Lecture 10: Vector Bundles

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1 Vector Bundles

1.1 Definitions

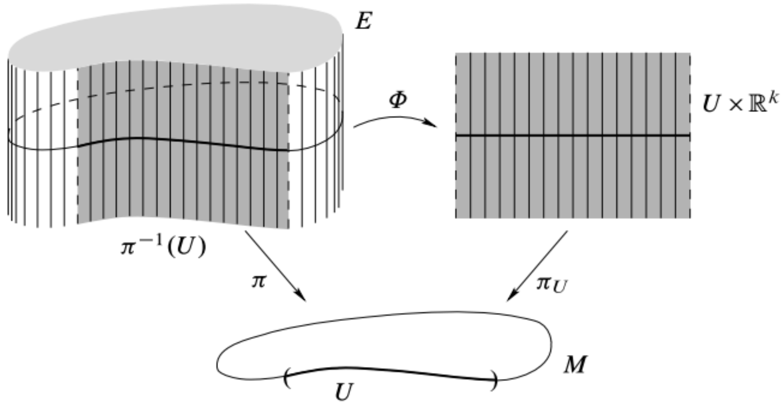

Figure 1: The local trivialization of vector bundle E over neighborhood U. [Lee, 2003.]

- **Definition** Let M be a *topological space*. A *(real)* ***vector bundle*** *of rank k over* M is a ***topological space*** E together with a ***surjective continuous*** *map* $\pi : E \rightarrow M$ satisfying the following conditions:

    1. For each $p \in M$, the ***fiber*** $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a *k-dimensional real* ***vector space***.

    2. For each $p \in M$, there exist a neighborhood U of p in M and a ***homeomorphism*** $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ (called *a* ***local trivialization*** *of* E *over* U), satisfying the following conditions (Fig. 1):

        (a) $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \rightarrow U$ is the ***projection***);

        (b) for each $q \in U$, the *restriction* of Φ to E_q is a ***vector space isomorphism*** from E_q to $\{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$.

    The space E is called ***the total space of the bundle***, M is called its ***base***, and π is its ***projection***.

- **Definition** If M and E are smooth manifolds with or without boundary, π is *a smooth map*, and *the local trivializations* can be chosen to be *diffeomorphisms*, then E is called *a* ***smooth vector bundle***. In this case, we call *any local trivialization* that is a *diffeomorphism* onto its image *a* ***smooth local trivialization***.

- **Remark** ***Vector bundle*** E is a ***generalization and abstraction*** of ***the tangent bundle*** $TM = \bigsqcup_{p \in M} T_pM$. Like the tangle bundle, the ***natural coordinates*** constructed on a vector bundle make it look, *locally*, like ***the Cartesian product*** of *an open subset* of M with $\mathbb{R}^n$.

- **Remark** The map π associates each *vector space* $\pi^{-1}(p)$ in the *vector bundle* to a point p in the topological space M. Since $\pi = \pi_U \circ \Phi$, we can think of it as a ***projection map*** after *local trivialization*.

- **Remark** The ***rank*** of a vector bundle is the ***dimension*** of *vector space* $\pi^{-1}(p)$ associated with each point p.

- **Remark** A ***rank-1 vector bundle*** is often called ***a (real) line bundle***. ***Complex vector bundles*** are defined similarly, with "real vector space" replaced by "complex vector space" and $\mathbb{R}^k$ replaced by $\mathbb{C}^k$ in the definition.

- **Remark** Strictly speaking, a vector bundle is a pair (E, π) of total space and the projection. Depending on what we wish to emphasize, we sometimes omit some of the ingredients from the notation, and write "E ***is a vector bundle over*** M," or "$E \to M$ ***is a vector bundle***," or "$\pi : E \to M$ ***is a vector bundle***".

- **Definition** If there exists *a local trivialization* of E over ***all of*** M (called *a* ***global trivialization*** *of* E), then E is said to be *a* ***trivial bundle***. In this case, E itself is ***homeomorphic*** to the *product space* $M \times \mathbb{R}^k$.

  If $E \to M$ is a *smooth bundle* that admits *a smooth global trivialization*, then we say that E is ***smoothly trivial***. In this case E is ***diffeomorphic*** to $M \times \mathbb{R}^k$, not just *homeomorphic*.

  For brevity, when we say that *a smooth bundle* is *trivial*, we always understand this to mean *smoothly trivial*, not just trivial in the topological sense.

- **Lemma 1.1** ***(Transition between Two Smooth Local Trivializations)***
  Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are ***two smooth local trivializations*** *of E with $U \cap V \neq \emptyset$. There exists a* ***smooth map*** *$\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form*

  $$\Phi \circ \Psi^{-1}(p, v) = (p,\, \tau(p)v),$$

  where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

  Note that the following diagram commute:

  $$\begin{tikzcd}
  (U \cap V) \times \mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
  & U \cap V &
  \end{tikzcd}$$

  Definition The smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ described in this lemma is called the ***transition function*** between the local trivializations Φ and Ψ.

  For example, if M is a smooth manifold and Φ and Ψ are the local trivializations of tangent bundle TM associated with two different smooth charts, then the transition function between them is ***the Jacobian matrix*** of the *coordinate transition map*.

- Like the tangent bundle, vector bundles are often most easily described by giving ***a collection of vector spaces***, one for each point of the base manifold. The next lemma shows that in order to construct a smooth vector bundle, it is sufficient to construct the local trivializations, as long as they overlap with smooth transition functions.

  Lemma 1.2 ***(Vector Bundle Chart Lemma)***. *[Lee, 2003.]*
  Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$ we are given a ***real vector space*** *E_p of some fixed dimension k. Let $E = \bigsqcup_{p \in M} E_p$, and let*

$\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following data:

1. *an **open cover** $\{U_\alpha\}_{\alpha \in A}$ of M*
2. *for each $\alpha \in A$, a **bijective** map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space **isomorphism** from E_p to $\{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$*
3. *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p,\, \tau_{\alpha,\beta}(p)v), \tag{1}$$

*Then E has a **unique topology** and **smooth structure** making it into **a smooth manifold** with or without boundary and a **smooth rank-k vector bundle over M**; with π as **projection** and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.*

1.2 Examples

- **Example** (***Product Bundles***).
One particularly simple example of a rank k *vector bundle* over *any space* M is ***the product space*** $E = M \times \mathbb{R}^k$ with $\pi = \pi_1 : M \times \mathbb{R}^k \to M$ as its projection. Any such bundle, called a ***product bundle***, is ***trivial*** (with the identity map as a *global trivialization*). If M is a *smooth manifold* with or without boundary, then $M \times \mathbb{R}^k$ is ***smoothly trivial***.

- **Example** (***The Möbius Bundle***).
Define an ***equivalence relation*** on $\mathbb{R}^2$ by declaring that $(x, y) \sim (x', y')$ if and only if $(x', y') = (x + n, (-1)^n y)$ for some $n \in \mathbb{Z}$. Let $E = \mathbb{R}^2/\sim$ denote ***the quotient space***, and let $q \in \mathbb{R}^2 \to E$ be ***the quotient map***.

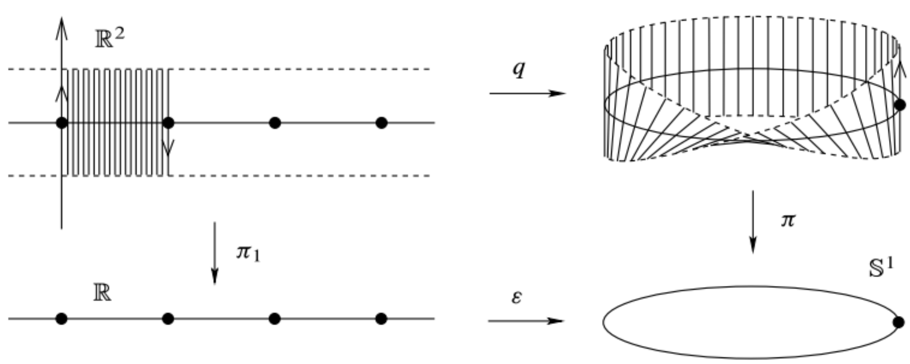

Figure 2: The Möbius Bundle. [Lee, 2003.]

To visualize E, let S denote the strip $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$. The restriction of q to S is *surjective* and *closed*, so it is a *quotient map*. The only nontrivial identifications made by $q|_S$ are on the *two boundary lines*, so we can think of E as the space obtained from S by giving the right-hand edge a half-twist to turn it upside-down, and then pasting it to the left-hand edge (Fig. 2). For any $r > 0$, the image under the quotient map q of the rectangle $[0, 1] \times [-r, r]$ is *a smooth* ***compact manifold*** *with boundary* called ***a Möbius band***; you can make a paper model of this space by pasting the ends of a strip of paper together with a half-twist.

Consider the following *commutative diagram*:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{q} & E \\ {\scriptstyle\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{R} & \xrightarrow{\epsilon} & \mathbb{S}^1, \end{array}$$

where π_1 is the ***projection*** onto the *first factor* and $\epsilon : \mathbb{R} \to \mathbb{S}^1$ is the ***smooth covering map*** $\epsilon(x) = \exp(2\pi\, jx)$. Because $\epsilon \circ \pi_1$ is ***constant*** on *each equivalence class*, it descends to ***a continuous map*** $\pi : E \to \mathbb{S}^1$.

A straightforward (if tedious) verification shows that E has a *unique smooth manifold structure* such that q is ***a smooth covering map*** and $\pi : E \to \mathbb{S}^1$ is ***a smooth real line bundle*** over $\mathbb{S}^1$, called ***the Möbius bundle***. ∎

- The most important examples of vector bundles are *tangent bundles*.

  Proposition 1.3 ***(The Tangent Bundle as a Vector Bundle).***
  Let M be a smooth n-manifold with or without boundary, and let TM be its tangent bundle. With its ***standard projection map****, its* ***natural vector space structure*** *on each* ***fiber****, and the* ***topology*** *and* ***smooth structure*** *constructed as in chapter 3, TM is a* ***smooth vector bundle*** *of rank n over M.*

  Proof: Given any smooth chart (U, φ) for M with coordinate functions (x^i), define a map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

  $$\Phi\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \left(p, (v^1, \ldots, v^n)\right)$$

  This is linear on fibers and satisfies $\pi_1 \circ \Phi = \pi$. The composite map

  $$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathrm{Id}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}^n$$

  is equal to the coordinate map $\widetilde{\varphi}$ constructed in chapter 3. Since both $\widetilde{\varphi}$ and $\varphi \times \mathrm{Id}_{\mathbb{R}^n}$ are *diffeomorphisms*, so is Φ. Thus, Φ satisfies all the conditions for a smooth local trivialization. ∎

- Another example is *the cotangent bundle* (its fiber is a dual space of tangent space) that will be defined in next chapter.

  Proposition 1.4 ***(The Cotangent Bundle as a Vector Bundle).***
  Let M be a smooth n-manifold with or without boundary. With its standard projection map and the natural vector space structure on each fiber, the ***cotangent bundle*** *T^*M has a* ***unique topology*** *and* ***smooth structure*** *making it into a* ***smooth rank-n vector bundle*** *over M for which all coordinate covector fields are* ***smooth local sections****.*

- **Example** (***Whitney Sums***).
  Given a smooth manifold M and smooth vector bundles $E' \to M$ and $E'' \to M$ of ranks k' and k'', respectively, we will construct a new vector bundle over M called ***the Whitney sum*** of E' and E'', whose ***fiber*** at each $p \in M$ is ***the direct sum*** $E'_p \oplus E''_p$. *The total space* is defined as $E' \oplus E'' = \bigsqcup_{p\in M} E'_p \oplus E''_p$, with the obvious projection $\pi : E' \oplus E'' \to M$. For each $p \in M$, choose a neighborhood U of p small enough that there exist *local trivializations*

(U, Φ') of E' and (U, Φ'') of E'', and define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{k'+k''}$ by

$$\Phi(v', v'') = \left(\pi'(v'), \left(\pi_{\mathbb{R}^{k'}} \circ \Phi'(v'), \pi_{\mathbb{R}^{k''}} \circ \Phi''(v'')\right)\right).$$

Suppose we are given *another* such pair of *local trivializations* $(\widetilde{U}, \widetilde{\Phi}')$ and $(\widetilde{U}, \widetilde{\Phi}'')$. Let $\tau' : (U \cap \widetilde{U}) \to GL(k', \mathbb{R})$ and $\tau'' : (U \cap \widetilde{U}'') \to GL(k'', \mathbb{R})$ be the corresponding *transition functions*. Then the ***transition function*** for $E' \oplus E''$ has the form

$$\widetilde{\Phi} \circ \Phi^{-1}(p, (v', v'')) = (p, \tau(p)(v', v'')),$$

where $\tau(p) = \tau'(p) \oplus \tau''(p) \in GL(k' + k'', \mathbb{R})$ is the ***block diagonal matrix***

$$\begin{bmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{bmatrix}.$$

Because this depends smoothly on p, it follows from the chart lemma that $E' \oplus E''$ is a smooth vector bundle over M. ∎

- **Example** (***Ambient Tangent Bundle***)
Suppose $\pi : E \to M$ is a rank-k vector bundle and $S \subseteq M$ is any subset. We define the ***restriction of*** E ***to*** S to be the set $E|_S = \bigcup_{p \in S} E_p$, with the projection $E|_S \to S$ obtained by **restricting** π. If $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a *local trivialization* of E over $U \subset M$; it restricts to a *bijective map* $\Phi|_U : (\pi|_S)^{-1}(U \cap S) \to (U \cap S) \times \mathbb{R}^k$, and it is easy to check that these form *local trivializations* for a ***vector bundle structure*** on $E|_S$. If E is a smooth vector bundle and $S \subseteq M$ is an *immersed* or *embedded submanifold*, it follows easily from the chart lemma that $E|_S$ is a smooth vector bundle. In particular, if $S \subseteq M$ is a *smooth (embedded or immersed) submanifold*, then the ***restricted bundle*** $TM|_S$ is called *the* ***ambient tangent bundle*** *over* M. ∎

2 Local and Global Sections of Vector Bundles

2.1 Local and Global Sections

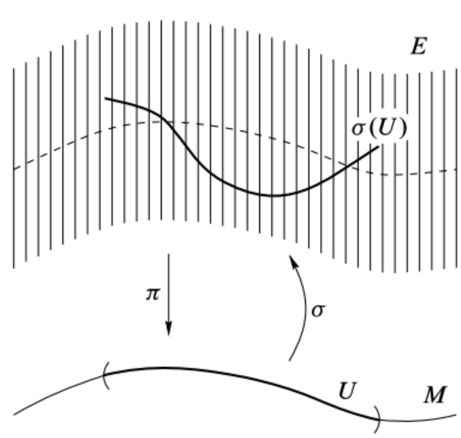

Figure 3: The local section of vector bundle E over neighborhood U. [Lee, 2003.]

- **Definition** Let $\pi : E \to M$ be a vector bundle. A ***section*** of E (sometimes called ***a cross section***) is a ***section*** of the map π, that is, a continuous map $\sigma : M \to E$ satisfying

$$\pi \circ \sigma = \mathrm{Id}_M.$$

This means that $\sigma(p)$ is an element of the *fiber* E_p for each $p \in M$.

- **Definition** More generally, ***a local section of*** E is a *continuous* map $\sigma : U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = \mathrm{Id}_U$. (See FIg 3.)

To emphasize the distinction, a section defined on *all of* M is sometimes called ***a global section***. Note that a *local section* of E over $U \subseteq M$ is the same as a *global section* of *the* ***restricted bundle*** $E|_U$.

- **Definition** If M is a smooth manifold with or without boundary and E is a ***smooth vector bundle***, a ***smooth (local or global) section*** *of* E is one that is a *smooth map* from its domain to E.

- **Remark** Just like a *vector bundle* E is a generalization of a *tangent bundle* TM, a ***section*** σ ***of vector bundle*** is a *generalization* of the ***vector fields*** X. $\sigma(p) \in E_p$ is an element of the vector space E_p and σ associates each point in space M with an element of the vector space E_p.

- **Definition** Define a ***rough (local or global) section*** of E over a set $U \subseteq M$ to be a map $\sigma : U \to E$ (*not necessarily continuous*) such that $\pi \circ \sigma = \mathrm{Id}_U$. A "section without further qualification always means a continuous section.

- **Definition** The ***zero section*** of E is the **global section** $\xi : M \to E$ defined by

$$\xi(p) = 0 \in E_p, \quad \forall p \in M.$$

- **Definition** As in the case of vector fields, the ***support*** of a section σ is the ***closure*** *of the set* $\{p \in M : \ \sigma(p) \neq 0\}$.

- **Example** (***Sections of Vector Bundles***).
Suppose M is a smooth manifold with or without boundary.

 1. Sections of TM are ***vector fields*** on M;

 2. Given an *immersed submanifold* $S \subseteq M$ with or without boundary, a section of ***the ambient tangent bundle*** $TM|_S \to S$ is called *a* ***vector field along*** S. It is a *continuous* map $X : S \to TM$ such that $X_p \in T_pM$ for each $p \in S$.

    This is different from *a vector field* ***on*** S, which satisfies $X_p \in T_pS$ at each point.

 3. If $E = M \times \mathbb{R}^k$ is a ***product bundle***, there is a *natural* ***one-to-one correspondence*** between sections of E and continuous functions from M to $\mathbb{R}^k$: a continuous function $F : M \to \mathbb{R}^k$ determines a ***section*** $\widetilde{F} : M \to M \times \mathbb{R}^k$ by $\widetilde{F}(x) = (x, F(x))$, and vice versa.

    If M is a smooth manifold with or without boundary, then the section $\widetilde{F}$ is smooth if and only if F is.

 4. The correspondence in the preceding paragraph yields a natural ***identification*** between the *space* $\mathcal{C}^\infty(M)$ and *the space of* ***smooth sections*** *of the* ***trivial line bundle*** $M \times \mathbb{R} \to M$

- **Definition** If $E \to M$ is a smooth vector bundle, the set of ***all smooth global sections of*** E is a ***vector space*** under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p)$$

  This vector space is usually ***denoted by*** $\Gamma(E)$. Note that for vector fields of tangent bundle TM, we use $\mathfrak{X}(M)$

- **Remark** Just like smooth vector fields, *smooth sections* of a *smooth bundle* $E \to M$ can be *multiplied* by *smooth real-valued functions*: if $f \in \mathcal{C}^\infty(M)$ and $\sigma \in \Gamma(E)$, we obtain *a* ***new section*** $f\sigma$ defined by

$$(f\sigma)(p) = f(p)\,\sigma(p).$$

- **Lemma 2.1** *(**Extension Lemma for Vector Bundles**).*
  *Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Suppose A is a **closed subset** of M, and $\sigma : A \to E$ is a section of $E|_A$ that is **smooth** in the sense that σ **extends** to a smooth local section of E in a neighborhood of each point. For each open subset $U \subseteq M$ containing A, there exists a **global smooth section** $\widetilde{\sigma} \in \Gamma(E)$ such that $\widetilde{\sigma}|_A = \sigma$ and supp$(\widetilde{\sigma}) \subseteq U$.*

2.2 Local and Global Frames

- **Definition** Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, a k***-tuple*** of ***local sections*** $(\sigma_1, \ldots, \sigma_k)$ of E over U is said to be ***linearly independent*** if their values $(\sigma_1(p), \ldots, \sigma_k(p))$ form a *linearly independent k-tuple in E_p* for each $p \in U$.

  Similarly, they are said to ***span*** E if *their values span E_p for each $p \in U$.*

- **Definition** *A **local frame** for E over U* is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of ***linearly independent*** *local sections* over U that ***span*** E; thus $(\sigma_1(p), \ldots, \sigma_k(p))$ is a ***basis*** for the *fiber* E_p for each $p \in U$.

  It is called a ***global frame*** if $U = M$.

- **Definition** If $E \to M$ is a smooth vector bundle, a *local or global frame* is a ***smooth frame*** if each σ_i is a *smooth section*. We often *denote a frame $(\sigma_1, \ldots, \sigma_k)$ by (σ_i)*.

- **Remark** The *(local or global) frames* for M that we defined in Chapter 8 are, in our new terminology, frames for the tangent bundle. We use both terms interchangeably depending on context: "***frame for*** M" and "***frame for*** TM" mean the same thing.

- **Proposition 2.2** *(**Completion of Local Frames for Vector Bundles**). [Lee, 2003.]*
  Suppose $\pi : E \to M$ is a smooth vector bundle of rank k.

  1. *If $(\sigma_1, \ldots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, with $1 \leq m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \ldots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \ldots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.*

  2. *If $(v_1, \ldots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$, with $1 \leq m \leq k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \ldots, m$.*

3. *If $A \subseteq M$ is a closed subset and $(\tau_1, \ldots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth in the sense described in Lemma 2.1, then there exists a smooth local frame $(\sigma_1, \ldots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for $i = 1, \ldots, k$.*

- **Remark** (***Local Frames Associated with Local Trivializations***).
Suppose $E \to M$ is a smooth vector bundle. If $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a smooth local trivialization of E, we can construct a local frame for E over U. Define maps $\sigma_1, \ldots, \sigma_k : U \to E$ by $\sigma_i(p) = \Phi^{-1}(p, e_i) = \Phi^{-1} \circ \widetilde{e}_i(p)$ as below:
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"'] \\ & U \arrow[ul, bend left, "\sigma_i"] \arrow[ur, bend right, "\widetilde{e}_i"'] & \end{tikzcd},$$
where $(e_1, \ldots, e_k)$ are the standard basis for $\mathbb{R}^k$ so that $\widetilde{e}_i$ is the frame such that $\widetilde{e}_i = (p, e_i)$. Then σ_i is *smooth* because Φ is a diffeomorphism, and the fact that $\pi_1 \circ \Phi = \pi$ implies that
$$\pi \circ \sigma_i(p) = \pi \circ \Phi^{-1}(p, e_i) = \pi_1(p, e_i) = p,$$
so σ_i is a *section*. To see that $(\sigma_i(p))$ forms *a basis for E_p*, just note that Φ restricts to an *isomorphism* from E_p to $\{p\} \times \mathbb{R}^k$, and $\Phi(\sigma_i(p)) = (p, e_i)$, so Φ takes $\sigma_i(p)$ to the standard basis for $\{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$. We say that ***this local frame*** (σ_i) ***is associated with*** Φ. $\blacksquare$

- **Proposition 2.3** *Every **smooth local frame** for a smooth vector bundle is associated with a **smooth local trivialization** constructed as above.*

- **Corollary 2.4** *A smooth vector bundle is **smoothly trivial** if and only if it admits a **smooth global frame**.*

- **Corollary 2.5** ***(The Coordinate Representation of Vector Bundle)***
Let $E \to M$ be a smooth vector bundle of rank k, let (V, φ) be a smooth chart on M with coordinate functions (x^i), and suppose there exists a smooth local frame (σ_i) for E over V. Define $\widetilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by
$$\widetilde{\varphi}\left(v^i \sigma_i(p)\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^k\right). \tag{2}$$
*Then $(\pi^{-1}(V), \widetilde{\varphi})$ is a **smooth coordinate chart** for E.*

- Just as *smoothness* of vector fields can be characterized in terms of their ***component functions*** in any smooth chart, *smoothness* of sections of vector bundles can be characterized in terms of ***local frames***.

- **Definition** Suppose (σ_i) is a smooth local frame for E over some open subset $U \subseteq M$. If $\tau : M \to E$ is a *rough section*, the value of τ at an arbitrary point $p \in U$ can be written $\tau(p) = \tau^i(p)\sigma_i(p)$ for some uniquely determined numbers $(\tau^1(p), \ldots, \tau^k(p))$. This defines k functions $\tau^i : U \to \mathbb{R}$, called *the **component functions** of τ with respect to the given local frame*.

- **Proposition 2.6** ***(Local Frame Criterion for Smoothness)***.
Let $\pi : E \to M$ be a smooth vector bundle, and let $\tau : M \to E$ be a rough section. If (σ_i) is a smooth local frame for E over an open subset $U \subseteq M$, then τ is smooth on U if and only if its component functions with respect to (σ_i) are smooth.

- **Proposition 2.7** ***(Uniqueness of the Smooth Structure on** TM**)***
*Let M be a smooth n-manifold with or without boundary. The topology and smooth structure on TM constructed in Proposition 3.18 are **the unique ones** with respect to which $\pi : TM \to M$ is a **smooth** vector bundle with the given vector space structure on the fibers, and such that **all coordinate vector fields are smooth local sections**.*

3 Bundle Homomorphisms

- **Definition** If $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles, a ***continuous map*** $F : E \to E'$ is called a ***bundle homomorphism*** if there exists a map $f : M \to M'$ satisfying $\pi' \circ F = f \circ \pi$,

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M', \end{tikzcd}$$

with the property that for each $p \in M$, the ***restricted map*** $F|_{E_p} : E_p \to E'_{f(p)}$ is ***linear***. The *relationship* between F and f is expressed by saying that ***F covers f***.

- **Remark** By definition, a bundle ***homomorphism*** is *not necessary bijective*, unlike the normal ***homemorphism*** definition.

- **Proposition 3.1** *Suppose $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles and $F : E \to E'$ is a bundle homomorphism **covering** $f : M \to M'$. Then f is **continuous** and is **uniquely determined** by F. If the bundles and F are all **smooth**, then f is **smooth** as well.*

- **Definition** A ***bijective bundle homomorphism*** $F : E \to E'$ whose *inverse* is also a *bundle homomorphism* is called a ***bundle isomorphism***; if F is also a *diffeomorphism*, it is called a ***smooth bundle isomorphism***. If there exists a *(smooth) bundle isomorphism* between E and E', the two bundles are said to be ***(smoothly) isomorphic***.

- **Definition** ***A bundle homomorphism over*** M is a *bundle homomorphism* covering the ***identity map*** of M; or in other words, a *continuous* map $F : E \to E'$ such that

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M, & \end{tikzcd}$$

and whose ***restriction to each fiber** is **linear***. If there exists a *bundle homomorphism* $F : E \to E'$ *over* M that is also a *(smooth) bundle isomorphism*, then we say that E and E' are ***(smoothly) isomorphic over*** M.

- The next proposition shows that it is not necessary to check smoothness of the inverse.

  Proposition 3.2 *Suppose E and E' are smooth vector bundles over a smooth manifold M with or without boundary, and $F : E \to E'$ is a **bijective smooth bundle homomorphism over** M. Then F is a smooth bundle **isomorphism**.*

- **Example** (**Bundle Homomorphisms**).

1. If $F : M \to N$ is a smooth map, ***the global differential*** $dF : TM \to TN$ is a ***smooth bundle homomorphism covering*** F.

2. If $E \to M$ is a smooth vector bundle and $S \subseteq M$ is an ***immersed submanifold*** with or without boundary, then the ***inclusion map*** $E|_S \hookrightarrow E$ is a ***smooth bundle homomorphism covering the inclusion of*** S ***into*** M.

- **Definition** Suppose $E \to M$ and $E' \to M'$ are smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth global sections. If $F : E \to E'$ is a ***smooth bundle homomorphism over*** M, then ***composition with*** F ***induces*** a map $\widetilde{F} : \Gamma(E) \to \Gamma(E')$ as follows:

$$\widetilde{F}(\sigma)(p) = (F \circ \sigma)(p) = F(\sigma(p)) \tag{3}$$

It is easy to check that $\widetilde{F}(\sigma)$ is a ***section*** of E', and it is ***smooth*** by composition.

- **Remark** Because a *bundle homomorphism* is ***linear on fibers***, the resulting map $\widetilde{F}$ on ***sections*** is ***linear*** over $\mathbb{R}$. In fact, it satisfies a stronger linearity property.

- **Definition** A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is said to be ***linear over*** $\mathcal{C}^\infty(M)$ if for any smooth functions $u_1, u_2 \in \mathcal{C}^\infty(M)$ and smooth sections $\sigma_1, \sigma_2 \in \Gamma(E)$,

$$\mathcal{F}(u_1\sigma_1 + u_2\sigma_2) = u_1\mathcal{F}(\sigma_1) + u_2\mathcal{F}(\sigma_2).$$

- It follows easily from the definition (3) that the map on sections induced by a *smooth bundle homomorphism* is ***linear*** *over* $\mathcal{C}^\infty(M)$. The next lemma shows that the converse is true as well.

Lemma 3.3 *(**Bundle Homomorphism Characterization Lemma**). [Lee, 2003.]*
*Let $\pi : E \to M$ and $\pi' : E' \to M'$ be smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth sections. A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is **linear over** $\mathcal{C}^\infty(M)$ **if and only if** there is a **smooth bundle homomorphism** $F : E \to E'$ over M such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

- **Example** (***Bundle Homomorphisms Over Manifolds***).

1. If M is a smooth manifold and $f \in \mathcal{C}^\infty(M)$, the map from $\mathfrak{X}(M)$ to itself defined by $X \mapsto fX$ is linear over $\mathcal{C}^\infty(M)$ because $f(u_1 X_1 + u_2 X_2) = u_1 f(X_1) + u_2 f(X_2)$, and thus defines ***a smooth bundle homomorphism*** over M from TM to itself.

2. If Z is a smooth vector field on $\mathbb{R}^3$, the ***cross product*** with Z defines a map from $\mathfrak{X}(\mathbb{R}^3)$ to itself: $X \mapsto X \times Z$. Since it is linear over $\mathcal{C}^\infty(\mathbb{R}^3)$ in X, it determines *a **smooth bundle homomorphism** over* $\mathbb{R}^3$ from $T\mathbb{R}^3$ to $T\mathbb{R}^3$.

3. Given $Z \in \mathfrak{X}(\mathbb{R}^n)$, the ***Euclidean dot product*** defines a map $X \mapsto X \cdot Z$ from $\mathfrak{X}(\mathbb{R}^n)$ to $\mathcal{C}^\infty(\mathbb{R}^n)$, which is linear over $\mathcal{C}^\infty(\mathbb{R}^n)$ and thus determines *a **smooth bundle homomorphism** over* $\mathbb{R}^n$ from $T\mathbb{R}^n$ to the *trivial line bundle* $\mathbb{R}^n \times \mathbb{R}$.

- **Remark** Because of *Bundle Homomorphism Characterization Lemma*, we usually dispense with the notation $\widetilde{F}$ and use ***the same symbol*** for both ***a bundle homomorphism*** $F : E \to E'$ over M and ***the linear map*** $F : \Gamma(E) \to \Gamma(E')$ that it induces on ***sections***, and we refer to a map of ***either*** *of these types* as ***a bundle homomorphism***.

4 Subbundles

5 Comparison of concepts

- By far, we have introduced a lot of abstract concepts that are generalization of our known concepts. Let us compare them in the following Table 1.

Table 1: Comparison between concepts

base	***Euclidean space*** $\mathbb{R}^n$	***smooth manifold*** M	***topological space*** M
element	p, **global coordinate** $\boldsymbol{x} = (x^1, \dots, x^n)$	p, **local coordinate** $\varphi(p) = (x^1, \dots, x^n)$	p
basis of base	coordinate vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$	the *local frame* for M	the *local frame* for M
vector space (fiber) at p	tangent space $T_{\boldsymbol{x}}\mathbb{R}^n \simeq \{\boldsymbol{x}\} \times \mathbb{R}^n \simeq \mathbb{R}^n$	**tangent space** $T_pM \simeq \{p\} \times \mathbb{R}^n$	**fiber** $E_p = \pi^{-1}(p)$; $E_p \simeq \{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$
dimension of vector space	n	n	k
basis of vector space	$\left(\frac{\partial}{\partial x^1}\Big\vert_p, \dots, \frac{\partial}{\partial x^n}\Big\vert_p\right) \equiv (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$	$\left(\frac{\partial}{\partial x^1}\Big\vert_p, \dots, \frac{\partial}{\partial x^n}\Big\vert_p\right)$	$(\sigma_1(p), \dots, \sigma_k(p))$
element in vector space	tangent vector $\boldsymbol{v} = v^i \boldsymbol{e}_i$	**tangent vector** $v = v^i \frac{\partial}{\partial x^i}\Big\vert_p$	$v = v^i \sigma_i(p)$
total space of bundle	tangent bundle $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$	**tangent bundle** $TM = \bigsqcup_{p \in M} T_pM$	**vector bundle** $E = \bigsqcup_{p \in M} E_p,$
element in bundle	$(x^1, \dots, x^n, v^1, \dots, v^n)$	$(x^1(p), \dots, x^n(p), v^1, \dots, v^n)$	$(x^1(p), \dots, x^n(p), v^1, \dots, v^k)$
section	**global vector field** $X = X^i \boldsymbol{e}_i \equiv X^i \frac{\partial}{\partial x^i}$	**local vector field** $X = X^i \frac{\partial}{\partial x^i}$ $X_p \in T_pM$	**local section** $\tau = \tau^i \sigma_i$ $\tau(p) \in E_p$
vector space of sections	$\mathfrak{X}(\mathbb{R}^n) \simeq \mathbb{R}^n$	$\mathfrak{X}(M) \equiv \Gamma(TM)$	$\Gamma(E)$
frame	**global frame** $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$	**basis vector fields** $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$	**local frame** $(\sigma_1, \dots, \sigma_k)$

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.