Lecture 3: Information Inequalities

Tianpei Xie

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Contents

1 Information Theory Basics

1.1 Entropy, Relative Entropy, and Mutual Information

- **Definition** (***Shannon Entropy***) [Cover and Thomas, 2006]
  Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X : \mathbb{R} \to \mathcal{X}$ be a random variable. Define $p(x)$ as *the probability density function* of X with respect to a base measure μ on $\mathcal{X}$. ***The Shannon Entropy*** is defined as

$$
\begin{aligned}
H(X) &:= \mathbb{E}_p\left[-\log p(X)\right] \\
&= \int_{\Omega} -\log p(X(\omega)) d\mathbb{P}(\omega) \\
&= -\int_{\mathcal{X}} p(x) \log p(x) d\mu(x)
\end{aligned}
$$

- **Definition** (***Conditional Entropy***) [Cover and Thomas, 2006]
  If a pair of random variables (X, Y) follows the joint probability density function $p(x, y)$ with respect to a base product measure μ on $\mathcal{X} \times \mathcal{Y}$. Then ***the joint entropy*** of (X, Y), denoted as $H(X, Y)$, is defined as

$$
H(X, Y) := \mathbb{E}_{X,Y}\left[-\log p(X, Y)\right] = -\int_{\mathcal{X}\times\mathcal{Y}} p(x, y) \log p(x, y) d\mu(x, y)
$$

  Then ***the conditional entropy*** $H(Y|X)$ is defined as

$$
\begin{aligned}
H(Y|X) &:= \mathbb{E}_{X,Y}\left[-\log p(Y|X)\right] = -\int_{\mathcal{X}\times\mathcal{Y}} p(x, y) \log p(y|x) d\mu(x, y) \\
&= \mathbb{E}_X\left[\mathbb{E}_Y\left[-\log p(Y|X)\right]\right] = \int_{\mathcal{X}} p(x) \left(-\int_{\mathcal{Y}} p(y|x) \log p(y|x) d\mu(y)\right) d\mu(x)
\end{aligned}
$$

- **Proposition 1.1** *(**Properties of Shannon Entropy**) [Cover and Thomas, 2006]*
  Let X, Y, Z be random variables.

  1. *(**Non-negativity**) $H(X) \geq 0$;*
  2. *(**Chain Rule**)*

$$
H(X, Y) = H(X) + H(Y|X)
$$

     Furthermore,

$$
H(X, Y|Z) = H(X|Z) + H(Y|X, Z)
$$

  3. *(**Sub-Additivity**)*

$$
H(X, Y) \leq H(X) + H(Y)
$$

  4. *(**Concavity**) $H(p) := \mathbb{E}_p\left[-\log p(X)\right]$ is a concave function in terms of p.d.f. p, i.e.*

$$
H(\lambda p_1 + (1-\lambda)p_2) \geq \lambda H(p_1) + (1-\lambda) H(p_2)
$$

     for any two p.d.fs p_1, p_2 on $\mathcal{X}$ and any $\lambda \in [0, 1]$.

- **Definition** (***Relative Entropy / Kullback-Leibler Divergence***) [Cover and Thomas, 2006]
  Suppose that P and Q are *probability measures* on a measurable space $\mathcal{X}$, and P is *absolutely continuous* with respect to Q, then ***the relative entropy*** or ***the Kullback-Leibler divergence*** is defined as

$$\mathbb{KL}\left(P \,\|\, Q\right) := \mathbb{E}_{P}\left[\log\left(\frac{dP}{dQ}\right)\right] = \int_{\mathcal{X}} \log\left(\frac{dP(x)}{dQ(x)}\right) dP(x)$$

  where $\frac{dP}{dQ}$ is *the Radon-Nikodym derivative* of P with respect to Q. Equivalently, the KL-divergence can be written as

$$\mathbb{KL}\left(P \,\|\, Q\right) = \int_{\mathcal{X}} \left(\frac{dP(x)}{dQ(x)}\right) \log\left(\frac{dP(x)}{dQ(x)}\right) dQ(x)$$

  which is *the entropy of P relative to Q*. Furthermore, if μ is a base measure on $\mathcal{X}$ for which densities p and q with $dP = p(x)d\mu$ and $dQ = q(x)d\mu$ exist, then

$$\mathbb{KL}\left(P \,\|\, Q\right) = \int_{\mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right) d\mu(x)$$

- **Definition** (***Mutual Information***) [Cover and Thomas, 2006]
  Consider two random variables X, Y on $\mathcal{X} \times \mathcal{Y}$ with joint probability distribution $P_{(X,Y)}$ and marginal distribution P_X and P_Y. ***The mutual information*** $I(X;Y)$ is *the relative entropy* between *the joint distribution* $P_{(X,Y)}$ and *the product distribution* $P_X \otimes P_Y$:

$$I(X;Y) = \mathbb{KL}\left(P_{(X,Y)} \,\|\, P_X \otimes P_Y\right) = \mathbb{E}_{P_{(X,Y)}}\left[\log \frac{dP_{(X,Y)}}{dP_X \otimes dP_Y}\right]$$

  If $P_{(X,Y)}$ has a probability density function $p(x,y)$ with respect to a base measure μ on $\mathcal{X} \times \mathcal{Y}$, then

$$I(X;Y) = \int_{\mathcal{X}\times\mathcal{Y}} p(x,y) \log\left(\frac{p(x,y)}{p_X(x)p_Y(y)}\right) d\mu(x,y)$$

- **Proposition 1.2** ***(Properties of Relative Entropy and Mutual Information)*** *[Cover and Thomas, 2006]*
  Let X, Y be random variables.

  1. *(**Non-negativity**) Let $p(x), q(x)$ be probability density function of P, Q.*

$$\mathbb{KL}\left(P \,\|\, Q\right) \geq 0$$

     with equality if and only if $p(x) = q(x)$ almost surely. Therefore, the mutual information is non-negative as well:

$$I(X;Y) \geq 0$$

     with equality if and only if X and Y are independent.

2. ***(Finite Cardinality Domain)*** *Let $|\mathcal{X}|$ be the number of elements in domain $\mathcal{X}$ and X is a discrete random variables in $\mathcal{X}$. Then the relative entropy of probability distribution p with respect to uniform distribution u on $\mathcal{X}$ is*

$$\begin{aligned} \mathbb{KL}\left(p \,\|\, u\right) &= \log|\mathcal{X}| - H(X) \ge 0 \\ \Rightarrow H(X) &\le \log|\mathcal{X}| \end{aligned}$$

3. ***(Symmetry)*** $I(X;Y) = I(Y;X)$

4. ***(Information Gain via Conditioning)*** *The mutual information $I(X;Y)$ is the reduction in the uncertainty of X due to the knowledge of Y (and vice versa)*

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \qquad (1) \\ &= H(Y) - H(Y|X) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned}$$

5. ***(Shannon Entropy as Self-Information)*** $I(X;X) = H(X)$

1.2 Chain Rules for Entropy, Relative Entropy, and Mutual Information

- **Proposition 1.3** ***(Conditioning Reduces Entropy)*** *[Cover and Thomas, 2006]*
  From non-negativity of mutual information, we see that the entropy of X is non-increasing when conditioning on Y

$$H(X|Y) \le H(X) \qquad (2)$$

  where equality holds if and only if X and Y are independent.

- **Proposition 1.4** ***(Chain Rule for Entropy)*** *[Cover and Thomas, 2006]*
  Let $X_1, X_2, \ldots, X_n$ be drawn according to $p(x_1, x_2, \ldots, x_n)$. Then

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1) \qquad (3)$$

- **Proposition 1.5** ***(Sub-Additivity of Entropy)*** *[Cover and Thomas, 2006]*
  Let $X_1, X_2, \ldots, X_n$ be drawn according to $p(x_1, x_2, \ldots, x_n)$. Then

$$H(X_1, X_2, \ldots, X_n) \le \sum_{i=1}^{n} H(X_i) \qquad (4)$$

  with equality if and only if the X_i are independent.

- **Proposition 1.6** ***(Chain Rule for Mutual Information)*** *[Cover and Thomas, 2006]*
  Let $X_1, X_2, \ldots, X_n, Y$ be drawn according to $p(x_1, x_2, \ldots, x_n, y)$. Then

$$I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} H(X_i; Y|X_{i-1}, \ldots, X_1) \qquad (5)$$

  where ***the conditional mutual information*** *is defined as*

$$I(X;Y|Z) := H(X|Z) - H(X|Y,Z) = \mathbb{KL}\left(P_{(X,Y|Z)} \,\|\, P_{X|Z} \otimes P_{Y|Z}\right)$$

- **Proposition 1.7** ***(Chain Rule for Relative Entropy)*** *[Cover and Thomas, 2006]*
  Let $P_{(X,Y)}$ and $Q_{(X,Y)}$ be two probability measures on product space $\mathcal{X}\times\mathcal{Y}$ and $P \ll Q$. Denote the marginal distributions P_X, Q_X and P_Y, Q_Y on $\mathcal{X}$ and $\mathcal{Y}$, respectively. $P_{Y|X}$ and $Q_{Y|X}$ are conditional distributions (Note that $P_{Y|X} \ll Q_{Y|X}$). Define ***the conditional relative entropy*** *as*

$$\mathbb{E}_X\left[\mathbb{KL}\left(P_{Y|X} \,\|\, Q_{Y|X}\right)\right] := \mathbb{E}_X\left[\mathbb{E}_{P_{Y|X}}\left[\log\left(\frac{dP_{Y|X}}{dQ_{Y|X}}\right)\right]\right].$$

  Then the relative entropy of joint distribution $P_{(X,Y)}$ with respect to $Q_{(X,Y)}$ is

$$\mathbb{KL}\left(P_{(X,Y)} \,\|\, Q_{(X,Y)}\right) = \mathbb{KL}\left(P_X \,\|\, Q_X\right) + \mathbb{E}_X\left[\mathbb{KL}\left(P_{Y|X} \,\|\, Q_{Y|X}\right)\right] \tag{6}$$

  In addition, let P and Q denote two joint distributions for $X_1, X_2, \ldots, X_n$, let $P_{1:i}$ and $Q_{1:i}$ denote the marginal distributions of $X_1, X_2, \ldots, X_i$ under P and Q, respectively. Let $P_{X_i|1\ldots i-1}$ and $Q_{X_i|1\ldots i-1}$ denote the conditional distribution of X_i with respect to $X_1, X_2, \ldots, X_{i-1}$ under P and under Q.

$$\mathbb{KL}\left(P \,\|\, Q\right) = \sum_{i=1}^{n} \mathbb{E}_{P_{1:i-1}}\left[\mathbb{KL}\left(P_{X_i|1\ldots i-1} \,\|\, Q_{X_i|1\ldots i-1}\right)\right] \tag{7}$$

1.3 Log-Sum Inequalities and Convexity

- **Proposition 1.8** ***(Log-Sum Inequalities)*** *[Cover and Thomas, 2006]*
  For non-negative numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\sum_{i=1}^{n} a_i \log\frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right)\log\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \tag{8}$$

  with equality if and only if $\frac{a_i}{b_i}$ is constant.

- **Proposition 1.9** ***(Joint Convexity of Relative Entropy)*** *[Cover and Thomas, 2006]*
  $\mathbb{KL}\left(p \,\|\, q\right)$ *is* ***convex*** *in the pair (p, q); that is, if (p_1, q_1) and (p_2, q_2) are two pairs of probability density functions, then for $\lambda \in [0, 1]$,*

$$\mathbb{KL}\left(\lambda p_1 + (1-\lambda)p_2 \,\|\, \lambda q_1 + (1-\lambda)q_2\right) \leq \lambda\mathbb{KL}\left(p_1 \,\|\, q_1\right) + (1-\lambda)\mathbb{KL}\left(p_2 \,\|\, q_2\right) \tag{9}$$

- **Proposition 1.10** *[Cover and Thomas, 2006]*
  Let $(X, Y) \sim p(x, y) = p(x)p(y|x)$. The mutual information $I(X; Y)$ is a ***concave*** *function of $p(x)$ for fixed $p(y|x)$ and a* ***convex*** *function of $p(y|x)$ for fixed $p(x)$.*

1.4 Data Processing Inequality

- **Definition** (*Data Processing Markov Chain*)
  Random variables X, Y, Z are said to ***form a Markov chain*** in that order (denoted by $X \to Y \to Z$) if the conditional distribution of Z depends only on Y and is ***conditionally independent*** of X. Specifically, X, Y, and Z form a Markov chain $X \to Y \to Z$ if the joint probability mass function can be written as

$$p(x, y, z) = p(x)p(y|x)p(z|y)$$

- **Proposition 1.11** *(**Data Processing Inequality**) [Cover and Thomas, 2006]*
  If $X \to Y \to Z$, then
$$I(X;Z) \le I(X;Y)$$

- **Corollary 1.12** *[Cover and Thomas, 2006]*
  In particular, if $Z = g(Y)$, we have
$$I(X;g(Y)) \le I(X;Y)$$

- **Corollary 1.13** *[Cover and Thomas, 2006]*
  If $X \to Y \to Z$, then
$$I(X;Y|Z) \le I(X;Y)$$
  *Thus, the dependence of X and Y is **decreased** (or remains unchanged) by the observation of a "**downstream**" random variable Z.*

1.5 Fano's Inequality

- **Remark** Suppose that we know a random variable Y and we wish to guess the value of a correlated random variable X. ***Fano's inequality*** relates ***the probability of error*** in guessing the random variable X to its *conditional entropy* $H(X|Y)$. It will be crucial in proving *the **converse** to Shannon's **channel capacity theorem***.

- **Proposition 1.14** *(**Fano's Inequality**)[Cover and Thomas, 2006]*
  Let X, Y be random variables on domain $\mathcal{X}, \mathcal{Y}$ and $\widehat{X} = g(Y)$ is an estimate of X where $g : \mathcal{Y} \to \mathcal{X}$ is measurable function. The probability of error is defined as
$$P_e = \mathbb{P}\left\{\widehat{X} \neq X\right\}.$$
  Then we have
$$H(P_e) + P_e \log|\mathcal{X}| \ge H(X|\widehat{X}) \ge H(X|Y) \tag{10}$$
  This inequality can be weakened to
$$1 + P_e \log|\mathcal{X}| \ge H(X|Y) \tag{11}$$
$$P_e \ge \frac{H(X|Y) - 1}{\log|\mathcal{X}|}. \tag{12}$$

- **Corollary 1.15** *[Cover and Thomas, 2006]*
  For any two random variables X, Y, let $p = \mathbb{P}\{X \neq Y\}$.
$$H(p) + p \log|\mathcal{X}| \ge H(X|Y) \tag{13}$$

- **Corollary 1.16** *[Cover and Thomas, 2006]*
  Let $P_e = \mathbb{P}\left\{\widehat{X} \neq X\right\}$, and let $\widehat{X} : \mathcal{Y} \to \mathcal{X}$; then
$$H(P_e) + P_e(\log|\mathcal{X}| - 1) \ge H(X|Y) \tag{14}$$

- **Lemma 1.17** ***(Bound of Error Probability via Shannon Entropy)*** *[Cover and Thomas, 2006]*
  If X, X' are independent identically distributed random variables with entropy $H(X)$,

$$\mathbb{P}\left\{X \neq X'\right\} \leq 1 - e^{-H(X)} \tag{15}$$

  with equality if and only if X has a uniform distribution.

- **Corollary 1.18** ***(Bound of Error Probability via Relative Entropy)*** *[Cover and Thomas, 2006]*
  If X, X' are independent random variables in $\mathcal{X}$ with distribution P and Q, respectively, and $P \ll Q$

$$\mathbb{P}\left\{X \neq X'\right\} \leq 1 - e^{-H(P) - \mathbb{KL}(P\|\,Q)}. \tag{16}$$

  Similarly, if $Q \ll P$, then

$$\mathbb{P}\left\{X' \neq X\right\} \leq 1 - e^{-H(Q) - \mathbb{KL}(Q\|\,P)}.$$

- **Remark** The error probability bound (15) states that *the* ***higher*** *the uncertainty* is (i.e. $H(X)$ increases), *the* ***lower*** *the probability that $X = X'$*. Or, equivalently, *the* ***lower*** *(the Shannon and relative)* ***entropy*** is, *the* ***lower*** *the* ***probability of error*** for an estimate X' of X.

  From *Fano's inequality* (10), we see that ***the probability of error*** for estimator $\widehat{X}$ based on observation Y is ***bounded below*** by *the conditional entropy* $H(X|Y)$ of state X given observation Y. That is, we *cannot achieve lower error* of the estimation if uncertainty of state given observation ($H(X|Y)$) is high.

2 Information Inequalities

2.1 Han's Inequality

- **Proposition 2.1** ***(Han's Inequality)*** *[Cover and Thomas, 2006, Boucheron et al., 2013]*
  Let $X_1, X_2, \ldots, X_n$ be random variables. Then

$$\begin{aligned} H(X_1, X_2, \ldots, X_n) &\leq \frac{1}{n-1}\sum_{i=1}^{n} H(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \qquad (17) \\ \Leftrightarrow H(X) &\leq \frac{1}{n-1}\sum_{i=1}^{n} H(X_{(-i)}) \end{aligned}$$

  Proof: For any $i = 1, \ldots, n$, by the definition of the conditional entropy and the fact that conditioning reduces entropy,

$$\begin{aligned} H(X_1, X_2, \ldots, X_n) &= H(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) + H(X_i|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \\ &\leq H(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) + H(X_i|X_1, \ldots, X_{i-1}). \end{aligned}$$

Summing these n inequalities and using the chain rule for entropy, we get

$$\begin{aligned}
nH(X_1, X_2, \dots, X_n) &\le \sum_{i=1}^{n} H(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) + \sum_{i=1}^{n} H(X_i|X_1, \dots, X_{i-1}) \\
&= \sum_{i=1}^{n} H(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) + H(X_1, X_2, \dots, X_n)
\end{aligned}$$

which is what we wanted to prove. ∎

- **Proposition 2.2** ***(Han's Inequality for Relative Entropy)*** *[Boucheron et al., 2013]* *Let $(\mathcal{X}, \mathscr{B})$ be a measurable space, and P and Q be probability measures on $\mathcal{X}^n$ such that $P = P_1 \otimes \dots \otimes P_n$ is a* ***product measure****. We denote the element of $\mathcal{X}^n$ by $x = (x_1, \dots, x_n)$ and write $x_{(-i)} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ for the $(n-1)$-vector obtained by* ***leaving out the i-th component of*** *x (i.e. the i-th Jackknife sample of x). Denote $Q_{(-i)}$ and $P_{(-i)}$ the marginal distributions of Q and P. Let $p_{(-i)}$ and $q_{(-i)}$ denote the corresponding probability density function with respect to base measure μ on $\mathcal{X}$.*

$$\begin{aligned}
q_{(-i)}(x_{(-i)}) &= \int_{y \in \mathcal{X}} q(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) d\mu(y) \\
p_{(-i)}(x_{(-i)}) &= \int_{y \in \mathcal{X}} p(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) d\mu(y) \\
&= \prod_{j \neq i} p_j(x_j).
\end{aligned}$$

Then

$$\mathbb{KL}\left(Q \,\|\, P\right) \ge \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{KL}\left(Q_{(-i)} \,\|\, P_{(-i)}\right) \tag{18}$$

or equivalently,

$$\mathbb{KL}\left(Q \,\|\, P\right) \le \sum_{i=1}^{n} \left(\mathbb{KL}\left(Q \,\|\, P\right) - \mathbb{KL}\left(Q_{(-i)} \,\|\, P_{(-i)}\right)\right) \tag{19}$$

Proof: From Han's inequality, we have

$$-H(Q) \ge -\frac{1}{n-1} \sum_{i=1}^{n} H(Q_{(-i)}).$$

Since

$$\mathbb{KL}\left(Q \,\|\, P\right) = -H(Q) + \mathbb{E}_Q\left[-\log P(X)\right]$$

and

$$\mathbb{KL}\left(Q_{(-i)} \,\|\, P_{(-i)}\right) = -H(Q_{(-i)}) + \mathbb{E}_{Q_{(-i)}}\left[-\log P_{(-i)}(X_{(-i)})\right],$$

it suffices to show that

$$\mathbb{E}_Q\left[-\log P(X)\right] = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}_{Q_{(-i)}}\left[-\log P_{(-i)}(X_{(-i)})\right].$$

This may be seen easily by noting that by the product property of P, we have $p(x) = p_{(-i)}(x_{(-i)})p_i(x_i)$ for all i, and also $p(x) = \prod_i p_i(x_i)$, and therefore

$$
\begin{aligned}
\mathbb{E}_Q\left[-\log P(X)\right] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_Q\left[-\log P_{(-i)}(X_{(-i)}) - \log P_i(X_i)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_Q\left[-\log P_{(-i)}(X_{(-i)})\right] + \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_Q\left[-\log P_i(X_i)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_Q\left[-\log P_{(-i)}(X_{(-i)})\right] + \frac{1}{n}\mathbb{E}_Q\left[-\log P(X)\right].
\end{aligned}
$$

Rearranging, we obtain

$$
\begin{aligned}
\mathbb{E}_Q\left[-\log P(X)\right] &= \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}_Q\left[-\log P_{(-i)}(X_{(-i)})\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}_{Q_{(-i)}}\left[-\log P_{(-i)}(X_{(-i)})\right]. \qquad \blacksquare
\end{aligned}
$$

2.2 Applications of Han's Inequality

2.2.1 Combinatorial Entropies

2.2.2 Edge Isoperimetric Inequality on the Binary Hypercube

2.3 Φ-Entropy

- **Definition** (***Φ-Entropy***)[Boucheron et al., 2013]
  Let $\Phi : [0, \infty) \to \mathbb{R}$ be a ***convex*** function, and assign, to every ***non-negative*** *integrable random variable* X, ***the Φ-entropy*** of X is defined as

$$
H_{\Phi}(X) = \mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]). \tag{20}
$$

- **Remark** The Φ-entropy is a ***functional*** of *distribution* P_X instead of a function of X.

- **Remark** By Jenson's inequality, the Φ-entropy is *non-negative*

$$
\begin{aligned}
&\Phi(\mathbb{E}\left[X\right]) \le \mathbb{E}\left[\Phi(X)\right] \\
&\Rightarrow H_{\Phi}(X) = \mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]) \ge 0.
\end{aligned}
$$

- **Example** (***Special Examples for Φ-Entropy***)

  1. For $\Phi(x) = x^2$, *the Φ-entropy* of X is *the* ***variance*** of X:

$$
H_{\Phi}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}\left[X\right])^2 = \mathrm{Var}(X).
$$

  2. For $\Phi(x) = -\log(x)$, *the Φ-entropy* of $Y = e^{\lambda X}$ is *the* ***logarithm of moment generating function*** of $X - \mathbb{E}\left[X\right]$:

$$
H_{\Phi}(e^{\lambda X}) = -\lambda\mathbb{E}\left[X\right] + \log\left(\mathbb{E}\left[e^{\lambda X}\right]\right) = \log\mathbb{E}\left[e^{\lambda(X - \mathbb{E}[X])}\right] := \psi_{X-\mathbb{E}[X]}(\lambda). \tag{21}
$$

3. For $\Phi(x) = x \log x$, *the* Φ-*entropy* of X is defined as the ***entropy*** of X

$$H_{\Phi}(X) = \mathrm{Ent}(X) := \mathbb{E}\left[X \log X\right] - \mathbb{E}\left[X\right] \log\left(\mathbb{E}\left[X\right]\right). \tag{22}$$

Let $(\Omega, \mathscr{B})$ be measurable space, and P and Q are probability measures on Ω with $P \ll Q$. Define a random variable X by the *Radon-Nikodym derivative* of P with respect to Q; that is,

$$X(\omega) := \begin{cases} \frac{dP}{dQ}(\omega) & Q(\omega) > 0 \\ 0 & \text{o.w.} \end{cases}.$$

We see that X is Q-measurable and $dP = X\, dQ$ with $\mathbb{E}_Q\left[X\right] = 1$. Then the entropy of X is the relative entropy of P with respect to Q.

$$\mathrm{Ent}(X) = \mathbb{KL}\left(P \,\|\, Q\right) \tag{23}$$

2.4 Sub-Additivity of Φ-Entropy

- **Remark** (***Sub-Additivity of Shannon Entropy***)
Let $X_1, X_2, \ldots, X_n$ be drawn according to $p(x_1, x_2, \ldots, x_n)$. Then

$$H(X_1, X_2, \ldots, X_n) \le \sum_{i=1}^{n} H(X_i)$$

with equality if and only if the X_i are independent.

- **Proposition 2.3** *(**Sub-Additivity of The Entropy**) [Boucheron et al., 2013]*
Let $\Phi(x) = x \log x$, for $x > 0$ and $\Phi(0) = 0$. Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables taking values in $\mathcal{X}$, and let $f : \mathcal{X}^n \to [0, \infty)$ be a measurable function. Letting $X = f(Z_1, Z_2, \ldots, Z_n)$ such that $\mathbb{E}\left[X \log X\right] < \infty$, we have

$$\mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])\right], \tag{24}$$

where $\mathbb{E}_{(-i)}\left[\cdot\right]$ is the conditional expectation operator conditioning on $Z_{(-i)}$. Introducing the notation $Ent_{(-i)}(X) = \mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])$, this can be re-written as

$$\mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]) \le \mathbb{E}\left[\sum_{i=1}^{n} Ent_{(-i)}(X)\right]. \tag{25}$$

Proof: The proposition is a direct consequence of Han's inequality for relative entropies. First note that if the inequality is true for a random variable X, then it is also true for cX where c is a positive constant. Hence, we may assume that $\mathbb{E}\left[X\right] = 1$. Now define the probability measure P on $\mathcal{X}^n$ by its probability density function p given by

$$p(z) = f(z)q(z), \quad \forall z \in \mathcal{X}^n$$

where q denote the probability density of $Z := (Z_1, Z_2, \ldots, Z_n)$ and Q the corresponding probability measure. Then

$$\mathrm{Ent}(X) := \mathbb{E}\left[X \log X\right] - \mathbb{E}\left[X\right] \log\left(\mathbb{E}\left[X\right]\right) = \mathbb{KL}\left(P \,\|\, Q\right)$$

which, by Han's inequality for relative entropy

$$\mathrm{Ent}(X) = \mathbb{KL}\left(P \,\|\, Q\right) \le \sum_{i=1}^{n} \left(\mathbb{KL}\left(P \,\|\, Q\right) - \mathbb{KL}\left(P_{(-i)} \,\|\, Q_{(-i)}\right)\right)$$

However, straightforward calculation shows that

$$\sum_{i=1}^{n} \left(\mathbb{KL}\left(P \,\|\, Q\right) - \mathbb{KL}\left(P_{(-i)} \,\|\, Q_{(-i)}\right)\right) = \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])\right]$$

and the statement follows. ∎

Proof: (***Alternative Proof via Duality Formulation of Entropy***)
Denote the conditional expectation operator $\mathbb{E}_{1:i}\left[\cdot\right] = \mathbb{E}\left[\cdot | Z_1, \ldots, Z_i\right]$ for $i = 1, \ldots, n$ and the convention $\mathbb{E}_0\left[\cdot\right] = \mathbb{E}\left[\cdot\right]$. Noting that the operator $\mathbb{E}_{1:n}\left[\cdot\right]$ is just identity when restricted to the set of $(Z_1, \ldots, Z_n)$-measurable and integrable random variables, we have the decomposition

$$X\left(\log X - \log\left(\mathbb{E}\left[X\right]\right)\right) = \sum_{i=1}^{n} X\left(\log\left(\mathbb{E}_{1:i}\left[X\right]\right) - \log\left(\mathbb{E}_{1:i-1}\left[X\right]\right)\right).$$

Note that since $Z_1, Z_2, \ldots, Z_n$ are independent, we have $\mathbb{E}_{(-i)}\left[\mathbb{E}_{1:i}\left[X\right]\right] = \mathbb{E}_{1:i-1}\left[X\right]$. Now the duality formula given in Theorem 2.7 yields

$$\mathbb{E}\left[X\left(\log(T) - \log\left(\mathbb{E}\left[T\right]\right)\right)\right] \le \mathrm{Ent}(X)$$

Setting $T := \mathbb{E}_{1:i}\left[X\right]$, and replacing expectation $\mathbb{E}\left[\cdot\right]$ by conditional expectation $\mathbb{E}_{(-i)}\left[\cdot\right]$

$$\mathbb{E}_{(-i)}\left[X\left(\log\left(\mathbb{E}_{1:i}\left[X\right]\right) - \log\left(\mathbb{E}_{(-i)}\left[\mathbb{E}_{1:i}\left[X\right]\right]\right)\right)\right] \le \mathrm{Ent}_{(-i)}(X).$$

Finally, taking expectations on both sides of the decomposition above yields

$$\begin{aligned}\mathbb{E}\left[X\left(\log X - \log\left(\mathbb{E}\left[X\right]\right)\right)\right] &= \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[X\left(\log\left(\mathbb{E}_{1:i}\left[X\right]\right) - \log\left(\mathbb{E}_{(-i)}\left[\mathbb{E}_{1:i}\left[X\right]\right]\right)\right)\right]\right] \\ &\le \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Ent}_{(-i)}(X)\right] \quad ∎\end{aligned}$$

- **Remark** The Efron-Stein inequality is the special case of the inequality when $\Phi(x) = x^2$,

$$\mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])\right].$$

$$\Rightarrow \mathrm{Var}(X) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Var}_{(-i)}(X)\right]$$

- **Remark** (***Han's inequality from Sub-additivity of Entropy***) [Boucheron et al., 2013]
It is interesting to notice that *Han's inequality* itself can be derived from *the sub-additivity of entropy.* In other words, for *discrete probability distributions*, the sub-additivity of entropy and Han's inequality are ***equivalent.***

- **Remark** (***Tensorization Property of Entropy***) [Wainwright, 2019]
The inequality in (24) or (25) is also called ***the tensoriztion property of entropy***.

 Let $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ where μ_i be the probability distribution of Z_i. Thus μ is the probability distribution of $Z = (Z_1, \ldots, Z_n)$ when Z_i are independent. *The sub-additivity of entropy* states that

$$\mathrm{Ent}_{\mu_1 \otimes \ldots \otimes \mu_n}(f) \le \mathbb{E}_{\mu_1 \otimes \ldots \otimes \mu_n}\left[\sum_{i=1}^{n} \mathrm{Ent}_{\mu_i}(f)\right]$$

 where the subscript μ_i indicates that the integration concerns the i-th variable only.

- **Proposition 2.4** ***(Sub-Additivity of Φ-Entropy)*** *[Boucheron et al., 2013]*
*Let $\mathcal{C}$ denote the class of functions $\Phi : [0,\infty) \to \mathbb{R}$ that are **continuous** and **convex** on $[0,\infty)$, **twice differentiable** on $(0,\infty)$, and such that either Φ is **affine** or Φ'' is **strictly positive** and $1/\Phi''$ is **concave**. For all $\Phi \in \mathcal{C}$, the **entropy functional** H_Φ is **sub-additive**. That is,*

$$\mathbb{E}\left[\Phi(X)\right] - \Phi(\mathbb{E}\left[X\right]) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])\right], \tag{26}$$
$$\Leftrightarrow H_\Phi(X) \le \mathbb{E}\left[\sum_{i=1}^{n} H_\Phi^{(-i)}(X)\right]$$

 where $H_\Phi^{(-i)}(X) := \mathbb{E}_{(-i)}\left[\Phi(X)\right] - \Phi(\mathbb{E}_{(-i)}\left[X\right])$ is the conditional entropy and, $\mathbb{E}_{(-i)}\left[\cdot\right]$ denotes conditional expectation conditioned on the $(n-1)$-vector $Z_{(-i)} := (Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n)$.

- **Remark** ***The sub-additivity property*** of H_Φ is equivalent to what we could call ***the Jensen property***

$$H_\Phi\left(\int f(z, Z_2) d\mu_1(z)\right) \le \int H_\Phi(f(z, Z_2)) d\mu_1(z)$$
$$\Leftrightarrow H_\Phi\left(\mathbb{E}_{Z_1}\left[f(Z_1, Z_2)\right]\right) \le \mathbb{E}_{Z_1}\left[H_\Phi\left(f(Z_1, Z_2)\right)\right] \tag{27}$$

 The proof of this property can be done by using the duality formulation of Φ-entropy in Theorem 2.12.

2.5 Duality and Variational Formulas

- **Lemma 2.5** *The **Legendre transform** (or **convex conjugate**) of $\Phi(x) = x\log(x)$ is e^{u-1}. That is,*

$$\sup_{x>0}\{u\,x - x\log(x)\} = e^{u-1}$$

 Proof: Solve the supremum on the left-hand side by taking derivative of the objective function and setting it as zero:

$$\nabla g(x) = u - \log(x) - 1 = 0$$
$$\Rightarrow x^* = e^{u-1}$$
$$\Rightarrow \sup_{x}\{u\,x - x\log(x)\} = g(x^*) = u\,e^{u-1} - e^{u-1}(u-1) = e^{u-1} \quad \blacksquare$$

- **Remark** If $\Phi(X) = X\log(X)$ is integrable, and $\mathbb{E}\left[e^U\right] = 1$, we have

$$UX \leq X\log(X) + \frac{1}{e}e^U.$$

Therefore, U_+X is integrable, and one can always define $\mathbb{E}[UX] = \mathbb{E}[U_+X] - \mathbb{E}[U_-X]$ for positive and negative part of U. Thus the $\mathbb{E}[UX]$ is well-defined.

- **Theorem 2.6** ***(Duality Formula of Entropy)*** *[Boucheron et al., 2013]*
Let X be a non-negative random variable defined on a probability space $(\Omega, \mathscr{A}, P)$ such that $\mathbb{E}[\Phi(X)] < \infty$. Then we have ***the duality formula***

$$Ent(X) = \sup_{U\in\mathcal{U}} \mathbb{E}[U\,X] \tag{28}$$

where the supremum is taken over the set $\mathcal{U}$ of all random variables $U : \Omega \to \mathbb{R}\cup\{\infty\}$ with $\mathbb{E}\left[e^U\right] = 1$. Moreover, if U is such that $\mathbb{E}[UX] \leq Ent(X)$ for all non-negative random variable X such that $\Phi(X)$ is integrable and $\mathbb{E}[X] = 1$, then $\mathbb{E}\left[e^U\right] \leq 1$.

Proof: Note that for any random variable U such that $\mathbb{E}\left[e^U\right] = 1$, we have

$$\begin{aligned}
\mathrm{Ent}(X) - \mathbb{E}_P[UX] &= \mathbb{E}_P[X\log(X)] - \mathbb{E}_P[X]\log(\mathbb{E}_P[X]) - \mathbb{E}_P[UX] \\
&= \mathbb{E}_P[X(\log(X) - U)] - \mathbb{E}_P[X]\log(\mathbb{E}_P[X]) \\
&= \mathbb{E}_P\left[X\log(Xe^{-U})\right] - \mathbb{E}_P[X]\log(\mathbb{E}_P[X]) \\
&= \mathbb{E}_{e^UP}\left[Xe^{-U}\log(Xe^{-U})\right] - \mathbb{E}_{e^UP}\left[Xe^{-U}\right]\log(\mathbb{E}_{e^UP}\left[Xe^{-U}\right]) \\
&= \mathrm{Ent}_{e^UP}(Xe^{-U})
\end{aligned}$$

Note that due to $\mathbb{E}\left[e^U\right] = 1$, $\int e^U dP = 1$, thus e^UP is a proper probability measure. This shows that

$$\begin{aligned}
&\mathrm{Ent}_{e^UP}(Xe^{-U}) \geq 0 \\
&\qquad \Rightarrow \mathrm{Ent}(X) \geq \mathbb{E}_P[UX]
\end{aligned}$$

with equality whenever $e^U = X/\mathbb{E}_P[X]$. This proves the duality formula.

Conversely, let U be such that $\mathbb{E}_P[UX] \leq \mathrm{Ent}(X)$ for all non-negative random variables such that $\Phi(X)$ is integrable. If $\mathbb{E}\left[e^U\right] = 0$, then there is nothing to prove Otherwise, given a positive integer n large enough to ensure that $x_n = \mathbb{E}\left[e^{\min\{U,n\}}\right] > 0$, one may define $X_n = e^{\min\{U,n\}}/x_n$, so that $\mathbb{E}[X_n] = 1$, which leads to

$$\mathbb{E}[UX_n] \leq \mathrm{Ent}(X_n),$$

and therefore

$$\begin{aligned}
\frac{1}{x_n}\mathbb{E}\left[Ue^{\min\{U,n\}}\right] &\leq \mathrm{Ent}(e^{\min\{U,n\}}/x_n) \\
&= \frac{1}{x_n}\left[\mathbb{E}\left[\min\{U,n\}\,e^{\min\{U,n\}}\right] - \log(x_n)\right]
\end{aligned}$$

Hence

$$\log(x_n) \leq 0$$

and taking the limit when $n \to \infty$, we show by monotonicity that $\mathbb{E}\left[e^U\right] \leq 1$. ∎

- **Theorem 2.7** ***(Alternative Duality Formula of Entropy)*** *[Boucheron et al., 2013]*

$$Ent(X) = \sup_{T} \mathbb{E}\left[X\left(\log(T) - \log\left(\mathbb{E}\left[T\right]\right)\right)\right] \tag{29}$$

where the supremum is taken over all non-negative and integrable random variables.

Proof: From (28), taking $U = \log \frac{T}{\mathbb{E}[T]}$, so that $\mathbb{E}\left[e^{U}\right] = \mathbb{E}\left[\frac{T}{\mathbb{E}[T]}\right] = 1$. This gives us (29). ∎

- **Corollary 2.8** ***(Duality Formula of Log Moment Generating Function)*** *[Cover and Thomas, 2006, Boucheron et al., 2013]*
Let X be a real-valued integrable random variable. Then for every $\lambda \in \mathbb{R}$,

$$\log \mathbb{E}_{Q}\left[e^{\lambda(X-\mathbb{E}[X])}\right] = \sup_{P \ll Q}\left\{\lambda\left(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]\right) - \mathbb{KL}\left(P \,\|\, Q\right)\right\}, \tag{30}$$

where the supremum is taken over all probability measures P absolutely continuous with respect to Q, and $\mathbb{E}_{P}\left[\cdot\right]$ denotes integration with respect to the measure P (recall that $\mathbb{E}_{Q}\left[\cdot\right]$ is integration with respect to Q).

Proof: Let $P \ll Q$. Taking $Y := \frac{dP}{dQ}$ and $U := \lambda(X - \mathbb{E}_{Q}\left[X\right]) - \psi_{X-\mathbb{E}_{Q}[X]}(\lambda)$ where $\psi_{X}(\lambda) := \log \mathbb{E}_{Q}\left[e^{\lambda X}\right]$. Note that $\mathbb{E}_{Q}\left[Y\right] = 1$ and $\mathbb{E}\left[e^{U}\right] = 1$. It follows from the duality formula that

$$\begin{aligned}\mathbb{KL}\left(P \,\|\, Q\right) = \mathrm{Ent}(Y) \geq \mathbb{E}\left[U\,Y\right] &= \mathbb{E}\left[\lambda(X - \mathbb{E}_{Q}\left[X\right])Y\right] - \psi_{X-\mathbb{E}_{Q}[X]}(\lambda)\\ &= \lambda(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]) - \psi_{X-\mathbb{E}_{Q}[X]}(\lambda)\end{aligned}$$

or equivalently

$$\psi_{X-\mathbb{E}_{Q}[X]}(\lambda) \geq \lambda(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]) - \mathbb{KL}\left(P \,\|\, Q\right),$$

therefore

$$\log \mathbb{E}_{Q}\left[e^{\lambda(X-\mathbb{E}_{Q}[X])}\right] \geq \sup_{P \ll Q}\left\{\lambda(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]) - \mathbb{KL}\left(P \,\|\, Q\right)\right\}.$$

Conversely, setting

$$U = \lambda\left(X - \mathbb{E}_{Q}\left[X\right]\right) - \sup_{P \ll Q}\left\{\lambda(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]) - \mathbb{KL}\left(P \,\|\, Q\right)\right\}$$

for every non-negative random variable Y such that $\mathbb{E}\left[Y\right] = 1$,

$$\mathbb{E}\left[UY\right] \leq \mathrm{Ent}(Y).$$

Hence, $\mathbb{E}\left[e^{U}\right] \leq 1$ by duality theorem, which means that

$$\log \mathbb{E}_{Q}\left[e^{\lambda\left(X-\mathbb{E}_{Q}[X]\right)}\right] \leq \sup_{P \ll Q}\left\{\lambda(\mathbb{E}_{P}\left[X\right] - \mathbb{E}_{Q}\left[X\right]) - \mathbb{KL}\left(P \,\|\, Q\right)\right\}. \quad ∎$$

- **Corollary 2.9** ***(Duality Formula of Kullback-Leibler Divergence)*** *[Cover and Thomas, 2006, Boucheron et al., 2013]*
Let P and Q be two probability distributions on the same space. Then

$$\mathbb{KL}\left(P \,\|\, Q\right) = \sup_{X}\left\{\mathbb{E}_{P}\left[X\right] - \log \mathbb{E}_{Q}\left[e^{X}\right]\right\}, \tag{31}$$

where the supremum is taken over all random variables such that $\mathbb{E}_{Q}\left[\exp\left(X\right)\right] < \infty$.

Proof: If $P \ll Q$, $\mathbb{KL}(P \,\|\, Q) = \mathrm{Ent}(dP/dQ)$ and the corollary follows from the alternative formulation of the duality formula. Let $Y = dP/dQ$ and $X = \log(T)$ so that

$$\begin{aligned}\mathbb{KL}(P \,\|\, Q) = \mathrm{Ent}(Y) &= \sup_{T} \mathbb{E}\left[dP/dQ \left(\log(T) - \log\left(\mathbb{E}\left[T\right]\right)\right)\right] \\ &= \sup_{X} \left\{\mathbb{E}_P\left[X\right] - \log \mathbb{E}_Q\left[e^X\right]\right\}.\end{aligned}$$

If $P \not\ll Q$, then there exists an event A such that $P(A) > 0 = Q(A)$, $\mathbb{KL}(P \,\|\, Q) = \infty$, and choosing $X_n = n\mathbb{1}\{A\}$ and letting n tend to infinity, we observe that the supremum on the right-hand side is infinite. ∎

- **Remark** This corollary asserts that if Q remains fixed, $\mathbb{KL}(P \,\|\, Q)$ is *the **convex dual** of the functional* $X \to \log \mathbb{E}_Q\left[e^X\right]$.

- **Theorem 2.10** ***(The Expected Value Minimizes Expected Bregman Divergence)*** *[Boucheron et al., 2013]*
*Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be **convex** and **differentiable**. For any $x, y \in I$, **the Bregman divergence** of f from x to y is $f(y) - f(x) - f'(x)(y - x)$. Let X be an I-valued random variable. Then*

$$\mathbb{E}\left[f(X) - f(\mathbb{E}\left[X\right])\right] = \inf_{a \in I} \mathbb{E}\left[f(X) - f(a) - f'(a)(X - a)\right] \tag{32}$$

- **Corollary 2.11** ***(Duality Formula of Entropy via Bregman Divergence)*** *[Boucheron et al., 2013]*
Let X be a non-negative random variable such that $\mathbb{E}\left[\Phi(X)\right] < \infty$. Then

$$Ent(X) = \inf_{u > 0} \mathbb{E}\left[X\left(\log(X) - \log(u)\right) - (X - u)\right] \tag{33}$$

- **Theorem 2.12** ***(Duality Formula of General Φ-Entropy)*** *[Boucheron et al., 2013]*
*Let $\mathcal{C}$ denote the class of functions $\Phi : [0, \infty) \to \mathbb{R}$ that are **continuous** and **convex** on $[0, \infty)$, **twice differentiable** on $(0, \infty)$, and such that either Φ is **affine** or Φ'' is **strictly positive** and $1/\Phi''$ is **concave**. Denote $conv(L_1^+)$ as **the convex set** of **non-negative** and **integrable** random variables X. Let $\Phi \in \mathcal{C}$ and $X \in conv(L_1^+)$. If $\Phi(X)$ is integrable, then*

$$H_\Phi(X) = \sup_{T \in conv(L_1^+), T \neq 0} \left\{\mathbb{E}\left[\left(\Phi'(T) - \Phi'(\mathbb{E}\left[T\right])\right)(X - T) + \Phi(T)\right] - \Phi(\mathbb{E}\left[T\right])\right\}. \tag{34}$$

The supremum is achieved when $T = X$ (or $T = 1$ if $X = 0$).

Another variational formulation of Φ-entropy via Bregman divergence is

$$H_\Phi(X) = \inf_{u > 0} \mathbb{E}\left[\Phi(X) - \Phi(u) - \Phi'(u)(X - u)\right]. \tag{35}$$

2.6 Wasserstein Distance and Transportation Cost Inequality

- **Proposition 2.13** ***(Wasserstein Distance and Transportation Cost Inequality)*** *[Boucheron et al., 2013]*
*Let X be a real-valued integrable random variable. Let ϕ be a **convex** and **continuously differentiable** function on a (possibly unbounded) interval $[0, b)$ and assume that $\phi(0) =$*

$\phi'(0) = 0$. Define, for every $x \ge 0$, ***the Legendre transform*** *$\phi^*(x) = \sup_{\lambda\in(0,b)}(\lambda x - \phi(\lambda))$, and let, for every $t \ge 0$, $\phi^{*-1}(t) = \inf\{x \ge 0 : \phi^*(x) > t\}$, i.e. the* ***the generalized inverse*** *of ϕ^*. Then the following two statements are equivalent:*

1. *for every $\lambda \in (0, b)$,*

$$\psi_{X-\mathbb{E}[X]}(\lambda) \le \phi(\lambda)$$

where $\psi_X(\lambda) := \log \mathbb{E}_Q\left[e^{\lambda X}\right]$ is the logarithm of moment generating function;

2. *for any probability measure P absolutely continuous with respect to Q such that $\mathbb{KL}\left(P \,\|\, Q\right) < \infty$,*

$$\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right] \le \phi^{*-1}\left(\mathbb{KL}\left(P \,\|\, Q\right)\right). \tag{36}$$

In particular, given $\nu > 0$, X follows a sub-Gaussian distribution, i.e.

$$\psi_{X-\mathbb{E}[X]}(\lambda) \le \frac{\nu\lambda^2}{2}$$

for every $\lambda > 0$ ***if and only if*** *for any probability measure P absolutely continuous with respect to Q and such that $\mathbb{KL}\left(P \,\|\, Q\right) < \infty$,*

$$\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right] \le \sqrt{2\nu\mathbb{KL}\left(P \,\|\, Q\right)}. \tag{37}$$

Proof: As a direct consequence of Corollary 2.8, we see that (1) holds if and only if for every distribution $P \ll Q$,

$$\begin{aligned}
&\psi_{X-\mathbb{E}[X]}(\lambda) \le \phi(\lambda) && \\
\Leftrightarrow\ &\lambda\left(\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right]\right) - \mathbb{KL}\left(P \,\|\, Q\right) \le \phi(\lambda), && \forall P \ll Q \\
\Leftrightarrow\ &\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right] \le \frac{\phi(\lambda) + \mathbb{KL}\left(P \,\|\, Q\right)}{\lambda}, && \forall P \ll Q, \lambda \in (0, b) \\
\Leftrightarrow\ &\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right] \le \inf_{\lambda\in(0,b)}\left\{\frac{\mathbb{KL}\left(P \,\|\, Q\right) + \phi(\lambda)}{\lambda}\right\} && \forall P \ll Q
\end{aligned}$$

Note that

$$\phi^{*-1}(t) = \inf_{\lambda\in(0,b)}\left[\frac{t + \phi(\lambda)}{\lambda}\right]$$

Setting $t = \mathbb{KL}\left(P \,\|\, Q\right)$, we have

$$\begin{aligned}
&\psi_{X-\mathbb{E}[X]}(\lambda) \le \phi(\lambda) \\
\Leftrightarrow\ &\mathbb{E}_P\left[X\right] - \mathbb{E}_Q\left[X\right] \le \phi^{*-1}\left(\mathbb{KL}\left(P \,\|\, Q\right)\right).
\end{aligned}$$

which shows that (i) is equivalent to (ii). Applying the previous result with $\phi(\lambda) = \lambda^2\nu/2$ for every $\lambda > 0$ leads to the stated special case of equivalence since then $\phi^{*-1}(t) = \sqrt{2\nu t}$. ∎

- **Remark** (***The Quadratic Transportation Cost Inequality / The Information Inequality***) [Boucheron et al., 2013, Wainwright, 2019]
The inequality (36) and (37) are called ***information inequality*** in [Wainwright, 2019] due to the role of Kullback-Leibler Divergence in information theory.

The inequality (37) is related to what is usually termed a ***quadratic transportation cost inequality***. If Ω is a *metric space*, the probability measure Q is said to satisfy a *quadratic transportation cost inequality* if the last inequality holds for every X which is *Lipschitz* on Ω with *Lipschitz norm* at most 1.

$$\mathcal{W}(P,Q) = \sup_{X \in \mathrm{Lip}_1} \{\mathbb{E}_P[X] - \mathbb{E}_Q[X]\} \le \sqrt{2\nu \mathbb{KL}(P \,\|\, Q)}. \tag{38}$$

where $Lip_1 = \left\{f \in \mathbb{R}^{\Omega} : |f(x) - f(y)| \le L\, d(x,y),\;\; L \le 1\right\}$ and d is the metric in Ω. Here $\mathcal{W}(P,Q)$ is ***the Wasserstein distance*** between P and Q induced by metric d.

2.7 Pinsker's Inequality

- **Definition** (***Total Variation / Variational Distance***)
  Let P, Q be two probability measures on measurable space $(\Omega, \mathscr{F})$. The ***total variation*** or ***variational distance*** between P and Q is defined by

$$V(P,Q) := \sup_{A \in \mathscr{F}} |P(A) - Q(A)| \tag{39}$$

- **Remark** (***Equivalent Formulation of Total Variation***)
  It is a well-known and simple fact that the total variation is half the L_1-distance, that is, if μ is a *common dominating measure* of P and Q and $p(x) = dP/d\mu$ and $q(x) = dQ/d\mu$ denote their respective densities, then

$$V(P,Q) := P(A^*) - Q(A^*) = \frac{1}{2}\int_{\Omega} |p(x) - q(x)|\, d\mu(x), \tag{40}$$

  where $A^* = \{x : p(x) \ge q(x)\}$.

- **Remark** (***Total Variation via Optimal Coupling of Two Measures***)
  We note that another important interpretation of *the variational distance* is related to *the best coupling of the two measures*

$$V(P,Q) = \min P\{X \neq Y\} \tag{41}$$

  where the minimum is taken over *all pairs of joint distributions* for the random variables (X, Y) whose marginal distributions are $X \sim P$ and $Y \sim Q$.

- **Remark** (***Applications of Pinsker's Inequality***)
  The importance of *Pinsker's inequality* in statistics stems from the fact that it provides ***a lower bound*** for *the* ***error*** *of certain hypothesis testing problems.*

  We use Pinsker's inequality for a completely different purpose, namely for establishing a transportation cost inequality that may be used to prove concentration inequalities.

- **Proposition 2.14** ***(Pinsker's Inequality)*** *[Cover and Thomas, 2006, Boucheron et al., 2013]*
  Let P, Q be two probability distributions on measurable space $(\Omega, \mathscr{F})$ such that $P \ll Q$. Then

$$V(P,Q)^2 \le \frac{1}{2}\mathbb{KL}(P \,\|\, Q). \tag{42}$$

Proof: Define the random variable X such that $dP = XdQ$ and let $A^* = \{X \geq 1\}$ be the set achieving the maximum in the definition of the total variation between P and Q. Then, setting $Z = \mathbb{1}\{A^*\}$,

$$V(P,Q) := P(A^*) - Q(A^*) = \mathbb{E}_P[Z] - \mathbb{E}_Q[Z].$$

It follows from Hoeffding's lemma that

$$\psi_{Z-\mathbb{E}[Z]}(\lambda) \leq \frac{\lambda^2}{8}$$

which by transportation cost inequality for sub-Gaussian variables we have

$$\mathbb{E}_P[Z] - \mathbb{E}_Q[Z] \leq \sqrt{\frac{1}{2}\mathbb{KL}(P \,\|\, Q)}. \quad \blacksquare$$

- **Remark** (***Total Variation as*** 1-***Wasserstein Distance***)
  The total variation between P and Q is ***the Wasserstein distance*** induced by ***the Hamming distance*** $d(x,y) = \#\{i : x_i \neq y_i\}$.

  $$V(P,Q) = \mathcal{W}_1(P,Q).$$

  Thus *the Pinsker's inequality* (42) is the special case of *transportation cost inequality* (36).

2.8 Birgé's Inequality and Multiple Testing Problem

- **Remark** We will use *the Pinsker's inequality* to derive a ***lower bound*** on ***the probability of error*** in *multiple testing problem.*

- **Proposition 2.15** ***(Sharper Information Inequality for Total Variation)*** *[Boucheron et al., 2013]*
  Let P, Q be two probability distributions on measurable space $(\Omega, \mathscr{F})$ such that $P \ll Q$.

  $$\sup_{A \in \mathscr{F}} h(P(A), Q(A)) \leq \mathbb{KL}(P \,\|\, Q) \tag{43}$$

  where $h(p,q) = \mathbb{KL}(p \,\|\, q) = q\log(q/p) + (1-q)\log((1-q)/(1-p))$ when $p, q \in [0,1]$ are parameters of Bernoulli random variables.

  Proof: For any $p \in [0,1]$, let

  $$\phi_p(\lambda) = \log\left(p\left(e^{\lambda} - 1\right) + 1\right)$$

  denote the logarithm of the moment generating function of the Bernoulli(p) distribution where $\lambda \in \mathbb{R}$. By the duality formulation of relative entropy, for $X = \mathbb{1}\{A\}$,

  $$\begin{aligned} &\mathbb{KL}(P \,\|\, Q) \geq \mathbb{E}_P[\lambda \mathbb{1}\{A\}] - \log \mathbb{E}_Q\left[e^{\lambda \mathbb{1}\{A\}}\right] \\ \Rightarrow\ &\mathbb{KL}(P \,\|\, Q) \geq \sup_{\lambda \geq 0}\left\{\lambda P(A) - \phi_{Q(A)}(\lambda)\right\}. \end{aligned}$$

  The proposition follows by noting that for any $a \in [0,1]$,

  $$h(a,p) = \sup_{\lambda \geq 0}\{\lambda a - \phi_p(\lambda)\}. \quad \blacksquare$$

- **Remark** Note that

$$h(P(A), Q(A)) \geq 2\left(P(A) - Q(A)\right)^2.$$

Thus the proposition above implies the Pinsker's inequality.

- **Remark** ***The variational representation*** *of* ***relative entropy*** may be used to establish ***lower bounds*** for ***the probability of error*** in *multiple testing problems*. The next result is a sharper version of *Fano's inequality*, a classical tool from information theory.

Proposition 2.16 ***(Birgé's Inequality)*** *[Boucheron et al., 2013]*
Let $P_0, P_1, \ldots, P_N$ be probability distributions on measurable space $(\Omega, \mathscr{F})$ and let $A_0, A_1, \ldots, A_N \in \mathscr{F}$ be pairwise disjoint events. If $a = \min_{i=0,\ldots,N} P_i(A_i) \geq 1/(N+1)$,

$$a \leq h\left(a, \frac{1-a}{N}\right) \leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{KL}\left(P_i \,\|\, P_0\right) \tag{44}$$

Proof: By the variational representation of relative entropy, for any $i = 0, \ldots, N$,

$$\sup_{\lambda > 0}\left\{\mathbb{E}_{P_i}\left[\lambda \mathbb{1}\left\{A_i\right\}\right] - \log \mathbb{E}_{P_0}\left[e^{\lambda \mathbb{1}\{A_i\}}\right]\right\} \leq \mathbb{KL}\left(P_i \,\|\, P_0\right).$$

See that

$$\begin{aligned}
1 - a &= 1 - \min_{i=0,\ldots,N} P_i(A_i) \\
&\geq 1 - P_0(A_0) \geq \sum_{i=1}^{N} P_0(A_i).
\end{aligned}$$

For any $\lambda > 0$,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N} \mathbb{KL}\left(P_i \,\|\, P_0\right) &\geq \frac{1}{N}\sum_{i=1}^{N}\left\{\lambda P_i(A_i) - \log \mathbb{E}_{P_0}\left[e^{\lambda \mathbb{1}\{A_i\}}\right]\right\} \\
&\geq \frac{1}{N}\sum_{i=1}^{N}\left\{\lambda a - \log\left(P_0(A_i)\left(e^{\lambda} - 1\right) + 1\right)\right\} \\
&= \lambda a - \frac{1}{N}\sum_{i=1}^{N}\log\left(P_0(A_i)\left(e^{\lambda} - 1\right) + 1\right) \\
&\geq \lambda a - \log\left(\frac{1}{N}\sum_{i=1}^{N}\left(P_0(A_i)\left(e^{\lambda} - 1\right) + 1\right)\right) \quad \text{(by convexity of } -\log(x)\text{)} \\
&= \lambda a - \log\left(\left(\frac{1}{N}\sum_{i=1}^{N} P_0(A_i)\right)\left(e^{\lambda} - 1\right) + 1\right) \\
&\geq \lambda a - \log\left(\frac{1 - P_0(A_0)}{N}\left(e^{\lambda} - 1\right) + 1\right) \\
&\geq \lambda a - \log\left(\frac{1 - a}{N}\left(e^{\lambda} - 1\right) + 1\right)
\end{aligned}$$

Note that the supremum of the right-hand side with respect to λ is $h\left(a, \frac{1-a}{N}\right)$. ∎

References

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