Lecture 0: Summary (part 4)

Tianpei Xie

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1 Riemannian Metrics and Riemannian Manifolds

1.1 Riemannian Metrics

- **Remark** The most important examples of symmetric tensors on a vector space are ***inner products***. Any inner product allows us to define ***lengths*** of vectors and ***angles*** between them, and thus to do Euclidean geometry.

- **Definition** Let M be a smooth manifold with or without boundary. ***A Riemannian metric*** on M is a smooth ***symmetric covariant 2-tensor field*** on M that is ***positive definite*** at each point.

  A Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M. One sometimes simply says "*M is a Riemannian manifold*" if M is understood to be endowed with *a specific Riemannian metric*. *A Riemannian manifold* ***with boundary*** is defined similarly.

- **Remark** If g is a Riemannian metric on M, then for each $p \in M$, *the 2-tensor g_p* is an ***inner product*** on T_pM. Because of this, we often use the notation $\langle v\,,\,w\rangle_g$ to denote the real number $g_p(v, w)$ for $v, w \in T_pM$.

- **Remark** (***Coordinate Representation of Riemannian Metric***)
  In any smooth local coordinates (x^i), ***a Riemannian metric*** can be written

$$g = g_{i,j}\; dx^i \otimes dx^j, \tag{1}$$

  where $(g_{i,j})$ is ***a symmetric positive definite matrix*** of smooth functions.

- **Remark** (***Alternative Coordinate Representation of Riemannian Metric***)
  The ***symmetry*** of g allows us to write g also in terms of ***symmetric products*** as follows:

$$g = \frac{1}{2} g_{i,j} dx^i\, dx^j \tag{2}$$

- **Remark** The followings are Riemannian metrics:

  1. The Euclidean metric $\bar{g} = \delta_{i,j} dx^i dx^j$, is a Riemannian metric.

  2. If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, we can define a Riemannian metric $\hat{g} = g \oplus \tilde{g}$ on the product manifold $M \times \widetilde{M}$, called ***the product metric***, as follows:

$$\hat{g}((v, \widetilde{v}), (w, \widetilde{w})) = g(v, w) + \widetilde{g}(\widetilde{v}, \widetilde{w}) \tag{3}$$

     for any $(v, \widetilde{v}), (w, \widetilde{w}) \in T_pM \times T_q\widetilde{M} \simeq T_{(p,q)}(M \times \widetilde{M})$.

- **Proposition 1.1** ***(Existence of Riemannian Metrics)****. [Lee, 2003., 2018]*
  Every smooth manifold *with or without boundary admits a* ***Riemannian metric****.*

- **Definition** The ***length*** or ***norm*** of a tangent vector $v \in T_pM$ is defined to be

$$|v|_g = \sqrt{g_p(v, v)} := \sqrt{\langle v\,,\,v\rangle_g}$$

- **Definition** The ***angle*** between two nonzero tangent vectors $v, w \in T_pM$ is the unique $\theta \in [0, \pi]$ satisfying:

$$\theta = \frac{\langle v\,,\, w\rangle_g}{|v|_g\,|w|_g}.$$

- **Definition** Tangent vectors $v, w \in T_pM$ are said to be ***orthogonal*** if $\langle v\,,\, w\rangle_g = 0$. This means either one or both vectors are zero, or the angle between them is $\pi/2$.

- **Definition** Let (M, g) be an n-dimensional Riemannian manifold with or without boundary. *A local frame* $(E_1, \ldots, E_n)$ for M on an open subset $U \subseteq M$ is an ***orthonormal frame*** if the vectors $(E_1|_p, \ldots, E_n|_p)$ form an ***orthonormal basis*** for T_pM at each point $p \in U$, or equivalently if $\langle E_i\,,\, E_j\rangle_g = \delta_{i,j}$.

1.2 Pullback Metrics and Riemannian Isometry

- **Definition** Suppose M, N are smooth manifolds with or without boundary, g is a Riemannian metric on N, and $F : M \to N$ is smooth. The ***pullback*** F^*g is a smooth 2-tensor field on M. If it is ***positive definite***, it is a Riemannian metric on M, called ***the pullback metric*** determined by F.

- **Proposition 1.2** ***(Pullback Metric Criterion)****. [Lee, 2003.]*
Suppose $F : M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a ***Riemannian metric*** *on M if and only if F is a* ***smooth immersion****.*

- **Definition** If (M, g) and $(\widetilde{M}, \widetilde{g})$ are both Riemannian manifolds, a smooth map $F : M \to \widetilde{M}$ is called a ***(Riemannian) isometry*** if it is a ***diffeomorphism*** that satisfies $F^*\widetilde{g} = g$. More generally, F is called ***a local isometry*** if every point $p \in M$ has a neighborhood U such that $F|_U$ is an *isometry* of U onto an open subset of $\widetilde{M}$; or equivalently, if F is a ***local diffeomorphism*** satisfying $F^*\widetilde{g} = g$.

  If there exists a *Riemannian isometry* between (M, g) and $(\widetilde{M}, \widetilde{g})$, we say that they are ***isometric*** as Riemannian manifolds. If each point of M has a *neighborhood* that is *isometric* to an *open subset* of $(\widetilde{M}, \widetilde{g})$, then we say that (M, g) is ***locally isometric*** to $(\widetilde{M}, \widetilde{g})$.

- **Definition** The study of properties of Riemannian manifolds that are ***invariant under (local or global) isometries*** is called ***Riemannian geometry***.

- **Definition** A Riemannian n-manifold (M, g) is said to be a ***flat Riemannian manifold***, and g is a ***flat metric***, if .(M, g) is **locally isometric** to $(\mathbb{R}^n, \bar{g})$.

- **Theorem 1.3** *For a Riemannian manifold (M, g), the following are equivalent:*

  1. *g is flat.*
  2. *Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{i,j} dx^i\, dx^j$.*
  3. *Each point of M is contained in the domain of a smooth coordinate chart in which* ***the coordinate frame*** *is* ***orthonormal.***
  4. *Each point of M is contained in the domain of a* ***commuting orthonormal frame.***

1.3 The Tangent-Cotangent Isomorphism

- **Definition** Given a Riemannian metric g on M, we define a ***bundle homomorphism*** $\widehat{g}: TM \to T^*M$ by setting

$$\widehat{g}(v)(w) = g_p(v,w)$$

for all $p \in M$ and $v, w \in T_pM$.

- **Remark** If X and Y are smooth vector fields on M, this yields

$$\widehat{g}(X)(Y) = g(X,Y).$$

$\widehat{g}(X)(Y)$ is ***linear*** over $\mathcal{C}^\infty(M)$ in Y and thus $\widehat{g}(X)$ is a ***smooth covector field*** by the tensor characterization lemma. On the other hand, the covector field $\widehat{g}(X)$ is ***linear*** over $\mathcal{C}^\infty(M)$ as a function of X, and thus $\widehat{g}$ is a ***smooth bundle homomorphism***. As usual, we use ***the same symbol*** for both the *pointwise bundle homomorphism* $\widehat{g}: TM \to T^*M$ and *the **linear map** on **sections*** $\widehat{g}: \mathfrak{X}(M) \to \mathfrak{X}^*(M)$. $\widehat{g}$ is also a ***bundle isomorphism***.

- **Definition** Given a smooth local frame (E_i) and its dual coframe (ϵ^i), let $g = g_{i,j}\epsilon^i\epsilon^j$ be the ***local expression*** for g. If $X = X^i\,E_i$ is a smooth vector field, the *covector field* $\widehat{g}(X)$ has the ***coordinate expression***:

$$\widehat{g}(X) = \left(g_{i,j}X^i\right)\epsilon^j := X_j\,\epsilon^j,$$

where the ***components*** of ***the covector field*** $\widehat{g}(X)$ is denoted by

$$X_j = g_{i,j}X^i. \tag{4}$$

We say that $\widehat{g}(X)$ *is obtained from* X ***by lowering an index***. And ***the covector field*** $\widehat{g}(X)$ is denoted by $X^\flat$ and called ***X flat***.

- **Remark** Because the matrix $(g_{i,j})$ is nonsingular at each point, the map $\widehat{g}$ is ***invertible***, and the matrix of $\widehat{g}^{-1}$ is just ***the inverse matrix of*** $(g_{i,j})$. We denote ***this inverse matrix*** by $(g^{i,j})$, so that $g^{i,j}g_{j,k} = g_{k,j}g^{j,i} = \delta^i_k$. The ***symmetry*** of $(g_{i,j})$ easily implies that $(g^{i,j})$ is also ***symmetric*** in i and j.

- **Definition** Given $\omega = \omega_j\,\epsilon^j$, the inverse map $\widehat{g}^{-1}$ is given by

$$\widehat{g}^{-1}(\omega) = \omega^i\,E_i$$

where

$$\omega^i = g^{i,j}\,\omega_j \tag{5}$$

If ω is a *covector field*, the ***vector field*** $\widehat{g}^{-1}(\omega)$ *is called* ω ***sharp*** and denoted by $\omega^\sharp$, and we say that *it is obtained from* ω *by* ***raising an index***.

Definition The *two* ***inverse isomorphisms*** $\flat$ and $\sharp$ are known as ***the musical isomorphisms***.

- **Definition** If g is a Riemannian metric on M and $f: M \to \mathbb{R}$ is a smooth function, the ***gradient*** of f is ***the vector field***

$$\text{grad } f = (df)^\sharp := \widehat{g}^{-1}(df)$$

obtained from df by ***raising an index***. It is also denoted as ∇f.

- **Remark** grad f is ***characterized*** by the fact that

$$df_p(w) = \langle \text{grad } f|_p \,,\, w \rangle_g \quad \forall p \in M,\, w \in T_pM, \tag{6}$$
$$\text{or } df(X) = \langle \text{grad } f \,,\, X \rangle_g \quad \forall X \in \mathfrak{X}(M),$$

and has the ***local basis expression***

$$\text{grad } f = \left(g^{i,j} E_i f\right) E_j. \tag{7}$$

Thus if (E_i) is an ***orthonormal frame***, then grad f is the *vector field* whose ***components are the same as the components of*** df; but *in other frames, this will not be the case.*

- **Remark** In smooth coordinates $(\partial/\partial x^i)$, we have

$$\text{grad } f = g^{i,j} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{8}$$

- **Definition** Suppose g is a Riemannian metric on M, and $x \in M$. We can define an ***inner product*** on ***the cotangent space*** T_x^*M by

$$\langle\, \omega \,,\, \eta \,\rangle_g = \langle\, \omega^\sharp \,,\, \eta^\sharp \,\rangle_g.$$

- **Remark** (***Coordinate Representation of Inner Product on Covectors***)
We see that under the formula for sharp operator

$$\begin{aligned}\langle\, \omega \,,\, \eta \,\rangle_g &= \langle\, \omega^\sharp \,,\, \eta^\sharp \,\rangle_g \\ &= g_{k,l} \left(g^{k,i}\, \omega_i\right) \left(g^{l,j}\, \eta_j\right) \\ &= \delta^i_l \omega_i \left(g^{l,j}\, \eta_j\right) \\ &= g^{i,j} \omega_i\, \eta_j.\end{aligned}$$

In other words, ***the inner product on covectors*** *is represented by* ***the inverse matrix*** $g^{i,j}$.

- Finally, there is a ***unique smooth fiber metric*** on each tensor bundle $T^{(k,l)}TM$ so that

$$\langle \alpha_1 \otimes \ldots \otimes \alpha_{k+l} \,,\, \beta_1 \otimes \ldots \otimes \beta_{k+l} \rangle = \langle \alpha_1 \,,\, \beta_1 \rangle \cdot \ldots \cdot \langle \alpha_{k+l} \,,\, \beta_{k+l} \rangle \tag{9}$$

2 The Levi-Civita Connection

2.1 Metric Connections

- **Definition** Let g be a *Riemannian or pseudo-Riemannian metric* on a smooth manifold M (with or without boundary). A connection ∇ on TM is said to be ***compatible with g***, or to be ***a metric connection***, if it satisfies the following *product rule* for all $X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_Z \langle X \,,\, Y \rangle = \langle \nabla_Z X \,,\, Y \rangle + \langle X \,,\, \nabla_Z Y \rangle \tag{10}$$
$$\Leftrightarrow Z \,\langle X \,,\, Y \rangle = \langle \nabla_Z X \,,\, Y \rangle + \langle X \,,\, \nabla_Z Y \rangle$$

- **Remark** More understanding of the equation (10):

  1. $\nabla_Z\langle X\,,\,Y\rangle = \nabla_Z(g(X,Y))$. Note that $\langle X\,,\,Y\rangle = g(X,Y) \in \mathcal{C}^{\infty}(M)$ is *a smooth function* since g is a ***covariant 2-tensor***. Thus $\nabla_Z\langle X\,,\,Y\rangle = Z\,\langle X\,,\,Y\rangle \in \mathcal{C}^{\infty}(M)$ since for $f \in \mathcal{C}^{\infty}(M)$, the directional derivative of f along direction of Z, $\nabla_Z f = Zf$. Intuitively, it measures ***the directional derivatives of the angle*** *between two vector fields X and Y along the direction of vector field Z.*

  2. $\langle \nabla_Z X\,,\,Y\rangle = g\,(\nabla_Z X, Y) \in \mathcal{C}^{\infty}(M)$ measures ***the angle between $\nabla_Z X$ and Y***; similarly, $\langle X\,,\,\nabla_Z Y\rangle = g\,(X, \nabla_Z Y)$ measures ***the angle between X and $\nabla_Z Y$***. In both terms, $\nabla_Z X$ is the ***directional derivative*** X along Z, which is *the difference* between X and *its infinitesimal parallel transport* along Z.

  3. The equation (10) states that "***the directional derivatives of the angle*** *between two vector fields X and Y along the direction of vector field Z* ***is equal to*** the ***sum of angles*** of *the* ***directional derivative of one vector field*** *along direction of Z* ***with respect to the other vector field***".

- **Proposition 2.1** ***(Characterizations of Metric Connections)****.*
  Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be a connection on TM. The following conditions are ***equivalent:***

  1. *∇ is* ***compatible*** *with g: $\nabla_Z\langle X\,,\,Y\rangle = \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle$.*

  2. *g is* ***parallel with respect to*** *∇: $\nabla g \equiv 0$.*

  3. *In terms of any smooth local frame (E_i), the* ***connection coefficients*** *of ∇ satisfy*
  $$\Gamma^{l}_{k,i} g_{l,j} + \Gamma^{l}_{k,j} g_{i,l} = E_k(g_{i,j}). \tag{11}$$

  4. *If V, W are smooth vector fields along any smooth curve γ, then*
  $$\frac{d}{dt}\langle V\,,\,W\rangle = \langle D_t V\,,\,W\rangle + \langle V\,,\,D_t W\rangle\,. \tag{12}$$

  5. *If V, W are* ***parallel*** *vector fields* ***along a smooth curve*** *γ in M, then $\langle V\,,\,W\rangle$ is* ***constant*** *along γ.*

  6. *Given any smooth curve γ in M, every* ***parallel transport map*** *along γ is a* ***linear isometry****.*

  7. *Given any smooth curve γ in M, every* ***orthonormal basis*** *at a point of γ can be* ***extended*** *to a* ***parallel orthonormal frame*** *along γ.*

- **Remark** From the proposition statement 5,6,7 above, we see that ***the metric connection*** ∇ that is compatible with g *defines the* ***parallel transport operation*** that maintains *the* ***angle*** *between two vector fields* ***unchanged***. In other word, ***the parallel transport defined by the metric connection*** is an ***isometry*** on the manifold.

- **Corollary 2.2** *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold with or without boundary, ∇ is a* ***metric connection*** *on M, and $\gamma : I \to M$ is a smooth curve.*

  1. *$|\gamma'(t)|$ is* ***constant*** *if and only if $D_t\gamma'(t)$ is* ***orthogonal*** *to $\gamma'(t)$ for all $t \in I$.*

  2. *If γ is a* ***geodesic****, then $|\gamma'(t)|$ is* ***constant****.*

- **Proposition 2.3** *If M is an embedded Riemannian or pseudo-Riemannian **submanifold** of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the **tangential connection** on M is **compatible** with the **induced** Riemannian or pseudo-Riemannian **metric**.*

2.2 Symmetric Connections

- **Definition** A ***connection*** ∇ on the tangent bundle of a smooth manifold M is ***symmetric*** if
$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{for all } X, Y \in \mathfrak{X}(M),$$
where $[X, Y]$ is the Lie bracket of two vector fields.

- **Definition** The ***torsion tensor*** of the *connection* ∇ is a ***smooth*** $(1, 2)$-***tensor field*** $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by
$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

- **Remark** Thus, a connection ∇ is ***symmetric*** if and only if its torsion ***vanishes*** identically $\tau \equiv 0$.

- **Remark** (***Coordinate Representation of Symmetric Connections***)
A connection is ***symmetric*** if and only if *its* ***connection coefficients*** in *every coordinate frame* is ***symmetric*** in ***lower two indices*** That is, $\underline{\Gamma_{i,j}^k = \Gamma_{j,i}^k}$ for all i, j.

- **Proposition 2.4** *If M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space, then the **tangential connection** on M is **symmetric**.*

2.3 The Levi-Civita Connections

- **Theorem 2.5** *(**Fundamental Theorem of Riemannian Geometry**). [Lee, 2018]*
*Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary). There exists a **unique connection** ∇ on TM that **is compatible with** g and **symmetric**. It is called the **Levi-Civita connection of** g (or also, when g is **positive definite**, the **Riemannian connection**).*

- **Corollary 2.6** *(**Formulas for the Levi-Civita Connection**). [Lee, 2018]*
*Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be its **Levi-Civita connection**.*

  1. *(**In Terms of Vector Fields**): If X, Y, Z are smooth vector fields on M, then*
  $$\langle \nabla_X Y\ ,\ Z \rangle = \frac{1}{2} \left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \right) \tag{13}$$
  *(This is known as **Koszul's formula**.)*

  2. *(**In Coordinates**): In any smooth coordinate chart for M, the **coefficients of the Levi-Civita connection** are given by*
  $$\Gamma_{i,j}^k = \frac{1}{2} g^{k,l} \left(\frac{\partial}{\partial x^i} g_{j,l} + \frac{\partial}{\partial x^j} g_{i,l} - \frac{\partial}{\partial x^l} g_{i,j} \right). \tag{14}$$

3. ***(In A Local Frame):*** *Let (E_i) be a smooth* ***local frame*** *on an open subset $U \subseteq M$, and let $c_{i,j}^{k} : U \to \mathbb{R}$ be the n^3 smooth functions defined by*

$$[E_i, E_j] = c_{i,j}^{k}\, E_k \tag{15}$$

Then ***the coefficients of the Levi-Civita connection*** *in this frame are*

$$\Gamma_{i,j}^{k} = \frac{1}{2} g^{k,l} \left(E_i\, g_{j,l} + E_j\, g_{i,l} - E_l\, g_{i,j} - g_{j,m} c_{i,l}^{m} - g_{l,m} c_{j,i}^{m} + g_{i,m} c_{l,j}^{m} \right). \tag{16}$$

4. ***(In A Local Orthonormal Frame):*** *If g is Riemannian, (E_i) is a smooth* ***local orthonormal frame****, and the functions $c_{i,j}^{k}$ are defined by (15), then*

$$\Gamma_{i,j}^{k} = \frac{1}{2} \left(c_{i,j}^{k} - c_{i,k}^{j} - c_{j,k}^{i} \right) \tag{17}$$

- **Remark** On every Riemannian or pseudo-Riemannian manifold, we will always use the Levi-Civita connection from now on without further comment.

- **Remark** *Geodesics* with respect to the Levi-Civita connection are called ***Riemannian geodesics***, or simply "*geodesics* as long as there is no risk of confusion.

- **Remark** The ***connection coefficients*** $\Gamma_{i,j}^{k}$ *of* ***the Levi-Civita connection*** in coordinates, given by (14), are called ***the Christoffel symbols of*** g.

- **Proposition 2.7** 1. *The Levi-Civita connection on a (pseudo-)****Euclidean space*** *is equal to the* ***Euclidean connection****.*

  2. *Suppose M is an* ***embedded*** *(pseudo-)Riemannian* ***submanifold*** *of a (pseudo-)Euclidean space. Then the Levi-Civita connection on M is equal to* ***the tangential connection*** $\nabla^{\top}$.

- **Proposition 2.8** ***(Naturality of the Levi-Civita Connection)****. [Lee, 2018]*
  Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and let ∇ denote the Levi-Civita connection of g and $\widetilde{\nabla}$ that of $\widetilde{g}$. If $\varphi : M \to \widetilde{M}$ is an isometry, then $\varphi^{*}\widetilde{g} = \nabla$.

  Remark An ***isometry*** φ between the manifold M and $\widetilde{M}$ can be used to define ***the pullback connection*** in M from *the Levi-Civita connection* $\widetilde{M}$. Recall that for general connections, we can only define a pullback connection if φ is a *diffeomorphism*.

- **Corollary 2.9** ***(Naturality of Geodesics)****.*
  Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a ***local isometry****. If γ is a* ***geodesic*** *in M, then $\varphi \circ \gamma$ is a* ***geodesic*** *in M.*

  Remark An ***isometry*** φ between the manifold M and $\widetilde{M}$ maps a ∇-geodesic in M to a $\widetilde{\nabla}$ -geodesic in $\widetilde{M}$ for both *Levi-Civita Connections* ∇ and $\widetilde{\nabla}$.

- **Proposition 2.10** *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold. The connection induced on each* ***tensor bundle*** *by the Levi-Civita connection is* ***compatible*** *with* ***the induced inner product on tensors****, in the sense that $X \langle F\,,\, G\rangle = \langle \nabla_X F\,,\, G\rangle + \langle F\,,\, \nabla_X G\rangle$ for every vector field X and every pair of smooth tensor fields $F, G \in T^{(k,l)}TM$.*

- **Proposition 2.11** ***(Volume Preseving under Parallel Transport)***
  Let (M, g) be an oriented Riemannian manifold. The Riemannian volume form of g is ***parallel*** *with respect to the Levi-Civita connection.*

- **Proposition 2.12** *The* ***musical isomorphisms commute with the total covariant derivative operator****: if F is any smooth tensor field with a* ***contravariant*** *i-th index position, and $\flat$ represents the operation of lowering the i-th index, then*

$$\nabla(F^{\flat}) = (\nabla F)^{\flat} \tag{18}$$

  Similarly, if G has a ***covariant*** *i-th position and $\sharp$ denotes raising the i-th index, then*

$$\nabla(G^{\sharp}) = (\nabla G)^{\sharp} \tag{19}$$

2.4 The Exponential Map

- **Lemma 2.13** ***(Rescaling Lemma)****.*
  For every $p \in M$, $v \in T_pM$, and $c, t \in \mathbb{R}$,

$$\gamma_{c\,v}(t) = \gamma_v(c\,t) \tag{20}$$

  whenever either side is defined.

- **Definition** Define a subset $\mathcal{E} \subseteq TM$, ***the domain of the exponential map***, by

$$\mathcal{E} = \{v \in TM : \gamma_v \text{ is defined on an interval containing [0,1]}\},$$

  and then define ***the exponential map*** $\exp : \mathcal{E} \to M$ by

$$\exp(v) = \gamma_v(1)$$

  For each $p \in M$, the ***restricted exponential map*** at p, denoted by $\exp_p$, is the restriction of exp to the set $\mathcal{E}_p = \mathcal{E} \cap T_pM$.

- **Remark** The ***exponential map*** *of a* ***Riemannian manifold*** should not be confused with *the* ***exponential map*** *of a* ***Lie group***. The two are closely related for ***bi-invariant metrics***, but in general they need not be.

- **Remark** Recall that a subset of a vector space V is said to be ***star-shaped*** *with respect to a point* $x \in S$ if for every $y \in S$, the *line segment* from x to y is contained in S.

- **Proposition 2.14** ***(Properties of the Exponential Map)****. [Lee, 2018]*
  Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let $\exp : \mathcal{E} \to M$ *be its exponential map.*

  1. *$\mathcal{E}$ is an* ***open*** *subset of TM containing the image of the* ***zero section****, and each set $\mathcal{E}_p \subseteq T_pM$ is* ***star-shaped with respect to*** *0.*
  2. *For each $v \in TM$, the* ***geodesic*** *γ_v is given by*

$$\gamma_v(t) = \exp(v\,t) \tag{21}$$

     for all t such that either side is defined.

3. *The exponential map is **smooth**.*
4. *For each point $p \in M$, the **differential** $d(\exp_p)_0 : T_0(T_pM) \simeq T_pM \to T_pM$ is **the identity map** of T_pM, under the usual identification of $T_0(T_pM)$ with T_pM.*

- **Proposition 2.15** ***(Naturality of the Exponential Map).***
*Suppose (M, g) and $\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a **local isometry**. Then for every $p \in M$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}_p & \xrightarrow{d\varphi_p} & \widetilde{\mathcal{E}}_{\varphi(p)} \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{\varphi(p)}} \\ M & \xrightarrow{\varphi} & \widetilde{M}, \end{array}$$

where $\mathcal{E}_p \subseteq T_pM$ and $\widetilde{\mathcal{E}}_{\varphi(p)} \subseteq T_{\varphi(p)}\widetilde{M}$ are the domains of the restricted exponential maps $\exp_p$ (with respect to g) and $\exp_{\varphi(p)}$ (with respect to $\tilde{g}$), respectively.

- **Remark** Under isometry transformation, the exponential map ***remain unchanged*** from TM to $T\widetilde{M}$.

- The following proposition shows that ***local isometries*** of connected manifolds are ***completely determined*** by their ***values*** and ***differentials*** at a single point.

  Proposition 2.16 *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian or pseudo-Riemannian manifolds, with M **connected**. Suppose $\varphi, \psi : M \to \widetilde{M}$ are **local isometries** such that for some point $p \in M$, we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

- **Definition** A Riemannian or pseudo-Riemannian manifold (M, g) is said to be ***geodesically complete*** if *every maximal geodesic* is defined for ***all*** $t \in \mathbb{R}$, or equivalently if *the domain of the exponential map is all of TM.*

2.5 Normal Neighborhoods and Normal Coordinates

- **Definition** Let (M, g) be a Riemannian or pseudo-Riemannian manifold of dimension n (without boundary). Recall that for every $p \in M$, the restricted exponential map $\exp_p$ maps the open subset $\mathcal{E}_p \subseteq T_pM$ smoothly into M. Because $d(\exp_p)_0$ is ***invertible***, the *inverse function theorem* guarantees that there exist a neighborhood V of the origin in T_pM and a neighborhood U of p in M such that $\exp_p : V \to U$ is a ***diffeomorphism***.

  A neighborhood U of $p \in M$ that is the ***diffeomorphic image*** under $\exp_p$ of *a star-shaped neighborhood of* $0 \in T_pM$ is called ***a normal neighborhood*** of p.

- **Definition** Every *orthonormal basis* (b_i) for T_pM determines ***a basis isomorphism*** $B : \mathbb{R}^n \to T_pM$ by $B(x^1, \ldots, x^n) = x^i\, b_i$. If $U = \exp_p(V)$ is ***a normal neighborhood*** of p, we can combine this *isomorphism* with *the exponential map* to get ***a smooth coordinate map*** $\varphi : B^{-1} \circ (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$:

$$\begin{tikzcd} T_pM \arrow[r, "B^{-1}"] & \mathbb{R}^n \\ U. \arrow[u, "(\exp_p|_V)^{-1}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Such coordinates are called ***(Riemannian or pseudo-Riemannian) normal coordinates*** centered at p.

- **Proposition 2.17** *(**Uniqueness of Normal Coordinates**). [Lee, 2018]*
  *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, p a point of M, and U a **normal neighborhood** of p. For every **normal coordinate chart** on U centered at p, the coordinate basis is **orthonormal** at p; and for every orthonormal basis (b_i) for T_pM, there is a **unique normal coordinate chart** (x^i) on U such that $\frac{\partial}{\partial x^i}|_p = b_i$ for $i = 1, \ldots, n$. In the Riemannian case, any two normal coordinate charts (x^i) and $(\widetilde{x}^j)$ are related by*

$$\widetilde{x}^j = A^j_i\, x^i \tag{22}$$

  for some (constant) matrix $A^j_i \in \mathcal{O}(n)$.

- **Proposition 2.18** *(**Properties of Normal Coordinates**). [Lee, 2018]*
  *Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, and let $(U, (x^i))$ be any **normal coordinate chart** centered at $p \in M$.*

  1. *The coordinates of p are $(0, \ldots, 0)$.*
  2. *The **components** of the **metric** at p are $g_{i,j} = \delta_{i,j}$ if g is **Riemannian**, and $g_{i,j} = \pm\delta_{i,j}$ otherwise.*
  3. *For every $v = v^i \frac{\partial}{\partial x^i}|_p \in T_pM$, the **geodesic** γ_v starting at p with **initial velocity** v is represented in **normal coordinates** by the line*

$$\gamma_v(t) = (tv^1, \ldots, tv^n), \tag{23}$$

     as long as t is in some interval I containing 0 such that $\gamma_v(I) \subseteq U$.
  4. *The **Christoffel symbols** in these coordinates **vanish** at p.*
  5. *All of the **first partial derivatives** of $g_{i,j}$ in these coordinates **vanish** at p.*

- **Remark** The *geodesics* ***starting at*** p and lying in a ***normal neighborhood*** of p are called ***radial geodesics***. (But be warned that geodesics that do not pass through p do not in general have a simple form in normal coordinates.)

3 Curvature

3.1 Flatness Criterion

- **Remark** Under the Euclidean connection, let us look more closely at the quantity $\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z$ when X, Y, and Z are smooth vector fields.

$$\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_Y Z &= \overline{\nabla}_X(Y(Z^k)\partial_k) = X\left(Y^j\partial_j(Z^k)\right)\partial_k = XY(Z^k)\partial_k \\
\overline{\nabla}_Y\overline{\nabla}_X Z &= YX(Z^k)\partial_k \\
\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z &= (XY - YX)(Z^k)\partial_k = [X,Y](Z^k)\partial_k = \overline{\nabla}_{[X,Y]}Z \\
\Rightarrow \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z &= \overline{\nabla}_{[X,Y]}Z.
\end{aligned}$$

Recall that a Riemannian manifold is said to be ***flat*** if it is ***locally isometric*** to a ***Euclidean space***, that is, if every point has a neighborhood that is ***isometric*** to an open set in $\mathbb{R}^n$ with its ***Euclidean metric***.

We say that a ***connection*** ∇ on a smooth manifold M satisfies ***the flatness criterion*** if whenever X, Y, Z are smooth vector fields defined on an open subset of M, the following identity holds:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z \tag{24}$$

- **Remark** The geometric interpretation of the term $\nabla_X \nabla_Y Z$ is the *two-step process*:

  1. First, ***parallel transport*** of Z along *the **flow** of vector field Y*;
  2. Then, ***parallel transport*** of Z along *the **flow** of vector field X*

  Then the resulting vector field is $\nabla_X \nabla_Y Z$.

- **Proposition 3.1** *If (M, g) is a **flat** Riemannian or pseudo-Riemannian manifold, then its **Levi-Civita connection** satisfies **the flatness criterion**.*

3.2 The Curvature Tensor

- **Definition** Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and define a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{25}$$

- The following proposition make sure this multilinear map defines a $(1, 3)$-tensor field

  Proposition 3.2 *The map R defined above is **multilinear** over $\mathcal{C}^\infty(M)$, and thus defines a $(1, 3)$-**tensor field** on M.*

- **Definition** For each pair of vector fields $X, Y \in \mathfrak{X}(M)$, the map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X, Y)Z$ is a ***smooth bundle endomorphism*** of TM, called ***the curvature endomorphism determined by X and Y***.

  The ***tensor field*** R itself is called ***the (Riemann) curvature endomorphism*** or the $(1, 3)$-***curvature tensor***.

- **Remark** (***Coordinate Representation of the Curvature Tensor***)
  We adopt the convention that ***the last index is the contravariant (upper) one***. This is contrary to our default assumption that *covector arguments come first*. Thus, for example, *the curvature endomorphism* can be written in terms of local coordinates (x^i) as

$$R = R_{i,j,k}^{l}\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l},$$

  where the coefficients $R_{i,j,k}^{l}$ are defined by

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{i,j,k}^{l}\, \frac{\partial}{\partial x^l}.$$

- **Remark** (***Understanding the Geometric Meaning of the*** $(1,3)$***-Curvature Tensor***)
  The $(1,3)$-tensor $R(X,Y)Z$ describes *the* ***difference*** *of resulting* ***vector fields*** after ***parallel transporting*** vector field Z through ***two different routes***:

  1. First ***parallel transporting*** along ***the flow of*** Y, then ***parallel transporting*** along ***the flow of*** X, the resulting vector field is $\nabla_X \nabla_Y Z$;

  2. First ***parallel transporting*** along ***the flow of*** X, then ***parallel transporting*** along ***the flow of*** Y, the resulting vector field is $\nabla_Y \nabla_X Z$;

  The last term $\nabla_{[X,Y]}Z$ provides additional ***correction*** if X and Y are ***not orthorgonal***.

  Thus $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is ***close related to*** *the* ***angle*** *of these* ***two resulting vector fields***. If the surface is ***flat***, this angle should be ***zero*** since *the vector field* ***does not rotate*** during the transport and it is ***regardless of the path it takes***. On the other hand, if ***the surface bends***, then the vector field *will rotate* during the parallel transport and thus traversing through different paths will cause the vector field *points to different directions* in final destination, i.e. the angle is not zero.

- **Proposition 3.3** ***(The Riemann Curvature via Coefficients of Connection)*** *[Lee, 2018]*
  Let (M,g) *be a Riemannian or pseudo-Riemannian manifold. In terms of any smooth local coordinates, the components of the* $(1,3)$*-curvature tensor are given by*

$$R_{i,j,k}^{l} = \partial_i \Gamma_{j,k}^{l} - \partial_j \Gamma_{i,k}^{l} + \Gamma_{j,k}^{m}\Gamma_{i,m}^{l} - \Gamma_{i,k}^{m}\Gamma_{j,m}^{l}. \tag{26}$$

- **Remark** The curvature endomorphism also measures ***the failure*** *of* ***second covariant derivatives*** *along* ***families*** *of curves to* ***commute***. Given a smooth one-parameter *family of curves* $\Gamma : J \times I \to M$, recall that the velocity fields $\partial_t \Gamma(s,t) = (\Gamma_s)'(t)$ and $\partial_s \Gamma(s,t) = (\Gamma^{(t)})'(s)$ are smooth vector fields along Γ.

  Proposition 3.4 *Suppose* (M,g) *is a smooth Riemannian or pseudo-Riemannian manifold and* $\Gamma : J \times I \to M$ *is a smooth one-parameter* ***family*** *of curves in* M*. Then for every smooth vector field* V *along* Γ*,*

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \partial_t \Gamma)V \tag{27}$$

- **Definition** We define the ***(Riemann) curvature tensor*** to be the $(0,4)$*-tensor field* $Rm = R^{\flat}$ (also denoted by *Riem* by some authors) obtained from the $(1,3)$-curvature tensor R by ***lowering its last index***. Its *action* on vector fields is given by

$$Rm(X,Y,Z,W) := \left\langle R(X,Y)Z\,,\,W \right\rangle_g \tag{28}$$

  This quanitity measures the angle between $R(X,Y)Z$ and W.

- **Remark** (***Coordinate Representation of the Riemann Curvature Tensor***)
  In terms of any smooth local coordinates, it is written

$$Rm = R_{i,j,k,l}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

  where $R_{i,j,k,l} = g_{l,m} R_{i,j,k}^{m}$. We also see that

$$R_{i,j,k,l} = g_{l,m}\left(\partial_i \Gamma_{j,k}^{m} - \partial_j \Gamma_{i,k}^{m} + \Gamma_{j,k}^{p}\Gamma_{i,p}^{m} - \Gamma_{i,k}^{p}\Gamma_{j,p}^{m}\right). \tag{29}$$

- **Proposition 3.5** *The* ***curvature tensor*** *is a* ***local isometry invariant****: if (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a local isometry, then $\varphi^* \widetilde{Rm} = Rm$.*

References

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