooth function is another smooth function.

- $fX \in \mathfrak{X}(M)$: This is a ***vector field***. At each point p, $(fX)_p = f(p)X_p$. f only multiplies the component function.

- $$\frac{\partial}{\partial x^i} \in \mathfrak{X}(M), \quad i = 1, \ldots, n$$

  forms a set of basis in $\mathfrak{X}(M)$. It is called ***the (local) coordinate frames of*** M. Note that it drops dependency on p.

- $X_pf \in \mathbb{R}$: The same as vf where $v = X_p \in T_pM$.

- $Xf(p)$: The same as X_pf. $Xf(p) = X_pf$.

- $fX(p) \in T_pM$: The same as $(fX)_p = f(p)X_p$. This is a tangent vector at p.
- XY: for both $X, Y \in \mathfrak{X}(M)$. It is ***a linear map*** $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ but it is ***not necessarily is a vector field*** since the product rule may not hold. That is, normally, $XY \notin \mathfrak{X}(M)$ since it contains a *second-order derivative term*.
- $XYf \in \mathcal{C}^\infty(M)$: It is a smooth function by linear map XY. It is Xg where $g = Yf$.
- $XYf(p) \in \mathbb{R}$: It is equal to $(XY)_pf = X_pY_pf$.
- $fXY(p)$: It is equal to $f(p)X_pY_p$. It is still *a smooth linear operator* $\mathcal{C}^\infty(M) \to \mathbb{R}$.
- $[X, Y] \in \mathfrak{X}(M)$: ***Lie bracket of vector fields*** X ***and*** Y. $[X, Y] = XY - YX$ is a vector field on M, even if neither XY nor YX is a vector field. The Lie bracket of vector fields X and Y is seen as ***the Lie direvative*** of Y along flow of X.
- $[X, Y]f \in \mathcal{C}^\infty(M)$: It is equal to $XYf - YXf$.
- $[X, Y]_pf \in \mathbb{R}$: It is equal to $(XY - YX)_pf = (XY)_pf - (YX)_pf = X_pY_pf - Y_pX_pf$.
- $f[X, Y] \in \mathfrak{X}(M)$
- $f[X, Y](p) \in T_pM$: It is $f(p)[X, Y]_p = f(p)X_pY_p - f(p)Y_pX_p$
- $[fX, gY] \in \mathfrak{X}(M)$: It is equal to $fg\,[X, Y]$.

1.4 Cotangent Bundle and Covector Field

- T^*M: The ***cotangent bundle*** on M. It is the ***union*** of *all cotangent spaces* for all $p \in M$. *Cotangent bundle* itself is a $2n$***-manifold***.

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

- π: The ***natural projection*** $\pi : T^*M \to M$ onto the manifold M. $\pi(p, \xi) = p$. It is a ***smooth surjective submersion***. Each cotangent space is ***the level set*** of π, i.e. $T_p^*M = \pi^{-1}(p)$.
- ω: A ***covector field***. It is a ***section*** of π, i.e. a continuous map $X : M \to T^*M$ so that $\pi \circ \omega = \mathrm{Id}_M$. That is, $\pi(\omega(p)) = p$. The value of ω at p is denoted as $\omega_p := \omega(p) \in T_p^*M$. ω_p is *a tangent covector vector* at p.
- $\mathfrak{X}^*(M) := \Gamma(T^*M)$: The ***vector space*** of all covector fields on M. $\Gamma(T^*M) =$ the vector space of all sections on cotangent bundle T^*M. This is a n***-dimensional space***.
- $f\omega \in \mathfrak{X}^*(M)$: at each point p, $(f\omega)_p = f(p)\omega_p$
- $f\omega(p) \in T_p^*M$: It is $(f\omega)_p$
-

$$dx^i \in \mathfrak{X}^*(M)$$

forms a set of dual basis in $\mathfrak{X}^*(M)$. It is called ***the (local) coordinate coframes of*** M. Note that it drops dependency on p.

- $\omega(X) \in \mathcal{C}^\infty(M)$: defines ***a smooth function*** on M, i.e. $\omega(X) : M \to \mathbb{R}$ for each $X \in \mathfrak{X}(M)$.
- $\omega(X)(p) \in \mathbb{R}$: It is equal to $\omega_p(X_p)$
- $df \in \mathfrak{X}^*(M)$: It is ***a differential 1-form*** and also is ***a covector field***.
- $g\, df \in \mathfrak{X}^*(M)$: This is the same as $g\,\omega$, where $\omega = df$.
- $df(X) \in \mathcal{C}^\infty(M)$: $df(X) = Xf$, it is also a *linear function*.
- $df(X)(p) \in \mathbb{R}$: $df_p(X_p) = X_p f$
- $Y(\omega(X)) \in \mathcal{C}^\infty(M)$: Note that $g := \omega(X) \in \mathcal{C}^\infty(M)$ is a smooth function on M for given X. Thus $Y(\omega(X)) = Y g$ is a smooth function on M
- $Y(\omega(X))(p)$: It is equal to $Y_p(\omega(X))$. That is, $Y_p\, f$, for smooth function $f = \omega(X)$.
- $Y(df(X))$: $Y(df(X)) = YXf$
- F^*: ***the pullback operator***: $T^*N \to T^*M$ for $F : M \to N$. It maps a covector field $\omega \in \mathfrak{X}^*(N)$ to a covector field $\eta = F^*\omega \in \mathfrak{X}^*(M)$
- $F^*\omega \in \mathfrak{X}^*(M)$: it is a covector field on M where ω is a covector field on N.
- $F^*\omega(p) \in T_p^*M$: It is $(F^*\omega)_p$, which is a *covector* on M
- $(F^*\omega)_p(v) \in \mathbb{R}$: $(F^*\omega)_p(v) = \omega_p(dF_p(v))$
- F^*df: This is equal to $F^*df = d(f \circ F)$
- F^*dy^j: $F^*df = d(y^j \circ F) = dF^j$

1.5 Vector Bundle and Section

- E: denote the ***vector bundle***. The definition of vector bundle is for a pair (E, π). A vector bundle is *a generalization of tangent bundle*,

$$E = \bigsqcup_{p \in M} E_p$$

  E is also called ***the total space of vector bundle*** and M is its ***base***.
- π: is the ***surjective continuous map (i.e. projection map)*** $\pi : E \to M$, which has two properties:
  1. its ***fiber*** $E_p = \pi^{-1}(p)$ is a ***vector space*** of (the same) dimension k
  2. There exists a ***local homemorphism*** from neighborhood $\pi^{-1}(U)$ in E to $U \times \mathbb{R}^k$; i.e. $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k \subseteq M \times \mathbb{R}^k$ such that $\pi = \pi_U \circ \Phi$. Moreover, $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is an ***isomorphism***.
- E_p: *the **fiber** of* π at $p \in M$, i.e. $E_p = \pi^{-1}(p)$. This is a k-dimensional vector space. It is a generalization of tangent space T_pM;
- k: the ***rank*** of vector bundle E, which is the ***dimension*** of each fiber.

- Φ: is called ***a local trivialization*** of E over $U \subseteq M$. It is a homemorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k \subseteq M \times \mathbb{R}^k$ and for $(p, v) \in E$, $\pi_U(\Phi(p, v)) = p$. Restricting the local trivialization in each fiber will have an ***isomorphism*** $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$. That is Φ will map each fiber to a k-dimensional Euclidean space. Φ is *a* ***tool to build a coordinate map*** *of* E.

  For smooth vector bundle, Φ is a ***diffeomorphism***. If $U = M$, then E is ***globally trivial*** since it admits a *global trivialization* over M.

- τ: is called ***the transition function*** between the local trivializations Φ and Ψ. It is the smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$, for U, V both neighborhoods in M corresponding to two local trivializations Φ and Ψ. The map $\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$.

- $\tau(p)$: for $p \in M$ is a ***generalization*** of ***the Jacobian matrix*** between two ***coordinate maps*** *in vector bundle.*

- σ: a ***section*** of vector bundle E, which is a section of π, i.e. a *continous* map $\sigma : M \to E$ so that $\pi \circ \sigma = \mathrm{Id}_M$. $\pi(\sigma(p)) = p$ for all $p \in M$. A section is a ***generalization*** of ***vector fields***.

- $\sigma(p)$: a ***vector*** in E_p. It is an abstraction of tangent vector in T_pM.

- $\Gamma(E)$: the ***vector space*** of all sections on E. For example, $\mathfrak{X}(M) = \Gamma(TM)$.

- $f\sigma \in \Gamma(E)$: *a section* on E. $(f\sigma)(p) = f(p)\sigma(p)$

- (σ_i): ***a local frame for*** E ***over*** $U \subseteq M$ is a k-tuple $(\sigma_1, \ldots, \sigma_k)$ in $\Gamma(E)$ such that $(\sigma_1(p), \ldots, \sigma_k(p))$ forms a *basis* for the fiber E_p at each point $p \in U$. $(\sigma_1, \ldots, \sigma_k)$ forms *the* ***basis*** *for all sections* $\Gamma(E)$. It is often abbreviated as "***a frame for*** M"

- $(\pi^{-1}(V), \widetilde{\varphi})$: ***the smooth chart for*** E, given smooth chart (V, φ), local frames (σ_i), $\widetilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ such that $\widetilde{\varphi}\left(v^i\sigma_i(p)\right) = (\varphi(p), v^1, \ldots, v^k)$

1.6 Submerision, Immersion and Embedding

- ι: The ***inclusion map*** $\iota : S \hookrightarrow M$. The ***canonical inclusion map*** is $\widehat{\iota}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0) \in \mathbb{R}^n$, i.e. to ***pad zeros*** until the output dimension matches. ι is an ***injective linear map***. All ***immersion*** F has representation locally as the *canonical inclusion map.*

- π: The ***projection map*** $\pi : S \subseteq N \to M$. The ***canonical projection map*** is $\widehat{\pi}(x^1, \ldots, x^m, \ldots, x^n) = (x^1, \ldots, x^m) \in \mathbb{R}^m$, i.e. to ***truncate*** until the output dimension matches. π is an ***surjective linear map***. All ***submersion*** F has representation locally as the *canonical projection map.*

- rank F at p: for ***smooth function*** $F : M \to N$. It is *the rank of* ***differential*** *of* F *at* p, i.e. rank dF_p or ***the rank of Jacobian matrix*** at p under coordinate representation. rank $F \leq \min\{\dim M, \dim N\}$. If the equality holds, then F is ***of full rank***.

1.7 Tensors

- $v_1 \otimes \ldots \otimes v_k$: for $v_i \in V_i$ vector space, $i = 1, \ldots, k$. This is a ***tensor product of k vectors***. It is a *k**-tuple*** $(v_1, \ldots, v_k)$ that also admits the ***multi-linearity property***. That is $(v_1 \otimes \ldots \otimes (a\, v_i + b\, v_i') \otimes \ldots \otimes v_k) = a\,(v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_k) + b\,(v_1 \otimes \ldots \otimes v_i' \otimes \ldots \otimes v_k)$ for all $1 \le i \le k$ and $a, b \in \mathbb{R}$.

  Therefore $v_1 \otimes \ldots \otimes v_k = \Pi(v_1, \ldots, v_k)$ for some ***(quotient) projection map*** Π.

- $V_1 \otimes \ldots \otimes V_k$: the ***tensor product of spaces*** $(V_i)_{i=1}^k$. The tensor product space $V_1 \otimes \ldots \otimes V_k$ can be obtained as ***the quotient space*** of $\mathcal{F}(V_1 \times \ldots \times V_k)/\mathcal{R}$ where $\mathcal{F}(S)$ is the set of all finite linear combinations of elements in S and $\mathcal{R} \subseteq \mathcal{F}(V_1 \times \ldots \times V_k)$ is the subspace spanned by the *multi-linearity equation*.

- $\omega^1 \otimes \ldots \otimes \omega^k$: for $\omega^i \in V_i^*$ dual vector space. This is a ***tensor product of k covectors***. This is also a ***multi-linear function*** $\alpha : V_1 \times \ldots \times V_k \to \mathbb{R}$. The value of $\omega^1 \otimes \ldots \otimes \omega^k(v_1, \ldots, v_k) = \prod_{i=1}^k \omega^i(v_i)$

- $V_1^* \otimes \ldots \otimes V_k^*$: the ***tensor product of dual spaces*** $(V_i^*)_{i=1}^k$.

- $T^k V$: is called ***the space of contravariant k-tensors*** on V. This is equal to $V_1 \otimes \ldots \otimes V_k$ where $V_i = V$ are all equal. The ***dimension*** of this vector space is n^k where $n = \dim V$.

- $T^k V^*$: is called ***the space of covariant k-tensors*** on V. This is equal to $V_1^* \otimes \ldots \otimes V_k^*$ where $V_i^* = V^*$ are all equal. The ***dimension*** of this vector space is n^k where $n = \dim V^*$.

- $\omega^1 \otimes \ldots \otimes \omega^k \in T^k V^*$: is called ***a covariant k-tensor***.

- k: is called *the **rank** of tensor*.

- $v_1 \otimes \ldots \otimes v_k \in T^k V$: is called ***a contravariant k-tensor***. It is also identifies as a ***multi-linear functions*** on $(\omega^1, \ldots, \omega^k)$.

- $T^{(k,l)} V$: is called ***the space of mixed*** (k,l)***-tensors*** on V. Its element is $v_1 \otimes \ldots \otimes v_k \otimes \omega^1 \otimes \ldots \otimes \omega^l$.

- $a_{i_1, \ldots, i_k} \in \mathbb{R}$: for $\alpha = a_{i_1, \ldots, i_k} \omega^{i_1} \otimes \ldots \otimes \omega^{i_k}$. It is a component for the covariant k-tensor. $a_{i_1, \ldots, i_k} = \alpha\,(E_1, \ldots, E_k)$ for (E_i) as basis for V_i.

- $T^k T_p^* M = T^k(T_p^* M)$: ***the space of all covariant k-tensors on*** $V = T_p M$ at p. This space has ***the finite dimension of*** n^k.

- $T^k T_p M = T^k(T_p M)$: ***the space of all contravariant k-tensors on*** $V = T_p M$ at p. This space has ***the finite dimension of*** n^k.

- $T^k T^* M$: is the ***vector bundle of all covariant k-tensors*** on M.

  $$T^k T^* M = \bigsqcup_{p \in M} T^k T_p^* M$$

  The vector bundle has a projection map $\pi : T^k T^* M \to M$.

- $T^k T M$: is the ***vector bundle of all contravariant k-tensors*** on M.

  $$T^k T M = \bigsqcup_{p \in M} T^k T_p M$$

The vector bundle has a projection map $\pi : T^kTM \to M$.

- $T^{(k,l)}TM$: is the ***vector bundle of all mixed*** (k,l)***-tensors*** on M.
$$T^{(k,l)}TM = \bigsqcup_{p \in M} T^{(k,l)}T_pM$$
- $T^{(0,0)}TM$: is equal to $M \times \mathbb{R}^k$
- $T^{(0,1)}TM$: is equal to *the cotangent bundle* T^*M.
- $T^{(1,0)}TM$: is equal to *the tangent bundle* TM.
- $T^{(k,0)}TM$: is equal to T^kTM.
- $T^{(0,k)}TM$: is equal to T^kT^*M.
- $\mathcal{T}^k := \Gamma(T^kT^*M)$: is called ***the space of all covariant k-tensor fields***. It is the vector space of all sections on T^kT^*M. This is an ***infinite dimensional space*** for $k > 1$.
- $\Gamma(T^kTM)$: is called ***the space of all contravariant k-tensor fields***. It is the vector space of all sections on T^kTM. This is an ***infinite dimensional space*** for $k > 1$.
- $\omega^1 \otimes \ldots \otimes \omega^k \in \Gamma(T^kT^*M)$: is ***a covariant*** k***-tensor field on*** M. Each $\omega^i \in \Gamma(T^*M) := \mathfrak{X}^*(M)$ is a *covector field* on M.
- $(\omega^1 \otimes \ldots \otimes \omega^k)_p \in T^kT_p^*M$: *a covariant* k*-tensor* at p. It is equal to $(\omega_p^1 \otimes \ldots \otimes \omega_p^k)$, which is ***a multi-linear function*** as *the tensor product of covectors at* p.
- $X_1 \otimes \ldots \otimes X_k \in \Gamma(T^kTM)$: is ***a contravariant*** k***-tensor field on*** M. Each $X_i \in \Gamma(TM) := \mathfrak{X}(M)$ is a *vector field* on M.
- $(X_1 \otimes \ldots \otimes X_k)(p) \in T^kT_pM$: is *a contravariant* k*-tensor* at p. It is equal to ***the tensor product of tangent vectors*** at p, i.e. $X_1(p) \otimes \ldots \otimes X_k(p)$.
- $dx^{i_1} \otimes \ldots \otimes dx^{i_k} \in \Gamma(T^kT^*M)$: is a covariant k-tensor field and is a ***basis*** for $\Gamma(T^kT^*M)$
- $(dx^{i_1} \otimes \ldots \otimes dx^{i_k})_p$: $= (dx_p^{i_1} \otimes \ldots \otimes dx_p^{i_k})$ is a basis for $T^kT_p^*M$
- $\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \in \Gamma(T^kTM)$: is a contravariant k-tensor field and is a ***basis*** for $\Gamma(T^kTM)$
- $\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}}(p)$: $= (\frac{\partial}{\partial x^{i_1}}\big|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}}\big|_p)$ is a basis for T^kT_pM
- $A(X_1, \ldots, X_k) \in \mathcal{C}^\infty(M)$: is a ***smooth function*** $M \to \mathbb{R}$ where $A \in \Gamma(T^kT^*M)$ is a covariant k-tensor field and $X_1, \ldots, X_k \in \mathfrak{X}(M)$ are smooth vector fields on M. It is a generalization of $\omega(X)$.
- $A(X_1, \ldots, X_k)(p) \in \mathbb{R}$: it is equal to $A_p(X_1|_p, \ldots, X_k|_p)$. This is close to $\omega(X)(p) = \omega_p(X_p)$.
- $\mathcal{A}$: is ***a multi-linear map*** over $\mathcal{C}^\infty$ induced by *the covariant* k*-tensor field* $A \in \Gamma(T^kT^*M)$. That is, $\mathcal{A} : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ where $\mathcal{A}(X_1, \ldots, X_k) = A(X_1, \ldots, X_k)$ as above.
- F^*: is ***the pullback operator*** on covariant tensor fields. $F^* : \Gamma(T^kT^*N) \to \Gamma(T^kT^*M)$ where $F : M \to N$.
- F^*A: is a *covariant* k*-tensor field* on M when A is a covariant k-tensor field on N

- $F^*A(X_1, \ldots, X_k)$: is a smooth function on M where A is a covariant k-tensor field on N and (X_i) are vector fields on M.

1.8 Symmetric Tensor Fields

- $\Sigma^k(V^*) \subseteq T^k V^*$: is the *vector space of all* ***symmetric covariant*** k***-tensors*** on V. A covariant k-tensor is ***symmetric*** if its value will not change when rearranging the order of its input vectors. It has ***dimension*** $\binom{n+k-1}{k}$ where $n = \dim V$.
- $\sigma \in S_k$: is a ***permutation*** of set $\{1, \ldots, k\}$. S_k is *the permutation group* for $\{1, \ldots, k\}$. $\sigma(i) = j$.
- $\mathrm{sgn}\,(\sigma) \in \{-1, +1\}$: is called the ***sign*** of permutation σ. Note that every permutation of a finite set can be expressed as the product of *transpositions*. $\mathrm{sgn}\,(\sigma) = (-1)$ if σ is composed of odd number of transpositions; $\mathrm{sgn}\,(\sigma) = 1$ if σ is is composed of even number of transpositions.
- ${}^{\sigma}\alpha \in T^k V^*$: is the covariant k-tensor ***after permutation on the indices*** of its input ${}^{\sigma}\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$.
- $\mathrm{Sym}\,(\alpha) \in \Sigma^k(V^*)$: is the ***symmetrization*** of a tensor α. $\mathrm{Sym}\,: T^k V^* \to \Sigma^k(V^*)$ is a projection of a covariant k-tensor α to its symmetrized version. It is the ***average*** of ${}^{\sigma}\alpha$ for all possible permutations in S_k. $\mathrm{Sym}\,(\alpha) = \frac{1}{k!}\sum_{\sigma \in S_k} {}^{\sigma}\alpha$.
- $\alpha\beta \in \Sigma^{k+l}(V^*)$: is ***the symmetric product*** between two covariant k-tensors $\alpha \in T^k V^*$ and $\beta \in T^l V^*$. $\alpha\beta = \mathrm{Sym}\,(\alpha \otimes \beta)$.
- $\Sigma^k(T^*M) \subseteq T^k T^*M$: is *the* ***subbundle*** *of* ***symmetric covariant*** k***-tensor fields*** on M.

$$\Sigma^k(T^*M) = \bigsqcup_{p \in M} \Sigma^k(T_p^*M)$$

- $\Gamma(\Sigma^k(T^*M)) \subseteq \Gamma(T^k T^*M)$: is *the vector space of all* ***symmetric covariant*** k***-tensor fields*** *on* M.
- $dx^i dx^j \in \Gamma(\Sigma^2(T^*M))$: is the ***symmetric product*** between ***two covector fields*** $dx^i, dx^j \in \Gamma(T^*M)$. $dx^i dx^j$ is a ***symmetric covariant*** 2***-tensor field*** on M, i.e. $dx^i dx^j = dx^j dx^i$. This is also one of basis in $\Sigma^2(T^*M)$.
- $g \in \Gamma(\Sigma^2(T^*M))$: is ***the Riemannian metric***. A Riemannian metric is a ***symmetric covariant*** 2***-tensor*** that is also ***positive definite***.
- $(g_{i,j})$: the *matrix (**component** function)* of *Riemannian metric*, i.e. $g = g_{i,j} dx^i\, dx^j$. This is a ***positive definite (PSD) matrix***.
- $(g^{i,j})$: the ***inverse*** of matrix $(g_{i,j})$.
- $\widehat{g}$: a ***bundle homemorphism*** $TM \to T^*M$. $\widehat{g}(X)$ is a covector field. And $\widehat{g}^{-1}(\omega)$ is a vector field.
- $\widehat{g}(X) = g(X, \cdot) \in \mathfrak{X}^*(M)$: is a ***covector field called*** $X^\flat$ so that $\widehat{g}(X)(Y) = g(X, Y)$.
- $\widehat{g}^{-1}(\omega) \in \mathfrak{X}(M)$: is a ***vector field called*** $\omega^\sharp$.

- $\flat$: is called the ***flat operator***. $\flat : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ is an ***isomorphism***, called ***musical isomorphism***. Its inverse is $\sharp$.
- $\sharp$: is called the ***sharp operator***. $\sharp : \mathfrak{X}^*(M) \to \mathfrak{X}(M)$ is an ***isomorphism***, called ***musical isomorphism***. Its inverse is $\flat$.
- $X^\flat \in \mathfrak{X}^*(M)$: is a ***covector field*** obtained from vector field X by ***lowering an index***. $X^\flat(\cdot) = \langle X\,,\,\cdot\rangle_g$.
- $X^\flat(Y) \in \mathcal{C}^\infty(M)$: $= g(X, Y) = \langle X\,,\,Y\rangle_g$.
- $\omega^\sharp \in \mathfrak{X}(M)$: is a ***vector field*** obtained from covector field ω by ***raising an index***. $\omega^\sharp(\cdot) = \langle \omega\,,\,\cdot\rangle_{g^{-1}}$.
- $\omega^\sharp f \in \mathcal{C}^\infty(M)$: is a smooth function since $\omega^\sharp$ is a vector field which is a derivation operator.
- $\text{grad}\, f \in \mathfrak{X}(M)$: is the ***gradient*** of f, i.e. $\text{grad}\, f = (df)^\sharp$. It is a ***vector field obtained from*** df ***by raising an index***.
- $(df)^\sharp$: $= \text{grad}\, f$ is a vector field as above.
- $F^\flat \in \Gamma(T^{(k-1,l+1)}TM)$: for (k,l)-tensor field F, this is a $(k-1, l+1)$-tensor field.
- $F^\sharp \in \Gamma(T^{(k+1,l-1)}TM)$: for (k,l)-tensor field F, this is a $(k+1, l-1)$-tensor field.

1.9 Differential Forms

- $\Lambda^k(V^*) \subseteq T^k V^*$: is the *vector space of all* ***alternating covariant*** k***-tensors*** on V. A covariant k-tensor is ***alternating*** if its value will change sign whenever two indices of its input vectors interchange. It has ***dimension*** $\binom{n}{k}$ where $n = \dim V$.
- $\text{Alt}\,(\alpha) \in \Lambda^k(V^*)$: is the ***alternation*** of a tensor α. $\text{Alt}\, : T^k V^* \to \Lambda^k(V^*)$ is a projection of a covariant k-tensor α to its alternating version. It is the ***signed average*** of ${}^\sigma\alpha$ for all possible permutations in S_k. $\text{Alt}\,(\alpha) = \frac{1}{k!}\sum_{\sigma\in S_k}(\text{sgn}\,(\sigma))({}^\sigma\alpha)$. *An alternating covariant* k*-tensor* is also called a ***k-covector***, ***exterior form***, or ***multicovector***.
- $\alpha \wedge \beta \in \Lambda^{k+l}(V^*)$: is ***the wedge product*** or ***exterior product*** between $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$. $\alpha\wedge\beta = \frac{(k+l)!}{k!l!}\text{Alt}\,(\alpha\otimes\beta)$.
- $I = (i_1\,,\ldots,i_k)$: is called a ***multi-index***. If $1 \le i_1 \le \ldots \le i_k \le n$, then it is called an ***increasing multi-index***.
- $I_\sigma = (i_{\sigma(1)}\,,\ldots,i_{\sigma(k)})$: is a *permutation* of multi-index.
- $\epsilon^I \in \Lambda^k(V^*)$: is equal to $\epsilon^{i_1\,,\ldots,i_k} = \epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$ where (ϵ^i) is dual basis in V^*.
- $a_I \in \mathbb{R}$: $:= a_{i_1\,,\ldots,i_k}$.
- $\epsilon^I(v_1\,,\ldots,v_k) \in \mathbb{R}$: is the ***determinant*** of $k\times k$ sub-matrix $\det(\epsilon^i(v_j))_{i\in I, j\in J}$
- $\delta^I_J \in \{-1, 0, 1\}$: is equal to $\text{sgn}\,(\sigma) = \pm 1$ if $J = I_\sigma$ for some $\sigma \in S_k$ and I, J do not have a repeated index; otherwise is equal to 0.
- $\sum'_I a_I\epsilon^I$: represent $\sum_{\{1\le i_1\le\ldots\le i_k\le n\}} a_I\epsilon^I$; that is, summation over ***all increasing multi-index***.

- IJ: $= (i_1, \ldots, i_k, j_1, \ldots, j_k)$ is the ***concatenation*** of two multi-index $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$.
- $\epsilon^{IJ} \in \Lambda^{k+l}(V^*)$: is a basis $(k+l)$-covector when $\epsilon^k \in \Lambda^k(V^*)$, $\epsilon^I \in \Lambda^l(V^*)$. We have formula $\epsilon^{IJ} = \epsilon^I \wedge \epsilon^J$.
- $\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) \in \mathbb{R}$: is the ***determinant*** of $k \times k$ sub-matrix $\det(\omega^i(v_j))_{i \in I, j \in J}$ where $i \in I$ is the row number, $j \in J$ is the column number.
- $\Lambda(V^*)$: is called ***the exterior algebra*** or ***Grassman algebra***. It is the ***direct sum*** of *vector space of all* ***alternating covariant tensors*** of ***rank*** $k \leq n$ on V.

$$\Lambda(V^*) = \oplus_{k=1}^n \Lambda^k(V^*)$$

  The *exterior product* $\wedge$ is an *operation* in this algebra. This algebra is ***graded*** and ***anticommutative***.
- ι_v: is called an ***interior product/multiplication operatior*** where $v \in V$. The map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ as $\iota_v(\omega)(v_2, \ldots, v_k) = \omega(v, v_2, \ldots, v_k)$. It is also denoted as $v \lrcorner \omega$ where $\omega \in \Lambda^k(V^*)$.
- $v \lrcorner \omega$: $= \iota_v(\omega)$ see above.
- $\Lambda^k(T^*M) \subseteq T^kT^*M$: is *the* ***subbundle*** *of* ***alternating covariant k-tensor fields***.

$$\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$$

- $\Omega^k(M) := \Gamma(\Lambda^k(T^*M)) \subseteq \Gamma(T^kT^*M)$: is *the vector space of all* ***alternating covariant k-tensor fields*** *on M*.
- $\omega \in \Omega^k(M)$: is called ***a differential k-form*** or just ***k-form***. It is an *alternating covariant k-tensor field*.
- $\Omega^1(M)$: $= \mathfrak{X}^*(M)$ is the space of covector fields on M.
- $\Omega^0(M)$: $= \mathcal{C}^\infty(M)$ is the space of all smooth functions on M.
- $df \in \Omega^1(M) = \mathfrak{X}^*(M)$: is a ***differential 1-form***.
- $\Omega^*(M)$: is ***the exterior algebra*** for ***all differential k-forms*** on M. It is ***the direct sum*** of all $\Omega^k(M)$. The exterior product $\wedge$ is an operation of this algebra.
- $\omega \wedge \eta \in \Omega^{k+l}(M)$: for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$.
- $dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^k(M)$: is ***a basis differential k-form*** when $i_1 \leq \ldots \leq i_k$.
- dx^I: $= dx^{i_1} \wedge \ldots \wedge dx^{i_k}$
- $F^*\omega \in \Omega^k(M)$: is the ***pullback*** of $\omega \in \Omega^k(N)$ by $F : M \to N$.
- F^*dy: $= d(y \circ F)$.
- $dx^I(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}) \in \{-1, 0, 1\}$: $= \delta^I_J$.
- ι_X: is ***the interior multiplication***: $\Omega^k(M) \to \Omega^{k-1}(M)$ for any $X \in \mathfrak{X}M$.

- $\iota_X(\omega) = X \lrcorner\, \omega \in \Omega^{k-1}(M)$: is *a differential* $(k-1)$*-form.*
- $\iota_X(\omega)(p) = (X \lrcorner\, \omega)_p \in \Lambda^k(T_p^*M)$: is a $(k-1)$*-covector.* $(X \lrcorner\, \omega)_p = X_p \lrcorner\, \omega_p$.
- d: is called ***the exterior derivative operation***. It is a ***linear*** map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ that satisfies:
    1. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
    2. $d \circ d \equiv 0$
    3. $df(X) = Xf$ for $f \in \Omega^0(M) := \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$.
- $d\omega \in \Omega^{k+1}(M)$: is a $(k+1)$-form, where ω is a k-form
- $d(u\, dx)$: $= du \wedge dx$.
- $F^*d\omega \in \Omega^{k+1}(M)$: is a $(k+1)$-form on M, where ω is a k-form on N.
- $d\,(F^*\omega)$: We have the formula $F^*d\omega = d\,(F^*\omega)$, which is called ***the naturality of the exterior derivative***.
- $F^*(\omega \wedge \eta) \in \Omega^{k+l}(M)$: $= F^*\omega \wedge F^*\eta$
- $F^*f \in \Omega^0(M) = \mathcal{C}^\infty(M)$: $= f \circ F$ where $f \in \mathcal{C}^\infty(N) = \Omega^0(N)$
- $d\omega(X,Y) \in \mathcal{C}^\infty(M)$: where $\omega \in \mathfrak{X}^*(M) = \Omega^1(M)$, and $X, Y \in \mathfrak{X}(M)$. Note that $d\omega \in \Omega^2(M)$ is a *differential* 2*-form.* For $\alpha = d\omega$, we know that $\alpha(X,Y) : M \to \mathbb{R}$ is a smooth function on M such that $\alpha(X,Y)(p) = \alpha_p(X_p, Y_p) \in \mathbb{R}$.
- $X(\omega(Y)) \in \mathcal{C}^\infty(M)$: Note that $\omega(Y) \in \mathcal{C}^\infty(M)$ is a smooth function since ω is a differential 1-form. Then $X(\omega(Y)) = Xf$ where $f = \omega(Y)$.
- $\omega([X,Y]) \in \mathcal{C}^\infty(M)$: Note that $[X,Y] \in \mathfrak{X}(M)$ is a vector field, so $\omega([X,Y]) := \omega(Z)$ where $Z = [X,Y]$. Thus it is a smooth function on M.

1.10 Connections

- ∇: the ***connection*** *symbol.* It is the (smooth) map $\mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that denotes ***the covariant derivative*** *of a section on vector bundle E along the (tangential) direction* specified by a vector field. A connection operation satisfies 3 rules: 1) linear over $\mathcal{C}^\infty(M)$ in its first argument; 2) linear over $\mathbb{R}$ in its second argument; 3) the product rule in its second argument.
- $\overline{\nabla}$: the ***Euclidean connection*** *symbol.* No rotation of axis during the directional derivative process.
- $\nabla^\top$: ***the tangential connection*** on embedded Riemannian submanifold as the tangential projection of the Euclidean connection.
- $\nabla_X Y \in \Gamma(E)$ or $\mathfrak{X}(M)$: ***the covariant derivative of*** Y ***in the direction of*** X. Note that it is not a $(1,2)$-tensor since it is not linear in $\mathcal{C}^\infty(M)$ in its second argument.
- $\nabla^a_X Y - \nabla^b_X Y \in \Gamma(T^{(1,2)}TM)$: it is a $(1,2)$-tensor.
- $\nabla_X f \in \mathcal{C}^\infty(M)$: $= Xf$, i.e. the covariant derivative of a smooth function $f \in \mathcal{C}^\infty(M)$

along direction of X.

- $\nabla_X Y|_p \in T_pM$: $= \nabla_{X_p} Y_p$. It is equal to the *covariant derivatives* of *vector field* $Y \in \mathfrak{X}(M)$ along the direction X_p *in* T_pM.
- $\nabla_{fX} Y \in \mathfrak{X}(M)$: the covariant derivative of Y along direction of fX. $\nabla_{fX} Y = f\,\nabla_X Y$.
- $\nabla_X(fY) \in \mathfrak{X}(M)$: $= X(fY) + f\,\nabla_X Y$.
- $\Gamma^k_{i,j}$: ***the coefficient for connection*** on TM. They are n^3 smooth functions $U \to \mathbb{R}$. The lower two indices i, j corresponds to the basis of direction vector field and the basis of the target vector field, and the upper index k corresponds to the basis for the resulting vector field. If the connection is a *metric connection*, then these functions are called ***the Christoffel Symbols***.
- $\nabla_{\partial_i} \partial_j \in \mathfrak{X}(M)$: $= \Gamma^k_{i,j} \partial_k$; It accounts for the rotation of basis vector ∂_j along the other basis direction ∂_i
- $\nabla_{(\nabla_X Y)} Z \in \mathfrak{X}(M)$: the covariant direvatives of Z along direction $W = \nabla_X Y$, which is also the directional derivatives of Y along X.
- $\nabla_{[X,Y]} Z \in \mathfrak{X}(M)$: the covariant direvatives of Z along direction $[X, Y]$. Note that if X, Y orthorgonal, then $[X, Y] = 0$, it will becomes 0.
- $\nabla_X \nabla_Y Z \in \mathfrak{X}(M)$: the ***covariant direvatives*** of Z **first** along direction Y and **then** taking *covariant derivatives* along X. (i.e. the ***second-order*** *derivatives* for two directions)
- $\nabla_Z \langle X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$: $= Z\,\langle X\,,\,Y\rangle$ it is the covariant direvatives of the inner product $\langle X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$ along direction Z
- $Z\,\langle X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$: Note that $\langle X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$. So this is just Zg where $g = \langle X\,,\,Y\rangle$.
- $\langle \nabla_Z X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$: This is the inner product $\langle W\,,\,Y\rangle$ where $W = \nabla_Z X$. For ***metric connection***, $Z\,\langle X\,,\,Y\rangle = \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle$
- $\nabla F \in \Gamma(T^{(k,l+1)}TM)$: is called ***the total covariant derivative of*** F. It is a $(k, l+1)$-tensor field for a (k, l)-tensor field F. $\nabla F(\;\ldots\;, Y) = (\nabla_Y F)(\;\ldots\;)$
- $(\nabla F)^\flat \in \Gamma(T^{(k-1,l+2)}TM)$: for a (k, l)-tensor field F, ∇F is a $(k, l+1)$-tensor, then this is a $(k-1, l+2)$-tensor field;
- $(\nabla F)^\sharp \in \Gamma(T^{(k+1,l)}TM)$: for a (k, l)-tensor field F, ∇F is a $(k, l+1)$-tensor, then this is a $(k+1, l)$-tensor field;
- $\nabla Y(X) \in \mathfrak{X}(M)$: $= \nabla_X Y$ where $Y \in \Gamma(T^{(1,0)}TM)$. $\nabla Y \in \Gamma(T^{(1,1)}TM)$
- $\nabla\omega(X) \in \mathfrak{X}^*(M)$: $= \nabla_X \omega$ where $\omega \in \Gamma(T^{(0,1)}TM)$, so $\nabla\omega \in \Gamma(T^{(0,2)}TM)$
- $\nabla_X \omega \in \mathfrak{X}^*(M)$: Here ∇ is ***the induced connection*** in T^*M from ∇ in TM.
- $\nabla\omega(Y, X) \in \mathcal{C}^\infty(M)$: $= (\nabla_X \omega)(Y)$ where $\omega \in \Gamma(T^{(0,1)}TM)$, so $\nabla\omega \in \Gamma(T^{(0,2)}TM)$
- $(\nabla_X \omega)(Y) \in \mathcal{C}^\infty(M)$: $= \langle \nabla_X \omega\,,\,Y\rangle \neq \nabla_X(\omega(Y))$. This is just a covector field $\eta = \nabla_X \omega$ act on a vector field Y. In fact $(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y)$.
- $\nabla_X(\omega(Y)) \in \mathcal{C}^\infty(M)$: $= \nabla_X \langle \omega\,,\,Y\rangle$. It is the covariant derivatives of function $\omega(Y) \in \mathcal{C}^\infty(M)$ along X. Also it is equal to $(\nabla_X \omega)(Y) + \omega(\nabla_X Y)$

- $\nabla^2 F \in \Gamma(T^{(k,l+2)}TM)$: is called ***the second covariant derivative of*** F. It is a $(k, l+2)$-tensor field for a (k,l)-tensor field F. $\nabla F(\ \dots\ , Y, X) = (\nabla^2_{X,Y} F)(\ \dots\ )$. Note that we have $\nabla^2_{X,Y} F = \nabla_X \nabla_Y F - \nabla_{(\nabla_X Y)} F$
- $\mathrm{tr}(F) \in \Gamma(T^{(k-1,l-1)}TM)$: is called *the* ***contraction*** *or* ***trace*** of F. For (k,l)-tensor F, this is equal to $(k-1, l-1)$-tensor . Note that $\mathrm{tr}(v \otimes \omega) = \omega(v)$ is the trace of the matrix representation of $v \otimes \omega = [\omega_i\, v^j]$
- $V(t) \in \mathfrak{X}(\gamma)$: is called ***the vector field*** V ***along curve*** γ. If $\widetilde{V}$ is the extension of V in M then $V(t) = \widetilde{V}_{\gamma(t)}$.
- $\mathfrak{X}(\gamma)$: is the vector space of all vector field $V(t)$ along curve γ.
- $\gamma'(t) \in \mathfrak{X}(\gamma)$: is ***the velocity vector field*** of curve γ, which is a vector field along the curve γ;
- $\nabla_{\gamma'(t)} V \in \mathfrak{X}(M)$: is the covariant derivative of V along the velocity vector field γ'. V restricted on image of γ will be the vector field along curve.
- D_t: is the ***covariant derivative along the curve*** γ. It is a map $\mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, and $D_t V(t) = \nabla_{\gamma'(t)} \widetilde{V}$
- $\nabla_{\gamma'(t)} \gamma'(t) \in \mathfrak{X}(M)$: is called ***the tangential acceleration***, when viewed as a vector field in M. It is the directional derivative of velocity vector field $\gamma'(t)$ along the direction of velocity vector field.
- $D_t V(t) \in \mathfrak{X}(\gamma)$: is the covariant derivative of vector field $V(t)$ along the curve γ. For ***parallel transport***, $D_t V(t) \equiv 0$ for all t.
- $D_t \gamma'(t) \in \mathfrak{X}(\gamma)$: is ***the tangential acceleration***, when viewed as the vector field along γ.
- γ_v: is ***the maximal geodesic curve*** γ with initial point $\gamma(0) = p$ and $\gamma'(0) = v$. Note that for geodesic curve $D_t \gamma'(t) \equiv 0$.
- $P^{\gamma}_{t_0,t_1}$: is the map of ***parallel transport*** along γ from $t = t_0$ to $t = t_1$. $P^{\gamma}_{t_0,t_1} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$.
- $P^{\gamma}_{t_0,t_1} V(t) \in \mathfrak{X}(M)$: is the resulting vector field after *parallel transport* of $V(t)$ along γ. $P^{\gamma}_{t_0,t_1} V(t) = (D_t V(t_0))|_{t_1}$.
- exp: is ***the exponential map***: $\mathcal{E} \subseteq TM \to M$ that maps from a tangent vector v to a point in M that reached by the geodesic passing 0 with initial velocity given. $\exp(v\,t) = \gamma_v(t)$
- $\exp_p$: is ***the exponential map restricted at*** $p : \mathcal{E}_p \subseteq T_p M \to M$.
- $d(\exp_p)_0$: is *the* ***differential*** *of the exponential map restricted at* p *evaluated at* 0. This is *an* ***identity map*** $T_0(T_p M) \simeq T_p M \to T_p M$.

1.11 Curvatures

- $R \in \Gamma(T^{(1,3)}TM)$: a $(1,3)$-tensor called ***Riemann curvature endomorphism***; $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.
- $R(X,Y)Z \in \mathfrak{X}(M)$: is a vector field that is $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Compare it with *the second order covariant derivative* $\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{(\nabla_X Y)} Z$

- $Rm \in \Gamma(T^{(0,4)}TM)$: is called the ***Riemann curvature tensor***. It is a $(0,4)$-tensor. $Rm = R^\flat$ is obtained from *the Riemann curvature endomorphism* by *lowering an index*.
- $Rm(X,Y,Z,W) \in \mathcal{C}^\infty(M)$: It is the inner product of the $(1,3)$-tensor $R(X,Y)Z$ with W. $Rm(X,Y,Z,W) = \langle R(X,Y)Z\,,\, W\rangle$

2 Definitions and Theorems

2.1 Tangent Space and Differential at p

- Given (U, φ), $p \in U$, the basis vector in T_pM is defined via partial derivatives in $\mathbb{R}^n$ via differential of parameterization map $d\varphi^{-1}$ at $\varphi(p)$

$$\frac{\partial}{\partial x^i}\bigg|_p \equiv d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) \tag{1}$$

- The basis vector at T_pM acts on a smooth function f is the partial derivatives of f at p

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}\bigg|_p = \frac{\partial f}{\partial x^i}(p)$$

- For $F : M \to N$, where M, N are smooth manifolds, $v \in T_pM$, $g \in \mathcal{C}^\infty(N)$, the differential of F at p as a linear map is defined as

$$dF_p(v)g = v(g \circ F). \tag{2}$$

So $dF_p(v)g|_q = v\left(g(F(p))\right)|_p$ where $q = F(p)$. $g \in \mathcal{C}^\infty(N)$ and $g \circ F \in \mathcal{C}^\infty(M)$.

Remark

$$\begin{tikzcd}
& dF_p(v) \arrow[r, "|_{F(p)}"] & dF_p(v)g \arrow[d, leftrightarrow, "="] \\
& v \arrow[u, "dF_p"'] \arrow[r, "|_p"] & v(g \circ F) \\
g \arrow[r, "\circ F"'] \arrow[uur, "\text{acted on by}"] & g \circ F \arrow[u, "\text{acted on by}"'] &
\end{tikzcd}$$

- For $\gamma : J \to M$, $f \in \mathcal{C}^\infty(M)$,

$$\gamma'(t) = d\gamma \left(\frac{d}{dt}\bigg|_t \right) \tag{3}$$

$$\Rightarrow \gamma'(t)f \equiv d\gamma \left(\frac{d}{dt}\bigg|_t \right) f$$

$$= \frac{d}{dt}\bigg|_t (f \circ \gamma) \tag{4}$$

- The ***chain rule*** of differentials

$$d\left(G \circ F\right)_p = dG_{F(p)} \circ dF_p$$

Think of two systems that connected sequentially. dG takes output of dF_p as input. It also computes the evaluation point $F(p)$.

- The ***product rule*** (*Leibniz's Law*) of derivations at p: $v \in T_pM$, $f, g \in \mathcal{C}^\infty(M)$

$$v(fg) = g(p)v(f) + f(p)\,v(g)$$

2.2 Cotangent Space

- Given $(\partial/\partial x^i|_p)$ as basis in T_pM, (dx^i_p) is its ***dual basis***

$$dx^i_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta^i_j \tag{5}$$

- For ***real-valued*** smooth function $f : M \to \mathbb{R}$, $df_p : T_pM \to \mathbb{R}$, $v \in T_pM$

$$df_p(v) := vf \tag{6}$$

  In other word, df_p *is a* ***linear functional*** on T_pM, i.e. $df_p \in T^*_pM$.

- Let df_p acts on $\gamma'(0) \in T_{\gamma(0)}M$ where $p = \gamma(0)$

$$df_p(\gamma'(0)) := \gamma'(0)f = \frac{d}{dt}\bigg|_{t=0} (f \circ \gamma)$$

- For $F : M \to N$, where M, N are smooth manifolds, the ***pullback*** of covector ω on $T_{F(p)}N$ by F is a covector on T_pM.

$$(F^*\omega)_p(v) = \omega(dF_p(v)) \tag{7}$$

  Remark pullback at point p

$$\begin{tikzcd}
 & & (F^*\omega)_p(v) \arrow[d, leftrightarrow, "="] \\
v \arrow[r, "dF_p"] \arrow[rru, bend left, "(F^*\omega)_p"] & dF_p(v) \arrow[r, "\omega"] & \omega(dF_p(v))
\end{tikzcd}$$

2.3 Tangent Bundle and Vector Field

- Every smooth vector field in $\mathfrak{X}(M)$ has a (local) coordinate representation based on the coordinate chart $(U, (x^i))$ and local coordinate frames

$$X = X^i \frac{\partial}{\partial x^i} = \nabla \cdot \boldsymbol{X}$$
$$\text{where } X^i = X(x^i)$$

  We have product rule

$$X(fg) = g\,X(f) + f\,X(g)$$

- X_p can be computed via plug-in coordinate $\varphi(p) = \boldsymbol{x}$ into the component function

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

- If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related, for $F : M \to N$, then

$$Y_{F(p)} = dF_p(X_p) \tag{8}$$

In particular, if F is a diffeomorphism, then $F_* : TM \to TN$ is the ***pushforward operator*** so that F_*X is F-related to X.

$$(F_*X)_q = dF_p(X_p) = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right)$$

where $q = F(p)$, i.e. $p = F^{-1}(q)$.

Remark The pushforward operation for vector field

$$\begin{tikzcd}
 & & & (F_*X)_q \arrow[d, leftrightarrow, "="] \\
q \arrow[r, "F^{-1}"] \arrow[urrr, bend left, "(F_*X)|_\cdot"] & p \arrow[r, "dF|_\cdot"] \arrow[d, "X|_\cdot"'] & dF_p \arrow[r] & dF_p(X_p) \\
 & X_p \arrow[ur, bend right, "\text{acted on}"'] & &
\end{tikzcd}$$

- For any smooth function $f \in \mathcal{C}^\infty(N)$ on N, Y is F-related to X, then

$$X(f \circ F) = (Yf) \circ F \tag{9}$$

2.4 Cotangent Bundle and Covector Field

- For any $\omega \in \mathfrak{X}^*(M)$, it can be represented via linear combination of its coframes (dx^i)

$$\omega = \omega_i dx^i$$
$$\text{where } \omega_i = \omega\left(\frac{\partial}{\partial x^i}\right)$$

Moreover, we have duality

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

- For $F : M \to N$, the pullback of ω by F is a linear map $F^* : T^*N \to T^*M$ so that

$$(F^*\omega)_p(X_p) = \omega_p\left(dF_p(X_p)\right)$$

Remark The pullback operation for covector field

$$\begin{tikzcd}
 & & (F^*\omega)_p(\cdot) \arrow[d, leftrightarrow, "="] \\
p \arrow[r, "\omega|_\cdot"] \arrow[urr, bend left, "(F_*\omega)|_\cdot"] \arrow[dr, "dF|_\cdot"'] & \omega_p \arrow[r] & \omega(dF_p(\cdot)) \\
 & dF_p \arrow[u, "\text{acted on}"'] &
\end{tikzcd}$$

2.5 Tensor

- Let $A \in \Gamma(T^kT^*M)$ be a ***covariant** k**-tensor field*** on M. A has the following coordinate representation:

$$A = a_{i_1\,,\ldots,\,i_k}\,dx^{i_1} \otimes \ldots \otimes dx^{i_k} \tag{10}$$

$$\text{where } a_{i_1\,,\ldots,\,i_k} = A\left(\frac{\partial}{\partial x^{i_1}}\,, \ldots\,, \frac{\partial}{\partial x^{i_k}}\right) \tag{11}$$

- Let $T \in \Gamma(T^kTM)$ be a ***contravariant** k**-tensor field*** on M. T has the following coordinate representation:

$$T = T^{i_1\,,\ldots,\,i_k}\,\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \tag{12}$$

$$\text{where } T^{i_1\,,\ldots,\,i_k} = T\left(x^{i_1}\,, \ldots\,, x^{i_k}\right) \tag{13}$$

2.6 Differential Form

- Let $\omega \in \Omega^k(M)$ be *a **differential** k**-form*** on M. ω has the following coordinate reprsentation:

$$\omega = \omega_{i_1\,,\ldots,\,i_k}\,dx^{i_1} \wedge \ldots \wedge dx^{i_k} \tag{14}$$

$$\text{where } \omega_{i_1\,,\ldots,\,i_k} = \omega\left(\frac{\partial}{\partial x^{i_1}}\,, \ldots\,, \frac{\partial}{\partial x^{i_k}}\right) \tag{15}$$

$$1 \le i_1 \le \ldots \le i_k \le n$$

Note that $\omega_{i_1\,,\ldots,\,i_k}$ is a determinant of $k \times k$ matrix whose row indexed by $(i_1\,, \ldots\,, i_k)$.

- Compute ***the pullback of a** n**-form by*** $F : M \to N$. If (x^i) and (y^j) are smooth coordinates locally, and u is a continuous real-valued function on V, then the following holds locally.

$$F^*\left(u\,dy^1 \wedge \ldots \wedge dy^n\right) = (u \circ F)\,(\det(DF))\,dx^1 \wedge \ldots \wedge dx^n \tag{16}$$

where DF represents **the Jacobian matrix of** F in these coordinates.

Note that *the **pullback operator** is equivalent to **"plug-in of** F **whenever you see coordinate in codomain*** (y^j)*"*. The ***determinant** of **Jacobian*** $\det(DF)$ is the result of converting differential of composite $y^i \circ F$ into the coframes (dx^i) in domain M.

$$F^*\left(u\,dy^1 \wedge \ldots \wedge dy^n\right) = (u \circ F)\,d\left(y^1 \circ F\right) \wedge \ldots \wedge d\left(y^n \circ F\right)$$

- The following ***invariant formula*** holds for all $\omega \in \Omega^1(M) = \mathfrak{X}^*(M)$ and $X, Y \in \mathfrak{X}(M)$,

$$d\omega\,(X, Y) = X\,(\omega\,(Y)) - Y\,(\omega\,(X)) - \omega\,([X, Y]) \tag{17}$$

Note that both LHS and RHS are ***smooth functions*** and they can be written in terms of its component functions:

$$\omega([X,Y]) = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\omega_i \quad \textbf{\textit{contains only}}\ w_i \text{ also 1-order derivative of } X_i, Y_i$$

$$X(\omega(Y)) = X^j \frac{\partial \omega_i}{\partial x^j} Y^i + X^j \frac{\partial Y^i}{\partial x^j}\omega_i \quad \text{contains mixed } 0,1\text{-order derivatives of } w_i, Y_i$$

$$Y(\omega(X)) = Y^j \frac{\partial \omega_i}{\partial x^j} X^i + Y^j \frac{\partial X^i}{\partial x^j}\omega_i \quad \text{contains mixed } 0,1\text{-order derivatives of } w_i, X_i$$

$$d\omega(X,Y) = \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) X^i Y^j \quad \textbf{\textit{contains only 1-order derivatives of}}\ w_i$$

2.7 Connections

- For ∇ a connection in TM, ∇ is a ***metric connection*** when

$$Z\langle X\,,\,Y\rangle = \nabla_Z\langle X\,,\,Y\rangle = \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle \tag{18}$$

- For ∇ a connection in TM, ∇ is a ***symmetric connection*** when

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{19}$$

- The ***second covariant derivatives*** is computed as

$$\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{(\nabla_X Y)} Z \tag{20}$$

- The ***Riemann curvature endomorphism*** is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{21}$$

3 Computation

3.1 Tangent Space and Differential at p

- The ***coordinate representation of tangent vector*** and the value after it acts on f:

$$
\begin{aligned}
v = v^i \frac{\partial}{\partial x^i}\Big|_p, \quad (v^i) \in \mathbb{R}^n \qquad (22) \\
\text{where } v^i = v\left(x^i\right) \\
\Rightarrow vf = v^i \frac{\partial}{\partial x^i}\Big|_p f = v^i \frac{\partial f}{\partial x^i}(p)
\end{aligned}
$$

- For $F : M \to N$, where M, N are smooth manifolds and $(U, (x^i))$ and $(V, (y^j))$ are coordinate charts for M and N. The ***coordinate representation*** of dF_p is

$$
dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)} \qquad (23)
$$

where $DF_p = [\frac{\partial F^j}{\partial x^i}(p)]_{i,j}$ is the ***Jacobian matrix*** of dF_p relative to the coordinates in M and N.

Then $dF_p(v)$ acts on g can be represented as

$$
dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) g = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial g}{\partial y^j}(F(p))
$$

- The ***change of coordinate*** formula between $(\widetilde{x}^j)$ and (x^i) on M

$$
\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \widetilde{x}^j}\Big|_p. \qquad (24)
$$

Then its component function

$$
\begin{aligned}
v = v^i \frac{\partial}{\partial x^i}\Big|_p = \widetilde{v}^i \frac{\partial}{\partial \widetilde{x}^j}\Big|_p \\
\Rightarrow \widetilde{v}^j = v\left(\widetilde{x}^j\right) = \left(v^i \frac{\partial}{\partial x^i}\Big|_p\right)\left(\widetilde{x}^j\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, v^i \qquad (25)
\end{aligned}
$$

- The ***product rule*** (*Leibniz's Law*) of derivations at p: $v \in T_pM,\ f, g \in \mathcal{C}^\infty(M)$

$$
v(fg) = g(p)v(f) + f(p)\, v(g)
$$

Thus for coordinate map (x^i), and $v = v^i \frac{\partial}{\partial x^i}|_p$, $\boldsymbol{x} = (x^1, \dots, x^n)$,

$$
v^i \frac{\partial}{\partial x^i}\Big|_p (f\, g) = g(\boldsymbol{x}) v^i \frac{\partial f}{\partial x^i}(\boldsymbol{x}) + f(\boldsymbol{x}) v^i \frac{\partial g}{\partial x^i}(\boldsymbol{x})
$$

3.2 Cotangent Space

- For any $\omega \in T_p^* M$, ***the coordinate representation*** of ω

$$\omega = \omega_i \, dx_p^i \tag{26}$$

$$\text{where } \omega_i = \omega \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) \in \mathbb{R} \tag{27}$$

- The *computation* of $\omega(v)$ via *its coordinate representation* is the inner product between their component functions:

$$\begin{aligned} \omega(v) &= \left(\omega_i \, dx_p^i\right) \left(v^i \left. \frac{\partial}{\partial x^i} \right|_p \right) \\ &= \omega_i \, v^i \\ &:= \langle \omega \, , \, v \rangle \end{aligned} \tag{28}$$

- ***A differential*** 1***-form at*** p, $df_p \in T_p^* M$, under dual basis (dx_p^i) is

$$df_p = \frac{\partial f}{\partial x^i}(p) \, dx_p^i \tag{29}$$

- The ***change of coordinate*** formula for covector between $(\widetilde{x}^j)$ and (x^i) on M

$$\begin{aligned} \omega = \widetilde{\omega}_j \, d\widetilde{x}_p^j &= \omega_i \, dx_p^i \\ \text{where } \omega_i &= \omega \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) \\ &= \omega \left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p) \left. \frac{\partial}{\partial \widetilde{x}^j} \right|_p \right) \\ \Rightarrow \omega_i &= \frac{\partial \widetilde{x}^j}{\partial x^i}(p) \, \widetilde{\omega}_j \end{aligned} \tag{30}$$

  Note that the covariant trends i.e. from $(\widetilde{x}^j)$ to (x^i) for both the basis and function transformation. This is the opposite as compared to (24).

- The ***pullback*** of ω by F under coordinate map (x^i) for M and (y^j) for N is

$$F^* \left(\omega_j \, dy_{F(p)}^j \right) = (\omega_j \circ F)_p \, d \left(y^j \circ F \right) |_p \tag{31}$$

$$= (\omega_j \circ F)_p \, dF_p^j \tag{32}$$

$$= (\omega_j \circ F)_p \, \frac{\partial F^j}{\partial x^i}(p) \, dx_p^i \tag{33}$$

- df_p acts on $v \in T_p M$ under standard basis vector is

$$df_p \left(v^i \left. \frac{\partial}{\partial x^i} \right|_p \right) = v^i \frac{\partial f}{\partial x^i}(p). \tag{34}$$

3.3 Tangent Bundle and Vector Field

- For given smooth chart $(U,(x^i))$ in M and $(V,(y^j))$ in N, for $p \in U \cap F^{-1}(V)$, when Y are F-related to X

$$
\begin{aligned}
X &= X^i \frac{\partial}{\partial x^i} \\
Y_{F(p)} &= dF_p \left(X^i(p) \frac{\partial}{\partial x^i}\Big|_p \right) \\
&= X^i(p)\, dF_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) \\
&= X^i(p) \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)}
\end{aligned}
\tag{35}
$$

That is, its component function

$$
Y^j \circ F = \frac{\partial F^j}{\partial x^i} X^i \tag{36}
$$

- Xf in a smooth function while fX is a vector field:

$$
\begin{aligned}
X &= X^i \frac{\partial}{\partial x^i} \\
\Rightarrow Xf &= X^i \frac{\partial f}{\partial x^i} \\
\text{and } fX &= fX^i \frac{\partial}{\partial x^i}
\end{aligned}
$$

- For any smooth function $f \in \mathcal{C}^\infty(N)$ on N, Y is F-related to X, then

$$
X(f \circ F) = (Yf) \circ F
$$

Note that $f \circ F \in \mathcal{C}^\infty(M)$, so the $X(f \circ F)$ is a smooth function on M. (Yf) is a smooth function on N. This equation implies that F can be "taken out of the bracket" while X is *pushforwarded* to become Y. Under the coordinates $(U,(x^i))$ in M and $(V,(y^j))$ in N

$$
\begin{aligned}
X(f \circ F) &= X^i \frac{\partial (f \circ F)}{\partial x^i} = X^i \left(\frac{\partial f}{\partial y^j} \circ F \right) \frac{\partial F^j}{\partial x^i} \\
&= \left(X^i \frac{\partial F^j}{\partial x^i} \right) \left(\frac{\partial f}{\partial y^j} \circ F \right) \\
&= \left(Y^j \circ F \right) \left(\frac{\partial f}{\partial y^j} \circ F \right) \quad \text{(by (36))} \\
&= \left(Y^j \frac{\partial f}{\partial y^j} \right) \circ F
\end{aligned}
\tag{37}
$$

- The coordinate representation of the pushforward of vector field X by F, i.e. $F_* X$ is

$$
\begin{aligned}
X &= X^i \frac{\partial}{\partial x^i} \\
\Rightarrow F_* X &= \left(\left(\frac{\partial F^j}{\partial x^i} X^i \right) \circ F^{-1} \right) \frac{\partial}{\partial y^j}
\end{aligned}
\tag{38}
$$

- If $X, Y \in \mathfrak{X}(M)$, ***the Lie bracket*** $[X, Y] \in \mathfrak{X}(M)$ has *the following* ***coordinate representation***:

$$
\begin{aligned}
X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j} \\
\Rightarrow [X, Y] &= XY - YX \\
&= \left(X^i \frac{\partial}{\partial x^i}\right)\left(Y^j \frac{\partial}{\partial x^j}\right) - \left(Y^j \frac{\partial}{\partial x^j}\right)\left(X^i \frac{\partial}{\partial x^i}\right) \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^i Y^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} - Y^j X^i \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \\
&= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}
\end{aligned} \tag{39}
$$

We also see that $XY \notin \mathfrak{X}(M)$, which can be seen in its coordinate representation.

$$
XY = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^i Y^j \frac{\partial^2}{\partial x^i \, \partial x^j}
$$

Note that every vector fields can be written as linear combination of $(\frac{\partial}{\partial x^j})$ but XY contains a ***second-order derivative*** term which does not belong to the tangent space at any point. In ***Lie bracket***, this second order mixed derivative term is ***cancelled out*** so it is a vector field. XY is still ***a linear smooth map*** though.

3.4 Cotangent Bundle and Covector Field

- Let $X \in \mathfrak{X}(M)$ be a vector field.

$$
\begin{aligned}
X &= X^i \frac{\partial}{\partial x^i} \\
dx^i (X) &= dx^i \left(X^i \frac{\partial}{\partial x^i}\right) \\
&= X^i
\end{aligned} \tag{40}
$$

- The differential 1-form is a covector field and its component function is the partial derivatives of f

$$
df = \frac{\partial f}{\partial x^i} dx^i
$$

- The *computation* of $\omega(X)$ via ***its coordinate representation*** is the inner product between their component functions:

$$
\begin{aligned}
X = X^i \frac{\partial}{\partial x^i}, \quad \omega = \omega_i \, dx^i \\
\omega(X) &= \left(\omega_i \, dx^i\right)\left(X^i \frac{\partial}{\partial x^i}\right) \\
&= \omega_i \, X^i
\end{aligned} \tag{41}
$$

- $df(X)$ is a continous function. In fact,

$$df(X) = Xf$$
$$\frac{\partial f}{\partial x^i} dx^i \left(X^j \frac{\partial}{\partial x^j} \right) = \frac{\partial f}{\partial x^i} X^i = \left(X^i \frac{\partial}{\partial x^i} \right) f$$

For $X = \gamma'(t)$ for smooth curve $\gamma : J \to M$, we have

$$df(\gamma'(t)) = \gamma'(t) f = \frac{d}{dt}(f \circ \gamma) = (f \circ \gamma)'$$

- The covector field ω *acts on* ***the Lie bracket*** $[X, Y]$ has the form

$$\omega = \omega_i dx^i \quad [X, Y] = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$
$$\begin{aligned} \Rightarrow \omega\left([X, Y]\right) &= \omega_i \, dx^i \left([X, Y]\right) \\ &= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \omega_i \end{aligned} \tag{42}$$

- We compute the representation of function $X(\omega(Y))$ where $X, Y \in \mathfrak{X}(M)$ and $\omega \in \mathfrak{X}^*(M)$

$$\begin{aligned} X(\omega(Y)) &= \left(X^j \frac{\partial}{\partial x^j} \right) \left(\omega_i \, dx^i \left(Y^s \frac{\partial}{\partial x^s} \right) \right) \\ &= \left(X^j \frac{\partial}{\partial x^j} \right) \left(\omega_i \, Y^i \right) \\ &= X^j \, \frac{\partial \omega_i}{\partial x^j} \, Y^i + X^j \, \frac{\partial Y^i}{\partial x^j} \, \omega_i \end{aligned}$$

3.5 Tensor

- Let A acts on $X_i \in \mathfrak{X}(M)$, $i = 1, \ldots, k$. The *Tensor Characterization Lemma* states that A induced a smooth function as

$$A = a_{i_1, \ldots, i_k} \, dx^{i_1} \otimes \ldots \otimes dx^{i_k}, \quad X_i = X_i^j \frac{\partial}{\partial x^j}. \quad i = 1, \ldots, k$$
$$\begin{aligned} \Rightarrow A\left(X_1, \ldots, X_k\right) &= \left(a_{i_1, \ldots, i_k} \, dx^{i_1} \otimes \ldots \otimes dx^{i_k}\right) \left(X_1, \ldots, X_k\right) \\ &= a_{i_1, \ldots, i_k} \, dx^{i_1}\left(X_1\right) \; \ldots \; dx^{i_k}\left(X_k\right) \\ &= a_{i_1, \ldots, i_k} \, X_1^{i_1} \; \ldots \; X_k^{i_k} \end{aligned} \tag{43}$$

Note that this is a composite of k derivations.

3.6 Differential Forms

- Let $\omega_p \in \Lambda^k(T_p^* M)$, and $v_1, \ldots, v_k \in T_p M$.

$$\text{If} \quad \omega_p = \omega_p^1 \wedge \ldots \wedge \omega_p^k,$$
$$\Rightarrow \quad \omega_p(v_1, \ldots, v_k) = \det \begin{bmatrix} \omega_p^1(v_1) & \ldots & \omega_p^1(v_k) \\ \vdots & \ddots & \vdots \\ \omega_p^k(v_1) & \ldots & \omega_p^k(v_k) \end{bmatrix}. \tag{44}$$

- Furthermore, assume that $\omega \in \Omega^k(M)$ is ***a differential** k**-form***, while each $\omega^s = dw^s = w^s_{i_s}\, dx^{i_s} \in \Omega^1(M)$ is a 1-*form* for $s = 1, \ldots, k$ and $v_j = v^{i_s}_j \frac{\partial}{\partial x^{i_s}} \in \mathfrak{X}(M)$ is a set of ***vector fields*** for $j = 1, \ldots, k$. $I := (i_1, \ldots, i_k) \subset \{1, \ldots, n\}$ is a multi-index of size k. The coordinate representation of (44) is

$$
\begin{aligned}
\omega &= \left(w^1_{i_s}\, dx^{i_s}\right) \wedge \ldots \wedge \left(w^k_{i_s}\, dx^{i_s}\right) \\
&= \sum_I{}' \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad 1 \le i_1 \le \ldots \le i_k \le n \\
\omega(v_1, \ldots, v_k) &= \left(\left(w^1_{i_s}\, dx^{i_s}\right) \wedge \ldots \wedge \left(w^k_{i_s}\, dx^{i_s}\right)\right)(v_1, \ldots, v_k) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) \left({}^{\sigma}\left(w^1_{i_s}\, dx^{i_s}\right) \otimes \ldots \otimes \left(w^k_{i_s}\, dx^{i_s}\right)\right)(v_1, \ldots, v_k) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) \prod_{j=1}^{k} \left(w^{\sigma(j)}_{i_s}\, dx^{i_s}\right)(v_j) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) \prod_{j=1}^{k} v^{i_s}_j w^{\sigma(j)}_{i_s} \\
&= \det \begin{bmatrix} v^{i_s}_1 w^1_{i_s} & \ldots & v^{i_s}_k w^1_{i_s} \\ \vdots & \ddots & \vdots \\ v^{i_s}_1 w^k_{i_s} & \ldots & v^{i_s}_k w^k_{i_s} \end{bmatrix} \\
&= \det\left(\boldsymbol{W}^T \boldsymbol{V}\right),
\end{aligned} \tag{45}
$$

where $\boldsymbol{V} : M \to \mathbb{R}^{n \times k}$ is a matrix of component functions of vector fields $(v_1, \ldots, v_k)$, and $\boldsymbol{W} : M \to \mathbb{R}^{n \times k}$ is a matrix of component functions of covector fields $(\omega^1, \ldots, \omega^k)$

$$
\begin{aligned}
\boldsymbol{V} &= [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k]_{n \times k} \\
\boldsymbol{W} &= [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^k]_{n \times k}
\end{aligned}
$$

- Thus we can compute ***the component function*** of a k-form as

$$
\begin{aligned}
\omega &= \left(w^1_{i_s}\, dx^{i_s}\right) \wedge \ldots \wedge \left(w^k_{i_s}\, dx^{i_s}\right) \\
&= \omega_{i_1, \ldots, i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad (1 \le i_1 \le \ldots \le i_k \le n) \\
\Rightarrow \omega_{i_1, \ldots, i_k} &= \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) \\
&= \det\left(\boldsymbol{W}^T [\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_k}]\right) = \det(\boldsymbol{W}^T_I)
\end{aligned}
$$

where $\boldsymbol{W}_I$ is a $k \times k$ submatrix of component matrix $\boldsymbol{W}$ by only selecting rows whose indices are in $I = \{(i_1, \ldots, i_k) : 1 \le i_1 \le \ldots \le i_k \le n\}$.

- The ***exterior derivative*** $d\omega \in \Omega^{k+1}(M)$ can be represented as

$$
\begin{aligned}
d\omega &= d\left(\omega_{i_1, \ldots, i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \\
&= d\omega_{i_1, \ldots, i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\
&= \left(\frac{\partial \omega_{i_1, \ldots, i_k}}{\partial x^s}\right) dx^s \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}
\end{aligned} \tag{46}
$$

For 1-form $\omega = \omega_j \, dx^j$, we have the 2-forms $d\omega$ can be written as

$$\begin{aligned}
d\omega &= d\left(\omega_j \, dx^j\right) \\
&= d\omega_j \wedge dx^j \\
&= \left(\frac{\partial \omega_j}{\partial x^i}\, dx^i\right) \wedge dx^j \quad \left(\text{note that } \; d\omega_j = \frac{\partial \omega_j}{\partial x^i}\, dx^i\right) \\
&= \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\
&= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j
\end{aligned} \tag{47}$$

- Let $X, Y \in \mathfrak{X}(M)$ be a vector field on M, $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$. Let $\omega = \omega_i dx^i$ as the 1-form. Then

$$\begin{aligned}
d\omega(X, Y) &= \left(\frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j\right)(X, Y) \\
&= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \otimes dx^j \,(X, Y) \\
&= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) X^i Y^j
\end{aligned} \tag{48}$$

  This is ***a differential operator*** that ***only*** contains ***the second-order derivative terms*** $X^i\, Y^j$ on smooth functions. The component function is in fact ***the determinant of a*** 2×2 ***submatrix*** of ***the Jacobian matrix*** $[\frac{\partial \omega_j}{\partial x^i}]_{j,i}$

- Let $X \in \mathfrak{X}(M)$ be a vector field on M. The ***interior product*** $X \lrcorner \left(dx^1 \wedge \ldots \wedge dx^k\right)$ has the following representation:

$$X = X^i \frac{\partial}{\partial x^i}$$

$$\begin{aligned}
X \lrcorner \left(dx^1 \wedge \ldots \wedge dx^k\right) &= \sum_{i=1}^{k} (-1)^{i-1} dx^i(X)\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^k \\
&= \sum_{i=1}^{k} (-1)^{i-1} X^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^k
\end{aligned} \tag{49}$$

  where $\widehat{dx^i}$ indicates that dx^i is ***omitted.***

  For $Y_2, \ldots, Y_k \in \mathfrak{X}(M)$, where $Y_j = Y_j^s \frac{\partial}{\partial x^s}$, $X = X^s \frac{\partial}{\partial x^s}$

$$\begin{aligned}
X \lrcorner \left(dx^1 \wedge \ldots \wedge dx^k\right)(Y_2, \ldots, Y_k) &= \left(dx^1 \wedge \ldots \wedge dx^k\right)(X, Y_2, \ldots, Y_k) \\
&= \det\left[\boldsymbol{X} \mid \boldsymbol{Y}\right]
\end{aligned} \tag{50}$$

  where $[\boldsymbol{X} \mid \boldsymbol{Y}] : M \to \mathbb{R}^{k \times k}$ is a matrix-valued function whose first column is from the component functions of X and the rest columns are component functions of (Y_j).

$$\boldsymbol{Y} = [\boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_k]_{k \times (k-1)}$$

The equation (49) corresponds to ***the expansion by minors*** along the first columns

$$X \lrcorner \left(dx^1 \wedge \ldots \wedge dx^k\right)(Y_2, \ldots, Y_k) = \sum_{i=1}^{k} (-1)^{i-1} X^i \det(\boldsymbol{Y}_{-i}) \tag{51}$$

where $\boldsymbol{Y}_{-i}$ is obtained by dropping i-th row of $\boldsymbol{Y}$.

- Compute ***the pullback of a*** n***-form by*** $F : M \to N$. If (x^i) and (y^j) are smooth coordinates locally, and u is a continuous real-valued function on V, then the following holds locally.

$$F^* \left(u\, dy^1 \wedge \ldots \wedge dy^n\right) = (u \circ F)\left(\det(DF)\right) dx^1 \wedge \ldots \wedge dx^n$$

where DF represents **the Jacobian matrix of** F in these coordinates.

3.7 Connections

- Given coordinate system (x^i), and ∇ is a connection on TM, the covariant derivative of $Y = Y^j \frac{\partial}{\partial x^j}$ in the direction of $X = X^i \frac{\partial}{\partial x^i}$ is

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} (Y^j \frac{\partial}{\partial x^j}) \\
&= X^i \nabla_{\frac{\partial}{\partial x^i}} (Y^j \frac{\partial}{\partial x^j}) \\
&= X^i \left(\left(\nabla_{\frac{\partial}{\partial x^i}} Y^j\right) \frac{\partial}{\partial x^j} + Y^j \nabla_{\frac{\partial}{\partial x^i}} (\frac{\partial}{\partial x^j}) \right) \\
&= X^i \left(\left(\frac{\partial}{\partial x^i} Y^j\right) \frac{\partial}{\partial x^j} + Y^j\, \Gamma^k_{i,j} \frac{\partial}{\partial x^k} \right) \\
&= X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j\, \Gamma^k_{i,j} \right) \frac{\partial}{\partial x^k} &(52)\\
&= \left(X(Y^k) + X^i\, Y^j\, \Gamma^k_{i,j} \right) \frac{\partial}{\partial x^k} &(53)
\end{aligned}$$

- The ***inner product*** of *covariant derivative of* Y *in direction of* X with Z gives:

$$\begin{aligned}
\langle \nabla_X Y \,,\, Z \rangle_g &= \left\langle \left(X(Y^k) + X^i\, Y^j\, \Gamma^k_{i,j} \right) \frac{\partial}{\partial x^k} \,,\, Z^l \frac{\partial}{\partial x^l} \right\rangle_g \\
&= \left(X(Y^k) + X^i\, Y^j\, \Gamma^k_{i,j} \right) Z^l \left\langle \frac{\partial}{\partial x^k} \,,\, \frac{\partial}{\partial x^l} \right\rangle_g \\
&= g_{k,l} \left(X(Y^k) + X^i\, Y^j\, \Gamma^k_{i,j} \right) Z^l &(54)\\
&:= g_{k,l} X(Y^k) Z^l + \Gamma_{i,j;l}\, X^i\, Y^j\, Z^l &(55)
\end{aligned}$$

Note that

$$\begin{aligned}
\Gamma_{i,j;k} := \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \,,\, \frac{\partial}{\partial x^k} \right\rangle_g &= \left\langle \Gamma^l_{i,j} \frac{\partial}{\partial x^l} \,,\, \frac{\partial}{\partial x^k} \right\rangle_g \\
&= g_{l,k} \Gamma^l_{i,j} &(56)
\end{aligned}$$

For metric connection ∇,

$$X\left<Y\,,\,Z\right> = \left<\nabla_X Y\,,\,Z\right> + \left<Y\,,\,\nabla_X Z\right>$$

Thus

$$\begin{aligned}
X\left(g_{k,l}Y^k\,Z^l\right) &= g_{k,l}X(Y^k)Z^l + g_{k,l}Y^k X(Z^l) + X(g_{k,l})Y^k\,Z^l \\
\left<\nabla_X Y\,,\,Z\right> &= g_{k,l}X(Y^k)Z^l + \Gamma_{i,j;l}\,X^i\,Y^j\,Z^l \\
\left<Y\,,\,\nabla_X Z\right> &= g_{k,l}X(Z^k)Y^l + \Gamma_{i,j;l}\,X^i\,Z^j\,Y^l \\
&\quad \text{since } \nabla \text{ is a metric connection, the equation holds} \\
X(g_{k,l})Y^k\,Z^l &= \Gamma_{i,k;l}\,X^i\,Y^k\,Z^l + \Gamma_{i,l;k}\,X^i\,Y^k\,Z^l \\
\partial_i(g_{k,l})X^iY^k\,Z^l &= \Gamma_{i,k;l}\,X^i\,Y^k\,Z^l + \Gamma_{i,l;k}\,X^i\,Y^k\,Z^l \\
&\quad \text{set } X^i = 1, Y^k = 1,\ Z^l = 1, \quad \forall i,k,l \\
\Rightarrow \frac{\partial}{\partial x^i}(g_{j,k}) &= \Gamma_{i,j;k} + \Gamma_{i,k;j} = g_{m,k}\Gamma^m_{i,j} + g_{m,j}\Gamma^m_{i,k}
\end{aligned}$$

- The ***difference*** between two covariant derivatives $\nabla_X Y - \nabla_Y X$:

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \left(X(Y^k) + X^i\,Y^j\,\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k} - \left(Y(X^k) + Y^i\,X^j\,\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k} \\
&= \left(X(Y^k) - Y(X^k) + \left(X^i\,Y^j - Y^i\,X^j\right)\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k} \\
&= [X,Y] + \left(\left(X^i\,Y^j - Y^i\,X^j\right)\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k}
\end{aligned} \tag{57}$$

Note that the Lie bracket is

$$[X,Y] = \left(X(Y^k) - Y(X^k)\right)\frac{\partial}{\partial x^k}$$

So the connection ∇ is ***symmetric*** if and only if $\Gamma^k_{i,j} = \Gamma^k_{j,i}$. If so, then

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \left[X(Y^k) - Y(X^k) + \left(X^i\,Y^j - Y^i\,X^j\right)\Gamma^k_{i,j}\right]\frac{\partial}{\partial x^k} \\
&\text{when } \Gamma^k_{i,j} = \Gamma^k_{j,i} \\
&= \left(X(Y^k) - Y(X^k)\right)\frac{\partial}{\partial x^k} + \left(X^i\,Y^j\,\Gamma^k_{i,j} - Y^iX^j\,\Gamma^k_{j,i}\right)\frac{\partial}{\partial x^k} \\
&= [X,Y] + \left(X^i\,Y^j\,\Gamma^k_{i,j} - Y^jX^i\,\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k} \\
&= [X,Y]
\end{aligned}$$

Thus the ***torsion tensor*** τ is computed as

$$\begin{aligned}
\tau(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \\
&= \left(X^i\,Y^j\,\Gamma^k_{i,j} - Y^jX^i\,\Gamma^k_{i,j}\right)\frac{\partial}{\partial x^k}
\end{aligned}$$

That is the connection ∇ is ***symmetric*** if and only if

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

- The covariant derivative of Z in direction of $[X, Y]$

$$
\begin{aligned}
\nabla_{[X,Y]} Z &= \nabla_{\left(\left(X(Y^k)-Y(X^k)\right)\frac{\partial}{\partial x^k}\right)} Z^l \frac{\partial}{\partial x^l} \\
&= \Big(X(Y^k) - Y(X^k)\Big) \nabla_{\frac{\partial}{\partial x^k}} \left(Z^l \frac{\partial}{\partial x^l}\right) \\
&= \Big(X(Y^k) - Y(X^k)\Big) \frac{\partial Z^l}{\partial x^k}\frac{\partial}{\partial x^l} + \Big(X(Y^k) - Y(X^k)\Big) Z^l (\nabla_{\partial_k}\partial_l) \\
&= \left(X(Y^i) - Y(X^i)\right) \frac{\partial Z^k}{\partial x^i}\frac{\partial}{\partial x^k} + \Gamma^k_{i,j} \left(X(Y^i) - Y(X^i)\right) Z^j \frac{\partial}{\partial x^k} \\
&= \left\{ \left(X(Y^i) - Y(X^i)\right) \frac{\partial Z^k}{\partial x^i} + \left(X(Y^i) - Y(X^i)\right) Z^j \Gamma^k_{i,j} \right\} \frac{\partial}{\partial x^k} \\
&= \left\{ [X,Y](Z^k) + [X,Y]^i \, Z^j \Gamma^k_{i,j} \right\} \frac{\partial}{\partial x^k}
\end{aligned} \tag{58}
$$

- The covariant derivative of ω in direction of X:

$$
\nabla_X \omega(Y) = X(\omega(Y)) - \omega\left(\nabla_X Y\right)
$$

Thus the coordinate representation is

$$
\nabla_X \omega = \left(X(\omega_k) - \omega_i \, X^j \, \Gamma^i_{j,k}\right) dx^k. \tag{59}
$$

- The total covariant derivative of a 1-form ω is

$$
\nabla \omega(Y, X) = (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega\left(\nabla_X Y\right)
$$

- We check the formula for total covariant derivative for tensor F

$$
\nabla_X F = \mathrm{tr}\left(\nabla F \otimes X\right)
$$

- **Example** (***The Covariant Hessian***).
  Let u be a smooth function on M.
  - The ***total covariant derivative of a smooth function is equal to its*** 1***-form*** $\nabla u = du \in \Omega^1(M) = \Gamma(T^{(0,1)}TM)$ since

    $$
    \nabla u(X) = \nabla_X u = Xu = du(X)
    $$

  - The 2-tensor $\nabla^2 u = \nabla(du)$ is called ***the covariant Hessian of*** u. Its action on smooth vector fields X, Y can be computed by the following formula:

    $$
    \nabla^2 u(Y, X) = \nabla^2_{X,Y} u = \nabla_X \nabla_Y u - \nabla_{(\nabla_X Y)} u = X(Yu) - (\nabla_X Y)(u) \tag{60}
    $$

    In any local coordinates, it is

    $$
    \nabla^2 u = u_{\,;i,j} \, dx^i \otimes dx^j
    $$

where

$$u_{\,;i,j} = \frac{\partial^2 u}{\partial x^j\,\partial x^i} - \Gamma^k_{j,i}\frac{\partial u}{\partial x^k}$$

If ∇ is symmetric, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ then

$$\begin{aligned}\nabla^2 u(Y, X) &= X(Yu) - (\nabla_X Y)(u)\\ &= (XY)u - [X, Y](u) - \nabla_Y X(u)\\ &= (YX)u - \nabla_Y X(u) := \nabla^2 u(X, Y)\end{aligned} \tag{61}$$

Thus $\nabla^2 u$ ***is a symmetric*** 2***-tensor if*** ∇ ***is symmetric***.

$$\nabla^2 u = u_{\,;i,j}\, dx^i\, dx^j$$

3.8 Geodesics and Parallel Transport

- The ***parallel transport*** of V along curve $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ is computed as

$$\begin{aligned}&\nabla_{\gamma'(t)} V = \nabla_{\dot\gamma^i(t)\partial_i} V \equiv 0\\ \Leftrightarrow\ &\left[\gamma'(t)(V^k) + \dot\gamma^i(t)\, V^j \Gamma^k_{i,j}\right]\partial_k \equiv 0\\ \Leftrightarrow\ &\gamma'(t)(V^k) + \dot\gamma^i(t)\, V^j \Gamma^k_{i,j}(\gamma(t)) = 0\\ \Leftrightarrow\ &\dot V^k(\gamma(t)) + \dot\gamma^i(t)\, V^j \Gamma^k_{i,j}(\gamma(t)) = 0, \quad k = 1, \ldots, n\end{aligned}$$

Let $V(t) = V_{\gamma(t)}$ be the vector field along curve γ, so that $V(t) = V^k(\gamma(t))\partial_k := V^k(t)\partial_k$.

$$\dot V^k = -\dot\gamma^i(t)\,\Gamma^k_{i,j}(\gamma(t)) V^j, \qquad k = 1, \ldots, n. \tag{62}$$

For fixed γ, this is *a system of n 1st-order* ***linear ODEs*** for $(V^1(t), \ldots, V^n(t))$.

- To obtain the geodesic equations, note that $\nabla_{\gamma'(t)} V \equiv 0$ for $V = \gamma'(t) = \dot\gamma^k(t)\partial_k$. Thus we have ***the geodesic equations***:

$$\ddot\gamma^k = -\dot\gamma^i\,\dot\gamma^j \Gamma^k_{i,j}(\gamma(t)), \qquad k = 1, \ldots, n. \tag{63}$$

This is a system of n ***2nd-order nonlinear ODEs*** for $(\gamma^1(t), \ldots, \gamma^n(t))$.

It can reduce to a system of $2n$ ***1st-order nonlinear ODEs***

$$\begin{aligned}\dot\gamma^k &= v^k\\ \dot v^k &= -v^i\, v^j \Gamma^k_{i,j}(\gamma(t)), \qquad k = 1, \ldots, n.\end{aligned} \tag{64}$$

3.9 Divergence of Vector Field

- From the formula, $d\left(X \lrcorner\, dV\right) = \operatorname{div}(X)\, dV$, we can derive the coordinate representation of divergence of vector field X

$$
\begin{aligned}
X &= X^i \frac{\partial}{\partial x^i} \\
dV &= \rho\, dx^1 \wedge \ldots \wedge dx^n \\
\Rightarrow X \lrcorner\, dV &= X \lrcorner \left(\rho\, dx^1 \wedge \ldots \wedge dx^n\right) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \rho X^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
\Rightarrow d\left(X \lrcorner\, dV\right) &= \sum_{i=1}^{n} (-1)^{i-1} d\left(\rho X^i\right) \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left(\sum_{s=1}^{n} \frac{\partial(\rho X^i)}{\partial x^s} dx^s\right) \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left(\frac{\partial(\rho X^i)}{\partial x^i} dx^i\right) \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \\
&= \frac{1}{\rho} \sum_{i=1}^{n} \left(\frac{\partial(\rho X^i)}{\partial x^i} \rho\, dx^1 \wedge \ldots \wedge dx^i \wedge \ldots \wedge dx^n\right) \\
&= \frac{1}{\rho} \sum_{i=1}^{n} \frac{\partial(\rho X^i)}{\partial x^i} dV = \left(\frac{1}{\rho} \sum_{i=1}^{n} \frac{\partial(\rho X^i)}{\partial x^i}\right) dV \\
\Rightarrow \operatorname{div}(X) &= \frac{1}{\rho} \sum_{i=1}^{n} \frac{\partial(\rho X^i)}{\partial x^i}
\end{aligned}
$$

References