Lecture 1: probability measure on infinite-dimensional space

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1 Weak Topology

1.1 Weak Topology

- **Definition** (***Weak Topology on a Set*** S) [Reed and Simon, 1980]
Let $\mathcal{F}$ be a family of functions from a set S to a topological vector space $(X, \mathscr{T})$. The ***$\mathcal{F}$-weak*** *(or simply* ***weak****)* ***topology*** on S is the weakest topology for which ***all the functions*** $f \in \mathcal{F}$ *are* ***continuous****.*

- **Remark** (***Construction of Weak Topology***) [Reed and Simon, 1980]
To construct a ***$\mathcal{F}$-weak topology*** on S, we take the family of *all* ***finite intersections*** *of sets* of the form $f^{-1}(U)$ where $f \in \mathcal{F}$ and $U \in \mathscr{T}$. The collections of these finite intersections of sets *form* ***a basis*** *of* ***the $\mathcal{F}$-weak topology****.*

  In other word, ***the subbasis*** for the *$\mathcal{F}$-weak topology* on S is of form
$$\mathscr{S} = \left\{ f^{-1}(U) : f \in \mathscr{F},\ \text{and } U \in \mathscr{T} \right\}$$
  And the basis of $\mathscr{T}$
$$\mathscr{B} = \left\{ f_1^{-1}(U_1) \cap \ldots \cap f_k^{-1}(U_k) : f_1, \ldots, f_k \in \mathscr{F},\ \ U_1, \ldots, U_k \in \mathscr{T},\ 1 \le k < \infty \right\}$$
$$\begin{aligned} B \in \mathscr{B} \Rightarrow B &= \{x : f_1(x) \in U_1, \ldots, f_k(x) \in U_k\},\ 1 \le k < \infty \\ &= \{x : (f_1(x), \ldots, f_k(x)) \in U\}. \end{aligned}$$
  The basis element is called a ***k-dimensional cylinder set****.*

- **Remark** Given a topology on Y and a family of functions in $Y^X = \{f : X \to Y\}$, $\mathscr{F}$-weak topolgy is ***a natural topology*** on X without additional information.

  A product topology on Y^ω can be seen as a $\mathscr{F}$-weak topology when $\mathscr{F} = \{\pi_\alpha : \prod_i Y_i \to Y_\alpha\}$.

- **Remark** A set S equipped with *$\mathcal{F}$-weak topology* ***has little knowledge on itself besides the output of functions*** $f \in \mathcal{F}$ from a family $\mathcal{F}$. The induced topology through a family of functions thus does not tell much besides the behavior of its output.

  For instance, S is the space of hidden states, $\mathcal{F} = \{f_1, \ldots, f_n\} \subset 2^S$ is a series of binary statistical tests, the weak topology on S *partition the domain according to the output of each test.*

- **Remark** By construction, the ***neighborhood base*** of each point $x \in S$ under the $\mathcal{F}$-weak topology is contained in the pre-images $\{f_n^{-1}(U_n)\}$ for ***finitely many*** of $(f_n) \in \mathcal{F}$.

- **Definition** (***Weak Topology on Banach Space***)
Let X be a ***Banach space*** with dual space X^*. The ***weak topology*** on X is *the weakest topology* on X so that $f(x)$ *is* ***continuous for all*** $f \in X^*$.

- **Remark** For infinite dimensional Banach spaces, ***the weak topology does not arise from a metric****.* This is one of the main reasons we have introduced topological spaces.

- **Remark** Thus a ***neighborhood*** *base at zero* for ***the weak topology*** is given by the sets of the form
$$N(f_1, \ldots, f_n; \epsilon) = \{x : |f_j(x)| < \epsilon;\ i = 1, \ldots, n\}$$

that is, neighborhoods of zero contain ***cylinders*** *with* ***finite-dimensional*** *open bases*. A net $\{x_\alpha\}$ converges *weakly* to x, written $x_\alpha \overset{w}{\to} x$, if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

- **Proposition 1.1** *[Reed and Simon, 1980]*

  1. *The weak topology is* ***weaker*** *than* ***the norm topology****, that is, every weakly open set is norm open.*

  2. *Every* ***weakly convergent*** *sequence is* ***norm bounded****.*

  3. *The weak topology is a Hausdorff topology.*

- **Proposition 1.2** *[Reed and Simon, 1980]*
  A linear functional f on a Banach space is ***weakly continuous*** *if and only if it is* ***norm continuous****.*

2 Cylindrical σ-Algebra on Weak Topology

2.1 Cylinder Set

- **Definition** Let X be locally convex space, a n-dimensional *cylinder set* as [Lifshits, 2013]

  $$C_A[f_1, \ldots, f_n] \equiv \{\boldsymbol{x} \mid (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})) \in A\ \} = \{\boldsymbol{x} \mid f_i(\boldsymbol{x}) \in A_i,\ 1 \le i \le n\}, n = 1, 2, \ldots,$$

  for any $A \in \mathcal{B}(\mathbb{R}^n)$, $A_i \in \mathcal{B}(\mathbb{R})$, $f_i \in X^* \subset \mathbb{R}^X$, the dual space of continuous linear functional on X.

- Note that we can define space of X^*-valued functions as $(X^*)^T$, where $f : T \to X^*$ denotes a trajectory on X^*. Define an evaluation map $\pi_k : (X^*)^T \to X^*$ so that $\pi_k(f) = f_k$. $C_A[f_1, \ldots, f_n] = (\pi_N(f))^{-1}(A)$.

- Define $\mathscr{C}_n$ consists of $C_A[f_1, \ldots, f_n]$ with all possible $A \in \mathcal{B}(\mathbb{R}^n)$, and $f_i \in X^* \subset \mathbb{R}^X$.

- Given $\{f_k, 1 \le k \le n\}$, define an equivalence relationship: $\boldsymbol{x}_1 \overset{R}{\sim} \boldsymbol{x}_2$, iff $f_k(\boldsymbol{x}_1) = f_k(\boldsymbol{x}_2), \forall 1 \le k \le n$.

  A cylinder set $C_A[f_1, \ldots, f_n]$ can be represented as the union of *cosets* in X corresponding to points in A, under the equivalence relationship R. Conversely, any union of cosets in X under this relationship forms a cylinder set.

  In particular, $\boldsymbol{x}_1, \boldsymbol{x}_2$ in the same coset iff their difference lies in the kernel space of the system of functions $f_k, 1 \le k \le n$; i.e.,

  $$\begin{aligned} &f_k(\boldsymbol{x}_1) - f_k(\boldsymbol{x}_2) = 0 \\ &\Rightarrow f_k(\boldsymbol{x}_1 - \boldsymbol{x}_2) = 0,\ 1 \le k \le n \quad \text{(linearity of } f_k) \\ &\boldsymbol{x}_1 - \boldsymbol{x}_2 \in \ker\{f_k\},\ 1 \le k \le n. \end{aligned}$$

  Note that $\boldsymbol{s} \in \ker\{f_k\}, 1 \le k \le n, \Leftrightarrow \boldsymbol{s} \in \ker\{\sum_k \alpha_k f_k\}, \forall \alpha_k$. Let $X_n^* \equiv X^*[f_1, \ldots, f_n]$ be the n-dimensional subspace in X^* spanned by $\{f_k, 1 \le k \le n\}$. Here the subspace X_n^* does not change by a change of basis $(f_1', \ldots, f_n')$.

- Define the *annihilator* X_0 *of* X_n^* in X as the linear subspace consisting of $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) = 0, \forall\, f \in X_n^*$, where X_n^* is the n-dimensional linear subspace in X^* spanned by $\{f_k, 1 \le k \le n\}$.

- Then an alternative *definition* of n-dimensional *cylinder set* is as following [Gel'fand and Vilenkin, 2014]:

  Definition Let X_n^* be the n-dimensional linear subspace in X^* spanned by $\{f_k, 1 \le k \le n\}$, X_0 be the linear subspace in X consisting of all $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 0, \forall f \in X_n^*$. Define the *quotient* map $q : X \to X/X_0$, which maps $\boldsymbol{x}$ to the *coset* $\boldsymbol{x} + X_0$ and X/X_0 is the *quotient space* induced by the relationship $\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \boldsymbol{x} - \boldsymbol{y} \in X_0$.

  Then a n-dimensional cylinder set $C_A[f_1, \ldots, f_n]$ is

  $$C_A[f_1, \ldots, f_n] \equiv C[A, X_0] = q^{-1}(A), \quad A \in X/X_0.$$

  It is called a *cylinder set with base A and generating subspace X_0*.

- Let $F_n \equiv F[f_1, \ldots, f_n] : \boldsymbol{x} \mapsto (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$ maps X to a n-dimension linear subspace in $\mathbb{R}^n$. $A \in \mathcal{B}(\mathbb{R}^n)$. Note that $X/X_0 \simeq F_n(X) = (X_n^*)^* \subset \mathbb{R}^n$ by the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "F_n=(f_1,\cdots,f_n)"'] & X/X_0 \arrow[d, dashed, "p"] \\ & F_n(X) \subset \mathbb{R}^n \end{tikzcd}$$

- Given X^* is locally convex linear topological space, X_n^* is linear subspace of X^*, then X/X_0 is the *adjoint space* of X_n^*.

  In other words, X/X_0 is a $n-$dimensional subspace.

- A cylinder set can be defined via various bases A and subsets X_0: [Gel'fand and Vilenkin, 2014]

  If $C[A_1, X_{1,0}] = C[A_2, X_{2,0}]$, then both cylinders can be generated by a common subspace $X_{3,0}$, which coincides with $X_{1,0} \cap X_{2,0}$ and is the annihilator $X_{3,0}$ of $X_{n,3}^*$ in X^*

  Since $X_{3,0} \subset X_{1,0}$, any coset w.r.t. $X_{3,0}$ corresponds to some coset w.r.t. $X_{1,0}$, so we can associate any coset w.r.t. $X_{3,0}$ with some coset w.r.t. $X_{1,0}$ that contains it. So it defines a linear mapping $T_1 : X/X_{3,0} \to X/X_{1,0}$. The consider the preimage $T_1^{-1}(A_1)$, then the cylinder set is defined as generated by $X_{3,0}$ with base $T_1^{-1}(A_1)$. Similarly, the cylinder set is defined as generated by $X_{3,0}$ with base $T_2^{-1}(A_2)$, where $T_2 : X/X_{3,0} \to X/X_{2,0}$ is the linear mapping.

  Note that two cylinder with the same generating subspace coincide iff their bases coincide, i.e., $T_1^{-1}(A_1) = T_2^{-1}(A_2)$.

- For a nondecreasing sequence of sets $\{A_n\}$, $A_n = \pi_N A_{n+1} \subset A_{n+1}$, $n \ge 1$ and $T_n \equiv [t_1, \ldots, t_n] \cup t_{n+1} = [t_1, \ldots, t_n, t_{n+1}] = T_{n+1}$, the cylinder sets $C_\xi[A_n; t_1, \ldots t_n] \supset C_\xi[A_{n+1}; t_1, \ldots t_{n+1}]$ is nonincreasing.

In other word, $C_n \equiv A_1 \times \dots A_n \times X \times X \dots \supseteq A_1 \times \dots A_n \times A_{n+1} \times X \times X \dots \equiv C_{n+1}$.

So for any $C_n \in \mathscr{C}_n \Rightarrow C_n \in \mathscr{C}_{n+1}$, i.e. $\mathscr{C}_{n+1}$ is finer than $\mathscr{C}_n$, or $\mathscr{C}_n \subset \mathscr{C}_{n+1}$.

2.2 Cylindrical σ-algebra

- Denote $\mathscr{C}_n$ as the collection of all $C[A, X_0]$, for all $A \in X/X_0 \simeq \mathcal{B}^n$ and X_0 as the annihilator of all possible n-dimensional subspace $X_n^* \subset X^*$.
- $\mathscr{C}_n$ forms an *algebra* and $\mathscr{C}_1 \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_n \subset \cdots$ form increasing nested sets.
- **Definition** The collection of all cylinder sets $\mathscr{C}_n$ for all finite dimensions $n \geq 1$ is referred as the *algebra of cylinder sets*, denoted as $\mathscr{C}_0$. That is, $\mathscr{C}_0 \equiv \bigvee_{n\in\mathbb{N}} \mathscr{C}_n$ and $\mathscr{A} \vee \mathscr{B} = \{A \cap B, A \in \mathscr{A}, B \in \mathscr{B}\}$. $\mathscr{C}_0$ is denotes as $\mathcal{B}^n \times X^* \times X^* \times X^* \times \cdots$.
- Similar as $\mathscr{C}_n$, $\mathscr{C}_0$ is an *algebra*. Note that for intersection of $C_1 \in \mathscr{C}_n$ and $C_2 \in \mathscr{C}_m, n \neq m$, we can always find some $\max\{m, n\} \leq s \leq m + n$ so that $C_1 \in \mathscr{C}_s$ and $C_2 \in \mathscr{C}_s$, then it shows the closure under finite intersection.
- In general, $X = \prod_{i\in A} X_i$, then a *n-dimensional cylinder set* in X is of form $U \times \prod_{i\notin S} X_i$ where $|S| = n$ is a finite subset of index set A, and $U \subset \prod_{i\in S} X_i$. Define the operation $\pi_j : X \to X_j$ as a projection on j-th coordinate, then a 1-dimensional cylinder is $\pi_j^{-1}(U)$, $U \subset X_j$. For a n-dimensional cylinder set, $\bigotimes_{j\in S} \pi_j^{-1}(U), U \subset \prod_{i\in S} X_i$. The cylinder sets are open sets if U is open in $\prod_{i\in S} X_i$.
- The collection of all cylinder sets forms an algebra but *not* σ-algebra.

- Note that on the algebra of cylinder sets, we can define a measure μ that is *finitely additive*, since for a union of a system of finite cylinder sets, we can find a common generating subspace so that the resulting bases is the finite union of individual bases. However, for the algebra of cylinder sets, the *countably additive* does not hold. This motivates the wider space with σ-algebra defined:

  Definition The σ-algebra $\mathscr{C} = \sigma(\mathscr{C}_0)$ generated from the algebra of cylinders $\mathscr{C}_0$ is called *cylinderical σ-algebra.*

  The cylinderical σ-algebra is the key ingredient in defining a measure on the topological vector space.
- For a nondecreasing sequence of sets $A_n \uparrow A$, $A_n = \pi_{T_n} A_{n+1} \subset A_{n+1}$, $n \geq 1$ and $T_n \uparrow T$, the cylinder sets $\lim_{n\to} C_\xi[A_n; T_n] = \bigcap_{n=1}^{\infty} C_\xi[A_n; T_n] = C_\xi[A; T]$.

- The cylinderical σ-algebra is *not* the Borel σ-algebra $\mathcal{B} \equiv \mathcal{B}(X)$, which is generated from all open sets in topology of X.
- Note that $\mathscr{C}_0 \subset \mathscr{C} \subset \mathcal{B}_W \subset \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra generated from topology in X and $\mathcal{B}_W$ is the Borel σ-algebra generated from weak topology in X.

3 Measure on infinite dimensional function space

- **Definition** Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{F}$ be a σ-algebra on X. Let $\mathcal{P}$ be a measure on $(X, \mathscr{F})$. A measurable subset A of X is said to be *inner regular* if

  $$\mathcal{P}(A) = \sup\{\mathcal{P}(F) \mid F \subseteq A, F \text{ compact and measurable}\}$$

  and said to be *outer regular* if

  $$\mathcal{P}(A) = \inf\{\mathcal{P}(G) \mid G \supseteq A, G \text{ open and measurable}\}$$

- A measure $\mathcal{P}$ is *inner regular* if all measurable set is inner regular; it is *outer regular*, if all measureable set is outer regular.

  $\mathcal{P}$ is *regular* if it is both inner regular and outer regular.

- Any Borel probability measure on a *locally compact Hausdorff space* with a *countable* base for its topology, or *compact metric space*, or Radon space, is regular.
- $\mathcal{P}$ is called *locally finite* if every point of X has a neighborhood U for which $\mathcal{P}(U)$ is finite.
- $\mathcal{P}$ is *Radon measure*, if it is locally finite and inner regular.
- $\mathcal{P}$ is *tight* if $B = X$ in above.
- A *Radon measure* on the locally compact Hausdorff space can be expressed in terms of *continuous linear functionals* on the space of *continuous functions with compact support.* (A Radon measure is real then it can be decomposed into the difference of two positive measures.)
- $\mathcal{P}$ is *Radon* $\Rightarrow$ $\mathcal{P}$ is *tight* and *regular.*
- $\mathcal{P}$ is finitely additive, regular, tight, then it is Radon.
- Let $\mathcal{Z}$ be an algebra of Borel subsets of Hausdorff topological space X, $\mathcal{Z} \subset \mathcal{B}$, where $\mathcal{B}$ denotes the Borel σ-algebra on X. The function $\mathcal{P} : \mathcal{Z} \to \mathbb{R}_+$ is called a *Radon function* if

  $$\mathcal{P}(B) = \sup\{\mathcal{P}(Z) \mid Z \subset B, Z \in \mathcal{Z}, Z \text{ is compact}\}.$$

- **Definition** The function $\mathcal{P}^*$defined for all $B \subset X$ is said to be *outer measure* if

  $$\mathcal{P}^*(B) = \inf\{\mathcal{P}(Z) \,|\, Z \supset B, Z \in \mathcal{Z}\}$$

  where $\mathcal{Z}$ is an *algebra* of Borel subsets in X.

- If $\mathcal{Z}$ is a σ-algebra and $\mathcal{P}$ is countably additive, then $\mathcal{P}$ is called a *measure.* A *probability measure* satisfies additionally $\mathcal{P}(X) = 1$.

  If $\mathcal{P}$ has more properties such as tight, regular, Radon, then $\mathcal{P}$ is called tight, regular, Radon, respectively.

- Any measure defined on the Borel σ-field in a complete separable metric space is Radon measure.
  - The Lebesgue measure on Euclidean space, restricted on Borel sets;

- Haar measure on any locally compact topological group;
- Gaussian measure on Euclidean space $\mathbb{R}^n$ with its Borel σ-algebra;
- Counting measure on Euclidean space is an example of a measure that is *not* a Radon measure, since it is not locally finite.

- Note that we can define a finitely additive measure on $\mathscr{C}_0$ and we need to extend it to countably additive, tight, regular (Radon) measure on $\mathscr{C}$.

  Here we need to verify that $\mathscr{C}$ contains a base for the weak topology (. in fact, the generating subspace X_0 is closed in weak topology of X); and see that $\mathcal{P}$ as a measure on $\mathscr{C}_0$ is finitely additive. Finally, *any* measure on $\mathscr{C}$ is regular, if X is Hausdorff, locally convex topological space.

  What we need additionally is the tightness of the outer measure.

- The topological support $\mathrm{supp}(\mathcal{P})$ of a measure $\mathcal{P}$ is defined to be the set of $\boldsymbol{x} \in X$ whose each neighborhood has a positive measure. The topological support is always closed set.

  The support of Radon measure $\mathcal{P}$ is well-defined, as the least closed set of full measure

$$\mathcal{P}(\mathrm{supp}(\mathcal{P})) = \mathcal{P}(X), \quad \mathrm{supp}(\mathcal{P}) = \bigcap \{A \,|\, \mathcal{P}(A) = \mathcal{P}(X), A \text{ is closed}\}\,.$$

4 Theorems

- **Proposition 4.1** *[Gel'fand and Vilenkin, 2014]*
  Let X_n^* be the n-dimensional linear subspace in X^* spanned by $\{f_k, 1 \le k \le n\}$, X_0 be the linear subspace in X consisting of all $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 0, \forall f \in X_n^*$. Define the quotient *map $q : X \to X/X_0$, which maps $\boldsymbol{x}$ to the* coset *$\boldsymbol{x} + X_0$ and X/X_0 is the* quotient space *induced by the relationship $\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \boldsymbol{x} - \boldsymbol{y} \in X_0$.*

  If X^* is locally convex linear topological space, then X/X_0 is the adjoint space *of X_n^*.*

  Proof: Given X^* is locally convex linear topological space, then any linear continuous functional defined on a subspace X_n^* can be extended to the linear continuous functional defined on the whole X^*.

  Define a linear continuous functional $G_{\boldsymbol{x}} : X_n^* \to \mathbb{R}$ as $f \mapsto f(\boldsymbol{x})$, for all $f \in X_n^*$. By duality, any $\boldsymbol{x} \in X$ is uniquely associated with a linear continuous functional $G_{\boldsymbol{x}}$ that is defined on X^* and thus on X_n^*. Two functionals $G_{\boldsymbol{x}}$ and $G_{\boldsymbol{y}}$ lies in the same coset iff $\boldsymbol{x} \sim \boldsymbol{y}$ relative to X_0; that is, they correspond to the same element in X/X_0. Thus for every $s \in X/X_0$, there corresponds a linear functional on X_n^* and to distinct elements in X/X_0, there corresponds distinct functionals.

  We show that the converse is true: For any linear functional $G_{\boldsymbol{x}}$ defined on X_n^*, it can be extended to X^*, and for all possible extension, since they coincide on X_n^*, should belong to the same coset relative to X_0 in X. Thus any linear functional on X_n^* corresponds to some element in X/X_0, which completes the proof. ∎

  Note that $X/X_0 \simeq F_n(X) = (X_n^*)^* \subset \mathbb{R}^n$ by the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, "q"] \arrow[dr, "F_n=(f_1,\cdots,f_n)"'] & X/X_0 \arrow[d, dashed, "p"] \\
 & F_n(X) \subset \mathbb{R}^n
\end{tikzcd}
$$

- **Proposition 4.2** *Denote $\mathscr{C}_n$ as the collection of all $C[A, X_0]$, for all $A \in X/X_0 \simeq (X_n^*)^* \subset \mathcal{B}^n$ and X_0 as the annihilator of all possible n-dimensional subspace $X_n^* \subset X^*$.*

  Then $\mathscr{C}_n$ forms an algebra.

  Proof: We check for the axiom of algebra:

  1. The complement: for given $C[A, X_0] \in \mathscr{C}_n$

$$
\begin{aligned}
X - C[A, X_0] &= X - \{\boldsymbol{x} \mid (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})) \in A\ \} \\
&= \{\boldsymbol{x} \mid (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})) \notin A\ \} \\
&= C[A^c, X_0] \in \mathscr{C}_n
\end{aligned}
$$

  2. Finite intersection: for any $C[A_1, X_{1,0}], C[A_2, X_{2,0}] \in \mathscr{C}_n$

$$
C[A_1, X_{1,0}]\ \cap\ C[A_2, X_{2,0}] = C[A_1 \cap A_2, X_{1,0} \cap X_{2,0}] \in \mathscr{C}_n
$$

  where $X_{1,0} \cap X_{2,0}$ is the annihilator of $X_{3,n}^*$ generated by $X_{1,n}^*$ and $X_{2,n}^*$ and $A_3 = A_1 \cap A_2$.

To clarify the finite intersection part, we consider the subspace $X_{3,0} = X_{1,0} \cap X_{2,0} \subset X$, which is seen as the annihilator $X_{3,0}$ of $X^*_{n,3}$ in X^* and a cylinder set C_3 generated from $X_{3,0}$ on the base $A_3 = A_1 \cap A_2$.

We show that $C_3[A_3, X_{3,0}] = C[A_1, X_{1,0}] \ \cap \ C[A_2, X_{2,0}]$.

$\Rightarrow$
We show $C_3[A_3, X_{3,0}] \subseteq C[A_1, X_{1,0}] \ \cap \ C[A_2, X_{2,0}]$. Note that $X_{3,0} \subset X_{1,0}$, so any $\boldsymbol{x} \in C_3[A_3, X_{3,0}]$, $\boldsymbol{x} \in f^{-1}(A_3) + \boldsymbol{s}, \forall f \in X^*_{n,3}$, where $\boldsymbol{s} \in X_{3,0} \subset X_{1,0}$ and $\boldsymbol{x}_3 \in f^{-1}(A_3) = f^{-1}(A_1 \cap A_2) \subset f^{-1}(A_1)$. Note that $X^*_{n,3} = (X/X_{3,0})^* \supset (X/X_{1,0})^* = X^*_{n,1}$, hence $\boldsymbol{x} \in f_1^{-1}(A_3) + \boldsymbol{s} \subset f_1^{-1}(A_1) + \boldsymbol{s}$, $\forall f_1 \in X^*_{n,1}, \Rightarrow \boldsymbol{x} \in C_1[A_1, X_{1,0}]$. Similarly, $\boldsymbol{x} \in C_2[A_2, X_{2,0}]$. So the left-inclusion is proved.

$\Leftarrow$
For arbitrary $\boldsymbol{x} \in C[A_1, X_{1,0}] \ \cap \ C[A_2, X_{2,0}]$, $\boldsymbol{x} \in f_1^{-1}(A_1) + \boldsymbol{s}$, $\forall f_1 \in X^*_{n,1}$, where $\boldsymbol{s} \in X_{1,0}$, and $\boldsymbol{x} \in f_2^{-1}(A_2) + \boldsymbol{s}$, $\forall f_2 \in X^*_{n,2}$, where $\boldsymbol{s} \in X_{2,0}$. Clearly, $\boldsymbol{s} \in X_{3,0} = X_{1,0} \cap X_{2,0}$ and $f_1, f_2 \in X^*_{n,3}$. Since $X_{3,0} \subset X_{1,0}$, any coset w.r.t. $X_{3,0}$ corresponds to some coset w.r.t. $X_{1,0}$, so we can associate any coset w.r.t. $X_{3,0}$ with some coset w.r.t. $X_{1,0}$ that contains it. Therefore, there exists a linear mapping $T_1 : X/X_{3,0} \to X/X_{1,0}$ as a inclusion map. Then consider the preimage $T_1^{-1}(A_1)$, then the set $C[A_1, X_{1,0}] \cap C[A_2, X_{2,0}]$ is by definition a cylinder set generated by $X_{3,0}$ with base $T_1^{-1}(A_1)$. Similarly, the cylinder set $C[A_1, X_{1,0}] \cap C[A_2, X_{2,0}]$ is defined as generated by $X_{3,0}$ with base $T_2^{-1}(A_2)$, where $T_2 : X/X_{3,0} \to X/X_{2,0}$ is the linear mapping. Finally, since two cylinders with the same generating subspace coincide iff their bases coincide, i.e., $T_1^{-1}(A_1) = T_2^{-1}(A_2)$, so $T_1^{-1}(A_1) = T_2^{-1}(A_2) = A_1 \cap A_2$. This shows $C_3[A_3, X_{3,0}] \supseteq C[A_1, X_{1,0}] \ \cap \ C[A_2, X_{2,0}]$.

This completes the proof. ∎

- **Proposition 4.3** *Let $\mathscr{C}_0$ be the algebra of cylinder sets defined on X and $\mathscr{C}$ is the σ-algebra that is generated from $\mathscr{C}_0$. Define $\mathscr{E}$ as the minimal σ-algebra generated from the collection of sets $\{\pi_t, t \geq 1\}$, where $\pi_t : X^T \to \mathbb{R}$ as $\pi_t(\xi) = \xi(t)$ is $(\mathcal{B}(X^T), \mathcal{B}(X))$.*

  Then $\mathscr{C} = \mathscr{E}$.

- **Theorem 4.4** *(Extension from finite additive, regular measure to Radon measure)*
  Assume that a function $\mathcal{P}$ is defined on some algebra of Borel sets $\mathcal{Z}$ of a Hausdorff topological space X, and the following conditions are satisfied:

  1. *The algebra $\mathcal{Z}$ contains a base of topology on X;*
  2. *The function $\mathcal{P}$ is finitely additive on $\mathcal{Z}$;*
  3. *The function $\mathcal{P}$ is regular on $\mathcal{Z}$;*
  4. *The outer measure $\mathcal{P}^*$ is* tight *on $\mathcal{Z}$, that is for any $\epsilon > 0$, there exists a* compact *set M such that*

$$\mathcal{P}^*(M) = \inf \{P(Z) \,|\, Z \supset M, Z \in \mathcal{Z}\} \geq P(X) - \epsilon.$$

Then the function $\mathcal{P}$ *can be* uniquely extended to a Radon measure *on the whole of the Borel* σ*-algebra of space* X*.*

5 Examples

- **Example** For Hilbert space X, which is self-dual, i.e., $X = X^*$, the cylinder set is defined as

  $$C[A, X_0] = \{\boldsymbol{x} | \{\langle \boldsymbol{x}, \boldsymbol{y}_k \rangle, 1 \le k \le n\} \in A\}, \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} \subset X, A \in \mathcal{B}^n.$$

  Note that $X_n^* = \operatorname{span}\{\langle \cdot, \boldsymbol{y}_1 \rangle, \ldots, \langle \cdot, \boldsymbol{y}_n \rangle\} = \operatorname{Img}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} \subset X$, which is the column space of matrix $[\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n]$. Define the linear mapping $F_n : X \to \mathbb{R}^n$ as $\boldsymbol{x} \mapsto \{\langle \boldsymbol{x}, \boldsymbol{y}_k \rangle, 1 \le k \le n\}$. So $X_n^* = F_n(X)$.

  Then the generating subspace X_0 is the kernel space of F_n, and $X_0 = (X_n^*)^\perp$ is the orthogonal complement of X_n^*.

  The quotient space $\dim(X/X_0) = \dim(X) - \dim(X_0) = \dim(X_n^*) = n$, which is *codimension* of X_0 in X. By first isomorphism theorem of linear algebra, $X/X_0 \simeq X_n^* = F_n(X)$.

  So $C[A, X_0] = A + (X_n^*)^\perp$, $A \subset \mathcal{B}(X_n^*)$. ∎

- **Example** Let X be an locally compact space and $\{f_j, 1 \le j \le n\}$ a sequence of elements of X. Consider the mapping $\hat{f} : (X, \mathscr{C}) \to (\mathbb{R}^\infty, \mathcal{B})$ defined by the formula $\hat{f}(\boldsymbol{x}) = \{f_j(\boldsymbol{x}), 1 \le j \le n\}$. Prove that the mapping $\hat{f}$ is measureable and continuous in the weak topology. Check that, for any set $C \subset \mathscr{C}$, one can choose $\hat{f}$ and a Borel set $A \subset \mathbb{R}^\infty$ such that $C = \hat{f}^{-1}(A)$.

- **Example** Show that any measure in $\mathbb{R}^\infty$ is a Radon measure. (Check that the space $\mathbb{R}^\infty$ is separable, and metrizable.)

- **Example** Let X be an locally compact space. Show that any measure in $\mathscr{C}$ is regular both in the original topology and in the weak topology in the space X. (Check that the space $\mathbb{R}^\infty$ is separable, and metrizable.)

- **Example** Consider the locally convex Hausdorff topological space Ω as the sample space, and the functional $\xi \in \Omega^*$ as random variables $\xi : \Omega \to \mathbb{R}$. $\mathcal{P}$ defined on the cylindrical algebra $\mathscr{C} \subset \mathcal{B}$ is the probability measure for the random function $\xi_\cdot : \Omega \times T \to \mathbb{R}$. In specific, any finitely-dimensional sample function $(\xi_1, \ldots, \xi_t) \subset \Omega^*$; i.e., for any $t \ge 1$

  $$\mathcal{P}((\xi_1, \ldots, \xi_t) \in A) = \mathcal{P}(\{\omega : (\xi_1(\omega), \ldots, \xi_t(\omega)) \in A\})$$

  where $\{\omega : (\xi_1(\omega), \ldots, \xi_t(\omega)) \in A\} = C[A, \xi_1, \ldots, \xi_t] \in \mathscr{C}$.

References

Izrail Moiseevich Gel'fand and N Ya Vilenkin. *Generalized functions: Applications of harmonic analysis*, volume 4. Academic press, 2014.

Mikhail Antolevich Lifshits. *Gaussian random functions*, volume 322. Springer Science & Business Media, 2013.

Michael Reed and Barry Simon. *Methods of modern mathematical physics: Functional analysis*, volume 1. Gulf Professional Publishing, 1980.