Lecture 1: Fundamental of Curves and Surface in $\mathbb{R}^3$

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Contents

1 Curves

1.1 Regular Curves in $\mathbb{R}^3$

- **Definition** A ***parameterized differentiable curve*** [do Carmo Valero, 1976] is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b) \subset \mathbb{R}$ to $\mathbb{R}^3$.

  The word *differentiable* in this definition means that α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such a way that the functions $x(t)$, $y(t)$, $z(t)$ are differentiable. The variable t is called the *parameter of the curve.* The word interval is taken in a generalized sense, so that we do not exclude the cases $a = -\infty$, $b = +\infty$.

  If we denote by $x'(t)$ the first derivative of x at the point t and use similar notations for the functions y and z, the vector $(x'(t), y'(t), z'(t)) = \alpha'(t) \in \mathbb{R}^3$ is called the ***tangent vector*** (or velocity vector) of the curve α at t. The image set $\alpha(I) \subset \mathbb{R}^3$ is called the *trace* of α.

- **Definition** A parameterized curve is said to be ***regular*** if $\alpha'(t) \neq 0$ for all $t \in I$.

- The *arc length* of a regular parameterized curve $\alpha : I \to \mathbb{R}^3$ from t_0 is defined as

$$s \equiv \int_{t_0}^{t} \left|\alpha'(t)\right| dt$$

  where $|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$. Note that a parameterized regular curve can be reparameterized by the arc length as $\alpha(s)$.

- Given the curve α parametrized by arc length $s \in (a, b)$, we may consider the curve β defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$, which has the same trace as the first one but is described in the *opposite direction.* We say, then, that these two curves differ by ***a change of orientation***.

1.2 Vector Product in $\mathbb{R}^3$

- Two ordered bases $e = [e_i]$ and $f = [f_i]$, $i = 1, \dots, n$, of an n-dimensional vector space V have ***the same orientation*** if the matrix of change of basis has positive determinant. We denote this relation by $e \sim f$. From elementary properties of determinants, it follows that $e \sim f$ is an ***equivalence relation***.

- Each of the equivalence classes determined by the above relation is called an ***orientation*** of V.

- In the case $V = \mathbb{R}^3$, there exists a natural ordered basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and we shall call the orientation corresponding to this basis the ***positive orientation*** of $\mathbb{R}^3$, the other one being the *negative orientation* (of course, this applies equally well to any $\mathbb{R}^n$).

  We also say that ***a given ordered basis*** of $\mathbb{R}^3$ is ***positive (or negative)*** if it belongs to the positive (or negative) orientation of $\mathbb{R}^3$. Thus, the ordered basis e_1, e_3, e_2 is a negative basis,since the matrix which changes this basis into e_1, e_2, e_3 has determinant equal to -1.

- The ***cross product*** (***vector product***) of two vectors u and v under the basis $\{e_1, e_2, e_3\}$ is

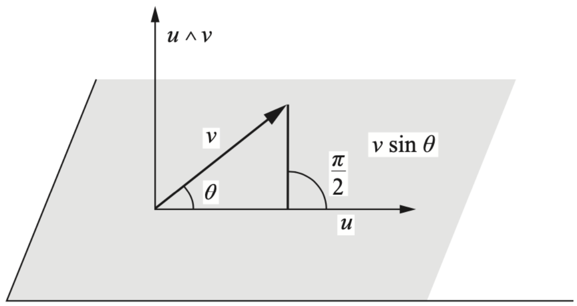


Figure 1: The basis of $\mathbb{R}^3$ formed by $u, v, u \wedge v$ [do Carmo Valero, 1976]

denoted as $u \wedge v$ and computed as

$$\langle u \wedge v \, , \, w \rangle = \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \equiv \det(u, v, w) \tag{1}$$

and

$$u \wedge v \equiv u \times v \equiv \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3 \tag{2}$$

- The following **properties** can easily be checked (actually they just express the usual *properties of **determinants***):
  1. (***anti-commutativity***) $u \wedge v = -v \wedge u$.
  2. $u \wedge v$ depends ***linearly*** on u and v; i.e., for any real numbers a, b, we have $(a\, u + b\, w) \wedge v = a\,(u \wedge v) + b\,(w \wedge v)$.
  3. $u \wedge v = 0$ if and only if u and v are ***linearly dependent***.
  4. $\langle u \wedge v \, , \, u \rangle = 0$, $\langle u \wedge v \, , \, v \rangle = 0$.
- We observe that $\langle u \wedge v \, , \, u \wedge v \rangle = |u \wedge v|^2 > 0$. This means that the determinant of the vectors u, v, $u \wedge v$ is *positive*; that is, $\{u, v, u \wedge v\}$ ***is a positive basis*** (Figure 1).
- The inner product of vector products is

$$\langle u \wedge v \, , \, x \wedge y \rangle = \det \begin{bmatrix} \langle u \, , \, x \rangle & \langle u \, , \, y \rangle \\ \langle v \, , \, x \rangle & \langle v \, , \, y \rangle \end{bmatrix}$$

  It follows that

$$\langle u \wedge v \, , \, u \wedge v \rangle = |u \wedge v|^2 = \det \begin{bmatrix} \langle u \, , \, u \rangle & \langle u \, , \, v \rangle \\ \langle v \, , \, u \rangle & \langle v \, , \, v \rangle \end{bmatrix} = |u|^2 \, |v|^2 \, \left(1 - \cos^2(\theta)\right) = A^2$$

  where θ is the ***angle*** of u and v, and A is the ***area*** of the ***parallelogram*** generated by u and v.
- Note that the vector product is **not associative**. In fact, we have the following identity:

$$(u \wedge v) \wedge w = (\langle u \, , \, w \rangle)\, v - (\langle v \, , \, w \rangle)\, u. \tag{3}$$

- Finally, let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) to $\mathbb{R}^3$, $t \in (a, b)$. It follows immediately from Eq. (2) that $u(t) \wedge v(t)$ is also differentiable and that

$$\frac{d}{dt}(u(t) \wedge v(t)) = \frac{du(t)}{dt} \wedge v(t) + u(t) \wedge \frac{dv(t)}{dt}$$

1.3 The Local Theory of Curves Parametrized by Arc Length

- The differential of $\alpha(s)$ as $t(s) \equiv \overrightarrow{t}(s) \equiv \alpha'(s)$ is the ***tangent vector*** (velocity) of $\alpha(s)$ at s.

- **Definition** The quantity $|\alpha''(s)| \equiv k(s)$ is referred as the ***curvature*** of $\alpha(s)$ at s. The curvature measures ***the rate of change*** of *the tangent line* along the curve.

  Notice that by a change of orientation, the tangent vector changes its direction; that is, if $\beta(-s) = \alpha(s)$, then

$$\frac{d\beta(-s)}{d(-s)} = -\frac{d\alpha(s)}{ds}$$

  Therefore, $\alpha''(s)$ and the curvature remain ***invariant*** under *a change of orientation.*

- For any ***closed*** parameterized ***convex*** curve, the curvature $k(s)$ is **nonnegative** and has **two** maxima and **two** minima or is constant everywhere.

- Moreover, the acceleration vector $\alpha''(s)$ is ***normal*** (orthogonal) to $\alpha'(s)$, because by differentiating $\langle \alpha'(s) \, , \, \alpha'(s) \rangle = 1$ we obtain $\langle \alpha''(s) \, , \, \alpha'(s) \rangle = 0$.

  Let $n(s)$ be the unit vector of $\alpha''(s)$ (i.e. $\alpha''(s) = k(s)\, \boldsymbol{n}(s)$), then $\boldsymbol{n}(s) \equiv \overrightarrow{n}(s)$ is the ***normal vector*** and it perpendicular to the tangent vector.

- The plane determined by the unit *tangent* and *normal vectors*, $\alpha'(s)$ and $n(s)$, is called the ***osculating plane*** at s

- We say that $s \in I$ is a *singular point* of order 1 if $\alpha''(s) = 0$.

- Define the vector $b(s) = t(s) \wedge n(s)$ as the ***binormal vector***. It is the normal vector of the $t - n$ plane and it is orthogonal to $(t(s), n(s))$.

- The differential of binormal vector $b'(s)$ characterize the ***strength*** *of the curve* to ***pull away from the plane*** where it currently lies. Its length $|b'(s)|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s.

- $b'(s)$ is ***parallel*** to $n(s)$ and is computed as $b'(s) = \tau(s)\, n(s)$ [do Carmo Valero, 1976]. (Some book use $b'(s) = -\tau(s)\, n(s)$.)

  Definition Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0$, $s \in I$. The number $\tau(s)$ defined by $b'(s) = \tau(s)\, n(s)$ is called the ***torsion*** of $\alpha(s)$ at s.

  If $\tau \equiv 0$, then the *curve will lies* ***entirely in a plane*** and vise versa. Note that $k(s) \neq 0$ is essential for above argument to hold.

  In contrast to the curvature, the torsion may be either positive or negative. The **sign** of torsion is related to the ***orientation*** of the curve relative to the ***osculating plane***.

- Both $k(s)$ and $\tau(s)$ are ***invariant*** to change of orientation.

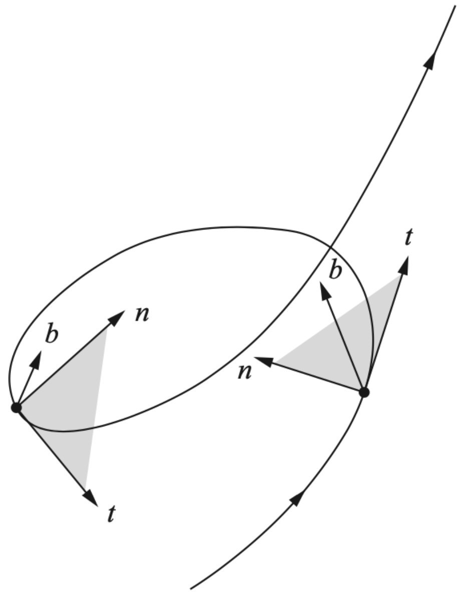


Figure 2: The osculating plane spanned by tangent vector $t(s) = \alpha'(s)$ and normal vector $n(s)$ with binormal vector $b(s)$ as its normal vector [do Carmo Valero, 1976]

- The three orthonormal vectors $(t(s), n(s), b(s))$ **form a basis** that uniquely characterizes the local behavior of a curve, and it is called the ***Frenet trihedron*** at s. The curvature k and the torsion τ will reveal information of curve α in the neighborhood of s.
- Given $\tau(s)$ and $k(s)$, the curve at s can be reparameterized via the trihedron $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$.
- The plane spanned by (t, n) is called ***osculating plane***. The plane spanned by (n, b) is called ***normal plane*** and the plane spanned by (t, b) is called ***rectifying plane***.
- The Frenet trihedron (t, n, b) at s can be computed via the *system of differential equations* as

$$\begin{aligned} t' &= k\, n \\ n' &= -k\, t - \tau\, b \\ b' &= \tau\, n \end{aligned} \tag{4}$$

called ***Frenet formula*** [do Carmo Valero, 1976], where $k(s) > 0$ and $\tau(s)$ are the curvature and torsion of a regular parameterized curve, respectively.

From theorem 1.1, we see that the curvature and torsion function determine a parameterized regular curve **up to a rigid transformation**. It is thus called *the fundamental theorem of the local theory of curves.*

- **Theorem 1.1** *(**The fundamental theorem of the local theory of curves**) [do Carmo Valero, 1976]*
*Given differentiable functions $k(s) > 0$ and $\tau(s)$ for $s \in I$, there exists a regular parameterized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature and $\tau(s)$ is the torsion. Moreover, any other curve $\hat{\alpha}$ satisfying the conditions above differ from α by a **rigid transformation** as $\hat{\alpha} = \rho \circ \alpha + c$ for ρ **an orthogonal transformation** and c **a translation vector**.*

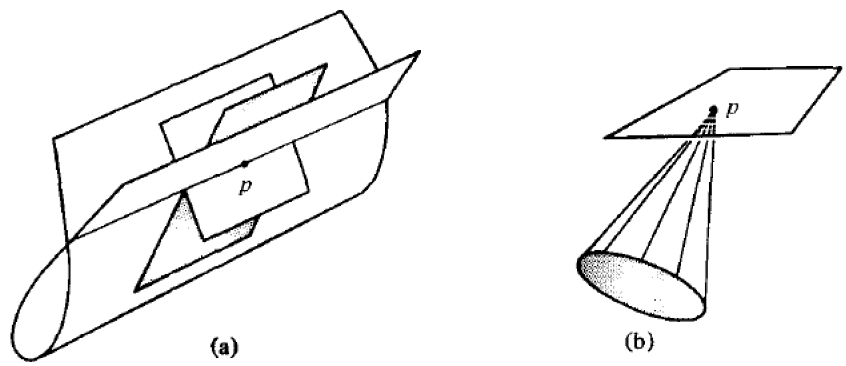


Figure 3: The situation avoided in the definition of regularity. (a) When the parameterization is not one-to-one, then the self-intersection of the surfaces will happen; (b) When the differential is not one-to-one at p, thus the tangent plane is not defined uniquely.

2 Regular Surfaces

2.1 Surfaces

- **Definition** A subset $\mathcal{S} \subset \mathbb{R}^3$ is a ***regular surface*** [do Carmo Valero, 1976], if for any $p \in \mathcal{S}$, there exists a neighborhood $V \subset \mathbb{R}^3$ and a map $\boldsymbol{x} : U \subset \boldsymbol{R}^2 \to V \cap \mathcal{S}$ of an open subset $U \subset \boldsymbol{R}^2$ onto $V \cap \mathcal{S} \subset \mathbb{R}^3$ such that

  1. $\boldsymbol{x} : (u, v) \in U \to (x(u, v), y(u, v), z(u, v))$ has ***differentials*** in U with all orders.
  2. $\boldsymbol{x}$ is a ***homemorphism***, i.e. $\boldsymbol{x}$ is a ***continuous bijection*** with ***continuous inverse*** $\boldsymbol{x}^{-1} : W \supset V \cap \mathcal{S} \to \mathbb{R}^2$.
  3. (***The regularity condition.***) For any $q \in U$, the differential $d\boldsymbol{x}_q$ is one-to-one, i.e. ***injective***.

  Let $q = (u_0, v_0)$. The vector e_1 is tangent to the curve $u \to (u, v_0)$ whose image under $\boldsymbol{x}$ is the curve

  $$u \to (x(u, v_0), y(u, v_0), z(u, v_0)).$$

  This image curve (called the *coordinate curve* $v = v_0$) lies on $\mathcal{S}$ and has at $\boldsymbol{x}(q)$ the tangent vector

  $$\frac{\partial \boldsymbol{x}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right).$$

  where the derivatives are computed at (u_0, v_0) and a vector is indicated by its components in the basis $\{f_1, f_2, f_3\}$. By the definition of differential

  $$d\boldsymbol{x}_q(e_1) \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = \frac{\partial \boldsymbol{x}}{\partial u}$$

  Similarly, for coordinate curve $u = u_0$,

  $$d\boldsymbol{x}_q(e_2) \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = \frac{\partial \boldsymbol{x}}{\partial v}$$

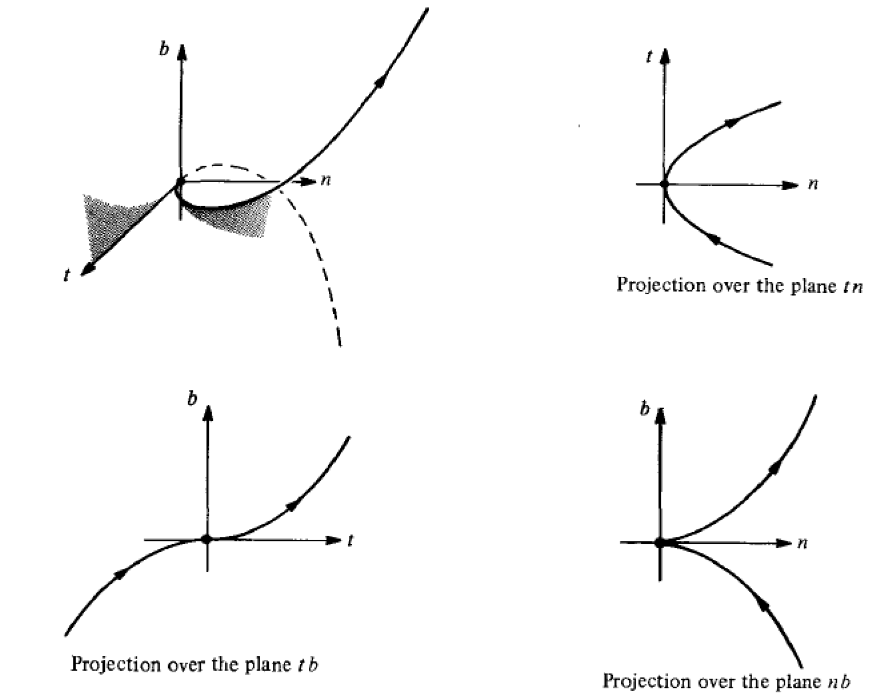


Figure 4: The local behavior of general regular curve. At the osculating plane, it is like a parabola. At the normal plane, it is like a cubic function.

Thus

$$dx_q \equiv \frac{\partial(x,y,z)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

. The condition 3 requires that $\frac{\partial(x,y,z)}{\partial(u,v)}$ has full column rank.

- **Definition** The map $\boldsymbol{x} : U \subset \mathbb{R}^2 \to V \cap \mathcal{S}$ is called a ***parameterization*** of the surface (at p). Its inverse $\boldsymbol{x}^{-1} : W \supset V \cap \mathcal{S} \to U$ is called a ***coordinate system***. We may write $\boldsymbol{x}^{-1} = (u, v)$, where u, v are smooth function on W and are called ***coordinate functions*** (*local coordinate* of surface at p as $p = (u, v)$). The neighborhood $V \cap \mathcal{S}$ of p is called the *coordinate neighborhood* of p in $\mathcal{S}$.

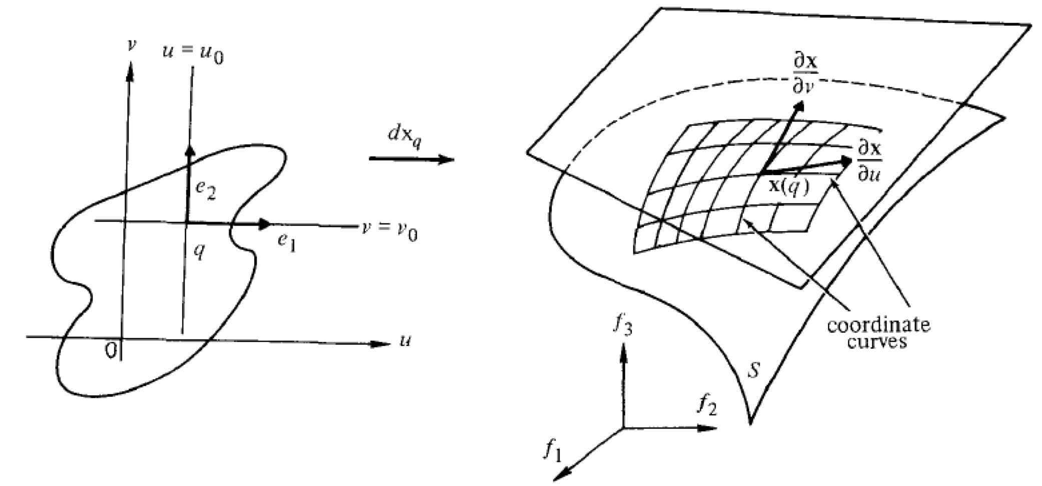
- **Theorem 2.1** *If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, $(u, v, f(u, v))$ is a regular surface in $\mathbb{R}^3$ for $(u, v) \in U$.*

- **Definition** Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, a point $p \in U$ is a ***critical point*** of F if dF_p is not surjective, i.e. $\frac{\partial F}{\partial(\xi_1,\ldots,\xi_m)} = \mathbf{0}$. The image of critical point is a ***critical value***. The value r that is not a critical value is called ***regular value*** of F. Note $dF_q \neq 0$ for all $q \in F^{-1}(r)$.

- **Definition** For p in a regular surface $\mathcal{S}$ and let one associated parameterization $\boldsymbol{x}$ that is *smooth* with *one-to-one differential*, then $\boldsymbol{x}$ is a ***homeomorphism***.

  $f : V \subset \mathcal{S} \to \mathbb{R}$ is differentiable at $p \in V \Leftrightarrow f \circ \boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\boldsymbol{x}^{-1}(p)$.

- **Theorem 2.2** *If $F : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^3$, and*



$r \in F(U)$ is a regular value of F, then the pre-image of F at r, $F^{-1}(r)$ is a regular surface in $\mathbb{R}^3$.

- **Proposition 2.3** *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface and $p \in \mathcal{S}$. Then there exists a neighborhood V of p in $\mathcal{S}$ such that V is the graph of a differentiable function which has one of the following three forms:*

$$z = f(x, y), \quad y = g(x, z), \quad x = h(y, z).$$

- A map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ is a diffeomorphism if ϕ is a smooth map and ϕ^{-1} is smooth as well.

2.2 Change of Parameters; Differentiable Functions on Surface

- **Proposition 2.4** *(**Change of Parameters**).*
*Let p be a point of a regular surface $\mathcal{S}$, and let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathcal{S}$, $y : V \subset \mathbb{R}^2 \to \mathcal{S}$ be two parametrizations of $\mathcal{S}$ such that $p \in \boldsymbol{x}(U) \cap \boldsymbol{y}(V) = W$. Then the "change of coordinates" $h = \boldsymbol{x}^{-1} \circ \boldsymbol{y} : \boldsymbol{y}^{-1}(W) \to \boldsymbol{x}^{-1}(W)$ is a **diffeomorphism**; that is, h is differentiable and has a differentiable inverse h^{-1}*

- In other words, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are given by

$$\boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$
$$\boldsymbol{y}(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta), z(\xi, \eta)), \quad (\xi, \eta) \in V,$$

then **the change of coordinates** h, given by

$$u = u(\xi, \eta), v = v(\xi, \eta), \quad (\xi, \eta) \in \boldsymbol{y}^{-1}(W),$$

has the property that the functions u and v have ***continuous*** partial derivatives *of all orders*, and the map h can be *inverted*, yielding

$$\xi = \xi(u, v), \eta = \eta(u, v), (u, v) \in \boldsymbol{x}^{-1}(W),$$

where the functions ξ and η also have partial derivatives of all orders. Since

$$\frac{\partial(u, v)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(u, v)} = 1$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.

- **Definition** Let $f : V \subset \mathcal{S} \to \mathbb{R}$ be a function defined in an open subset V of a regular surface $\mathcal{S}$. Then f is said to be ***differentiable*** at $p \in V$ if, for some ***parametrization*** $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathcal{S}$ with $p \in \boldsymbol{x}(U) \subset V$, the composition $f \circ \boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\boldsymbol{x}^{-1}(p)$. f is differentiable in V if it is differentiable at all points of V.

  It follows immediately from the last proposition that the definition given **does not depend on the choice of the parametrization $\boldsymbol{x}$**. In fact, if $\boldsymbol{y} : V \subset \mathbb{R}^2 \to \mathcal{S}$ is another parametrization with $p \in \boldsymbol{y}(V)$, and if $h = \boldsymbol{x}^{-1} \circ \boldsymbol{y}$, then $f \circ \boldsymbol{y} = f \circ \boldsymbol{x} \circ h$ is also differentiable, whence the asserted independence.

- We shall frequently make the notational abuse of indicating f and $f \circ \boldsymbol{x}$ by the same symbol $f(u, v)$, and say that $f(u, v)$ is the expression of f in the system of coordinates $\boldsymbol{x}$. This is equivalent to identifying $\boldsymbol{x}(U)$ with U and thinking of (u, v), indifferently, as a point of U and as a point of $\boldsymbol{x}(U)$ with coordinates (u, v). From now on, abuses of language of this type will be used without further comment.

- The definition of differentiability can be easily extended to mappings between surfaces.

  Definition A *continuous map* $\varphi : V_1 \subset \mathcal{S}_1 \to \mathcal{S}_2$ of an open set V_1 of a regular surface $\mathcal{S}_1$ to a regular surface $\mathcal{S}_2$ is said to be ***differentiable*** at $p \in V_1$ if, given parametrizations $\boldsymbol{x}_1 : U_1 \subset \mathbb{R}^2 \to \mathcal{S}_1$, $\boldsymbol{x}_2 : U_2 \subset \mathbb{R}^2 \to \mathcal{S}_2$, with $p \in \boldsymbol{x}_1(U)$ and $\varphi(\boldsymbol{x}_1(U_1)) \subset \boldsymbol{x}_2(U_2)$, the map $\boldsymbol{x}^{-1} \circ \varphi \circ \boldsymbol{x}_1 : U_1 \to U_2$ is differentiable at $q = \boldsymbol{x}^{-1}(p)$.

  In other words, φ is differentiable if when expressed in *local coordinates* as $\varphi(u_1, v_1) = (\varphi_1(u_1, v_1), \varphi_2(u_1, v_1))$ the functions φ_1 and φ_2 have *continuous* partial derivatives of all orders.

- We should mention that the natural notion of *equivalence* associated with differentiability is the notion of ***diffeomorphism***.

  Two regular surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are ***diffeomorphic*** if there exists a *differentiable* map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ with a ***differentiable inverse*** $\varphi^{-1} : \mathcal{S}_2 \to \mathcal{S}_1$.Such a φ is called a *diffeomorphism* from $\mathcal{S}_1$ to $\mathcal{S}_2$.

- **Definition** A ***parametrized surface*** $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a **differentiable map $\boldsymbol{x}$** from an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The set $\boldsymbol{x}(U) \subset \mathbb{R}^3$ is called the ***trace*** of $\boldsymbol{x}$. $\boldsymbol{x}$ is *regular* if the differential $d\boldsymbol{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is **one-to-one** for all $q \in U$ (i.e., the vectors $\frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}$ are linearly independent for all $q \in U$). A point $p \in U$ where $d\boldsymbol{x}_p$ is not one-to-one is called a *singular point* of $\boldsymbol{x}$.

2.3 The Tangent Plane and Differential of a Map

- The ***tangent vector*** to a *regular surface* $\mathcal{S}$ at p is the tangent vector $\alpha'(0)$ of a differentiable parameterized curve $\alpha : I = (-\epsilon, \epsilon) \to \mathcal{S}$ on $\mathcal{S}$ with $\alpha(0) = p$.

- The ***tangent plane*** to $\mathcal{S}$ at p consists of **all tangent vector** $\alpha'(0)$ for all differentiable parameterized curve α on $\mathcal{S}$ that pass through $p \in \mathcal{S}$. Denote the tangent space at $p \in \mathcal{S}$ as T_pS

- By proposition 2.5, the tangent space at T_pS has basis $(\frac{\partial \boldsymbol{x}}{\partial u}(p), \frac{\partial \boldsymbol{x}}{\partial v}(p)) \equiv (\frac{\partial}{\partial u}(p), \frac{\partial}{\partial v}(p))$ [Amari and Nagaoka, 2007]. The tangent space T_pS does not depend on the parameterization.

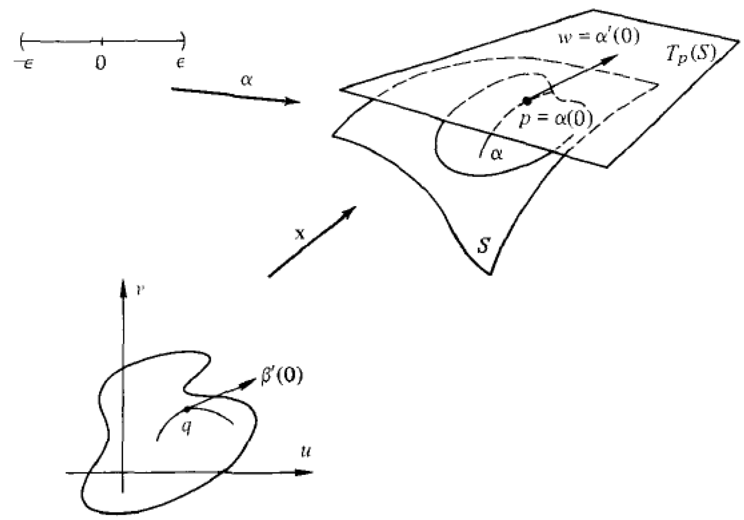


Figure 5: The tangent plane as the subspace of tangent vector of embedded curves. Find the coordinate of the tangent vector in tangent space.

- **Definition** The ***differential*** of a map $\varphi : V \subset \mathcal{S}_1 \to \mathcal{S}_2$ at $p \in \mathcal{S}_1$ is a linear map $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$, where $d\varphi_p(\boldsymbol{w}) = \beta'(0)$ for $\boldsymbol{w} \in T_pS_1$ with the curve on $\mathcal{S}_2$ as $\beta = \varphi \circ \alpha$ and $\alpha : (-\epsilon, \epsilon) \to V$ is the curve on $\mathcal{S}_1$.

- **Proposition 2.5** *Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathcal{S}$ be a parameterization of a regular surface $\mathcal{S}$ and let $q \in U$. The **tangent plane** to $\mathcal{S}$ at $\boldsymbol{x}(q)$ is given as*

$$d\boldsymbol{x}_q\left(\mathbb{R}^2\right) \subset \mathbb{R}^3$$

as a 2−dimensional linear subspace.

  By the above proposition, the plane $d\boldsymbol{x}_q\left(\mathbb{R}^2\right)$, which passes through $\boldsymbol{x}(q) = p$, does not depend on the parametrization $\boldsymbol{x}$.

- (***Tangent vector via basis***)
  For $\alpha'(0) \equiv \boldsymbol{w} \in T_pS$, for some $\alpha = \boldsymbol{x} \circ \beta$, where $\beta : (-\epsilon, \epsilon) \to U$ by $\beta(t) = (u(t), v(t))$, with $\beta(0) = q = \boldsymbol{x}^{-1}(p)$. Then

$$\alpha'(0) = \frac{d}{dt}(\boldsymbol{x} \circ \beta)(0) = \frac{d}{dt}\boldsymbol{x}(u(t), v(t))(0) \tag{5}$$
$$= \boldsymbol{x}_u u'(0) + \boldsymbol{x}_v v'(0) \tag{6}$$

  Thus under the basis $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ of T_pS, the coordinate of $\boldsymbol{w}$ in T_pS is $(u'(0), v'(0))$, and $\boldsymbol{w}$ is the velocity of the curve α is represented as $(u(t), v(t))$ in parameterization $\boldsymbol{x}$ at $t = 0$.

- (***Differential of map via basis***)
  If $\boldsymbol{w} = (u'(0), v'(0))$ in $T_p(S_1)$, and $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v))$, with $\alpha(t) = (u(t), v(t))$, then the tangent of β at $\varphi(p)$ is given via the differential of map of $\boldsymbol{w}$ at p is given in its own coordinates as

$$\beta'(0) = d\varphi_p(\boldsymbol{w}) = \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial u} & \dfrac{\partial \varphi_1}{\partial v} \\ \dfrac{\partial \varphi_2}{\partial u} & \dfrac{\partial \varphi_2}{\partial v} \end{bmatrix} \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix} \tag{7}$$

Thus $d\varphi_p$ as a linear mapping under coordinates $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ in T_pS and $(\boldsymbol{x}'_{u'}, \boldsymbol{x}'_{v'})$ in T_pS is given as the matrix $\begin{bmatrix} \frac{\partial\varphi_1}{\partial u} & \frac{\partial\varphi_1}{\partial v} \\ \frac{\partial\varphi_2}{\partial u} & \frac{\partial\varphi_2}{\partial v} \end{bmatrix}$.

- **Definition** A map $\varphi : U \subset \mathcal{S}_1 \to \mathcal{S}_2$ is a ***local diffeomorphism*** at $p \in U$ if there exists a neighborhood $V \subset U$ of p such that ϕ restricted on V is a diffeomorphism onto an open subset $\varphi(V) \subset \mathcal{S}_2$.

- **Theorem 2.6** *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two regular surfaces and $\varphi : U \subset \mathcal{S}_1 \to \mathcal{S}_2$ is a differentiable mapping of an open subset $U \subset \mathcal{S}_1$ such that the differential $d\varphi_p$ of φ at p is an isomorphism, then φ is a* ***local diffeomorphism*** *at p.*

- **Definition** The (unit) vectors that are normal to the tangent plane at p is called *the (unit)* ***normal vectors*** at p, denoted as $N(p)$. It can be defined by the rule

$$N(p) = \frac{\boldsymbol{x}_u \wedge \boldsymbol{x}_v}{|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|}(p)$$

- The angle between two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ at the intersecting point p is defined as the angle btw two tangent plane at p or the angle btw the normal vectors at p.

2.4 The First Fundamental Form and Area

- The *inner product* $\langle \cdot\, , \cdot \rangle$ on the tangent space T_pS is induced from $\mathbb{R}^3$.

- **Definition** The ***first fundamental form*** of a regular surface $\mathcal{S} \subset \mathbb{R}^3$ at $p \in \mathcal{S}$ is defined as a **quadratic form**, $I_p : T_pS \to \mathbb{R}$ given by

$$I_p(\boldsymbol{w}) = \langle w\, , \, w \rangle_p = \|w\|_2^2 \geq 0 \;\; \boldsymbol{w} \in T_pS. \tag{8}$$

- For orthogonal basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$, the first fundamental form is the ***Pythagorean theorem*** in $\mathcal{S}$.

- (***The first fundamental form via basis***)
Under the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ associated with $\boldsymbol{x}(u, v)$ at p, the first fundamental form can be formulated explicitly. Since $\boldsymbol{w} = \alpha'(0)$ for $\alpha : (-\epsilon, \epsilon) \to \mathcal{S}$ with $\alpha(t) = (u(t), v(t))$ and $p = \alpha(0) = \boldsymbol{x}(u(0), v(0))$, thus

$$\begin{aligned} I_p(\alpha'(0)) &= \left\langle \boldsymbol{x}_u u'(0) + \boldsymbol{x}_v v'(0)\, , \, \boldsymbol{x}_u u'(0) + \boldsymbol{x}_v v'(0) \right\rangle \\ &= \langle \boldsymbol{x}_u\, , \, \boldsymbol{x}_u \rangle \left(u'(0)\right)^2 + 2\, \langle \boldsymbol{x}_u\, , \, \boldsymbol{x}_v \rangle \left(u'(0)v'(0)\right) + \langle \boldsymbol{x}_v\, , \, \boldsymbol{x}_v \rangle \left(v'(0)\right)^2 \\ &= E\, \left(u'(0)\right)^2 + 2\, F\, \left(u'(0)v'(0)\right) + G\, \left(v'(0)\right)^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} E(u(0), v(0)) &= \langle \boldsymbol{x}_u\, , \, \boldsymbol{x}_u \rangle_p \\ F(u(0), v(0)) &= \langle \boldsymbol{x}_u\, , \, \boldsymbol{x}_v \rangle_p \\ G(u(0), v(0)) &= \langle \boldsymbol{x}_v\, , \, \boldsymbol{x}_v \rangle_p \end{aligned} \tag{10}$$

are ***coefficients of the first fundamental form*** in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. Note that $p = \boldsymbol{x}(u, v)$ runs in the coordinate neighborhood, the quantities $E(u, v), F(u, v), G(u, v)$ are differentiable function on U.

- Also, we can compute the angle btw two parameterized regular curve $\alpha(t)$ and $\beta(t)$ on $\mathcal{S}$ that intersects at $t = t_0$ as

$$\cos(\theta) = \frac{\langle \alpha'(t_0)\,,\, \beta'(t_0)\rangle}{\|\alpha'(t_0)\|_2\;\|\beta'(t_0)\|_2}.$$

Then the angle ϕ btw two coordinate curves of a parameterization $\boldsymbol{x}(u,v)$ is given by

$$\cos(\phi) = \frac{\langle \boldsymbol{x}_u\,,\, \boldsymbol{x}_v\rangle}{\|\boldsymbol{x}_u\|_2\;\|\boldsymbol{x}_v\|_2} = \frac{F}{\sqrt{E\,G}}. \tag{11}$$

It follows that the coordinate curves of a parametrization are *orthogonal* if and only if $F(u,v) = 0$ for all (u,v). Such a parametrization is called an ***orthogonal parametrization***.

- The **matrix** of first fundamental form is given as

$$\boldsymbol{J} \equiv \begin{bmatrix} \langle \boldsymbol{x}_u\,,\, \boldsymbol{x}_u\rangle_p & \langle \boldsymbol{x}_u\,,\, \boldsymbol{x}_v\rangle_p \\ \langle \boldsymbol{x}_v\,,\, \boldsymbol{x}_u\rangle_p & \langle \boldsymbol{x}_v\,,\, \boldsymbol{x}_v\rangle_p \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{12}$$

and for $\boldsymbol{w} = \alpha'(0) = (u'(0), v'(0))$,

$$I_p(\boldsymbol{w}) = \boldsymbol{w}^T\,\boldsymbol{J}\,\boldsymbol{w}$$

- Given the first fundamental form $I(\alpha'(t))$ on T_pS, we can evaluate the the **arc length** without using its coordinate in $\mathbb{R}^3$

$$\begin{aligned} s &= \int_0^t \sqrt{I(\alpha'(t))}dt \\ &= \int_0^t \sqrt{E(u'(t))^2 + 2\,F\,(u'(t)v'(t)) + G\,(v'(t))^2}dt \\ &= \int_0^t \sqrt{\frac{\partial \boldsymbol{\alpha}}{\partial t}^T \boldsymbol{J}\,\frac{\partial \boldsymbol{\alpha}}{\partial t}}dt \end{aligned} \tag{13}$$

- Another metric question that can be treated by the first fundamental form is the computation (or definition) of the *area* of a bounded region of a regular surface $\mathcal{S}$.

Definition A ***(regular) domain*** of $\mathcal{S}$ is an ***open*** and ***connected*** subset of $\mathcal{S}$ such that its ***boundary*** is the image in $\mathcal{S}$ of a circle by a *differentiable homeomorphism* which is regular (that is, its *differential* is nonzero) except at a finite number of points. A ***region*** of $\mathcal{S}$ is the union of a domain with its boundary. A region of $\mathcal{S} \subset \mathbb{R}^3$ is ***bounded*** if it is contained in some ball of $\mathbb{R}^3$.

- **Definition** Let $\mathcal{R} \subset \mathcal{S}$ be a *bounded region* of a regular surface contained in the coordinate neighborhood of the parametrization $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathcal{S}$. The positive number

$$\int\int_Q |\boldsymbol{x}_u \wedge \boldsymbol{x}_v|\,dudv, \quad Q = \boldsymbol{x}^{-1}(\mathcal{R}) \tag{14}$$

is called the ***area*** of $\mathcal{R}$. Here $|\boldsymbol{x}_u \wedge \boldsymbol{x}_v| = \sqrt{EG - F^2}$.

3 Examples and exercises

3.1 Curves

3.2 Surfaces

References

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