# Lecture 4: The Entropy Methods

Tianpei Xie

Jan. 19th., 2023

## Contents

# 1 Logarithmic Sobolev Inequality

## 1.1 Bernoulli Logarithmic Sobolev Inequality

- **Remark** (*Setting*)
  Consider a **uniformly distributed binary vector** $Z = (Z_1, \ldots, Z_n)$ on the hypercube $\{-1, +1\}^n$. In other words, the components of $X$ are *independent, identically distributed* **random sign (Rademacher) variables** with $\mathbb{P}\{Z_i = -1\} = \mathbb{P}\{Z_i = +1\} = 1/2$ (i.e. *symmetric Bernoulli random variables*).

  Let $f : \{-1, +1\}^n \to \mathbb{R}$ be a real-valued function on **binary hypercube**. $X := f(Z)$ is an induced real-valued random variable. Define $\widetilde{Z}^{(i)} = (Z_1, \ldots, Z_{i-1}, Z_i', Z_{i+1}, \ldots, Z_n)$ be the sample $Z$ with $i$-th component replaced by an *independent copy* $Z_i'$. Since $Z, \widetilde{Z}^{(i)} \in \{-1, +1\}^n$, $\widetilde{Z}^{(i)} = (Z_1, \ldots, Z_{i-1}, -Z_i, Z_{i+1}, \ldots, Z_n)$, i.e. *the $i$-th sign is* **flipped**. Also denote the $i$-th *Jackknife sample* as $Z_{(i)} = (Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n)$ by *leaving out* the $i$-th component. $\mathbb{E}_{(-i)}[X] := \mathbb{E}\left[X | Z_{(i)}\right]$.

  Denote the $i$-th component of **discrete gradient** of $f$ as

  $$\nabla_i f(z) := \frac{1}{2}\left(f(z) - f(\widetilde{z}^{(i)})\right)$$

  and $\nabla f(z) = (\nabla_1 f(z), \ldots, \nabla_n f(z))$

- **Remark** (*Jackknife Estimate of Variance*)
  Recall that *the **Jackknife estimate of variance***

  $$\mathcal{E}(f) := \mathbb{E}\left[\sum_{i=1}^{n}\left(f(Z) - \mathbb{E}_{(-i)}\left[f(\widetilde{Z}^{(i)})\right]\right)^2\right]$$
  $$= \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}\left(f(Z) - f(\widetilde{Z}^{(i)})\right)^2\right].$$

  Using the notation of discrete gradient of $f$, we see that

  $$\mathcal{E}(f) := 2\mathbb{E}\left[\|\nabla f(Z)\|_2^2\right]$$

- **Remark** (*Entropy Functional*)
  Recall that the entropy functional for $f$ is defined as

  $$H_\Phi(f(Z)) = \text{Ent}(f) := \mathbb{E}\left[f(Z)\log f(Z)\right] - \mathbb{E}\left[f(Z)\right]\log\left(\mathbb{E}\left[f(Z)\right]\right).$$

- **Proposition 1.1** (*Logarithmic Sobolev Inequality for Rademacher Random Variables*). [Boucheron et al., 2013]
  *If $f : \{-1, +1\}^n \to \mathbb{R}$ be an arbitrary real-valued function defined on the $n$-dimensional **binary hypercube** and assume that $Z$ is **uniformly distributed** over $\{-1, +1\}^n$. Then*

  $$Ent(f^2) \leq \mathcal{E}(f) \tag{1}$$
  $$\Leftrightarrow Ent(f^2(Z)) \leq 2\mathbb{E}\left[\|\nabla f(Z)\|_2^2\right] \tag{2}$$

**Proof:** The key is to apply the tensorization property of $\Phi$-entropy. Let $X = f(Z)$. By tensorization property,

$$\mathrm{Ent}(X^2) \leq \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Ent}_{(-i)}(X^2)\right]$$

where $\mathrm{Ent}_{(-i)}(X^2) := \mathbb{E}_{(-i)}\left[X^2 \log X^2\right] - \mathbb{E}_{(-i)}\left[X^2\right]\log\left(\mathbb{E}_{(-i)}\left[X^2\right]\right)$.

It thus suffice to show that for all $i = 1, \ldots, n$,

$$\mathrm{Ent}_{(-i)}(X^2) \leq \frac{1}{2}\mathbb{E}_{(-i)}\left[\left(f(Z) - f(\widetilde{Z}^{(i)})\right)^2\right].$$

Given any fixed realization of $Z_{(-i)}$, $X = f(Z) = \widetilde{f}(Z_i)$ can only takes two different values with equal probability. Call these two values $a$ and $b$. See that

$$\mathrm{Ent}_{(-i)}(X^2) = \frac{1}{2}a^2 \log a^2 + \frac{1}{2}b^2 \log b^2 - \frac{1}{2}(a^2 + b^2)\log\left(\frac{a^2 + b^2}{2}\right)$$

$$\frac{1}{2}\mathbb{E}_{(-i)}\left[\left(f(Z) - f(\widetilde{Z}^{(i)})\right)^2\right] = \frac{1}{2}(a - b)^2.$$

Thus we need to show

$$\frac{1}{2}a^2 \log a^2 + \frac{1}{2}b^2 \log b^2 - \frac{1}{2}(a^2 + b^2)\log\left(\frac{a^2 + b^2}{2}\right) \leq \frac{1}{2}(a - b)^2.$$

By symmetry, we may assume that $a \geq b$. Since $(|a| - |b|)^2 \leq (a - b)^2$, without loss of generality, we may further assume that $a, b \geq 0$.

Define

$$h(a) := \frac{1}{2}a^2 \log a^2 + \frac{1}{2}b^2 \log b^2 - \frac{1}{2}(a^2 + b^2)\log\left(\frac{a^2 + b^2}{2}\right) - \frac{1}{2}(a - b)^2$$

for $a \in [b, \infty)$. $h(b) = 0$. It suffice to check that $h'(b) = 0$ and that $h$ is concave on $[b, \infty)$. Note that

$$h'(a) = a \log a^2 + 1 - a \log\left(\frac{a^2 + b^2}{2}\right) - 1 - (a - b)$$

$$= a \log \frac{2a^2}{(a^2 + b^2)} - (a - b).$$

So $h'(b) = 0$. Moreover,

$$h''(a) = \log \frac{2a^2}{(a^2 + b^2)} + 1 - \frac{2a^2}{(a^2 + b^2)} \leq 0$$

due to inequality $\log(x) + 1 \leq x$. ∎

- **Remark (*Logarithmic Sobolev Inequality* $\Rightarrow$ *Efron-Stein Inequality*).** [Boucheron et al., 2013]
  Note that for $f$ non-negative,

$$\mathrm{Var}(f(Z)) \leq \mathrm{Ent}(f^2(Z)).$$

3

Thus *logarithmic Sobolev inequality* (1) implies

$$\text{Var}(f(Z)) \leq \mathcal{E}(f)$$

which is *the Efron-Stein inequality*.

- **Corollary 1.2** (***Logarithmic Sobolev Inequality for Asymmetric Bernoulli Random Variables***). *[Boucheron et al., 2013]*
  *If $f : \{-1, +1\}^n \to \mathbb{R}$ be an arbitrary real-valued function and $Z = (Z_1, \ldots, Z_n) \in \{-1, +1\}^n$ with $p = \mathbb{P}\{Z_i = +1\}$. Then*

$$Ent(f^2) \leq c(p)\mathbb{E}\left[\|\nabla f(Z)\|_2^2\right] \tag{3}$$

  *where*

$$c(p) = \frac{1}{1 - 2p} \log \frac{1 - p}{p}$$

  *Note that $\lim_{p \to 1/2} c(p) = 2$.*

## 1.2   Gaussian Logarithmic Sobolev Inequality

- **Proposition 1.3** (***Gaussian Logarithmic Sobolev Inequality***). *[Boucheron et al., 2013]*
  *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **continuous differentiable** function and let $Z = (Z_1, \ldots, Z_n)$ be a vector of $n$ **independent standard Gaussian** random variables. Then*

$$Ent(f^2(Z)) \leq 2\mathbb{E}\left[\|\nabla f(Z)\|_2^2\right]. \tag{4}$$

**Proof:** We first prove for $n = 1$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous differentiable and $Z$ is standard Gaussian distribution. Without loss of generality, assume that $\mathbb{E}\left[f'(Z)\right] < \infty$ since it is trivial when $\mathbb{E}\left[f'(Z)\right] = \infty$. By density argument, it suffice to prove the proposition when $f$ is *twice differentiable with bounded support*.

Now let $\epsilon_1, \ldots, \epsilon_n$ be *independent Rademacher random variables* and introduce

$$S_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \epsilon_j.$$

Note that $\epsilon_i \in \{-1, +1\}$ with equal probability, thus

$$\mathbb{E}_{(-i)}[S_n] = \frac{1}{2}\left[\left(\frac{1}{\sqrt{n}} \sum_{j \neq i} \epsilon_j + \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} \sum_{j \neq i} \epsilon_j - \frac{1}{\sqrt{n}}\right)\right]$$

$$= \frac{1}{2}\left[\left(S_n + \frac{1 - \epsilon_i}{\sqrt{n}}\right) + \left(S_n - \frac{1 + \epsilon_i}{\sqrt{n}}\right)\right].$$

In the proof of Gaussian Poincaré inequality, we show that by *central limit theorem*,

$$\limsup_{n \to \infty} \mathbb{E}\left[\sum_{i=1}^{n} \left|f(S_n) - f\left(S_n - \frac{2\epsilon_i}{\sqrt{n}}\right)\right|^2\right] = 4\mathbb{E}\left[(f'(Z))^2\right].$$

4

On the other hands, for any *continuous uniformly bounded function* $f$, by *central limit theorem*,

$$\lim_{n \to \infty} \text{Ent}\left(f^2(S_n)\right) = \text{Ent}(f^2(Z))$$

The proof is then completed by invoking *the logarithmic Sobolev inequality* for *Rademacher random variables*

$$\text{Ent}\left(f^2(S_n)\right) \leq \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}\left|f\left(S_n\right) - f\left(S_n - \frac{2\epsilon_i}{\sqrt{n}}\right)\right|^2\right]$$

$$\Rightarrow \lim_{n \to \infty} \text{Ent}\left(f^2(S_n)\right) \leq \frac{1}{2}\lim_{n \to \infty}\mathbb{E}\left[\sum_{i=1}^{n}\left|f\left(S_n\right) - f\left(S_n - \frac{2\epsilon_i}{\sqrt{n}}\right)\right|^2\right]$$

$$\Rightarrow \text{Ent}(f^2(Z)) \leq 2\mathbb{E}\left[(f'(Z))^2\right].$$

The extension of the result to dimension $n \geq 1$ follows easily from *the sub-additivity of entropy* which states that

$$\text{Ent}(f^2) \leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}_{(-i)}\left[f^2(Z)\log f^2(Z)\right] - \mathbb{E}_{(-i)}\left[f^2(Z)\right]\log\mathbb{E}_{(-i)}\left[f^2(Z)\right]\right]$$

where $\mathbb{E}_{(-i)}\left[\cdot\right]$ denotes the integration with respect to $i$-th variable $Z_i$ only. Thus by induction, for all $i$

$$\mathbb{E}_{(-i)}\left[f^2(Z)\log f^2(Z)\right] - \mathbb{E}_{(-i)}\left[f^2(Z)\right]\log\mathbb{E}_{(-i)}\left[f^2(Z)\right] \leq 2\mathbb{E}_{(-i)}\left[(\partial_i f(Z))^2\right].$$

Thus

$$\text{Ent}(f^2) \leq 2\mathbb{E}\left[\mathbb{E}_{(-i)}\left[\sum_{i=1}^{n}(\partial_i f(Z))^2\right]\right] = 2\mathbb{E}\left[\|\nabla f(Z)\|_2^2\right]. \quad \blacksquare$$

- **Remark** (*Dimension Free Property*).
  *The Gaussian logarithmic Sobolev inequality* has a constant $C = 2$ that is ***independent of dimension*** $n$:

  $$\mathbb{E}_\mu\left[f^2\right] \leq 2\mathbb{E}_\mu\left[\|\nabla f\|_2^2\right].$$

  This *dimension-free property* is related to *the **concentration** of **Gaussian measure*** $\mu$. As a consequence, this inequality can be extended to functions of *Gaussian measure* on ***infinite dimensional space***, such as Gibbs measure, *Gaussian process* etc.

- **Remark** (*Equivalent Form of Gaussian Logarithmic Sobolev Inequality*)
  Assume $f : \mathbb{R}^n \to (0, \infty)$ and $\int_{\mathbb{R}^n} f d\mu = 1$ under Gaussian measure $\mu$. Substituting $f \to \sqrt{f}$, *the logarithmic Sobolev inequality* becomes

  $$\text{Ent}_\mu(f) = \int f \log f d\mu \leq \frac{1}{2}\int \frac{\|\nabla f\|_2^2}{f}d\mu \tag{5}$$

- **Remark** (*Gaussian Logarithmic Sobolev Inequality ⇒ Gaussian Poincaré Inequality*). [Boucheron et al., 2013]
  Recall that *the Gaussian Poincaré inequality*

  $$\mathrm{Var}(f(Z)) \leq \mathbb{E}\left[\|\nabla f(Z)\|_2^2\right]$$

  Since

  $$(1+t)\log(1+t) = t + \frac{t^2}{2} + o(t^2)$$

  as $t \to 0$, we can get for Gaussian measures,

  $$\mathrm{Ent}_\mu(1+\epsilon h) = \frac{\epsilon^2}{2}\mathrm{Var}_\mu(h) + o(\epsilon^2).$$

  Similarly,

  $$\int \frac{\|\nabla(1+\epsilon h)\|_2^2}{1+\epsilon h} d\mu = \epsilon^2 \int \|\nabla h\|_2^2 \, d\mu + o(\epsilon^2).$$

  Thus from *the Gaussian logarithmic Sobolev inequality*,

  $$\mathrm{Ent}_\mu(1+\epsilon h) \leq \frac{1}{2}\int \frac{\|\nabla(1+\epsilon h)\|_2^2}{1+\epsilon h} d\mu$$

  $$\Leftrightarrow \frac{\epsilon^2}{2}\mathrm{Var}_\mu(h) + o(\epsilon^2) \leq \frac{\epsilon^2}{2}\int \|\nabla h\|_2^2 \, d\mu + o(\epsilon^2)$$

  $$\Leftrightarrow \mathrm{Var}(f(Z)) \leq \mathbb{E}\left[\|\nabla f(Z)\|_2^2\right] \qquad \text{as } \epsilon \to 0.$$

  Thus *the Gaussian logarithmic Sobolev inequality* implies *the Gaussian Poincaré inequality*.

## 1.3 Information Theory Interpretation

- **Remark** (*Information Interpretation of Gaussian Logarithmic Sobolev Inequality*)
  Let $\nu, \mu$ be two *probability measures* on $(\mathcal{X}, \mathscr{F})$ and $\nu \ll \mu$. Define $f := \frac{d\nu}{d\mu}$ be *the Radon-Nikodym derivative* of $\nu$ with respect to $\mu$ (i.e $f$ is the *probability density function* of $\nu$ with respect to $\mu$). Then the entropy becomes **the relative entropy**

  $$\mathrm{Ent}_\mu(f) := \mathbb{E}_\mu\left[f\log f\right] = \mathbb{KL}\left(\nu \parallel \mu\right)$$

  since $\mathbb{E}_\mu[f] = \int_{\mathcal{X}^n} f d\mu = 1$.

  On the other hand, **the (relative) Fisher information** is defined as

  $$I(\nu \parallel \mu) := \mathbb{E}_\nu\left[\|\nabla \log f\|_2^2\right]$$

  $$= \int \left\|\frac{\nabla f}{f}\right\|_2^2 d\nu = \int \frac{\|\nabla f\|_2^2}{f^2} d\nu$$

  $$= \int \frac{\|\nabla f\|_2^2}{f} d\mu$$

  Thus **the information interpretation** of *the Gaussian logarithmic Sobolev inequality* is

  $$\mathbb{KL}\left(\nu \parallel \mu\right) \leq \frac{1}{2}I(\nu \parallel \mu) \tag{6}$$

  Note that *the Fisher information metric is* **the Riemannian metric** *induced by the relative entropy.*

## 1.4 Logarithmic Sobolev Inequality for General Probability Measures

- **Remark** (*The Sobolev Inequality*)
  **The Sobolev inequality** states for smooth function $f : \mathbb{R}^n \to \mathbb{R}$ where $n \geq 3$ and $p = \frac{2n}{n-2} > 2$

$$\|f\|_p^2 \leq C_n \int_{\mathbb{R}^n} |\nabla f|^2 \, dx$$

  where the inequality is sharp when the constant

$$C_n := \frac{1}{\pi n(n-2)} \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{2/n}$$

- **Proposition 1.4** *(Euclidean Logarithmic Sobolev Inequality).*
  *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $m$ be Lebesgue measure on $\mathbb{R}^n$, then*

$$Ent_m(f^2) \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \mathbb{E}_m \left[ \|\nabla f\|_2^2 \right] \right) \tag{7}$$

$$\Leftrightarrow \int f^2 \log \left( \frac{f^2}{\int f^2 dx} \right) dx \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \int |\nabla f|^2 \, dx \right)$$

- **Definition** (*Logarithmic Sobolev Inequality for General Probability Measure*).
  *A probability measure $\mu$ on $\mathbb{R}^n$ is said to satisfy the* **logarithmic Sobolev inequality** *for some constant $C > 0$ if for any smooth function $f$*

$$\text{Ent}_\mu(f^2) \leq C \, \mathbb{E}_\mu \left[ \|\nabla f\|_2^2 \right] \tag{8}$$

  holds for any **continuous differentiable** function $f : \mathbb{R}^n \to \mathbb{R}$. The left-hand side is called **the entropy functional**, which is defined as

$$\text{Ent}(f^2) := \mathbb{E}_\mu \left[ f^2 \log f^2 \right] - \mathbb{E}_\mu \left[ f^2 \right] \log \mathbb{E}_\mu \left[ f^2 \right]$$

$$= \int f^2 \log \left( \frac{f^2}{\int f^2 d\mu} \right) d\mu.$$

  The right-hand side is defined as

$$\mathbb{E}_\mu \left[ \|\nabla f\|_2^2 \right] = \int \|\nabla f\|_2^2 \, d\mu.$$

  Thus we can rewrite *the logarithmic Sobolev inequality* in *functional form*

$$\int f^2 \log \left( \frac{f^2}{\int f^2 d\mu} \right) d\mu \leq C \int \|\nabla f\|_2^2 \, d\mu \tag{9}$$

- **Remark** (*Logarithmic Sobolev Inequality*)
  For non-negative function $f$, we can replace $f \to \sqrt{f}$, so that *the logarithmic Sobolev inequality* becomes

$$\text{Ent}_\mu(f) \leq \frac{1}{2} \int \frac{\|\nabla f\|_2^2}{f} d\mu \tag{10}$$

- **Remark** (*Modified Logarithmic Sobolev Inequality via Convex Cost and Duality*)
  For some **convex non-negative cost** $c : \mathbb{R}^n \to \mathbb{R}_+$, **the convex conjugate** of $c$ (Legendre transform of $c$) is defined as

$$c^*(x) := \sup_y \left\{ \langle x\,,\, y \rangle - c(y) \right\}$$

Then we can obtain **the modified logarithmic Sobolev inequality**

$$\mathrm{Ent}_\mu(f) \leq \int f^2 \, c^* \left( \frac{\nabla f}{f} \right) d\mu \tag{11}$$

## 1.5   Variants of Logarithmic Sobolev Inequalities

- **Proposition 1.5** (*A Modified Logarithmic Sobolev Inequalities for Moment Generating Function*) *[Boucheron et al., 2013]*
  *Consider independent random variables $Z_1, \dots, Z_n$ taking values in $\mathcal{X}$, a real-valued function $f : \mathcal{X}^n \to \mathbb{R}$ and the random variable $X = f(Z_1, \dots, Z_n)$. Also denote $Z_{(-i)} = (Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n)$ and $X_{(-i)} = f_i(Z_{(-i)})$ where $f_i : \mathcal{X}^{n-1} \to \mathbb{R}$ is an arbitrary function. Let $\phi(x) = e^x - x - 1$. Then for all $\lambda \in \mathbb{R}$,*

$$\lambda \mathbb{E}\left[ X e^{\lambda X} \right] - \mathbb{E}\left[ e^{\lambda X} \right] \log \mathbb{E}\left[ e^{\lambda X} \right] \leq \sum_{i=1}^n \mathbb{E}\left[ e^{\lambda X} \phi(-\lambda(X - X_{(-i)})) \right] \tag{12}$$

- **Proposition 1.6** (*Symmetrized Modified Logarithmic Sobolev Inequalities*) *[Boucheron et al., 2013]*
  *Consider independent random variables $Z_1, \dots, Z_n$ taking values in $\mathcal{X}$, a real-valued function $f : \mathcal{X}^n \to \mathbb{R}$ and the random variable $X = f(Z_1, \dots, Z_n)$. Also denote $\widetilde{X}^{(i)} = f(Z_1, \dots, Z_{i-1}, Z_i', Z_{i+1}, \dots, Z_n)$. Let $\phi(x) = e^x - x - 1$. Then for all $\lambda \in \mathbb{R}$,*

$$\lambda \mathbb{E}\left[ X e^{\lambda X} \right] - \mathbb{E}\left[ e^{\lambda X} \right] \log \mathbb{E}\left[ e^{\lambda X} \right] \leq \sum_{i=1}^n \mathbb{E}\left[ e^{\lambda X} \phi(-\lambda(X - \widetilde{X}^{(i)})) \right] \tag{13}$$

*Moreover, denoting $\tau(x) = x(e^x - 1)$, for all $\lambda \in \mathbb{R}$,*

$$\lambda \mathbb{E}\left[ X e^{\lambda X} \right] - \mathbb{E}\left[ e^{\lambda X} \right] \log \mathbb{E}\left[ e^{\lambda X} \right] \leq \sum_{i=1}^n \mathbb{E}\left[ e^{\lambda X} \tau(-\lambda(X - \widetilde{X}^{(i)})_+) \right],$$

$$\lambda \mathbb{E}\left[ X e^{\lambda X} \right] - \mathbb{E}\left[ e^{\lambda X} \right] \log \mathbb{E}\left[ e^{\lambda X} \right] \leq \sum_{i=1}^n \mathbb{E}\left[ e^{\lambda X} \tau(\lambda(\widetilde{X}^{(i)} - X)_-) \right].$$

## 1.6   Poisson Logarithmic Sobolev Inequality

- **Proposition 1.7** (*Modified Logarithmic Sobolev Inequality for Bernoulli Random Variable*). *[Boucheron et al., 2013]*
  *Let $f : \{0,1\} \to (0,\infty)$ be a **non-negative** real-valued function defined on the binary set $\{0,1\}$. Define **the discrete derivative** of $f$ at $x \in \{0,1\}$ by*

$$\nabla f := f(1-x) - f(x).$$

*Let $X$ be a Bernoulli random variable with parameter $p \in (0,1)$ (i.e. $\mathbb{P}\{X = 1\} = p$). Then*

$$Ent(f(X)) \leq (p(1-p))\mathbb{E}\left[\nabla f(X) \nabla \log f(X)\right]. \tag{14}$$

*and*

$$Ent(f(X)) \leq (p(1-p))\mathbb{E}\left[\frac{|\nabla f(X)|^2}{f(X)}\right]. \tag{15}$$

- **Proposition 1.8 (*Poisson Logarithmic Sobolev Inequality*).** *[Boucheron et al., 2013] Let $f : \mathbb{N} \to (0, \infty)$ be a **non-negative** real-valued function defined on the set of non-negative integers $\mathbb{N}$. Define **the discrete derivative** of $f$ at $x \in \mathbb{N}$ by*

$$\nabla f := f(x+1) - f(x).$$

*Let $X$ be a Poisson random variable. Then*

$$Ent(f(X)) \leq (\mathbb{E}\left[X\right])\mathbb{E}\left[\nabla f(X) \nabla \log f(X)\right]. \tag{16}$$

*and*

$$Ent(f(X)) \leq (\mathbb{E}\left[X\right])\mathbb{E}\left[\frac{|\nabla f(X)|^2}{f(X)}\right]. \tag{17}$$

## 1.7 Applications

# 2 The Entropy Methods

## 2.1 Tensorization Property of $\Phi$-Entropy

- **Remark** Recall that the $\Phi$-entropy for $\Phi(x) = x \log(x)$ as

$$H_\Phi(X) = \text{Ent}(X) := \mathbb{E}\left[X \log X\right] - \mathbb{E}\left[X\right] \log\left(\mathbb{E}\left[X\right]\right).$$

The variational formulation of $H_\Phi(X)$ is

$$\text{Ent}(X) = \sup_{T} \left\{X\left(\log(T) - \log(\mathbb{E}\left[T\right])\right)\right\}$$

-

## 2.2 Herbst's Argument

## 2.3 Bounded Difference Inequality

## 2.4 Modified Logarithmic Sobolev Inequalities

## 2.5 Concentration of Convex Lipschitz Functions

## 2.6 Exponential Tail Bounds for Self-Bounding Functions

# References

Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence.* Oxford university press, 2013.