

Lecture 4: Empirical Processes

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1 Uniform Law of Large Numbers

1.1 Motivations

- **Remark** (*Unbiased Estimator of Cumulative Distribution Function*)

The law of any scalar random variable X can be fully specified by its **cumulative distribution function (CDF)**, whose value at any point $t \in \mathbb{R}$ is given by $F(t) := \mathcal{P}[X \leq t]$. Now suppose that we are given a collection $\{X_i\}_{i=1}^n$ of n i.i.d. samples, each drawn according to the law specified by F . A natural *estimate* of F is **the empirical CDF** given by

$$\hat{F}_n(t) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, t]}(X_i), \quad (1)$$

where $\mathbb{1}_{(-\infty, t]}(x)$ is a $\{0, 1\}$ -valued indicator function for the event $\{x \leq t\}$. Since **the population CDF** can be written as $F(t) = \mathbb{E} [\mathbb{1}_{(-\infty, t]}(X)]$, the empirical CDF is an **unbiased estimate**.

For each $t \in \mathbb{R}$, **the strong law of large numbers** suggests that

$$\hat{F}_n(t) \rightarrow F(t), \quad \text{a.s.}$$

A natural goal is to strengthen *this pointwise convergence* to a form of **uniform convergence**. The reason why uniform convergence of $\hat{F}_n(t)$ to $F(t)$ is important is that it can be used to prove the **consistency** of **plug-in estimator** for *functionals of distribution function*.

- **Example** (*Expectation Functionals*)

Given some integrable function g , we may define **the expectation functional** γ_g via

$$\gamma_g(F) := \int g(x) dF(x). \quad (2)$$

For any g , *the plug-in estimate* is given by $\gamma_g(\hat{F}_n) = \frac{1}{n} \sum_{i=1}^n g(X_i)$, corresponding to **the sample mean** of $g(X)$.

- **Example** (*Quantile Functionals*)

For any $\alpha \in [0, 1]$, **the quantile functional** Q_α is given by

$$Q_\alpha(F) := \inf \{t \in \mathbb{R} : F(t) \geq \alpha\}. \quad (3)$$

The **median** corresponds to the special case $\alpha = 0.5$. *The plug-in estimate* is given by

$$Q_\alpha(\hat{F}_n) := \inf \left\{ t \in \mathbb{R} : \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, t]}(X_i) \geq \alpha \right\} \quad (4)$$

and corresponds to estimating the α -th quantile of the distribution by *the α -th sample quantile*. In the special case $\alpha = 0.5$, this estimate corresponds to *the sample median*. In this case, $Q_\alpha(\hat{F}_n)$ is a fairly complicated, *nonlinear function of all the variables*, so that this convergence does not follow immediately by a classical result such as the law of large numbers.

- **Example** (*Goodness-of-fit Functionals*)

It is frequently of interest to test the hypothesis of whether or not a given set of data has

been drawn from a known distribution F_0 . Such tests can be performed using *functionals that measure the distance* between F and the target CDF F_0 , including the *sup-norm distance* $\|F - F_0\|_\infty$, or other distances such as *the Cramer-von Mises criterion* based on the functional

$$\gamma_g(F) := \int_{-\infty}^{+\infty} (F(x) - F_0(x))^2 dF_0(x)$$

- **Remark (Consistency of Plug-In Estimate)**

For any *plug-in estimator* $\gamma_g(\hat{F}_n)$, an important question is to understand when it is *consistent* – that is, when does $\gamma_g(\hat{F}_n)$ converge to $\gamma_g(F)$ in *probability* (or *almost surely*)?

We can define *the continuity of a functional* γ with respect to *the supremum norm*: more precisely, we say that the functional γ is *continuous at F in the sup-norm* if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|G - F\|_\infty := \sup_{t \in \mathbb{R}} |G(t) - F(t)| \leq \delta \quad \text{implies that} \quad |\gamma(G) - \gamma(F)| \leq \epsilon.$$

Thus for any *continuous functional*, it reduces the *consistency* question for *the plug-in estimator* $\gamma_g(\hat{F}_n)$ to the issue of whether or not the random variable $\|\hat{F}_n - F\|_\infty$ *converges to zero*.

1.2 Glivenko-Cantelli Theorem

- **Theorem 1.1 (Glivenko-Cantelli Theorem)** [Wellner and van der Vaart, 2013, Wainwright, 2019, Giné and Nickl, 2021]

For any distribution, the empirical CDF

$$\hat{F}_n(t) := \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{(-\infty, t]}(X_i)$$

is a *strongly consistent estimator* of the population CDF in *the uniform norm*, meaning that

$$\left\| \hat{F}_n - F \right\|_\infty := \sup_{t \in \mathbb{R}} \left| \hat{F}_n(t) - F(t) \right| \rightarrow 0, \quad a.s. \quad (5)$$

- **Remark (Uniform Law of Large Numbers)**

The Glivenko-Cantelli theorem generalizes the strong law of large numbers to stochastic process. It confirms that the *convergence* of sample mean $\mathcal{P}_n f$ to its expectation $\mathcal{P}f$ is true in function space \mathcal{F} not only in *pointwise topology* but also in *uniform topology*. Thus, *the Glivenko-Cantelli theorem* is also called *the uniform law of large numbers*.

2 Empirical Processes

2.1 Definitions

- **Definition (Empirical Measure)** [Wellner and van der Vaart, 2013, Giné and Nickl, 2021]
Let $(\mathcal{X}, \mathcal{F}, \mathcal{P})$ be a *probability space*, and let $X_i, i \in \mathbb{N}$, be the *coordinate functions* of the

infinite product probability space $(\Omega, \mathcal{B}, \mathbb{P}) := (\mathcal{X}^\infty, \mathcal{F}^\infty, \mathcal{P}^\infty)$, $X_i : \mathcal{X}^\infty \rightarrow \mathcal{X}$, which are **independent identically distributed** \mathcal{X} -valued random variables with law \mathcal{P} .

The empirical measure corresponding to the ‘observations’ X_1, \dots, X_n , for any $n \in \mathbb{N}$, is defined as **the random discrete probability measure**

$$\mathcal{P}_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \quad (6)$$

where δ_x is *Dirac measure* at x , that is, unit mass at the point x . In other words, for each event A , $\mathcal{P}_n(A)$ is the **proportion of observations** X_i , $i = 1, \dots, n$, that fall in A ; that is,

$$\mathcal{P}(A) = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}(A) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_i \in A\}, \quad A \in \mathcal{F}.$$

- **Remark (*Probability Measure with Operator Notation*)** [Wellner and van der Vaart, 2013, Giné and Nickl, 2021]

For any measure μ and μ -integrable function f , we will use the following **operator notation** for the integral of f with respect to μ :

$$\mu f \equiv \mu(f) = \int_{\Omega} f d\mu.$$

This is valid since there exists an isomorphism between *the space of probability measure* and *the space of bounded linear functional* on $\mathcal{C}_0(\Omega)$ by Riesz-Markov representation theorem (assuming Ω is *locally compact*). By this notion the expectation $\mathcal{P}f = \mathbb{E}_{\mathcal{P}}[f]$.

- **Definition (*Empirical Process*)** [Wellner and van der Vaart, 2013, Giné and Nickl, 2021]
Let \mathcal{F} be a *collection of \mathcal{P} -integrable functions* $f : \mathcal{X} \rightarrow \mathbb{R}$, usually infinite. For any such class of functions \mathcal{F} , **the empirical measure** defines a **stochastic process**

$$f \rightarrow \mathcal{P}_n f, \quad f \in \mathcal{F} \quad (7)$$

which we may call **the empirical process indexed by \mathcal{F}** , although we prefer to reserve the notation ‘*empirical process*’ for the **centred and normalised process**

$$f \rightarrow \nu_n(f) := \sqrt{n}(\mathcal{P}_n f - \mathcal{P}f), \quad f \in \mathcal{F}. \quad (8)$$

- **Remark** An explicit notion of (*centered and normalized*) *empirical process* is

$$\sqrt{n}(\mathcal{P}_n f - \mathcal{P}f) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (f(X_i) - \mathbb{E}_{\mathcal{P}}[f(X)]), \quad f \in \mathcal{F}.$$

where $X_1, \dots, X_n \sim \mathcal{P}$ are i.i.d random variables. Note that it is a stochastic process since *the function f is changing* in \mathcal{F} , i.e. the process $(\mathcal{P}_n - \mathcal{P})f$ is indexed by function $f \in \mathcal{F}$ not finite dimensional variable.

- **Remark (*Random Measure on Function Space \mathcal{F}*)**

Normally we assume that data are sampled from some distribution \mathcal{P} and the data itself is random. However, the empirical measure

$$\mathcal{P}_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

itself is considered as a **random** probability measure. That is, *the sampling mechanism itself contains randomness* and it is not sampling from one distribution but **a system of distributions depending on the choice of dataset** X_1, \dots, X_n , which in turn were sampled from some *prior* \mathcal{P} . Due to this randomness, $\mathcal{P}_n f = \mathbb{E}_{\mathcal{P}_n} [f]$ is not a fixed expectation number but a random variable. For given $f \in \mathcal{F}$, this is the empirical mean (i.e. sample mean)

$$\mathcal{P}_n f = \mathbb{E}_{\mathcal{P}_n} [f] = \frac{1}{n} \sum_{i=1}^n f(X_i).$$

The critical difference between *empirical process* vs. *sample mean* is that the **latter** assume that f is **fixed**, while the former is defined with respect to **a class of functions** \mathcal{F} .

- **Remark** In probability theory, an **empirical process** is a **stochastic process** that describes the **proportion** of objects in a system in a **given state**. Applications of the theory of empirical processes arise in **non-parametric statistics**.

- **Remark (Object of Empirical Process Theory)**

The **object** of empirical process theory is to study the **properties** of the **approximation** of $\mathcal{P}f$ by $\mathcal{P}_n f$, **uniformly in** \mathcal{F} , concretely, to obtain both **probability estimates** for the random quantities

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathcal{P}_n f - \mathcal{P}f|$$

and **probabilistic limit theorems** for the processes $\{(\mathcal{P}_n - \mathcal{P})(f) : f \in \mathcal{F}\}$.

Note that the quantity $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$ is a **random variable** since \mathcal{P}_n is a **random measure**.

- **Remark (Measurability Problem)**

There may be a **measurability problem** for

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathcal{P}_n f - \mathcal{P}f|$$

since *the uncountable suprema* of measurable functions *may not be measurable*.

However, there are many situations where this is actually a **countable supremum**. For instance, for probability distribution on \mathbb{R}

$$\|\mathcal{P}_n - \mathcal{P}\|_{\infty} := \sup_{t \in \mathbb{R}} |(\mathcal{P}_n - \mathcal{P})(-\infty, t)| = \sup_{t \in \mathbb{Q}} |F_n(t) - F(t)| = \sup_{t \in \mathbb{Q}} |(\mathcal{P}_n - \mathcal{P})(-\infty, t)|$$

where $F(t) = \mathcal{P}(-\infty, t)$ is the cumulative distribution function. If \mathcal{F} is *countable* or if there exists \mathcal{F}_0 *countable* such that

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} = \|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}_0}, \quad \text{a.s.}$$

then the measurability problem disappears. For the next few sections we will simply **assume** that the class \mathcal{F} is **countable**.

- **Remark (Bounded Assumption)**

If we assume that

$$\sup_{f \in \mathcal{F}} |f(x) - \mathcal{P}f| < \infty, \quad \forall x \in \mathcal{X}, \quad (9)$$

then the maps from \mathcal{F} to \mathbb{R} ,

$$f \rightarrow f(x) - \mathcal{P}f, \quad x \in \mathcal{X},$$

are **bounded functionals** over \mathcal{F} , and therefore, so is $f \rightarrow (\mathcal{P}_n - \mathcal{P})(f)$. That is,

$$\mathcal{P}_n - \mathcal{P} \in \ell_\infty(\mathcal{F}),$$

where $\ell_\infty(\mathcal{F})$ is **the space of bounded real functionals** on \mathcal{F} , a *Banach space* if we equip it with the supremum norm $\|\cdot\|_{\mathcal{F}}$.

A large literature is available on *probability in separable Banach spaces*, but unfortunately, $\ell_\infty(\mathcal{F})$ is **only separable** when the class \mathcal{F} is **finite**, and **measurability problems** arise because *the probability law* of the process $\{(\mathcal{P}_n - \mathcal{P})(f) : f \in \mathcal{F}\}$ **does not extend to the Borel σ -algebra of $\ell_\infty(\mathcal{F})$** even in simple situations.

- **Remark** This chapter addresses **three main questions** about the empirical process:
 1. The first question has to do with **concentration** of $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$ about *its mean* when \mathcal{F} is **uniformly bounded**. Recall that $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$ is a random variable itself, due to randomness of the empirical measure. We mainly use the *non-asymptotic analysis* to obtain *the exponential bound for concentration*.
 2. The second question is do **good estimates** for **mean** $\mathbb{E} [\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}]$ exist? We will examine two main techniques that give answers to this question, both related to **metric entropy** and **chaining**. One of them, called **bracketing**, uses *chaining* in combination with *truncation* and *Bernstein's inequality*. The other one applies to **Vapnik-Cervonenkis (VC) classes of functions**.
 3. Finally, the last question about the empirical process refers to **limit theorems**, mainly **the uniform law of large numbers** and the **central limit theorem**, in fact, the analogues of **the classical Glivenko-Cantelli** and *Donsker theorems* for the empirical distribution function.

Formulation of *the central limit theorem* will require some more *measurability* because we will be considering **convergence in law** of random elements in **not necessarily separable Banach spaces**.

2.2 Glivenko-Cantelli Class

- **Definition** (**Glivenko-Cantelli Class**) [Wellner and van der Vaart, 2013, Wainwright, 2019, Giné and Nickl, 2021]
We say that \mathcal{F} is a **Glivenko-Cantelli class** for \mathcal{P} if

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathcal{P}_n f - \mathcal{P}f| \rightarrow 0$$

in probability as $n \rightarrow \infty$.

This notion can also be defined in a *stronger* sense, requiring **almost sure convergence** of $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$, in which case we say that \mathcal{F} satisfies a **strong Glivenko-Cantelli law**.

- **Example (*Empirical CDFs and Indicator Functions*)**

Consider the function class

$$\mathcal{F} := \{\mathbb{1}_{(-\infty, t]}(\cdot), t \in \mathbb{R}\} \quad (10)$$

where $\mathbb{1}_{(-\infty, t]}$ is the $\{0, 1\}$ -valued indicator function of the interval $(-\infty, t]$. For each fixed $t \in \mathbb{R}$, we have the equality $\mathbb{E} [\mathbb{1}_{(-\infty, t]}(X)] = \mathcal{P}[X \leq t] = F(t)$, so that the classical *Glivenko-Cantelli theorem* is equivalent to a **strong uniform law for the class** (10),

2.3 Tail bounds for Empirical Processes

- **Remark** Consider the *suprema of empirical process*:

$$Z := \sup_{f \in \mathcal{F}} \{\mathcal{P}_n f\} = \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n f(X_i) \right\} \quad (11)$$

where (X_1, \dots, X_n) are independent random variables drawn from $\mathcal{P} := \otimes_{i=1}^n \mathcal{P}_i$, each \mathcal{P}_i is supported on some set $\mathcal{X}_i \subseteq \mathcal{X}$. \mathcal{F} is a family of real-valued functions $f : \mathcal{X} \rightarrow \mathbb{R}$. The primary goal of this section is to derive a number of *upper bounds* on the *tail event* $\{Z \geq \mathbb{E}[Z] + t\}$.

- **Theorem 2.1 (*Functional Hoeffding Inequality*)** [Wainwright, 2019, Boucheron et al., 2013]

For each $f \in \mathcal{F}$ and $i = 1, \dots, n$, assume that there are real numbers $a_{i,f} \leq b_{i,f}$ such that $f(x) \in [a_{i,f}, b_{i,f}]$ for all $x \in \mathcal{X}_i$. Then for all $t \geq 0$, we have

$$\mathcal{P} \{Z \geq \mathbb{E}[Z] + t\} \leq \exp \left(-\frac{nt^2}{4L^2} \right) \quad (12)$$

where $Z := \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n f(X_i) \right\}$, and $L^2 := \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n (a_{i,f} - b_{i,f})^2 \right\}$.

- **Theorem 2.2 (*Functional Bernstein Inequality, Talagrand Concentration for Empirical Processes*)** [Wainwright, 2019, Boucheron et al., 2013]

Consider a **countable** class of functions \mathcal{F} **uniformly bounded** by b . Then for all $t > 0$, the *suprema of empirical process* Z as defined in (11) satisfies the upper tail bound

$$\mathcal{P} \{Z \geq \mathbb{E}[Z] + t\} \leq \exp \left(-\frac{nt^2}{8e\Sigma^2 + 4bt} \right) \quad (13)$$

where $\Sigma^2 := \mathbb{E} \left[\sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n f^2(X_i) \right\} \right]$ is **the weak variance**.

- **Remark** As opposed to control only in terms of **bounds on the function values**, the inequality (13) **also** brings a notion of **variance** into play.

- **Remark** We will prove the bound in next section:

$$\Sigma^2 \leq \sigma^2 + 2b\mathbb{E}[Z]$$

where $\sigma^2 := \sup_{f \in \mathcal{F}} \mathbb{E} [f^2(X)]$. Then, the functional Bernstein inequality (13) can be formulated as

$$\mathcal{P} \left\{ Z \geq \mathbb{E}[Z] + c_0\gamma\sqrt{t} + c_1bt \right\} \leq e^{-nt} \quad (14)$$

for some constant c_0, c_1 and $\gamma^2 := \sigma^2 + 2b\mathbb{E}[Z]$. We can have an alternative form of this bound (14) for any $\epsilon > 0$,

$$\mathcal{P}\left\{Z \geq (1 + \epsilon)\mathbb{E}[Z] + c_0\sigma\sqrt{t} + (c_1 + c_0^2/\epsilon)bt\right\} \leq e^{-nt}. \quad (15)$$

- **Theorem 2.3** (*Bousquet's Inequality, Functional Bennet Inequality*) [Boucheron et al., 2013]

Let X_1, \dots, X_n be **independent identically distributed** random vectors. Assume that $\mathbb{E}[f(X_i)] = 0$, and that $f(X_i) \leq 1$ for all $f \in \mathcal{F}$. Let

$$\gamma^2 = \sigma^2 + 2\mathbb{E}[Z],$$

where $\sigma^2 := \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E}[f^2(X_i)] \right\}$ is **the wimpy variance**. Let $\phi(u) = e^u - u - 1$ and $h(u) = (1 + u) \log(1 + u) - u$, for $u \geq -1$. Then for all $\lambda \geq 0$,

$$\log \mathbb{E} \left[e^{\lambda(Z - \mathbb{E}[Z])} \right] \leq n\gamma^2 \phi(\lambda).$$

Also, for all $t \geq 0$,

$$\mathcal{P}\{Z \geq \mathbb{E}[Z] + t\} \leq \exp\left(-n\gamma^2 h\left(\frac{t}{\gamma^2}\right)\right). \quad (16)$$

2.4 Symmetrization and Contraction Principle

- **Definition** (*Symmetrized Empirical Process*)

Let X_1, \dots, X_n be independent random variables on \mathcal{X} and \mathcal{F} be a class of measurable functions on \mathcal{X} . Consider the symmetrized process

$$f \rightarrow \mathcal{P}_n^\epsilon f := \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i), \quad \forall f \in \mathcal{F} \quad (17)$$

where $\epsilon := (\epsilon_1, \dots, \epsilon_n)$ are **independent Rademacher random variables** taking values in $\{-1, +1\}$ with equal probability and ϵ_i 's are independent from $X = (X_1, \dots, X_n)$. **The supremum norm of symmetrized process** is defined as

$$\|\mathcal{P}_n^\epsilon\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \right|$$

- **Definition** (*Rademacher Process*)

Let $\epsilon := (\epsilon_1, \dots, \epsilon_n)$ be **independent Rademacher random variables** taking values in $\{-1, +1\}$ with equal probability. The Rademacher process is defined as

$$t \rightarrow \frac{1}{n} \sum_{i=1}^n \epsilon_i t_i, \quad t := (t_1, \dots, t_n) \in T \subset \mathbb{R}^n. \quad (18)$$

So the symmetrized empirical process (17) is a Rademacher process conditioning on $X = (X_1, \dots, X_n)$.

- **Remark (*Symmetrization*)**

The technique that replaces the empirical process $(\mathcal{P}_n - \mathcal{P})f$ by the symmetrized version $\mathcal{P}_n^\epsilon f$ is called ***symmetrization***. The idea is that, for fixed (X_1, \dots, X_n) , the symmetrized empirical measure (17) is a *Rademacher process*, hence a ***sub-Gaussian process***.

Proposition 2.4 (*Symmetrization Inequalities*). [Wellner and van der Vaart, 2013, Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]

For every ***nondecreasing, convex*** $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ and class of measurable functions \mathcal{F} ,

$$\mathbb{E}_{X,\epsilon} \left[\Phi \left(\frac{1}{2} \|\mathcal{P}_n^\epsilon\|_{\overline{\mathcal{F}}} \right) \right] \leq \mathbb{E}_X [\Phi (\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}})] \leq \mathbb{E}_{X,\epsilon} [\Phi (2 \|\mathcal{P}_n^\epsilon\|_{\mathcal{F}})] \quad (19)$$

where $\overline{\mathcal{F}} := \{f - \mathbb{E}_{\mathcal{P}}[f] : f \in \mathcal{F}\}$ is the ***recentered function class***.

Proof: We first prove the upper bound. Let Y be i.i.d. samples with the same distribution as X . For fixed f , $\mathbb{E}_X [f(X)] = \mathbb{E}_Y [\frac{1}{n} \sum_{i=1}^n f(Y_i)]$.

$$\begin{aligned} \mathbb{E}_X [\Phi (\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}})] &= \mathbb{E}_X \left[\Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) - \mathbb{E}_X [f(X)]) \right| \right) \right] \\ &= \mathbb{E}_X \left[\Phi \left(\sup_{f \in \mathcal{F}} \left| \mathbb{E}_Y \left[\frac{1}{n} \sum_{i=1}^n (f(X_i) - f(Y_i)) \right] \right| \right) \right] \\ &\leq \mathbb{E}_{X,Y} \left[\Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) - f(Y_i)) \right| \right) \right] \\ &= \mathbb{E}_{X,Y,\epsilon} \left[\Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i (f(X_i) - f(Y_i)) \right| \right) \right] \end{aligned}$$

The first inequality is due to Jensen's inequality since Φ is non-decreasing and convex. The last equality is due to the fact that $\epsilon_i(f(X_i) - f(Y_i))$ and $f(X_i) - f(Y_i)$ have the same joint distribution. Next by triangle inequality and Jensen's inequality we have

$$\begin{aligned} \dots &\leq \mathbb{E}_{X,Y,\epsilon} \left[\Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \right| + \sup_{f \in \mathcal{F}} \left| -\frac{1}{n} \sum_{i=1}^n \epsilon_i f(Y_i) \right| \right) \right] \\ &\leq \frac{1}{2} \mathbb{E}_{X,\epsilon} \left[\Phi \left(2 \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \right| \right) \right] + \frac{1}{2} \mathbb{E}_{Y,\epsilon} \left[\Phi \left(2 \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i f(Y_i) \right| \right) \right] \\ &= \mathbb{E}_{X,\epsilon} \left[\Phi \left(2 \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \right| \right) \right] \end{aligned}$$

which proves the upper bound. To prove the lower bound, we have

$$\begin{aligned} \mathbb{E}_{X,\epsilon} \left[\Phi \left(\frac{1}{2} \|\mathcal{P}_n^\epsilon\|_{\overline{\mathcal{F}}} \right) \right] &= \mathbb{E}_{X,\epsilon} \left[\Phi \left(\frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i (f(X_i) - \mathbb{E}_{Y_i} [f(Y_i)]) \right| \right) \right] \\ &\leq \mathbb{E}_{X,Y,\epsilon} \left[\Phi \left(\frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i (f(X_i) - f(Y_i)) \right| \right) \right] \\ &= \mathbb{E}_{X,Y} \left[\Phi \left(\frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) - f(Y_i)) \right| \right) \right] \end{aligned}$$

where the first inequality is due to convexity of Φ and Jensen's inequality and equality follows since $\epsilon_i(f(X_i) - f(Y_i))$ and $f(X_i) - f(Y_i)$ have the same joint distribution. Note that by triangle inequality

$$\begin{aligned} \frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) - f(Y_i)) \right| &= \frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) + \mathbb{E}_X[f(X)] - \mathbb{E}_Y[f(Y)] - f(Y_i)) \right| \\ &\leq \frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) + \mathbb{E}_X[f(X)]) \right| + \frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(Y_i) + \mathbb{E}_X[f(Y)]) \right| \end{aligned}$$

Since Φ is convex and non-decreasing,

$$\begin{aligned} \Phi \left(\frac{1}{2} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) - f(Y_i)) \right| \right) &\leq \frac{1}{2} \Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(X_i) + \mathbb{E}_X[f(X)]) \right| \right) \\ &\quad + \frac{1}{2} \Phi \left(\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f(Y_i) + \mathbb{E}_X[f(Y)]) \right| \right) \end{aligned}$$

The claim follows by taking expectations and using the fact that X and Y are identically distributed. \blacksquare

- **Proposition 2.5 (Contraction Principle, Simple Version)** [Boucheron et al., 2013, Vershynin, 2018]

Let x_1, \dots, x_n be vectors whose real-valued components are indexed by T , that is, $x_i = (x_{i,s})_{s \in T}$. Let $\alpha_i \in [0, 1]$ for $i = 1, \dots, n$. Let $\epsilon_1, \dots, \epsilon_n$ be independent Rademacher random variables. Then

$$\mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i \alpha_i x_{i,s} \right] \leq \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right] \quad (20)$$

Proof: Define $\Psi : (\mathbb{R}^T)^n \rightarrow \mathbb{R}$ as the right hand side:

$$\Psi(x_1, \dots, x_n) = \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right].$$

The function Ψ is **convex** since it is a linear combination of suprema of linear functions. It is also *invariant* under sign change in the sense that for all $(\eta_1, \dots, \eta_n) \in \{-1, 1\}^n$,

$$\Psi(\eta_1 x_1, \dots, \eta_n x_n) = \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i \eta_i x_{i,s} \right] = \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right] = \Psi(x_1, \dots, x_n).$$

Fix $(x_1, \dots, x_n) \in (\mathbb{R}^T)^n$. Consider the restriction of Ψ to the **convex hull** of the 2^n points of the form $(\eta_1 x_1, \dots, \eta_n x_n)$, with $(\eta_1, \dots, \eta_n) \in \{-1, 1\}^n$. The **supremum** of Ψ is achieved at one of the **vertices** $(\eta_1 x_1, \dots, \eta_n x_n)$. The sequence of vectors $(\alpha_1 x_1, \dots, \alpha_n x_n)$ lies inside the convex hull of $(\eta_1 x_1, \dots, \eta_n x_n)$ and therefore

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i \alpha_i x_{i,s} \right] &= \Psi(\alpha_1 x_1, \dots, \alpha_n x_n) \\ &\leq \Psi(\eta_1 x_1, \dots, \eta_n x_n) = \Psi(x_1, \dots, x_n) \\ &= \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right]. \quad \blacksquare \end{aligned}$$

- **Remark** For arbitrary $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, the contraction principle becomes [Vershynin, 2018]

$$\mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i \alpha_i x_{i,s} \right] \leq \|\alpha\|_\infty \mathbb{E} \left[\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right] \quad (21)$$

- To prove the following general contraction principle, we need the following lemma

Lemma 2.6 [Boucheron et al., 2013, Vershynin, 2018]

Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ denote a **convex non-decreasing function**. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a 1-Lipschitz function such that $\varphi_i(0) = 0$. Let $T \subset \mathbb{R}^2$. Then

$$\Psi \left(\sup_{s \in T} \{s_1 + \varphi(s_2)\} \right) + \Psi \left(\sup_{s \in T} \{s_1 - \varphi(s_2)\} \right) \leq \Psi \left(\sup_{s \in T} \{s_1 + s_2\} \right) + \Psi \left(\sup_{s \in T} \{s_1 - s_2\} \right)$$

(Hint: For non-decreasing convex function Ψ , we have

$$\Psi(d) - \Psi(c) \leq \Psi(b) - \Psi(a)$$

for $0 \leq d - c \leq b - a$ and $c \leq a$. It suffice to show that

$$\begin{aligned} \Psi(s_1^* + \varphi(s_2^*)) + \Psi(t_1^* - \varphi(t_2^*)) &\leq \Psi(s_1^* + s_2^*) + \Psi(t_1^* - t_2^*) \\ \Rightarrow \Psi(t_1^* - \varphi(t_2^*)) - \Psi(t_1^* - t_2^*) &\leq \Psi(s_1^* + s_2^*) - \Psi(s_1^* + \varphi(s_2^*)) \end{aligned}$$

where $s^* = (s_1^*, s_2^*)$ and $t^* = (t_1^*, t_2^*)$ are optimal solution for the first and second term on the left-hand side of inequality.)

- **Proposition 2.7 (Talagrand's Contraction Principle)** [Boucheron et al., 2013, Vershynin, 2018]

Let x_1, \dots, x_n be vectors whose real-valued components are indexed by T , that is, $x_i = (x_{i,s})_{s \in T}$. For each $i = 1, \dots, n$, let $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ be a 1-Lipschitz function such that $\varphi_i(0) = 0$. Let $\epsilon_1, \dots, \epsilon_n$ be independent Rademacher random variables, and let $\Psi : [0, \infty) \rightarrow \mathbb{R}$ be a **non-decreasing convex function**. Then

$$\mathbb{E} \left[\Psi \left(\sup_{s \in T} \sum_{i=1}^n \epsilon_i \varphi_i(x_{i,s}) \right) \right] \leq \mathbb{E} \left[\Psi \left(\sup_{s \in T} \sum_{i=1}^n \epsilon_i x_{i,s} \right) \right] \quad (22)$$

and

$$\mathbb{E} \left[\Psi \left(\frac{1}{2} \sup_{s \in T} \left| \sum_{i=1}^n \epsilon_i \varphi_i(x_{i,s}) \right| \right) \right] \leq \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left| \sum_{i=1}^n \epsilon_i x_{i,s} \right| \right) \right]. \quad (23)$$

Proof: We show the first inequality. It suffices to prove that, if $T \subset \mathbb{R}^n$ is a finite set of vectors $s = (s_1, \dots, s_n)$, then

$$\mathbb{E} \left[\Psi \left(\sup_{s \in T} \sum_{i=1}^n \epsilon_i \varphi_i(s_i) \right) \right] \leq \mathbb{E} \left[\Psi \left(\sup_{s \in T} \sum_{i=1}^n \epsilon_i s_i \right) \right]$$

The key step is that for an arbitrary function $A : T \rightarrow \mathbb{R}$

$$\mathbb{E} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^n \epsilon_i \varphi_i(s_i) \right\} \right) \right] \leq \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^n \epsilon_i s_i \right\} \right) \right]$$

For $n = 1$, using the above lemma

$$\mathbb{E} \left[\Psi \left(\sup_{u \in U} \{u_1 + \epsilon \varphi(u_2)\} \right) \right] \leq \mathbb{E} \left[\Psi \left(\sup_{u \in U} \{u_1 + \epsilon u_2\} \right) \right]$$

where $U = \{(A(s), s), s \in T\}$.

We prove by induction on n . Assume that the hypothesis hold true for all $1, \dots, n-1$. Then

$$\begin{aligned} & \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^n \epsilon_i \varphi_i(s_i) \right\} \right) \right] \\ &= \mathbb{E}_{\epsilon_1, \dots, \epsilon_{n-1}} \left[\mathbb{E}_{\epsilon_n} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^{n-1} \epsilon_i \varphi_i(s_i) + \epsilon_n \varphi_i(s_n) \right\} \right) \middle| \epsilon_1, \dots, \epsilon_{n-1} \right] \right] \\ & \text{by hypothesis on } n=1 \\ &\leq \mathbb{E}_{\epsilon_1, \dots, \epsilon_{n-1}} \left[\mathbb{E}_{\epsilon_n} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^{n-1} \epsilon_i \varphi_i(s_i) + \epsilon_n s_n \right\} \right) \middle| \epsilon_1, \dots, \epsilon_{n-1} \right] \right] \\ &= \mathbb{E}_{\epsilon_n} \left[\mathbb{E}_{\epsilon_1, \dots, \epsilon_{n-1}} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^{n-1} \epsilon_i \varphi_i(s_i) + \epsilon_n s_n \right\} \right) \middle| \epsilon_n \right] \right] \\ & \text{by hypothesis on } n-1 \\ &\leq \mathbb{E}_{\epsilon_n} \left[\mathbb{E}_{\epsilon_1, \dots, \epsilon_{n-1}} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^{n-1} \epsilon_i s_i + \epsilon_n s_n \right\} \right) \middle| \epsilon_n \right] \right] \\ &= \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left\{ A(s) + \sum_{i=1}^n \epsilon_i s_i \right\} \right) \right] \end{aligned}$$

which proves the first inequality. For the second inequality, we see that

$$\begin{aligned} \mathbb{E} \left[\Psi \left(\frac{1}{2} \sup_{s \in T} \left| \sum_{i=1}^n \epsilon_i \varphi_i(x_{i,s}) \right| \right) \right] &= \mathbb{E} \left[\Psi \left(\frac{1}{2} \sup_{s \in T} \left(\sum_{i=1}^n \epsilon_i \varphi_i(x_{i,s}) \right)_+ + \frac{1}{2} \sup_{s \in T} \left(\sum_{i=1}^n -\epsilon_i \varphi_i(x_{i,s}) \right)_+ \right) \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left(\sum_{i=1}^n \epsilon_i \varphi_i(x_{i,s}) \right)_+ \right) \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\Psi \left(\sup_{s \in T} \left(\sum_{i=1}^n -\epsilon_i \varphi_i(x_{i,s}) \right)_+ \right) \right] \end{aligned}$$

The second inequality in the theorem now follows by invoking twice the first inequality and noting that the function $\Psi((x)_+)$ is *convex and non-decreasing*. \blacksquare

- **Remark** Let $\varphi_i = \varphi$ for all i and replace $x_{i,s} \rightarrow f(X_i)$ and $s \in T \rightarrow f \in \mathcal{F}$. We obtain the contraction principle for symmetrized empirical process indexed by function class \mathcal{F} .

$$\begin{aligned} \mathbb{E} \left[\Psi \left(\sup_{g \in \varphi \circ \mathcal{F}} \mathcal{P}_n^\epsilon g \right) \right] &\leq \mathbb{E} \left[\Psi \left(\sup_{f \in \mathcal{F}} \mathcal{P}_n^\epsilon f \right) \right] \tag{24} \\ \text{and } \mathbb{E} \left[\Psi \left(\frac{1}{2} \|\mathcal{P}_n^\epsilon\|_{\varphi \circ \mathcal{F}} \right) \right] &\leq \mathbb{E} [\Psi (\|\mathcal{P}_n^\epsilon\|_{\mathcal{F}})]. \end{aligned}$$

2.5 Rademacher Complexity and Gaussian Complexity

- **Definition (*Empirical Rademacher Complexity*)**

Let \mathcal{F} be a family of functions on \mathcal{X} and $\mathcal{D} = (X_1, \dots, X_n)$ a fixed *sample* of size n with elements in \mathcal{X} . Then, the empirical Rademacher complexity of \mathcal{F} with respect to the sample \mathcal{D} is defined as:

$$\hat{\mathfrak{R}}_{\mathcal{D}}(\mathcal{F}) = \mathbb{E}_{\epsilon} \left[\sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \right] = \mathbb{E}_{\epsilon} \left[\sup_{f \in \mathcal{F}} \mathcal{P}_n^{\epsilon} f \right] \quad (25)$$

where $\epsilon := (\epsilon_1, \dots, \epsilon_n)$ are *independent uniform random variables* taking values in $\{-1, +1\}$. The random variables ϵ_i are called Rademacher variables.

- **Definition (*Rademacher Complexity*)**

For any integer $n \geq 1$, the Rademacher complexity of \mathcal{F} is defined as the *expectation* of the empirical Rademacher complexity over all samples \mathcal{D}_n of size n drawn according to $\mathcal{P} = \otimes_{i=1}^n \mathcal{P}_i$:

$$\mathfrak{R}_n(\mathcal{F}) = \mathbb{E}_{\mathcal{D}_n \sim \mathcal{P}} \left[\hat{\mathfrak{R}}_{\mathcal{D}_n}(\mathcal{F}) \right].$$

- By *symmetrization inequality* (19) and *bounded difference inequality*, we can obtain the following upper and lower bounds on *supremum norm of centered empirical process*

Proposition 2.8 (*Uniform Upper Bound via Rademacher Complexity*) [Wainwright, 2019]

Let \mathcal{F} be a class of *b-uniformly bounded functions*, i.e. $\|f\|_{\infty} \leq b$ for all $f \in \mathcal{F}$. Then, for any positive $n \geq 1$, any $\delta > 0$, **with \mathcal{P} -probability at least $1 - \delta$** :

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} \leq 2\mathfrak{R}_n(\mathcal{F}) + \sqrt{\frac{2b^2 \log(1/\delta)}{n}} \quad (26)$$

Consequently, as long as $\mathfrak{R}_n(\mathcal{F}) = o(1)$, we have $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} \xrightarrow{a.s.} 0$.

- **Proposition 2.9 (*Uniform Lower Bound via Rademacher Complexity*)** [Wainwright, 2019]

Let \mathcal{F} be a class of *b-uniformly bounded functions*, i.e. $\|f\|_{\infty} \leq b$ for all $f \in \mathcal{F}$. Then, for any positive $n \geq 1$, any $\delta > 0$, **with \mathcal{P} -probability at least $1 - \delta$** :

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} \geq \frac{1}{2}\mathfrak{R}_n(\mathcal{F}) - \frac{\sup_{f \in \mathcal{F}} |\mathbb{E}_{\mathcal{P}}[f]|}{2\sqrt{n}} - \sqrt{\frac{2b^2 \log(1/\delta)}{n}} \quad (27)$$

As a consequence, if the Rademacher complexity $\mathfrak{R}_n(\mathcal{F})$ remains **bounded away from zero**, then $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$ cannot converge to zero in probability.

- **Remark** From both Proposition 2.8 and Proposition 2.9, we have shown that *the Rademacher complexity* provides a necessary and sufficient condition for a uniformly bounded function class \mathcal{F} to be **Glivenko-Cantelli**.
- The following result follows from the *Talagrand's contraction principle* (24)

Lemma 2.10 (*Talagrand's Lemma*) [Mohri et al., 2012]

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an L -Lipschitz. Then, for any hypothesis set \mathcal{F} of real-valued functions, the following inequality holds:

$$\hat{\mathfrak{R}}_{\mathcal{D}}(\varphi \circ \mathcal{F}) \leq L \hat{\mathfrak{R}}_{\mathcal{D}}(\mathcal{F}). \quad (28)$$

3 Variance of Suprema of Empirical Process

3.1 General Upper Bounds for the Variance

- **Definition** (*Variances of Empirical Process*)

Let X_1, \dots, X_n be independent random variables taking values in \mathcal{X} . Depending on **ordering** of the **expectation**, **suprema** and **summation** operator, we define *three different types of variance* associated with empirical process

$$\mathcal{P}_n f = \mathbb{E}_{\mathcal{P}_n} [f] = \frac{1}{n} \sum_{i=1}^n f(X_i).$$

1. *The strong variance* is defined as

$$V := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\sup_{f \in \mathcal{F}} f^2(X_i) \right] \quad (29)$$

2. *The weak variance* is defined as

$$\Sigma^2 := \mathbb{E} \left[\sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n f^2(X_i) \right\} \right] \quad (30)$$

3. *The wimpy variance* is defined as

$$\sigma^2 := \sup_{f \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E} [f^2(X_i)] \right\} \quad (31)$$

By Jensen's inequality,

$$\sigma^2 \leq \Sigma^2 \leq V$$

In general, there may be *significant gaps between any two of these quantities*. A notable difference is the case of **Rademacher averages** when $\sigma^2 = \Sigma^2$.

3.2 Bounding the Weak Variance via Wimpy Variance

3.3 Maximal Inequalities

4 Expected Value of Suprema of Empirical Process

4.1 Covering Number, Packing Number and Metric Entropy

4.2 Chaining and Dudley's Entropy Integral

4.3 Vapnik-Chervonenkis Class

4.4 Comparison Theorems

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