Lecture 6: Littlewood's Principles

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Contents

1 Principles

- **Proposition 1.1** *(**Littlewood's Three Principles**)[Royden and Fitzpatrick, 1988, Tao, 2011]:*

  1. *Every (**measurable**) set is nearly a **finite sum** of **intervals**;*
  2. *Every (**absolutely integrable**) function is nearly **continuous**; and*
  3. *Every **(pointwise) convergent sequence of functions** is nearly **uniformly convergent***

- **Remark** ***The Littlewood's 1st and 2nd principles are shown only for Euclidean space*** $\mathbb{R}^d$, since it relies on such concepts as "*elementary set*" or "*continuous function*" defined for *an abstract measure space*. In other word, ***the necessary condition*** these two principles to hold is that *the measure space* $(X, \mathscr{F})$ *is a **topological space** with **Borel σ-algebra*** $\mathscr{B}$ *included in* $\mathscr{F}$.

  The ***Littlewood's 3rd principles***, i.e., *the Egorov's theorem*, holds for a ***finite measure space*** $(X, \mathscr{F}, \mu)$ *in which* $\mu(X) < \infty$. There are cases in which $m(X) = \infty$ and the theorem does not hold. [Tao, 2011]

1.1 Every Measurable Set is Nearly a Finite Sum of Intervals

- The ***First Principle***:

  Proposition 1.2 *(Criteria for measurability [Tao, 2011])*
  The followings are equivalent:

  1. *E is Lebesgue measureable.*
  2. *(**Outer approximation by open**) For every $\epsilon > 0$, one can contain E in an open set U with $m^*(U - E) \le \epsilon$.*
  3. *(**Almost open**) For every $\epsilon > 0$, one can find an open set U such that $m^*(U \Delta E) \le \epsilon$, where $U \Delta E = (U - E) \cup (E - U) = U \cup E - U \cap E$ is the symmetric difference. (In other words, E differs from an open set by a set of outer measure at most ϵ.)*

- **Remark** For E finite Lebesgue measureable, E differs from a ***bounded*** *open set* by a set of *arbitrarily small Lebesgue outer measure*. This bounded open set can be *decomposed* as a finite union of open cubes in $\mathbb{R}^d$. [Royden and Fitzpatrick, 1988].

1.2 Every Pointwise Convergent Sequence of Functions is Nearly Uniformly Convergent

- **Theorem 1.3** *(**Approximation of*** L^1 ***functions**).*
  Let $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$.

  1. *There exists an absolutely integrable simple function g such that*

  $$\|f - g\|_{L^1(\mathbb{R}^d)} \le \epsilon;$$

2. *There exists a* step function g *(, i.e. g is represented as a finite linear combination of indicator functions of boxes) such that* $\|f-g\|_{L^1(\mathbb{R}^d)} \leq \epsilon$;

3. *There exists a* ***continuous, compactly supported*** g *such that* $\|f-g\|_{L^1(\mathbb{R}^d)} \leq \epsilon$.

Proof:

– When f is unsigned, we see from the definition of the lower Lebesgue integral that there exists an unsigned simple function g such that $g \leq f$ and

$$\int_{\mathbb{R}^d} f(x)dx \leq \int_{\mathbb{R}^d} g(x)dx + \epsilon$$

i.e. $\|f-g\|_{L^1(\mathbb{R}^d)} \leq \epsilon$. For $f \in L^1(\mathbb{R}^d)$, we just choose g_+ for f_+ and g_- for f_- as above. Therefore $g = g_+ - g_-$ absolutely integrable simple function, and we have

$$\begin{aligned}
\|f-g\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |f(x)-g(x)|\,dx \\
&\leq \left| \int_{\mathbb{R}^d} f(x)dx - \int_{\mathbb{R}^d} g(x)dx \right| \\
&\leq \left| \int_{\mathbb{R}^d} f_+(x)dx - \int_{\mathbb{R}^d} g_+(x)dx \right| + \left| \int_{\mathbb{R}^d} f_-(x)dx - \int_{\mathbb{R}^d} g_-(x)dx \right| \\
&\leq 2\epsilon
\end{aligned}$$

– See that $\|f-h\|_{L^1(\mathbb{R}^d)} \leq \|f-g\|_{L^1(\mathbb{R}^d)} + \|g-h\|_{L^1(\mathbb{R}^d)}$, where g is the absolutely integrable simple function and h is step function. Thus we only need to show that $\|g-h\|_{L^1(\mathbb{R}^d)} \leq \epsilon$. Note that by triangle inequality and linearity of g, we just need to show this when $g = \mathbb{1}\{x \in E\}$ for some finite measureable set $E \subset \mathbb{R}^d$. Then there exists some box B, such that the set $E \subset B$ and $m^*(E \Delta B) < \epsilon$. So $\|g-h\|_{L^1(\mathbb{R}^d)} \leq \epsilon$ holds.

– Again by triangle inequality, we only need to show that $\|h-g\|_{L^1(\mathbb{R}^d)} \leq \epsilon$, where g is the continuous function, $h = \mathbb{1}\{x \in E\}$ is the step function. For E measureable in a normal space $\mathbb{R}^d$, we can find an open box $F \supseteq \overline{E}$ such that $m^*(F - E) \leq \epsilon$. Then following the well-known *Urysohn's lemma*, there exists a continuous function g on $\mathbb{R}^d$ such that $g(\overline{E}) = 1$ and $g(F^c) = 0$ and $0 \leq g(x) \leq 1$ otherwise. Therefore g is the continuous, compactly supported function and $g(x) \geq \mathbb{1}\{x \in E\} = h(x)$ with $\|h-g\|_{L^1(\mathbb{R}^d)} \leq \epsilon$. ∎

- **Definition** (***Locally uniform convergence***).
  A sequence of functions $f_n : \mathbb{R}^d \to \mathbb{C}$ converges *locally uniformly* to a limit $f : \mathbb{R}^d \to \mathbb{C}$ if, for every *bounded* subset E of $\mathbb{R}^d$, f_n converges *uniformly* to f on E. In other words, for every bounded $E \subset \mathbb{R}^d$ and any $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \epsilon$ for all $n \geq N$ and $x \in E$.

- **Remark** Recall the following convergence definitions:

  1. (***Pointwise convergence***)
     For every $x \in \mathbb{R}^d$, any $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \epsilon$ for all $n \geq N$.

  2. (***Pointwise almost everywhere convergence***)
     For *almost every* $x \in \mathbb{R}^d$, any $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \epsilon$ for all $n \geq N$.

3. (***Uniform convergence***)
   For any $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \le \epsilon$ for all $n \ge N$ and $x \in \mathbb{R}^d$.

- **Remark** Uniform convergence $\Rightarrow$ Locally Uniform convergence $\Rightarrow$ Pointwise convergence $\Rightarrow$ Pointwise almost everywhere convergence.

- The ***Third Principle***:

  Theorem 1.4 ***(Egorov's theorem).***
  Let $f_n : \mathbb{R}^d \to \mathbb{C}$ be a sequence of measurable functions that ***converge pointwise almost everywhere*** *to another function $f : \mathbb{R}^d \to \mathbb{C}$, and let $\epsilon > 0$. Then there exists a Lebesgue measurable set A of measure at most ϵ, such that f_n* ***converges locally uniformly*** *to f outside of A.*

  Proof: [Tao, 2011]
  By modifying f_n and f on a set of measure zero (that can be absorbed into A at the end of the argument) we may assume that f_n converges pointwise everywhere to f, thus for every $x \in \mathbb{R}^d$ and $m > 0$ there exists $N \ge 0$ such that $|f_n(x) - f(x)| \le 1/m,$ for all $n \ge N$.

  Denote $E_{N,m} \equiv \left\{x \in \mathbb{R}^d \,:\, |f_n(x) - f(x)| > 1/m; \text{ for some } n \ge N\right\}$. Then the above statement is equivalent to

$$\bigcap_{N=1}^{\infty} E_{N,m} = \emptyset$$

  for each $m \ge 1$. Note that the $E_{N,m}$ are Lebesgue measurable, and are decreasing in N. Applying downward monotone convergence, we see that for any radius $R > 0$,

$$\lim_{N \to \infty} m\left(E_{N,m} \cap B_R(0)\right) = 0$$

  (The restriction to the ball $B_R(0)$ is necessary, because the downward monotone convergence property only works when the sets involved have finite measure.) In particular, for any $m \ge 1$, we can find N_m such that

$$m\left(E_{N,m} \cap B_m(0)\right) \le \frac{\epsilon}{2^m}$$

  for any $N > N_m$. (See that now N_m does not depend on the point x.)

  Now let $A \equiv \bigcup_{m=1}^{\infty} E_{N_m,m} \cap B_m(0)$. Then A is Lebesgue measurable, and by countable subadditivity, we have

$$\begin{aligned} m(A) &\le \sum_{m=1}^{\infty} m\left(E_{N_m,m} \cap B_m(0)\right) \\ &\le \epsilon \end{aligned}$$

  By construction, we have

$$|f_n(x) - f(x)| \le 1/m,$$

  for all $m \ge 1$, all $x \in B_m(0)/A$ (i.e. $x \in \mathbb{R}^d/A$, $|x| \le m$,) and all $n \ge N_m$. In particular, we see for any ball $B_{m_0}(0)$ with an integer radius, f_n converges uniformly to f on $B_{m_0}(0)/A$. Since every bounded set is contained in such a ball, the claim follows. ∎

  Note that the exceptional set A in Egorov's theorem cannot be taken to have zero measure, at least if one uses the bounded-set definition of local uniform convergence as above.

1.3 Every Absolutely Integrable Function is Nearly Continuous

- The ***Second Principle***:

  Theorem 1.5 ***(Lusin's theorem).***
  *Let $f : \mathbb{R}^d \to \mathbb{C}$ be **absolutely integrable**, and let $\epsilon > 0$. Then there exists a Lebesgue measurable set $E \subset \mathbb{R}^d$ of measure at most ϵ such that the **restriction** of f to the **complementary** set $\mathbb{R}^d \setminus E$ is **continuous** on that set.*

  Proof: [Tao, 2011]
  By L^1 approximations, for any $n \geq 1$ one can find a *continuous, compactly supported* function f_n such that $\|f - f_n\|_{L^1(\mathbb{R}^d)} \leq \epsilon/4^n$. By Markov's inequality, that implies that $|f(x) - f_n(x)| \leq 1/2^{n-1}$ for all x outside of a Lebesgue measurable set A_n of measure at most $\epsilon/2^{n+1}$.

  Letting $A = \bigcup_{n=1}^{\infty} A_n$, we conclude that A is Lebesgue measurable with measure at most $\epsilon/2$, and f_n converges uniformly to f outside of A. But the uniform limit of continuous functions is continuous, and the same is true for local uniform limits (because continuity is itself a local property). We conclude that the restriction f to $\mathbb{R}^d \setminus E$ is continuous, as required. ∎

- **Remark** This theorem does not imply that the *unrestricted* function f is continuous on $\mathbb{R}^d \setminus E$. For instance, the absolutely integrable function $\mathbb{1}\{\mathbb{Q}\} : \mathbb{R} \to \mathbb{C}$ is nowhere continuous, so is certainly not continuous on $\mathbb{R} \setminus E$ for any E of finite measure; but on the other hand, if one deletes the measure zero set $E \equiv \mathbb{Q}$ from the reals, then the restriction of f to $\mathbb{R} \setminus E$ is identically zero and thus continuous. [Tao, 2011]

- **Remark** When dealing with unsigned measurable functions such as $f : \mathbb{R}^d \to [0, +\infty]$, then Lusin's theorem ***does not apply directly*** because f could be ***infinite*** on a set of positive measure, which is clearly in contradiction with the conclusion of Lusin's theorem (unless one allows the continuous function to also take values in the extended non-negative reals $[0, +\infty]$ with the extended topology). However, if one knows already that f ***is almost everywhere finite*** (which is for instance the case when f is absolutely integrable), then *Lusin's theorem applies* (since one can simply zero out f on the null set where it is infinite, and add that null set to the exceptional set of Lusin's theorem).

2 Examples

- **Example** [Tao, 2011]

  1. The Taylor partial sum e.g. $\sum_{k=0}^{k} \frac{x^k}{k!}$ is locally uniform convergent to $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ around bounded neighborhood of 0.

  2. The functions $x \mapsto x/n$ on $\mathbb{R}$ for $n = 1, 2, \ldots$ converge locally uniformly (and hence pointwise) to zero on $\mathbb{R}$, but do not converge uniformly.

  3. The functions $f_n(x) = \frac{1}{nx}\mathbb{1}\{x > 0\}$, for $n = 1, 2, \ldots$ (with the convention that $f_n(0) = 0$) converge pointwise everywhere to zero, but do not converge locally uniformly (not in a bounded set, it does locally converge if the convergence is w.r.t. open neighborhood).

- **Example** (*[Tao, 2011]* ***The non-zero measure set of non-convergent points***)
  Consider the moving bump example $f_n = \mathbb{1}\{[n, n+1]\}$ on $\mathbb{R}$, which "escapes to horizontal

infinity". This sequence converges pointwise (and locally uniformly) to the zero function $f = 0$. However, for any $0 < \epsilon < 1$ and any n, we have $|f_n(x) - f(x)| > \epsilon$ on a set of measure 1, namely on the interval $[n, n+1]$. Thus, if one wanted f_n to converge uniformly to f outside of a set A, then that set A has to contain a set of measure 1. In fact, the non-convergent set A must contain the intervals $[n, n+1]$ for all sufficiently large n and must therefore have infinite measure.

- **Exercise 2.1** *[Tao, 2011] Show that the hypothesis that f is absolutely integrable in Lusin's theorem can be relaxed to being* locally *absolutely integrable (i.e. absolutely integrable on every bounded set), and then relaxed further to that of being* measurable *(but still finite everywhere or almost everywhere). (To achieve the latter goal, one can replace f locally with a horizontal truncation $f\mathbb{1}\{|f| \leq n\}$; alternatively, one can replace f with a bounded variant, such as $\frac{f}{(1+|f|^2)^{1/2}}$.)*

- **Exercise 2.2** (**Littlewood-like Principles**). *[Tao, 2011]*
The following facts are not, strictly speaking, instances of any of Littlewood's three principles, but are in a similar spirit.

  1. *(**Absolutely integrable functions** almost have **bounded support**)*
  Let $f : \mathbb{R}^d \to \mathbb{C}$ be an absolutely integrable function, and let $\epsilon > 0$. Show that there exists a ball $B_R(0)$ outside of which f has an L^1 norm of at most ϵ, or in other words that $\int_{\mathbb{R}^d/B_R(0)} f(x)dx \leq \epsilon$.

  2. *(**Measurable functions** are almost **locally bounded**)*
  Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function supported on a set of finite measure, and let $\epsilon > 0$. Show that there exists a measurable set $E \subset \mathbb{R}^d$ of measure at most ϵ outside of which f is locally bounded, or in other words that for every $R > 0$ there exists $M < 1$ such that $|f(x)| \leq M$ for all $x \in B_R(0)/E$.

  Note: it is important in the second part of the exercise that f is known to be finite everywhere (or at least almost everywhere); the result would of course fail if f was, say, unsigned but took the value $+\infty$ on a set of positive measure.

- **Exercise 2.3** *[Tao, 2011]*
Show that a function $f : \mathbb{R}^d \to \mathbb{C}$ is measurable if and only if it is the pointwise almost everywhere limit of continuous functions $f_n : \mathbb{R}^d \to \mathbb{C}$. (Hint: if $f : \mathbb{R}^d \to \mathbb{C}$ is measurable and $n \geq 1$, show that there exists a continuous function $f_n : \mathbb{R}^d \to \mathbb{C}$ for which the set $\{x \in B_n(0) : |f(x) - f_n(x)| \geq 1/n\}$ has measure at most $1/2^n$. Use this (and Egorov's theorem) to give an alternate proof of Lusin's theorem for arbitrary measurable *functions.*

References

Halsey Lawrence Royden and Patrick Fitzpatrick. *Real analysis*, volume 198. Prentice Hall, Macmillan New York, 1988.

Terence Tao. *An introduction to measure theory*, volume 126. American Mathematical Soc., 2011.