Lecture 5: Metrization Theorems and Paracompactness

Tianpei Xie

Nov. 7th., 2022

Contents

1 Paracompactness

1.1 Local Finiteness

- **Definition** (***Local Finiteness***)
  Let X be a topological space. A *collection* $\mathscr{A}$ of *subsets* of X is said to be ***locally finite in*** X if every point of X has a *neighborhood* that ***intersects*** *only* ***finitely many elements*** of $\mathscr{A}$.

- **Remark** (***Understanding Locally Finite***)
  A locally finite collection of subsets in a topological space is ***evenly spread across the space***. In other word, there exists ***no cluster point*** $x \in X$ for these subsets so that *every neighborhood of* x *will intersect with infinitely many subsets in the collection.*

  Local finiteness describe *the* ***distribution*** *of the given collection of subsets* in X. We can think of $\mathscr{A}$ as the result of "***uniform sampling***" of subsets across the space.

- **Example** (***Locally Finite Collections in*** $\mathbb{R}$)
  The collection of intervals
  $$\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}$$
  is *locally finite* in the topological space $\mathbb{R}$.

  On the other hand, the collection
  $$\mathscr{B} = \{(0, 1/n) : n \in \mathbb{Z}\}$$
  has a cluster point $0 \in \mathbb{R}$ so it is not locally finite in $\mathbb{R}$. However, it is locally finite for $(0, 1)$.

- **Lemma 1.1** *(**Properties of Locally Finiteness**) [Munkres, 2000]*
  Let $\mathscr{A}$ be a locally finite collection of subsets of X. Then:

  1. *Any* ***subcollection*** *of $\mathscr{A}$ is locally finite.*
  2. *The collection $\mathscr{B} = \{\bar{A}\}_{A\in\mathscr{A}}$ of the* ***closures*** *of the elements of A is locally finite.*
  3. $\overline{\bigcup_{A\in\mathscr{A}} A} = \bigcup_{A\in\mathscr{A}} \bar{A}$.

  Proof: To prove (2), note that *any* ***open set*** U *that intersects the set $\bar{A}$ necessarily intersects* A. To prove (3), let Y denote the union of the elements of $\mathscr{A}$:
  $$Y = \bigcup_{A\in\mathscr{A}} A.$$
  In general, $\bigcup_{A\in\mathscr{A}} \bar{A} \subseteq \bar{Y}$; we prove the *reverse inclusion*, under the assumption of *local finiteness.*

  Let $x \in \bar{Y}$; let U be a neighborhood of x that *intersects only finitely many elements* of $\mathscr{A}$, say $A_1, \ldots, A_k$. We assert that x belongs to one of the sets $\bar{A}_1, \ldots, \bar{A}_k$, and hence belongs to $\bigcup_{A\in\mathscr{A}} \bar{A}$. For otherwise, the set $U \setminus \bigcup_{i=1}^{k} \bar{A}_i$ would be a neighborhood of x that ***intersects no element*** *of $\mathscr{A}$ and hence does not intersect Y*, contrary to the assumption that $x \in \bar{Y}$. ∎

- **Definition** (***Locally Finite Indexed Family***)
  The indexed family $\{A_\alpha\}_{\alpha\in J}$ is said to be ***locally finite indexed family in*** X if every $x \in X$ has a neighborhood that *intersects* A_α for only ***finitely many values*** of α.

- **Remark** $\{A_\alpha\}_{\alpha \in J}$ is a ***locally finite indexed family*** if and only if it is ***locally finite*** as *a collection of sets* and each *nonempty* subset A of X equals A_α for *at most finitely many values* of α.

- **Definition** (***Countably Local Finiteness***)
  A collection $\mathscr{B}$ of subsets of X is said to be ***countably locally finite*** if $\mathscr{B}$ can be written as ***the countable union*** of collections $\mathscr{B}_n$, each of which is ***locally finite***.

$$\mathscr{B} = \bigcup_{n \in \mathbb{Z}_+} \mathscr{B}_n$$

  Countably locally finite is also called ***σ-locally finite***.

- **Remark** Note that both *a* ***countable*** *collection* and *a* ***locally finite*** *collection* are ***countably locally finite***.

- **Remark** We can consider a ***countably locally finite*** collection as the result of ***superposition*** of ***countable layers*** of *uniform sampling* of subsets in a topological space.

- **Definition** (***Refinement of Collection***)
  Let $\mathscr{A}$ be a collection of subsets of the space X. A collection $\mathscr{B}$ of subsets of X is said to be a ***refinement of*** $\mathscr{A}$ (or is said to ***refine*** $\mathscr{A}$) if for each element B of $\mathscr{B}$, there is an element A of $\mathscr{A}$ ***containing*** B.

  If the elements of $\mathscr{B}$ are ***open sets***, we call $\mathscr{B}$ an ***open refinement of*** $\mathscr{A}$; if they are ***closed sets***, we call $\mathscr{B}$ a ***closed refinement***.

- **Exercise 1.2** *Let $\mathscr{A}$ be the following collection of subsets of $\mathbb{R}$:*

$$\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}.$$

  Which of the following collections refine $\mathscr{A}$?

$$\begin{aligned} \mathscr{B} &= \{(x, x+1) : x \in \mathbb{R}\}, \\ \mathscr{C} &= \{(n, n+3/2) : n \in \mathbb{Z}\}, \\ \mathscr{D} &= \{(x, x+3/2) : x \in \mathbb{R}\} \end{aligned}$$

  Solution: $\mathscr{B}$ is a refinement of $\mathscr{A}$. For each $x \in \mathbb{R}$, there exsits some $n \in \mathbb{Z}$ such that $n \le x < n+1$. Thus $n+1 \le x+1 < n+2$. So for every $(x, x+1)$ we can find corresponding n such that $(x, x+1) \subset (n, n+2)$.

  $\mathscr{C}$ is a refinement of $\mathscr{A}$. Obviously, for given n, $(n, n+3/2) \subset (n, n+2)$.

  $\mathscr{D}$ is not a refinement of $\mathscr{A}$. Choose $(\sqrt{3}, \sqrt{3}+3/2) \in \mathscr{D}$. Suppose $(\sqrt{3}, \sqrt{3}+3/2) \subseteq (n, n+2)$ for some n, i.e. $\sqrt{3} \ge n$ and $\sqrt{3}+3/2 \le n+2$ or $\sqrt{3} \le n+1/2$. This is not possible since the closet integer to $\sqrt{3}$ is $n = 1$, but $\sqrt{3} > 1.5$. ∎

- **Remark** (***Finer $\Rightarrow$ Smaller Subsets***)
  $\mathscr{B}$ is a ***refinement*** of $\mathscr{A}$ $\Rightarrow \forall B \in \mathscr{B}$, B is a subset of some element in $\mathscr{A}$.

  Note that there may exists some $A \in \mathscr{A}$ does not intersect with any $B \in \mathscr{B}$.

- **Theorem 1.3** *[Munkres, 2000]*
  Let X be a ***metrizable*** *space. If $\mathscr{A}$ is an open covering of X, then there is an* ***open covering*** *$\mathscr{E}$ of X* ***refining*** *$\mathscr{A}$ that is* ***countably locally finite****.*

- **Remark** For ***metrizable*** space X, *every **open covering** has a **countable locally finite refinement** that also **covers** X.*

1.2 Paracompactness

- **Definition** (***Compactness in terms of Refinement***)
  A space X is ***compact*** if every *open covering* $\mathscr{A}$ of X has *a **finite open refinement*** $\mathscr{B}$ that ***covers*** X.

- We generalize the definition of compactness by relaxing the finiteness to locally finiteness

  Definition (***Paracompactness***)
  A space X is ***paracompact*** if *every open covering* $\mathscr{A}$ of X has a ***locally finite open refinement*** $\mathscr{B}$ that ***covers*** X.

- **Remark** (***Compactness** vs. **Paracompactness***)
  *Paracompactness is a **generalization** of compactness*, i.e. *all compact space is paracompact.*

  Both compactness and paracompactness assert *the **existence** of an **open subcovering** with **some structure***. But ***the constraint on the structure*** is different:

  1. *Compactness* ***controls the cardinality*** of *subcovering*, i.e. to be ***finite***.
  2. *Paracompactness* ***controls the distribution*** of *subcovering*, i.e. to be ***evenly distributed*** *across space without cluster point* or to be ***locally finite***.

- **Example** ($\mathbb{R}^n$)
  The space $\mathbb{R}^n$ is ***paracompact***. Let $X = \mathbb{R}^n$. Let $\mathscr{A}$ be an *open covering* of X. Let $B_0 = \emptyset$, and for each *positive integer* m, let $B_m = B(0, m)$ denote *the open ball of* ***radius*** m ***centered at the origin***. Note that $B_m \subseteq B_{m+1}$ for all m and its closure $\bar{B}_m$ is a *compact subset* of $\mathbb{R}^n$.

  Given m, choose ***finitely many elements*** of $\mathscr{A}$ that ***cover*** $\bar{B}_m$ (since $\bar{B}_m$ is compact) and ***intersect*** *each one* with ***the open set*** $X \setminus \bar{B}_{m-1}$; let this *finite collection* of open sets be denoted $\mathscr{C}_m$. That is $\mathscr{C}_m = \{A_i \cap (X \setminus \bar{B}_{m-1}) : A_i \in \mathscr{A}, \bar{B}_m \subseteq \bigcup_i^k A_i, 1 \leq i \leq k\}$. Then the collection $\mathscr{C} = \bigcup_m \mathscr{C}_m$ is a *refinement* of $\mathscr{A}$.

  It is clearly *locally finite*, for the open set B_m intersects only *finitely many elements* of $\mathscr{C}$, namely those elements belonging to the *collection* $\mathscr{C}_1 \cup \ldots \cup \mathscr{C}_m$. Finally, $\mathscr{C}$ covers X. For, given x, let m be the *smallest integer* such that $x \in \bar{B}_m$. Then x belongs to an element of $\mathscr{C}_m$, by definition. ∎

- **Example** (***k-Dimensional Topological Manifold***)
  *Every k-dimensional topological manifold is **paracompact**.*

- **Theorem 1.4** *[Munkres, 2000]*
  *Every **paracompact Hausdorff** space X is **normal**.*

- **Proposition 1.5** ***(Paracompactness by Closed Subspace)*** *[Munkres, 2000]*
  *Every **closed** subspace of a paracompact space is paracompact*

- **Remark** ***A paracompact subspace*** of a *Hausdorff space* X ***need not be closed*** in X.

Indeed, the open interval $(0, 1)$ is *paracompact*, being *homeomorphic* to $\mathbb{R}$, but it is *not closed* in $\mathbb{R}$.

- **Remark** ***The product of two paracompact spaces need not be paracompact.***

  The space $\mathbb{R}_\ell$ is *paracompact*, for it is *regular and Lindelöf.* However, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is *not paracompact*, for it is *Hausdorff* but ***not normal***.

- **Remark** ***A subspace of a paracompact space need not be paracompact.***

  The space $\bar{S}_\Omega \times \bar{S}_\Omega$ is *compact* and, therefore, ***paracompact***. But the *subspace* $S_\Omega \times \bar{S}_\Omega$ is ***not paracompact***, for it is *Hausdorff* but *not normal.*

- **Lemma 1.6** *[Munkres, 2000]*
  *Let X be **regular**. Then the following conditions on X are **equivalent:** Every open covering of X has a **refinement** that is:*

  1. *An **open** covering of X and **countably locally finite**.*
  2. *A **covering** of X and **locally finite**.*
  3. *A **closed** covering of X and **locally finite**.*
  4. *An **open** covering of X and **locally finite**.*

- **Remark** Given *regularity* (T_3 *axioms of separation*), "*open subcovering that is countably locally finite*" ="*open subcovering that is locally finite*"

- **Theorem 1.7** *[Munkres, 2000]*
  *Every **metrizable** space is paracompact.*

- **Proposition 1.8** *[Munkres, 2000]*
  *Every **regular Lindelöf space** is paracompact.*

- **Example** ($\mathbb{R}^\omega$ *with **Product** and **Uniform Topologies***)
  The space $\mathbb{R}^\omega$ is ***paracompact*** in both *the **product** and **uniform** topologies.* This result follows from the fact that $\mathbb{R}^\omega$ is ***metrizable*** in these topologies.

  It is *not known* whether $\mathbb{R}^\omega$ is *paracompact* in *the box topology.*

- **Example** ($\mathbb{R}^J$ *for **Uncountable Product is Not Paracompact***)
  For $\mathbb{R}^J$ is *Hausdorff* but ***not normal***.

1.3 Partition of Unity

- **Remark** *One of **the most useful properties*** that a *paracompact space* X possesses has to do with *the **existence** of **partitions of unity*** on X.

2 Metrization Theorems

2.1 The Nagata-Smirnov Metrization Theorem

- **Theorem 2.1** *(**Nagata-Smirnov Metrization Theorem**). [Munkres, 2000]*
  *A space X is **metrizable** **if and only if** X is **regular** and has a **basis** that is **countably locally finite**.*

2.2 The Smirnov Metrization Theorem

- **Definition** (***Locally Metrizable***)
  A space X is ***locally metrizable*** if every point x of X has a ***neighborhood*** U that is ***metrizable*** in *the subspace topology*.

- **Theorem 2.2** *(**Smirnov Metrization Theorem**). [Munkres, 2000]*
  *A space X is **metrizable** **if and only if** it is a **paracompact Hausdorff** space that is **locally metrizable**.*

- **Remark** (***Sufficient and Necessary Conditions for Metrization***)

$$
\begin{tikzcd}
\textbf{\textit{metrizable}}\ \textit{space} \arrow[rr, bend left] \arrow[drr, leftrightarrow, "\textit{countable locally finite basis}"] \arrow[ddr, leftrightarrow, "\textit{locally metrizable}"'] & & \textbf{\textit{Hausdorff}}\ \textit{space} \\
& & \textbf{\textit{regular}}\ \textit{space} \arrow[u] \\
& \textbf{\textit{paracompact Hausdorff}}\ \textit{space} \arrow[r] & \textbf{\textit{normal}}\ \textit{space} \arrow[u] \\
& \textbf{\textit{compact Hausdorff}}\ \textit{space} \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

- **Example** (***Locally Convex Space is Metrizable***)

  Definition (***Locally Convex Space***)
  A topological vector space $(X, \mathscr{T})$ is called ***locally convex space*** if its topology $\mathscr{T}$ is *the weakest topology* for which all ***semi-norms*** $\{q_\theta, \theta \in \Theta\}$ are *continuous*. $\mathscr{T}$ is generated by ***the convex basis*** $U_{\boldsymbol{x},r,\theta} = \{\boldsymbol{y} \in X \mid q_\theta(\boldsymbol{y} - \boldsymbol{x}) \leq r\} \in \mathscr{B}, \boldsymbol{x} \in X, r > 0$.

  From the *Smirnov Metrization Theorem*, we see that ***the locally convex space is metrizable***.

References

James R Munkres. *Topology, 2nd*. Prentice Hall, 2000.