Lecture 2: random functions and functional analysis

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Jul. 9th., 2015

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1 Definitions

1.1 Random functions

- A family of random variables $\xi \equiv \{\xi_t, t \in T\}$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *random function*. For $T \subset \mathbb{R}$, it is called a *random process*, whereas for $T \subset \mathbb{R}^n$, it is called a *random field*.
- Note that each random variable is a function $\xi_t : T \times \Omega \to \mathbb{R}$, and it is a measureable mapping from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}^1)$.

  For fixed $\omega \in \Omega$, $\xi_{\cdot}(\omega)$ a function on T. It is called a *sample function* of the random function [Lifshits, 2013], or a *sample path* of the random process for $T \subset \mathbb{R}$.
- It is assumed that all $\xi_t, \forall t$ are well-defined on a *common* subset $\Omega_0 \subset \Omega$. Then $\xi : T \times \Omega_0 \to \mathbb{R}$ is a *modification* of the random function above.

  Different modifications defines a different property about the sample paths (measureabilty, boundedness, continuity etc.) That is, a random process ξ_t process the corresponding property, if an appropriate modification of this random function is considered.

- The joint distributions of random vectors $(\xi_{t_1}, \ldots, \xi_{t_n})$ for all possible $(t_1, \ldots, t_n)$ are called the *finite-dimensional distributions* of the random functions ξ.
- $\{\xi_t, t \in T\}$ is called *(Strictly Sense) Stationary (SSS)*, if its finite-dimensional distributions remain unaltered upon a parameter shift; i.e., $(\xi_{t_1}, \ldots, \xi_{t_n})$ and $(\xi_{t_1+\tau}, \ldots, \xi_{t_n+\tau})$ are identical distributed, for all $(t_1, \ldots, t_n)$, $\tau \in \mathbb{R}^1$.

  $\{\xi_t, t \in T\}$ is called *stationary increments* if $(\xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}})$ and $(\xi_{t_2+\tau} - \xi_{t_1+\tau}, \ldots, \xi_{t_n+\tau} - \xi_{t_{n-1}+\tau})$ are identical distributed, for all $(t_1, \ldots, t_n)$, $\tau \in \mathbb{R}^1$.

  $\{\xi_t, t \in T\}$ is called *uncorrelated increments* if $\xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are pairwise uncorrelated, for all $(t_1, \ldots, t_n)$, $\tau \in \mathbb{R}^1$.

  If these variables are jointly independent, it is called *independent increments*.

- Given the metric topology on $T = (T, \rho)$ and a random function ξ on it, a modification of ξ is called $\rho-$*separable*, if there exists a countable subset $T_c \subset T$ such that, for any open set $V \subset T$, the equalities

$$\sup_{t \in V} \xi_t = \sup_{t \in V \cap T_c} \xi_t; \qquad \inf_{t \in V} \xi_t = \inf_{t \in V \cap T_c} \xi_t;$$

  holds with probability one. The subset T_c is called the *separant* of the modification ξ.

  Note we always deals with the countable index set T or within the separant T_c of the uncountable set.

- $\{\xi_t, t \in T\}$ is called *Wide-Sense Stationary (WSS)* if the covariance function is the function of increment of index

$$
\begin{aligned}
K(s,t) &\equiv \operatorname{cov}(\xi_t\,,\,\xi_s) \\
&= K(t-s), \quad s,t \in T.
\end{aligned}
$$

SSS process is WSS process. The covariance function of WSS process has spectral representation.

1.2 Topology and functional analysis

- A vector space X endowed with a topology is called a *topological vector space*, denoted as $(X, \mathscr{T})$, if the addition $+ : X \times X \to X$ and scale multiplication $\cdot : \mathbb{R} \times X \to X$ are continuous.
- A topological vector space is *locally convex space*, if V is open and $x \in V$, then one can find a *convex open* set $U \subset X$ such that $x \in U \subset V$. That is, there exists a base of convex sets $\mathscr{B}$ that generates the topology.
- A *semi-norm* on a vector space X is a mapping $q : X \to \mathbb{R}_+$ satisfying the homogeneity condition, i.e. $q(\gamma x) = |\gamma|\, q(x)$ and the triangle inequality, $q(x+y) \leq q(x) + q(y)$. If furthermore $q(x) = 0 \Rightarrow x = 0$, then q is a *norm*.
- The smallest topology $\mathscr{T}$ induced by the set of semi-norms $\{q_\theta, \theta \in \Theta\}$ is generated by the convex basis $U_{x,r,\theta} = \{y \in X \mid q_\theta(y-x) \leq r\} \in \mathscr{B}, x \in X, r > 0$. The topological vector space $(X, \mathscr{T})$ is thus locally convex space.

  If $\{q_\theta, \theta \in \Theta\}$ is a set of norms, then $(X, \mathscr{T})$ is a *normed space*.
- Given the inner product (*duality*) $\langle \cdot\,,\,\cdot \rangle_d$ defined a product space $X \times X'$, a set of semi-norm is defined as $\{q_{\boldsymbol{v}}(\cdot) \equiv \langle \cdot\,,\,\boldsymbol{v} \rangle_d \mid \boldsymbol{v} \in X'\}$.

- In a topological vector space X, the *dual* space X^* is the set of all *linear continuous* real-valued *functionals* on X.

  The dual space X^* is a vector space.
- For a Hausdorff locally convex space X, for any $\boldsymbol{x} \in X, \boldsymbol{x} \neq 0$, there exists a linear functional $f \in X^*$ such that $f(x) = 1$.
- The dual space can be made a Hausdorff locally convex space as well, by defining the *weak topology* in X^*. The weak topology in X^* is induced by the norm $q_{\boldsymbol{x}}(f) = |f(\boldsymbol{x})|$, for all $f \in X^*$, $\boldsymbol{x} \in X$.

  The weak topology can also be introduced into X by $q_f(\boldsymbol{x}) = |f(\boldsymbol{x})|$, for all $\boldsymbol{x} \in X$, $f \in X^*$.
- Another topology in X^* is given by norm $q_\Delta(f) = \sup_{\boldsymbol{x} \in \Delta} |f(\boldsymbol{x})|$, for any Δ strongly convex compact subset of X. Denote the topology as $\mathscr{T}_{X,X^*}$, which is no weaker than the topology above.

  If a linear functional of functional $L : X^* \to \mathbb{R}$ is continuous in $\mathscr{T}_{X,X^*}$, then there exists a vector $\boldsymbol{v} \in X$, such that $L(f) = f(\boldsymbol{v})$ for all $f \in X^*$. In other words, $(X^*, \mathscr{T}_{X,X^*})^* = X$, the dual space of dual with $\mathscr{T}_{X,X^*}$ is the primal space.

- A *duality* is naturally induced btw X and X' in that a topology induced by $\{q_{\boldsymbol{v}}(\cdot) \equiv \langle \cdot \, , \, \boldsymbol{v} \rangle \mid \boldsymbol{v} \in X'\}$ in X and similarly in X'. The dual space $(X^*, \mathscr{T}_{X,X^*}) \simeq (X', \mathscr{T}_q)$.
- In dual form, a linear functional can be uniquely represented as $f(\cdot) = \langle \cdot \, , \, \boldsymbol{v} \rangle_d \, , \boldsymbol{v} \in X'$.
- The weak topology $\mathscr{T}_{X,X^*}$ coincides with the topology induced by duality.
- As an example, for $X = \mathbb{R}^\infty$, $X^* \equiv c_0 \subset \mathbb{R}^\infty$ be the subspace of finite sequences. The weak topology is induced by the semi-norm $\{q_j = |x_j| \, , j \in \mathbb{N}\}$ and it defines the point-wise convergence.
- For X Hausdorff, locally convex, $X' = X^*$ and $\langle X \, , \, X^* \rangle_d$ is a dual pair, so $\langle f \, , \, \boldsymbol{x} \rangle_d \equiv f(\boldsymbol{x}), \boldsymbol{x} \in X, f \in X^*$.
- If X is Hilbert space, then $X' = X$ and $\langle \, , \, \rangle_d$ is given by the $\langle \, , \, \rangle_X$ defined in X.

1.3 Gaussian process as measure on space of functions

- A random function ξ is *Gaussian* if *all* its finite-dimensional distributions are Gaussian. For a Gaussian random function, a covariance function

$$\mathrm{cov}\,(\xi_s \, , \, \xi_t) = K(s,t), \quad \forall s \, t \in T$$

is well defined. K is a covariance function of a Gaussian random function if and only if K is positive definite.

- For Gaussian process, WSS $\Leftrightarrow$ SSS.
- A random function is Gaussian if its distribution $\mathcal{P}$ that defined on $(X, \mathcal{B})$ is Gaussian. Here X is the linear space of functions on T, which is infinite-dimensional.

2 Theorems

- **Theorem 2.1** *(The representation of stationary kernel:* Bochner*'s theorem)*
  A complex-valued function K on $\mathbb{R}^D$ is the covariance function of a weakly stationary mean square continuous complex-valued random process on $\mathbb{R}^D$ if and only if it can be represented as

  $$K(\boldsymbol{x}, \boldsymbol{x}') = K(\boldsymbol{x} - \boldsymbol{x}') = \int_{\mathbb{R}^D} \exp\left(2\pi j\, \boldsymbol{s}^T(\boldsymbol{x} - \boldsymbol{x}')\right) d\mu(\boldsymbol{s}), \tag{1}$$

  where μ is a positive finite measure, which is called the spectral measure *of this process [Lifshits, 2013].*

  The covariance function of a stationary process can be represented as the Fourier transform of a positive finite measure.

  For the spectral density exists as $S(\boldsymbol{s})$,

  $$\begin{aligned} K(\boldsymbol{x} - \boldsymbol{x}') = K(\boldsymbol{\tau}) &= \int_{\mathbb{R}^D} S(\boldsymbol{s}) \exp\left(2\pi j\, \boldsymbol{s}^T \boldsymbol{\tau}\right) d\boldsymbol{s}, \\ S(\boldsymbol{s}) &= \int_{\mathbb{R}^D} K(\boldsymbol{\tau}) \exp\left(-2\pi j\, \boldsymbol{s}^T \boldsymbol{\tau}\right) d\boldsymbol{\tau}. \end{aligned} \tag{2}$$

- **Theorem 2.2** *Let T be arbitrary set, $K : T \times T \to \mathbb{R}$ a positive definite function. Then there exists a probability space and a Gaussian random function defined on that space, whose covariance function is K.*

3 Computations and examples

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References

Mikhail Antolevich Lifshits. *Gaussian random functions*, volume 322. Springer Science & Business Media, 2013.