Lecture 8: Signed Measures and Radon-Nikodym Derivative

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Contents

1 Signed Measure

1.1 Definitions

- **Definition** (***Signed Measure***)
  Let $(X, \mathscr{B})$ be a measure space. A ***signed measure*** on $(X, \mathscr{B})$ is a function $\nu : \mathscr{B} \to [-\infty, +\infty]$ such that
  1. (***Emptyset***) $\nu(\emptyset) = 0$;
  2. (***Finiteness in One Direction***) ν assumes at most one of the values $\pm\infty$;
  3. (***Countable Additivity***) if $\{E_j\}$ is a sequence of disjoint sets in $\mathscr{B}$, then $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$, where the latter converges absolutely if $\nu\left(\bigcup_{j=1}^{\infty} E_j\right)$ is finite.
- **Definition** A measure μ is ***finite***, if $\mu(X) < \infty$; μ is σ***-finite***, if $X = \bigcup_{k=1}^{\infty} U_k$, $\mu(U_k) < \infty$.
- **Example** If μ_1, μ_2 are measures on $\mathscr{B}$ and *at least one of them* is *finite*, then $\nu = \mu_1 - \mu_2$ is *a signed measure.*
- **Example** If μ is a measure on $\mathscr{B}$ and $f : X \to [-\infty, +\infty]$ is a measurable function such that *at least one of* $\int_X f_+ d\mu$ *and* $\int_X f_- d\mu$ *is finite* (in which case, f is called an ***extended*** μ***-integrable function***). Then ν defined as $\nu(E) = \int_X f \mathbb{1}_E d\mu$ is a *signed measure.*
- **Remark** ***Every signed measure*** can be represented as one of these two forms
  1. $\nu = \mu_+ - \mu_-$, where at least one of μ_+, μ_- is a finite measure;
  2. μ is measure on $\mathscr{B}$, and $f : X \to [-\infty, +\infty]$ is *extended μ-integrable* with at least one of f_+ and f_- finite integrable. Then $\nu(A) = \int_X f \mathbb{1}\{A\}\, d\mu$ is a signed measure.
- Like unsigned measure, we have monotone downward and upward convergence:

  Proposition 1.1 *Let ν be a **signed measure** on $(X, \mathscr{B})$.*
  1. *(**Upwards monotone convergence**) If $E_1 \subseteq E_2 \subseteq \ldots$ are $\mathscr{B}$-measurable, then*
  $$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \nu(E_n) = \sup_n \nu(E_n). \tag{1}$$
  2. *(**Downwards monotone convergence**) If $E_1 \supseteq E_2 \supseteq \ldots$ are $\mathscr{B}$-measurable, and $\nu(E_n) < \infty$ for **at least one** n, then*
  $$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \nu(E_n) = \inf_n \nu(E_n). \tag{2}$$
- **Definition** (***Positive Measure***)
  If ν is a signed measure on $(X, \mathscr{B})$, a ***set*** $E \in \mathscr{B}$ is called ***positive*** (resp. ***negative***, ***null***) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$, $\nu(F) = 0$) for ***all*** $\mathscr{B}$***-measurable subset*** of E (i.e. $F \in \mathscr{B}$ such that $F \subseteq E$).

In other word, E is ν***-positive***, ν***-negative***, ν***-null*** if and only if $\nu(E \cap M) > 0$, $\nu(E \cap M) < 0$, $\nu(E \cap M) = 0$ ***for any*** M. Thus if $\nu(E) = \int_X f \mathbb{1}\{E\}\, d\mu$, then it corresponds to $f \geq 0$, $f \leq 0$ and $f = 0$ for μ***-almost everywhere*** $x \in E$.

- **Lemma 1.2** *[Folland, 2013]*
Any ***measureable subset*** *of a positive set is positive, and the* ***union*** *of any* ***countable*** *positive set is positive.*

  Proof: The first part is clear from the definition. Let $P_1, \ldots,$ be countable collection of positive sets. Note that any finite collection $\bigcup_{k=1}^{n-1} P_k$ is positive by definition. Consider $Q_n = P_n - \bigcup_{k=1}^{n-1} P_k$. Since $Q_n \subset P_n$, Q_n is positive and $\bigcup_{k=1}^{\infty} Q_k = \bigcup_{k=1}^{\infty} P_k$ with disjoint Q_k. Hence for any $E \subset \bigcup_{k=1}^{\infty} P_k$, then $\nu(E) = \nu\left(\bigcup_{k=1}^{\infty}(E \cap Q_k)\right) = \sum_{k=1}^{\infty} \nu(E \cap Q_k) > 0$. ▮

- **Remark** For two measures μ, ν on $(X, \mathscr{B})$ among which at least one of them is finite, the expression $\mu \geq \nu$ on E means that for every $F \subseteq E \in \mathscr{B}$, $(\mu - nu)(F) \geq 0$. That is, E is *a positive set* of $(\mu - nu)$.

1.2 Decomposition of Signed Measure

- **Remark** Given a signed measure ν, we can ***partition*** the space X into positive set (i.e. all of its measurable subsets have positive measure) and negative set (i.e. all of its measurable subsets have negative measure).

- **Theorem 1.3** ***(The Hahn Decomposition Theorem)****[Folland, 2013]*
If ν is a ***signed measure*** *on $(X, \mathscr{B})$, there exists a* ***positive set*** *P and a* ***negative set*** *N for ν such that $P \cup N = X$ and $P \cap N = \emptyset$. If P', N' is another such pair, then $P \Delta P' = N \Delta N'$ is* ***null*** *w.r.t. ν.*

  Proof: Without loss of generality, assume that ν does not take value $+\infty$. Let m be the supremum of $\nu(E)$ as E ranges over all positive sets; thus there is a sequence $\{P_j\}$ of positive sets such that $\nu(P_j) \to m$. Let $P = \bigcup_{j=1}^{\infty} P_j$. By the lemma and the proposition above, P is positive and $\nu(P) = m$, which is finite. We claim that $N = X - P$ is negative. To this end, we assume that N is not negative, and derive for a contradiction.

  First, notice that N cannot contain any nonnull positive sets. Indeed, if $E \subset N$ is positive, then $\nu(E) > 0$, and $E \cup P$ is positive with $\nu(E \cup P) = \nu(E) + \nu(P) > m$, which violates the assumption.

  Second, if $A \subset N$, $\nu(A) > 0$, there exists $B \subset A$, with $\nu(B) > \nu(A)$. Indeed, since A cannot be positive, there exists $C \subset A$ with $\nu(C) < 0$; thus if $B = A - C$, we have $\nu(B) = \nu(A) - \nu(C) > \nu(A)$.

  If N is not negative, then we can specify a sequence of subsets $\{A_j\}$ of N and a sequence of positive integers $\{n_j\}$ as follows: n_1 is the smallest integer for which there exists a set $B \subset N$ with $\nu(B) > 1/n_1$, and let A_1 be the set as defined above. And n_j is the smallest integer for which there exists a set $B \subset A_{j-1}$ with $\nu(B) \geq \nu(A_{j-1}) + 1/n_j$ and A_j is such a set.

  Let $A = \bigcap_{j=1}^{\infty} A_j$. Then $\infty > \nu(A) = \lim_{j \to \infty} \nu(A_j) > \sum_{j=1}^{\infty} \frac{1}{n_j}$ with $n_j \to \infty$. But once again, there exists $B \subset A$ with $\nu(B) \geq \nu(A) + 1/n$ for some integer n. For j sufficiently large, we have $n < n_j$, and $B \subset A_{j-1}$, which violates the construction of A_{j-1}. So N is not negative is untenable.

Finally, if P', N' is another pair of sets as stated, we have $P - P' \subset P$ and $P - P' \subset N'$, so that $P - P'$ is both positive and negative, thus it is a null set. $\blacksquare$

- **Definition** [Folland, 2013, Resnick, 2013]
  The decomposition of $X = P \cup N$ as X is a ***disjoint union*** *of a* ***positive set*** *and a* ***negative set*** is called a ***Hahn decomposition for*** ν.

- **Remark** Note that the Hahn decomposition is usually ***not unique*** as the ν-null set can be transferred between subparts P and N. To find unique decomposition, we need the following concepts:

- **Definition** [Folland, 2013]
  Two *signed measures* μ, ν on $(X, \mathscr{B})$ are ***mutually singular***, or that ν is ***singular*** w.r.t. to μ, or vice versa, if and only if there exists a ***partition*** $E, F \in \mathscr{B}$ of X such that $E \cap F = \emptyset$ and $E \cup F = X$, E ***is null for*** μ and F ***is null for*** ν. Informal speaking, ***mutual singular*** means that μ and ν "***live on disjoint sets***". We describe it using perpendicular sign

$$\mu \perp \nu$$

- **Theorem 1.4** *(**The Jordan Decomposition Theorem**)[Folland, 2013]*
  *If ν is a signed measure on $(X, \mathscr{B})$, there exists **unique positive measure** ν_+ and ν_- such that*

$$\nu = \nu_+ - \nu_- \qquad \text{and} \qquad \nu_+ \perp \nu_-.$$

  Proof: Let $X = P \cup N$ be the *Hahn decomposition* for ν and define $\nu_+(E) = \nu(E \cap P)$ and $\nu_-(F) = -\nu(F \cap N)$. Then clearly, $\nu = \nu_+ - \nu_-$ and $\nu_+ \perp \nu_-$.

  If also $\nu = \mu_+ - \mu_-$ and $\mu_+ \perp \mu_-$, let $E, F \in \mathscr{B}$ be a partition of X as $E \cap F = \emptyset$ and $E \cup F = X$, and $\mu_+(F) = \mu_-(E) = 0$. Then $X = E \cup F$ is another Hahn decomposition, so $P \Delta E$ is ν-null. Therefore, for any $A \in \mathscr{B}$, $\mu_+(A) = \nu(A \cap E) = \nu(A \cap P) = \nu_+(A)$ and likewise $\nu_- = \mu_-$. $\blacksquare$

- **Definition** The two positive measures ν_+, ν_- are called the ***positive*** and ***negative variations*** of ν, and $\nu = \nu_+ - \nu_-$ is called the ***Jordan decomposition*** of ν.

  Furthermore, define the ***total variations*** of ν as the measure $|\nu|$ such that

$$|\nu| = \nu_+ + \nu_-.$$

- **Proposition 1.5** *Let ν, μ be signed measures on $(X, \mathscr{B})$ and $|\nu|$ is the total variations of ν. Then*

  1. *$E \in \mathscr{B}$ is ν-null if and only if $|\nu|(E) = 0$*
  2. *$\nu \perp \mu$ **if and only if** $|\nu| \perp \mu$ if and only if $(\nu_+ \perp \mu) \wedge (\nu_- \perp \mu)$.*

- **Proposition 1.6** *If ν_1, ν_2 are signed measures that both omit $\pm\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$*

- **Exercise 1.7** *Let ν be a signed measure on $(X, \mathscr{B})$.*

  1. *$L^1(\nu) = L^1(|\nu|)$;*

2. *If* $f \in L^1(\nu)$*, then*

$$\left|\int_X f d\nu\right| \le \int_X |f|\, d\,|\nu|$$

3. *If* $E \in \mathscr{B}$*, then*

$$|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| : \; |f| \le 1\right\}$$

- **Remark** We recall that ν assume at most one of values on $\pm\infty$:

  1. If ν does not take $+\infty$, then $\nu_+(X) = \nu(P) < \infty$ ***is a finite measure***;

  2. if ν does not take $-\infty$, then $\nu_-(X) = -\nu(N) < \infty$ ***is a finite measure***.

  In particular, if the range of ν is contained in $\mathbb{R}$, then ν is *bounded*.

- **Remark** We observe that ν is *of form* $\nu(E) = \int_E f d\mu$ where $|\nu| = \mu$ and $f = \mathbb{1}_P - \mathbb{1}_N$ and $X = P \cup N$ being a *Hahn decomposition* for ν.

- **Remark** (***Integration with respect to Signed Measure***)
  Let ν be signed measures on $(X, \mathscr{B})$ and $\nu = \nu_+ - \nu_-$ is *the Jordan decomposition* of ν then

$$\int_X f d\nu = \int_X f d\nu_+ - \int_X f d\nu_-$$

  for all $f \in L^1(X, \nu)$.

- **Definition** *A signed measure* ν is called ***σ-finite*** if $|\nu|$ is *σ-finite*.

1.3 Lebesgue-Radon-Nikodym Theorem

- **Definition** [Folland, 2013]
  Suppose ν is *a* ***signed measure*** on $(X, \mathscr{B})$ and μ is *a* ***positive measure*** on $(X, \mathscr{B})$. Then ν is said to be ***absolutely continuous w.r.t.*** μ and write

$$\nu \ll \mu$$

  if $\nu(E) = 0$ for *every* $E \in \mathscr{B}$ *for which* $\mu(E) = 0$.

- **Proposition 1.8** *Suppose* ν *is a signed measure on* $(X, \mathscr{B})$*,* ν_+, ν_- *are positive and negative variation of* ν *and* $|\nu|$ *is the total variation. Then* $\nu \ll \mu$ ***if and only if*** $|\nu| \ll \mu$ ***if and only if*** $(\nu_+ \ll \mu) \wedge (\nu_- \ll \mu)$*.*

- **Remark** ***Absolutly continuity*** is in a sense ***antithesis*** (i.e. *direct opposite*) of ***mutual singularity***. More precisely, if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$, since E, F are disjoint sets such that $E \cup F = X$, and $\mu(E) = |\nu|(F) = 0$, then $\nu \ll \mu$ implies that $|\nu|(E) = 0$. One can *extend* the notion of absolute continuity to the case where μ *is a signed measure* (namely, $\nu \ll \mu$ iff $\nu \ll |\mu|$), but we shall have no need of this more general definition.

- **Theorem 1.9** *(ϵ-δ* ***Language of Absolute Continuity of Measures****)*
  Let ν *is a* ***finite signed measure*** *and* μ *is a* ***positive*** *measure on* $(X, \mathscr{B})$*. Then* $\nu \ll \mu$ *if and only if for every* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $|\nu(E)| < \epsilon$*,* ***whenever*** $\mu(E) < \delta$*.*

Proof: Since $\nu \ll \mu$ iff $|\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, it suffices to assume that $\nu = |\nu|$ is positive.

"$\Leftarrow$", it is clear.

"$\Rightarrow$", if the $\epsilon - \delta$ condition is not satisfied, there exists $\epsilon > 0$, for all $n \in \mathbb{N}$ we can find $E_n \in \mathscr{B}$, with $\mu(E_n) < \frac{1}{2^n}$ and $\nu(E_n) \geq \epsilon$.

Let $F_k = \bigcup_{n=k}^{\infty} E_n$ and $F = \bigcap_{k \geq 1} F_k$. Then $\mu(F_k) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = 2^{1-k}$, so $\mu(F) = 0$. But $\nu(F_k) \geq \epsilon$ for all k, and hence since ν is finite, $\nu(F) = \lim_{k\to\infty} \nu(F_k) \geq \epsilon$. Thus it is false that $\nu \ll \mu$. ∎

- **Remark** If μ is a *measure* and f is ***extended μ-integrable***, then ***the signed measure*** ν *defined via* $\nu(E) = \int_E f d\mu$ *is* ***absolutely continuous*** *w.r.t.* μ; it is ***finite*** if and only if f is ***absolutely integrable***. For any complex-valued $f \in L^1(\mu)$, the preceding theorem can be applied to $\Re(f)$ and $\Im(f)$.

- **Corollary 1.10** *If $f \in L^1(X, \mu)$, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.*

- **Definition** For *a* ***signed measure*** ν defined via $\nu(E) = \int_E f d\mu$ for all $E \in \mathscr{B}$, we use the notation to express the relationship

$$d\nu = f\, d\mu.$$

Sometimes, by a slight abuse of language, we shall refer to "***the signed measure*** $f\, d\mu$"

- **Lemma 1.11** *[Folland, 2013]*
Suppose that ν and μ are ***finite measures*** *on $(X, \mathscr{B})$. Either $\nu \perp \mu$, or there exists $\epsilon > 0$ and $E \in \mathscr{B}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E, i.e. E is a* ***positive set for*** $\nu - \epsilon\mu$.

Proof: Let $X = P_n \cup N_n$ be a Hahn decomposition on $(X, \mathscr{B})$ for $\nu - n^{-1}\mu$ and let $P = \bigcup_{n=1}^{\infty} P_n$ and $N = \bigcap_{n=1}^{\infty} N_n$. Then N is a negative set for $\nu - n^{-1}\mu$ for all n, i.e., $0 \leq \nu(N) \leq n^{-1}\mu(N)$ for all n, so $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$; if $\mu(P) > 0$, then $\mu(P_n) > 0$ for some n, and P_n is positive set for $\nu - n^{-1}\mu$. ∎

- **Theorem 1.12** ***(Lebesgue-Radon-Nikodym Theorem)****[Folland, 2013]*
Let ν be a <u>***σ-finite signed*** *measure*</u> *and μ be a* <u>***σ-finite positive*** *measure*</u> *on $(X, \mathscr{B})$. There exists* <u>***unique σ-finite signed measure***</u> *λ, ρ on $(X, \mathscr{B})$ such that*

$$\lambda \perp \mu\,, \quad \textit{and} \quad \rho \ll \mu\,, \quad \textit{and} \quad \nu = \lambda + \rho.$$

In particular, if $\nu \ll \mu$, then

$$d\nu = f d\mu, \qquad \textit{for some } f.$$

Proof: We proof it under different cases:

  - ***Case 1***: Suppose that ν, μ are ***finite positive measures*** and let

$$\mathcal{F} \equiv \left\{ f : X \to [0, \infty] : \int_E f d\mu \leq \nu(E),\ \forall\, E \in \mathscr{B} \right\}.$$

Note that $0 \in \mathcal{F}$. Also for $g, f \in \mathcal{F}$, then $h = \max\{f, g\} \in \mathcal{F}$, since for $A = \{x : f(x) \geq g(x)\}$,

$$\int_E h d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E).$$

Let $a = \sup\left\{\int f d\mu \mid f \in \mathcal{F}\right\}$, ***noting that*** $a < \nu(X) < \infty$. *There exists a sequence of functions* $\{f_n, n \geq 1\} \subset \mathcal{F}$ such that $\lim_{n \to \infty} \int f_n d\mu \to a$. Let $g_n = \max\{f_1, \ldots, f_n\}$ and $f = \sup_{n \geq 1} f_n$. Clearly, $g_n \in \mathcal{F}$ and $g_n \to f$, μ-a.e. Also $\int g_n d\mu \geq \int f_n d\mu$. Since $\{g_n\}$ is monotone increasing, by monotone convergence theorem, $\lim_{n \to \infty} \int g_n d\mu = \int f d\mu = a$ and $f \in \mathcal{F}$.

We claim that $d\lambda = d\nu - f d\mu$ (, which is a positive finite measure since $f \in \mathcal{F}$), is ***singular*** w.r.t. μ. By the lemma above, if not, then there exist a set E and $\epsilon > 0$ such that $\mu(E) > 0$ and $\lambda(E) \geq \epsilon\mu(E)$. Then $\epsilon \mathbb{1}_E \, d\mu \leq \mathbb{1}_E \, d\lambda \leq d\lambda = d\nu - f d\mu$ and the function $(f + \epsilon \mathbb{1}_E) \in \mathcal{F}$. But $\int (f + \epsilon \mathbb{1}_E) d\mu = a + \epsilon\mu(E) > a$, which violates the assumption on a.

Thus the *existence* of λ, f and $d\rho = f d\mu$ is proved. For *uniqueness*, if also $d\nu = d\lambda' + f' d\mu$, we have that $d\lambda - d\lambda' = (f' - f) d\mu$. But $(\lambda - \lambda') \perp \mu$, while $(f' - f) d\mu \ll d\nu$; hence $d\lambda - d\lambda' = 0$ and $f' = f$ μ-a.e. Thus we are done in the finite measure cases.

- ***Case 2***: suppose that ν, μ are ***σ-finite positive measures***. Then X is a *countable disjoint union of μ-finite sets* and a countable disjoint union of ν-finite sets; by taking their intersections, we have a disjoint collection $\{A_j\} \subset \mathscr{B}$ such that $\mu(A_j)$ and $\nu(A_j)$ are both finite and $X = \bigcup_j A_j$.

  Define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$. Use the prove above, $d\nu_j = d\lambda_j + f_j d\mu_j$, where $\lambda_j \perp \mu_j$. Since $\mu_j(A_j^c) = \nu_j(A_j^c) = 0$, then we have $\lambda_j(A_j^c) = \nu_j(A_j^c) - \int_{A_j^c} f_j d\mu_j = 0$, and we may assume that $f_j = 0$ on A_j^c.

  Let $\lambda = \sum_{j=1}^{\infty} \lambda_j$ and $f = \sum_{j=1}^{\infty} f_j$. Then $d\nu = d\lambda + f d\mu$, $\lambda \perp \mu$, and $d\lambda$ and $f d\mu$ are σ-finite, as desired. Uniqueness follows as above.

- ***General Case***: If ν is a signed measure, just apply the preceding argument to ν_+, ν_- and subtract the results. ∎

- **Definition** The decomposition $\nu = \rho + \lambda$, where $\lambda \perp \mu$ and $\rho \ll \mu$, is called the ***Lebesgue decomposition*** *of ν with respect to μ.*

- **Definition** If $\nu \ll \mu$, then according to *the Lebesgue-Radon-Nikodym theorem*, $d\nu = f d\mu$ for some f, where f is called the ***Radon-Nikodym derivative*** *of ν w.r.t.* μ and is denoted as

$$f := \frac{d\nu}{d\mu} \quad \Rightarrow \quad d\nu = \frac{d\nu}{d\mu} d\mu.$$

- **Remark** By Lebesgue decomposition, *a signed measure ν* can be represented as

$$d\nu = d\lambda + f d\mu$$

- **Remark** (***Jordan Decomposition vs. Lebesgue Decomposition***)
  We see ***two unique decompositions***: the Jordan decomposition and the Lebesgue decomposition. We can make a comparison:

1. Both of these two are *decompositions* of *a* ***signed*** *measure* ν.
2. Both of these two decompositions seperate ν into two ***mutually signular*** sub-measures of ν.
3. Both of these two decompositions are ***unique***

On the other hand,

1. ***The Jordan decomposition*** is to split *a signed measure* ν ***itself*** into ***two positive measures***, i.e. ν_+ and ν_- that are ***mutually singular*** ($\nu_+ \perp \nu_-$).
2. ***The Lebesgue decomposition*** is to split *a signed measure* ν ***with respect to a postive measure*** μ. The result is *two-fold*: 1) *two mutually singular sub-measures* $\lambda \perp \rho$ 2) their relationship with μ is ***opposite***: $\lambda \perp \mu$, i.e. their support do not overlap; $\rho \ll \mu$, i.e. its support lies within support of μ.
3. Note that λ, ρ from *the Lebesgue decomposition* is **not** *necessarily* ***positive***. But both ν and μ need to be σ***-finite*** which is *not required* for the Jordan decomposition.

- **Proposition 1.13** *[Folland, 2013]*
Suppose ν *is* σ***-finite signed measure*** *and* λ, μ *are* σ***-finite measure*** *on* $(X, \mathscr{B})$ *such that* $\nu \ll \mu$ *and* $\mu \ll \lambda$.

1. If $g \in L^1(X, \nu)$, *then* $g\left(\frac{d\nu}{d\mu}\right) \in L^1(X, \mu)$ *and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

2. We have $\nu \ll \lambda$, *and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \quad \lambda\text{-a.e.}$$

Proof: 1. By Radon-Nikodym theorem, the expression holds if $g = \mathbb{1}\{E\}$ for any E ν-measureable,i.e.

$$\nu(E) = \int \mathbb{1}\{E\}\, d\nu = \int \mathbb{1}\{E\}\left(\frac{d\nu}{d\mu}\right) d\mu.$$

Note that for any simple function $g = \sum_{s=1}^{m} g_s \mathbb{1}\{E_s\}$ for finitely many ν-measureable set E, due to the linearity, the expression

$$\int g d\nu = \sum_{s=1}^{m} g_s \int \mathbb{1}\{E_s\}\, d\nu = \sum_{s=1}^{m} g_s \int \mathbb{1}\{E_s\}\left(\frac{d\nu}{d\mu}\right) d\mu = \int g\left(\frac{d\nu}{d\mu}\right) d\mu$$

hold. Then for any nonnegative integrable function g, there exists a monotone increasing sequence $g_n \le g_{n+1}$ converges to g ν-a.e.

$$\int g d\nu = \limsup_{\substack{g_n \le g,\\ g_n \text{ simple}}} \int g_n d\nu = \limsup_{\substack{g_n \le g,\\ g_n \text{ simple}}} \int g_n \left(\frac{d\nu}{d\mu}\right) d\mu = \int g\left(\frac{d\nu}{d\mu}\right) d\mu$$

The last equality comes from monotone convergence theorem. For absolutely integrable function $g = g_+ - g_-$ with g_+, g_- both nonnegative integrable function. The expression hold by linearity.

2. Let $g = \mathbb{1}\{E\}\left(\frac{d\nu}{d\mu}\right)$ and replace ν, μ with μ, λ, we have

$$\int \mathbb{1}\{E\}\, d\nu = \int \mathbb{1}\{E\}\left(\frac{d\nu}{d\mu}\right) d\mu = \int \mathbb{1}\{E\}\left(\frac{d\nu}{d\mu}\right)\frac{d\mu}{d\lambda} d\lambda$$

for any E measureable. Therefore,

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \quad \lambda\text{-}a.e. \qquad \blacksquare$$

- **Corollary 1.14** *If $\mu \ll \lambda$ and $\lambda \ll \mu$, then $(d\lambda/d\mu)(d\mu/d\lambda) = 1$ a.e. (with respect to either λ or μ).*

- **Proposition 1.15** *If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathscr{B})$, then there exists a measure μ such that $\mu_i \ll \mu$ for all $i = 1, \ldots, n$, namely, $\mu = \sum_{i=1}^{n} \mu_i$.*

- **Exercise 1.16** *(**Conditional Expectation**)*
  *Let $(X, \mathscr{B}, \mu)$ be a **finite measure space**, $\mathscr{F}$ is a sub-σ-algebra of $\mathscr{B}$, and $\nu = \mu|_{\mathscr{F}}$. Show that if $f \in L^1(X, \mu)$, there exists $g \in L^1(X, \nu)$ (thus g is $\mathscr{F}$**-measureable**) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathscr{F}$; if g' is another such function then $g = g'$ ν-a.e.*

  *In **probability theory**, where $(X, \mathscr{B}) \equiv (\Omega, \mathscr{A})$, $f \equiv X$ is a **random variable**, then $g \equiv \mathbb{E}[X|\mathscr{F}]$ is called **the conditional expectation of X on $\mathscr{F}$**, which is $\mathscr{F}$-measure random variable.*

  Proof: We can define a signed measure λ on $(X, \mathscr{B})$ as $d\lambda = f d\mu$, i.e. $\lambda \ll \mu$. We claim that $\lambda|_{\mathscr{F}} \ll \nu = \mu|_{\mathscr{F}}$. Then by Radon-Nikodym theorem, there exists a $\mathscr{F}$-measureable function

  $$g = \frac{\lambda|_{\mathscr{F}}}{\mu|_{\mathscr{F}}},$$

  so that for every $E \in \mathscr{F}$,

  $$\begin{aligned}\lambda(E) = \lambda|_{\mathscr{F}}(E) &= \int_E \frac{\lambda|_{\mathscr{F}}}{\mu|_{\mathscr{F}}}\, d\mu|_{\mathscr{F}} \\ &= \int_E g d\nu.\end{aligned}$$

  and $\lambda(E) = \int_E f d\mu$, which shows the result.

  To show the claim is true, we see that $\nu(E) = \mu(E)$ and $\lambda|_{\mathscr{F}}(E) = \lambda(E)$ for every $E \in \mathscr{F}$ and $\lambda \ll \mu$, so for any $\epsilon > 0$, there exists $\delta > 0$, such that if $\mu(E) < \delta$, then $\lambda(E) < \epsilon$. It is equivalent to say $\nu(E) < \delta$ implies $\lambda|_{\mathscr{F}}(E) < \epsilon$, which proves the claim. $\blacksquare$

- **Remark** Note that similarly, the ***conditional distribution*** $P(A|\mathscr{F}) = \mathbb{E}[\mathbb{1}_A \,|\, \mathscr{F}]$ is *a **random variable***. Also, $\mathbb{E}[X|Y] = \mathbb{E}[X|\sigma(Y)]$, where $\sigma(Y)$ is the sub-σ-algebra induced by $Y^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathscr{B}$.

  $$\mathbb{E}[X|Y](\omega_y) = \int_{\Omega_x} K(\omega_y, d\omega_x) Z_{\omega_y}(\omega_x) P(d\omega_x)$$

  where $K : \Omega_Y \times \mathscr{B}_X \to [0, 1]$ is *the transition kernel*, $Z_{\omega_y}(\omega_x) = Z(\omega_x, \omega_y) = (X(\omega_x), Y(\omega_y))$.

2 Exercise

- **Exercise 2.1** *Show that if* λ *is a signed measure and* μ *is a positive measure on* $(X, \mathscr{B})$*, then* $\lambda \ll \mu$ *implies that* λ_+, λ_- *and* $|\lambda|$ *are absolutely continuous with respect to* μ*.*

- **Exercise 2.2** *Show that if* λ *is a signed measure and* μ *is a positive measure on* $(X, \mathscr{B})$*, then* $|\lambda| \perp \mu$ *implies that* λ_+, λ_- *are singular with respect to* μ*.*

- **Exercise 2.3** *Let* $X = [0, 1]$ *and* $\mathscr{B}$ *be the Borel* σ*-algebra. If* μ *is the counting measure on* $\mathscr{B}$ *and* λ *is the Lebesgue measure on* $\mathscr{B}$*, then* λ *is a finite measure and* $\lambda \ll \mu$*, but the Radon-Nikodym theorem fails.*

References

Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

Sidney I Resnick. *A probability path.* Springer Science & Business Media, 2013.