Lecture 5: Concentration of Measure and Isoperimetry

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Contents

1 The Classic Isoperimetry Inequalities

1.1 Brunn-Minkowski Inequality

- **Definition** (***Minkowski Sum of Sets***)
  Consider sets $A, B \subseteq \mathbb{R}^n$ and define ***the Minkowski sum*** of A and B as the set of all vectors in $\mathbb{R}^n$ formed by sums of elements of A and B:

  $$A + B := \{x + y : x \in A, y \in B\}$$

  Similarly, for $c \in \mathbb{R}$, let $cA = \{cx : x \in A\}$. Denote by $\mathrm{Vol}(A)$ the ***Lebesgue measure*** of a *(measurable) set* $A \subset \mathbb{R}^n$.

- **Theorem 1.1** *(**Brunn-Minkowski Inequality**) [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  *Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0, 1]$,*

  $$Vol\,(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda\, Vol(A)^{\frac{1}{n}} + (1-\lambda)\, Vol(B)^{\frac{1}{n}}. \tag{1}$$

  Note: a convex body in $\mathbb{R}^n$ is closed and compact set.

  Proof: (***Part 1,*** $n = 1$)
  Note that if $A \subset \mathbb{R}$, and $c \geq 0$ then $\mathrm{Vol}(cA) = c\mathrm{Vol}(A)$. Thus it suffice to prove

  $$\mathrm{Vol}\,(A + B) \geq \mathrm{Vol}(A) + \mathrm{Vol}(B).$$

  To see this, observe that none of the three volumes involved changes if the sets A and B are ***translated*** arbitrarily. Since A, B are compact subsets in $\mathbb{R}$, it is closed and bounded. Let $a = \max\{a' : a' \in A\}$ and $b = \min\{b' : b' \in B\}$. Let $A' = A + \{-a\}$ and $B' = B + \{-b\}$ so that $A' \subset (-\infty, 0]$ and $B' \subset [0, +\infty)$. Also $\mathrm{Vol}(A') = \mathrm{Vol}(A)$ and $\mathrm{Vol}(B') = \mathrm{Vol}(B)$. However,

  $$\begin{aligned} A' \cup B' &\subset A' + B' \\ \Rightarrow \mathrm{Vol}(A') + \mathrm{Vol}(B') = \mathrm{Vol}(A' \cup B') &\leq \mathrm{Vol}(A' + B') \end{aligned}$$

  This prove the 1-dimensional case for *the Brunn-Minkowski inequality.* ∎

  To prove $n > 1$ case, we need the following inequalities:

- **Theorem 1.2** *(**The Prékopa-Leindler Inequality**). [Boucheron et al., 2013, Wainwright, 2019]*
  *Let $\lambda \in (0, 1)$, and let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be **non-negative measurable functions** such that for all $x, y \in \mathbb{R}^n$,*

  $$h\,(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} g(y)^{1-\lambda}.$$

  Then

  $$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-\lambda}. \tag{2}$$

  Proof: The proof goes by induction with respect to the dimension n.

1. ($n=1$ ***case***). Consider measurable non-negative functions f, g, h satisfying the condition of the theorem. By *the monotone convergence theorem*, it suffices to prove the statement for ***bounded functions*** f and g. Without loss of generality, assume that $\sup_{x\in\mathbb{R}^n} f(x) = \sup_{x\in\mathbb{R}^n} g(x) = 1$. Then

$$\int_{\mathbb{R}} f(x)dx = \int_0^1 \mathrm{Vol}\,\{x : f(x) \geq t\}\,dt$$
$$\int_{\mathbb{R}} g(x)dx = \int_0^1 \mathrm{Vol}\,\{x : g(x) \geq t\}\,dt.$$

For any fixed $t \in [0,1]$, if $f(x) \geq t$ and $g(y) \geq t$, then by the hypothesis of the theorem, $h\left(\lambda x + (1-\lambda)y\right) \geq t$. This implication may be re-written as

$$\lambda\,\{x : f(x) \geq t\} + (1-\lambda)\,\{x : g(x) \geq t\} \subset \{x : h(x) \geq t\}\,.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}} h(x)dx &= \int_0^\infty \mathrm{Vol}\,\{x : h(x) \geq t\}\,dt \\
&\geq \int_0^1 \mathrm{Vol}\,\{x : h(x) \geq t\}\,dt \\
&\geq \int_0^1 \mathrm{Vol}\left(\lambda\,\{x : f(x) \geq t\}\right) + \mathrm{Vol}\left((1-\lambda)\,\{x : g(x) \geq t\}\right)dt \\
&\quad (\text{ by 1-dimensional } \textit{Brunn-Minkowski inequality}) \\
&\geq \lambda \int_0^1 \mathrm{Vol}\left(\{x : f(x) \geq t\}\right)dt + (1-\lambda)\int_0^1 \mathrm{Vol}\left(\{x : g(x) \geq t\}\right)dt \\
&= \lambda \int_{\mathbb{R}} f(x)dx + (1-\lambda)\int_{\mathbb{R}} g(x)dx \\
&\geq \left(\int_{\mathbb{R}} f(x)dx\right)^{\lambda}\left(\int_{\mathbb{R}} g(x)dx\right)^{1-\lambda} \quad (\text{by the } \textit{arithmetic-geometric mean inequality})
\end{aligned}$$

2. For the induction step, assume that the theorem holds for all dimensions $1, \ldots, n-1$ and let $f, g, h : \mathbb{R}^n \to [0,\infty)$, $\lambda \in (0,1)$ be such that they satisfy the assumption of the theorem. Now let $x, y \in \mathbb{R}^{n-1}$ and $a, b \in \mathbb{R}$. Then

$$h\left(\lambda\,(x,a) + (1-\lambda)\,(y,b)\right) \geq f\left((x,a)\right)^{\lambda} g((y,b))^{1-\lambda},$$

so by the inductive hypothesis

$$\int_{\mathbb{R}^{n-1}} h\left((x, \lambda a + (1-\lambda)b)\right)dx \geq \left(\int_{\mathbb{R}^{n-1}} f\left((x,a)\right)dx\right)^{\lambda}\left(\int_{\mathbb{R}^{n-1}} g((x,b))dx\right)^{1-\lambda}$$

In other words, introducing

$$F(a) := \int_{\mathbb{R}^{n-1}} f\left((x,a)\right)dx, \quad G(b) := \int_{\mathbb{R}^{n-1}} g((x,b))dx$$
$$H((\lambda a + (1-\lambda)b)) := \int_{\mathbb{R}^{n-1}} h\left((x, \lambda a + (1-\lambda)b)\right)dx.$$

We have

$$H((\lambda a+(1-\lambda)b)) \geq (F(a))^{\lambda}(G(b))^{1-\lambda},$$

so by *Fubini's theorem* and the one-dimensional inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)dx = \int_{\mathbb{R}} H(a)da &\geq \left(\int_{\mathbb{R}} F(a)da\right)^{\lambda}\left(\int_{\mathbb{R}} G(a)da\right)^{1-\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(x)dx\right)^{\lambda}\left(\int_{\mathbb{R}^n} g(x)dx\right)^{1-\lambda}. \quad \blacksquare\end{aligned}$$

- **Corollary 1.3** *(**Weaker Brunn-Minkowski Inequality**) [Boucheron et al., 2013, Wainwright, 2019]*
  *Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0,1]$,*

$$Vol(\lambda A+(1-\lambda)B) \geq Vol(A)^{\lambda}\, Vol(B)^{1-\lambda}. \tag{3}$$

  Proof: We apply *the Prékopa-Leindler inequality* with $f(x) = \mathbb{1}\{x \in A\}$, $g(x) = \mathbb{1}\{x \in B\}$ and $h(x) = \mathbb{1}\{x \in \lambda A+(1-\lambda)B\}$. We see that

$$h(\lambda x+(1-\lambda)y) = \mathbb{1}\{\lambda x+(1-\lambda)y \in \lambda A+(1-\lambda)B\} \geq \mathbb{1}\{x \in A, y \in B\} = f(x)^{\lambda}g(y)^{1-\lambda}.$$

  Thus the hypothesis of *the Prékopa-Leindler inequality* holds. $\blacksquare$

- **Proof:** ($n > 1$ ***case for Brunn-Minkowski Inequality***). First observe that it suffices to prove that for all *nonempty compact sets* A and B,

$$\mathrm{Vol}(A+B)^{\frac{1}{n}} \geq \mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}$$

  since $\mathrm{Vol}(cA)^{1/n} = c\mathrm{Vol}(A)^{1/n}$ for any $c \in \mathbb{R}$ and $A \subset \mathbb{R}^n$. Also notice that we may assume that $\mathrm{Vol}(A), \mathrm{Vol}(B) > 0$ because otherwise the inequality holds trivially. Defining $A' = \mathrm{Vol}(A)^{-\frac{1}{n}}A$ and $B' = \mathrm{Vol}(B)^{-\frac{1}{n}}B$, we have $\mathrm{Vol}(A') = \mathrm{Vol}(B') = 1$. By *weaker Brunn-Minkowski inequality*, for $\lambda \in (0,1)$,

$$\mathrm{Vol}\left(\lambda A' + (1-\lambda)B'\right) \geq 1.$$

  Finally, we apply this *inequality* with the choice

$$\lambda = \frac{\mathrm{Vol}(A)^{\frac{1}{n}}}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}}$$

  obtaining

$$\begin{aligned}\mathrm{Vol}\left(\frac{\mathrm{Vol}(A)^{\frac{1}{n}}A'}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}} + \frac{\mathrm{Vol}(B)^{\frac{1}{n}}B'}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}}\right) &\geq 1 \\ \Rightarrow \mathrm{Vol}\left(\frac{A}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}} + \frac{B}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}}\right) &\geq 1 \\ \Rightarrow \mathrm{Vol}\left(\frac{A+B}{\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}}\right) &\geq 1 \\ \Rightarrow \frac{\mathrm{Vol}(A+B)}{\left(\mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}\right)^{n}} &\geq 1\end{aligned}$$

  which proves the theorem. $\blacksquare$

Figure 5.1 Isoperimetric inequality in $\mathbb{R}^n$ states that among all sets A of given volume, the Euclidean balls minimize the volume of the ε-neighborhood A_ε.

Figure 1: Isoperimetry in $\mathbb{R}^n$ [Vershynin, 2018]

1.2 The Blowup of Sets and Classical Isoperimetry Theorem

- **Definition** (***Blowup of Sets***)
  For any $t > 0$, and any (measurable) sets $A \subset \mathbb{R}^n$, ***the t-blowup (or, t-enlargement)*** *of* A is defined by

  $$A_t := \{x \in \mathbb{R}^n : d(x, A) < t\} = A + t\,B$$

  where $B = \{x \in \mathbb{R}^n : d(0, x) < 1\}$ is an *open unit ball* and $d(x, A) = \inf_{y \in A} d(x, y)$.

- **Definition** (***Surface Area of Sets***)
  let $A \subset \mathbb{R}^n$ be a measurable set and denote by $\mathrm{Vol}(A)$ its *Lebesgue measure. The* ***surface area*** *of* A is defined by

  $$\mathrm{Vol}(\partial A) = \lim_{t \to 0} \frac{\mathrm{Vol}(A_t) - \mathrm{Vol}(A)}{t}.$$

  provided that the limit exists. Here A_t denotes *the t-blowup* of A.

- **Remark** (***Isoperimetry Theorem***)
  The classical isoperimetric theorem in $\mathbb{R}^n$ states that, among all sets with ***a given volume***, ***the Euclidean unit ball minimizes the surface area***. This theoerm can be formally stated as below:

- **Theorem 1.4** *(**Isoperimetry Theorem**) [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  Let $A \subset \mathbb{R}^n$ be such that $Vol(A) = Vol(B)$ where $B := \{x \in \mathbb{R}^n : d(0, x) < 1\}$ is an unit ball. Then for any $t > 0$,

  $$Vol(A_t) \geq Vol(B_t) \tag{4}$$

  Moreover, if $Vol(\partial A)$ exists, then

  $$Vol(\partial A) \geq Vol(\partial B). \tag{5}$$

  Proof: By *the Brunn-Minkowski inequality*,

  $$\begin{aligned} \mathrm{Vol}(A_t)^{1/n} = \mathrm{Vol}(A + tB)^{1/n} &\geq \mathrm{Vol}(A)^{1/n} + t\mathrm{Vol}(B)^{1/n} \\ &= (1 + t)\mathrm{Vol}(B)^{1/n} \\ &= \mathrm{Vol}(B_t)^{1/n}, \end{aligned}$$

establishing the first statement. The second follows simply because

$$\mathrm{Vol}(A_t) - \mathrm{Vol}(A) \geq \mathrm{Vol}(B)((1+t)^n - 1) \geq nt\mathrm{Vol}(B)$$

where $(1+t)^n \geq 1 + nt$ for $t \geq 0$. Thus $\mathrm{Vol}(\partial A) \geq n\mathrm{Vol}(B)$. The isoperimetric theorem now follows from the fact that $\mathrm{Vol}(\partial B) = n\mathrm{Vol}(B)$. ∎

2 Concentration via Isoperimetry

2.1 Levy's Inequalities

- **Remark** We can generalize the classical isoperimety problem to a probability space $(\mathcal{X}, \mathcal{B}[\mathcal{X}], \mathbb{P})$ where $\mathcal{X}$ is a *metric space* with metric d, $\mathcal{B}[\mathcal{X}]$ is the Borel σ-algrebra and $\mathbb{P}$ is a probability measure on $\mathcal{B}[\mathcal{X}]$. Let $B := \{x \in \mathbb{R}^n : d(0, x) < 1\}$. The classical isoperimetry problem aims at finding the set $A^* \subset \mathcal{X}$ that ***minimizes the surface area***

  $$\mathbb{P}(\partial A) = \lim_{t \to 0} \frac{\mathbb{P}(A_t) - \mathbb{P}(A)}{t}$$

  This is equivalent to find subset A in $\mathcal{X}$ with ***minimal t-blowup*** for given p, and for all $t > 0$

  $$A^* := \inf_{A \subset \mathcal{X}: \mathbb{P}(A) \geq p} \mathbb{P}(A_t), \quad \forall t > 0$$

  where

  $$A_t = A + tB = \{x \in \mathcal{X} : \exists y \in A \text{ s.t. } d(x, y) < t\} = \left\{x \in \mathcal{X} : \inf_{y \in A} d(x, y) := d(x, A) < t\right\}.$$

  We write the definition formally.

- **Definition** (***Isoperimetry Problem***) [Boucheron et al., 2013]
  Given a ***metric space*** $\mathcal{X}$ with corresponding *distance* d, consider ***the measure space*** formed by $\mathcal{X}$, *the σ-algebra* of all ***Borel sets of*** $\mathcal{X}$, and a probability measure $\mathbb{P}$. Let X be a *random variable* taking values in $\mathcal{X}$, distributed according to $\mathbb{P}$.

  The isoperimetric problem in this case is the following: given $p \in (0, 1)$ and $t > 0$, ***determine the sets*** A with $\mathbb{P}[X \in A] \geq p$ for which *the measure*

  $$\mathbb{P}[d(X, A) \geq t]$$

  is ***maximal***.

- **Remark** (***Isoperimetric Inequalities***)
  Even though the exact solution is only known in a few special cases, useful *bounds* for $\mathbb{P}[d(X, A) \geq t]$ can be derived under remarkably general circumstances. *Such bounds are usually referred to as* ***isoperimetric inequalities***.

- **Definition** (***Concentration Function***) [Boucheron et al., 2013, Wainwright, 2019]
  The concentration function $\alpha : [0, \infty) \to \mathbb{R}_+$ associated with ***metric measure space*** $((\mathcal{X}, d), \mathbb{P})$ is given by

  $$\alpha_{\mathbb{P},(\mathcal{X},d)}(t) := \sup_{A \subset \mathcal{X}:\, \mathbb{P}(A) \geq \frac{1}{2}} \mathbb{P}[d(X, A) \geq t] = \sup_{A \subset \mathcal{X}:\, \mathbb{P}(A) \geq \frac{1}{2}} \mathbb{P}(A_t^c)$$

where $A_t := A + tB = \{x \in \mathcal{X} : d(x, A) < t\}$ is *the t-blowup* of $A \subset \mathcal{X}$. We simply denote it as $\alpha(t)$.

Thus the optimal A^* for isoperimetry problem is the one that attains the $\alpha(t) = \mathbb{P}(A_t^c)$.

- **Example** (***Concentration Function of Lebesgue Measure in $\mathbb{R}^n$ and Isoperimetric Inequality***)
  Note that the volume of a *t-ball in* $\mathbb{R}^n$ is

$$\mathrm{Vol}(tB) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} t^n \equiv c_n t^n$$

  Thus the radius of ball B with the same volume of A is

$$r := \left(\frac{\mathrm{Vol}(A)}{c_n}\right)^{\frac{1}{n}}.$$

  The classical isoperimetric inequality states that

$$\begin{aligned} \mathrm{Vol}(A_t)) &\geq \left((r+t)\mathrm{Vol}(B)^{1/n}\right)^n \\ \Leftrightarrow \mathrm{Vol}(A_t) &\geq c_n \left(\left(\frac{\mathrm{Vol}(A)}{c_n}\right)^{\frac{1}{n}} + t\right)^n \\ \Leftrightarrow \left(\frac{\mathrm{Vol}(A_t)}{c_n}\right)^{\frac{1}{n}} &\geq \left(\frac{\mathrm{Vol}(A)}{c_n}\right)^{\frac{1}{n}} + t \end{aligned} \tag{6}$$

  Define ***the isoperimetric function*** of *the Lebesgue measure space* $(\mathbb{R}^n, \mu)$ as

$$\lambda(u) := \left(\frac{u}{c_n}\right)^{\frac{1}{n}}$$

  so *the classical isoperimetric inequality* is equivalent to *the concentration of Lebesgue measure*

$$\lambda\left(\mu(A_t)\right) \geq \lambda\left(\mu(A)\right) + t.$$

- **Theorem 2.1** ***(Levy's Inequalities)****[Boucheron et al., 2013, Wainwright, 2019]*
  For any Lipschitz function $f : \mathcal{X} \to \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left\{f(X) \geq Med(f(X)) + t\right\} &\leq \alpha_{\mathbb{P}}(t) \\ \mathbb{P}\left\{f(X) \leq Med(f(X)) - t\right\} &\leq \alpha_{\mathbb{P}}(t). \end{aligned} \tag{7}$$

  where $Med(f(X))$ is ***the median*** *of $f(X)$, i.e.*

$$\mathbb{P}\left\{f(X) \leq Med(f(X)\right\} \geq \frac{1}{2}, \quad and \quad \mathbb{P}\left\{f(X) \geq Med(f(X)\right\} \geq \frac{1}{2}.$$

  Proof: Consider the set $A = \{x : f(x) \leq \mathrm{Med}(f(X))\}$. By the definition of a *median*, $\mathbb{P}(A) \geq \frac{1}{2}$. On the other hand, by *the Lipschitz property* of f, for any $x, y \in \mathcal{X}$,

$$|f(x) - f(y)| \leq d(x, y).$$

So for all $y \in A$, $f(y) \le \mathrm{Med}(f(X))$

$$
\begin{aligned}
f(x) - \mathrm{Med}(f(X)) &\le f(x) - f(y) \le d(x,y) \\
\Rightarrow f(x) - \mathrm{Med}(f(X)) &\le \inf_{y \in A} d(x,y) := d(x,A).
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
A_t := \{x \in \mathcal{X} : d(x,A) < t\} &\subseteq \{x \in \mathcal{X} : f(x) < \mathrm{Med}(f(X)) + t\} \\
\mathbb{P}(A_t^c) &\ge \mathbb{P}\left\{f(X) \ge \mathrm{Med}(f(X)) + t\right\}
\end{aligned}
$$

The first inequality now follows from the definition of the concentration function. The second inequality follows from the first by considering f. ∎

- **Remark** For L-Lipschitz function f, the inequality becomes

$$
\mathbb{P}\left\{f(X) - \mathrm{Med}(f(X)) \ge t\right\} \le \alpha\left(\frac{t}{L}\right), \quad \mathbb{P}\left\{f(X) - \mathrm{Med}(f(X)) \le -t\right\} \le \alpha\left(\frac{t}{L}\right).
$$

- **Theorem 2.2** ***(Converse of Levy's Inequalities)****[Boucheron et al., 2013, Wainwright, 2019]*
If $\beta : \mathbb{R}_+ \to [0,1]$ is a function such that for ***every Lipschitz function*** *$f : \mathcal{X} \to \mathbb{R}$*

$$
\mathbb{P}\left\{f(X) - \mathit{Med}(f(X)) \ge t\right\} \le \beta(t). \tag{8}
$$

then $\beta(t) \ge \alpha_{\mathbb{P}}(t)$.

Proof: Note that for any $A \subset \mathcal{X}$, the function f_A defined by $f_A(x) = d(x,A)$ is *Lipschitz* since

$$
|f_A(x) - f_A(y)| = |d(x,A) - d(y,A)| \le d(x,y).
$$

Also, if $P(A) \ge 1/2$, then 0 is a median of $f_A(X)$, since

$$
\mathbb{P}\left\{f_A(x) \le 0\right\} = \mathbb{P}\left\{d(X,A) \le 0\right\} = \mathbb{P}(A) \ge \frac{1}{2}.
$$

Therefore

$$
\alpha(t) := \sup_{A \subset \mathcal{X} : \mathbb{P}(A) \ge 1/2} \mathbb{P}\left\{f_A(x) - \mathrm{Med}(f_A(X)) \ge t\right\} \le \beta(t). \quad ∎
$$

- **Proposition 2.3** ***(Levy's Inequalities for Mean)****[Boucheron et al., 2013, Wainwright, 2019]*
If $\beta : \mathbb{R}_+ \to [0,1]$ is a function such that for ***every Lipschitz function*** *$f : \mathcal{X} \to \mathbb{R}$*

$$
\mathbb{P}\left\{f(X) - \mathbb{E}\left[f(X)\right] \ge t\right\} \le \beta(t). \tag{9}
$$

then $\beta(t) \ge \alpha_{\mathbb{P}}(t/2)$.

- **Remark** (***Isoperimetric Inequalities*** $\Leftrightarrow$ ***Concentration of Lipschitz Functions***)
The first result points out that *isoperimetric inequalities* (more precisely, ***upper bounds*** *for* ***the concentration function***) imply *concentration of Lipschitz functions.*

The converse shows that *concentration of Lipschitz functions* implies an *isoperimetric inequality.* In other word, *among all upper bounds* of $\mathbb{P}(A_t^c)$ for fixed A_t,

- **Corollary 2.4** *(**Concentration of Measure on Hamming Metric Space**) [Boucheron et al., 2013]*
  Consider independent random variables $Z_1, \ldots, Z_n$ *taking their values in a* (measurable) set $\mathcal{X}$ *and denote the vector of these variables by* $Z = (Z_1, \ldots, Z_n)$ *taking its value in* $\mathcal{X}^n$*. For an arbitrary (measurable) set* $A \subset \mathcal{X}^n$*, we write* $\mathbb{P}(A) = \mathbb{P}(Z \in A)$*. The **Hamming distance*** $d_H(x, y)$ *between the vectors* $x, y \in \mathcal{X}^n$ *is defined as **the number of coordinates** in which* x *and* y ***differ**. Then for any* $t > 0$,

$$\mathbb{P}\left\{d_H(x, A) \geq \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}} + t\right\} \leq \exp\left(-\frac{2t^2}{n}\right) \tag{10}$$

  Proof: As we shown in previous proof, $f_A(x) = d_H(x, A)$ is a Lipschitz function with respect to Hamming distance d_H. It follows from the definition that

$$\sup_{x \in \mathcal{X}^n, y_i \in \mathcal{X}} \left| f_A(x) - f_A(\widetilde{x}^{(i)}) \right| \leq d_H(x, \widetilde{x}^{(i)}) = 1$$

  where $\widetilde{x}^{(i)} = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$, so f_A has the bounded difference property. By bounded difference inequality,

$$\mathbb{P}\left\{\mathbb{E}\left[f_A(Z)\right] - f_A(Z) \geq t\right\} \leq \exp\left(-\frac{2t^2}{n}\right).$$

  Taking $t = \mathbb{E}\left[f_A(Z)\right] = \mathbb{E}\left[d_H(Z, A)\right]$, the left-hand side becomes $\mathbb{P}\left\{f_A(Z) \leq 0\right\} = \mathbb{P}\left\{d_H(Z, A) \leq 0\right\} = \mathbb{P}(A)$. Then the inequality becomes

$$\mathbb{P}(A) \leq \exp\left(-\frac{2}{n}\left(\mathbb{E}\left[d_H(Z, A)\right]\right)^2\right)$$
$$\Rightarrow \mathbb{E}\left[d_H(Z, A)\right] \leq \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}}.$$

  Then, by using the bounded difference inequality again, we obtain

$$\mathbb{P}\left\{d_H(Z, A) \geq \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}} + t\right\} \leq \mathbb{P}\left\{d_H(Z, A) \geq \mathbb{E}\left[d_H(Z, A)\right] + t\right\} \leq \exp\left(-\frac{2t^2}{n}\right). \quad \blacksquare$$

- **Remark** (***Equivalent Form***)
  From above isoperimetric inequality,

$$\mathbb{P}\left\{d_H(x, A) \geq \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}} + t\right\} \leq \exp\left(-\frac{2t^2}{n}\right)$$

  Denote $u := \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}}$. By change of variable, for any $t \geq u$,

$$\mathbb{P}\left\{d_H(x, A) \geq t\right\} \leq \exp\left(-\frac{2\left(t - u\right)^2}{n}\right).$$

  On the one hand, if $t \leq 2u = \sqrt{-2n\log\mathbb{P}(A)}$, then $\mathbb{P}(A) \leq \exp(-t^2/(2n))$. On the other hand, since $(t - u)^2 \geq t^2/4$ for $t \geq 2u = \sqrt{-2n\log\mathbb{P}(A)}$. the inequality above implies

$\mathbb{P}\{d_H(x, A) \geq t\} \leq \exp(-t^2/(2n))$. Thus, for all $t > 0$, we have ***the concentration of measure in Hamming metric space***:

$$\mathbb{P}(A)\mathbb{P}\{d_H(x, A) \geq t\} \leq \min\{\mathbb{P}(A), \mathbb{P}\{d_H(x, A) \geq t\}\} \leq \exp\left(-\frac{t^2}{2n}\right) \tag{11}$$

- **Remark** (***Concentration of Measure***)
To interpret the result in (10), we see that on the left-hand side we have the measure of the set of points whose Hamming distance is at least $t + \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}(A)}}$ away from A. This inequality means that for A with ***small measure*** $\mathbb{P}(A)$, the measure of points whose ***Hamming distance*** from A *is less than* $O(\sqrt{n})$ is ***extremely large***. In other words, ***product measure*** *on Hamming metric space are* ***concentrated*** *on* ***extremely small sets***. This phenomenon is called "***concentration of measure***".

- **Example** (***Bounded Difference Property*** $\Leftrightarrow$ ***Lipschitz Condition w.r.t. Hamming Distance***)
Note that any function with ***bounded difference property*** is ***Lipschitz function*** with respect to ***Hamming distance***.

$$\begin{aligned}
&\sup_{x\in\mathcal{X}^n, y_i\in\mathcal{X}} |f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \\
&\leq c_i = c_i\, d_H((x_1, \ldots, x_n), (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)), \quad 1 \leq i \leq n \\
\Rightarrow |f(x) - f(y)| &= \left|\sum_{i=1}^{n}(f(x_{(i-1)}) - f(x_{(i)}))\right| \\
&\leq \sum_{i=1}^{n}\left|f(x_{(i-1)}) - f(x_{(i)})\right| \\
&\leq \sum_{i=1}^{n} c_i \mathbb{1}\left\{x_{(i-1)}[i] \neq x_{(i)}[i]\right\} \\
&= d_{H,c}(x, y)
\end{aligned}$$

where $x_{(i)}$ is replicate of $x_{(i-1)}$ except for i-th component, which is replaced by y_i. Note that $x_{(0)} = x$ and $x_{(n)} = y$. Therefore, *the bounded difference inequality* can be seen as *an isoperimetry inequality* for *Lipschitz function with respect to Hamming distance*.

$$\mathbb{P}\{f(Z) - \mathbb{E}[f(Z)] \geq t\} \leq \exp\left(-\frac{2t^2}{n}\right)$$

2.2 Isoperimetric Inequalities on the Unit Sphere

- **Definition** (***Spherical Cap and its t-Blowup***)
Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the $(n-1)$-*dimensional* ***unit sphere***. The ***intersection*** of a ***half-space*** and $\mathbb{S}^{n-1}$ is called a ***spherical cap***. In particular, for some $y \in \mathbb{R}^n$, denote the associated spherical cap as

$$H_y := \left\{x \in \mathbb{S}^{n-1} : \langle x\,,\, y\rangle \leq 0\right\}$$

With some simple geometry, it can be shown that its t-*blowup* corresponds to the set

$$H_y^t := \left\{x \in \mathbb{S}^{n-1} : \langle x\,,\, y\rangle < \sin(t)\right\}$$

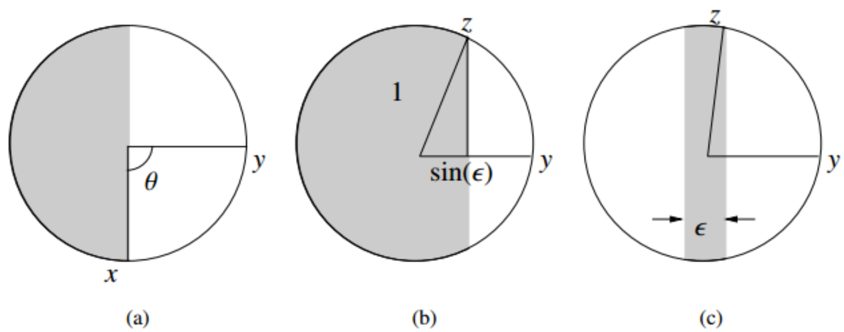

Figure 3.1 (a) Idealized illustration of the sphere $\mathbb{S}^{n-1}$. Any vector $y \in \mathbb{S}^{n-1}$ defines a hemisphere $H_y = \{x \in \mathbb{S}^{n-1} \mid \langle x, y \rangle \le 0\}$, corresponding to those vectors whose angle $\theta = \arccos \langle x, y \rangle$ with y is at least $\pi/2$ radians. (b) The ϵ-enlargement of the hemisphere H_y. (c) A central slice $T_y(\epsilon)$ of the sphere of width ϵ.

Figure 2: spherical cap and t-blowup. [Wainwright, 2019]

- **Theorem 2.5** ***(Isoperimetry Theorem on Unit Sphere)*** *[Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  Let A be a subset of the sphere $\mathbb{S}^{n-1}$, and let σ denote the ***normalized area*** *on that sphere. Let $t > 0$. Then, among all sets $A \subset \mathbb{S}^{n-1}$ with given area $\sigma(A)$, the* ***spherical caps minimize the area of the neighborhood*** *$\sigma(A_t)$, where*

$$A_t := \left\{x \in \mathbb{S}^{n-1} : \exists y \in A \textit{ such that } \|x - y\| < t\right\}$$

- **Remark** Define a *metric* ρ on sphere $\mathbb{S}^{n-1}$ as

$$\rho(x, y) := \arccos(\langle x\,,\, y\rangle)$$

  Thus $(\mathbb{S}^{n-1}, \rho)$ is a ***metric space***. Let $\mathbb{P}$ be uniform distribution on $\mathbb{S}^{n-1}$ so that $((\mathbb{S}^{n-1}, \rho), \mathbb{P})$ is a probability space.

- **Proposition 2.6** ***(Isoperimetric Inequalities for Uniform Distribution over Sphere)*** *[Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the $(n-1)$-dimensional ***unit sphere****. For any $t \in [0, 1]$,*

$$\alpha_{\mathbb{S}^{n-1}}(t) \le c \exp\left(-\frac{nt^2}{2}\right) \tag{12}$$

  for some constant c.

  Proof: Consider spherical cap

$$C(y, 0) := \left\{x \in \mathbb{S}^{n-1} : \langle x\,,\, y\rangle \ge 0\right\}$$

  and its *t-blowup*

$$C(y, t) := \left\{x \in \mathbb{S}^{n-1} : \langle x\,,\, y\rangle \ge t\right\}.$$

  According to *the isoperimetry theorem on unit sphere*, the concentration function for uniform distribution over $\mathbb{S}^{n-1}$

$$\alpha_{\mathbb{S}^{n-1}}(t) = \mathbb{P}(C(y, t)).$$

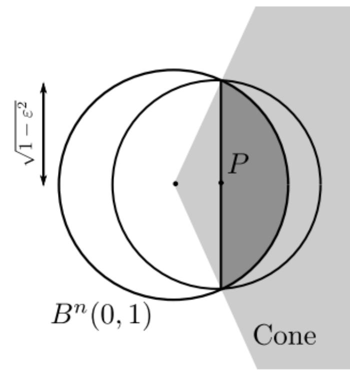

Figure 2: Small ε.

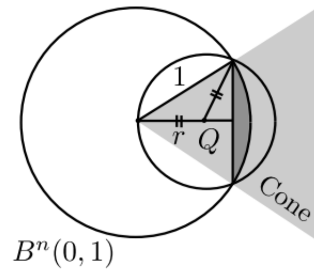

Figure 3: Large ε. By the congruence $\frac{1/2}{r} = \frac{\varepsilon}{1}$.

Figure 3: proof for upper bound of area of spherical cap (left) for small t (right) for large t

Note that $\mathbb{P}\left(C(y,0)\right) \leq 1/2$. In order to bound the concentration function from above, consider for small $t \in [0, 1/\sqrt{2}]$,

$$\begin{aligned}
\alpha_{\mathbb{S}^{n-1}}(t) = \mathbb{P}\left(C(y,t)\right) &= \frac{\mathrm{Vol}(B^n(0,1) \cap \mathrm{Cone})}{\mathrm{Vol}(B^n(0,1))} \\
&\leq \frac{\mathrm{Vol}(B^n(P, \sqrt{1-t^2}))}{\mathrm{Vol}(B^n(0,1))} \\
&= (\sqrt{1-t^2})^n \\
&\leq \exp\left(-\frac{nt^2}{2}\right)
\end{aligned}$$

For $t \in [1/\sqrt{2}, 1)$, it is enough to consider a different auxiliary ball which includes the set $\mathrm{Cone} \cap B^n(0,1)$. We obtain

$$\begin{aligned}
\alpha_{\mathbb{S}^{n-1}}(t) = \mathbb{P}\left(C(y,t)\right) &\leq \frac{\mathrm{Vol}(B^n(Q,r))}{\mathrm{Vol}(B^n(0,1))} \\
&= r^n = \left(\frac{1}{2t}\right)^n \\
&\leq \exp\left(-\frac{nt^2}{2}\right)
\end{aligned}$$

where the last inequality is from $e^{x^2/2} \leq 2x$ for $x \in [1/\sqrt{2}, 1]$. Due to convexity, this is only to be checked at the boundary of our interval $[1/\sqrt{2}, 1]$, $\blacksquare$

- By Levy's inequality, we have the following proposition

  Proposition 2.7 ***(Lipschitz Function on*** $\mathbb{S}^{n-1}$***)*** *[Wainwright, 2019]*
  For any 1*-Lipschitz function* f *defined on the sphere* $\mathbb{S}^{n-1}$*, we have the two-sided bound*

$$\mathbb{P}\left\{|f(Z) - Med(f(Z))| \geq t\right\} \leq \sqrt{2\pi}\exp\left(-\frac{nt^2}{2}\right) \tag{13}$$

Moreover, replacing median by the mean, we have

$$\mathbb{P}\left\{|f(Z) - \mathbb{E}\left[f(Z)\right]| \geq t\right\} \leq 2\sqrt{2\pi}\exp\left(-\frac{nt^2}{8}\right) \tag{14}$$

- **Exercise 2.8** ***(The Blow-Up Phenomenon)***
  Let A be a subset of the sphere $\sqrt{n}\mathbb{S}^{n-1}$ such that

  $$\mathbb{P}\left(A\right) > 2\exp\left(-cs^2\right) \;\; \textit{for some } s > 0;$$

  1. *Prove that $\mathbb{P}(A_s) > 1/2$.*
  2. *Deduce from this that for any $t \geq s$,*

     $$\mathbb{P}\left(A_{2t}\right) > 1 - 2\exp\left(-ct^2\right).$$

     Here $c > 0$ is the absolute constant in upper bound of concentration function.

- **Remark** (***Zero-One Law for Independent Variables***) [Vershynin, 2018]
  The blow-up phenomenon we just saw may be quite *counter-intuitive* at first sight. How can an exponentially small set A undergo such a dramatic transition to an exponentially large set A_{2t} under such a small perturbation $2t$? (Remember that t can be much smaller than the radius $\sqrt{n}$ of the sphere.)

  However perplexing this may seem, this is *a typical phenomenon in* ***high dimensions***. It is *reminiscent* of ***zero-one laws*** in *probability theory*, which basically state that *events that are determined by many random variables* tend to have *probabilities either zero or one.*

2.3 Gaussian Isoperimetric Inequalities and Concentration of Gaussian Measure

- **Remark** (***Gaussian Isoperimetric Problem***)
  The Gaussian isoperimetric problem is to determine which (Borel) sets A have ***minimal Gaussian boundary measure*** among all sets in $\mathbb{R}^n$ with a *given probability p*.

  The Gaussian isoperimetric theorem states the beautiful fact that ***the extremal sets*** are ***linear half-spaces*** *in all dimensions* and *for all p*.

- **Definition** (***Gaussian Isoperimetric Function***)
  Denote *the cumulative distribution function of standard Normal distribution*:

  $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt := \int_{-\infty}^{x} \varphi(t)dt$$

  where $\varphi(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} = (\Phi(x))'$ is *the probability density function* of standard normal distribution. $\Phi^{-1}(x)$ is *the quantile function of normal distribution.*

  Define ***the Gaussian isoperimetric function*** as

  $$\gamma(x) := \varphi\left(\Phi^{-1}(x)\right), \quad x \in (0, 1).$$

  Also we define $\gamma(0) = \gamma(1) = 0$.

- **Remark** Note that

$$x = \Phi(\Phi^{-1}(x))$$
$$\Rightarrow 1 = \varphi(\Phi^{-1}(x))(\Phi^{-1}(x))' = \gamma(x)(\Phi^{-1}(x))'$$
$$\Leftrightarrow 1/\gamma(x) = (\Phi^{-1}(x))'.$$

The quantity $1/\gamma(x) = (\Phi^{-1}(x))'$ is known as ***quantile-density function*** *of normal distribution.*

- **Proposition 2.9** *(**Basic Property of the Gaussian Isoperimetric Function**) [Boucheron et al., 2013]*
The Gaussian isoperimetric function γ satisfies:

1.
$$\gamma'(x) = -\Phi^{-1}(x), \quad \textit{for all } x \in (0,1),$$

2.
$$\gamma(x)\gamma''(x) = -1, \quad \textit{for all } x \in (0,1),$$

3. $(\gamma')^2$ *is convex over* $(0,1)$.

Proof: 1. See that

$$\varphi'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-\frac{x^2}{2}} = (-x)\varphi(x)$$
$$\varphi''(x) = (x^2-1)\varphi(x)$$

Thus

$$\gamma(x)' = \left(\varphi\left(\Phi^{-1}(x)\right)\right)' = \frac{d\varphi}{dy}(\Phi^{-1}(x))\frac{d\Phi^{-1}}{dx}(x) = (-\Phi^{-1}(x))\left(\Phi^{-1}(x)\right)'\gamma(x) = -\Phi^{-1}(x),$$

since $\left(\Phi^{-1}(x)\right)'\gamma(x) = 1$, we have the result.

2.
$$\gamma''(x) = (\gamma'(x))' = -\left(\Phi^{-1}(x)\right)' = -\frac{1}{\gamma(x)}$$
$$\gamma(x)\gamma''(x) = -1$$

3. Since $\gamma > 0$ for $x \in (0,1)$,

$$\gamma''(x) = -\frac{1}{\gamma(x)} < 0.$$

Thus $\gamma(x)$ is concave function in $(0,1)$. ∎

- **Lemma 2.10** *(**Asymptotic Behavior of Gaussian Isoperimetric Function**) [Boucheron et al., 2013]*
For all $x \in [0,1/2]$,

$$x\sqrt{\frac{1}{2}\log\frac{1}{x}} \le \gamma(x) \le x\sqrt{2\log\frac{1}{x}}.$$

Moreover,

$$\lim_{x\to 0} \frac{\gamma(x)}{x\sqrt{2\log\frac{1}{x}}} = 1$$

- **Proposition 2.11** ***(Bobkov's Inequality)*** *[Boucheron et al., 2013]*
  Suppose Z is uniformly distributed over $\{-1,1\}^n$. Then for all $n \geq 1$ and for all functions $f : \{-1,1\}^n \to [0,1]$,

$$\gamma\left(\mathbb{E}\left[f(Z)\right]\right) \leq \mathbb{E}\left[\sqrt{\gamma(f(Z))^2 + \|\nabla f(Z)\|_2^2}\right] \tag{15}$$

- **Proposition 2.12** ***(Bobkov's Gaussian Inequality)*** *[Boucheron et al., 2013]*
  *Let $Z := (Z_1, \ldots, Z_n)$ be a vector of **independent standard Gaussian** random variables. Let $f : \mathbb{R}^n \to [0,1]$ be a differentiable function with gradient ∇f. Then*

$$\gamma\left(\mathbb{E}\left[f(X)\right]\right) \leq \mathbb{E}\left[\sqrt{\gamma(f(X))^2 + \|\nabla f(X)\|_2^2}\right] \tag{16}$$

  *where $\gamma = \varphi \circ \Phi^{-1}$ is **the Gaussian isoperimetric function**.*

- **Theorem 2.13** ***(Gaussian Isoperimetric Theorem)*** *[Boucheron et al., 2013] [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  *Let $\mathbb{P}$ be the **standard Gaussian measure** on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a Borel set. Then*

$$\liminf_{t\to 0} \frac{\mathbb{P}(A_t) - \mathbb{P}(A)}{t} \geq \gamma\left(\mathbb{P}(A)\right), \tag{17}$$

  *where $A_t := \{x : d(x, A) < t\}$ be the t-blowup of A. Moreover, if A is a **half-space** defined by $A := \{x \in \mathbb{R}^n : x_1 \leq z\}$, then*

$$\liminf_{t\to 0} \frac{\mathbb{P}(A_t) - \mathbb{P}(A)}{t} = \gamma\left(\mathbb{P}(A)\right) = \varphi(z), \tag{18}$$

  *where $\gamma := \varphi \circ \Phi^{-1}$ is **the Gaussian isoperimetric function**.*

- **Proposition 2.14** ***(Differentiablity of Measure of t-Blowup)*** *[Boucheron et al., 2013]*
  *If A is a **finite union** of **open** balls in $\mathbb{R}^n$, then $\mathbb{P}(A_t)$ is a **differentiable** function of $t > 0$.*

- Next we describe ***an equivalent version*** of ***the Gaussian isoperimetric theorem*** in the manner of *measure concentration*:

  Theorem 2.15 ***(Gaussian Concentration Theorem)*** *[Boucheron et al., 2013] [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  *Let $\mathbb{P}$ be the **standard Gaussian measure** on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a Borel set. Then for all $t \geq 0$,*

$$\begin{aligned} \mathbb{P}(A_t) &\geq \Phi\left(\Phi^{-1}\left(\mathbb{P}(A)\right) + t\right). \\ \Leftrightarrow \Phi^{-1}(\mathbb{P}(A_t)) &\geq \Phi^{-1}\left(\mathbb{P}(A)\right) + t \end{aligned} \tag{19}$$

  *Equality holds if A is a **half-space**.*

Proof: We call *a Borel set* $A \subset \mathbb{R}^n$ ***smooth*** if $\mathbb{P}(A_t)$ is a *differentiable function* of t on $(0, \infty)$.

1. Observe that if A is *smooth*, then

$$\begin{aligned}
\frac{d}{dt}\Phi^{-1}(\mathbb{P}(A_t)) &= \left[(\Phi^{-1})'(\mathbb{P}(A_t))\right]\frac{d}{dt}\mathbb{P}(A_t) \\
&= \frac{1}{\gamma(\mathbb{P}(A_t))}\frac{d}{dt}\mathbb{P}(A_t) \\
&\geq \frac{1}{\frac{d}{dt}\mathbb{P}(A_t)}\left(\frac{d}{dt}\mathbb{P}(A_t)\right) = 1
\end{aligned}$$

   The last inequality is due to *the Gaussian isoperimetric inequality*

$$\frac{d}{dt}\mathbb{P}(A_t) \geq \liminf_{s\to 0}\frac{\mathbb{P}(A_{t+s}) - \mathbb{P}(A_t)}{s} \geq \gamma\left(\mathbb{P}(A_t)\right).$$

   Therefore, by integration

$$\begin{aligned}
\Phi^{-1}(\mathbb{P}(A_t)) &= \Phi^{-1}(\mathbb{P}(A)) + \int_0^t \frac{d}{ds}\Phi^{-1}(\mathbb{P}(A_s))ds \\
&\geq \Phi^{-1}(\mathbb{P}(A)) + \int_0^t ds = \Phi^{-1}(\mathbb{P}(A)) + t.
\end{aligned}$$

   Hence, the theorem holds for all smooth sets. The remaining work is to extend this to all *Borel sets.*

2. Note first that if $\mathbb{P}(A) = 0$, the theorem is automatically satisfied and therefore we may focus on Borel sets A with *positive probability.* By Proposition 2.14, the concentration property holds for ***any finite union of open balls***.

3. Now let A be *any Borel set* with $\mathbb{P}(A) > 0$. Let $0 < \epsilon < t$. Then by ***Vitali's covering theorem***, there exists a *countable* collection of *disjoint open balls* $\{B_1, B_2, \ldots\}$, all intersecting A and *diameter at most* ϵ, such that $P(A - \bigcup_{n=1}^{\infty} B_n) = 0$. But then

$$\begin{aligned}
\mathbb{P}(A_t) &\geq \mathbb{P}\left(\bigcup_{n=1}^{\infty}(B_n)_{t-\epsilon}\right) \\
&= \lim_{n\to\infty}\mathbb{P}\left(\bigcup_{i=1}^{n}(B_i)_{t-\epsilon}\right) \\
&\geq \lim_{n\to\infty}\Phi\left(\Phi^{-1}\left(\mathbb{P}\left(\bigcup_{i=1}^{n}(B_i)_{t-\epsilon}\right)\right) + t - \epsilon\right) \\
&= \Phi\left(\Phi^{-1}\left(\mathbb{P}\left(\bigcup_{i=1}^{\infty}(B_i)_{t-\epsilon}\right)\right) + t - \epsilon\right) \\
&\geq \Phi\left(\Phi^{-1}\left(\mathbb{P}\left(A\right)\right) + t - \epsilon\right)
\end{aligned}$$

   The argument is completed by taking ϵ to 0. ∎

- **Remark** (***Gaussian Concentration Theorem*** $\equiv$ ***Gaussian Isoperimetric Theorem***)
  The Gaussian concentration theorem is equivalent to *the Gaussian isoperimetric theorem* since

$$\begin{aligned}\liminf_{t\to 0}\frac{\mathbb{P}(A_t)-\mathbb{P}(A)}{t} &\ge \liminf_{t\to 0}\frac{\Phi\left(\Phi^{-1}\left(\mathbb{P}(A)\right)+t\right)-\Phi\left(\Phi^{-1}\left(\mathbb{P}(A)\right)\right)}{t}\\ &= \Phi'(\Phi^{-1}(\mathbb{P}(A)))\\ &= \varphi(\Phi^{-1}(\mathbb{P}(A)))\\ &= \gamma(\mathbb{P}(A)).\end{aligned}$$

- **Exercise 2.16** *(**From Isoperimetry to Concentration**) [Boucheron et al., 2013]*
  Assume that a probability distribution $\mathbb{P}$ on $\mathbb{R}^n$ satisfies, for all Borel sets $A \subset \mathbb{R}^n$,

$$\liminf_{t\to 0}\frac{\mathbb{P}(A_t)-\mathbb{P}(A)}{t} \ge c\, f\left(F^{-1}(\mathbb{P}(A))\right),$$

  where $c \in (0,1]$ is a constant, F is a continously differentiable distribution function and $f = F'$ is its derivative. Prove that for all Borel set A and all $t \ge 0$,

$$\mathbb{P}(A_t) \ge F\left(F^{-1}(\mathbb{P}(A)) + c\,t\right).$$

- As a direct consequence of *the Gaussian isoperimetric inequality*, we have the improved result for *Gaussian concentration inequality*:

  Theorem 2.17 *(**Gaussian Concentration Inequality, Sharp Bound**) [Boucheron et al., 2013, Wainwright, 2019]*
  *Let $Z = (Z_1, \ldots, Z_n)$ be a vector of n **independent standard normal** random variables. Let $f : \mathbb{R}^n \to \mathbb{R}$ denote an L-**Lipschitz function**. Then, for all $t > 0$,*

$$\mathbb{P}\left\{f(Z) - Med(f(Z)) \ge t\right\} \le 1 - \Phi\left(\frac{t}{L}\right). \tag{20}$$

  where $\Phi(t)$ is the cumulative distribution function of standard normal random variable.

- **Remark** Note that by ***Gordon's inequality***

$$1 - \Phi(t) \le \left(\frac{1}{\sqrt{2\pi}}\right)\frac{1}{t}e^{-\frac{t^2}{2}} = \frac{1}{t}\varphi(t)$$

  The Gaussian concentration inequality fails to capture the corrective factor t^{-1}. The inequality above cannot be improved in general as for $f(x) = n^{-1/2}\sum_{i=1}^{n} x_i$, equality is achieved for all $t > 0$.

2.4 Convex Distance Inequality

- **Definition** (***Weighted Hamming Distance***)
  Given $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \ge 0$, ***the weighed Hamming distance*** between $x, y \in \mathcal{X}^n$ is defined as

$$d_\alpha(x,y) = \sum_{i=1}^{n}\alpha_i \mathbb{1}\left\{x_i \ne y_i\right\}.$$

- **Remark** (***Measure Concentration in Weighted Hamming Distance Space***)
  Similar to the inequality (10), for *metric measure space* $\mathcal{X}^n$ with respect to *weighted Hamming distance*, we have the measure concentration inequality for $A \subset \mathcal{X}^n$

$$\mathbb{P}\left\{ d_\alpha(x, A) \geq \sqrt{\frac{\|\alpha\|_2}{2} \log \frac{1}{\mathbb{P}(A)}} + t \right\} \leq \exp\left(-\frac{2t^2}{\|\alpha\|_2} \right)$$

  where $\|\alpha\|_2 = \sqrt{\sum_{i=1}^n \alpha_i^2}$. Assume $\|\alpha\|_2 = 1$

$$\mathbb{P}\left\{ d_\alpha(x, A) \geq \sqrt{\frac{1}{2} \log \frac{1}{\mathbb{P}(A)}} + t \right\} \leq \exp\left(-2t^2 \right)$$

  Following the same argument, we can find *an equivalent form* as in (11)

$$\sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2 = 1} \mathbb{P}(A) \mathbb{P}\left\{ d_\alpha(x, A) \geq t \right\} \leq \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2 = 1} \min\left\{ \mathbb{P}(A), \mathbb{P}\left\{ d_\alpha(x, A) \geq t \right\} \right\} \leq \exp\left(-\frac{t^2}{2} \right)$$

  A key contribution for ***convex distance inequality*** is that the above inequality remains true even if the ***supremum*** is taken ***within the probability***; i.e.

$$\mathbb{P}(A) \mathbb{P}\left\{ \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2 = 1} d_\alpha(x, A) \geq t \right\} \leq \exp\left(-\frac{t^2}{4} \right).$$

- **Definition** (***Convex Distance***)
  For any $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$, ***the convex distance*** of x from the set A by

$$d_T(x, A) := \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2 = 1} d_\alpha(x, A)$$

- **Theorem 2.18** ***(Convex Distance Inequality)*** *[Boucheron et al., 2013]*
  For any subset $A \subset \mathcal{X}^n$ and $t > 0$,

$$\mathbb{P}(A) \mathbb{P}\left\{ d_T(X, A) \geq t \right\} = \mathbb{P}(A) \mathbb{P}\left\{ \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2 = 1} d_\alpha(X, A) \geq t \right\} \leq \exp\left(-\frac{t^2}{4} \right). \quad (21)$$

- With convex distance inequality, we can improve *the concentration bound* for *convex Lipschitz functions*. First, we relate convex distance with the minimal distance to convex set

  Lemma 2.19 ***(Convex Distance vs. Distance to Convex Set)*** *[Boucheron et al., 2013]*
  Let $A \subset [0, 1]^n$ be a ***convex set*** *and let $x = (x_1, \ldots, x_n) \in [0, 1]^n$. Then*

$$d(x, A) := \inf_{y \in A} \|x - y\|_2 \leq d_T(x, A). \quad (22)$$

- **Theorem 2.20** ***(Concentration of Quasi-Convex Lipschitz Functions)*** *[Boucheron et al., 2013]*

Let $Z := (Z_1, \ldots, Z_n)$ be independent random variables taking values in the interval $[0,1]$ and let $f : [0,1]^n \to \mathbb{R}$ be a ***quasi-convex function****; that is*

$$\{z : f(z) \le s\} \;\; \textit{is convex set for all } s \in \mathbb{R}.$$

Moreover, f is Lipschitz function satisfying

$$|f(x) - f(y)| \le \|x - y\| \quad \textit{for all } x, y \in [0,1]^n.$$

Then $X = f(Z_1, \ldots, Z_n)$ satisfies, for all $t > 0$,

$$\mathbb{P}\{f(Z) \ge Med(f(Z)) + t\} \le 2\exp\left(-\frac{t^2}{4}\right), \tag{23}$$
$$\mathbb{P}\{f(Z) \le Med(f(Z)) - t\} \le 2\exp\left(-\frac{t^2}{4}\right).$$

Proof: For some $s \in \mathbb{R}$, define the set $A_s = \{z : f(z) \le s\} \subset [0,1]^n$. Because of *quasi-convexity*, A_s is *convex*. By *the Lipschitz property* and *Lemma* 2.19, for all $z \in [0,1]^n$,

$$f(z) \le s + d(z, A) \le s + d_T(z, A).$$

So by *convex distance inequality*,

$$\mathbb{P}\{f(Z) \le s\}\,\mathbb{P}\{f(Z) \ge s + t\} \le \exp\left(-\frac{t^2}{4}\right)$$

Take $s = \mathrm{Med}(f(Z))$ to get the *upper tail inequality* and $s = \mathrm{Med}(f(Z)) - t$ to get *the lower tail inequality.* ∎

2.5 Edge Isoperimetric Inequality on the Binary Hypercube

2.6 Vertex Isoperimetric Inequality on the Binary Hypercube

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