Lecture 3: Topology Review

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Contents

1 Review of Topology

1.1 Set Theory Basis

- **Definition** Given a set X, the collection of all subsets of X, denoted as 2^X, is defined as

$$2^X := \{E : E \subseteq X\}$$

- **Remark** The followings are basic operation on 2^X: For $A, B \in 2^X$,

  1. ***Inclusion***: $A \subseteq B$ if and only if $\forall x \in A$, $x \in B$.
  2. ***Union***: $A \cup B = \{x : x \in A \vee x \in B\}$.
  3. ***Intersection***: $A \cap B = \{x : x \in A \wedge x \in B\}$.
  4. ***Difference***: $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
  5. ***Complement***: $A^c = X \setminus A = \{x : x \in X \wedge x \notin A\}$.
  6. ***Symmetric Difference***: $A \Delta B = (A \setminus B) \cup (B \setminus A) = \{x \in X : x \notin A \vee x \notin B\}$.

  We have ***deMorgan's laws***:

$$\left(\bigcup_{a \in A} U_a\right)^c = \bigcap_{a \in A} U_a^c, \quad \left(\bigcap_{a \in A} U_a\right)^c = \bigcup_{a \in A} U_a^c$$

- **Remark** Note that the following equality is useful:

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

- **Definition** A ***rule of assignment*** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate ***at most one ordered pair belonging to*** r. Thus, a subset r of $C \times D$ is *a rule of assignment* if

$$[(c, d) \in r \text{ and } (c, d') \in r] \Rightarrow [d = d'].$$

  Given a rule of assignment r, ***the domain*** of r is defined to be the *subset* of C consisting of *all first coordinates of elements* of r, and ***the image*** set of r is defined as the *subset* of D consisting of *all second coordinates of elements* of r.

  A ***function*** f is *a rule of assignment* r, together with a set B that *contains the image set* of r.

  The ***pre-image*** of f is defined as

$$f^{-1}(E) = \{x \in X : f(x) \in E\}.$$

- **Remark** The pre-image operation ***commutes*** with ***all basic set operations***:

$$
\begin{aligned}
A \subseteq B \Rightarrow f^{-1}(A) &\subseteq f^{-1}(B) \\
f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) &= \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \\
f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) &= \bigcap_{\alpha\in A} f^{-1}(E_\alpha) \\
f^{-1}(A\setminus B) &= f^{-1}(A)\setminus f^{-1}(B) \\
f^{-1}(E^c) &= \left(f^{-1}(E)\right)^c
\end{aligned}
$$

- **Remark** The image operation ***commutes*** with only ***inclusion*** *and* ***union*** *operations*:

$$
\begin{aligned}
A \subseteq B \Rightarrow f(A) &\subseteq f(B) \\
f\left(\bigcup_{\alpha\in A} E_\alpha\right) &= \bigcup_{\alpha\in A} f(E_\alpha)
\end{aligned}
$$

  For the other operations:

$$
\begin{aligned}
f\left(\bigcap_{\alpha\in A} E_\alpha\right) &\subseteq \bigcap_{\alpha\in A} f(E_\alpha) \\
f(A\setminus B) &\supseteq f(A)\setminus f(B)
\end{aligned}
$$

- **Definition** A map $f : X \to Y$ is ***surjective, or, onto***, if for every $y \in Y$, there exists a $x \in X$ such that $y = f(x)$. In set theory notation:

$$f : X \to Y \text{ is surjective } \Leftrightarrow f^{-1}(Y) \subseteq X.$$

  A map $f : X \to Y$ is ***injective***, if for every $x_1 \neq x_2 \in X$, their map $f(x_1) \neq f(x_2)$, or equivalently, $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

  If a map $f : X \to Y$ is both *surjective* and *injective*, we say f is a ***bijective***, or there exists an ***one-to-one correspondence*** between X and Y. Thus $Y = f(X)$.

- **Remark**

$$
\begin{aligned}
f^{-1}(f(B)) &\supseteq B, \quad \forall B \subseteq X \\
f(f^{-1}(E)) &\subseteq E, \quad \forall E \subseteq Y \\
f : X \to Y \text{ is surjective } &\Leftrightarrow f^{-1}(Y) \subseteq X. \\
&\Rightarrow f(f^{-1}(E)) = E. \\
f : X \to Y \text{ is injective } &\Rightarrow f^{-1}(f(B)) = B \\
&\Rightarrow f\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f(E_\alpha) \\
&\Rightarrow f(A\setminus B) = f(A)\setminus f(B)
\end{aligned}
$$

- **Proposition 1.1** *The following statements for composite functions are true:*

  1. *If f, g are both injective, then $g \circ f$ is injective.*
  2. *If f, g are both surjective, then $g \circ f$ is surjective.*
  3. *Every **injective** map $f : X \to Y$ can be writen as $f = \iota \circ f_R$ where $f_R : X \to f(X)$ is a **bijective** map and ι is the **inclusion map**.*
  4. *Every **surjective** map $f : X \to Y$ can be writen as $f = f_p \circ \pi$ where $\pi : X \to (X/\sim)$ is **a quotient map** (projection $x \mapsto [x]$) for the equivalent relation $x \sim y \Leftrightarrow f(x) = f(y)$ and $f_p : (X/\sim) \to Y$ is defined as $f_p([x]) = f(x)$ **constant** in each coset $[x]$.*
  5. *If $g \circ f$ is **injective**, then f is **injective**.*
  6. *If $g \circ f$ is **surjective**, then g is **surjective**.*

- **Definition** A ***relation*** on a set A is a *subset* R of *the cartesian product* $A \times A$.

  If R is a relation on A, we use the notation xRy to mean the same thing as $(x, y) \in R$. We read it "x is in the relation R to y."

- **Remark** ***A rule of assignment*** r for a function $f : A \to A$ is also a *subset* of $A \times A$. But it is a subset of a *very* ***special*** *kind*: namely, one such that ***each element*** of A appears as the ***first coordinate*** of an element of r ***exactly once***. ***Any subset*** *of $A \times A$ is a relation on A.*

- **Definition** ***An equivalence relation*** on X is a relation R on X such that

  1. (***Reflexivity***): xRx for all $x \in X$;
  2. (***Symmetry***): xRy if and only if yRx for all $x, y \in X$;
  3. (***Transitivity***): xRy and yRz then xRz for all $x, y, z \in X$.

  We usually denote the equivalence relation R as $\sim$.

- **Definition** (***Equivalence Class***)
  The equivalence class of an element x is denoted as $[x] := \{y \in X : xRy\}$.

- **Definition** A relation C on a set A is called ***an order relation*** (or ***a simple order***, or ***a linear order***) if it has the following properties:

  1. (***Comparability***) For every x and y in A for which $x \neq y$, either xCy or yCx.
  2. (***Nonreflexivity***) For no x in A does the relation xCx hold.
  3. (***Transitivity***) If xCy and yCz, then xCz.

  We denote order relation as $>$ or $<$. We shall use the notation $x \leq y$ to stand for the statement "either $x < y$ or $x = y$"; and we shall use the notation $y > x$ to stand for the statement "$x < y$." We write $x < y < z$ to mean "$x < y$ and $y < z$"

- **Definition** Suppose that A is a set ordered by the relation $<$. Let A_0 be a subset of A. We say that the element b is *the **largest element** of A_0* if $b \in A_0$ and $x \leq b$ for every $x \in A_0$.

  Similarly, we say that a is *the **smallest element*** of A_0 if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$.

- **Remark** It is easy to see that a set has ***at most one*** largest element and *at most one* smallest element.

- **Definition** (***The Upper Bound and The Supremum of Subset***)
  We say that *the subset A_0 of A is* ***bounded above*** if there is *an element b of A* such that $x \leq b$ for every $x \in A_0$; the element $b \in A$ is called ***an upper bound for*** A_0.

  If *the set of all upper bounds* for A_0 has *a* ***smallest element***, that element is called ***the least upper bound***, or ***the supremum***, of A_0. It is denoted by $\sup A_0$, it may or may not belong to A_0. If it *does*, it is ***the largest element*** of A_0.

- **Definition** (***The Lower Bound and The Infimum of Subset***)
  Similarlly, we say that *the subset A_0 of A is* ***bounded below*** if there is *an element a of A* such that $a \leq x$ for every $x \in A_0$; the element $a \in A$ is called ***a lower bound for*** A_0.

  If *the set of all lower bounds* for A_0 has *a* ***largest element***, that element is called ***the greatest lower bound***, or ***the infimum***, of A_0. It is denoted by $\inf A_0$, it may or may not belong to A_0. If it *does*, it is ***the smallest element*** of A_0.

- **Definition** (***The Least Upper Bound Property*** *and* ***The Greatest Lower Bound Property***)
  An ordered set A is said to have ***the least upper bound property*** if *every nonempty* subset A_0 of A that is *bounded above* has *a least upper bound.*

  Analogously, the set A is said to have ***the greatest lower bound property*** if *every nonempty* subset A_0 of A that is *bounded below* has *a greatest lower bound.*

- **Definition** (***Well-Ordered Set***)
  A set A with an order relation $<$ is said to be ***well-ordered*** if *every nonempty subset* of A has a ***smallest element***.

- **Definition** (***Strict Partial Order***)
  Given a set A, a relation $\prec$ on A is called a ***strict partial order*** on A if it has the following two properties;

  1. (***Nonreflexivity***) The relation $a \prec a$ never holds.
  2. (***Transitivity***) If $a \prec b$ and $b \prec c$, then $a \prec c$.

  Moreover, suppose that we define $a \preceq b$ either $a \prec b$ or $a = b$. Then the relation $\preceq$ is called ***a partial order*** on A.

- **Definition** (***Upper Bound and Maximal Element for Strict Partial Order***)
  Let A be a set and let $\prec$ be a *strict partial order* on A. If B is a subset of A, ***an upper bound*** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

  A maximal element of A is an element m of A such that for ***no element*** *a of A does the relation $m \prec a$ hold.*

- **Theorem 1.2** *(Zorn's Lemma). [Munkres, 2000]*
  Let A be a set that is ***strictly partially ordered****. If every* ***simply ordered subset*** *of A has an* ***upper bound in*** *A, then A has a* ***maximal element****.*

- **Principle 1.3** *(The Axiom of Choice).*
  If $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets, then $\prod_{\alpha \in A} X_\alpha$ is non-empty.

- **Corollary 1.4** *If* $\{X_\alpha\}_{\alpha\in A}$ *is a* ***disjoint*** *collection of nonempty sets, there is a set* $Y \subset \bigcup_{\alpha\in A} X_\alpha$ *such that* $Y \cap X_\alpha$ *contains* ***precisely one element*** *for each* $\alpha \in A$*.*

1.2 Topological Space

- **Definition** Let X be a set. *A **topology*** on X is *a collection* $\mathscr{T}$ of *subsets* of X, called ***open subsets***, satisfying

  1. X and $\emptyset$ are *open*.
  2. The ***union*** of ***any family*** of open subsets is open.
  3. The ***intersection*** of *any* ***finite*** *family* of open subsets is open.

  A pair $(X, \mathscr{T})$ consisting of a set X together with a topology $\mathscr{T}$ on X is called ***a topological space***.

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Definition** A ***continuous bijective*** map $F : X \to Y$ with ***continuous inverse*** is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Definition** Suppose X is a topological space. A collection $\mathscr{B}$ of open subsets of X is said to be ***a basis*** for *the topology of* X (plural: ***bases***) if every open subset of X is the *union of some collection of elements* of $\mathscr{B}$.

  More generally, suppose X is merely a set, and $\mathscr{B}$ is a collection of *subsets* of X satisfying the following conditions:

  1. $X = \bigcup_{B\in\mathscr{B}} B$.
  2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B2$.

  Then *the collection of **all unions** of elements of* $\mathscr{B}$ is a *topology* on X, called ***the topology generated by*** $\mathscr{B}$, and $\mathscr{B}$ is a ***basis** for this topology*.

- **Lemma 1.5** *(**Obtaining Basis from Given Topology**). [Munkres, 2000]*
  Let X *be a topological space. Suppose that* $\mathscr{C}$ *is a collection of open sets of* X *such that for each open set* U *of* X *and each* x *in* U*, there is an element* C *of* $\mathscr{C}$ *such that* $x \in C \subset U$*. Then* C *is a basis for the topology of* X*.*

- **Lemma 1.6** *(**Topology Comparison via Bases**). [Munkres, 2000]*
  Let $\mathscr{B}$ *and* $\mathscr{B}'$ *be bases for the topologies* $\mathscr{T}$ *and* $\mathscr{T}'$*, respectively, on* X*. Then the following are equivalent:*

  1. $\mathscr{T}'$ *is* ***finer*** *than* $\mathscr{T}$*.*
  2. *For each* $x \in X$ *and each basis element* $B \in \mathscr{B}$ *containing* x*, there is a basis element* $B' \in \mathscr{B}'$ *such that* $x \in B' \subset B$*.*

- **Definition** (***Subbasis***)
  A subbasis $\mathscr{S}$ for a *topology* on X is a collection of subsets of X whose union equals X.

The topology generated by the *subbasis* $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of ***all unions*** *of* ***finite intersections*** *of elements of* $\mathscr{S}$.

- **Remark** (***Basis from Subbasis***)
For a *subbasis* $\mathscr{S}$, the collection $\mathscr{B}$ of ***all finite intersections*** of elements of $\mathscr{S}$ is a ***basis***,

1.3 Limit Point and Closure

- **Definition** A subset A of a topological space X is said to be ***closed*** if the set $X \setminus A$ is *open*.

- **Definition** Given a subset A of a topological space X, ***the interior of*** A is defined as *the union of all open sets* ***contained*** in A, and ***the closure of*** A is defined as *the intersection of all closed sets* ***containing*** A.

  The interior of A is denoted by Int A or by $\mathring{A}$ and ***the closure of*** A is denoted by CI A or by $\bar{A}$. Obviously $\mathring{A}$ is *an open set* and $\bar{A}$ is *a closed set*; furthermore,
$$\mathring{A} \subseteq A \subseteq \bar{A}.$$
  If A is ***open***, $A = \mathring{A}$; while if A is ***closed***, $A = \bar{A}$.

- **Proposition 1.7** *(**Characterization of Closure in terms of Basis**) [Munkres, 2000]*
Let A be a subset of the topological space X.

  1. *Then $x \in \bar{A}$ if and only if every* ***open*** *set U* ***containing*** *x* ***intersects*** *A.*
  2. *Supposing the topology of X is given by a* ***basis****, then $x \in \bar{A}$ if and only if every basis element B* ***containing*** *x* ***intersects*** *A.*

- **Remark** We can say "*U is a **neighborhood** of x*" if "*U is an open set containing x*".

- **Definition** (***Limit Point***)
If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point*** (or "***cluster point***," or "***point of accumulation***") of A if ***every neighborhood*** *of x* ***intersects*** *A in some point* ***other than*** *x itself.*

  Said differently, x is ***a limit point*** of A if it belongs to ***the closure of*** $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.

- **Theorem 1.8** *(**Decomposition of Closure**)*
Let A be a subset of the topological space X; let A' be the set of ***all limit points*** *of A. Then*
$$\bar{A} = A \cup A'.$$

- **Corollary 1.9** *A subset of a topological space is* ***closed*** *if and only if it contains all its* ***limit points****.*

1.4 Subspace, Product and Quotient Topologies

1.4.1 Subspace Topology

- **Definition** If X is a topological space and $S \subseteq X$ is an arbitrary subset, we define ***the subspace topology*** on S (sometimes called *the* ***relative topology***) by declaring a subset

$U \subseteq S$ to be *open* in S *if and only* if there exists an open subset $V \subseteq X$ such that $U = V \cap S$.

Any subset of X endowed with the subspace topology is said to be ***a subspace of*** X.

- **Lemma 1.10** ***(Basis of Subspace Topology)***
If $\mathscr{B}$ is a basis for the topology of X then the collection

$$\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$$

is a ***basis*** *for the subspace topology on $S \subset X$.*

- **Proposition 1.11** *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

- **Proposition 1.12** ***(Closure in Subspace Topology)***
Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

1.4.2 Product Topology

- **Definition** (***J-tuples***)
Let J be an index set. Given a set X, we define a ***J-tuple*** *of elements* of X to be a function $x : J \to X$. If α is an element of J, we often denote ***the value of*** X ***at*** α by X_α rather than $x(\alpha)$; we call it ***the α-th coordinate*** of x. And we often *denote the function x itself* by the symbol

$$(x_\alpha)_{\alpha \in J}$$

which is as close as we can come to a "*tuple notation*" for an arbitrary index set J. We denote ***the set of all*** J***-tuples*** of elements of X by X^J.

- **Definition** (***Arbitrary Cartesian Products***)
Let $\{A_\alpha\}_{\alpha \in J}$ be an *indexed* family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. *The* ***cartesian product*** *of this indexed family*, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

- **Definition** (***Projection Mapping or Coordinate Projection***)
Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its β-th coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$

it is called ***the projection mapping*** *associated with the index β.*

- **Definition** (***Product Topology***)
  Let $\mathscr{S}_\beta$ denote the collection

$$\mathscr{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\right\},$$

  and let $\mathscr{S}$ denote *the union of these collections*,

$$\mathscr{S} = \bigcup_{\beta \in J} \mathscr{S}_\beta.$$

  The topology generated by *the* ***subbasis*** S is called ***the product topology***. In this topology $\prod_{\alpha\in J} X_\alpha$ is called ***a product space***.

- **Remark** (***Product Topology = Weak Topology by Coordinate Projections***)
  The product topology on $\prod_{\alpha\in J} X_\alpha$ is ***the weak topology*** generated by *a family of projection mappings* $(\pi_\beta)_{\beta\in J}$. It is ***the coarest (weakest) topology*** *such that* $(\pi_\beta)_{\beta\in J}$ *are* ***continuous***.

  A typical element of the basis from *the product topology* is ***the finite intersection*** *of subbasis* where the *index is different*:

$$\pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \ldots \cap \pi_{\beta_n}^{-1}(V_{\beta_n})$$

  Thus a ***neighborhood*** of x in ***the product topology*** is

$$N(x) = \{(x_\alpha)_{\alpha\in J} : x_{\beta_1} \in V_{\beta_1}, \ldots, x_{\beta_n} \in V_{\beta_n}\}$$

  where there is ***no restriction*** for $\alpha \in \{\beta_1, \ldots, \beta_n\}$.

  Note that for ***the box topology***, a *neighborhood* of x is

$$N_b(x) = \{(x_\alpha)_{\alpha\in J} : x_\alpha \in U_\alpha,\ \forall \alpha \in J\} \subset N(x)$$

  Thus ***the box topology*** is ***finer*** than ***the product topology***. Moreover, *for* ***finite product*** $\prod_{\alpha=1}^{n} X_\alpha$, the box topology and the product topology is the ***same***.

- **Definition** If X and Y are topological spaces, a continuous injective map $F : X \to Y$ is called a ***topological embedding*** if it is a ***homeomorphism*** onto its image $F(X) \subseteq Y$ in the subspace topology.

1.4.3 Quotient Topology

- **Definition** (***Quotient Map***)
  Let X and Y be topological spaces; let $\pi : X \to Y$ be a ***surjective map***. The map π is said to be ***a quotient map*** provided a subset U of Y is ***open*** in Y ***if and only if*** $\pi^{-1}(U)$ is ***open*** in X.

- **Remark** (***Quotient Map = Strong Continuity***)
  The condition of quotient map is ***stronger*** than continuity (it is called ***strong continuity*** in some literature).

$$\begin{aligned}
\text{continuity}: &\quad U \text{ is open in } Y \Rightarrow \pi^{-1}(U) \text{ is open in } X\\
\text{open map}: &\quad \pi(V) \text{ is open in } Y \Leftarrow V \text{ is open in } X\\
\text{quotient map}: &\quad U \text{ is open in } Y \Leftrightarrow \pi^{-1}(U) \text{ is open in } X
\end{aligned}$$

An equivalent condition is to require that a subset A of K be ***closed*** in Y *if and only if* $\pi^{-1}(A)$ is ***closed*** in X. Equivalence of the two conditions follows from equation

$$\pi^{-1}(Y \setminus B) = X \setminus \pi^{-1}(B).$$

- **Definition** (***Saturated Set and Fiber***)
  If $\pi : X \to Y$ is a ***surjective*** *map*, a subset $U \subseteq X$ is said to be ***saturated*** with respect to π if U contains every set $\pi^{-1}(\{y\})$ that it ***intersects***. Thus U is ***saturated*** if it equals to the ***entire preimage*** of its ***image***: $U = \pi^{-1}(\pi(U))$.

  Given $y \in Y$, the ***fiber*** of π over y is the set $\pi^{-1}(\{y\})$.

- **Definition** (***Quotient Map via Saturated Set***)
  A surjective map $\pi : X \to Y$ is a ***quotient map*** if π is ***continuous*** and π maps ***saturated open sets*** of X to ***open sets*** of Y (or *saturated closed sets* of X to *closed sets* of Y).

- **Definition** (***Open Map*** *and* ***Closed Map***)
  A map $f : X \to Y$ (continuous or not) is said to be an ***open map*** if for every *open* subset $U \subseteq X$, the image set $f(U)$ is *open* in Y, and a ***closed map*** if for every *closed* subset $K \subseteq X$, the image $f(K)$ is *closed* in Y .

- **Definition** (***Quotient Topology***)
  If X is a space and A is a set and if $\pi : X \to A$ is a **surjective** map, then there exists **exactly one topology** $\mathscr{T}$ on A relative to which π is a quotient map; it is called ***the quotient topology*** induced by π.

- **Definition** (***Quotient Space***)
  Suppose X is a topological space and $\sim$ is *an equivalence relation* on X. Let $X/\sim$ denote ***the set of equivalence classes*** in X, and let $\pi : X \to X/\sim$ be the ***natural projection*** sending each *point* to its *equivalence class*. Endowed with ***the quotient topology*** determined by π, the space $X/\sim$ is called ***the quotient space*** (or *identification space*) of X determined by π.

1.5 Constructing Continuous Functions

- **Proposition 1.13** *(**Rules for Constructing Continuous Functions**). [Munkres, 2000] Let X, Y, and Z be topological spaces.*

  1. *(**Constant Function**) If $f : X \to Y$ maps all of X into the **single point** y_0 of Y, then f is **continuous**.*
  2. *(**Inclusion**) If A is a subspace of X, the **inclusion function** $\iota : A \overset{X}{\hookrightarrow}$ is **continuous**.*
  3. *(**Composites**) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous*
  4. *(**Restricting the Domain**) If $f : X \to Y$ is **continuous**, and if A is a subspace of X, then **the restricted function** $f|_A : A \to Y$ is continuous.*
  5. *(**Restricting or Expanding the Range**) Let $f : X \to Y$ be **continuous**. If Z is a **subspace** of Y containing the **image** set $f(X)$, then the function $g : X \to Z$ obtained*

*by **restricting the range** of f is **continuous**. If Z is a space having Y as a **subspace,** then the function $h : X \to Z$ obtained by **expanding the range** of f is **continuous**.*

6. *(**Local Formulation of Continuity**) The map $f : X \to Y$ is **continuous** if X can be written as the **union of open sets** U_α such that $f|_{U_\alpha}$ is **continuous** for each α.*

- **Theorem 1.14** ***(The Pasting Lemma / Gluing Lemma)**. [Munkres, 2000]*
  *Let $X = A \cup B$, where A and B are **closed** in X. Let $f : A \to Y$ and $g : B \to Y$ be **continuous**. If $f(x) = g(x)$ for **every** $x \in A \cap B$, then f and g combine to give a **continuous function** $h : X \to Y$, defined by setting $h|_A = f$, and $h|_B = g$.*

- **Remark** The set A and B can be open sets, and the gluing lemma comes "***Local Formulation of Continuity***".

- **Remark** Notice the condition for *the gluing lemma*:
  1. The domain X is a union of two ***closed sets (or open sets)*** A and B
  2. The two functions f and g are ***continuous*** each of closed domain sets, respectively
  3. f and g ***agree on the intersection*** of two sets $A \cap B$.

- **Theorem 1.15** ***(Maps into Products)**. [Munkres, 2000]*
  Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

  *Then f is **continuous** if and only if the functions*

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

  *are **continuous**. The maps f_1 and f_2 are called **the coordinate functions** of f.*

1.6 Metric Topology

- **Definition** (***Metric Space***)
  A ***metric space*** is a set M and a real-valued function $d(\cdot,\cdot) : M \times M \to \mathbb{R}$ which satisfies:
  1. (***Non-Negativity***) $d(x, y) \geq 0$
  2. (***Definiteness***) $d(x, y) = 0$ if and only if $x = y$
  3. (***Symmetric***) $d(x, y) = d(y, x)$
  4. (***Triangle Inequality***) $d(x, z) \leq d(x, y) + d(y, z)$

  The function d is called a ***metric*** on M. The metric space M equipped with metric d is denoted as (M, d).

- **Definition** (ϵ-***Ball***)
  Given a metric d on X, the number $d(x, y)$ is often called *the **distance** between x and y in the metric d*. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called ***the ϵ-ball centered at** x*. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

- **Definition** (***Metric Topology***)
  If d is a *metric* on the set X, then *the collection of all ϵ-balls $B_d(x, \epsilon)$*, for $x \in X$ and $\epsilon > 0$, is a ***basis*** for a *topology* on X, called ***the metric topology*** *induced by d*.

- **Definition** (***Metrizability***)
  If X is a topological space, X is said to be ***metrizable*** if *there exists a metric d* on the set X that *induces the topology* of X. ***A metric space*** is *a metrizable space* X together with a specific metric d that *gives the topology of X*.

- **Remark** (***Metrizability as Inverse Problem***)
  Given a *metric* d on X, we can generate *a metric topology* using ϵ-balls as basis. ***Conversely***, ***given a topology** $\mathscr{T}$ on X, **is** $\mathscr{T}$ **a metric topology for some unknown metric** d **?***

  This is the question that ***the metrization theory*** is trying to answer.

- **Theorem 1.16** *(ϵ-δ **Definition of Continuous Function in Metric Space**). [Munkres, 2000]*
  *Lei $f : X \to Y$; let X and Y be **metrizable** with metrics d_x and d_y, respectively. Then **continuity** of f is **equivalent** to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) < \epsilon.$$

- **Remark** To use ϵ-δ definition, *both **domain** and **codomain*** need to be ***metrizable***.

- **Lemma 1.17** *(**The Sequence Lemma**). [Munkres, 2000]*
  *Let X be a topologicaJ space; let $A \subseteq X$. If there is a sequence of points of A **converging** to x, then $x \in \bar{A}$; the **converse** holds if X is **metrizable**.*

- **Proposition 1.18** *Let $f : X \to Y$. If the function f is **continuous**, then for every **convergent** sequence $x_n \to x$ in X, the sequence $f(x_n)$ **converges** to $f(x)$. The **converse** holds if X is **metrizable**.*

- **Remark** To show the converse part, i.e. "*if $x_n \to x \Rightarrow f(x_n) \to f(x)$ then f is continuous*", we just need the space X to be ***first countable***. That is, at each point x, there is ***a countable collection*** $(U_n)_{n \in \mathbb{Z}_+}$ of ***neighborhoods*** of x such that any neighborhood U of x *contains* at least one of the sets U_n.

- **Proposition 1.19** *(**Arithmetic Operations of Continuous Functions**).*
  If X is a topological space, and if $f, g : X \to Y$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

- **Definition** (***Uniform Convergence***)
  Let $f_n : X \to Y$ be a sequence of functions from the ***set*** X to ***the metric space*** Y. Let d be the metric for Y. We say that the sequence (f_n) ***converges uniformly*** to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

  for all $n > N$ and ***all** x **in** X*.

- **Theorem 1.20** ***(Uniform Limit Theorem)***. *[Munkres, 2000]*
  *Let $f_n : X \to Y$ be a sequence of **continuous** functions from the **topological** space X to the **metric space** Y. If (f_n) converges **uniformly** to f, then f is **continuous**.*

1.7 Connectedness and Compactness

Remark ***Connectedness*** and ***compactness*** are basic ***topological properties***. Both of them are defined based on a collection of open subsets.

1. ***Connectedness*** is a ***global topological property***: a topological space is *connected* if it cannot be partitioned by two *disjoint nonempty open* subsets. *Connectedness* reveals the information of ***entire space*** *not just within a neighborhood.* Connectedness is ***compatible*** with the ***continuity*** of functions as it implies ***the intermediate value theorem***, which in turn, can be used to construct *inverse function*. Moreover, *connectedness* defines ***an equivalence relationship*** which allows a ***partition*** of the space into ***components***.

2. ***Connectedness*** is a ***local-to-global topological property***: a topological space is *compact* if every open cover have a finite sub-cover. Using ***finite sub-cover***, ***local properties*** defined *within each neighborhood* can be ***generalized globally*** to entire space. Concept of functions that are closely related to compactness is ***the uniformly continuity*** and ***the maximum value theorem***. The compactness allows us to drop dependency on each individual point x.

Compared to *connectedness*, ***compactness*** is usually a ***strong condition*** on the topological space.

1.7.1 Connectedness and Local Connectedness

- ***Concepts Related to Connectedness***

$$\begin{tikzcd}
\textbf{\textit{path-connected}}\ \textit{space} \arrow[r] \arrow[d, "\text{only one}"] & \textbf{\textit{connected}}\ \textit{space} \arrow[d, "\text{only one}"] \\
\textbf{\textit{path components}} \arrow[r, "\subseteq"] & \textbf{\textit{components}} \arrow[l, dashed, bend left, "\text{if locally path connected}"] \\
\textbf{\textit{path components}}\ \textit{of every open subset} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \textbf{\textit{components}}\ \textit{of every open subset} \arrow[u, "\subseteq"'] \\
\textbf{\textit{locally path connected}}\ \textit{space} \arrow[u, "\text{open}"] & \textbf{\textit{locally connected}}\ \textit{space}. \arrow[u, "\text{open}"']
\end{tikzcd}$$

- **Definition** (***Separation*** *and* ***Connectedness***)
  Let X be a topological space. A ***separation*** of X is a pair U, V of ***disjoint nonempty open*** *subset*s of X whose union is X.

  The space X is said to be ***connected*** if there *does not exist a separation* of X.

- **Definition** Equivalently, X is ***connected*** if and only if the only subsets of X that are ***both open and closed*** are $\emptyset$ and X itself.

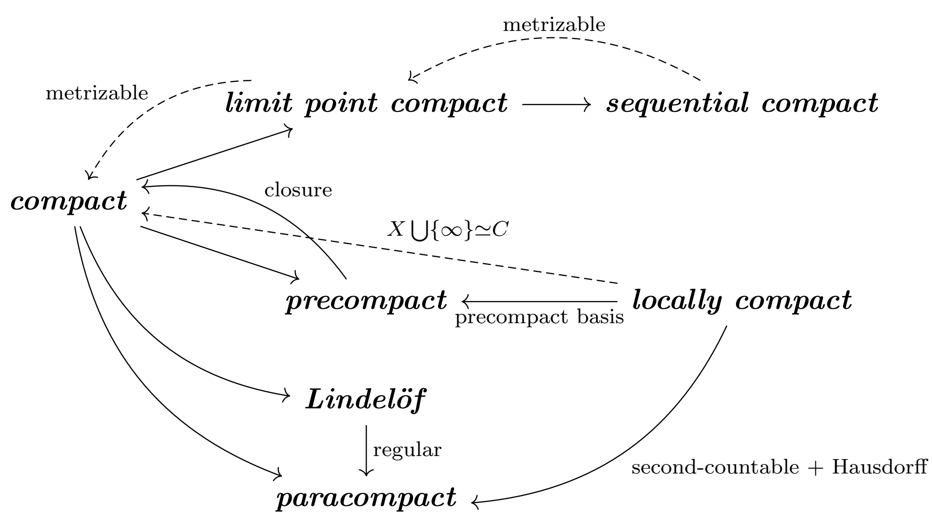
- **Remark** (***Proof of Connectedness***)
  As the definition suggests, the proof of connectedness is done ***by contradition***. One first assume that the set X has a ***seperation***; it can be separated into two ***disjoint nonempty open*** sets such that $X = A \cup B$. Then we proof by contradiction using ***existing connectedness conditions*** and the ***property of open subsets (basis, continuity etc.)***.

- **Definition** Recall that a topological space X is
  - ***connected*** if there do not exist two *disjoint*, *nonempty*, *open* subsets of X whose union is X;
  - ***path-connected*** if every pair of points in X can be ***joined by a path*** in X, and
  - ***locally path-connected*** if X has a ***basis*** of *path-connected open subsets*.

- **Definition** A ***maximal connected subset*** of X (i.e., a connected subset that is not properly contained in any larger connected subset) is called a ***component*** (or ***connected component***) of X.

1.7.2 Compactness and Local Compactness

- ***Concepts Related to Compactness***



- **Definition** (***Covering of Set*** *and* ***Open Covering of Topological Set***)
  A collection $\mathscr{A}$ of subsets of a space X is said to ***cover** X*, or to be *a* ***covering*** *of* X, if the union of the elements of $\mathscr{A}$ is equal to X.

  It is called an ***open covering of** X* if its elements are *open subsets* of X.

- **Definition** (***Compactness***)
  A topological space X is said to be ***compact*** if *every open covering $\mathscr{A}$* of X contains a ***finite*** *subcollection* that also *covers* X.

- To prove *compactness*, the following property is useful:

  Definition (***Finite Intersection Property***)
  A collection $\mathscr{C}$ of subsets of X is said to have ***the finite intersection property*** if for *every*

finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathscr{C}$, the ***intersection*** $C_1 \cap \dots \cap C_n$ is ***nonempty***.

- **Proposition 1.21** ***(Equivalent Definition of Compactness)*** *[Munkres, 2000]*
Let X be a topological space. Then X is ***compact if and only if*** *for every collection $\mathscr{C}$ of* ***closed*** *sets in X having* ***the finite intersection property****, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is* ***nonempty****.*

- **Definition** If X and Y are topological spaces, a map $F : X \to Y$ (continuous or not) is said to be ***proper*** if for every ***compact*** set $K \subseteq Y$, the ***preimage*** $F^{-1}(K)$ is ***compact***.

- **Definition** A topological space X is said to be ***locally compact*** if every point has a ***neighborhood*** contained in a ***compact subset*** of X.

  A subset of X is said to be ***precompact*** in X if its ***closure*** in X is *compact*.

- If X is *not a compact Hausdorff space*, then *under what conditions* is X *homeomorphic* with *a* ***subspace*** *of a compact Hausdorff space* ?

  Theorem 1.22 ***(Unique One-Point Compactification)*** *[Munkres, 2000]*
  Let X be a space. Then X is ***locally compact Hausdorff*** *if and only if there exists a space Y satisfying the following conditions:*

  1. *X is a subspace of Y.*
  2. *The set $Y \setminus X$ consists of* ***a single point*** *(which is the limit point of X).*
  3. *Y is a* ***compact Hausdorff*** *space.*

  If Y and Y' are two spaces satisfying these conditions, then there is a ***homeomorphism*** *of Y with Y' that equals* ***the identity map*** *on X.*

- **Definition** (***One-Point Compactification***)
If Y is a ***compact Hausdorff*** space and X is a proper *subspace* of Y whose ***closure*** equals Y, then Y is said to be a ***compactification*** of X.

  If $Y \setminus X$ equals *a single point*, then Y is called ***the one-point compactification*** of X.

- **Proposition 1.23** ***(Locally Compact Hausdorff = Precompact Basis)*** *[Munkres, 2000]*
Let X be a ***Hausdorff*** *space. Then X is* ***locally compact if and only if*** *given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is* ***compact*** *and $\bar{V} \subseteq U$.*

- **Corollary 1.24** ***(Closed or Open Subspace)*** *[Munkres, 2000]*
Let X be locally compact Hausdorff; let A be a subspace of X. If A is ***closed*** *in X or* ***open*** *in X, then A is locally compact.*

- **Corollary 1.25** *[Munkres, 2000]*
A space X is ***homeomorphic*** *to an* ***open*** *subspace of a* ***compact Hausdorff*** *space* ***if and only if*** *X is* ***locally compact Hausdorff****.*

- For a ***Hausdorff space*** X, the following are equivalent:

  1. X is ***locally compact***.

2. Each point of X has a ***precompact*** neighborhood.

3. X has a basis of ***precompact*** open subsets.

- **Theorem 1.26** ***(Tychonoff Theorem)***. *[Munkres, 2000]*
An ***arbitrary product*** *of* ***compact*** *spaces is* ***compact*** *in the* ***product topology***.

1.8 Countability and Separability

1.8.1 Countability Axioms

- ***Concepts Related to Countablity Axioms***

$$
\begin{tikzcd}
 & \textbf{\textit{first-countable}} \text{ \textit{space}} & \textbf{\textit{metrizable}} \text{ \textit{space}} \arrow[l] \\
\textbf{\textit{second-countable}} \text{ \textit{space}} \arrow[ur, bend left] \arrow[r] \arrow[dr] & \textbf{\textit{separable}} \text{ \textit{space}} \arrow[l, dashed, bend right, "metrizable"'] & \\
 & \textbf{\textit{Lindelöf}} \text{ \textit{space}} \arrow[ul, dashed, bend left, "metrizable"] &
\end{tikzcd}
.
$$

- **Definition** (***Countability***)
A topological space X is said to be

1. ***first-countable*** if there is a ***countable neighborhood basis*** at each point,

2. ***second-countable*** if there is ***a countable basis*** for its topology.

- **Proposition 1.27** ***(Limit Point Detected by Convergent Sequence)*** *[Munkres, 2000]*
Let X be a topological space.

1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the* ***converse*** *holds if X is* ***first-countable***.

2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The* ***converse*** *holds if X is* ***first-countable***.

- **Definition** (***Dense Subset***)
A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$. (That is, *every point in X is a limit point of A.*)

- **Definition** (***Separability***)
A topological space X is called ***separable*** if and only if it has a ***countable dense set***.

- **Definition** (***Lindelöf Space***)
A space for which *every open covering* contains *a* ***countable*** *subcovering* is called a ***Lindelöf space***.

- **Proposition 1.28** ***(Properties of Second-Countability)*** *[Munkres, 2000]*
Suppose that X has a ***countable basis***. *Then:*

1. *Every* ***open covering*** *of X contains a* ***countable*** *subcollection covering X. (X is* ***Lindelöf space****)*

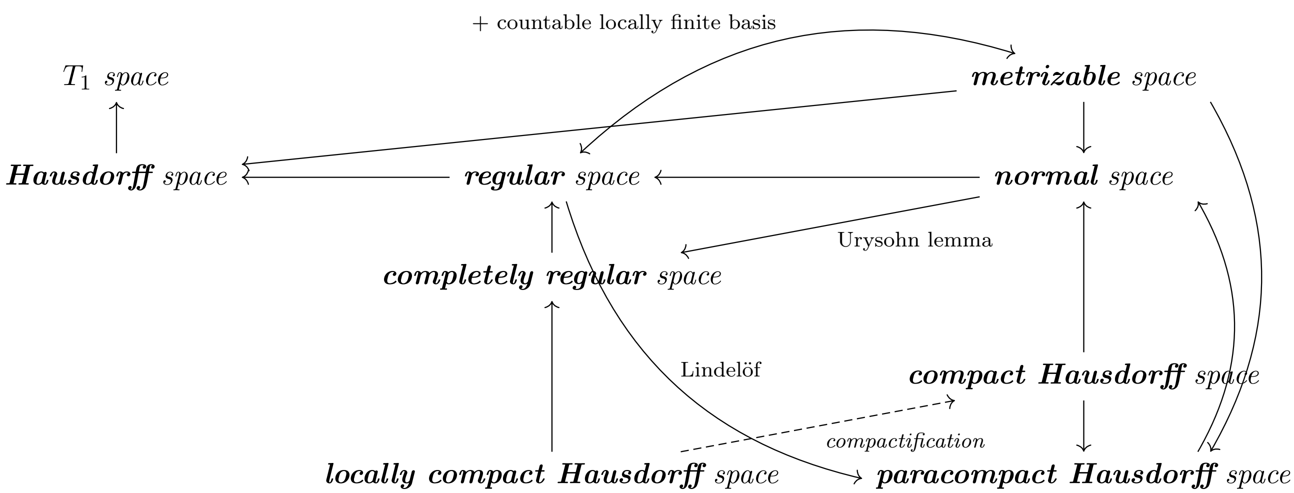
2. *There exists a **countable** subset of X that is **dense** in X. (X is **separable**)*

- **Proposition 1.29** ***(Metric Space Countablility and Separablility)***

  1. *Every **metric space** is **first countable**.*
  2. *A metric space is **second countable** if and only if it is **separable**.*
  3. *Any **second countable** topological space is **separable**.*

1.8.2 Separability Axioms

- ***Concepts Related to Separation Axioms***



- **Definition** (***Separation Axioms***)

  1. A topological space is called a T_1 ***space*** if and only if for all x and y, $x \neq y$, there is an ***open set*** U with $y \in U$, $x \notin U$.

     Equivalently, a space is T_1 *if and only if* $\{x\}$ is ***closed*** for each x.

  2. A topological space is called ***Hausdorff*** (or T_2) if and only if for all all x and y, $x \neq y$, there are ***open sets*** U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

  3. A topological space is called ***regular*** (or T_3) if and only if it is T_1 and for all x and C, ***closed***, with $x \notin C$, there are ***open sets*** U, V such that $x \in U$, $C \subset V$, and $U \cap V = \emptyset$.

     Equivalently, a space is T_3 if *the **closed neighborhoods** of any point are a **neighborhood base**.*

  4. A topological space is called ***normal*** (or T_4) if and only if it is T_1 and for all C_1, C_2, ***closed***, with $C_1 \cap C_2 = \emptyset$, there are ***open sets*** U, V with $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \emptyset$.

- **Proposition 1.30**

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$$

- **Proposition 1.31** *(**Limit Point in T_1 Axiom**). [Munkres, 2000]*
  *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is **a limit point** of A if and only if every **neighborhood** of x contains **infinitely many points** of A.*

- **Proposition 1.32** *(**Limit Point is Unique in Hausdorff Space**). [Munkres, 2000]*
  *If X is a **Hausdorff space**, then a sequence of points of X **converges to at most one point** of X.*

- **Lemma 1.33** *Let X be a topological space. Let one-point sets in X be closed.*

  1. *X is **regular** if and only if given a point x of X and a neighborhood U of x, there is a **neighborhood** V of x such that $\bar{V} \subseteq U$.*

  2. *X is **normal** if and only if given a **closed** set A and an open set U containing A, there is an **open set** V containing A such that $\bar{V} \subseteq U$.*

- **Proposition 1.34** *[Munkres, 2000]*
  *Every **locally compact Hausdorff** space is **regular**.*

1.9 Important Results and Theorems on Normal Space

- **Theorem 1.35** *(**Regular** + **Second-Countable** $\Rightarrow$ **Normal**)[Munkres, 2000]*
  *Every **regular** space with a **countable basis** is **normal**.*

- **Proposition 1.36** *(**Regular** + **Lindelöf** $\Rightarrow$ **Normal**)[Munkres, 2000]*
  *Every **regular Lindelöf** space is **normal**.*

- **Theorem 1.37** *[Munkres, 2000]*
  *Every **metrizable** space is **normal**.*

- **Theorem 1.38** *[Munkres, 2000, Reed and Simon, 1980]*
  *Every **compact Hausdorff** space X is **normal**.*

- **Theorem 1.39** *[Munkres, 2000]*
  *Every **well-ordered** set X is **normal** in the **order topology**.*

- **Theorem 1.40** *(**Urysohn Lemma**). [Munkres, 2000]*
  *Let X be a **normal** space; let A and B be **disjoint closed subsets** of X. Let $[a, b]$ be a **closed interval** in the real line. Then there exists a **continuous** map*

  $$f : X \to [a, b]$$

  *such that $f(x) = a$ for **every** x in A, and $f(x) = b$ for **every** x in B.*

- **Definition** (***Separation by Continuous Function***)
  If A and B are two subsets of the topological space X, and if there is a **continuous** function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be ***separated by a continuous function***.

- **Definition** (***Completely Regular***)
  A space X is ***completely regular*** if *one-point sets* are *closed* in X and if for each point x_0 and each ***closed*** set A not containing x_0, there is a ***continuous function*** $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

- **Remark**

$$\text{normal} \;\Rightarrow\; \text{completely regular} \;\Rightarrow \text{regular}$$

- **Theorem 1.41** *(**Urysohn Lemma, Locally Compact Version**). [Folland, 2013]*
*Let X be a **locally compact Hausdorff** space and $K \subseteq U \subseteq X$ where K is **compact** and U is **open**. Then there exists a **continuous** map*

$$f : X \to [0,1]$$

*such that $f(x) = 1$ for **every** $x \in K$, and $f(x) = 0$ for x outside a **compact subset** of U.*

- **Corollary 1.42** *[Folland, 2013]*
*Every **locally compact Hausdorff** space is **completely regular**.*

- **Proposition 1.43** *[Reed and Simon, 1980]*
*Let $\mathcal{C}(X)$ be the set of all complex-valued **continuous functions** on X and $\mathcal{C}_{\mathbb{R}}(X) \subseteq \mathcal{C}(X)$ be the set of all **real-valued continuous functions** on X. Also define $\mathcal{C}^b(X)$ as the set of all complex-valued **bounded continuous functions** on X. When X is **a compact space**, $\mathcal{C}^b(X) = \mathcal{C}(X)$. Define the norm as*

$$\|f\|_{\infty} = \sup_{x \in X} |f(x)| \,.$$

*Then for **compact Hausdorff space** X, $\mathcal{C}(X)$ is a (complex) **Banach space** and $\mathcal{C}(X)$ is **a (real) Banach space**.*

- **Theorem 1.44** *(**Embedding Theorem**). [Munkres, 2000]*
*Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of **continuous** functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is **positive** at x_0 and **vanishes outside** U. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

*is a **topological embedding** of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$ for each α then F **embeds** X in $[0,1]^J$.*

- **Definition** (***Separation** of **Points** From **Closed** Set by **Continuous** Functions*)
A family of continuous functions that satisfies the hypotheses of *the embedding theorem above* is said to ***separate points from closed sets in*** X.

  The existence of such a family is readily seen to be *equivalent*, for a space X in which one-point sets are *closed*, to the requirement that X be *completely regular*.

- **Corollary 1.45** *(**Embedding Equivalent Definition** of **Completely Regular**) [Munkres, 2000]*
*A space X is **completely regular if and only if** it is **homeomorphic** to a subspace of $[0,1]^J$ for some J.*

- **Theorem 1.46** *(**Tietze Extension Theorem**) [Munkres, 2000, Reed and Simon, 1980]*
*Let X be a **normal space**; let A be a **closed subspace** of X.*

1. *Any **continuous** map of A into the **closed interval** $[a,b]$ of $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $[a,b]$.*

2. *Any **continuous** map of A into $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $\mathbb{R}$.*

- **Theorem 1.47** ***(Tietze Extension Theorem, Locally Compact Version)*** *[Folland, 2013]*
*Let X be a **locally compact Hausdorff space**; let K be a **compact subspace** of X. If $f \in \mathcal{C}(K)$ is a **continuous** map of K into $\mathbb{R}$, there exists a **continuous** extension $F \in \mathcal{C}(X)$ of **all of** X into $\mathbb{R}$ such that $F|_K = f$. Moreover, F may be taken to **vanish outside a compact set**.*

1.10 Metrization

- **Theorem 1.48** ***(The Urysohn Metrization Theorem)***. *[Munkres, 1975, Folland, 2013]*
*Every **second countable normal** space is **metrizable**.*

1.11 Nets and Convergence in Topological Space

- **Definition** (***Directed System of Index Set***)
A ***directed system*** is *an index set I* together with an ***ordering*** $\prec$ which satisfies:

  1. If $\alpha, \beta \in l$ then there exists $\gamma \in I$ so that $\gamma \succ \alpha$ and $\gamma \succ \beta$.

  2. $\prec$ is a ***partial ordering***.

- **Definition** (***Net***)
A ***net*** in a topological space X is a mapping from a *directed system* I to X; we denote it by $\{x_\alpha\}_{\alpha \in I}$

- **Remark** (*Net vs. Sequence*)
Net is a generalization and abstraction of ***sequence***. The directed system I is ***not necessarily countable***. So $\{x_\alpha\}_{\alpha \in I}$ may not be a countable sequence. *A sequence is a net with countable index set $I \subseteq \mathbb{N}$.* The directed system can be any set e.g. a graph.

- **Definition** If $P(\alpha)$ is a ***proposition*** depending on an ***index*** α in a *directed set* I we say ***$P(\alpha)$ is eventually true*** if there is a β in I with $P(\alpha)$ *true* if *for all* $\alpha \succ \beta$.

  We say ***$P(\alpha)$ is frequently true*** if it is ***not eventually false***, that is, if for any β *there exists* an $\alpha \succ \beta$ with $P(\alpha)$ *true.*

- **Definition** (***Convergence***)
A ***net*** $\{x_\alpha\}_{\alpha \in I}$ in a topological space X is said to ***converge*** to a point $x \in X$ (written $x_\alpha \to x$) if for ***any neighborhood*** N of x, ***there exists*** a $\beta \in l$ so that $x_\alpha \in N$ if $\alpha \succ \beta$. The point x that being converged to is called ***the limit point*** of x_α.

  Note that if $x_\alpha \to x$, then x_α is ***eventually in all neighborhoods of*** x. If x_α is ***frequently in any neighborhood of*** x, we say that x is a ***cluster point*** of x_α.

- **Remark** This definition *generalizes* the ϵ-δ language for convergence in metric space. Notice that the notions of *limit* and *cluster point* generalize the same notions for sequences in a metric space..

- **Proposition 1.49** *[Reed and Simon, 1980]*
*Let A be a set in a topological space X. Then, a point x is in the **closure** of A if and only if there is a net $\{x_\alpha\}_{\alpha\in I}$ with $x_\alpha \in A$, So that $x_\alpha \to x$.*

- **Proposition 1.50** *[Reed and Simon, 1980]*

  1. *(**Continuous Function**): A function f from a topological space X to a topological space Y is **continuous** if and only if for **every convergent net** $\{x_\alpha\}_{\alpha\in I}$ **in** X, with $x_\alpha \to x$, the net $\{f(x_\alpha)\}_{\alpha\in I}$ **converges in** Y to $f(x)$.*

  2. *(**Uniqueness of Limit Point for Hausdorff Space**): Let X be a **Hausdorff space**. Then a net $\{x_\alpha\}_{\alpha\in I}$ in X can have **at most one limit**; that is, if $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.*

- **Definition** A net $\{x_\alpha\}_{\alpha\in I}$ is a ***subnet*** of a net $\{y_\beta\}_{\beta\in J}$ if and only if there is a function $F : I \to J$ such that

  1. $x_\alpha = y_{F(\alpha)}$ for each $\alpha \in I$.

  2. For all $\beta' \in J$, there is an $\alpha' \in I$ such that $\alpha \succ \alpha'$ implies $F(\alpha) \succ \beta'$ (that is, $F(\alpha)$ is ***eventually larger*** *than any fixed* $\beta \in J$).

- **Proposition 1.51** *A point x in a topological space X is a **cluster point** of a **net** $\{x_\alpha\}_{\alpha\in I}$ if and only if **some subnet** of $\{x_\alpha\}_{\alpha\in I}$ **converges** to x.*

- **Theorem 1.52** ***(The Bolzano-Weierstrass Theorem)*** *[Reed and Simon, 1980]*
*A space X is **compact** if and only if **every net** in X **has a convergent subnet**.*

2 Special Space

- **Remark** (***Metric Space*** *and* ***Compact Hausdorff Space***)
Two of the most well-behaved classes of spaces to deal with in mathematics are ***the metrizable spaces*** and ***the compact Hausdorff spaces***.

  1. ***Metrizable space*** (X, d):

     – ***subspace*** of *metrizable* space is *meterizable*;

     – ***compact subspace*** of *metric* space is ***bounded*** in that metric and is ***closed***;

     – *every metrizable space* is ***normal*** (T_4);

     – ***compactness*** = ***sequential compactness*** = ***limit point compactness***;

     – ***sequence lemma***: for $A \subset X$, $x \in \bar{A}$ if and only if there exists a squence of points in A that converges to x. ($\Rightarrow$ need X being metric space);

     – f is ***continuous*** at x if and only if $x_n \to x$ leads to $f(x_n) \to f(x)$ ($\Leftarrow$ part holds for metric space)

     – ***unform limit theorem***: If the *range* of f_n is a *metric space* and f_n are *continuous*, then $f_n \to f$ *uniformly* means that f is a *continuous* function.

– ***unform continuity theorem***: if f is a *countinous* map between two *metric spaces*, and the domain is ***compact***, then f is ***uniformly continuous***.

– every metric space is ***first-countable***.

2. ***Compact Hausdorff Space***:

– ***subspace*** of *compact Hausdorff space* is *compact Hausdorff* if and only if it is ***closed***.

– ***closed subspace*** of *compact* space is ***compact***;

– ***compact subspace*** of *Hausdorff* space is ***closed***;

– *compact Hausdorff space* X is ***normal*** (T_4), thus it is ***completely regular***;

– ***arbitrary product*** of *compact (Hausdorff)* space is *compact (Hausdorff)*;

– ***compactness*** $\Rightarrow$ ***sequential compactness***;

– ***compactness*** $=$ ***net compactness***, i.e. every *net* has a convergence *subnet*;

– ***image*** of *compact* space under continuous map f is *compact*;

– ***continuous bijection*** between two *compact Hausdorff* spaces is a ***homemorphism*** (and is a ***closed map***);

– ***closed graph theorem***: f is ***continuous*** if and only if its ***graph*** is ***closed***;

– ***uncountability***: for *compact Hausdorff space*, if the space has *no isolated points*, then it is *uncountable*;

– if compact Hausdorff space is ***second-countable***, then it is ***metrizable***.

3 Summary of Preservation of Topological Properties

Table 1: Summary of Preservation of Topological Properties Under Transformations

	subspace	***product space***	***image of continuous function***
connected	✓	✓ under ***product topology***	✓
locally connected	if ***open and connected*** subspace, ✓	if ***all but finitely many*** *of spaces are* ***connected***, ✓	in general ×
compact	if ***closed*** subspace, ✓;	for ***arbitrary*** product, ✓	✓
locally compact	if ***closed*** *or* ***open*** subspace and Hausdorff, ✓	if ***finite*** product, ✓; if ***infinite*** product ×	if f is a ***perfect map***, then ✓; in general ×
first-countable	✓	if ***countable*** product, ✓	if f is a ***open map***, then ✓; in general ×
second-countable	✓	if ***countable*** product, ✓	if f is a ***open map or perfect map***, then ✓; in general ×
separable	if metrizable, then ✓; in general ×	if ***countable*** product, ✓	✓
Lindelöf	if metrizable, then ✓; in general ×	×	✓
T_1 ***axiom***	✓	for ***arbitrary*** product, ✓	in general ×
Hausdorff T_2	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓; in general ×
regular T_3	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓ ; in general ×
completely regular	✓	for ***arbitrary*** product, ✓	in general ×
normal T_4	×	×	×
paracompact	if ***closed*** subspace, ✓;	×	×
topologically complete	for ***closed*** *or* ***open*** subspace, ✓	if ***countable*** product, ✓	×

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