Lecture 5: Abstract Integrations

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1 Recall

- **Definition** Let $\mathscr{B}$ be a *Boolean algebra* on a space X. An (unsigned) ***finitely additive measure*** μ on $\mathscr{B}$ is a map $\mu : \mathscr{B} \to [0, +\infty]$ that obeys the following axioms

  1. $\mu(\emptyset) = 0$;
  2. ***Finite union***: for any ***disjoint sets*** $A, B \in \mathscr{B}$,

  $$\mu\left(A \cup B\right) = \mu(A) + \mu(B).$$

- **Proposition 1.1** *(**Properties of Finitely Additive Measure**) [Tao, 2011]*
  Let $\mu : \mathscr{B} \to [0, +\infty]$be a finitely additive measure on a Boolean σ-algebra $\mathscr{B}$.

  1. *(**Monotonicity**) If E, F are $\mathscr{B}$-measurable and $E \subseteq F$, then*

  $$\mu(E) \leq \mu(F).$$

  2. *(**Finite additivity**) If k is a natural number, and $E_1, \ldots, E_k$ are $\mathscr{B}$-measurable and **disjoint**, then*

  $$\mu(E_1 \cup \ldots \cup E_k) = \mu(E_1) + \ldots + \mu(E_k).$$

  3. *(**Finite subadditivity**) If k is a natural number, and $E_1, \ldots, E_k$ are $\mathscr{B}$-measurable, then*

  $$\mu(E_1 \cup \ldots \cup E_k) \leq \mu(E_1) + \ldots + \mu(E_k).$$

  4. *(**Inclusion-exclusion for two sets**) If E, F are $\mathscr{B}$-measurable, then*

  $$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F).$$

  (Caution: remember that the cancellation law $a + c = b + c \Rightarrow a = b$ does not hold in [0; +1] if c is infinite, and so the use of cancellation (or subtraction) should be avoided if possible.)

- **Example** (***Dirac measure***).
  Let $x \in X$ and $\mathscr{B}$ be an arbitrary *Boolean algebra* on X. Then ***the Dirac measure*** δ_x at x, defined by setting $\delta_x(E) := \mathbb{1}\{x \in E\}$, is ***finitely additive***.

- **Example** (***Zero measure***).
  The ***zero measure*** $0 : E \mapsto 0$ is a *finitely additive measure* on any Boolean algebra.

- **Example** (***Linear combinations of measures***).
  If $\mathscr{B}$ is a Boolean algebra on X, and $\mu, \nu : \mathscr{B} \to [0, +\infty]$ are *finitely additive measures* on $\mathscr{B}$, then $\mu + \nu : E \mapsto \mu(E) + \nu(E)$ is also a ***finitely additive measure***, as is $c\mu : E \mapsto c \times \mu(E)$ for any $c \in [0, +\infty]$. Thus, for instance, the sum of Lebesgue measure and a Dirac measure is also a finitely additive measure on the Lebesgue algebra (or on any of its sub-algebras).

  In other word, ***the space of all finitely additive measures** on $\mathscr{B}$ is a **vector space***.

- **Example** (***Restriction of a measure***).
  If $\mathscr{B}$ is a Boolean algebra on X, $\mu : \mathscr{B} \to [0, +\infty]$ is a *finitely additive measure*, and Y is a $\mathscr{B}$-measurable subset of X, then ***the restriction*** $\mu|_Y : \mathscr{B}|_Y \to [0, +\infty]$ of $\mathscr{B}$ to Y, defined by setting $\mu|_Y(E) := \mu(E)$ whenever $E \in \mathscr{B}|_Y$ (i.e. if $E \in \mathscr{B}$ and $E \subseteq Y$), is also a ***finitely additive measure***.

- **Example** (***Counting measure***).
  If $\mathscr{B}$ is a Boolean algebra on X, then the function $\# : \mathscr{B} \to [0, +\infty]$ defined by setting $\#(E)$ to be the ***cardinality*** of E if E is *finite*, and $\#(E) := +\infty$ if E is infinite, is a ***finitely additive measure***, known as ***counting measure***.

- **Proposition 1.2** ***(Finitely Additive Measures on Atomic Algebra)***
  *Let $\mathscr{B}$ be a **finite** Boolean algebra, generated by a finite family $A_1, \ldots, A_k$ of non-empty **atoms**. For every **finitely additive measure** μ on $\mathscr{B}$ there exists $c_1, \ldots, c_k \in [0, +\infty]$ such that*
  $$\mu(E) = \sum_{1 \le j \le k : A_j \subseteq E} c_j.$$
  Equivalently, if x_j is a point in A_j for each $1 \le j \le k$, then
  $$\mu = \sum_{j=1}^{k} c_j \, \delta_{x_j}.$$
  *where $c_1, \ldots, c_k$ are **uniquely** determined by μ.*

- **Definition** Let $(X, \mathscr{B})$ be a measurable space. *An (unsigned)* ***countably additive measure*** μ on $\mathscr{B}$, or ***measure*** for short, is a map $\mu : \mathscr{B} \to [0, +\infty]$ that obeys the following axioms:

  1. (***Empty set***) $\mu(\emptyset) = 0$.
  2. (***Countable additivity***) Whenever $E_1, E_2, \ldots \in \mathscr{B}$ are a ***countable sequence*** of ***disjoint*** *measurable sets*, then
     $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

  A triplet $(X, \mathscr{B}, \mu)$, where $(X, \mathscr{B})$ is a ***measurable space*** and $\mu : \mathscr{B} \to [0, +\infty]$ is a ***countably additive measure***, is known as ***a measure space***.

- **Remark** Note the distinction between a ***measure space*** and a ***measurable space***. The latter has the ***capability*** to be equipped with a *measure*, but the former is ***actually*** equipped with a *measure*.

- **Definition** [Folland, 2013]
  Let $(X, \mathscr{B}, \mu)$ be a measure space.

  – If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \mathscr{B}$), then μ is called ***finite***.

  – If $X = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathscr{B}$ and $\mu(E_j) < \infty$, then μ is called ***σ-finite***. More generally, if $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathscr{B}$ and $\mu(E_j) < \infty$, then E is said to be ***σ-finite*** for μ.

  – If for each $E \in \mathscr{B}$ with $\mu(E) = \infty$ there exists $F \in \mathscr{B}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, then μ is called ***semi-finite***.

- **Proposition 1.3** *Let $(X, \mathscr{B}, \mu)$ be a **measure space**.*

  1. *(**Countable subadditivity**) If $E_1, E_2, \ldots$ are $\mathscr{B}$-measurable, then*
  $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

  2. *(**Upwards monotone convergence**) If $E_1 \subseteq E_2 \subseteq \ldots$ are $\mathscr{B}$-measurable, then*
  $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \sup_n \mu(E_n). \tag{1}$$

  3. *(**Downwards monotone convergence**) If $E_1 \supseteq E_2 \supseteq \ldots$ are $\mathscr{B}$-measurable, and $\underline{\mu(E_n) < \infty \text{ for } \textbf{at least one } n}$, then*
  $$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \inf_n \mu(E_n). \tag{2}$$

- **Proposition 1.4** *(**Dominated convergence for sets**). [Tao, 2011]*
  *Let $(X, \mathscr{B}, \mu)$ be a measure space. Let $E_1, E_2, \ldots$ be a sequence of $\mathscr{B}$-measurable sets that **converge** to another set E, in the sense that $\mathbb{1}_{E_n}$ converges **pointwise** to $\mathbb{1}_E$. Then*

  1. *E is also $\mathscr{B}$-measurable.*

  2. *If there exists a $\mathscr{B}$-measurable set F of **finite measure** (i.e. $\mu(F) < \infty$) that **contains all of the** E_n, then*
  $$\lim_{n\to\infty} \mu(E_n) = \mu(E).$$
  (Hint: Apply downward monotonicity to the sets $\bigcup_{n>N}(E_n \Delta E)$.)

  3. *The previous part of this proposition can **fail** if the hypothesis that all the E_n are contained in a set of finite measure is **omitted**.*

- **Exercise 1.5** *(**Countably Additive Measures on Countable Set with Discrete σ-Algebra**)*
  *Let X be an at most **countable** set with **the discrete σ-algebra**. Show that every measure μ on this measurable space can be uniquely represented in the form*
  $$\mu = \sum_{x\in X} c_x \, \delta_x$$
  for some $c_x \in [0, +\infty]$, thus
  $$\mu(E) = \sum_{x\in E} c_x$$
  *for all $E \subseteq X$. (This claim fails in the **uncountable** case, although showing this is slightly tricky.)*

- **Definition** (***Completeness***). [Tao, 2011]
  A ***null set*** of a measure space $(X, \mathscr{B}, \mu)$ is defined to be a $\mathscr{B}$-measurable set of ***measure zero***. A ***sub-null*** set is any subset of a null set.

  A measure space is said to be ***complete*** if *every sub-null set is a null set.*

- **Theorem 1.6** *The* ***Lebesgue measure space*** $(\mathbb{R}^d, \mathcal{L}[\mathbb{R}^d], m)$ *is* **complete***, but* ***the Borel measure space*** $(\mathbb{R}^d, \mathcal{B}[\mathbb{R}^d], m)$ *is* ***not.***

- Completion is a convenient property to have in some cases, particularly when dealing with properties that hold almost everywhere. Fortunately, it is fairly easy to modify any measure space to be complete:

  Proposition 1.7 *(***Completion***).*
  Let $(X, \mathscr{B}, \mu)$ be a measure space. There exists a ***unique refinement*** *$(X, \overline{\mathscr{B}}, \overline{\mu})$, known as* ***the completion*** *of $(X, \mathscr{B}, \mu)$, which is the* ***coarsest*** *refinement of $(X, \mathscr{B}, \mu)$ that is* ***complete.*** *Furthermore, $\overline{\mathscr{B}}$ consists precisely of those sets that differ from a $\mathscr{B}$-measurable set by* ***a $\mathscr{B}$-subnull set.***

- **Definition** (***Abstract outer measure***). [Tao, 2011]
  Let X be a set. *An* ***abstract outer measure*** (or ***outer measure*** for short) is a map $\mu^* : 2^X \to [0, +\infty]$ that assigns an *unsigned extended real number* $\mu^*(E) \in [0, +\infty]$ to every set $E \subseteq X$ which obeys the following axioms:

  1. (***Empty set***) $\mu^*(\emptyset) = 0$.
  2. (***Monotonicity***) If $E \subseteq F$, then $\mu^*(E) \le \mu^*(F)$.
  3. (***Countable subadditivity***) If $E_1, E_2, \ldots \subseteq X$ is a countable sequence of subsets of X, then
  $$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu^*(E_n).$$

  Outer measures are also known as ***exterior measures***.

- **Definition** (***Carathéodory measurability***).
  Let μ^* be an *outer measure* on a set X. A *set* $E \subseteq X$ is said to be ***Carathéodory measurable*** *with respect to* μ^* (or, μ^****-measurable***) if one has
  $$\mu^*(A) = \mu^*(A \setminus E) + \mu^*(A \cap E)$$
  for every set $A \subseteq X$.

- **Example** (***Null sets are Carathéodory measurable***).
  Suppose that E is a ***null set*** for *an* ***outer measure*** μ^* (i.e. $\mu^*(E) = 0$). Then that E *is Carathéodory measurable with respect to* μ^*.

- **Example** (***Compatibility with Lebesgue measurability***). A set $E \subseteq \mathbb{R}^d$ is *Carathéodory measurable with respect to Lebesgue outer measurable* if and only if it is *Lebesgue measurable.*

- **Theorem 1.8** *(***Carathéodory extension theorem***). [Tao, 2011]*
  Let $\mu^* : 2^X \to [0, +\infty]$ be an outer measure on a set X, let $\mathscr{B}$ be the collection of all subsets of X that are ***Carathéodory measurable with respect to*** *μ^*, and let $\mu : \mathscr{B} \to [0, +\infty]$ be the*

restriction *of μ^* to $\mathscr{B}$ (thus $\mu(E) := \mu^*(E)$ whenever $E \in \mathscr{B}$). Then $\mathscr{B}$* ***is a σ-algebra,*** *and μ* ***is a measure****.*

- **Definition** (***Pre-measure***).
  <u>A **pre-measure**</u> on a ***Boolean algebra*** $\mathscr{B}_0$ is a function $\mu_0 : \mathscr{B}_0 \to [0, +\infty]$ that satisfies the conditions:

  1. (***Empty Set***): $\mu_0(\emptyset) = 0$
  2. (***Countably Additivity***): IF $E_1, E_2, \ldots \in \mathscr{B}_0$ are *disjoint sets* such that $\bigcup_{n=1}^{\infty} E_n$ is in $\mathscr{B}_0$,

$$\mu_0\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

- **Remark** A *pre-measure* μ_0 is a ***finitely additive measure*** that ***already*** is *countably additive* ***within*** a Boolean algebra $\mathscr{B}_0$.

- **Remark** *The countably additivity condition* for pre-measure can be releaxed to be *the countably subadditivity* $\mu_0(\cup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu_0(E_n)$ without affecting the definition of a pre-measure.

- **Proposition 1.9** *Let $\mathscr{B} \subset 2^X$ and $\mu_0 : \mathscr{B} \to [0, +\infty]$ be such that $\emptyset, X \in \mathscr{B}$, and $\mu_0(\emptyset) = 0$. For any $A \subseteq X$, define*

$$\mu^*(A) := \inf\left\{\sum_{j=1}^{\infty} \mu_0(E_j) : E_j \in \mathscr{B}, \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j\right\}.$$

  Then μ^* is an outer measure.

- **Theorem 1.10** ***(Hahn-Kolmogorov Theorem).***
  Every ***pre-measure*** *$\mu_0 : \mathscr{B}_0 \to [0, +\infty]$ on a Boolean algebra $\mathscr{B}_0$ in X can be* ***extended*** *to a* ***countably additive measure*** *$\mu : \mathscr{B} \to [0, +\infty]$.*

- **Remark** We can construct an *outer measure* μ^* according to Proposition 1.9. Let $\mathscr{B}$ be the *collection* of all sets $E \subseteq X$ that are *Carathéodory measurable with respect to μ^* (μ^*-measurable)*, and let μ be the *restriction* of μ^* to $\mathscr{B}$. The tuple $(X, \mathscr{B}, \mu)$ is what we want in *Hahn-Kolmogorov theorem.*

  The measure μ constructed in this way is called <u>***the Hahn-Kolmogorov extension***</u> ***of the pre-measure*** μ_0.

- **Proposition 1.11** ***(Uniqueness of the Hahn-Kolmogorov Extension)***
  Let $\mu_0 : \mathscr{B}_0 \to [0, +\infty]$ be a ***pre-measure****, let $\mu : \mathscr{B} \to [0, +\infty]$ be the* ***Hahn-Kolmogorov extension*** *of μ_0, and let $\mu' : \mathscr{B}' \to [0, +\infty]$ be* ***another*** *countably additive extension of μ_0. Suppose also that μ_0 is σ-**finite**, which means that one can express the whole space X as the countable union of sets $E_1, E_2, \ldots \in \mathscr{B}_0$ for which $\mu_0(E_n) < \infty$ for all n. Then μ and μ' agree on their common domain of definition. In other words, show that $\mu(E) = \mu'(E)$ for all $E \in \mathscr{B} \cap \mathscr{B}'$.*

2 Measurable Functions, and Integration on a Measure Space

2.1 Measurable Functions

- **Definition** Let $(X, \mathscr{B})$ be a measurable space, and let $f : X \to [0, +\infty]$ or $f : X \to \mathbb{C}$ be an ***unsigned*** or ***complex-valued function***. We say that f is ***measurable*** if $f^{-1}(U)$ is ***$\mathscr{B}$-measurable*** for every ***open subset*** U of $[0, +\infty]$ or $\mathbb{C}$.

- **Remark** The inverse image of a Lebesgue measurable set by a *measurable function* need not remain Lebesgue. measurable. This is due to the definition of above measureable function. The pre-image of E is Lebesgue measureable, if if E has a slightly stronger measurability property than Lebesgue measurability, namely ***Borel measurability***.

- In general, we have the following

  Definition For $f : X \to Y$, and $X \equiv (X, \mathscr{F})$, $Y \equiv (Y, \mathscr{B})$ are measurable spaces, then f is called ***$(\mathscr{F}, \mathscr{B})$ measureable*** (or $(\mathscr{F}/\mathscr{B})$ measureable or, simply, *measureble*), if $f^{-1}(E) \in \mathscr{F}$ for every $E \in \mathscr{B}$.

- **Definition** Note that if $\{(Y_\alpha, \mathscr{B}_\alpha)\}$ is a family of measureable spaces, and $\{f_\alpha\}$ for $f_\alpha : X \to Y_\alpha$, then there is a ***unique smallest*** σ-algebra on X so that $\{f_\alpha\}$ are all measureable. It is generated by $f_\alpha^{-1}(E_\alpha), E_\alpha \in \mathscr{B}_\alpha$. It is called the ***$\sigma$-algebra generated by*** $\{f_\alpha\}$.

  In particular, $X = \prod_\alpha Y_\alpha$ has ***product σ-algebra*** that is *generated by coordinate functions* $\{\pi_\alpha\}$.

- **Proposition 2.1** *Let $(X, \mathscr{B})$ be a measurable space.*

  1. $f : X \to [0, +\infty]$ *is **measurable** if and only if the **level sets*** $\{x \in X : f(x) > \lambda\}$ *are $\mathscr{B}$-measurable.*

  2. *The **indicator function** $\mathbb{1}_E$ of a set $E \subseteq X$ is **measurable** if and only if E itself is **$\mathscr{B}$-measurable**.*

  3. $f : X \to [0, +\infty]$ *or* $f : X \to \mathbb{C}$ *is measurable if and only if* $f^{-1}(E)$ *is $\mathscr{B}$-measurable for every **Borel-measurable** subset E of* $[0, +\infty]$ *or* $\mathbb{C}$.

  4. $f : X \to \mathbb{C}$ *is measurable if and only if its real and imaginary parts are measurable.*

  5. $f : X \to \mathbb{R}$ *is measurable if and only if the **magnitudes*** $f_+ := \max\{f, 0\}$, $f_- := \max\{-f, 0\}$ *of its **positive** and **negative** parts are **measurable**.*

  6. *If* $f_n : X \to [0, +\infty]$ *are a sequence of **measurable** functions that converge **pointwise** to a limit* $f : X \to [0, +\infty]$*, then f is also **measurable**. The same claim holds if* $[0, +\infty]$ *is replaced by* $\mathbb{C}$.

  7. *If* $f : X \to [0, +\infty]$ *is measurable and* $\varphi : [0, +\infty] \to [0, +\infty]$ *is **continuous**, the composite $\varphi \circ f$ is measurable. The same claim holds if* $[0, +\infty]$ *is replaced by* $\mathbb{C}$.

  8. *The **sum** or **product** of two **measurable** functions in* $[0, +\infty]$ *or* $\mathbb{C}$ *is still measurable.*

- **Definition** A function $f : (X, \mathscr{F}) \to (Y, \mathscr{B})$ is ***simple*** if it only takes ***finitely many*** *different values* $s_1, \cdots, s_k \in Y$.

Then the σ-algebra $f^{-1}(\mathscr{B})$ reduce to $\sigma\left(\left\{f^{-1}(\{s_\alpha\})\right\}_{\alpha=1}^{k}\right)$, the ***finite σ-algebra*** generated by *atomic algebra* with atoms $E_\alpha \equiv f^{-1}(\{s_\alpha\})$. The ***canonical representation*** of f is

$$f = \sum_{\alpha=1}^{k} s_\alpha \mathbb{1}\left\{E_\alpha\right\},$$

which is determined up to a reordering.

- **Proposition 2.2** ***(Measurable Function with respect to Atomic Algebra is Simple)***
*Let $(X, \mathscr{B})$ be a measurable space that is **atomic**, thus $\mathscr{B} = \mathscr{A}((A_\alpha)_{\alpha\in I})$ for some partition $X = \bigcup_{\alpha\in I} A_\alpha$ of X into disjoint non-empty atoms. A function $f : X \to [0, +\infty]$ or $f : X \to \mathbb{C}$ is measurable if and only if it is **constant** on each atom, or equivalently if one has a **representation of the form***

$$f(x) = \sum_{\alpha\in I} c_\alpha \mathbb{1}\left\{x \in A_\alpha\right\},$$

for some constants $c_\alpha \in [0; +\infty]$ or in $\mathbb{C}$ as appropriate. Furthermore, the c_α are uniquely determined by f.

- **Theorem 2.3** ***(Egorov's theorem)****. [Tao, 2011]*
*Let $(X, \mathscr{B}, \mu)$ be a **finite measure space** (so $\mu(X) < \infty$), and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions that **converge pointwise almost everywhere** to a limit $f : X \to \mathbb{C}$. For $\epsilon > 0$, there exists a measurable set E of measure **at most** ϵ such that f_n converges **uniformly** to f outside of E.*

- **Remark** Give an example to show that the claim can **fail** when the measure μ is not finite.

2.2 Simple Integral of Simple Functions

- **Definition** (***Simple integral***).
Let $(X, \mathscr{B}, \mu)$ be a measure space with $\mathscr{B}$ ***finite*** (i.e., its *cardinality* is *finite* and there are only *finitely many measurable sets*). X can then be partitioned into a finite number of atoms $A_1, \cdots, A_n$. If $f : X \to [0, +\infty]$ is measurable, it has ***a unique representation*** of the form

$$f(x) = \sum_{\alpha\in I} c_\alpha \mathbb{1}\left\{x \in A_\alpha\right\},$$

for some constants $c_\alpha \in [0; +\infty]$. We then define the ***simple integral*** $\text{simp}\int_X f d\mu$ of f by the formula

$$\text{simp}\int_X f d\mu \equiv \sum_{\alpha\in I} c_\alpha \mu(A_\alpha)$$

- **Remark** Note that the precise decomposition into atoms *does not affect* the definition of the simple integral.

Proposition 2.4 ***(Simple integral unaffected by refinements)****. [Tao, 2011]*
*Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $(X, \mathscr{B}', \mu')$ be a **refinement** of $(X, \mathscr{B}, \mu)$, which*

*means that $\mathscr{B}'$ contains $\mathscr{B}$ and $\mu' : \mathscr{B}' \to [0,+\infty]$ **agrees** with $\mu : \mathscr{B} \to [0,+\infty]$ on $\mathscr{B}$. Suppose that both $\mathscr{B}, \mathscr{B}'$ are **finite**, and let $f : \mathscr{B} \to [0,+\infty]$ be measurable. We have*

$$\textit{simp} \int_X f d\mu = \textit{simp} \int_X f d\mu'.$$

Proof: Since taking simple integrals both w.r.t. $\mathscr{B}$ and $\mathscr{B}'$ implies that f is both $\mathscr{B}$-measurable and $\mathscr{B}'$-measurable, we see that for the finite values $a_1, \ldots, a_k$ of f we have $f^{-1}(a_i) \in \mathscr{B}$.

$$\text{simp} \int_X f d\mu' = \sum_{i=1}^{k} a_i \mu'(f^{-1}(a_i)) = \sum_{i=1}^{k} a_i \mu'|_{\mathscr{B}}(f^{-1}(a_i)) = \sum_{i=1}^{k} a_i \mu(f^{-1}(a_i)) = \text{simp} \int_X f d\mu. \quad \blacksquare$$

- The above proposition allows one to extend the *simple integral* to *simple functions*:

  Definition (***Integral of simple functions***).
  An ***(unsigned) simple function*** $f : X \to [0,+\infty]$ on a measurable space $(X, \mathscr{B})$ is a *measurable function* that takes on ***finitely many values*** $a_1, \cdots, a_k$. Note that such a function is then automatically *measurable* with respect to *at least one* ***finite sub-σ-algebra*** $\mathscr{B}'$ of $\mathscr{B}$, namely *the σ-algebra $\mathscr{B}'$* ***generated by the preimages*** $f^{-1}\{a_1\}, \cdots, f^{-1}\{a_k\}$ of $a_1, \cdots, a_k$.

  We then define the ***simple integral*** simp $\int_X f d\mu$ by the formula

  $$\begin{aligned}\text{simp} \int_X f d\mu &\equiv \text{simp} \int_X f \; d\mu|_{\mathscr{B}'} \\ &= \sum_{i=1}^{k} a_i \mu\left(f^{-1}\{a_k\}\right)\end{aligned}$$

  where $\mu|_{\mathscr{B}'} : \mathscr{B}' \to [0,+\infty]$ is the ***restriction*** of $\mu : \mathscr{B} \to [0,+\infty]$ to $\mathscr{B}'$.

- **Remark** Note that there could be ***multiple finite σ-algebras*** with respect to which f is *measurable*, but all such extensions will give the same simple integral. Indeed, if f were measurable with respect to two separate finite sub-σ-algebras $\mathscr{B}'$ and $\mathscr{B}''$ of $\mathscr{B}$, then it would also be *measurable* with respect to their ***common refinement*** $\mathscr{B}' \vee \mathscr{B}'' := (\mathscr{B}' \cup \mathscr{B}'')$, which is also *finite* and then by Proposition 2.4, $\int_X f d\mu|_{\mathscr{B}'}$ and $\int_X f d\mu|_{\mathscr{B}''}$ are both equal to $\int_X f d\mu|_{\mathscr{B}' \vee \mathscr{B}''}$, and hence equal to each other.

- **Remark** As with the Lebesgue theory, we say that a property $P(x)$ of an element $x \in X$ of a measure space $(X, \mathscr{B}, \mu)$ ***holds μ-almost everywhere*** if it ***holds outside*** of a ***sub-null set***, i.e. $\mu(\{P(x) \text{ does not hold}\}) = 0$.

- **Proposition 2.5** *Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f, g : X \to [0,+\infty]$ be simple unsigned functions.*

  1. *(**Monotonicity**) If $f \leq g$ then simp $\int_X f d\mu \leq$ simp $\int_X g d\mu$.*
  2. *(**Compatibility with measure**) For every $\mathscr{B}$-measurable set E, we have simp $\int_X \mathbb{1}_E d\mu = \mu(E)$.*
  3. *(**Homogeneity**) For every $c \in [0,+\infty]$, one has simp $\int_X (cf) d\mu = c \times$ simp $\int_X f d\mu$.*

4. ***(Finite additivity)*** *We have simp* $\int_X (f+g)d\mu = $ *simp* $\int_X f d\mu +$ *simp* $\int_X g d\mu$.
5. ***(Insensitivity to refinement)*** *Let* $(X, \mathscr{B}, \mu)$ *be a measure space, and let* $(X, \mathscr{B}', \mu')$ *be its refinement, which means that* $\mathscr{B}'$ *contains* $\mathscr{B}$ *and* $\mu' : \mathscr{B}' \to [0, +\infty]$ ***agrees*** *with* $\mu : \mathscr{B} \to [0, +\infty]$ *on* $\mathscr{B}$. *Suppose that both* $\mathscr{B}, \mathscr{B}'$ *are* ***finite***, *and let* $f : \mathscr{B} \to [0, +\infty]$ *be measurable. We have*
$$\text{simp} \int_X f d\mu = \text{simp} \int_X f d\mu'.$$
6. ***(Almost everywhere equivalence)*** *If* μ*-almost everywhere* $f = g$, *then simp* $\int_X f d\mu =$ *simp* $\int_X g d\mu$
7. ***(Finiteness)*** *simp* $\int_X f d\mu < \infty$ *if and only if* f *is* ***finite*** ***μ-almost everywhere*** *and is* ***supported*** *on a set of* ***finite measure***.
8. ***(Vanishing)*** *simp* $\int_X f d\mu = 0$ *if and only if* $f = 0$ μ*-almost everywhere.*

- **Exercise 2.6** ***(Inclusion-exclusion principle)***.
Let $(X, \mathscr{B}, \mu)$ *be a measure space, and let* $A_1, \ldots, A_n$ *be* $\mathscr{B}$*-measurable sets of* ***finite measure***. *Show that*
$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{J \subseteq [1:n], J \neq \emptyset} (-1)^{|J|-1} \mu\left(\bigcap_{i \in J} A_i\right)$$
(Hint: Compute simp $\int_X (1 - \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i})) d\mu$ *in two different ways.)*

- **Remark** The simple integral could also be defined on *finitely additive measure spaces*, rather than *countably additive ones*, and all the above properties would still apply. However, on a finitely additive measure space one would have difficulty extending the integral beyond simple functions.

2.3 Unsigned Integral

- **Definition** Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f : X \to [0, +\infty]$ be *(unsigned) measurable*. Then we define the ***unsigned integral*** $\int_X f d\mu$ of f by the formula
$$\int_X f d\mu \equiv \sup_{\substack{0 \le g \le f, \\ g \text{ simple}}} \text{simp} \int_X g d\mu$$

- **Proposition 2.7** ***(Properties of the unsigned integral)***.
Let $(X, \mathscr{B}, \mu)$ *be a measure space, and let* $f, g : X \to [0, +\infty]$ *be measurable.*
  1. ***(Almost everywhere equivalence)*** *If* $f = g$ μ*-almost everywhere, then* $\int_X f d\mu = \int_X g d\mu$
  2. ***(Monotonicity)*** *If* $f \le g$ μ*-almost everywhere, then* $\int_X f d\mu \le \int_X g d\mu$.
  3. ***(Homogeneity)*** *We have* $\int_X (cf) d\mu = c \int_X f d\mu$ *for every* $c \in [0, +\infty]$.
  4. ***(Superadditivity)*** *We have* $\int_X (f+g) d\mu \ge \int_X f d\mu + \int_X g d\mu$.

5. *(**Compatibility with the simple integral**) If f is **simple**, then $\int_X f d\mu = simp\ \int_X f d\mu$.*

6. *(**Markov's inequality**) For any $0 < \lambda < 1$, one has*

$$\mu\left(\{x \in X : f(x) \geq \lambda\}\right) \leq \frac{1}{\lambda}\int_X f d\mu$$

In particular, if $\int_X f d\mu < \infty$, then the sets $\{x \in X : f(x) \geq \lambda\}$ have finite measure for each $\lambda > 0$.

7. *(**Finiteness**) If $\int_X f d\mu < \infty$, then $f(x)$ is **finite** for μ-**almost every** x.*

8. *(**Vanishing**) If $\int_X f d\mu = 0$, then $f(x)$ is zero for μ-almost every x.*

9. *(**Vertical truncation**) We have*

$$\lim_{n\to\infty}\int_X \min\{f, n\}\, d\mu = \int_X f d\mu$$

10. *(**Horizontal truncation**) If $E_1 \subseteq E_2 \subseteq \ldots$ is an **increasing sequence** of $\mathscr{B}$-measurable sets, then*

$$\lim_{n\to\infty}\int_X f\mathbb{1}_{E_n} d\mu = \int_X f\mathbb{1}_{\cup_{n=1}^{\infty}E_n} d\mu.$$

11. *(**Restriction**) If Y is a measurable subset of X, then*

$$\int_X f\mathbb{1}_Y d\mu = \int_Y f|_Y d\mu|_Y,$$

*where $f|_Y : Y \to [0, +\infty]$ is the **restriction** of $f : X \to [0, +\infty]$ to Y, and $\mu|_Y$ is the restriction μ on Y. We will often abbreviate $\int_Y f|_Y d\mu|_Y$ (by slight abuse of notation) as $\int_Y f d\mu$.*

- **Theorem 2.8** *Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f, g : X \to [0, +\infty]$ be measurable. Then*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

- **Proposition 2.9** *(**Linearity in** μ).*
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f : X \to [0, +\infty]$ be measurable.

  1. *$\int_X f d(c\mu) = c \times \int_X f d\mu$ for every $c \in [0, +\infty]$.*

  2. *If $\mu_1, \mu_2, \ldots$ are a sequence of measures on $\mathscr{B}$,*

$$\int_X f d\left(\sum_{n=1}^{\infty}\mu_n\right) = \sum_{n=1}^{\infty}\int_X f d\mu_n.$$

- **Proposition 2.10** *(**Pushforward Measure**).*
*Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $\varphi : X \to Y$ be $(\mathscr{B}, \mathscr{C})$ measureable from $(X, \mathscr{B})$ to another measurable space $(Y, \mathscr{C})$. Define the **pushforward** $\phi_*\mu : \mathscr{C} \to [0, +\infty]$ of μ **by** φ by the formula*

$$\varphi_*\mu(E) := \mu(\phi^{-1}(E)).$$

1. $\varphi_* \mu$ *is a* ***measure*** *on* $\mathscr{C}$, *so that* $(Y, \mathscr{C}, \phi_* \mu)$ *is a measure space.*

2. *(**Change of variables formula**). If* $f : Y \to [0, +\infty]$ *is* $\mathscr{C}$*-measurable, then*

$$\int_Y f d(\phi_* \mu) = \int_X (f \circ \phi) d\mu.$$

- **Corollary 2.11** *Let* $T : \mathbb{R}^d \to \mathbb{R}^d$ *be an invertible linear transformation, and let* m *be Lebesgue measure on* $\mathbb{R}^d$. *Then* $T_* m = \frac{1}{|\det T|} m$, *where* $T_* m$ *is* ***the pushforward of*** m.

- **Example** (***Sums as integrals***). Let X be an arbitrary set (with the ***discrete*** σ***-algebra***), let $\#$ be ***counting measure***, and let $f : X \to [0, +\infty]$ be an arbitrary unsigned function. Then f is ***measurable*** with

$$\int_X f d\# = \sum_{x \in X} f(x).$$

2.4 Absolutely Convergent Integral

- **Definition** (***Absolutely convergent integral***).
Let $(X, \mathscr{F}, \mu)$ be a measure space. A *measurable function* $f : X \to \mathbb{C}$ is said to be ***absolutely integrable*** if the *unsigned integral*

$$\|f\|_{L^1(X)} \equiv \int_X |f| \, d\mu$$

is ***finite***. We refer to this quantity $\|f\|_{L^1(X)}$ as ***the*** $L^1(X)$ ***norm of*** f, and use $L^1(X)$ or $L^1(X, \mathscr{F}, \mu)$ or $L^1(\mu)$ to denote the space of absolutely integrable functions. If f is *real-valued* and absolutely integrable, we define ***the Lebesgue integral*** $\int_X f d\mu$ by the formula

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu$$

where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ are the magnitudes of the positive and negative components of f. (note that the two unsigned integrals on the right-hand side are finite, as f_+, f_- are pointwise dominated by $|f|$). If f is *complex-valued* and absolutely integrable, we define ***the Lebesgue integral*** $\int_X f(x) d\mu$ by the formula

$$\int_X f d\mu = \int_X \Re(f) d\mu + i \int_X \Im(f) d\mu,$$

where the two integrals on the right are interpreted as real-valued absolutely integrable Lebesgue integrals.

- **Remark** Sometimes $\int_X f d\mu$ is also denoted as $\int_X f(x) \mu(dx)$ or $\int_X f(x) d\mu(x)$, where $X \subseteq \mathbb{R}^d$ and $\mu(E) = \int_E \mu dx$.

- **Proposition 2.12** *(**Properties of absolutly convergent integral**)*
Let $(X, \mathscr{B}, \mu)$ *be a measure space.*

1. $L^1(X, \mathscr{B}, \mu)$ *is a* ***complex vector space***.

2. *The integration map $f \mapsto \int_X f d\mu$ is a* ***complex linear map*** *from $L^1(X, \mathscr{B}, \mu)$ to $\mathbb{C}$.*
3. *The* ***triangle inequality***
$$\|f + g\|_{L^1(\mu)} \le \|f\|_{L^1(\mu)} + \|g\|_{L^1(\mu)}$$
and the ***homogeneity property***
$$\|c\, f\|_{L^1(\mu)} = |c|\, \|f\|_{L^1(\mu)}$$
hold for all $f, g \in L^1(X, \mathscr{B}, \mu)$ and $c \in \mathbb{C}$.
4. *If $f, g \in L^1(X, \mathscr{B}, \mu)$ are such that $f(x) = g(x)$ for μ-almost every $x \in X$, then $\int_X f d\mu = \int_X g d\mu$.*
5. *If $f \in L^1(X, \mathscr{B}, \mu)$, and $(X, \mathscr{B}', \mu')$ is a* ***refinement*** *of $(X, \mathscr{B}, \mu)$, then $f \in L^1(X, \mathscr{B}', \mu')$, and*
$$\int_X f d\mu' = \int_X f d\mu.$$
(Hint: it is easy to get one inequality. To get the other inequality, first work in the case when f is both bounded and has finite measure support (i.e. is both vertically and horizontally truncated).)
6. *If $f \in L^1(X, \mathscr{B}, \mu)$, then $\|f\|_{L^1(\mu)} = 0$ if and only if f is zero μ-almost everywhere.*
7. *If $Y \subseteq X$ is $\mathscr{B}$-measurable and $f \in L^1(X, \mathscr{B}, \mu)$, then $f|_Y \in L^1(Y, \mathscr{B}|_Y, \mu|_Y)$ and*
$$\int_Y f|_Y \, d\mu|_Y = \int_X f \mathbb{1}_Y \, d\mu.$$
As before, by abuse of notation we write $\int_Y f d\mu$ for $\int_Y f|_Y \, d\mu|_Y$.

2.5 The Convergence Theorems

- **Proposition 2.13** *(**Uniform Convergence on a Finite Measure Space**). [Tao, 2011] Suppose that $(X, \mathscr{B}, \mu)$ is a* ***finite measure space*** *(so $\mu(X) < \infty$), and $f_n : X \to [0, +\infty]$ (resp. $f_n : X \to \mathbb{C}$) are a sequence of unsigned measurable functions (resp. absolutely integrable functions) that* ***converge uniformly*** *to a limit f. Then $\int_X f_n d\mu$* ***converges*** *to $\int_X f d\mu$.*

  Proof: Since $f_n \to f$ uniformly, we have for all $\epsilon > 0$, $\exists N$, for $n \ge N$, $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon$, thus
$$\sup_{x \in X} \left(||f_n(x)| - |f(x)||\right) \le \sup_{x \in X} \left(|f_n(x) - f(x)|\right) < \epsilon$$
  So $|f_n| \to |f|$ and f is absolutely integrable if f_n is absolutely integrable
$$\int_X |f| \, d\mu = \int_X |f - f_n + f_n| \, d\mu \le \int_X |f - f_n| \, d\mu + \int_X |f_n| \, d\mu \le \epsilon \mu(X) + \int_X |f_n| < \infty$$
  To prove convergence, see that we can choose N so that $\sup_{x \in X} |f_n(x) - f(x)| < \epsilon / \mu(X)$ since $\mu(X) < \infty$, then
$$\left| \int_X f_n d\mu - \int_X f d\mu \right| = \left| \int_X (f_n - f) d\mu \right| \le \int_X |f_n - f| \, d\mu \le \epsilon / \mu(X) \int_X d\mu = \epsilon. \qquad \blacksquare$$

- **Theorem 2.14** *(**Monotone Convergence Theorem**). [Tao, 2011]*
  *Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $0 \le f_1 \le f_2 \le \dots$ be a **monotone non-decreasing** sequence of **unsigned** measurable functions on X. Then we have*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \left(\lim_{n\to\infty} f_n \right) d\mu$$

Proof: Let $f \equiv \lim_{n\to\infty} f_n = \sup_n f_n$, then $f : X \to [0, +\infty]$ is measurable. Since the f_n are non-decreasing to f, we see from monotonicity that $\int_X f_n d\mu$ are non-decreasing and bounded above $\int_X f d\mu$, which gives the bound

$$\lim_{n\to\infty} \int_X f_n d\mu \le \int_X f d\mu.$$

It remains to establish the reverse inequality

$$\int_X f d\mu \le \lim_{n\to\infty} \int_X f_n d\mu.$$

By definition, it is equivalent to show that

$$\int_X g d\mu \le \lim_{n\to\infty} \int_X f_n d\mu.$$

for any $0 \le g \le f$ simple function. By horizontal truncation we may assume without loss of generality that g is also finite everywhere. Thus by canonical representation

$$g = \sum_{i=1}^{m} c_i \mathbb{1}\left\{E_i\right\}$$

for some m, $c_1, \dots, c_m \in (0, \infty)$ and $E_1, \dots, E_m$ being $\mathscr{F}$-measureable. The integral

$$\int_X g d\mu = \sum_{i=1}^{m} c_i \mu\left\{E_i\right\}.$$

Let $0 < \epsilon < 1$ be arbitrary. Then we have $f(x) = \sup_n \{f_n(x)\} > (1-\epsilon)c_i$ for all $x \in E_i$. Thus, if we define the sets

$$E_{i,n} = \{x \in E_i : f_n(x) > (1-\epsilon)c_i\}$$

then the $E_{i,n}$ increase to E_i and are measurable. By upwards monotonicity of measure, we conclude that

$$\lim_{n\to\infty} \mu(E_{i,n}) = \mu(E_i),\ 1 \le i \le m.$$

On the other hand, observe the pointwise bound

$$f_n(x) \ge (1-\epsilon) \sum_{i=1}^{m} c_i \mathbb{1}\left\{E_{i,n}\right\}$$

hold for any n. Integrate both sides,

$$\int_X f_n d\mu \geq (1-\epsilon) \sum_{i=1}^{m} c_i \mu\left(E_{i,n}\right).$$

and take the limits $n \to \infty$,

$$\begin{aligned}\lim_{n\to\infty} \int_X f_n d\mu &\geq (1-\epsilon) \sum_{i=1}^{m} c_i \lim_{n\to\infty} \mu\left(E_{i,n}\right) \\ &= (1-\epsilon) \sum_{i=1}^{m} c_i \mu(E_i).\end{aligned}$$

Finally, send $\epsilon \to 0$, we have the required formula. ∎

- **Remark** Note that in the special case when each f_n is an indicator function $f_n = \mathbb{1}\{E_n\}$, this theorem collapses to the upwards monotone convergence property. Conversely, the upwards monotone convergence property will play a key role in the proof of this theorem.

- **Remark** Note that the result still holds if the monotonicity $f_n \leq f_{n+1}$ only holds almost everywhere rather than everywhere.

- **Corollary 2.15** ***(Tonelli's Theorem for Sums and Integrals)***
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots$ be a sequence of ***unsigned*** *measurable functions on X. Then*

$$\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu$$

- **Exercise 2.16** *Give an example to show that this corollary can fail if the f_n are assumed to be absolutely integrable rather than unsigned measurable, even if the sum $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for each x. (Hint: think about the three escapes to infinity.)*

- **Lemma 2.17** ***(Borel-Cantelli Lemma)***. *[Tao, 2011, Resnick, 2013]*
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $E_1, E_2, \ldots$ be a sequence of $\mathscr{B}$-measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then

$$\mu\left\{\limsup_{n\to\infty} E_n\right\} = 0.$$

That is, almost every $x \in X$ is contained in ***at most finitely many*** *of the E_n (i.e. $\{n \in \mathbb{N} : x \in E_n\}$ is finite for almost every $x \in X$).*

(Hint: Apply *Tonelli's theorem* to the indicator functions $\mathbb{1}_{E_n}$.)

Proof: Consider the indicator function $f_n = \mathbb{1}\{x \in E_n\}$, which is unsigned $\mathscr{B}$-measurable functions since E_n is $\mathscr{B}$-measurable. By Tonelli's theorem,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(E_n) &= \sum_{n=1}^{\infty} \int_X \mathbb{1}\left\{x \in E_n\right\} d\mu \\
&= \int_X \left(\sum_{n=1}^{\infty} \mathbb{1}\left\{x \in E_n\right\} \right) d\mu \\
&= \int_X \left(\mathbb{1}\left\{ x \in \bigcup_{n=1}^{\infty} E_n \right\} \right) d\mu \\
&= \mu\left(\bigcup_{n=1}^{\infty} E_n \right)
\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$. Then by *the downwards monotone convergence* property

$$\begin{aligned}
\mu\left(\limsup_{n\to\infty} E_n \right) &= \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n \right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} E_n \right) \\
&= \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) \\
&\leq \limsup_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) = 0,
\end{aligned}$$

since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\sum_{n=N}^{\infty} \mu(E_n) \to 0$ as $N \to \infty$. ∎

- When one *does not have monotonicity*, one can at least obtain an important inequality, known as *Fatou's lemma*:

  Corollary 2.18 ***(Fatou's Lemma).***
  Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to [0, \infty]$ be a sequence of unsigned measurable functions. Then

  $$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

  Proof: Write $F_N \equiv \inf_{n\geq N} f_n$ for each N. Then the F_N are measurable and non-decreasing, and hence by the monotone convergence theorem

  $$\begin{aligned}
  \int_X \left(\lim_{N\to\infty} F_N \right) d\mu &= \lim_{N\to\infty} \int_X F_N d\mu \\
  \Rightarrow \int_X \left(\liminf_{n\to\infty} f_n \right) d\mu &= \lim_{N\to\infty} \int_X \inf_{n\geq N} f_n d\mu \\
  &\leq \lim_{N\to\infty} \inf_{n\geq N} \int_X f_n d\mu \\
  &= \liminf_{n\to\infty} \int_X f_n d\mu
  \end{aligned}$$

The second last inequality holds since

$$\int_X \inf_{n\geq N} f_n d\mu \leq \int_X f_n d\mu, \ \ \forall\, n \geq N$$
$$\Rightarrow \int_X \inf_{n\geq N} f_n d\mu \leq \inf_{n\geq N} \int_X f_n d\mu,$$

which completes the proof. ∎

Remark Informally, *Fatou's lemma* tells us that when taking ***the pointwise limit*** of ***unsigned functions*** f_n, that mass $\int_X f_n d\mu$ can be ***destroyed in the limit*** (as was the case in the three key moving bump examples), *but* it ***cannot be created in the limit***. Of course the unsigned hypothesis is necessary here.

While this lemma was stated only for pointwise limits, the same general ***principle*** (*that mass can be destroyed, but not created, by the process of taking limits*) tends to hold for other "*weak*" notions of convergence.

- Finally, we give the other major way to shut down loss of mass via *escape to infinity*, which is to *dominate* all of the functions involved by an *absolutely convergent one*. This result is known as *the dominated convergence theorem*:

Theorem 2.19 ***(Dominated Convergence Theorem).***
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to \mathbb{C}$ be a sequence of measurable functions that converge ***pointwise μ-almost everywhere*** *to a measurable limit $f : X \to \mathbb{C}$. Suppose that there is an* ***unsigned absolutely integrable*** *function $G : X \to [0, +\infty]$ such that $|f_n|$ are pointwise μ-almost everywhere* ***bounded*** *by G for each n. Then we have*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof: By modifying f_n, f on a null set, we may assume without loss of generality that the f_n converge to f *pointwise everywhere* rather than *μ-almost everywhere*, and similarly we can assume that $|f_n|$ are *bounded* by G *pointwise everywhere* rather than *μ-almost everywhere*.

By taking real and imaginary parts we may assume without loss of generality that f_n, f are *real*, thus $-G \leq f_n \leq G$ *pointwise*. Of course, this implies that $-G \leq f \leq G$ *pointwise* also.

If we apply Fatou's lemma to the unsigned functions $f_n + G$, we see that

$$\int_X \left(\liminf_{n\to\infty} f_n + G\right) d\mu \leq \liminf_{n\to\infty} \int_X (f_n + G)\, dx$$
$$\Rightarrow \int_X (f + G)\, d\mu \leq \liminf_{n\to\infty} \int_X (f_n + G)\, dx$$
$$\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n dx \quad (\text{since } \int_X G dx < \infty)$$

Similarly, if we apply that lemma to the unsigned functions $G - f_n$, we obtain

$$-\int_X f d\mu \leq \liminf_{n\to\infty} -\int_X f_n dx \quad (\text{since } \int_X G dx < \infty)$$
$$\Rightarrow \int_X f d\mu \geq \limsup_{n\to\infty} \int_X f_n dx$$

It concludes that $\limsup_{n\to\infty} \int_X f_n dx = \liminf_{n\to\infty} \int_X f_n dx = \lim_{n\to\infty} \int_X f_n dx = \int_X f d\mu$. ∎

- **Remark** From the moving bump examples we see that this statement *fails* if there is no *absolutely integrable dominating function* G.

- **Remark** Note also that when each of the fn is an indicator function $f_n = \mathbb{1}_{E_n}$, the dominated convergence theorem collapses to *dominated convergence for sets* in previous chapter.

- **Proposition 2.20** ***(Almost dominated convergence)****.*
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to \mathbb{C}$ be a sequence of measurable functions that converge pointwise μ-almost everywhere to a measurable limit $f : X \to \mathbb{C}$. Suppose that there is an unsigned absolutely integrable functions $G, g_1, g_2, \ldots X \to [0, +\infty]$ such that the $|f_n|$ are pointwise μ-almost everywhere bounded by $G + g_n$, and that $\int_X g_n d\mu \to 0$ as $n \to \infty$. Then
$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

- **Exercise 2.21** ***(Defect version of Fatou's lemma)****.*
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to [0, +\infty]$ be a sequence of ***unsigned absolutely integrable functions*** *that converges* ***pointwise*** *to an absolutely integrable limit f. Show that*
$$\int_X f_n d\mu - \int_X f d\mu - \|f - f_n\|_{L^1(\mu)} \to 0$$
as $n \to \infty$. (Hint: Apply the dominated convergence theorem to $\min(f_n, f)$.) Informally, this result tells us that the gap between the left and right hand sides of Fatous lemma can be measured by the quantity $\|f - f_n\|_{L^1(\mu)}$.

- **Proposition 2.22** *Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $g : X \to [0, +\infty]$ be measurable. Then the function $\mu_g : \mathscr{B} \to [0, +\infty]$ defined by the formula*
$$\mu_g(E) := \int_X g\, \mathbb{1}_E d\mu = \int_E g d\mu$$
is a measure.

- *The monotone convergence theorem* is, in some sense, *a* ***defining property*** *of the unsigned integral*:

  Proposition 2.23 ***(Characterisation of the unsigned integral)****.*
  Let $(X, \mathscr{B})$ be a measurable space. $I : f \mapsto I(f)$ be a map from the space $U(X, \mathscr{B})$ of ***unsigned measurable functions*** *$f : X \to [0, +\infty]$ to $[0, +\infty]$ that obeys the following axioms:*

  1. ***(Homogeneity)*** *For every $f \in U(X, \mathscr{B})$ and $c \in [0, +\infty]$, one has $I(c\, f) = c\, I(f)$.*
  2. ***(Finite additivity)*** *For every $f, g \in U(X, \mathscr{B})$, one has $I(f + g) = I(f) + I(g)$.*
  3. ***(Monotone convergence)*** *If $0 \le f_1 \le f_2 \le \ldots$ are a* ***nondecreasing*** *sequence of unsigned measurable functions, then $I(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} I(f_n)$.*

  Then there exists a ***unique measure*** *μ on $(X, \mathscr{B})$ such that*
  $$I(f) = \int_X f d\mu, \quad \text{for all } f \in U(X, \mathscr{B}).$$
  Furthermore, μ is given by the formula $\mu(E) := I(\mathbb{1}_E)$ for all $\mathscr{B}$-measurable sets E.

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