Lecture 15: Connections

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Contents

1 Differentiating Vector Fields

- **Remark** There are *two alternatives* for the definition of ***geodesics***:
  - Geodesics is the *"**shortest**" path* that connects two points on the surface; This definition is hard since the definition of manifold is abstract.
  - Geodesics is the curve on the surface that has ***zero tangential acceleration***. This is the motivation to introduce the concept of *connections*.
- **Remark** ***A connection*** is a ***coordinate-independent*** set of rules for taking ***directional derivatives of vector fields***.
- **Remark** (***Defining Directional Derivatives of Vector Fields on*** $\mathbb{R}^n$) [Lee, 2018]
  Let $I \subseteq \mathbb{R}$ be an interval and $\gamma : I \to \mathbb{R}^n$ a smooth curve, written in standard coordinates as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. The ***velocity*** $\dot{\gamma}$ and ***acceleration*** $\ddot{\gamma}$ at each $t \in I$, computed by ***differentiating the components***:

$$\dot{\gamma}(t) = \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \tag{1}$$

$$\ddot{\gamma}(t) = \ddot{\gamma}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \tag{2}$$

  A curve γ in $\mathbb{R}^n$ is a ***straight line*** if and only if it has a parametrization for which $\ddot{\gamma}(t) = 0$.

  We can define *the directional derivative of a vector field* similarly by *differentiating the components of the vector field*. Given a vector field $Y \in \mathfrak{X}(\mathbb{R}^n)$ and a vector $v \in T_p\mathbb{R}^n$, we define ***the Euclidean directional derivative of*** Y ***in the direction*** v by the formula

$$\overline{\nabla}_v Y := v(Y^i)\frac{\partial}{\partial x^i}\Big|_p$$

  where its ***component*** is *the* ***directional derivative of the component function*** Y^i ***along direction*** v

$$v(Y^i) := v\,Y^i = v^i\frac{\partial Y^i}{\partial x^i}(p)$$

  Note that $v = v^i\frac{\partial}{\partial x^i}|_p$.

  We can further generalize this definition by replacing the tangent vector $v \in T_p\mathbb{R}^n$ by *another vector field* $X \in \mathfrak{X}(\mathbb{R}^n)$. Thus ***the directional derivative of*** Y ***along*** X is written as

$$\overline{\nabla}_X Y := X(Y^i)\frac{\partial}{\partial x^i} \tag{3}$$

- **Remark** (***Directional Derivatives of Vector Fields on embedded submanifold*** $M \subseteq \mathbb{R}^n$) [Lee, 2018]
  Suppose $M \subseteq \mathbb{R}^n$ is an ***embedded submanifold***, and consider a smooth curve $\gamma : I \to M$. We want to think of a *geodesic* in M as a curve in M that is "*as straight as possible.* Of course, if M itself is curved, then $\dot{\gamma}(t)$ (thought of as a vector in $\mathbb{R}^n$) will probably have to vary, or else the curve will leave M. But we can try to insist that *the velocity not change any more than necessary for the curve to stay in M.*

One way to do this is to compute the Euclidean acceleration $\ddot{\gamma}(t)$ as above, and then apply the tangential projection $\pi^{\top} : T_{\gamma(t)}\mathbb{R}^n \to T_{\gamma(t)}M$. This yields a vector $\ddot{\gamma}^{\top}(t) = \pi^{\top}(\ddot{\gamma}(t))$ tangent to M , which we call ***the tangential acceleration*** of γ. It is reasonable to say that γ *is as straight as it is possible for a curve in M to be if its tangential acceleration is zero.*

Similarly, suppose Y is a smooth vector field on (an open subset of) M, and we wish to ask ***how much Y is varying in M in the direction of a vector*** $v \in T_pM$. As in the case of velocity vectors, if we look at it from the point of view of $\mathbb{R}^n$, the vector field Y might be forced to vary just so that it can *remain tangent to M*. One plausible way is to extend Y to a smooth vector field $\widetilde{Y}$ on an open subset of $\mathbb{R}^n$, compute *the Euclidean directional derivative of $\widetilde{Y}$* in the direction v, and then *project orthogonally onto T_pM*. Let us define *the **tangential directional derivative** of Y in the direction v* to be

$$\overline{\nabla}_v^{\top} Y := \pi^{\top}\left(\overline{\nabla}_v \widetilde{Y}\right) \tag{4}$$

- **Remark** (***Defining Directional Derivatives of Vector Fields on M ?***)
  Without the amibient Euclidean space $\mathbb{R}^n$, how to define the directional derivatives of vector fields on an abstract manifold M ? Still, consider a smooth curve $\gamma : I \to M$. Its *velocity* $\dot{\gamma}(t)$ is well defined on $T_{\gamma(t)}M$ via $\dot{\gamma}(t) = d\gamma(\frac{d}{dt})$. ***But the acceleration is not well defined*** since in order to define it, we need compute $\dot{\gamma}(t + \Delta)$ which does not lie in the space $T_{\gamma(t)}M$.

  The acceleration in Euclidean space is a special case since ***the tangent space $T_p\mathbb{R}^n$ at every point p is the same as*** $\mathbb{R}^n$. This is ***not the case*** for general smooth manifold M.

  The velocity vector $\dot{\gamma}(t)$ is an example of ***a vector field along a curve***. To interpret the acceleration of a curve in a manifold, what we need is some ***coordinate-independent way*** to *differentiate vector fields along curves.*

- **Remark** To do so, we need a way to ***compare values of the vector field at different points***, or intuitively, to "***connect***" ***nearby tangent spaces***. This is where a connection comes in: it will be an additional piece of data on a manifold, *a **rule*** for *computing directional derivatives of vector fields.*

2 Connections

2.1 Definitions

- **Definition** Let $\pi : E \to M$ be a *smooth vector bundle* over *a smooth manifold M* with or without boundary, and let $\Gamma(E)$ denote the space of *smooth sections* of E. A ***<u>connection</u>*** in E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

  written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

  1. $\nabla_X Y$ is ***linear** over* $\mathcal{C}^{\infty}(M)$ ***in*** X: for $f_1, f_2 \in \mathcal{C}^{\infty}(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{(f_1\, X_1 + f_2\, X_2)} Y = f_1\, \nabla_{X_1} Y + f_2\, \nabla_{X_2} Y$$

2. $\nabla_X Y$ is ***linear*** *over* $\mathbb{R}$ ***in*** Y: for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$$

3. ∇ satisfies the following ***product rule***: for $f \in \mathcal{C}^\infty(M)$,

$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y$$

The symbol ∇ is read "***del***" or "***nabla***," and $\nabla_X Y$ is called ***the covariant derivative of*** Y ***in the direction*** X.

- **Remark** There is *a variety of types of connections* that are useful in different circumstances. The type of connection we have defined here is sometimes called ***a Koszul connection*** to distinguish it from other types.

- **Remark** In definition, we see that ***the first argument is always a vector field on*** M, while *the second argument* can be any sections for any vector bundle E on M. By definition, the nabla operator ∇ is ***not symmetric***, since the first argument specifies the direction along which the second argument changes.

- **Remark** The notion of "*covariant*" reflects the fact that the *components of the covariant derivative* have a transformation law that "*varies correctly*" to give a well-defined meaning *independent of coordinates.*

- **Remark** $\nabla_X Y$ is ***linear*** over smooth function space $\mathcal{C}^\infty(M)$ in ***its first argument*** X but ***not in its second argument*** Y due to *the product rule.* Following this argument, we see that $\nabla_X Y$ is ***not a tensor*** since it is ***not bilinear*** due to not being linear over $\mathcal{C}^\infty(M)$ in second argument.

- **Remark** Although a connection is defined by its action on *global sections*, it follows from the definitions that it is actually a ***local operator***.

  Lemma 2.1 ***(Locality)****. [Lee, 2018]*
  Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$, and $p \in M$, the covariant derivative $\nabla_X Y|_p$ depends ***only*** *on the values of X and Y in an arbitrarily* ***small neighborhood*** *of p. More precisely, if $X = \widetilde{X}$ and $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.*

- **Proposition 2.2** ***(Restriction of a Connection)****.[Lee, 2018]*
  Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every open subset $U \subseteq M$, there is a ***unique connection*** *∇^U on the* ***restricted bundle*** *$E|_U$ that satisfies the following relation for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$:*

$$\nabla^U_{(X|_U)}(Y|_U) = (\nabla_X Y)|_U. \tag{5}$$

- **Proposition 2.3** *Under the hypotheses of Lemma 2.1, $\nabla_X Y|_p$ depends* ***only*** *on the* ***values*** *of Y in a* ***neighborhood*** *of p and the* ***value*** *of X* ***at*** *p.*

- **Remark** In the situation of these two propositions, we typically just refer to the *restricted connection* as ∇ instead of ∇^U; the proposition guarantees that there is no ambiguity in doing so. Thus if X is a vector field defined in a neighorhood of p,

$$\nabla_v Y = \nabla_X Y|_p, \quad \text{for } v = X_p.$$

- **Remark** Note that ***the Lie derivative*** is defined as

$$(\mathscr{L}_X Y)_p = \frac{d}{dt}\bigg|_{t=0} d\left(\theta_{-t}\right)_{\theta_t(p)}\left(Y_{\theta_t(p)}\right),$$

where θ is the ***flow of*** X in the neighborhood of p. Comparing $\mathscr{L}_X Y$ to connections $\nabla_X Y$, we see that *the Lie derivative depends on* ***the value of*** X ***is the neighorhood of*** p, while the connection $\nabla_X Y$ just depends on X **at** p:

$$\nabla_v Y = \nabla_X Y|_p, \quad \text{for } v = X_p.$$

On the other hand, *the Lie derivative is also defined as a* ***directional derivative*** *of a vector field* that is ***coordinate invariant***, just as the connection.

Another difference is that ***the Lie derivative*** does ***not*** requires the ***additional geometric structure*** (e.g. the definition of the abstract connections) and it applies to all smooth manifolds. However, ***its extension to tensor fields*** is not straightforward without specifyinig how to extend that tangent vector to a vector field.

2.2 Connections in the Tangent Bundle

- We focus on the connection in tangent bundle.

  Definition Suppose M is a smooth manifold with or without boundary. By the definition we just gave, *a connection in* TM is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

  satisfying properties (1)-(3) above. *A connection in the tangent bundle* TM is often called simply ***a connection on M***. (The terms ***affine connection*** and ***linear connection*** are also sometimes used in this context.)

- **Definition** For computations, we need to examine how a connection appears in terms of *a local frame*. Let (E_i) be a *smooth local frame* for TM on an open subset $U \subseteq M$. For every choice of the indices i and j, we can expand the vector field $\nabla_{E_i} E_j$ in terms of this same frame:

$$\nabla_{E_i} E_j = \Gamma_{i,j}^{k}\, E_k. \tag{6}$$

  As i, j, and k range from 1 to $n = \dim M$, this defines n^3 smooth functions $\Gamma_{i,j}^{k} : U \to \mathbb{R}$, called ***the connection coefficients*** *of* ∇ ***with respect to the given frame***.

- The following proposition shows that the connection is completely determined in U by its connection coefficients.

  Proposition 2.4 ***(Coordinate Representation of Connection)*** *[Lee, 2018]*
  Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a smooth local frame over an open subset $U \subseteq M$, and let $\left\{\Gamma_{i,j}^{k}\right\}$ be the connection coefficients of ∇ with respect to this frame. For smooth vector fields $X, Y \in \mathfrak{X}(M)$, written in terms of the frame as $X = X^i\, E_i$, $Y = Y^j\, E_j$, one has

$$\nabla_X Y = \left(X(Y^k) + X^i\, Y^j \Gamma_{i,j}^{k}\right) E_k. \tag{7}$$

- **Remark** The n^3 functions $\{\Gamma_{i,j}^k\}$ are called ***the Christoffel symbols*** under *the metric connections.* [do Carmo Valero, 1976]

- **Remark** The smooth function $\Gamma_{i,j}^k \in \mathcal{C}^\infty(M)$ has three indices: *two lower indices* (i, j) cooresponds to the index of ***component*** X^i ***for the directional vector field*** X, and the index of ***component*** Y^j ***for the differentiated vector field*** Y in $\nabla_X Y$; *the one upper index* k correponds to the index of the ***basis*** vector field $\partial/\partial x^k$ which spans the space of vector fields.

- **Remark** Compare these two coordinate representation:

$$\nabla_X Y = \left(X(Y^k) + X^i\, Y^j \Gamma_{i,j}^k\right)\, E_k \qquad \text{for } X, Y \in \mathfrak{X}(M)$$

$$\overline{\nabla}_X Y = X(Y^k)\, E_k \qquad \text{for } X, Y \in \Gamma(T\mathbb{R}^n).$$

  The connection coefficients $\{\Gamma_{i,j}^k\}$ account for an **additional "*rotation*" of basis** vector when moving Y from one tangent space to another along the direction of X. For Euclidean space, the basis is fixed when moving along the tangent direction (i.e. no *rotation* just *translation*).

- **Proposition 2.5** ***(Transformation Law for Connection Coefficients)****. [Lee, 2018]* *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose we are given two smooth local frames (E_i) and $(\widetilde{E}_j)$ for TM on an open subset $U \subseteq M$, related by $\widetilde{E}_i = A_i^j E_j$ for some matrix of functions (A_i^j). Let $\Gamma_{i,j}^k$ and $\widetilde{\Gamma}_{i,j}^k$ denote the connection coefficients of ∇ with respect to these two frames. Then*

$$\widetilde{\Gamma}_{i,j}^k = (A^{-1})_t^k A_i^r A_j^s \Gamma_{r,s}^t + (A^{-1})_t^k A_i^s E_s(A_j^t) \tag{8}$$

2.3 Existence of Connections

- **Example** (***The Euclidean Connection***).
  In $T\mathbb{R}^n$, define ***the Euclidean connection*** $\overline{\nabla}$ by formula

$$\overline{\nabla}_X Y = X(Y^i)\, \frac{\partial}{\partial x^i} \qquad \text{for } X, Y \in \Gamma(T\mathbb{R}^n).$$

  It is easy to check that this satisfies the required properties for a connection, and that its *connection coefficients* in the standard coordinate frame are all zero

- **Example** (***The Tangential Connection on a Submanifold of*** $\mathbb{R}^n$).
  Let $M \subseteq \mathbb{R}^n$ be an *embedded submanifold.* Define a connection $\nabla^\top$ on TM by setting

$$\nabla_X^\top Y := \pi^\top \left(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}\big|_M\right)$$

  where $\pi^\top$ is the ***orthogonal projection*** onto TM, $\overline{\nabla}$ is the Euclidean connection on $\mathbb{R}^n$, and $\widetilde{X}$ and $\widetilde{Y}$ are smooth extensions of X and Y to an open set in $\mathbb{R}^n$. $\nabla^\top$ is called ***the tangential connection***.

  Since the value of $\overline{\nabla}_{\widetilde{X}} \widetilde{Y}$ at a point $p \in M$ depends only on $\widetilde{X}_p = X_p$, this just boils down to defining $(\overline{\nabla}_X \widetilde{Y})_p$ to be equal to the tangential directional derivative $\overline{\nabla}_{X_p} \widetilde{Y}$ that we defined in (4) above.

- **Lemma 2.6** *Suppose M is a smooth n-manifold with or without boundary, and M admits **a global frame** (E_i). Formula (7) gives a **one-to-one correspondence** between connections in TM and choices of n^3 smooth real-valued functions $\{\Gamma_{i,j}^k\}$ on M.*

- **Proposition 2.7** *The tangent bundle of every smooth manifold with or without boundary admits a connection.*

- **Proposition 2.8** *(**The Difference Tensor**).*
Let M be a smooth manifold with or without boundary. For any two connections ∇^0 and ∇^1 in TM, define a map $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$D(X, Y) = \nabla^0_X Y - \nabla^1_X Y.$$

*Then D is **bilinear** over $\mathcal{C}^\infty(M)$, and thus defines a $(1,2)$**-tensor field** called **the difference tensor between ∇^0 and ∇^1**.*

- **Theorem 2.9** *Let M be a smooth manifold with or without boundary, and let ∇^0 be any connection in TM. Then **the set $\mathcal{A}(TM)$ of all connections** in TM is equal to the following **affine space**:*

$$\mathcal{A}(TM) = \left\{ \nabla^0 + D : \; D \in \Gamma(T^{(1,2)}TM) \right\},$$

where $D \in \Gamma(T^{(1,2)}TM)$ is interpreted as a map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$, and $\nabla^0 + D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$(\nabla^0 + D)(X, Y) = \nabla^0_X Y + D(X, Y).$$

3 Covariant Derivatives of Tensor Fields

3.1 Extension of ∇ From Tangent Bundle to Tensor Bundles

- **Remark** Given the connection ∇ on *tangent bundle*, we can induce a connection on each ***tensor bundle*** of all ranks. Note that connection is a set of rules, which also are compatible to tensor space.

- **Proposition 3.1** *(**Induced Connection on Tensor Bundle**)*
*Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Then ∇ **uniquely** determines a **connection** in **each tensor bundle** $T^{(k,l)}TM$, also denoted by ∇, such that the following four conditions are satisfied.*

  1. *In $T^{(1,0)}TM = TM$, ∇ **agrees** with the given connection.*

  2. *In $T^{(0,0)}TM = M \times \mathbb{R}$, ∇ is given by ordinary **differentiation of functions**:*

  $$\nabla_X f = Xf.$$

  3. *∇ obeys the following **product rule** with respect to **tensor products**:*

  $$\nabla_X F \otimes G = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$$

4. ∇ ***commutes*** *with all* ***contractions****: if "tr" denotes a* ***trace*** *on any pair of indices, one* ***covariant*** *and one* ***contravariant****, then*

$$\nabla_X(\mathrm{tr}\,(F)) = \mathrm{tr}\,(\nabla_X F):$$

This connection also satisfies the following ***additional properties****:*

(1) ∇ *obeys the following* ***product rule*** *with respect to the* ***natural pairing*** *between a* ***covector field*** ω *and a* ***vector field*** Y*:*

$$\nabla_X\langle\omega\,,\,Y\rangle = \langle\nabla_X\omega\,,\,Y\rangle + \langle\omega\,,\,\nabla_X Y\rangle\,.$$

Note that $\langle\omega\,,\,Y\rangle = \omega(Y)$*.*

(2) For all $F \in \Gamma(T^{(k,l)}TM)$*, smooth* 1*-forms* $\omega_1\,,\ldots,\omega_k$*, and smooth vector fields* $Y_1\,,\ldots,Y_l$*,*

$$\begin{aligned}(\nabla_X F)\left(\omega^1\,,\ldots,\omega^k,Y_1\,,\ldots,Y_l\right) = X\left(F\left(\omega^1\,,\ldots,\omega^k,Y_1\,,\ldots,Y_l\right)\right)\\ -\sum_{i=1}^{k} F\left(\omega^1\,,\ldots,(\nabla_X\omega^i)\,,\ldots,\omega^k,Y_1\,,\ldots,Y_l\right)\\ -\sum_{j=1}^{l} F\left(\omega^1\,,\ldots,\omega^k,Y_1\,,\ldots,(\nabla_X Y_j)\,,\ldots,Y_l\right)\end{aligned} \tag{9}$$

- **Remark** We have the formula for a (k,l)-tensor field F

$$F(\omega^1\,,\ldots,\omega^k,V_1\,,\ldots,V_l) = \underbrace{\mathrm{tr}\circ\ldots\circ\mathrm{tr}}_{k+l}\left(F\otimes\omega^1\otimes\ldots\otimes\omega^k\otimes V_1\otimes\ldots\otimes V_l\right), \tag{10}$$

where each trace operator acts on an upper index of F and the lower index of the corresponding 1-form, or a lower index of F and the upper index of the corresponding vector field.

For instance, for covariant 2-tensor field $g = \omega^1\otimes\omega^2$:

$$\begin{aligned}g(X,Y) &= \mathrm{tr}\left(\mathrm{tr}(\omega^1\otimes\omega^2\otimes X\otimes Y)\right)\\ &= \mathrm{tr}\left(\mathrm{tr}(\omega^2\otimes Y)\,\omega^1\otimes X\right)\\ &= \mathrm{tr}\left((\omega^2(Y))\,\omega^1\otimes X\right)\\ &= (\omega^2(Y))\,\mathrm{tr}\left(\omega^1\otimes X\right)\\ &= (\omega^2(Y))\,(\omega^1(X))\end{aligned}$$

- **Remark** For a covariant 2-tensor field $g = g_{i,j}dx^i\otimes dx^j$, the covariant derivative of g in direction of Z is

$$(\nabla_{(Z)}g)(X,Y) = Z\left(g(X,Y)\right) - g\left(\nabla_Z X,Y\right) - g\left(X,\nabla_Z Y\right)$$

- **Remark** Observe that condition 2 and additional property (1) imply that ***the covariant derivative of every*** 1***-form*** ω can be computed by

$$\begin{aligned}\langle\nabla_X\omega\,,\,Y\rangle &= \nabla_X\langle\omega\,,\,Y\rangle - \langle\omega\,,\,\nabla_X Y\rangle\\ \Rightarrow(\nabla_X\omega)(Y) &= X(\omega(Y)) - \omega(\nabla_X Y).\end{aligned} \tag{11}$$

It follows that the ***connection on*** 1***-forms*** is ***uniquely*** determined by *the original connection in* TM*.*

- **Proposition 3.2** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a local frame for M , (ϵ^j) is its dual coframe, and $\{\Gamma^k_{i,j}\}$ are the connection coefficients of ∇ with respect to this frame. Let X be a smooth vector field, and let $X^i\, E_i$ be its local expression in terms of this frame.*

  1. *The* ***covariant derivative of a** 1**-form*** $\omega = \omega_i \epsilon^i$ *is given locally by*
  $$\nabla_X \omega = \left(X(\omega_k) - X^j \omega_i \Gamma^i_{j,k} \right) \epsilon^k \tag{12}$$
  2. *If $F \in \Gamma(T^{(k,l)}TM)$ is a smooth mixed tensor field of any rank, expressed locally as*
  $$F = F^{j_1\,,\dots,\, j_l}_{i_1\,,\dots,\, i_k}\, E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_l}$$
  then the covariant derivative of F is given locally by
  $$\nabla_X F = \left(X \left(F^{j_1\,,\dots,\, j_l}_{i_1\,,\dots,\, i_k} \right) + \sum_{s=1}^{k} X^m F^{j_1\,,\dots,\, p\,,\dots,\, j_l}_{i_1\,,\dots,\, i_k} \Gamma^{i_s}_{m,p} - \sum_{s=1}^{l} X^m F^{j_1\,,\dots,\, j_l}_{i_1\,,\dots,\, p\,,\dots,\, i_k} \Gamma^{p}_{m,j_s} \right) \times$$
  $$E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_l}.$$

- **Proposition 3.3** *(**The Total Covariant Derivative**). [Lee, 2018]*
  Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every $F \in \Gamma(T^{(k,l)}TM)$, the map
  $$\nabla F : \underbrace{\Omega^1(M) \times \ldots \times \Omega^1(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{l+1} \to \mathcal{C}^\infty(M)$$
  given by
  $$\nabla F \left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X \right) = (\nabla_X F) \left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l \right) \tag{13}$$
  *defines a **smooth** $(k, l{+}1)$**-tensor field** on M called **the total covariant derivative of** F.*

- **Remark** The total covariant derivative of $Y \in \mathfrak{X}(M) := \Gamma(T^{(1,0)}TM)$ is a $(1,1)$-***tensor field***
  $$\nabla Y(\omega, X) = (\nabla_X Y)(\omega) = \omega\left(\nabla_X Y\right).$$
  Similarly, the total covariant derivative of $\omega \in \mathfrak{X}^*(M) = \Omega^1(M) = \Gamma(T^{(0,1)}TM)$ is a $(0,2)$-***tensor field***
  $$\nabla\omega(Y, X) = (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega\left(\nabla_X Y\right)$$
  Note that we can compare it with the ***invariant formula*** *for* ***exterior derivatives***:
  $$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

- **Remark** When we write the components of a total covariant derivative in terms of a local frame, it is standard practice to use a ***semicolon*** to *separate indices resulting from differentiation from the preceding indices.*

Thus, for example, if Y is a vector field written in coordinates as $Y = Y^i\, E_i$, the components of the $(1,1)$-tensor field ∇Y are written $Y^i_{\ ;j}$, so that

$$\nabla Y = Y^i_{\ ;j}\, E_i \otimes \epsilon^j,$$

with

$$Y^i_{\ ;j} = (E_j Y^i + Y^k \Gamma^i_{j,k})$$

For a 1-form ω, the formulas read

$$\nabla \omega = \omega_{i\ ;j} \epsilon^i \otimes \epsilon^j$$

with

$$\omega_{i,\ ;j} = E_j \omega_i - \omega_k \Gamma^k_{j,i}.$$

- **Proposition 3.4** *Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a local frame for M , (ϵ^j) is its dual coframe, and $\{\Gamma^k_{i,j}\}$ are the connection coefficients of ∇ with respect to this frame. The components of the total covariant derivative of a (k,l)-tensor field F with respect to this frame are given by*

$$F^{j_1\,,\ldots,\,j_l}_{i_1\,,\ldots,\,i_k\ ;m} = E_m\left(F^{j_1\,,\ldots,\,j_l}_{i_1\,,\ldots,\,i_k}\right) + \sum_{s=1}^{k} F^{j_1\,,\ldots,\,p\,,\ldots,\,j_l}_{i_1\,,\ldots,\,i_k} \Gamma^{i_s}_{m,p} - \sum_{s=1}^{l} F^{j_1\,,\ldots,\,j_l}_{i_1\,,\ldots,\,p\,,\ldots,\,i_k} \Gamma^p_{m,j_s}.$$

- **Remark** It can be verified that the following formula for total covariant derivative holds

$$\nabla_Y F = \mathrm{tr}\,(\nabla F \otimes Y) \tag{14}$$

3.2 Second Covariant Derivatives

- **Definition** Given vector fields $X, Y \in \mathfrak{X}(M)$, let us introduce the notation $\nabla^2_{X,Y} F$ for the (k,l)-tensor field obtained by inserting X, Y in the last two slots of $\nabla^2 F = \nabla(\nabla F)$:

$$\nabla^2_{X,Y} F(\ \ldots\ ) = \nabla^2 F(\ldots, Y, X)$$

- **Proposition 3.5** *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth vector field or tensor field F,*

$$\nabla^2_{X,Y} F = \nabla_X\,(\nabla_Y F) - \nabla_{(\nabla_X Y)} F. \tag{15}$$

- **Example** (***The Covariant Hessian***).
  Let u be a smooth function on M.
  - The <u>***total covariant derivative of** u **is equal to its** 1**-form***</u> $\nabla u = du \in \Omega^1(M) = \Gamma(T^{(0,1)}TM)$ since

$$\nabla u(X) = \nabla_X u = Xu = du(X)$$

– The 2-tensor $\nabla^2 u = \nabla(du)$ is called ***the covariant Hessian of*** u. Its action on smooth vector fields X, Y can be computed by the following formula:

$$\nabla^2 u(Y,X) = \nabla^2_{X,Y} u = \nabla_X \nabla_Y u - \nabla_{(\nabla_X Y)} u = X(Yu) - (\nabla_X Y)(u) \tag{16}$$

In any local coordinates, it is

$$\nabla^2 u = u_{\,;i,j}\, dx^i \otimes dx^j$$

where

$$u_{\,;i,j} = \frac{\partial}{\partial x^j} \frac{\partial u}{\partial x^i} - \Gamma^k_{j,i} \frac{\partial u}{\partial x^k}$$

References

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