Lecture 0: Summary (Part 2)

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Contents

1 Hilbert Space

- **Remark** (***Hilbert Space vs. Banach Space***)
  Hilbert space is a special Banach space equipped with inner product. Historically, Hilbert space appears eariler. The theory of inner product and Hilbert spaces is richer than that of general normed and Banach spaces. ***Distinguishing features*** are

  1. ***representations*** of $\mathcal{H}$ as a ***direct sum*** of a ***closed subspace*** and its ***orthogonal complement*** (section 2.3),
  2. ***orthonormal sets*** and sequences and corresponding ***representations*** of elements of $\mathcal{H}$ (section 2.5),
  3. ***the Riesz representation*** of ***bounded linear functionals*** by inner products, (section 2.4)
  4. ***the Hilbert-adjoint operator*** T^* of a bounded linear operator T (section 2.10).

1.1 Inner Product Space

- **Remark** Finite-dimensional vector spaces have *three kinds of properties* whose generalizations we will study in the next four chapters:

  1. ***linear*** *properties*,
  2. ***metric*** *properties*,
  3. and ***geometric*** *properties*.

  A ***Hilbert space*** generalizes the ***geometric*** property of a finite-dimensional vector space to *infinite-dimensional* via definition of inner product.

- **Definition** A *complex vector space* V is called ***an inner product space*** if there is a *complex-valued function* $\langle \cdot\,,\,\cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following four conditions for an $x, y, z \in V$ and $a, b \in \mathbb{C}$:

  1. (***Positive Definiteness***): $\langle x\,,\,x \rangle \ge 0$ and $\langle x\,,\,x \rangle = 0$ if and only if $x = 0$
  2. (***Linearity***): $\langle a\,x + b\,y\,,\,z \rangle = a\,\langle x\,,\,z \rangle + b\,\langle y\,,\,z \rangle$
  3. (***Hermitian***): $\langle x\,,\,y \rangle = \overline{\langle y\,,\,x \rangle}$

  The function $\langle \cdot\,,\,\cdot \rangle$ is called ***an inner product***.

- **Remark** Without "condition $\langle x\,,\,x \rangle = 0$ if and only if $x = 0$", we have ***semi-inner product*** [Conway, 2019].

- **Remark** From *Hermitian property*, we have $\langle x\,,\,a\,y + b\,z \rangle = \overline{a}\,\langle x\,,\,y \rangle + \overline{b}\,\langle x\,,\,z \rangle$.

- **Remark** For *real vector space*, an inner product is a ***symmetric covariant*** 2***-tensor***, or a ***symmetric bilinear form***.

- **Remark** Some books [Reed and Simon, 1980] define inner product via ***linearity in second argument***; while others [Kreyszig, 1989, Luenberger, 1997, Conway, 2019] defines it in terms of ***linearity in first argument***. The difference is the position of conjugate.

- **Proposition 1.1** *Every **inner product space** V is a **normed linear space** with the norm $\|x\| = \sqrt{\langle x\,,\, x\rangle}$.*

- **Remark** We denote $\|x\| = \sqrt{\langle x\,,\, x\rangle}$ as *the* ***length*** of a vector. With the definition of length, we can define the ***distance*** d as

$$d(x, y) := \|x - y\| = \sqrt{\langle x - y\,,\, x - y\rangle}.$$

  As a consequence of *the Pythagorean Theorem*, d satisfies the triangle inequality so it is a *metric*. Thus ***every inner product space is a metric space***.

- **Proposition 1.2** ***(Parallelogram Law)***
  For any $x, y \in (V, \langle\cdot\,,\,\cdot\rangle)$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

- **Remark** The followings are other versions of ***Parallelogram Law***:

$$\begin{aligned}
\Re\,\langle x\,,\, y\rangle &= \frac{1}{2}\left(\|x + y\|^2 - \|x\|^2 - \|y\|^2\right) \\
\Re\,\langle x\,,\, y\rangle &= \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x - y\|^2\right) \\
\Re\,\langle x\,,\, y\rangle &= \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right) \\
\langle x\,,\, y\rangle &= \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\,\|x + iy\|^2 - i\,\|x - iy\|^2\right) \\
&= \Re\,\langle x\,,\, y\rangle + i\Re\,\langle x\,,\, iy\rangle
\end{aligned}$$

- The converse holds true as well.

  Proposition 1.3 *In a **normed space** $(V, \|\cdot\|)$, if **the parallelogram law***

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

  *holds, then there exists a **unique inner product** $\langle\cdot,\ \cdot\rangle$ on V such that $\|x\| = \sqrt{\langle x\,,\, x\rangle}$ for all $x \in V$.*

- **Remark** The inner product defines the concept of ***angle*** (and *orthorgonality*), and ***distance***. Hence it allows the ***geometric property*** of Euclidean space to be generalized.

- **Definition** Two vectors, x and y, in an inner product space V are said to be ***orthogonal*** if $\langle x\,,\, y\rangle = 0$. A collection $\{x_n\}$ of vectors in V is called ***an orthonormal set*** if $\langle x_i\,,\, x_i\rangle = 1$ for all i, and $\langle x_i\,,\, x_j\rangle = 0$ if $i \neq j$.

- **Theorem 1.4** ***(Pythagorean Theorem)***
  *Let $\{x_i\}_{i=1}^n$ be an **orthonormal** set in an inner product space V. Then for all $x \in V$,*

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x_i\,,\, x\rangle|^2 + \left\|x - \sum_{i=1}^{n}\langle x_i\,,\, x\rangle\, x_i\right\|^2$$

- **Corollary 1.5** ***(Bessel's inequality)***
  Let $\{x_i\}_{i=1}^n$ be an ***orthonormal*** *set in an inner product space V. Then for all $x \in V$,*

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x_i \, , \, x \rangle|^2$$

- **Corollary 1.6** ***(Cauchy-Schwartz's inequality)***
  Let V be an inner product space. For $x, y \in V$,

$$|\langle x \, , \, y \rangle| \leq \|x\| \; \|y\| \, .$$

1.2 Hilbert Space

- **Definition** A ***complete*** *inner product space* is called ***a Hilbert space***.

  Inner product spaces are sometimes called ***pre-Hilbert spaces***.

- **Definition** Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be ***isomorphic*** if there is a ***surjective linear*** *operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\langle Ux \, , \, Uy \rangle_{\mathcal{H}_2} = \langle x \, , \, y \rangle_{\mathcal{H}_1}$$

  for all $x, y \in \mathcal{H}_1$. Such an operator is called ***unitary***.

  item

  Example ($\mathcal{L}^2[a, b]$)
  Define $\mathcal{L}^2([a, b])$ to be the set of complex-valued measurable functions on $[a, b]$, a finite interval, that satisfy $\int_{[a,b]} |f(x)|^2 \, dx < \infty$. We define an inner product by

$$\langle f \, , \, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

  $\mathcal{L}^2([a, b])$ is a complete metric space. Actually, $\mathcal{L}^2([a, b])$ is a completion of $\mathcal{C}^0([a, b])$ with finite $\mathcal{L}^2$ norm

$$\|f\|_{\mathcal{L}^2} = \left(\int_a^b |f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

  Thus $\mathcal{L}^2([a, b])$ is a *Hilbert space.*

- **Example** (ℓ^2)
  Define ℓ^2 to be *the set of sequences $(x_n)_{n=1}^{\infty}$ of complex numbers* which satisfy $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ with the inner product

$$\langle (x_n)_{n=1}^{\infty} \, , \, (y_n)_{n=1}^{\infty} \rangle = \sum_{n=1}^{\infty} \overline{x_n} \, y_n.$$

  ℓ^2 is a complete metric space with ℓ^2 norm

$$\|(x_n)_{n=1}^{\infty}\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} .$$

So ℓ^2 is a *Hilbert space.*

We will see that any Hilbert space that has a ***countable dense set*** and is ***not finite dimensional*** is ***isomorphic*** to ℓ^2 In this sense, ℓ^2 is *the canonical example* of a Hilbert space.

- **Example** $(\mathcal{L}^2(\mathbb{R}^n, \mu))$
Define μ to be a *Borel measure* on $\mathbb{R}^n$ and $\mathcal{L}^2(\mathbb{R}^n, \mu)$ to be the set of complex-valued measurable functions on $\mathbb{R}^n$ that satisfy $\int_{\mathbb{R}^n} |f(x)|^2 \, d\mu < \infty$. We define an inner product by

$$\langle f \,,\, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)} d\mu$$

$\mathcal{L}^2(\mathbb{R}^n, \mu)$ is a *Hilbert space.*

1.3 The Projection Theorem

- **Remark** ***Orthogonality*** is the central concept of Hilbert space. In the presence of closed subspaces, the orthogonality allows us to decompose the Hilbert space into the direct sum of the *subspace* and its *orthogonal complement.*

- **Definition** (***Direct Sum***)
Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. Then the set of *pairs* (x, y) with $x \in \mathcal{H}_1, y \in \mathcal{H}_2$ is a *Hilbert space* with *inner product*

$$\langle (x_1, y_1) \,,\, (x_2, y_2) \rangle := \langle x_1 \,,\, x_2 \rangle_{\mathcal{H}_1} + \langle y_1 \,,\, y_2 \rangle_{\mathcal{H}_2}$$

This space is called ***the direct sum*** *of the spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ and is denoted by $\mathcal{H}_1 \oplus \mathcal{H}_2$.

- **Definition** (***Orthogonal Complement***)
Let $\mathcal{M} \subseteq \mathcal{H}$ is a ***closed*** *linear subspace* of Hilbert space $\mathcal{H}$ with *induced inner product* $\langle \,,\, \rangle$ (i.e. $\langle x \,,\, y \rangle_{\mathcal{M}} = \langle x \,,\, y \rangle_{\mathcal{H}}$ for all $x, y \in \mathcal{M}$). $\mathcal{M}$ is also a *Hilbert space.*

We denote by $\mathcal{M}^{\perp}$ the set of vectors in $\mathcal{H}$ which are *orthogonal* to $\mathcal{M}$; $\mathcal{M}^{\perp}$ is called ***the orthogonal complement of*** $\mathcal{M}$. It follows from the linearity of the inner product that $\mathcal{M}^{\perp}$ is a *linear subspace* of $\mathcal{H}$ and an elementary argument shows that $\mathcal{M}^{\perp}$ is *closed.* So $\mathcal{M}^{\perp}$ is also a *Hilbert space.*

- **Remark** The following theorem is going to show that

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp} = \left\{ x + y : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, \text{ i.e. } \langle x \,,\, y \rangle = 0 \right\}.$$

This important geometric property is one of the main reasons that Hilbert spaces are ***easier*** to handle than Banach spaces.

- **Lemma 1.7** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace of $\mathcal{H}$, and suppose $x \in \mathcal{H}$. Then there exists in $\mathcal{M}$ a* ***unique*** *element z* ***closest*** *to x.*

- **Theorem 1.8** ***(The Projection Theorem)***
Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace. Then every $x \in \mathcal{H}$ can be ***uniquely*** *written $x = z + w$ where $z \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$.*

- **Remark** The projection theorem sets up a natural *isomorphism* $\mathcal{M} \oplus \mathcal{M}^{\perp} \to \mathcal{H}$ given by

$$(z, w) \mapsto z + w$$

We will often suppress the isomorphism and simply write $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

1.4 Orthonormal Bases

- **Definition** (***Complete Orthonormal Basis***)
If S is *an orthonormal set* in a Hilbert space $\mathcal{H}$ and *no other orthonormal set* contains S as a proper subset, then S is called *an* ***orthonormal basis*** (or a ***complete orthonormal system***) for $\mathcal{H}$.

- **Theorem 1.9** ***(Existence of Orthonormal Basis)***
Every Hilbert space $\mathcal{H}$ has an ***orthonormal basis****.*

- **Proposition 1.10** ***(Orthogonal Representation of Element in Hilbert Space)***
Let $\mathcal{H}$ be a Hilbert space and $S = (x_{\alpha})_{\alpha \in A}$ an ***orthonormal basis****. Then for each $y \in \mathcal{H}$,*

$$y = \sum_{\alpha \in A} \langle y \,,\, x_{\alpha} \rangle \, x_{\alpha} \tag{1}$$

and

$$\|y\|_{\mathcal{H}} = \sum_{\alpha \in A} |\langle y \,,\, x_{\alpha} \rangle|^2 \tag{2}$$

The equality in (1) *means that the sum on the right-hand side converges (independent of order) to y in $\mathcal{H}$.* ***Conversely****, if $\sum_{\alpha \in A} |c_{\alpha}|^2 < \infty$, $c_{\alpha} \in \mathbb{C}$, then $\sum_{\alpha \in A} c_{\alpha} x_{\alpha}$ converges to an element of $\mathcal{H}$.*

- **Remark** From Bessel's inequality, we already seen that for any finite collection A' of x_{α}, we have $\sum_{\alpha \in A'} |\langle y \,,\, x_{\alpha} \rangle|^2 \le \|y\|_{\mathcal{H}}$. The main difficulty is on how to prove convergence of $\sum_{n=1}^{N} |\langle y \,,\, x_n \rangle|^2$ as $N \to \infty$. Similarly we need to prove that $y - \sum_{n=1}^{m} \langle y \,,\, x_{\alpha_n} \rangle \, x_{\alpha_n}$ is still orthogonal to x_{α} as $m \to \infty$.

- **Remark** The unique coefficients $(\langle y \,,\, x_{\alpha} \rangle)$ is called ***the Fourier coefficients of*** *y with respect to basis (x_{α})*.

- **Remark** (***Gram-Schmidt Orthogonalization***)
Given *any set of independent vectors* $(v_1, v_2, \ldots)$. we can construct *an orthonormal basis* $(b_1, b_2, \ldots)$ via

$$\begin{aligned} b_1 &= \frac{v_1}{\|v_1\|} \\ b_j &= \frac{v_j - \sum_{i=1}^{j-1} \langle v_j \,,\, b_i \rangle \, b_i}{\left\| v_j - \sum_{i=1}^{j-1} \langle v_j \,,\, b_i \rangle \, b_i \right\|}, \quad j \ge 2 \end{aligned}$$

Thus $\text{span}\,\{v_1, \,, \ldots, v_m\} = \text{span}\,\{b_1, \,, \ldots, b_m\}$ for all $m \ge 1$.

1.5 Separability

- **Definition** (***Separability***)
  A *metric space* which has a ***countable dense subset*** is said to be ***separable***.

- **Remark** Most Hilbert space we have seen is separable.

- **Proposition 1.11** *(**Canonical Hilbert Space**)*
  *A Hilbert space $\mathcal{H}$ is **separable** if and only if it has a **countable orthonormal basis** S. If there are $N < \infty$ elements in S, then $\mathcal{H}$ is **isomorphic** to $\mathbb{C}^N$, If there are **countably many** elements in S, then $\mathcal{H}$ is **isomorphic** to ℓ^2.*

- **Remark** Consider the map $v \mapsto (\langle v\,,\, x_n\rangle)_{n=1}^{\infty}$ for orthonormal basis $(x_n)_{n=1}^{\infty}$ as the isomorphism $\mathcal{H} \to \ell^2$.

- **Remark** Notice that in the separable case, *the Gram-Schmidt process* anows us to construct an orthonormal basis without using *Zorn's lemma*.

2 Bounded Linear Operator

2.1 The Riesz Representation Theorem

- **Definition** (***Bounded Linear Operator***)
  A ***bounded linear transformation*** (or ***bounded operator***) is a mapping $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ from a normed linear space X to a normed linear space Y that satisfies

  1. (***Linearity***) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$
  2. (***Boundedness***) $\|Tx\|_Y \le C\, \|x\|_X$ for small $C \ge 0$.

  The smallest such C is called *the **norm** of T*, written $\|T\|$ or $\|T\|_{X,Y}$. Thus

$$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y$$

- **Remark** Denote the space of ***all bounded linear operator*** between Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ as $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. The space $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is linear space with norm

$$\|T\| := \sup_{\|x\|_{\mathcal{H}_1} = 1} \|Tx\|_{\mathcal{H}_2}, \quad \forall T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$

  It can be shown that $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is a *complete normed space (i.e. a Banach space).*

- **Definition** (***Dual Space***)
  The space $\mathcal{L}(\mathcal{H}, \mathbb{C})$ is called the ***dual space*** of $\mathcal{H}$ and is denoted by $\mathcal{H}^*$. The elements of $\mathcal{H}^*$ are called ***continuous linear functionals***. That is, the dual space $\mathcal{H}^*$ is the space of *continuous linear functionals* on $\mathcal{H}$.

- **Remark** The *dual space* $\mathcal{H}^*$ is also called ***covector space*** with respect to a vector space $\mathcal{H}$ and the linear functionals are called ***covectors***. This terms are mostly used in *differential geometry* when the vector space is *the tangent space.*

- **Theorem 2.1** *(**The Riesz Representation Theorem**) [Reed and Simon, 1980, Kreyszig, 1989, Conway, 2019]*
  *For each $T \in \mathcal{H}^*$, there is a **unique** $y_T \in \mathcal{H}$ such that*

  $$T(x) = \langle x \, , \, y_T \rangle$$

  for all $x \in \mathcal{H}$. In addition $\|y_T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}^*}$.

- **Remark** *The Riesz Representation Theorem* [Conway, 2019, Kreyszig, 1989] is also called ***The Riesz Lemma*** [Reed and Simon, 1980].

- **Remark** We note that *the Cauchy-Schwarz inequality* shows that the ***converse*** of *the Riesz Representation Theorem* is ***true***. Namely, each $y \in \mathcal{H}$ defines *a continuous linear functional* T_y on $\mathcal{H}^*$ by

  $$T_y(x) = \langle x \, , \, y \rangle \, .$$

  Thus *the Riesz Representation Theorem* together with *the Cauchy-Schwarz inequality* defines an ***isomorphism*** $\mathcal{H}^* \to \mathcal{H}$ between a Hilbert space $\mathcal{H}$ and its dual $\mathcal{H}^*$. In other words, unlike the case in Banach space, the bounded linear functional on Hilbert space has a simple form.

- **Corollary 2.2** *(**The Riesz Representation for Sesquilinear Form**)*
  Let $B(\cdot, \cdot)$ be a function from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$ which satisfies:

  1. *(**Linearity**)* $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$
  2. *(**Conjugate Linearity**)* $B(x, \alpha y + \beta z) = \overline{\alpha} B(x, y) + \overline{\beta} B(x, z)$
  3. *(**Boundedness**)* $|B(x, y)| \le C \, \|x\|_{\mathcal{H}} \, \|y\|_{\mathcal{H}}$

  *for all $x, y, z \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$. Then there is a **unique bounded linear transformation** $A : \mathcal{H} \to \mathcal{H}$ so that*

  $$B(x, y) = \langle x \, , \, Ay \rangle$$

  *for all $x, y \in \mathcal{H}$. The **norm** of A is the smallest constant C such that (3) holds.*

- **Remark** A bilinear function on $\mathcal{H}$ obeying (1) and (2) is called a ***sesquilinear form*** (as a generalization of ***bilinear form*** in complex vector space).

  In terms of this, an inner product in complex vector space is ***a complex Hermitian form*** (also called a ***symmetric sesquilinear form***).

2.2 Hilbert Adjoint Operator

- **Definition** (***Hilbert Space Adjoint***)
  Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a *bounded linear operator*, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are *Hilbert spaces*. Then ***the Hilbert-adjoint operator*** T^* of T is the operator

  $$T^* : \mathcal{H}_2 \to \mathcal{H}_1$$

  such that for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

  $$\langle Tx \, , \, y \rangle_{\mathcal{H}_2} = \langle x \, , \, T^* y \rangle_{\mathcal{H}_1} \tag{3}$$

- **Proposition 2.3** *(**Existence of Adjoint Operator**) [Kreyszig, 1989]*
*The Hilbert-adjoint operator T^* of T **exists**, is **unique** and is a **bounded linear operator** with norm*

$$\|T^*\| = \|T\| .$$

- **Lemma 2.4** *(**Zero operator**). [Kreyszig, 1989] Let X and Y be inner product spaces and $Q : X \to Y$ a bounded linear operator. Then:*

  1. *$Q = 0$ if and only if $\langle Qx \,, \, y\rangle = 0$ for all $x \in X$ and $y \in Y$.*
  2. *If $Q : X \to X$, where X is complex, and $\langle Qx \,, \, x\rangle = 0$ for all $x \in X$, then $Q = 0$.*

- **Proposition 2.5** *(**Properties of Hilbert-adjoint operators**). [Reed and Simon, 1980, Kreyszig, 1989]*
Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, $S : \mathcal{H}_1 \to \mathcal{H}_2$ and $T : \mathcal{H}_1 \to \mathcal{H}_2$ bounded linear operators and α any scalar. Then we have

  1. *$\langle T^*y \,, \, x\rangle = \langle y \,, \, Tx\rangle$, $(x \in H_1, y \in \mathcal{H}_2)$*
  2. *$(S + T)^* = S^* + T^*$*
  3. *$(\alpha T)^* = \alpha T^*$*
  4. *$(T^*)^* = T$*
  5. *$\|T^*T\| = \|TT^*\| = \|T\|^2$*
  6. *$T^*T = 0 \Leftrightarrow T = 0$*
  7. *$(ST)^* = T^*S^*$ (assuming $\mathcal{H}_2 = \mathcal{H}_1$)*
  8. *If T has a **bounded inverse**, T^{-1}, then T^* has a **bounded inverse** and $(T^*)^{-1} = (T^{-1})^*$.*

2.3 Self-Adjoint, Unitary and Normal Operators

- **Definition** A ***bounded linear operator*** $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is said to be

  1. ***self-adjoint*** or ***Hermitian*** if

  $$T^* = T \quad \Leftrightarrow \quad \langle Tx \,, \, y\rangle = \langle x \,, \, Ty\rangle$$

  2. ***unitary*** if T is *bijective* and

  $$T^* = T^{-1}$$

  3. ***normal*** if

  $$T^*T = TT^*$$

- **Definition** (***Projection Operator***)
If $P \in \mathcal{L}(\mathcal{H})$ and $P^2 = P$, then P is called a ***projection***. If in addition $P = P^*$, then P is called an ***orthogonal projection***.

- **Remark** If T is ***self-adjoint*** and ***unitary***, then T is ***normal***.

- **Remark** If a basis for $\mathbb{C}^n$ is given and *a* ***linear operator*** on $\mathbb{C}^n$ is represented by a certain ***matrix***, then its ***Hilbert-adjoint operator*** is represented by the ***complex conjugate transpose*** of that matrix. For $\mathbb{R}^n$, then the ***Hilbert-adjoint operator*** is represented by the ***transpose*** of that matrix

- **Remark** Similarly we have

  1. The matrix representation for *self-adjoint operator* is ***Hermitian*** or ***Symmetric***.

  $$T^* = T \quad \Leftrightarrow \quad \boldsymbol{T}^H = \boldsymbol{T} \text{ (or for real vector space } \boldsymbol{T}^T = \boldsymbol{T})$$

  2. The matrix representation for *unitary operator* is ***unitary*** or ***orthogonal***.

  $$T^* = T^{-1} \quad \Leftrightarrow \quad \boldsymbol{T}^H = \boldsymbol{T}^{-1} \text{ (or for real vector space } \boldsymbol{T}^T = \boldsymbol{T}^{-1})$$

  3. The matrix representation for *normal operator* is ***normal***.

  $$T^*T = TT^* \quad \Leftrightarrow \quad \boldsymbol{T}^H\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}^H \text{ (or for real vector space } \boldsymbol{T}^T\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}^T)$$

- **Proposition 2.6** *(**Self-adjointness**). [Kreyszig, 1989]*
  Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then:

  1. *If T is **self-adjoint**, $\langle Tx\,,\,x\rangle$ is **real** for all $x \in \mathcal{H}$.*
  2. *If $\mathcal{H}$ is complex and $\langle Tx\,,\,x\rangle$ is **real** for all $x \in \mathcal{H}$, the operator T is **self-adjoint***

- **Proposition 2.7** *(**Self-adjointness of product**). [Kreyszig, 1989]*
  *The product of two bounded **self-adjoint** linear operators S and T on a Hilbert space $\mathcal{H}$ is **self-adjoint** if and only if the operators **commute**,*

  $$ST = TS.$$

- **Proposition 2.8** *(**Sequences of self-adjoint operators**). [Kreyszig, 1989]*
  *Let (T_n) be a sequence of **bounded self-adjoint** linear operators $T_n : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. Suppose that (T_n) converges, say,*

  $$T_n \to T, \qquad i.e.\ \ \|T_n - T\| \to 0$$

  *where $\|\cdot\|$ is the norm on the space $\mathcal{L}(\mathcal{H}, \mathcal{H})$. Then the limit operator T is a **bounded self-adjoint** linear operator on H.*

- **Proposition 2.9** *(**Unitary operator**). [Kreyszig, 1989]*
  *Let the operators $U : \mathcal{H} \to \mathcal{H}$ and $V : \mathcal{H} \to \mathcal{H}$ be **unitary**; here, $\mathcal{H}$ is a Hilbert space. Then:*

  1. *U is **isometric**; thus $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$;*
  2. *$\|U\| = 1$, provided $\mathcal{H} \neq \{0\}$,*
  3. *$U^{-1} = U^*$ is **unitary**,*
  4. *UV is unitary,*

5. *U is normal.*

6. *A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is* ***unitary*** *if and only if T is* ***isometric*** *and* ***surjective****.*

- **Remark** Note that an ***isometric operator*** need not be *unitary* since it may fail to be ***surjective***. An example is the *right shift operator* $T : \ell^2 \to \ell^2$ given by

$$(\xi_1, \xi_2, \xi_3, \ldots) \mapsto (0, \xi_1, \xi_2, \xi_3, \ldots).$$

3 Spectrum of Bounded Linear Operator in Hilbert Space

3.1 Finite Dimensional Case

- **Remark** (***Eigenvalues*** *of Linear Transformation in* ***Finite Dimensional Space***)
If is a linear transformation on $\mathbb{C}^n$, then the ***eigenvalues*** of are the complex numbers λ such that the ***determinant*** (called ***the characteristic determinant***)

$$\det(\lambda I - T) = 0.$$

The set of such λ is called ***the spectrum of*** T. It can consist of ***at most*** *n points*, since $\det(\lambda I - T)$ is *a* ***polynomial*** *of degree n*, called ***the characteristic polynomial*** *of* T.

- **Remark** If λ is ***not an eigenvalue***, then $\lambda I - T$ ***has an inverse*** since

$$\det(\lambda I - T) \neq 0.$$

- **Proposition 3.1** *(**Invariance of Eigenvalue under Change of Basis**) [Kreyszig, 1989]*
All matrices representing a given linear operator $T : X \to X$ on a ***finite dimensional normed space*** *X relative to various bases for X have the* ***same eigenvalues****.*

- **Theorem 3.2** *(**The Existence of Eigenvalues**). [Kreyszig, 1989]*
A linear operator on a ***finite dimensional*** *complex normed space $X \neq \{0\}$ has* ***at least one eigenvalue****.*

3.2 Infinite Dimensional Case

- **Definition** (***Resolvent*** *and* ***Spectrum***)
Let $T \in \mathcal{L}(X)$. A complex number λ is said to be in ***the resolvent set*** *$\rho(T)$ of T* if

$$\lambda I - T$$

is a ***bijection*** with a ***bounded inverse***.

$$R_\lambda(T) := (\lambda I - T)^{-1}$$

is called ***the resolvent*** *of T at λ*. Note that $R_\lambda(T)$ is defined on $\mathrm{Ran}(\lambda I - T)$.

If $\lambda \notin \rho(T)$, then λ is said to be in the ***spectrum*** $\sigma(T)$ ***of*** T.

- **Remark** The name "***resolvent***" is appropriate, since $R_\lambda(T)$ helps to solve the equation $(\lambda I - T)\, x = y$. Thus, $x = (\lambda I - T)^{-1}\, y = R_\lambda(T) y$ provided $R_\lambda(T)$ exists.

- **Definition** (***Point Spectrum***, ***Continuous Spectrum*** *and* ***Residual Spectrum***)
  Let $T \in \mathcal{L}(X)$

  1. ***Point Spectrum***: An $x \neq 0$ which satisfies

  $$\begin{aligned} Tx &= \lambda x \\ \text{or }\; (\lambda I - T)\, x &= 0, \quad \text{for some } \lambda \in \mathbb{C} \end{aligned}$$

     is called an ***eigenvector*** *of* T; λ is called ***the corresponding eigenvalue***.

     If λ is an *eigenvalue*, then $(\lambda I - T)$ is ***not injective*** (i.e. $\mathrm{Ker}\,(\lambda I - T) \neq \{0\}$) so λ is *in the spectrum of* T. ***The set of all eigenvalues*** is called ***the point spectrum of*** T. It is denoted as $\sigma_p(T)$.

  2. ***Continuous Spectrum***: If λ is ***not an eigenvalue*** and if $\mathrm{Ran}\,(\lambda I - T)$ is ***dense*** but the resolvent $R_\lambda(T)$ is ***unbounded***, then λ is said to be in ***the continuous spectrum***. It is denoted as $\sigma_c(T)$.

  3. ***Residual Spectrum***: If λ is ***not an eigenvalue*** and if $\mathrm{Ran}\,(\lambda I - T)$ is ***not dense***, then λ is said to be in ***the residual spectrum***. It is denoted as $\sigma_r(T)$.

- **Remark** (***Pure Point Spectrum for Finite Dimensional Case***)
  If X is ***finite dimensional*** *normed linear space*, $T \in \mathcal{L}(X)$ then $\sigma_c(T) = \sigma_r(T) = \emptyset$.

Table 1: Comparison between different subset of spectrums and resolvent set

	point spectrum $\sigma_p(T)$	***continuous spectrum*** $\sigma_c(T)$	***residual spectrum*** $\sigma_r(T)$	***resolvent set*** $\rho(T)$
$R_\lambda(T)$ ***exists***	$\times$	✓	✓	✓
$R_\lambda(T)$ *is* ***bounded***	$\times$	$\times$	$-$	✓
$R_\lambda(T)$ *is defined in a* ***dense*** *subset of* Y	$\times$	✓	$\times$	✓

- **Remark** (***Partition*** *of Complex Space* $\mathbb{C}$)
  All four sets above are disjoint and they forms a partition of $\mathbb{C}$

  $$\begin{aligned} \mathbb{C} &= \rho(T) \cup \sigma(T) \\ &= \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T). \end{aligned}$$

  We will prove this later.

- **Remark** (***Some Special Case***)

  1. If X ***finite dimensional***, $\mathbb{C} = \rho(T) \cup \sigma_p(T)$ since $\sigma_c(T) = \sigma_r(T) = \emptyset$.
  2. If $T \in \mathcal{L}(\mathcal{H})$ and T is ***self-adjoint***, $\mathbb{C} = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T)$ since $\sigma_r(T) = \emptyset$.
  3. If $T \in \mathcal{L}(\mathcal{H})$ and T is ***self-adjoint and compact***, $\mathbb{C} = \rho(T) \cup \sigma_p(T)$

- **Remark** If X is a function space, the *eigenvectors* of *linear operator* T is called the ***eigenfunctions*** of T.

- **Definition** (***Eigenspace of Linear Operator***)
  The subspace of domain $D(T)$ consisting of $\{0\}$ and ***all eigenvectors*** of T corresponding to *an eigenvalue* λ of T is called ***the eigenspace of T*** corresponding to that eigenvalue λ.

4 Spectrum of Compact Operator

4.1 Compact Operators

- **Definition** (***Kernel of Integral Operator***)
  Consider the simple operator T_K, defined in $\mathcal{C}[0,1]$ by

$$(T_K f)(x) = \int_0^1 K(x,y) f(y) dy,$$

  where the function $K(x,y)$ is *continuous* on the square $0 \le x, y \le 1$. T_K is called an ***integral kernel operator*** and $K(x,y)$ is called *the* ***kernel*** *of the integral operator* T_K.

- **Remark** (***Properties of Integral Kernel Operator***)
  We summary some important property of the integral kernel operator T_K:

  1. T_K is ***bounded linear operator*** on $\mathcal{C}[0,1]$.

$$\begin{aligned} |(T_K f)(x)| &\le \left(\sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)| \right) \left(\sup_{y\in[0,1]} |f(y)| \right) \\ \Rightarrow \|T_K f\|_\infty &\le \left(\sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)| \right) \|f\|_\infty \end{aligned}$$

  2. For $K^*(x,y) := \overline{K(y,x)}$,

$$(T_K)^* = T_{K^*}$$

  3. Let B_M denote the functions f in $\mathcal{C}[0,1]$ such that $\|f\|_\infty \le M$, i.e. closed $\|\|_\infty$-ball in $\mathcal{C}[0,1]$

$$B_M := \{f \in \mathcal{C}[0,1] : \|f\|_\infty \le M\}$$

     The set of functions $T_K(B_M) := \{T_K f : f \in B_M\}$ is ***equicontinuous***.

     Proof: Since $K(x,y)$ is *continuous* on the *compact* set $[0,1]\times[0,1]$, $K(x,y)$ is *uniformly continuous*. Thus, given an $\epsilon > 0$, we can find $\delta > 0$ such that $|x - x'| < \delta$ implies $|K(x,y) - K(x',y)| < \epsilon$ for all $y \in [0,1]$. Thus, for all $f \in B_M$

$$\begin{aligned} \left|(T_K f)(x) - (T_K f)(x')\right| &\le \left(\sup_{(x,y)\in[0,1]\times[0,1]} \left|K(x,y) - K(x',y)\right| \right) \|f\|_\infty \\ &\le \epsilon M. \quad \blacksquare \end{aligned}$$

4. Moreover, $\underline{T_K(B_M) := \{T_K f : f \in B_M\}}$ is ***precompact*** in $\mathcal{C}[0,1]$, i.e. its closure $\overline{T_K(B_M)}$ is ***compact***. In other word, for every sequence $f_n \in B_M$, the *sequence* $T_K f_n$ has a ***convergent subsequence***.

   This follows from the fact that $T_K(B_M)$ is *equicontinuous* and *uniformly bounded* by $\|T_K\|\, M$. So by *the Ascoli's theorem*, we have the result.

5. The *operator norm* of T_K is *bounded above* by the L^2 *norm* of kernel function K

$$\|T_K\| \le \|K\|_{L^2}$$

6. The eigenfunctions of T_K $\{\varphi_n\}_{n=1}^{\infty}$ forms a complete orthonormal basis in $L^2(M, \mu)$.

$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$$

   where λ_n is the eigenvalue corresponding to eigenfunction φ_n.

- **Definition** (***Compact Operator***)
  Let X and Y be *Banach spaces*. An operator $T \in \mathcal{L}(X, Y)$ is called ***compact*** (or ***completely continuous***) if T takes ***bounded sets*** in X into ***precompact sets*** in Y.

  Equivalently, T is ***compact*** if and only if for every ***bounded*** *sequence* $\{x_n\} \subseteq X$, $\{Tx_n\}$ has a ***subsequence convergent*** in Y.

- **Example** (***Finite Rank Operators***)
  Suppose that *the **range** of T is **finite dimensional**.* That is, every vector in the range of T can be written

$$Tx = \sum_{i=1}^{n} \alpha_i y_i,$$

  for some fixed family $\{y_i\}_{i=1}^{n}$ in Y. If x_n is any *bounded sequence* in X, the corresponding $\alpha_i^{(n)}$ are *bounded* since T is *bounded*. The usual subsequence trick allows one to extract a *convergent subsequence* from $\{Tx_n\}$ which proves that T is *compact*. $\blacksquare$

- An important property of the compact operator is

  Theorem 4.1 *(**Weakly Convergent** + **Compact Operator** = **Uniformly Convergent**)* *[Reed and Simon, 1980]*
  *A **compact** operator maps **weakly convergent** sequences into **norm convergent** sequences; i.e. if $T \in \mathcal{L}(X)$ is compact, then*

$$x_n \stackrel{w}{\to} x \quad \Rightarrow \quad Tx_n \stackrel{norm}{\to} Tx.$$

  *The converse holds true if X is **reflective**.*

- **Proposition 4.2** *[Reed and Simon, 1980]*
  *Let X and Y be **Banach spaces**, $T \in \mathcal{L}(X, Y)$.*

  1. *If $\{T_n\}$ are **compact** and $T_n \to T$ in the **norm topology**, then T is **compact**.*
  2. *T is **compact** if and only if T' is **compact**.*

3. *If $S \in \mathcal{L}(Y, Z)$ with Z a Banach space and if T **or** S is **compact**, then ST is **compact**.*

- The proposition above shows that the space of compact operators on $\mathcal{H}$ is a ***closed subspace*** of $\mathcal{L}(\mathcal{H})$, thus it is *a Banach space too.*

  Definition (***Space of Compact Operators***)
  Now assume that $\mathcal{H}$ is a ***separable Hilbert space***. We denote *the Banach space of* ***compact operators*** on *a separable Hilbert space* by $\text{Com}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$.

- **Theorem 4.3** *(**Compact Operator Approximated by Finite Rank Operator**)[Reed and Simon, 1980]*
  *Let $\mathcal{H}$ be a **separable Hilbert space**. Then every **compact operator** on $\mathcal{H}$ is the **norm limit** of a sequence of operators of **finite rank**.*

4.2 Fredholm Alternative

- **Remark** (***Fredholm Alternative***)
  The basic principle which makes compact operators important is ***the Fredholm alternative***: If A is ***compact***, then ***exactly one*** *of the following two statements holds true*:

  1. $$A\varphi = \varphi \text{ has a solution;}$$

  2. $$(I - A)^{-1} \text{ exists.}$$

  From *the Fredhold alternative*, we see that if ***for any*** φ there is ***at most one*** ψ (***uniqueness*** *statement*) such that
  $$(I - A)\,\psi = \varphi$$
  then there is ***always exactly one*** (*i.e.* ***existence*** *statement*). That is, ***compactness*** *and* ***uniqueness*** *together imply* ***existence***.

- **Theorem 4.4** *(**Analytic Fredholm Theorem**) [Reed and Simon, 1980]*
  *Let D be an **open connected** subset of $\mathbb{C}$. Let $f : D \to \mathcal{L}(\mathcal{H})$ be an **analytic operator-valued function** such that $f(z)$ is **compact** for each $z \in D$. Then, either*

  1. *$(I - f(z))^{-1}$ exists for **no** $z \in D$; or*
  2. *$(I - f(z))^{-1}$ exists for **all** $z \in D \setminus S$ where S is a **discrete** subset of D (i.e. S is a set which has no limit points in D.) In this case, $(I - f(z))^{-1}$ is **meromorphic** in D, **analytic** in $D \setminus S$, the **residues** at the poles are **finite rank operators**, and if $z \in S$ then*
     $$f(z)\varphi = \varphi$$
     *has a **nonzero solution** in $\mathcal{H}$*

- **Corollary 4.5** *(**The Fredholm Alternative**) [Reed and Simon, 1980]*
  *If A is a **compact** operator on $\mathcal{H}$, then **either** $(I - A)^{-1}$ exists **or** $\varphi = \varphi$ has a solution.*

- **Theorem 4.6** *(**Riesz-Schauder Theorem**) [Reed and Simon, 1980]*
  *Let A be a **compact** operator on $\mathcal{H}$, then $\sigma(A)$ is a **discrete set** having **no limit points except** perhaps $\lambda = 0$.*

  *Further, any **nonzero** $\lambda \in \sigma(A)$ is an **eigenvalue** of **finite multiplicity** (i.e. the corresponding space of eigenvectors is **finite dimensional**).*

- **Remark** (***Compact** Operator has only **Nonzero Point Spectrum** with **Finite Dimensional Eigenspace***)
  Riesz-Schauder Theorem states that *the **spectrum** for **compact** operator on **Hilbert** space* consists of *only* the point spectrum besides $\lambda = 0$.

  Moreover, *the **eigenspace*** corresponding to *each **nonzero eigenvalue*** is *finite dimensional.*

- **Theorem 4.7** *(**The Hilbert-Schmidt Theorem**) [Reed and Simon, 1980]*
  *Let A be a **self-adjoint compact operator** on $\mathcal{H}$. Then, there is a **complete orthonormal basis**, $\{\phi_n\}_{n=1}^{\infty}$, for $\mathcal{H}$ so that*

  $$A\phi_n = \lambda_n \phi_n$$

  and $\lambda_n \to 0$ as $n \to \infty$.

- **Remark** (***Eigendecomposition** of Hilbert Space based on **Self-Adjoint Compact Operator***)
  In other word, given a self-adjoint compact operator A on $\mathcal{H}$, *the HIlbert space $\mathcal{H}$ is the direct sum of eigenspaces of A.*

  $$\mathcal{H} = \bigoplus_{\lambda_n \in \sigma(A) \subset \mathbb{R}} \mathrm{Ker}\,(\lambda_n I - A)$$

  A ***self-adjoint compact operator*** on $\mathcal{H}$ is the closest counterpart of ***Hermitian matrix** / **Symmetric Real matrix*** in infinite dimensional space.

- **Theorem 4.8** *(**Canonical Form for Compact Operators**) [Reed and Simon, 1980]*
  *Let A be a **compact** operator on $\mathcal{H}$. Then there exist (**not necessarily complete**) **orthonormal sets** $\{\psi_n\}_{n=1}^{N}$ and $\{\phi_n\}_{n=1}^{N}$ and **positive real numbers** $\{\lambda_n\}_{n=1}^{N}$ with $\lambda_n \to 0$ so that*

  $$A = \sum_{n=1}^{N} \lambda_n \left\langle \cdot\,, \psi_n \right\rangle \phi_n \tag{4}$$

  *The sum in (4), which may be finite or infinite, **converges in norm**. The numbers, $\{\lambda_n\}_{n=1}^{N}$, are called the **singular values of** A.*

- **Remark** (***SVD for Compact Operator***)
  Recall for *finite dimensional case*, *the **singular value decomposition (SVD)***

  $$A = \sum_{n=1}^{N} \lambda_n \phi_n \psi_n^T.$$

  The *singular value decomposition* is a generalization for *the spectral decomposition* for *self-adjoint operator*. But it only exists for ***compact operator***.

4.3 The Trace Class

- We generalize the definition of *trace* of linear operator from finite dimensional space to infinite dimensional space:

  Definition (***Trace of Positive Semi-Definite Operator***)
  Let $\mathcal{H}$ be a ***separable Hilbert space***, $\{\phi_n\}_{n=1}^{\infty}$ an ***orthonormal basis*** Then for any ***positive semi-definite*** operator $A \in \mathcal{L}(\mathcal{H})$, we define

  $$\mathrm{tr}\,(A) = \sum_{n=1}^{\infty} \langle A\phi_n\,,\,\phi_n\rangle$$

  The number $\mathrm{tr}\,(A)$ is called ***the trace of*** A.

- **Proposition 4.9** *(**Properties** of **Trace**) [Reed and Simon, 1980]*
  *Let $\mathcal{H}$ be a separable Hilbert space, $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis. Then for any **positive semi-definite** operator $A \in \mathcal{L}(\mathcal{H})$, its trace $\mathrm{tr}\,(A)$ as defined above is **independent** of the orthonormal basis chosen. The trace has the following properties:*

  1. *(**Linearity**):* $\mathrm{tr}\,(A + B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)\,.$
  2. *(**Positive Homogeneity**):* $\mathrm{tr}\,(\lambda A) = \lambda \mathrm{tr}\,(A)$ *for all* $\lambda \geq 0$*.*
  3. *(**Unitary Invariance**):* $\mathrm{tr}\,\left(U\,A\,U^{-1}\right) = \mathrm{tr}\,(A)$ *for any **unitary** operator* U*.*
  4. *(**Monotonicity**): if* $B \succeq A \succeq 0$*, then* $\mathrm{tr}\,(B) \geq \mathrm{tr}\,(A)$

- **Remark** (***Trace of General Linear Operator***)
  Let $A \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator on separable Hilbert space. Instead of considering *the trace of* A, we consider *the trace of modulus of* A,

  $$\mathrm{tr}\,(|A|) = \mathrm{tr}\left(\sqrt{A^{*}A}\right).$$

- **Definition** (***Trace Class***)
  An operator $A \in \mathcal{L}(\mathcal{H})$ is called ***trace class*** if and only if

  $$\mathrm{tr}\,(|A|) = \mathrm{tr}\left(\sqrt{A^{*}A}\right) < \infty.$$

  The family of all trace class operators is denoted by $\mathcal{B}_1(\mathcal{H})$.

- The following lemma is used in proof of part 2 in next proposition

  Lemma 4.10 *Every $B \in \mathcal{L}(\mathcal{H})$ can be written as a linear combination of **four unitary operators**.*

- **Proposition 4.11** *(**Space of Trace Class Operator**) [Reed and Simon, 1980]*
  *The family of all trace class operators $\mathcal{B}_1(\mathcal{H})$ is a **$*$-ideal** in $\mathcal{L}(\mathcal{H})$, that is,*

  1. *$\mathcal{B}_1(\mathcal{H})$ is a **vector space**.*
  2. *(**Operator Multiplication**) If $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$, then $AB \in \mathcal{B}_1(\mathcal{H})$ and $BA \in \mathcal{B}_1(\mathcal{H})$.*
  3. *(**Adjoint**) If $A \in \mathcal{B}_1(\mathcal{H})$ then $A^{*} \in \mathcal{B}_1(\mathcal{H})$.*

- **Remark Definition** (***-Algebra***)
  An ***algebra*** $\mathcal{A}$ *over field* K is a K***-vector space*** together with a ***binary product*** $(a, b) \mapsto ab$ satisfying

  1. $a(bc) = (ab)c$,
  2. $\lambda(ab) = (\lambda a)b = a(\lambda b)$,
  3. $a(b + c) = ab + ac$,
  4. $(a + b)c = ac + bc$,

  for all $a, b, c \in \mathcal{A}$ and $\lambda \in K$.

  A $*$***-algebra*** $\mathcal{A}$ is a ***algebra*** over $\mathbb{C}$ with a unary ***involution*** $* : a \mapsto a^*$ such that

  1. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$,
  2. $(ab)^* = b^* a^*$,
  3. $(a^*)^* = a$,

  for all $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$.

  Example (***Hilbert Adjoint as*** $*$***-Operation***)
  For $\mathcal{L}(\mathcal{H})$, let the $*$-operation be *the* ***Hilbert adjoint***, i.e. $\langle Tx\,,\, y\rangle = \langle x\,,\, T^* y\rangle$ so $\mathcal{L}(\mathcal{H})$ *is a* $*$***-algebra*** with *operator addition* and *operator multiplication.*

  Definition (***Left Ideal***)
  For an arbitrary ***ring*** $(R, +, \cdot)$, let $(R, +)$ be its ***additive group***. A *subset* I is called a ***left ideal*** of R if it is an *additive subgroup* of R that "*absorbs multiplication from the left by elements of* R"; that is, I is a *left ideal* if it satisfies the following two conditions:

  1. $(I, +)$ is a *subgroup* of $(R, +)$,
  2. For every $r \in R$ and every $x \in I$, the *product* rx is in I.

- **Proposition 4.12** *(**Norm of Trace Class**) [Reed and Simon, 1980]*
  Let $\|\cdot\|_1$ *be defined in* $\mathcal{B}_1(\mathcal{H})$ *by*

  $$\|A\|_1 = \mathrm{tr}\left(|A|\right).$$

  Then $\mathcal{B}_1(\mathcal{H})$ *is a* ***Banach space*** *with norm* $\|\cdot\|_1$ *and*

  $$\|A\| \le \|A\|_1$$

- **Remark** $\mathcal{B}_1(\mathcal{H})$ is ***not closed*** under *the operator norm* $\|\cdot\|$ in $\mathcal{L}(\mathcal{H})$.

- **Proposition 4.13** *(**Compactness**) [Reed and Simon, 1980]*
  Every $A \in \mathcal{B}_1(\mathcal{H})$ *is* ***compact***. *A* ***compact*** *operator* A *is in* $\mathcal{B}_1(\mathcal{H})$ ***if and only if***

  $$\sum_{n=1}^{\infty} \lambda_n < \infty$$

  where $\{\lambda_n\}$ *are the* ***singular values*** *of* A.

- **Corollary 4.14** *(**Finite Rank Approximation**) [Reed and Simon, 1980]*
  *The finite rank operators are $\|\cdot\|_1$-**dense** in $\mathcal{B}_1(\mathcal{H})$.*

- **Proposition 4.15** *[Reed and Simon, 1980]*
  *If $A \in \mathcal{B}_1(\mathcal{H})$ and $\{\varphi_n\}_{n=1}^{\infty}$ is **any** orthonormal basis, then*

$$\sum_{n=1}^{\infty} \langle A\phi_n \, , \, \phi_n \rangle$$

  *converges **absolutely** and the limit is **independent** of the choice of basis.*

4.4 Hilbert-Schmidt Operator

- **Definition** (***Hilbert-Schmidt Operator***)
  An operator $T \in \mathcal{L}(\mathcal{H})$ is called ***Hilbert-Schmidt*** if and only if

$$\mathrm{tr}\,(T^{*}T) < \infty.$$

  The family of all Hilbert-Schmidt operators is denoted by $\mathcal{B}_2(\mathcal{H})$ or $\mathcal{B}_{HS}(\mathcal{H})$.

- **Proposition 4.16** *(**Space of Hilbert-Schmidt Operator**) [Reed and Simon, 1980]*
  1. *The space of all Hilbert-Schmidt operators $\mathcal{B}_2(\mathcal{H})$ is a $*$-**ideal** in $\mathcal{L}(\mathcal{H})$,*
  2. *(**Inner Product**): If $A, B \in \mathcal{B}_2(\mathcal{H})$, then for **any orthonormal basis** $\{\varphi_n\}_{n=1}^{\infty}$,*

$$\sum_{n=1}^{\infty} \langle A^{*}B\varphi_n \, , \, \varphi_n \rangle$$

     *is **absolutely summable**, and its **limit**, denoted by $\langle A \, , \, B \rangle_{HS}$, is **independent** of the orthonormal basis chosen, i.e.*

$$\langle A \, , \, B \rangle_{HS} = \mathrm{tr}\,(A^{*}B)$$

  3. *$\mathcal{B}_2(\mathcal{H})$ with inner product $\langle \cdot \, , \, \cdot \rangle_{HS}$ is a **Hilbert space**.*
  4. *(**Norm**): Let $\|\cdot\|_2$ be defined in $\mathcal{B}_2(\mathcal{H})$ by*

$$\|A\|_2 := \sqrt{\langle A \, , \, A \rangle}_{HS} = \sqrt{\mathrm{tr}\,(A^{*}A)}.$$

     Then

$$\|A\| \le \|A\|_2 \le \|A\|_1 \, , \quad \textit{and} \quad \|A\|_2 = \|A^{*}\|_2$$

  5. *(**Compactness**) Every $A \in \mathcal{B}_2(\mathcal{H})$ is **compact** and a **compact operator**, A, is in $\mathcal{B}_2(\mathcal{H})$ **if and only if***

$$\sum_{n=1}^{\infty} \lambda_n^2 < \infty$$

     *where $\{\lambda_n\}$ are the **singular values** of A.*

6. ***(Finite Rank Approximation)*** *The* ***finite rank operators*** *are* $\|\cdot\|_2$-***dense*** *in* $\mathcal{B}_2(\mathcal{H})$.
7. $A \in \mathcal{B}_2(\mathcal{H})$ ***if and only if***
$$\{\|A\varphi_n\|\}_{n=1}^{\infty} \in \ell^2$$
for ***some*** *orthonormal basis* $\{\varphi_n\}_{n=1}^{\infty}$.
8. $A \in \mathcal{B}_1(\mathcal{H})$ *if and only if* $A = BC$ *with* $B, C \in \mathcal{B}_2(\mathcal{H})$.
9. $\mathcal{B}_2(\mathcal{H})$ *is not* $\|\cdot\|$*-closed in* $\mathcal{L}(\mathcal{H})$.

- **Theorem 4.17** ***(Hilbert-Schmidt Operator of*** L^2 ***Space)*** *[Reed and Simon, 1980]*
Let (M, μ) *be a* ***measure space*** *and* $\mathcal{H} = L^2(M, \mu)$. *Then* $T \in \mathcal{L}(\mathcal{H})$ *is* ***Hilbert-Schmidt if and only if*** *there is a function*
$$K \in L^2(M \times M, \mu \otimes \mu)$$
with
$$(Tf)(x) = \int_M K(x, y) f(y) d\mu(y),$$
Moreover,
$$\|T\|_2^2 = \int_{M \times M} |K(x, y)|^2 \, d\mu(x) d\mu(y).$$

- **Remark** A ***Hilbert-Schmidt*** operator T on a ***square integrable*** *space* $L^2(M, \mu)$ is a ***integral kernel operator***.

In other word, for $T \in \mathcal{L}(\mathcal{H})$, if $\mathrm{tr}\,(T^*T) < \infty$, then T is a ***compact operator***. If, in particular, $\mathcal{H} = L^2(M, \mu)$, then T can be written as the *integral kernel operator*
$$(Tf)(x) = \int_M K(x, y) f(y) d\mu(y),$$

- **Theorem 4.18** ***(Mercer's Theorem)*** *[Borthwick, 2020].*
Suppose Ω *is a* ***compact domain*** *and* T *is a* ***positive Hilbert-Schmidt operator*** *on* $L^2(\Omega)$. *If the integral kernel* $K(\cdot, \cdot)$ *is* ***continuous*** *on* $\Omega \times \Omega$, *then the* ***eigenfunction*** φ_k *is* ***continuous*** *on* Ω *if* $\lambda_k > 0$, *and the expansion*
$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$$
converges ***uniformly*** *on* ***compact*** *sets.*

4.5 Trace of Linear Operator

- **Definition** (***Trace***)
The map $\mathrm{tr} : \mathcal{B}_1(\mathcal{H}) \to \mathbb{C}$ given by
$$\mathrm{tr}\,(A) = \sum_{n=1}^{\infty} \langle A\phi_n \,,\, \phi_n \rangle$$
where $\{\phi_n\}_{n=1}^{\infty}$ is *any orthonormal basis* in $\mathcal{H}$ is called ***the trace***.

- **Remark** For $A \in \mathcal{B}_1(\mathcal{H})$, $\sum_{n=1}^{\infty} |\langle A\phi_n\,,\,\phi_n\rangle| < \infty$ for *any orthonormal basis* $\{\phi_n\}_{n=1}^{\infty}$.
- **Remark** (***Decomposition of Self-Adjoint operator***)
For any $A \in \mathcal{L}(\mathcal{H})$ and A being self-adjoint,

$$A = A_+ - A_-$$

where both A_+ and A_- are ***positive*** and $A_+A_- = 0$.

Not surprisingly, $A \in \mathcal{B}_1(\mathcal{H})$ if and only if

$$\mathrm{tr}\,(A_+) < \infty, \;\; \mathrm{tr}\,(A_-) < \infty,$$

and

$$\mathrm{tr}\,(A) = \mathrm{tr}\,(A_+) - \mathrm{tr}\,(A_-)\,.$$

- Finally, we collect the property of trace for linear operators:

Proposition 4.19 *(**Properties of Trace**) [Reed and Simon, 1980]*

1. $\mathrm{tr}\,(\cdot)$ *is linear.*
2. $\mathrm{tr}\,(A^*) = \overline{\mathrm{tr}\,(A)}$.
3. $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ *if* $A \in \mathcal{B}_1(\mathcal{H})$ *and* $B \in \mathcal{L}(\mathcal{H})$.

- **Remark** If $A \in \mathcal{B}_1(\mathcal{H})$, the map

$$B \mapsto \mathrm{tr}\,(AB)$$

is a ***linear functional*** on $\mathcal{L}(\mathcal{H})$. We can also hold $B \in \mathcal{L}(\mathcal{H})$ *fixed* and obtain a ***linear functional*** on $\mathcal{B}_1(\mathcal{H})$ given by the map

$$A \mapsto \mathrm{tr}\,(BA)\,.$$

The set of these functionals is just ***the dual of*** $\mathcal{B}_1(\mathcal{H})$.

- **Proposition 4.20** *(**Dual Space of Compact Operators**) [Reed and Simon, 1980]*

1. $\mathcal{B}_1(\mathcal{H}) = (Com(\mathcal{H}))^*$. *That is, the map* $A \mapsto \mathrm{tr}\,(A\cdot)$ *is an **isometric isomorphism** of* $\mathcal{B}_1(\mathcal{H})$ *onto* $(Com(\mathcal{H}))^*$.
2. $\mathcal{L}(\mathcal{H}) = (\mathcal{B}_1(\mathcal{H}))^*$. *That is, the map* $B \mapsto \mathrm{tr}\,(B\cdot)$ *is an **isometric isomorphism** of* $\mathcal{L}(\mathcal{H})$ *onto* $(\mathcal{B}_1(\mathcal{H}))^*$.

5 Spectrum of Bounded Self-Adjoint Operator in Hilbert Space

5.1 General Properties

- **Proposition 5.1** *(**Spectral Radius Calculation**) [Reed and Simon, 1980]*
*Let X be a **Hilbert space**, $T \in \mathcal{L}(X)$ and T is **self-adjoint**. Then*

$$r(T) = \|T\|$$

- **Theorem 5.2** ***(Spectrum and Resolvent of Adjoint) (Phillips)*** *[Reed and Simon, 1980]*
*If X is a **Hilbert space** and $T \in \mathcal{L}(X)$, then*

$$\sigma(T) = \sigma(T^*) \;\; \text{and} \;\; R_\lambda(T^*) = (R_\lambda(T))^*.$$

- **Proposition 5.3** ***(Spectrum of Self-Adjoint Operator)*** *[Reed and Simon, 1980]*
*Let be a **self-adjoint operator** on a **Hilbert space** $\mathcal{H}$. Then,*

  1. *T has **no residual spectrum**, i.e. $\sigma_r(T) = \emptyset$.*

  2. *$\sigma(T)$ is a subset of $\mathbb{R}$.*

  3. ***Eigenvectors** corresponding to **distinct eigenvalues** of T are **orthogonal**.*

- **Remark** (***Resemblence to Symmetric or Hermitian Matrix***)
This property is the same as the *spectrum* for *symmetric* real matrix or *Hermitian matrix* in *finite dimensional case*. That is,

  1. *the **eigenvalues** of* symmetric real matrices *or* Hermitian matrices *are all **real-valued***;

  2. the ***eigenspaces*** corresponds to ***distinct eigenvalue***s are ***orthorgonal*** to each other.

5.2 Positive Semidefinite Operators and the Polar Decomposition

- **Definition** (***Positive-Semidefinite Operator***)
Let $\mathcal{H}$ be a ***Hilbert space***. An operator $B \in \mathcal{L}(\mathcal{H})$ is called ***positive-semidefinite*** if

$$\langle Bx\,,\, x\rangle \geq 0 \text{ for all } x \in \mathcal{H}.$$

We write $B \succeq 0$ if is *positive-semidefinite* and $B \succeq A$ if $(B - A) \succeq 0$.

Similarly, B is called ***positive-definite*** if

$$\langle Bx\,,\, x\rangle > 0 \text{ for all } x \neq 0 \in \mathcal{H}.$$

The *positive semidefinite operator* is sometimes called ***positive*** *operator*.

- **Proposition 5.4** ***(Positive Semi-Definiteness $\Rightarrow$ Self-Adjoint)*** *[Reed and Simon, 1980]*
*Every (bounded) **positive semidefinite** operator on a **complex Hilbert space** is **self-adjoint**.*

  Proof: Notice that $\langle Ax\,,\, x\rangle$ takes only real value, so

$$\langle Ax\,,\, x\rangle = \overline{\langle Ax\,,\, x\rangle} = \langle x\,,\, Ax\rangle$$

By *the polarization identity*,

$$\langle Ax\,,\, y\rangle = \langle x\,,\, Ay\rangle$$

if $\langle Ax\,,\, x\rangle = \langle x\,,\, Ax\rangle$ for all x. Thus, if A is positive, it is self-adjoint. ∎

- **Remark** (***Square Root of Positive Semidefinite Operator***)
For any $A \in \mathcal{L}(\mathcal{H})$ notice that ***the normal operator*** *is* ***positive semi-definite***

$$A^*A \succeq 0$$

since

$$\langle A^*Ax\,,\,x\rangle = \|Ax\|^2 \geq 0.$$

Just as $|z| = \sqrt{\bar{z}z}$, we want to find the modulus of a linear operator as

$$|A| := \sqrt{A^*A}$$

To show the square root of positive semidefinite operator makes sense, we have the following lemma

Lemma 5.5 *The power series for* $\sqrt{1-z}$ *about zero converges* ***absolutely*** *for all complex numbers* z *satisfying* $|z| \leq 1$.

Theorem 5.6 ***(Square Root Lemma)*** *[Reed and Simon, 1980]*
Let $A \in \mathcal{L}(\mathcal{H})$ *and* $A \succeq 0$. *Then there is a* ***unique*** $B \in \mathcal{L}(\mathcal{H})$ *with* $B \succeq 0$ *and* $B^2 = A$. *Furthermore,* B ***commutes*** *with every bounded operator which commutes with* A.

- **Definition** For $A \in \mathcal{L}(\mathcal{H})$, we can define ***absolute value*** *of* A as the square root of its normal operation

$$|A| := \sqrt{A^*A}$$

- **Remark** For $|\cdot|$ operation on linear operator A:

  1. $|\lambda A| = |\lambda|\,|A|$
  2. $|\cdot|$ is ***norm continuous*** on $\mathcal{L}(\mathcal{H})$
  3. in general the following equations ***do not hold***

$$|AB| = |A|\,|B|\,, \quad |A| = |A^*|$$

- **Definition** (***Partial Isometry***)
An operator $U \in \mathcal{L}(\mathcal{H})$ is called an ***isometry*** if

$$\|Ux\| = \|x\|\,, \quad \text{all } x \in \mathcal{H}.$$

U is called a ***partial isometry*** if U is an *isometry* when ***restricted*** to the *closed subspace* $(\mathrm{Ker}(U))^{\perp}$.

- **Remark** (***Partial Isometry*** $=$ ***Unitary*** $(\boldsymbol{Ker}(U))^{\perp} \to \boldsymbol{Ran}(U)$)
If U is a ***partial isometry***, $\mathcal{H}$ can be written as

$$\mathcal{H} = (\mathrm{Ker}(U)) \oplus (\mathrm{Ker}(U))^{\perp}, \quad \mathcal{H} = (\mathrm{Ran}(U)) \oplus (\mathrm{Ran}(U))^{\perp}$$

and U is a ***unitary operator*** between $(\mathrm{Ker}(U))^{\perp}$, *the* ***initial subspace*** *of* U, and $\mathrm{Ran}(U)$, *the* ***final subspace*** of U.

Moreover, its *adjoint* is its *inverse*, $U^* = (U_{(\mathrm{Ker}(U))^{\perp}})^{-1} : \mathrm{Ran}(U) \to (\mathrm{Ker}(U))^{\perp}$.

- **Proposition 5.7** ***(Projection Operators by Partial Isometry)*** *[Reed and Simon, 1980]*
Let U *be a* ***partial isometry***. *Then* $P_i = U^*U$ *and* $P_f = UU^*$ *are respectively the* ***projections*** *onto the* ***initial*** *and* ***final subspaces*** *of* U, *i.e.*

$$P_i := U^*U = P_{(Ker(U))^{\perp}}, \quad P_f := UU^* = P_{Ran(U)},$$

Conversely, if $U \in \mathcal{L}(\mathcal{H})$ *with* U^*U *and* UU^* ***projections***, *then* U *is a* ***partial isometry***.

- **Theorem 5.8** ***(Polar Decomposition)*** *[Reed and Simon, 1980]*
  Let A be a bounded linear operator on a ***Hilbert space****. Then there is a* ***partial isometry*** *U such that*

$$A = U\,|A|$$

  U is ***uniquely*** *determined by the condition that $Ker(U) = Ker(A)$. Moreover, $Ran(U) = \overline{Ran(A)}$.*

5.3 Spectral Theorem for Finite Dimensional Case

- **Definition** (***Similarity***) [Horn and Johnson, 2012]
  Let $A, B \in M_n$ be given $n \times n$ matrices. We say that B ***is similar to*** A if there exists a ***nonsingular*** $S \in M_n$ such that

$$B = S^{-1}AS$$

  The transformation $A \to S^{-1}AS$ is called a ***similarity transformation*** by the *similarity matrix* S.

- **Definition** (***Normal Matrix***) [Horn and Johnson, 2012]
  A matrix $A \in M_n$ is ***normal*** if

$$AA^* = A^*A,$$

  that is, if A ***commutes*** with its ***conjugate transpose (adjoint)***.

- **Definition** (***Diagonalizable***) [Horn and Johnson, 2012]
  If $A \in M_n$ is *similar* to a *diagonal matrix*, then A is said to be ***diagonalizable***.

- **Definition** (***Unitary Similarity***) [Horn and Johnson, 2012]
  Let $A, B \in M_n$ be given. We say that A is ***unitarily similar*** to B if there is a ***unitary*** $U \in M_n$ such that

$$A = UBU^*$$

  We say that A is ***unitarily diagonalizable*** if it is ***unitarily similar*** to a diagonal matrix.

  We say that A is ***orthogonally similar*** to B if there is a ***unitary (real orthorgonal)*** $U \in M_n(\mathbb{R})$ such that

$$A = UBU^T$$

  We say that A is ***orthogonally diagonalizable*** if it is ***orthogonally similar*** to a diagonal matrix.

- **Theorem 5.9** ***(Spectral Theorem of Normal Matrix)*** *[Horn and Johnson, 2012]*
  Let $A = [a_{i,j}] \in M_n$ have ***eigenvalues*** *$\lambda_1, \ldots, \lambda_n$. The following statements are* ***equivalent:***

  1. *A is* ***normal****.*
  2. *A is* ***unitarily diagonalizable****, i.e. there exists unitary matrix $U \in M_n$ such that*

$$A = U\Lambda U^*$$

     where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$.

3. $\sum_{i,j=1}^{n} |a_{i,j}|^2 = \sum_{i=1}^{n} \lambda_i^2$

4. *A has n* ***orthonormal eigenvectors***

- **Definition** (***Spectral Decomposition***)
  A *representation* of a ***normal matrix*** $A \in M_n$ as $A = U \Lambda U^*$, in which U is ***unitary*** and Λ is ***diagonal***, is called a ***spectral decomposition of*** A.

- The Hermitian matrix is normal matrix, so the following theorem is a special case of the spectral theorem for normal matrix.

  Theorem 5.10 ***(Spectral Theorem for Hermitian Matrices)*** *[Horn and Johnson, 2012]*
  Let $A \in M_n$ be ***Hermitian*** *and have eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. Then*

  1. *$\lambda_1, \ldots, \lambda_n$ are* ***real*** *numbers.*

  2. *A is* ***unitarily diagonalizable***

  3. *There is a* ***unitary*** *$U \in M_n$ such that*

  $$A = U \Lambda U^*$$

- **Remark** This is equivalent to say that for ***self-adjoint bounded linear operator*** A on finite dimensional space V, there exists ***unitary operator*** $U : \mathbb{C}^n \to V$ such that

  $$[U^{-1} A U f]_k = \lambda_k f_k$$

  for any $f = (f_k)_{k=1}^{n} \in \mathbb{C}^n$.

5.4 Spectral Theorem

5.4.1 The Continuous Functional Calculus

- **Remark** (***Spectral Theorem for Self-Adjoint Bounde Linear Operator in Hilbert Space***)
  Given a *bounded self-adjoint operator* $A \in \mathcal{L}(\mathcal{H})$ on *Hilbert space* $\mathcal{H}$, we can find a ***measure*** μ on a *measure space* $\mathcal{M}$ and a ***unitary operator*** $U : L^2(\mathcal{M}, \mu) \to \mathcal{H}$ so that

  $$[U^{-1} A U f](x) = F(x) f(x)$$

  for some ***bounded real-valued measurable function*** F on $\mathcal{M}$.

  In practice, $\mathcal{M}$ will be a *union of copies of* $\mathbb{R}$ and F will be x, so the ***core*** of the proof of the theorem will be ***the construction of certain measures*** μ.

- **Remark** (***Functional Calculus***) [Borthwick, 2020]
  In operator theory, the term "***functional calculus***" refers to the ability to *apply a function to an operator.*

  For $A \in \mathcal{L}(\mathcal{H})$, one need to make sense of $f(A)$ for some continuous function f. For instance, If $f(x) = \sum_{j=0}^{n} a_j x^j$ is a *polynomial*, we want

  $$f(A) = \sum_{j=0}^{n} a_j A^j.$$

Similarly, suppose that $f(x) = \sum_{j=0}^{\infty} c_j x^j$ is a *power series* with *radius of convergence* R. If $\|A\| < R$, then $\sum_{j=0}^{\infty} c_j A^j$ *converges* in $\mathcal{H}$ so it is natural to set

$$f(A) = \sum_{j=0}^{\infty} a_j A^j.$$

- In particular, we have

  Lemma 5.11 ***(Spectrum of Polynomial of Operators)*** *[Reed and Simon, 1980]*
  Let $P(x) = \sum_{n=0}^{N} a_n x^n$ *and* $P(A) = \sum_{n=0}^{N} a_n A^n$*. Then*

  $$\sigma\left(P(A)\right) = \{P(\lambda) : \lambda \in \sigma(A)\}$$

- **Lemma 5.12** ***(Norm of Polynomial of Bounded Self-Adjoint Operators)*** *[Reed and Simon, 1980]*
  Let A *be a* ***bounded self-adjoint*** *operator. Then*

  $$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$$

- **Theorem 5.13** ***(Continuous Functional Calculus)*** *[Reed and Simon, 1980]*
  Let A *be a* ***self-adjoint*** *operator on a* ***Hilbert space*** $\mathcal{H}$*. Then there is a* ***unique*** *map* $\phi : \mathcal{C}(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ *with the following properties:*

  1. ϕ *is an* ***algebraic*** $*$***-homomorphism****, that is,*
     - ***(Preserve Operator Product)*** $\phi(fg) = \phi(f)\phi(g)$
     - ***(Preserve Scalar Product)*** $\phi(\lambda f) = \lambda \phi(f)$
     - ***(Preserve Identity)*** $\phi(1) = I$
     - ***(Preserve Adjoint/Conjugacy)*** $\phi(\bar{f}) = \phi(f)^*$
  2. ϕ *is* ***continuous****, that is,*

     $$\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} \le C\,\|f\|_{\infty}.$$

  3. *Let* f *be the function* $f(x) = x$*; then* $\phi(f) = A$*. Moreover,* ϕ *has the* ***additional*** *properties:*
  4. *If* $A\psi = \lambda\psi$*, then*

     $$\phi(f)\psi = f(\lambda)\psi \tag{5}$$

  5. ***(Spectral Mapping Theorem)***

     $$\sigma(\phi(f)) = \{f(\lambda) : \lambda \in \sigma(A)\} \tag{6}$$

  6. ***(Preserve Positivity)*** *If* $f \ge 0$*, then* $\phi(f) \succeq 0$.

7. ***(Preserve Norm)*** *(This strengthens the (2)).*

$$\|\phi(f)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_{\infty} \tag{7}$$

We sometimes write $f(A)$ or $\phi_A(f)$ for $\phi(f)$ to emphasize the dependency on A.

- **Remark** Note that the continuous function f in defining $f(A)$ is defined on $\sigma(A)$, i.e. ***the spectrum of operator*** A, so f ***is a spectral domain transformation function***. In the map,

$$\phi : f \mapsto \phi(f) := f(A) : \mathcal{H} \to \mathcal{H}.$$

  1. So in equation

$$\begin{aligned} \phi(fg) = \phi(f)\phi(g) &\Leftrightarrow (fg)(A) = f(A)g(A) \\ \phi(\lambda f) = \lambda\phi(f) &\Leftrightarrow (\lambda f)(A) = \lambda\, f(A) \\ \phi(1) = I &\Leftrightarrow 1(A) = I \\ \phi(\bar{f}) = \phi(f)^* &\Leftrightarrow (\bar{f})(A) = (f(A))^* \\ \phi(\mathrm{Id}) = \mathrm{Id} &\Leftrightarrow (\mathrm{id})(A) = A \end{aligned}$$

  the LHS of first equation is an operator corresponding to the ***product of two functions***, while the RHS of first equation is ***the product of two operators***, each corresponding to one function.

  2. The equation (5) makes sure that *the spectral decomposition* of $f(A)$ and that of A ***shares the same set of eigenfunctions***.

  3. The spectral mapping theorem in (6) actually defines $f(A)$ as the operator whose spectrum is transformed by f. In other words, $f(A)$ ***is the operator obtained by spectral domain transformation via*** f.

  In signal processing, $f(A)$ corresponds to ***the spectral filtering*** of A.

- **Remark** There are some more remarks:

  1. $\phi(f) \succeq 0$ ***if and only if*** $f \geq 0$.

  2. (***Abelian C^*-Algebra***)
  Since $fg = gf$ for all f, g,

$$\{f(A) : f \in \mathcal{C}(\sigma(A))\}$$

  forms an ***abelian algebra*** closed under ***adjoints***. Since $\|\phi(f)\| = \|f\|_{\infty}$ and $\mathcal{C}(\sigma(A))$ is ***complete***, $\{f(A) : f \in \mathcal{C}(\sigma(A))\}$ is ***norm-closed***. It is thus an ***abelian C^*-algebra*** of *operators*.

  3. (***C^*-Algebra Generated by*** A)
  The image of ϕ, i.e. $\{f(A) : f \in \mathcal{C}(\sigma(A))\}$ is actually the ***C^*-algebra generated by*** A, that is, the ***smallest*** *C^*-algebra containing A.*

  4. This result shows that *the space of* ***continuous function on spectrum of*** A, $\mathcal{C}(\sigma(A))$ and *the C^*-algebra generated by A* are ***isometrically isomorphic***.

$$\mathcal{C}(\sigma(A)) \simeq \mathrm{Ran}\;\phi = \{f(A) : f \in \mathcal{C}(\sigma(A))\}\,.$$

5. The property (1) and (3) ***uniquely*** *determines* the mapping ϕ.

- **Example** (***Existence of Square Root for Positive Operator***)
For $A \succeq 0$, $\sigma(A) \geq 0$ and $\sigma(A) \subset \mathbb{R}$, so let $f(x) = \sqrt{x}$, then

$$A = (f(A))^2.$$

- **Example** For $f(x) = (\lambda - x)^{-1}$,

$$\left\|(A - \lambda I)^{-1}\right\| = \sup_{x \in \sigma(A)} |x - \lambda|^{-1} = \frac{1}{\text{dist}\ (\lambda, \sigma(A))}$$

for A bounded and $\lambda \notin \sigma(A)$.

5.4.2 Spectral Measure

- **Remark** (***Positive Linear Functional on*** $\mathcal{C}(\sigma(A))$)
For each $\psi \in \mathcal{H}$, *the quadratic form* below defines a *bounded linear functional* on $\mathcal{L}(\mathcal{H})$

$$\widetilde{I}_{\psi} : A \mapsto \langle \psi \,,\, A\psi \rangle_{\mathcal{H}} \,.$$

Then by continuous functional calculus, we can define a map $I_{\psi} = \widetilde{I}_{\psi} \circ \phi : \mathcal{C}(\sigma(A)) \to \mathbb{R}$, which is seen as a ***positive linear functional*** on $\mathcal{C}(\sigma(A))$, i.e. $\forall \psi \in \mathcal{H}$,

$$I_{\psi}(f) := \langle \psi \,,\, f(A)\psi \rangle \geq 0 \text{ whenever } f \geq 0.$$

For *a* ***bounded self-adjoint operator*** A, the *spectrum* $\sigma(A) \subset \mathbb{R}$ is a ***closed bounded subset*** of $\mathbb{R}$ so it is ***compact***. Thus $\mathcal{C}(\sigma(A))$ is a space of continuous functions on compact domain, so, by Riesz-Markov theorem, $(\mathcal{C}(\sigma(A)))^* \simeq \mathcal{M}(\sigma(A))$, *the space of* ***complex signed Radon measures*** *on* $\sigma(A)$. In other word, for each $\psi \in \mathcal{H}$, there ***exists some*** ***positive Radon measure on spectral domain*** $\mu_{\psi} \in \mathcal{M}(\sigma(A))$ so that

$$I_{\psi}(f) := \langle \psi \,,\, f(A)\psi \rangle = \int_{\sigma(A)} f d\mu_{\psi}. \tag{8}$$

Let $f = \bar{g}g$, the equation (8) becomes

$$\begin{aligned}
\|g(A)\psi\|_{\mathcal{H}}^2 = \langle \psi \,,\, \bar{g}g(A)\psi \rangle_{\mathcal{H}} = \int_{\sigma(A)} \bar{g}g d\mu_{\psi} &= \int_{\sigma(A)} |g(\lambda)|^2 \, d\mu_{\psi}(\lambda) \\
\Rightarrow \|g(A)\psi\|_{\mathcal{H}}^2 &= \int_{\sigma(A)} |g(\lambda)|^2 \, d\mu_{\psi}(\lambda),
\end{aligned} \tag{9}$$

which confirms that ***the energy in time-domain should match the energy in spectral domain***.

- **Definition** (***Spectral Measure***)
For each $\psi \in \mathcal{H}$, the measure $\mu_{\psi} \in \mathcal{M}(\sigma(A))$ defined in (8) is called *the* ***spectral measure associated with the vector*** ψ.

5.4.3 Spectral Theorem in Functional Calculus Form

- **Remark** (***Extension to Bounded Borel Functions on*** $\mathbb{R}$) [Reed and Simon, 1980]
  The first and simplest application of the μ_{ψ} is to allow us to ***extend*** *the functional calculus to* $B(\mathbb{R})$, *the* ***bounded Borel measurable functions*** *on* $\mathbb{R}$.

  1. Note that *the double dual of* $\mathcal{C}(X)$ on *compact* metric space X is *the space of bounded Borel measurable function* $B(X) = L^{\infty}(X, \mu)$ [Lax, 2002].

     $$B(X) \simeq (\mathcal{C}(X))^{**}$$

     In other word, for fixed bounded self-adjoint operator A and $\psi \in \mathcal{H}$, the map

     $$I_{\psi} : g \mapsto \int_{\sigma(A)} g d\mu_{\psi}$$

     is well-defined for $g \in B(\sigma(A))$. Extending to $B(\mathbb{R})$ is natural since $\mathbb{R}$ is *locally compact.*

  2. Use *the polarization identity*

     $$\Re \langle x\, ,\, y \rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2),$$

     we can construct *the bilinear form* for any $\psi, \varphi \in \mathcal{H}$

     $$F(\psi, \varphi) = \frac{1}{2}(I_{(\psi+\varphi)}(g) - I_{(\psi)}(g) - I_{(\varphi)}(g))$$

  3. By *Riesz representation theorem*, there exists a unique linear operator $\widetilde{A}_g$ on $\mathcal{H}$ so that

     $$F(\psi, \varphi) = \left\langle \varphi\, ,\, \widetilde{A}_g \psi \right\rangle = \frac{1}{2}(I_{(\psi+\varphi)}(g) - I_{(\psi)}(g) - I_{(\varphi)}(g))$$

     Thus we identifies $g(A) \equiv \widetilde{A}_g$ for any $g \in B(\mathbb{R})$ so that

     $$\langle \psi\, ,\, g(A)\psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}} g d\mu_{\psi}.$$

  This shows that ***the functional calculus can be extended to all bounded Borel functions***.

- **Theorem 5.14** ***(Spectral Theorem, Functional Calculus Form)*** *[Reed and Simon, 1980]*
  Let A be a ***bounded self-adjoint*** *operator on $\mathcal{H}$. There is a* ***unique map*** $\widehat{\phi} : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ *so that*

  1. $\widehat{\phi}$ *is an* ***algebraic $*$-homomorphism***.
  2. $\widehat{\phi}$ *is* ***norm continuous***:

     $$\|\widehat{\phi}(f)\|_{\mathcal{L}(\mathcal{H})} \le C \, \|f\|_{\infty}.$$

  3. *Let f be the function $f(x) = x$; then $\widehat{\phi}(f) = A$.*

4. ***(Pointwise Convergence $\Rightarrow$ Strong Convergence)***
   Suppose $f_n(x) \to f(x)$ for each x and $\|f_n\|_\infty$ is bounded. Then $\widehat{\phi}(f_n) \to \widehat{\phi}(f)$ ***strongly****.*
   Moreover $\widehat{\phi}$ has the properties :

5. *If $A\psi = \lambda\psi$, then*

$$\widehat{\phi}(f)\psi = f(\lambda)\psi \tag{10}$$

6. ***(Preserve Positivity)*** *If $f \geq 0$, then $\widehat{\phi}(f) \succeq 0$.*

7. ***(Preserve Commutative)*** *If $BA = AB$, then $B\widehat{\phi}(f) = \widehat{\phi}(f)B$.*

- **Remark** The proof of (4) is via dominated convergence theorem.

- **Remark** ***The norm equality*** of *the continuous functional calculus* carries over if we define $\|f\|'_\infty$ to be *the L^∞-norm with respect to a suitable notion* of "***almost everywhere***." Namely, pick *an orthonormal basis* $\{\varphi_n\}$ and say that a property is true a.e. if it is true a.e. *with respect to each μ_{φ_n}*. Then $\|\widehat{\phi}(f)\|_{L^2(\mathcal{H})} = \|f\|'_\infty$.

5.4.4 Spectral Theorem in Multiplication Operator Form

- **Definition** (***Cyclic Vector***)
  A vector $\psi \in \mathcal{H}$ is called a ***cyclic vector*** *for A* if *finite linear combinations* of the elements $\{A^n\psi\}_{n=0}^\infty$ are ***dense*** in $\mathcal{H}$.

- **Remark** Not all operators have cyclic vectors.

- Recall the following theorem for normed vector space

  Theorem 5.15 ***(Bounded Linear Transformation Theorem)*** *[Reed and Simon, 1980]*
  Suppose T is a ***bounded*** *linear transformation from a* ***normed vector space*** *$(V_1, \|\|_1)$ to a* ***complete normed vector space*** *$(V_2, \|\|_2)$. Then T can be* ***uniquely extended*** *to a bounded linear transformation (with the same bound), $\widetilde{T}$, from the* ***completion*** *of V_1 to $(V_2, \|\|_2)$*

- **Lemma 5.16** ***(Spectral Theorem for Bounded Self-Adjoint Operator with Cyclic Vector)*** *[Reed and Simon, 1980]*
  Let A be a ***bounded self-adjoint operator*** *with* ***cyclic vector*** *ψ. Then, there is a* ***unitary operator*** *$U : L^2(\sigma(A), \mu_\psi) \to \mathcal{H}$ with*

$$[U^{-1}AUf](\lambda) = \lambda f(\lambda)$$

  Equality is in the sense of elements of $L^2(\sigma(A), \mu_\psi)$.

- **Lemma 5.17** ***(Direct Sum Decomposition of Hilbert Space via Invariant Subspaces)*** *[Reed and Simon, 1980]*
  Let A be a ***self-adjoint*** *operator on a* ***separable Hilbert space*** *$\mathcal{H}$. Then there is a* ***direct sum decomposition***

$$\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$$

  with $N = 1, 2, \ldots,$ or ∞ so that:

1. $\mathcal{H}_n$ is ***invariant*** under operator A; that is, for any $\psi \in \mathcal{H}_n$, $A\psi \in \mathcal{H}_n$.
2. For each n, there exists a $\psi_n \in \mathcal{H}_n$ that is **cyclic** for $A|_{\mathcal{H}_n}$, i.e.

$$\mathcal{H}_n = \overline{\{f(A)\psi_n : f \in \mathcal{C}(\sigma(A))\}}.$$

- **Theorem 5.18** *(**Spectral theorem, Multiplication Operator Form**) [Reed and Simon, 1980]*
*Let A be a **bounded self-adjoint** operator on $\mathcal{H}$, a **separable Hilbert space**. Then, there exist **measures** $\{\mu_{\psi_n}\}_{n=1}^{N}$ ($N = 1, 2, \ldots,$ or ∞) on $\sigma(A)$ and a **unitary** operator*

$$U : \bigoplus_{n=1}^{N} L^2(\mathbb{R}, \mu_{\psi_n}) \to \mathcal{H}$$

so that

$$[U^{-1}AU\psi]_n(\lambda) = \lambda\psi_n(\lambda) \tag{11}$$

*where we write an element $\psi \in \bigoplus_{n=1}^{N} L^2(\sigma(A), \mu_{\psi_n})$ as an N-tuple $(\psi_1(\lambda), \ldots, \psi_N(\lambda))$. This realization of A is called a **spectral representation**.*

- **Remark** (***Self-Adjoint Bounded Operator = Mulitplication Operator in Spectral Domain***)
This theorem tells us that ***every bounded self-adjoint operator is a multiplication operator on a suitable measure space***; what changes as *the operator changes* are *the underlying measures*.

- **Remark** (***Multiplication Operator***)
Define ***the multiplication operator*** $M_f : v \mapsto fv$ on L^2 for $f \in L^2$, so (11) becomes

$$U^{-1}AU = M_\alpha \tag{12}$$

where $\alpha(x) = x$.

- **Corollary 5.19** *(**Spectral theorem, Single Spectral Measure**) [Reed and Simon, 1980]*
*Let A be a **bounded self-adjoint** operator on a **separable Hilbert space** $\mathcal{H}$. Then there exists a **finite measure space** (M, μ), a **bounded function** F on M, and a **unitary map**, $U : L^2(M, \mu) \to \mathcal{H}$, so that*

$$[U^{-1}AUf]_n(m) = F(m)f(m)$$

- **Example** (***Self-Adjoint Operator on Finite Dimensional Space***)
Let A be an $n \times n$ ***self-adjoint (Hermitian)*** *matrix*. The ***finite dimensional spectral theorem*** says that A has a *complete orthonormal set* of **eigenvectors**, $\psi_1, \ldots, \psi_n$, with

$$A\psi_i = \lambda_i\psi_i.$$

Suppose first that *the eigenvalues are **distinct***. The *spectral measure* is just *the sum of **Dirac measures***,

$$\mu = \sum_{i=1}^{n} \delta_{\lambda_i}, \tag{13}$$

and $L^2(\mathbb{R},\mu)$ is just $\mathbb{C}^n$ since $f \in L^2$ is ***determined*** by

$$(f(\lambda_1), \ldots, f(\lambda_n)).$$

Clearly, the function λf corresponds to the n-tuple $(\lambda_1 f(\lambda_1), \ldots, \lambda_n f(\lambda_n))$, so A is ***multiplication*** by λ on $L^2(\mathbb{R},\mu)$.

If we take

$$\bar{\mu} = \sum_{i=1}^{n} a_i \delta_{\lambda_i},$$

with $a_1, \ldots, a_n > 0$, *A can also be represented as* ***multiplication*** *by λ on $L^2(\mathbb{R},\bar{\mu})$.* Thus, we explicitly see *the* ***nonuniqueness*** *of the* ***measure*** in this case.

We can also see when ***more than one measure is needed***: *one can represent a finite-dimensional self-adjoint operator as multiplication on $L^2(\mathbb{R},\mu)$ with* ***only one measure if and only if*** *A has* ***no repeated eigenvalues***. ∎

- **Example** (***Self-Adjoint Compact Operator***)
  Let A be ***compact*** and ***self-adjoint***. *The Hilbert-Schmidt theorem* tells us there is a *complete orthonormal set of* ***eigenvectors*** $\{\psi_n\}_{n=1}^{\infty}$, with

  $$A\psi_n = \lambda_n \psi_n.$$

  If there is *no repeated eigenvalue*,

  $$\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{\lambda_n} \tag{14}$$

  works as a ***spectral measure***. ∎

- **Example** (***Fourier Transform***)
  Note that for $f \in L^2(\mathbb{R}, dx)$, the Fourier transform of f is written as

  $$\mathcal{F}f(\lambda) := F(\lambda) = \frac{1}{(2\pi)^{-1}} \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx$$
  $$f(x) = \int_{\mathbb{R}} F(\lambda) e^{i\lambda x} d\lambda$$

  The Fourier transform $\mathcal{F}$ can be seen as a unitary map $\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, \mu(d\lambda))$, which is the inverse of U where $e^{i\lambda x} d\lambda = \mu(d\lambda)$.

  Consider $A = \frac{1}{i}\frac{d}{dx}$ on $L^2(\mathbb{R}, dx)$, which is *self-adjoint* but ***unbounded***. The Fourier transform of A gives

  $$\mathcal{F}\left(\frac{1}{i}\frac{d}{dx} f\right)(\lambda) = \lambda\, \mathcal{F}f(\lambda)$$
  $$\Leftrightarrow (U^{-1}AUF)(\lambda) = \lambda\, F(\lambda)$$

  where the unitary map $U : L^2(\mathbb{R}, \mu(d\lambda)) \to L^2(\mathbb{R}, dx)$ is ***the inverse Fourier transform***

  $$(UF)(x) = f(x) = \int_{\mathbb{R}} F(\lambda) e^{i\lambda x} d\lambda.$$

- **Definition** (***Essential Range***)
  Let F be a real-valued function on a measure space (X, μ). We say λ is in ***the essential range of F if and only if*** for all $\epsilon > 0$,

$$\mu\{x : F(x) \in (\lambda - \epsilon, \lambda + \epsilon)\} = \mu \circ F^{-1}(B(\lambda, \epsilon)) > 0.$$

- **Proposition 5.20** *(**Spectrum of Multiplication Operator via Essential Range**) [Reed and Simon, 1980]*
  *Let F be a **bounded real-valued** function on a measure space (X, μ). Let M_F be the multiplication operator on $L^2(X, \mu)$ given by*

$$(M_F g)(x) = F(x)g(x)$$

  *Then $\sigma(M_F)$ is **the essential range** of F.*

5.4.5 Spectral Theorem in Spectral Projection Form

- **Definition** (***Spectral Projection***)
  Let A be a ***bounded self-adjoint*** operator and S a ***Borel set*** of $\mathbb{R}$.

$$P_S := \mathbb{1}_S(A) = \widehat{\phi}(\mathbb{1}_S)$$

  is called a ***spectral projection of** A*. It is result of applying the ***characteristic function of set** R*, $\mathbb{1}_S(x)$, on operator A via *functional calculus*.

- **Remark** (***Spectral Projection is Orthorgonal Projection***)
  P_S is an ***orthogonal projection*** since for each x

$$\mathbb{1}_S^2(x) = \mathbb{1}_S(x) = \bar{\mathbb{1}}_S(x).$$

  It is equivalent to a 0-1 ***test*** *to check if each point of spectrum of A is in S.*

- **Proposition 5.21** *(**Properties of Spectral Projection**) [Reed and Simon, 1980]*
  *The family $\{P_S\}$ of **spectral projections** of a **bounded self-adjoint** operator, A, has the following properties:*

  1. *Each P_S is an **orthogonal projection**.*
  2. *$P_\emptyset = 0$; $P_{(-a,a)} = 1$ for **some** a.*
  3. *(**Countable Disjoint Union**) If $S = \bigcup_{n=1}^\infty S_n$ with $S_n \cap S_m = \emptyset$ for all $n \neq m$, then in norm topology*

$$P_S = \sum_{n=1}^\infty P_{S_n}$$

  4. *$P_{S_1} P_{S_2} = P_{S_1 \cap S_2}$*

- **Definition** (***Projection-Valued Measure***)
  A family of ***projections*** obeying (1)-(3) is called a ***(bounded) projection-valued measure (p.v.m.)***.

- **Remark** For a family of projections $\{P_S : S \in \mathcal{B}(\mathbb{R})\}$, we have this mapping

$$P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}).$$

  P as a set function is finite i.e. $P(\mathbb{R}) = 1$ and $P(\emptyset) = 0$ and countably additive, therefor P is a ***vector-valued*** *Borel measure on spectral domain* $\mathcal{B}(\mathbb{R})$.

- **Remark** We can obtain a *spectral measure* $\mu_{\psi,S}$ from P_S via

$$\langle \psi\,,\, P_S\psi \rangle = \int_{\sigma(A)} \mathbb{1}_S d\mu_\psi = \mu_\psi(S \cap \sigma(A)) = \int_{\sigma(A)} d\mu_{\psi,S}$$

  for any $\psi \in \mathcal{H}$. We will use the ***symbol*** $d\langle \psi\,,\, P_S\psi \rangle$ to mean ***integration*** with respect to this measure $d\mu_{\psi,S} = \mathbb{1}_S d\mu_\psi$.

  By *standard Riesz representation theorem* methods, there is a ***unique*** operator with

$$\langle \psi\,,\, B\psi \rangle = \int f(\lambda)\, d\langle \psi\,,\, P_S\psi \rangle$$

- **Proposition 5.22** ***(Linear Operator Corresponding to Projection-Value Measure)*** *[Reed and Simon, 1980]*
  If P_S is a ***projection-valued measure*** *and f a* ***bounded*** *Borel function on supp(P_S), then there is a* ***unique*** *operator B such that*

$$\langle \psi\,,\, B\psi \rangle = \int f(\lambda)\, d\langle \psi\,,\, P_S\psi \rangle\,.$$

  We denote

$$B := \int f(\lambda) dP_S(\lambda).$$

$$\Rightarrow \left\langle \psi\,,\, \left(\int f(\lambda) dP_S(\lambda)\right) \psi \right\rangle = \int f(\lambda)\, d\langle \psi\,,\, P_S\psi \rangle$$

- **Theorem 5.23** ***(Spectral Theorem, Projection-Valued Measure Form)*** *[Reed and Simon, 1980]*
  There is a ***one-one correspondence*** *between* ***(bounded) self-adjoint operators*** *A and* ***(bounded) projection valued measures*** *$\{P_S\}$. In particular:*

  1. *Given A, each projection-valued measure P_S can be obtained as*

$$P_S := \mathbb{1}_S(A) = \widehat{\phi}(\mathbb{1}_S)$$

  2. *Given $\{P_S : S \subset \mathbb{R},\ \text{Borel set}\}$, the operator A can be obtained as*

$$A = \int_{\mathbb{R}} \lambda\, dP_\lambda \tag{15}$$

     and

$$f(A) = \int_{\mathbb{R}} f(\lambda)\, dP_\lambda \tag{16}$$

- **Remark** (***Understand Integration w.r.t. Projection-Valued Measure***)
As always, we can develop the integration with respect to projection-valued measure from simple function $f \in \mathcal{L}^2(\sigma(A), \mu_\psi)$:

$$f(\lambda) = \sum_{n=1}^{N} c_n \mathbb{1}_{S_n}(\lambda)$$

where $S_n := f^{-1}(\{c_n\})$, $\sigma(A) = \bigcup_{n=1}^{N} S_n$ and $S_n \cap S_m = \emptyset$. Using $\widehat{\phi} : \mathcal{L}^2(\sigma(A), \mu_\psi) \to \mathcal{L}(\mathcal{H})$, we can apply *functional calculus* on A to have

$$f(A) = \sum_{n=1}^{N} c_n \mathbb{1}_{S_n}(A) := \sum_{n=1}^{N} c_n P_{S_n} = \widehat{\phi}\left(\sum_{n=1}^{N} c_n \mathbb{1}_{S_n}\right).$$

Recall that when we define integration of simple function we have

$$\text{simp} \int f(\lambda) d\lambda = \sum_{n=1}^{N} c_n \mu_\psi(S_n) = \sum_{n=1}^{N} c_n \left\langle \psi\,,\, P_{S_n}\psi \right\rangle.$$

Equivalently, we can have integration of simple function with respect to the projection-valued measure $\{P_{S_n}\}$

$$\text{simp} \int f(\lambda) dP_\lambda = \sum_{n=1}^{N} c_n P(S_n) = \sum_{n=1}^{N} c_n P_{S_n} = f(A).$$

Then for unsigned function $f \geq 0$,

$$\underline{\int} f(\lambda) dP_\lambda = \sup_{g \text{ simple, } 0 \leq g \leq f} \text{simp} \int g(\lambda) dP_\lambda$$

and for any absolutely integrable function $f = f_+ - f_-$,

$$\int f(\lambda) dP_\lambda = \underline{\int} f_+(\lambda) dP_\lambda - \underline{\int} f_-(\lambda) dP_\lambda.$$

Finally we see that $P_{B(\lambda,\epsilon)} = 0$ if $\lambda \notin \sigma(A)$ so this integral is well-defined all over $\mathbb{R}$.

- **Remark** (***Bounded Real-Valued Measurable Function*** $\Leftrightarrow$ ***Bounded Self-Adjoint Operator***) [Halmos, 2017]
*The **essence** of spectral theorem* (in *functional calculus form* and in *spectral projection form*):

The ***analogs*** of ***bounded, real-valued, measurable*** *function* in Hilbert space thoery are ***bounded, self-adjoint linear operators***. Since a function is the *characteristic function of a set if and only if* it is ***idempotent***, it is clear on the algebraic gounds that the analogs of ***characteristic functions*** are ***projections***. The ***approximability*** of functions by ***simple functions*** corresponds in the analogy to the *approximability* of self-adjoint operators by ***real, finite linear combinations of projections***.

- **Remark** (***Comparison of Spectral Projection***)
  Consider the spectral theorem in projection form

$$A = \int_{\mathbb{R}} \lambda dP_{\lambda} \qquad \textbf{\textit{general}}\ \textit{self-adjoint}$$

$$A = \sum_{i=1}^{n} \lambda_i \varphi_i \varphi_i^T = \sum_{i=1}^{n} \lambda_i P_{\mathcal{H}_i} \qquad \textbf{\textit{finite dimensional}}$$

$$A = \sum_{i=1}^{\infty} \lambda_i P_{\mathcal{H}_i} \qquad \textbf{\textit{compact}}\ \textit{self-adjoint}$$

  where $\mathcal{H}_i = \mathrm{Ker}\,(\lambda_i I - A) = \mathrm{span}\,\{A^n \varphi_i : n = 0, 1, \ldots\}$ is ***the invariant subspace***, φ_i is ***cyclic vector*** *as the eigenvectors / eigenfunctions* corresponding to λ_i. For finite dimensional and compact operator case, $\mathcal{H}_i$ is *finite dimensional*.

  The decomposition of spectrum tells us that for general bounded self-adjoint operator

$$A = \int_{\mathbb{R}} \lambda dP_{\lambda} = \sum_{\{i:\lambda_i \in \sigma_{disc}(A)\}} \lambda_i P_{\mathcal{H}_i} + \int_{\sigma_{ess}(A)} \lambda dP_{\lambda} \tag{17}$$

  where $\mathcal{H}_i = \mathrm{Ker}\,(\lambda_i I - A)$ is ***the invariant subspace (eigenspace)*** and $\mathcal{H}_i$ is ***finite dimensional***.

5.4.6 Understanding Spectrum via Spectral Measures

- **Definition** (***Support of a Family of Measures***)
  If $\{\mu_n\}_{n=1}^N$ is a *family of measures*, <u>***the support of*** $\{\mu_n\}_{n=1}^N$</u> is the *complement* of *the largest open set* with $\mu_n(B) = 0$ for all n; so

$$supp(\{\mu_n\}_{n=1}^N) = \overline{\bigcup_{n=1}^{N} supp(\mu_n)}$$

- **Proposition 5.24** ***(Support of All Spectral Measures = the Spectrum)*** *[Reed and Simon, 1980]*
  Let A be a ***self-adjoint operator*** *and $\{\mu_n\}_{n=1}^N$ a family of* ***spectral measures****. Then*

$$\sigma(A) = \mathrm{supp}(\{\mu_n\}_{n=1}^N).$$

- **Remark** (***Multiple Ways to Decompose the Spectrum***)
  The recall the ***partition*** of spectrum by ***point spectrum***, ***continuous spectrum*** *and* ***residual spectrum***. We see that

  1.
$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

     This is related to the ***resolvent*** $R_{\lambda}(A) = (A - \lambda I)^{-1}$: its ***existence***, its ***range*** (***dense*** or not) and its ***boundedness***. These subsets are *disjoint*. Importantly, this decomposition is ***general*** and it applies to ***all linear operator***.

2.

$$\sigma(A) = \overline{\sigma_{pp}(A)} \cup \sigma_{ac}(A) \cup \sigma_{sing}(A).$$

This is related to the ***decompose*** of ***spectral measure*** μ_{ψ} *with respect to Lebesgue measure* and the ***pure point set***. These sets *may not be disjoint*. Both this and the one below are related to ***spectral measure*** of ***self-adjoint operator***.

3.

$$\sigma(A) = \sigma_{disc}(A) \cup \sigma_{ess}(A).$$

This is related to the ***dimensionality of image set*** of ***spectral projection*** $P_{B(\lambda,\epsilon)}$ on any open intervals around λ. It is related to the multiplicity of the kernel $\mathrm{Ker}\{A - \lambda I\}$. These sets *are disjoint*.

- **Definition** (***Pure Point of Measure***)
Given measure space (X, μ), a collection of ***closed one-point sets*** with *nonzero measure* is called ***the pure point set of measure*** μ. That is,

$$P := \{x \in X : \mu(\{x\}) > 0\}.$$

For $X = \mathbb{R}$ and μ is Borel measure, the pure point set is ***countable***.

- **Definition** (***Pure Point Measure and Continuous Measure***)
The pure point measure is defined as *the restriction of* μ *on the pure point set* P *of that measure*. For Borel measure μ on $\mathbb{R}$, and any ***Borel set*** $S \in \mathcal{B}(\mathbb{R})$,

$$\mu_{pp}(S) = \mu(S \cap P) = \sum_{x \in S \cap P} \mu(\{x\}).$$

A measure $\mu = \mu_{cont}$ is ***continuous*** if it has ***no pure point***, i.e. $\mu(\{x\}) = 0$ for any $\{x\} \in \mathcal{B}(\mathbb{R})$.

By definition, the following decomposition of measure μ holds:

$$\mu = \mu_{pp} + \mu_{cont}, \quad \mu_{pp} \perp \mu_{cont}$$

- **Remark** (***Decomposition of Borel Measure with respect to Lebesgue Measure***)
Recall from Lebesgue decomposition theorem, given λ as the Lebesgue measure on $\mathbb{R}$, any measure μ on $\mathbb{R}$ can be decomposed into two mutually singular parts:

$$\mu = \mu_{ac} + \mu_{sing}, \quad \mu_{ac} \perp \mu_{sing}$$

where $\mu_{ac} \ll \lambda$ and $\mu_{sing} \perp \lambda$. Combining with decomposition of pure point measure and continuous measure, we have the decomposition of any measure on $\mathbb{R}$ with respect to Lebesgue measure on $\mathbb{R}$,

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing} \tag{18}$$

where μ_{pp} is ***the pure point measure***, μ_{ac} is the part of ***continuous*** *measure* that is ***absolutely continuous*** *with respect to Lebesgue measure*, and μ_{sing} is the part of ***continuous*** *measure* that is ***singular*** *with respect to Lebesgue measure*.

- **Remark** (***Decomposition of Invariant Subspace***)
We apply above decomposition to spectral measure μ. Since these parts are mutually singular to each other, we have

$$L^2(\mathbb{R}, \mu) = L^2(\mathbb{R}, \mu_{pp}) \oplus L^2(\mathbb{R}, \mu_{ac}) \oplus L^2(\mathbb{R}, \mu_{sing}). \tag{19}$$

We can verify that any $\psi \in L^2(\mathbb{R}, \mu)$ has an ***absolutely continuous spectral measure*** μ_{ac} *with respect to Lebesgue measure* ***if and only if***

$$\psi \in L^2(\mathbb{R}, \mu_{ac}) \Leftrightarrow \int_{\mathbb{R}} |\psi|^2 \, d\mu_{ac} = \int_{\mathbb{R}} |\psi|^2 \, p \, d\lambda < \infty$$

where $p = d\mu_{ac}/d\lambda$ a.e.. Similarly for *pure point* and *singular measures.*

- **Definition** Let A be a ***bounded self-adjoint*** operator on $\mathcal{H}$. Let
  1. $\mathcal{H}_{pp} := \{\psi \in \mathcal{H} : \mu_\psi$ ***is a pure point measure***$\}$
  2. $\mathcal{H}_{ac} := \{\psi \in \mathcal{H} : \mu_\psi$ ***has no pure point and*** $\mu_\psi \ll \lambda$ ***Lebesgue measure***$\}$
  3. $\mathcal{H}_{sing} := \{\psi \in \mathcal{H} : \mu_\psi$ ***has no pure point and*** $\mu_\psi \perp \lambda$ ***Lebesgue measure***$\}$

- **Proposition 5.25** ***(Direct Sum Decomposition of Hilbert Space via Spectral Measure Decomposition)*** *[Reed and Simon, 1980]*
Let A be a ***bounded self-adjoint*** *operator on separable Hilbert space $\mathcal{H}$. For any $\psi \in \mathcal{H}$, μ_ψ is the spectral measure on $\sigma(A)$ corresponding to ψ. Then the following direct sum decompositon holds*

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$$

Moreover,
  1. *Each of these subspaces is* ***invariant*** *under A, i.e. for any ψ in these subspaces, $A\psi$ is in the same subspace.*
  2. *$A|_{\mathcal{H}_{pp}}$ has a* ***complete set of eigenvectors;***
  3. *$A|_{\mathcal{H}_{ac}}$ has* ***only absolutely continuous spectral measures***
  4. *$A|_{\mathcal{H}_{sing}}$ has* ***only continuous singular spectral measures****.*

- **Definition** (***Partition of Spectrum***)
We define the following *subsets of spectrum $\sigma(A)$*:
  1. ***Pure Point Spectrum***: $\sigma_{pp}(A) := \{\lambda \in \sigma(A) : \lambda$ ***is an eigenvalue of*** $A\}$
  2. ***Absolutely Continuous Spectrum***: $\sigma_{ac}(A) := \sigma\left(A|_{\mathcal{H}_{ac}}\right)$
  3. ***(Continuous) Singular Spectrum***: $\sigma_{sing}(A) := \sigma\left(A|_{\mathcal{H}_{sing}}\right)$

We can also defines ***the continuous spectrum*** as $\sigma_{cont}(A) := \sigma\left(A|_{\mathcal{H}_{ac} \oplus \mathcal{H}_{sing}}\right)$.

- **Remark** These *spectrums* are ***spectrum*** of *the linear operator A* ***restricted*** *in* ***each invariant subspace***. They are also the ***support*** of corresponding ***spectral measure***.

- **Remark** Unlike pure point spectrum, the singular spectrum $\sigma_{sing}(A)$ may contains spectral measure that is singular to Lebesgue measure but still without pure point.

- **Proposition 5.26** *[Reed and Simon, 1980]*

$$\begin{aligned}\sigma(A) &= \overline{\sigma_{pp}(A)} \cup \sigma_{ac}(A) \cup \sigma_{sing}(A) \\ &= \overline{\sigma_{pp}(A)} \cup \sigma_{cont}(A)\end{aligned}$$

- **Remark** The sets ***need not be disjoint***, however. The reader should be warned that $\sigma_{sing}(A)$ may have nonzero Lebesgue measure.

- **Proposition 5.27** ***(Criterion for Spectrum)*** *[Reed and Simon, 1980]*
  $\lambda \in \sigma(A)$ ***if and only if***

$$P_{B(\lambda,\epsilon)}(A) = P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \neq 0$$

  for any $\epsilon > 0$.

- **Definition** (***Essential Spectrum and Discrete Spectrum***)

  1. We say $\lambda \in \sigma_{ess}(A)$, ***the essential spectrum of*** A, *if and only if*

  $$P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \textit{ is infinite dimensional}$$

     for all $\epsilon > 0$. P is *infinite dimensional* means $\overline{\mathrm{Ran}(P)}$ is *infinite dimensional.*

  2. If $\lambda \in \sigma(A)$, but

  $$P_{(\lambda-\epsilon,\lambda+\epsilon)}(A) \textit{ is finite dimensional}$$

     for some $\epsilon > 0$, we say $\lambda \in \sigma_{disc}(A)$, ***the discrete spectrum of*** .

- **Proposition 5.28** *[Reed and Simon, 1980]*
  $\sigma_{ess}(A)$ *is always* ***closed.***

- **Proposition 5.29** *[Reed and Simon, 1980]*
  $\lambda \in \sigma_{disc}(A)$ ***if and only if*** **both** *the following hold:*

  1. *λ is an* ***isolated*** *point of $\sigma(A)$, that is, for some ϵ, $(\lambda-\epsilon,\lambda+\epsilon) \cap \sigma(A) = \{\lambda\}$.*

  2. *λ is an* ***eigenvalue*** *of* ***finite multiplicity****, i.e.,*

  $$dim\{\varphi : A\varphi = \lambda\varphi\} = dim\ Ker\{A - \lambda I\} < \infty.$$

- **Proposition 5.30** $\lambda \in \sigma_{ess}(A)$ ***if and only if*** **at least one** *of the following holds:*

  1. *$\lambda \in \sigma_{cont}(A) = \sigma_{ac}(A) \cup \sigma_{sing}(A)$.*

  2. *λ is a* ***limit point*** *of $\sigma_{pp}(A)$.*

  3. *λ is an* ***eigenvalue*** *of* ***infinite multiplicity****.*

- **Theorem 5.31** ***(Weyl's Criterion)*** *[Reed and Simon, 1980]*
  Let A be a ***bounded self-adjoint*** *operator. Then $\lambda \in \sigma(A)$* ***if and only if*** *there exists $\{\psi_n\}_{n=1}^{\infty}$ so that $\|\psi_n\| = 1$ and*

$$\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0.$$

  $\lambda \in \sigma_{ess}(A)$ ***if and only if*** *the above $\{\psi_n\}_{n=1}^{\infty}$ can be chosen to be* ***orthogonal****.*

- **Remark** *The essential spectrum* ***cannot be removed*** by ***essentially finite dimensional perturbations***.

  A general implies that $\sigma_{ess}(A) = \sigma_{ess}(B)$ if $A - B$ is ***compact***.

- **Remark** Finally, we discuss one useful formula relating the resolvent and spectral projections. It is a matter of computation to see that the box on $[a, b]$

$$f_\epsilon(x) = \begin{cases} 0 & x \notin [a, b] \\ \frac{1}{2} & x = a \text{ or } x = b \\ 1 & x \in (0, 1) \end{cases}$$

We can find

$$f_\epsilon(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \left(\frac{1}{x - \lambda - i\epsilon} - \frac{1}{x - \lambda + i\epsilon} \right) d\lambda$$

Moreover, $|f_\epsilon(x)|$ is ***bounded uniformly*** in ϵ. Applying the functional calculus on A, we have

Theorem 5.32 ***(Stone's formula)*** *[Reed and Simon, 1980]*
Let A be a ***bounded self-adjoint*** *operator. Then*

$$\begin{aligned} \frac{1}{2} \left(P_{[a,b]} + P_{(a,b)} \right) &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \left[(A - \lambda - i\epsilon)^{-1} - (A - \lambda + i\epsilon)^{-1} \right] d\lambda \qquad (20) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \left[R_{\lambda + i\epsilon}(A) - R_{\lambda - i\epsilon}(A) \right] d\lambda \end{aligned}$$

for $R_\lambda(A) = (A - \lambda)^{-1}$, the ***resolvent*** *of A.*

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