Lecture 9: Integral Curves and Flows

Tianpei Xie

Oct. 16th., 2022

Contents

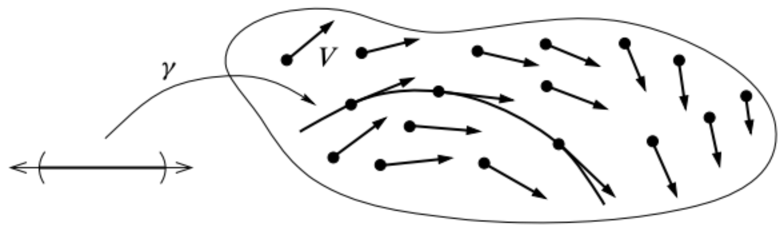


Figure 1: An integral curve of a vector field [Lee, 2003.]

1 Integral Curves

- Suppose M is a smooth manifold with or without boundary. If $\gamma : J \to M$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}M$.

- **Definition** Suppose M is a smooth manifold with or without boundary and V is a *vector field* on M. An ***integral curve*** of V is a *differentiable curve* $\gamma : J \to M$ whose ***velocity*** at each point is equal to the ***value of*** V at that point:

$$\gamma'(t) = V_{\gamma(t)}, \quad \forall t \in J.$$

  (See Fig. 1) If $0 \in J$, the point $\gamma(0)$ is called ***the starting point of*** γ.

- **Example** (**Integral Curves**)

  1. Let (x, y) be *standard coordinates* on $\mathbb{R}^2$, and let $V = \frac{\partial}{\partial x}$ be the ***first coordinate vector field***. It is easy to check that the integral curves of V are precisely ***the straight lines*** parallel to the x-axis, with parametrizations of the form $\gamma(t) = (a + t, b)$ for constants a and b. (Fig, 2 (a))

  2. Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ (Fig. 2(b)). If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (x(t), y(t))$, then the condition $\gamma'(t) = W_{\gamma(t)}$ for γ to be an integral curve translates to

$$x'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y'(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} = x(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} - y(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)}.$$

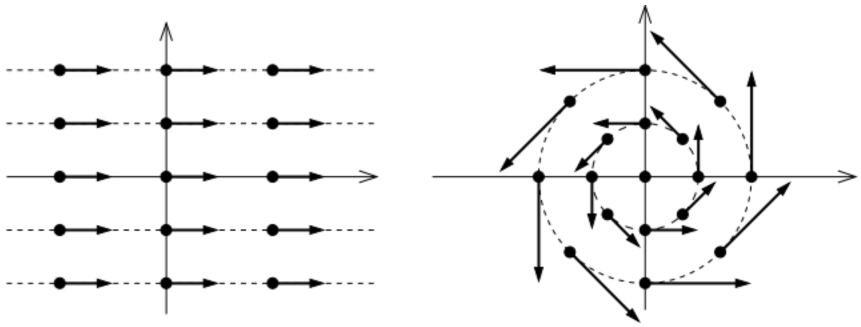
     Comparing the components of these vectors, we see that this is equivalent to the system of ordinary differential equations

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t). \end{aligned}$$

     These equations have the solutions:

$$\begin{aligned} x(t) &= a\cos(t) - b\sin(t) \\ y(t) &= a\sin(t) + b\cos(t) \end{aligned}$$

     for arbitrary constants a and b, and thus each curve of the form $\gamma(t) = (a\cos(t) - b\sin(t), a\sin(t) + b\cos(t))$ is an integral curve of W.



(a) $V = \partial/\partial x$. (b) $W = x\,\partial/\partial y - y\,\partial/\partial x$.

Figure 2: Vector fields and their integral curves [Lee, 2003.]

- **Remark** Finding integral curves boils down to solving *a system of ordinary differential equations* in a smooth chart. Suppose $\gamma : J \to M$ is a smooth curve and V is a smooth vector field on M. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for to be an integral curve of V can be written

$$\dot{\gamma}^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}, \tag{1}$$

which reduces to the following ***autonomous system of ordinary differential equations (ODEs)***:

$$\dot{\gamma}^i(t) = V^i(\gamma^1(t), \ldots, \gamma^n(t)), \qquad i = 1, \ldots, n. \tag{2}$$

- The fundamental fact about such systems is **the *existence*, *uniqueness*, and *smoothness theorem*** from ODE theory [Amann, 2011, Hirsch et al., 2012].

  Proposition 1.1 *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of V starting at p.*

- The next two lemmas show how affine reparametrizations affect integral curves.

  Lemma 1.2 ***(Rescaling Lemma)**. [Lee, 2003.]*
  Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma} : \widetilde{J} \to M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t : at \in J\}$.

- **Lemma 1.3** ***(Translation Lemma)**. [Lee, 2003.]*
  Let V, M, J, and γ be as in the preceding lemma. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma} : \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t + b)$ is also an integral curve of V, where $\widehat{J} = \{t : t + b \in J\}$.

- **Proposition 1.4** ***(Naturality of Integral Curves)**. [Lee, 2003.]*
  *Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related **if and only if** F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

  Proof: Suppose first that X and Y are F-related, and $\gamma : J \to M$ is an integral curve of X.

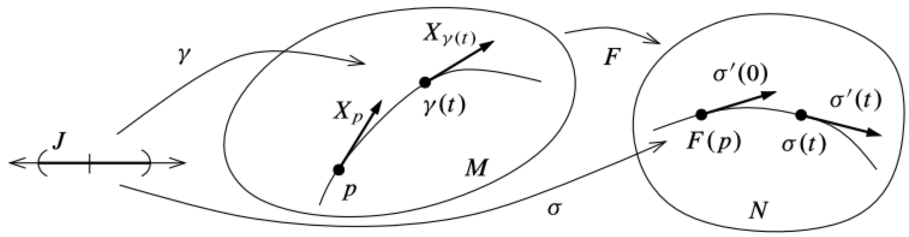


Figure 3: Flows for F-related vector fields [Lee, 2003.]

If we define $\sigma : J \to N$ by $\sigma = F \circ \gamma$ (see Fig. 3), then

$$\sigma'(t) = (F \circ \gamma)'(t) = dF_{\gamma(t)}(\gamma'(t)) = dF_{\gamma(t)}(X_{\gamma(t)}) = Y_{F(\gamma(t))} = Y_{\sigma(t)},$$

so σ is an integral curve of Y.

Conversely, suppose F takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma : (-\epsilon, \epsilon) \to M$ be an integral curve of X starting at p. Since $F \circ \gamma$ is an integral curve of Y starting at $F(p)$, we have

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p(\gamma'(0)) = dF_p(X_p)$$

which shows that X and Y are F-related. ∎

2 Flows

2.1 Global Flows

- **Definition** A ***global flow on** M* (also called ***a one-parameter group action***) is defined as a ***continuous left** $\mathbb{R}$**-action on*** M; that is, a *continuous map* $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \theta(0, p) = p \tag{3}$$

- For a global flow θ on M, we define two collections of maps as follows:
  - **Definition** For each $t \in \mathbb{R}$, ***define*** a continuous map $\theta_t : M \to M$ by

$$\theta_t(p) = \theta(t, p).$$

    The defining properties in (3) are equivalent to ***the group laws***:

$$\theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{Id}_M \tag{4}$$

  - **Definition** For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by

$$\theta^{(p)}(t) = \theta(t, p).$$

    The image of this curve is ***the orbit of** p **under the group action***.

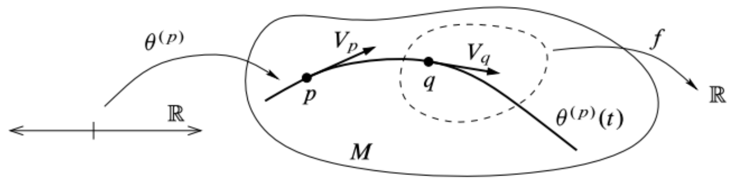


Figure 4: The infinitesimal generator of the globla flow [Lee, 2003.]

- **Definition** If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a *tangent vector* $V_p \in T_pM$ by

$$V_p = (\theta^{(p)})'(0) = d\theta^{(p)} \left(\frac{d}{dt}\bigg|_{t=0} \right).$$

The assignment $p \mapsto V_p$ is ***a (rough) vector field*** on M; which is called ***the infinitesimal generator of the global flow*** θ.

Remark V is the *infinitesimal generator* of the flow $\theta \Leftrightarrow \theta$ is the *integral curve* of V.

- **Proposition 2.1** *Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M; and each curve $\theta^{(p)}$ is* ***an integral curve*** *of V.*

This means that $(\theta^{(p)})'(t) = V_{\theta^{(p)}(t)}$ for all $p \in M$ and all $t \in \mathbb{R}$.

2.2 The Fundamental Theorem on Flows

- We have seen that ***every smooth global flow*** gives rise to ***a smooth vector field*** whose ***integral curves*** are precisely *the curves defined by the flow.*

Conversely, however, it is **not true** that every smooth vector field is the infinitesimal generator of a smooth global flow.

- **Definition** If M is a manifold, a ***flow domain*** for M is an open subset $\mathfrak{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathfrak{D}^{(p)} = \{t \in \mathbb{R} : (t,p) \in \mathfrak{D}\}$ is an *open interval* ***containing*** 0 (Fig. 5).

A ***flow*** on M is a continuous map $\theta : \mathfrak{D} \to M$; where $\mathfrak{D} \subseteq \mathbb{R} \times M$ is *a flow domain*, that satisfies the following ***group laws***:

$$\theta(0,p) = p, \quad \forall p \in M \tag{5}$$

$$\theta(t, \theta(s,p)) = \theta(t+s,p), \quad \forall s \in \mathfrak{D}^{(p)},\ t \in \mathfrak{D}^{(\theta(s,p))},\ (\text{i.e. } t+s \in \mathfrak{D}^{(p)}) \tag{6}$$

We sometimes call θ ***a local flow*** to distinguish it from *a global flow* as defined earlier. The unwieldy term ***local one-parameter group action*** is also used.

- **Definition** If θ is a flow, we define $\theta_t(p) = \theta^{(p)}(t) = \theta(t,p)$ whenever $(t,p) \in \mathfrak{D}$, just as for a global flow. For each $t \in \mathbb{R}$, we also define

$$M_t = \{p \in M : (t,p) \in \mathfrak{D}\} \tag{7}$$

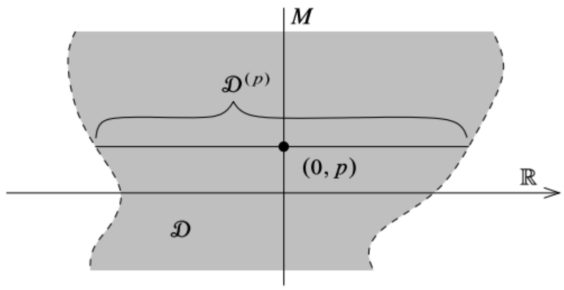


Figure 5: The flow domain [Lee, 2003.]

so that

$$p \in M_t \Leftrightarrow t \in \mathfrak{D}^{(p)} \Leftrightarrow (t,p) \in \mathfrak{D}.$$

If θ is smooth, ***the infinitesimal generator*** of θ is defined by $V_p = (\theta^{(p)})'(0)$.

- **Proposition 2.2** *If $\theta : \mathfrak{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

- **Definition** *A* ***maximal integral curve*** is one that cannot be extended to an integral curve on *any larger open interval*, and *a* ***maximal flow*** is a flow that admits *no extension* to a flow on a *larger flow domain.*

- **Theorem 2.3** *(**Fundamental Theorem on Flows**). [Lee, 2003.]*
  Let V be a smooth vector field on a smooth manifold M. There is a ***unique smooth maximal flow*** *$\theta : \mathfrak{D} \to M$ whose* ***infinitesimal generator*** *is V. This flow has the following properties:*

  1. *For each $p \in M$, the curve $\theta^{(p)} : \mathfrak{D}^{(p)} \to M$ is the* ***unique maximal integral curve*** *of V starting at p.*
  2. *If $s \in \mathfrak{D}^{(p)}$, then $\mathfrak{D}^{(\theta(s,p))}$ is the interval $\mathfrak{D}^{(p)} - s = \{t - s : t \in \mathfrak{D}^{(p)}\}$.*
  3. *For each $t \in \mathbb{R}$, the set M_t is* ***open*** *in M; and $\theta_t : M_t \to M_{-t}$ is a* ***diffeomorphism*** *with* ***inverse*** *θ_{-t}.*

- **Remark** The flow whose ***existence*** and ***uniqueness*** are asserted in the fundamental theorem is called ***the flow generated by*** V , or just <u>***the flow of*** V</u>.

- **Proposition 2.4** *(**Naturality of Flows**). [Lee, 2003.]*
  Suppose M and N are smooth manifolds, $F : M \to N$ is a smooth map, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ {\scriptstyle \theta_t}\downarrow & & \downarrow{\scriptstyle \eta_t} \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

- **Corollary 2.5** *(**Diffeomorphism Invariance of Flows**).*
  Let $F : M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the ***flow of pushforward*** *F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.*

2.3 Complete Vector Fields

- As we observed earlier in this chapter, not every smooth vector field generates a *global flow*. The ones that do are important enough to deserve a name.

  Definition We say that *a smooth vector field* is ***complete*** if it generates a ***global flow***, or equivalently if *each* of its *maximal integral curves* is defined for ***all*** $t \in \mathbb{R}$.

- We will show below that *all compactly supported smooth vector fields, and therefore all smooth vector fields on a compact manifold, are complete.* The proof will be based on the following lemma.

  Lemma 2.6 *(**Uniform Time Lemma**).*
  *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is **a positive number** ϵ such that for **every** $p \in M$, the domain of $\theta^{(p)}$ contains $(-\epsilon, \epsilon)$. Then V is complete.*

- **Theorem 2.7** *Every **compactly supported** smooth vector field on a smooth manifold is **complete**.*

- **Corollary 2.8** *On a **compact smooth manifold**, **every** smooth vector field is **complete**.*

- *Left-invariant vector fields on Lie groups* form another class of vector fields that are always complete.

  Theorem 2.9 *Every **left-invariant** vector field on a **Lie group** is complete.*

- Here is another useful property of integral curves.

  Lemma 2.10 *(**Escape Lemma**).*
  *Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in **any compact subset** of M.*

3 Flowouts

-

4 Flows and Flowouts on Manifolds with Boundary

5 Lie Derivatives

- The directional derivatives of a smooth function on M is obtained via vf where v is a tangent vector operator $v \in T_pM$. What about the directional derivative of a vector field?

- **Remark** In *Euclidean space*, it makes sense to ask this question. We can define *the directional derivatives of a vector field W at point p* as below:
$$D_v W(p) := \lim_{t \to 0} \frac{W_{p+t\,v} - W_p}{t} = \frac{d}{dt}\Big|_{t=0} W_{p+t\,v}.$$

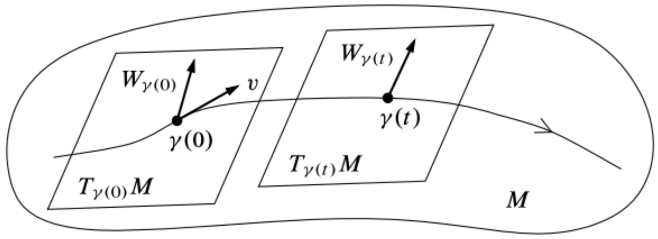


Figure 6: The directional derivative of vector fields [Lee, 2003.]

An easy calculation shows that $D_v W(p)$ can be evaluated by applying D_v to each component of W separately (See Fig 6.):

$$D_v W(p) = D_v W^i(p) \left.\frac{\partial}{\partial x^i}\right|_p.$$

Unfortunately, this definition is heavily dependent upon the fact that $\mathbb{R}^n$ is a ***vector space***, so that *the tangent vectors W_{p+tv} and W_p can* ***both*** *be viewed as elements* of $\mathbb{R}^n$.

- **Remark** For a manifold M, the vector field W_{p+tv} may not be well-defined since we do not know if $p + tv \in M$. Therefore, we replace $p + tv$ by the curve $\gamma(t) = \theta(p, t)$ where $\gamma(0) = p$ and $\gamma'(0) = v$. On the other hand, the vector field $W_{\gamma(0)}$ and $W_{\gamma(t)}$ are *not in the same tangent space* (one in $T_{\gamma(0)}M$ and the other $T_{\gamma(t)}M$). We got away with it in Euclidean space because there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself; this is not true for general smooth manifold M.

  This problem can be circumvented if we replace the vector $v \in T_pM$ with a ***vector field*** $V \in \mathfrak{X}(M)$, so we can use the ***flow*** of V to ***push values of*** W ***back to*** p and then differentiate.

- **Definition** Suppose M is a smooth manifold, V is a *smooth vector field* on M; and θ is the ***flow of*** V. For any smooth vector field W on M, define ***a rough vector field*** on M, *denoted by* $\mathscr{L}_V W$ and called the ***Lie derivative*** *of W with respect to V*, by

$$\begin{aligned}(\mathscr{L}_V W)_p &= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) - W_p}{t} \\ &= \left.\frac{d}{dt}\right|_{t=0} d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right),\end{aligned} \tag{8}$$

  provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighborhood of p, and θ_{-t} is the inverse of θ_t, so both $d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$ and W_p are elements of T_pM (Fig 7).

- **Remark** If M has nonempty boundary, this definition of $\mathscr{L}_V W$ makes sense as long as V is tangent to ∂M so that its flow exists.

- **Lemma 5.1** *Suppose M is a smooth manifold with or without boundary, and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume in addition that V is tangent to ∂M. Then $(\mathscr{L}_V W)_p$ exists for every $p \in M$, and $\mathscr{L}_V W$ is a* ***smooth vector field****.*

  Proof: Let θ be the flow of V. For arbitrary $p \in M$, let $(U, (x^i))$ be a smooth chart containing p. Choose an open interval J_0 containing 0 and an open subset $U_0 \subseteq U$ containing p such

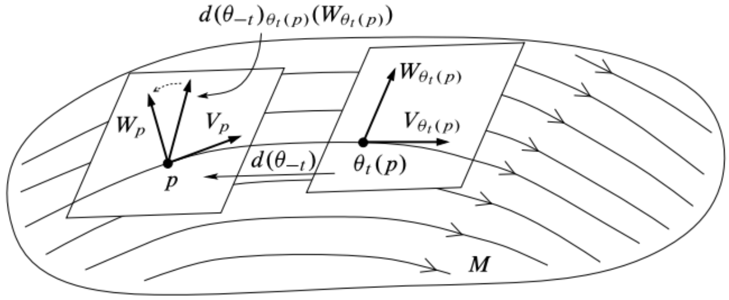


Figure 7: The Lie derivative of vector fields [Lee, 2003.]

that θ maps $J_0 \times U_0$ into U. For $(t, x) \in J_0 \times U_0$, write the component functions of θ as $(\theta^1(t, x), \ldots, \theta^n(t, x))$. Then for any $(t, x) \in J_0 \times U_0$, the matrix of $d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)}M \to T_xM$ is

$$\left(\frac{\partial \theta^i}{\partial x^j}(-t, \theta(t, x))\right).$$

Therefore,

$$d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t, x)) W^j(\theta(t, x)) \left.\frac{\partial}{\partial x^i}\right|_x.$$

Because θ^i and W^j are smooth functions, the coefficient of $\partial/\partial x^i|_x$ depends smoothly on (t, x). It follows that $(\mathscr{L}_V W)_x$, which is obtained by taking the derivative of this expression with respect to t and setting $t = 0$, exists for each $x \in U_0$ and depends smoothly on x. ∎

- The following theorem is critical to understand the ***Lie derivatives*** and ***Lie bracket***.

Theorem 5.2 *If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathscr{L}_V W = [V, W]$.*

Proof: Suppose $V, W \in \mathfrak{X}(M)$, and let $\mathcal{R}(V) \subseteq M$ be the set of *regular points* of V (the set of points $p \in M$ such that $V_p \neq 0$). Note that $\mathcal{R}(V)$ is *open* in M by continuity, and its *closure* is the support of V. We will show that $(\mathscr{L}_V W)_p = [V, W]_p$ for all $p \in M$, by considering three cases.

- $p \in \mathcal{R}(V)$. In this case, we can choose smooth coordinates (u^i) on a neighborhood of p in which V has the coordinate representation $V = \partial/\partial u^1$ (Theorem 9.22). In these coordinates, the *flow* of V is $\theta_t(u) = (u^1 + t, u^2, \ldots, u^n)$. For each fixed t, the matrix of $d(\theta_{-t})_{\theta_t(x)}$ in these coordinates (*the Jacobian matrix* of θ_{-t}) is the *identity at every point.* Consequently, for any $u \in U$,

$$\begin{aligned} d\left(\theta_{-t}\right)_{\theta_t(u)}\left(W_{\theta_t(u)}\right) &= d\left(\theta_{-t}\right)_{\theta_t(u)}\left(W^j(u^1 + t, u^2, \ldots, u^n) \left.\frac{\partial}{\partial u^j}\right|_{\theta_t(u)}\right) \\ &= W^j(u^1 + t, u^2, \ldots, u^n) \left.\frac{\partial}{\partial u^j}\right|_u. \end{aligned}$$

Using the definition of the Lie derivative, we obtain

$$(\mathscr{L}_V W)_u = \left.\frac{d}{dt}\right|_{t=0} W^j(u^1 + t, u^2, \ldots, u^n) \left.\frac{\partial}{\partial u^j}\right|_u = \frac{\partial W^j}{\partial u^1}(u^1, u^2, \ldots, u^n) \left.\frac{\partial}{\partial u^j}\right|_u.$$

On the other hand, by virtue of formula (note that $V_i = 0$ for all $i \neq 1$ and $V_1 = 1$.)

$$\begin{aligned}[V, W] &= \left(V^i \frac{\partial W^j}{\partial u^i} - W^i \frac{\partial V^j}{\partial u^i}\right) \frac{\partial}{\partial u^j} \\ &= \left(\frac{\partial W^j}{\partial u^1}\right) \frac{\partial}{\partial u^j},\end{aligned} \tag{9}$$

for the Lie bracket in coordinates, $[V, W]_u$ is easily seen to be equal to the same expression.

  – $p \in \text{supp}(V)$. Because supp$V$ is the closure of $\mathcal{R}(V)$, it follows by continuity from Case 1 that $(\mathscr{L}_V W)_p = [V, W]_p$ for $p \in \text{supp}(V)$.

  – $p \in M \setminus \text{supp}(V)$. In this case, $V \equiv 0$ on a neighborhood of p. On the one hand, this implies that θ_t is equal to the identity map in a neighborhood of p for all t, so $d\,(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = W_p$, which implies $(\mathscr{L}_V W)_p = 0$. On the other hand, $[V, W]_p = 0$ by formula (9). ∎

- **Remark** This theorem allows us to extend the definition of the ***Lie derivative*** to arbitrary *smooth vector fields* on a smooth manifold M with boundary. Given $V, W \in \mathfrak{X}(M)$ we define $(\mathscr{L}_V W)_p$ for $p \in \partial M$ by embedding M in a smooth manifold $\widetilde{M}$ without boundary (such as the double of M), extending V and W to smooth vector fields on $\widetilde{M}$, and computing the Lie derivative there. By virtue of the preceding theorem, $(\mathscr{L}_V W)_p = [V, W]_p$ is independent of the choice of extension.

- **Remark** This thoerem also gives us a ***geometric interpretation*** of *the Lie bracket of two vector fields*: it is the ***directional derivative*** *of the second vector field* along *the* ***flow*** *of the first*.

- **Corollary 5.3** *Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.*

  1. *(**Anti-symmetric**) $\mathscr{L}_V W = -\mathscr{L}_W V$.*
  2. *$\mathscr{L}_V [W, X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$.*
  3. *(**Lie Bracket definition**) $\mathscr{L}_{[V,W]} X = \mathscr{L}_V \mathscr{L}_W X - \mathscr{L}_W \mathscr{L}_V X$.*
  4. *If $g \in \mathcal{C}^\infty(M)$, then $\mathscr{L}_V (gW) = (Vg)\, W + g\, \mathscr{L}_V W$.*
  5. *(**Pushforward**) If $F : M \to N$ is a **diffeomorphism**, then $F_*(\mathscr{L}_V X) = \mathscr{L}_{F_* V} F_* X$.*

- **Remark** Note that the Lie derivative is ***not linear over $\mathcal{C}^\infty(M)$ in V***, i.e.

$$\mathscr{L}_{fV} W \neq f\, \mathscr{L}_V W$$

- **Remark** If V and W are vector fields on M and θ is the flow of V, the Lie derivative $(\mathscr{L}_V W)_p$, by definition, expresses the t-derivative of *the* ***time-dependent vector*** $d\,(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) \in T_pM$ at $t = 0$. The next proposition shows how it can also be used to compute the derivative of this expression at other times.

- **Proposition 5.4** *Suppose M is a smooth manifold with or without boundary and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any*

(t_0, p) *in the domain of* θ,

$$\left.\frac{d}{dt}\right|_{t=t_0} d\left(\theta_{-t}\right)_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = d(\theta_{-t_0})\left((\mathscr{L}_V W)_{\theta_{t_0}(p)}\right). \tag{10}$$

6 Commuting Vector Fields

6.1 Commuting Vector Fields

- **Definition** Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. We say that V and W ***commute*** if $VW\,f = WV\,f$ for every smooth function f, or equivalently if $[V, W] \equiv 0$.

- **Definition** If $\theta : \mathfrak{D} \to M$ is a ***smooth flow***, a vector field W is said to be ***invariant under*** θ if W is θ_t***-related to itself*** for each t; more precisely, this means that $W|_{M_t}$ is θ_t -related to $W|_{M_{-t}}$ for each t, or equivalently that

$$d(\theta_t)_p(W_p) = W_{\theta_t(p)}, \quad \forall\,(t, p) \in \mathfrak{D}$$

- **Theorem 6.1** *For smooth vector fields V and W on a smooth manifold M, the following are equivalent:*

  1. *V and W **commute**.*
  2. *W is **invariant** under the **flow** of V.*
  3. *V is **invariant** under the **flow** of W.*

- **Corollary 6.2** *Every smooth vector field is invariant under its own flow.*

  Note that $[V, V] \equiv 0$.

- **Definition** If θ and ψ are flows on M, we say that θ and ψ ***commute*** if the following condition holds for every $p \in M$: whenever J and K are open intervals containing 0 such that *one of the expressions* $\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined for ***all*** $(s, t) \in J \times K$, ***both are defined*** and ***they are equal***.

  For global flows, this is the same as saying that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all s and t.

- **Theorem 6.3** ***Smooth vector fields commute*** *if and only if* ***their flows commute***.

6.2 Commuting Frames

7 Time-Dependent Vector Fields

References

Herbert Amann. *Ordinary differential equations: an introduction to nonlinear analysis*, volume 13. Walter de gruyter, 2011.

Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012.

John Marshall Lee. *Introduction to smooth manifolds*. Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.