Lecture 4: Compactness in Function Spaces

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Contents

1 Complete Metric Spaces and Function Spaces

1.1 Complete Metric Space

- **Definition** (***Cauchy Net in Topological Vector Space***)
A *net* $\{x_\alpha\}_{\alpha \in I}$ in ***toplogocial vector space*** X is called ***Cauchy*** if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta)\in I\times I}$ ***converges*** *to zero.* (Here $I \times I$ is ***directed*** in the usual way: $(\alpha, \beta) \prec (\alpha', \beta')$ if and only if $\alpha \prec \alpha'$ and $\beta \prec \beta'$.)

- **Definition** (***Completeness***)
A toplogocial vector space X is ***complete*** if every Cauchy net converges.

- **Proposition 1.1** *(**Complete First Countable Topological Vector Space**)*
*If X is a **first-countable topological vector space** and every **Cauchy sequence** in X converges, then every **Cauchy net** in X converges.*

- **Proposition 1.2** *(**Completeness of Euclidean Space**) [Munkres, 2000]*
*Euclidean space $\mathbb{R}^k$ is **complete** in either of its usual **metrics**, the **euclidean metric** d or the **square metric** ρ.*

- **Lemma 1.3** *(**Convergence in Product Space is Weak Convergence**) [Munkres, 2000]*
Let X be the product space $X = \prod_\alpha X_\alpha$; let x_n be a sequence of points of X. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α.

- **Proposition 1.4** *(**Completeness of Countable Product Space**) [Munkres, 2000]*
*There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is **complete**.*

- **Definition** (***Uniform Metric in Function Space***)
Let (Y, d) be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the ***standard bounded metric*** on Y derived from d. If $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha\in J}$ are points of the cartesian product Y^J, let

$$\bar{\rho}(x, y) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\}.$$

It is easy to check that $\bar{\rho}$ is a metric; it is called ***the uniform metric*** on Y^J corresponding to the metric d on Y.

Note that ***the space of all functions*** $f : J \to Y$, ***denoted*** as Y^J, is a subset of the product space $J \times Y$. We can define uniform metric in the function space: if f, $g : J \to Y$, then

$$\bar{\rho}(f, g) = \sup\left\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\right\}.$$

- **Proposition 1.5** *(**Completeness of Function Space Under Uniform Metric**) [Munkres, 2000]*
*If the space Y is **complete** in the metric d, then the space Y^J is **complete** in the **uniform metric** $\bar{\rho}$ corresponding to d.*

- **Definition** (***Space of Continuous Functions and Bounded Functions***)
Let Y^X be *the space of all functions* $f : X \to Y$, where X is a *topological space* and Y is a *metric space with metric* d. Denote the ***subspace*** of Y^X consisting of all ***continuous functions*** f as $\mathcal{C}(X, Y)$.

Also denote *the set of all* ***bounded functions*** $f : X \to Y$ as $\mathcal{B}(X, Y)$. (A function f is said to be ***bounded*** if its image $f(X)$ is a ***bounded subset*** of *the metric space* (Y, d).)

- **Proposition 1.6** *(**Completeness of** $\mathcal{C}(X,Y)$ **and** $\mathcal{B}(X,Y)$ **Under Uniform Metric**) [Munkres, 2000]*
  *Let X be a topological space and let (Y, d) be a metric space. The set $\mathcal{C}(X, Y)$ of **continuous** functions is **closed** in Y^X under the **uniform metric**. So is the set $\mathcal{B}(X, Y)$ of **bounded functions**. Therefore, if Y is **complete**, these spaces are **complete** in the **uniform metric**.*

- **Definition** (***Sup Metric on Bounded Functions***)
  If (Y, d) is a metric space, one can define another metric *on the set* $\mathcal{B}(X, Y)$ *of* ***bounded functions*** from X to Y by the equation

$$\rho(x, y) = \sup \{d(f(x), g(x)) : x \in X\}.$$

  It is easy to see that ρ is well-defined, for the set $f(X) \cup g(X)$ is ***bounded*** if both $f(X)$ and $g(X)$ are. The metric ρ is called ***the sup metric***.

- **Theorem 1.7** *(**Existence of Completion**) [Munkres, 2000]*
  *Let (X, d) be a metric space. There is an **isometric embedding** of X into a **complete** metric space.*

- **Definition** (***Completion***)
  Let X be a *metric space*. If $h : X \to Y$ is an ***isometric embedding*** of X into a ***complete*** *metric space* Y, then the ***subspace*** $h(X)$ of Y is a *complete metric space*. It is called ***the completion of*** X.

1.2 Compactness in Metric Spaces

- **Remark** (***Compactness*** *and* ***Completeness***)
  How is ***compactness*** of a metric space X related to ***completeness*** of X?

  The followings is from *the sequential compactness* and definition of *completeness*:

  Proposition 1.8 *Every **compact** metric space is **complete**.*

  The *converse* does not hold – ***a complete metric space need not be compact***. It is reasonable to ask what ***extra condition*** one needs to impose on a complete space to be assured of its compactness. Such a condition is the one called *total boundedness*.

- **Definition** (***Total Boundedness***)
  A metric space (X, d) is said to be ***totally bounded*** if for every $\epsilon > 0$, there is a ***finite covering*** *of* X *by* ϵ***-balls***.

- **Theorem 1.9** *[Munkres, 2000]*
  *A metric space (X, d) is **compact** if and only if it is **complete** and **totally bounded**.*

- **Remark** We now apply this result to find ***the compact subspaces*** of the space $\mathcal{C}(X, \mathbb{R}^n)$, *in the* ***uniform topology***. We know that a subspace of $\mathbb{R}^n$ is compact if and only if it is ***closed*** and ***bounded***.

  One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. ***But*** it does not, even if X is *compact*. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies *an* ***additional***

condition, called ***equicontinuity***.

- **Definition** (***Equicontinuity***) [Reed and Simon, 1980, Munkres, 2000]
Let (Y, d) be a *metric space*. Let $\mathscr{F}$ be a *subset* of the function space $\mathcal{C}(X, Y)$ (i.e. $f \in \mathscr{F}$ is continuous). If $x_0 \in X$, *the set* $\mathscr{F}$ *of functions* is said to be ***equicontinuous at*** x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that *for all* $x \in U$ and ***all*** $f \in \mathscr{F}$,
$$d(f(x), f(x_0)) < \epsilon.$$
If the set $\mathscr{F}$ is *equicontinuous* at x_0 for each $x_0 \in X$, it is said simply to be ***equicontinuous*** or $\mathscr{F}$ is an ***equicontinuous family***.

  We say $\mathscr{F}$ is a ***uniformly equicontinuous family*** if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ whenever $p(x, x') < \delta$ for all $x, x' \in X$ and ***every*** $f \in \mathscr{F}$.

- **Remark** An *equicontinuous family* of functions is *a family of continuous functions*.

- **Remark** ***Continuity*** of the function f at x_0 means that ***given*** f and given $\epsilon > 0$, there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$. ***Equicontinuity* of $\mathscr{F}$ means that *a single neighborhood* U can be chosen that will work for all the functions** f in the collection $\mathscr{F}$.

- **Lemma 1.10** *(**Total Boundedness** $\Rightarrow$ **Equicontinuous**) [Munkres, 2000]*
*Let X be a **space**; let (Y, d) be a **metric** space. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is **totally bounded** under the **uniform metric** corresponding to d, then $\mathscr{F}$ is **equicontinuous** under d.*

- **Lemma 1.11** *(**Equicontinuous** + **Compactness** $\Rightarrow$ **Total Boundedness**) [Munkres, 2000]*
*Let X be a space; let (Y, d) be a metric space; assume X and Y are **compact**. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is **equicontinuous** under d, then $\mathscr{F}$ is **totally bounded** under the **uniform** and **sup** metrics corresponding to d.*

- **Definition** (***Pointwise Bounded***)
If (Y, d) is a *metric space*, a *subset* $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is said to be ***pointwise bounded*** under d if for each $x \in X$, the subset
$$F_a = \{f(a) : f \in \mathscr{F}\}$$
of Y is ***bounded*** under d.

- **Theorem 1.12** *(**Ascoli's Theorem, Classical Version**). [Munkres, 2000]*
*Let X be a **compact** space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding **uniform topology**. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has **compact closure if and only if** $\mathscr{F}$ is **equicontinuous** and **pointwise bounded** under d.*

- **Corollary 1.13** *Let X be **compact**; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding **uniform topology**. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is **compact if and only if** it is **closed, bounded** under the **sup metric** ρ, and **equicontinuous** under d.*

- **Remark** (***Ascoli's Theorem, Sequence Version***) [Reed and Simon, 1980]
*Let $\{f_n\}$ be a family of **uniformly bounded equicontinuous functions** on $[0, 1]$. Then **some subsequence** $\{f_{n,m}\}$ converges **uniformly** on $[0, 1]$.*

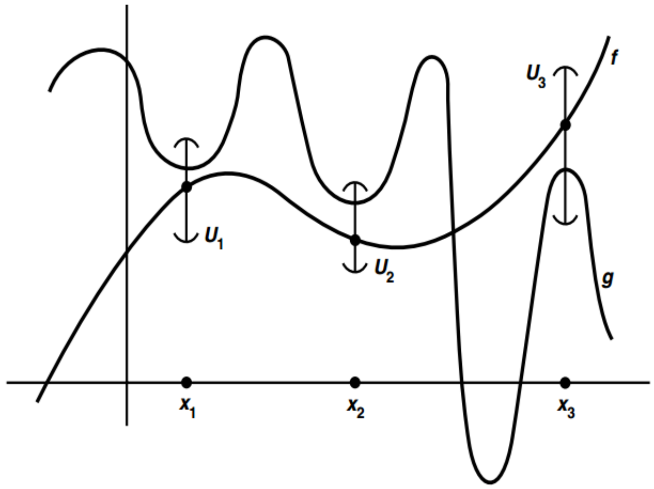

Figure 1: The function g in neighborhood of f in topology of pointwise convergence. [Munkres, 2000]

1.3 Pointwise and Compact Convergence

- **Definition** (***Topology of Pointwise Convergence / Point-Open Topology***)
  Given a point x of the set X and an open set U of the space Y, let
  $$S(x,U) = \left\{f : f \in Y^X \text{ and } f(x) \in U\right\}.$$
  The sets $S(x,U)$ are a ***subbasis*** for *topology* on Y^X, which is called ***the topology*** of ***pointwise convergence*** (or ***the point-open topology***)

- **Remark** (***Basis of Point-Open Topology***)
  The general *basis element* for this topology is a *finite intersection* of subbasis elements $S(x,U)$. Thus a typical ***basis element*** about the function f *consists of all functions g that are **"close"** to f **at finitely many points***. Such a *neighborhood* is illustrated in Figure 1; it consists of all functions g whose graphs *intersect the three vertical intervals* pictured.

- **Remark** ***The topology of pointwise convergence on Y^X is the product topology***.

  If we replace X by J and denote the general element of J by α to make it look more familiar, then the set $S(\alpha,U)$ of all functions $x : J \to Y$ such that $x(\alpha) \in U$ is just the subset $\pi_\alpha^{-1}(U)$ of Y^J, which is *the standard subbasis element* for the product topology.

- **Proposition 1.14** ***(Pointwise Convergence Topology)****[Munkres, 2000]*
  *A sequence f_n of functions **converges** to the function f in the **topology of pointwise convergence if and only if** for **each** x in X, the sequence $f_n(x)$ of **points of** Y converges to the point $f(x)$.*

- **Remark** Compare the *subbasis* of *the point-open topology* on function space Y^X and *the weak topology* on space X
  $$\begin{aligned} S(x,U) &= \left\{f : f \in Y^X \text{ and } f(x) \in U\right\} && \textit{point-open topology}. \\ B(f,U) &= \{x : x \in X \text{ and } f(x) \in U\} && \textit{weak topology}. \end{aligned}$$

- **Example** (**Pointwise Convergence Does Not Preserve Continuity**)

Consider the space $\mathbb{R}^I$, where $I = [0, 1]$. The sequence (f_n) of continuous functions given by $f_n(x) = x^n$ *converges* in the ***topology of pointwise convergence*** to the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases},$$

This example shows that the subspace $\mathcal{C}(I, \mathbb{R})$ of continuous functions is ***not closed*** in $\mathbb{R}^I$ *in the topology of pointwise convergence.* Note that $\mathcal{C}(I, \mathbb{R})$ is ***closed*** *in* $\mathbb{R}^I$ *under* ***uniform topology*** due to *Uniform Limit theorem.*

- **Definition** (***Topology of Compact Convergence***)
  Let (Y, d) be a *metric space*; let X be a *topological space*. Given an element f of Y^X, a ***compact subspace*** C of X, and a number $\epsilon > 0$, let $B_C(f, \epsilon)$ denote the set of all those elements g of Y^X for which

  $$\sup\{d(f(x), g(x)) : x \in C\} < \epsilon.$$

  The sets $B_C(f, \epsilon)$ form a ***basis*** for a topology on Y^X. It is called the ***topology of compact convergence*** (or sometimes the "***topology of uniform convergence on compact sets***").

- **Proposition 1.15** *(**Topology of Uniform Convergence in Compact Sets**) [Munkres, 2000]*
  *A sequence $f_n : X \to Y$ of functions converges to the function f in the **topology of compact convergence** if and only if for **each compact subspace** C of X, the sequence $f_n|_C$ converges **uniformly** to $f|_C$.*

- **Definition** A space X is said to be ***compactly generated*** if it satisfies the following condition: A set A is ***open*** in X if $A \cap C$ is ***open*** in C for each **compact subspace** C of X.

- **Lemma 1.16** *[Munkres, 2000]*
  *If X is **locally compact**, or if X satisfies **the first countability axiom**, then X is **compactly generated**.*

- The crucial fact about compactly generated spaces is the following:

  Lemma 1.17 *(**Continuous Extension on Compact Generated Space**) [Munkres, 2000]*
  *If X is compactly generated, then a function $f : X \to Y$ is **continuous** if for each **compact subspace** C of X, the restricted function $f|_C$ is **continuous**.*

- **Theorem 1.18** *(**$\mathcal{C}(X, Y)$ on Compact Generated Space**) [Munkres, 2000]*
  *Let X be a **compactly generated space**: let (Y, d) be a metric space. Then $\mathcal{C}(X, Y)$ is **closed** in Y^X in the **topology of compact convergence**.*

- **Corollary 1.19** *(**Compact Convergence Limit**) [Munkres, 2000]*
  *Let X be a **compactly generated space**; let (Y, d) be a **metric** space. If a sequence of **continuous** functions $f_n : X \to Y$ converges to f in the **topology of compact convergence**, then f is **continuous**.*

- **Remark** (***Useful Topologies on*** Y^X)

1. ***Uniform Topology***: generated by the ***basis***

$$B_U(f,\epsilon) = \left\{ g \in Y^X : \sup_{x\in X} \bar{d}(f(x), g(x)) < \epsilon \right\}$$

   It corresponds to ***the uniform convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *uniform topology*, following *the Uniform Limit Theorem.*

2. ***Topology of Pointwise Convergence***: generated by the ***basis***

$$\begin{aligned} B_{U_1,\ldots,U_n}(x_1,\ldots,x_n,\epsilon) &= \bigcap_{i=1}^{n} S(x_i, U_i) \\ &= \left\{ f \in Y^X : f(x_1) \in U_1, \ldots, f(x_n) \in U_n \right\}, \quad 1 \le n < \infty. \end{aligned}$$

   It corresponds to ***the pointwise convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***not closed*** in Y^X under the *topology of pointwise convergence*

3. ***Topology of Compact Convergence***: generated by the ***basis***

$$B_C(f,\epsilon) = \left\{ g \in Y^X : \sup_{x\in C} d(f(x), g(x)) < \epsilon \right\}.$$

   It corresponds to ***the uniform convergence*** of f_n to f in Y^X for $x \in C$. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *topology of compact convergence* ***if X is compactly generated***.

- **Theorem 1.20** ***(Relationship between Topologies on Y^X)*** *[Munkres, 2000]*
  Let X be a space; let (Y,d) be a metric space. For the function space Y^X , one has the following ***inclusions of topologies:***

  (uniform) $\supseteq$ ***(compact convergence)*** $\supseteq$ ***(pointwise convergence)***.

  If X is ***compact***, *the* ***first two*** *coincide, and if X is* ***discrete***, *the* ***second two*** *coincide.*

- **Remark** Note that both *uniform topology* and *topology of compact convergence* rmade specific use of the metric d for the space Y, i.e. it can only be defined when the image of function Y is a metric space.

  But ***the topology of pointwise convergence*** does *not use the definition of metric* d in Y. In fact, ***it is defined for any image space*** Y.

- **Definition** (***Compact-Open Topology on Continuous Function Space***)
  Let X and Y be topological spaces. If C is a ***compact subspace*** of X and U is an *open* subset of Y, define

$$S(C,U) = \{ f \in \mathcal{C}(X,Y) : f(C) \subseteq U \}.$$

  The sets $S(C,U)$ form a ***subbasis*** for a *topology* on $\mathcal{C}(X,Y)$ that is called ***the compact-open topology***.

- **Proposition 1.21** *(**Compact-Open on $\mathcal{C}(X,Y)$ = Compact Convergence**) [Munkres, 2000]*
  Let X be a space and let (Y,d) be a metric space. On the set $\mathcal{C}(X,Y)$, the ***compact-open topology*** *and the* ***topology of compact convergence coincide***.

- **Corollary 1.22** ***(Compact Convergence on*** $\mathcal{C}(X,Y)$ ***Need Not*** d*)* *[Munkres, 2000]*
  Let Y be a metric space. The ***compact convergence topology*** *on $\mathcal{C}(X,Y)$ does* ***not*** *depend on the* ***metric*** *of Y. Therefore if X is* ***compact****, the* ***uniform topology*** *on $\mathcal{C}(X,Y)$ does not depend on the metric of Y.*

- **Remark** The fact that the definition of ***the compact-open topology*** does not involve a ***metric*** is just one of its useful features.

  Another is the fact that it satisfies the requirement of "***joint continuity***. Roughly speaking, this means that the expression $f(x)$ is *continuous* not only in the *single*"variable x, but is ***continuous jointly*** *in* ***both*** *the x and f .*

- **Theorem 1.23** ***(Compact-Open Topology*** $\Rightarrow$ ***Joint Continuity for*** x ***and*** f*)*
  Let X be ***locally compact Hausdorff****; let $\mathcal{C}(X,Y)$ have the* ***compact-open topology****. Then the map*

  $$e : X \times \mathcal{C}(X,Y) \to Y$$

  defined by the equation

  $$e(x,f) = f(x)$$

  is ***continuous****. The map e is called* ***the evaluation map****.*

- **Definition** Given a function $f : X \times Z \to Y$, there is a corresponding function $F : Z \to \mathcal{C}(X,Y)$, defined by the equation

  $$(F(z))(x) = f(x,z).$$

  Conversely, given $F : Z \to \mathcal{C}(X,Y)$, this equation defines a corresponding function $f : X \times Z \to Y$. We say that F is the map of Z into $\mathcal{C}(X,Y)$ that is induced by f.

- **Proposition 1.24** *Let X and Y be spaces; give $\mathcal{C}(X,Y)$ the* ***compact-open topology****. If $f : X \times Z \to Y$ is* ***continuous****, then* ***so is*** *the induced function $F : Z \to \mathcal{C}(X,Y)$. The* ***converse*** *holds if X is* ***locally compact Hausdorff****.*

- **Theorem 1.25** ***(Ascoli's Theorem, General Version)****. [Munkres, 2000]*
  Let X be a space and let (Y,d) be a ***metric*** *space. Give $\mathcal{C}(X,Y)$ the* ***topology of compact convergence****; let $\mathcal{F}$ be a subset of $\mathcal{C}(X,Y)$.*

  1. *If $\mathcal{F}$ is* ***equicontinuous*** *under d and the set*

     $$F_a = \{f(a) : f \in \mathcal{F}\}$$

     has ***compact closure*** *for each $a \in X$, then $\mathcal{F}$ is* ***contained*** *in a* ***compact subspace*** *of $\mathcal{C}(X,Y)$.*

  2. *The* ***converse*** *holds if X is* ***locally compact Hausdorff****.*

- **Remark** Compare with classical version, we see generalizations:

  1. X need not to be ***compact***; $\Rightarrow$ does not even need X to be topological. $\Leftarrow$ holds when X is ***locally compact Hausdorff***.

2. $\mathcal{C}(X, Y)$ is under ***compact-open topology*** which is ***weaker*** than ***uniform topology***, i.e. we does not require convergence of sequence *uniformly* but only *uniformly in a compact subset*.

3. $\mathcal{F}$ does not need to be ***pointwise bounded*** under d. In other word, the set

$$F_a = \{f(a) : f \in \mathcal{F}\}$$

need not to be ***bounded*** but need to have ***compact closure*** for each $a \in X$. Note that for metric space Y, if Y is finite dimensional, it is the same requirement as boundness. But compact closure is stronger than bounded.

- **Proposition 1.26** *(**Equicontinuity** + **Pointwise Convergence** $\Rightarrow$ **Compact Convergence**) [Munkres, 2000]*
*Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of **continuous** functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f in the **topology of pointwise convergence**. If $\{f_n\}$ is **equicontinuous**, then f is **continuous** and f_n converges to f in the **topology of compact convergence**.*

2 Compactness in Banach Space

Remark (***Compactness in Function Space***)
The importance of ***compactness*** in analysis is well-known, and the fact tha *closed bounded sets* are *compact* in *finite dimensional spaces* lies at the heart of much of the analysis on these spaces. ***Unfortunately***, as we have seen, this is *not true* in *infinite dimensional spaces*.

There are ***two main compactness results*** in *function space*:

1. The ***Ascoli's theorem***: Let X be a *compact Hausdorff space*; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding ***uniform topology***. *A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is **compact** if and only if it is **closed, bounded** under the **sup metric** ρ, and **equicontinuous** under d.*

2. The ***Banach-Alaoglu theorem***: Let X be a *Banach space. The **unit ball** in X^*, $\{f \in X^* : \|f\| \le 1\}$ is **compact** in the **weak* topology**.*

   In this section we will show that a *partial analogue* of this result can be obtained in ***infinite dimensions*** if we adopt a *weaker definition of the convergence* of a sequence than the usual definition.

2.1 Strong and Weak Convergence

- **Definition** (***Strong Convergence***). [Kreyszig, 1989]
A sequence (x_n) in a normed space X is said to be ***strongly convergent*** (or ***convergent in the norm***) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

This is written $\lim_{n\to\infty} x_n = x$ or simply $x_n \to x$ is called *the **strong limit** of* (x_n), and we say that (x_n) *converges **strongly** to* x.

- **Definition** (***Weak Convergence***). [Kreyszig, 1989]
  A sequence (x_n) in a normed space X is said to be ***weakly convergent*** if there is an $x \in X$ such that *for **every** $f \in X^*$*,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

  This is written $x_n \xrightarrow{w} x$ or $x_n \rightharpoonup x$. The element x is called ***the weak limit*** *of* (x_n), and we say that (x_n) ***converges weakly*** *to* x*.*

- **Remark** For weak convergence, we see it as convergence of *real numbers* $s_n = f(x_n)$ in $\mathbb{R}$.

- **Remark** (***Weak Convergence Analysis is Common***)
  Weak convergence has various applications throughout analysis (for instance, in the *calculus of variations*, *the general theory of differential equations* and *probability theory*).

  The concept illustrates ***a basic principle of functional analysis***, namely, the fact that ***the investigation of spaces is often related to that of their dual spaces***, i.e. *probing a variable by using a test functional.*

- **Remark** In *Hilbert space* $\mathcal{H}$, we say $x_n \xrightarrow{w} x$ if there exists an $x \in \mathcal{H}$ such that for all $y \in \mathcal{H}$

$$\lim_{n\to\infty} \langle x_n\,,\, y\rangle = \langle x\,,\, y\rangle\,.$$

  Note that given a set of orthonormal basis (e_n), we have $f(e_n) := \langle e_n\,,\, y\rangle$ and from Bessel inequality

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle e_n\,,\, y\rangle|^2 \le \|y\|^2 < \infty \\ \Rightarrow \lim_{n\to\infty} |\langle e_n\,,\, y\rangle| \to 0 \\ \Rightarrow e_n \xrightarrow{w} 0.\end{aligned}$$

  But $\|e_n - e_m\| \not\to 0$, (e_n) does not converge in norm (strongly).

- **Lemma 2.1** *(**Weak Convergence**).*
  *Let (x_n) be a **weakly convergent** sequence in a normed space X, say, $x_n \xrightarrow{w} x$. Then:*

  1. *The weak limit x of (x_n) is **unique**.*
  2. *Every **subsequence** of (x_n) converges weakly to x.*
  3. *The sequence $(\|x_n\|)$ is **bounded**.*

- **Proposition 2.2** *(**Strong and Weak Convergence**). [Kreyszig, 1989]*
  Let (x_n) be a sequence in a normed space X. Then:

  1. ***Strong convergence** implies **weak convergence** with the same limit.*
  2. *The converse of (1) is **not** generally true.*
  3. *If dim $X < \infty$, then **weak convergence** implies **strong convergence**.*

- **Remark** From above, we see that in ***finite dimensional normed spaces*** the distinction between ***strong*** and ***weak convergence*** disappears completely.

2.2 Weak Topology

- **Remark** The weak convergence, $x_n \overset{w}{\to} x$, can be considered as *convergence of net* $\{x_n\}_{n=1}^{\infty}$ *in the* ***weak topology***.

- **Definition** (***Weak Topology on a Set*** S) [Reed and Simon, 1980]
Let $\mathcal{F}$ be a family of functions from a set S to a topological vector space $(X, \mathscr{T})$. The ***$\mathcal{F}$-weak*** *(or simply* ***weak****)* ***topology*** on S is the weakest topology for which ***all the functions*** $f \in \mathcal{F}$ *are* ***continuous***.

- **Remark** (***Construction of Weak Topology***) [Reed and Simon, 1980]
To construct a ***$\mathcal{F}$-weak topology*** on S, we take the family of *all* ***finite intersections*** *of sets* of the form $f^{-1}(U)$ where $f \in \mathcal{F}$ and $U \in \mathscr{T}$. The collections of these finite intersections of sets *form* ***a basis*** *of* ***the $\mathcal{F}$-weak topology***.

  In other word, ***the subbasis*** for the *$\mathcal{F}$-weak topology* on S is of form
$$\mathscr{S} = \left\{ f^{-1}(U) : f \in \mathscr{F}, \text{ and } U \in \mathscr{T} \right\}$$
  And the basis of $\mathscr{T}$
$$\begin{aligned} \mathscr{B} &= \left\{ f_1^{-1}(U_1) \cap \ldots \cap f_k^{-1}(U_k) : f_1, \ldots, f_k \in \mathscr{F},\; U_1, \ldots, U_k \in \mathscr{T},\; 1 \le k < \infty \right\} \\ B \in \mathscr{B} \Rightarrow B &= \{x : f_1(x) \in U_1, \ldots, f_k(x) \in U_k\},\; 1 \le k < \infty \\ &= \{x : (f_1(x), \ldots, f_k(x)) \in U\}. \end{aligned}$$
  The basis element is called a ***k-dimensional cylinder set***.

- **Remark** Given a topology on Y and a family of functions in $Y^X = \{f : X \to Y\}$, $\mathscr{F}$-weak topolgy is ***a natural topology*** on X without additional information.

  A product topology on Y^{ω} can be seen as a $\mathscr{F}$-weak topology when $\mathscr{F} = \{\pi_{\alpha} : \prod_i Y_i \to Y_{\alpha}\}$.

- **Remark** A set S equipped with *$\mathcal{F}$-weak topology* ***has little knowledge on itself besides the output of functions*** $f \in \mathcal{F}$ from a family $\mathcal{F}$. The induced topology through a family of functions thus does not tell much besides the behavior of its output.

  For instance, S is the space of hidden states, $\mathcal{F} = \{f_1, \ldots, f_n\} \subset 2^S$ is a series of binary statistical tests, the weak topology on S *partition the domain according to the output of each test*.

- **Remark** By construction, the ***neighborhood base*** of each point $x \in S$ under the $\mathcal{F}$-weak topology is contained in the pre-images $\{f_n^{-1}(U_n)\}$ for ***finitely many*** of $(f_n) \in \mathcal{F}$.

- **Definition** (***Weak Topology on Banach Space***)
Let X be a ***Banach space*** with dual space X^*. The ***weak topology*** on X is *the weakest topology* on X so that $f(x)$ *is* ***continuous for all*** $f \in X^*$.

- **Remark** For infinite dimensional Banach spaces, ***the weak topology does not arise from a metric***. This is one of the main reasons we have introduced topological spaces.

- **Remark** Thus a ***neighborhood*** *base at zero* for ***the weak topology*** is given by the sets of the form
$$N(f_1, \ldots, f_n; \epsilon) = \{x : |f_j(x)| < \epsilon;\; j = 1, \ldots, n\}$$

that is, neighborhoods of zero contain ***cylinders*** *with* ***finite-dimensional*** *open bases*. A net $\{x_\alpha\}$ converges *weakly* to x, written $x_\alpha \xrightarrow{w} x$, if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

- **Proposition 2.3** *[Reed and Simon, 1980]*

  1. *The weak topology is* ***weaker*** *than* ***the norm topology****, that is, every weakly open set is norm open.*
  2. *Every* ***weakly convergent*** *sequence is* ***norm bounded****.*
  3. *The weak topology is a* ***Hausdorff*** *topology.*

- **Proposition 2.4** ***(Weak Topology on Hilbert Space)*** *[Reed and Simon, 1980]*
  Let $\mathcal{H}$ be a ***Hilbert space****. Let $\{\varphi_\alpha\}_{\alpha \in I}$ be an* ***orthonormal basis*** *for $\mathcal{H}$. Given a sequence $\psi_n \in \mathcal{H}$, let*

  $$\psi_n^{(\alpha)} = \langle \psi_n \, , \, \varphi_\alpha \rangle$$

  be the coordinates of ψ_n. Then $\psi_n \to \psi$ in the ***weak topology*** *(or $\psi_n \xrightarrow{w} \psi$)* ***if and only if***

  1. *$\psi_n^{(\alpha)} \to \psi^{(\alpha)}$ for each α; and*
  2. *$\|\psi_n\|$ is* ***bounded****.*

  Proof: Suppose $\psi_n \xrightarrow{w} \psi$; then (1) follows by definition and (2) comes from the fact that *every weakly convergent sequence is norm bounded.*

  On the other hand, let (1) and (2) hold and let $\mathcal{F} \subset \mathcal{H}$ be the subspace of *finite linear combinations* of the φ_α. By (1), $\langle \psi_n \, , \, \varphi_\alpha \rangle \to \langle \psi \, , \, \varphi_\alpha \rangle$ if $\varphi \in \mathcal{F}$. Using the fact that $\mathcal{F}$ is dense, (2), and an $\epsilon/3$ argument, the weak convergence follows. ∎

- **Proposition 2.5** ***(Weak Topology of $\mathcal{C}(X)$ on Compact Hausdorff Space)*** *[Reed and Simon, 1980]*
  Let X be a ***compact Hausdorff*** *space and consider the* ***weak topology on*** *$\mathcal{C}(X)$ (i.e. $\mathcal{C}(X, \mathbb{R})$). Let $\{f_n\}$ be a sequence in $\mathcal{C}(X)$. Then $f_n \to f$ in the* ***weak topology*** *(or $f_n \xrightarrow{w} f$)* ***if and only if***

  1. *$f_n(x) \to f(x)$ for each $x \in X$; and*
  2. *$\|f_n\|$ is* ***bounded****.*

  Proof: For if $f_n \xrightarrow{w} f$, then (1) holds since $f \to f(x)$ is an element of $\mathcal{C}(X)^*$ and (2) comes from the fact that *every weakly convergent sequence is norm bounded.*

  On the other hand, if (1) and (2) hold, then

  $$|f_n(x)| \leq \sup_n \|f_n\|_\infty$$

  which is L^1 with respect to any *Baire measure* μ. Thus, by the *dominated convergence theorem*, for any $\mu \in \mathcal{M}_+(X)$, $\int f_n d\mu \to \int f d\mu$. Since any $\lambda \in \mathcal{M}(X) = \mathcal{C}(X)^*$ is a *finite linear combination* of measures in $\mathcal{M}_+(X)$, we conclude that $f_n \to f$ weakly. ∎

- **Proposition 2.6** ***(Banach Space Weak Continuity = Norm Continuity)*** *[Reed and Simon, 1980]*
  A linear functional f on a ***Banach space*** *is* ***weakly continuous if and only if*** *it is* ***norm continuous****.*

2.3 Weak* Topology

- **Definition** (***Weak*** ***Topology on Banach Space***)
Let X be a *normed vector space* and X^* be its dual space. The ***weak*** ***topology*** on X^* is *the weakest topology* on X^* so that $f(x)$ *is* ***continuous for all*** $x \in X$.

- **Remark** ***The weak*** ***topology*** can be considered as a topology induced by $x \in X$ on dual space X^*, i.e. a topology on functional space on X induced by point in X.

  In fact, ***the weak*** ***topology*** is *the* ***topology*** *of* ***pointwise convergence***:
$$f_\alpha \to f \quad \Leftrightarrow \quad f_\alpha(x) \to f(x) \text{ for all } x \in X.$$
  Moreover, ***the weak*** ***topology*** is *the* ***product topology*** *on product space* $\mathbb{R}^X$.

- **Definition** (Y***-Weak Topology*** $\sigma(X, Y)$)
Let X be a *vector space* and let Y be a *family of* ***linear functionals*** on X which ***separates points*** of X. That is, for any $x_1 \neq x_2$ in X, there exists a $f \in Y$ so that $f(x_1) \neq f(x_2)$. Then ***the*** Y***-weak topology on*** X, written $\sigma(X, Y)$, is *the weakest topology* on X for which all the *functionals* in Y are *continuous.*

- **Remark** Y*-weak topology* $\sigma(X, Y)$ is *the* $\mathscr{F}$*-weak topology* when *domain* of $\mathscr{F}$ is a *vector space* and $\mathscr{F}$ is a family of *linear functionals.*

- **Remark** Because Y is assumed to *separate points*, $\sigma(X, Y)$ is *a* ***Hausdorff topology*** on X. Note that

  1. *the weak topology* on X is *the* $\sigma(X, X^*)$ *topology*
  2. *the weak** *topology* on X^* is *the* $\sigma(X^*, X)$ *topology*

  The $\sigma(X, Y)$ *topology depends* only on ***the vector space generated by*** Y so we henceforth suppose that Y *is a vector space.*

- **Remark** Notice that ***the weak*** ***topology*** *is even* ***weaker*** *than* ***the weak topology***.
$$\textbf{the norm topology} \subset \textbf{the weak topology} \subseteq \textbf{the weak}^* \textbf{ topology}$$

- **Remark** As one might expect, X is ***reflexive*** if and only if the ***weak*** and ***weak*** ***topologies coincide***, and many *characterizations* of *reflexivity* depend on relations involving the *weak* and *weak** *topologies.*

- **Proposition 2.7** *(*$\sigma(X, Y)$ ***Topology*** = ***Pointwise Convergence Topology on*** X*) [Reed and Simon, 1980]*
The $\sigma(X, Y)$*-****continuous*** *linear functionals on* X *are* ***precisely*** Y*, in particular the only* ***weak*** ***continuous functionals*** *on* X^* *are the* ***elements*** *of* X*.*

- **Theorem 2.8** ***(The Banach-Alaoglu Theorem)*** *[Reed and Simon, 1980]*
Let X^* *be the dual of some Banach space,* X*. Then* ***the unit ball*** *in* X^**,* $\{f \in X^* : \|f\| \leq 1\}$ *is* ***compact*** *in the* ***weak*** ***topology***.

- **Corollary 2.9** ***(The Banach-Alaoglu Theorem, Sequential Version)*** *[Rynne and Youngson, 2007]*
If X *is* ***separable*** *and* $\{f_n\}$ *is a* ***bounded*** *sequence in* X^**, then* $\{f_n\}$ *has a* ***weak*** ***convergent subsequence***.

- **Theorem 2.10** ***(Kakutani's Theorem)*** *[Rynne and Youngson, 2007]*
  *X is **reflexive** Banach space **if and only if the unit ball** in X, $\{x \in X : \|x\| \leq 1\}$ is **compact** in the **weak topology**.*

- **Corollary 2.11** *[Rynne and Youngson, 2007]*
  *If X is **reflexive** Banach space and $\{x_n\}$ is a **bounded** sequence in X, then $\{x_n\}$ has a **weakly convergent subsequence**.*

- **Corollary 2.12** *[Rynne and Youngson, 2007]*
  *If X is **reflexive** Banach space and $M \subseteq X$ is **bounded, closed** and **convex**, then any sequence in M has a **subsequence** which is **weakly convergent** to an element of M.*

- **Exercise 2.13** *[Rynne and Youngson, 2007]*
  *Suppose that X is **reflexive** Banach space, M is a **closed, convex** subset of X, and $y \in X \setminus M$. Show that there is a point $y_M \in M$ such that*

  $$y - y_M = \inf\{y - x : x \in M\}.$$

  *Show that this result is **not true** if the assumption that M is **convex** is omitted.*

- **Example** (***Convergence in Distribution***)
  Convergence in distribution is also called ***weak convergence*** in probability theory [Folland, 2013]. In functional analysis, however, ***weak convergence*** is actually reserved for a different mode of convergence, while ***the convergence in distribution*** is ***the weak**** ***convergence*** on $\mathcal{M}(X)$.

  In general, it is actually ***not*** *a mode of* ***convergence of functions*** f_n ***itself*** but instead is *the* ***convergence*** *of* ***bounded linear functionals*** $\int f d\mu_n$. Equivalently, it is *the* ***convergence of measures*** F_n *on* $\mathscr{B}(\mathbb{R})$.

  $$\text{weak convergence} \qquad \int f_n d\mu \to \int f d\mu, \quad \forall \mu \in \mathcal{M}(X),$$
  $$\text{convergence in distribution} \qquad \int f d\mu_n \to \int f d\mu, \quad \forall f \in \mathcal{C}_0(X)$$

  Definition (***Cumulative Distribution Function***) [Van der Vaart, 2000]
  Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. Given any real-valued measurable function $\xi : \Omega \to \mathbb{R}$, we define the ***cumulative distribution function*** $F : \mathbb{R} \to [0, \infty]$ of ξ to be the function

  $$F_\xi(\lambda) := \mu\left(\{x \in X : \xi(x) \leq \lambda\}\right) = \int_X \mathbb{1}\left\{\xi(x) \leq \lambda\right\} d\mu(x).$$

  Definition (***Converge in Distribution***) [Van der Vaart, 2000]
  Let $\xi_n : \Omega \to \mathbb{R}$ be a sequence of real-valued *measurable functions*, and $\xi : \Omega \to \mathbb{R}$ be another measurable function. We say that ξ_n ***converges in distribution*** to ξ if *the cumulative distribution function* $F_n(\lambda)$ of ξ_n ***converges pointwise*** to the cumulative distribution function $F(\lambda)$ of ξ at all $\lambda \in \mathbb{R}$ for which F is continuous. Denoted as $\xi_n \xrightarrow{F} \xi$ or $\xi_n \xrightarrow{d} \xi$ or $\xi_n \rightsquigarrow \xi$.

  $$\xi_n \xrightarrow{d} \xi \;\Leftrightarrow\; F_n(\lambda) \to F(\lambda), \text{ for all } \lambda \in \mathbb{R}$$

  Theorem 2.14 ***(The Portmanteau Theorem)****. [Van der Vaart, 2000]*
  The following statements are equivalent.

1. $X_n \rightsquigarrow X$.

2. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *that are non-zero only on a* ***closed*** *and* ***bounded*** *set.*

3. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***bounded continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$.

4. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***bounded measurable functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *for which* $\mathbb{P}(X \in \{x : h \text{ is continuous at } x\}) = 1$.

We can reformulate the definition of *convergence in distribution* as below:

Definition [Wellner et al., 2013]
Let $(\mathcal{X}, d)$ be a *metric space*, and $(\mathcal{X}, \mathscr{B})$ be *a measurable space*, where $\mathscr{B}$ is ***the Borel σ-field on $\mathcal{X}$***, the smallest σ-field containing *all the open balls* (as the basis of *metric topology* on $\mathcal{X}$). Let $\{\mathcal{P}_n\}$ and $\mathcal{P}$ be ***Borel probability measures*** on $(\mathcal{X}, \mathscr{B})$.

Then the sequence $\mathcal{P}_n$ ***converges in distribution*** to $\mathcal{P}$, which we write as $\mathcal{P}_n \rightsquigarrow \mathcal{P}$, if and only if

$$\int_\Omega f d\mathcal{P}_n \to \int_\Omega f d\mathcal{P}, \qquad \text{for all } f \in \mathcal{C}_b(\mathcal{X}).$$

Here $\mathcal{C}_b(\mathcal{X})$ denotes the set of *all* ***bounded, continuous****, real functions on* $\mathcal{X}$.

We can see that ***the convergence in distribution*** is actually ***a weak* convergence***. That is, it is ***the weak convergence*** of ***bounded linear functionals*** $I_{\mathcal{P}_n} \stackrel{w^*}{\to} I_{\mathcal{P}}$ on *the space of all probability measures* $\mathcal{P}(\mathcal{X}) \simeq (\mathcal{C}_b(\mathcal{X}))^*$ on $(\mathcal{X}, \mathscr{B})$ where

$$I_{\mathcal{P}} : f \mapsto \int_\Omega f d\mathcal{P}.$$

Note that the $I_{\mathcal{P}_n} \stackrel{w^*}{\to} I_{\mathcal{P}}$ is equivalent to $I_{\mathcal{P}_n}(f) \to I_{\mathcal{P}}(f)$ *for all* $f \in \mathcal{C}_b(\mathcal{X})$.

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