

Summary Part 1: Probabilistic Methods for Non-Asymptotic Analysis

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1 Basic Inequalities

1.1 Arithmetic, Calculus and Algebra

- Remark (*Basic Inequalities*)

1. *Arithmetic Mean-Geometric Mean Inequality:*

$$\sum_{i=1}^n \alpha_i x_i \geq \prod_{i=1}^n x_i^{\alpha_i}$$

where $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$. Simple case, $\frac{a+b}{2} \geq \sqrt{ab}$

2. $a^2 + b^2 \geq \pm 2ab$; Also $a + b \geq 2\sqrt{ab}$

3. $(a + b)^2 \leq 2(a^2 + b^2)$; Also $(\sqrt{a} + \sqrt{b})^2 \leq 2(a + b)$

4. $x + 1 \leq e^x$

5. $\log x \leq x - 1$ for $x > 0$

6. $e^s - e^t \leq e^t(s - t)$ for $s \geq t$

- 7.

$$h(x) := (x + 1) \log(x + 1) - x \geq \frac{x^2}{2(1 + x/3)}, \quad \text{for } x > 0$$

- 8.

$$-\log(1 - x) - x \leq \frac{x^2}{2(1 - x)}, \quad \text{for } x \in (0, 1)$$

- 9.

$$h_1(x) := 1 + x - \sqrt{1 + 2x} \geq \frac{x^2}{2(1 + x)}, \quad \text{for } x > 0.$$

10. *Log-Sum Inequality:*

For non-negative numbers a_1, \dots, a_n and b_1, \dots, b_n ,

$$\sum_{i=1}^n a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i \right) \log \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$$

with equality if and only if $\frac{a_i}{b_i}$ is constant.

11. Taylor series:

$$\begin{aligned}
e^x &:= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots, \quad \forall x \\
\log(1+x) &:= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \dots, \quad |x| < 1 \\
\frac{1}{1-x} &:= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots, \quad |x| < 1 \\
(1+x)^\alpha &:= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n
\end{aligned}$$

1.2 Function Space, Convexity and Duality

- **Proposition 1.1 (*Jensen's inequality*)** [Vershynin, 2018]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $f : \Omega \rightarrow \mathbb{R}$ be a \mathbb{P} -measurable function and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be **convex function**. Then

$$\varphi(\mathbb{E}[X]) := \varphi\left(\int X d\mathbb{P}\right) \leq \int \varphi \circ X d\mathbb{P} := \mathbb{E}[\varphi(X)]. \quad (1)$$

- **Remark** As a simple consequence of Jensen's inequality, $\|X\|_{L^p}$ is an **increasing function in p** , that is

$$\|X\|_{L^p} \leq \|X\|_{L^q} \quad \text{for any } 1 \leq p \leq q \leq \infty \quad (2)$$

This inequality follows since $\varphi(x) = x^{q/p}$ is a *convex function* if $q/p \geq 1$.

- **Proposition 1.2 (*Minkowski's inequality*)** [Vershynin, 2018]

For any $p \in [1, \infty]$, $X, Y \in L^p(\Omega, \mathbb{P})$,

$$\|X + Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}, \quad (3)$$

which implies that $\|\cdot\|_{L^p}$ is a norm.

- **Proposition 1.3 (*Cauchy-Schwarz inequality*)** [Vershynin, 2018]

For any random variables $X, Y \in L^2(\Omega, \mathbb{P})$, the following inequality is satisfied:

$$|\langle X, Y \rangle_{L^2}| := |\mathbb{E}[XY]| \leq \|X\|_{L^2} \|Y\|_{L^2}. \quad (4)$$

This inequalities can be extended to *conjugate spaces* L^p and L^q

- **Proposition 1.4 (*Hölder's inequality*)** [Vershynin, 2018]

For $p, q \in (1, \infty)$, $1/p + 1/q = 1$, then the random variables $X \in L^p(\Omega, \mathbb{P})$, $Y \in L^q(\Omega, \mathbb{P})$ satisfy

$$|\langle X, Y \rangle_{L^2}| := |\mathbb{E}[XY]| \leq \|X\|_{L^p} \|Y\|_{L^q}. \quad (5)$$

- **Definition (*Legendre Transform*)**

Let $\mathcal{X} \subset \mathbb{R}^n$ be a **convex set**, and $f : \mathcal{X} \rightarrow \mathbb{R}$ a **convex** function; then its **Legendre transform** is the function $f^* : \mathcal{X}^* \rightarrow \mathbb{R}$ defined by

$$f^*(x^*) = \sup_{x \in \mathcal{X}} (\langle x^*, x \rangle - f(x)), \quad x^* \in \mathcal{X}^*$$

where \sup denotes the supremum, and the domain \mathcal{X}^* is

$$\mathcal{X}^* = \left\{ x^* \in \mathbb{R}^n : \sup_{x \in \mathcal{X}} (\langle x^*, x \rangle - f(x)) < \infty \right\}.$$

The function f^* is called the **convex conjugate function** of f .

- **Theorem 1.5 (*Fenchel's Inequality / Fenchel-Young Inequality*)**

Suppose $f^* : \mathcal{X}^* \rightarrow \mathbb{R}$ is the convex conjugate of function $f : \mathcal{X} \rightarrow \mathbb{R}$. For every $x \in \mathcal{X}$ and $p \in \mathcal{X}^*$, i.e., independent (x, p) pairs,

$$\langle p, x \rangle \leq f(x) + f^*(p). \quad (6)$$

- **Theorem 1.6 (*Young's Convolution Inequality*)**

Suppose f is in the Lebesgue space $L^p(\mathbb{R}^d)$ and g is in $L^q(\mathbb{R}^d)$ and $1/p + 1/q = 1/r + 1$ with $1 \leq p, q, r \leq \infty$. Then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \quad (7)$$

Here the star denotes **convolution**:

$$(f * g)(t) = \int_{\mathbb{R}^d} f(t - \tau)g(\tau)d\tau$$

1.3 Probability Theory

- Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : \Omega \rightarrow \mathbb{R}$ is a real-valued measurable function on Ω .
- For a random variable X , the **expectation** and **variance** are denoted as

$$\begin{aligned} \mathbb{E}[X] &= \int X d\mathbb{P} \\ \text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \end{aligned}$$

- The **moment generating function** of X and its **logarithm** are denoted as

$$\begin{aligned} M_X(\lambda) &:= \mathbb{E}[e^{\lambda X}] \\ \psi_X(\lambda) &:= \log \mathbb{E}[e^{\lambda X}] \end{aligned}$$

- For $p > 0$, the **p -th moment** of X is defined as $\mathbb{E}[X^p]$, and the **p -th absolute moment** is $\mathbb{E}[|X|^p]$.

- The L^p **norm** of X is

$$\|X\|_{L^p} := \mathbb{E} [|X|^p]^{1/p}$$

where $1 \leq p < \infty$. Note that the L^p space is a *Banach space*, which is defined as

$$L^p(\Omega, \mathbb{P}) := \{X : \|X\|_{L^p} < \infty\}.$$

- The **essential supremum** of $|X|$ is the L^∞ **norm** of X

$$\|X\|_{L^\infty} := \text{ess sup } |X|$$

Similarly, L^∞ is a Banach space as well

$$L^\infty(\Omega, \mathbb{P}) := \{X : \|X\|_{L^\infty} < \infty\}.$$

- For $p = 2$, L^2 space is a *Hilbert space* with inner product between random variables $X, Y \in L^2(\Omega, \mathbb{P})$

$$\langle X, Y \rangle_{L^2} := \mathbb{E} [XY] = \int XY d\mathbb{P}$$

The **standard deviation** is

$$\sigma(X) = (\text{Var}(X))^{1/2} = \|X - \mathbb{E}[X]\|_{L^2}.$$

The **covariance** is defined as

$$\begin{aligned} \text{cov}(X, Y) &:= \langle X - \mathbb{E}[X], Y - \mathbb{E}[Y] \rangle \\ &= \mathbb{E} [(X - \mathbb{E}[X]) (Y - \mathbb{E}[Y])] \end{aligned}$$

When we consider random variables as vectors in the Hilbert space L^2 , the identity above gives a **geometric interpretation of the notion of covariance**. The more the vectors $X - \mathbb{E}[X]$ and $Y - \mathbb{E}[Y]$ are aligned with each other, the bigger their inner product and covariance are.

- The **cumulative distribution function (CDF)** is defined as

$$F_X(t) := \mathbb{P}[X \leq t], \quad t \in \mathbb{R}.$$

The following result is important

Lemma 1.7 (Integral Identity). [Vershynin, 2018]

Let X be a **non-negative** random variable. Then

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > t] dt. \tag{8}$$

The two sides of this identity are either finite or infinite simultaneously.

- **Theorem 1.8** (*Central Limit Theorem, Linderberg-Levy*)

Let X_1, \dots, X_n be **independent identically distributed** random variables with mean $\mathbb{E}[X_i] = 0$ and variance $\text{Var}(X_i) = 1$. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \xrightarrow{d} N(0, 1) \quad (9)$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \leq t \right\} - \Phi(t) \right| = 0$$

where $\Phi(t) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \mathbb{P}\{g \leq t\}$ for some Gaussian variable g .

- **Theorem 1.9** (*Central Limit Theorem, Nonasymptotic, Berry-Esseen*) [Vershynin, 2018]

Let X_1, \dots, X_n be **independent identically distributed** random variables with mean $\mathbb{E}[X_i] = 0$, variance $\text{Var}(X_i) = \sigma^2$ and $\rho := \mathbb{E}[|X_i|^3] < \infty$. Then with some constant $C > 0$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n X_i \leq t \right\} - \Phi(t) \right| \leq \frac{C}{\sigma^3\sqrt{n}} \rho \quad (10)$$

where $\Phi(t) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \mathbb{P}\{g \leq t\}$ for some Gaussian variable g .

- **Remark** The *Berry-Esseen* version of central limit theorem is **non-asymptotic** and it has a bound

$$\mathbb{P} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \leq t \right\} \leq \mathbb{P}\{g \leq t\} + \frac{C}{\sqrt{n}} \rho = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du + \frac{C}{\sqrt{n}} \rho$$

This bound is **sharp**, i.e. the equality is attained when $X_i \sim \text{Bernoulli}(1/2)$.

- **Theorem 1.10** (*Poisson Limit Theorem*). [Vershynin, 2018]

Let $X_{N,i}$, $1 \leq i \leq N$, be independent random variables $X_{N,i} \sim \text{Ber}(p_{N,i})$, and let $S_N = \sum_{i=1}^N X_{N,i}$. Assume that, as $N \rightarrow \infty$

$$\max_{i \leq N} p_{N,i} \rightarrow 0 \quad \text{and} \quad \mathbb{E}[S_N] = \sum_{i=1}^N p_{N,i} \rightarrow \lambda < \infty,$$

Then, as $N \rightarrow \infty$,

$$S_N = \sum_{i=1}^N X_{N,i} \xrightarrow{d} \text{Pois}(\lambda)$$

1.4 Information Theory

- **Definition** (*Shannon Entropy*) [Cover and Thomas, 2006]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \mathbb{R} \rightarrow \mathcal{X}$ be a random variable. Define $p(x)$ as the

probability density function of X with respect to a base measure μ on \mathcal{X} . **The Shannon Entropy** is defined as

$$\begin{aligned} H(X) &:= \mathbb{E}_p [-\log p(X)] \\ &= \int_{\Omega} -\log p(X(\omega)) d\mathbb{P}(\omega) \\ &= - \int_{\mathcal{X}} p(x) \log p(x) d\mu(x) \end{aligned}$$

- **Definition (Conditional Entropy)** [Cover and Thomas, 2006]

If a pair of random variables (X, Y) follows the joint probability density function $p(x, y)$ with respect to a base product measure μ on $\mathcal{X} \times \mathcal{Y}$. Then **the joint entropy** of (X, Y) , denoted as $H(X, Y)$, is defined as

$$H(X, Y) := \mathbb{E}_{X, Y} [-\log p(X, Y)] = - \int_{\mathcal{X} \times \mathcal{Y}} p(x, y) \log p(x, y) d\mu(x, y)$$

Then **the conditional entropy** $H(Y|X)$ is defined as

$$\begin{aligned} H(Y|X) &:= \mathbb{E}_{X, Y} [-\log p(Y|X)] = - \int_{\mathcal{X} \times \mathcal{Y}} p(x, y) \log p(y|x) d\mu(x, y) \\ &= \mathbb{E}_X [\mathbb{E}_Y [-\log p(Y|X)]] = \int_{\mathcal{X}} p(x) \left(- \int_{\mathcal{Y}} p(y|x) \log p(y|x) d\mu(y) \right) d\mu(x) \end{aligned}$$

- **Proposition 1.11 (Properties of Shannon Entropy)** [Cover and Thomas, 2006]

Let X, Y, Z be random variables.

1. (**Non-negativity**) $H(X) \geq 0$;
2. (**Concavity**) $H(p) := \mathbb{E}_p [-\log p(X)]$ is a concave function in terms of p.d.f. p , i.e.

$$H(\lambda p_1 + (1 - \lambda)p_2) \geq \lambda H(p_1) + (1 - \lambda)H(p_2)$$

for any two p.d.fs p_1, p_2 on \mathcal{X} and any $\lambda \in [0, 1]$.

- **Definition (Relative Entropy / Kullback-Leibler Divergence)** [Cover and Thomas, 2006]

Suppose that P and Q are probability measures on a measurable space \mathcal{X} , and P is absolutely continuous with respect to Q , then **the relative entropy** or **the Kullback-Leibler divergence** is defined as

$$\text{KL}(P \parallel Q) := \mathbb{E}_P \left[\log \left(\frac{dP}{dQ} \right) \right] = \int_{\mathcal{X}} \log \left(\frac{dP(x)}{dQ(x)} \right) dP(x)$$

where $\frac{dP}{dQ}$ is the Radon-Nikodym derivative of P with respect to Q . Equivalently, the KL-divergence can be written as

$$\text{KL}(P \parallel Q) = \int_{\mathcal{X}} \left(\frac{dP(x)}{dQ(x)} \right) \log \left(\frac{dP(x)}{dQ(x)} \right) dQ(x)$$

which is the entropy of P relative to Q . Furthermore, if μ is a base measure on \mathcal{X} for which densities p and q with $dP = p(x)d\mu$ and $dQ = q(x)d\mu$ exist, then

$$\text{KL}(P \parallel Q) = \int_{\mathcal{X}} p(x) \log \left(\frac{p(x)}{q(x)} \right) d\mu(x)$$

- **Definition (Mutual Information)** [Cover and Thomas, 2006]

Consider two random variables X, Y on $\mathcal{X} \times \mathcal{Y}$ with joint probability distribution $P_{(X,Y)}$ and marginal distribution P_X and P_Y . **The mutual information** $I(X; Y)$ is the relative entropy between the joint distribution $P_{(X,Y)}$ and the product distribution $P_X \otimes P_Y$:

$$I(X; Y) = \mathbb{KL}(P_{(X,Y)} \parallel P_X \otimes P_Y) = \mathbb{E}_{P_{(X,Y)}} \left[\log \frac{dP_{(X,Y)}}{dP_X \otimes dP_Y} \right]$$

If $P_{(X,Y)}$ has a probability density function $p(x, y)$ with respect to a base measure μ on $\mathcal{X} \times \mathcal{Y}$, then

$$I(X; Y) = \int_{\mathcal{X} \times \mathcal{Y}} p(x, y) \log \left(\frac{p(x, y)}{p_X(x)p_Y(y)} \right) d\mu(x, y)$$

- **Proposition 1.12 (Properties of Relative Entropy and Mutual Information)** [Cover and Thomas, 2006]

Let X, Y be random variables.

1. **(Non-negativity)** Let $p(x), q(x)$ be probability density function of P, Q .

$$\mathbb{KL}(P \parallel Q) \geq 0$$

with equality if and only if $p(x) = q(x)$ almost surely. Therefore, the mutual information is non-negative as well:

$$I(X; Y) \geq 0$$

with equality if and only if X and Y are independent.

2. **(Symmetry)** $I(X; Y) = I(Y; X)$
3. **(Information Gain via Conditioning)** The mutual information $I(X; Y)$ is the reduction in the uncertainty of X due to the knowledge of Y (and vice versa)

$$\begin{aligned} I(X; Y) &= H(X) - H(X|Y) \\ &= H(Y) - H(Y|X) \\ &= H(X) + H(Y) - H(X, Y) \end{aligned} \tag{11}$$

4. **(Shannon Entropy as Self-Information)** $I(X; X) = H(X)$
5. **(Joint Convexity of Relative Entropy)** The relative entropy $\mathbb{KL}(p \parallel q)$ is **convex** in the pair (p, q) ; that is, if (p_1, q_1) and (p_2, q_2) are two pairs of probability density functions, then for $\lambda \in [0, 1]$,

$$\mathbb{KL}(\lambda p_1 + (1 - \lambda)p_2 \parallel \lambda q_1 + (1 - \lambda)q_2) \leq \lambda \mathbb{KL}(p_1 \parallel q_1) + (1 - \lambda) \mathbb{KL}(p_2 \parallel q_2) \tag{12}$$

- **Proposition 1.13 (Conditioning Reduces Entropy)** [Cover and Thomas, 2006]

From non-negativity of mutual information, we see that the entropy of X is non-increasing when conditioning on Y

$$H(X|Y) \leq H(X) \tag{13}$$

where equality holds if and only if X and Y are independent.

- **Proposition 1.14** (*Chain Rule for Entropy*) [Cover and Thomas, 2006]
Let X_1, X_2, \dots, X_n be drawn according to $p(x_1, x_2, \dots, x_n)$. Then

$$H(X_1, X_2, \dots, X_n) = \sum_{i=1}^n H(X_i | X_{i-1}, \dots, X_1) \quad (14)$$

- **Proposition 1.15** (*Sub-Additivity of Entropy*) [Cover and Thomas, 2006]
Let X_1, X_2, \dots, X_n be drawn according to $p(x_1, x_2, \dots, x_n)$. Then

$$H(X_1, X_2, \dots, X_n) \leq \sum_{i=1}^n H(X_i) \quad (15)$$

with equality if and only if the X_i are independent.

- **Proposition 1.16** (*Chain Rule for Relative Entropy*) [Cover and Thomas, 2006]
Let $P_{(X,Y)}$ and $Q_{(X,Y)}$ be two probability measures on product space $\mathcal{X} \times \mathcal{Y}$ and $P \ll Q$. Denote the marginal distributions P_X, Q_X and P_Y, Q_Y on \mathcal{X} and \mathcal{Y} , respectively. $P_{Y|X}$ and $Q_{Y|X}$ are conditional distributions (Note that $P_{Y|X} \ll Q_{Y|X}$). Define **the conditional relative entropy** as

$$\mathbb{E}_X [\text{KL}(P_{Y|X} \parallel Q_{Y|X})] := \mathbb{E}_X \left[\mathbb{E}_{P_{Y|X}} \left[\log \left(\frac{dP_{Y|X}}{dQ_{Y|X}} \right) \right] \right].$$

Then the relative entropy of joint distribution $P_{(X,Y)}$ with respect to $Q_{(X,Y)}$ is

$$\text{KL}(P_{(X,Y)} \parallel Q_{(X,Y)}) = \text{KL}(P_X \parallel Q_X) + \mathbb{E}_X [\text{KL}(P_{Y|X} \parallel Q_{Y|X})] \quad (16)$$

In addition, let P and Q denote two joint distributions for X_1, X_2, \dots, X_n , let $P_{1:i}$ and $Q_{1:i}$ denote the marginal distributions of X_1, X_2, \dots, X_i under P and Q , respectively. Let $P_{X_i|1\dots i-1}$ and $Q_{X_i|1\dots i-1}$ denote the conditional distribution of X_i with respect to X_1, X_2, \dots, X_{i-1} under P and under Q .

$$\text{KL}(P \parallel Q) = \sum_{i=1}^n \mathbb{E}_{P_{1:i-1}} [\text{KL}(P_{X_i|1\dots i-1} \parallel Q_{X_i|1\dots i-1})] \quad (17)$$

- **Proposition 1.17** (*Han's Inequality*) [Cover and Thomas, 2006, Boucheron et al., 2013]
Let X_1, X_2, \dots, X_n be random variables. Then

$$\begin{aligned} H(X_1, X_2, \dots, X_n) &\leq \frac{1}{n-1} \sum_{i=1}^n H(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \\ &\Leftrightarrow H(X) \leq \frac{1}{n-1} \sum_{i=1}^n H(X_{(-i)}) \end{aligned} \quad (18)$$

2 Summary: General Proof Stratgy for Concentration Problem

There are many proof techniques introduced. We can summarize them as follows:

1. *The Cramér-Chernoff Method:*

This class of methods essentially apply *the Markov inequality* on *exponential transform* $e^{\lambda X}$ with parameter λ . The key is to *bound* the *log-moment generating function* from above and then use *the Legendre transform* to find the concentration bound.

Specifically, for a real-valued random variable X , any $\lambda \geq 0$, the following inequality holds

$$\mathbb{P}\{X \geq t\} = \mathbb{P}\{e^{\lambda X} \geq e^{\lambda t}\} \leq e^{-\lambda t} \mathbb{E}[e^{\lambda X}] = \exp(-\lambda t + \psi_X(\lambda))$$

where $\psi_X(\lambda) := \log \mathbb{E}[e^{\lambda X}]$. One can choose optimal λ^* that *minimizes the upper bound above*. Since $\psi_X(\lambda)$ is a *convex function*, we can define its *Legendre transform*

$$\psi_X^*(t) := \sup_{\lambda \in \mathbb{R}} \{\lambda t - \psi_X(\lambda)\}.$$

The expression of the right-hand side is known as the *convex conjugate* of ψ_X . The Legendre transform of log-moment generating function is also its *convex conjugate*. Thus we have

$$\mathbb{P}\{X \geq t\} \leq \exp\{-\psi_X^*(t)\}$$

The lower bound can be found by applying above formula to $-X$.

In other word, in order to prove concentration around mean

$$\mathbb{P}\{f(X) \geq \mathbb{E}[f(X)] + t\} \text{ or } \mathbb{P}\{f(X) \leq \mathbb{E}[f(X)] - t\}$$

using *the Cramér-Chernoff Method*, we just need to find *the upper bound* $\phi(\lambda)$ of *the logarithmic moment generating function* $\psi(\lambda)$

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda(f(X) - \mathbb{E}[f(X)])}] \leq \phi(\lambda).$$

Remark (*Advantages and Disadvantages of Cramér-Chernoff Method*)

There are several advantages for this method:

- (a) The derivation is *distribution-free*, since *Markov inequality* is based on fundamental properties of *measure and integration theory*. Moreover, *the bounds on logarithmic moment generating function* $\psi(\lambda)$ can be used to *characterize different distributions* in terms of *their concentration behavior*.
- (b) This method is *widely applicable*. Most of techniques we learned here is to compute *the upper bound* for $\psi(\lambda)$ and then apply the *Cramér-Chernoff method*.
- (c) The formula is *easy to compute* if the *simple bounds* on *logarithmic moment generating function* is *computed*. Then it will compute the rate via *Legendre transform of upper bound of* $\psi(\lambda)$.
- (d) The function $\psi(\lambda)$ easily handles *product measures* $\mathbb{P} = \otimes_{k=1}^n \mathbb{P}_k$ (i.e. *independent variables*).

$$\psi_Z(\lambda) = \log \mathbb{E}[e^{\lambda \sum_{i=1}^n X_i}] = \log \prod_{i=1}^n \mathbb{E}[e^{\lambda X_i}] = n \psi_X(\lambda)$$

and consequently,

$$\psi_Z^*(t) = n \psi_X^* \left(\frac{t}{n} \right).$$

For ***martingale difference sequence***, we see that by conditioning on previous input

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ \lambda \left(\sum_{k=1}^n D_k \right) \right\} \right] &= \mathbb{E} \left[\mathbb{E} \left[\exp \left\{ \lambda \left(\sum_{k=1}^n D_k \right) \right\} \mid \mathcal{B}_{n-1} \right] \right] \\ &= \mathbb{E} \left[\exp \left\{ \lambda \left(\sum_{k=1}^{n-1} D_k \right) \right\} \mathbb{E} \left[\exp \{ \lambda D_n \} \mid \mathcal{B}_{n-1} \right] \right] \end{aligned}$$

If we can control each martingale difference by

$$\log \mathbb{E} \left[\exp \{ \lambda D_n \} \mid \mathcal{B}_{n-1} \right] \leq \phi(\lambda)$$

then we have

$$\begin{aligned} \psi_Z(\lambda) &\leq \log \mathbb{E} \left[\exp \left\{ \lambda \left(\sum_{k=1}^{n-1} D_k \right) \right\} \right] + \phi(\lambda) \\ &\leq \dots \\ &\leq n\phi(\lambda). \end{aligned}$$

The main disadvantage is that the Chernoff bound is ***not necessarily sharp***, since the Markov inequality is not necessarily sharp.

2. ***Entropy Method:***

The entropy method focus on the ***tensorization property*** of the ***entropy functional*** $\text{Ent}(X)$

$$\text{Ent}(X) := \mathbb{E} [X \log X] - \mathbb{E} [X] \log (\mathbb{E} [X]).$$

Specifically, let Z_1, Z_2, \dots, Z_n be *independent random variables* taking values in \mathcal{X} , and let $f : \mathcal{X}^n \rightarrow [0, \infty)$ be a measurable function. Letting $X = f(Z_1, Z_2, \dots, Z_n)$ such that $\mathbb{E} [X \log X] < \infty$, we have

$$\text{Ent}(X) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}_{(-i)}(X) \right].$$

where $\mathbb{E}_{(-i)}[\cdot]$ is the conditional expectation operator conditioning on $Z_{(-i)}$, which is equal to Z after dropping i -component. In other word, the ***key strategy*** for proving concentration using entropy method is to find ***the upper bound*** for each ***single variable entropy functional***

$$\text{Ent}_{(-i)}(X) := \mathbb{E}_{(-i)} [X \log X] - \mathbb{E}_{(-i)} [X] \log (\mathbb{E}_{(-i)} [X]) \equiv H_\Phi(\mathbb{P}_i).$$

Note that for independent random variables Z , this term ***depends only on distribution of*** Z_i , since the rest $Z_{(-i)}$ are ***controlled*** by the conditioning. For distributions such as *Gaussian, Bernoulli and Poisson*, one can use ***the logarithmic Sobolev inequalities*** to derive ***the upper bound of the entropy functional*** via ***norm of gradients***.

To obtain *the concentraion bound*, we use **the Herbst's argument**; that is, the find the bound

$$\text{Ent}(e^{\lambda X}) \leq \mathbb{E} \left[e^{\lambda X} \right] \phi(\lambda)$$

and using the differential equation for the log-moment generating function ψ

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E} [e^{\lambda Z}]} = \lambda \psi'(\lambda) - \psi(\lambda) = \lambda^2 \left(\frac{\psi(\lambda)}{\lambda} \right)',$$

we can obtain the upper bound for $\psi(\lambda)$:

$$\begin{aligned} \left(\frac{\psi(\lambda)}{\lambda} \right)' &\leq \lambda^{-2} \phi(\lambda) \\ \left(\frac{\psi(\lambda)}{\lambda} \right) &\leq \lim_{\lambda \rightarrow 0} \left(\frac{\psi(\lambda)}{\lambda} \right) + \int_0^\lambda s^{-2} \phi(s) ds \\ \psi(\lambda) &\leq \lambda \left(\mathbb{E} [X] + \int_0^\lambda s^{-2} \phi(s) ds \right). \end{aligned}$$

Finally, we apply *the Chernoff bound*.

In general, **the key advantage** of the **entropy method** is that the tensorization property allows us to **generalize the concentration result from 1-dimensional distribution to n-dimensional product distribution**.

The main effort is to find a concentration inequality for **entropy of single variable distribution**. One way to find such concentration is to use **the logarithmic Sobolev inequalities**.

3. **Transportation Method:**

The transportation method is closed related to various *statistical divergence* esp. **the Kullback-Leibler divergence** and **the information inequality**. The central part of the proof is to show that for *given distribution* \mathbb{P} of concern, **the transportation cost inequality** holds:

$$\mathcal{W}_1^d(\mathbb{Q}, \mathbb{P}) := \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \mathbb{E}_\gamma [d(Y, X)] \leq \phi^{*-1}(\text{KL}(\mathbb{Q} \parallel \mathbb{P})) \quad \forall \text{ distribution } \mathbb{Q}$$

where $\Pi(\mathbb{Q}, \mathbb{P}) = \{\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{X}) : Y_\# \gamma = \mathbb{Q}, X_\# \gamma = \mathbb{P}\}$ i.e. γ is a **coupling** of marginal distribution \mathbb{Q} and \mathbb{P} . And, for every $s \geq 0$,

$$\phi^{*-1}(s) = \inf\{t \in \text{dom}(\phi^*) : \phi^*(t) > s\}$$

is defined as the **the generalized inverse** of the Legendre transform $\phi^* = \sup_{\lambda \in (0, b)} (\lambda x - \phi(\lambda))$.

There are *two ways to proceed*:

- (a) Based on *the duality of 1-Wasserstein distance*, this *transportation cost inequality* implies that for any 1-Lipschitz function $f : \mathcal{X} \rightarrow \mathbb{R}$ with respect to metric d

$$\mathbb{E}_\mathbb{Q} [f(Y)] - \mathbb{E}_\mathbb{P} [f(X)] = \mathbb{E}_\gamma [f(Y) - f(X)] \leq \mathcal{W}_1^d(\mathbb{Q}, \mathbb{P}) \leq \phi^{*-1}(\text{KL}(\mathbb{Q} \parallel \mathbb{P})).$$

(b) Or, we use *the Cauchy-Schwartz inequality*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[f(Y)] - \mathbb{E}_{\mathbb{P}}[f(X)] &= \mathbb{E}_{\gamma}[f(Y) - f(X)] \leq \sum_{i=1}^n \alpha_i \mathbb{E}_{\gamma}[d(Y_i, X_i)] \\ &\leq \left(\sum_{i=1}^n \alpha_i^2 \right)^{1/2} \left(\sum_{i=1}^n (\mathbb{E}_{\gamma}[d(Y_i, X_i)])^2 \right)^{1/2}\end{aligned}$$

If we can show that the quadratic of transportation cost

$$\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n (\mathbb{E}_{\gamma}[d(Y_i, X_i)])^2 \leq \varphi(\text{KL}(\mathbb{Q} \parallel \mathbb{P}))$$

Then

$$\mathbb{E}_{\mathbb{Q}}[f(Y)] - \mathbb{E}_{\mathbb{P}}[f(X)] \leq \left(\left(\sum_{i=1}^n \alpha_i^2 \right) \varphi(\text{KL}(\mathbb{Q} \parallel \mathbb{P})) \right)^{1/2}$$

Finally by the **transportation lemma**, we can show that

$$\psi_{f(X)}(\lambda) := \mathbb{E}_{\mathbb{P}} \left[e^{\lambda(f(X) - \mathbb{E}[f(X)])} \right] \leq \phi(\lambda).$$

The concentration follows from *Chernoff bound* with rate function $\phi^*(t)$.

Note that the transportation cost inequality has **the tensorization property** as well. This allows us to generalize the inequality from 1-dimension distribution to product distributions.

Remark (*Advantages and Disadvantages of Transportation Method*)

There are several advantages for this method:

- (a) *The optimal transport problem and the Wasserstein distance* is closely related to **the information geometry** of probability space $\mathcal{P}(\mathcal{X})$. In particular, the transportation cost inequality relates the optimal transport cost to the relative entropy:

$$\mathcal{W}_p^d(\mathbb{Q}, \mathbb{P}) \leq \varphi(\text{KL}(\mathbb{Q} \parallel \mathbb{P})).$$

This provides an alternative **information theoretical interpretation** of the concentration behavior of independent random variables.

- (b) *The low optimal transportation cost* is closely associated with **the concentration of measure** in $\mathbb{P} \in \mathcal{P}(\mathcal{X})$. In fact, we can bound the concentration function $\alpha_{\mathbb{P},(\mathcal{X},d)}(t)$ from above by the upper bound of optimal transport cost.
- (c) **The dual formulation** naturally leads to *the concentration of Lipschitz function* or other *strong uniform continuous functions*.
- (d) The concept of **coupling** $\gamma \in \Pi(\mathbb{Q}, \mathbb{P})$ allows us to extend the concentration results to **dependent variables**, such as *Markov chains*, *Markov random field* etc. In those cases, we can separate the conditional distribution $\mathbb{P}(X_i | X_{1:i-1})$ and the marginal distributions $\mathbb{P}(X_{1:i-1})$.

4. **Concentration of Measure and Isoperimetric Inequalities:**

The applicability of **isoperimetric Inequalities** are *limited* to a few cases such as **Gaussian measure**, **Bernoulli measure** (or the uniform distribution on **binary hypercube**, compact manifolds, Lebesgue measure on \mathbb{R}^n , graph vertex and edge boundaries etc.).

The key is to derive the upper bound for **the concentration function**:

$$\alpha_{\mathbb{P},(\mathcal{X},d)} := \sup \left\{ \mathbb{P} \{A_t^c\} : A \subset \mathcal{X}, \mathbb{P}(A) \geq \frac{1}{2} \right\}.$$

Note that for $d(x, A) := \inf_{y \in A} d(x, y)$, the t -blowup of A is defined as

$$\mathbb{P} \{A_t^c\} := \mathbb{P} \{d(X, A) \geq t\}.$$

Then the goal is to find **the isoperimetric inequality**

$$\alpha_{\mathbb{P},(\mathcal{X},d)} \leq \exp(-\phi(t)) \quad \Leftrightarrow \quad \mathbb{P}(A)\mathbb{P}(A_t^c) := \mathbb{P}(A)\mathbb{P}\{d(X, A) \geq t\} \leq \exp(-\phi(t)).$$

By Levy's inequality, for *Lipschitz function* $f : \mathcal{X} \rightarrow \mathbb{R}$, let $A := \{x : f(x) \leq \text{Med}(f(X))\}$, so that $\mathbb{P}(A) \geq 1/2$, and the complement of t -blowup of A becomes

$$\mathbb{P}(A_t^c) = \mathbb{P}\{f(X) \geq \text{Med}(f(X)) + t\}$$

Then the above isoperimetric inequality is equivalent to

$$\begin{aligned} \mathbb{P}\{f(X) \geq \text{Med}(f(X)) + t\} &\leq 2 \exp(-\phi(t)) \\ \mathbb{P}\{f(X) \leq \text{Med}(f(X)) - t\} &\leq 2 \exp(-\phi(t)). \end{aligned}$$

Besides the existing result for *Gaussian and Bernoulli random variables*, the Talagrand's **convex distance inequality** is very useful to derive such isoperimetric inequality based on *weighted Hamming distance*.

$$\mathbb{P}(A)\mathbb{P}\left\{\sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2=1} \inf_{y \in A} \sum_{i=1}^n \alpha_i \mathbb{1}\{x_i \neq y_i\} \geq t\right\} \leq \exp\left(-\frac{t^2}{4}\right).$$

Note that if A is a convex set,

$$d(x, A) := \inf_{y \in A} \|x - y\|_2 \leq d_T(x, A) := \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2=1} \inf_{y \in A} \sum_{i: x_i \neq y_i} \alpha_i$$

If **isoperimetric theorem** exists **for given distribution or space**, then the derived concentration bound is known to be **sharp** due to **the concentration of measure phenomenon**. This is the main **advantage** of using isoperimetric inequalities. However, proving **isoperimetric theorem** is extremely hard and thus is not widely available.

Transportation methods and *logarithmic Sobolev inequalities* can also use to show *isoperimetric inequalities*.

3 Summary: Distribution-Free Concentration Inequality

- **Remark (*Distribution-Free Concentration Inequality*)**

Some concentration results are based on **assumption on specific underlying distributions** such as *Gaussian, Bernoulli, Poisson, sub-Gaussian, sub-Gamma* etc. On the other hand, some concentration results are based on assumption on specific function class such as *bounded (actually is sub-Gaussian), Lipschitz function, bounded difference, convex function* etc. The latter results do not rely on specific distribution assumption, so it is called **the distribution-free concentration inequality**.

We list out several important inequalities:

1. **Theorem 3.1 (*Markov's Inequality*)**. [Vershynin, 2018]

For any **non-negative** random variable X and $t > 0$, we have

$$\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}[X]}{t}$$

2. **Theorem 3.2 (*Chebyshev's Inequality*)**. [Vershynin, 2018]

Let X be a random variable with mean μ and variance σ^2 . Then, for any $t > 0$, we have

$$\mathbb{P}\{|X - \mu| \geq t\} \leq \frac{\sigma^2}{t^2}.$$

3. **Theorem 3.3 (*Chernoff's inequality*)** [Boucheron et al., 2013]

Let X be a real-valued random variable. For $\lambda \geq 0$, $\psi_X(\lambda)$ is the **the logarithm of moment generating function** of X and $\psi_X^*(t)$ is its **Legendre (Cramér) transform**. Then

$$\mathbb{P}\{X \geq t\} \leq \exp(-\psi_X^*(t)).$$

4. **Theorem 3.4 (*Hoeffding's inequality*)** [Boucheron et al., 2013]

Let X_1, \dots, X_n be independent random variables such that X_i takes its values in $[a_i, b_i]$ **almost surely** for all $i \leq n$. Then for every $t > 0$,

$$\mathbb{P}\left\{\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \geq t\right\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

5. **Corollary 3.5 (*Azuma-Hoeffding Inequality*)**[Wainwright, 2019]

Let $\{(D_k, \mathcal{B}_k), k \geq 1\}$ be a **martingale difference sequence** for which there are constants $\{(a_k, b_k)\}_{k=1}^n$ such that $D_k \in [a_k, b_k]$ **almost surely** for all $k = 1, \dots, n$. Then, for all $t \geq 0$,

$$\mathbb{P}\left\{\left|\sum_{k=1}^n D_k\right| \geq t\right\} \leq 2 \exp\left(-\frac{2t^2}{\sum_{k=1}^n (b_k - a_k)^2}\right)$$

6. **Theorem 3.6 (*McDiarmid's Inequality / Bounded Differences Inequality*)**[Boucheron et al., 2013, Wainwright, 2019]

Suppose that f satisfies **the bounded difference property** (50) with parameters (L_1, \dots, L_n) i.e. for each index $k = 1, 2, \dots, n$,

$$|f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq L_k, \quad \text{for all } x, x' \in \mathcal{X}^n.$$

Assume that the random vector $X = (X_1, X_2, \dots, X_n)$ has **independent** components. Then

$$\mathbb{P}\{|f(X) - \mathbb{E}[f(X)]| \geq t\} \leq 2 \exp\left(-\frac{2t^2}{\sum_{k=1}^n L_k^2}\right).$$

Note that functions with bounded difference property are **Lipschitz function** with respect to **Hamming distance**.

7. Theorem 3.7 (Concentration of Separately Convex Lipschitz Functions) [Boucheron et al., 2013]

Let $Z := (Z_1, \dots, Z_n)$ be independent random variables, each taking values in the interval $[a_i, b_i]$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a **separately convex function** (i.e. f is **convex in each coordinate** while the **others** are **fixed**) such that

$$|f(x) - f(y)| \leq L \|x - y\| \quad \text{for all } x, y \in [0, 1]^n.$$

Then $X = f(Z_1, \dots, Z_n)$ satisfies, for all $t > 0$,

$$\mathbb{P}\{f(Z) - \mathbb{E}[f(Z)] \geq t\} \leq \exp\left(-\frac{t^2}{2L^2 \sum_{k=1}^n (b_k - a_k)^2}\right).$$

Convex Lipschitz assumption is **stronger** than bounded difference assumption.

8. Theorem 3.8 (Concentration of Quasi-Convex Lipschitz Functions) [Boucheron et al., 2013]

Let $Z := (Z_1, \dots, Z_n)$ be independent random variables taking values in the interval $[0, 1]$ and let $f : [0, 1]^n \rightarrow \mathbb{R}$ be a **quasi-convex function**; that is

$$\{z : f(z) \leq s\} \text{ is convex set for all } s \in \mathbb{R}.$$

Moreover, f is Lipschitz function satisfying

$$|f(x) - f(y)| \leq \|x - y\| \quad \text{for all } x, y \in [0, 1]^n.$$

Then $X = f(Z_1, \dots, Z_n)$ satisfies, for all $t > 0$,

$$\begin{aligned} \mathbb{P}\{f(Z) \geq \text{Med}(f(Z)) + t\} &\leq 2 \exp\left(-\frac{t^2}{4}\right), \\ \mathbb{P}\{f(Z) \leq \text{Med}(f(Z)) - t\} &\leq 2 \exp\left(-\frac{t^2}{4}\right). \end{aligned}$$

4 Comparison: Gaussian Tail Bound vs. Poisson Tail Bound

- **Remark** (*Gaussian Tail Bound vs. Poisson Tail Bound*)

Based on Chernoff bound, we can derive the tail bound for two important class of distributions:

1. **Gaussian tail bound:** for any $t > 0$,

$$\mathbb{P}\{X \geq t\} \leq \exp\left(-\frac{t^2}{2\nu}\right),$$

where $\nu > 0$.

2. **Poisson tail bound:** for any $t > \mathbb{E}[X] = \nu$,

$$\mathbb{P}\{X \geq t\} \leq \exp\left(-\nu h\left(\frac{t}{\nu}\right)\right) = e^{-\nu} \left(\frac{e\nu}{t+\nu}\right)^{t+\nu}$$

where $h(x) = (1+x)\log(1+x) - x$ for all $x \geq -1$ and $\nu > 0$.

Note that for **small deviation** around the mean, the tail of Poisson distribution $\text{Pois}(\nu)$ behaves like *Gaussian* $\mathcal{N}(\nu, \nu)$:

$$\mathbb{P}\{|X - \nu| \geq t\} \leq 2 \exp\left(-\frac{ct^2}{\nu}\right)$$

But in the **large deviation** regime, the Poisson tail is **heavier** than Gaussian. Such distribution is a **sub-Gamma distribution**.

- **Remark** *The Bennet inequality* captures *the Poisson tail behavior*: for sum of n independent random variables X_i such that $X_i \leq b$ almost surely with zero mean, finite variance $\nu = \sum_{i=1}^n \mathbb{E}[X_i^2]$. The Bennet inequality provides a tail bound as

$$\mathbb{P}\left\{\sum_{i=1}^n X_i \geq t\right\} \leq \exp\left(-\frac{\nu}{b^2} h\left(\frac{bt}{\nu}\right)\right)$$

In small deviation regime, where $u := bt/\nu \ll 1$, $h(u) \approx u^2$, the Bennet inequality gives a Gaussian tail bound $\approx \exp(-t^2/\nu)$.

In large deviation regime, $u \gg bt/\nu \geq 2$, $h(u) \geq \frac{1}{2}u \log(u)$, thus the Bennet inequality gives a Poisson tail bound $\approx (\nu/b^2 t)^{t/(2b^2)}$

5 The Cramér-Chernoff Method

5.1 From Markov Inequality to Cramér-Chernoff Method

- **Proposition 5.1 (Markov's Inequality).** [Vershynin, 2018]
For any **non-negative** random variable X and $t > 0$, we have

$$\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}[X]}{t} \quad (19)$$

- **Proposition 5.2 (Chebyshev's Inequality).** [Vershynin, 2018]
Let X be a random variable with mean μ and variance σ^2 . Then, for any $t > 0$, we have

$$\mathbb{P}\{|X - \mu| \geq t\} \leq \frac{\sigma^2}{t^2}. \quad (20)$$

- **Remark (Cramér-Chernoff Method)**

In this section we describe and formalize the Cramér-Chernoff bounding method. This method determines *the best possible bound* for a **tail probability** that one can possibly obtain using *Markov's inequality* with an exponential function $\phi(t) = e^{\lambda t}$.

Recall that for a real-valued random variable X , any $\lambda \geq 0$, the following inequality holds

$$\mathbb{P}\{X \geq t\} \leq e^{-\lambda t} \mathbb{E}[e^{\lambda X}] = \exp(-\lambda t + \psi_X(\lambda))$$

where $\psi_X(\lambda) := \log \mathbb{E}[e^{\lambda X}]$. One can choose optimal λ^* that **minimizes the upper bound above**. Since $\psi_X(\lambda)$ is a **convex function**, we can define its **Legendre transform**

$$\psi_X^*(t) := \sup_{\lambda \in \mathbb{R}} \{\lambda t - \psi_X(\lambda)\}.$$

The expression of the right-hand side is known as the **Fenchel-Legendre dual function** (or the **convex conjugate**) of ψ_X . The Legendre transform of log-moment generating function is also its convex conjugate.

In other word, in order to prove concentration around mean

$$\mathbb{P}\{f(X) \geq \mathbb{E}[f(X)] + t\} \text{ or } \mathbb{P}\{f(X) \leq \mathbb{E}[f(X)] - t\}$$

using **the Cramér-Chernoff Method**, we just need to find **the upper bound of the logarithmic moment generating function**

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda(f(X) - \mathbb{E}[f(X)])}] \leq \phi(\lambda)$$

- **Proposition 5.3 (Chernoff's inequality)** [Boucheron et al., 2013]
Let X be a real-valued random variable. For $\lambda \geq 0$, $\psi_X(\lambda)$ is the **the logarithm of moment generating function** of X and $\psi_X^*(t)$ is its **Legendre (Cramér) transform**. Then

$$\mathbb{P}\{X \geq t\} \leq \exp(-\psi_X^*(t)). \quad (21)$$

- **Remark** The *Legendre transform* is also called *the Cramér transform* [Boucheron et al., 2013].

Since $\psi_X(0) = 0$, its *Legendre transform* $\psi_X^*(t)$ is *nonnegative*.

- **Definition** (*The Rate Function*)

The rate function is defined as *the Legendre transformation* of the logarithm of the moment generating function of a random variable. That is,

$$\psi_X^*(t) := \sup_{\lambda \in \mathbb{R}} \{ \lambda t - \psi_X(\lambda) \}, \quad (22)$$

where $\psi_X(\lambda) := \log \mathbb{E} [e^{\lambda X}]$. Thus, by *Chernoff's inequality*, we can bound the tail probabilities of random variables via its rate function.

- **Remark** (*Sums of independent random variables*)

The reason why Chernoff's inequality became popular is that it is very simple to use when applied to a sum of independent random variables. As an illustration, assume that $Z := X_1 + \dots + X_n$ where X_1, \dots, X_n are *independent and identically distributed* real-valued random variables. Denote the logarithm of the moment-generating function of the X_i by $\psi_X(\lambda) = \log \mathbb{E} [e^{\lambda X_i}]$, and the corresponding *Legendre transform* by $\psi_X^*(t)$. Then, by independence, for all λ for which $\psi_X(\lambda) < \infty$,

$$\psi_Z(\lambda) = \log \mathbb{E} [e^{\lambda \sum_{i=1}^n X_i}] = \log \prod_{i=1}^n \mathbb{E} [e^{\lambda X_i}] = n \psi_X(\lambda)$$

and consequently,

$$\psi_Z^*(t) = n \psi_X^*\left(\frac{t}{n}\right).$$

Thus the *Chernoff's inequality* states that

$$\mathbb{P} \{ Z \geq t \} \leq \exp(-\psi_Z^*(t)) = \exp\left(-n \psi_X^*\left(\frac{t}{n}\right)\right).$$

- **Example** (*Normal Distribution*)

Let X be a *centered normal random variable* with variance σ^2 . Then

$$\psi_X(\lambda) = \frac{\lambda^2 \sigma^2}{2}, \quad \lambda_t = \frac{t}{\sigma^2}$$

and, therefore for every $t > 0$,

$$\psi_X^*(t) = \frac{t^2}{2\sigma^2}.$$

Hence, *Chernoff's inequality* implies, for all $t > 0$,

$$\mathbb{P} \{ X \geq t \} \leq \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

Chernoff's inequality appears to be quite sharp in this case. In fact, one can show that it cannot be improved uniformly by more than a factor of 1/2. ■

- **Example (*Poisson Distribution*)**

Let X be a **Poisson random variable** with parameter ν , that is, $\mathbb{P}\{X = k\} = \frac{1}{k!}e^{-\nu}\nu^k$ for all $k = 0, 1, 2, \dots$. Let $Z = X - \nu$ be the *corresponding centered variable*. Then by direct calculation,

$$\psi_Z(\lambda) = \nu(e^\lambda - \lambda - 1), \quad \lambda_t = \log\left(1 + \frac{t}{\nu}\right)$$

Therefore *the Legendre transform* equals, for every $t > 0$,

$$\psi_Z^*(t) = \nu h\left(\frac{t}{\nu}\right).$$

where the function h is defined, for all $x \geq -1$, by $h(x) = (1+x)\log(1+x) - x$. Similarly, for every $t \leq \nu$,

$$\psi_{-Z}^*(t) = \nu h\left(-\frac{t}{\nu}\right).$$

- **Example (*Bernoulli Distribution*)**

Let X be a **Bernoulli random variable** with probability of success p , that is, $\mathbb{P}\{X = 1\} = p$ and $\mathbb{P}\{X = 0\} = 1 - p$. Let $Z = X - p$ be the *corresponding centered variable*. If $0 < t < 1 - p$, we have

$$\psi_Z(\lambda) = \log(pe^\lambda + 1 - p) - p\lambda, \quad \lambda_t = \log\frac{(1-p)(p+t)}{p(1-p-t)}$$

and therefore, for every $t \in (0, 1 - p)$,

$$\psi_Z^*(t) = (1 - p - t) \log \frac{1 - p - t}{1 - p} + (p + t) \log \frac{p + t}{p}.$$

Equivalently, setting $a = t + p$ for every $a \in (p, 1)$,

$$\psi_Z^*(t) = h_p(a) = (1 - a) \log \frac{1 - a}{1 - p} + a \log \frac{a}{p}.$$

We note here that $h_p(a)$ is just the **Kullback-Leibler divergence** $\text{KL}(\mathbb{P}_a \parallel \mathbb{P}_p)$ between a Bernoulli distribution \mathbb{P}_a of parameter a and a Bernoulli distribution \mathbb{P}_p of parameter p .

$$\mathbb{P}\{X \geq t\} \leq \exp(-\text{KL}(\mathbb{P}_{p+t} \parallel \mathbb{P}_p))$$

5.2 Sub-Gaussian Random Variables

- **Definition (*Sub-Gaussian Random Variable*)**

A **centered** random variable X is said to be **sub-Gaussian with variance factor ν** if

$$\psi_X(\lambda) \leq \frac{\lambda^2 \nu}{2}, \quad \text{for every } \lambda \in \mathbb{R}. \quad (23)$$

We denote the collection of such random variables by $\mathcal{G}(\nu)$.

- **Proposition 5.4 (Moment Characterization of Sub-Gaussian Random Variables)**
[Boucheron et al., 2013]

Let X be a random variable with $\mathbb{E}[X] = 0$. If for some $\nu > 0$

$$\mathbb{P}\{X > t\} \vee \mathbb{P}\{-X > t\} \leq \exp\left(-\frac{t^2}{2\nu}\right), \quad \text{for all } t > 0 \quad (24)$$

then for every integer $q \geq 1$,

$$\mathbb{E}[X^{2q}] \leq 2q!(2\nu)^q \leq q!(4\nu)^q. \quad (25)$$

Conversely, if for some positive constant C

$$\mathbb{E}[X^{2q}] \leq q!C^q,$$

then $X \in \mathcal{G}(4C)$ (and therefore (25) holds with $\nu = 4C$).

- **Proposition 5.5 (Equivalent Definitions for Sub-Gaussian Random Variables)**.
[Vershynin, 2018]

Let X be a random variable. Then the following properties are **equivalent**; the parameters $K_i > 0$ appearing in these properties differ from each other by at most an absolute constant factor.

1. The **tails** of X satisfy

$$\mathbb{P}\{|X| \geq t\} \leq 2 \exp(-t^2/K_1^2) \quad \text{for all } t \geq 0.$$

2. The **moments** of X satisfy

$$\|X\|_{L^p} = (\mathbb{E}[|X|^p])^{1/p} \leq K_2 \sqrt{p} \quad \text{for all } p \geq 1.$$

3. The **moment-generating function (MGF)** of X^2 satisfies

$$\mathbb{E}[\exp(\lambda^2 X^2)] \leq \exp(K_3^2 \lambda^2) \quad \text{for all } \lambda \text{ such that } |\lambda| \leq \frac{1}{K_3}$$

4. The **MGF** of X^2 is **bounded** at some point, namely

$$\mathbb{E}[\exp(X^2/K_4^2)] \leq 2.$$

Moreover, if $\mathbb{E}[X] = 0$ then properties (1)-(4) are also **equivalent** to the following one.

5. The **MGF** of X satisfies

$$\mathbb{E}[\exp(\lambda X)] \leq \exp(K_5^2 \lambda^2) \quad \text{for all } \lambda \in \mathbb{R}.$$

- **Definition (Sub-Gaussian Norm)**

The sub-gaussian norm of X , denoted $\|X\|_{\psi_2}$, is defined to be the **smallest** K_4 that satisfies

$$\mathbb{E}[\exp(X^2/K_4^2)] \leq 2.$$

In other words, we define

$$\|X\|_{\psi_2} = \inf \{t > 0 : \mathbb{E}[\exp(X^2/t^2)] \leq 2\}. \quad (26)$$

- **Remark** (*Sub-Gaussian Characterizations via Sub-Gaussian Norm*)

We can restate the properties of sub-gaussian random variables in terms of sub-gaussian norm:

$$\begin{aligned}\mathbb{P}\{|X| \geq t\} &\leq 2 \exp\left(-ct^2 / \|X\|_{\psi_2}^2\right) \quad \text{for all } t \geq 0; \\ \|X\|_{L^p} &\leq C \|X\|_{\psi_2} \sqrt{p} \quad \text{for all } p \geq 1; \\ \mathbb{E}\left[\exp(X^2 / \|X\|_{\psi_2}^2)\right] &\leq 2; \\ \text{if } \mathbb{E}[X] = 0, \quad \text{then } \mathbb{E}[\exp(\lambda X)] &\leq \exp(C\lambda^2 \|X\|_{\psi_2}^2) \quad \text{for all } \lambda \in \mathbb{R}.\end{aligned}$$

- **Example** Here are some classical examples of sub-gaussian distributions.

1. (**Gaussian**): As we already noted, $X \sim N(0, 1)$ is a sub-gaussian random variable with $\|X\|_{\psi_2} \leq C$, where C is an absolute constant. More generally, if $X \sim N(0, \sigma^2)$ then X is sub-gaussian with

$$\|X\|_{\psi_2} \leq C\sigma \quad (27)$$

2. (**Bernoulli**): Let X be a random variable with *symmetric Bernoulli distribution*. Since $|X| = 1$, it follows that X is a sub-gaussian random variable with

$$\|X\|_{\psi_2} \leq \frac{1}{\sqrt{\log 2}} \quad (28)$$

3. (**Bounded**): More generally, any *bounded random variable* X is sub-gaussian with

$$\|X\|_{\psi_2} \leq C \|X\|_{\infty} \quad (29)$$

where $C = 1/\sqrt{\log 2}$.

5.3 Sub-Exponential and Sub-Gamma Random Variables

- **Remark** For *exponential distribution* $X \sim \exp(a)$ with rate a (inverse of scale parameter), the p.d.f. and moment generating function

$$\begin{aligned}f_X(x) &= ae^{-ax}, \quad x > 0 \\ M_X(\lambda) &= \frac{1}{1 - \lambda/a}, \quad 0 < \lambda < a\end{aligned}$$

For *Gamma distribution* $X \sim \Gamma(a, 1/b)$ with shape parameter a and scale parameter b , the p.d.f. and the moment generating function

$$\begin{aligned}f_X(x) &= \frac{1}{\Gamma(a) b^a} x^{a-1} e^{-x/b}, \quad x > 0 \\ M_X(\lambda) &= \left(\frac{1}{1 - b\lambda}\right)^a, \quad 0 < \lambda < 1/b\end{aligned}$$

Also $\mathbb{E}[X] = ab$ and $\text{Var}(X) = ab^2$.

- **Definition (*Sub-Exponential Random Variables*)**

A **nonnegative** random variable X has a **sub-exponential distribution** if there exists a constant $a > 0$ such that

$$\mathbb{E} \left[e^{\lambda X} \right] \leq \frac{1}{1 - \lambda/a} \quad \text{for every } \lambda \text{ such that } 0 < \lambda < a$$

or $\psi_X(\lambda) \leq \log \left(\frac{1}{1 - \lambda/a} \right)$

- **Definition (*Sub-Gamma Random Variables*)**

A real-valued **centered** random variable X is said to be **sub-gamma on the right tail** with **variance factor** ν and **scale parameter** c if

$$\psi_X(\lambda) \leq \frac{\lambda^2 \nu}{2(1 - c\lambda)} \quad \text{for every } \lambda \text{ such that } 0 < \lambda < 1/c$$

We denote the collection of such random variables by $\Gamma_+(\nu, c)$.

Similarly, a real-valued centered random variable X is said to be **sub-gamma on the left tail** with **variance factor** ν and **scale parameter** c if $-X$ is **sub-gamma on the right tail** with variance factor ν and tail parameter c . We denote the collection of such random variables by $\Gamma_-(\nu, c)$.

Finally, X is simply said to be **sub-gamma with variance factor** ν and **scale parameter** c if X is sub-gamma **both** on the right and left tails with **the same** variance factor ν and scale parameter c . The collection of such random variables is denoted by $\Gamma(\nu, c)$.

Observe that $\Gamma(\nu, 0) = \mathcal{G}(\nu)$.

- **Remark** To derive the definition for sub-gamma distribution, we see that *the variance factor* $\nu := ab^2$ and $c := b$. Also $\mathbb{E}[X] = ab$. The logarithmic moment generating function of Gamma distribution $\Gamma(a, 1/b) = \Gamma(\nu/c^2, 1/c)$ is

$$\psi_{X - \mathbb{E}[X]}(\lambda) = a \log \left(\frac{1}{1 - b\lambda} \right) - \lambda ab \leq \frac{\lambda^2 b^2 a}{2(1 - b\lambda)} \equiv \frac{\lambda^2 \nu}{2(1 - c\lambda)}$$

The last inequality is due to

$$\log \left(\frac{1}{1 - u} \right) - u \leq \frac{u^2}{2(1 - u)}$$

- **Remark** Note that the sum of n i.i.d. random variables with exponential distribution $\exp(1/b)$ have the Gamma distribution $\Gamma(n, 1/b)$. So *the sub-gamma distributed* random variable follows *the sub-exponential distribution* as well (with shape parameter = 1).
- **Proposition 5.6 (*Equivalent Definitions for Sub-Exponential Random Variables*)**. [Vershynin, 2018]

Let X be a random variable. Then the following properties are **equivalent**; the parameters $K_i > 0$ appearing in these properties differ from each other by at most an absolute constant factor.

1. The **tails** of X satisfy

$$\mathbb{P} \{|X| \geq t\} \leq 2 \exp(-t/K_1) \quad \text{for all } t \geq 0.$$

2. The **moments** of X satisfy

$$\|X\|_{L^p} = (\mathbb{E}[|X|^p])^{1/p} \leq K_2 p \quad \text{for all } p \geq 1.$$

3. The **moment-generating function (MGF)** of $|X|$ satisfies

$$\mathbb{E}[\exp(\lambda |X|)] \leq \exp(K_3 \lambda) \quad \text{for all } \lambda \text{ such that } 0 \leq \lambda \leq \frac{1}{K_3}$$

4. The **MGF** of $|X|$ is **bounded** at some point, namely

$$\mathbb{E}[\exp(|X|/K_4)] \leq 2.$$

Moreover, if $\mathbb{E}[X] = 0$ then properties (1)-(4) are also **equivalent** to the following one.

5. The **MGF** of X satisfies

$$\mathbb{E}[\exp(\lambda X)] \leq \exp(K_5^2 \lambda^2) \quad \text{for all } \lambda \text{ such that } |\lambda| \leq \frac{1}{K_5}.$$

- **Definition (Sub-Exponential Norm)**

The **sub-exponential norm** of X , denoted $\|X\|_{\psi_1}$, is defined to be the **smallest** K_4 that satisfies

$$\mathbb{E}[\exp(|X|/K_4)] \leq 2.$$

In other words, we define

$$\|X\|_{\psi_1} = \inf \{t > 0 : \mathbb{E}[\exp(|X|/t)] \leq 2\}. \quad (30)$$

- **Remark** Sub-gaussian and sub-exponential distributions are closely related.

1. First, *any sub-gaussian distribution is clearly sub-exponential.*
2. Second, *the square of a sub-gaussian random variable is sub-exponential:*

Lemma 5.7 (Sub-exponential is Sub-gaussian Squared). [Vershynin, 2018]
A random variable X is **sub-gaussian** if and only if X^2 is **sub-exponential**. Moreover,

$$\|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2$$

More generally, *the product of two sub-gaussian random variables is sub-exponential:*

Lemma 5.8 (Product of Sub-Gaussians is Sub-Exponential). [Vershynin, 2018]
Let X and Y be **sub-gaussian** random variables. Then XY is **sub-exponential**. Moreover,

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_2} \|Y\|_{\psi_2}.$$

- **Proposition 5.9 (Moment Characterization of Sub-Exponential Random Variables)**
[Boucheron et al., 2013]

Let X be a nonnegative random variable. If X is sub-exponential distributed with parameter $a > 0$ then for every integer $q \geq 1$,

$$\mathbb{E}[X^q] \leq 2^{q+1} \frac{q!}{a^q}. \quad (31)$$

Conversely, if there exists a constant $a > 0$ in order that for every positive integer q ,

$$\mathbb{E}[X^q] \leq \frac{q!}{a^q},$$

then X is sub-exponential. More precisely, for any $0 < \lambda < a$,

$$\mathbb{E}[e^{\lambda X}] \leq \frac{1}{1 - \lambda/a}.$$

• **Remark (Concentration Inequalities for Sub-Gamma Distribution)**

Similarly to the *sub-Gaussian property*, the **sub-gamma property** can be characterized in terms of *tail or moment conditions*. We start by computing **the Fenchel-Legendre dual function** of

$$\psi(\lambda) = \frac{\lambda^2 \nu}{2(1 - c\lambda)}.$$

Setting

$$h_1(u) = 1 + u - \sqrt{1 + 2u} \text{ for } u > 0,$$

it follows by elementary calculation that for every $t > 0$,

$$\psi^*(t) = \sup_{\lambda \in (0, 1/c)} \left\{ t\lambda - \frac{\lambda^2 \nu}{2(1 - c\lambda)} \right\} = \frac{\nu}{c^2} h_1\left(\frac{ct}{\nu}\right).$$

Since h_1 is an increasing function from $(0, \infty)$ onto $(0, \infty)$ with **inverse function**

$$h^{-1}(u) = u + \sqrt{2u} \text{ for } u > 0,$$

we finally get

$$\psi^{*-1}(u) = \sqrt{2\nu u} + cu.$$

Hence, *Chernoff's inequality* implies that whenever X is a *sub-gamma random variable on the right tail* with *variance factor* ν and *scale parameter* c , for every $t > 0$, we have

$$\mathbb{P}\{X > t\} \leq \exp\left(\frac{\nu}{c^2} h_1\left(\frac{ct}{\nu}\right)\right), \quad (32)$$

or equivalently, for every $t > 0$,

$$\mathbb{P}\left\{X > \sqrt{2\nu t} + ct\right\} \leq e^{-t}. \quad (33)$$

Therefore, if X belongs to $\Gamma(\nu, c)$, then for every $t > 0$,

$$\mathbb{P}\left\{X > \sqrt{2\nu t} + ct\right\} \vee \mathbb{P}\left\{-X > \sqrt{2\nu t} + ct\right\} \leq e^{-t}. \quad \blacksquare$$

5.4 Hoeffding's Inequality

- **Remark (*Bounded Variables*)**

Bounded variables are an important class of *sub-Gaussian random variables*. The *sub-Gaussian property* of *bounded random variables* is established by the following lemma:

- **Lemma 5.10 (*Hoeffding's Lemma*)** [Boucheron et al., 2013]

Let X be a random variable with $\mathbb{E}[X] = 0$, taking values in a **bounded interval** $[a, b]$ and let $\psi_X(\lambda) := \log \mathbb{E}[e^{\lambda X}]$. Then

$$\psi_X''(\lambda) \leq \frac{(b-a)^2}{4}$$

and $X \in \mathcal{G}((b-a)^2/4)$.

- **Proposition 5.11 (*Hoeffding's inequality*)** [Boucheron et al., 2013]

Let X_1, \dots, X_n be independent random variables such that X_i takes its values in $[a_i, b_i]$ **almost surely** for all $i \leq n$. Let

$$S = \sum_{i=1}^n (X_i - \mathbb{E}[X_i]).$$

Then for every $t > 0$,

$$\mathbb{P}\{S \geq t\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \quad (34)$$

- **Proposition 5.12 (*General Hoeffding's inequality*)** [Vershynin, 2018]

Let X_1, \dots, X_n be **independent sub-gaussian** random variables. Let

$$S = \sum_{i=1}^n (X_i - \mathbb{E}[X_i]).$$

Then for every $t > 0$,

$$\mathbb{P}\{S \geq t\} \leq \exp\left(-\frac{ct^2}{\sum_{i=1}^n \|X_i\|_{\psi_2}^2}\right). \quad (35)$$

5.5 Bernstein's Inequality

- **Definition (*Bernstein's Condition*)**

Given a random variable X with mean $\mu = \mathbb{E}[X]$ we say that **Bernstein's condition** with parameter ν, c holds if the *variance* $\text{Var}(X) = \mathbb{E}[X^2] - \mu^2 \leq \nu$, and

$$\sum_{i=1}^n \mathbb{E}[(X - \mu)_+^q] \leq \frac{q!}{2} \nu c^{q-2}, \quad \text{for all integers } q \geq 2,$$

where $(x)_+ = \max\{x, 0\}$.

- **Remark** If X is bounded, then it satisfies the Bernstein's condition.

If X satisfies the Bernstein's condition, X follows a **sub-gamma distribution**.

- **Proposition 5.13** (**Bernstein's Condition \Rightarrow Sub-Gamma Distribution**). [Boucheron et al., 2013]

Let X_1, \dots, X_n be independent real-valued random variables and each X_i satisfies **the Bernstein's condition** with parameter ν and c . If $S = \sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, then for all $\lambda \in (0, 1/c)$ and $t > 0$

$$\psi_S(\lambda) \leq \frac{\lambda^2 \nu}{2(1 - c\lambda)}$$

and

$$\psi_S^*(t) \geq \frac{\nu}{c^2} h_1\left(\frac{ct}{\nu}\right),$$

where $h_1(u) = 1 + u - \sqrt{1 + 2u}$ for $u > 0$. In particular, for all $t > 0$,

$$\mathbb{P}\{S \geq \sqrt{2\nu t} + ct\} \leq e^{-t}. \quad (36)$$

- **Proposition 5.14** (**Bernstein's Inequality**). [Boucheron et al., 2013]

Let X_1, \dots, X_n be independent real-valued random variables satisfying **the Bernstein's conditions** above and let $S = \sum_{i=1}^n (X_i - \mathbb{E}[X_i])$. Then for all $t > 0$,

$$\mathbb{P}\{S \geq t\} \leq \exp\left(-\frac{t^2}{2(\nu + ct)}\right). \quad (37)$$

- **Corollary 5.15** (**Bernstein's Inequality for Bounded Distributions**). [Vershynin, 2018]

Let X_1, \dots, X_n be **independent, mean zero** random variables, such that $|X_i| \leq b$ all i . Then, for every $t \geq 0$, we have

$$\mathbb{P}\left\{\left|\frac{1}{n} \sum_{i=1}^n X_i\right| \geq t\right\} \leq 2 \exp\left(-\frac{t^2}{2(\nu + bt/3)}\right). \quad (38)$$

Here $\nu = \sum_{i=1}^n \mathbb{E}[X_i^2]$ is the variance of the sum.

- **Corollary 5.16** (**Bernstein's Inequality**). [Vershynin, 2018]

Let X_1, \dots, X_n be **independent, mean zero, sub-exponential** random variables. Then, for every $t \geq 0$, we have

$$\mathbb{P}\left\{\left|\sum_{i=1}^n X_i\right| \geq t\right\} \leq 2 \exp\left[-c \min\left\{\frac{t^2}{\sum_{i=1}^n \|X_i\|_{\psi_2}^2}, \frac{t}{\max_i \|X_i\|_{\psi_1}}\right\}\right] \quad (39)$$

where $c > 0$ is an absolute constant.

- **Proposition 5.17** (**Bernstein's Inequality, Linear Combination Form**). [Vershynin, 2018]

Let X_1, \dots, X_n be **independent, mean zero, sub-exponential random variables**, and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Then, for every $t \geq 0$, we have

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n a_i X_i \right| \geq t \right\} \leq 2 \exp \left[-c \min \left\{ \frac{t^2}{K^2 \|a\|_2^2}, \frac{t}{K \|a\|_\infty} \right\} \right] \quad (40)$$

where $c > 0$ is an absolute constant and $K = \max_i \|X_i\|_{\psi_1}$.

- **Corollary 5.18 (Bernstein's Inequality, Average Form).** [Vershynin, 2018]
Let X_1, \dots, X_n be **independent, mean zero, sub-exponential random variables**. Then, for every $t \geq 0$, we have

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n X_i \right| \geq t \right\} \leq 2 \exp \left[-c \min \left\{ \frac{t^2}{K^2}, \frac{t}{K} \right\} n \right] \quad (41)$$

where $K = \max_i \|X_i\|_{\psi_1}$.

5.6 Bennett's Inequality

- **Remark** Our starting point is the fact that *the logarithmic moment-generating function of an independent sum equals the sum of the logarithmic moment-generating functions of the centered summands*, that is,

$$\psi_S(\lambda) = \sum_{i=1}^n \left(\log \mathbb{E} \left[e^{\lambda X_i} \right] - \lambda \mathbb{E} [X_i] \right).$$

Using $\log u \leq u - 1$ for $u > 0$,

$$\psi_S(\lambda) \leq \sum_{i=1}^n \mathbb{E} \left[e^{\lambda X_i} - \lambda X_i - 1 \right]. \quad (42)$$

Both Bennett's and Bernstein's inequalities may be derived from this bound, under different integrability conditions for the X_i .

- **Proposition 5.19 (Bennett's Inequality)** [Boucheron et al., 2013]
Let X_1, \dots, X_n be independent random variables with **finite variance** such that $X_i \leq b$ for some $b > 0$ **almost surely** for all $i \leq n$. Let

$$S = \sum_{i=1}^n (X_i - \mathbb{E} [X_i])$$

and $\nu = \sum_{i=1}^n \mathbb{E} [X_i^2]$. If we write $\phi(u) = e^u - u - 1$ for $u \in \mathbb{R}$, then, for all $\lambda > 0$,

$$\log \mathbb{E} \left[e^{\lambda S} \right] \leq n \log \left(1 + \frac{\nu}{nb^2} \phi(b\lambda) \right) \leq \frac{\nu}{b^2} \phi(b\lambda),$$

and for any $t > 0$,

$$\mathbb{P} \{ S \geq t \} \leq \exp \left(-\frac{\nu}{b^2} h \left(\frac{bt}{\nu} \right) \right) \quad (43)$$

where $h(u) = (1 + u) \log(1 + u) - u$ for $u > 0$.

- **Remark** This bound can be analyzed in two different regimes:

1. In the **small deviation regime**, where $u := bt/\nu \ll 1$, we have asymptotically $h(u) \approx u^2$ and Bennett's inequality gives approximately the Gaussian tail bound $\approx \exp(-t^2/\nu)$.
2. In the **large deviations regime**, say where $u := bt/\nu \geq 2$, we have $h(u) \geq \frac{1}{2}u \log u$, and Bennett's inequality gives a **Poisson-like tail** $(\nu/bt)^{t/2b}$.

5.7 The Johnson-Lindenstrauss Lemma

- **Remark (Overview of The Johnson-Lindenstrauss Lemma)**

The celebrated **Johnson-Lindenstrauss lemma** states roughly that, given an arbitrary set of n points in a (high-dimensional) Euclidean space, there exists a **linear embedding** of these points in a d -dimensional Euclidean space such that **all pairwise distances are preserved** within a factor of $1 \pm \epsilon$ if d is proportional to $(\log n)/\epsilon^2$. It is remarkable that this result does not involve the dimension of the space to which the n points belong. In fact, the dimension of this space may even be *infinite*.

- **Definition (ϵ -Isometry)**

Consider an arbitrary set $A \subset \mathbb{R}^D$ or $A \subset \mathcal{H}$ for separable Hilbert space \mathcal{H} . Given $\epsilon \in (0, 1)$, a map $f : \mathbb{R}^D \rightarrow \mathbb{R}^d$ is called an ϵ -isometry on A if for every pair $a, a' \in A$, we have

$$(1 - \epsilon) \|a - a'\|_2^2 \leq \|f(a) - f(a')\|_2^2 \leq (1 + \epsilon) \|a - a'\|_2^2.$$

- **Remark (Problem Statement)**

A natural question is to find **the smallest possible value of d** for which a linear ϵ -isometry exists on A . The Johnson-Lindenstrauss lemma, stated and proved below, ensures that when A is a **finite set** with cardinality n , a linear ϵ -isometry exists whenever $d \geq \kappa \epsilon^{-2} \log(n)$, where κ is an absolute constant.

- **Remark (Gaussian Random Projection)**

The basic idea is to construct a **random projection** $W : \mathbb{R}^D \rightarrow \mathbb{R}^d$ (i.e. a linear mapping) that is an exact **isometry “in expectation,”** that is, for every $\alpha \in \mathbb{R}^D$,

$$\mathbb{E} [\|W(\alpha)\|_2^2] = \mathbb{E} [\|\alpha\|_2^2].$$

In other words, denoting by $L^{2,d}$ the space of square-integrable \mathbb{R}^d -valued random vectors, W is an **isometry** from \mathbb{R}^D into $L^{2,d}$.

To construct W , let $X_{i,j}$, $i = 1, \dots, d$, $j = 1, \dots, D$ be independent and identically distributed real-valued random variables such that $\mathbb{E}[X_{i,j}] = 0$ and $\text{Var}(X_{i,j}) = 1$. For every $\alpha = (\alpha_1, \dots, \alpha_D) \in \mathbb{R}^D$ and $i \in \{1, \dots, d\}$, define

$$W_i(\alpha) := \sum_{j=1}^D \alpha_j X_{i,j}$$

$W_i(\alpha)/\sqrt{d}$ is the i -th component of the random vector $W(\alpha)$, that is, W is defined by

$$W(\alpha) := \left(\frac{1}{\sqrt{d}} \sum_{j=1}^D \alpha_j X_{i,j} \right)_{i=1}^d$$

$$\Rightarrow W(\alpha) := \frac{1}{\sqrt{d}} X \alpha^T.$$

Observe that by independence of the $X_{i,j}$, for every $i = 1, \dots, d$,

$$\mathbb{E} [W_i(\alpha)^2] = \mathbb{E} \left[\left(\sum_{j=1}^D \alpha_j X_{i,j} \right)^2 \right] = \sum_{j=1}^D \alpha_j^2 \mathbb{E} [X_{i,j}^2] = \mathbb{E} [\|\alpha\|_2^2].$$

Therefore, for every $\alpha \in \mathbb{R}^D$,

$$\mathbb{E} [\|W(\alpha)\|_2^2] = \frac{1}{d} \sum_{i=1}^d \mathbb{E} [W_i(\alpha)^2] = \mathbb{E} [\|\alpha\|_2^2].$$

and indeed, W is an *isometry* from \mathbb{R}^D into $L^{2,d}$.

- **Theorem 5.20 (The Johnson-Lindenstrauss Lemma)** [Boucheron et al., 2013]
Let A be a **finite subset** of \mathbb{R}^D with cardinality n . Assume that for some $\nu \geq 1$, $X = [X_{i,j}]$ where $X_{i,j}$ are i.i.d with zero mean sub-Gaussian random variables with variance less than or equal to ν^2 , i.e. $X_{i,j} \in \mathcal{G}(\nu)$ and let $\epsilon, \delta \in (0, 1)$. If

$$d \geq 100 \frac{\nu^2 \log \left(\frac{n}{\sqrt{\delta}} \right)}{\epsilon^2},$$

then with probability at least $1 - \delta$, W is an ϵ -**isometry** on A . That is, for every pair $a, a' \in A$, with probability $1 - \delta$ we have

$$(1 - \epsilon) \|a - a'\|_2^2 \leq \left\| \frac{1}{\sqrt{d}} X a^T - \frac{1}{\sqrt{d}} X (a')^T \right\|_2^2 \leq (1 + \epsilon) \|a - a'\|_2^2.$$

6 Martingale Method

6.1 Martingale and Martingale Difference Sequence

- **Definition (Martingale)** [Resnick, 2013]
Let $\{X_n, n \geq 0\}$ be a stochastic process on (Ω, \mathcal{F}) and $\{\mathcal{F}_n, n \geq 0\}$ be a **filtration**; that is, $\{\mathcal{F}_n, n \geq 0\}$ is an *increasing sub σ -fields* of \mathcal{F}

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}.$$

Then $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is a **martingale (mg)** if

1. X_n is **adapted** in the sense that for each n , $X_n \in \mathcal{F}_n$; that is, X_n is \mathcal{F}_n -measurable.
2. $X_n \in L_1$; that is $\mathbb{E} [|X_n|] < \infty$ for $n \geq 0$.

3. For $0 \leq m < n$

$$\mathbb{E}[X_n | \mathcal{F}_m] = X_m, \quad \text{a.s.} \quad (44)$$

If the equality of (44) is replaced by \geq ; that is, things are getting better on the average:

$$\mathbb{E}[X_n | \mathcal{F}_m] \geq X_m, \quad \text{a.s.} \quad (45)$$

then $\{X_n\}$ is called a **sub-martingale (submg)** while if things are getting worse on the average

$$\mathbb{E}[X_n | \mathcal{F}_m] \leq X_m, \quad \text{a.s.} \quad (46)$$

$\{X_n\}$ is called a **super-martingale (supermg)**.

- **Remark** $\{X_n\}$ is **martingale** if it is both a **sub** and **supermartingale**. $\{X_n\}$ is a **super-martingale** if and only if $\{-X_n\}$ is a **submartingale**.
- **Remark** If $\{X_n\}$ is a **martingale**, then $\mathbb{E}[X_n]$ is *constant*. In the case of a **submartingale**, the mean increases and for a **supermartingale**, the mean decreases.
- **Proposition 6.1** [Resnick, 2013]
If $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is a **(sub, super) martingale**, then

$$\{(X_n, \sigma(X_0, X_1, \dots, X_n)), n \geq 0\}$$

is also a **(sub, super) martingale**.

- **Definition (Martingale Differences)**. [Resnick, 2013]
 $\{(d_j, \mathcal{B}_j), j \geq 0\}$ is a **(sub, super) martingale difference sequence** or a **(sub, super) fair sequence** if
 1. For $j \geq 0$, $\mathcal{B}_j \subset \mathcal{B}_{j+1}$.
 2. For $j \geq 0$, $d_j \in L_1$, $d_j \in \mathcal{B}_j$; that is, d_j is absolutely integrable and \mathcal{B}_j -measurable.
 3. For $j \geq 0$,

$$\begin{aligned} \mathbb{E}[d_{j+1} | \mathcal{B}_j] &= 0, & (\text{martingale difference / fair sequence}); \\ &\geq 0, & (\text{submartingale difference / subfair sequence}); \\ &\leq 0, & (\text{supermartingale difference / supfair sequence}) \end{aligned}$$

- **Proposition 6.2 (Construction of Martingale From Martingale Difference)**[Resnick, 2013]
If $\{(d_j, \mathcal{B}_j), j \geq 0\}$ is **(sub, super) martingale difference sequence**, and

$$X_n = \sum_{j=0}^n d_j,$$

then $\{(X_n, \mathcal{B}_n), n \geq 0\}$ is a **(sub, super) martingale**.

- **Proposition 6.3** (*Construction of Martingale Difference From Martingale*) [Resnick, 2013]

Suppose $\{(X_n, \mathcal{B}_n), n \geq 0\}$ is a (*sub, super*) *martingale*. Define

$$\begin{aligned} d_0 &:= X_0 - \mathbb{E}[X_0] \\ d_j &:= X_j - X_{j-1}, \quad j \geq 1. \end{aligned}$$

Then $\{(d_j, \mathcal{B}_j), j \geq 0\}$ is a (*sub, super*) *martingale difference sequence*.

- **Proposition 6.4** (*Orthogonality of Martingale Differences*). [Resnick, 2013]

If $\{(X_n, \mathcal{B}_n), n \geq 0\}$ is a *martingale* where X_n can be decomposed as

$$X_n = \sum_{j=0}^n d_j,$$

d_j is \mathcal{B}_j -measurable and $\mathbb{E}[d_j^2] < \infty$ for $j \geq 0$, then $\{d_j\}$ are *orthogonal*:

$$\mathbb{E}[d_i d_j] = 0 \quad i \neq j.$$

- **Example** (*Smoothing as Martingale*)

Suppose $X \in L_1$ and $\{\mathcal{B}_n, n \geq 0\}$ is an increasing family of sub σ -algebra of \mathcal{B} . Define for $n \geq 0$

$$X_n := \mathbb{E}[X | \mathcal{B}_n].$$

Then (X_n, \mathcal{B}_n) is a *martingale*. From this result, we see that $\{(d_n, \mathcal{B}_n), n \geq 0\}$ is a *martingale difference sequence* when

$$d_n := \mathbb{E}[X | \mathcal{B}_n] - \mathbb{E}[X | \mathcal{B}_{n-1}], \quad n \geq 1. \quad (47)$$

- **Example** (*Sums of Independent Random Variables*)

Suppose that $\{Z_n, n \geq 0\}$ is an *independent* sequence of integrable random variables satisfying for $n \geq 0$, $\mathbb{E}[Z_n] = 0$. Set

$$\begin{aligned} X_0 &:= 0, \\ X_n &:= \sum_{i=1}^n Z_i, \quad n \geq 1 \\ \mathcal{B}_n &:= \sigma(Z_0, \dots, Z_n). \end{aligned}$$

Then $\{(X_n, \mathcal{B}_n), n \geq 0\}$ is a *martingale* since $\{(Z_n, \mathcal{B}_n), n \geq 0\}$ is a *martingale difference sequence*.

- **Example** (*Likelihood Ratios*).

Suppose $\{Y_n, n \geq 0\}$ are *independent identically distributed* random variables and suppose the true density of Y_n is f_0 (The word “density” can be understood with respect to some fixed reference measure μ .) Let f_1 be some other probability density. For simplicity suppose $f_0(y) > 0$, for all y . For $n \geq 0$, define the likelihood ratio

$$\begin{aligned} X_n &:= \frac{\prod_{i=0}^n f_1(Y_i)}{\prod_{i=0}^n f_0(Y_i)} \\ \mathcal{B}_n &:= \sigma(Y_0, \dots, Y_n) \end{aligned}$$

Then (X_n, \mathcal{B}_n) is a *martingale*.

6.2 Bernstein Inequality for Martingale Difference Sequence

- **Proposition 6.5** (*Bernstein Inequality, Martingale Difference Sequence Version*) [Wainwright, 2019]

Let $\{(D_k, \mathcal{B}_k), k \geq 1\}$ be a **martingale difference sequence**, and suppose that

$$\mathbb{E} [\exp(\lambda D_k) | \mathcal{B}_{k-1}] \leq \exp\left(\frac{\lambda^2 \nu_k^2}{2}\right)$$

almost surely for any $|\lambda| < 1/\alpha_k$. Then the following hold:

1. The sum $\sum_{k=1}^n D_k$ is **sub-exponential** with **parameters** $(\sqrt{\sum_{k=1}^n \nu_k^2}, \alpha_*)$ where $\alpha_* := \max_{k=1, \dots, n} \alpha_k$. That is, for any $|\lambda| < 1/\alpha_*$,

$$\mathbb{E} \left[\exp \left\{ \lambda \left(\sum_{k=1}^n D_k \right) \right\} \right] \leq \exp \left(\frac{\lambda^2 \sum_{k=1}^n \nu_k^2}{2} \right)$$

2. The sum satisfies **the concentration inequality**

$$\mathbb{P} \left\{ \left| \sum_{k=1}^n D_k \right| \geq t \right\} \leq \begin{cases} 2 \exp \left(-\frac{t^2}{2 \sum_{k=1}^n \nu_k^2} \right) & \text{if } 0 \leq t \leq \frac{\sum_{k=1}^n \nu_k^2}{\alpha_*} \\ 2 \exp \left(-\frac{t}{\alpha_*} \right) & \text{if } t > \frac{\sum_{k=1}^n \nu_k^2}{\alpha_*}. \end{cases} \quad (48)$$

6.3 Azuma-Hoeffding Inequality

- **Corollary 6.6** (*Azuma-Hoeffding Inequality*) [Wainwright, 2019]

Let $\{(D_k, \mathcal{B}_k), k \geq 1\}$ be a **martingale difference sequence** for which there are constants $\{(a_k, b_k)\}_{k=1}^n$ such that $D_k \in [a_k, b_k]$ almost surely for all $k = 1, \dots, n$. Then, for all $t \geq 0$,

$$\mathbb{P} \left\{ \left| \sum_{k=1}^n D_k \right| \geq t \right\} \leq 2 \exp \left(-\frac{2t^2}{\sum_{k=1}^n (b_k - a_k)^2} \right) \quad (49)$$

6.4 Bounded Difference Inequality

- An important application of *Azuma-Hoeffding Inequality* concerns functions that satisfy a **bounded difference property**.

Definition (*Functions with Bounded Difference Property*)

Given vectors $x, x' \in \mathcal{X}^n$ and an index $k \in \{1, 2, \dots, n\}$, we define a new vector $x^{(-k)} \in \mathcal{X}^n$ via

$$x_j^{(-k)} = \begin{cases} x_j & j \neq k \\ x'_k & j = k \end{cases}$$

With this notation, we say that $f : \mathcal{X}^n \rightarrow \mathbb{R}$ satisfies **the bounded difference inequality** with parameters (L_1, \dots, L_n) if, for each index $k = 1, 2, \dots, n$,

$$\left| f(x) - f(x^{(-k)}) \right| \leq L_k, \quad \text{for all } x, x' \in \mathcal{X}^n. \quad (50)$$

- **Corollary 6.7** (*McDiarmid's Inequality / Bounded Differences Inequality*) [Wainwright, 2019]
Suppose that f satisfies **the bounded difference property** (50) with parameters (L_1, \dots, L_n) and that the random vector $X = (X_1, X_2, \dots, X_n)$ has **independent** components. Then

$$\mathbb{P}\{|f(X) - \mathbb{E}[f(X)]| \geq t\} \leq 2 \exp\left(-\frac{2t^2}{\sum_{k=1}^n L_k^2}\right). \quad (51)$$

7 Bounding Variance

7.1 Mean-Median Deviation

- **Definition** (*Median of Random Variable*)
The **median** of a random variable $X \in \mathcal{X}$ with distribution \mathbb{P} is a constant m such that

$$\mathbb{P}\{X \geq m\} \geq \frac{1}{2} \quad \wedge \quad \mathbb{P}\{X \leq m\} \geq \frac{1}{2}$$

- **Proposition 7.1** (*Mean-Median Deviation, Variance Bound*) [Boucheron et al., 2013]
Let $X \in \mathcal{X}$ be a random variable with distribution \mathbb{P} , m be the **median** of X and $\mu = \mathbb{E}[X]$ be the **mean** of X . If $\text{Var}(X) = \sigma^2 < \infty$, then

$$|m - \mu| \leq \sqrt{\text{Var}(X)} = \sigma \quad (52)$$

(proof by Jensen's inequality $|m - \mu| = |\mathbb{E}[X - m]| \leq \mathbb{E}[|X - m|] \leq \mathbb{E}[|X - \mu|] \leq \sqrt{\mathbb{E}[|X - \mu|^2]}$)

- **Exercise 7.2** (*Mean-Median Deviation via Concentration Inequality*) [Boucheron et al., 2013]
Let X be a random variable with **median** m such that positive constants a and b exist so that for all $t > 0$,

$$\mathbb{P}\{|X - m| \geq t\} \leq a \exp\left(-\frac{t^2}{b}\right)$$

Show that

$$|m - \mu| \leq \min\left\{\sqrt{ab}, \frac{a}{2}\sqrt{b\pi}\right\}.$$

- **Exercise 7.3** (*Concentration Inequality Around Medians and Means*) [Wainwright, 2019]
Given a scalar random variable X , suppose that there are positive constants c_1, c_2 such that for all $t \geq 0$,

$$\mathbb{P}\{|X - \mathbb{E}[X]| \geq t\} \leq c_1 \exp(-c_2 t^2) \quad (53)$$

1. Prove that $\text{Var}(X) \leq \frac{c_1}{c_2}$

2. Let m_X be the a **median** of X . Show that **whenever the mean concentration bound (53) holds**, then for **any median** m_X , we have, for all $t \geq 0$, **the median concentration**

$$\mathbb{P}\{|X - m_X| \geq t\} \leq c_3 \exp(-c_4 t^2) \quad (54)$$

where $c_3 := 4c_1$ and $c_4 := \frac{c_2}{8}$.

3. Conversely, show that **whenever the median concentration bound (54) holds**, then **mean concentration (53) holds** with $c_1 = 2c_3$ and $c_2 = \frac{c_4}{4}$.

7.2 The Efron-Stein Inequality and Jackknife Estimation

- **Remark (Variance of Smoothing Martingale Difference Sequence)**

Suppose $X \in L_1$ and $\{\mathcal{B}_n, n \geq 0\}$ is an increasing family of sub σ -algebra of \mathcal{B} formed by

$$\mathcal{B}_n := \sigma(Z_1, \dots, Z_n).$$

For $n \geq 1$, define

$$\begin{aligned} d_0 &:= \mathbb{E}[X] \\ d_n &:= \mathbb{E}[X|\mathcal{B}_n] - \mathbb{E}[X|\mathcal{B}_{n-1}] \\ &= \mathbb{E}[X|Z_1, \dots, Z_n] - \mathbb{E}[X|Z_1, \dots, Z_{n-1}]. \end{aligned}$$

From (47) we see that (d_n, \mathcal{B}_n) is a martingale difference sequence. By *orthogonality of martingale difference*, we see that

$$\mathbb{E}[d_i d_j] = 0 \quad i \neq j.$$

Therefore, based on the decomposition

$$X - EX = \sum_{i=1}^n d_i$$

we have

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}\left[\left(\sum_{i=1}^n d_i\right)^2\right] = \sum_{i=1}^n \mathbb{E}[d_i^2] + 2 \sum_{i>j} \mathbb{E}[d_i d_j] \\ &= \sum_{i=1}^n \mathbb{E}[d_i^2]. \end{aligned} \quad (55)$$

- **Remark (Variance of General Functions of Independent Random Variables)**

Then above formula (55) holds when $X = f(Z_1, \dots, Z_n)$ for general function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with n independent random variables (Z_1, \dots, Z_n) . By *Fubini's theorem*,

$$\mathbb{E}[X|Z_1, \dots, Z_i] = \int_{\mathcal{Z}^{n-i}} f(Z_1, \dots, Z_i, z_{i+1}, \dots, z_n) d\mu_{i+1}(z_{i+1}) \dots d\mu_n(z_n)$$

where μ_j is the probability distribution of Z_j for $j \geq 1$.

Let $Z_{(-i)} := (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n)$ be all random variables (Z_1, \dots, Z_n) **except for** Z_i . Denote $\mathbb{E}_{(-i)}[\cdot]$ as the conditional expectation of X given $Z_{(-i)}$

$$\begin{aligned}\mathbb{E}_{(-i)}[X] &:= \mathbb{E}[X|Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n] \\ &= \int_{\mathcal{Z}} f(Z_1, \dots, Z_{i-1}, z_i, Z_{i+1}, \dots, Z_n) d\mu_i(z_i).\end{aligned}$$

Then, again by *Fubini's theorem (smoothing properties of conditional expectation)*,

$$\mathbb{E}[\mathbb{E}_{(-i)}[X] | Z_1, \dots, Z_i] = \mathbb{E}[X | Z_1, \dots, Z_{i-1}] \quad (56)$$

• **Proposition 7.4 (Efron-Stein Inequality)** [Boucheron et al., 2013]

Let Z_1, \dots, Z_n be **independent random variables** and let $X = f(Z)$ be a square-integrable function of $Z = (Z_1, \dots, Z_n)$. Then

$$\text{Var}(X) \leq \sum_{i=1}^n \mathbb{E}[(X - \mathbb{E}_{(-i)}[X])^2] := \nu. \quad (57)$$

Moreover, if Z'_1, \dots, Z'_n are **independent** copies of Z_1, \dots, Z_n and if we define, for every $i = 1, \dots, n$,

$$X'_i := f(Z_1, \dots, Z_{i-1}, Z'_i, Z_{i+1}, \dots, Z_n),$$

then

$$\nu = \frac{1}{2} \sum_{i=1}^n \mathbb{E}[(X - X'_i)^2] = \sum_{i=1}^n \mathbb{E}[(X - X'_i)_+^2] = \sum_{i=1}^n \mathbb{E}[(X - X'_i)_-^2]$$

where $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$ denote the **positive** and **negative** parts of a real number x . Also,

$$\nu = \inf_{X_i} \sum_{i=1}^n \mathbb{E}[(X - X_i)^2],$$

where the infimum is taken over the class of all $Z_{(-i)}$ -measurable and square-integrable variables X_i , $i = 1, \dots, n$.

• **Example (The Jackknife Estimate)**

We should note here that the Efron-Stein inequality was first motivated by the study of the so-called **jackknife estimate of statistics**.

To describe this estimate, assume that Z_1, \dots, Z_n are i.i.d. random variables and one wishes to *estimate a functional θ of the distribution* of the Z_i by a function $X = f(Z_1, \dots, Z_n)$ of the data. The quality of the estimate is often measured by its bias $\mathbb{E}[X] - \theta$ and its variance $\text{Var}(X)$. Since the distribution of the Z_i 's is unknown, one needs to *estimate* the bias and variance **from the same sample**. The jackknife estimate of the bias is defined by

$$(n-1) \left(\frac{1}{n} \sum_{i=1}^n X_i - X \right) \quad (58)$$

where X_i is an appropriately defined function of $Z_{(-i)} := (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n)$. $Z_{(-i)}$ is often called *the i -th jackknife sample* while X_i is the so-called *jackknife replication* of X . In an analogous way, the jackknife estimate of the variance is defined by

$$\sum_{i=1}^n (X - X_i)^2 \quad (59)$$

Using this language, *the Efron-Stein inequality* simply states that *the jackknife estimate of the variance is always positively biased*. In fact, this is how Efron and Stein originally formulated their inequality.

7.3 Functions with Bounded Differences

- **Corollary 7.5** [Boucheron et al., 2013]
If f has the **bounded differences property** with parameters (L_1, \dots, L_n) , then

$$\text{Var}(f(Z)) \leq \frac{1}{4} \sum_{i=1}^n L_i^2.$$

7.4 Convex Poincaré Inequality

- **Theorem 7.6 (Convex Poincaré Inequality)** [Boucheron et al., 2013]
Let Z_1, \dots, Z_n be **independent** random variables taking values in the interval $[0, 1]$ and let $f : [0, 1]^n \rightarrow \mathbb{R}$ be a **separately convex function** whose partial derivatives exist; that is, for every $i = 1, \dots, n$ and fixed $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n$, f is a convex function of its i -th variable. Then $f(Z) = f(Z_1, \dots, Z_n)$ satisfies

$$\text{Var}(f(Z)) \leq \mathbb{E} \left[\|\nabla f(Z)\|_2^2 \right]. \quad (60)$$

7.5 Gaussian Poincaré Inequality

- **Theorem 7.7 (Gaussian Poincaré Inequality)** [Boucheron et al., 2013]
Let $Z = (Z_1, \dots, Z_n)$ be a vector of **i.i.d. standard Gaussian** random variables (i.e. Z is a Gaussian vector with **zero mean** vector and **identity covariance matrix**). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be any **continuously differentiable** function. Then

$$\text{Var}(f(Z)) \leq \mathbb{E} \left[\|\nabla f(Z)\|_2^2 \right]. \quad (61)$$

8 Entropy Method

8.1 Entropy Functional and Φ -Entropy

- **Definition (Φ -Entropy)** [Boucheron et al., 2013]
Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be a **convex** function, and assign, to every **non-negative integrable**

random variable X , the Φ -entropy of X is defined as

$$H_\Phi(X) = \mathbb{E} [\Phi(X)] - \Phi(\mathbb{E} [X]). \quad (62)$$

- **Remark** The Φ -entropy is a **functional** of *distribution* P_X instead of a function of X .
- **Remark** By Jensen's inequality, the Φ -entropy is *non-negative*

$$\begin{aligned} \Phi(\mathbb{E} [X]) &\leq \mathbb{E} [\Phi(X)] \\ \Rightarrow H_\Phi(X) &= \mathbb{E} [\Phi(X)] - \Phi(\mathbb{E} [X]) \geq 0. \end{aligned}$$

- **Example** (*Special Examples for Φ -Entropy*)

1. For $\Phi(x) = x^2$, the Φ -entropy of X is the **variance** of X :

$$H_\Phi(X) = \mathbb{E} [X^2] - (\mathbb{E} [X])^2 = \text{Var}(X).$$

2. For $\Phi(x) = -\log(x)$, the Φ -entropy of $Y = e^{\lambda X}$ is the **logarithm of moment generating function** of $X - \mathbb{E} [X]$:

$$H_\Phi(e^{\lambda X}) = -\lambda \mathbb{E} [X] + \log \left(\mathbb{E} [e^{\lambda X}] \right) = \log \mathbb{E} [e^{\lambda(X - \mathbb{E} [X])}] := \psi_{X - \mathbb{E} [X]}(\lambda). \quad (63)$$

3. For $\Phi(x) = x \log x$, the Φ -entropy of X is defined as the **entropy functional** of X

$$H_\Phi(X) = \text{Ent}(X) := \mathbb{E} [X \log X] - \mathbb{E} [X] \log (\mathbb{E} [X]). \quad (64)$$

Let (Ω, \mathcal{B}) be measurable space, and P and Q are probability measures on Ω with $P \ll Q$. Define a random variable X by the *Radon-Nikodym derivative* of P with respect to Q ; that is,

$$X(\omega) := \begin{cases} \frac{dP}{dQ}(\omega) & Q(\omega) > 0 \\ 0 & \text{o.w.} \end{cases}.$$

We see that X is Q -measurable and $dP = X dQ$ with $\mathbb{E}_Q [X] = 1$. Then the entropy of X is the relative entropy of P with respect to Q .

$$\text{Ent}(X) = \text{KL}(P \parallel Q) \quad (65)$$

8.2 Dual Formulation

- **Lemma 8.1** The **Legendre transform** (or **convex conjugate**) of $\Phi(x) = x \log(x)$ is e^{u-1} . That is,

$$\sup_{x>0} \{u x - x \log(x)\} = e^{u-1}$$

- **Proposition 8.2** (**Duality Formula of Entropy**) [Boucheron et al., 2013]
Let X be a non-negative random variable defined on a probability space (Ω, \mathcal{A}, P) such that $\mathbb{E} [\Phi(X)] < \infty$. Then we have **the duality formula**

$$\text{Ent}(X) = \sup_{U \in \mathcal{U}} \mathbb{E} [U X] \quad (66)$$

where the supremum is taken over the set \mathcal{U} of all random variables $U : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ with $\mathbb{E}[e^U] = 1$. Moreover, if U is such that $\mathbb{E}[UX] \leq \text{Ent}(X)$ for all non-negative random variable X such that $\Phi(X)$ is integrable and $\mathbb{E}[X] = 1$, then $\mathbb{E}[e^U] \leq 1$.

- **Corollary 8.3 (Alternative Duality Formula of Entropy)** [Boucheron et al., 2013]

$$\text{Ent}(X) = \sup_T \mathbb{E}[X(\log(T) - \log(\mathbb{E}[T]))] \quad (67)$$

where the supremum is taken over all non-negative and integrable random variables.

- **Corollary 8.4 (Duality Formula of Log-MGF)** [Cover and Thomas, 2006, Boucheron et al., 2013]

Let X be a real-valued integrable random variable. Then for every $\lambda \in \mathbb{R}$,

$$\log \mathbb{E}_{\mathbb{P}}[e^{\lambda(X - \mathbb{E}[X])}] = \sup_{\mathbb{Q} \ll \mathbb{P}} \{\lambda(\mathbb{E}_{\mathbb{Q}}[X] - \mathbb{E}_{\mathbb{P}}[X]) - \text{KL}(\mathbb{Q} \parallel \mathbb{P})\}, \quad (68)$$

where the supremum is taken over all probability measures \mathbb{Q} absolutely continuous with respect to \mathbb{P} .

- **Corollary 8.5 (Duality Formula of Kullback-Leibler Divergence)** [Cover and Thomas, 2006, Boucheron et al., 2013]

Let \mathbb{P} and \mathbb{Q} be two probability distributions on the same space. Then

$$\text{KL}(\mathbb{Q} \parallel \mathbb{P}) = \sup_X \{\mathbb{E}_{\mathbb{Q}}[X] - \log \mathbb{E}_{\mathbb{P}}[e^X]\}, \quad (69)$$

where the supremum is taken over all random variables such that $\mathbb{E}_{\mathbb{P}}[\exp(X)] < \infty$.

- **Definition (Bregman Divergence)**

Let $F : \mathcal{X} \rightarrow \mathbb{R}$ be a continuously-differentiable, **strictly convex** function defined on a convex set \mathcal{X} . The **Bregman divergence** associated with F for points $p, q \in \mathcal{X}$ is the difference between the value of F at point p and the value of the *first-order Taylor expansion* of F around point q evaluated at point p :

$$\mathbb{D}^F(p \parallel q) = F(p) - F(q) - \langle \nabla F(q), p - q \rangle \quad (70)$$

- **Theorem 8.6 (The Expected Value Minimizes Expected Bregman Divergence)** [Boucheron et al., 2013]

Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \rightarrow \mathbb{R}$ be **convex** and **differentiable**. For any $x, y \in I$, the **Bregman divergence** of f from x to y is $f(y) - f(x) - f'(x)(y - x)$. Let X be an I -valued random variable. Then

$$\mathbb{E}[f(X) - f(\mathbb{E}[X])] = \inf_{a \in I} \mathbb{E}[f(X) - f(a) - f'(a)(X - a)] \quad (71)$$

- **Corollary 8.7 (Duality Formula of Entropy via Bregman Divergence)** [Boucheron et al., 2013]

Let X be a non-negative random variable such that $\mathbb{E}[\Phi(X)] < \infty$. Then

$$\text{Ent}(X) = \inf_{u > 0} \mathbb{E}[X(\log(X) - \log(u)) - (X - u)] \quad (72)$$

8.3 Tensorization Property

- **Proposition 8.8** (*Sub-Additivity of The Entropy / Tensorization Property*) [Boucheron et al., 2013]

Let $\Phi(x) = x \log x$, for $x > 0$ and $\Phi(0) = 0$. Let Z_1, Z_2, \dots, Z_n be independent random variables taking values in \mathcal{X} , and let $f : \mathcal{X}^n \rightarrow [0, \infty)$ be a measurable function. Letting $X = f(Z_1, Z_2, \dots, Z_n)$ such that $\mathbb{E}[X \log X] < \infty$, we have

$$\text{Ent}(X) := \mathbb{E}[\Phi(X)] - \Phi(\mathbb{E}[X]) \leq \sum_{i=1}^n \mathbb{E}[\mathbb{E}_{(-i)}[\Phi(X)] - \Phi(\mathbb{E}_{(-i)}[X])], \quad (73)$$

where $\mathbb{E}_{(-i)}[\cdot]$ is the conditional expectation operator conditioning on $Z_{(-i)}$. Introducing the notation $\text{Ent}_{(-i)}(X) = \mathbb{E}_{(-i)}[\Phi(X)] - \Phi(\mathbb{E}_{(-i)}[X])$, this can be re-written as

$$\text{Ent}(X) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}_{(-i)}(X) \right]. \quad (74)$$

8.4 Herbst's Argument

- **Remark** (*Entropy Functional for Moment Generating Function*)

Let $X = e^{\lambda Z}$ where Z is a random variable. The entropy function of X becomes

$$\text{Ent}(e^{\lambda Z}) = \mathbb{E}[\lambda Z e^{\lambda Z}] - \mathbb{E}[e^{\lambda Z}] \log(\mathbb{E}[e^{\lambda Z}])$$

Denote $\psi_{Z-\mathbb{E}[Z]}(\lambda) := \log \mathbb{E}[e^{\lambda(Z-\mathbb{E}[Z])}]$. Then we have

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} = \lambda \psi'_{Z-\mathbb{E}[Z]}(\lambda) - \psi_{Z-\mathbb{E}[Z]}(\lambda). \quad (75)$$

Our strategy is based on using (75) *the sub-additivity of entropy* and then univariate calculus to derive *upper bounds for the derivative of $\psi(\lambda)$* . By solving the obtained *differential inequality*, we obtain tail bounds via *Chernoff's bounding*.

For example, if

$$\begin{aligned} \frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} &\leq \frac{\nu \lambda^2}{2} \\ \Leftrightarrow \lambda \psi'_{Z-\mathbb{E}[Z]}(\lambda) - \psi_{Z-\mathbb{E}[Z]}(\lambda) &\leq \frac{\nu \lambda^2}{2}, \\ \Leftrightarrow \frac{1}{\lambda} \psi'_{Z-\mathbb{E}[Z]}(\lambda) - \frac{1}{\lambda^2} \psi_{Z-\mathbb{E}[Z]}(\lambda) &\leq \frac{\nu}{2}. \end{aligned}$$

Setting $G(\lambda) = \lambda^{-1} \psi_{Z-\mathbb{E}[Z]}(\lambda)$, we see that the differential inequality becomes

$$G'(\lambda) \leq \frac{\nu}{2}.$$

Since $G(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$, which implies that

$$G(\lambda) \leq \frac{\nu \lambda}{2},$$

and the result follows.

- **Proposition 8.9 (*Herbst's Argument*)** [Boucheron et al., 2013, Wainwright, 2019]
Let Z be an integrable random variable such that for some $\nu > 0$, we have, for every $\lambda > 0$,

$$\frac{\text{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} \leq \frac{\nu \lambda^2}{2} \quad (76)$$

Then, for every $\lambda > 0$, the logarithmic moment generating function of centered random variable $(Z - \mathbb{E}[Z])$ satisfies

$$\psi_{Z - \mathbb{E}[Z]}(\lambda) := \log \mathbb{E} \left[e^{\lambda(Z - \mathbb{E}[Z])} \right] \leq \frac{\nu \lambda^2}{2}.$$

8.5 Association Inequalities

- **Proposition 8.10 (*Chebyshev's Association Inequalities*)** [Boucheron et al., 2013]
Let f and g be **nondecreasing** real-valued functions defined on the real line. If X is a real-valued random variable and Y is a **nonnegative** random variable, then

$$\mathbb{E}[Y] \mathbb{E}[Y f(X) g(X)] \geq \mathbb{E}[Y f(X)] \mathbb{E}[Y g(X)] \quad (77)$$

If f is **nonincreasing** and g is **nondecreasing** then

$$\mathbb{E}[Y] \mathbb{E}[Y f(X) g(X)] \leq \mathbb{E}[Y f(X)] \mathbb{E}[Y g(X)] \quad (78)$$

- **Proposition 8.11 (*Harris's Inequalities*)** [Boucheron et al., 2013]
Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be **nondecreasing** functions. Let Z_1, \dots, Z_n be **independent** real-valued random variables and define the random vector $Z = (Z_1, \dots, Z_n)$ taking values in \mathbb{R}^n . Then

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)] \mathbb{E}[g(X)] \quad (79)$$

If f is **nonincreasing** and g is **nondecreasing** then

$$\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)] \mathbb{E}[g(X)] \quad (80)$$

8.6 Connection to Variance Bounds

- **Proposition 8.12 (*A Modified Logarithmic Sobolev Inequalities for Moment Generating Function*)** [Boucheron et al., 2013]
Consider independent random variables Z_1, \dots, Z_n taking values in \mathcal{X} , a real-valued function $f : \mathcal{X}^n \rightarrow \mathbb{R}$ and the random variable $X = f(Z_1, \dots, Z_n)$. Also denote $Z_{(-i)} = (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n)$ and $X_{(-i)} = f_i(Z_{(-i)})$ where $f_i : \mathcal{X}^{n-1} \rightarrow \mathbb{R}$ is an arbitrary function. Let $\phi(x) = e^x - x - 1$. Then for all $\lambda \in \mathbb{R}$,

$$\text{Ent}(e^{\lambda X}) := \mathbb{E}[\lambda X e^{\lambda X}] - \mathbb{E}[e^{\lambda X}] \log \mathbb{E}[e^{\lambda X}] \leq \sum_{i=1}^n \mathbb{E} \left[e^{\lambda X} \phi(-\lambda(X - X_{(-i)})) \right] \quad (81)$$

- **Proposition 8.13 (*Symmetrized Modified Logarithmic Sobolev Inequalities*)** [Boucheron et al., 2013]

Consider independent random variables Z_1, \dots, Z_n taking values in \mathcal{X} , a real-valued function $f : \mathcal{X}^n \rightarrow \mathbb{R}$ and the random variable $X = f(Z_1, \dots, Z_n)$. Also denote $\tilde{X}^{(i)} = f(Z_1, \dots, Z_{i-1}, Z'_i, Z_{i+1}, \dots, Z_n)$. Let $\phi(x) = e^x - x - 1$. Then for all $\lambda \in \mathbb{R}$,

$$\lambda \mathbb{E} [X e^{\lambda X}] - \mathbb{E} [e^{\lambda X}] \log \mathbb{E} [e^{\lambda X}] \leq \sum_{i=1}^n \mathbb{E} [e^{\lambda X} \phi(-\lambda(X - \tilde{X}^{(i)}))] \quad (82)$$

Moreover, denoting $\tau(x) = x(e^x - 1)$, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \lambda \mathbb{E} [X e^{\lambda X}] - \mathbb{E} [e^{\lambda X}] \log \mathbb{E} [e^{\lambda X}] &\leq \sum_{i=1}^n \mathbb{E} [e^{\lambda X} \tau(-\lambda(X - \tilde{X}^{(i)})_+)] , \\ \lambda \mathbb{E} [X e^{\lambda X}] - \mathbb{E} [e^{\lambda X}] \log \mathbb{E} [e^{\lambda X}] &\leq \sum_{i=1}^n \mathbb{E} [e^{\lambda X} \tau(\lambda(\tilde{X}^{(i)} - X)_-)] . \end{aligned}$$

- **Remark** In the proof, we have

$$\begin{aligned} \text{Ent}_{\mu_i}(e^{\lambda X}) &\leq \mathbb{E}_{\mu_i} [e^{\lambda X} (\log e^{\lambda X} - \log e^{\lambda X'_i}) - (e^{\lambda X} - e^{\lambda X'_i})] \\ &= \mathbb{E}_{\mu_i} [e^{\lambda X} (\lambda(X - X'_i) - (e^{\lambda X} - e^{\lambda X'_i}))] \\ &\leq \mathbb{E}_{\mu_i} [(e^{\lambda X} - e^{\lambda X'_i})(\lambda(X - X'_i)_+)] \\ &\leq \lambda^2 \mathbb{E}_{\mu_i} [(X - X'_i)_+^2] \end{aligned}$$

Using the convexity of e^x , we have $e^s - e^t \leq e^t(s - t)$ if $s > t$. Thus

$$\text{Ent}(e^{\lambda X}) \leq \lambda^2 \sum_{i=1}^n \mathbb{E} [(X - X'_i)_+^2] .$$

From Efron-Stein inequality, if we can bound

$$\sum_{i=1}^n \mathbb{E} [(X - X'_i)_+^2] \leq \nu ,$$

then we can bound both the variance $\text{Var}(X)$ and entropy $\text{Ent}(e^{\lambda X})$.

9 Transportation Method

9.1 Optimal Transport, Wasserstein Distance and its Dual

- **Definition (*Pushforward Measure*)** [Peyr and Cuturi, 2019]

Let $(\mathcal{X}, \mathcal{B}_X)$ and $(\mathcal{Y}, \mathcal{B}_Y)$ be two topological measurable spaces. Denote the spaces of *general (Radon) measures* on \mathcal{X}, \mathcal{Y} as $\mathcal{M}(\mathcal{X})$ and $\mathcal{M}(\mathcal{Y})$. Also let $\mathcal{C}(\mathcal{X})$ be space of continuous functions on \mathcal{X} . For a *continous* map $T : \mathcal{X} \rightarrow \mathcal{Y}$, the **push-forward operator** is defined as $T_{\#} : \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{Y})$ that satisfies

$$\forall h \in \mathcal{C}(\mathcal{X}), \quad \int_{\mathcal{Y}} h(y) d(T_{\#}\alpha)(y) = \int_{\mathcal{X}} h(T(x)) d\alpha(x). \quad (83)$$

$$\text{or equivalently,} \quad (T_{\#}\alpha)(B) := \alpha(\{x : T(x) \in B \subset \mathcal{Y}\}) = \alpha(T^{-1}(B)) \quad (84)$$

where the *push-forward measure* $\beta := T_{\#}\alpha \in \mathcal{M}(\mathcal{Y})$ of some $\alpha \in \mathcal{M}(\mathcal{X})$, $T^{-1}(\cdot)$ is the pre-image of T .

- **Remark (*Density Function of Pushforward Measure*)**

Assume that (α, β) have densities $(\rho_{\alpha}, \rho_{\beta})$ with respect to a fixed measure, and $\beta = T_{\#}\alpha$. We see that $T_{\#}$ acts on a density ρ_{α} linearly to a density ρ_{β} as a change of variable, i.e.

$$\begin{aligned} \rho_{\alpha}(\mathbf{x}) &= |\det(T'(\mathbf{x}))| \rho_{\beta}(T(\mathbf{x})) \\ |\det(T'(\mathbf{x}))| &= \frac{\rho_{\alpha}(\mathbf{x})}{\rho_{\beta}(T(\mathbf{x}))} \end{aligned} \tag{85}$$

- **Definition (*Optimal Transport Problem, Monge Problem*)** [Villani, 2009, Santambrogio, 2015, Peyr and Cuturi, 2019]

Let $(\mathcal{X}, \mathcal{B}_X)$ and $(\mathcal{Y}, \mathcal{B}_Y)$ be two measurable spaces, where \mathcal{X} and \mathcal{Y} are *complete separable metric spaces*. Denote $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ as the space of probability measures on \mathcal{X} and \mathcal{Y} . Define a **cost function** $c : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}_+$ as non-negative real-valued measurable functions on $\mathcal{X} \times \mathcal{Y}$. The optimal transport problem by *Monge* (i.e. **Monge Problem**) is defined as follows: given two probability measures $\mathbb{P} \in \mathcal{P}(\mathcal{X})$ and $\mathbb{Q} \in \mathcal{P}(\mathcal{Y})$, find a *continuous measurable map* $T : \mathcal{X} \rightarrow \mathcal{Y}$ so that

$$\begin{aligned} \inf_T \int_{\mathcal{X}} c(x, T(x)) d\mathbb{P}(x) \\ \text{s.t. } \mathbb{Q} = T_{\#}\mathbb{P} \end{aligned}$$

The optimal solution T is also called an *optimal transportation plan*.

- **Definition (*Optimal Transport Problem, Kantorovich Relaxation*)** [Villani, 2009, Santambrogio, 2015, Peyr and Cuturi, 2019]

The optimal transport problem by *Kantorovich* (i.e. **Kantorovich Relaxation**) is defined as follows: given two probability measures $\mathbb{P} \in \mathcal{P}(\mathcal{X})$ and $\mathbb{Q} \in \mathcal{P}(\mathcal{Y})$, find a *joint probability measure* $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$ so that

$$\begin{aligned} \inf_{\gamma} \int_{\mathcal{X} \times \mathcal{Y}} c(x, y) d\gamma(x, y) \\ \text{s.t. } \gamma \in \Pi(\mathbb{P}, \mathbb{Q}) := \{\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : \pi_{\mathcal{X}, \#}\gamma = \mathbb{P}, \pi_{\mathcal{Y}, \#}\gamma = \mathbb{Q}\} \end{aligned}$$

where $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is the space of joint probability measure on $\mathcal{X} \times \mathcal{Y}$, $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are the coordinate projection onto \mathcal{X} and \mathcal{Y} . $\pi_{\mathcal{X}, \#}\gamma = \mathbb{P}$ means that \mathbb{P} is the marginal distribution of γ on \mathcal{X} . Similarly \mathbb{Q} is the marginal distribution of γ on \mathcal{Y} .

Equivalently, let X and Y are *random variables* taking values in \mathcal{X} and \mathcal{Y} . The *joint distribution* of (X, Y) is γ with marginal distribution of X and Y being \mathbb{P} and \mathbb{Q} . Then the problem is

$$\min_{\gamma \in \Pi(\mathbb{P}, \mathbb{Q})} \mathbb{E}_{\gamma} [c(X, Y)]$$

The joint distribution $\gamma \in \Pi(\mathbb{P}, \mathbb{Q})$ such that $X_{\#}\gamma = \mathbb{P}$ and $Y_{\#}\gamma = \mathbb{Q}$ is called a *coupling*.

- **Definition (*Dual Problem of Kantorovich Problem*)** [Villani, 2009, Santambrogio, 2015, Peyr and Cuturi, 2019]

The **dual problem** of *Kantorovich problem* is described as below:

$$\begin{aligned} \mathcal{L}_c(\mathbb{P}, \mathbb{Q}) = & \max_{(\varphi, \psi) \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})} \int_{\mathcal{X}} \varphi(x) d\mathbb{P}(x) + \int_{\mathcal{Y}} \psi(y) d\mathbb{Q}(y) \\ \text{s.t. } & \varphi(x) + \psi(y) \leq c(x, y), \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}, \end{aligned}$$

Here, (φ, ψ) is a pair of *continuous functions* on \mathcal{X} and \mathcal{Y} respectively and they are also the **Kantorovich potentials**. The feasible region is

$$\mathcal{R}(c) := \{(\varphi, \psi) \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y}) : \varphi \oplus \psi \leq c\}$$

where $(\varphi \oplus \psi)(x, y) = \varphi(x) + \psi(y)$.

In other words, the dual optimization problem is

$$\max_{(\varphi, \psi) \in \mathcal{R}(c)} \mathbb{E}_{\mathbb{P}}[\varphi(X)] + \mathbb{E}_{\mathbb{Q}}[\psi(Y)]$$

- **Proposition 9.1 (Strong Duality)** [Santambrogio, 2015]

Let \mathcal{X}, \mathcal{Y} be **complete separable spaces**, and $c : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}_+$ be **lower semi-continuous and bounded from below**. Then the optimal value of primal and dual problems are the same

$$\min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[c(X, Y)] = \mathcal{L}_c(\mathbb{P}, \mathbb{Q}) = \max_{(\varphi, \psi) \in \mathcal{R}(c)} \mathbb{E}_{\mathbb{P}}[\varphi(X)] + \mathbb{E}_{\mathbb{Q}}[\psi(Y)].$$

- **Definition (Wasserstein Distance)**

Let $((\mathcal{X}, d), \mathcal{B})$ be a *metric measurable space* with *Borel σ -algebra* induced by metric d . Let X, Y be two random variables taking values in \mathcal{X} with distribution \mathbb{P} and \mathbb{Q} . **The Wasserstein distance** between probability distributions \mathbb{P} and \mathbb{Q} induced by d is defined as

$$\mathcal{W}_1(\mathbb{P}, \mathbb{Q}) \equiv \mathcal{W}_d(\mathbb{P}, \mathbb{Q}) := \min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[d(X, Y)] \quad (86)$$

In general, for $p \in [1, \infty)$, we can define **Wasserstein p -distance** as

$$\mathcal{W}_p(\mathbb{P}, \mathbb{Q}) \equiv \mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) := \left(\min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[(d(X, Y))^p] \right)^{1/p}. \quad (87)$$

- **Remark** Not to confuse the **2-Wasserstein distance** with **the Wasserstein distance induced by L_2 norm**:

$$\begin{aligned} \mathcal{W}_{\|\cdot\|_2}(\mathbb{P}, \mathbb{Q}) & \equiv \mathcal{W}_{1, \|\cdot\|_2}(\mathbb{P}, \mathbb{Q}) := \min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[\|X - Y\|_2] \\ \mathcal{W}_2(\mathbb{P}, \mathbb{Q}) & \equiv \mathcal{W}_{2,d}(\mathbb{P}, \mathbb{Q}) := \sqrt{\min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[d(X, Y)^2]} \end{aligned}$$

- **Remark (Wasserstein p -Distance is a Metric in $\mathcal{P}(\mathcal{X})$)**

The **Wasserstein p -distance** $\mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) := (\min_{X \sim \mathbb{P}, Y \sim \mathbb{Q}} \mathbb{E}[(d(X, Y))^p])^{1/p}$ is a well-defined metric in $\mathcal{P}(\mathcal{X})$: for all $\mathbb{P}, \mathbb{Q}, \mathbb{M} \in \mathcal{P}(\mathcal{X})$,

1. (Non-Negativity): $\mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) \geq 0$.

2. (*Definiteness*): $\mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) = 0$ iff $\mathbb{P} = \mathbb{Q}$
3. (*Symmetric*): $\mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) = \mathcal{W}_{p,d}(\mathbb{Q}, \mathbb{P})$
4. (*Triangular inequality*): $\mathcal{W}_{p,d}(\mathbb{P}, \mathbb{Q}) \leq \mathcal{W}_{p,d}(\mathbb{P}, \mathbb{M}) + \mathcal{W}_{p,d}(\mathbb{M}, \mathbb{Q})$

• **Definition (*Total Variation / Variational Distance*)**

Let P, Q be two probability measures on measurable space (Ω, \mathcal{F}) . The ***total variation*** or ***variational distance*** between P and Q is defined by

$$V(P, Q) := \sup_{A \in \mathcal{F}} |P(A) - Q(A)| \quad (88)$$

• **Remark (*Equivalent Formulation of Total Variation*)**

It is a well-known and simple fact that the total variation is half the L_1 -distance, that is, if μ is a *common dominating measure* of P and Q and $p(x) = dP/d\mu$ and $q(x) = dQ/d\mu$ denote their respective densities, then

$$V(P, Q) := P(A^*) - Q(A^*) = \frac{1}{2} \int_{\Omega} |p(x) - q(x)| d\mu(x), \quad (89)$$

where $A^* = \{x : p(x) \geq q(x)\}$.

• **Remark (*Total Variation via Optimal Coupling of Two Measures*)**

We note that another important interpretation of the *variational distance* is related to the *best coupling of the two measures*

$$V(P, Q) = \min P\{X \neq Y\} \quad (90)$$

where the minimum is taken over *all pairs of joint distributions* for the random variables (X, Y) whose marginal distributions are $X \sim P$ and $Y \sim Q$.

• **Proposition 9.2 (*Pinsker's Inequality*)** [Cover and Thomas, 2006, Boucheron et al., 2013]

Let P, Q be two probability distributions on measurable space (Ω, \mathcal{F}) such that $P \ll Q$. Then

$$V(P, Q)^2 \leq \frac{1}{2} \text{KL}(P \parallel Q). \quad (91)$$

• **Remark (*Total Variation as 1-Wasserstein Distance*)**

The total variation between P and Q is the ***Wasserstein distance*** induced by the ***Hamming distance*** $d(x, y) = \# \{i : x_i \neq y_i\}$.

$$V(P, Q) = \mathcal{W}_1(P, Q).$$

Thus the *Pinsker's inequality* (91) is the special case of *transportation cost inequality* (93).

• **Theorem 9.3 (*Kantorovich-Rubenstein Duality*)** [Villani, 2009]

Let \mathcal{X} be a ***Polish space***, i.e. \mathcal{X} a ***complete separable metric space*** equipped with a Borel σ -algebra induced by metric d , and \mathbb{P} and \mathbb{Q} be probability measures on \mathcal{X} . For fixed $p \in [1, \infty)$, let Lip_1 be the space of all 1-***Lipschitz*** function with respect to metric d such that

$$\|f\|_L := \sup_{x, y \in \mathcal{X}} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \right\} \leq 1.$$

Then

$$\mathcal{W}_d(\mathbb{P}, \mathbb{Q}) \equiv \mathcal{W}_{1,d}(\mathbb{P}, \mathbb{Q}) = \sup_{f \in \text{Lip}_1} \{\mathbb{E}_{\mathbb{P}}[f(X)] - \mathbb{E}_{\mathbb{Q}}[f(Y)]\}. \quad (92)$$

9.2 Concentration via Transportation Cost

- **Lemma 9.4 (Transportation Lemma)** [Boucheron et al., 2013]

Let X be a real-valued integrable random variable. Let ϕ be a **convex** and **continuously differentiable** function on a (possibly unbounded) interval $[0, b)$ and assume that $\phi(0) = \phi'(0) = 0$. Define, for every $x \geq 0$, **the Legendre transform** $\phi^*(x) = \sup_{\lambda \in (0, b)} (\lambda x - \phi(\lambda))$, and let, for every $t \geq 0$, $\phi^{*-1}(t) = \inf\{x \geq 0 : \phi^*(x) > t\}$, i.e. **the generalized inverse** of ϕ^* . Then the following two statements are equivalent:

1. for every $\lambda \in (0, b)$,

$$\psi_{X - \mathbb{E}[X]}(\lambda) \leq \phi(\lambda)$$

where $\psi_X(\lambda) := \log \mathbb{E}_Q [e^{\lambda X}]$ is the logarithm of moment generating function;

2. for any probability measure P absolutely continuous with respect to Q such that $\text{KL}(P \parallel Q) < \infty$,

$$\mathbb{E}_P[X] - \mathbb{E}_Q[X] \leq \phi^{*-1}(\text{KL}(P \parallel Q)). \quad (93)$$

In particular, given $\nu > 0$, X follows a sub-Gaussian distribution, i.e.

$$\psi_{X - \mathbb{E}[X]}(\lambda) \leq \frac{\nu \lambda^2}{2}$$

for every $\lambda > 0$ **if and only if** for any probability measure P absolutely continuous with respect to Q and such that $\text{KL}(P \parallel Q) < \infty$,

$$\mathbb{E}_P[X] - \mathbb{E}_Q[X] \leq \sqrt{2\nu \text{KL}(P \parallel Q)}. \quad (94)$$

- **Remark (Transportation Method)**

Let $\mathbb{P} = \otimes_{i=1}^n \mathbb{P}_i$ be the product measure for $Z := (Z_1, \dots, Z_n)$ on \mathcal{X}^n and $f : \mathcal{X}^n \rightarrow \mathbb{R}$ be 1-Lipschitz function. Consider a probability measure \mathbb{Q} on \mathcal{X}^n , absolutely continuous with respect to \mathbb{P} and let Y be a random variable (defined on the same probability space as \mathcal{X}) such that Y has distribution \mathbb{Q} .

The lemma above suggests that one may prove *sub-Gaussian concentration inequalities* for $X = f(Z_1, \dots, Z_n)$ by proving a “*transportation*” inequality as above. The key to achieving this relies on *coupling*. In particular, the Kantorovich-Rubenstein duality for $\mathcal{W}_{1,d}$ suggests that

$$\mathbb{E}_{\mathbb{Q}}[f(Y)] - \mathbb{E}_{\mathbb{P}}[f(Z)] \leq \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \mathbb{E}_{\gamma}[d(Y, Z)] := \mathcal{W}_{1,d}(\mathbb{Q}, \mathbb{P})$$

Thus, it suffices to upper bound the 1-Wasserstein distance between \mathbb{Q} and \mathbb{P} .

- **Definition (d -Transportation Cost Inequality)** [Wainwright, 2019]

Let (\mathcal{X}, d) be a metric space with metric d , and $(\mathcal{X}, \mathcal{B})$ be a measurable space, where \mathcal{B} is the Borel σ -algebra induced by metric d , **the probability measure** \mathbb{P} is said to satisfy a **d -transportation cost inequality** with parameter $\nu > 0$ if

$$\mathcal{W}_{1,d}(\mathbb{Q}, \mathbb{P}) \leq \sqrt{2\nu \text{KL}(\mathbb{Q} \parallel \mathbb{P})} \quad (95)$$

for all probability measure $\mathbb{Q} \ll \mathbb{P}$ on \mathcal{B} .

- **Theorem 9.5 (Isoperimetric Inequality via Transportation Cost)** [Wainwright, 2019]
Consider a metric measure space $(\mathcal{X}, \mathcal{B}, \mathbb{P})$ with metric d , and suppose that \mathbb{P} satisfies the d -transportation cost inequality in (95). Then its **concentration function** satisfies the bound

$$\alpha_{\mathbb{P},(\mathcal{X},d)}(t) \leq \exp\left(-\frac{(t-t_0)_+^2}{2\nu}\right), \text{ for } t \geq t_0 \quad (96)$$

where $t_0 := \sqrt{2\nu \log 2}$. Moreover, for any $Z \sim \mathbb{P}$ and any L -Lipschitz function $f : \mathcal{X} \rightarrow \mathbb{R}$, we have the **concentration inequality**

$$\mathbb{P}\{|f(Z) - \mathbb{E}[f(Z)]| \geq t\} \leq 2 \exp\left(-\frac{t^2}{2\nu L^2}\right). \quad (97)$$

9.3 Tensorization for Transportation Cost

- **Proposition 9.6 (Tensorization for Transportation Cost)** [Boucheron et al., 2013]
Suppose that, for each $k = 1, 2, \dots, n$, the univariate distribution \mathbb{P}_k satisfies a d_k -transportation cost inequality with parameter ν_k . Then the **product distribution** $\mathbb{P} = \otimes_{k=1}^n \mathbb{P}_k$ satisfies the transportation cost inequality

$$\mathcal{W}_{1,d}(\mathbb{Q}, \mathbb{P}) = \sqrt{2 \left(\sum_{k=1}^n \nu_k \right) \text{KL}(\mathbb{Q} \parallel \mathbb{P})}, \quad \text{for all distributions } \mathbb{Q} \ll \mathbb{P} \quad (98)$$

where the Wasserstein metric is defined using the distance $d(x, y) := \sum_{k=1}^n d_k(x_k, y_k)$.

9.4 Induction Lemma

- **Lemma 9.7** [Boucheron et al., 2013]
Let $\mathbb{P} = \otimes_{i=1}^n \mathbb{P}_i$ be a **product probability measure** on a product measurable space \mathcal{X}^n and let \mathbb{Q} be a probability measure absolutely continuous with respect to \mathbb{P} (i.e. $\mathbb{Q} \ll \mathbb{P}$). Let $w : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ be a measurable function and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a **convex function**. Suppose that for every $i = 1, \dots, n$ and for every probability measure $\nu \ll \mathbb{P}_i$ which is absolutely continuous with respect to \mathbb{P}_i ,

$$\min_{\gamma \in \Pi(\mathbb{P}_i, \nu)} \phi(\mathbb{E}_\gamma[w(X_i, Y_i)]) \leq \text{KL}(\nu \parallel \mathbb{P}_i)$$

Then

$$\min_{\gamma \in \Pi(\mathbb{P}, \mathbb{Q})} \sum_{i=1}^n \phi(\mathbb{E}_\gamma[w(X_i, Y_i)]) \leq \text{KL}(\mathbb{Q} \parallel \mathbb{P}).$$

9.5 Marton's Transportation Inequality

- **Theorem 9.8 (Marton's Transportation Inequality)** [Boucheron et al., 2013]
Let $\mathbb{P} = \otimes_{k=1}^n \mathbb{P}_k$ be a product probability measure on \mathcal{X}^n , and let \mathbb{Q} be a probability measure

absolutely continuous with respect to \mathbb{P} . Define two random vectors $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ in \mathcal{X}^n with distribution \mathbb{P} and \mathbb{Q} respectively. Then

$$\begin{aligned} \mathcal{W}_{2,d_H}(\mathbb{Q}, \mathbb{P}) &:= \sqrt{\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \gamma^2 \{X_i \neq Y_i\}} \leq \sqrt{\frac{1}{2} \text{KL}(\mathbb{Q} \parallel \mathbb{P})} \\ &\Leftrightarrow \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \gamma^2 \{X_i \neq Y_i\} \leq \frac{1}{2} \text{KL}(\mathbb{Q} \parallel \mathbb{P}) \end{aligned} \quad (99)$$

- **Theorem 9.9 (Marton's Conditional Transportation Inequality)** [Boucheron et al., 2013]

Let $\mathbb{P} = \otimes_{k=1}^n \mathbb{P}_k$ be a product probability measure on \mathcal{X}^n , and let \mathbb{Q} be a probability measure absolutely continuous with respect to \mathbb{P} . Define two random vectors $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ in \mathcal{X}^n with distribution \mathbb{P} and \mathbb{Q} respectively. Then

$$\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \mathbb{E}_\gamma \left[\sum_{i=1}^n (\gamma^2 \{X_i \neq Y_i | X_i\} + \gamma^2 \{X_i \neq Y_i | Y_i\}) \right] \leq 2 \text{KL}(\mathbb{Q} \parallel \mathbb{P}) \quad (100)$$

- **Proposition 9.10 (Concentration of Lipschitz Function with Function Weighted Hamming Distance)** [Boucheron et al., 2013]

Let $f : \mathcal{X}^n \rightarrow \mathbb{R}$ be a measurable function and let Z_1, \dots, Z_n be independent random variables taking their values in \mathcal{X} . Define $X = f(Z_1, \dots, Z_n)$. Assume that there exist **measurable functions** $c_i : \mathcal{X}_n \rightarrow [0, \infty)$ such that for all $x, y \in \mathcal{X}^n$,

$$f(y) - f(z) \leq \sum_{i=1}^n c_i(z) \mathbb{1} \{y_i \neq z_i\}.$$

Setting

$$\nu = \mathbb{E} \left[\sum_{i=1}^n c_i^2(Z) \right] \quad \text{and} \quad \nu_\infty = \sup_{z \in \mathcal{X}^n} \sum_{i=1}^n c_i^2(z)$$

for all $\lambda > 0$, we have

$$\psi_{X - \mathbb{E}[X]}(\lambda) \leq \frac{\nu \lambda^2}{2} \quad \text{and} \quad \psi_{-X + \mathbb{E}[X]}(\lambda) \leq \frac{\nu_\infty \lambda^2}{2}$$

In particular, for all $t > 0$,

$$\begin{aligned} \mathbb{P} \{X \geq \mathbb{E}[X] + t\} &\leq \exp \left(-\frac{t^2}{2\nu} \right) \\ \mathbb{P} \{X \leq \mathbb{E}[X] - t\} &\leq \exp \left(-\frac{t^2}{2\nu_\infty} \right). \end{aligned} \quad (101)$$

- **Remark** The condition in above proposition covers

1. *Lipschitz functions* such as *functions with bounded difference*,

2. **self-bounding functions** including **configuration functions**: Let f be such a configuration function. For any $z \in \mathcal{X}^n$, fix a *maximal sub-sequence* $(z_{i,1}, \dots, z_{i,m})$ satisfying property Π (so that $f(z) = m$). Let $c_i(z)$ denote the indicator that z_i belongs to the sub-sequence $(z_{i,1}, \dots, z_{i,m})$. Thus,

$$\sum_{i=1}^n c_i^2(z) = \sum_{i=1}^n c_i(z) = f(z).$$

It follows from the definition of a configuration function that for all $z, y \in \mathcal{X}^n$,

$$f(y) \geq f(z) - \sum_{i=1}^n c_i(z) \mathbb{1}\{z_i \neq y_i\}$$

So $g = -f$ satisfies the condition in above proposition.

3. **weakly self-bounding functions**

4. **convex distance function**

$$d_T(z, A) := \sup_{\alpha \in \mathbb{R}_+^n : \|\alpha\|_2=1} \inf_{y \in A} \sum_{i=1}^n \alpha_i \mathbb{1}\{z_i \neq y_i\}$$

Denote by $c(z) = (c_1(z), \dots, c_n(z)) = \alpha^*$ the vector of nonnegative components in the unit ball for which the supremum is achieved. Thus

$$\begin{aligned} d_T(z, A) - d_T(y, A) &\leq \inf_{z' \in A} \sum_{i=1}^n c_i(z) \mathbb{1}\{z_i \neq z'_i\} - \inf_{y' \in A} \sum_{i=1}^n c_i(z) \mathbb{1}\{y_i \neq y'_i\} \\ &\leq \sum_{i=1}^n c_i(z) \mathbb{1}\{z_i \neq y_i\} \end{aligned}$$

9.6 Talagrand's Gaussian Transportation Inequality

- **Theorem 9.11 (Talagrand's Gaussian Transportation Inequality)** [Boucheron et al., 2013]

Let \mathbb{P} be the standard Gaussian probability measure on \mathbb{R}^n , and let \mathbb{Q} be a probability measure absolutely continuous with respect to \mathbb{P} . Define two random vectors $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ in \mathcal{X}^n with distribution \mathbb{P} and \mathbb{Q} respectively. Then

$$\begin{aligned} \mathcal{W}_{2,d}(\mathbb{Q}, \mathbb{P}) &:= \sqrt{\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \mathbb{E}_\gamma [(X_i - Y_i)^2] \leq \sqrt{2\text{KL}(\mathbb{Q} \parallel \mathbb{P})}} \\ &\Leftrightarrow \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \mathbb{E}_\gamma [(X_i - Y_i)^2] \leq 2\text{KL}(\mathbb{Q} \parallel \mathbb{P}) \end{aligned} \tag{102}$$

10 Proofs of Bounded Difference Inequality

- **Theorem 10.1 (McDiarmid's Inequality / Bounded Differences Inequality)** [Boucheron et al., 2013, Wainwright, 2019]

Suppose that f satisfies **the bounded difference property** (50) with parameters (L_1, \dots, L_n) i.e. for each index $k = 1, 2, \dots, n$,

$$|f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq L_k, \quad \text{for all } x, x' \in \mathcal{X}^n.$$

Assume that the random vector $X = (X_1, X_2, \dots, X_n)$ has **independent** components. Then

$$\mathbb{P}\{|f(X) - \mathbb{E}[f(X)]| \geq t\} \leq 2 \exp\left(-\frac{2t^2}{\sum_{k=1}^n L_k^2}\right).$$

10.1 Martingale Method

- **Proof:** Consider the associated *martingale difference sequence*

$$D_k := \mathbb{E}[f(X)|X_1, \dots, X_k] - \mathbb{E}[f(X)|X_1, \dots, X_{k-1}].$$

We claim that D_k lies in an interval of length at most L_k almost surely. In order to prove this claim, define the random variables

$$\begin{aligned} A_k &:= \inf_x \{\mathbb{E}[f(X)|X_1, \dots, X_{k-1}, x]\} - \mathbb{E}[f(X)|X_1, \dots, X_{k-1}] \\ B_k &:= \sup_x \{\mathbb{E}[f(X)|X_1, \dots, X_{k-1}, x]\} - \mathbb{E}[f(X)|X_1, \dots, X_{k-1}]. \end{aligned}$$

On one hand, we have

$$D_k - A_k = \mathbb{E}[f(X)|X_1, \dots, X_k] - \inf_x \{\mathbb{E}[f(X)|X_1, \dots, X_{k-1}, x]\},$$

so that $D_k \geq A_k$ almost surely. A similar argument shows that $D_k \leq B_k$ almost surely. We now need to show that $B_k - A_k \leq L_k$ almost surely. Observe that by the independence of $\{X_k\}_{k=1}^n$, we have

$$\mathbb{E}[f(X) | x_1, \dots, x_k] = \mathbb{E}_{(k+1)}[f(x_1, \dots, x_k, X_{k+1}, \dots, X_n)], \text{ for any } (x_1, \dots, x_k),$$

where $\mathbb{E}_{(k+1)}[\cdot]$ denote the expectation over (X_{k+1}, \dots, X_n) . Consequently, we have

$$\begin{aligned} B_k - A_k &= \sup_x \mathbb{E}_{(k+1)}[f(X_1, \dots, X_{k-1}, x, X_{k+1}, \dots, X_n)] \\ &\quad - \inf_x \mathbb{E}_{(k+1)}[f(X_1, \dots, X_{k-1}, x, X_{k+1}, \dots, X_n)] \\ &\leq \sup_{x, y} \{\mathbb{E}_{(k+1)}[f(X_{1:k-1}, x, X_{k+1:n})] - \mathbb{E}_{(k+1)}[f(X_{1:k-1}, y, X_{k+1:n})]\} \\ &\leq L_k, \end{aligned}$$

using the bounded differences assumption. Thus, the variable D_k lies within an interval of length L_k at most surely, so that the claim follows as a corollary of the Azuma-Hoeffding inequality. ■

10.2 Entropy Method

- **Proof:** Recall that for a random variable Y taking its values in $[a, b]$, then we know from *Hoeffding's Lemma* that the logarithmic moment generating functions $\psi(\lambda)$ satisfies

$$\psi(\lambda)'' = \text{Var}(Y) \leq \frac{(b-a)^2}{4}$$

for every $\lambda \in \mathbb{R}$. Hence, Hoeffding's inequality is obtained since

$$\frac{\text{Ent}(e^{\lambda Y})}{\mathbb{E}[e^{\lambda Y}]} = \lambda \psi'(\lambda) - \psi(\lambda) = \int_0^\lambda s \psi''(s) ds \leq \frac{(b-a)^2}{4} \int_0^\lambda s ds = \frac{(b-a)^2 \lambda^2}{8},$$

Note that by the bounded differences assumption, given $X_{(-i)}$, $f(X)$ is a random variable whose range is in an interval of length at most L_i , so

$$\frac{\text{Ent}_{(-i)}(e^{\lambda f(X)})}{\mathbb{E}_{(-i)}[e^{\lambda f(X)}]} \leq \frac{L_i^2 \lambda^2}{8}$$

From the tensorization property of entropy, we can bound the entropy of total function

$$\begin{aligned} \text{Ent}(e^{\lambda f(X)}) &\leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}_{(-i)}(e^{\lambda f(X)}) \right] \leq \sum_{i=1}^n \frac{L_i^2 \lambda^2}{8} \mathbb{E} \left[\mathbb{E}_{(-i)}[e^{\lambda f(X)}] \right] \\ \frac{\text{Ent}(e^{\lambda f(X)})}{\mathbb{E}[e^{\lambda f(X)}]} &\leq \frac{\sum_{i=1}^n L_i^2 \lambda^2}{8} \equiv \frac{\nu \lambda^2}{2}, \end{aligned}$$

where $\nu := \frac{1}{4} \sum_{i=1}^n L_i^2$. Using *Herbst's argument*, it leads to the bound of logarithmic moment generating function:

$$\psi_{f(X)}(\lambda) \leq \frac{\nu \lambda^2}{2}.$$

Finally, we apply *the Chernoff's inequality*

$$\mathbb{P}\{f(X) - \mathbb{E}[f(X)] \geq t\} \leq \inf_{\lambda > 0} \exp(\psi_{f(X)}(\lambda) - \lambda t) \leq \exp\left(-\frac{t^2}{2\nu}\right). \quad \blacksquare$$

10.3 Isoperimetric Inequality on Binary Hypercube

- **Definition (*Vertex Boundary of Graph*)** [Boucheron et al., 2013]
Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let $\mathcal{A} \subset \mathcal{V}$ be a set of its vertices. **The vertex boundary** of \mathcal{A} is defined as *the set of those vertices, not in \mathcal{A} , which are **connected** to some vertex in \mathcal{V} by an edge*. We denote the vertex boundary of \mathcal{A} by $\partial V(\mathcal{A})$.
- **Remark (*Binary Hypercube as Nearest Neighbor Graph with Hamming Distance*)**
Consider the graph as binary hypercube $\{-1, +1\}^n$ in which two vertices are connected by an edge if and only if their **Hamming distance equals 1**. Define the *norm* as the *Hamming distance* to $-1^n = (-1, \dots, -1)$

$$\|x\|_H := \sum_{i=1}^n \mathbb{1}\{x_i = 1\} = d_H(x, -1^n)$$

Theorem 10.2 (*Harp's Vertex Isoperimetric Theorem*) [Boucheron et al., 2013]
For $N = \sum_{i=0}^k \binom{n}{i}$, for $k = 0, \dots, n$, let S_N is a **Hamming ball** centered at the vector $-1^n = (-1, \dots, -1)$, i.e.

$$S_N = \{x \in \{-1, +1\}^n : d_H(x, -1^n) \leq k\} = B_H(-1^n, k).$$

For any subset $A \subset \{-1, +1\}^n$, where $|A| = |S_N|$,

$$|\partial V(A)| \geq |\partial V(S_N)|$$

- **Definition** (*t-Blowup of Set A in Binary Hypercube*)

For any $A \subset \{-1, +1\}^n$ and $x \in \{-1, +1\}^n$, let $d_H(x, A) = \min_{y \in A} d_H(x, y)$ be the *Hamming distance* of x to the set A . Also, denote by

$$A_t := \{x \in \{-1, +1\}^n : d_H(x, A) < t\}$$

the *t-blowup* of the set A , that is, the set of points whose Hamming distance from A is at most t .

- **Corollary 10.3** (*Isoperimetric Inequality in Binary Hypercube*) [Boucheron et al., 2013]

Let $A \subset \{-1, +1\}^n$ such that $|A| \geq \sum_{i=0}^k \binom{n}{i}$. Then for any $t = 1, 2, \dots, n - k + 1$,

$$|A_t| \geq \sum_{i=0}^{k+1-t} \binom{n}{i}. \quad (103)$$

In particular, if $|A|/2^n \geq 1/2$ then we may take $k = \lfloor n/2 \rfloor$ in the corollary above and

$$\frac{|A_t|}{2^n} \geq \mathbb{P}\{X < \mathbb{E}[X] + t\} \geq 1 - \exp\left(-\frac{2t^2}{n}\right) \quad (104)$$

where $X \sim \text{Ber}(1/2)$ is a symmetric Bernoulli random variable taking values in $\{-1, +1\}$ with $\mathbb{P}\{X = 1\} = \mathbb{P}\{X = -1\} = 1/2$.

- **Proof:** (*Proof of Bounded Difference Inequality on Binary Hypercube*)

Note that any function with **bounded difference property** is **Lipschitz function** with respect to **Hamming distance**.

$$\begin{aligned} & \sup_{x \in \mathcal{X}^n, y_i \in \mathcal{X}} |f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| \\ & \leq c_i = c_i d_H((x_1, \dots, x_n), (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)), \quad 1 \leq i \leq n \\ \Rightarrow |f(x) - f(y)| &= \left| \sum_{i=1}^n (f(x_{(i-1)}) - f(x_{(i)})) \right| \\ & \leq \sum_{i=1}^n |f(x_{(i-1)}) - f(x_{(i)})| \\ & \leq \sum_{i=1}^n L_i \mathbb{1}\{x_{(i-1)}[i] \neq x_{(i)}[i]\} \\ & \leq \left(\sum_i^n L_i^2 \right)^{1/2} d_H(x, y) \end{aligned}$$

where $x_{(i)}$ is replicate of $x_{(i-1)}$ except for i -th component, which is replaced by y_i . Note that $x_{(0)} = x$ and $x_{(n)} = y$.

The Harp's isoperimetric theorem suggests that the concentration function

$$\alpha_{\mathbb{P},(\{-1,+1\}^n, d_{H,L})}(t) := \sup_{A: \mathbb{P}\{A\} \geq 1/2} \mathbb{P}\{A_t\} \leq \exp\left(-\frac{2t^2}{\sum_i^n L_i^2}\right)$$

where \mathbb{P} is uniform distribution on $\{-1, +1\}^n$. Thus by *Levy's inequality*, we prove that for $Z \in \{-1, 1\}^n$ and Lipschitz function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ with respect to weighted Hamming distance $d_{H,L}$,

$$\mathbb{P}\{|f(Z) - \text{Med}(f(Z))| \geq t\} \leq 2 \exp\left(-\frac{2t^2}{\sum_i^n L_i^2}\right). \quad \blacksquare$$

10.4 Transportation Method

- **Proof:** Any function with *bounded difference property* is *Lipschitz function* with respect to *Hamming distance*. This implies that for all $x, y \in \mathcal{X}^n$,

$$f(y) - f(x) \leq \sum_{i=1}^n L_i \mathbb{1}\{x_i \neq y_i\} \equiv d_{H,L}(x, y).$$

Note that for coupling $\gamma \in \Pi(\mathbb{Q}, \mathbb{P})$ where $Y \sim \mathbb{Q}$ and $X \sim \mathbb{P}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[f(Y)] - \mathbb{E}_{\mathbb{P}}[f(X)] &= \mathbb{E}_{\gamma}[f(Y) - f(X)] \\ &\leq \sum_{i=1}^n L_i \mathbb{E}_{\gamma}[\mathbb{1}\{X_i \neq Y_i\}] \\ &\leq \left(\sum_{i=1}^n L_i^2\right)^{1/2} \left(\sum_{i=1}^n (\mathbb{E}_{\gamma}[\mathbb{1}\{X_i \neq Y_i\}])^2\right)^{1/2} \end{aligned}$$

We want to prove the concentration using transportation cost inequality. That is, to bound the term

$$\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n (\mathbb{E}_{\gamma}[\mathbb{1}\{X_i \neq Y_i\}])^2 = \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \gamma^2\{X_i \neq Y_i\}.$$

We have shown that

$$\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \gamma\{X \neq Y\} = \mathcal{W}_{1, d_H}(\mathbb{Q}, \mathbb{P}) = \sup_{A \in \mathcal{X}} |\mathbb{Q}(A) - \mathbb{P}(A)| \equiv \|\mathbb{Q} - \mathbb{P}\|_{TV}.$$

For each independent variable X_i, Y_i , and their marginal distribution $\mathbb{P}_i, \mathbb{Q}_i$ where $\mathbb{Q}_i \ll \mathbb{P}_i$, by Pinsker's inequality,

$$\begin{aligned} \left[\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \gamma\{X_i \neq Y_i\} \right]^2 &= \|\mathbb{Q} - \mathbb{P}\|_{TV}^2 \leq \frac{1}{2} \text{KL}(\mathbb{Q}_i \| \mathbb{P}_i) \\ \min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \gamma^2\{X_i \neq Y_i\} &\leq \left[\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \gamma\{X_i \neq Y_i\} \right]^2 \leq \frac{1}{2} \text{KL}(\mathbb{Q}_i \| \mathbb{P}_i) \end{aligned}$$

Thus by induction lemma,

$$\min_{\gamma \in \Pi(\mathbb{Q}, \mathbb{P})} \sum_{i=1}^n \gamma^2 \{X_i \neq Y_i\} \leq \frac{1}{2} \text{KL}(\mathbb{Q} \parallel \mathbb{P})$$

which is the *Marton's transportation inequality*. Finally, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[f(Y)] - \mathbb{E}_{\mathbb{P}}[f(X)] &\leq \left(\sum_{i=1}^n L_i^2 \right)^{1/2} \left(\sum_{i=1}^n (\mathbb{E}_{\gamma} [\mathbf{1}_{\{X_i \neq Y_i\}}])^2 \right)^{1/2} \\ &\leq \sqrt{\frac{(\sum_{i=1}^n L_i^2)}{2} \text{KL}(\mathbb{Q} \parallel \mathbb{P})}. \end{aligned}$$

Then we can apply the transportation lemma with $\nu := \frac{1}{4} \sum_{i=1}^n L_i^2$, which proves the bounded difference inequality. \blacksquare

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