Lecture 2: Topological Space and Continuous Functions

Tianpei Xie

Nov. 7th., 2022

Contents

1 Topological Spaces

1.1 Definitions

- **Definition** [Munkres, 2000]
  Let X be a set. ***A topology*** on X is *a collection* $\mathscr{T}$ of *subsets* of X, called ***open subsets***, satisfying

  1. X and $\emptyset$ are *open*.
  2. The ***union*** of ***any family*** of open subsets is open.
  3. The ***intersection*** of *any* ***finite*** *family* of open subsets is open.

  A pair $(X, \mathscr{T})$ consisting of a set X together with a topology $\mathscr{T}$ on X is called ***a topological space***.

- **Example** (***Discrete and Trivial Topology***)
  If X is any set, *the collection of* **all subsets** of X is a *topology* on X; it is called ***the discrete topology***.

  The collection consisting of X *and* 0 *only* is also a *topology* on X; we shall call it *the* ***indiscrete topology***, or ***the trivial topology***.

- **Example** (***The Finite Complement Topology***)
  Let X be a set; let $\mathscr{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ *either is* **finite** *or is* ***all of*** X. Then $\mathscr{T}_f$ is a topology on X, called ***the finite complement topology***.

  Both X and $\emptyset$ are in $\mathscr{T}_f$, since $X \setminus X = \emptyset$ is finite and $X \setminus \emptyset$ is all of X. If $\{U_\alpha\}$ is an *indexed family of nonempty elements* of $\mathscr{T}_f$, to show that $\cup_\alpha U_\alpha$ is in $\mathscr{T}_c$, we compute

  $$X \setminus \bigcup_{\alpha} U_\alpha = \bigcap_{\alpha} (X \setminus U_\alpha)$$

  The latter set is *finite* because each set $X \setminus U_\alpha$ is *finite*. If $U_1, \ldots, U_n$ are nonempty elements of $\mathscr{T}_f$, to show that $\cap_i U_i$ is in $\mathscr{T}_f$, we compute

  $$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

  The latter set is a *finite union of finite sets* and, therefore, *finite*. ∎

- **Definition** (***Comparable Topologies on the Same Set***)
  Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set X. If $\mathscr{T}' \supseteq \mathscr{T}$, we say that $\mathscr{T}'$ is ***finer*** (or ***stronger***) than $\mathscr{T}$; if $\mathscr{T}'$ **properly** *contains* $\mathscr{T}$, we say that $\mathscr{T}'$ is ***strictly finer*** than $\mathscr{T}$.

  We also say that $\mathscr{T}$ is ***coarser*** (or ***weaker***) than $\mathscr{T}'$, or ***strictly coarser***, in these two respective situations. We say $\mathscr{T}$ is ***comparable*** with $\mathscr{T}'$ if either $\mathscr{T}' \subseteq \mathscr{T}$ or $\mathscr{T} \subseteq \mathscr{T}'$.

- **Remark** ***Topology*** of a set X defines ***all local information*** we know regarding a set. For each point $x \in X$, it specifies what do we mean by a "***neighborhood***" U of x. Thus properties that relies on the ***local characteristic*** of the space likely depend on the topology of the space. Examples include *the continuity* of function, *the convergence properties* of sequence and *differential properties* of function.

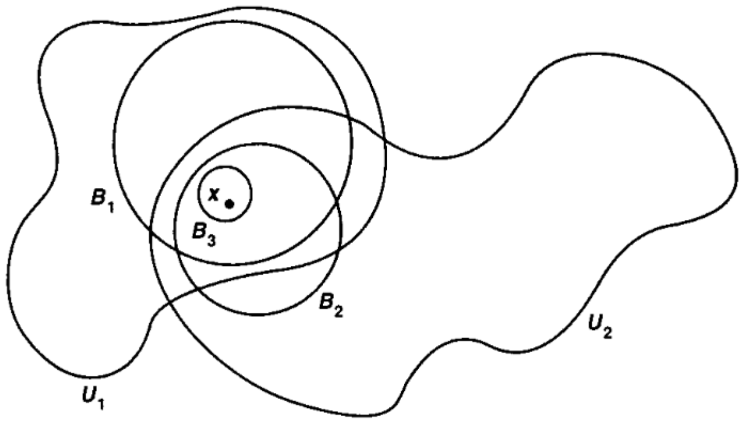

Figure 1: The basis of a topology [Munkres, 2000]

1.2 Basis for a Topology

- **Definition** Suppose X is a topological space. A collection $\mathscr{B}$ of open subsets of X is said to be ***a basis*** for *the topology of* X (plural: ***bases***) if every open subset of X is the *union of some collection of elements* of $\mathscr{B}$.

  More generally, suppose X is merely a set, and $\mathscr{B}$ is a collection of *subsets* of X satisfying the following conditions:

  1. $X = \bigcup_{B \in \mathscr{B}} B$.
  2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B2$.

  Then *the collection of* ***all unions*** *of elements of* $\mathscr{B}$ is a *topology* $\mathscr{T}$ on X, called ***the topology*** $\mathscr{T}$ ***generated by*** $\mathscr{B}$, and $\mathscr{B}$ is a ***basis*** for this *topology*.

- **Remark** (***Basis Element in Each Neighborhood***)
  By definition, a subset U of X is said to be ***open*** in X (that is, to be an element of $\mathscr{T}$) if for each $x \in U$, there exists a ***basis element*** $B \in \mathscr{B}$ such that $x \in B \subset U$. Note that each basis element is itself an element of $\mathscr{T}$.

- **Lemma 1.1** *Let X be a set; let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. Then $\mathscr{T}$ equals the collection of* ***all unions*** *of elements of $\mathscr{B}$.*

- **Remark** This lemma states that every open set U in X can be expressed as a *union* of *basis elements*. This expression for U is ***not***, however, ***unique***.

- **Lemma 1.2** ***(Obtaining Basis from Given Topology)****. [Munkres, 2000]*
  Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then C is a basis for the topology of X.

- **Lemma 1.3** ***(Topology Comparison via Bases)****. [Munkres, 2000]*
  Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively, on X. Then the following are equivalent:

  1. $\mathscr{T}'$ is ***finer*** *than $\mathscr{T}$.*

2. *For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.*

- **Remark** The ***basis element*** of ***finer*** *topology* are always ***smaller*** than *the basis element* of *coaser topology* so the finer basis element should be included in coaser basis element.

- **Example** (***Topology*** *in* $\mathbb{R}$)
If $\mathscr{B}$ is the collection of all ***open intervals*** in the real line,

$$(a, b) = \{x : a < x < b\},$$

the topology generated by $\mathscr{B}$ is called ***the standard topology*** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

If $\mathscr{B}'$ is the collection of all *half-open* intervals of the form

$$[a, b) = \{x : a \leq x < b\},$$

where $a < b$, the topology generated by $\mathscr{B}'$ is called ***the lower limit topology*** *on* $\mathbb{R}$. When $\mathbb{R}$ is given *the lower limit topology*, we denote it by $\mathbb{R}_\ell$.

Finally let K denote *the set of all numbers of the form* $1/n$, for $n \in \mathbb{Z}_+$,and let $\mathscr{B}''$ be the collection of *all open intervals* (a, b), along with *all sets of the form* $(a, b) \setminus K$. The topology generated by $\mathscr{B}$" will be called ***the K-topology*** *on* $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote $\mathbb{R}_K$

Lemma 1.4 *The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are* ***strictly finer*** *than* ***the standard topology*** *on $\mathbb{R}$, but are not comparable with one another.*

- **Definition** (***Subbasis***)
A subbasis $\mathscr{S}$ for a *topology* on X is a collection of subsets of X whose union equals X. The topology generated by the *subbasis* $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of ***all unions*** *of* ***finite intersections*** *of elements of* $\mathscr{S}$.

- **Remark** (***Basis from Subbasis***)
For a *subbasis* $\mathscr{S}$, the collection $\mathscr{B}$ of ***all finite intersections*** of elements of $\mathscr{S}$ is a ***basis***,

1.3 The Order Topology

- **Example** (***Order Topology***)
If X is ***a simply ordered set***, there is a ***standard topology*** for X, defined using the order relation. It is called ***the order topology***. The order topology is generated by ***intervals***.

- **Definition** (***Intervals based on Simple Order Relation***)
Suppose that X is a set having *a simple order relation* $<$. Given elements a and b of X such that $a < b$, there are *four subsets* of X that are called ***the intervals*** determined by a and b. They are the following :

$$(a, b) = \{x : a < x < b\},$$
$$(a, b] = \{x : a < x \leq b\},$$
$$[a, b) = \{x : a \leq x < b\},$$
$$[a, b] = \{x : a \leq x \leq b\}.$$

A set of the *first* type is called ***an open interval*** in X, a set of the *last* type is called ***a closed interval*** in X, and sets of *the second and third* types are called ***half-open intervals***.

- **Definition** Let X be a set with a ***simple order relation***; assume X has more than one element. Let $\mathscr{B}$ be the collection of all sets of the following types:

  1. ***All open intervals*** (a, b) in X.
  2. ***All intervals of the form*** $[a_0, b)$, where a_0 is the ***smallest element*** (if any) of X.
  3. ***All intervals of the form*** $(a, b_0]$, where b_0 is the ***largest element*** (if any) of X.

  The collection $\mathscr{B}$ is a basis for a topology on X, which is called ***the order topology***.

- **Definition** (***Rays***)
  If X is an ordered set, and a is an element of X, there are four subsets of X that are called ***the rays*** *determined by* a. They are the following:

  $$(a, +\infty) = \{x : x > a\},$$
  $$(-\infty, a) = \{x : x < a\},$$
  $$[a, +\infty) = \{x : x \geq a\},$$
  $$(-\infty, a] = \{x : x \leq a\}.$$

  Sets of the first two types are called ***open rays***, and sets of the last two types are called ***closed rays***.

- **Remark** The ***open rays*** in X are *open sets* in ***the order topology***. In fact, ***the open rays*** *form a* ***subbasis*** *for* ***the order topology*** *on* X.

1.4 The Product Topology

- **Definition** (***Product Topology***)
  Let X and Y be topological spaces. ***The product topology*** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

- **Proposition 1.5** ***(Basis of Product Topology)***
  If $\mathscr{B}$ is a **basis** *for the topology of X and $\mathscr{C}$ is a* **basis** *for the topology of Y, then the collection*

  $$\mathscr{D} = \{B \times C : B \in \mathscr{B},\ \text{and } C \in \mathscr{C}\}$$

  is a **basis** *for the topology of $X \times Y$.*

- It is sometimes useful to express the product topology in terms of a *subbasis*. To do this, we first define certain functions called *projections*.

  Definition Let $\pi_1 : X \times Y \to X$ be defined by the equation

  $$\pi_1(x, y) = x;$$

  $\pi_2 : X \times Y \to Y$ he defined by the equation

  $$\pi_2(x, y) = y.$$

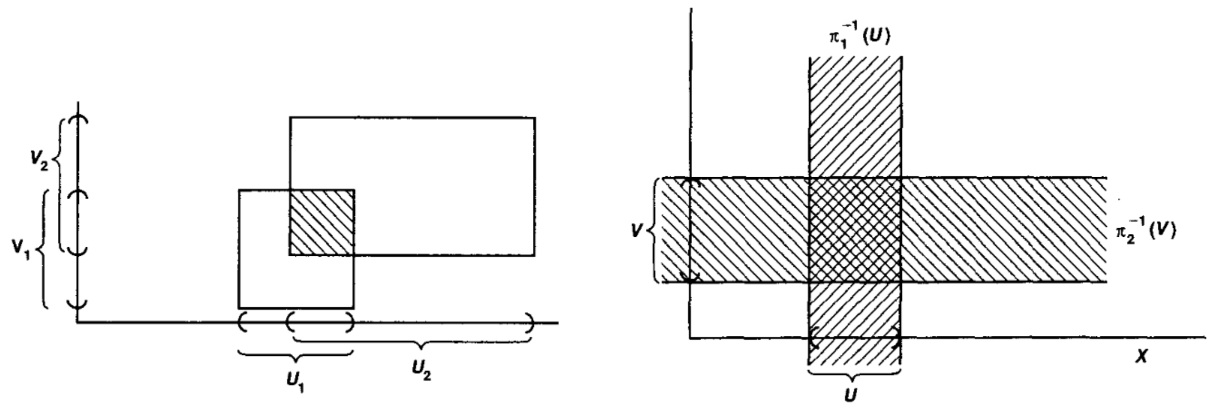

Figure 2: (Left) The basis of a product topology (Right) The subbasis of a product topology [Munkres, 2000]

The maps π_1 and π_2 are called *the* ***projections*** *of* $X \times Y$ ***onto*** *its* ***first*** *and* ***second*** *factors*, respectively.

- **Remark** Both π_1 and π_2 are ***surjective***. If U is an ***open*** subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is ***open*** in $X \times Y$.

  Similarly, if V is ***open*** in Y, then $\pi_2^{-1}(V) = X \times V$ which is also ***open*** in $X \times Y$.

- **Proposition 1.6** ***(Subbasis of Product Topology)***
  The collection

$$\mathscr{S} = \left\{\pi_1^{-1}(U) : U \textit{ open in } X\right\} \cup \left\{\pi_2^{-1}(V) : V \textit{ open in } Y\right\}$$

  is a **subbasis** *for* ***the product topology*** *on* $X \times Y$*.*

1.5 The Subspace Topology

- **Definition** If $(X, \mathscr{T})$ is a topological space and $S \subseteq X$ is an arbitrary subset, we define ***the subspace topology*** on S (sometimes called *the* ***relative topology***) as

$$\mathscr{T}_S = \{S \cap U : U \in \mathscr{T}\}$$

  That is, a subset $U \subseteq S$ to be *open* in S *if and only* if there exists an *open* subset $V \subseteq X$ such that $U = V \cap S$. Any subset of X endowed with *the subspace topology* is said to be ***a subspace of*** X.

- **Lemma 1.7** ***(Basis of Subspace Topology)***
  If $\mathscr{B}$ *is a basis for the topology of* X *then the collection*

$$\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$$

  is a ***basis*** *for the subspace topology on* $S \subset X$*.*

- **Remark** (***Open Relative to Which Set ?***)
  When dealing with a space X and a *subspace* Y, one needs to be careful when one uses the term "*open set*". Does one mean *an element of* ***the topology of*** Y or *an element of* ***the***

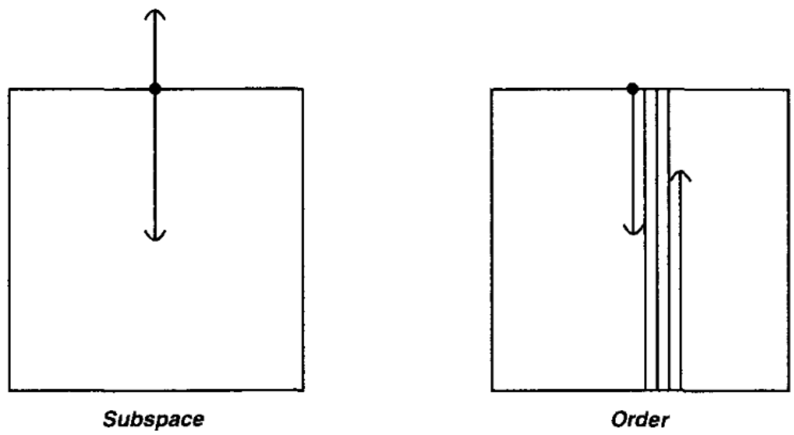

Figure 3: (Left) The subspace topology inherited from amient space (Right) The order topology on subspace [Munkres, 2000]

topology of X ? We make the following definition : If Y is a subspace of X, we say that ***a set*** U ***is open in*** Y (or *open* ***relative*** *to* Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U ***is open in*** X if it belongs to the topology of X.

- **Lemma 1.8** ***(Open Subspace)***
  Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

- **Proposition 1.9** ***(Product of Subspace Equal to Subspace of Product)*** *[Munkres, 2000]*
  If A is a subspace of X and B is a subspace of Y, then ***the product topology*** *on $A \times B$ is the same as the topology $A \times B$ inherits as a* ***subspace*** *of $X \times Y$.*

- **Remark** (***Subspace Topology $\neq$ Order Topology on Subspace***)
  Now let X be an ordered set in *the order topology*, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. *However, the resulting order topology on Y* ***need not be the same*** *as the topology that Y inherits as a subspace of X.*

  Let $I = [0, 1]$. The *dictionary order* on $I \times I$ is just *the restriction to $I \times I$ of the dictionary order* on the plane $\mathbb{R} \times \mathbb{R}$. However, ***the dictionary order topology*** on $I \times I$ is ***not the same*** as ***the subspace topology*** on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

  For example, the set $\{1/2\} \times (1/2, 1]$ is *open* in $I \times I$ in *the subspace topology*, but *not* in *the order topology*, as you can check. See Figure 3. The set $I \times I$ in *the dictionary order topology* will be called ***the ordered square***, and denoted by l_o^2.

- **Definition** Given an ordered set X, let us say that a subset Y of X is ***convex*** in X if for each pair of points $a < b$ of Y, ***the entire interval*** (a, b) of points of X *lies in* Y. Note that *intervals* and *rays* in X are *convex* in X.

- **Proposition 1.10** ***(Convex Subspace Preserve Order Topology)****[Munkres, 2000]*
  Let X be an ordered set in the order topology; let Y be a subset of X that is ***convex*** *in X.*

*Then **the order topology on** Y is the same as the topology Y inherits as a **subspace** of X.*

2 Closed Sets and Limit Points

2.1 Closed Sets

- **Definition** A subset A of a topological space X is said to be ***closed*** if the set $X \setminus A$ is *open*.
- **Proposition 2.1** *Let X be a topological space. Then the following conditions hold:*
  1. *$\emptyset$ and X are **closed**.*
  2. ***Arbitrary intersections** of **closed** sets are **closed**.*
  3. ***Finite unions** of **closed** sets are **closed**.*
- **Remark** When dealing with ***subspaces***, one needs to be careful in using the term "***closed set***." If Y is a subspace of X, we say that a set A is ***closed in*** Y if A is a subset of Y and if A is ***closed*** in ***the subspace topology*** of Y (that is, if $Y \setminus A$ is *open* in Y).

  Proposition 2.2 ***(Closed Set in Subspace Topology)***
  Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.
- **Remark** A set A that is ***closed in*** the subspace Y may or may ***not be closed in*** the larger space X.

  Proposition 2.3 *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

2.2 Closure and Interior of a Set

- **Definition** Given a subset A of a topological space X, ***the interior of** A* is defined as *the union of all open sets **contained*** in A, and ***the closure of** A* is defined as *the intersection of all closed sets **containing*** A.

  The interior of A is denoted by Int A or by $\mathring{A}$ and ***the closure of*** A is denoted by CI A or by $\bar{A}$. Obviously $\mathring{A}$ is *an open set* and $\bar{A}$ is *a closed set*; furthermore,

  $$\mathring{A} \subseteq A \subseteq \bar{A}.$$

  If A is ***open***, $A = \mathring{A}$; while if A is ***closed***, $A = \bar{A}$.
- **Proposition 2.4** ***(Closure in Subspace Topology)****[Munkres, 2000]*
  Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.
- **Remark** The definition of the closure of a set does not give us a convenient way for actually finding the closures of specific sets, since the collection of all closed sets in X, like the collection of all open sets, is usually much too big to work with. In the following theorem, we describe it using only the basis: Note that *a set A **intersects** a set B* if the intersection $A \cap B$ is *not empty.*

Proposition 2.5 ***(Characterization of Closure in terms of Basis)*** *[Munkres, 2000]*
Let A be a subset of the topological space X.

1. *Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.*
2. *Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.*

- **Remark** We can say "*U is a **neighborhood** of x*" if "*U is an open set containing x*".

2.3 Limit Points

- **Definition** (***Limit Point***)
  If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point*** (or "***cluster point***," or "***point of accumulation***") of A if ***every neighborhood*** *of x* ***intersects*** *A in some point* ***other than*** *x itself.*

  Said differently, x is ***a limit point*** of A if it belongs to ***the closure of*** $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.

- **Theorem 2.6** ***(Decomposition of Closure)***
  Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then

$$\bar{A} = A \cup A'.$$

- **Corollary 2.7** *A subset of a topological space is* ***closed*** *if and only if it contains all its* ***limit points.***

- **Definition** A topological space is called ***Hausdorff*** (or T_2) if and only if for all all x and y, $x \neq y$, there are ***open sets*** U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

- **Proposition 2.8** *Every* ***finite*** *point set in a* ***Hausdorff*** *space X is* ***closed****.*

- **Proposition 2.9** ***(Limit Point in T_1 Axiom)****. [Munkres, 2000]*
  Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is ***a limit point*** *of A if and only if every* ***neighborhood*** *of x contains* ***infinitely many points*** *of A.*

- **Proposition 2.10** ***(Limit Point is Unique in Hausdorff Space)****. [Munkres, 2000]*
  If X is a ***Hausdorff space****, then a sequence of points of X* ***converges to at most one point*** *of X.*

3 Continuous Functions

3.1 Continuity of a Function

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Remark** ***Continuity of a function*** depends *not only upon* ***the function** f **itself***, but also *on the* ***topologies*** *specified for* ***its domain*** *and* ***range***. If we wish to emphasize this fact, we can say that *f is* ***continuous relative to*** *specific topologies on X and Y.*

- **Remark** (***Prove Continuity via Basis***)
  If the topology of ***the range space*** Y is given by a ***basis*** $\mathscr{B}$, then to prove ***continuity of*** f it suffices to show that *the* ***inverse image*** *of every* ***basis element*** *is* ***open***: The arbitrary open set V of Y can be written as *a union of basis elements*

$$V = \bigcup_{\alpha \in J} B_\alpha$$
$$\Rightarrow f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$$

- **Remark** (***Prove Continuity via Subbasis***)
  If the topology on Y is given by ***a subbasis*** $\mathscr{S}$, to prove continuity of f it will even suffice to show that ***the inverse image*** *of each* ***subbasis*** *element is* ***open***: The arbitrary basis element B for Y can be written as ***a finite intersection*** $S_1 \cap \ldots \cap S_n$ of subbasis elements; it follows from the equation

$$f^{-1}(B) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n)$$

  that the inverse image of every basis element is *open*.

- **Example** (***$\mathscr{F}$-Weak Topology using Continuity Only***)
  One can ***define a topology just*** based on ***the notion of continuity*** from a family of functions. Let $\mathscr{F}$ be a family of functions from a set S to a topological space $(X, \mathscr{T})$. The ***$\mathscr{F}$-weak*** *(or simply* ***weak****)* ***topology*** on S is *the* ***coarest topology*** for which ***all the functions*** $f \in \mathscr{F}$ *are* ***continuous***.

  The ***$\mathscr{F}$-weak*** *topology* $\mathscr{T}$ is generated by ***subbasis*** $\mathscr{S}$ of the preimage sets $S = f^{-1}(U)$ where $f \in \mathscr{F}$ and $U \in \mathscr{T}$. And the basis of $\mathscr{T}$ is *the collection* of ***all finite intersections*** of preimages $f^{-1}(U)$ for $f \in \mathscr{F}$ and $U \in \mathscr{T}$.

- **Proposition 3.1** ***(Equivalent Definition of Continuity)*** *[Munkres, 2000]*
  Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

  1. *f is* ***continuous***.
  2. *For every subset A of X, one has $f(\bar{A}) \subseteq \overline{f(A)}$.*
  3. *For every* ***closed*** *set B of Y, the set $f^{-1}(B)$ is* ***closed*** *in X.*
  4. *For* ***each*** *$x \in X$ and each* ***neighborhood*** *V of $f(x)$, there is a* ***neighborhood*** *U of X such that $f(U) \subseteq V$.*

  If the condition in (4) holds for the point x of X, we say that ***f is continuous at the point x***.

3.2 Homeomorphisms

- **Definition** (***Homemorphism***)
  A ***continuous bijective*** map $f : X \to Y$ with ***continuous inverse***

$$f^{-1} : Y \to X$$

  is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Remark** (***Homemorphism is Topological Equivalence***)
  A ***homeomorphism*** $f : X \to Y$ gives us a *bijective correspondence* not only between X and Y but between *the collections of open sets of X and of Y*. As a result, *any* ***property*** *of X that is* ***entirely expressed in terms of the topology*** *of X* (that is, in terms of *the open sets* of X) ***yields***, via the correspondence f, *the* ***corresponding property*** *for the space Y*.

  Such a *property* of X is called *a* ***topological property*** *of X*. A *homemorphism* is an ***isomorphism*** between topological space, i.e. it ***preserves the topological structure*** during the transformation.

- **Remark** (***Isomorphism***)
  For vector space, an ***(linear) isomorphism*** is a ***bijective linear mapping*** from *one vector spaces* to *another vector space* that ***preserve*** the ***structure*** of that vector space. However, depending on definition of specific structure, we can have various different definition of isomorphisms:

  1. For ***metric space***, an *isomorphism* is a *bijective linear operator* that ***preserves the metric***. It is often called an ***isometry***.

  2. For ***inner product space***, an *isomorphism* is a *surjective linear operator* that ***preserves the inner product***. It is often called an ***surjective isometry***.

  3. For ***linear algebra***, an *isomorphism* is a ***bijective linear mapping*** that ***preserves all algebraic operations*** (i.e. *the vector addition* and *scalar multiplication*).

  In general, ***isomorphism*** is a ***structure-preserving mapping*** between two structures of the same type that *can be* ***reversed*** by ***an inverse mapping***. It means that "***two spaces are essentially of the same form***". For instance, the followings are also called ***isomorphism*** depending on the context:

  1. ***homemorphism*** between *topological spaces*,

  2. ***diffeomorphism*** between *smooth manifolds*,

  3. ***bijective homomorphism*** between *algebraic groups / rings / fields*,

  4. ***graph isomorphism*** between *graphs* that preseves the edge structure,

  Also an isomorphism is called a ***transformation*** in ***geometry***, e.g. *rigid transformation*, *affine transformation* etc.

- **Definition** (***Topological Embedding***)
  If X and Y are topological spaces, a ***continuous injective*** map $f : X \to Y$ is called a ***topological embedding*** if it is a ***homeomorphism*** *onto* its image $f(X) \subseteq Y$ *in the subspace topology* (i.e. $f^{-1}|_{f(X)} : f(X) \to X$ is *continuous in Y*).

- **Remark** (***Smooth Embedding***)
  If X and Y are smooth manifoolds, ***a smooth embedding*** $f : X \to Y$ when it is a **topological embedding**, and it is *smooth map* with *injective differential df_x* for all $x \in X$ (called a ***smooth immersion***).

3.3 Constructing Continuous Functions

- **Proposition 3.2** ***(Rules for Constructing Continuous Functions)****. [Munkres, 2000]* *Let X, Y, and Z be topological spaces.*

  1. ***(Constant Function)*** *If $f : X \to Y$ maps* ***all*** *of X into the* ***single point*** *y_0 of Y, then f is* ***continuous****.*

  2. ***(Inclusion)*** *If A is a subspace of X, the* ***inclusion function*** *$\iota : A \hookrightarrow X$ is* ***continuous****.*

  3. ***(Composites)*** *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous*

  4. ***(Restricting the Domain)*** *If $f : X \to Y$ is continuous, and if A is a subspace of X, then* ***the restricted function*** *$f|_A : A \to Y$ is continuous.*

  5. ***(Restricting or Expanding the Range)*** *Let $f : X \to Y$ be continuous. If Z is a* ***subspace*** *of Y containing the* ***image*** *set $f(X)$, then the function $g : X \to Z$ obtained by* ***restricting the range*** *of f is continuous. If Z is a space having Y as a* ***subspace****, then the function $h : X \to Z$ obtained by* ***expanding the range*** *of f is continuous.*

  6. ***(Local Formulation of Continuity)*** *The map $f : X \to Y$ is* ***continuous*** *if X can be written as the* ***union of open sets*** *U_α such that $f|_{U_\alpha}$ is* ***continuous*** *for each α.*

- **Theorem 3.3** ***(The Pasting Lemma / Gluing Lemma)****. [Munkres, 2000]* *Let $X = A \cup B$, where A and B are* ***closed*** *in X. Let $f : A \to Y$ and $g : B \to Y$ be* ***continuous****. If $f(x) = g(x)$ for* ***every*** *$x \in A \cap B$, then f and g combine to give a* ***continuous function*** *$h : X \to Y$, defined by setting $h|_A = f$, and $h|_B = g$.*

- **Remark** The set A and B can be open sets, and the gluing lemma comes "***Local Formulation of Continuity***".

- **Remark** Notice the condition for *the gluing lemma*:

  1. The domain X is a union of two ***closed sets (or open sets)*** A and B

  2. The two functions f and g are ***continuous*** each of closed domain sets, respectively

  3. f and g ***agree on the intersection*** of two sets $A \cap B$.

- **Theorem 3.4** ***(Maps into Products)****. [Munkres, 2000]* *Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a)).$$

  Then f is ***continuous*** *if and only if the functions*

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

  are ***continuous****. The maps f_1 and f_2 are called* ***the coordinate functions*** *of f.*

- **Remark** There is no useful criterion for the *continuity* of a map $f : A \times B \to X$ whose ***domain is a product space***. One might conjecture that f *is continuous if it is continuous* "*in each variable separately,*" but ***this conjecture is not true***.

4 Topological Spaces (Continued.)

4.1 The Product Topology

- **Definition** (***J-tuples***)
  Let J be an index set. Given a set X, we define a ***J-tuple*** *of elements* of X to be a function $x : J \to X$. If α is an element of J, we often denote ***the value of*** X ***at*** α by X_α rather than $x(\alpha)$; we call it ***the*** α***-th coordinate*** of x. And we often *denote the function* x *itself* by the symbol

$$(x_\alpha)_{\alpha \in J}$$

  which is as close as we can come to a "*tuple notation*" for an arbitrary index set J. We denote ***the set of all*** J***-tuples*** of elements of X by X^J.

- **Definition** (***Arbitrary Cartestian Products***)
  Let $\{A_\alpha\}_{\alpha \in J}$ be an *indexed* family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. *The* ***cartesian product*** *of this indexed family*, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

  is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

  such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

- **Remark** The existance of just construction is due to *the Axioms of Choice* since J is an arbitrary set.

- **Remark** If $A_\alpha = X$ for all $\alpha \in J$, then we use the notation X^J to represent the cartestian product $\prod_{\alpha \in J} X$

- **Definition** (***Box Topology***)
  Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of ***topological spaces***. Let us take as a ***basis*** for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

  the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

  where U_α is ***open*** in X_α, for each $\alpha \in J$. The topology generated by this ***basis*** is called ***the box topology***.

- **Definition** (***Projection Mapping***)
  Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its β-th coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$

it is called ***the projection mapping*** *associated with the index* β.

- **Definition** (***Product Topology***)
  Let $\mathscr{S}_\beta$ denote the collection

$$\mathscr{S}_\beta = \left\{ \pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta \right\},$$

and let $\mathscr{S}$ denote *the union of these collections*,

$$\mathscr{S} = \bigcup_{\beta \in J} \mathscr{S}_\beta.$$

The topology generated by *the* ***subbasis*** S is called ***the product topology***. In this topology $\prod_{\alpha \in J} X_\alpha$ is called ***a product space***.

- **Remark** (***Product Topology = Weak Topology by Coordinate Projections***)
  The product topology on $\prod_{\alpha \in J} X_\alpha$ is ***the weak topology*** generated by *a family of projection mappings* $(\pi_\beta)_{\beta \in J}$. It is ***the coarest (weakest) topology*** *such that* $(\pi_\beta)_{\beta \in J}$ *are* ***continuous***.

  A typical element of the basis from *the product topology* is ***the finite intersection*** *of subbasis* where the *index is different*:

$$\pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \ldots \cap \pi_{\beta_n}^{-1}(V_{\beta_n})$$

Thus a ***neighborhood*** of x in ***the product topology*** is

$$N(x) = \{(x_\alpha)_{\alpha \in J} : x_{\beta_1} \in V_{\beta_1}, \ldots, x_{\beta_n} \in V_{\beta_n}\}$$

where there is ***no restriction*** for $\alpha \in \{\beta_1, \ldots, \beta_n\}$.

Note that for ***the box topology***, a *neighborhood* of x is

$$N_b(x) = \{(x_\alpha)_{\alpha \in J} : x_\alpha \in U_\alpha, \ \forall \alpha \in J\} \subset N(x)$$

Thus ***the box topology*** is ***finer*** than ***the product topology***. Moreover, *for* ***finite product*** $\prod_{\alpha=1}^n X_\alpha$, the box topology and the product topology is the ***same***.

- **Proposition 4.1** *(**Comparison of the Box and Product Topologies**). [Munkres, 2000]* *The box topology on $\prod_{\alpha \in J} X_\alpha$ has as basis all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is* ***open*** *in X_α* ***for each*** *α. The product topology on $\prod_{\alpha \in J} X_\alpha$ has as basis all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is* ***open*** *in X_α for each α and U_α* ***equals*** *X_α* ***except for finitely many values*** *of a.*

- **Remark** Whenever we consider the product $\prod_{\alpha \in J} X_\alpha$, we shall ***assume*** it is given ***the product topology*** unless we specifically state otherwise.

- **Proposition 4.2** ***(Basis for Box and Product Topology)***
  Suppose the topology on each space X_α is given by a basis $\mathscr{B}_\alpha$. The collection of all sets of the form

  $$\prod_\alpha B_\alpha$$

  where $B_\alpha \in \mathscr{B}_\alpha$ for ***each*** *α, will serve as a* ***basis*** *for* ***the box topology*** *on $\prod_{\alpha\in J} X_\alpha$.*

  The collection of all sets of the same form, where $B_\alpha \in \mathscr{B}_\alpha$ for ***finitely many indices*** *α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a* ***basis*** *for* ***the product topoiogy*** *$\prod_{\alpha\in J} X_\alpha$.*

- **Proposition 4.3** *Let A_α be a* ***subspace*** *of X_α, for each $\alpha \in J$. Then $\prod_\alpha A_\alpha$ is a* ***subspace*** *of $\prod_\alpha X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

- **Proposition 4.4** *If each space X_α is a* ***Hausdorff space****, then $\prod_\alpha X_\alpha$ is a* ***Hausdorff*** *space in both the box and product topologies.*

- **Proposition 4.5** *Let (X_α) be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod_\alpha X_\alpha$ is given either the product or the box topology, then*

  $$\prod_\alpha \bar{A}_\alpha = \overline{\prod_\alpha A_\alpha}$$

- **Proposition 4.6** ***(Maps into Aribitrary Products****). [Munkres, 2000]*
  Let $f : A \to \prod_\alpha X_\alpha$ is given by the equation

  $$f(x) = (f_\alpha(x))_{\alpha\in J}$$

  where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_\alpha X_\alpha$ be the ***product topology****. Then the function f is* ***continuous*** *if and only if each function f_α is* ***continuous****.*

- **Example** (***Maps into Aribitrary Products Not Hold in Box Topology***)
  Consider $\mathbb{R}^{omega}$, the *countabiy infinite product* of $\mathbb{R}$ with itself. Recall that

  $$\mathbb{R}^\omega = \prod_{n\in\mathbb{Z}_+} X_n$$

  where $X_n = \mathbb{R}$ for each n. Let us define a function $f : \mathbb{R} \to \mathbb{R}^\omega$ by the equation

  $$f(t) = (t, t, \ldots)$$

  the n-th coordinate function of f is the function $f_n(t) = t$. Each of the coordinate functions $f_n : \mathbb{R} \to \mathbb{R}$ is *continuous*; therefore, the function f is *continuous* if $\mathbb{R}^\omega$ is *given* ***the product topology***. But f is not continuous if $\mathbb{R}^{omega}$ is *given* ***the box topology***. Consider, for example, the basis element

  $$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \ldots$$

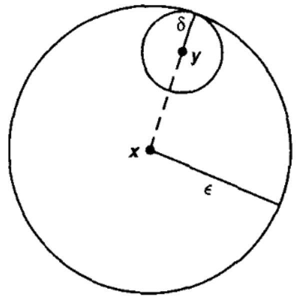

Figure 4: $B(y, \delta) \subset B(x, \epsilon)$ **for** $y \in B(x, \epsilon)$ **due to triangle inequality. [Munkres, 2000]**

11 11 for *the box topology*. We assert that $f^{-1}(B)$ is ***not open*** in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta, \delta)) \subset B$, so that, applying π_n to *both sides of the inclusion*,

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subset (-\frac{1}{n}, \frac{1}{n})$$

*for **all** n.* a contradiction. ∎

4.2 The Metric Topology

4.2.1 Metric Topology and Metrizability

- **Definition** (***Metric Space***)
  A ***metric space*** is a set M and a real-valued function $d(\cdot, \cdot) : M \times M \to \mathbb{R}$ which satisfies:

  1. (***Non-Negativity***) $d(x, y) \geq 0$
  2. (***Definiteness***) $d(x, y) = 0$ if and only if $x = y$
  3. (***Symmetric***) $d(x, y) = d(y, x)$
  4. (***Triangle Inequality***) $d(x, z) \leq d(x, y) + d(y, z)$

  The function d is called a ***metric*** on M. The metric space M equipped with metric d is denoted as (M, d).

- **Definition** (ϵ-***Ball***)
  Given a metric d on X, the number $d(x, y)$ is often called *the **distance** between x and y in the metric d.* Given $\epsilon > 0$, consider the set

  $$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

  of all points y whose distance from x is less than ϵ. It is called ***the ϵ-ball centered at x***. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

- **Definition** (***Metric Topology***)
  If d is a *metric* on the set X, then *the collection of all ϵ-balls $B_d(x, \epsilon)$*, for $x \in X$ and $\epsilon > 0$, is a ***basis*** for a *topology* on X, called ***the metric topology*** *induced by d.*

- **Remark** (***The Triangle Inequality is Necessary for Basis***)
  The triangle inequality conditon is a necessary condition for the ϵ-balls to form a *basis*. It guarantees that for any $y \in B(x, \epsilon)$, there exists a neighborhood of y, $B(y, \delta)$ such that $B(y, \delta) \subset B(x, \epsilon)$.

  Definition (***Open Set in Metric Topology***)
  A set U is ***open*** *in the metric topology* induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

- **Remark** (***Other "Metric-Like" Functions***)
  Among all algebraic properties that define a metric, ***the triangle inequality is the strongest***. There are many "*metric-like*" functions that do not satisfy the triangle inquality.

  1. ***divergence function***: $D = \mathbb{D}\left(\cdot \,\|\, \cdot\right) : M \times M \to \mathbb{R}_+$ satisfying for any $p, q \in M$

     $$\mathbb{D}\left(p \,\|\, q\right) > 0, \text{ and } \mathbb{D}\left(p \,\|\, q\right) = 0, \text{ iff } p = q.$$

     Divergence function ***does not satiesfy*** neither *the* ***symmetric*** *property* nor *the* ***triangle inequality*** *property*. But it satisfies *the* ***positive definiteness*** *property*. A divergence can act like ***a measure of closeness*** *between two points*.

  2. ***inner product***: for *a vector space* M, $\langle \cdot \,,\, \cdot \rangle : M \times M \to \mathbb{R}_+$ is a ***bilinear form*** that ***satiesfies*** both *the* ***symmetric*** *property* and *the* ***positive definiteness*** *property*. This makes it *almost a metric*. In fact, *every inner product can induce a* norm $\|x\| = \sqrt{\langle x \,,\, x \rangle}$ and thus it can induce a metric via norm $\|x - y\|$

  3. ***semi-norm***: for *a vector space* M, a semi-norm $q : M \to \mathbb{R}_+$ is a mapping that ***satiesfies*** both *the* ***homogeneity*** *property* and *the* ***triangle inequality*** *property*. But it does not satisfies ***the positive definiteness condition***. $q(x-y)$ is thus not a metric. It can also be used to *measure the closeness* between two points but *lack of power to tell if these two points are the same*.

- **Remark** (***Metric Topology is Quantitative***)
  A ***metric*** provides a measurement on the ***closeness*** between two points. ***The metric topology*** generated by open balls thus provides a ***quantitative descripition of the neighborhood*** and it answers the question "*how close the neighborhood of x is* ?" On the other hand, ***the general topology*** answer this question using ***qualitative description*** via ***comparison*** with other neighborhoods via the *inclusion* operation $\subset$. Note that inclusion $\subset$ is ***partially ordered***, while the metric maps onto the real line where $<$ is ***simply ordered***.

  The study of ***topology*** is to acknowledge that *in many areas of research,* ***there might not exist a properly defined metric*** *in the set of interest*. On the other hand, *the study of* ***analysis*** mainly focus on *the space equipped with metric topology*.

- **Definition** (***Metrizability***)
  If X is a topological space, X is said to be ***metrizable*** if *there exists a metric d* on the set X that *induces the topology* of X. ***A metric space*** is *a metrizable space X* together with a specific metric d that *gives the topology of X*.

- **Remark** (***Metrizability as Inverse Problem***)
  Given a *metric* d on X, we can generate *a metric topology* using ϵ-balls as basis. ***Conversely***, ***given a topology $\mathscr{T}$ on X, is $\mathscr{T}$ a metric topology for some unknown metric d ?***

This is the question that ***the metrization theory*** is trying to answer.

- **Remark** (***Metrizability is Valuable***)
  Many of the spaces important for mathematics are metrizable, but some are not. ***Metrizability*** is always a highly desirable attribute for a space to possess, for the existence of a *metric* gives one *a valuable tool* for *proving theorems* about the space.

- **Definition** Let X be a metric space with metric d. A subset A of X is said to be ***bounded*** if there is some number M such that

$$d(a_1, a_2) \leq M$$

  for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the ***diameter*** of A is defined to be the number

$$\operatorname{diam} A = \sup \{d(a_1, a_2) : \forall a_1, a_2 \in A\}.$$

- **Remark** The boundedness property depends on specific metric topology, thus it is not a topological property.

  For instance, the following metric guarantee that every open set is bounded.

  Definition (***Standard Bounded Metric***)
  Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

  Then $\bar{d}$ is a *metric* that induces ***the same topology*** as d.

  The metric $\bar{d}$ is called ***<u>the standard bounded metric</u>*** *corresponding to* d.

- **Definition** (***Euclidean Metric and Square Metric***)
  Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the ***norm*** of x by the equation

$$\|x\|_2 = \left(x_1^2 + \ldots + x_n^2\right)^{1/2};$$

  and we define ***<u>the euclidean metric</u>*** d on $\mathbb{R}^n$ by the equation

$$d(x, y) = \|x - y\|_2 = \left((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right)^{1/2}.$$

  We define ***<u>the square metric</u>*** ρ by the equation

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

- **Lemma 4.7** *Let d and d' be two metrics on the set X; let $\mathscr{T}$ and $\mathscr{T}'$ be the topologies they induce, respectively. Then $\mathscr{T}'$ is* ***finer*** *than $\mathscr{T}$ if and only if for each x in X and each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon).$$

- **Proposition 4.8** *The topologies on $\mathbb{R}^n$ induced by* ***the euclidean metric*** *d and* ***the square metric*** *ρ are the* ***same*** *as the* ***product topology*** *on $\mathbb{R}^n$.*

- **Remark** (***Finite Dimensional* Vector Space has Only One Meaningful Topology**)
  In ***finite dimensional*** vector space, ***all norms are equivalent***, and *all norm-induced metric topologies* are *the same*. For infinite dimenisonal space, these topologies are different.

- **Definition** (***Uniform Topology on Infinite Dimensional Space***)
  Given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(x, y) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\},$$

  where $\bar{d}$ is ***the standard bounded metric*** on $\mathbb{R}$, It is easy to check that $\bar{p}$ is indeed a metric; it is called ***the uniform metric*** *on* $\mathbb{R}^J$, and the topology it induces is called ***the uniform topology***.

- **Proposition 4.9** *The **uniform topology** on $\mathbb{R}^J$ is **finer** than the product topology and **coarser** than the box topology; these three topologies are all different if J is infinite.*

$$\mathscr{T}_{\boldsymbol{product}} \subset \mathscr{T}_{\boldsymbol{uniform}} \subset \mathscr{T}_{\boldsymbol{box}}$$

- **Theorem 4.10** *(**Countable Product Space with Product Topology is Metrizable**). [Munkres, 2000]*
  *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the **standard bounded metric** on $\mathbb{R}$. If x and y are two points of W", define*

$$D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

  *Then D is a metric that induces **the product topology** on $\mathbb{R}^\omega$.*

4.2.2 Constructing Continuous Functions on Metric Space

- The followings are some important facts about the metric topology:
  1. **Proposition 4.11** *Every **metric space** (X, d) is **Hausdorff**.*
  2. **Proposition 4.12** *Every **subspace** of metric space (X, d) is a metric space. That is, if A is a subspace of the topological space X and d is a metric for X, then the restriction of d on $A \times A$ is a metric for the topology of A.*

- **Theorem 4.13** *(ϵ-δ **Definition of Continuous Function in Metric Space**). [Munkres, 2000]*
  *Lei $f : X \to Y$; let X and Y be **metrizable** with metrics d_x and d_y, respectively. Then **continuity** of f is **equivalent** to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) < \epsilon.$$

- **Remark** The ϵ-δ definition of continuous function is equivalent to

$$f(B(x, \delta)) \subset B(f(x), \epsilon) \quad \Leftrightarrow \quad B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$$

- **Remark** To use ϵ-δ definition, *both **domain** and **codomain*** need to be ***metrizable***.

- **Lemma 4.14** *(**The Sequence Lemma**). [Munkres, 2000]*
*Let X be a topologicaJ space; let $A \subseteq X$. If there is a sequence of points of A **converging** to x, then $x \in \bar{A}$; the **converse** holds if X is **metrizable**.*

- **Proposition 4.15** *Let $f : X \to Y$. If the function f is **continuous**, then for every **convergent** sequence $x_n \to x$ in X, the sequence $f(x_n)$ **converges** to $f(x)$. The **converse** holds if X is **metrizable**.*

- **Remark** To show the converse part, i.e. "*if $x_n \to x \Rightarrow f(x_n) \to f(x)$ then f is continuous*", we just need the space X to be ***first countable***. That is, at each point x, there is ***a countable collection*** $(U_n)_{n \in \mathbb{Z}_+}$ of ***neighborhoods*** of x such that any neighborhood U of x *contains* at least one of the sets U_n.

- **Proposition 4.16** *(**Arithmetic Operations of Continuous Functions**).*
If X is a topological space, and if $f, g : X \to Y$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

- **Definition** (***Uniform Convergence***)
Let $f_n : X \to Y$ be a sequence of functions from the ***set*** X to ***the metric space*** Y. Let d be the metric for Y. We say that the sequence (f_n) ***converges uniformly*** to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that
$$d(f_n(x), f(x)) < \epsilon$$
for all $n > N$ and ***all** x **in** X*.

- **Theorem 4.17** *(**Uniform Limit Theorem**). [Munkres, 2000]*
*Let $f_n : X \to Y$ be a sequence of **continuous** functions from the **topological** space X to the **metric space** Y. If (f_n) converges **uniformly** to f, then f is **continuous**.*

- **Remark** (***Uniform Convergence** = **Convergence of Functions in Uniform Metric***)
A sequence of functions $f_n : X \to \mathbb{R}$ ***converges uniformly*** to $f : X \to \mathbb{R}$ ***if and only if*** the sequence (f_n) converges to f when they are considered as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$, where $\mathbb{R}^X$ is the space of all real-valued functions on X and $\bar{\rho}$ is ***the unform metric*** defined before.

- **Example** The ***countable product space*** $\mathbb{R}^\omega$ in the ***box topology*** is ***not metrizable***. (on the other hand, it is metrizable *in product topology*).

- **Example** An ***uncountable product*** of $\mathbb{R}$ with itself is ***not metrizable***.

4.3 The Quotient Topology

4.3.1 Definitions and Properties

- **Remark** (***Quotient Topology as "Cut-and-Paste"***)
One motivation of ***quotient topology*** comes from geometry, where one often has occasion to use "***cut-and-paste***" techniques to construct such geometric objects as surfaces.:

  1. The ***torus*** (surface of a doughnut), for example, can be constructed by taking a ***rectangle*** and "*pasting*" its edges together appropriately

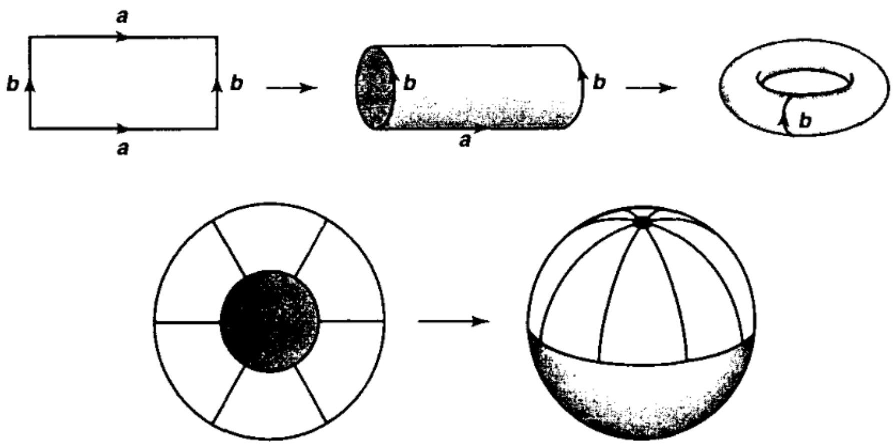

Figure 5: (Upper) The torus can be constructed via cut-and-paste along the rectangle edges. (Lower) The sphere can be constructed via by taking a disc and collapsing its entire boundary to a single point. [Munkres, 2000]

2. The ***sphere*** (surface of a ball) can be constructed by taking a ***disc*** and *collapsing* its entire boundary to a single point;

See Figure 5.

- **Definition** (***Quotient Map***)
  Let X and Y be topological spaces; let $\pi : X \to Y$ be a ***surjective map***. The map π is said to be ***a quotient map*** provided a subset U of Y is ***open*** in Y ***if and only if*** $\pi^{-1}(U)$ is ***open*** in X.

- **Remark** (***Quotient Map = Strong Continuity***)
  The condition of quotient map is ***stronger*** than continuity (it is called ***strong continuity*** in some literature).

$$\begin{aligned} \text{continuity :} \quad & U \text{ is open in } Y \Rightarrow \pi^{-1}(U) \text{ is open in } X \\ \text{quotient map :} \quad & U \text{ is open in } Y \Leftrightarrow \pi^{-1}(U) \text{ is open in } X \end{aligned}$$

  An equivalent condition is to require that a subset A of K be ***closed*** in Y *if and only if* $\pi^{-1}(A)$ is ***closed*** in X. Equivalence of the two conditions follows from equation

$$\pi^{-1}(Y \setminus B) = X \setminus \pi^{-1}(B).$$

- **Definition** (***Saturated Set and Fiber***)
  If $\pi : X \to Y$ is a ***surjective*** *map*, a subset $U \subseteq X$ is said to be ***saturated*** with respect to π if U contains every set $\pi^{-1}(\{y\})$ that it ***intersects***. Thus U is ***saturated*** if it equals to the ***entire preimage*** of its ***image***: $U = \pi^{-1}(\pi(U))$.

  Given $y \in Y$, the ***fiber*** of π over y is the set $\pi^{-1}(\{y\})$.

- **Definition** (***Quotient Map via Saturated Set***)
  A surjective map $\pi : X \to Y$ is a ***quotient map*** if π is ***continuous*** and π maps ***saturated open sets*** of X to ***open sets*** of Y (or *saturated closed sets* of X to *closed sets* of Y).

- **Definition** (***Open Map*** *and* ***Closed Map***)
  A map $f : X \to Y$ (continuous or not) is said to be an ***open map*** if for every *open* subset $U \subseteq X$, the image set $f(U)$ is *open* in Y, and a ***closed map*** if for every *closed* subset $K \subseteq X$, the image $f(K)$ is *closed* in Y .

- **Proposition 4.18** *If $\pi : X \to Y$ is a **surjective continuous** map that is **either open** or **closed**, then π is a **quotient** map.*

  Remark There are *quotient maps* that are ***neither*** *open* nor *closed*. See Exercise in [Munkres, 2000].

- **Example** (***Coordinate Projection as Quotient Map***)
  Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be ***projection onto the first coordinate***; then π_1 is *continuous* and *surjective*. Furthermore, π_1 is an ***open map***. For if $U \times V$ is a nonempty *basis element* for $\mathbb{R} \times \mathbb{R}$, then $\pi_1(U \times V) = U$ is *open* in $\mathbb{R}$; it follows that π_1 carries open sets of $\mathbb{R} \times \mathbb{R}$ to open sets of $\mathbb{R}$. That is, π_1 is a ***quotient map***.

  However, π_1 is ***not*** *a* ***closed map***. The subset

  $$C = \{(x, y) : x \cdot y = 1\}$$

  of $\mathbb{R} \times \mathbb{R}$ is *closed*, but $\pi_1(C) = \mathbb{R} \setminus \{0\}$, which is *not closed* in $\mathbb{R}$.

- **Definition** (***Quotient Topology***)
  If X is a space and A is a set and if $\pi : X \to A$ is a ***surjective*** map, then there exists ***exactly one topology*** $\mathscr{T}$ ***on*** A relative to which π is a ***quotient map***; it is called ***the quotient topology*** induced by π. *The quotient topology* $\mathscr{T}$ on A is defined as

  $$\mathscr{T} = \left\{U \subset A : \pi^{-1}(U) \text{ is open in } X\right\}$$

- **Definition** (***Quotient Space***)
  Suppose X is a topological space and $\sim$ is *an equivalence relation* on X. Let $X/\sim$ denote ***the set of equivalence classes*** in X, and let $\pi : X \to X/\sim$ be the ***natural projection*** sending each *point* to its *equivalence class*. Endowed with ***the quotient topology*** determined by π, the space $X/\sim$ is called ***the quotient space*** (or *identification space*) of X determined by π.

  Definition [Munkres, 2000]
  Let X be a topological space, and let X^* be a ***partition*** of X into *disjoint subsets whose union is* X. Let $\pi : X \to X^*$ be the ***surjective*** map that carries each point of X to the element of X^* *containing* it. In ***the quotient topology*** induced by π, the space X^* is called a ***quotient space*** *of* X.

- **Remark** (***Understanding Topology of Quotient Space***)
  We can describe the topology of $X/\sim$ in another way. A *subset* U of $X/\sim$ is ***a collection of equivalence classes***, and the set $\pi^{-1}(U)$ is just ***the union*** *of* ***the equivalence classes belonging to*** U.

  Thus the typical ***open set*** of $X/\sim$ is ***a collection*** *of* ***equivalence classes*** whose ***union*** is ***an open set*** of X.

  $$V \text{ open in } X/\sim \quad \Leftrightarrow \quad U := \pi^{-1}(V) = \bigcup_{[y] \in V} [y] \text{ open in } X$$

- **Remark** (***Geometrical Understanding of Quotient Space***)
*A set of points in X in the **same equivalence class** $[y]$ is considered as **one point** in quotient space $X/\sim$.* Geometrically, it is seen as ***collapsing a set of points** into **one*** if this set of points are in a connected neighborhood, or, it is seen as ***cut-and paste** a set of points in boundary* with another set of points in boundary.

- **Example** ($\mathbb{D}^2/\sim = \mathbb{S}^2$)
Let X be the ***closed unit ball***

$$X = \mathbb{D}^2 := \left\{(x, y) : x^2 + y^2 \le 1\right\}$$

in $\mathbb{R}^2$, and let $X/\sim$ be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$. One can show that $X/\sim$ is *homeomorphic* with the subspace of $\mathbb{R}^3$ called ***the unit 2-sphere***, defined by

$$\left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}$$

- **Proposition 4.19** ***(Restricting Quotient Map to Subspace)**. [Munkres, 2000]*
*Let $\pi : X \to Y$ be a **quotient map**; let A be a subspace of X that is **saturated** with respect to π; let $q : A \to \pi(A)$ be the map obtained by restricting π.*

  1. *If A is either **open** or **closed** in X, then q is a **quotient map**.*
  2. *If π is either an **open map** or a **closed map**, then q is a **quotient map**.*

- **Remark** (***Composite of Quotient Maps is Quotient Map***).
Composites of maps behave nicely; it is easy to check that the *composite of two quotient maps is a quotient map*; this fact follows from the equation

$$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U).$$

- **Remark** (***Product of Quotient Maps Need Not to be Quotient Map***).
On the other hand, products of maps do not behave well; *the cartesian product of two quotient maps **need not** be a quotient map.*

  One needs further conditions on either the maps or the spaces in order for this statement to be true.

  1. One such, a condition on the spaces, is called ***local compactness***; we shall study it later.
  2. Another, a condition on the *maps*, is the condition that ***both** the maps p and q be **open maps***. In that case, it is easy to see that $p \times q$ is also ***an open map***, so it is a quotient map.

- **Remark** (***Quotient Space of Haudorff Space Need Not to be Hausdorff***)
The Hausdorff condition does not behave well; *even if X is Hausdorff, there is no reason that the quotient space $X/\sim$ needs to be Hausdorff.* There is a simple condition for $X/\sim$ to satisfy the T_1 axiom; one simply requires that ***each element** of the partition $X/\sim$ be a **closed subset** of X*. Conditions that will ensure $X/\sim$ is *Hausdorff* are harder to find.

4.3.2 Constructing Continuous Function on Quotient Space

- We want to know if $f : (X/\sim) \to Z$ is *continuous function*.

- **Theorem 4.20** ***(Passing Continuity to the Quotient)***. *[Munkres, 2000]*
  *Let $\pi : X \to Y$ be a **quotient map**. Let Z be a space and let $g : X \to Z$ be a map that is **constant** on **each fiber** $\pi^{-1}(\{y\})$, for $y \in Y$. Then g **induces** a map $f : Y \to Z$ such that $f \circ \pi = g$. The induced map f is **continuous** if and only if g is **continuous**: f is a **quotient map** if and only if g is a **quotient map**.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z. \end{tikzcd}$$

- **Corollary 4.21** *Let $g : X \to Z$ be a **surjective continuous** map. Let $X/\sim$ be the following collection of subsets of X:*

$$X/\sim := \left\{ g^{-1}(\{z\}) : z \in Z \right\},$$

  *Given $X/\sim$ the **quotient topology**,*

  1. *The map g induces a **bijective continuous map** $f : (X/\sim) \to Z$, which is a **homeomorphism if and only if** g is a **quotient map**.*
  2. *If Z is **Hausdorff**, so is $X/\sim$.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ (X/\sim) \arrow[r, dashed, "f"'] & Z. \end{tikzcd}$$

- **Example** (***The Product of two Quotient Maps Need Not be a Quotient Map***).
  Let $X = \mathbb{R}$ and let $X/\sim$ be the *quotient space* obtained from X by ***identifying*** the subset $\mathbb{Z}_+$ to a point b; let $\pi : X \to X/\sim$ be the quotient map. Let $\mathbb{Q}$ be the subspace of $\mathbb{R}$ consisting of *the rational numbers*; let $i : \mathbb{Q} \to \mathbb{Q}$ be the *identity map*. We show that

$$\pi \times i : X \times \mathbb{Q} \to (X/\sim) \times \mathbb{Q}$$

  is ***not a quotient map***.

  For each n, let $c_n = \sqrt{2}/n$, and consider *the straight lines* in $\mathbb{R}^2$ with slopes 1 and -1, respectively, through the point (n, c_n). Let U_n consist of *all points of $X \times \mathbb{Q}$ that lie **above both** of these lines **or beneath both** of them*, and also *between the vertical lines $x = n - 1/4$ and $x = n + 1/4$*. Then U_n is ***open*** in $X \times \mathbb{Q}$; it contains the set $\{n\} \times \mathbb{Q}$ because c_n is ***not rational***.

  Let U be the ***union*** of the sets U_n; then U is ***open*** in $X \times \mathbb{Q}$. It is ***saturated*** with respect to $\pi \times i$ because it *contains the entire set $\mathbb{Z}_+ \times \{q\}$) for each $q \in \mathbb{Q}$*. We assume that $U' := (\pi \times i)(U)$ is ***open*** in $(X/\sim) \times \mathbb{Q}$ and derive a *contradiction*.

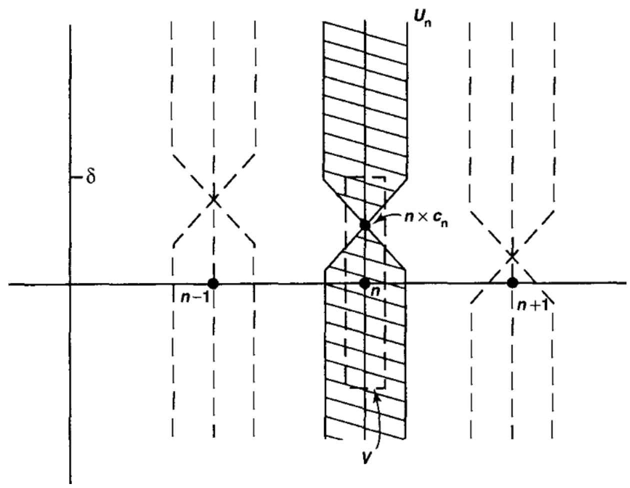

Figure 6: The product of two quotient maps need not be a quotient map. [Munkres, 2000]

Because U contains, in particular, the set $\mathbb{Z}_+ \times \{0\}$, the set U' contains the point $(b, 0)$. Hence U' contains an *open set* of the form $W \times I_\delta$, where W is a neighborhood of b in $X/\sim$ and I_δ consists of all *rational numbers* y with $|y| < \delta$. Then

$$\pi^{-1}(W) \times I_\delta \subset U.$$

Choose n large enough that $c_n < \delta$. Then since $\pi^{-1}(W)$ is *open* in X and contains $\mathbb{Z}_+$, we can choose $\epsilon < 1/4$ so that the interval $(n-\epsilon, n+\epsilon)$ is contained in $\pi^{-1}(W)$. Then U contains the subset $V = (n-\epsilon, n+\epsilon) \times I_\delta$ of $X \times \mathbb{Q}$. But the figure makes clear that there are many points (x, y) of V that do not lie in U'. (One such is the point (x, y), where $x = n + \frac{1}{2}\epsilon$ and y is a ***rational number*** with $|y - c_n| < \frac{1}{2}\epsilon$.)

5 Topological Groups

- **Definition** (***Topological Group***)
  A ***topological group*** G is a ***group*** that is also a ***topological space*** satisfying ***the*** T_1 ***axiom***, such that ***the multiplication map*** $m : G \times G \to G$ and ***inversion map*** $i : G \to G$, given by

  $$m(x, y) = xy, \quad i(x) = x^{-1}.$$

  are both ***continuous maps***. Here, $G \times G$ is viewed as a *topological space* by using *the product topology.*

- **Example** (***Common Topological Groups***)
  The following are topological groups:

  1. $(\mathbb{Z}, +)$
  2. $(\mathbb{R}, +)$
  3. $(\mathbb{R}_+, \cdot)$

4. $(\mathbb{S}^1, \cdot)$, where we take $\mathbb{S}^1$ to be *the space of* ***all complex numbers*** z for which $|z| = 1$

- **Example** (***Lie Groups***)

  Definition (***Lie Group***) [Lee, 2003.]
  A ***Lie group*** is a ***smooth manifold*** $\mathcal{G}$ (without boundary) that is also *a* ***group*** in the *algebraic sense*, with the property that *the multiplication map* $m : G \times G \to G$ and *inversion map* $i : G \to G$, given by

  $$m(g, h) = gh, \quad i(g) = g^{-1}.$$

  are both ***smooth***.

  A Lie group is a topological group. The followings are all ***Lie groups***:

  1. ***The general linear group*** $GL(n, \mathbb{R})$ is the set of ***invertible*** $n \times n$ *matrices* with real entries.

     $$GL(n, \mathbb{R}) \equiv \left\{ \boldsymbol{A} \in \mathbb{R}^{n\times n} : \ \det(\boldsymbol{A}) \neq 0 \right\}.$$

     It is a *group* under ***matrix multiplication***, and it is an *open submanifold* of the vector space $M(n, \mathbb{R}) \simeq \mathbb{R}^{n\times n}$. *Multiplication is smooth* because *the matrix entries* of a product matrix AB are *polynomials* in the entries of A and B. Inversion is *smooth* by *Cramer's rule.*

  2. Let $GL_+(n, \mathbb{R})$ denote the subset of $GL(n, \mathbb{R})$ consisting of matrices with ***positive determinant***. Because $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = (\det(A))^{-1}$, it is a subgroup of $GL(n, \mathbb{R})$; and because it is the *preimage* of $(0, +\infty)$ under *the continuous determinant function*, it is an open subset of $GL(n, \mathbb{R})$ and therefore an n^2-dimensional manifold. The *group operations* are the restrictions of those of $GL(n, \mathbb{R})$, so they are smooth. Thus $GL_+(n, \mathbb{R})$ is a *Lie group.*

  3. The ***special linear group*** $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ consisting of matrices with a ***determinant of*** 1.

     $$SL(n, \mathbb{R}) \equiv \left\{ \boldsymbol{A} \in \mathbb{R}^{n\times n} : \ \det(\boldsymbol{A}) = 1 \right\}.$$

     It is a *Lie group* with dimension $\dim SL(n, \mathbb{R}) = n^2 - 1$.

  4. The ***orthogonal group*** *of dimension* n, denoted $\mathcal{O}(n)$, is the group of ***distance-preserving transformations*** of a Euclidean space of dimension n that preserve a fixed point, where the group operation is given by composing transformations. Also, $(\mathcal{O}(n), \cdot)$ is the group of $n \times n$ ***orthogonal matrices***, where the group operation $(\cdot)$ is given by matrix multiplication, and an orthogonal matrix is a real matrix whose inverse equals its transpose. *The orthogonal group is a Lie group* with *dimension* $n(n-1)/2$.

     $$\mathcal{O}(n) \equiv \left\{ \boldsymbol{Q} \in GL(n, \mathbb{R}) : \ \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}_n \right\}.$$

  5. The ***special orthogonal group*** $\mathcal{SO}(n)$ is the group of *the* ***orthogonal matrices*** *of* ***determinant*** 1. This group is also called the ***rotation group***

     $$\mathcal{SO}(n) \equiv \{ \boldsymbol{Q} \in \mathcal{O}(n) : \ \det(\boldsymbol{Q}) = 1 \}.$$

     It is an open subgroup of $\mathcal{O}(n)$, which is a *Lie group* of dimension $\dim \mathcal{SO}(n) = \dim \mathcal{O}(n) = n(n-1)/2$.

6. ***The complex general linear group*** $GL(n, \mathbb{C})$ is the group of *invertible complex* $n \times n$ *matrices* under matrix multiplication. It is an open submanifold of $M(n, \mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold, and it is a *Lie group* because *matrix products and inverses* are *smooth functions* of the real and imaginary parts of the matrix entries.

7. If V is any *real or complex vector space*, $GL(V)$ denotes *the set of* ***invertible linear maps*** from V to itself. It is a *group under composition*. If V has ***finite dimension*** n, any basis for V determines an *isomorphism* of $GL(V)$ with $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, so $GL(V)$ is a *Lie group*.

8. $(\mathbb{Z}, +)$

9. $(\mathbb{R}, +)$

10. The set $\mathbb{R}^*$ of *nonzero real numbers* is a 1-***dimensional Lie group*** *under multiplication.* (In fact, it is exactly $GL(1, \mathbb{R})$ if we identify a 1×1 matrix with the corresponding real number.) The subset $\mathbb{R}_+$ of ***positive real numbers*** is an *open subgroup*, and is thus itself a 1-*dimensional Lie group.*

11. The set $\mathbb{C}^*$ of ***nonzero complex numbers*** is a 2-*dimensional Lie group* under complex multiplication, which can be identified with $GL(1, \mathbb{C})$.

12. The ***circle*** $\mathbb{S}^1 \subset \mathbb{C}^*$ is a smooth manifold and a group under complex multiplication. With appropriate ***angle functions*** as *local coordinates* on open subsets of $\mathbb{S}^1$, *multiplication and inversion* have the *smooth coordinate expressions* $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$ and $\theta \mapsto -theta$, and therefore $\mathbb{S}^1$ is a Lie group, called <u>***the circle group***</u>.

13. The <u>***n-torus***</u> $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is an n-***dimensional abelian Lie group***.

- **Example** (***Discrete Group***)
  Any ***group*** with ***the discrete topology*** is a *topological group*, called a <u>***discrete group***</u>. If in addition the group is *finite* or *countably infinite*, then it is a ***zero-dimensional Lie group***, called a <u>***discrete Lie group***</u>.

- **Definition** (***Homogeneous Space***)
  A topological space G is a <u>***homogeneous space***</u> if for every pair $x, y \in G$, there exists a *homemorphism* $f : G \to G$ such that $f(x) = y$.

- **Proposition 5.1** *(**Topological Groups Are Homogeneous**)*
  Every ***topological group*** *is a* ***homogeneous*** *space; in particular, define map $h_\alpha : G \to G$ as $h_\alpha(x) = \alpha \cdot x$ and $g_\alpha : G \to G$ as $g_\alpha(x) = x \cdot \alpha$, for $\alpha \in G$. Then h_α, g_α are* ***homemorphisms****.*

- **Proposition 5.2** *(**Subgroup of Topological Group**)*
  Let H be a ***subspace*** *of topological group G. If H is also a* ***subgroup*** *of G, then both H and its closure $\bar{H}$ are* ***topological groups****.*

- **Definition** (***Left Coset*** *and* ***Right Coset***)
  For $H \subset G$ as the *subgroup* of G, define the <u>***left coset***</u> as $xH = \{x \cdot h : h \in H\}$. Similarly, define <u>***the right coset***</u> as $Hx = \{h \cdot x : h \in H\}$

- **Definition** (***Quotient Group***)
  The collection of *left cosets* defines a <u>***quotient group***</u> $G/H = \{xH \,|\, x \in G\}$ with *the group operation* $xH \cdot yH = (x \cdot y)H$.

- **Proposition 5.3** *Let G be a topological group.*

  1. *If $\alpha \in G$, the map $f_\alpha : x \mapsto \alpha \cdot x$ induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$. Thus G/H is a* ***homogeneous space****.*
  2. *If H is a* ***closed*** *set in the topology of G, then* ***one-point sets*** *are* ***closed*** *in G/H.*
  3. *The* ***quotient map*** *$\pi : G \to G/H$ is* ***open****.*
  4. *If H is* ***closed*** *in the topology of G and is a* ***normal subgroup*** *of G, then the (left) quotient group G/H under quotient topology is a* ***topological group****.*
  5. *If H is* ***compact subgroup*** *of G and $\pi : G \to G/H$ is closed, then G/H is* ***compact****.*

- **Example** $(GL(n,\mathbb{R})/SL(n,\mathbb{R}) \simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\})$.
  Given the *generalized linear group $GL(n,\mathbb{R})$, the special linear group $SL(n,\mathbb{R})$* is a subgroup of $GL(n,\mathbb{R})$. The *quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$* $\simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

  Proof: Let $G = GL(n,\mathbb{R})$ and $H = SL(n,\mathbb{R}) = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} :\ \det(\boldsymbol{A}) = 1\}$. Define $\det : GL(n,\mathbb{R}) \to \mathbb{R}^*$. The map det is constant on each left coset

$$xH = (\det)^{-1}(r);$$

  where $x \in G$ with $\det(x) = r \neq 0$. Note that for all $\boldsymbol{A} \in xH$, $\boldsymbol{A} = x\boldsymbol{S}$ where $\boldsymbol{S}$ is the matrix in $SL(n,\mathbb{R})$. so $\det(\boldsymbol{A}) = \det(x)\det(\boldsymbol{S}) = \det(x) = r$ since $\det \boldsymbol{S} = 1$. Moreover det is a *surjective continuous* map. Now we show that det is an open map, therefore $\det : GL(n,\mathbb{R}) \to \mathbb{R}^*$ is a quotient map.

  To prove that det(any open subset of $GL(n,\mathbb{R})$) is an open set in $\mathbb{R}^*$, consider the matrix $\boldsymbol{A} \in GL(n,\mathbb{R})$, note that by expansion by minor, the determinant of $\boldsymbol{A}$ can be written as

$$\begin{aligned}
\det(\boldsymbol{A}) &= \sum_{i=1}^{m} (-1)^{i-1} a_{1,i} \det(\boldsymbol{A}_{-i}) \\
\det(\boldsymbol{A} + \delta \boldsymbol{E}_k) &= \sum_{i=1}^{m} (-1)^{i-1} (a_{1,i} + \delta\, \mathbb{1}_{i,k}) \det(\boldsymbol{A}_{-i}) \\
&= \det(\boldsymbol{A}) + (-1)^{k-1} \delta \det(\boldsymbol{A}_{-k}) \\
|\det(\boldsymbol{A} + \delta \boldsymbol{E}_k) - \det(\boldsymbol{A})| &\le \epsilon
\end{aligned}$$

  From the corollary above, there exists a bijective continuous map $f : G/H \to \mathbb{R}^*$ so that the following diagram commutes

$$\begin{tikzcd}
G \arrow[d, "\pi"'] \arrow[dr, "\det"] & \\
(G/H) \arrow[r, dashed, "f"'] & \mathbb{R}^*.
\end{tikzcd}$$

  Since $\det : GL(n,\mathbb{R}) \to \mathbb{R}^*$ is a quotient map, f is a *homemorphism*. ∎

- **Example** $(GL(n,\mathbb{R})/SL(n,\mathbb{R}) \simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\})$.
  Given the *generalized linear group $GL(n,\mathbb{R})$, the special linear group $SL(n,\mathbb{R})$* is a subgroup of $GL(n,\mathbb{R})$. The *quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$* $\simeq \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

- **Example** $(\mathcal{O}(n)/\mathcal{SO}(n) \simeq \mathbb{Z}_2 = \{-1, 1\} = \mathbb{Z}/2\mathbb{Z})$.
  The quotient group of orthogonal group $\mathcal{O}(n)$ over the special orthorognal group $\mathcal{SO}(n) = \{\boldsymbol{Q} \in \mathcal{O}(n) : \det(\boldsymbol{Q}) = 1\}$ is homemorphic to $\mathbb{Z}_2 = \{-1, 1\}$.

- **Example** $(\mathcal{O}(n)/\mathcal{O}(n-1) \simeq \mathbb{S}^{n-1})$.
  The quotient group of n-dimensional orthogonal group $\mathcal{O}(n)$ over $(n-1)$-dimensional orthogonal group $\mathcal{O}(n-1)$ is homemorphic to $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$.

- **Definition** (***Topological Group Action***)
  An ***action*** of a ***topological group*** G on a ***topological space*** X is a ***continuous map*** $\phi : G \times X \to X$ such that for $g(x) := \phi(g, x)$,

$$
\begin{aligned}
(g_1 \cdot g_2)(x) &= g_1(g_2(x)), && \forall g_1, g_2 \in G, x \in X \\
1_G(x) &= \mathrm{Id}_X(x) = x, && \forall x \in X
\end{aligned}
$$

  where 1_G is the unit element of group G. Together with the group action, X is called a ***G-space***.

- **Remark** The map $x \mapsto g(x)$ is a ***continuous map*** on X for each $g \in G$. This map has ***inverse map*** $x \mapsto g^{-1}(x)$ which is continuous as well. Thus the map $x \mapsto g(x)$ is a ***homemorphism***.

- **Example** *The topological group* $\mathcal{O}(n)$ *acts* on $\mathbb{R}^n$ is the rotation transformation of vectors in $\mathbb{R}^n$. Similarly, $\mathcal{O}(n)$ *acts* on $\mathbb{S}^1$ is the rotation of circle $\mathbb{S}^1$.

- **Definition** (***Orbit under Topological Group Actions***)
  If *the topological group G acts* on topological space X, and $x \in X$, then ***the orbit** of x* is defined as

$$
G(x) = \{g(x) : g \in G\}
$$

- **Definition** The ***stablizer*** of x under group actions G is defined as

$$
G_x = \{g \in G : g(x) = x\}
$$

- **Definition** (***Orbit Space*** X/G)
  Let G be a topological group and X be a G-*space* so that G acts on X. ***The orbit space*** is the set of *all orbits of action* with *quotient topology.* The quotient map $\pi : x \mapsto G(x)$ maps x to its orbit. *The orbit space* is often called ***the quotient of X by group actions G***, i.e.

$$
X/G = \{G(x) : x \in X\}.
$$

- **Proposition 5.4** *(**Orbit Space by Compact Group**)*
  *Let G be a **compact** topological group and X be a topological space so that G acts on X. Let X/G be the **orbit space**, i.e. the quotient space of X by group actions G. Then*

  1. *X/G is **Hausdorff** if X is **Hausdorff**;*
  2. *X/G is **regular** if X is **regular**;*
  3. *X/G is **normal** if X is **normal**;*

4. *X/G is **locally compact** if X is **locally compact**;*

5. *X/G is **second countable** if X is **second countable**;*

- **Example** (***Global Flow on Smooth Manifold***) [Lee, 2003.]

  Definition A ***global flow on*** M (also called ***a one-parameter group action***) is defined as a ***continuous left*** $\mathbb{R}$***-action on*** M; that is, a *continuous map* $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

  $$\begin{aligned} \theta_{t+s}(p) &= \theta_t \circ \theta_s(p), \\ \theta_0(p) &= p \end{aligned}$$

  where $\theta_t = \theta(t, \cdot) : M \to M$ is a *continuous map* and $\theta_0 = \mathrm{Id}_M$.

  As we can see that, ***the global flow is topological group action*** *of* $(\mathbb{R}, +)$ *on the smooth manifold M (a topological space).*

  Definition For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by

  $$\theta^{(p)}(t) = \theta(t, p).$$

  The image of this curve is ***the orbit of*** p ***under the group action***.

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.

James R Munkres. *Topology, 2nd.* Prentice Hall, 2000.