Lecture 2: Smooth Maps

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Oct. 14th., 2022

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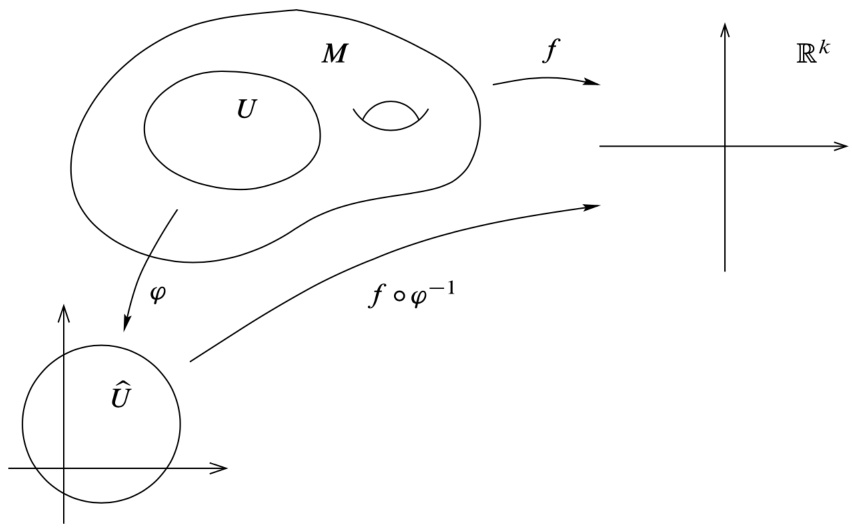

Figure 1: A smooth function on manifold [Lee, 2003.]

1 Smooth Functions and Smooth Maps

1.1 Smooth Functions on Manifolds

- **Definition** Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a *smooth chart* (U, φ) for M whose domain contains p and such that the *composite function* $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 1).

  If M is a smooth manifold *with boundary*, the definition is exactly the same, except that $\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, and in the latter case we interpret smoothness of $f \circ \varphi^{-1}$ to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ *extends to a smooth function* in the ordinary sense.

- The most important special case is that of ***smooth real-valued functions*** $f : M \to \mathbb{R}$ the set of all such functions is denoted by $\mathcal{C}^\infty(M)$. Because sums and constant multiples of smooth functions are smooth, $\mathcal{C}^\infty(M)$ is a vector space over $\mathbb{R}$.

- **Definition** Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the ***coordinate representation*** of f.

  By definition, f is *smooth* **if and only** if its coordinate representation is *smooth* in some smooth chart around *each point. Smooth functions have smooth coordinate representations in* ***every smooth chart***.

1.2 Smooth Maps Between Manifolds

- The definition of smooth functions generalizes easily to maps between manifolds.

  Definition Let M, N be *smooth manifolds*, and let $F : M \to N$ be any map. We say that

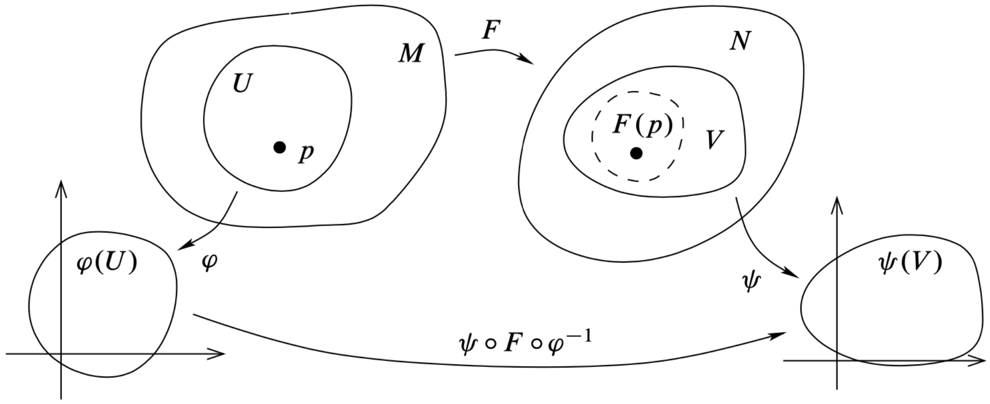

Figure 2: A smooth map between manifolds [Lee, 2003.]

F is a ***smooth map*** if for every $p \in M$, there exist *smooth charts* (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is ***smooth*** from $\varphi(U)$ to $\psi(V)$. (See Fig. 2)

If M and N are smooth manifolds *with boundary*, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

- **Proposition 1.1** *Every smooth map is* ***continuous****.*

- **Proposition 1.2** *(**Equivalent Characterizations of Smoothness**) [Lee, 2003.]*
  Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is ***smooth*** *if and only if either of the following conditions is satisfied:*

  1. *For every $p \in M$, there exist* ***smooth charts*** *(U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is* ***open*** *in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is* ***smooth*** *from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.*

  2. *F is continuous and there exist* ***smooth atlases*** *$\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for* ***each*** *α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.*

- **Proposition 1.3** *(**Smoothness Is Local**) [Lee, 2003.]*
  Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

  1. *If every point $p \in M$ has a neighborhood U such that the* ***restriction*** *$F|_U$ is smooth, then F is smooth.*

  2. *Conversely, if F is smooth, then its restriction to* ***every open subset*** *is smooth.*

- **Corollary 1.4** *(**Gluing Lemma for Smooth Maps**) [Lee, 2003.]*
  Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an ***open cover*** *of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a* ***unique smooth map*** *$F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$, for each $\alpha \in A$.*

- **Definition** If $F : M \to N$ is a *smooth map*, and (U, φ) and (V, ψ) are any smooth charts for M and N, respectively, we call $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ the ***coordinate representation*** of F with respect to the given coordinates. It maps the set $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

- **Proposition 1.5** *Let M, N and P be smooth manifolds with or without boundary.*

  1. *Every* ***constant map*** *$c : M \to N$ is smooth.*

  2. *The* ***identity map*** *of M is smooth.*

  3. *If $U \subseteq M$ is an* ***open submanifold*** *with or without boundary, then the* ***inclusion map*** *$U \hookrightarrow M$ is smooth.*

  4. *If $F : M \to N$ and $G : N \to P$ are smooth, then so is the* ***composite map*** *$G \circ F : M \to P$.*

- **Proposition 1.6** *Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has* ***nonempty boundary****. For each i, let $\pi_i : M_1 \times \ldots \times M_k \to M_i$ denote the* ***projection*** *onto the M_i factor. A map $F : N \to M_1 \times \ldots \times M_k$ is smooth if and only if each of the* ***component maps*** *$F_i = \pi_i \circ F : N \to M_i$ is smooth.*

1.3 Examples of Smooth Map

- **Example** Any map from a zero-dimensional manifold into a smooth manifold with or without boundary is automatically smooth, because each coordinate representation is **constant**.

- **Example** If the *circle* $\mathbb{S}^1$ is given its *standard smooth structure*, the map $\epsilon : \mathbb{R} \to \mathbb{S}^1$ defined by $\epsilon(t) = \exp(2\pi i t)$ is *smooth*, because with respect to any angle coordinate θ for $\mathbb{S}^1$ it has a coordinate representation of the form $\widehat{\epsilon}(t) = 2\pi t + c$ for some constant c, as you can check.

- **Example** The map $\epsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ defined by $\epsilon^n(t) = \left(\exp\left(2\pi i x^1\right), \ldots, \exp\left(2\pi i x^n\right)\right)$ is *smooth* since *n-torus* $\mathbb{T}^n = \mathbb{S} \times \ldots \times \mathbb{S}$.

- **Example** Now consider the n-sphere $\mathcal{S}^n$ with its *standard smooth structure*. The ***inclusion map*** $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is certainly *continuous*, because it is the inclusion map of a topological subspace. It is a smooth map because its coordinate representation with respect to any of the graph coordinates

$$\widehat{\iota} = \iota \circ (\varphi_i^{\pm})^{-1}(u^1, \ldots, u^n) = \left(u^1, \ldots, u^{i-1}, \sqrt{1 - \|u\|_2}, u^{i+1}, \ldots, u^n\right)$$

  which is *smooth* on its domain.

- **Example** The ***quotient map*** $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ used to define $\mathbb{RP}^n$ is *smooth*, because its coordinate representation in terms of any of the coordinates for $\mathbb{RP}^n$ constructed in Example before and standard coordinates on $\mathbb{R}^{n+1} \setminus \{0\}$ is

$$\begin{aligned}\widehat{\pi}(x^1, \ldots, x^{n+1}) &= \varphi_i \circ \pi(x^1, \ldots, x^{n+1}) = \varphi_i[x^1, \ldots, x^{n+1}] \\ &= \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).\end{aligned}$$

- **Example** Define $q : \mathbb{S}^n \to \mathbb{RP}^n$ as the restriction of $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ to $\mathbb{S}^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$.

It is a *smooth* map, because it is the composition $q = \pi \circ \iota$ of the maps in the preceding two examples.

- **Example** If $M_1, \ldots, M_k$ are smooth manifolds, then each projection map $\pi_i : M_1 \times \ldots \times M_k \to M_i$ is *smooth*, because its coordinate representation with respect to any of the product charts of Example 1.8 is just a coordinate projection.

1.4 Diffeomorphisms

- **Definition** If M and N are smooth manifolds with or without boundary, a ***diffeomorphism*** from M to N is a ***smooth bijective map*** $F : M \to N$ that has a ***smooth inverse***. We say that M and N are ***diffeomorphic*** if there exists a *diffeomorphism* between them. Sometimes this is symbolized by $M \approx N$.

- **Example** If M is any smooth manifold and (U, φ) is a smooth coordinate chart on M, then $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a ***diffeomorphism***. (In fact, it has an identity map as a coordinate representation.)

- **Proposition 1.7** ***(Properties of Diffeomorphisms)***

  1. *Every* **composition** *of diffeomorphisms is a diffeomorphism.*
  2. *Every* **finite product** *of diffeomorphisms between smooth manifolds is a diffeomorphism.*
  3. *Every diffeomorphism is a* **homeomorphism** *and an* **open map***.*
  4. *The* **restriction** *of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.*
  5. *"Diffeomorphic" is an* **equivalence relation** *on the class of all smooth manifolds. with or without boundary.*

- The following theorem is a weak version of *invariance of dimension*, which suffices for many purposes.

  Theorem 1.8 ***(Diffeomorphism Invariance of Dimension)****.*
  A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.

- **Theorem 1.9** ***(Diffeomorphism Invariance of the Boundary)****.*
  Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\mathrm{Int}\, M$ to $\mathrm{Int}\, N$.

- Just as two topological spaces are considered to be "the same" if they are ***homeomorphic***, two smooth manifolds with or without boundary are essentially indistinguishable if they are ***diffeomorphic***.

- The ***central concern*** of smooth manifold theory is the study of properties of smooth manifolds that are ***preserved by diffeomorphisms***. (This includes properties that are invariant under change of variables since the coordination itself is a diffemorphism.)

- It is natural to wonder whether the smooth structure on a given topological manifold is *unique*. This straightforward version of the question is easy to answer: we observed in Example before that every zero-dimensional manifold has a unique smooth structure, but each positive-dimensional manifold admits ***many distinct smooth structures*** *as soon as it admits one.*

2 Partitions of Unity

2.1 Theorems

- Recall the gluing lemma in topology

  Lemma 2.1 *(**Gluing Lemma for Continuous Maps**).*
  Let X and Y be topological spaces, and suppose one of the following conditions holds:

  1. *$B_1, \ldots, B_n$ are **finitely** many **closed** subsets of X whose union is X.*

  2. *$\{B_i\}_{i \in A}$ is a collection of **open** subsets of X whose union is X.*

  *Suppose that for all i we are given **continuous** maps $F_i : B_i \to Y$ that **agree on overlaps**: $F_i|_{B_i \cap B_j} = F_j|_{B_i \cap B_j}$. Then there exists a **unique continuous map** $F : X \to Y$ whose restriction to each B_i is equal to F_i.*

  Comparing with the Gluing Lemma for smooth maps, we see that it does not hold for *the finitely many **closed** subsets case.*

  Corollary 2.2 *(**Gluing Lemma for Smooth Maps**) [Lee, 2003.]*
  *Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an **open cover** of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a **unique smooth map** $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$, for each $\alpha \in A$.*

- **Remark** A ***disadvantage*** of Corollary above is that in order to use it, we must construct maps that agree exactly on relatively large subsets of the manifold, which is ***too restrictive*** or some purposes. In this section we introduce ***partitions of unity***, which are tools for "blending together" local smooth objects into global ones ***without necessarily assuming that they agree on overlaps***.

- **Lemma 2.3** *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

  $$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

  *is **smooth**.*

- **Lemma 2.4** *Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a **smooth** function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) \equiv 1$ for $t \le r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) = 0$ for $t > r_2$.*

  A function with the properties of h in the preceding lemma is usually called ***a cutoff function***. Let $h = f(r_2 - t)/(f(r_2 - t) + f(t - r_1))$ where f is define in preivous lemma.

- **Lemma 2.5** *Given any positive real numbers $r_1 < r_2$, there is a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H \equiv 1$ on $\bar{B}_{r_1}(0)$, $0 < H(x) < 1$ for all $x \in B_{r_2}(0) \backslash \bar{B}_{r_1}(0)$, and $H = 0$ on $\mathbb{R}^n \backslash B_{r_2}(0)$.*

  Let $H(x) = h(\|x\|)$ where h is the cutoff function as above.

- **Definition** The function H constructed in this lemma is an example of ***a smooth bump function***, a smooth real-valued function that is ***equal to 1*** on a ***specified set*** and is ***zero outside a specified neighborhood*** of that set.

- **Definition** If f is any real-valued or vector-valued function on a topological space M, ***the support of*** f, denoted by supp f, is the ***closure*** of the set of points where f is ***nonzero***:

$$\text{supp } f = \overline{\{p \in M : f(p) \neq 0\}}$$

(For example, if H is the function constructed in the preceding lemma, then supp $H = \bar{B}_{r_2}(0)$.) If supp f is contained in some set $U \subseteq M$, we say that f ***is supported in*** U. A function f is said to be ***compactly supported*** if supp f is a *compact set*. Clearly, every function *on a compact space* is *compactly supported*.

- **Definition** Suppose M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary ***open cover of*** M, indexed by a set A. ***A partition of unity subordinate to*** $\mathcal{X}$ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of ***continuous functions*** $\psi : M \to \mathbb{R}$ with the following properties:

  1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.
  2. supp $\psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
  3. The family of supports (supp $\psi_\alpha)_{\alpha \in A}$ is ***locally finite***, meaning that every point has a *neighborhood* that *intersects* supp ψ_α for ***only finitely many values*** of α.
  4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

  If M is a smooth manifold with or without boundary, ***a smooth partition of unity*** is one for which each of the functions ψ_α is ***smooth***.

- **Theorem 2.6** ***(Existence of Partitions of Unity)***. *[Lee, 2003.]*
*Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed **open cover** of M. Then there **exists** a **smooth partition of unity** subordinate to $\mathcal{X}$.*

2.2 Applications of Partitions of Unity

- **Definition** If M is a topological space, $A \subseteq M$ is a ***closed*** subset, and $U \subseteq M$ is an ***open*** subset containing A, a ***continuous*** function $\psi : M \to \mathbb{R}$ is called ***a bump function for*** A ***supported in*** U if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and supp $\psi \subseteq U$.

- **Proposition 2.7** ***(Existence of Smooth Bump Functions)***. *[Lee, 2003.]*
*Let M be a smooth manifold with or without boundary. For any **closed** subset $A \subseteq M$ and any **open** subset U containing A, there **exists** a **smooth bump function for** A **supported in** U.*

- **Definition** Suppose M and N are smooth manifolds with or without boundary, and $A \subseteq M$ is an arbitrary subset. We say that a map $F : A \to N$ is ***smooth on*** A if it has a ***smooth extension*** *in a neighborhood of each point*: that is, if for every $p \in A$ there is *an open subset* $W \subseteq M$ containing p and ***a smooth map*** $\widetilde{F} : W \to N$ whose ***restriction*** *to* $W \cap A$ ***agrees*** *with* F .

- **Lemma 2.8** ***(Extension Lemma for Smooth Functions)***. *[Lee, 2003.]*
*Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a **closed** subset, and*

$f : A \to \mathbb{R}^k$ *is a* ***smooth*** *function. For any open subset* U *containing* A*, there exists a smooth function* $\widetilde{f} : M \to \mathbb{R}^k$ *such that* $\widetilde{f}|_A = f$ *and supp* $\widetilde{f} \subseteq U$.

- **Definition** If M is a topological space, ***an exhaustion function*** for M is a *continuous function* $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ (called ***a sublevel set of*** f) is ***compact*** for each $c \in \mathbb{R}$.

  Example of exhaustion function

$$f(x) = \|x\|^2, \quad f(x) = \frac{1}{1 - \|x\|_2^2}$$

- **Proposition 2.9** *(**Existence of Smooth Exhaustion Functions**). [Lee, 2003.]*
  Every smooth manifold with or without boundary admits a ***smooth positive exhaustion function****.*

- **Theorem 2.10** *(**Level Sets of Smooth Functions**). [Lee, 2003.]*
  Let M *be a smooth manifold. If* K *is any* ***closed subset*** *of* M*, there is a* ***smooth nonnegative function*** $f : M \to \mathbb{R}$ *such that* $f^{-1}(0) = K$.

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.