Lecture 14: Differential Forms

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Oct. 30th., 2022

Contents

1 The Algebra of Alternating Tensors

1.1 The Exterior Form

- **Remark** Let V be a finite-dimensional (real) vector space. Recall that a covariant k-tensor on V is said to be ***alternating*** if its value changes sign whenever two arguments are interchanged, or equivalently if any permutation of the arguments causes its value to be multiplied by the *sign* of the permutation.

- **Definition** *Alternating covariant k-tensors* are also called ***exterior forms***, ***multicovectors***, or ***k-covectors***. The vector space of all k-covectors on V is denoted by $\Lambda^k(V^*)$.

- **Lemma 1.1** *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

  1. *α is **alternating**.*
  2. *$\alpha(v_1, \ldots, v_k) = 0$ whenever the k-tuple $(v_1, \ldots, v_k)$ is **linearly dependent**.*
  3. *α gives the value zero whenever **two of its arguments** are **equal**:*

  $$\alpha(v_1, \ldots, w, \ldots, w, v_k) = 0.$$

- **Definition** We define a projection Alt $: T^k(V^*) \to \Lambda^k(V^*)$, called ***alternation***, as follows:

  $$\text{Alt } \alpha = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\{\sigma\}) ({}^{\sigma}\alpha)$$

  where S_k is *the symmetric group on k elements*. More explicitly, this means

  $$\text{Alt } \alpha(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\{\sigma\})\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

- **Example** If α is any 1-tensor, then Alt $\alpha = \alpha$. If β is a 2-tensor, then

  $$\text{Alt } \beta(v, w) = \frac{1}{2} (\beta(v, w) - \beta(w, v)).$$

  For a 3-tensor γ,

  $$\text{Alt } \gamma(v, w, x) = \frac{1}{6} (\gamma(v, w, x) + \gamma(w, x, v) + \gamma(x, v, w) - \gamma(w, v, x) - \gamma(v, x, w) - \gamma(x, w, v)).$$

- **Proposition 1.2** ***(Properties of Alternation)**.*
  Let α be a covariant tensor on a finite-dimensional vector space.

  1. *Alt α is alternating.*
  2. *Alt $\alpha = \alpha$ if and only if α is alternating.*

1.2 Elementary Alternating Tensors

- **Definition** Given a positive integer k, ***an ordered k-tuple*** $I = (i_1, \ldots, i_k)$ of *positive integers* is called *a* ***multi-index*** *of length* k. If I is such a multi-index and $\sigma \in S_k$ is a permutation of $\{1, \ldots, k\}$, we write I for the following multi-index:

  $$I_\sigma = \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right).$$

  Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$.

- **Definition** Let V be an n-dimensional vector space, and suppose $(\epsilon^1, \ldots, \epsilon^n)$ is any *basis* for V^*. We now define a collection of *k-covectors* on V that generalize ***the determinant function*** on $\mathbb{R}^n$.

  For each multi-index $I = (i_1, \ldots, i_k)$ of length k such that $1 \le i_1 \le \ldots \le i_k \le n$, define ***a covariant k-tensor*** $\epsilon^I = \epsilon^{i_1, \ldots, i_k}$ by

  $$\epsilon^I(v_1, \ldots, v_k) = \det \begin{bmatrix} \epsilon^{i_1}(v_1) & \ldots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \ldots & \epsilon^{i_k}(v_k) \end{bmatrix} = \det \begin{bmatrix} v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \ldots & v_k^{i_k} \end{bmatrix}. \tag{1}$$

  In other words, if $\boldsymbol{V}$ denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \ldots, v_k$ with respect to the basis (E_i) dual to (ϵ^i), then $\epsilon^I(v_1, \ldots, v_k)$ is the ***determinant of the $k \times k$ submatrix*** consisting of rows $i_1, \ldots, i_k$ of $\boldsymbol{V}$. Because the determinant changes sign whenever two columns are interchanged, it is clear that ϵ^I is *an alternating k-tensor*. We call ϵ^I ***an elementary alternating tensor*** or ***elementary k-covector***.

- **Definition** If I and J are multiindices of length k, we define the Kronecker delta function:

  $$\delta^I_J = \det \begin{bmatrix} v_{j_1}^{i_1} & \ldots & v_{j_k}^{i_1} \\ \vdots & \ddots & \vdots \\ v_{j_1}^{i_k} & \ldots & v_{j_k}^{i_k} \end{bmatrix}$$

  (I represent the row number, J represent the column number.)

- **Remark** The following is the property of Kronecker delta

  $$\delta^I_J = \begin{cases} \text{sign}\{\sigma\} & \text{if neither } I \text{ nor } J \text{ has a repeated index, } J = I_\sigma,\ \sigma \in S_k \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I \end{cases}$$

- **Lemma 1.3** ***(Properties of Elementary k-Covectors).***
  Let (E_i) be a basis for V, let (ϵ^i) be the dual basis for V^*, and let ϵ^I be as defined above.

  1. *If I has a repeated index, then $\epsilon^I = 0$.*
  2. *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\epsilon^I = \text{sign}\{\sigma\}\, \epsilon^J$.*
  3. *The result of evaluating ϵ^I on a sequence of basis vectors is*

     $$\epsilon^I\left(E_{j_1}, \ldots, E_{j_k}\right) = \delta^I_J.$$

- **Definition** A multi-index $I = (i_1, \ldots, i_k)$ is said to be ***increasing*** if $i_1 < \ldots < i_k$. It is useful to use a primed summation sign to denote a sum over *only increasing multi-indices*

$$\sum_I{}' a_I \epsilon^I = \sum_{\{I : i_1 < \ldots < i_k\}} a_I \epsilon^I.$$

- **Proposition 1.4** *(**A Basis for** $\Lambda^k(V^*)$)*
Let V be an n-dimensional vector space. If (ϵ^i) is any basis for V^*, then for each positive integer $k \leq n$, the collection of k-covectors

$$\underline{\mathcal{E} = \left\{\epsilon^I : I \textit{ is an increasing multi-index of length } k\right\}}$$

*is **a basis for** $\Lambda^k(V^*)$. Therefore,*

$$dim\ \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If $k > n$, then $dim\ \Lambda^k(V^*) = 0$.

- **Remark** In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V^*)$ is 1-***dimensional*** and is spanned by $\epsilon^{1,\ldots,n}$.

- **Proposition 1.5** *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T : V \to V$ is any **linear map** and $v_1, \ldots, v_n$ are arbitrary vectors in V, then*

$$\omega\left(Tv_1, \ldots, Tv_n\right) = (\det T)\, \omega\left(v_1, \ldots, v_n\right). \tag{2}$$

1.3 The Wedge Product

- **Definition** Let V be a finite-dimensional real vector space. Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their ***wedge product*** or ***exterior product*** to be the following $(k+l)$-*covector*:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \mathrm{Alt}\left(\omega \otimes \eta\right) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \left(\mathrm{sign}\left\{\sigma\right\}\right) \left({}^{\sigma}\left(\omega \otimes \eta\right)\right) \tag{3}$$

- The coefficients come from the following lemma:

  Lemma 1.6 *Let V be an n-dimensional vector space and let $(\epsilon^1, \ldots, \epsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$,*

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ} \tag{4}$$

  *where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is obtained by **concatenating** I and J.*

- **Proposition 1.7** *(**Properties of the Wedge Product**).*
*Suppose $\omega, \omega', \eta, \eta'$ and ξ are **multicovectors** on a finite-dimensional vector space V.*

  1. *(**Bilinearity**): For $a, a; \in \mathbb{R}$,*

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

2. ***(Associativity):***

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$$

3. ***(Anticommutativity):*** *For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega \tag{5}$$

4. *If (ϵ^i) is any basis for V^* and $I = (i_1, \ldots, i_k)$ is any multi-index, then*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^I \tag{6}$$

5. *For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$,*

$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \ldots, v_k) = \det\big(\omega^j(v_i)\big) \tag{7}$$

- **Remark** Because of part (4) of this lemma, henceforth we generally use the notations ϵ^I and $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ ***interchangeably***

- **Definition** A k-covector η is said to be ***decomposable*** if it can be expressed in the form $\eta = \omega^1 \wedge \ldots \wedge \omega^k$, where $\omega^1, \ldots, \omega^k$ are *covectors*.

- **Remark** It is important to be aware that not every k-covector is decomposable when $k > 1$; however, it follows from Proposition 1.4 and above Lemma part (4) that ***every k-covector*** *can be written as a* ***linear combination*** *of* ***decomposable ones***.

- **Definition** For any n-dimensional vector space V, define **a vector space** $\Lambda(V^*)$ by

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

It follows from Proposition 1.4 that $\dim \Lambda(V^*) = 2^n$. The wedge product turns $\Lambda(V^*)$ into an ***associative algebra***, called ***the exterior algebra*** (or ***Grassmann algebra***) of V.

An algebra A is said to be ***graded*** if it has a direct sum decomposition $A = \bigoplus_{k \in \mathbb{Z}} A^k$ such that the product satisfies $(A^k)(A^l) \subseteq A^{k+l}$ for each k and l. A graded algebra is ***anticommutative*** if the product satisfies $ab = (-1)^{kl} ba$ for $a \in A^k$, $b \in A^l$. So $\Lambda(V^*)$ ***is an anticommutative graded algebra***.

- **Remark** There are two ***conventions*** to define the wedge product:

  1. ***The determinant convention***:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\omega \otimes \eta)$$

  In this way, we have

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^I,$$
$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \ldots, v_k) = \det\big(\omega^j(v_i)\big)$$

2. ***The Alt convention***

$$\omega \,\overline{\wedge}\, \eta = \mathrm{Alt}\,(\omega \otimes \eta)$$

In this way, we have to multiply the coefficient in front of basis and deterimant

$$\begin{aligned}\epsilon^{I} \,\overline{\wedge}\, \epsilon^{J} &= \frac{k!\, l!}{(k+l)!}\epsilon^{IJ},\\ (\omega^{1} \,\overline{\wedge}\, \ldots \,\overline{\wedge}\, \omega^{k})(v_1, \ldots, v_k) &= \frac{1}{k!}\det\left(\omega^{j}(v_i)\right)\end{aligned}$$

- **Remark** For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$, ***the exterior product*** is considered as the ***determinant function*** of a $k \times k$ submatrix

$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \ldots, v_k) = \det \begin{bmatrix} \omega^1(v_1) & \ldots & \omega^1(v_k) \\ \vdots & \ddots & \vdots \\ \omega^k(v_1) & \ldots & \omega^k(v_k) \end{bmatrix}$$

where ***vectors*** $v_1, \ldots, v_k$ *forms* ***column vector***, and ***covectors*** $\omega^1, \ldots, \omega^k$ *form the* ***row vector***.

In other words, we can think of ***exterior product of covectors*** as ***an abstraction of determinant operation***.

1.4 Interior Multiplication

- **Definition** Let V be a finite-dimensional vector space. For each $v \in V$, we define a *linear map* $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$, called ***interior multiplication (interior product)*** by v, as follows:

$$(\iota_v \omega)(w_1, \ldots, w_{k-1}) = \omega\,(v, w_1, \ldots, w_{k-1})\,.$$

In other words, $(\iota_v \omega)$ is obtained from ω by ***inserting*** v ***into the first slot***. By convention, we interpret $(\iota_v \omega)$ to be ***zero*** when ω is a 0-***covector*** (i.e., a ***number***). Another common notation is

$$v \lrcorner\, \omega = (\iota_v \omega).$$

This is often read "v ***into*** ω."

- **Proposition 1.8** *Let V be a finite-dimensional vector space and $v \in V$.*

  1. $\iota_v \circ \iota_v = 0$.
  2. *If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\iota_v(\omega \wedge \eta) = \iota_v(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_v(\eta) \tag{8}$$

- **Remark** It is easy to verify the following form

$$\iota_v\left(\omega^1\wedge\ldots\wedge\omega^k\right) = v\,\lrcorner\left(\omega^1\wedge\ldots\wedge\omega^k\right) = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(v)\;\left(\omega^1\wedge\ldots\wedge\widehat{\omega}^i\wedge\ldots\wedge\omega^k\right) \tag{9}$$

$$\Leftrightarrow\left(\omega^1\wedge\ldots\wedge\omega^k\right)(v,v_2,\ldots,v_k) = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(v)\;\left(\omega^1\wedge\ldots\wedge\widehat{\omega}^i\wedge\ldots\wedge\omega^k\right)(v_2,\ldots,v_k)$$

where the hat indicates that ω^i is ***omitted***. In *determiant form*, it can be written as

$$\det\boldsymbol{V} = \sum_{i=1}^{k}(-1)^{i-1}\omega^i(v)\;\det\boldsymbol{V}_1^i \tag{10}$$

where $\boldsymbol{V}_j^i$ denote the $(k-1)\times(k-1)$ submatrix of $\boldsymbol{V}$ obtained by ***deleting*** the i-th row and j-th column. This is just ***the expansion of*** $\det\boldsymbol{V}$ ***by minors*** *along the first column*, and therefore is equal to $\det\boldsymbol{v}$.

- **Remark** The *exterior product* ***increase*** the *rank* of tensor, while the *interior product* ***decrease*** the *rank* of tensor by 1.

2 Differential Forms on Manifolds

- **Definition** Let T^kT^*M be the *bundle* of all covariant k-tensors on M. The subset of T^kT^*M consisting of ***alternating tensors*** is denoted by $\Lambda^k(T^*M)$:

$$\Lambda^k(T^*M) = \bigsqcup_{p\in M}\Lambda^k(T_p^*M).$$

$\Lambda^k(T^*M)$ is a *smooth subbundle* of T^kT^*M, so it is a *smooth vector bundle of rank* $\binom{n}{k}$.

- **Remark** $\Lambda^k(T^*M)$ is ***the bundle of all alternating covariant k-tensors (exterior forms, k-covectors)*** on M.

- **Definition** A ***section*** of $\Lambda^k(T^*M)$ is called ***a differential k-form***, or just a ***k-form***; this is a *(continuous) tensor field* whose value at each point is an *alternating tensor*. The integer k is called *the* ***degree*** *of the form*. We denote the *vector space* of ***smooth k-forms*** by

$$\Omega^k(M) = \Gamma\left(\Lambda^k(T^*M)\right).$$

- **Remark** *A k-form is just an alternating covariant k-tensor fields.*

- **Remark** The ***wedge product*** of *two differential forms* is defined ***pointwise***: $(\omega\wedge\eta)_p = \omega_p\wedge\eta_p$. Thus, ***the wedge product of a k-form with an l-form is a $(k+l)$-form***. If f is a 0-form (i.e. a smooth function) and ω is a k-form, we interpret the wedge product $f\wedge\omega$ to mean the ordinary product $f\omega$.

- **Remark** The direct sum of all vector spaces of smooth k-forms for $k\leq n$ is

$$\Omega^*(M) = \bigoplus_{k=0}^{n}\Omega^k(M). \tag{11}$$

Then the wedge product turns $\Omega^*(M)$ into *an* ***associative, anticommutative graded algebra***.

- **Remark** (***Duality of Basis***)
  The basis of differential k-forms $(dx^{i_1} \wedge \ldots \wedge dx^{i_k})$ in $\Gamma\left(\Lambda^k(T^*M)\right)$ acts on the local coordinate frames $(\partial/\partial x^i)$ in TM

$$\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J$$

- **Remark** (***Coordinate Representation of k-Forms***)
  In any smooth chart, a k-form ω can be written locally as

$$\omega = \sum_I{}' \omega_I dx^I := \sum_I{}' \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

  where the coefficients ω^I are ***continuous functions*** defined on the coordinate domain, and we use dx^I as an abbreviation for $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (not to be mistaken for the differential of a real-valued function x^I). Also $\sum_I' \epsilon^I$ means that sum with increasing multi-indices. ***The component function*** ω_I is computed as

$$\omega_I = \omega\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right).$$

  Note that ω_I is ***the determinant of a $k \times k$ principal sub-matrix (i.e. principal minors)*** whose *rows and columns* are indexed by increasing multi-index I.

- **Example** The followings are some basic differential k-forms:
  1. Any smooth function $f \in \mathcal{C}^\infty(M)$ is a 0-***form***;
  2. ***A differential*** 1-***form*** is ***the covariant vector field*** df

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

  3. ***A differential*** 2-***form*** is written as

$$\omega = \sum_{i<j} \omega_{i,j} \, dx^i \wedge dx^j$$

- **Definition** If $F : M \to N$ is a smooth map and ω is ***a differential form*** on N, *the* ***pullback*** F^* *is a* ***differential form on*** M; defined as for any covariant tensor field:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_p\left(dF_p(v_1), \ldots, dF_p(v_k)\right).$$

- **Lemma 2.1** *Suppose $F : M \to N$ is smooth.*
  1. $F^* : \Omega^k(N) \to \Omega^k(M)$ *is* **linear** *over* $\mathbb{R}$.
  2. $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

3. In any smooth chart,

$$F^{*}\left(\sum_{I}{}' \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_{I}{}' \left(\omega_I \circ F\right) \, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F) \tag{12}$$

- **Example** Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Thinking of the transformation to polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$\begin{aligned} dx \wedge dy &= d\left(r\cos\theta\right) \wedge d\left(r\sin\theta\right) \\ &= \left(\cos\theta \, dr - r\sin\theta d\theta\right) \wedge \left(\sin\theta dr + r\cos\theta d\theta\right) \\ &= r dr \wedge d\theta. \end{aligned}$$

- **Proposition 2.2** ***(Pullback Formula for Top-Degree Forms).***
Let $F : M \to N$ be a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^{*}\left(u \, dy^1 \wedge \ldots \wedge dy^n\right) = \left(u \circ F\right) \left(\det(DF)\right) dx^1 \wedge \ldots \wedge dx^n \tag{13}$$

where DF represents ***the Jacobian matrix of*** *F in these coordinates.*

Note that $d(y^i \circ F) = dF^i = \det(DF)^i_j dx^j$

- **Corollary 2.3** ***(Change of Coordinates for Differential Forms)***
If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \widetilde{U}$:

$$d\widetilde{x}^1 \wedge \ldots \wedge d\widetilde{x}^n = \det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n. \tag{14}$$

- **Remark** The equation (13) provides a computational formula for pullback of differential forms under coordinate systems for domain and codomain. And the equation (14) provides the fomula for change of variables of differential forms.

- **Definition** ***Interior multiplication*** also extends naturally to ***vector fields*** and ***differential forms***, simply by letting it act *pointwise*: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$-form $X \lrcorner \, \omega = \iota_X \omega$ by

$$(X \lrcorner \, \omega)_p = X_p \lrcorner \, \omega_p.$$

3 Exterior Derivatives

3.1 Definitions

- **Remark** An important question for differential k-form ω is that under what condition there exists a function f so that $\omega = df$, i.e, the tensor field ω is *exact*. A necessary condition is that ω is *closed*, i.e.

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}.$$

In other word, ω is closed if and only if $d\omega = 0$.

- **Remark** For each smooth manifold M with or without boundary, we will show that there is ***a differential operator*** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω. Thus, it will follow that *a necessary condition* for a smooth k-form ω to be equal to $d\eta$ for some $(k-1)$-form η is that $d\omega = 0$.

- **Definition** If $\omega = \sum_J' \omega_J dx^J$ is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, we define its ***<u>exterior derivative</u>*** $d\omega$ to be the following <u>$(k+1)$-form</u>:

$$d\omega := d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J, \tag{15}$$

where $d\omega_J$ is the differential of the function ω_J. In somewhat more detail, this is

$$d\omega := d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}. \tag{16}$$

- **Remark** The *exterior derivatives* of a k-form is ***a linear combination*** of $(k+1)$*-forms.* It component function is ***the principal minior*** *of* ***Jacobian matrix of component functions*** $(\frac{\partial \omega_j}{\partial x^i})$.

- **Remark** When ω is a 1-form, this becomes

$$\begin{aligned} d\omega = d\left(\sum_j \omega_j dx^j\right) &= \sum_j d\omega_j \wedge dx^j \\ &= \sum_j \sum_i \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j. \end{aligned}$$

Note that the component is the determinant of a 2×2 sub-matrix of Jacobian $(\frac{\partial \omega_j}{\partial x^i})$.

- **Proposition 3.1** ***(Properties of the Exterior Derivative on*** $\mathbb{R}^n$***).***

  1. *d is **linear** over $\mathbb{R}$.*
  2. *If ω is a smooth k-form and η is a smooth l-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, then*
  $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$
  3. *$d \circ d \equiv 0$.*
  4. *d **commutes** with **pullbacks**: if U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, V is an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$, $F : U \to V$ is a smooth map, and $\omega \in \Omega^k(V)$, then*
  $$F^*(d\omega) = d\left(F^*\omega\right). \tag{17}$$

- These results allow us to transplant the definition of the exterior derivative to manifolds.

  Theorem 3.2 ***(Existence and Uniqueness of Exterior Differentiation).***
  *Suppose M is a smooth manifold with or without boundary. There are **unique operators** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called **exterior differentiation**, satisfying the following four properties:*

  1. *d is **linear** over $\mathbb{R}$.*
  2. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*
  $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$
  3. *$d \circ d \equiv 0$.*
  4. *For $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

  In any smooth coordinate chart, d is given by (15).

- **Remark** The ***exterior differentiation*** defines the differential of k-form. It is an ***extension*** of ***differentiation*** to ***determinant function***.

- **Definition** If $A = \oplus_k A^k$ is a *graded algebra*, a *linear map* $T : A \to A$ is said to be a map ***of degree*** m if $T(A^k) \subseteq A^{k+m}$ for each k. It is said to be an ***antiderivation*** if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x \in A^k$ and $y \in A^l$.

- **Remark** (***The Exterior Differentiation vs. The Interior Multiplication***)

  1. The ***exterior differentiation*** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an ***antiderivation*** of *degree* $+1$ whose ***square is zero***.
  2. On the other hand, the ***interior multiplication*** $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is an ***antiderivation*** of *degree* -1 whose ***square is zero***, where $X \in \mathfrak{X}(M)$.

- Another important feature of the exterior derivative is that *it **commutes** with all pullbacks.*

  Proposition 3.3 ***(Naturality of the Exterior Derivative).***
  *If $F : M \to N$ is a smooth map, then for each k **the pullback map** $F^* : \Omega^k(N) \to \Omega^k(M)$ **commutes** with d: for all $\omega \in \Omega^k(N)$,*
  $$F^*(d\omega) = d\,(F^*\omega)\,. \tag{18}$$

3.2 An Invariant Formula for the Exterior Derivative

- **Proposition 3.4** ***(Exterior Derivative of a 1-Form).***
  For any smooth 1-form ω and smooth vector fields X and Y,
  $$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{19}$$

  Proof: Since any smooth 1-form can be expressed locally as a sum of terms of the form $u\,dv$ for smooth functions u and v, it suffices to consider that case. Suppose $\omega = u\,dv$, and X, Y are smooth vector fields. The LHS of (19)
  $$\begin{aligned} d\,(u\,dv)\,(X, Y) &= (du \wedge dv)(X, Y) = du(X)dv(Y) - du(Y)dv(X) \\ &= X(u)Y(v) - X(v)Y(u) \end{aligned}$$

The RHS is

$$
\begin{aligned}
&= X(u\, dv(Y)) - Y(u\, dv(X)) - u\, dv([X,Y]) \\
&= X(u\, Y(v)) - Y(u\, X(v)) - u\, [X,Y](v) \\
&= X(u)Y(v) + u\, XY(v) - Y(u)X(v) - u\, YX(v) - u\ ((XY - YX)v) \\
&= X(u)Y(v) - Y(u)X(v) + u\ (XY(v) - YX(v)) - u\ (XY(v) - YX(v)) \\
&= X(u)Y(v) - Y(u)X(v).
\end{aligned}
$$

Thus (19) holds. $\blacksquare$.

- **Proposition 3.5** *Let M be a smooth n-manifold with or without boundary, let (E_i) be a smooth local frame for M, and let (ϵ^i) be the dual coframe. For each i, let $b^i_{j,k}$ denote the **component functions** of **the exterior derivative** of ϵ^i in this frame, and for each j,k, let $c^i_{j,k}$ be the **component functions** of the **Lie bracket** $[E_j, E_k]$:*

$$
d\epsilon^i = \sum_{j<k} b^i_{j,k} \epsilon^j \wedge \epsilon^k; \quad [E_j, E_k] = c^i_{j,k} E_i
$$

Then $b^i_{j,k} = -c^i_{j,k}$.

- **Proposition 3.6** ***(Invariant Formula for the Exterior Derivative).***
Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields $X_1, \ldots, X_{k+1}$ on M,

$$
\begin{aligned}
d\omega\left(X_1, \ldots, X_{k+1}\right) = &\sum_{1 \le i \le k+1} (-1)^i X_i \left(\omega\left(X_1, \ldots, \widehat{X}_i \ldots, X_{k+1}\right)\right) \\
&+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}\right), \qquad (20)
\end{aligned}
$$

*where the hats indicate **omitted** arguments.*

- **Remark** The proof of formula (20) and (19) is only based on the definition of k-form and vector fields, and it does not involve any specific coordinate system. Thus it can be used to give an ***invariant definition*** of ***the exterior differentiation*** *d*.

3.3 Lie Derivatives of Differential Forms