Lecture 6: Martingale

Tianpei Xie

Feb.2nd., 2023

Contents

1 Conditional Expectation

- **Definition** (***Conditional Expectation***) [Resnick, 2013]
  Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra. Suppose $X \in L^1(\Omega, \mathscr{F}, \mathcal{P})$. There exists a function $\mathbb{E}[X|\mathscr{G}]$, called the ***conditional expectation*** of X ***with respect to*** $\mathscr{G}$ such that

  1. $\mathbb{E}[X|\mathscr{G}]$ is ***$\mathscr{G}$-measureable*** and ***integrable*** *with respect to* $\mathcal{P}$.
  2. $\mathbb{E}[X|\mathscr{G}]$ satisfies ***the functional equation***:

  $$\int_G X d\mathcal{P} = \int_G \mathbb{E}[X|\mathscr{G}]\, d\mathcal{P}, \quad \forall\, G \in \mathscr{G}.$$

- **Remark** To *prove the **existence*** of such a random variable,

  1. consider first the case of ***nonnegative*** X. Define a measure ν on $\mathscr{G}$ by

     $$\nu(G) = \int_G X d\mathcal{P} = \int_\Omega X \mathbb{1}_G \, d\mathcal{P}.$$

     This measure is *finite* because X is *integrable*, and it is ***absolutely continuous*** with respect to $\mathcal{P}$. By the *Lebesgue-Radon-Nikodym Theorem*, there is a $\mathscr{G}$-measurable function f such that

     $$\nu(G) = \int_G f d\mathcal{P}.$$

     This f has properties (1) and (2).

  2. If X is *not necessarily nonnegative*, $\mathbb{E}[X_+|\mathscr{G}] - \mathbb{E}[X_-|\mathscr{G}]$ clearly has the required properties.

- **Remark** As $\mathscr{G}$ increases, condition (1) becomes ***weaker*** and condition (2) becomes ***stronger***.

- **Remark** Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space, with $\mathscr{G} \subset \mathscr{F}$ a sub-σ-algebra, define

  $$\mathcal{P}[A|\mathscr{G}] = \mathbb{E}[\mathbb{1}_A|\mathscr{G}]$$

  for all $A \in \mathscr{F}$.

- **Remark** By definition, the conditional expectation is a ***Radon-Nikodym derivative*** of $d\nu|_{\mathscr{G}} = X d\mathcal{P}|_{\mathscr{G}}$ w.r.t. $d\mathcal{P}|_{\mathscr{G}}$ within $\mathscr{G}$.

  $$\mathbb{E}[X|\mathscr{G}] := \frac{X d\mathcal{P}|_{\mathscr{G}}}{d\mathcal{P}|_{\mathscr{G}}} = X|_{\mathscr{G}}.$$

  Thus $\mathbb{E}[X|\mathscr{G}]$ *is the **projection** of X on **sub σ-algebra** $\mathscr{G}$*.

- **Remark** (***Conditioning on Random Variables***)
  By definition, conditioning on random variables $(X_t, t \in T)$ on $(\Omega, \mathscr{B})$ can be expressed as

  $$\mathbb{E}[X|X_t, t \in T] \equiv \mathbb{E}[X|\sigma(X_t, t \in T)],$$

  where $\sigma(X_t, t \in T)$ is the σ-algebra generated by the cylinder set

  $$C_n[A] \equiv \{\omega : (X_t(\omega), 1 \le t \le n) \in A\} \in \mathscr{B}, \quad A \in \mathcal{B}(\mathbb{R}^n), \forall\, n$$

- **Remark** (***σ*-Algebra Generated by Partition of Sample Space**)
  As above, assume that the sub σ-algebra $\mathscr{G}$ is generated by a ***partition*** $B_1, B_2, \ldots$ of Ω, then for $X \in L^1(\Omega, \mathscr{F}, \mathcal{P})$,

$$\mathbb{E}\left[X|B_i\right] = \int X d\mathcal{P}(X|B_i) = \int_{B_i} X d\mathcal{P}/\mathcal{P}(B_i)$$

  where $\mathcal{P}(X|B_i)$ is *the conditional probability* defined in previous section. If $\mathcal{P}(B_i) = 0$, then $\mathbb{E}\left[X|B_i\right] = 0$. We claim that

  1. $$\mathbb{E}\left[X|\mathscr{G}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[X|B_i\right] \mathbb{1}_{B_i}, \quad a.s.$$

  2. For any $A \in \mathscr{F}$,
  $$\mathcal{P}(A|\mathscr{G}) = \sum_{i=1}^{\infty} \mathcal{P}(A|B_i) \mathbb{1}_{B_i}, \quad a.s.$$

- **Remark** Both $P[A|\mathscr{F}]$ and $\mathbb{E}\left[X|\mathscr{F}\right]$ are *random variables* from $\Omega \to \mathbb{R}$. Formally speaking,

$$\begin{aligned}
P\left[(X,Y) \in A|\sigma(X)\right]_{\omega} &\equiv P\left[(X(\omega), Y) \in A\right] \\
&= P\left\{\omega' : (X(\omega), Y(\omega')) \in A\right\} \\
&\equiv f(X(\omega)) \\
&= \nu|_{\sigma(X)}(A) \\
\mathbb{E}\left[(X,Y)|\sigma(X)\right]_{\omega} &= \lim_{\substack{m(A)\to 0 \\ \omega \in A \in \sigma(X)}} \frac{P\left\{\omega' : (X(\omega), Y(\omega')) \in A\right\}}{m(A)}
\end{aligned}$$

  It is the expected value of X for someone who knows for each $E \in \mathscr{F}$, whether or not $\omega \in E$, which E itself remains unknown.

- **Proposition 1.1** ***(Properties of Conditional Expectation)*** *[Resnick, 2013]*
  Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space and $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra. Suppose $X, Y \in L^1(\Omega, \mathscr{F}, \mathcal{P})$ and $\alpha, \beta \in \mathbb{R}$.

  1. ***(Linearity)**: $\mathbb{E}\left[\alpha X + \beta Y|\mathscr{G}\right] = \alpha\mathbb{E}\left[X|\mathscr{G}\right] + \beta\mathbb{E}\left[Y|\mathscr{G}\right]$;*

  2. ***(Projection)**: If X is **$\mathscr{G}$-measurable**, then $\mathbb{E}\left[X|\mathscr{G}\right] = X$ almost surely.*

  3. ***(Conditioning on Indiscrete σ-Algebra)**:*
  $$\mathbb{E}\left[X|\left\{\emptyset, \Omega\right\}\right] = \mathbb{E}\left[X\right].$$

  4. ***(Monotonicity)**: If $X \geq 0$, then $\mathbb{E}\left[X|\mathscr{G}\right] \geq 0$ almost surely. Similarly, if $X \geq Y$, then $\mathbb{E}\left[X|\mathscr{G}\right] \geq \mathbb{E}\left[Y|\mathscr{G}\right]$ almost surely.*

  5. ***(Modulus Inequality)**:*
  $$|\,\mathbb{E}\left[X|\mathscr{G}\right]| \leq \mathbb{E}\left[|X|\;|\mathscr{G}\right].$$

6. *(**Monotone Convergence Theorem**): If $\{X_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathscr{F}, \mathcal{P})$, $0 \le X_1 \le X_2 \le \dots$ is a **monotone sequence** of **non-negative** random variables and $X_n \to X$ then*

$$\lim_{n\to\infty} \mathbb{E}\left[X_n|\mathscr{G}\right] = \mathbb{E}\left[\lim_{n\to\infty} X_n \Big| \mathscr{G}\right] = \mathbb{E}\left[X|\mathscr{G}\right].$$

7. *(**Fatou Lemma**): If $\{X_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathscr{F}, \mathcal{P})$, and $X_n \ge 0$ for all n, then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n \Big| \mathscr{G}\right] \le \liminf_{n\to\infty} \mathbb{E}\left[X_n|\mathscr{G}\right]$$

8. *(**Dominated Convergence Theorem**): If $\{X_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathscr{F}, \mathcal{P})$ and $|X_n| \le Z$, where $Z \in L^1(\Omega, \mathscr{F}, \mathcal{P})$ is a random variable, $X_n \to X$ almost surely, then*

$$\lim_{n\to\infty} \mathbb{E}\left[X_n|\mathscr{G}\right] = \mathbb{E}\left[\lim_{n\to\infty} X_n \Big| \mathscr{G}\right] = \mathbb{E}\left[X|\mathscr{G}\right], \quad a.s.$$

9. *(**Product Rule**): If Y is $\mathscr{G}$-measurable,*

$$\mathbb{E}\left[X\,Y|\mathscr{G}\right] = Y\,\mathbb{E}\left[X|\mathscr{G}\right], \quad a.s.$$

10. *(**Smoothing**): For $\mathscr{F}_1 \subset \mathscr{F}_0 \subset \mathscr{F}$,*

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[X|\mathscr{F}_0\right]|\mathscr{F}_1\right] &= \mathbb{E}\left[X|\mathscr{F}_1\right]\\ \mathbb{E}\left[\mathbb{E}\left[X|\mathscr{F}_1\right]|\mathscr{F}_0\right] &= \mathbb{E}\left[X|\mathscr{F}_1\right].\end{aligned}$$

*Note that $\mathbb{E}\left[X|\mathscr{F}_1\right]$ is **smoother** than $\mathbb{E}\left[X|\mathscr{F}_0\right]$. Moreover*

$$\mathbb{E}\left[X\right] = \mathbb{E}\left[X|\{\emptyset,\Omega\}\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathscr{F}_0\right]|\{\emptyset,\Omega\}\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathscr{F}_0\right]\right].$$

11. *(**The Conditional Jensen's Inequality**). Let ϕ be a **convex** function, $\phi(X) \in L^1(\Omega, \mathscr{F}, \mathcal{P})$. Then almost surely*

$$\phi\left(\mathbb{E}\left[X|\mathscr{G}\right]\right) \le \mathbb{E}\left[\phi(X)|\mathscr{G}\right]$$

2 Martingale

References

Sidney I Resnick. *A probability path.* Springer Science & Business Media, 2013.