Lecture 17: The Levi-Civita Connection

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Contents

1 The Tangential Connection Revisited

- Suppose $\gamma : I \to M \subseteq \mathbb{R}^n$ is a smooth curve. Then γ can be regarded as either a smooth curve in M or a smooth curve in $\mathbb{R}^n$, and ***a smooth vector field*** V ***along*** γ that takes its values in TM can be regarded as either a vector field along γ in M or a vector field along γ in $\mathbb{R}^n$. Let $\bar{D}_t(V)$ denote *the covariant derivative of* V *along* γ (***as a curve in*** $\mathbb{R}^n$) with respect to ***the Euclidean connection*** $\bar{\nabla}$, and let $D_t^{\top}(V)$ denote its covariant derivative along γ (***as a curve in*** M) with respect to ***the tangential connection*** $\nabla^{\top}$.

$$\nabla_X^{\top} Y = \pi^{\top}\left(\bar{\nabla}_{\widetilde{X}}\widetilde{Y}\right)$$

  where $X, Y \in \mathfrak{X}(M)$ and $\widetilde{X}, \widetilde{Y}$ are smooth extension of X, Y to a neighborhood of M.

- Then we have the following proposition

  Proposition 1.1 *Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold, $\gamma : I \to M$ a smooth curve in M , and V a smooth vector field along γ that takes its values in TM . Then for each $t \in I$,*

$$D_t^{\top}(V(t)) = \pi^{\top}\left(\bar{D}_t(V(t))\right).$$

- **Corollary 1.2** *Suppose $M \subseteq \mathbb{R}^n$ is an embedded submanifold. A smooth curve $\gamma : I \to M$ is a* ***geodesic*** *with respect to the* ***tangential connection*** *on M if and only if its* ***ordinary acceleration*** *$\gamma''(t)$ is* ***orthogonal*** *to $T_{\gamma(t)}M$ for all $t \in I$.*

- **Remark** Let $(\mathbb{R}^{r,s}, \bar{q}^{r,s})$ be the *pseudo-Euclidean space of signature* (r, s). If $M \subseteq \mathbb{R}^{r,s}$ is an *embedded Riemannian or pseudo-Riemannian submanifold*, then for each $p \in M$, the tangent space $T_p\mathbb{R}^{r,s}$ decomposes as a ***direct sum*** $T_pM \oplus N_pM$, where $N_pM = (T_pM)^{\perp}$ is the *orthogonal complement* of T_pM with respect to $\bar{q}^{r,s}$. We let $\pi^{\top} : T_p\mathbb{R}^{r,s} \to T_pM$ be the $\bar{q}^{r,s}$*-orthogonal projection*, and define ***the tangential connection*** $\nabla^{\top}$ on M by

$$\nabla_X^{\top} Y = \pi^{\top}\left(\bar{\nabla}_{\widetilde{X}}\widetilde{Y}\right)$$

  , where $\widetilde{X}, \widetilde{Y}$ are smooth extensions of X and Y to a neighborhood of M, and $\bar{\nabla}$ is the ordinary Euclidean connection on $\mathbb{R}^{r,s}$. This is a well-defined connection on M.

2 Connections on Abstract Riemannian Manifolds

2.1 Metric Connections

- **Remark** For Euclidean connection, we have the following equation from the product rule

$$Z(\langle X\,,\,Y\rangle) = \bar{\nabla}_Z\langle X\,,\,Y\rangle = \left\langle \bar{\nabla}_Z X\,,\,Y\right\rangle + \left\langle X\,,\,\bar{\nabla}_Z Y\right\rangle, \quad \forall X, Y, Z \in \mathfrak{X}(\mathbb{R}^n).$$

It can be verified easily by computing in terms of the standard basis. For $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ and $Z = Z^i \frac{\partial}{\partial x^i}$

$$
\begin{aligned}
\langle X\,,\,Y\rangle &= \sum_{j=1}^{n} X^j\,Y^j \\
\text{LHS } &= Z\left(\langle X\,,\,Y\rangle\right) = Z^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} X^j, Y^j\right) \\
&= \sum_{j=1}^{n} Z^i \frac{\partial X^j}{\partial x^i} Y^j + \sum_{j=1}^{n} Z^i \frac{\partial Y_j}{\partial x^i} X^j \\
\text{RHS } &= \left\langle Z(X^j)\frac{\partial}{\partial x^j}\,,\,Y\right\rangle + \left\langle X\,,\,Z(Y^j)\frac{\partial}{\partial x^j}\right\rangle \\
&= \left\langle Z^i \frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j}\,,\,Y\right\rangle + \left\langle X\,,\,Z^i \frac{\partial Y_j}{\partial x^i}\frac{\partial}{\partial x^j}\right\rangle \\
&= \sum_{j=1}^{n} Z^i \frac{\partial X^j}{\partial x^i} Y^j + \sum_{j=1}^{n} Z^i \frac{\partial Y_j}{\partial x^i} X^j \\
\Rightarrow \text{LHS } &= \text{RHS} \quad \blacksquare
\end{aligned}
$$

- **Definition** Let g be a *Riemannian or pseudo-Riemannian metric* on a smooth manifold M (with or without boundary). A connection ∇ on TM is said to be ***compatible with g***, or to be ***a metric connection***, if it satisfies the following *product rule* for all $X, Y, Z \in \mathfrak{X}(M)$:

$$
\begin{aligned}
\nabla_Z \langle X\,,\,Y\rangle &= \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle \qquad (1)\\
\Leftrightarrow Z\,\langle X\,,\,Y\rangle &= \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle
\end{aligned}
$$

- **Remark** More understanding of the equation (1):

  1. $\nabla_Z \langle X\,,\,Y\rangle = \nabla_Z(g(X,Y))$. Note that $\langle X\,,\,Y\rangle = g(X,Y) \in \mathcal{C}^\infty(M)$ is *a smooth function* since g is a ***covariant 2-tensor***. Thus $\nabla_Z \langle X\,,\,Y\rangle = Z\,\langle X\,,\,Y\rangle \in \mathcal{C}^\infty(M)$ since for $f \in \mathcal{C}^\infty(M)$, the directional derivative of f along direction of Z, $\nabla_Z f = Zf$. Intuitively, it measures ***the directional derivatives of the angle*** *between two vector fields X and Y along the direction of vector field Z.*

  2. $\langle \nabla_Z X\,,\,Y\rangle = g\left(\nabla_Z X, Y\right) \in \mathcal{C}^\infty(M)$ measures ***the angle between $\nabla_Z X$ and Y***; similarly, $\langle X\,,\,\nabla_Z Y\rangle = g\left(X, \nabla_Z Y\right)$ measures ***the angle between X and $\nabla_Z Y$***. In both terms, $\nabla_Z X$ is the ***directional derivative*** X along Z, which is *the difference* between X and *its infinitesimal parallel transport* along Z.

  3. The equation (1) states that "***the directional derivatives of the angle*** *between two vector fields X and Y along the direction of vector field Z* ***is equal to*** the ***sum of angles*** of *the* ***directional derivative of one vector field*** *along direction of Z* ***with respect to the other vector field***".

- **Proposition 2.1** ***(Characterizations of Metric Connections)***.
  Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be a connection on TM. The following conditions are ***equivalent:***

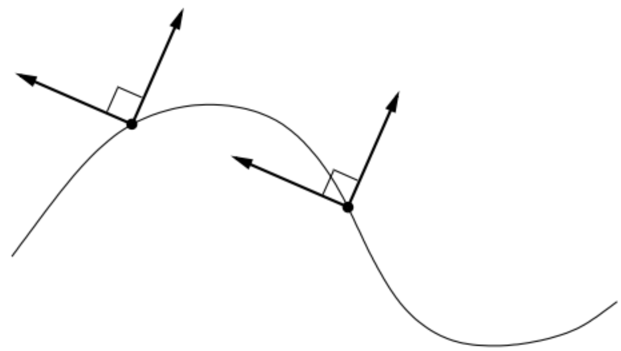

Figure 1: A parallel orthonormal frame [Lee, 2018]

1. *∇ is **compatible** with g: $\nabla_Z \langle X\,,\,Y\rangle = \langle \nabla_Z X\,,\,Y\rangle + \langle X\,,\,\nabla_Z Y\rangle$.*
2. *g is **parallel with respect to** ∇: $\nabla g \equiv 0$.*
3. *In terms of any smooth local frame (E_i), the **connection coefficients** of ∇ satisfy*

$$\Gamma^l_{k,i} g_{l,j} + \Gamma^l_{k,j} g_{i,l} = E_k(g_{i,j}). \tag{2}$$

4. *If V, W are smooth vector fields along any smooth curve γ, then*

$$\frac{d}{dt}\langle V\,,\,W\rangle = \langle D_t V\,,\,W\rangle + \langle V\,,\,D_t W\rangle\,. \tag{3}$$

5. *If V, W are **parallel** vector fields **along a smooth curve** γ in M, then $\langle V\,,\,W\rangle$ is **constant** along γ.*
6. *Given any smooth curve γ in M, every **parallel transport map** along γ is a **linear isometry**.*
7. *Given any smooth curve γ in M, every **orthonormal basis** at a point of γ can be **extended** to a **parallel orthonormal frame** along γ (Fig. 1)*

- **Remark** From the proposition statement 5,6,7 above, we see that ***the metric connection*** ∇ that is compatible with g *defines the **parallel transport operation*** that maintains *the **angle** between two vector fields **unchanged**.* In other word, ***the parallel transport defined by the metric connection*** is an ***isometry*** on the manifold.
- **Corollary 2.2** *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold with or without boundary, ∇ is a **metric connection** on M, and $\gamma : I \to M$ is a smooth curve.*
  1. *$|\gamma'(t)|$ is **constant** if and only if $D_t\gamma'(t)$ is **orthogonal** to $\gamma'(t)$ for all $t \in I$.*
  2. *If γ is a **geodesic**, then $|\gamma'(t)|$ is **constant**.*
- **Proposition 2.3** *If M is an embedded Riemannian or pseudo-Riemannian **submanifold** of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$, the **tangential connection** on M is **compatible** with the **induced** Riemannian or pseudo-Riemannian **metric**.*

2.2 Symmetric Connections

- **Remark** For $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$, the Lie bracket between X and Y is

$$
\begin{aligned}
[X, Y] &= X(Y^i)\frac{\partial}{\partial x^i} - Y(X^i)\frac{\partial}{\partial x^i} \\
\text{since } \bar{\nabla}_X Y &= X(Y^i)\frac{\partial}{\partial x^i}, \\
\bar{\nabla}_Y X &= Y(X^i)\frac{\partial}{\partial x^i} \\
\Rightarrow [X, Y] &= \bar{\nabla}_X Y - \bar{\nabla}_Y X
\end{aligned}
$$

- **Definition** A ***connection*** ∇ on the tangent bundle of a smooth manifold M is ***symmetric*** if

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

  where $[X, Y]$ is the Lie bracket of two vector fields.

- **Definition** The ***torsion tensor*** of the *connection* ∇ is a ***smooth*** $(1, 2)$***-tensor field*** $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

- **Remark** Thus, a connection ∇ is ***symmetric*** if and only if its torsion ***vanishes*** identically $\tau \equiv 0$.

- **Remark** (***Coordinate Representation of Symmetric Connections***)
  A connection is ***symmetric*** if and only if *its* ***connection coefficients*** in *every coordinate frame* is ***symmetric*** in ***lower two indices*** That is, $\Gamma^k_{i,j} = \Gamma^k_{j,i}$ for all i, j.

- **Proposition 2.4** *If M is an embedded (pseudo-)Riemannian submanifold of a (pseudo-)Euclidean space, then the* ***tangential connection*** *on M is* ***symmetric****.*

2.3 The Levi-Civita Connection

- **Remark** The last two propositions show that if we wish to single out a connection on each Riemannian or pseudo-Riemannian manifold in such a way that it matches the tangential connection when the manifold is presented as an embedded submanifold of $\mathbb{R}^n$ or $\mathbb{R}^{r,s}$ with the induced metric, then we must require at least that ***the connection be compatible with the metric*** and ***symmetric***.

- **Theorem 2.5** *(**Fundamental Theorem of Riemannian Geometry**). [Lee, 2018]*
  Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary). There exists a ***unique connection*** *∇ on TM that* ***is compatible with*** *g and* ***symmetric****. It is called the* ***Levi-Civita connection of*** *g (or also, when g is* ***positive definite****, the* ***Riemannian connection****).*

- **Corollary 2.6** *(**Formulas for the Levi-Civita Connection**). [Lee, 2018]*
  Let (M, g) be a Riemannian or pseudo-Riemannian manifold (with or without boundary), and let ∇ be its ***Levi-Civita connection****.*

1. ***(In Terms of Vector Fields)****: If X, Y, Z are smooth vector fields on M, then*

$$\langle \nabla_X Y\,,\, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\right) \tag{4}$$

(This is known as ***Koszul's formula****.)*

2. ***(In Coordinates)****: In any smooth coordinate chart for M, the* ***coefficients of the Levi-Civita connection*** *are given by*

$$\Gamma^k_{i,j} = \frac{1}{2} g^{k,l}\left(\frac{\partial}{\partial x^i} g_{j,l} + \frac{\partial}{\partial x^j} g_{i,l} - \frac{\partial}{\partial x^l} g_{i,j}\right). \tag{5}$$

3. ***(In A Local Frame)****: Let (E_i) be a smooth* ***local frame*** *on an open subset $U \subseteq M$, and let $c^k_{i,j} : U \to \mathbb{R}$ be the n^3 smooth functions defined by*

$$[E_i, E_j] = c^k_{i,j}\, E_k \tag{6}$$

Then ***the coefficients of the Levi-Civita connection*** *in this frame are*

$$\Gamma^k_{i,j} = \frac{1}{2} g^{k,l}\left(E_i\, g_{j,l} + E_j\, g_{i,l} - E_l\, g_{i,j} - g_{j,m} c^m_{i,l} - g_{l,m} c^m_{j,i} + g_{i,m} c^m_{l,j}\right). \tag{7}$$

4. ***(In A Local Orthonormal Frame)****: If g is Riemannian, (E_i) is a smooth* ***local orthonormal frame****, and the functions $c^k_{i,j}$ are defined by (6), then*

$$\Gamma^k_{i,j} = \frac{1}{2}\left(c^k_{i,j} - c^j_{i,k} - c^i_{j,k}\right) \tag{8}$$

- **Remark** On every Riemannian or pseudo-Riemannian manifold, we will always use the Levi-Civita connection from now on without further comment.

- **Remark** *Geodesics* with respect to the Levi-Civita connection are called ***Riemannian geodesics***, or simply "*geodesics* as long as there is no risk of confusion.

- **Remark** The ***connection coefficients*** $\Gamma^k_{i,j}$ *of* ***the Levi-Civita connection*** in coordinates, given by (5), are called ***the Christoffel symbols of*** g.

- **Proposition 2.7** *1. The Levi-Civita connection on a (pseudo-)****Euclidean space*** *is equal to the* ***Euclidean connection****.*

  2. Suppose M is an ***embedded*** *(pseudo-)Riemannian* ***submanifold*** *of a (pseudo-)Euclidean space. Then the Levi-Civita connection on M is equal to* ***the tangential connection*** *$\nabla^{\top}$.*

- **Proposition 2.8** ***(Naturality of the Levi-Civita Connection)****. [Lee, 2018]*
  Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and let ∇ denote the Levi-Civita connection of g and $\widetilde{\nabla}$ that of $\widetilde{g}$. If $\varphi : M \to \widetilde{M}$ is an isometry, then $\varphi^*\widetilde{g} = \nabla$.

  Remark An ***isometry*** φ between the manifold M and $\widetilde{M}$ can be used to define ***the pullback connection*** in M from *the Levi-Civita connection* $\widetilde{M}$. Recall that for general connections, we can only define a pullback connection if φ is a *diffeomorphism*.

- **Corollary 2.9** ***(Naturality of Geodesics).***
  Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds with or without boundary, and $\varphi : M \to \widetilde{M}$ is a ***local isometry****. If γ is a* ***geodesic*** *in M, then $\varphi \circ \gamma$ is a* ***geodesic*** *in M.*

  Remark An ***isometry*** φ between the manifold M and $\widetilde{M}$ maps a ∇-geodesic in M to a $\widetilde{\nabla}$ -geodesic in $\widetilde{M}$ for both *Levi-Civita Connections* ∇ and $\widetilde{\nabla}$.

- **Proposition 2.10** *Suppose (M, g) is a Riemannian or pseudo-Riemannian manifold. The connection induced on each* ***tensor bundle*** *by the Levi-Civita connection is* ***compatible*** *with* ***the induced inner product on tensors****, in the sense that $X \langle F\,,\, G\rangle = \langle \nabla_X F\,,\, G\rangle + \langle F\,,\, \nabla_X G\rangle$ for every vector field X and every pair of smooth tensor fields $F, G \in T^{(k,l)}TM$.*

- **Proposition 2.11** ***(Volume Preseving under Parallel Transport)***
  Let (M, g) be an oriented Riemannian manifold. The Riemannian volume form of g is ***parallel*** *with respect to the Levi-Civita connection.*

- **Proposition 2.12** *The* ***musical isomorphisms commute with the total covariant derivative operator****: if F is any smooth tensor field with a* ***contravariant*** *i-th index position, and $\flat$ represents the operation of lowering the i-th index, then*

  $$\nabla(F^{\flat}) = (\nabla F)^{\flat} \tag{9}$$

  Similarly, if G has a ***covariant*** *i-th position and $\sharp$ denotes raising the i-th index, then*

  $$\nabla(G^{\sharp}) = (\nabla G)^{\sharp} \tag{10}$$

3 The Exponential Map

- **Remark** It is shown above that each initial point $p \in M$ and each initial velocity vector $v \in T_pM$ determine a ***unique maximal geodesic*** γ_v. How do geodesics change if we vary the initial point or the initial velocity ? The dependence of geodesics on the initial data is encoded in a map from the tangent bundle into the manifold, called ***the exponential map***, whose properties are fundamental to the further study of Riemannian geometry.

- **Lemma 3.1** ***(Rescaling Lemma).***
  For every $p \in M$, $v \in T_pM$, and $c, t \in \mathbb{R}$,

  $$\gamma_{c\,v}(t) = \gamma_v(c\,t) \tag{11}$$

  whenever either side is defined.

- **Definition** Define a subset $\mathcal{E} \subseteq TM$, ***the domain of the exponential map***, by

  $$\mathcal{E} = \{v \in TM : \gamma_v \text{ is defined on an interval containing } [0,1]\}\,,$$

  and then define ***the exponential map*** $\exp : \mathcal{E} \to M$ by

  $$\exp(v) = \gamma_v(1)$$

  For each $p \in M$, the ***restricted exponential map*** at p, denoted by $\exp_p$, is the restriction of exp to the set $\mathcal{E}_p = \mathcal{E} \cap T_pM$.

- **Remark** The ***exponential map*** *of a* ***Riemannian manifold*** should not be confused with *the* ***exponential map*** *of a* ***Lie group***. The two are closely related for ***bi-invariant metrics***, but in general they need not be.

- **Remark** Recall that a subset of a vector space V is said to be ***star-shaped*** *with respect to a point* $x \in S$ if for every $y \in S$, the *line segment* from x to y is contained in S.

- **Proposition 3.2** *(**Properties of the Exponential Map**). [Lee, 2018]*
Let (M, g) be a Riemannian or pseudo-Riemannian manifold, and let $\exp : \mathcal{E} \to M$ be its exponential map.

  1. *$\mathcal{E}$ is an **open** subset of TM containing the image of the **zero section**, and each set $\mathcal{E}_p \subseteq T_pM$ is **star-shaped with respect to** 0.*

  2. *For each $v \in TM$, the **geodesic** γ_v is given by*
$$\gamma_v(t) = \exp(v\,t) \tag{12}$$
for all t such that either side is defined.

  3. *The exponential map is **smooth**.*

  4. *For each point $p \in M$, the **differential** $d(\exp_p)_0 : T_0(T_pM) \simeq T_pM \to T_pM$ is **the identity map** of T_pM, under the usual identification of $T_0(T_pM)$ with T_pM.*

- **Remark** The ***geodesic equation*** *under the initial boundary condition* can be written in the following form:
$$\dot{x}^k(t) = v^k(t) \tag{13}$$
$$\dot{v}^k(t) = -v^i(t)v^j(t)\,\Gamma_{i,j}^k(x(t)) \tag{14}$$
Treating $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ as coordinates on $U \times \mathbb{R}^n$, we can recognize (14) as the equations for the ***flow*** of ***the vector field*** $G \in \mathfrak{X}(U \times \mathbb{R}^n)$ given by
$$G_{(x,v)} = v^k \frac{\partial}{\partial x^k}\Big|_{(x,v)} - v^i v^j\,\Gamma_{i,j}^k(x)\frac{\partial}{\partial v^k}\Big|_{(x,v)}. \tag{15}$$
The importance of G stems from the fact that it actually defines ***a global vector field on the total space of*** TM, called ***the geodesic vector field***. It can be verified that the components of G under a change of coordinates *take the same form* in *every coordinate chart.*

  Note that G acts on a function $f \in \mathcal{C}^\infty(U \times \mathbb{R}^n)$ as
$$Gf(p, v) = \frac{d}{dt}\Big|_{t=0} f(\gamma_v(t), \gamma_v'(t)). \tag{16}$$

- **Proposition 3.3** *(**Naturality of the Exponential Map**).*
*Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a **local isometry**. Then for every $p \in M$, the following diagram commutes:*
$$\begin{array}{ccc} \mathcal{E}_p & \xrightarrow{d\varphi_p} & \widetilde{\mathcal{E}}_{\varphi(p)} \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{\varphi(p)}} \\ M & \xrightarrow{\varphi} & \widetilde{M}, \end{array}$$

where $\mathcal{E}_p \subseteq T_pM$ and $\widetilde{\mathcal{E}}_{\varphi(p)} \subseteq T_{\varphi(p)}\widetilde{M}$ are the domains of the restricted exponential maps $\exp_p$ *(with respect to g) and* $\exp_{\varphi(p)}$ *(with respect to $\tilde{g}$), respectively.*

- **Remark** Under isometry transformation, the exponential map ***remain unchanged*** from TM to $T\widetilde{M}$.

- The following proposition shows that ***local isometries*** of connected manifolds are ***completely determined*** by their ***values*** and ***differentials*** at a single point.

  Proposition 3.4 *Let (M, g) and $\widetilde{M}, \widetilde{g})$ be Riemannian or pseudo-Riemannian manifolds, with M* ***connected****. Suppose $\varphi, \psi : M \to \widetilde{M}$ are* ***local isometries*** *such that for some point $p \in M$, we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

- **Definition** A Riemannian or pseudo-Riemannian manifold (M, g) is said to be ***geodesically complete*** if *every maximal geodesic* is defined for ***all*** $t \in \mathbb{R}$, or equivalently if *the domain of the exponential map is all of TM.*

4 Normal Neighborhoods and Normal Coordinates

- **Definition** Let (M, g) be a Riemannian or pseudo-Riemannian manifold of dimension n (without boundary). Recall that for every $p \in M$, the restricted exponential map $\exp_p$ maps the open subset $\mathcal{E}_p \subseteq T_pM$ smoothly into M. Because $d(\exp_p)_0$ is ***invertible***, the *inverse function theorem* guarantees that there exist a neighborhood V of the origin in T_pM and a neighborhood U of p in M such that $\exp_p : V \to U$ is a ***diffeomorphism***.

  A neighborhood U of $p \in M$ that is the ***diffeomorphic image*** under $\exp_p$ of *a star-shaped neighborhood of* $0 \in T_pM$ is called ***a normal neighborhood*** of p.

- **Definition** Every *orthonormal basis* (b_i) for T_pM determines ***a basis isomorphism*** $B : \mathbb{R}^n \to T_pM$ by $B(x^1, \ldots, x^n) = x^i\, b_i$. If $U = \exp_p(V)$ is ***a normal neighborhood*** of p, we can combine this *isomorphism* with *the exponential map* to get ***a smooth coordinate map*** $\varphi : B^{-1} \circ (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$:

$$\begin{tikzcd} T_pM \arrow[r, "B^{-1}"] & \mathbb{R}^n \\ U. \arrow[u, "(\exp_p|_V)^{-1}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

  Such coordinates are called ***(Riemannian or pseudo-Riemannian) normal coordinates*** centered at p.

- **Proposition 4.1** ***(Uniqueness of Normal Coordinates)****. [Lee, 2018]*
  Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, p a point of M, and U a ***normal neighborhood*** *of p. For every* ***normal coordinate chart*** *on U centered at p, the coordinate basis is* ***orthonormal*** *at p; and for every orthonormal basis (b_i) for T_pM, there is a* ***unique normal coordinate chart*** *(x^i) on U such that $\frac{\partial}{\partial x^i}|_p = b_i$ for $i = 1, \ldots, n$. In the Riemannian case, any two normal coordinate charts (x^i) and $(\widetilde{x}^j)$ are related by*

$$\widetilde{x}^j = A^j_i\, x^i \tag{17}$$

  for some (constant) matrix $A^j_i \in \mathcal{O}(n)$.

- **Proposition 4.2** ***(Properties of Normal Coordinates)***. *[Lee, 2018]*
  Let (M, g) be a Riemannian or pseudo-Riemannian n-manifold, and let $(U, (x^i))$ be any ***normal coordinate chart*** *centered at $p \in M$.*

  1. *The coordinates of p are $(0, \ldots, 0)$.*
  2. *The* ***components*** *of the* ***metric*** *at p are $g_{i,j} = \delta_{i,j}$ if g is* ***Riemannian****, and $g_{i,j} = \pm\delta_{i,j}$ otherwise.*
  3. *For every $v = v^i \frac{\partial}{\partial x^i}|_p \in T_pM$, the* ***geodesic*** *$\gamma_v$ starting at p with* ***initial velocity*** *v is represented in* ***normal coordinates*** *by the line*
     $$\gamma_v(t) = (tv^1, \ldots, tv^n), \tag{18}$$
     as long as t is in some interval I containing 0 such that $\gamma_v(I) \subseteq U$.
  4. *The* ***Christoffel symbols*** *in these coordinates* ***vanish*** *at p.*
  5. *All of the* ***first partial derivatives*** *of $g_{i,j}$ in these coordinates* ***vanish*** *at p.*

- **Remark** The *geodesics* ***starting at*** p and lying in a ***normal neighborhood*** of p are called ***radial geodesics***. (But be warned that geodesics that do not pass through p do not in general have a simple form in normal coordinates.)

5 Tubular Neighborhoods and Fermi Coordinates

5.1 Tubular Neighborhoods

5.2 Fermi Coordinates

6 Geodesics of the Model Spaces

6.1 Euclidean Space

6.2 Spheres

6.3 Hyperbolic Spaces

References

John M Lee. *Introduction to Riemannian manifolds*, volume 176. Springer, 2018.