Lecture 0: Summary (part 2)

Tianpei Xie

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1 Vector Space

1.1 Topological Vector Space

- **Definition** A ***vector space*** over a field F is a set V together with two operations, the ***(vector) addition*** $+ : V \times V \to V$ and ***scalar multiplication*** $\cdot : F \times V \to V$, that satisfy the eight axioms listed below: for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, $\alpha, \beta \in F$,

  1. The *associativity* of addition: $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$;
  2. The *commutativity* of addition: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$;
  3. The *identity* of addition: $\exists\, \mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{x} = \boldsymbol{x}$;
  4. The *inverse* of addition: $\forall\, \boldsymbol{x} \in V$, $\exists\, -\boldsymbol{x} \in V$, so that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$;
  5. *Compatibility* of scalar multiplication with field multiplication: $\alpha(\beta \cdot \boldsymbol{x}) = (\alpha\beta) \cdot \boldsymbol{x}$;
  6. The *identity* of scalar multiplication: $\exists\, 1 \in F$, such that $1 \cdot \boldsymbol{x} = \boldsymbol{x}$;
  7. The *distributivity* of scalar multiplication with respect to vector addition: $\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = \alpha \cdot \boldsymbol{x} + \alpha \cdot \boldsymbol{y}$;
  8. The *distributivity* of scalar multiplication with respect to field addition: $(\alpha + \beta) \cdot \boldsymbol{x} = \alpha \cdot \boldsymbol{x} + \beta \cdot \boldsymbol{x}$.

  Elements of V are commonly called *vectors*. Elements of F are commonly called *scalars*. When $F = \mathbb{R}$, we say V is ***a real vector space***.

- **Definition** A vector space X endowed with a topology is called a ***topological vector space***, denoted as $(X, \mathscr{T})$, if the addition $+ : X \times X \to X$ and scale multiplication $\cdot : F \times X \to X$ are continuous.

- **Definition** A topological vector space is ***locally convex space***, if V is open and $\boldsymbol{x} \in V$, then one can find a *convex open* set $U \subset X$ such that $\boldsymbol{x} \in U \subset V$. That is, there exists a base of convex sets $\mathscr{B}$ that generates the topology $\mathscr{T}$.

- **Definition** Let V and W be *real vector spaces*. A map $T : V \to W$ is ***linear*** if $T(a\,v + b\,w) = a\,T(v) + b\,T(w)$ for all vectors $v, w \in V$ and all scalars a, b.

  In the special case $W = F$, a linear map from V to F is usually called ***a linear functional*** on V.

- **Definition** If $T : V \to W$ is a linear map, the ***kernel*** or ***null space*** of T, denoted by Ker$T$ or $T^{-1}(0)$, is the set $\{v \in V : T(v) = 0\}$, and the ***image*** of T, denoted by ImT or $T(V)$, is the set $\{w \in W : w = T(v) \text{ for some } v \in V\}$.

- **Definition** If V and W are vector spaces, a ***bijective linear map*** $T : V \to W$ is called an ***isomorphism***.

  In this case, there is a *unique inverse map* $T^{-1} : W \to V$, and T^{-1} is also *linear*:

  $$a\,T^{-1}(v) + b\,T^{-1}(w) = T^{-1}\,(a\,v + b\,w)$$

  For this reason, a *bijective linear map* is also said to be ***invertible***. If there exists an *isomorphism* $T : V \to W$, then V and W are said to be ***isomorphic***.

1.2 Dual Vector Spaces and Covectors

- **Definition** Let V be a *finite-dimensional real vector space.* We define a ***covector*** on V to be a ***real-valued linear functional*** on V, that is, a ***linear map*** $\omega : V \to F$.

  The space of all covectors on V is itself a *real vector space* under the obvious operations of *pointwise addition* and *scalar multiplication.* It is denoted by V^* and called the ***dual space*** of V.

- **Proposition 1.1** ***(Duality between Vector Space and Covector Space)***
  Let V be a finite-dimensional vector space. Given any basis $(E_1, \ldots, E_n)$ for V, let $\epsilon^1, \ldots, \epsilon^n \in V^*$ be the covectors defined by

$$\epsilon^i(E_j) = \delta^i_j$$

  where δ^i_j is the Kronecker delta symbol. Then $\epsilon^1, \ldots, \epsilon^n$ is a ***basis*** *for V^*, called the* ***dual basis*** *to (E_j). Therefore, $\dim V^* = \dim V$.*

- **Example** For example, we can apply this to ***the standard basis*** $(e_1, \ldots, e_n)$ for $\mathbb{R}^n$. The *dual basis* is denoted by $(e^1, \ldots, e^n)$ (note the *upper indices*), and is called ***the standard dual basis***. These basis *covectors* are the *linear functionals* on $\mathbb{R}^n$ given by

$$e^i(v) = e^i(v^1, \ldots, v^n) = v^i. \tag{1}$$

  In other words, e^i is the linear functional that *picks out the i-th component of a vector*.

  In **matrix notation**, a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by a $1 \times n$ matrix, called a ***row matrix***. The ***basis covectors*** can therefore also be thought of as the linear functionals represented by the *row matrices*

$$e^i = (0, \ldots, 1, \ldots, 0), \quad i = 1, \ldots, n \tag{2}$$

  where i-th element is 1 and the others are all zeros.

- **Remark** (***Coordinate Representation of Covectors***)
  More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the ***dual basis*** (ϵ^i) as

$$\omega = \omega_i \epsilon^i \tag{3}$$

  where the components are determined by $\omega_i = \omega(E_i)$.

- **Remark** Covector acts on vector to obtain a real number, which is ***the inner product*** between the component functions (coordinates in V^*) of covector and the component function (coordinates in V) of vector. This is ***the duality principle***.

$$\omega(v) = \left(\omega_i \epsilon^i\right)\left(v^j E_j\right) = \omega_i v^j \epsilon^i\left(E_j\right) = \omega_i v^j \delta^i_j = \omega_i v^i. \tag{4}$$

- **Definition** Suppose V and W are vector spaces and $A : V \to W$ is a *linear map.* We define a *linear map* $A^* : W^* \to V^*$, called ***the dual map*** or ***transpose of*** A, by

$$(A^* \omega)(v) = \omega\left(A\, v\right), \quad \forall\, \omega \in W^*,\ v \in V. \tag{5}$$

- **Definition** Apart from the fact that the dimension of V^* is the same as that of V, the second most important fact about dual spaces is the following characterization of the ***second dual space*** $V^{**} = (V^*)^*$.

  For each vector space V there is a natural, ***basis-independent map*** $\xi : V \to V^{**}$, defined as follows. For each vector $v \in V$, define a ***linear functional*** $\xi(v) : V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v), \quad \forall \omega \in V^*. \tag{6}$$

- **Proposition 1.2** *For any finite-dimensional vector space V, the map $\xi : V \to V^{**}$ is an* ***isomorphism****.*

- **Remark** When a covector ω acts on a vector v as $\omega(v)$, it is ***equivalent*** to say that the vector ξ_v acts on covector ω as $\xi_v(\omega)$. The isomorphism $v \mapsto \xi_v$ indicates that ***a vector can be seen as a linear functional on space of linear functionals itself***.

- **Remark** Some of important things to note:

  - The preceding proposition shows that when V is finite-dimensional, we can unambiguously ***identify*** V^{**} with V itself, because the map ξ is *canonically defined*, without reference to any basis.

  - It is important to observe that although V^* is *also* ***isomorphic*** to V (for the simple reason that any two finite-dimensional vector spaces of the same dimension are isomorphic), there is ***no canonical isomorphism*** $V \simeq V^*$.

  - Because of Proposition above, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more ***symmetric-looking notations*** $\langle \omega \,, v \rangle$ and $\langle v \,, \omega \rangle$, both expressions can be thought of either as ***the action of the covector*** $\omega \in V^*$ ***on the vector*** $v \in V$, or as ***the action of the linear functional*** $\xi(v) \in V^{**}$ ***on the element*** $\omega \in V^*$.

    There should be no cause for confusion with the use of the same angle bracket notation for inner products: *whenever one of the arguments is a* ***vector*** *and the other a* ***covector***, the notation $\langle \omega \,, v \rangle$ is always to be interpreted as the ***natural pairing*** between vectors and covectors, *not as an inner product*. We typically omit any mention of the map ξ, and think of $v \in V$ *either as a* ***vector*** or as *a* ***linear functional*** on V^*, depending on the context.

  - There is also a ***symmetry*** between ***bases*** and ***dual bases*** for a finite-dimensional vector space V: any *basis* for V ***determines*** a *dual basis* for V^*, and ***conversely***, any *basis* for V^* determines a *dual basis* for $V^{**} = V$.

    If (ϵ^i) is the basis for V^* *dual* to a basis (E_j) for V, then (E_j) is the basis *dual* to (ϵ^i), because both statements are equivalent to the relation $\left\langle \epsilon^i \,, E_j \right\rangle = \delta^i_j$.

2 Tangent Vector and Cotangent Vector

2.1 Tangent Vectors and Differentials at p

- **Remark** An element in Euclidean space $(x^1, \ldots, x^n) \in \mathbb{R}^n$ has two distinct roles:

  1. As a ***point*** in space, whose only property is its ***location*** $(x^1, \ldots, x^n)$;
  2. As a ***vector***, which are objects that have ***magnitude*** and ***direction***, but whose location is irrelevant.

  These two roles are ***split*** in the settings of a ***smooth manifold*** M: The first one corresponds to a ***point*** $p \in M$ and the second one corresponds to ***the tangent vector*** $v \in T_pM$. The point p and its associated tangent vector v are *independent*.

- **Definition** If a is a point of $\mathbb{R}^n$, a map $w : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a ***derivation at a*** if it is ***linear*** over $\mathbb{R}$ and satisfies the following ***product rule (Leibniz rule)***:

$$w(f\,g) = f(a)\,w(g) + g(a)\,w(f), \quad \forall\, f, g \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{7}$$

- **Remark** Let $T_a\mathbb{R}^n$ denote the ***set of all derivations*** of $\mathcal{C}^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is a *vector space* under the operations

$$(w_1 + w_2)(f) = w_1(f) + w_2(f), \quad (c\,w)(f) = c\,w(f).$$

- **Remark** For vector space, its tangent space coincides with its self. That is, derivations at a point are in *one-to-one correspondence* with geometric tangent vectors.

  Proposition 2.1 *Let $a \in \mathbb{R}^n$.*

  1. *For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by following*

$$D_v|_a(f) = D_v(f(a)) = \left.\frac{d}{dt}\right|_{t=0} f(a + t\,v). \tag{8}$$

     is a derivation at a.

  2. *The map $v_a \to D_v|_a$ is an **isomorphism** from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.*

- **Corollary 2.2** *For any $a \in \mathbb{R}^n$, the n **derivations***

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a \quad \textit{defined by} \quad \left.\frac{\partial}{\partial x^i}\right|_a(f) := \frac{\partial f}{\partial x^i}(a). \tag{9}$$

  *form a **basis** for $T_a\mathbb{R}^n$, which therefore has dimension n.*

- **Definition** Let M be a smooth manifold with or without boundary, and let p be a point of M. A ***linear*** *map* $v : \mathcal{C}^\infty(M) \to \mathbb{R}$ is called a ***derivation at p*** if it satisfies the *Product rule*:

$$v(f\,g) = f(a)\,v(g) + g(a)\,v(f), \quad \forall\, f, g \in \mathcal{C}^\infty(M) \tag{10}$$

  The set of all derivations of $\mathcal{C}^\infty(M)$ at p, denoted by T_pM, is a ***vector space*** called the ***tangent space*** to M at p. An element of T_pM is called a ***tangent vector*** at p.

- **Remark** Each tangent vector $v \in T_pM$ has ***two roles***:
  1. An ***element (vector)*** in tangent space T_pM;
  2. A ***linear functional*** $v : \mathcal{C}^\infty(M) \to \mathbb{R}$ that act on a smooth function f by taking directional derivatives of f along direction of v

- **Definition** If M and N are *smooth* manifolds with or without boundary and $F : M \to N$ is a *smooth* map, for each $p \in M$ we define a map
$$dF_p : T_pM \to T_{F(p)}N,$$
called the ***differential*** of F at p, as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the ***derivation** at $F(p)$* that ***acts*** on $f \in \mathcal{C}^\infty(N)$ by the rule
$$dF_p(v)(f) = v(f \circ F). \tag{11}$$
Note that if $f \in \mathcal{C}^\infty(N)$, then $f \circ F \in \mathcal{C}^\infty(M)$, so $v(f \circ F)$ makes sense.

- **Remark** $dF_p(v) : \mathcal{C}^\infty(N) \to \mathbb{R}$ is a ***linear operator*** because v is, and is a ***derivation*** at $F(p)$ because for any $f, g \in \mathcal{C}^\infty(N)$ we have the product rule
$$\begin{aligned} dF_p(v)(fg) &= v((f\,g) \circ F) = v((f \circ F)\,(g \circ F)) \\ &= (f \circ F)(p)\,v(g \circ F) + (g \circ F)(p)\,v(f \circ F) \\ &= f(F(p))\,dF_p(v)(g) + g(F(p))\,dF_p(v)(f) \end{aligned}$$

- **Remark** The *differential* at p, dF_p is a ***linear operator*** that maps *a* ***linear functional*** on $\mathcal{C}^\infty(M)$ to *another* ***linear functional*** $\mathcal{C}^\infty(N)$. This reflects the impact of smooth map $F : M \to N$.

- **Proposition 2.3** ***(Properties of Differentials)**.*
Let $M, N,$ and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.
  1. $dF_p : T_pM \to T_{F(p)}N$ *is* ***linear***.
  2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{(G \circ F)(p)}P$.
  3. $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$.
  4. *If F is a* ***diffeomorphism***, *then $dF_p : T_pM \to T_{F(p)}N$ is an* ***isomorphism***, *and* $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

- **Remark** The property 4 is critical. Note that given a smooth chart (U, φ), $\varphi : U \to \mathbb{R}^n$ is a *diffeomorphism* so ***the differential of coordinate map*** $d\varphi_p$ is an ***isomorphism*** between T_pU and $T_{\varphi(p)}\mathbb{R}^n$.

- **Remark** Get familar with these following expressions:
  1. $vf \in \mathbb{R}$ where $v \in T_pM$. This is to compute the directional derivatives of f along direction of v at point p;
  2. $dF_p(v) \in T_{F(p)}N$, where $v \in T_pM$. This is a linear operator that maps a tangent vector in T_pM to a tangent vector in $T_{F(p)}N$. Note that the base point $p \mapsto F(p)$.
  3. $dF_p(v)g \in \mathbb{R}$ where $g \in \mathcal{C}^\infty(N)$. This is to compute the directional derivatives of g along direction of $dF_p(v)$ at point $F(p)$;

2.2 Coordinate Representation of Tangent Vector and Differentials

- **Remark** By Corollary 2.2, the derivations $\frac{\partial}{\partial x^1}\big|_{\varphi(p)}, \ldots, \frac{\partial}{\partial x^n}\big|_{\varphi(p)}$ form a basis for $T_{\varphi(p)}\mathbb{R}^n$. Therefore, ***the preimages of these vectors under the* isomorphism $d\varphi_p$ *form a basis for*** T_pM

$$\frac{\partial}{\partial x^i}\bigg|_p := (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) \tag{12}$$

- **Remark** Unwinding the definitions (12), we see that $\frac{\partial}{\partial x^i}\big|_p$ acts on a function $f \in \mathcal{C}^\infty(U)$ by

$$\frac{\partial}{\partial x^i}\bigg|_p(f) = \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}(f \circ \varphi^{-1}) \equiv \frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\widehat{f} \tag{13}$$

  where $\widehat{f} = f \circ \varphi^{-1}$ is the ***coordinate representation*** of f, and $\widehat{p} = (p^1, \ldots, p^n) = \varphi(p)$ is the ***coordinate representation*** of p.

  In other word, using same coordinate representation φ^{-1} we can convert the derivation of function along coordinate basis in tangent space of manifold to a partial derivatives of that parameterized function along coordinate axis in Euclidean space.

  Definition $\displaystyle\frac{\partial}{\partial x^i}\bigg|_p$ is the ***derivation*** that takes the *i-th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p*. The vectors $\frac{\partial}{\partial x^i}\big|_p$ are called the ***coordinate vectors at p*** *associated with the given coordinate system*.

- We summarize our discussion as below proposition.

  Proposition 2.4 *Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart (U, φ) containing p, the* ***coordinate vectors*** $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ ***form a basis*** *for T_pM.*

- **Definition** (***Coordinate Representation of Tangent Vector***)
  A tangent vector $v \in T_pM$ can be written ***uniquely*** as a linear combination

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{14}$$

  where we use the *Einstein summation convention* as usual.

  The *ordered basis* $(\frac{\partial}{\partial x^i}\big|_p)$ is called a ***coordinate basis*** for T_pM, and the numbers $(v^1, \ldots, v^n)$ are called the ***components*** of v *with respect to the coordinate basis.*

- **Remark** (***Coordinate Representation of dF_p between Euclidean spaces***)

  Definition The action of ***differential*** of $F : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ *on a typical basis vector* can be represented as

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\,\frac{\partial}{\partial y^j}\bigg|_{F(p)} \tag{15}$$

  where $(x^1, \ldots, x^n)$ is the coordinates of U and $(y^1, \ldots, y^m)$ is the coordinate of V.

Here, the ***matrix*** of dF_p in terms of the coordinate bases is

$$\begin{bmatrix} \dfrac{\partial F^1}{\partial x^1}(p) & \dots & \dfrac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^m}{\partial x^1}(p) & \dots & \dfrac{\partial F^m}{\partial x^n}(p) \end{bmatrix}_{m\times n} \tag{16}$$

This matrix is none other than ***<u>the Jacobian matrix</u>*** of F at p, which is the ***matrix representation*** of the ***total derivative*** $DF_p : \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to ***the total derivative*** $DF_p : \mathbb{R}^n \to \mathbb{R}^m$, under our usual identification of Euclidean spaces with their tangent spaces. The same calculation applies if U is an open subset of $\mathbb{H}^n$ and V is an open subset of $\mathbb{H}^m$.

- (***Coordinate Representation of*** dF_p ***between Manifolds***)

  Definition For a smooth map $F : M \to N$ between smooth manifolds with or without boundary, the action of ***differential*** dF_p on a typical basis vector can be represented as

  $$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}, \tag{17}$$

  where $\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is the ***coordinate representation*** of F under smooth charts (U, φ) for M and (V, ψ) for N. Also $\widehat{p} = \varphi(p)$ is the coordinate representation of p. The matrix for $[\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})]_{j,i}$ is the Jacobian matrix.

  That is dF_p is represented ***in coordinate bases*** by the ***Jacobian matrix*** of (*the coordinate representative of*) F.

- **Remark** Unlike the Euclidean space, the *Jacobian matrix* $[\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})]_{j,i}$ based on ***local representation*** of F and p under smooth charts (U, φ) for M and (V, ψ) for N. Thus the Jacobian matrix for differential dF ***is dependent on the point*** p.

- **Remark** Get familar with these following expressions:

  1. Notice that for coordinate basis, the directional derivatives coincides with the partial derivatives after converting to coordinate representation.

     $$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}) = \frac{\partial \widehat{f}}{\partial x^i}(x^1, \dots, x^n) \in \mathbb{R} \;\text{ is}$$

     – *the coordinate basis vector* $\left.\frac{\partial}{\partial x^i}\right|_p$ *act on* f.

     – the i-th partial derivatives of (coordinate representation) $\widehat{f}$ evaluated at $\widehat{p} = \varphi(p)$

  2. The diferential of basis in T_pM is the linear map (via transition matrix) of the basis $T_{F(p)}N$

     $$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)} \in T_{F(p)}N.$$

  3. Notice that how differential dF_p act on the coordinate basis operator to obtain a new differential operator through the composition of F. Under coordinate (x^i) in M and (y^j)

in N, the following is just ***the chain rule***

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)g = \left.\frac{\partial}{\partial x^i}\right|_p (g\circ F) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)} g = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\,\frac{\partial g}{\partial y^j}(y^1,\ldots,y^m),$$

2.3 Change of Coordinates

- Suppose (U,φ) and (V,ψ) are two smooth charts on M, and $p\in U\cap V$. Let us denote the coordinate functions of φ by (x^i) and those of ψ by $(\widetilde{x}^i)$. Any tangent vector at p can be represented with respect to either basis $(\left.\frac{\partial}{\partial x^i}\right|_p)$ or $(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_p)$.

- **Remark** Given two smooth charts (U,φ) and (V,ψ) on M, the ***change of coordinates*** between basis vectors $(\left.\frac{\partial}{\partial x^i}\right|_p)$ (of φ) and $(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_p)$ (of ψ) is obtained via

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p \tag{18}$$

  where $\widehat{p}=\varphi(p)$ is the coordinate representation of p under φ.

2.4 Parameterized Curves

- **Definition** If M is a manifold with or without boundary, we define a ***curve*** in M to be a *continuous* map $\gamma: J\to M$ where $J\subseteq\mathbb{R}$ is an interval.

- **Definition** Let M be a smooth manifold with or without boundary. Our definition of tangent spaces leads to a natural interpretation of *velocity vectors*: given a smooth curve $\gamma: J\to M$ and $t_0\in J$, we define the ***velocity of*** γ ***at*** t_0, denoted by $\gamma'(t_0)$, to be the vector

$$\gamma'(t_0) = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)\in T_{\gamma(t_)}M$$

  where $\left.\frac{d}{dt}\right|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

- **Remark** This tangent vector ***acts*** on functions by

$$\gamma'(t_0)\,f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)\,f = \left.\frac{d}{dt}\right|_{t_0}(f\circ\gamma) = (f\circ\gamma)'(t_0).$$

  In other words, $\gamma'(t_0)$ is the ***derivation*** at $\gamma(t_0)$ obtained by taking the *derivative of a function along* γ.

  If t_0 is an endpoint of J, this still holds, provided that we interpret the derivative with respect to t as a *one-sided derivative*, or equivalently as the derivative of any smooth extension of $f\circ\gamma$ to an open subset of $\mathbb{R}$.

- **Remark** Now let (U,φ) be a smooth chart with coordinate functions (x^i). If $\gamma(t_0)\in U$, we can write the ***coordinate representation*** of γ as $\gamma(t)=(\gamma^1(t),\ldots,\gamma^n(t))$, at least for t sufficiently close to t_0, and then the ***coordinate formula*** for the differential yields

$$\gamma'(t_0) := d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) = \frac{d\gamma^i}{dt}(t_0)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)} \tag{19}$$

This means that $\gamma'(t_0)$ is given by essentially the same formula as it would be in *Euclidean space*: it is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

- **Proposition 2.5** ***(The Velocity of a Composite Curve)*** *[Lee, 2003.]*
Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF\left(\gamma'(t_0)\right). \tag{20}$$

- **Corollary 2.6** ***(Computing the Differential Using a Velocity Vector)*** *[Lee, 2003.]*
Suppose $F : M \to N$ is a smooth map, $p \in M$, and $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0) \tag{21}$$

for any smooth curve $\gamma : J \to M$ such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

- **Proposition 2.7** ***(Derivative of a Function Along a Curve)****.*
Suppose M is a smooth manifold with or without boundary, $\gamma : J \to M$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the ***derivative*** *of the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)). \tag{22}$$

- **Remark** Therefore we have

$$\gamma'(t_0)f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) f = (f \circ \gamma)'(t_0) = df_{\gamma(t_0)}\left(\gamma'(t_0)\right)$$

This shows the *duality* between the tangent vector $\gamma'(t_0) \in T_{\gamma(t_0)}M$ and the differential $df_{\gamma(t_0)}$.

2.5 Tangent Covectors on Manifolds

- **Definition** Let M be a smooth manifold with or without boundary. For each $p \in M$, we define the ***cotangent space*** at p, denoted by T_p^*M, to be the ***dual space*** to the *tangent space* T_pM:

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called ***tangent covectors at** p*, ***cotangent vectors at** p* , or just ***covectors at** p*. $\omega \in T_p^*M$ is a ***linear functional*** on tangent space T_pM.

- **Remark** (***Coordinate Representation of Covectors***) [Lee, 2003.]
Given smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ the coordinate basis $(\frac{\partial}{\partial x^i}\big|_p)$ gives rise to a dual basis for T_p^*M, which we denote for the moment by $(\lambda^i\big|_p)$. (In a short while, we will come up with a better notation.)

Any covector $\omega \in T_p^*M$ can thus be written ***uniquely*** as $\omega = \omega_i\, \lambda^i\big|_p$ where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right). \tag{23}$$

- **Remark** (***Change of Coordinates for Covectors***) [Lee, 2003.]
  Suppose now that $(\widetilde{x}^i)$ is *another set of smooth coordinates* whose domain contains p, and let $(\widetilde{\lambda}^j\big|_p)$ denote the basis for T_p^*M dual to $(\frac{\partial}{\partial \widetilde{x}^j}\big|_p)$. We can compute the *components* of the same covector ω with respect to the *new coordinate system* as follows.

  First observe that the computations in (18) show that the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p.$$

  Writing ω in both systems as $\omega = \omega_i\, \lambda^i\big|_p = \widetilde{\omega}_j\, \widetilde{\lambda}^j\big|_p$, we can use (**??**) to compute the components ω_i in terms of $\widetilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j.$$

  In sum, we have ***the change of coordinate formula for covectors***

$$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\,\widetilde{\omega}_j. \tag{24}$$

- **Definition** Let f be a *smooth real-valued function* on a *smooth manifold* M with or without boundary. (As usual, all of this discussion applies to functions defined on an open subset $U \subseteq M$; simply by *replacing* M with U throughout.) We define a ***covector field*** df , called ***the differential of*** f, by

$$df_p(v) = v\, f, \quad \forall v \in T_pM.$$

- **Remark** (***Coordinate Representation of differential of*** f)
  Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (λ^i) be the corresponding *coordinate coframe* on U. Write df in coordinates as $df_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$, then the definition of df implies

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p).$$

  This yields the following formula for ***the coordinate representation of*** df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\; \lambda^i|_p \tag{25}$$

  Thus, the ***component functions*** of df in any smooth coordinate chart are ***the partial derivatives of*** f *with respect to those coordinates*. Because of this, we can think of df as *an analogue of the classical gradient*, reinterpreted in a way that makes *coordinate-independent sense* on a manifold.

- **Remark** (***Basis of Tangent Covector Space*** T_p^*M)
  Let (x^j) be a set of ***coordinate functions***, where $x^j : U \to \mathbb{R}$ has coordinate representation as $(x^j \circ \varphi^{-1})(x^1, \ldots, x^n) = x^j$. According to (25), we can represent the ***differential of coordinate function*** x^j as

$$dx^j|_p = \frac{\partial x^j}{\partial x^i}(p)\; \lambda^i|_p = \delta_i^j\; \lambda^i|_p = \lambda^j|_p.$$

In other words, ***the coordinate covector field*** λ^j *is none other than* ***the differential*** dx^j. Therefore, the formula (25) for df_p can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)\, dx^i|_p. \tag{26}$$

or as ***an equation between covector fields*** instead of covectors. The ***coordinate representation of differential*** df is

$$df = \frac{\partial f}{\partial x^i}\, dx^i. \tag{27}$$

Thus, we have recovered the familiar classical expression for the differential of a function f in coordinates. Henceforth, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

- **Remark** (***Coordinate Representation of Tangent Covectors***)
  The ***coordinate representation of tangent covector*** $\omega_p \in T_pM^*$ is

$$\begin{aligned} \omega_p &= \omega_i\, dx^i|_p \\ \text{where } \omega_i &= \omega_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \end{aligned} \tag{28}$$

- **Remark** (***Duality of Basis***)
  The basis of tangent space T_pM is $(\partial/\partial x^j|_p)$ and the basis for the cotangent space T_p^*M is $(dx^i|_p)$. Thus we have ***the duality principle*** on basis

$$dx^i|_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j, \quad \forall i, j = 1, \ldots, n, \;\; p \in M \tag{29}$$

  In other word, $dx^i|_p$ is the *linear functional that* ***picks out the*** i***-th component of a tangent vector*** at p.

- **Remark** Just like a tangent vector $v \in T_pM$ has two roles: an element in vector space and a linear functional on $\mathcal{C}^\infty(M)$, **a cotangent vector** $df_p \in T_p^*M$ has two roles as well:

  1. ***A linear map (operator)*** from T_pM to $T_{f(p)}\mathbb{R}$. That is, it maps a tangent vector in T_pM to a tangent vector in $T_{f(p)}\mathbb{R}$: $v \mapsto df_p(v)$. Therefore, $df_p(v)$ ***is a linear functional*** can act on a function $\mathbb{R}$.
  2. ***An element (covector)*** in dual vector space T_p^*M. Each element in this dual space is ***a linear functional*** on T_pM. In this sense, $df_p(v)$ ***is a real number***.

- **Remark** Note that a nonzero linear functional $\omega_p \in T_p^*M$ is completely determined by two pieces of data: its ***kernel***, which is a linear hyperplane in T_pM (*a codimension-*1 *linear subspace*); and the set of vectors v for which $\omega_p(v) = 1$, which is *an* ***affine hyperplane parallel to the kernel***. The value of $\omega_p(v)$ for any other vector v is then obtained by linear interpolation or extrapolation.

- ***One very important property*** of the differential is the following characterization of smooth functions with vanishing differentials.

  Proposition 2.8 ***(Functions with Vanishing Differentials)****. [Lee, 2003.]*
  If f is a smooth real-valued function on a smooth manifold M with or without boundary, then $df = 0$ if and only if f is ***constant*** *on* ***each component*** *of M .*

- **Remark** Be familiar with the following expressions:

  1. The ***differential*** 1***-form of*** f is ***a covector field***

  $$df = \frac{\partial f}{\partial x^i} dx^i$$

  where (dx^i) are the coordinate covector fields.

  2. A *covector* $\omega \in T_p^* M$ acts on a *tangent vector* $v \in T_p M$ results in "*the inner product*"

  $$\omega(v) = \left(\omega_i\, dx^i|_p\right) \left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = \omega_i v^i = \langle \omega\,,\, v\rangle \in \mathbb{R}$$

  Note that $\langle \omega\,,\, v\rangle$ is not actually inner product in normal sense, since the first term is a cotangent vector and the second term is a tangent vector.

  3. Since $(T_pM)^{**} \simeq T_pM$, we can identify a linear functional that associated with each tangent vector to *act* on covector. The linear functional ξ in $(T_pM)^{**}$ that ***identifies*** with the coordinate vector $\partial/\partial x^j$ is ***the component function*** ω_j of the covectors.

  $$\xi \left(\frac{\partial}{\partial x^j}\Big|_p \right) (\omega_p) = \omega_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \omega_j$$

  4. For $df_p(v) \in T_{f(p)}\mathbb{R}$, it can acts on a one-parameter function $g \in \mathcal{C}^\infty(\mathbb{R})$

  $$df_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) g = \frac{\partial}{\partial x^j}\Big|_p (g \circ f) = \frac{dg}{ds}(f(p))\, \frac{\partial f}{\partial x^j}(x^1, \ldots, x^n)$$

  5. Do not confuse $df_p : T_pM \to T_{f(p)}\mathbb{R}$ with differential of parameterized curve $d\gamma : T_0\mathbb{R} \to T_{\gamma(0)}M$.

3 Bundles

3.1 Tangent Bundle, Frames and Vector Field

3.1.1 Tangent Bundle

- Often it is useful to consider the set of *all tangent vectors at all points* of a manifold.

  Definition Given a smooth manifold M with or without boundary, the ***tangent bundle*** of M, denoted by TM, is defined as the ***disjoint union*** of *the tangent spaces* ***at all points*** of M:

  $$TM = \bigsqcup_{p \in M} T_pM.$$

- **Definition** The tangent bundle comes equipped with a ***natural projection map*** $\pi : TM \to M$, which sends each vector in T_pM to the point p at which it is tangent: $\pi(p, v) = p$.

- **Remark** The ***tangent bundle*** TM is ***a global extension*** of *the local tangent space* T_pM. It plays a critical role when we want to ***generalize*** a concept on local tangent space ***globally*** (i.e. ***dropping the dependency*** *on point* p). These concepts include:

  1. From ***tangent vector*** $v \in T_pM$ to ***vector field*** $X : M \to TM$, where $X_p \in T_pM$;
  2. From ***derivation*** *at* $p : \mathcal{C}^\infty(M) \to \mathbb{R}$, to ***derivation operator*** $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$;
  3. From ***differential of*** F at p, $dF_p : T_pM \to T_pN$, to ***the global differential of*** F, $dF : TM \to TN$.
  4. From ***basis*** *tangent vectors* in T_pM to ***local frames*** *of manifold* M.

- **Remark** Intuitively, the natural projection map $\pi : TM \to M$ helps us to ***locate*** for local information given the global structure. Its preimage also confine the region of interest in the global structure. Each tangnet space $T_pM = \pi^{-1}(p)$ is the preimage of π at p, called the ***fiber*** at p

- **Proposition 3.1** ***(Tangent Bundle Is a Manifold)*** *[Lee, 2003.]*
  For any smooth n-manifold M*, the tangent bundle* TM *has a* ***natural topology*** *and* ***smooth structure*** *that make it into a* ***2n-dimensional smooth manifold****. With respect to this structure, the projection* $\pi : TM \to M$ *is* ***smooth****.*

- **Definition** Given any smooth chart $(U, (x^i))$ for M, the coordinates (x^i, v^i) given by

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p\right) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).$$

  are called ***natural coordinates*** on TM. Here, the coordinate map $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$.

- **Remark** From the coordinate of TM, we can see that ***the tangent vector*** v *is considered as* ***free variables*** as opposed to be considered as associated with the position p in TM as $v \in T_pM$.

- **Proposition 3.2** *If* M *is a smooth n-manifold with or without boundary, and* M *can be covered by* ***a single smooth chart****, then* TM *is diffeomorphic to* $M \times \mathbb{R}^n$*.*

- **Remark** The above proposition states that for a manifold that has a global coordinate system, its tangent bundle also have *a global structure* as $M \times \mathbb{R}^n$. Note that normally since a tangent space is diffeomorphic to $\{p\} \times \mathbb{R}^n$, the tangent bundle is only defined locally.

- **Definition** By putting together the ***differentials*** of F ***at all points*** of M, we obtain a ***globally*** *defined map* between ***tangent bundles***, called ***the global differential*** or ***global tangent map*** and denoted by $dF : TM \to TN$.

  This is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p.

- *One important feature* of the smooth structure we have defined on TM is that it makes the differential of a smooth map into **a smooth map between tangent bundles**.

  Proposition 3.3 *If* $F : M \to N$ *is a smooth map, then its global differential* $dF : TM \to TN$ *is a smooth map.*

- The following properties of tangent bundle comes from Proposition 2.3:

  Corollary 3.4 ***(Properties of the Global Differential)*** *[Lee, 2003.]*

Suppose $F : M \to N$ and $G : N \to P$ are smooth maps.

1. *$d(G \circ F) = dG \circ dF : TM \to TP$.*
2. *$d(Id_M) = Id_{TM} : TM \to TM$.*
3. *If F is a* ***diffeomorphism****, then $dF : TM \to TN$ is also a* ***diffeomorphism****, and $(dF)^{-1} = d(F^{-1})$*

3.1.2 Vector Fields

- **Definition** If M is a smooth manifold with or without boundary, a ***vector field*** on M is a ***section*** of the map $\pi : TM \to M$. More concretely, a *vector field* is a ***continuous*** map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \mathrm{Id}_M, \tag{30}$$

  or *equivalently*, $X_p \in T_pM$ for each $p \in M$.

- **Remark** We write the ***value of** X **at** p* as X_p instead of $X(p)$ to be consistent with our notation for elements of the tangent bundle, as well as to avoid conflict with the notation $v(f)$ for the action of a vector on a function.

- **Definition** When the map $X : M \to TM$ is *smooth* and the tangent bundle TM is given a *smooth manifold structure*, X is a ***smooth vector field***.

  In addition, for some purposes it is useful to consider maps from M to TM that would be vector fields except that they *might not be continuous*. A ***rough vector field*** on M is a (*not necessarily continuous*) map $X : M \to TM$ satisfying (30).

- **Remark** (***Coordinate Representation of Vector Field At a Point***)
  Suppose M is a smooth n-manifold (with or without boundary). If $X : M \to TM$ is a *rough vector field* and $(U, (x^i))$ is any *smooth coordinate chart* for M, we can write the ***value** of X at any point* $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p. \tag{31}$$

  This defines n ***functions*** $X^i : U \to \mathbb{R}$, called the ***component functions*** of X in the given chart.

- **Proposition 3.5** ***(Smoothness Criterion for Vector Fields)*** *[Lee, 2003.]*
  Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the ***restriction*** *of X to U is* ***smooth*** *if and only if its* ***component functions*** *with respect to this chart are* ***smooth****.*

- **Remark** (***The Space of all Vector Fields on a Manifold is a Vector Space***)
  If M is a smooth manifold with or without boundary, it is standard to use the ***notation*** $\mathfrak{X}(M)$ (or ***equivalently*** $\Gamma(TM)$) to denote ***the set of all smooth vector fields on M***.

  $\mathfrak{X}(M)$ *is a* ***vector space*** under pointwise addition and scalar multiplication:

  1. For any $a, b \in \mathbb{R}$ and any $X, Y \in \mathfrak{X}(M)$,

$$(a\,X + b\,Y)_p = a\,X_p + b\,Y_p.$$

2. The *zero element* of this vector space is the ***zero vector field***, whose value at each $p \in M$ is $0 \in T_pM$.

In addition, *smooth vector fields* can be multiplied by *smooth real-valued functions*: if $f \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX : M \to TM$ by

$$(fX)_p = f(p)\, X_p.$$

- **Remark** (***Coordinate Representation of Vector Field***)
  We can generalize the formula (31) as the coordinate representation of the vector field X

$$X = X^i \frac{\partial}{\partial x^i}. \tag{32}$$

  where $(\partial/\partial x^i)$ are *the coordinate vector fields*, which are ***basis*** for $\mathfrak{X}(M)$ and X^i is the i-th component function of X in the given coordinates.

  In partial differential equations (PDEs), we usually write (32) in *dot-product form*

$$X = \boldsymbol{X} \cdot \nabla = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \tag{33}$$

$$\text{where } \boldsymbol{X} = [X^1, \ldots, X^n], \quad \nabla := \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right).$$

  ∇ (*the nabla symbol*) is also called ***gradient operator***.

- An essential property of vector fields is that they define ***operators*** on the space of smooth real-valued functions.

  Definition If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf : U \to \mathbb{R}$, defined by

$$(X\, f)_p = X_p\, f$$

  Note that $v\, f \equiv v(f)$ as we omit the parenthesis.

- **Remark** Please do not *confuse* with these two terms:

  1. *A function* ***mulitiplies*** *a vector field is a vector field*, i.e. $fX \in \mathfrak{X}(M)$. Thus $(fX)_p = f(p)\, X_p \in T_pM$.
  2. *A vector field* ***acts*** *on a function is a function*, i.e. $Xf \in \mathcal{C}^\infty(M)$. Thus $(Xf)_p = X_p\, f \in \mathcal{C}^\infty(M)$

- **Definition** Define a map $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is called a ***derivation*** (as distinct from a *derivation at p*, defined in Chapter 3) if it is **linear** over $\mathbb{R}$ and satisfies the *Product rule*

$$X(fg) = f\, X(g) + g\, X(f), \qquad \forall\, f, g \in \mathcal{C}^\infty(M) \tag{34}$$

  Note $fX\, g \equiv (fX)(g)$ is a function f multiplying the vector field X *then* acts on function g.

- **Remark** Please not be confused by the following notions:

  1. $\left.\dfrac{\partial}{\partial x^i}\right|_p f \in \mathbb{R}$ is a real number. Similarly $vf \equiv v(f) \in \mathbb{R}$.

2. $X\,f \equiv X(f) \in \mathcal{C}^{\infty}(M)$ is a smooth function. At each point p, $X_p f \equiv (X\,f)_p \in \mathbb{R}$.

3. $(Xf)_p = X_p f$ is the directional derivative of f along the direction X_p. In other word, to compute the function $g(p)$ evaluated at p, we can first assign p to vector field X to "simplify" it as X_p, and then to compute the directional derivatives $X_p f$. This is usually simpler than computing $g = Xf$ first and then assign p to value.

   An example

$$X = y^2 \frac{\partial}{\partial x} + \cos(x)\frac{\partial}{\partial y}, \quad f(x,y) = \sin(x)\,y$$

$$\begin{aligned} g(0,1) = (Xf)_{(0,1)} &= \left(y^2\Big|_{(0,1)} \frac{\partial}{\partial x}\Big|_{(0,1)} + \cos(x)\Big|_{(0,1)} \frac{\partial}{\partial y}\Big|_{(0,1)} \right) (\sin(x)\,y) \\ &= \left(\frac{\partial}{\partial x}\Big|_{(0,1)} + \frac{\partial}{\partial y}\Big|_{(0,1)} \right) (\sin(x)\,y) \\ &= (y\cos(x) + \sin(x))|_{(0,1)} = 1 \end{aligned}$$

- **Definition** Suppose $F : M \to N$ is *smooth* and X is a *vector field* on M, and suppose there happens to be a *vector field* Y on N with the property that for each $p \in M$,

$$dF_p(X_p) = Y_{F(p)}.$$

  In this case, we say *the **vector fields** X and Y are **F-related***.

- **Remark** The ***differential*** dF_p is defined *locally*, and it ***does not guarantee to map a vector field (a global concept) to a vector field***. For example, if F is *not surjective*, there is no way to decide what vector to assign to a point $q \in N \setminus F(M)$. If F is *not injective*, then for some points of N there may be several different vectors obtained by applying dF to X at different points of M.

3.1.3 Local and Global Frames

- **Definition** Suppose M is a smooth n-manifold with or without boundary. An *ordered k-tuple* $(X_1, \ldots, X_k)$ of ***vector fields*** defined on some subset $A \subseteq M$ is said to be ***linearly independent*** if $(X_1|_p, \ldots, X_k|_p)$ is a *linearly independent* k-tuple in T_pM for each $p \in A$, and is said to ***span the tangent bundle*** if the k-tuple $(X_1|_p, \ldots, X_k|_p)$ spans T_pM at each $p \in A$.

- **Definition** A ***local frame*** for M is *an ordered n-tuple of* ***vector fields*** $(E_1, \ldots, E_n)$ defined on an **open subset** $U \subseteq M$ that is ***linearly independent*** and ***spans the tangent bundle***; thus the vectors $(E_1|_p, \ldots, E_n|_p)$ form a basis for T_pM at each $p \in U$.

  $(E_1, \ldots, E_n)$ is called a ***global frame*** if $U = M$, and a ***smooth frame*** if each of the vector fields E_i is *smooth*.

  We often use the shorthand notation (E_i) to denote a frame $(E_1, \ldots, E_n)$.

- **Remark** The concept of *frames* is an extension of *the basis vector and coordinate system* to *manifold*. Frames are a set of *linearly independent* ***vector fields***, which form the *basis* of space of all vector fields $\mathfrak{X}(M)$. Note that ***the concept of linear independent vector fields is defined locally at each tangent space***.

- **Definition** A k-tuple of vector fields $(E_1, \ldots, E_k)$ defined on some subset $A \subseteq \mathbb{R}^n$ is said to be ***orthonormal*** if for each $p \in A$, the vectors $(E_1|_p, \ldots, E_k|_p)$ are ***orthonormal*** with respect to the Euclidean dot product (where we identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ in the usual way).

  A *(local or global) frame* consisting of *orthonormal vector fields* is called an ***orthonormal frame***.

- **Lemma 3.6** ***(Gram-Schmidt Algorithm for Frames).***
  Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over an open subset $U \subseteq \mathbb{R}^n$. Then there is a smooth orthonormal frame (E_j) over U such that span$(E_1|_p, \ldots, E_j|_p) =$ span$(X_1|_p, \ldots, X_j|_p)$ for each $j = 1, \ldots, n$ and each $p \in U$.

- Although *smooth local frames* are plentiful, *global ones* are not.

  Definition A smooth manifold with or without boundary is said to be ***parallelizable*** if it admits a **smooth global frame**.

- **Example** These are some examples of parallizable or non-parallizable manifolds:
  - $\mathbb{R}^n$, $\mathbb{S}^1$ and $\mathbb{T}^n$ are all *parallelizable manifold.*
  - *All **Lie groups** are parallelizable.*
  - Most smooth manifolds are *not parallelizable.* The simplest example of a *nonparallelizable manifold* is $\mathbb{S}^2$. (In fact, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the ***only*** spheres that are parallelizable.)

3.1.4 Integral Curves and Flows

- **Definition** Suppose M is a smooth manifold with or without boundary and V is a *vector field* on M. An ***integral curve*** of V is a *differentiable curve* $\gamma : J \to M$ whose ***velocity*** at each point is equal to the ***value of*** V at that point:

$$\gamma'(t) = V_{\gamma(t)}, \quad \forall t \in J.$$

  If $0 \in J$, the point $\gamma(0)$ is called ***the starting point of*** γ.

- **Remark** Finding integral curves boils down to solving *a system of ordinary differential equations* in a smooth chart. Suppose $\gamma : J \to M$ is a smooth curve and V is a smooth vector field on M. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for to be an integral curve of V can be written

$$\dot{\gamma}^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}, \tag{35}$$

  which reduces to the following ***autonomous system of ordinary differential equations (ODEs)***:

$$\dot{\gamma}^i(t) = V^i(\gamma^1(t), \ldots, \gamma^n(t)), \qquad i = 1, \ldots, n. \tag{36}$$

- The fundamental fact about such systems is **the *existence*, *uniqueness*, and *smoothness theorem*** from ODE theory [Amann, 2011, Hirsch et al., 2012].

Proposition 3.7 *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of V starting at p.*

- **Remark** The followings show how how ***affine reparametrizations*** affect integral curves:

  1. **Lemma 3.8** *(**Rescaling Lemma**). [Lee, 2003.]*
     Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma} : \widetilde{J} \to M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t : at \in J\}$.

  2. **Lemma 3.9** *(**Translation Lemma**). [Lee, 2003.]*
     Let V, M, J, and γ be as in the preceding lemma. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma} : \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t + b)$ is also an integral curve of V, where $\widehat{J} = \{t : t + b \in J\}$.

  3. **Proposition 3.10** *(**Naturality of Integral Curves**). [Lee, 2003.]*
     *Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related **if and only if** F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

- **Definition** A ***global flow on** M* (also called ***a one-parameter group action***) is defined as a ***continuous left** $\mathbb{R}$**-action on** M*; that is, a *continuous map* $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \theta(0, p) = p \tag{37}$$

- **Definition** If M is a manifold, a ***flow domain*** for M is an open subset $\mathfrak{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathfrak{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathfrak{D}\}$ is an *open interval* ***containing*** 0.

  A ***flow*** on M is a continuous map $\theta : \mathfrak{D} \to M$; where $\mathfrak{D} \subseteq \mathbb{R} \times M$ is *a flow domain*, that satisfies the following ***group laws***:

$$\theta(0, p) = p, \quad \forall p \in M \tag{38}$$

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \forall s \in \mathfrak{D}^{(p)},\ t \in \mathfrak{D}^{(\theta(s,p))},\ (\text{i.e. } t + s \in \mathfrak{D}^{(p)}) \tag{39}$$

  We sometimes call θ ***a local flow*** to distinguish it from *a global flow* as defined earlier. The unwieldy term ***local one-parameter group action*** is also used.

- For a global flow θ on M, we define two collections of maps as follows:

  - **Definition** For each $t \in \mathbb{R}$, ***define*** a continuous map $\theta_t : M \to M$ by

$$\theta_t(p) = \theta(t, p).$$

    The defining properties in (37) are equivalent to ***the group laws***:

$$\theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{Id}_M \tag{40}$$

  - **Definition** For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by

$$\theta^{(p)}(t) = \theta(t, p).$$

    The image of this curve is ***the orbit of** p **under the group action***.

- **Definition** If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a *tangent vector* $V_p \in T_pM$ by

$$V_p = (\theta^{(p)})'(0) = d\theta^{(p)} \left(\frac{d}{dt}\Big|_{t=0} \right).$$

  The assignment $p \mapsto V_p$ is ***a (rough) vector field*** on M; which is called ***the infinitesimal generator of the global flow*** θ. That is, θ is the integral curve of V.

- **Definition** If θ is a flow, we define $\theta_t(p) = \theta^{(p)}(t) = \theta(t,p)$ whenever $(t,p) \in \mathfrak{D}$, just as for a global flow. For each $t \in \mathbb{R}$, we also define

$$M_t = \{p \in M : (t,p) \in \mathfrak{D}\} \tag{41}$$

  so that

$$p \in M_t \Leftrightarrow t \in \mathfrak{D}^{(p)} \Leftrightarrow (t,p) \in \mathfrak{D}.$$

  If θ is smooth, ***the infinitesimal generator*** of θ is defined by $V_p = (\theta^{(p)})'(0)$.

- **Definition** *A* ***maximal integral curve*** is one that cannot be extended to an integral curve on *any larger open interval*, and *a* ***maximal flow*** is a flow that admits *no extension* to a flow on a *larger flow domain.*

- **Remark** Given a smooth flow, we can define a vector field as the infinitesimal generator of it. Conversely, for every smooth vector field, there exists a unique maximal smooth flow which is the ***integral curve*** *of the vector field* ***locally***. The flow is ***time-reversible***.

  1. **Proposition 3.11** *If $\theta : \mathfrak{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

  2. **Theorem 3.12** *(**Fundamental Theorem on Flows**). [Lee, 2003.]*
  Let V be a smooth vector field on a smooth manifold M. There is a ***unique smooth maximal flow*** *$\theta : \mathfrak{D} \to M$ whose* ***infinitesimal generator*** *is V. This flow has the following properties:*

     (a) For each $p \in M$, the curve $\theta^{(p)} : \mathfrak{D}^{(p)} \to M$ is the ***unique maximal integral curve*** *of V starting at p.*

     (b) If $s \in \mathfrak{D}^{(p)}$, then $\mathfrak{D}^{(\theta(s,p))}$ is the interval $\mathfrak{D}^{(p)} - s = \left\{t - s : t \in \mathfrak{D}^{(p)}\right\}$.

     (c) For each $t \in \mathbb{R}$, the set M_t is ***open*** *in M; and $\theta_t : M_t \to M_{-t}$ is a* ***diffeomorphism*** *with* ***inverse*** *θ_{-t}.*

3.2 Vector Bundle, Frames and Section

- **Definition** Let M be a *topological space.* A *(real)* ***vector bundle*** *of rank k over M* is a ***topological space*** E together with a ***surjective continuous*** *map* $\pi : E \to M$ satisfying the following conditions:

  1. For each $p \in M$, the ***fiber*** $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a *k-dimensional real* ***vector space***.

2. For each $p \in M$, there exist a neighborhood U of p in M and a ***homeomorphism*** $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called *a **local trivialization** of E over U*), satisfying the following conditions:
   (a) $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is the ***projection***);
   (b) for each $q \in U$, the *restriction* of Φ to E_q is a ***vector space isomorphism*** from E_q to $\{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$.

The space E is called ***the total space of the bundle***, M is called its ***base***, and π is its ***projection***.

- **Definition** If M and E are smooth manifolds with or without boundary, π is *a smooth map*, and *the local trivializations* can be chosen to be *diffeomorphisms*, then E is called *a **smooth vector bundle***. In this case, we call *any local trivialization* that is a *diffeomorphism* onto its image *a **smooth local trivialization***.

- **Remark** The ***rank*** of a vector bundle is the ***dimension*** of *vector space* $\pi^{-1}(p)$ associated with each point p.

- **Remark** The idea of local trivialization provides a way to map a fiber $\pi^{-1}(p)$ in vector bundle to a Euclidean space. This is critical to make sure the vector bundle itself is a *topological manifold*.

- **Definition** If there exists *a local trivialization* of E over ***all of*** M (called *a **global trivialization** of E*), then E is said to be *a **trivial bundle***. In this case, E itself is ***homeomorphic*** to the *product space* $M \times \mathbb{R}^k$.

  If $E \to M$ is a *smooth bundle* that admits *a smooth global trivialization*, then we say that E is ***smoothly trivial***. In this case E is ***diffeomorphic*** to $M \times \mathbb{R}^k$, not just *homeomorphic*.

  For brevity, when we say that *a smooth bundle* is *trivial*, we always understand this to mean *smoothly trivial*, not just trivial in the topological sense.

- **Lemma 3.13** ***(Transition between Two Smooth Local Trivializations)***
  *Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are **two smooth local trivializations** of E with $U \cap V \neq \emptyset$. There exists a **smooth map** $\tau : U \cap V \to GL(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form*

$$\Phi \circ \Psi^{-1}(p, v) = (p,\, \tau(p)v),$$

  where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

  Note that the following diagram commute:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U \cap V &
\end{tikzcd}$$

  Definition The smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ described in this lemma is called the ***transition function*** between the local trivializations Φ and Ψ.

For example, if M is a smooth manifold and Φ and Ψ are the local trivializations of tangent bundle TM associated with two different smooth charts, then the transition function between them is ***the Jacobian matrix*** of the *coordinate transition map.*

- Like the tangent bundle, vector bundles are often most easily described by giving ***a collection of vector spaces***, one for each point of the base manifold. The next lemma shows that in order to construct a smooth vector bundle, it is sufficient to construct the local trivializations, as long as they overlap with smooth transition functions.

  Lemma 3.14 ***(Vector Bundle Chart Lemma)****. [Lee, 2003.]*
  Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$ we are given a ***real vector space*** *E_p of some fixed dimension k. Let $E = \bigsqcup_{p \in M} E_p$, and let $\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following data:*

  1. *an* ***open cover*** *$\{U_\alpha\}_{\alpha \in A}$ of M*
  2. *for each $\alpha \in A$, a* ***bijective*** *map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space* ***isomorphism*** *from E_p to $\{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$*
  3. *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form*

  $$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p,\, \tau_{\alpha,\beta}(p)v), \tag{42}$$

  Then E has a ***unique topology*** *and* ***smooth structure*** *making it into* ***a smooth manifold*** *with or without boundary and a* ***smooth rank-k vector bundle over M****; with π as* ***projection*** *and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.*

- **Definition** Let $\pi : E \to M$ be a vector bundle. A ***section*** of E (sometimes called ***a cross section***) is a ***section*** of the map π, that is, a continuous map $\sigma : M \to E$ satisfying

  $$\pi \circ \sigma = \mathrm{Id}_M.$$

  This means that $\sigma(p)$ is an element of the *fiber* E_p for each $p \in M$.

- **Definition** More generally, ***a local section of*** E is a *continuous* map $\sigma : U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = \mathrm{Id}_U$.

  To emphasize the distinction, a section defined on *all of* M is sometimes called ***a global section***. Note that a *local section* of E over $U \subseteq M$ is the same as a *global section* of *the* ***restricted bundle*** $E|_U$.

- **Definition** If M is a smooth manifold with or without boundary and E is a ***smooth vector bundle***, a ***smooth (local or global) section*** *of* E is one that is a *smooth map* from its domain to E.

- **Definition** Define a ***rough (local or global) section*** of E over a set $U \subseteq M$ to be a map $\sigma : U \to E$ (*not necessarily continuous*) such that $\pi \circ \sigma = \mathrm{Id}_U$. A "section without further qualification always means a continuous section.

- **Definition** The ***zero section*** of E is the **global section** $\xi : M \to E$ defined by

  $$\xi(p) = 0 \in E_p, \quad \forall p \in M.$$

- **Definition** As in the case of vector fields, the ***support*** of a section σ is the ***closure*** *of the set* $\{p \in M : \sigma(p) \neq 0\}$.

- **Definition** If $E \to M$ is a smooth vector bundle, the set of ***all smooth global sections of*** E is a ***vector space*** under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p)$$

  This vector space is usually ***denoted by*** $\Gamma(E)$. Note that for vector fields of tangent bundle TM, we use $\mathfrak{X}(M)$

- **Remark** Just like smooth vector fields, *smooth sections* of a *smooth bundle* $E \to M$ can be *multiplied* by *smooth real-valued functions*: if $f \in \mathcal{C}^{\infty}(M)$ and $\sigma \in \Gamma(E)$, we obtain *a* ***new section*** $f\sigma$ defined by

$$(f\sigma)(p) = f(p)\,\sigma(p).$$

- **Definition** Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, a ***k-tuple*** of ***local sections*** $(\sigma_1, \ldots, \sigma_k)$ of E over U is said to be ***linearly independent*** if their values $(\sigma_1(p), \ldots, \sigma_k(p))$ form a *linearly independent k-tuple in E_p* for each $p \in U$.

  Similarly, they are said to ***span*** E if *their values span E_p for each $p \in U$.*

- **Definition** *A* ***local frame*** *for E over U* is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of ***linearly independent*** *local sections* over U that ***span*** E; thus $(\sigma_1(p), \ldots, \sigma_k(p))$ is a ***basis*** for the *fiber* E_p for each $p \in U$.

  It is called a ***global frame*** if $U = M$.

- **Definition** If $E \to M$ is a smooth vector bundle, a *local or global frame* is a ***smooth frame*** if each σ_i is a *smooth section*. We often *denote a frame $(\sigma_1, \ldots, \sigma_k)$ by (σ_i)*.

- **Remark** The *(local or global) frames* for M that we defined in Chapter 8 are, in our new terminology, frames for the tangent bundle. We use both terms interchangeably depending on context: "***frame for*** M" and "***frame for*** TM" mean the same thing.

- **Corollary 3.15** ***(The Coordinate Representation of Vector Bundle)***
  Let $E \to M$ be a smooth vector bundle of rank k, let (V, φ) be a smooth chart on M with coordinate functions (x^i), and suppose there exists a smooth local frame (σ_i) for E over V. Define $\widetilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by

$$\widetilde{\varphi}\left(v^i\sigma_i(p)\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^k\right). \tag{43}$$

  Then $(\pi^{-1}(V), \widetilde{\varphi})$ is a ***smooth coordinate chart*** *for E.*

- **Definition** Suppose (σ_i) is a smooth local frame for E over some open subset $U \subseteq M$. If $\tau : M \to E$ is a *rough section*, the value of τ at an arbitrary point $p \in U$ can be written $\tau(p) = \tau^i(p)\sigma_i(p)$ for some uniquely determined numbers $(\tau^1(p), \ldots, \tau^k(p))$. This defines k functions $\tau^i : U \to \mathbb{R}$, called *the* ***component functions*** *of τ with respect to the given local frame.*

- **Proposition 3.16** ***(Local Frame Criterion for Smoothness).***
  Let $\pi : E \to M$ be a smooth vector bundle, and let $\tau : M \to E$ be a rough section. If (σ_i) is a smooth local frame for E over an open subset $U \subseteq M$, then τ is smooth on U if and only if its component functions with respect to (σ_i) are smooth.

3.3 Comparison of Concepts for Bundles

- By far, we have introduced a lot of abstract concepts that are generalization of our known concepts. Let us compare them in the following Table 1.

Table 1: Comparison between concepts

base	***Euclidean space*** $\mathbb{R}^n$	***smooth manifold*** M	***topological space*** M
element	p, **global coordinate** $\boldsymbol{x} = (x^1, \ldots, x^n)$	p, **local coordinate** $\varphi(p) = (x^1, \ldots, x^n)$	p
basis of base	coordinate vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$	the *local frame* for M	the *local frame* for M
vector space (***fiber***) at p	tangent space $T_{\boldsymbol{x}}\mathbb{R}^n \simeq \{\boldsymbol{x}\} \times \mathbb{R}^n \simeq \mathbb{R}^n$	**tangent space** $T_pM \simeq \{p\} \times \mathbb{R}^n$	**fiber** $E_p = \pi^{-1}(p)$; $E_p \simeq \{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$
dimension of vector space	n	n	k
basis of vector space	$\left(\frac{\partial}{\partial x^1}\Big\rvert_p, \ldots, \frac{\partial}{\partial x^n}\Big\rvert_p\right) \equiv (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$	$\left(\frac{\partial}{\partial x^1}\Big\rvert_p, \ldots, \frac{\partial}{\partial x^n}\Big\rvert_p\right)$	$(\sigma_1(p), \ldots, \sigma_k(p))$
element in vector space	tangent vector $\boldsymbol{v} = v^i \boldsymbol{e}_i$	**tangent vector** $v = v^i \frac{\partial}{\partial x^i}\Big\rvert_p$	$v = v^i \sigma_i(p)$
total space of ***bundle***	tangent bundle $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$	**tangent bundle** $TM = \bigsqcup_{p \in M} T_pM$	**vector bundle** $E = \bigsqcup_{p \in M} E_p,$
element in bundle	$(x^1, \ldots, x^n, v^1, \ldots, v^n)$	$(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n)$	$(x^1(p), \ldots, x^n(p), v^1, \ldots, v^k)$
section	**global vector field** $X = X^i \boldsymbol{e}_i \equiv X^i \frac{\partial}{\partial x^i}$	**local vector field** $X = X^i \frac{\partial}{\partial x^i}$ $X_p \in T_pM$	**local section** $\tau = \tau^i \sigma_i$ $\tau(p) \in E_p$
vector space of sections	$\mathfrak{X}(\mathbb{R}^n) \simeq \mathbb{R}^n$	$\mathfrak{X}(M) \equiv \Gamma(TM)$	$\Gamma(E)$
frame	**coordinate basis** or **global frame** $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$	**coordinate vector fields** $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$	**local frame** $(\sigma_1, \ldots, \sigma_k)$

3.4 Bundle Homomorphism

- **Definition** If $\pi : E \to M$ and $\pi' : E' \to M'$ are vector bundles, a ***continuous map*** $F : E \to E'$ is called a ***bundle homomorphism*** if there exists a map $f : M \to M'$ satisfying $\pi' \circ F = f \circ \pi$,
$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M', \end{array}$$
with the property that for each $p \in M$, the ***restricted map*** $F|_{E_p} : E_p \to E'_{f(p)}$ is ***linear***. The *relationship* between F and f is expressed by saying that ***F covers f***.

- **Definition** A ***bijective bundle homomorphism*** $F : E \to E'$ whose *inverse* is also a *bundle homomorphism* is called a ***bundle isomorphism***; if F is also a *diffeomorphism*, it is called a ***smooth bundle isomorphism***. If there exists a *(smooth) bundle isomorphism* between E and E', the two bundles are said to be ***(smoothly) isomorphic***.

- **Definition** ***A bundle homomorphism over*** M is a *bundle homomorphism* covering the ***identity map*** of M; or in other words, a *continuous* map $F : E \to E'$ such that
$$\begin{array}{ccc} E & \xrightarrow{\quad F \quad} & E' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & M, & \end{array}$$
and whose ***restriction to each fiber*** *is* ***linear***. If there exists a *bundle homomorphism* $F : E \to E'$ *over* M that is also a *(smooth) bundle isomorphism*, then we say that E and E' are ***(smoothly) isomorphic over*** M.

- **Definition** Suppose $E \to M$ and $E' \to M'$ are smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth global sections. If $F : E \to E'$ is a ***smooth bundle homomorphism over*** M, then ***composition with*** F ***induces*** a map $\widetilde{F} : \Gamma(E) \to \Gamma(E')$ as follows:
$$\widetilde{F}(\sigma)(p) = (F \circ \sigma)(p) = F(\sigma(p)) \tag{44}$$
It is easy to check that $\widetilde{F}(\sigma)$ is a ***section*** of E', and it is ***smooth*** by composition.

- **Definition** A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is said to be ***linear over*** $\mathcal{C}^\infty(M)$ if for any smooth functions $u_1, u_2 \in \mathcal{C}^\infty(M)$ and smooth sections $\sigma_1, \sigma_2 \in \Gamma(E)$,
$$\mathcal{F}(u_1\sigma_1 + u_2\sigma_2) = u_1\mathcal{F}(\sigma_1) + u_2\mathcal{F}(\sigma_2).$$

- **Lemma 3.17** ***(Bundle Homomorphism Characterization Lemma).*** *[Lee, 2003.]*
Let $\pi : E \to M$ and $\pi' : E' \to M'$ be smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth sections. A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is ***linear over*** $\mathcal{C}^\infty(M)$ ***if and only if*** *there is a* ***smooth bundle homomorphism*** $F : E \to E'$ *over M such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

- **Remark** Because of *Bundle Homomorphism Characterization Lemma*, we usually dispense with the notation $\widetilde{F}$ and use ***the same symbol*** for both ***a bundle homomorphism*** $F : E \to E'$ over M and ***the linear map*** $F : \Gamma(E) \to \Gamma(E')$ that it induces on ***sections***, and we refer to a map of ***either*** *of these types* as ***a bundle homomorphism***.

3.5 Cotangent Bundle, Coframes and Covector Field

- **Definition** For any smooth manifold M with or without boundary, *the disjoint union*

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

is called the ***cotangent bundle of*** M. It has a ***natural projection map*** $\pi : T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$.

- **Definition** Given any smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ we can show that the ***basis*** for T_p^*M dual to $(\frac{\partial}{\partial x^i}\big|_p)$ is *the differential of coordinate function* $(dx^i\big|_p)$. This defines n maps $dx^1, \ldots, dx^n : U \to T^*M$, called ***coordinate covector fields***.

- **Definition** As in the case of the *tangent bundle*, smooth local coordinates for M yield smooth local coordinates for its *cotangent bundle*. If (x^i) are *smooth coordinates* on an open subset $U \subseteq M$, the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\Phi\left(\xi_i \, dx^i\big|_p\right) = (x^1(p), \ldots, x^n(p), \xi_1, \ldots, \xi_n)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) the ***natural coordinates*** for $T * M$ associated with (x^i).

- **Remark** Here ξ_i is the *fiber coordinates* for the covectors ω in T_p^*M.

- **Definition** A ***(local or global) section*** of T^*M is called a ***covector field*** or a ***(differential) 1-form***.

- **Remark** (***Representation of Covector Field via Coordinate Fields***) In any smooth local coordinates on an open subset $U \subseteq M$; ***a (rough) covector field*** $\omega \in \Gamma(T^*M)$ can be written in terms of ***the coordinate covector fields*** (dx^i) as

$$\omega = \omega_i \, dx^i$$

where n *functions* $\omega_i : U \to \mathbb{R}$ are called the ***component functions*** of ω. They are characterized by

$$\omega_i = \omega_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

- **Remark** If ω is a ***(rough) covector field*** and X is a ***vector field*** on M, then we can form a ***function*** $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega_p(X_p), \quad p \in M.$$

If we write $\omega = \omega_i \, \lambda^i$ and $X = X^j \frac{\partial}{\partial x_j}$ in terms of *local coordinates*, then $\omega(X)$ has the ***local coordinate representation*** $\omega(X) = \omega_i \, X^i$.

- **Remark** Recall that $\omega_p(v) \in \mathbb{R}$ and $\omega(X) \in \mathcal{C}^\infty(M)$.

- **Definition** Let M be a smooth manifold with or without boundary, and let $U \subseteq M$ be an open subset. ***A local coframe*** for M over U is an ordered n-tuple of covector fields $(\epsilon^1, \ldots, \epsilon^n)$ defined on U such that $(\epsilon^i|_p)$ forms a basis for T_p^*M at each point $p \in U$. If $U = M$, it is called ***a global coframe***. (***A local coframe*** for M is just a local frame for the vector bundle T^*M

- **Example** (***Coordinate Coframes***).
  For any smooth chart $(U, (x^i))$, the ***coordinate covector fields*** (dx^i) constitute a local coframe over U, called ***a coordinate coframe***. Every coordinate frame is ***smooth***, because its ***component functions*** in the given chart are ***constants***.

- **Definition** Given a local frame $E_1, \ldots, E_n)$ for TM over an open subset U, there is a ***uniquely determined (rough) local coframe*** $(\epsilon^1, \ldots, \epsilon^n)$ over U such that $\epsilon_i|_p$ is *the **dual basis** to* $E_i|_p$ for each $p \in U$, or equivalently $\epsilon^i(E_j) = \delta^i_j$. This coframe is called *the **coframe dual to*** (E_i). *Conversely*, if we start with a local coframe (ϵ^i) over an open subset $U \subseteq M$, there is a uniquely determined local frame (E_i), called *the **frame dual to*** (ϵ^i), determined by $\epsilon^i(E_j) = \delta^i_j$.

- **Remark** The coframe dual to $(\partial/\partial x^i)$ is (dx^i) and the frame dual to (dx^i) is $(\partial/\partial x^i)$.

- **Remark** We denote *the real vector space of **all smooth covector fields** on M by* $\mathfrak{X}^*(M)$ (or $\Gamma(T^*M)$). As smooth sections of a vector bundle, elements of $\mathfrak{X}^*(M)$ can be ***multiplied*** by smooth real-valued functions: if $f \in \mathcal{C}^\infty(M)$ and $\omega \in \mathfrak{X}^*(M)$, the covector field $f\omega$ is defined by

$$(f\omega)_p = f(p)\,\omega_p. \tag{45}$$

  Because it is the space of smooth sections of a vector bundle, $\mathfrak{X}^*(M)$ is a *module* over $\mathcal{C}^\infty(M)$.

- **Remark** Note that a nonzero linear functional $\omega_p \in T_p^*M$ is completely determined by two pieces of data: its ***kernel***, which is a linear hyperplane in T_pM (*a codimension-*1 *linear subspace*); and the set of vectors v for which $\omega_p(v) = 1$, which is *an **affine hyperplane parallel to the kernel*** The value of $\omega_p(v)$ for any other vector v is then obtained by linear interpolation or extrapolation.

- **Remark** (***Visualize the Vector Fields and the Covector Fields***)

  1. A vector field on M can be considered as an arrow attached to each point of M.

  2. A covector field on M can be considered as defining ***a pair of hyperplanes*** in each tangent space, *one* ***through the origin*** and *another* ***parallel*** *to it*, and varying continuously from point to point.

     Where the covector field is small, one of the hyperplanes becomes *very far from the kernel*, eventually disappearing altogether at points where the covector field takes the value zero.

- **Definition** Let f be a *smooth real-valued function* on a *smooth manifold* M with or without boundary. (As usual, all of this discussion applies to functions defined on an open subset $U \subseteq M$; simply by *replacing* M with U throughout.) We define a ***covector field*** df, called ***the differential of*** f, by

$$df_p(v) = v\,f, \quad \forall v \in T_pM.$$

- **Remark** (***Coordinate Representation of differential of*** f)
  Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (dx^i) be the corresponding *coordinate coframe* on U. Then ***the coordinate representation of*** df:

$$df = \frac{\partial f}{\partial x^i}\,dx^i \tag{46}$$

Thus, the ***component functions*** of df in any smooth coordinate chart are ***the partial derivatives of*** f *with respect to those coordinates.* Because of this, we can think of df as *an analogue of the classical gradient*, reinterpreted in a way that makes *coordinate-independent sense* on a manifold.

- **Remark** (***The Differential*** df_p ***is the Best Linear Approximation of Function*** f ***Near*** p)
Suppose M is a smooth manifold and $f \in \mathcal{C}^{\infty}(M)$, and let p be a point in M. By choosing smooth coordinates on a neighborhood of p, we can think of f as a function on an open subset $U \subseteq \mathbb{R}^n$. Recall that $dx^i|_p$ is the *linear functional that picks out the i-th component of a tangent vector* at p. Writing $\Delta f = f(p+v) - f(p)$ for $v \in \mathbb{R}^n$, Taylors theorem shows that f is well approximated when v is small by
$$\Delta f = f(p+v) - f(p) \approx \frac{\partial f}{\partial x^i}(p)v^i = \frac{\partial f}{\partial x^i}(p)dx^i(v) = df_p(v).$$
In other words, df_p ***is the linear functional that best approximates*** f ***near*** p.

  The great power of the concept of the differential comes from the fact that we can define df ***invariantly on any manifold***, without resorting to vague arguments involving *infinitesimals.*

3.6 Pushforward and Pullback

- **Definition** Suppose $F : M \to N$ is *smooth* and X is a *vector field* on M, and suppose there happens to be a *vector field* Y on N with the property that for each $p \in M$,
$$dF_p(X_p) = Y_{F(p)}.$$
In this case, we say *the* ***vector fields*** X *and* Y *are* ***F-related***.

- **Remark** The ***differential*** dF_p is defined *locally*, and it ***does not guarantee to map a vector field (a global concept) to a vector field***. For example, if F is *not surjective*, there is no way to decide what vector to assign to a point $q \in N \setminus F(M)$. If F is *not injective*, then for some points of N there may be several different vectors obtained by applying dF to X at different points of M.

- **Proposition 3.18** *Suppose* $F : M \to N$ *is a smooth map between manifolds with or without boundary,* $X \in \mathfrak{X}(M)$*, and* $Y \in \mathfrak{X}(N)$*. Then* X *and* Y *are* F***-related if and only if*** *for* ***every smooth real-valued function*** f *defined on an open subset of* N*,*
$$X(f \circ F) = (Yf) \circ F \tag{47}$$

- **Proposition 3.19** *Suppose* M *and* N *are smooth manifolds with or without boundary, and* $F : M \to N$ *is a* ***diffeomorphism****. For every* $X \in \mathfrak{X}(M)$*, there is a* ***unique*** *smooth vector field on* N *that is* F*-related to* X*.*

- **Definition** Suppose M and N are *smooth manifolds with or without boundary*, and $F : M \to N$ is a ***diffeomorphism***. For every $X \in \mathfrak{X}(M)$, there is a ***unique*** *smooth vector field* Y on N that is F-related to X. We denote the ***unique vector field*** that is F***-related*** to X by F_*X, and call it the ***pushforward of*** X ***by*** F. And F_*X is defined explicitly by the formula
$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}), \quad \forall q \in N. \tag{48}$$

- **Corollary 3.20** *Suppose $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. For any $f \in \mathcal{C}^{\infty}(N)$,*

$$(F_{*}X\, f) \circ F = X(f \circ F)$$

- **Definition** Let $F : M \to N$ be a *smooth map* between smooth manifolds with or without boundary, and let $p \in M$ be arbitrary. The differential $dF_p : T_pM \to T_{F(p)}N$ yields *a* ***dual linear map***

$$dF_p^{*} : T_{F(p)}^{*}N \to T_p^{*}M,$$

called ***the (pointwise) pullback by** F **at** p*, or ***the cotangent map* of** F. Unraveling the definitions, we see that dF_p^{*} is characterized by

$$dF_p^{*}(\omega)(v) = \omega(dF_p(v)), \quad \omega \in T_{F(p)}^{*}N,\ v \in T_p^{*}M.$$

- **Definition** Given a smooth map $F : M \to N$ and a *covector field* ω on N , define a ***rough covector field*** $F^{*}\omega$ on M, called the ***pullback of** ω **by** F*, by

$$(F^{*}\omega)_p = dF_p^{*}\left(\omega_{F(p)}\right) \tag{49}$$

We also denote *the pullback of ω by F as $F^{\#}\omega$.*

- **Remark** ***Pushforward operator*** F_{*} is *more restricted* than ***Pullback operator*** F^{*} on F. The former acts on a vector field on M to produce a vector field on N and the latter acts on a covector field (a differential 1-form) on N to produce a covector field on M.

- **Remark** Get familiar with the following expressions:

  1. For $g \in \mathcal{C}^{\infty}(N)$, $q = F(p) \in N$ so that $p = F^{-1}(q) \in M$,

  $$(F_{*}X)_q\, g = dF_p(X_p)g = X_p\,(g \circ F)$$

  2. For $p \in M$, $X_p \in T_pM$, $q = F(p) \in N$, $\omega_q \in T_q^{*}N$,

  $$(F^{*}\omega)_p(X_p) = \left(dF_p^{*}\omega_q\right)(X_p) = \omega_q\left(dF_p(X_p)\right)$$

  The last equality use the definition of dual map $(A^{*}w)(v) = w(A\,v)$

  3. Given the coordinate representation of covector $\omega = \omega_j dy^j$, the pullback of a covector field can also be written in the following way:

  $$\begin{aligned} F^{*}\omega &= F^{*}(\omega_j dy^j) = (\omega_j \circ F)F^{*}(dy^j) \\ &= (\omega_j \circ F)\, d\left(y^j \circ F\right) && (50) \\ &= (\omega_j \circ F)\, d\,F^j && (51) \end{aligned}$$

  $F^{*}\omega$ is computed as follows: whereaver you see y^i in the expression for B, just substitute the ith component function of F and expand.

  4. For a diffeomorphism F, $(F^{*})^{-1} = F_{*}$. That is ***the inverse of pullback operation is the pushforward operation***.

3.7 Compare the Tangent and Cotangent Bundles

Table 2: Comparison between tangent space and cotangent space

base	***smooth manifold*** M	***smooth manifold*** M
element	$\varphi(p) = (x^1, \dots, x^n)$	$\varphi(p) = (x^1, \dots, x^n)$
vector space (***fiber***) at p	**tangent space** T_pM	**cotangent space** $T_p^*M = (T_pM)^*$
dimension of vector space	n	n
basis of vector space	$\left(\frac{\partial}{\partial x^1}\Big\|_p, \dots, \frac{\partial}{\partial x^n}\Big\|_p\right)$	$(dx^1\|_p, \dots, dx^n\|_p)$
element in vector space	**tangent vector** $: \mathcal{C}^\infty(M) \to \mathbb{R}$ $v = v^i \frac{\partial}{\partial x^i}\Big\|_p$	**cotangent vector** $: T_pM \to \mathbb{R}$ $\omega = \xi_i \; dx^i\|_p$
total space of ***bundle***	**tangent bundle** $TM = \bigsqcup_{p \in M} T_pM$	**cotangent bundle** $T^*M = \bigsqcup_{p \in M} T_p^*M,$
element in bundle	$(x^1(p), \dots, x^n(p), v^1, \dots, v^n)$	$(x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n)$
section	**local vector field** $X = X^i \frac{\partial}{\partial x^i}$ $X_p \in T_pM$	**local covector field** $\omega = \xi_i dx^i$ $\omega_p \in T_p^*M$
vector space of sections	$\mathfrak{X}(M) \equiv \Gamma(TM)$	$\mathfrak{X}^*(M) \equiv \Gamma(T^*M)$
frame	**coordinate vector fields** $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$	**coordinate covector fields** $(dx^1, \dots, dx^n)$
duality	$\xi\left(\frac{\partial}{\partial x^i}\Big\|_p\right)(dx^j\|_p) = \delta_i^j$	$dx^j\|_p \left(\frac{\partial}{\partial x^i}\Big\|_p\right) = \delta_i^j$
change of coordinates	**contravariant** $\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, v^i$	**covariant** $\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\, \widetilde{\omega}_j$
functions	$F : M \to N$ ***diffeomorphism*** $dF_p : T_pM \to T_{F(p)}N$ ***Pushforward***: $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ $(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}),\ q \in N$	$dF_p^* : T_{F(p)}^*N \to T_p^*M$ ***dual map of*** dF_p ***Pullback***: $F^* : \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$ $(F^*\omega)_p = dF_p^*\left(\omega_{F(p)}\right),\ p \in M$

References

Herbert Amann. *Ordinary differential equations: an introduction to nonlinear analysis*, volume 13. Walter de gruyter, 2011.

Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos.* Academic press, 2012.

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.