Lecture 3: Countability and Separation Axioms

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1 The Countability and Separation Axioms

1.1 The Countability Axioms

- **Definition** (***First-Countable***)
  A space X is said to have a ***countable basis at*** x if there is a ***countable*** *collection* $\mathscr{B}$ of ***neighborhoods*** of x such that *each neighborhood* of x ***contains at least one*** of the elements of $\mathscr{B}$.

  A space that has *a* ***countable basis*** *at each of its points* is said to satisfy ***the first countability axiom***, or to be ***first-countable***.

- **Remark** *Every metric space is first-countable.*

- **Proposition 1.1** ***(Limit Point Detected by Convergent Sequence)*** *[Munkres, 2000]*
  Let X be a topological space.

  1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the* ***converse*** *holds if X is* ***first-countable****.*

  2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The* ***converse*** *holds if X is* ***first-countable****.*

- **Definition** (***Second-Countable***)
  If a space X has ***a countable basis*** *for its topology*, then X is said to satisfy ***the second countability axiom***, or to be ***second-countable***.

- **Example** ($\mathbb{R}$)
  The real line $\mathbb{R}$ has a ***countable basis***, which is the collection of all *open intervals* (a, b) with ***rational end points***.

- **Example** ($\mathbb{R}^n$ and $\mathbb{R}^\omega$ under *product topology*)

  1. *The finite dimensional space* $\mathbb{R}^n$ has a ***countable basis***, which is the collection of all *product of intervals* with ***rational end points***.

  2. *The countable infinite dimensional space* $\mathbb{R}^\omega$ has a ***countable basis***, which is the collection of all products $\prod_{n \in \mathbb{Z}_+} U_n$, where U_n is an *open interval with* ***rational end points*** *for* ***finitely many*** *values* of n, and $U_n = \mathbb{R}$ for *all other values* of n.

- **Example** ($\mathbb{R}^\omega$ *under* ***Uniform Topology Not Second-Countable***)
  In the ***uniform topology***, $\mathbb{R}^\omega$ satisfies *the first countability axiom (being metrizable)*. However, it *does* ***not satisfy the second***.

  To verify this fact, we first show that if X is a space having a countable basis $\mathscr{B}$, then ***any discrete subspace*** A of X must be ***countable***. Choose, for each $a \in A$, a basis element B_a that *intersects* A in the point a ***alone***. If a and b are distinct points of A, the sets B_a and B_b are *different*, since the first *contains* a and the second does not. It follows that the map $a \mapsto B_a$ is an ***injection*** of A into $\mathscr{B}$, so A *must be countable* (as $\mathscr{B}$ being countable).

  Now we note that the subspace A of $\mathbb{R}^\omega$ consisting of *all sequences of* 0*'s and* 1*'s is* ***uncountable***; and it has the ***discrete topology*** because $\bar{\rho}(a, b) = 1$ for any two distinct points a and b of A. Therefore, in the uniform topology $\mathbb{R}^\omega$ *does not have a countable basis.* ∎

- **Example** (***Topological Manifolds***)

  Definition Suppose M is a ***topological space***. We say that M is a ***topological manifold*** of *dimension* n or a ***topological*** n***-manifold*** if it has the following properties:

  1. M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.

  2. M is ***second-countable***: there exists a ***countable basis*** for the topology of M.

  3. M is ***locally Euclidean of dimension*** n: each point of M has a neighborhood that is ***homeomorphic*** to an open subset of $\mathbb{R}^n$.

- Both countability axioms are well behaved with respect to the operations of taking subspaces or countable products:

  Proposition 1.2 *(**Subspaces and Countable Product**) [Munkres, 2000]*
  *A **subspace** of* _____

  1. *a first-countable space is first-countable;*

  2. *a second-countable space is second-countable.*

  *And a **countable product** of* _____

  1. *first-countable spaces is first-countable;*

  2. *second-countable spaces is second-countable.*

- **Definition** (***Dense Subset***)
  A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$. (That is, *every point in X is a limit point of A.*)

- **Definition** (***Separability***)
  A topological space X is called ***separable*** if and only if it has a ***countable dense set***.

- **Definition** (***Lindelöf Space***)
  A space for which *every open covering* contains *a* ***countable*** *subcovering* is called a ***Lindelöf space***.

- **Proposition 1.3** *(**Properties of Second-Countability**) [Munkres, 2000]*
  *Suppose that X has a **countable basis**. Then:*

  1. *Every **open covering** of X contains a **countable** subcollection covering X. (X is **Lindelöf space**)*

  2. *There exists a **countable** subset of X that is **dense** in X. (X is **separable**)*

- **Proposition 1.4** *(**Metric Space Equivalence**) [Munkres, 2000]*
  *Suppose that X is a **metrizable spcae**. The following statements are equivalent:*

  1. *X has a **countable basis** (**second-countable**).*

  2. *X has a **countable dense subset** (**separable**).*

  3. *Every **open covering** of X contains a **countable** subcollection covering X. (**Lindelöf space**).*

- **Proposition 1.5** *(**Compact Metrizable Space**) [Munkres, 2000]*

Every ***compact metrizable*** *space X has a countable basis (i.e.* ***second-countable****).*

[Hint: Let $\mathscr{A}_n$ be a finite covering of X by $1/n$-balls.]

- **Proposition 1.6** ***(Preservation by Continuity)*** *[Munkres, 2000]*
  Let $f : X \to Y$ be continuous.

  1. *If X is* ***Lindelöf*** *, then $f(X)$ is* ***Lindelöf*** *;*
  2. *if X has a* ***countable dense subset****, then $f(X)$ satisfies the same condition.*

- **Proposition 1.7** ***(Preservation by Product)*** *[Munkres, 2000]*
  If X is a ***countable product*** *of spaces having countable dense subsets (****separable****), then X has a countable dense subset (****separable****).*

- **Example** (***The Product of two Lindelöf Spaces Need Not be Lindelöf***)
  The space $\mathbb{R}_\ell$ is *Lindelöf*, but the product space $\mathbb{R}_\ell^2$ is not. $\mathbb{R}_\ell^2$ is called ***the Sorgenfrey plane***.

  The space $\mathbb{R}_\ell^2$ has as basis all sets of the form $[a, b) \times [c, d)$. To show it is not *Lindelöf*, consider the subspace

  $$L = \{(x, -x) : x \in \mathbb{R}_\ell\}.$$

  It is easy to check that L is ***closed*** in $\mathbb{R}_\ell^2$. Let us cover $\mathbb{R}_\ell^2$ by ***the open set*** $\mathbb{R}_\ell^2 \setminus L$ and by all *basis elements* of the form

  $$[a, b) \times [-a, d).$$

  Each of these open sets intersects L in ***at most one point***. Since L is ***uncountable***, *no countable subcollection covers $\mathbb{R}_\ell^2$.* $\blacksquare$

- **Example** (***The Subspace of Lindelöf Space Need Not be Lindelöf***)
  The ***ordered square*** I_o^2 is ***compact***; therefore it is *Lindelöf*, trivially. However, the *subspace* $A = I \times (0, 1)$ is ***not Lindelöf***. For A is the union of the disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in A. This collection of sets is ***uncountable***, and ***no proper subcollection covers*** A. $\blacksquare$

- **Proposition 1.8** ***(Preservation by Continuous Open Map)*** *[Munkres, 2000]*
  Let $f : X \to Y$ be ***continuous open map****. If X satisfies* ***the first*** *or* ***the second countability axiom****, then $f(X)$ satisfies the same axiom.*

- **Remark** (*Relationship of Several Topological Properties*)

$$\begin{tikzcd}
 & \textbf{\textit{first-countable}}\ \textit{space} \\
\textbf{\textit{second-countable}}\ \textit{space} \arrow[ur, bend left] \arrow[r] \arrow[dr] & \textbf{\textit{separable}}\ \textit{space} \arrow[l, dashed, bend right, "\textit{metrizable}"'] \\
 & \textbf{\textit{Lindelöf}}\ \textit{space.} \arrow[ul, dashed, bend left, "\textit{metrizable}"]
\end{tikzcd}$$

- **Definition** (G_δ ***Set***)
  A ***G_δ set*** in a space X is a set A that equals a ***countable intersection*** of *open* sets of X.

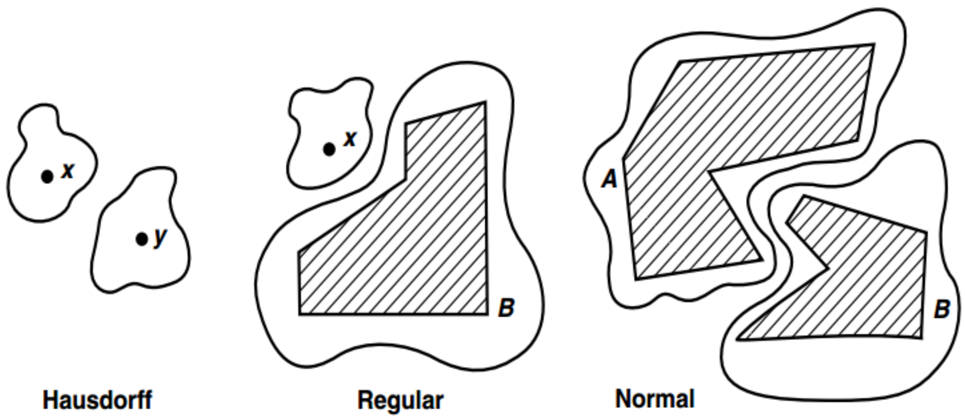

Figure 1: The separation axioms [Munkres, 2000]

- **Remark** By definition of topology, G_δ is ***neither open*** *nor* ***closed***.

- **Exercise 1.9** *Show that in a first-countable T_1 space, every one-point set is a G_δ set.*

1.2 The Separation Axioms

- **Definition** (***Regular Space*** *and* ***Normal Space***)
  Suppose that one-point sets are *closed* in X (i.e. X is T_1 space). Then X is said to be ***regular (**T_3**)*** if for each pair consisting of *a point x and* ***a closed set*** B ***disjoint*** *from x*, there exist ***disjoint open sets*** containing x and B, respectively.

  The space X is said to be ***normal (**T_4**)*** if for each pair A, B of ***disjoint closed sets*** of X, there exist ***disjoint open sets*** containing A and B, respectively.

- **Proposition 1.10**

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$$

- **Remark** (***Separation axioms*** $\neq$ ***Discounnected Space***)
  These axioms are called ***separation axioms*** for the reason that they involve "*separating* certain kinds of *sets from one another* by ***disjoint open sets***.

  We have used the word "*separation*" before, of course, when we studied *connected spaces*. But in that case, we were trying to find *disjoint open sets* ***whose union was the entire space***.

- **Lemma 1.11** *Let X be a topological space. Let one-point sets in X be closed.*
  1. *X is* ***regular*** *if and only if given a point x of X and a neighborhood U of x, there is a* ***neighborhood*** *V of x such that $\bar{V} \subseteq U$.*
  2. *X is* ***normal*** *if and only if given a* ***closed*** *set A and an open set U containing A, there is an* ***open set*** *V containing A such that $\bar{V} \subseteq U$.*

- **Remark** X is ***regular*** $\Leftrightarrow$ Each point of X has a ***closed neighborhood***

Note that X is ***locally compact Hausdorff*** $\Leftrightarrow$ Each point of X has a ***precompact neighborhood*** i.e. it has a closed neighborhood and *the closure is compact.*

- **Proposition 1.12** ***(Simply Ordered Set is Hausdorff)*** *[Munkres, 2000]*
  Every ***simply ordered set*** *is a* ***Hausdorff*** *space in the* ***order topology****.*

- **Proposition 1.13** ***(Order Topology is Regular)*** *[Munkres, 2000]*
  Every ***order topology*** *is a* ***regular****.*

- **Remark** It can be shown actually that every ***order topology*** is a ***normal***, which includes all of these two previous results.

- **Proposition 1.14** ***(Preservation of Hausdorff and Regular Axioms)***

  1. *The* **product** *of two Hausdorff/regular spaces is a Hausdorff/regular space.*

  2. *A* ***subspace*** *of a Hausdorff/regular space is a Hausdorff/regular space.*

- **Example** ($\mathbb{R}_K$ ***is Hausdorff but Not Regular***)
  The space $\mathbb{R}_K$ is ***Hausdorff*** but ***not regular***. Recall that $\mathbb{R}_K$ denotes the reals in the topology having as *basis all open intervals* (a, b) *and all sets of the form* $(a, b) \setminus K$, where $K = \{1/n : n \in \mathbb{Z}_+\}$. This space is *Hausdorff*, because any two distinct points have *disjoint open intervals* containing them.

  But it is ***not regular***. The set K is ***closed*** in $\mathbb{R}_K$, and it does *not contain the point* 0. Suppose that there exist *disjoint open sets* U and V containing 0 and K, respectively. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) \setminus K$, since each basis element of the form (a, b) containing 0 intersects K. Choose n large enough that $1/n \in (a, b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d). Finally, choose z so that $z < 1/n$ and $z > \max\{c, 1/(n + 1)\}$. Then z belongs to both U and V, so they are not disjoint. ∎

- **Example** ($\mathbb{R}_\ell$ ***is Normal***)
  The space $\mathbb{R}_\ell$ is ***normal***. Recall that $\mathbb{R}_\ell$ is $\mathbb{R}$ with ***lower limit topology***. (i.e. the basis element is *the half-interval* $[a, b)$.) It is immediate that *one-point sets are closed* in $\mathbb{R}_\ell$, since the topology of $\mathbb{R}_\ell$ is *finer* than that of $\mathbb{R}$.

  To check ***normality***, suppose that A and B are *disjoint closed sets* in $\mathbb{R}_\ell$. For each point a of A choose a basis element $[a, x_a)$ *not intersecting* B; and for each point b of B choose a basis element $[b, x_b)$ *not intersecting* A. The open sets

  $$U = \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x_b)$$

  are ***disjoint open sets*** about A and B, respectively. ∎

- **Example** (***The Sorgenfrey plane*** $\mathbb{R}_\ell^2$ ***is Not Normal***)
  The space $\mathbb{R}_\ell$ is regular, so the product space $\mathbb{R}_\ell^2$ is regular. Thus this example serves *two purposes*. It shows that ***a regular space need not be normal***, and it shows that ***the product of two normal spaces need not be normal***.

- **Definition** (***Perfect Map***)
  A ***closed continuous surjective*** *map* $p : X \to Y$ is called a ***perfect map*** if $p^{-1}(\{y\})$ is ***compact*** for each $y \in Y$.

- **Remark** *A perfect map* is a *quotient map.*

- **Proposition 1.15** ***(Preservation Properties of Perfect Map)*** *[Munkres, 2000]*
  *Let $p : X \to Y$ be a **perfect map**, i.e. it is a **closed continuous surjective** map who preimage of one point set is **compact**. Then*

  1. *If X is **Hausdorff**, then so is Y.*
  2. *If X is **regular**, then so is Y.*
  3. *If X is **locally compact**, then so is Y.*
  4. *If X is **second-countable**, then so is Y.*

- **Theorem 1.16** ***(Preservation Properties of Orbit Space)*** *[Munkres, 2000]*
  *Let G be a **compact topological group**; let X be a topological space; let α be an **action** of G on X. The orbit space X/G is the quotient space under equivalence relationship $x \sim \alpha(x)$.*

  1. *If X is **Hausdorff**, then so is X/G.*
  2. *If X is **regular**, then so is X/G.*
  3. *If X is **normal**, then so is X/G.*
  4. *If X is **locally compact**, then so is X/G.*
  5. *If X is **second-countable**, then so is X/G.*

1.3 Normal Spaces

- **Remark** As we have seen, unlike its name suggested, normal spaces are *not as well-behaved* as one might wish. On the other hand, ***most of the spaces*** with which we are familiar do *satisfy this axiom*, as we shall see.

  Its ***importance*** comes from the fact that the results one can prove ***under the hypothesis of normality*** are central to much of topology. The ***Urysohn metrization theorem*** and ***the Tietze extension theorem*** are two such results

- **Proposition 1.17** *[Munkres, 2000]*
  *Every **locally compact Hausdorff** space is **regular**.*

- **Theorem 1.18** ***(Regular + Second-Countable ⇒ Normal)****[Munkres, 2000]*
  *Every **regular** space with a **countable basis** is **normal**.*

- **Proposition 1.19** ***(Regular + Lindelöf ⇒ Normal)****[Munkres, 2000]*
  *Every **regular Lindelöf** space is **normal**.*

- **Theorem 1.20** *[Munkres, 2000]*
  *Every **metrizable** space is **normal**.*

- **Theorem 1.21** *[Munkres, 2000, Reed and Simon, 1980]*
  *Every **compact Hausdorff** space X is **normal**.*

- **Theorem 1.22** *[Munkres, 2000]*
  *Every **well-ordered** set X is **normal** in the **order topology**.*

In fact, a stronger theorem holds:

Theorem 1.23 *Every **order** topology is normal*

- **Example** (***The Uncountable Product of Normal Spaces Need Not be Normal***)
  If J is ***uncountable***, the product space $\mathbb{R}^J$ is ***not normal***.

  This example serves *three purposes*. It shows that ***a regular space*** $\mathbb{R}^J$ ***need not be normal***. It shows that ***a subspace of a normal space need not be normal***, for $\mathbb{R}^J$ is *homeomorphic* to the subspace $(0,1)^J$ of $[0,1]^J$, which (assuming *the Tychonoff theorem*) is ***compact Hausdorff*** and therefore *normal*. And it shows that ***an uncountable product of normal spaces need not be normal***. It leaves unsettled the question as to whether *a finite or a countable product of normal spaces might be normal*.

- **Example** (***The Finite Product of Normal Spaces Need Not be Normal***).
  Recall $S_\Omega = \{x : x \in X \text{ and } x < \Omega\}$ is the ***uncountable section*** of a ***well-ordered set*** X by Ω where Ω is the ***largest element*** of X (called ***the minimal uncountable well-ordered set***).

  Consider ***the well-ordered set*** $\bar{S}_\Omega$, *in the order topology*, and consider the subset S_Ω, *in the subspace topology* (which is the same as the order topology). *Both spaces are* ***normal***, but the product space $S_\Omega \times \bar{S}_\Omega$ is ***not normal***.

  his example serves *three purposes*. First, it shows that ***a regular space need not be normal***, for $S_\Omega \times \bar{S}_\Omega$ is a *product of regular spaces* and therefore regular. Second, it shows that ***a subspace of a normal space need not be normal***, for $S_\Omega \times \bar{S}_\Omega$ is a *subspace* of $\bar{S}_\Omega \times \bar{S}_\Omega$, which is a ***compact Hausdorff*** *space* and therefore ***normal***. Third, it shows that ***the product of two normal spaces need not be normal***.

2 Important Theorems

2.1 The Urysohn Lemma

- **Theorem 2.1** *(**Urysohn Lemma**). [Munkres, 2000]*
  *Let X be a **normal** space; let A and B be **disjoint closed subsets** of X. Let $[a, b]$ be a **closed interval** in the real line. Then there exists a **continuous** map*

  $$f : X \to [a, b]$$

  *such that $f(x) = a$ for **every** x in A, and $f(x) = b$ for **every** x in B.*

- **Corollary 2.2** *(**Urysohn Lemma for** G_δ). [Munkres, 2000]*
  *Let X be a **normal** space. Then there exists a **continuous** map*

  $$f : X \to [0, 1]$$

  *such that $f(x) = 0$ for **every** $x \in A$, and $f(x) > 0$ for **every** $x \notin A$ **if and only if** A is a G_δ set, i.e. it equal to a countable intersection of open sets in X.*

- **Theorem 2.3** *(**Strong Form of Urysohn Lemma**). [Munkres, 2000]*
  *Let X be a **normal** space. Then there exists a **continuous** map*

  $$f : X \to [0, 1]$$

Table 1: Comparison the Urysohn Lemma and Geometric Hahn-Banach Theorem

	Urysohn's Lemma	*Geometric Hahn-Banach Theorem*
space	***normal*** *topological space* T_4	***normed linear*** *space*
weaker space	***completely regular*** *topological space*	***locally convex*** *space*
objects	*two* ***closed*** *subsets* A, B	*two* ***convex*** *subsets* A, B
separation pre-condition	***closed*** *subsets are* ***disjoint***	***convex*** *sets are* ***disjoint***
separating function	***continuous function*** $f : X \to [0, 1]$	***a hyperplane defined by linear functional*** $\ell(x) = a$
conclusion	*two* ***closed sets*** *can be* ***separated*** *by* f	*two* ***convex sets*** *can be* ***separated*** *by* ***hyperplane***
conclusion in math	$f(A) = \{0\}$ and $f(B) = \{1\}$	$\sup_{a\in A} \ell(a) \le a \le \inf_{b\in B} \ell(b)$

such that $f(x) = 0$ *for* $x \in A$, ***and*** $f(x) = 1$ *for* $x \in B$, *and* $0 < f(x) < 1$ ***otherwise if and only if*** A *and* B *are disjoint closed* G_δ *set in* X.

- **Definition** (***Separation by Continuous Function***)
  If A and B are two subsets of the topological space X, and if there is a **continuous** function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be ***separated by a continuous function***.

- **Remark** *The Urysohn lemma* says that if *every pair of* ***disjoint closed sets*** in X can be *separated by disjoint open sets*, then each such pair can be ***separated by a continuous function***. The ***converse*** is trivial, for if $f : X \to [0, 1]$ is the function, then $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ are *disjoint open sets* containing A and B, respectively.

- **Remark** (***Separation by Continuous Function*** *vs* ***Separation by Linear Function***)
  We can compare *the Urysohn lemma* with *the geometric Hahn-Banach theorem* which seperate two ***convex sets*** with *linear functional*. See Table 1. The *geometric Hahn-Banach theorem* can be seen as a generalization of *the Urysohn lemma* in ***normed linear space***.

- **Remark** (***Continuous Function in Compact Hausdorff Space***) [Reed and Simon, 1980]
  The Urysohn lemma suggests that there are ***a lot of continuous functions*** in *normal space*. *The space of all real-valued continuous functions* $\mathcal{C}_{\mathbb{R}}(X)$ on a ***compact Hausdorff space*** X (which is normal space) has a ***dense subset*** since any real-valued continuous functions on $[0, 1]$ is a *uniform limit* of ***polynomials***.

- **Definition** (***Completely Regular***)
  A space X is ***completely regular*** if *one-point sets* are *closed* in X and if for each point x_0 and each ***closed*** set A not containing x_0, there is a ***continuous function*** $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

- **Remark**

$$\text{normal} \;\Rightarrow\; \text{completely regular} \;\Rightarrow \text{regular}$$

Proposition 2.4 *A **subspace** of a completely regular space is completely regular.*

*A **product** of completely regular spaces is completely regular.*

- **Example** ($S_\Omega \times \bar{S}_\Omega$ ***is Completely Regular but Not Normal***).
$S_\Omega \times \bar{S}_\Omega$ is ***not normal*** but it is the product space of two completely regular spaces.

- **Theorem 2.5** *(**Urysohn Lemma, Locally Compact Version**). [Folland, 2013]*
*Let X be a **locally compact Hausdorff** space and $K \subseteq U \subseteq X$ where K is **compact** and U is **open**. Then there exists a **continuous** map*

$$f : X \to [0, 1]$$

*such that $f(x) = 1$ for **every** $x \in K$, and $f(x) = 0$ for x outside a **compact subset** of U.*

- **Corollary 2.6** *[Folland, 2013]*
*Every **locally compact Hausdorff** space is **completely regular**.*

- **Remark** (***Dual Space of $\mathcal{C}_c(X)$ on Locally Compact Hausdorff Space***) [Reed and Simon, 1980, Folland, 2013]
The famous ***Riesz-Markov theorem*** shows that the ***dual space*** of $\mathcal{C}_c(X)$, the space of compactly supported continuous function on *locally compact Hausdorff space* X is isomorphic to *the space of **signed regular Borel measures*** on X, i.e. $(\mathcal{C}_c(X))^* \simeq \mathcal{M}(X)$. The proof of *the Riesz-Markov theorem* is based on ***the Urysohn lemma*** *for locally compact space.*

2.2 The Urysohn Metrization Theorem

- **Theorem 2.7** *(**Urysohn Metrization Theorem**). [Munkres, 2000]*
*Every **regular** space X with a **countable basis** is **metrizable**.*

- **Theorem 2.8** *(**Embedding Theorem**). [Munkres, 2000]*
*Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha\in J}$ is an indexed family of **continuous** functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is **positive** at x_0 and **vanishes outside** U. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha\in J}$$

*is a **topological embedding** of X in $\mathbb{R}^J$. If f_α maps X into $[0, 1]$ for each α then F **embeds** X in $[0, 1]^J$.*

- **Definition** (***Separation*** *of* ***Points*** *From* ***Closed*** *Set by* ***Continuous*** *Functions*)
A family of continuous functions that satisfies the hypotheses of *the embedding theorem above* is said to ***separate points from closed sets in*** X.

The existence of such a family is readily seen to be *equivalent*, for a space X in which one-point sets are *closed*, to the requirement that X be *completely regular*.

Table 2: Comparison Tietze Extension Theorem and Hahn-Banach Theorem

	Tietze Extension Theorem	***Hahn-Banach Theorem***
space	***normal*** *topological space* T_4	***normed linear*** *space*
subspace	***topological*** *subspace*	***linear*** *subspace*
function to be extended	***real-valued continuous function***	***linear functional***
additional constraint	*the subspace is* ***closed***	*the functional* ***bounded above by a sublinear functional***
conclusion	*the domain of continuous function can be* ***extended to entire space***	*the domain of linear functional can be* ***extended to entire space***

- **Corollary 2.9** *(**Embedding Equivalent Definition** of **Completely Regular**) [Munkres, 2000]*
  *A space X is **completely regular if and only if** it is **homeomorphic** to a subspace of $[0, 1]^J$ for some J.*

2.3 The Tietze Extension Theorem

- **Theorem 2.10** *(**Tietze Extension Theorem**) [Munkres, 2000, Reed and Simon, 1980]*
  *Let X be a **normal space**; let A be a **closed subspace** of X.*

  1. *Any **continuous** map of A into the **closed interval** $[a, b]$ of $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $[a, b]$.*
  2. *Any **continuous** map of A into $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $\mathbb{R}$.*

- **Theorem 2.11** *(**Tietze Extension Theorem, Locally Compact Version**) [Folland, 2013]*
  *Let X be a **locally compact Hausdorff space**; let K be a **compact subspace** of X. If $f \in \mathcal{C}(K)$ is a **continuous** map of K into $\mathbb{R}$, there exists a **continuous** extension $F \in \mathcal{C}(X)$ of **all of** X into $\mathbb{R}$ such that $F|_K = f$. Moreover, F may be taken to **vanish outside a compact set**.*

- **Remark** (***Extension of Continuous Function vs. Extension of Linear Functional***)
  We can compare *the Tietze extension theorem* with *the Hahn-Banach theorem* in normed linear space. See from Table 2 that the Hahn-Banach theorem generalize the Tietze extension theorem from normal topological space to the normed linear space (which is metrizable so normal).

3 Embeddings of Manifolds

4 Summary of Preservation of Topological Properties

Table 3: Summary of Preservation of Topological Properties Under Transformations

	subspace	***product space***	***image of continuous function***
connected	✓	if finite product, ✓; if countable product, ✓ under product topology	✓
locally connected	if ***open and connected*** subspace, ✓	if ***all but finitely many*** *of spaces are* ***connected***, ✓	in general ×
compact	if ***closed*** subspace, ✓;	for ***arbitrary*** product, ✓	✓
locally compact	if ***closed*** *or* ***open*** subspace and Hausdorff, ✓	if ***finite*** product, ✓; if infinite product ×	if f is a ***perfect map***, then ✓; in general ×
first-countable	✓	if ***countable*** product, ✓	if f is a ***open map***, then ✓; in general ×
second-countable	✓	if ***countable*** product, ✓	if f is a ***open map or perfect map***, then ✓; in general ×
separable	if metrizable, then ✓; in general ×	if ***countable*** product, ✓	✓
Lindelöf	if metrizable, then ✓; in general ×	×	✓
T_1 ***axiom***	✓	for ***arbitrary*** product, ✓	in general ×
Hausdorff T_2	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓; in general ×
regular T_3	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓ ; in general ×
completely regular	✓	for ***arbitrary*** product, ✓	in general ×
normal T_4	×	×	×
paracompact	if ***closed*** subspace, ✓;	×	×
topologically complete	for ***closed*** *or* ***open*** subspace, ✓	if ***countable*** product, ✓	×

5 Summary of Counterexamples for Topological Properties

Table 4: Summary of Counterexamples for Topological Properties

	$\mathbb{R}^\omega$ $\mathscr{T}_{prod}$	$\mathbb{R}^\omega$ $\mathscr{T}_{box}$	$\mathbb{R}^\omega$ $\mathscr{T}_{unif}$	$\mathbb{R}_K$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell^2$	I_o^2	S_Ω	$\bar{S}_\Omega$	$S_\Omega \times \bar{S}_\Omega$	$(x, \sin(1/x))$
connected	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
path connected	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
locally connected	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$
locally path connected	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$
limit point compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$		$\checkmark$
sequentially compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$		$\checkmark$
locally compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
paracompact	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$
first-countable	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	
second-countable	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	
separable	$\checkmark$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	
Lindelöf	$\checkmark$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$
T_1 ***axiom***	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Hausdorff T_2	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
regular T_3	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
completely regular	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
normal T_4	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	
locally metrizable	$\checkmark$	$\times$	$\checkmark$	$\times$			$\times$	$\checkmark$	$\times$	$\times$	
metrizable	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$

1. $(\mathbb{R}^\omega, \mathscr{T}_{prod})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***product topology***. Note that under product topology, the ***basis*** is of form $\prod_{n\in\mathbb{Z}_+} U_n$ where there exists some N so that for all $n \geq N$, $U_n = \mathbb{R}$.

2. $(\mathbb{R}^\omega, \mathscr{T}_{box})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***box topology***. Note that under box topology, the ***basis*** is of form $\prod_{n\in\mathbb{Z}_+} U_n$ where $U_n \neq \mathbb{R}$ for all n.

3. $(\mathbb{R}^\omega, \mathscr{T}_{unif})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***uniform topology***. Note that the uniform topology is induced by ***the uniform metric*** $\bar{\rho}$ on $\mathbb{R}^\omega$, which is defined by the equation

$$\bar{\rho}((x_n)_{n\in\mathbb{Z}_+}, (y_n)_{n\in\mathbb{Z}_+}) = \sup\left\{\bar{d}(x_n, y_n) : n \in \mathbb{Z}_+\right\},$$

   where $\bar{d}$ is ***the standard bounded metric*** on $\mathbb{R}$.

4. $\mathbb{R}_K$: the real line $\mathbb{R}$ equipped with the ***K-topology***. The K-topology is ***generated*** by *all open intervals* (a, b) **and** *all sets **of the form***

$$(a, b) \setminus K \text{ where } K = \{1/n : n \in \mathbb{Z}_+\}.$$

5. $\mathbb{R}_\ell$: the real line $\mathbb{R}$ equipped with the ***lower limit topology***. The basis of lower limit topology is the collection of all ***half-open intervals*** of the form

$$[a, b) = \{x : a \leq x < b\},$$

   where $a < b$. $\mathbb{R}_\ell$ is also called ***the Sorgenfrey line***.

6. $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$: is called ***the Sorgenfrey plane***.

7. I_o^2: is called ***ordered square*** where $I = [0, 1]$. It is the set $[0, 1] \times [0, 1]$ in ***the dictionary order topology***. In dictionary order relationship, $(x_1, x_2) < (y_1, y_2)$ if and only if $x_1 < y_1$ or $(x_1 = y_1) \wedge (x_2 < y_2)$. In dictionary order topology, open intervals are of the form

$$\{(x_1, x_2) : x_1 \in (a, b) \text{ or } (x_1 = c) \wedge (x_2 \in (d, e))\} = ((a, b) \times I) \cup (c \times (d, e)).$$

8. S_Ω: is ***the uncountable ordinal space***. If A is a ***well-ordered set*** then A itself contains a ***smallest element*** which we will denote by a_0. For each element x in a *well-ordered set* A, ***the section at*** x is defined to be the subset

$$S_x = (-\infty, x) = [a_0, x) = \{y \in A : y < x\}.$$

   The uncountable ordinal space S_Ω *is an* ***uncountable well-ordered set*** in which *each section* S_x *is* ***countable***. This description of S_Ω is justified by the following:

   Lemma 5.1 *There exists an uncountable well-ordered set A such that S_x is countable for each $x \in A$, and any two uncountable well-ordered sets satisfying this property are* ***order isomorphic*** *(that is, they have the same order type).*

9. $\bar{S}_\Omega$: is ***the closed uncountable ordinal space***. It is defined by $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ with ***the well-ordering*** given by: (a) if $x, y \in S_\Omega$ then $x < y$ in $\bar{S}_\Omega$ iff $x < y$ in S_Ω, and (b) if $x \in S_\Omega$ then $x < \Omega$. Notice that Ω is a ***maximal element*** in $\bar{S}_\Omega$ (but S_Ω *does not have a maximal element*). S_Ω is the section of Ω in $\bar{S}_\Omega$.

10. $S_\Omega \times \bar{S}_\Omega$

11. $\bar{S}$: is called ***the topologist's sine curve***. It is the closure of the graph

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

That is $\bar{S} = S \cup \{(x, y) : x = 0\}$.

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