Lecture 5: Submanifolds

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Contents

1 Embedded Submanifolds

1.1 Defintions and Examples

- **Definition** Suppose M is a smooth manifold with or without boundary. ***An embedded submanifold*** of M is a subset $S \subseteq M$ that is a *manifold* (without boundary) in the ***subspace topology***, endowed with a ***smooth structure*** with respect to which the ***inclusion map*** $S \hookrightarrow M$ is a ***smooth embedding***. Embedded submanifolds are also called ***regular submanifolds***.

- **Definition** If S is an *embedded submanifold* of M, the ***difference*** $\dim M - \dim S$ is called ***the codimension*** of S in M, and the ***containing manifold*** M is called ***the ambient manifold*** for S.

  An embedded ***hypersurface*** is an embedded submanifold of codimension 1. The *empty set* is an embedded submanifold of *any dimension*.

- **Proposition 1.1** *(**Open Submanifolds**). [Lee, 2003.]*
  *Suppose M is a smooth manifold. The embedded submanifolds of **codimension 0** in M are exactly the **open submanifolds**.*

- There are several other ways to create submanifolds:

  Proposition 1.2 *(**Images of Embeddings as Submanifolds**). [Lee, 2003.]*
  *Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is a **smooth embedding**. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a **unique smooth structure** making it into an **embedded submanifold** of M with the property that F is a **diffeomorphism** onto its image.*

- **Proposition 1.3** *(**Slices of Product Manifolds**). [Lee, 2003.]*
  *Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$ (called **a slice of the product manifold**) is an **embedded submanifold** of $M \times N$ diffeomorphic to M.*

- **Proposition 1.4** *(**Graphs as Submanifolds**). [Lee, 2003.]*
  *Suppose M is a smooth m-manifold (without boundary), N is a smooth n-manifold with or without boundary, $U \subseteq M$ is open, and $f : U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote **the graph of** f:*
  $$\Gamma(f) = \{(x, y) \in M \times N : x \in U, y = f(x)\}.$$
  *Then $\Gamma(f)$ is an **embedded m-dimensional** submanifold of $M \times N$*

- **Definition** An embedded submanifold $S \subseteq M$ is said to be ***properly embedded*** if the inclusion $S \hookrightarrow M$ is a ***proper map***.

- **Proposition 1.5** *Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedded submanifold. Then S is **properly** embedded if and only if it is a **closed** subset of M.*

- **Corollary 1.6** *Every **compact** embedded submanifold is **properly** embedded.*

- **Proposition 1.7** *(**Global Graphs Are Properly Embedded**). [Lee, 2003.]*
  *Suppose M is a smooth manifold, N is a smooth manifold with or without boundary, and $f : M \to N$ is a **smooth** map. With the smooth manifold structure as above, the graph of f $\Gamma(f)$ is properly embedded in $M \times N$.*

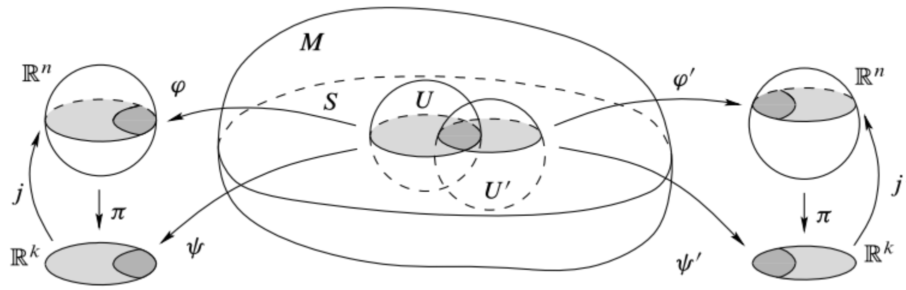

Figure 1: Smooth compatibility of slice charts [Lee, 2003.]

1.2 Slice Charts for Embedded Submanifolds

- **Definition** if U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$, a ***k-dimensional slice*** of U (or simply a k-slice) is any subset of the form
$$S = \left\{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n\right\}$$
for some constants $c^{k+1}, \ldots, c^n$. (When $k = n$, this just means $S = U$.) Clearly, ***every k-slice is homeomorphic to an open subset of*** $\mathbb{R}^k$.

- **Remark** Sometimes it is convenient to consider slices defined by setting some subset of the coordinates other than the last ones equal to constants.

- **Remark** Although in general we allow our slices to be defined by arbitrary constants $c^{k+1}, \ldots, c^n$, it is sometimes useful to have slice coordinates for which the constants are ***all zero***, which can easily be achieved by subtracting a constant from each coordinate function.

- **Definition** Let M be a smooth n-manifold, and let (U, φ) be a smooth chart **on** M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that ***S is a k-slice of U***.

- **Definition** Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S ***satisfies the local k-slice condition*** if *each point* of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ *is a* ***single*** ***k-slice*** *in* U. Any such chart is called ***a slice chart for S in M*** , and the corresponding coordinates $(x^1, \ldots, x^n)$ are called ***slice coordinates***.

- **Remark** The key to understand the ***the local k-slice condition*** for $S \subseteq M$:

  1. It is a condition on the ***subset*** S only; it does ***not presuppose*** any particular ***topology*** or ***smooth structure*** on S. All it needs is the topology and smooth structure from the ambient manifold M.

  2. The *local neighborhood* $U \subseteq M$ is a ***neighborhood*** *of p in the* ***ambient manifold*** M not a neigborhood in S (since we do not define such topology);

  3. The k-slice representation is for the ***intersection*** $S \cap U$ under ***the smooth chart*** (U, φ) of ***the ambient manifold*** M.

- **Theorem 1.8** ***(Local Slice Criterion for Embedded Submanifolds)*** *[Lee, 2003.].*
Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. ***Conversely****, if $S \subseteq M$ is a subset that* ***satisfies the local k-slice condition****, then with the* ***subspace topology****, S is a topological manifold of*

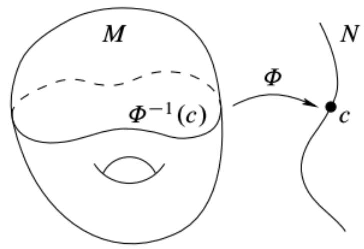

Figure 2: A level set [Lee, 2003.]

dimension k, and it ***has a smooth structure*** *making it into a k-dimensional embedded submanifold of* M *.*

- **Theorem 1.9** *If* M *is a smooth n-manifold with boundary, then with the subspace topology,* ∂M *is a topological* $(n-1)$*-dimensional manifold (without boundary), and has a smooth structure such that it is a properly* ***embedded submanifold*** *of* M*.*

- **Example** (***Spheres as Submanifolds***).
For any $n \geq 0$, $\mathbb{S}^n$ ***is an embedded submanifold of*** $\mathbb{R}^{n+1}$, because it is *locally* the *graph* of *a smooth function*: the intersection of $\mathbb{S}^n$ with the open subset $\left\{x : x^i > 0\right\}$ is the graph of the smooth function

$$x^i = f(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}),$$

where $f : \mathbb{B}^n \to \mathbb{R}$ is given by $f(u) = \sqrt{1 - |u|^2}$. Similarly, the intersection of $\mathbb{S}^n$ with $\left\{x : x^i < 0\right\}$ is the graph of $-f$. Since every point in $\mathbb{S}^n$ is in one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is thus an embedded submanifold of $\mathbb{R}^{n+1}$.

1.3 Level Sets

- **Remark** In practice, embedded submanifolds are most often presented as ***solution sets*** of *equations or systems of equations.*

- **Definition** If $\Phi : M \to N$ is any map and c is any point of N, we call the set $\Phi^{-1}(c)$ ***a level set of*** Φ (Fig. 2). (In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called ***the zero set of*** Φ.)

- **Remark** It is easy to find *level sets of smooth functions* that are *not smooth submanifolds.*

$$\Theta(x, y) = x^2 - y, \quad \Phi(x, y) = x^2 - y^2, \quad \Psi(x, y) = x^2 - y^3.$$

(Note that the zero set $\Theta^{-1}(0)$ is an embedded submanifolds in $\mathbb{R}^2$ but not for others.) In fact, ***every closed subset of*** M can be expressed as ***the zero set*** of some smooth real-valued function.

- **Theorem 1.10** ***(Constant-Rank Level Set Theorem)****. [Lee, 2003.]*
Let M *and* N *be smooth manifolds, and let* $\Phi : M \to N$ *be a smooth map* ***with constant rank*** r*.* ***Each level set*** *of* Φ *is a properly embedded submanifold of* ***codimension*** r *in* M*.*

Proof: Write $m = \dim M$, $n = \dim N$, and $k = m - r$. Let $c \in \mathbb{N}$ be arbitrary, and let S denote the level set $\Phi^{-1}(c) \subseteq M$. From the rank theorem, for each $p \in S$ there are smooth

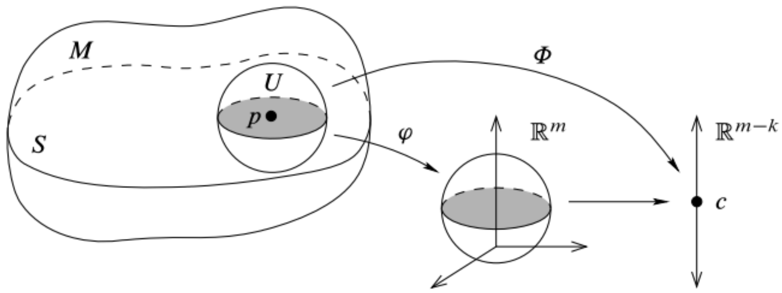

Figure 3: An embedded submanifold is locally a level set [Lee, 2003.]

charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form (4.1), and therefore $S \cap U$ is the slice

$$\left\{(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \in U : x^1 = \ldots = x^r = 0\right\}$$

Thus S satisfies the local k-slice condition, so it is an embedded submanifold of dimension k. It is closed in M by continuity, so it is properly embedded by Proposition 1.8. ∎

- **Corollary 1.11** ***(Submersion Level Set Theorem)***. *[Lee, 2003.]*
  If M and N are smooth manifolds and $\Phi : M \to N$ is a ***smooth submersion****, then each level set of Φ is a* ***properly embedded*** *submanifold whose* ***codimension*** *is equal to the* ***dimension of*** *N.*

- **Remark** This result should be compared to the corresponding result in linear algebra: if $L : \mathbb{R}^m \to \mathbb{R}^r$ is a surjective linear map, then the kernel of L is a linear subspace of codimension r by ***the rank-nullity law***. The vector equation $Lx = 0$ is equivalent to r linearly independent scalar equations, each of which can be thought of as cutting down one of the degrees of freedom in $\mathbb{R}^m$, leaving a subspace of codimension r.

  In the context of smooth manifolds, the analogue of *a surjective linear map* is ***a smooth submersion***, ***each*** of whose ***(local) component functions cuts down the dimension by one***.

- **Definition** If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be ***a regular point*** of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is ***surjective***; it is ***a critical point*** of Φ otherwise.

  This means, in particular, that ***every point*** of M is ***critical*** if dim $M <$ dim N, and every point is ***regular*** if and only if Φ is a ***submersion***.

- **Definition** A point $c \in N$ is said to be ***a regular value*** of Φ if ***every point*** *of the level set* $\Phi^{-1}(c)$ *is a regular point*, and ***a critical value*** otherwise. In particular, if $\Phi^{-1}(c) = \emptyset$, then c is *a regular value*. Finally, a level set $\Phi^{-1}(c)$ is called ***a regular level set*** if c is a regular value of Φ; in other words, a regular level set is a level set consisting ***entirely*** *of regular points* of Φ (points p such that $d\Phi_p$ is surjective).

- **Remark** If Φ is a ***smooth immersion***, *every point is a critical point of* Φ. A level set from a smooth immersion is a critical level set.

- **Remark** *Every properly embedded submanifold $M = \Phi^{-1}(c)$ is a regular level set.* The following theorem shows that the converse is true as well.

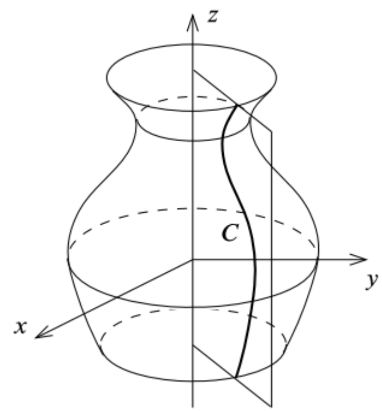

Figure 4: A surface of revolution [Lee, 2003.]

- **Theorem 1.12** *(**Regular Level Set Theorem**). [Lee, 2003.]*
  Every regular level set *of a smooth map between smooth manifolds is a* ***properly embedded*** *submanifold whose codimension is equal to the dimension of the codomain.*

- **Example** (***Spheres***). Now we can give a much easier proof that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$. The sphere is a regular level set of the smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$, since $df_x(v) = 2\sum_i x^i v^i$, which is surjective except at the origin.

- **Proposition 1.13** *(**Local Level Set Criterion for Smooth Embedded Submanifolds**)* *Let S be a subset of a smooth m-manifold M. Then S is an* ***embedded k-submanifold*** *of M if and only if every point of S has* ***a neighborhood*** *<u>U in M</u> such that <u>$U \cap S$</u> is a* ***level set*** *of a* ***smooth submersion*** *$\Phi : U \to \mathbb{R}^{m-k}$.*

- **Definition** If $S \subseteq M$ is an embedded submanifold, *a smooth map* $\Phi : M \to N$ such that S is ***a regular level set of*** Φ is called <u>***a defining map for S***</u>. In the special case $N = \mathbb{R}^{m-k}$ (so that Φ is a real-valued or vector-valued function), it is usually called ***a defining function***.

  More generally, if U is an open subset of M and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called ***a local defining map (or local defining function) for*** S.

  Remark The above proposition says *every embedded submanifold admits a local defining function in a neighborhood of each of its points.*

- **Example** (***Surfaces of Revolution***).
  Let $\mathbb{H}$ be the half-plane $\{(r, z) : r > 0\}$, and suppose $C \subseteq \mathbb{H}$ is an ***embedded 1-dimensional submanifold***. The <u>***surface of revolution determined by*** C</u> is the subset $S_C \subseteq \mathbb{R}^3$ given by

$$S_C = \left\{(x, y, z) : \left(\sqrt{x^2 + y^2}, z\right) \in C\right\}.$$

  The set C is called its <u>***generating curve***</u> (see Fig. 4). If $\varphi : U \to \mathbb{R}$ is any ***local defining function*** for C in $\mathbb{H}$, we get a ***local defining function*** Φ for S_C by

$$\Phi(x, y, z) = \varphi\left(\sqrt{x^2 + y^2}, z\right),$$

defined on the open subset

$$\widetilde{U} = \left\{(x,y,z) : \left(\sqrt{x^2+y^2}, z\right) \in U\right\} \subseteq \mathbb{R}^3$$

A computation shows that the Jacobian matrix of Φ is

$$D\Phi(x,y,z) = \left(\frac{x}{r}\frac{\partial \varphi}{\partial r}(r,z), \frac{y}{r}\frac{\partial \varphi}{\partial r}(r,z), \frac{\partial \varphi}{\partial z}(r,z)\right)$$

where we have written $r = \sqrt{x^2+y^2}$. At any point $(x,y,z) \in S_C$, at least one of the components of $D\Phi(x,y,z)$ is *nonzero*, so S_C ***is a regular level set of*** Φ and is thus ***an embedded*** 2***-dimensional submanifold of*** $\mathbb{R}^3$.

For a specific example, the *doughnut-shaped* ***torus*** of revolution D is the *surface of revolution* obtained from the circle $(r-2)^2 + z^2 = 1$. It is a regular level set of the function $\Phi(x,y,z) = (\sqrt{x^2+y^2} - 2)^2 + z^2$, which is smooth on $\mathbb{R}^3$ minus the z-axis. ∎

2 Immersed Submanifolds

2.1 Definitions and Examples

- **Definition** Let M be a smooth manifold with or without boundary. ***An <u>immersed</u> submanifold*** of M is a subset $S \subseteq M$ endowed with a *topology* (***not necessarily the subspace topology***) with respect to which it is ***a topological manifold*** (without boundary), and *a smooth structure* with respect to which ***the inclusion map*** $S \hookrightarrow M$ is ***a smooth immersion.***

  As for embedded submanifolds, we define the ***codimension*** of S in M to be $\dim M - \dim S$.

  Remark This terms can be generalized to the ***immersed topological submanifold of*** M to be a subset $S \subseteq M$ endowed with a topology such that it is a topological manifold and such that the *inclusion map is a topological immersion.* It is an ***embedded topological submanifold*** if the inclusion is a *topological embedding.*

- **Remark** *Every embedded submanifold is also an immersed submanifold.* Because immersed submanifolds are the more general of the two types of submanifolds, we adopt the convention that the term ***smooth submanifold*** without further qualification means an immersed one, which includes an embedded submanifold as a special case. Similarly, the term ***smooth hypersurface*** without qualification means an immersed submanifold of ***codimension*** 1.

- The immersed submanifolds arise in natural way:

  Proposition 2.1 *(**Images of Immersions as Submanifolds**). [Lee, 2003.]*
  *Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is an **injective smooth immersion**. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a **smooth submanifold** of M and such that $F : N \to S$ is a **diffeomorphism** onto its image.*

- **Example** (***Immersed Submanifold but Not an Embedded Submanifold***)
  Both examples of ***The Figure-Eight*** and ***the Dense Curve on the Torus*** are *images*

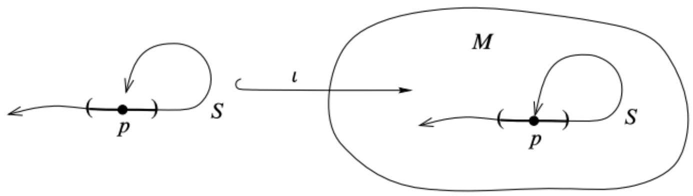

Figure 5: An immersed submanifold is locally embedded submanifold. [Lee, 2003.]

of injective smooth immersions, they are ***immersed submanifolds*** when given appropriate topologies and smooth structures. As smooth manifolds, they are ***diffeomorphic*** to $\mathbb{R}$. *They are* ***not embedded submanifolds***, because ***neither*** one has ***the subspace topology***. In fact, their image sets cannot be made into embedded submanifolds even if we are allowed to change their topologies and smooth structures. $\blacksquare$

- **Remark** Suppose M is a smooth manifold and $S \subseteq M$ is ***an immersed submanifold***. It can be shown that every subset of S that is ***open*** in the ***subspace topology*** is also ***open*** in its given ***submanifold topology***; and the **converse** is true if and only if S is ***embedded***.

- **Proposition 2.2** ***(Criterion for Immersed Submanifold to be Embedded Submanifold)***
Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an ***immersed submanifold****. If any of the following holds, then S is* ***embedded****.*

    1. *S has* ***codimension*** *0 in M.*
    2. *The inclusion map $S \hookrightarrow M$ is* ***proper****.*
    3. *S is* ***compact****.*

- **Proposition 2.3** ***(Immersed Submanifolds Are Locally Embedded)****. [Lee, 2003.]*
If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an ***immersed submanifold****, then for each $p \in S$ there exists a neighborhood U of p* ***in S*** *that is an* ***embedded submanifold*** *of M.*

    Note that a smooth immersion is locally a smooth embedding.

- **Remark** It is important to be clear about what this proposition does and does not say: given an immersed submanifold $S \subseteq M$ and a point $p \in S$, it is possible to find *a neighborhood U of p* ***in S*** such that $\underline{U}$ is *embedded*; but it may not be possible to find *a neighborhood V of p* ***in M*** such that $\underline{V \cap S}$ is embedded. (Fig 5)

- **Definition** Suppose $S \subseteq M$ is an immersed k-dimensional submanifold. ***A local parametrization*** of S is a continuous map $X : U \to M$ whose domain is an ***open subset*** $U \subseteq \mathbb{R}^k$, whose image is an ***open subset*** of S, and which, considered as a map into S, is a ***homeomorphism*** *onto its image*. It is called a ***smooth local parametrization*** if it is a ***diffeomorphism*** onto its image (with respect to Ss smooth manifold structure). If the image of X is all of S, it is called ***a global parametrization***.

- **Remark** For *a smooth chart* (U, φ) of M, $\varphi : U \to \widehat{U} \subseteq \mathbb{R}^n$ is a *diffeomorphism*, *its inverse* $\varphi^{-1} : \widehat{U} \to U \subseteq M$ is ***a smooth local parameterization*** (in fact $X = \mathrm{Id}_M \circ \varphi^{-1}$).

- **Proposition 2.4** *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an*

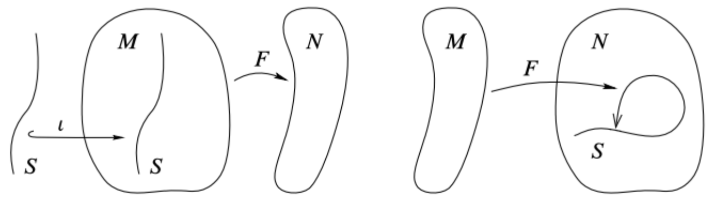

Figure 6: (Left) Restricting the domain. (Right) Restricting the codomain. [Lee, 2003.]

*immersed k-submanifold, $\iota : S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X : U \to M$ is a **smooth local parametrization** of S **if and only if** there is a smooth coordinate chart (V, φ) for S such that $\underline{X = \iota \circ \varphi^{-1}}$. Therefore, every point of S is in the image of some local parametrization.*

- **Example** (***Graph Parametrizations***).
Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function. The map $\gamma_f : U \to \mathbb{R}^n \times \mathbb{R}^k$ given by
$$\gamma_f(u) = (u, f(u))$$
is a ***smooth global parametrization*** of $\Gamma(f)$, called ***a graph parametrization***. Its ***inverse*** is ***the graph coordinate map*** $\varphi : \Gamma(f) \to U$
$$\varphi(x, y) = x, \quad \forall (x, y) \in \Gamma(f).$$
For example, the map $F : \mathbb{B}^2 \to \mathbb{R}^3$ given by
$$F(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$
is a *smooth local parametrization* of $\mathbb{S}^2$ whose image is the open upper hemisphere, and whose *inverse* is one of the *graph coordinate maps*.

3 Restricting Maps to Submanifolds

3.1 Theorems

- **Remark** Given a smooth map $F : M \to N$, it is important to know whether F is still smooth when its domain or codomain is restricted to a submanifold. See Fig. 6.

- **Theorem 3.1** ***(Restricting the Domain of a Smooth Map)****. [Lee, 2003.]*
*If M and N are smooth manifolds with or without boundary, $F : M \to N$ is a smooth map, and $S \subseteq M$ is an **immersed or embedded submanifold**, then $F|_S : S \to N$ is smooth.*

- The next theorem gives sufficient conditions for a map to be smooth when its codomain is restricted to an immersed submanifold.It shows that the failure of continuity is the only thing that can go wrong.

Theorem 3.2 ***(Restricting the Codomain of a Smooth Map)****. [Lee, 2003.]*
*Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an **immersed submanifold**, and $F : N \to M$ is a smooth map whose **image is contained in** S. If F is **continuous** as a map from N to S, then $F : N \to S$ is smooth.*

- **Corollary 3.3** ***(Embedded Case).***
  *Let M be a smooth manifold and $S \subseteq M$ be an **embedded submanifold**. Then every smooth map $F : N \to M$ whose **image is contained in** S is also **smooth** as a map from N to S.*

- **Definition** If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be ***weakly embedded*** in M if every smooth map $F : N \to M$ ***whose image lies in*** S is ***smooth*** as a map from N to S. (*Weakly embedded submanifolds* are called ***initial submanifolds*** by some authors.)

- **Remark** Corollary above shows that *every embedded submanifold is weakly embedded.*

3.2 Uniqueness of Smooth Structures on Submanifolds

- **Theorem 3.4** *Suppose M is a smooth manifold and $S \subseteq M$ is an **embedded submanifold**. The subspace topology on S and the smooth structure from the local k-slice condition are **the only topology and smooth structure** with respect to which S is an embedded or immersed submanifold.*

- **Remark** Thanks to this uniqueness result, we now know that a *subset* $S \subseteq M$ is an *embedded submanifold* ***if and only if*** it satisfies *the local slice condition*, and if so, its topology and smooth structure are ***uniquely determined***.

  Because the local slice condition is ***a local condition***, if every point $p \in S$ has a neighborhood $\underline{U \subseteq M}$ such that $\underline{U \cap S}$ is an embedded k-submanifold $\underline{\text{of } U}$, then S is an embedded k-submanifold of M.

- **Theorem 3.5** *Suppose M is a smooth manifold and $S \subseteq M$ is an **immersed submanifold**. For the **given topology** on S, there is **only one smooth structure** making S into an immersed submanifold.*

- **Theorem 3.6** *If M is a smooth manifold and $S \subseteq M$ is a **weakly embedded submanifold**, then S has **only one topology and smooth structure** with respect to which it is an immersed submanifold.*

3.3 Extending Functions from Submanifolds

- **Remark** Complementary to the restriction problem is the problem of extending smooth functions from a submanifold to the ambient manifold. Here we say $f \in \mathcal{C}^{\infty}(S)$ for submanifold $S \subseteq M$, when f is considered as a function on the manifold S.

- **Lemma 3.7** ***(Extension Lemma for Functions on Submanifolds).***
  Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in \mathcal{C}^{\infty}(S)$.

  1. *If S is **embedded**, then there exist a **neighborhood** U of S in M and a smooth function $\widetilde{f} \in \mathcal{C}^{\infty}(U)$ such that $\widetilde{f}|_S = f$.*
  2. *If S is **properly embedded**, then the neighborhood U above can be taken to be **all** of M.*

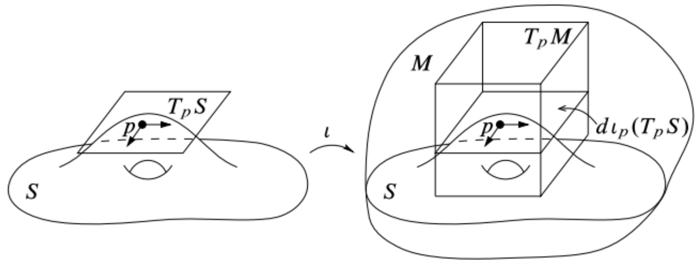

Figure 7: The tangent space of a submanifold. [Lee, 2003.]

4 The Tangent Space to a Submanifold

- **Remark** The ***tangent space to a smooth submanifold*** of an abstract smooth manifold can be viewed as *a* ***subspace*** of ***the tangent space to the ambient manifold***, once we make appropriate identifications. The following proof is based on the ***differential*** of ***the inclusion map*** as a smooth immersion.

  Proof: Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota : S \hookrightarrow M$ is a ***smooth immersion***, at each point $p \in S$ we have an *injective linear map* $d\iota_p : T_pS \to T_pM$. In terms of ***derivations***, this injection works in the following way: for any vector $v \in T_pS$, the image vector $\widetilde{v} = d\iota_p(v) \in T_pM$ acts on smooth functions on M by

  $$\widetilde{v}f = d\iota_p(v)f = v\,(f \circ \iota) = v\,(f|_S)\,.$$

  We adopt the convention of ***identifying*** T_pS with ***its image*** *under this map*, thereby thinking of T_pS as a certain linear subspace of T_pM (Fig. 7). This identification makes sense regardless of whether S is *embedded or immersed*. ∎

- There are several *alternative* ways to *characterize* the tangent space of a submanifold

  1. ***Smooth curve*** *on* ***submanifold***.

     Proposition 4.1 *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose* ***image is contained in*** *S, and which is also* ***smooth*** *as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.*

  2. ***Derivations*** *on functions whose* ***restriction on submanifold are constant zero***.

     Proposition 4.2 *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by*

     $$T_pS = \{v \in T_pM : vf = 0 \textit{ whenever } f \in \mathcal{C}^\infty(M) \textit{ and } f|_S = 0\}\,.$$

  3. ***Kernel subspace*** *of* ***differential map of local defining map***.

     Proposition 4.3 *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any* ***local defining map*** *for S, then $T_pS =$* ***Ker*** *$(d\Phi_p) : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.*

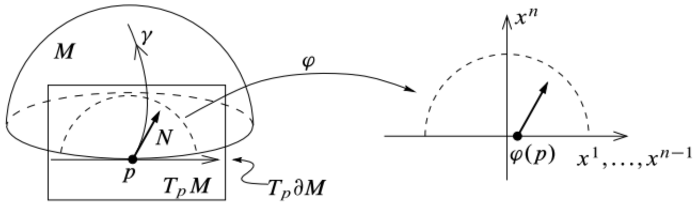

Figure 8: An inward-pointing vector. [Lee, 2003.]

Note that $S \cap U = (\Phi \circ \iota)^{-1}(c)$ is the level set of $\Phi \circ \iota$ thus it is constant for $\Phi \circ \iota$. So $d\Phi_p \circ d\iota_p = 0$.

Corollary 4.4 *Suppose $S \subseteq M$ is a **level set** of a **smooth submersion** $\Phi = (\Phi^1, \ldots, \Phi^k) : M \to \mathbb{R}^k$. A vector $v \in T_pM$ is tangent to S if and only if $v\Phi^1 = \ldots = v\Phi^k = 0$.*

- **Remark** If M is a smooth manifold ***with boundary*** and $p \in \partial M$, it is intuitively evident that the vectors in T_pM can be separated into ***three classes***:

  1. those ***tangent to the boundary***;
  2. those pointing ***inward***; See Fig 8.
  3. those pointing ***outward***.

  Definition If $p \in \partial M$, a vector $v \in T_pM \setminus T_p\partial M$ is said to be ***inward-pointing*** if for some $\epsilon > 0$ there exists a smooth curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is ***outward-pointing*** if there exists such a curve whose domain is $(-\epsilon, 0]$.

  Proposition 4.5 ***(Characterization of Tangent Vectors on Boundary using Component Functions)***
  *Suppose M is a smooth n-dimensional manifold with boundary, $p \in \partial M$, and (x^i) are any smooth boundary coordinates defined on a neighborhood of p. The **inward-pointing vectors** in T_pM are precisely those with **positive** x^n**-component**, the **outward-pointing** ones are those with **negative** x^n**-component**, and the ones **tangent to** ∂M are those with **zero** x^n**-component**. Thus, T_pM is the **disjoint union** of $T_p\partial M$, the set of inward-pointing vectors, and the set of outwardpointing vectors, and $v \in T_pM$ is inward-pointing if and only if $-v$ is outward-pointing.*

- **Definition** If M is a smooth manifold with boundary, a ***boundary defining function*** for M is a smooth function $f : M \to [0, \infty)$ such that $f^{-1}(0) = \partial M$ and $df_p \neq 0$ for all $p \in \partial M$. For example, $f(x) = \sqrt{1 - |x|^2}$ is a boundary defining function for the closed unit ball $\overline{\mathbb{B}}^n$.

- **Proposition 4.6** *Every smooth manifold with boundary admits a **boundary defining function**.*

5 Submanifolds with Boundary

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.