Lecture 4: Vector Fields

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1 Vector field on Euclidean space and on surface

1.1 Field of directions and vector field

- **Definition** A ***vector field*** $\boldsymbol{w}$ in an open set U of Euclidean space $\mathbb{R}^2$ is a map which assign to each $q \in U$ a vector $\boldsymbol{w}(q) \in \mathbb{R}^2$. The vector field is said to be *differentiable* if writing $q = (x, y)$ and $\boldsymbol{w}(q) = (a(x, y), b(x, y))$, the functions a, b are differentiable function in U.

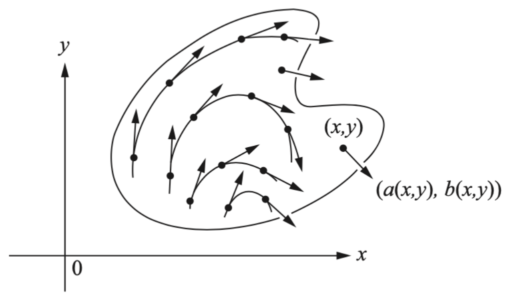

Figure 1: A vector field

- **Definition** A ***(tangent) vector field*** $\boldsymbol{w}$ in an open set $U \subset \mathcal{S}$ of a regular surface $\mathcal{S}$ is a correspondence which assigns to each $p \in U$ a vector $\boldsymbol{w}(p) \in T_pS$. The vector field $\boldsymbol{w}$ is *differentiable* at $p \in U$ if, for some parameterization $\boldsymbol{x}(u, v)$ at p, the functions $a(u, v)$ and $b(u, v)$ given by

$$\boldsymbol{w}(p) = a(u, v)\boldsymbol{x}_u + b(u, v)\boldsymbol{x}_v$$

are differentiable functions at p; it is clear that this definition does not depends on the choice of $\boldsymbol{x}$.

- **Definition** A ***trajectory*** of a vector field $\boldsymbol{w}$ is a differentiable parameterized curve $\alpha(t) = (x(t), y(t)), t \in I$ such that $\alpha'(t) = \boldsymbol{w}(\alpha(t))$.

- The vector field $\boldsymbol{w}$ determines *a system of differential equations*,

$$\frac{dx}{dt} = a(x, y),$$
$$\frac{dy}{dt} = b(x, y),$$

and that a trajectory of $\boldsymbol{w}$ is a solution to the above system of equations.

- **Theorem 1.1** *Let $\boldsymbol{w}$ be a vector field in an open set $U \subset \mathbb{R}^2$. Given $p \in U$, there exists a trajectory $\alpha : I \to U$ of $\boldsymbol{w}$, i.e. $\alpha'(t) = \boldsymbol{w}(\alpha(t)), t \in I$ with $\alpha(0) = p$. This trajectory is unique in the following sense: Any other trajectory $\beta : J \to U$ with $\beta(0) = p$ agrees with α in $I \cap J$.*

This gives the existence and uniqueness of trajectory in local neighborhood.

- **Theorem 1.2** *Let $\boldsymbol{w}$ be a vector field in an open set $U \subset \mathbb{R}^2$. Given $p \in U$, there exists a neighborhood $V \subset U$ of p, an interval I, and a mapping $\alpha : V \times I \to U$ such that*

  - *For a fixed $p \in V$, the curve $\alpha(p, t), t \in I$, is the trajectory of $\boldsymbol{w}$ passing through p; that is,*

$$\alpha(q, 0) = p, \quad \frac{\partial \alpha}{\partial t}(p, t) = \boldsymbol{w}\left(\alpha(p, t)\right)$$

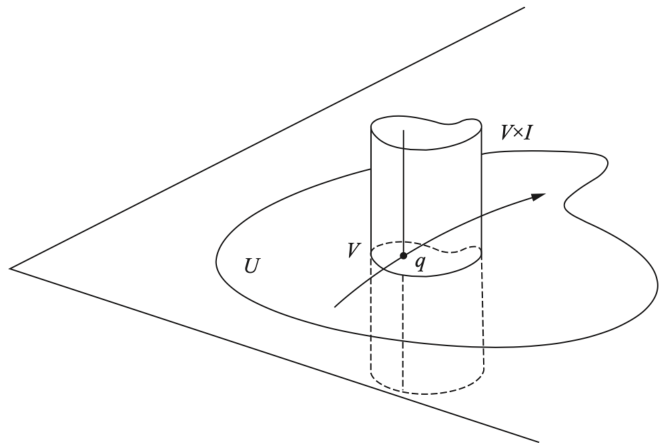

Figure 2: All trajectories which pass p in a neighborhood V can be represented by α

– *α is differentiable.*

This means that the trajectory depends differentiable on initial point p.

Geometrically Theorem 1.2 means that all trajectories which pass, for $t = 0$, in a certain neighborhood V of p may be "collected" into a single differentiable map. It is in this sense that we say that the trajectories depend differentiably on p.

- **Definition** The collection of trajectories $\alpha(q, t)$ passing through a neighborhood V of p is called a ***(local) flow*** of $\boldsymbol{w}$ at p.

- Given the parameterization $\boldsymbol{x}(u, v)$ at p, the differentiable vector field $\boldsymbol{w}$ and the curve $\alpha(t) = \boldsymbol{x}(u(t), v(t))$ on $\mathcal{S}$ with $\alpha(0) = p$, $\dot{\alpha}(0) = \boldsymbol{y}$, the vector field can be represented as

$$\begin{aligned} \boldsymbol{w}(t) &= a(u(t), v(t))\boldsymbol{x}_u + b(u(t), v(t))\boldsymbol{x}_v \\ &= a(t)\boldsymbol{x}_u + b(t)\boldsymbol{x}_v \end{aligned} \tag{1}$$

- **Lemma 1.3** ***(The existence of first integral)***
*Let $\boldsymbol{w}$ be a vector field in an open set $U \subset \mathbb{R}^2$ and let $p \in U$ such that $\boldsymbol{w}(p) \neq 0$. Then there exists a neighborhood $W \subset U$ of p and a differentiable function $f : W \to \mathbb{R}$ such that f is **constant along each trajectory** of $\boldsymbol{w}$ and $df_q \neq 0$ for all $q \in W$.*

Proof: Choose the Cartesian coordinate system in $\mathbb{R}^2$ such that $p = (0, 0)$ and $\boldsymbol{w}(p)$ is in direction of $x-$axis. Let the $\alpha : V \times I \to U$ be a local flow at p, $V \subset U, t \in I$, and let the $\hat{\alpha}$ be the restriction of α to the rectangle

$$(V \times I) \cap \{(x, y, t), x = 0\}$$

By definition of the local flow, $d\hat{\alpha}_p$ maps the unit vector of the t axis into $\boldsymbol{w}$ and maps the unit vector of y-axis into itself. Thus $d\hat{\alpha}_p \neq 0$. It follows that there exists a neighborhood $W \subset U$ of p, where $\hat{\alpha}^{-1}$ is defined and differentiable. The projection of $\hat{\alpha}^{-1}(x, y)$ onto the $y-$axis is a differentiable function $\xi = f(x, y)$, which has the same value ξ for all points of the trajectory passing through $(0, \xi)$.

In other word, note that $\hat{\alpha}(0, y, t)$ is the point obtained by "walking" in the trajectory of $(0, y)$ an time t. On the other hand, $\hat{\alpha}^{-1}(x, y)$ are the points of the form $(0, y', t)$ for some y' and

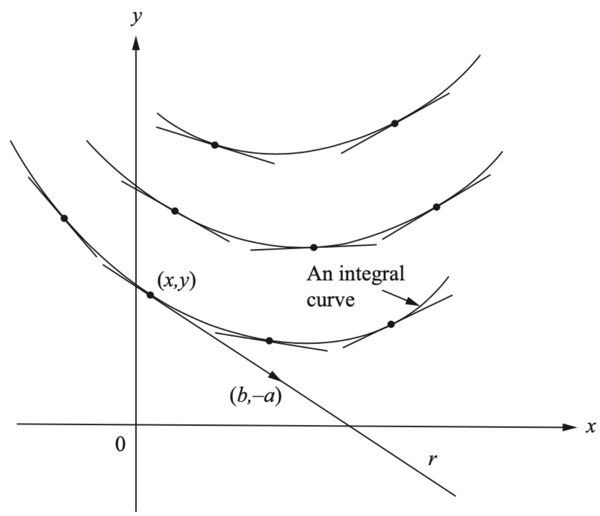

Figure 3: An integral curve of differential equations

some $t \in I$. The projection of $\hat{\alpha}^{-1}(x, y)$ onto the $y-$axis is the intersection of the trajectory passing through (x, y) with the $y-$axis. By the uniqueness of the trajectory, if you take (x, y) and (x_1, y_1) in the same trajectory, they must pass through the same position y-axis, so the function $\hat{\alpha}^{-1}(x, y)$ is constant on trajectories.

Since $d\hat{\alpha}_p \neq 0$, W may be taken sufficiently small so that $df_q \neq 0$ for all $q \in W$. f is the function we required. ■

- **Definition** The function $f : W \to \mathbb{R}$ above is called a *(local) first integral of a vector field* of $\boldsymbol{w}$ in a neighborhood W of p. In other word, for f to be ***the first integral of vector field*** $\boldsymbol{w}$, $\alpha(t)$ be the trajectory of the vector field, then

$$\begin{aligned}\frac{df(\alpha(t))}{dt} &= a(u, v)\frac{\partial f}{\partial u} + b(u, v)\frac{\partial f}{\partial v} \\ &\equiv \boldsymbol{w}(f) = 0\end{aligned}$$

In other word, the **curve** $f(\alpha(t)) = const$ is seen as ***one solution*** for the system of differential equations.

- **Definition** A ***field of directions*** r is an open set $U \subset \mathbb{R}^2$ is a correspondence which assigns to each $p \in U$ a ***line*** $r(p)$ in $\mathbb{R}^2$ passing through p.

  r is said to be ***differentiable*** at $p \in U$ if there exists *nonzero differentiable vector field* $\boldsymbol{w}$ defined in a neighborhood $V \subset U$ of p, such that for each $q \in V$, $\boldsymbol{w}(q) \neq 0$ is a ***basis*** of $r(q)$; r is *differentiable* in U, if it is *differentiable* in every $p \in U$.

  Definition In differential equations, a ***field of directions*** is given by

$$a(x, y)\frac{dx}{dt} + b(x, y)\frac{dy}{dt} = 0$$

  The above form is also called $1-$***form differentials***.

- Note that for each differentiable $\boldsymbol{w}$ in U there exists a differentiable field of directions r with $r(p) =$ line generated by $\boldsymbol{w}(p)$.

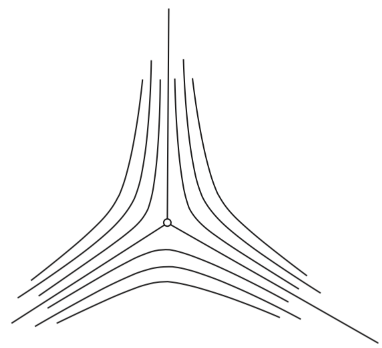

Figure 4: Field of directions and the integral curve.

- **Definition** A *regular connected* curve $\mathcal{C} \subset U$ is an ***integral curve of a field of directions*** r defined in U if $r(q)$ is the ***tangent line*** to $\mathcal{C}$ at q for all $q \in \mathcal{C}$.

  It is clear that given r in U, there passes, for each $q \in U$ an integral curve of r.

- The ***difference*** between *field of directions* and the *vector field* is that for $\boldsymbol{w}_2 = \lambda \boldsymbol{w}_1$ with $\lambda \neq 0$ constant, they corresponds to the ***same field of direction*** r (i.e. up to scale).

  Conversely, if two vectors belong to the same straight line passing through p they are considered equivalent. Thus for every $p \in U$, $r(p) = (r_1, r_2)$ with r_1, r_2 being two real numbers and $(r_1, r_2) \sim (\lambda r_1, \lambda r_2)$

1.2 Vector fields in local coordinates and derivative of functions

- **Theorem 1.4** *Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ are two vector fields in an open subset $U \subset \mathcal{S}$, which are linearly independent at some point $p \in U$. Then it is possible to* **parameterize** *a neighborhood $V \subset U$ of p in a way that for each $q \in V$ the coordinate lines of this parameterization passing through q are* ***tangent*** *to the lines determined by $\boldsymbol{w}_1(q)$ and $\boldsymbol{w}_2(q)$.*

  (*Note that not necessary to be the tangent line.*)

  Proof: Let W be a neighborhood of p where the first integrals f_1 and f_2 of $\boldsymbol{w}_1, \boldsymbol{w}_2$, respectively, are defined. Define a map $\varphi : W \to \mathbb{R}^2$ as

$$\varphi(q) = (f_1(q), f_2(q)), \quad q \in W.$$

  Since f_1 is constant on the trajectory of $\boldsymbol{w}_1$ and $df_1 \neq 0$, we have at p

$$d\varphi_p(\boldsymbol{w}_1) = ((df_1)_q(\boldsymbol{w}_1), (df_2)_q(\boldsymbol{w}_1)) = (0, a),$$

  where $a = (df_2)_q(\boldsymbol{w}_1) \neq 0$, since $\boldsymbol{w}_1, \boldsymbol{w}_2$ are linearly independent. Similarly, see that

$$d\varphi_p(\boldsymbol{w}_2) = (b, 0),$$

  where $b = (df_1)_q(\boldsymbol{w}_2) \neq 0$. It follows that $d\varphi_p \neq 0$ and hence φ is a local diffeomorphism. There exist, therefore, a neighborhood $\bar{U} \subset \mathbb{R}^2$ of $\varphi(p)$ which is mapped diffeomorphically by $\boldsymbol{x} = \varphi^{-1}$ onto a neighborhood $V = \boldsymbol{x}(\bar{U})$ of p; that is, $\boldsymbol{x}$ is a parameterization of $\mathcal{S}$ at p, whose coordinate curve is given by

$$f_1(q) = const. \qquad\qquad f_2(q) = const.$$

  are tangent at q to the lines determined by $\boldsymbol{w}_1(q)$ and $\boldsymbol{w}_2(q)$. ∎

- **Corollary 1.5** *Given two fields of directions r_1, r_2 in an open set $U \subset \mathcal{S}$ such that at $p \in U$, $r_1(p) \neq r_2(p)$, there exists a* ***parameterization*** $\boldsymbol{x}$ *in a neighborhood of p such that the* ***coordinate curves*** *of* $\boldsymbol{x}$ *are the* ***integral curves*** *of r_1, r_2.*

- **Corollary 1.6** *(**The existence of the orthogonal parameterization**).*
For all $p \in U$, there exists a ***parameterization*** $\boldsymbol{x}(u, v)$ *is a neighborhood V of p such that the coordinate curve $u = const.$ and $v = const.$ intersects* ***orthogonally*** *for each $q \in V$ (such that* $\boldsymbol{x}$ *is called an* ***orthogonal parameterization****).*

- It thus represent the **basis vector field** as $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, and

$$\boldsymbol{w}(u, v) = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}$$

- **Corollary 1.7** *Let $p \in \mathcal{S}$ be a hyperbolic point of $\mathcal{S}$.Then it is possible to parametrize a neighborhood of p in such a way that the coordinate curves of this parametrization are the* ***asymptotic curves*** *of $\mathcal{S}$.*

- **Corollary 1.8** *Let $p \in \mathcal{S}$ be a non-umbilical point of $\mathcal{S}$.Then it is possible to parametrize a neighborhood of p in such a way that the coordinate curves of this parametrization are the* ***lines of curvature*** *of $\mathcal{S}$.*

- **Definition** Define *the* ***derivative*** $\boldsymbol{w}(f)$ *of a differentiable function* $f : U \subset \mathcal{S} \to \mathbb{R}$ *relative to a* ***vector field*** $\boldsymbol{w}$ in U by

$$\boldsymbol{w}(f)(q) = \frac{d}{dt}(f \circ \alpha)\bigg|_{t=0}, \quad q \in U$$

where $\alpha : I \to \mathcal{S}$ is the ***trajectory of*** $\boldsymbol{w}$ passing through q such that $\alpha(0) = q, \alpha'(0) = \boldsymbol{w}(q)$.

- Thus the vector field $\boldsymbol{w}$ can also be considered as a ***differential operator*** *on* ***space of continuous functions*** $\mathbb{C}^{\infty}$ as $\boldsymbol{w} : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ as

$$\boldsymbol{w}(f) = \text{directional derivative of } f \text{ along trajectory } \alpha \text{ of } \boldsymbol{w}.$$

Then

$$\begin{aligned} \boldsymbol{w}(f) &= \left(a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v}\right)(f) \\ &= a(u, v)\frac{\partial f}{\partial u} + b(u, v)\frac{\partial f}{\partial v} \end{aligned}$$

- The composition of two vector fields $\boldsymbol{w}, \boldsymbol{v}$ together gives

$$\begin{aligned} \boldsymbol{w}\boldsymbol{v}(fg) \equiv \boldsymbol{w}\left(\boldsymbol{v}(fg)\right) &= \boldsymbol{w}\left(\boldsymbol{v}(f)g\right) + \boldsymbol{w}\left(f\boldsymbol{v}(g)\right) \\ &= \boldsymbol{w}\left(\boldsymbol{v}(f)\right)g + \boldsymbol{v}\left(f\right)\boldsymbol{w}\left(g\right) + \boldsymbol{w}\left(f\right)\boldsymbol{v}(g) + f\boldsymbol{w}\left(\boldsymbol{v}\left(g\right)\right) \\ \boldsymbol{v}\boldsymbol{w}(fg) &= \boldsymbol{v}\left(\boldsymbol{w}(f)\right)g + \boldsymbol{w}\left(f\right)\boldsymbol{v}\left(g\right) + \boldsymbol{v}\left(f\right)\boldsymbol{w}(g) + f\boldsymbol{v}\left(\boldsymbol{w}\left(g\right)\right) \\ \left[\boldsymbol{w}\,\boldsymbol{v} - \boldsymbol{v}\,\boldsymbol{w}\right](fg) &= \left(\left[\boldsymbol{w}\,\boldsymbol{v} - \boldsymbol{v}\,\boldsymbol{w}\right](f)\right)g + f\left(\left[\boldsymbol{w}\,\boldsymbol{v} - \boldsymbol{v}\,\boldsymbol{w}\right](g)\right) \\ \left[\boldsymbol{w},\,\boldsymbol{v}\right](fg) &= \left(\left[\boldsymbol{w},\,\boldsymbol{v}\right](f)\right)g + f\left(\left[\boldsymbol{w},\,\boldsymbol{v}\right](g)\right) \end{aligned}$$

where the operator

$$[\boldsymbol{w},\,\boldsymbol{v}] \equiv [\boldsymbol{w}\,\boldsymbol{v} - \boldsymbol{v}\,\boldsymbol{w}]$$

is called the ***Lie bracket***.

References