Lecture 11: The Cotangent Bundle

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1 Covectors

1.1 Covectors on Vector Space

- *Tangent covectors* are *linear functionals* on the tangent space at a point $p \in M$. The space of all covectors at p is a vector space called the *cotangent space* at p; in linear-algebraic terms, it is the *dual space* to tangent space T_pM. [Lee, 2003.]

- Whereas *tangent vectors* give us a *coordinate-free* interpretation of derivatives of curves, it turns out that derivatives of *real-valued functions* on a manifold are most naturally interpreted as *tangent covectors*.

- **Definition** Let V be a *finite-dimensional real vector space*. We define a ***covector*** on V to be a ***real-valued linear functional*** on V, that is, a *linear map* $\omega : V \to \mathbb{R}$.

  The space of all covectors on V is itself a *real vector space* under the obvious operations of *pointwise addition* and *scalar multiplication*. It is denoted by V^* and called the ***dual space*** of V.

- Note that V can be a space of vector space of functionals itself. And a functional of functionals is a functional, since a *functional* is a function of functions.

- **Proposition 1.1** *Let V be a finite-dimensional vector space. Given any basis $(E_1, \ldots, E_n)$ for V, let $\epsilon^1, \ldots, \epsilon^n \in V^*$ be the covectors defined by*

  $$\epsilon^i(E_j) = \delta^i_j$$

  *where δ^i_j is the Kronecker delta symbol. Then $\epsilon^1, \ldots, \epsilon^n$ is a **basis** for V^*, called the **dual basis** to (E_j). Therefore, $dim V^* = dim V$.*

- **Example** For example, we can apply this to ***the standard basis*** $(e_1, \ldots, e_n)$ for $\mathbb{R}^n$. The *dual basis* is denoted by $(e^1, \ldots, e^n)$ (note the *upper indices*), and is called ***the standard dual basis***. These basis *covectors* are the *linear functionals* on $\mathbb{R}^n$ given by

  $$e^i(v) = e^i(v^1, \ldots, v^n) = v^i. \tag{1}$$

  In other words, e^i is the linear functional that *picks out the i-th component of a vector*.

  In **matrix notation**, a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by a $1 \times n$ matrix, called a ***row matrix***. The ***basis covectors*** can therefore also be thought of as the linear functionals represented by the *row matrices*

  $$e^i = (0, \ldots, 1, \ldots, 0), \quad i = 1, \ldots, n \tag{2}$$

  where i-th element is 1 and the others are all zeros.

- In general, if (E_j) is a basis for V and (ϵ^i) is its dual basis, then for any vector $v = v^j E_j \in V$, we have (using the summation convention)

  $$\epsilon^i(v) = \epsilon^i(v^j E_j) = v^j \epsilon^i(E_j) = v^j \delta^i_j = v^i$$

  Thus,just as in the case of $\mathbb{R}^n$, the i-th basis covector ϵ^i picks out the i-th component of a vector with respect to the basis (E_j).

- More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the *dual basis* as

$$\omega = \omega_i \epsilon^i \tag{3}$$

where the components are determined by $\omega_i = \omega(E_i)$.

The ***action*** of ω on a vector $v = v^j E_j$ is

$$\omega(v) = \omega_i \epsilon^i(v) = \omega_i \, v^i \tag{4}$$

- Note that we always write ***basis covectors*** *with* ***upper indices***, and ***components*** of a covector *with* ***lower indices***, because this helps to ensure that mathematically meaningful summations such as (3) and (4) always follow our index conventions.

- **Definition** Suppose V and W are vector spaces and $A : V \to W$ is a *linear map*. We define a linear map $A^* : W^* \to V^*$, called ***the dual map*** or ***transpose of*** A, by

$$(A^* \, \omega)(v) = \omega \, (A \, v) \, , \quad \forall \, \omega \in W^*, \; v \in V. \tag{5}$$

- **Proposition 1.2** *The* ***dual map*** *satisfies the following properties:*

  1. $(A \circ B)^* = B^* \circ A^*$.
  2. $(Id_V)^* : V^* \to V^*$ *is the identity map of* V^*.

- **Corollary 1.3** *The assignment that sends a vector space to its dual space and a linear map to its dual map is a* ***contravariant functor*** *from the category of real vector spaces to itself.*

- **Definition** Apart from the fact that the dimension of V^* is the same as that of V, the second most important fact about dual spaces is the following characterization of the ***second dual space*** $V^{**} = (V^*)^*$.

For each vector space V there is a natural, ***basis-independent map*** $\xi : V \to V^{**}$, defined as follows. For each vector $v \in V$, define a ***linear functional*** $\xi(v) : V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v), \quad \forall \omega \in V^*. \tag{6}$$

- **Proposition 1.4** *For any finite-dimensional vector space* V*, the map* $\xi : V \to V^{**}$ *is an* ***isomorphism****.*

- **Remark** Some of important things to note:

  – The preceding proposition shows that when V is finite-dimensional, we can unambiguously ***identify*** V^{**} with V itself, because the map ξ is *canonically defined*, without reference to any basis.

  – It is important to observe that although V^* is *also* ***isomorphic*** to V (for the simple reason that any two finite-dimensional vector spaces of the same dimension are isomorphic), there is ***no canonical isomorphism*** $V \simeq V^*$.

  – Because of Proposition above, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more ***symmetric-looking notations*** $\langle \omega \, , \, v \rangle$ and $\langle v \, , \, \omega \rangle$, both expressions can be thought of either as ***the action of the***

covector $\omega \in V^*$ ***on the vector*** $v \in V$, or as ***the action of the linear functional*** $\xi(v) \in V^{**}$ ***on the element*** $\omega \in V^*$.

There should be no cause for confusion with the use of the same angle bracket notation for inner products: *whenever one of the arguments is a* ***vector*** *and the other a* ***covector***, the notation $\langle \omega\,,\, v\rangle$ is always to be interpreted as the ***natural pairing*** between vectors and covectors, *not as an inner product*. We typically omit any mention of the map ξ, and think of $v \in V$ *either as a* ***vector*** or as *a* ***linear functional*** on V^*, depending on the context.

– There is also a ***symmetry*** between ***bases*** and ***dual bases*** for a finite-dimensional vector space V: any *basis* for V determines a *dual basis* for V^*, and ***conversely***, any *basis* for V^* determines a *dual basis* for $V^{**} = V$.

If (ϵ^i) is the basis for V^* *dual* to a basis (E_j) for V, then (E_j) is the basis *dual* to (ϵ^i), because both statements are equivalent to the relation $\left\langle \epsilon^i\,,\, E_j \right\rangle = \delta^i_j$.

- Just like $\mathbb{R}^n$, any element in a finite-dimensional vector space V can either be
  - a ***vector***, i.e. a single point in the vector space V;
  - a ***linear functional***, which act on functions that defined on space V.

1.2 Tangent Covectors on Manifolds

- **Definition** Let M be a smooth manifold with or without boundary. For each $p \in M$, we define the ***cotangent space*** at p, denoted by T_p^*M, to be the ***dual space*** to the *tangent space* T_pM:

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called ***tangent covectors at*** p, or just ***covectors at*** p.

- **Remark** (***Coordinate Representation of Covectors***) [Lee, 2003.]
Given smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ the coordinate basis $(\frac{\partial}{\partial x^i}\big|_p)$ gives rise to a dual basis for T_p^*M, which we denote for the moment by $(\lambda^i\big|_p)$. (In a short while, we will come up with a better notation.)

Any covector $\omega \in T_p^*M$ can thus be written ***uniquely*** as $\omega = \omega_i\, \lambda^i\big|_p$ where

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right). \tag{7}$$

- **Remark** (***Change of Coordinates for Covectors***) [Lee, 2003.]
Suppose now that $(\widetilde{x}^i)$ is *another set of smooth coordinates* whose domain contains p, and let $(\widetilde{\lambda}^j\big|_p)$ denote the basis for T_p^*M dual to $(\frac{\partial}{\partial \widetilde{x}^j}\big|_p)$. We can compute the *components* of the same covector ω with respect to the *new coordinate system* as follows.

First observe that the computations in Chapter 3 show that the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p. \tag{8}$$

(Here we use the same notation p to denote either a point in M or its coordinate representation as appropriate.)

Writing ω in both systems as $\omega = \omega_i \, \lambda^i\big|_p = \widetilde{\omega}_j \, \widetilde{\lambda}^j\big|_p$, we can use (8) to compute the components ω_i in terms of $\widetilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j.$$

In sum, we have ***the change of coordinate formula for covectors***

$$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j. \tag{9}$$

- **Remark** (***The Origin of The Name "Covector"***) [Lee, 2003.]
  In the early days of smooth manifold theory, before most of the abstract coordinate-free definitions we are using were developed, mathematicians tended to think of a ***tangent vector at a point*** p as an ***assignment*** of an n-tuple of real numbers to *each smooth coordinate system*, with the property that the n-tuples $(v^1, \ldots, v^n)$ and $(\widetilde{v}^1, \ldots, \widetilde{v}^n)$ assigned to two different coordinate systems (x^i) and $(\widetilde{x}^j)$ were related by the transformation law that we derived in Chapter 3:

  $$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,v^i. \tag{10}$$

  (See that the change of $v^i \to \widetilde{v}^j$ using partial derivatives $\widetilde{x}^j \to x^i$.)

  Similarly, a ***tangent covector*** was thought of as an n-tuple $(\omega_1, \ldots, \omega_n)$ that transforms, by virtue of (9), according to the following slightly different rule:

  $$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j \tag{11}$$

  (See that the change of $\widetilde{\omega}_j \to \omega_i$ using partial derivatives $\widetilde{x}^j \to x^i$.)

  Since the transformation law (8) for the ***coordinate partial derivatives*** follows directly from the chain rule, it can be thought of as *fundamental*. Thus it became customary to call *tangent covectors* ***<u>covariant vectors</u>*** because ***their components transform in the same way as ("vary with") the coordinate partial derivatives***, with the **Jacobian matrix** $\frac{\partial \widetilde{x}^j}{\partial x^i}(p)$ multiplying the objects associated with the "*new*" coordinates $(\widetilde{x}^j)$ to obtain those associated with the "*old*" coordinates (x^i).

  Analogously, *tangent vectors* were called ***<u>contravariant vectors</u>***, because ***their components transform in the opposite way***. (Remember, it was the component n-tuples that were thought of as the objects of interest.) Admittedly, these terms do not make a lot of sense, but by now they are well entrenched.

1.3 Covector Fields

- **Definition** For any smooth manifold M with or without boundary, *the disjoint union*

  $$T^*M = \coprod_{p \in M} T_p^*M$$

is called the ***cotangent bundle of*** M. It has a ***natural projection map*** $\pi : T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$.

- **Definition** Given any smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ we denote the ***basis*** for T_p^*M dual to $(\frac{\partial}{\partial x^i}\big|_p)$ by $(\lambda^i\big|_p)$. This defines n maps $\lambda^1, \ldots, \lambda^n : U \to T^*M$, called ***coordinate covector fields***.

- **Proposition 1.5** *(The Cotangent Bundle as a Vector Bundle).*
Let M be a smooth n-manifold with or without boundary. With its standard projection map and the natural vector space structure on each fiber, the ***cotangent bundle*** *T^*M has a* ***unique topology*** *and* ***smooth structure*** *making it into a* ***smooth rank-n vector bundle*** *over M for which all coordinate covector fields are* ***smooth local sections****.*

  Proof: Given a smooth chart (U, φ) on M; with coordinate functions (x^i), define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(\xi_i \lambda^i\big|_p\right) = (x^1(p), \ldots, x^n(p), \xi_1, \ldots, \xi_n)$$

  where λ^i is the i-th coordinate covector field associated with (x^i). Suppose $(\widetilde{U}, \widetilde{\varphi})$ is another smooth chart with coordinate functions $(\widetilde{x}^j)$, and let $\widetilde{\Phi} : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \widetilde{U})$, it follows from (9) that

$$\Phi \circ \widetilde{\Phi}^{-1}(p, (\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n)) = \left(p, \left(\frac{\partial \widetilde{x}^j}{\partial x^1}(p)\widetilde{\xi}_j, \ldots, \frac{\partial \widetilde{x}^j}{\partial x^n}(p)\widetilde{\xi}_j\right)\right)$$

  The $GL(n, \mathbb{R})$-valued function $(\partial \widetilde{x}^j / \partial x^i)$ is smooth, so it follows from the vector bundle chart lemma that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are smooth local trivializations. Uniqueness follows as in the proof of Proposition 10.24. ∎

- **Definition** As in the case of the *tangent bundle*, smooth local coordinates for M yield smooth local coordinates for its *cotangent bundle*. If (x^i) are *smooth coordinates* on an open subset $U \subseteq M$, the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i \lambda^i\big|_p \mapsto (x^1(p), \ldots, x^n(p), \xi_1, \ldots, \xi_n)$$

  is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) the ***natural coordinates*** for $T * M$ associated with (x^i).

- **Definition** A ***(local or global) section*** of T^*M is called a ***covector field*** or a ***(differential) 1-form***.

  Like sections of other bundles, covector fields without further qualification are assumed to be merely ***continuous***; when we make different assumptions, we use the terms ***rough covector field*** and ***smooth covector field*** with the obvious meanings.

- As we did with vector fields, we write *the* ***value*** *of a covector field* ω **at a point** $p \in M$ as ω_p instead of $\omega(p)$, to avoid conflict with the notation for *the action of a covector on a vector.* If ω itself has subscripts or superscripts, we usually use the notation $\omega|_p$ instead.

- **Remark** (***Representation of Covector Field via Coordinate Fields***)
In any smooth local coordinates on an open subset $U \subseteq M$; *a (rough) covector field* ω can be

written in terms of ***the coordinate covector fields*** (λ^i) as $\omega_i\,\lambda^i$ for n *functions* $\omega_i : U \to \mathbb{R}$ called the ***component functions*** of ω. They are characterized by

$$\omega_i = \omega_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

- If ω is a ***(rough) covector field*** and X is a ***vector field*** on M, then we can form a function $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega_p(X_p), \quad p \in M.$$

  If we write $\omega = \omega_i\,\lambda^i$ and $X = X^j \frac{\partial}{\partial x_j}$ in terms of *local coordinates*, then $\omega(X)$ has the ***local coordinate representation*** $\omega(X) = \omega_i\,X^i$.

- **Proposition 1.6** ***(Smoothness Criteria for Covector Fields)*** *[Lee, 2003.]*
  Let M be a smooth manifold with or without boundary, and let $\omega : M \to T^*M$ be a ***rough covector field****. The following are* ***equivalent****:*

  1. *ω is smooth.*
  2. *In* ***every smooth*** *coordinate chart, the* ***component functions*** *of ω are smooth.*
  3. *Each point of M is contained in* ***some coordinate chart*** *in which ω has smooth component functions.*
  4. *For* ***every*** *smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.*
  5. *For* ***every open subset*** *$U \subseteq M$ and* ***every smooth vector field*** *X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth on U.*

1.4 Coframes

- **Definition** Let M be a smooth manifold with or without boundary, and let $U \subseteq M$ be an open subset. ***A local coframe*** for M over U is an ordered n-tuple of covector fields $(\epsilon^1, \ldots, \epsilon^n)$ defined on U such that $(\epsilon^i|_p)$ forms a basis for T_p^*M at each point $p \in U$. If $U = M$, it is called ***a global coframe***. (***A local coframe*** for M is just a local frame for the vector bundle T^*M

- **Example** (***Coordinate Coframes***).
  For any smooth chart $(U, (x^i))$, the ***coordinate covector fields*** (λ^i) defined above constitute a local coframe over U, called ***a coordinate coframe***. Every coordinate frame is ***smooth***, because its ***component functions*** in the given chart are ***constants***.

- **Definition** Given a local frame $E_1, \ldots, E_n)$ for TM over an open subset U, there is a ***uniquely determined (rough) local coframe*** $(\epsilon^1, \ldots, \epsilon^n)$ over U such that $\epsilon_i|_p$ is *the* ***dual basis*** *to* $E_i|_p$ for each $p \in U$, or equivalently $\epsilon^i(E_j) = \delta^i_j$. This coframe is called *the* ***coframe dual to*** (E_i). *Conversely*, if we start with a local coframe (ϵ^i) over an open subset $U \subseteq M$, there is a uniquely determined local frame (E_i), called *the* ***frame dual to*** (ϵ^i), determined by $\epsilon^i(E_j) = \delta^i_j$.

- **Remark** The coframe dual to $(\partial/\partial x^i)$ is (dx^i) and the frame dual to (dx^i) is $(\partial/\partial x^i)$.

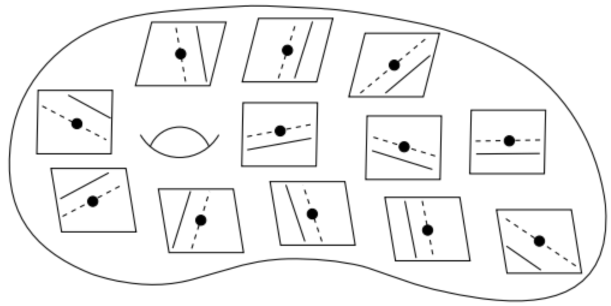

Figure 1: A covector field

- **Lemma 1.7** *Let M be a smooth manifold with or without boundary. If (E_i) is a rough local frame over an open subset $U \subseteq M$ and (ϵ^i) is its **dual coframe**, then (E_i) is smooth if and only if (ϵ^i) is smooth.*

- **Remark** Given a local coframe (ϵ^i) over an open subset $U \subseteq M$, every (rough) covector field ω on U can be expressed in terms of the coframe as $\omega = \omega_i \epsilon^i$ for some functions $\omega_1, \ldots, \omega_n : U \to \mathbb{R}$, called ***the component functions*** of ω with respect to the given *coframe*. The component functions are determined by $\omega_i = \omega(E_i)$, where (E_i) is the *frame dual to* (ϵ^i).

- **Proposition 1.8** *(**Coframe Criterion for Smoothness of Covector Fields**).*
Let M be a smooth manifold with or without boundary, and let ω be a rough covector field on M. If (ϵ^i) is a smooth coframe on an open subset $U \subseteq M$, then ω is smooth on U if and only if its component functions with respect to (ϵ^i) are smooth.

- **Remark** We denote *the real vector space of **all smooth covector fields** on M by $\mathfrak{X}^*(M)$* (or $\Gamma(T^*M)$). As smooth sections of a vector bundle, elements of $\mathfrak{X}^*(M)$ can be ***multiplied*** by smooth real-valued functions: if $f \in \mathcal{C}^\infty(M)$ and $\omega \in \mathfrak{X}^*(M)$, the covector field $f\omega$ is defined by

$$(f\omega)_p = f(p)\,\omega_p. \tag{12}$$

  Because it is the space of smooth sections of a vector bundle, $\mathfrak{X}^*(M)$ is a *module* over $\mathcal{C}^\infty(M)$.

- **Remark** Note that a nonzero linear functional $\omega_p \in T_p^*M$ is completely determined by two pieces of data: its ***kernel***, which is a linear hyperplane in T_pM (*a codimension*-1 *linear subspace*); and the set of vectors v for which $\omega_p(v) = 1$, which is *an **affine hyperplane parallel to the kernel*** (Fig. 1) The value of $\omega_p(v)$ for any other vector v is then obtained by linear interpolation or extrapolation.

- **Remark** (***Visualize the Vector Fields and the Covector Fields***)

  1. A vector field on M can be considered as an arrow attached to each point of M.

  2. A covector field on M can be considered as defining ***a pair of hyperplanes*** in each tangent space, *one* ***through the origin*** and *another* ***parallel*** *to it*, and varying continuously from point to point.

     Where the covector field is small, one of the hyperplanes becomes *very far from the kernel*, eventually disappearing altogether at points where the covector field takes the value zero.

2 The Differential of a Function

- **Remark** Although ***the partial derivatives of a smooth function*** cannot be interpreted in a *coordinate-independent way* as the *components* of *a vector field*, it turns out that they *can* be interpreted as the ***components of a covector field***. This is *the most important application* of covector fields.

- **Definition** Let f be a *smooth real-valued function* on a *smooth manifold* M with or without boundary. (As usual, all of this discussion applies to functions defined on an open subset $U \subseteq M$; simply by *replacing* M with U throughout.) We define a ***covector field*** df , called ***the differential of*** f, by

$$df_p(v) = v\, f, \quad \forall v \in T_pM.$$

- **Proposition 2.1** *The differential of a smooth function is a smooth covector field.*

- **Remark** Similar to comparison between *tangent vector* $v \in T_pM$ and *tangent vector field* X, the *differential of* f is a ***covector field***, i.e. a smooth function that maps a point p to covector df_p, the differential of f *at* p. df can be seen as a global concept that summarizes information of differential maps across the manifold.

- **Remark** (***Coordinate Representation of differential of*** f)
  Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (λ^i) be the corresponding *coordinate coframe* on U. Write df in coordinates as $df_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$, then the definition of df implies

$$A_i(p) = df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p).$$

  This yields the following formula for ***the coordinate representation of*** df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\ \lambda^i|_p \tag{13}$$

  Thus, the ***component functions*** of df in any smooth coordinate chart are ***the partial derivatives of*** f *with respect to those coordinates.* Because of this, we can think of df as *an analogue of the classical gradient*, reinterpreted in a way that makes *coordinate-independent sense* on a manifold.

  If we apply (13) to the special case in which f is one of the *coordinate functions* $x^j : U \to \mathbb{R}$, we obtain

$$dx^j|_p = \frac{\partial x^j}{\partial x^i}(p)\ \lambda^i|_p = \delta^j_i\ \lambda^i|_p = \lambda^j|_p.$$

  In other words, ***the coordinate covector field*** λ^j *is none other than* ***the differential*** dx^j. Therefore, the formula (13) for df_p can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)\ dx^i|_p. \tag{14}$$

or as ***an equation between covector fields*** instead of covectors. The ***coordinate representation of differential*** df is

$$df = \frac{\partial f}{\partial x^i}\, dx^i. \tag{15}$$

Thus, we have recovered the familiar classical expression for the differential of a function f in coordinates. Henceforth, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

The ***coordinate representation of covector field*** ω is

$$\omega = \omega_i\, dx^i \tag{16}$$

$$\text{where } dx^i\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j, \quad \forall i, j = 1, \ldots, n, p \in M$$

- **Remark** The equation (16) should be considered as *the* ***equation*** *between* ***covector field*** df and *the corresponding* ***coordinate covector fields*** $dx^i, i = 1, \ldots, n$. This is derived without using the total differential equation from *the multivariate calculus.* The linear cooefficients for the combination is *the partial derivatives of* f.

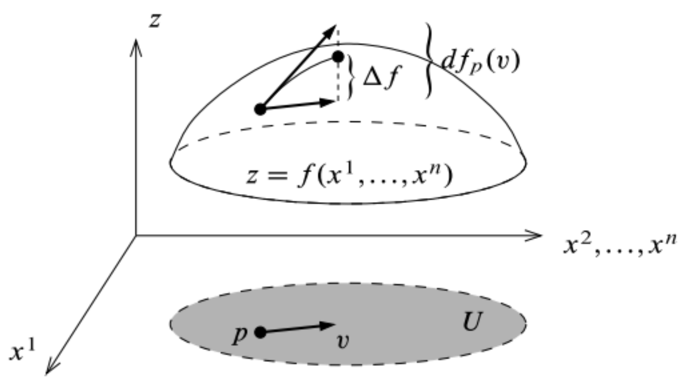

Figure 2: The differential df can be seen as rate of changes of f along curve γ

- **Proposition 2.2** ***(Properties of the Differential).***
  Let M be a smooth manifold with or without boundary, and let $f, g \in \mathcal{C}^\infty(M)$.

  1. *If a and b are constants, then $d(a\,f + b\,g) = a\,df + b\,dg$.*
  2. *$d(f\,g) = f\,dg + g\,df$.*
  3. *$d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.*
  4. *If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\, df$.*
  5. *If f is constant, then $df = 0$.*

- ***One very important property*** of the differential is the following characterization of smooth functions with vanishing differentials.

  Proposition 2.3 ***(Functions with Vanishing Differentials).*** *[Lee, 2003.]*
  If f is a smooth real-valued function on a smooth manifold M with or without boundary, then $df = 0$ if and only if f is ***constant*** *on* ***each component*** *of M .*

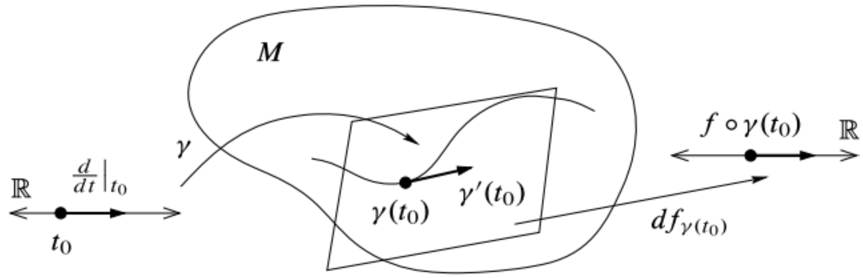

Figure 3: The derivative of f along curve γ

- **Remark** Suppose M is a smooth manifold and $f \in \mathcal{C}^\infty(M)$, and let p be a point in M. By choosing smooth coordinates on a neighborhood of p, we can think of f as a function on an open subset $U \subseteq \mathbb{R}^n$. Recall that $dx^i|_p$ is the *linear functional that picks out the i-th component of a tangent vector* at p. Writing $\Delta f = f(p+v) - f(p)$ for $v \in \mathbb{R}^n$, Taylors theorem shows that f is well approximated when v is small by

$$\Delta f = f(p+v) - f(p) \approx \frac{\partial f}{\partial x^i}(p) v^i = \frac{\partial f}{\partial x^i}(p) dx^i(v) = df_p(v).$$

In other words, df_p ***is the linear functional that best approximates*** f ***near*** p***.*** (See Fig 2).

The great power of the concept of the differential comes from the fact that we can define df ***invariantly*** on any manifold, without resorting to vague arguments involving *infinitesimals*.

- **Proposition 2.4** ***(Derivative of a Function Along a Curve)****.*
Suppose M is a smooth manifold with or without boundary, $\gamma : J \to M$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the ***derivative*** *of the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)). \tag{17}$$

Proof: For any $t_0 \in J$

$$\begin{aligned} df_{\gamma(t_0)}(\gamma'(t_0)) &= \gamma'(t_0) f \quad \text{(by definition of } df\text{)} \\ &= d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) f \quad \text{(by definition of } \gamma'\text{)} \\ &= \frac{d}{dt}\bigg|_{t_0} (f \circ \gamma) \quad \text{(by definition of } d\gamma\text{)} \\ &= (f \circ \gamma)'(t_0). \quad \blacksquare \end{aligned}$$

- **Remark** You may have noticed that for a smooth real-valued function $f : M \to \mathbb{R}$, we now have ***two different definitions*** for *the differential of* f at a point $p \in M$.

  1. we defined df_p as ***a linear map*** from T_pM to $T_{f(p)}\mathbb{R}$, i.e. ***a linear operator*** that maps a tangent vector in T_pM to another tangent vector in $T_{f(p)}\mathbb{R}$.

  2. we defined df_p as ***a covector at*** p, which is to say a linear map from T_pM to $\mathbb{R}$, i.e. ***a linear functional*** on T_pM.

- **Remark** Similarly, if γ is a smooth curve in M, we have ***two different meanings*** for the expression $(f \circ \gamma)'(t)$:

  1. $(f \circ \gamma)$ can be interpreted as a smooth curve in $\mathbb{R}$, and thus $(f \circ \gamma)'(t)$ is its ***velocity*** at the point $f \circ \gamma(t)$, which is an element of ***the tangent space*** $T_{f\circ\gamma(t)}\mathbb{R}$. By (17), we see that it equal to $df_{\gamma(t)}(\gamma'(t))$, as ***a tangent vector***.
  2. $(f \circ \gamma)$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ***ordinary derivative***. By (17), we see that it equal to $df_{\gamma(t)}(\gamma'(t))$, ***as a real number***.

3 Pullbacks of Covector Fields

3.1 Definitions

- Recall that for diffeomorphism $F : M \to N$, the ***pushforward of a vector field*** X ***by*** F, denoted as $F_* X$ or $F_\# X$ is *the unique vector field* obtained by differential of F acting on X.

$$(F_* X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right), \quad q \in N.$$

  Dualizing this process leads to a linear map on covectors going *in the opposite direction*.

- **Definition** Let $F : M \to N$ be a *smooth map* between smooth manifolds with or without boundary, and let $p \in M$ be arbitrary. The differential $dF_p : T_pM \to T_{F(p)}N$ yields *a* ***dual linear map***

$$dF_p^* : T^*_{F(p)}N \to T^*_pM,$$

  called ***the (pointwise) pullback by*** F ***at*** p, or ***the cotangent map* of** F. Unraveling the definitions, we see that dF_p^* is characterized by

$$dF_p^*(\omega)(v) = \omega(dF_p(v)), \quad \omega \in T^*_{F(p)}N,\ v \in T^*_pM.$$

- **Definition** Given a smooth map $F : M \to N$ and a *covector field* ω on N , define a ***rough covector field*** $F^*\omega$ on M, called the ***pullback of*** ω ***by*** F, by

$$(F^*\omega)_p = dF_p^*\left(\omega_{F(p)}\right)$$

  We also denote *the pullback of* ω *by* F *as* $F^\#\omega$.

- **Remark** Observe that the assignments $(M, p) \mapsto T^*_pM$ and $F \mapsto dF_p^*$ yield a *contravariant functor* from the category of pointed smooth manifolds to the category of real vector spaces.

- **Remark** Although the *pushforward of a vector field by* F, $F_\# X$, is only uniquely defined for *diffeomorphism* F, the *pullback of a covector field by* F, $F^\#\omega$, is ***always unique***.

  It acts on a vector $v \in T_pM$ by

$$(F^\#\omega)_p(v) = dF_p^\#(\omega_{F(p)})(v) = \omega_{F(p)}(dF_p(v))$$

- **Proposition 3.1** *Let $F : M \to N$ be a smooth map between smooth manifolds with or without boundary. Suppose u is a* continuous real-valued *function on N, and ω is a covector field on N. Then*

$$F^*(u\,\omega) = (u \circ F) F^* \omega \tag{18}$$

If in addition u is smooth, then

$$F^* du = d(u \circ F) \tag{19}$$

Proof: To proof (18), we compute:

$$\begin{aligned}
(F^*(u\,\omega))_p &= dF_p^*((u\,\omega)_{F(p)}) \quad \text{(by definition of pullback of covector field)} \\
&= dF_p^*(u(F(p))\,\omega_{F(p)}) \quad \text{(by smooth function times vector field)} \\
&= u(F(p))\,dF_p^*(\omega_{F(p)}) \\
&= (u \circ F)(p)\,(F^*\omega)_p = ((u \circ F)F^*\omega)_p
\end{aligned}$$

To proof (19), we compute for every $p \in M, v \in T_pM$

$$\begin{aligned}
(F^* du)_p(v) &= dF_p^*(du_{F(p)})(v) \quad \text{(by definition of pullback of covector field)} \\
&= du_{F(p)}\,(dF_p(v)) \quad \text{(by definition of pullback of covector at } p\text{)} \\
&= dF_p(v)u \quad \text{(by definition of } du\text{)} \\
&= v(u \circ F) \quad \text{(by definition of } dF_p\text{)} \\
&= d(u \circ F)_p(v) \quad \text{(by definition of } d(u \circ F)\text{)}
\end{aligned}$$

∎

- **Proposition 3.2** *Suppose $F : M \to N$ is a smooth map between smooth manifolds with or without boundary, and let ω be a covector field on N. Then $F^*\omega$ is a **(continuous) covector field** on M. If ω is smooth, then so is $F^*\omega$.*

- **Remark** (***Coordinate Representation of Pullback Covector Fields***)
  Given the coordinate representation of covector $\omega = \omega_j dy^j$, the pullback of a covector field can also be written in the following way:

$$\begin{aligned}
F^*\omega &= F^*(\omega_j dy^j) \\
&= (\omega_j \circ F) F^*(dy^j) \\
&= (\omega_j \circ F)\, d\left(y^j \circ F\right) \qquad (20) \\
&= (\omega_j \circ F)\, d\,F^j \qquad (21)
\end{aligned}$$

  where F^j is the j th component function of F in these coordinates. Using either of these formulas, the computation of pullbacks in coordinates is exceedingly simple.

  In other words, to compute $F^*\omega$, all you need to do is ***substitute*** *the* ***component functions*** *of F for the* ***coordinate functions*** *of N everywhere they appear in ω.*

- **Remark** Get familiar with the following expressions:

  1. For $g \in \mathcal{C}^\infty(N)$, $q = F(p) \in N$ so that $p = F^{-1}(q) \in M$,

$$(F_* X)_q\, g = dF_p(X_p) g = X_p\,(g \circ F)$$

2. For $p \in M$, $X_p \in T_pM$, $q = F(p) \in N$, $\omega_q \in T_q^*N$,

$$(F^*\omega)_p(X_p) = \left(dF_p^*\omega_q\right)(X_p) = \omega_q\left(dF_p(X_p)\right)$$

   The last equality use the definition of dual map $(A^*w)(v) = w(A\,v)$

3. For a diffeomorphism F, $(F^*)^{-1} = F_*$. That is ***the inverse of pullback operation is the pushforward operation***.

3.2 Restricting Covector Fields to Submanifolds

- **Remark** Compare to restricting vector fields to submanifolds, the restriction of covector fields to submanifolds is much simpler.

- **Remark** (***The Pullback of Covector Field by the Inclusion Map is a Covector Field on Submanifold***)
  Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an ***immersed submanifold*** with or without boundary, and $\iota : S \hookrightarrow M$ is *the inclusion map*. If ω is any smooth covector field on M, ***the pullback by*** ι ***yields a smooth covector field*** $\iota^*\omega$ ***on*** S.

  To see what this means, let $v \in T_pS$ be arbitrary, and compute

$$(\iota^*\omega)_p(v) = \omega_p\left(d\iota_p(v)\right) = \omega_p(v).$$

  since $d\iota_p : T_pS \to T_pM$ is just the inclusion map, under our usual identification of T_pS with a subspace of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called ***the restriction of*** ω ***to*** S.

  Be warned, however, that $\iota^*\omega$ might equal ***zero*** at a given point of S, even though ***considered as a covector field on*** M, ω ***might not vanish there***.

- **Example** ($\omega \neq 0$ but $\iota^*\omega = 0$)
  Let $\omega = dy$ on $\mathbb{R}^2$, and let S be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is ***nonzero*** everywhere, because one of its component functions is ***always*** 1. However, the restriction $\iota^*\omega$ is ***identically zero***, because y vanishes identically on S:

$$\iota^*\omega = \iota^*dy = d(y \circ \iota) = 0.$$

- **Remark** One usually says that "ω ***vanishes along*** S" or "ω ***vanishes at points of*** S" if $\omega_p = 0$ for every point $p \in S$.

  The ***weaker condition*** that $\iota^*\omega = 0$ is expressed by saying that "***the restriction of*** ω ***to*** S ***vanishes***", or "***the pullback of*** ω ***to*** S ***vanishes***".

4 Compare Tangent Bundle and Cotangent Bundle

Table 1: Comparison between tangent space and cotangent space

base	***smooth manifold*** M	***smooth manifold*** M
element	$\varphi(p) = (x^1, \dots, x^n)$	$\varphi(p) = (x^1, \dots, x^n)$
vector space (***fiber***) at p	**tangent space** T_pM	**cotangent space** $T_p^*M = (T_pM)^*$
dimension of vector space	n	n
basis of vector space	$\left(\frac{\partial}{\partial x^1}\Big\vert_p, \dots, \frac{\partial}{\partial x^n}\Big\vert_p \right)$	$(dx^1\big\vert_p, \dots, dx^n\big\vert_p)$
element in vector space	**tangent vector** $: \mathcal{C}^\infty(M) \to \mathbb{R}$ $v = v^i \frac{\partial}{\partial x^i}\Big\vert_p$	**cotangent vector** $: T_pM \to \mathbb{R}$ $\omega = \xi_i \, dx^i\big\vert_p$
total space of ***bundle***	**tangent bundle** $TM = \bigsqcup_{p \in M} T_pM$	**cotangent bundle** $T^*M = \bigsqcup_{p \in M} T_p^*M,$
element in bundle	$(x^1(p), \dots, x^n(p), v^1, \dots, v^n)$	$(x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n)$
section	**local vector field** $X = X^i \frac{\partial}{\partial x^i}$ $X_p \in T_pM$	**local covector field** $\omega = \xi_i dx^i$ $\omega_p \in T_p^*M$
vector space of sections	$\mathfrak{X}(M) \equiv \Gamma(TM)$	$\mathfrak{X}^*(M) \equiv \Gamma(T^*M)$
frame	**coordinate vector fields** $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right)$	**coordinate covector fields** $\left(dx^1, \dots, dx^n \right)$
duality	$\xi \left(\frac{\partial}{\partial x^i}\Big\vert_p \right) (dx^j\vert_p) = \delta_i^j$	$dx^j\vert_p \left(\frac{\partial}{\partial x^i}\Big\vert_p \right) = \delta_i^j$
change of coordinates	**contravariant** $\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, v^i$	**covariant** $\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \widetilde{\omega}_j$
functions	$F : M \to N$ ***diffeomorphism*** $dF_p : T_pM \to T_{F(p)}N$ ***Pushforward***: $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ $(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}),\ q \in N$	$dF_p^* : T_{F(p)}^*N \to T_p^*M$ ***dual map of*** dF_p ***Pullback***: $F^* : \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$ $(F^*\omega)_p = dF_p^* \left(\omega_{F(p)}\right),\ p \in M$

5 Line Integrals

6 Conservative Covector Fields

- **Definition** A *smooth covector field* ω on a smooth manifold M with or without boundary is said to be ***exact*** (or *an* ***exact differential***) on M if there is a function $f \in \mathcal{C}^{\infty}(M)$ such that $\omega = df$. In this case, the function f is called ***a potential*** *for* ω.

- **Definition** A curve $\gamma : [a, b] \to M$ is a ***closed curve segment*** if $\gamma(a) = \gamma(b)$. The integral of df over γ is ***zero***.

- **Definition** A smooth covector field ω is said to be ***conservative*** if the line integral of ω over *every piecewise smooth* ***closed*** *curve segment* is ***zero***.

- **Proposition 6.1** *A smooth covector field ω is conservative if and only if its line integrals are* ***path-independent****, in the sense that $\int_{\gamma} \omega = \int_{\widetilde{\gamma}} \omega$ whenever γ and $\widetilde{\gamma}$ are piecewise smooth curve segments with the* ***same*** *starting and ending points.*

- **Theorem 6.2** *Let M be a smooth manifold with or without boundary. A smooth covector field on M is* ***conservative*** *if and only if it is* ***exact***.

- **Remark** To check whether a given covector field is exact, there is a very simple ***necessary condition***, which follows from the fact that ***partial derivatives*** *of smooth functions can be taken in* ***any order***.

- **Remark** Suppose $\omega \in \mathfrak{X}^{*}(M)$ is ***exact***. Let f be any potential function for ω, and let $(U, (x^i))$ be any smooth chart on M. Because f is smooth, it satisfies the following identity on U:

$$\frac{\partial f}{\partial x^i \partial x^j} = \frac{\partial f}{\partial x^j \partial x^i} \tag{22}$$

Writing $\omega = \omega_i dx^i$ in coordinates, we see that $\omega = df$ is equivalent to $\omega_i = \partial f / \partial x^i$. Substituting this into (22), we find that the component functions of ω satisfy the following identity *for* ***each pair*** *of indices i and j*:

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}. \tag{23}$$

We say that a *smooth covector field* ω is ***closed*** if its components in every smooth chart satisfy (23). The following proposition summarizes the computation above.

Proposition 6.3 *Every exact covector field is closed.*

- **Proposition 6.4** *Let ω be a smooth covector field on a smooth manifold M with or without boundary. The following are equivalent:*

  1. *ω is* ***closed***.
  2. *ω satisfies* (23) *in* ***some smooth chart*** *around* ***every point***.
  3. *For any open subset $U \subseteq M$ and smooth vector fields $X, Y \in \mathfrak{X}(U)$,*

$$X\left(\omega\left(Y\right)\right) - Y\left(\omega\left(X\right)\right) = \omega\left(\left[X, Y\right]\right). \tag{24}$$

- **Corollary 6.5** *Suppose* $F : M \to N$ *is a* ***local diffeomorphism****. Then the* ***pullback*** $F^* : \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$ *takes* ***closed*** *covector fields to* ***closed*** *covector fields, and* ***exact*** *ones to* ***exact*** *ones.*

References

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