

Lecture 5: Concentration of Measure and Isoperimetry

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Contents

1	The Classic Isoperimetry Inequalities	2
1.1	Brunn-Minkowski Inequality	2
1.2	The Classical Isoperimetry Theorem	5
2	Concentration via Isoperimetry	6
2.1	Isoperimetry Problem and Levy's Inequalities	6
2.2	Isoperimetric Inequalities on the Unit Sphere	8
2.3	Gaussian Isoperimetric Inequalities and Concentration of Gaussian Measure	9
2.4	Edge Isoperimetric Inequality on the Binary Hypercube	9
2.5	Vertex Isoperimetric Inequality on the Binary Hypercube	9
2.6	Convex Distance Inequality	9

1 The Classic Isoperimetry Inequalities

1.1 Brunn-Minkowski Inequality

- **Definition** (*Minkowski Sum of Sets*)

Consider sets $A, B \subseteq \mathbb{R}^n$ and define the Minkowski sum of A and B as the set of all vectors in \mathbb{R}^n formed by sums of elements of A and B :

$$A + B := \{x + y : x \in A, y \in B\}$$

Similarly, for $c \in \mathbb{R}$, let $cA = \{cx : x \in A\}$. Denote by $\text{Vol}(A)$ the **Lebesgue measure** of a (measurable) set $A \subset \mathbb{R}^n$.

- **Theorem 1.1** (*Brunn-Minkowski Inequality*) [*Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019*]

Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0, 1]$,

$$\text{Vol}(\lambda A + (1 - \lambda)B)^{\frac{1}{n}} \geq \lambda \text{Vol}(A)^{\frac{1}{n}} + (1 - \lambda) \text{Vol}(B)^{\frac{1}{n}}. \quad (1)$$

Note: a convex body in \mathbb{R}^n is closed and compact set.

Proof: (*Part 1, $n = 1$*)

Note that if $A \subset \mathbb{R}$, and $c \geq 0$ then $\text{Vol}(cA) = c\text{Vol}(A)$. Thus it suffice to prove

$$\text{Vol}(A + B) \geq \text{Vol}(A) + \text{Vol}(B).$$

To see this, observe that none of the three volumes involved changes if the sets A and B are **translated** arbitrarily. Since A, B are compact subsets in \mathbb{R} , it is closed and bounded. Let $a = \max\{a' : a' \in A\}$ and $b = \min\{b' : b' \in B\}$. Let $A' = A + \{-a\}$ and $B' = B + \{-b\}$ so that $A' \subset (-\infty, 0]$ and $B' \subset [0, +\infty)$. Also $\text{Vol}(A') = \text{Vol}(A)$ and $\text{Vol}(B') = \text{Vol}(B)$. However,

$$\begin{aligned} A' \cup B' &\subset A' + B' \\ \Rightarrow \text{Vol}(A') + \text{Vol}(B') &= \text{Vol}(A' \cup B') \leq \text{Vol}(A' + B') \end{aligned}$$

This prove the 1-dimensional case for *the Brunn-Minkowski inequality*. ■

To prove $n > 1$ case, we need the following inequalities:

- **Theorem 1.2** (*The Prékopa-Leindler Inequality*). [*Boucheron et al., 2013, Wainwright, 2019*]

Let $\lambda \in (0, 1)$, and let $f, g, h : \mathbb{R}^n \rightarrow [0, \infty)$ be **non-negative measurable functions** such that for all $x, y \in \mathbb{R}^n$,

$$h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}.$$

Then

$$\int_{\mathbb{R}^n} h(x) dx \geq \left(\int_{\mathbb{R}^n} f(x) dx \right)^\lambda \left(\int_{\mathbb{R}^n} g(x) dx \right)^{1-\lambda}. \quad (2)$$

Proof: The proof goes by induction with respect to the dimension n .

1. ($n = 1$ **case**). Consider measurable non-negative functions f, g, h satisfying the condition of the theorem. By *the monotone convergence theorem*, it suffices to prove the statement for **bounded functions** f and g . Without loss of generality, assume that $\sup_{x \in \mathbb{R}^n} f(x) = \sup_{x \in \mathbb{R}^n} g(x) = 1$. Then

$$\begin{aligned}\int_{\mathbb{R}} f(x) dx &= \int_0^1 \text{Vol} \{x : f(x) \geq t\} dt \\ \int_{\mathbb{R}} g(x) dx &= \int_0^1 \text{Vol} \{x : g(x) \geq t\} dt.\end{aligned}$$

For any fixed $t \in [0, 1]$, if $f(x) \geq t$ and $g(y) \geq t$, then by the hypothesis of the theorem, $h(\lambda x + (1 - \lambda)y) \geq t$. This implication may be re-written as

$$\lambda \{x : f(x) \geq t\} + (1 - \lambda) \{x : g(x) \geq t\} \subset \{x : h(x) \geq t\}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} h(x) dx &= \int_0^\infty \text{Vol} \{x : h(x) \geq t\} dt \\ &\geq \int_0^1 \text{Vol} \{x : h(x) \geq t\} dt \\ &\geq \int_0^1 \text{Vol} (\lambda \{x : f(x) \geq t\} + (1 - \lambda) \{x : g(x) \geq t\}) dt \\ &\quad (\text{by 1-dimensional Brunn-Minkowski inequality}) \\ &\geq \lambda \int_0^1 \text{Vol} (\{x : f(x) \geq t\}) dt + (1 - \lambda) \int_0^1 \text{Vol} (\{x : g(x) \geq t\}) dt \\ &= \lambda \int_{\mathbb{R}} f(x) dx + (1 - \lambda) \int_{\mathbb{R}} g(x) dx \\ &\geq \left(\int_{\mathbb{R}} f(x) dx \right)^\lambda \left(\int_{\mathbb{R}} g(x) dx \right)^{1-\lambda} \quad (\text{by the arithmetic-geometric mean inequality})\end{aligned}$$

2. For the induction step, assume that the theorem holds for all dimensions $1, \dots, n - 1$ and let $f, g, h : \mathbb{R}^n \rightarrow [0, \infty)$, $\lambda \in (0, 1)$ be such that they satisfy the assumption of the theorem. Now let $x, y \in \mathbb{R}^{n-1}$ and $a, b \in \mathbb{R}$. Then

$$h(\lambda(x, a) + (1 - \lambda)(y, b)) \geq f((x, a))^\lambda g((y, b))^{1-\lambda},$$

so by the inductive hypothesis

$$\int_{\mathbb{R}^{n-1}} h((x, \lambda a + (1 - \lambda)b)) dx \geq \left(\int_{\mathbb{R}^{n-1}} f((x, a)) dx \right)^\lambda \left(\int_{\mathbb{R}^{n-1}} g((x, b)) dx \right)^{1-\lambda}$$

In other words, introducing

$$\begin{aligned}F(a) &:= \int_{\mathbb{R}^{n-1}} f((x, a)) dx, \quad G(b) := \int_{\mathbb{R}^{n-1}} g((x, b)) dx \\ H((\lambda a + (1 - \lambda)b)) &:= \int_{\mathbb{R}^{n-1}} h((x, \lambda a + (1 - \lambda)b)) dx.\end{aligned}$$

We have

$$H((\lambda a + (1 - \lambda)b)) \geq (F(a))^\lambda (G(b))^{1-\lambda},$$

so by *Fubini's theorem* and the one-dimensional inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^n} h(x) dx &= \int_{\mathbb{R}} H(a) da \geq \left(\int_{\mathbb{R}} F(a) da \right)^\lambda \left(\int_{\mathbb{R}} G(a) da \right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(x) dx \right)^\lambda \left(\int_{\mathbb{R}^n} g(x) dx \right)^{1-\lambda}. \quad \blacksquare \end{aligned}$$

- **Corollary 1.3 (*Weaker Brunn-Minkowski Inequality*)** [*Boucheron et al., 2013, Wainwright, 2019*]

Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0, 1]$,

$$\text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda \text{Vol}(B)^{1-\lambda}. \quad (3)$$

Proof: We apply the *Prékopa-Leindler inequality* with $f(x) = \mathbb{1}\{x \in A\}$, $g(x) = \mathbb{1}\{x \in B\}$ and $h(x) = \mathbb{1}\{x \in \lambda A + (1 - \lambda)B\}$. We see that

$$h(\lambda x + (1 - \lambda)y) = \mathbb{1}\{\lambda x + (1 - \lambda)y \in \lambda A + (1 - \lambda)B\} \geq \mathbb{1}\{x \in A, y \in B\} = f(x)^\lambda g(y)^{1-\lambda}.$$

Thus the hypothesis of the *Prékopa-Leindler inequality* holds. \blacksquare

- **Proof: ($n > 1$ case for *Brunn-Minkowski Inequality*)**. First observe that it suffices to prove that for all *nonempty compact sets* A and B ,

$$\text{Vol}(A + B)^{\frac{1}{n}} \geq \text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}$$

since $\text{Vol}(cA)^{1/n} = c \text{Vol}(A)^{1/n}$ for any $c \in \mathbb{R}$ and $A \subset \mathbb{R}^n$. Also notice that we may assume that $\text{Vol}(A), \text{Vol}(B) > 0$ because otherwise the inequality holds trivially. Defining $A' = \text{Vol}(A)^{-\frac{1}{n}} A$ and $B' = \text{Vol}(B)^{-\frac{1}{n}} B$, we have $\text{Vol}(A') = \text{Vol}(B') = 1$. By *weaker Brunn-Minkowski inequality*, for $\lambda \in (0, 1)$,

$$\text{Vol}(\lambda A' + (1 - \lambda)B') \geq 1.$$

Finally, we apply this *inequality* with the choice

$$\lambda = \frac{\text{Vol}(A)^{\frac{1}{n}}}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}}$$

obtaining

$$\begin{aligned} &\text{Vol}\left(\frac{\text{Vol}(A)^{\frac{1}{n}} A'}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}} + \frac{\text{Vol}(B)^{\frac{1}{n}} B'}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}}\right) \geq 1 \\ \Rightarrow &\text{Vol}\left(\frac{A}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}} + \frac{B}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}}\right) \geq 1 \\ \Rightarrow &\text{Vol}\left(\frac{A + B}{\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}}\right) \geq 1 \\ \Rightarrow &\frac{\text{Vol}(A + B)}{\left(\text{Vol}(A)^{\frac{1}{n}} + \text{Vol}(B)^{\frac{1}{n}}\right)^n} \geq 1 \end{aligned}$$

which proves the theorem. \blacksquare



Figure 5.1 Isoperimetric inequality in \mathbb{R}^n states that among all sets A of given volume, the Euclidean balls minimize the volume of the ε -neighborhood A_ε .

Figure 1: Isoperimetry in \mathbb{R}^n [Vershynin, 2018]

1.2 The Classical Isoperimetry Theorem

- **Definition (*Blowup of Sets*)**

For any $t > 0$, and any (measurable) sets $A \subset \mathbb{R}^n$, the t -blowup of A is defined by

$$A_t := \{x \in \mathbb{R}^n : d(x, A) < t\} = A + tB$$

where $B = \{x \in \mathbb{R}^n : d(0, x) < 1\}$ is an *open unit ball* and $d(x, A) = \inf_{y \in A} d(x, y)$.

- **Definition (*Surface Area of Sets*)**

let $A \subset \mathbb{R}^n$ be a measurable set and denote by $\text{Vol}(A)$ its *Lebesgue measure*. The surface area of A is defined by

$$\text{Vol}(\partial A) = \lim_{t \rightarrow 0} \frac{\text{Vol}(A_t) - \text{Vol}(A)}{t}.$$

provided that the limit exists. Here A_t denotes *the t -blowup* of A .

- **Remark (*Isoperimetry Theorem*)**

The classical isoperimetric theorem in \mathbb{R}^n states that, among all sets with **a given volume**, the Euclidean unit ball minimizes the surface area. This theorem can be formally stated as below:

- **Theorem 1.4 (*Isoperimetry Theorem*)** [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]

Let $A \subset \mathbb{R}^n$ be such that $\text{Vol}(A) = \text{Vol}(B)$ where $B := \{x \in \mathbb{R}^n : d(0, x) < 1\}$ is an unit ball. Then for any $t > 0$,

$$\text{Vol}(A_t) \geq \text{Vol}(B_t) \tag{4}$$

Moreover, if $\text{Vol}(\partial A)$ exists, then

$$\text{Vol}(\partial A) \geq \text{Vol}(\partial B). \tag{5}$$

Proof: By the *Brunn-Minkowski inequality*,

$$\begin{aligned} \text{Vol}(A_t)^{1/n} &= \text{Vol}(A + tB)^{1/n} \geq \text{Vol}(A)^{1/n} + t\text{Vol}(B)^{1/n} \\ &= (1 + t)\text{Vol}(B)^{1/n} \\ &= \text{Vol}(B_t)^{1/n}, \end{aligned}$$

establishing the first statement. The second follows simply because

$$\text{Vol}(A_t) - \text{Vol}(A) \geq \text{Vol}(B)((1+t)^n - 1) \geq nt\text{Vol}(B)$$

where $(1+t)^n \geq 1+nt$ for $t \geq 0$. Thus $\text{Vol}(\partial A) \geq n\text{Vol}(B)$. The isoperimetric theorem now follows from the fact that $\text{Vol}(\partial B) = n\text{Vol}(B)$. ■

2 Concentration via Isoperimetry

2.1 Isoperimetry Problem and Levy's Inequalities

- **Remark** We can generalize the classical isoperimetry problem to a probability space $(\mathcal{X}, \mathcal{B}[\mathcal{X}], \mathbb{P})$ where \mathcal{X} is a *metric space* with metric d , $\mathcal{B}[\mathcal{X}]$ is the Borel σ -algebra and \mathbb{P} is a probability measure on $\mathcal{B}[\mathcal{X}]$. Let $B := \{x \in \mathbb{R}^n : d(0, x) < 1\}$. The classical isoperimetry problem aims at finding the set $A^* \subset \mathcal{X}$ that **minimizes the surface area**

$$\mathbb{P}(\partial A) = \lim_{t \rightarrow 0} \frac{\mathbb{P}(A_t) - \mathbb{P}(A)}{t}$$

This is equivalent to find subset A in \mathcal{X} with **minimal t -blowup** for given p , and for all $t > 0$

$$A^* := \inf_{A \subset \mathcal{X}: \mathbb{P}(A) \geq p} \mathbb{P}(A_t), \quad \forall t > 0$$

where

$$A_t = A + tB = \{x \in \mathcal{X} : \exists y \in A \text{ s.t. } d(x, y) < t\} = \left\{x \in \mathcal{X} : \inf_{y \in A} d(x, y) := d(x, A) < t\right\}.$$

We write the definition formally.

- **Definition (*Isoperimetry Problem*)** [Boucheron et al., 2013]
Given a *metric space* \mathcal{X} with corresponding *distance* d , consider **the measure space** formed by \mathcal{X} , the σ -algebra of all **Borel sets** of \mathcal{X} , and a probability measure \mathbb{P} . Let X be a *random variable* taking values in \mathcal{X} , distributed according to \mathbb{P} .

The isoperimetric problem in this case is the following: given $p \in (0, 1)$ and $t > 0$, **determine the sets** A with $\mathbb{P}[X \in A] \geq p$ for which **the measure**

$$\mathbb{P}[d(X, A) \geq t]$$

is **maximal**.

- **Remark (*Isoperimetry Inequalities*)**
Even though the exact solution is only known in a few special cases, useful *bounds* for $\mathbb{P}[d(X, A) \geq t]$ can be derived under remarkably general circumstances. *Such bounds are usually referred to as isoperimetric inequalities*.
- **Definition (*Concentration Function*)** [Boucheron et al., 2013, Wainwright, 2019]
The concentration function is defined as follows: for all $t > 0$,

$$\alpha(t) := \sup_{A \subset \mathcal{X}: \mathbb{P}(A) \geq \frac{1}{2}} \mathbb{P}[d(X, A) \geq t] = \sup_{A \subset \mathcal{X}: \mathbb{P}(A) \geq \frac{1}{2}} \mathbb{P}(A_t^c)$$

where $A_t := A + tB = \{x \in \mathcal{X} : d(x, A) < t\}$ is the t -blowup of $A \subset \mathcal{X}$.

Thus the optimal A^* for isoperimetry problem is the one that attains the $\alpha(t) = \mathbb{P}(A_t^c)$.

- **Theorem 2.1 (Levy's Inequalities)** [Boucheron et al., 2013, Wainwright, 2019]
For any Lipschitz function $f : \mathcal{X} \rightarrow \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\{f(X) \geq \text{Med}(f(X)) + t\} &\leq \alpha(t) \\ \mathbb{P}\{f(X) \leq \text{Med}(f(X)) - t\} &\leq \alpha(t).\end{aligned}\tag{6}$$

where $\text{Med}(f(X))$ is the median of $f(X)$, i.e.

$$\mathbb{P}\{f(X) \leq \text{Med}(f(X))\} \geq \frac{1}{2}, \quad \text{and} \quad \mathbb{P}\{f(X) \geq \text{Med}(f(X))\} \geq \frac{1}{2}.$$

Proof: Consider the set $A = \{x : f(x) \leq \text{Med}(f(X))\}$. By the definition of a *median*, $\mathbb{P}(A) \geq \frac{1}{2}$. On the other hand, by the Lipschitz property of f , for any $x, y \in \mathcal{X}$,

$$|f(x) - f(y)| \leq d(x, y).$$

So for all $y \in A$, $f(y) \leq \text{Med}(f(X))$

$$\begin{aligned}f(x) - \text{Med}(f(X)) &\leq f(x) - f(y) \leq d(x, y) \\ \Rightarrow f(x) - \text{Med}(f(X)) &\leq \inf_{y \in A} d(x, y) := d(x, A).\end{aligned}$$

Equivalently,

$$\begin{aligned}A_t &:= \{x \in \mathcal{X} : d(x, A) < t\} \subseteq \{x \in \mathcal{X} : f(x) < \text{Med}(f(X)) + t\} \\ \mathbb{P}(A_t^c) &\geq \mathbb{P}\{f(X) \geq \text{Med}(f(X)) + t\}\end{aligned}$$

The first inequality now follows from the definition of the concentration function. The second inequality follows from the first by considering f . ■

- **Remark** For L -Lipschitz function f , the inequality becomes

$$\mathbb{P}\{f(X) - \text{Med}(f(X)) \geq t\} \leq \alpha\left(\frac{t}{L}\right), \quad \mathbb{P}\{f(X) - \text{Med}(f(X)) \leq -t\} \leq \alpha\left(\frac{t}{L}\right).$$

- **Theorem 2.2 (Converse of Levy's Inequalities)** [Boucheron et al., 2013, Wainwright, 2019]

If $\beta : \mathbb{R}_+ \rightarrow [0, 1]$ is a function such that for **every Lipschitz function** $f : \mathcal{X} \rightarrow \mathbb{R}$

$$\mathbb{P}\{f(X) - \text{Med}(f(X)) \geq t\} \leq \beta(t).\tag{7}$$

then $\beta(t) \geq \alpha(t)$.

Proof: Note that for any $A \subset \mathcal{X}$, the function f_A defined by $f_A(x) = d(x, A)$ is Lipschitz since

$$|f_A(x) - f_A(y)| = |d(x, A) - d(y, A)| \leq d(x, y).$$

Also, if $\mathbb{P}(A) \geq 1/2$, then 0 is a median of $f_A(X)$, since

$$\mathbb{P}\{f_A(x) \leq 0\} = \mathbb{P}\{d(X, A) \leq 0\} = \mathbb{P}(A) \geq \frac{1}{2}.$$

Therefore

$$\alpha(t) := \sup_{A \subset \mathcal{X} : \mathbb{P}(A) \geq 1/2} \mathbb{P}\{f_A(X) - \text{Med}(f_A(X)) \geq t\} \leq \beta(t). \quad \blacksquare$$

- **Remark (*Isoperimetry Inequalities* \Leftrightarrow *Concentration of Lipschitz Functions*)**

The first result points out that *isoperimetric inequalities* (more precisely, **upper bounds for the concentration function**) imply concentration of Lipschitz functions.

The converse shows that *concentration of Lipschitz functions* implies an *isoperimetric inequality*. In other word, among all upper bounds of $\mathbb{P}(A_t^c)$ for fixed A_t ,

- **Corollary 2.3 (*Isoperimetry Inequality for Metric Space by Hamming Distance*)** [Boucheron et al., 2013]

Consider independent random variables Z_1, \dots, Z_n taking their values in a (measurable) set \mathcal{X} and denote the vector of these variables by $Z = (Z_1, \dots, Z_n)$ taking its value in \mathcal{X}^n . For an arbitrary (measurable) set $A \subset \mathcal{X}^n$, we write $\mathbb{P}(A) = \mathbb{P}(Z \in A)$. The **Hamming distance** $d_H(x, y)$ between the vectors $x, y \in \mathcal{X}^n$ is defined as **the number of coordinates in which x and y differ**. Then for any $t > 0$,

$$\mathbb{P} \left\{ d_H(x, A) \geq \sqrt{\frac{n}{2} \log \frac{1}{\mathbb{P}(A)}} + t \right\} \leq \exp \left(-\frac{2t^2}{n} \right) \quad (8)$$

Proof:

- **Example (*Bounded Difference Inequality as Isoperimetry Inequality*)**

Note that function with **bounded difference property** is **Lipschitz function** with respect to **Hamming distance**:

$$\begin{aligned} & \sup_{x \in \mathcal{X}^n, y_i \in \mathcal{X}} |f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| \\ & \leq 1 = d_H((x_1, \dots, x_n), (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)), \quad 1 \leq i \leq n \\ \Rightarrow |f(x) - f(y)| &= \left| \sum_{i=1}^n (f(x_{(i-1)}) - f(x_{(i)})) \right| \\ & \leq \sum_{i=1}^n |f(x_{(i-1)}) - f(x_{(i)})| \\ & \leq \sum_{i=1}^n \mathbb{1} \{x_{(i-1)}[i] \neq x_{(i)}[i]\} \\ & = d_H(x, y) \end{aligned}$$

where $x_{(i)}$ is replicate of $x_{(i-1)}$ except for i -th component, which is replaced by y_i . Note that $x_{(0)} = x$ and $x_{(n)} = y$. Therefore, the *bounded difference inequality* can be seen as an *isoperimetry inequality* for Lipschitz function with respect to Hamming distance.

$$\mathbb{P} \{f(Z) - \mathbb{E}[f(Z)] \geq t\} \leq \exp \left(-\frac{2t^2}{n} \right)$$

2.2 Isoperimetric Inequalities on the Unit Sphere

- **Remark (*Volume Ratio of Unit Balls and its Interior*)** [Vershynin, 2018]

Let $B(0, 1) := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the unit ball in \mathbb{R}^n . The volume ratio between $B(0, 1)$

and its ϵ -interior $B(0, 1 - \epsilon)$ is

$$\frac{\text{Vol}(B(0, 1 - \epsilon))}{\text{Vol}(B(0, 1))} = (1 - \epsilon)^n \leq \exp(-n\epsilon)$$

The inequality is due to $1 - x \leq e^{-x}$.

As $n \rightarrow \infty$, the above ratio goes to 0. In other words, most of volume in $B(0, 1)$ is ***concentrated*** in the ***boundary*** $\partial B = \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. This phenomenon is called “***the curse of dimensionality***”.

2.3 Gaussian Isoperimetric Inequalities and Concentration of Gaussian Measure

2.4 Edge Isoperimetric Inequality on the Binary Hypercube

2.5 Vertex Isoperimetric Inequality on the Binary Hypercube

2.6 Convex Distance Inequality

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