Lecture 0: Summary (Part 1)

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Dec. 15th., 2022

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1 Topology Basis

1.1 Topology, Basis and Subbasis

- **Definition** Let X be a set. *A* ***topology*** on X is *a collection* $\mathscr{T}$ of *subsets* of X, called ***open subsets***, satisfying

  1. X and $\emptyset$ are *open*.
  2. The ***union*** of ***any family*** of open subsets is open.
  3. The ***intersection*** of *any* ***finite*** *family* of open subsets is open.

  A pair $(X, \mathscr{T})$ consisting of a set X together with a topology $\mathscr{T}$ on X is called ***a topological space***.

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Definition** A ***continuous bijective*** map $F : X \to Y$ with ***continuous inverse*** is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Definition** Suppose X is a topological space. A collection $\mathscr{B}$ of open subsets of X is said to be ***a basis*** for *the topology of* X (plural: ***bases***) if every open subset of X is the *union of some collection of elements* of $\mathscr{B}$.

  More generally, suppose X is merely a set, and $\mathscr{B}$ is a collection of *subsets* of X satisfying the following conditions:

  1. $X = \bigcup_{B\in\mathscr{B}} B$.
  2. If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B2$.

  Then *the collection of* ***all unions*** *of elements of* $\mathscr{B}$ is a *topology* on X, called ***the topology generated by*** $\mathscr{B}$, and $\mathscr{B}$ is a ***basis*** for this *topology*.

- **Lemma 1.1** *(**Obtaining Basis from Given Topology**). [Munkres, 2000]*
  Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then C is a basis for the topology of X.

- **Lemma 1.2** *(**Topology Comparison via Bases**). [Munkres, 2000]*
  Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively, on X. Then the following are equivalent:

  1. *$\mathscr{T}'$ is **finer** than $\mathscr{T}$.*
  2. *For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.*

- **Definition** (***Subbasis***)
  A subbasis $\mathscr{S}$ for a *topology* on X is a collection of subsets of X whose union equals X. The topology generated by the *subbasis* $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of ***all unions*** *of* ***finite intersections*** *of elements of* $\mathscr{S}$.

- **Remark** (***Basis from Subbasis***)
  For a *subbasis* $\mathscr{S}$, the collection $\mathscr{B}$ of ***all finite intersections*** of elements of $\mathscr{S}$ is a ***basis***,

1.2 Limit Point and Closure

- **Definition** A subset A of a topological space X is said to be ***closed*** if the set $X \setminus A$ is *open*.
- **Definition** Given a subset A of a topological space X, ***the interior of*** A is defined as *the union of all open sets* ***contained*** in A, and ***the closure of*** A is defined as *the intersection of all closed sets* ***containing*** A.

  The interior of A is denoted by Int A or by $\mathring{A}$ and ***the closure of*** A is denoted by CI A or by $\bar{A}$. Obviously $\mathring{A}$ is *an open set* and $\bar{A}$ is *a closed set*; furthermore,

  $$\mathring{A} \subseteq A \subseteq \bar{A}.$$

  If A is ***open***, $A = \mathring{A}$; while if A is ***closed***, $A = \bar{A}$.

- **Proposition 1.3** *(**Characterization of Closure in terms of Basis**) [Munkres, 2000]*
  Let A be a subset of the topological space X.
  1. *Then $x \in \bar{A}$ if and only if every* ***open*** *set U* ***containing*** *x* ***intersects*** *A.*
  2. *Supposing the topology of X is given by a* ***basis***, *then $x \in \bar{A}$ if and only if every basis element B* ***containing*** *x* ***intersects*** *A.*
- **Remark** We can say "*U is a* ***neighborhood*** *of x*" if "*U is an open set containing x*".
- **Definition** (***Limit Point***)
  If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point*** (or "***cluster point***," or "***point of accumulation***") of A if ***every neighborhood*** *of x* ***intersects*** *A in some point* ***other than*** *x itself.*

  Said differently, x is ***a limit point*** of A if it belongs to ***the closure of*** $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.
- **Theorem 1.4** *(**Decomposition of Closure**)*
  Let A be a subset of the topological space X; let A' be the set of ***all limit points*** *of A. Then*

  $$\bar{A} = A \cup A'.$$

- **Corollary 1.5** *A subset of a topological space is* ***closed*** *if and only if it contains all its* ***limit points***.

1.3 Subspace, Product and Quotient Topologies

1.3.1 Subspace Topology

- **Definition** If X is a topological space and $S \subseteq X$ is an arbitrary subset, we define ***the subspace topology*** on S (sometimes called *the* ***relative topology***) by declaring a subset $U \subseteq S$ to be *open* in S *if and only* if there exists an open subset $V \subseteq X$ such that $U = V \cap S$.

Any subset of X endowed with the subspace topology is said to be ***a subspace of*** X.

- **Lemma 1.6** ***(Basis of Subspace Topology)***
If $\mathscr{B}$ is a basis for the topology of X then the collection
$$\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$$
*is a **basis** for the subspace topology on $S \subset X$.*

- **Proposition 1.7** *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

- **Proposition 1.8** ***(Closure in Subspace Topology)***
Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

1.3.2 Product Topology

- **Definition** (***J-tuples***)
Let J be an index set. Given a set X, we define a ***J-tuple*** *of elements* of X to be a function $x : J \to X$. If α is an element of J, we often denote ***the value of*** X ***at*** α by X_α rather than $x(\alpha)$; we call it ***the α-th coordinate*** of x. And we often *denote the function x itself* by the symbol
$$(x_\alpha)_{\alpha \in J}$$
which is as close as we can come to a "*tuple notation*" for an arbitrary index set J. We denote ***the set of all J-tuples*** of elements of X by X^J.

- **Definition** (***Arbitrary Cartestian Products***)
Let $\{A_\alpha\}_{\alpha \in J}$ be an *indexed* family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. *The **cartesian product** of this indexed family*, denoted by
$$\prod_{\alpha \in J} A_\alpha$$
is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions
$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$
such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

- **Definition** (***Projection Mapping or Coordinate Projection***)
Let
$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$
be the function assigning to each element of the product space its β-th coordinate,
$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta;$$
it is called ***the projection mapping*** *associated with the index β.*

- **Definition** (***Product Topology***)
  Let $\mathscr{S}_\beta$ denote the collection

$$\mathscr{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\right\},$$

  and let $\mathscr{S}$ denote *the union of these collections*,

$$\mathscr{S} = \bigcup_{\beta\in J} \mathscr{S}_\beta.$$

  The topology generated by *the* ***subbasis*** S is called ***the product topology***. In this topology $\prod_{\alpha\in J} X_\alpha$ is called ***a product space***.

- **Remark** (***Product Topology = Weak Topology by Coordinate Projections***)
  The product topology on $\prod_{\alpha\in J} X_\alpha$ is ***the weak topology*** generated by *a family of projection mappings* $(\pi_\beta)_{\beta\in J}$. It is ***the coarest (weakest) topology*** *such that* $(\pi_\beta)_{\beta\in J}$ *are* ***continuous***.

  A typical element of the basis from *the product topology* is ***the finite intersection*** *of subbasis* where the *index is different*:

$$\pi_{\beta_1}^{-1}(V_{\beta_1}) \cap \ldots \cap \pi_{\beta_n}^{-1}(V_{\beta_n})$$

  Thus a ***neighborhood*** of x in ***the product topology*** is

$$N(x) = \{(x_\alpha)_{\alpha\in J} : x_{\beta_1} \in V_{\beta_1}, \ldots, x_{\beta_n} \in V_{\beta_n}\}$$

  where there is ***no restriction*** for $\alpha \in \{\beta_1, \ldots, \beta_n\}$.

  Note that for ***the box topology***, a *neighborhood* of x is

$$N_b(x) = \{(x_\alpha)_{\alpha\in J} : x_\alpha \in U_\alpha,\ \forall \alpha \in J\} \subset N(x)$$

  Thus ***the box topology*** is ***finer*** than ***the product topology***. Moreover, *for* ***finite product*** $\prod_{\alpha=1}^{n} X_\alpha$, the box topology and the product topology is the ***same***.

- **Definition** If X and Y are topological spaces, a continuous injective map $F : X \to Y$ is called a ***topological embedding*** if it is a ***homeomorphism*** onto its image $F(X) \subseteq Y$ in the subspace topology.

1.3.3 Quotient Topology

- **Definition** (***Quotient Map***)
  Let X and Y be topological spaces; let $\pi : X \to Y$ be a ***surjective map***. The map π is said to be ***a quotient map*** provided a subset U of Y is ***open*** in Y ***if and only if*** $\pi^{-1}(U)$ is ***open*** in X.

- **Remark** (***Quotient Map = Strong Continuity***)
  The condition of quotient map is ***stronger*** than continuity (it is called ***strong continuity*** in some literature).

$$\begin{aligned} \text{continuity :} &\quad U \text{ is open in } Y \Rightarrow \pi^{-1}(U) \text{ is open in } X \\ \text{open map :} &\quad \pi(V) \text{ is open in } Y \Leftarrow V \text{ is open in } X \\ \text{quotient map :} &\quad U \text{ is open in } Y \Leftrightarrow \pi^{-1}(U) \text{ is open in } X \end{aligned}$$

An equivalent condition is to require that a subset A of K be ***closed*** in Y *if and only if* $\pi^{-1}(A)$ is ***closed*** in X. Equivalence of the two conditions follows from equation

$$\pi^{-1}(Y \setminus B) = X \setminus \pi^{-1}(B).$$

- **Definition** (***Saturated Set and Fiber***)
  If $\pi : X \to Y$ is a ***surjective*** *map*, a subset $U \subseteq X$ is said to be ***saturated*** with respect to π if U contains every set $\pi^{-1}(\{y\})$ that it ***intersects***. Thus U is ***saturated*** if it equals to the ***entire preimage*** of its ***image***: $U = \pi^{-1}(\pi(U))$.

  Given $y \in Y$, the ***fiber*** of π over y is the set $\pi^{-1}(\{y\})$.

- **Definition** (***Quotient Map via Saturated Set***)
  A surjective map $\pi : X \to Y$ is a ***quotient map*** if π is ***continuous*** and π maps ***saturated open sets*** of X to ***open sets*** of Y (or *saturated closed sets* of X to *closed sets* of Y).

- **Definition** (***Open Map*** *and* ***Closed Map***)
  A map $f : X \to Y$ (continuous or not) is said to be an ***open map*** if for every *open* subset $U \subseteq X$, the image set $f(U)$ is *open* in Y, and a ***closed map*** if for every *closed* subset $K \subseteq X$, the image $f(K)$ is *closed* in Y .

- **Definition** (***Quotient Topology***)
  If X is a space and A is a set and if $\pi : X \to A$ is a ***surjective*** map, then there exists **exactly one topology** $\mathscr{T}$ on A relative to which π is a quotient map; it is called ***the quotient topology*** induced by π.

- **Definition** (***Quotient Space***)
  Suppose X is a topological space and $\sim$ is *an equivalence relation* on X. Let $X/\sim$ denote ***the set of equivalence classes*** in X, and let $\pi : X \to X/\sim$ be the ***natural projection*** sending each *point* to its *equivalence class*. Endowed with ***the quotient topology*** determined by π, the space $X/\sim$ is called ***the quotient space*** (or *identification space*) of X determined by π.

1.4 Continuous Function

1.4.1 Definitions

- **Definition** A map $F : X \to Y$ is said to be ***continuous*** if for every open subset $U \subseteq Y$, the ***preimage*** $F^{-1}(U)$ is ***open*** in X.

- **Remark** ***Continuity of a function*** depends *not only upon* ***the function*** f ***itself***, but also *on the* ***topologies*** *specified for* ***its domain*** *and* ***range***. If we wish to emphasize this fact, we can say that f *is* ***continuous relative to*** *specific topologies on* X *and* Y.

- **Remark** (***Prove Continuity via Basis***)
  If the topology of ***the range space*** Y is given by a ***basis*** $\mathscr{B}$, then to prove ***continuity of*** f it suffices to show that *the* ***inverse image*** *of every* ***basis element*** *is* ***open***: The arbitrary

open set V of Y can be written as *a union of basis elements*

$$
\begin{aligned}
V &= \bigcup_{\alpha \in J} B_{\alpha} \\
\Rightarrow f^{-1}(V) &= \bigcup_{\alpha \in J} f^{-1}(B_{\alpha})
\end{aligned}
$$

- **Remark** (***Prove Continuity via Subbasis***)
  If the topology on Y is given by ***a subbasis*** $\mathscr{S}$, to prove continuity of f it will even suffice to show that ***the inverse image*** *of each* ***subbasis*** *element is* ***open***: The arbitrary basis element B for Y can be written as ***a finite intersection*** $S_1 \cap \ldots \cap S_n$ of subbasis elements; it follows from the equation

  $$
  f^{-1}(B) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n)
  $$

  that the inverse image of every basis element is *open*.

- **Example** (***$\mathscr{F}$-Weak Topology using Continuity Only***)
  One can ***define a topology just*** based on ***the notion of continuity*** from a family of functions. Let $\mathscr{F}$ be a family of functions from a set S to a topological space $(X, \mathscr{T})$. The ***$\mathscr{F}$-weak*** *(or simply* ***weak****)* ***topology*** on S is *the* ***coarest topology*** for which ***all the functions*** *$f \in \mathscr{F}$ are* ***continuous***.

  The ***$\mathscr{F}$-weak*** *topology* $\mathscr{T}$ is generated by ***subbasis*** $\mathscr{S}$ of the preimage sets $S = f^{-1}(U)$ where $f \in \mathscr{F}$ and $U \in \mathscr{T}$. And the basis of $\mathscr{T}$ is *the collection* of ***all finite intersections*** of preimages $f^{-1}(U)$ for $f \in \mathscr{F}$ and $U \in \mathscr{T}$.

- **Proposition 1.9** *(**Equivalent Definition of Continuity**) [Munkres, 2000]*
  Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

  1. *f is **continuous**.*
  2. *For every subset A of X, one has $f(\bar{A}) \subseteq \overline{f(A)}$.*
  3. *For every **closed** set B of Y, the set $f^{-1}(B)$ is **closed** in X.*
  4. *For **each** $x \in X$ and each **neighborhood** V of $f(x)$, there is a **neighborhood** U of X such that $f(U) \subseteq V$.*

  *If the condition in (4) holds for the point x of X, we say that f **is continuous at the point** x.*

1.4.2 Homemorphism

- **Definition** (***Homemorphism***)
  A ***continuous bijective*** map $f : X \to Y$ with ***continuous inverse***

  $$
  f^{-1} : Y \to X
  $$

  is called a ***homeomorphism***. If there exists a *homeomorphism* from X to Y, we say that X and Y are ***homeomorphic***.

- **Definition** (***Topological Embedding***)
  If X and Y are topological spaces, a ***continuous injective*** map $f : X \to Y$ is called

a ***topological embedding*** if it is a ***homeomorphism*** *onto* its image $f(X) \subseteq Y$ *in the subspace topology* (i.e. $f^{-1}|_{f(X)} : f(X) \to X$ is *continuous in* Y).

- **Remark** (***Smooth Embedding***)
  If X and Y are smooth manifoolds, ***a smooth embedding*** $f : X \to Y$ when it is a ***topological embedding***, and it is *smooth map* with *injective differential* df_x for all $x \in X$ (called a ***smooth immersion***).

1.4.3 Constructing Continuous Functions

- **Proposition 1.10** *(**Rules for Constructing Continuous Functions**). [Munkres, 2000] Let X, Y, and Z be topological spaces.*
  1. *(**Constant Function**) If $f : X \to Y$ maps all of X into the **single point** y_0 of Y, then f is **continuous**.*
  2. *(**Inclusion**) If A is a subspace of X, the **inclusion function** $\iota : A \overset{X}{\hookrightarrow}$ is **continuous**.*
  3. *(**Composites**) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous*
  4. *(**Restricting the Domain**) If $f : X \to Y$ is **continuous**, and if A is a subspace of X, then **the restricted function** $f|_A : A \to Y$ is continuous.*
  5. *(**Restricting or Expanding the Range**) Let $f : X \to Y$ be **continuous**. If Z is a **subspace** of Y containing the **image** set $f(X)$, then the function $g : X \to Z$ obtained by **restricting the range** of f is **continuous**. If Z is a space having Y as a **subspace**, then the function $h : X \to Z$ obtained by **expanding the range** of f is **continuous**.*
  6. *(**Local Formulation of Continuity**) The map $f : X \to Y$ is **continuous** if X can be written as the **union of open sets** U_α such that $f|_{U_\alpha}$ is **continuous** for each α.*

- **Theorem 1.11** *(**The Pasting Lemma / Gluing Lemma**). [Munkres, 2000] Let $X = A \cup B$, where A and B are **closed** in X. Let $f : A \to Y$ and $g : B \to Y$ be **continuous**. If $f(x) = g(x)$ for **every** $x \in A \cap B$, then f and g combine to give a **continuous function** $h : X \to Y$, defined by setting $h|_A = f$, and $h|_B = g$.*

- **Remark** The set A and B can be open sets, and the gluing lemma comes "***Local Formulation of Continuity***".

- **Remark** Notice the condition for *the gluing lemma*:
  1. The domain X is a union of two ***closed sets (or open sets)*** A and B
  2. The two functions f and g are ***continuous*** each of closed domain sets, respectively
  3. f and g ***agree on the intersection*** of two sets $A \cap B$.

- **Theorem 1.12** *(**Maps into Products**). [Munkres, 2000] Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a)).$$

*Then f is **continuous** if and only if the functions*

$$f_1 : A \to X \quad \textit{and} \quad f_2 : A \to Y$$

*are **continuous**. The maps f_1 and f_2 are called **the coordinate functions** of f.*

1.5 Metric Topology

- **Definition** (***Metric Space***)
  A ***metric space*** is a set M and a real-valued function $d(\cdot, \cdot) : M \times M \to \mathbb{R}$ which satisfies:

  1. (***Non-Negativity***) $d(x, y) \geq 0$
  2. (***Definiteness***) $d(x, y) = 0$ if and only if $x = y$
  3. (***Symmetric***) $d(x, y) = d(y, x)$
  4. (***Triangle Inequality***) $d(x, z) \leq d(x, y) + d(y, z)$

  The function d is called a ***metric*** on M. The metric space M equipped with metric d is denoted as (M, d).

- **Definition** (***ϵ-Ball***)
  Given a metric d on X, the number $d(x, y)$ is often called *the **distance** between x and y in the metric d*. Given $\epsilon > 0$, consider the set

  $$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

  of all points y whose distance from x is less than ϵ. It is called ***the ϵ-ball centered at x***. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

- **Definition** (***Metric Topology***)
  If d is a *metric* on the set X, then *the collection of all ϵ-balls $B_d(x, \epsilon)$*, for $x \in X$ and $\epsilon > 0$, is a ***basis*** for a *topology* on X, called ***the metric topology*** *induced by d*.

- **Definition** (***Metrizability***)
  If X is a topological space, X is said to be ***metrizable*** if *there exists a metric d* on the set X that *induces the topology* of X. ***A metric space*** is *a metrizable space X* together with a specific metric d that *gives the topology of X*.

- **Theorem 1.13** *(**ϵ-δ Definition of Continuous Function in Metric Space**). [Munkres, 2000]*
  *Lei $f : X \to Y$; let X and Y be **metrizable** with metrics d_x and d_y, respectively. Then **continuity** of f is **equivalent** to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

  $$d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) < \epsilon.$$

- **Remark** To use ϵ-δ definition, *both **domain** and **codomain*** need to be ***metrizable***.

- **Lemma 1.14** *(**The Sequence Lemma**). [Munkres, 2000]*
  *Let X be a topological space; let $A \subseteq X$. If there is a sequence of points of A **converging** to x, then $x \in \bar{A}$; the **converse** holds if X is **metrizable**.*

- **Proposition 1.15** *Let $f : X \to Y$. If the function f is* ***continuous****, then for every* ***convergent*** *sequence $x_n \to x$ in X, the sequence $f(x_n)$* ***converges*** *to $f(x)$. The* ***converse*** *holds if X is* ***metrizable****.*

- **Remark** To show the converse part, i.e. "*if $x_n \to x \Rightarrow f(x_n) \to f(x)$ then f is continuous*", we just need the space X to be ***first countable***. That is, at each point x, there is ***a countable collection*** $(U_n)_{n \in \mathbb{Z}_+}$ of ***neighborhoods*** of x such that any neighborhood U of x *contains* at least one of the sets U_n.

- **Proposition 1.16** *(**Arithmetic Operations of Continuous Functions**).*
If X is a topological space, and if $f, g : X \to Y$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

- **Definition** (***Uniform Convergence***)
Let $f_n : X \to Y$ be a sequence of functions from the ***set*** X to ***the metric space*** Y. Let d be the metric for Y. We say that the sequence (f_n) ***converges uniformly*** to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that
$$d(f_n(x), f(x)) < \epsilon$$
for all $n > N$ and ***all*** x ***in*** X.

- **Theorem 1.17** *(**Uniform Limit Theorem**). [Munkres, 2000]*
Let $f_n : X \to Y$ be a sequence of ***continuous*** *functions from the* ***topological*** *space X to the* ***metric space*** *Y. If (f_n) converges* ***uniformly*** *to f, then f is* ***continuous****.*

1.6 Connectedness and Local Connectedness

- **Remark** ***Connectedness*** and ***compactness*** are basic ***topological properties***. Both of them are defined based on a collection of open subsets.

    1. ***Connectedness*** is a ***global topological property***: a topological space is *connected* if it cannot be partitioned by two *disjoint nonempty open* subsets. *Connectedness* reveals the information of ***entire space*** *not just within a neighborhood*. Connectedness is ***compatible*** with the ***continuity*** of functions as it implies ***the intermediate value theorem***, which in turn, can be used to construct *inverse function*. Moreover, *connectedness* defines ***an equivalence relationship*** which allows a ***partition*** of the space into ***components***.

    2. ***Connectedness*** is a ***local-to-global topological property***: a topological space is *compact* if every open cover have a finite sub-cover. Using ***finite sub-cover***, ***local properties*** defined *within each neighborhood* can be ***generalized globally*** to entire space. Concept of functions that are closely related to compactness is ***the uniformly continuity*** and ***the maximum value theorem***. The compactness allows us to drop dependency on each individual point x.

    Compared to *connectedness*, ***compactness*** is usually a ***strong condition*** on the topological space.

- **Definition** (***Separation*** *and* ***Connectedness***)
Let X be a topological space. A ***separation*** of X is a pair U, V of ***disjoint nonempty open*** *subset*s of X whose union is X.

The space X is said to be ***connected*** if there *does not exist a separation* of X.

- **Definition** Equivalently, X is ***connected*** if and only if the only subsets of X that are ***both open and closed*** are $\emptyset$ and X itself.

- **Definition** Recall that a topological space X is
  - ***connected*** if there do not exist two *disjoint*, *nonempty*, *open* subsets of X whose union is X;
  - ***path-connected*** if every pair of points in X can be ***joined by a path*** in X, and
  - ***locally path-connected*** if X has a ***basis*** of *path-connected open subsets*.

- **Theorem 1.18** ***(Intermediate Value Theorem)****. [Munkres, 2000]*
  Let $f : X \to Y$ be a **continuous** *map, where X is a* **connected** *space and Y is an ordered set in the* **order topology***. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there* **exists** *a point c of X such that $f(c) = r$.*

- ***Concepts Related to Connectedness***

$$\begin{tikzcd}
\textbf{\textit{path-connected}}\ \textit{space} \arrow[r] \arrow[d, "\text{only one}"] & \textbf{\textit{connected}}\ \textit{space} \arrow[d, "\text{only one}"] \\
\textbf{\textit{path components}} \arrow[r, "\subseteq"] & \textbf{\textit{components}} \arrow[l, dashed, bend left, "\text{if locally path connected}"] \\
\textbf{\textit{path components}}\ \textit{of every open subset} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \textbf{\textit{components}}\ \textit{of every open subset} \arrow[u, "\subseteq"'] \\
\textbf{\textit{locally path connected}}\ \textit{space} \arrow[u, "\text{open}"] & \textbf{\textit{locally connected}}\ \textit{space}. \arrow[u, "\text{open}"']
\end{tikzcd}$$

1.7 Compactness and Local Compactness

- ***Concepts Related to Compactness***

$$\begin{tikzcd}
 & \textbf{\textit{limit point compact}} \arrow[r] \arrow[dl, dashed, bend right, "\text{metrizable}"'] & \textbf{\textit{sequential compact}} \arrow[l, dashed, bend right, "\text{metrizable}"'] \\
\textbf{\textit{compact}} \arrow[ur] \arrow[dr] \arrow[ddr, bend right] \arrow[dddr, bend right] & & \\
 & \textbf{\textit{precompact}} \arrow[ul, bend right, "\text{closure}"'] & \textbf{\textit{locally compact}} \arrow[l, "\text{precompact basis}"] \arrow[ull, dashed, "X\bigcup\{\infty\}\simeq C"'] \arrow[ddl, bend left, "\text{second-countable + Hausdorff}"] \\
 & \textbf{\textit{Lindelöf}} \arrow[d, "\text{regular}"] & \\
 & \textbf{\textit{paracompact}} &
\end{tikzcd}$$

- **Definition** (***Covering of Set*** *and* ***Open Covering of Topological Set***)
  A collection $\mathscr{A}$ of subsets of a space X is said to ***cover*** X, or to be *a* ***covering*** *of* X, if the union of the elements of $\mathscr{A}$ is equal to X.

  It is called an ***open covering of*** X if its elements are *open subsets* of X.

- **Definition** (***Compactness***)
  A topological space X is said to be ***compact*** if *every open covering $\mathscr{A}$* of X contains a ***finite*** *subcollection* that also *covers* X.

- To prove *compactness*, the following property is useful:

  Definition (***Finite Intersection Property***)
  A collection $\mathscr{C}$ of subsets of X is said to have ***the finite intersection property*** if for *every finite subcollection*

  $$\{C_1, \ldots, C_n\}$$

  of $\mathscr{C}$, the ***intersection*** $C_1 \cap \ldots \cap C_n$ is ***nonempty***.

- **Proposition 1.19** ***(Equivalent Definition of Compactness)*** *[Munkres, 2000]*
  Let X be a topological space. Then X is ***compact if and only if*** *for every collection $\mathscr{C}$ of* ***closed*** *sets in X having* ***the finite intersection property****, the intersection $\bigcap_{C\in\mathscr{C}} C$ of all the elements of $\mathscr{C}$ is* ***nonempty****.*

- **Definition** If X and Y are topological spaces, a map $F : X \to Y$ (continuous or not) is said to be ***proper*** if for every ***compact*** set $K \subseteq Y$, the ***preimage*** $F^{-1}(K)$ is ***compact***.

- **Corollary 1.20** ***(Closed Interval in Real Line is Compact)****[Munkres, 2000]*
  Every ***closed interval*** *in $\mathbb{R}$ is* ***compact****.*

- **Proposition 1.21** ***(Closed and Bounded in Euclidean Metric = Compact)****[Munkres, 2000]*
  A subspace A of $\mathbb{R}^n$ is ***compact*** *if and only if it is* ***closed*** *and is* ***bounded*** *in the* ***euclidean metric*** *d or the* ***square metric*** *ρ*

- **Theorem 1.22** ***(Extreme Value Theorem)****. [Munkres, 2000]*
  Let $f : X \to Y$ be ***continuous****, where Y is an* ***ordered set*** *in the order topology. If X is* ***compact****, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

- **Definition** (***Uniform Continuity***)
  A function $f : (X, d_X) \to (Y, d_Y)$ is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

  $$d_X(x_0, x_1) < \delta \quad \Rightarrow \quad d_Y(f(x_0), f(x_1)) < \epsilon.$$

- **Theorem 1.23** ***(Uniform Continuity Theorem)****. [Munkres, 2000]*
  Let $f : X \to Y$ be a ***continuous*** *map of the* ***compact*** *metric space (X, d_X) to the metric space (Y, d_Y). Then f is* ***uniformly continuous****.*

- **Definition** (***Limit Point Compactness***)
  A space X is said to be ***limit point compact*** if every infinite subset of X has a ***limit point***.

- **Proposition 1.24** ***(Compactness $\Rightarrow$ Limit Point Compactness)*** *[Munkres, 2000]*
  Compactness *implies* ***limit point compactness****, but* ***not conversely****.*

- **Example** (**Limit Point Compactness $\not\Rightarrow$ Compactness**)
  Let Y consist of ***two points***; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is ***limit point compact***, for *every nonempty subset of X has a **limit point***. It is ***not compact***, for the *covering* of X by the open sets $U_n = \{n\} \times Y$ has *no finite subcollection covering X*. $\blacksquare$

- **Definition** (***Sequential Compactness***)
  Let X be a topological space. If (x_n) is a *sequence* of points of X, and if

  $$n_1 < n_2 < \ldots < n_i < \ldots$$

  is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n).

  The space X is said to be ***sequentially compact*** if *every sequence of points of X has a **convergent subsequence***.

- **Theorem 1.25** ***(Equivalent Definitions of Compactness in Metric Space)*** *[Munkres, 2000]*
  *Let X be a **metrizable space**. Then the following are **equivalent**:*

  1. *X is **compact**.*
  2. *X is **limit point compact**.*
  3. *X is **sequentially compact**.*

- **Definition** A topological space X is said to be ***locally compact*** if every point has a ***neighborhood*** contained in a ***compact subset*** of X.

  A subset of X is said to be ***precompact*** in X if its ***closure*** in X is *compact*.

- If X is *not a compact Hausdorff space*, then *under what conditions* is X *homeomorphic* with *a **subspace** of a compact Hausdorff space* ?

  Theorem 1.26 ***(Unique One-Point Compactification)*** *[Munkres, 2000]*
  *Let X be a space. Then X is **locally compact Hausdorff** if and only if there exists a space Y satisfying the following conditions:*

  1. *X is a subspace of Y.*
  2. *The set $Y \setminus X$ consists of **a single point** (which is the limit point of X).*
  3. *Y is a **compact Hausdorff** space.*

  *If Y and Y' are two spaces satisfying these conditions, then there is a **homeomorphism** of Y with Y' that equals **the identity map** on X.*

- **Definition** (***One-Point Compactification***)
  If Y is a ***compact Hausdorff*** space and X is a proper *subspace* of Y whose ***closure*** equals Y, then Y is said to be a ***compactification*** of X.

  If $Y \setminus X$ equals *a single point*, then Y is called ***the one-point compactification*** of X.

- **Proposition 1.27** ***(Locally Compact Hausdorff = Precompact Basis)*** *[Munkres, 2000]*
  *Let X be a **Hausdorff** space. Then X is **locally compact if and only if** given x in X,*

*and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is **compact** and $\bar{V} \subseteq U$.*

- **Corollary 1.28** ***(Closed or Open Subspace)*** *[Munkres, 2000]*
  *Let X be locally compact Hausdorff; let A be a subspace of X. If A is **closed** in X or **open** in X, then A is locally compact.*

- **Corollary 1.29** *[Munkres, 2000]*
  *A space X is **homeomorphic** to an **open** subspace of a **compact Hausdorff** space **if and only if** X is **locally compact Hausdorff**.*

- For a ***Hausdorff space*** X, the following are equivalent:

  1. X is ***locally compact***.
  2. Each point of X has a ***precompact*** neighborhood.
  3. X has a basis of ***precompact*** open subsets.

- **Theorem 1.30** ***(Tychonoff Theorem).*** *[Munkres, 2000]*
  *An <u>**arbitrary product**</u> of **compact** spaces is **compact** in the <u>**product topology**</u>.*

1.8 Countability and Separability

1.8.1 Countability Axioms

- ***Concepts Related to Countablity Axioms***

$$\begin{tikzcd}
 & \textit{\textbf{first-countable} space} & \textit{\textbf{metrizable} space} \arrow[l] \\
\textit{\textbf{second-countable} space} \arrow[ur, bend left] \arrow[r] \arrow[dr] & \textit{\textbf{separable} space} \arrow[l, dashed, bend right, "metrizable"'] & \\
 & \textit{\textbf{Lindelöf} space} \arrow[ul, dashed, bend left, "metrizable"] &
\end{tikzcd}$$

.

- **Definition** (***Countability***)
  A topological space X is said to be

  1. ***first-countable*** if there is a ***countable neighborhood basis*** at each point,
  2. <u>***second-countable***</u> if there is ***a countable basis*** for its topology.

- **Proposition 1.31** ***(Limit Point Detected by Convergent Sequence)*** *[Munkres, 2000]*
  Let X be a topological space.

  1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the **converse** holds if X is **first-countable**.*
  2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The **converse** holds if X is **first-countable**.*

- **Definition** (***Dense Subset***)
A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$. (That is, *every point in X is a limit point of A.*)

- **Definition** (***Separability***)
A topological space X is called ***separable*** if and only if it has a ***countable dense set***.

- **Definition** (***Lindelöf Space***)
A space for which *every open covering* contains *a* ***countable*** *subcovering* is called a ***Lindelöf space***.

- **Proposition 1.32** ***(Properties of Second-Countability)*** *[Munkres, 2000]*
Suppose that X has a ***countable basis***. *Then:*

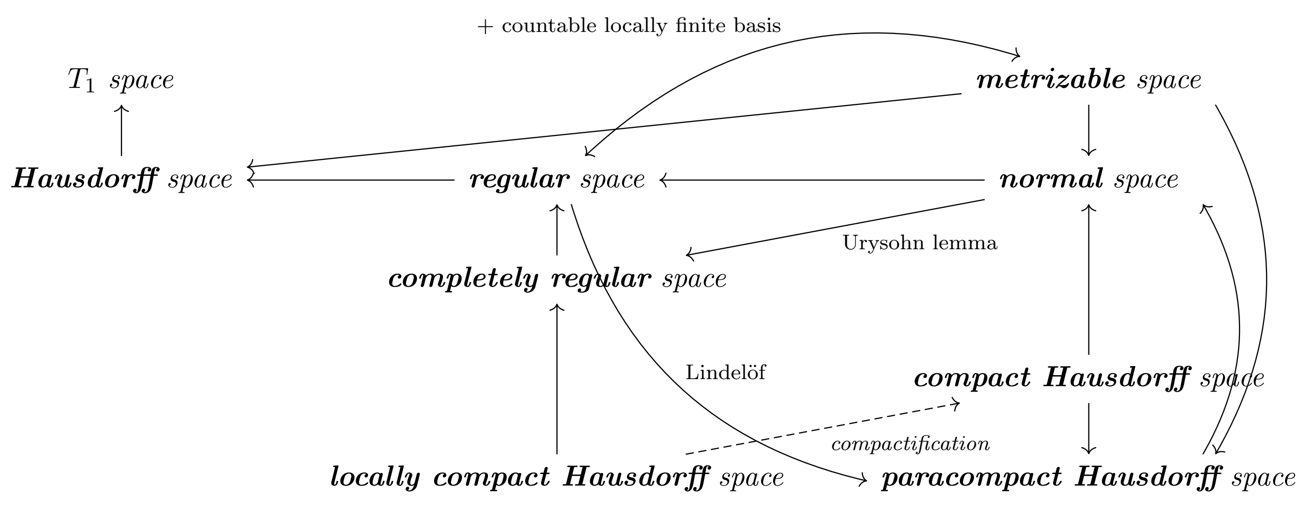
  1. *Every* ***open covering*** *of X contains a* ***countable*** *subcollection covering X. (X is* ***Lindelöf space****)*
  2. *There exists a* ***countable*** *subset of X that is* ***dense*** *in X. (X is* ***separable****)*

- **Proposition 1.33** ***(Metric Space Countablility and Separablility)***

  1. *Every* ***metric space*** *is* ***first countable***.
  2. *A metric space is* ***second countable*** *if and only if it is* ***separable***.
  3. *Any* ***second countable*** *topological space is* ***separable***.

1.8.2 Separability Axioms

- ***Concepts Related to Separation Axioms***



- **Definition** (***Separation Axioms***)

  1. A topological space is called a ***T_1 space*** if and only if for all x and y, $x \neq y$, there is an ***open set*** U with $y \in U$, $x \notin U$.

     Equivalently, a space is T_1 *if and only if* $\{x\}$ is ***closed*** for each x.
  2. A topological space is called ***Hausdorff*** (or T_2) if and only if for all all x and y, $x \neq y$, there are ***open sets*** U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

3. A topological space is called ***regular*** (or T_3) if and only if it is T_1 and for all x and C, ***closed***, with $x \notin C$, there are ***open sets*** U, V such that $x \in U$, $C \subset V$, and $U \cap V = \emptyset$.

   Equivalently, a space is T_3 if *the **closed neighborhoods** of any point are a **neighborhood base***.

4. A topological space is called ***normal*** (or T_4) if and only if it is T_1 and for all C_1, C_2, ***closed***, with $C_1 \cap C_2 = \emptyset$, there are ***open sets*** U, V with $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \emptyset$.

- **Proposition 1.34**

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$$

- **Proposition 1.35** *(**Limit Point in T_1 Axiom**). [Munkres, 2000]*
  *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is **a limit point** of A if and only if every **neighborhood** of x contains **infinitely many points** of A.*

- **Proposition 1.36** *(**Limit Point is Unique in Hausdorff Space**). [Munkres, 2000]*
  *If X is a **Hausdorff space**, then a sequence of points of X **converges to at most one point** of X.*

- **Lemma 1.37** *Let X be a topological space. Let one-point sets in X be closed.*

  1. *X is **regular** if and only if given a point x of X and a neighborhood U of x, there is a **neighborhood** V of x such that $\bar{V} \subseteq U$.*

  2. *X is **normal** if and only if given a **closed** set A and an open set U containing A, there is an **open set** V containing A such that $\bar{V} \subseteq U$.*

- **Proposition 1.38** *[Munkres, 2000]*
  *Every **locally compact Hausdorff** space is **regular**.*

1.9 Important Results and Theorems on Normal Space

- **Theorem 1.39** *(**Regular + Second-Countable $\Rightarrow$ Normal**)[Munkres, 2000]*
  *Every **regular** space with a **countable basis** is **normal**.*

- **Theorem 1.40** *[Munkres, 2000]*
  *Every **metrizable** space is **normal**.*

- **Theorem 1.41** *[Munkres, 2000, Reed and Simon, 1980]*
  *Every **compact Hausdorff** space X is **normal**.*

- **Theorem 1.42** *(**Urysohn Lemma**). [Munkres, 2000]*
  *Let X be a **normal** space; let A and B be **disjoint closed subsets** of X. Let $[a, b]$ be a **closed interval** in the real line. Then there exists a **continuous** map*

$$f : X \rightarrow [a, b]$$

  *such that $f(x) = a$ for **every** x in A, and $f(x) = b$ for **every** x in B.*

- **Theorem 1.43** *(**Urysohn Lemma, Locally Compact Version**). [Folland, 2013]*
  *Let X be a **locally compact Hausdorff** space and $K \subseteq U \subseteq X$ where K is **compact** and*

*U is **open**. Then there exists a **continuous** map*

$$f : X \to [0, 1]$$

*such that $f(x) = 1$ for **every** $x \in K$, and $f(x) = 0$ for x outside a **compact subset** of U.*

- **Theorem 1.44** ***(Tietze Extension Theorem)*** *[Munkres, 2000, Reed and Simon, 1980]*
  *Let X be a **normal space**; let A be a **closed subspace** of X.*

  1. *Any **continuous** map of A into the **closed interval** $[a, b]$ of $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $[a, b]$.*
  2. *Any **continuous** map of A into $\mathbb{R}$ may be **extended** to a **continuous** map of **all of** X into $\mathbb{R}$.*

- **Theorem 1.45** ***(Tietze Extension Theorem, Locally Compact Version)*** *[Folland, 2013]*
  *Let X be a **locally compact Hausdorff space**; let K be a **compact subspace** of X. If $f \in \mathcal{C}(K)$ is a **continuous** map of K into $\mathbb{R}$, there exists a **continuous** extension $F \in \mathcal{C}(X)$ of **all of** X into $\mathbb{R}$ such that $F|_K = f$. Moreover, F may be taken to **vanish outside a compact set**.*

- **Theorem 1.46** ***(The Urysohn Metrization Theorem)**. [Munkres, 1975, Folland, 2013]*
  *Every **second countable normal** space is **metrizable**.*

1.10 Nets

- **Definition** (***Directed System of Index Set***)
  A ***directed system*** is *an index set* I together with an ***ordering*** $\prec$ which satisfies:

  1. If $\alpha, \beta \in l$ then there exists $\gamma \in I$ so that $\gamma \succ \alpha$ and $\gamma \succ \beta$.
  2. $\prec$ is a ***partial ordering***.

- **Definition** (***Net***)
  A ***net*** in a topological space X is a mapping from a *directed system* I to X; we denote it by $\{x_\alpha\}_{\alpha \in I}$

- **Remark** (*Net vs. Sequence*)
  Net is a generalization and abstraction of ***sequence***. The directed system I is ***not necessarily countable***. So $\{x_\alpha\}_{\alpha \in I}$ may not be a countable sequence. *A sequence is a net with countable index set* $I \subseteq \mathbb{N}$. The directed system can be any set e.g. a graph.

- **Definition** If $P(\alpha)$ is a ***proposition*** depending on an ***index*** α in a *directed set* I we say $P(\alpha)$ ***is eventually true*** if there is a β in I with $P(\alpha)$ *true* if *for all* $\alpha \succ \beta$.

  We say $P(\alpha)$ ***is frequently true*** if it is ***not eventually false***, that is, if for any β *there exists* an $\alpha \succ \beta$ with $P(\alpha)$ *true*.

- **Definition** (***Convergence***)
  A ***net*** $\{x_\alpha\}_{\alpha \in I}$ in a topological space X is said to ***converge*** to a point $x \in X$ (written $x_\alpha \to x$) if for ***any neighborhood*** N of x, ***there exists*** a $\beta \in l$ so that $x_\alpha \in N$ if $\alpha \succ \beta$. The point x that being converged to is called ***the limit point*** of x_α.

Note that if $x_\alpha \to x$, then x_α is ***eventually in all neighborhoods of*** x. If x_α is ***frequently in any neighborhood of*** x, we say that x is a ***cluster point*** of x_α.

- **Proposition 1.47** *[Reed and Simon, 1980]*
  Let A be a set in a topological space X. Then, a point x is in the ***closure*** *of A if and only if there is a net $\{x_\alpha\}_{\alpha \in I}$ with $x_\alpha \in A$, So that $x_\alpha \to x$.*

- **Proposition 1.48** *[Reed and Simon, 1980]*

  1. *(**Continuous Function**): A function f from a topological space X to a topological space Y is* ***continuous*** *if and only if for* ***every convergent net*** *$\{x_\alpha\}_{\alpha \in I}$* ***in*** *$X$, with $x_\alpha \to x$, the net $\{f(x_\alpha)\}_{\alpha \in I}$* ***converges in*** *$Y$ to $f(x)$.*

  2. *(**Uniqueness of Limit Point for Hausdorff Space**): Let X be a* ***Hausdorff space****. Then a net $\{x_\alpha\}_{\alpha \in I}$ in X can have* ***at most one limit****; that is, if $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.*

- **Definition** A net $\{x_\alpha\}_{\alpha \in I}$ is a ***subnet*** of a net $\{y_\beta\}_{\beta \in J}$ if and only if there is a function $F : I \to J$ such that

  1. $x_\alpha = y_{F(\alpha)}$ for each $\alpha \in I$.

  2. For all $\beta' \in J$, there is an $\alpha' \in I$ such that $\alpha \succ \alpha'$ implies $F(\alpha) \succ \beta'$ (that is, $F(\alpha)$ is ***eventually larger*** *than any fixed $\beta \in J$*).

- **Proposition 1.49** *A point x in a topological space X is a* ***cluster point*** *of a* ***net*** *$\{x_\alpha\}_{\alpha \in I}$ if and only if* ***some subnet*** *of $\{x_\alpha\}_{\alpha \in I}$* ***converges*** *to x.*

- **Theorem 1.50** *(**The Bolzano-Weierstrass Theorem**) [Reed and Simon, 1980]*
  A space X is ***compact*** *if and only if* ***every net*** *in X* ***has a convergent subnet****.*

2 Topology in Function Space

2.1 Complete Metric Space

- **Definition** (***Cauchy Net in Topological Vector Space***)
  A *net* $\{x_\alpha\}_{\alpha \in I}$ in ***toplogocial vector space*** X is called ***Cauchy*** if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta) \in I \times I}$ ***converges*** *to zero*. (Here $I \times I$ is ***directed*** in the usual way: $(\alpha, \beta) \prec (\alpha', \beta')$ if and only if $\alpha \prec \alpha'$ and $\beta \prec \beta'$.)

- **Definition** (***Completeness***)
  A toplogocial vector space X is ***complete*** if every Cauchy net converges.

- **Proposition 2.1** *(**Complete First Countable Topological Vector Space**)*
  If X is a ***first-countable topological vector space*** *and every* ***Cauchy sequence*** *in X converges, then every* ***Cauchy net*** *in X converges.*

- **Proposition 2.2** *(**Completeness of Euclidean Space**) [Munkres, 2000]*
  Euclidean space $\mathbb{R}^k$ is ***complete*** *in either of its usual* ***metrics****, the* ***euclidean metric*** *d or the* ***square metric*** *ρ.*

- **Lemma 2.3** *(**Convergence in Product Space is Weak Convergence**) [Munkres, 2000]*

Let X be the product space $X = \prod_\alpha X_\alpha$; let x_n be a sequence of points of X. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α.

- **Proposition 2.4** ***(Completeness of Countable Product Space)*** *[Munkres, 2000]*
  There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is ***complete.***

- **Definition** (***Uniform Metric in Function Space***)
  Let (Y, d) be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the ***standard bounded metric*** on Y derived from d. If $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ are points of the cartesian product Y^J, let

  $$\bar{\rho}(x, y) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\}.$$

  It is easy to check that $\bar{\rho}$ is a metric; it is called ***the uniform metric*** on Y^J corresponding to the metric d on Y.

  Note that ***the space of all functions*** $f : J \to Y$, ***denoted*** as Y^J, is a subset of the product space $J \times Y$. We can define uniform metric in the function space: if $f,\ g : J \to Y$, then

  $$\bar{\rho}(f, g) = \sup\left\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\right\}.$$

- **Proposition 2.5** ***(Completeness of Function Space Under Uniform Metric)*** *[Munkres, 2000]*
  If the space Y is ***complete*** *in the metric d, then the space Y^J is* ***complete*** *in the* ***uniform metric*** *$\bar{\rho}$ corresponding to d.*

- **Definition** (***Space of Continuous Functions and Bounded Functions***)
  Let Y^X be *the space of all functions* $f : X \to Y$, where X is a *topological space* and Y is a *metric space with metric d*. Denote the ***subspace*** of Y^X consisting of all ***continuous functions*** f as $\mathcal{C}(X, Y)$.

  Also denote *the set of all* ***bounded functions*** $f : X \to Y$ as $\mathcal{B}(X, Y)$. (A function f is said to be ***bounded*** if its image $f(X)$ is a ***bounded subset*** of *the metric space* (Y, d).)

- **Proposition 2.6** ***(Completeness of $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ Under Uniform Metric)*** *[Munkres, 2000]*
  Let X be a topological space and let (Y, d) be a metric space. The set $\mathcal{C}(X, Y)$ of ***continuous*** *functions is* ***closed*** *in Y^X under the* ***uniform metric****. So is the set $\mathcal{B}(X, Y)$ of* ***bounded functions****. Therefore, if Y is* ***complete****, these spaces are* ***complete*** *in the* ***uniform metric****.*

- **Definition** (***Sup Metric on Bounded Functions***)
  If (Y, d) is a metric space, one can define another metric *on the set $\mathcal{B}(X, Y)$ of* ***bounded functions*** from X to Y by the equation

  $$\rho(x, y) = \sup\{d(f(x), g(x)) : x \in X\}.$$

  It is easy to see that ρ is well-defined, for the set $f(X) \cup g(X)$ is ***bounded*** if both $f(X)$ and $g(X)$ are. The metric ρ is called ***the sup metric***.

- **Theorem 2.7** ***(Existence of Completion)*** *[Munkres, 2000]*
  Let (X, d) be a metric space. There is an ***isometric embedding*** *of X into a* ***complete*** *metric space.*

- **Definition** (***Completion***)
  Let X be a *metric space*. If $h : X \to Y$ is an ***isometric embedding*** of X into a ***complete*** *metric space* Y, then the ***subspace*** $h(X)$ of Y is a *complete metric space*. It is called ***the completion of*** X.

- **Definition** (***Topological Complete***)
  A space X is said to be ***topologically complete*** if there *exists* a metric for the *topology* of X relative to which X is *complete*.

- **Proposition 2.8** *(**Properties of Topological Complete**) [Munkres, 2000]*
  The followings are properties of topological completeness:

  1. *A **closed** subspace of a topologically complete space is topologically complete.*
  2. *A **countable product** of topologically complete spaces is topologically complete (in the **product topology**).*
  3. *An **open** subspace of a topologically complete space is topologically complete.*
  4. *A G_δ **set** in a topologically complete space is topologically complete.*

2.2 Compactness in Metric Spaces

- **Remark** (***Compactness*** *and* ***Completeness***)
  How is ***compactness*** of a metric space X related to ***completeness*** of X?

  The followings is from *the sequential compactness* and definition of *completeness*:

  Proposition 2.9 *Every **compact** metric space is **complete**.*

  The *converse* does not hold – ***a complete metric space need not be compact***. It is reasonable to ask what ***extra condition*** one needs to impose on a complete space to be assured of its compactness. Such a condition is the one called *total boundedness*.

- **Definition** (***Total Boundedness***)
  A metric space (X, d) is said to be ***totally bounded*** if for every $\epsilon > 0$, there is a ***finite covering*** *of* X *by* ***ϵ-balls***.

- **Theorem 2.10** *[Munkres, 2000]*
  *A metric space (X, d) is **compact** if and only if it is **complete** and **totally bounded**.*

- **Remark** We now apply this result to find ***the compact subspaces*** of the space $\mathcal{C}(X, \mathbb{R}^n)$, *in the **uniform topology***. We know that a subspace of $\mathbb{R}^n$ is compact if and only if it is ***closed*** and ***bounded***.

  One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. ***But*** it does not, even if X is *compact*. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies *an **additional condition***, called ***equicontinuity***.

- **Definition** (***Equicontinuity***) [Reed and Simon, 1980, Munkres, 2000]
  Let (Y, d) be a *metric space*. Let $\mathscr{F}$ be a *subset* of the function space $\mathcal{C}(X, Y)$ (i.e. $f \in \mathscr{F}$ is continuous). If $x_0 \in X$, *the set $\mathscr{F}$ of functions* is said to be ***equicontinuous at*** x_0 if given

$\epsilon > 0$, there is a neighborhood U of x_0 such that *for all* $x \in U$ and ***all*** $f \in \mathscr{F}$,

$$d(f(x), f(x_0)) < \epsilon.$$

If the set $\mathscr{F}$ is *equicontinuous* at x_0 for each $x_0 \in X$, it is said simply to be ***equicontinuous*** or $\mathscr{F}$ is an ***equicontinuous family***.

We say $\mathscr{F}$ is a ***uniformly equicontinuous family*** if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ whenever $p(x, x') < \delta$ for all $x, x' \in X$ and ***every*** $f \in \mathscr{F}$.

- **Remark** An *equicontinuous family* of functions is *a family of continuous functions.*

- **Remark** ***Continuity*** of the function f at x_0 means that ***given*** f and given $\epsilon > 0$, there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$. ***Equicontinuity*** **of** $\mathscr{F}$ **means that** ***a single neighborhood*** U **can be chosen that will work for all the functions** f in the collection $\mathscr{F}$.

- **Lemma 2.11** ***(Total Boundedness*** $\Rightarrow$ ***Equicontinuous)*** *[Munkres, 2000]*
Let X be a ***space****; let (Y, d) be a* ***metric*** *space. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is* ***totally bounded*** *under the* ***uniform metric*** *corresponding to d, then $\mathscr{F}$ is* ***equicontinuous*** *under d.*

- **Lemma 2.12** ***(Equicontinuous*** + ***Compactness*** $\Rightarrow$ ***Total Boundedness)*** *[Munkres, 2000]*
Let X be a space; let (Y, d) be a metric space; assume X and Y are ***compact****. If the subset $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is* ***equicontinuous*** *under d, then $\mathscr{F}$ is* ***totally bounded*** *under the* ***uniform*** *and* ***sup*** *metrics corresponding to d.*

- **Definition** (***Pointwise Bounded***)
If (Y, d) is a *metric space*, a *subset* $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is said to be ***pointwise bounded*** under d if for each $x \in X$, the subset

$$F_a = \{f(a) : f \in \mathscr{F}\}$$

of Y is ***bounded*** under d.

- **Theorem 2.13** ***(Ascolis Theorem, Classical Version)****. [Munkres, 2000]*
Let X be a ***compact*** *space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding* ***uniform topology****. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has* ***compact closure if and only if*** *$\mathscr{F}$ is* ***equicontinuous*** *and* ***pointwise bounded*** *under d.*

- **Corollary 2.14** *Let X be* ***compact****; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding* ***uniform topology****. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is* ***compact if and only if*** *it is* ***closed, bounded*** *under the* ***sup metric*** *ρ, and* ***equicontinuous*** *under d.*

- **Remark** (***Ascoli's Theorem, Sequence Version***) [Reed and Simon, 1980]
Let $\{f_n\}$ be a family of ***uniformly bounded equicontinuous functions*** *on* $[0, 1]$*. Then* ***some subsequence*** $\{f_{n,m}\}$ *converges* ***uniformly*** *on* $[0, 1]$.

- **Definition** (***Continuous Functions that Vanish At Infinity*** $\mathcal{C}_0(X, \mathbb{R})$)
Let X be a space. A subset $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R})$ is said to ***vanish uniformly at infinity*** if given $\epsilon > 0$, there is a ***compact subspace*** C of X such that $|f(x)| < \epsilon$ for $x \in X \setminus C$ and $f \in \mathcal{F}$.

If $\mathcal{F}$ consists of a single function f, we say simply that ***f vanishes at infinity***. Let $\mathcal{C}_0(X, \mathbb{R})$ denote *the set of continuous functions $f : X \to \mathbb{R}$ that* ***vanish at infinity***.

- **Corollary 2.15** *[Munkres, 2000]*
Let X be ***locally compact Hausdorff****; give $\mathcal{C}_0(X, \mathbb{R})$ the uniform topology. A subset $\mathcal{F}$ of $\mathcal{C}_0(X, \mathbb{R})$ has* ***compact closure if and only if*** *it is* ***pointwise bounded***, ***equicontinuous***, *and* ***vanishes uniformly at infinity***.

2.3 Pointwise and Compact Convergence

- **Remark** (***Useful Topologies on Y^X***)
  1. ***Uniform Topology***: generated by the ***basis***
  $$B_U(f, \epsilon) = \left\{ g \in Y^X : \sup_{x \in X} \bar{d}(f(x), g(x)) < \epsilon \right\}$$
  It corresponds to ***the uniform convergence*** of f_n to f in Y^X. $\mathcal{C}(X, Y)$ is ***closed*** in Y^X under the *uniform topology*, following *the Uniform Limit Theorem.*
  2. ***Topology of Pointwise Convergence***: generated by the ***basis***
  $$\begin{aligned} B_{U_1, \ldots, U_n}(x_1, \ldots, x_n, \epsilon) &= \bigcap_{i=1}^{n} S(x_i, U_i) \\ &= \left\{ f \in Y^X : f(x_1) \in U_1, \ldots, f(x_n) \in U_n \right\}, \quad 1 \le n < \infty. \end{aligned}$$
  It corresponds to ***the pointwise convergence*** of f_n to f in Y^X. $\mathcal{C}(X, Y)$ is ***not closed*** in Y^X under the *topology of pointwise convergence*. Note that *the topology of poinwise convergence* is *the* ***product topology*** *of* Y^X.
  3. ***Topology of Compact Convergence***: generated by the ***basis***
  $$B_C(f, \epsilon) = \left\{ g \in Y^X : \sup_{x \in C} d(f(x), g(x)) < \epsilon \right\}, \ C \text{ is compact set.}$$
  It corresponds to ***the uniform convergence*** of f_n to f in Y^X for $x \in C$. $\mathcal{C}(X, Y)$ is ***closed*** in Y^X under the *topology of compact convergence* ***if X is compactly generated***.

  On $\mathcal{C}(X)$, the topology of compact convergence is equal to the compact-open topology:

  Definition (***Compact-Open Topology on Continuous Function Space***)
  Let X and Y be topological spaces. If C is a ***compact subspace*** of X and U is an *open* subset of Y, define
  $$S(C, U) = \{ f \in \mathcal{C}(X, Y) : f(C) \subseteq U \}.$$
  The sets $S(C, U)$ form a ***subbasis*** for a *topology* on $\mathcal{C}(X, Y)$ that is called ***the compact-open topology***.

  We see that the *uniform topology* is the *finest* among them all and the *topology of pointwise convergence* is the *coarest.*

  (uniform) $\supseteq$ (compact convergence) $\supseteq$ (pointwise convergence).

- **Proposition 2.16** *(**Topology of Uniform Convergence in Compact Sets**) [Munkres, 2000]*
  *A sequence $f_n : X \to Y$ of functions converges to the function f in the **topology of compact convergence** if and only if for **each compact subspace** C of X, the sequence $f_n|_C$ converges **uniformly** to $f|_C$.*

- **Definition** (***Compactly Generated Space***)
  A space X is said to be ***compactly generated*** if it satisfies the following condition: A set A is ***open (or closed)*** in X if $A \cap C$ is ***open (or closed)*** in C for each ***compact subspace*** C of X.

- **Lemma 2.17** *[Munkres, 2000]*
  *If X is **locally compact**, or if X satisfies **the first countability axiom**, then X is **compactly generated**.*

- **Proposition 2.18** *Let X and Y be spaces; give $\mathcal{C}(X, Y)$ the **compact-open topology**. If $f : X \times Z \to Y$ is **continuous**, then **so is** the induced function $F : Z \to \mathcal{C}(X, Y)$. The **converse** holds if X is **locally compact Hausdorff**.*

- **Theorem 2.19** *(**Ascoli's Theorem, General Version**). [Munkres, 2000]*
  *Let X be a space and let (Y, d) be a **metric** space. Give $\mathcal{C}(X, Y)$ the **topology of compact convergence**; let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$.*

  1. *If $\mathcal{F}$ is **equicontinuous** under d and the set*
  $$F_a = \{f(a) : f \in \mathcal{F}\}$$
  *has **compact closure** for each $a \in X$, then $\mathcal{F}$ is **contained** in a **compact subspace** of $\mathcal{C}(X, Y)$.*
  2. *The **converse** holds if X is **locally compact Hausdorff**.*

- **Proposition 2.20** *(**Equicontinuity + Pointwise Convergence $\Rightarrow$ Compact Convergence**) [Munkres, 2000]*
  *Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of **continuous** functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f in the **topology of pointwise convergence**. If $\{f_n\}$ is **equicontinuous**, then f is **continuous** and f_n converges to f in the **topology of compact convergence**.*

2.4 Subspaces of Continuous Functions

- **Definition** (***Subspace of Continuous Functions***)
  Let $\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R})$ be *the space of **continuous** real-valued functions* on topological space X and $\mathcal{B}(X) := \mathcal{B}(X, \mathbb{R})$ be *the space of **bounded** real-valued functions* on X.

  1. The intersection of $\mathcal{B}(X)$ and $\mathcal{C}(X)$ is the space of all ***bounded continuous*** *functions*
  $$\mathcal{BC}(X) := \mathcal{BC}(X, \mathbb{R}) = \mathcal{B}(X, \mathbb{R}) \cap \mathcal{C}(X, \mathbb{R})$$
  Note that $\mathcal{BC}(X) \subseteq \mathcal{B}(X)$ is a ***closed subspace***.
  2. Define the ***support*** of a function f, $\text{supp}(f)$ as *the **smallest closed set** outside of which f vanishes.* The subset $\mathcal{C}_c(X) \subseteq \mathcal{C}(X)$ is the space of all *continuous functions*

with ***compact support***

$$\mathcal{C}_c(X) = \{f \in \mathcal{C}(X, \mathbb{R}) : \text{supp }(f) \;\textit{is compact}\}\,.$$

Note that by *Tietze Extension Theorem, the locally compact Hausdorff space* X has a rich supply of continuous functions that vanishes outside a compact set.

3. Recall also that $\mathcal{C}_0(X)$ is the space of *continuous functions* on X that ***vanishes at infinity***, i.e. for all $\epsilon > 0$, $|f(x)| < \epsilon$ if $x \in X \setminus C$ for some ***compact subset*** $C \subseteq X$.

$$\mathcal{C}_0(X) = \{f \in \mathcal{C}(X, \mathbb{R}) : f \;\textit{vanishes at infinity}\}\,.$$

Note that

$$\mathcal{C}_c(X) \subseteq \mathcal{C}_0(X) \subseteq \mathcal{BC}(X) \subseteq \mathcal{C}(X)$$

- The crucial fact about compactly generated spaces is the following:

  Lemma 2.21 *(**Continuous Extension on Compact Generated Space**) [Munkres, 2000]* *If X is compactly generated, then a function $f : X \to Y$ is **continuous** if for each **compact subspace** C of X, the restricted function $f|_C$ is **continuous**.*

- **Theorem 2.22** *(**$\mathcal{C}(X, Y)$ on Compact Generated Space**) [Munkres, 2000]* *Let X be a **compactly generated space**: let (Y, d) be a metric space. Then $\mathcal{C}(X, Y)$ is **closed** in Y^X in the **topology of compact convergence**.*

- Recall that

  Proposition 2.23 *If X is a **locally compact Hausdorf** space, $\mathcal{C}(X)$ is a **closed** subspace of $\mathbb{R}^X$ in the **topology of compact convergence**.*

- **Proposition 2.24** *[Folland, 2013]* *If X is a topological space, $\mathcal{BC}(X)$ is a **closed** subspace of $\mathcal{B}(X)$ in the **uniform metric**; in particular, $\mathcal{BC}(X)$ is **complete**.*

- **Proposition 2.25** *[Folland, 2013]* *If X is a **locally compact Hausdorf** space, $\mathcal{C}_0(X)$ is a **closure** of $\mathcal{C}_c(X)$ in the **uniform metric**.*

- **Remark** Note that $\mathcal{C}_0(X) = \overline{\mathcal{C}_c(X)}$ is the ***completion*** of $\mathcal{C}_c(X)$ *under uniform metrc.*

2.5 Baire Category

- **Remark** (***Empty Interior = Complement is Dense***) Recall that if A is a subset of a space X, the ***interior*** of A is defined as *the union of all open sets of X that are contained in A.*

  To say that A has ***empty interior*** is to say then that A ***contains no open set of*** X *other than the empty set.* ***Equivalently***, A has ***empty interior*** if every point of A is *a **limit point** of the **complement** of A*, that is, if *the **complement** of A is **dense** in X*.

$$\mathring{A} = \emptyset \quad\Leftrightarrow\quad A^c \text{ is dense in } X$$

In [Reed and Simon, 1980], if a subset $\overline{A}$ of X has *empty interior*, A is said to be ***nowhere dense*** in X.

- **Example** Some examples:

  1. The set $\mathbb{Q}$ of *rationals* has ***empty interior*** as a subset of $\mathbb{R}$
  2. The *interval* $[0, 1]$ has ***nonempty interior***.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Space***)
  A space X is said to be a ***Baire space*** if the following condition holds: Given ***any countable*** collection $\{A_n\}$ of ***closed*** sets of X each of which has ***empty interior*** in X, their ***union*** $\bigcup_{n=1}^{\infty} A_n$ also has ***empty interior*** in X.

- **Example** Some examples:

  1. The space $\mathbb{Q}$ of *rationals* is ***not a Baire space***. For each one-point set in $\mathbb{Q}$ is *closed and has empty interior* ***in*** $\mathbb{Q}$; and $\mathbb{Q}$ is *the countable union of its one-point subsets*.
  2. The space $\mathbb{Z}_+$, on the other hand, does form *a* ***Baire space***. Every subset of $\mathbb{Z}_+$ is *open*, so that there exist *no subsets* of $\mathbb{Z}_+$ having *empty interior*, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Category***)
  A subset A of a space X was said to be of ***the first category in*** X if it ***was contained*** *in the* ***union*** *of a* ***countable*** *collection of* ***closed*** *sets of X having* ***empty interiors*** *in X*; ***otherwise***, it was said to be of ***the second category in*** X.

- **Remark** *A space X is a* ***Baire space*** *if and only if every* ***nonempty open*** *set in X is of* ***the second category***.

- **Lemma 2.26** *(**Open Set Definition of Baire Space**) [Munkres, 2000]*
  X is a ***Baire space if and only if*** *given any* ***countable*** *collection $\{U_n\}$ of* ***open*** *sets in X, each of which is* ***dense*** *in X, their* ***intersection*** *$\bigcap_{n=1}^{\infty} U_n$ is also* ***dense*** *in X.*

- **Theorem 2.27** *(**Baire Category Theorem**). [Munkres, 2000]*
  If X is a ***compact Hausdorff*** *space or a* ***complete metric space****, then X is a* ***Baire space***.

- **Remark** In other word, neither ***compact Hausdorff*** space or a ***complete metric space*** is a *countable union of closed subsets with empty interior (that are nowhere dense).*

- **Lemma 2.28** *[Munkres, 2000]*
  Let $C_1 \supset C_2 \supset \ldots$ be a ***nested*** *sequence of* ***nonempty closed sets*** *in the* ***complete metric space*** *X. If diam $C_n \to 0$, then $\bigcap_n C_n = \emptyset$.*

- **Lemma 2.29** *[Munkres, 2000]*
  Any ***open*** *subspace Y of a Baire space X is itself a Baire space.*

3 Locally Convex Topological Space

3.1 Topological Vector Space

- **Definition** (***Topological Vector Space***)
  A vector space X endowed with a topology $\mathscr{T}$ is called a ***topological vector space***, denoted as $(X, \mathscr{T})$, if the addition $+ : X \times X \to X$ and scale multiplication $\cdot : \mathbb{R} \times X \to X$ are ***continuous***.

- **Theorem 3.1** *[Treves, 2016]*
  *Every **locally compact Hausdorff** topological vector space is **finite-dimensional**.*

3.2 Locally Convex Topological Vector Space

- **Definition** (***Locally Convex Space***)
  A topological vector space X is a ***locally convex topological vector space*** (or just ***locally convex space***), if V is open and $x \in V$, then one can find a *convex open* set $U \subset X$ such that $x \in U \subset V$. That is, there exists *a **base of convex sets** $\mathscr{B}$ that **generates the topology** $\mathscr{T}$.*

- **Remark** The most common way of defining locally convex topologies on vector spaces is in terms of *semi-norms*.

- **Definition** (***Semi-Norm***)
  A ***semi-norm*** on a vector space X is a mapping $q : X \to \mathbb{R}_+$ satisfying the following conditions:

  1. *homogeneity*: $q(\gamma \boldsymbol{x}) = |\gamma|\, q(\boldsymbol{x})$;
  2. the *triangle inequality*: $q(\boldsymbol{x} + \boldsymbol{y}) \le q(\boldsymbol{x}) + q(\boldsymbol{y})$.

  If furthermore $q(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = 0$, then q is a ***norm***.

- **Remark** A ***metric*** $d : X \times X \to \mathbb{R}_+$ that ***induced*** from a norm is given by $d_\theta(\boldsymbol{x}, \boldsymbol{y}) = q_\theta(\boldsymbol{y} - \boldsymbol{x})$, $\forall \boldsymbol{x}, \boldsymbol{y} \in X$.

- **Proposition 3.2** *A normed space $(X, \mathscr{T})$ induced by $\{q_\theta, \theta \in \Theta\}$ is* Hausdorff *if and only if for any $\boldsymbol{x} \neq 0, \boldsymbol{x} \in X$, $\exists \theta \in \Theta$, such that $q_\theta(\boldsymbol{x}) > 0$.*

- **Definition** (***Locally Convex Space** generated by **Semi-Norms***)
  The ***smallest topology*** $\mathscr{T}$ induced by the set of ***semi-norms*** $\{q_\theta, \theta \in \Theta\}$ is generated by ***the convex basis*** $U_{\boldsymbol{x},r,\theta} = \{\boldsymbol{y} \in X \,|\, q_\theta(\boldsymbol{y} - \boldsymbol{x}) \le r\} \in \mathscr{B}, \boldsymbol{x} \in X, r > 0$. The topological vector space $(X, \mathscr{T})$ is thus ***locally convex space***.

  If $\{q_\theta, \theta \in \Theta\}$ is a set of ***norms***, then $(X, \mathscr{T})$ is a ***normed space***.

- **Remark** The most commonly seen *topological vector space* are ***the normed linear space***. It is a vector space X equipped with norm $\|\cdot\|$ and the topology generated by the norm induced metric d. It is denoted as $(X, \|\cdot\|)$.

  The ***locally convex space*** is seen as a generalization of *normed vector space*.

- **Proposition 3.3** ***(Continuous Linear Operator)*** *[Folland, 2013]*

*Suppose X and Y are vector spaces with topologies defined, respectively, by the families $\{p_\alpha\}_{\alpha\in A}$ and $\{q_\beta\}_{\beta\in B}$ of **semi-norms**, and $T : X \to Y$ is a linear map. Then T is **continuous if and only if** for each $\beta \in B$, there exists $\alpha_1, \ldots, \alpha_k \in A$ and $C > 0$ such that $q_\beta(Tx) \le C \sum_{i=1}^{k} p_{\alpha_i}(x)$.*

- **Remark** If the semi-norms are *norms*, then the condition above is *the bounded condition* for continuous linear operator.

- **Proposition 3.4** *[Folland, 2013]*
  *Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of **seminorms**.*

  1. *X is **Hausdorff** if and only if for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.*

  2. *If X is **Hausdorff** and A is **countable**, then X is **metrizable** with a **translation invariant metric** (i.e., $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$).*

- **Definition** (***Fréchet Space***)
  A ***complete Hausdorff*** ***topological vector space*** X whose topology is defined by a ***countable*** family of *seminorms* $\{q_\theta, \theta \in \Theta\}$ is called a ***Fréchet space***.

- **Example** 1. *A **Fréchet space*** is a ***complete locally convex space***.

  2. *A **Banach space** is a **Fréchet space**.*

- **Example** (***Locally Integrable Functions*** $L^1_{loc}(X, \mu)$)
  The space of all ***locally integrable functions*** *on* $\mathbb{R}$, $L^1_{loc}(\mathbb{R})$, *is a* ***Fréchet space*** with the topology defined by the ***semi-norms***

  $$p_k(f) = \int_{|x|\le k} |f(x)|\, dx.$$

  Completeness follows easily from the completeness of L^1. An obvious *generalization* of this construction yields a ***locally convex topological vector space*** $L^1_{loc}(X, \mu)$ where X is any *locally convex Hausdorff (LCH) space* and μ is a *Borel measure* on X that is *finite* on *compact sets.*

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