Lecture 6: Locally Convex Topological Vector Space

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1 Topological Vector Space

1.1 Vector Space

- **Definition** A ***vector space*** over a ***field*** F is a set V together with two operations, the ***(vector) addition*** $+ : V \times V \to V$ and ***scale multiplication*** $\cdot : \mathbb{R} \times V \to V$, that satisfy the eight axioms listed below: for all $x, y, z \in V$, $\alpha, \beta \in F$,

  1. The ***associativity*** of ***vector addition***: $x + (y + z) = (x + y) + z$;
  2. The ***commutativity*** of ***vector addition***: $x + y = y + x$;
  3. The ***identity*** of ***vector addition***: $\exists\, 0 \in V$ such that $0 + x = x$;
  4. The ***inverse*** of ***vector addition***: $\forall\, x \in V$, $\exists\, -x \in V$, so that $x + (-x) = 0$;
  5. ***Compatibility*** of ***scalar multiplication*** with ***field multiplication***: $\alpha(\beta \cdot x) = (\alpha\beta) \cdot x$;
  6. The ***identity*** of ***scalar multiplication***: $\exists\, 1 \in F$, such that $1 \cdot x = x$;
  7. The ***distributivity*** of ***scalar multiplication*** with respect to ***vector addition***: $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;
  8. The ***distributivity*** of ***scalar multiplication*** with respect to ***field addition***: $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

  Elements of V are commonly called ***vectors***. Elements of F are commonly called ***scalars***.

- **Definition** (***Topological Vector Space***)
  A vector space X endowed with a topology $\mathscr{T}$ is called a ***topological vector space***, denoted as $(X, \mathscr{T})$, if the addition $+ : X \times X \to X$ and scale multiplication $\cdot : \mathbb{R} \times X \to X$ are ***continuous***.

- **Theorem 1.1** *[Treves, 2016]*
  Every ***locally compact Hausdorff*** *topological vector space is* ***finite-dimensional***.

2 Locally Convex Topological Vector Space

- **Definition** (***Locally Convex Space***)
  A topological vector space X is a ***locally convex topological vector space*** (or just ***locally convex space***), if V is open and $x \in V$, then one can find a *convex open* set $U \subset X$ such that $x \in U \subset V$. That is, there exists *a* ***base of convex sets*** $\mathscr{B}$ *that* ***generates the topology*** $\mathscr{T}$.

- **Remark** The most common way of defining locally convex topologies on vector spaces is in terms of *semi-norms*.

- **Definition** (***Semi-Norm***)
  A ***semi-norm*** on a vector space X is a mapping $q : X \to \mathbb{R}_+$ satisfying the following conditions:

1. *homogeneity*: $q(\gamma \boldsymbol{x}) = |\gamma|\, q(\boldsymbol{x})$;

2. the *triangle inequality*: $q(\boldsymbol{x}+\boldsymbol{y}) \le q(\boldsymbol{x}) + q(\boldsymbol{y})$.

If furthermore $q(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = 0$, then q is a ***norm***.

- **Remark** A ***metric*** $d : X \times X \to \mathbb{R}_+$ that ***induced*** from a norm is given by $d_\theta(\boldsymbol{x},\boldsymbol{y}) = q_\theta(\boldsymbol{y}-\boldsymbol{x})$, $\forall \boldsymbol{x}, \boldsymbol{y} \in X$.

- **Proposition 2.1** *A normed space $(X, \mathscr{T})$ induced by $\{q_\theta, \theta \in \Theta\}$ is* Hausdorff *if and only if for any $\boldsymbol{x} \neq 0, \boldsymbol{x} \in X$, $\exists \theta \in \Theta$, such that $q_\theta(\boldsymbol{x}) > 0$.*

- **Definition** (***Locally Convex Space*** *generated by* ***Semi-Norms***)
The ***smallest topology*** $\mathscr{T}$ induced by the set of ***semi-norms*** $\{q_\theta, \theta \in \Theta\}$ is generated by ***the convex basis*** $U_{\boldsymbol{x},r,\theta} = \{\boldsymbol{y} \in X \,|\, q_\theta(\boldsymbol{y}-\boldsymbol{x}) \le r\} \in \mathscr{B}, \boldsymbol{x} \in X, r > 0$. The topological vector space $(X, \mathscr{T})$ is thus ***locally convex space***.

  If $\{q_\theta, \theta \in \Theta\}$ is a set of ***norms***, then $(X, \mathscr{T})$ is a ***normed space***.

- **Remark** The most commonly seen *topological vector space* are ***the normed linear space***. It is a vector space X equipped with norm $\|\cdot\|$ and the topology generated by the norm induced metric d. It is denoted as $(X, \|\cdot\|)$.

  The ***locally convex space*** is seen as a generalization of *normed vector space*.

- **Proposition 2.2** *(**Continuous Linear Operator**) [Folland, 2013]*
*Suppose X and Y are vector spaces with topologies defined, respectively, by the families $\{p_\alpha\}_{\alpha\in A}$ and $\{q_\beta\}_{\beta\in B}$ of **semi-norms**, and $T : X \to Y$ is a linear map. Then T is **continuous if and only if** for each $\beta \in B$, there exists $\alpha_1, \ldots, \alpha_k \in A$ and $C > 0$ such that $q_\beta(Tx) \le C \sum_{i=1}^{k} p_{\alpha_i}(x)$.*

- **Remark** If the semi-norms are *norms*, then the condition above is *the bounded condition* for continuous linear operator.

- **Proposition 2.3** *[Folland, 2013]*
*Let X be a vector space equipped with the topology defined by a family $\{p_\alpha\}_{\alpha\in A}$ of **seminorms**.*

  1. *X is **Hausdorff** if and only if for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.*

  2. *If X is **Hausdorff** and A is **countable**, then X is **metrizable** with a **translation invariant metric** (i.e., $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$).*

- **Definition** (***Fréchet Space***)
A ***complete Hausdorff topological vector space*** X whose topology is defined by a ***countable*** family of *seminorms* $\{q_\theta, \theta \in \Theta\}$ is called a ***Fréchet space***.

- **Example** 1. *A **Fréchet space*** is a ***complete locally convex space***.

  2. *A **Banach space** is a **Fréchet space**.*

- **Example** (***Locally Integrable Functions*** $L^1_{loc}(X, \mu)$)
*The space of all **locally integrable functions** on $\mathbb{R}$, $L^1_{loc}(\mathbb{R})$, is a **Fréchet space*** with the topology defined by the ***semi-norms***

$$p_k(f) = \int_{|x|\le k} |f(x)|\, dx.$$

Completeness follows easily from the completeness of L^1. An obvious *generalization* of this construction yields a ***locally convex topological vector space*** $L^1_{loc}(X, \mu)$ where X is any *locally convex Hausdorff (LCH) space* and μ is a *Borel measure* on X that is *finite* on *compact sets.*

References

Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

François Treves. *Topological Vector Spaces, Distributions and Kernels: Pure and Applied Mathematics, Vol. 25*, volume 25. Elsevier, 2016.