Lecture 0: Summary (part 3)

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Contents

1 Tensors

- **Definition** Suppose $V_1, \ldots, V_k$, and W are *vector spaces*. A map $F : V_1 \times \ldots \times V_k \to W$ is said to be ***multilinear*** if it is ***linear*** as a function of *each variable* ***separately*** when the others are held ***fixed***: for each i,

$$F(v_1, \ldots,\, av_i + a'v_i',\, \ldots, v_k) = a\,F(v_1, \ldots,\, v_i,\, \ldots, v_k) + a'F(v_1, \ldots,\, v_i',\, \ldots, v_k).$$

  A multilinear function of ***one variable*** is just ***a linear function***, and a multilinear function of ***two variables*** is generally called ***bilinear***.

- **Remark** Let us write $L(V_1, \ldots, V_k; W)$ for ***the set of all multilinear maps from*** $V_1 \times \ldots \times V_k$ ***to*** W. It is a ***vector space*** under the usual operations of *pointwise addition* and *scalar multiplication*:

$$\begin{aligned}(F' + F)(v_1, \ldots,\, v_i,\, \ldots, v_k) &= F(v_1, \ldots,\, v_i,\, \ldots, v_k) + F'(v_1, \ldots,\, v_i,\, \ldots, v_k),\\ (aF)(v_1, \ldots,\, v_i,\, \ldots, v_k) &= a\,F(v_1, \ldots,\, v_i,\, \ldots, v_k).\end{aligned}$$

- **Example** (***Some Familiar Multilinear Functions***).

  1. The ***dot product***, $\langle \cdot\,, \cdot \rangle$ in $\mathbb{R}^n$ is a ***scalar-valued bilinear function*** of two vectors, used to compute ***lengths*** of vectors and ***angles*** between them.
  2. The ***cross product***, $(\cdot \times \cdot)$ in $\mathbb{R}^3$ is a ***vector-valued bilinear function*** of two vectors, used to compute ***areas*** of parallelograms and to find a third vector ***orthogonal*** to two given ones.
  3. The ***determinant***, $\det(\cdot)$ is a ***real-valued multilinear function*** of n *vectors* in $\mathbb{R}^n$, used to detect ***linear independence*** and to compute the ***volume*** of the parallelepiped spanned by the vectors.
  4. The ***bracket in a Lie algebra*** $\mathfrak{g}$, $[\cdot, \cdot]$ is a $\mathfrak{g}$***-valued bilinear function*** of two elements of $\mathfrak{g}$.

- **Definition** let $V_1, \ldots, V_k, W_1, \ldots, W_l$ be real vector spaces, and suppose $F \in L(V_1, \ldots, V_k; \mathbb{R})$ and $G \in L(W_1, \ldots, W_l; \mathbb{R})$. Define a function $F \otimes G : V_1 \times \ldots \times V_k \times W_1 \times \ldots \times W_l \to \mathbb{R}$ by

$$(F \otimes G)(v_1, \ldots,\, v_k,\, w_1 \ldots, w_l) = F(v_1, \ldots,\, v_k)\, G(w_1 \ldots, w_l) \tag{1}$$

  It follows from the multilinearity of F and G that $(F \otimes G)(v_1, \ldots,\, v_k,\, w_1 \ldots, w_l)$ depends *linearly* on each argument v_i or w_j *separately*, so $F \otimes G$ is an element of $L(V_1, \ldots, V_k, W_1, \ldots, W_l; \mathbb{R})$ called ***the tensor product of*** F ***and*** G.

- **Remark** If $\omega^j \in V_j^*$ for $j = 1, \ldots, k$, then $\omega^1 \otimes \ldots \otimes \omega^k \in L(V_1, \ldots, V_k; \mathbb{R})$ is the ***multilinear function*** given by

$$(\omega^1 \otimes \ldots \otimes \omega^k)(v_1, \ldots, v_k) = \omega^1(v_1)\, \ldots \omega^k(v_k). \tag{2}$$

  We can see that $\omega^1 \otimes \ldots \otimes \omega^k$ is a multilinear extension of the linear functional ω.

- **Proposition 1.1** ***(A Basis for the Space of Multilinear Functions).***
  Let $V_1, \ldots, V_k$ be real vector spaces of dimensions $n_1, \ldots, n_k$, respectively. For each $j \in set 1, \ldots, k$, let $(E_1^{(j)}, \ldots, E_{n_j}^{(j)})$ be a **basis** *for V_j, and let $(\epsilon_{(j)}^1, \ldots, \epsilon_{(j)}^{n_j})$ be the corresponding* ***dual basis*** *for V_j^*. Then the set*

$$\mathfrak{B} = \left\{ \epsilon_{(1)}^{i_1} \otimes \ldots \otimes \epsilon_{(k)}^{i_k} : \; 1 \le i_j \le n_j, j = 1, \ldots, k \right\}$$

  is a basis *for $L(V_1, \ldots, V_k; \mathbb{R})$, which therefore has* ***dimension equal to*** *$n_1 \ldots n_k$.*

1.1 Abstract Tensor Product

- **Remark** Intuitively, we want to define the tensor product $v_1 \otimes \ldots \otimes v_k$ by concatenating all vectors into k-tuple $(v_1, \ldots, v_k)$. But this naive construction is not enough. We have the following challenges:

  1. The product space $V_1 \times \ldots \times V_k$ is *not necessarily a* ***vector space*** since we have not define the addition and scalar product for k-tuple $(v_1, \ldots, v_k)$
  2. We want the ***multilinearity holds*** for the operator on k-tuple $(v_1, \ldots, v_k)$, i.e. we want

  $$(v_1, \ldots, a\, v_i' + b\, v_i'', v_k) = a\,(v_1, \ldots, v_i', v_k) + b\,(v_1, \ldots, v_i'', v_k) \tag{3}$$

     for any $i \in \{1, \ldots, k\}$ and any $a, b \in \mathbb{R}$.

  The above constructions aim to solve these challenges:

  1. Instead of defining the algebraic structure on product space $V_1 \times \ldots \times V_k$, we extend it to ***the free vector space*** $\mathcal{F}(V_1 \times \ldots \times V_k)$, the set of *all linear combinations* of k-tuples $(v_1, \ldots, v_k)$. By construction $\mathcal{F}(V_1 \times \ldots \times V_k) \supseteq V_1 \times \ldots \times V_k$ and *it is a vector space without defining the algebraic structure* since it use an indicator function to map to $\mathbb{R}$.
  2. Instead of enforcing the *multilinearity* to hold, we ***partition the space*** $\mathcal{F}(V_1 \times \ldots \times V_k)$ ***according to the multilinearitiy rule***. That is, the set of tuples $(v_1, \ldots, a\, v_i' + b\, v_i'', v_k)$ and $(v_1, \ldots, v_i', v_k), (v_1, \ldots, v_i'', v_k)$ that satisfies the equation (3) *will be grouped together* via the equivalence relationship. The rule is actually a set of linear combinations of (difference of) tuples, denoted as $\mathcal{R}$.

     Now we instead focusing on the equivalent class itself. By construction, ***the equivalence class will satisfies the multilinear rule*** (3) (The representer of the equivalence class follow the rule). Thsu $V_1 \otimes \ldots \otimes V_k = \mathcal{F}(V_1 \times \ldots \times V_k)/\mathcal{R}$ is the tensor product space that we wants.

- **Definition** Now let $V_1, \ldots, V_k$ be real vector spaces. We begin by forming ***the free vector space*** $\mathcal{F}(V_1 \times \ldots \times V_k)$, which is the set of all finite formal linear combinations of k-tuples $(v_1, \ldots, v_k)$ with $v_i \in V_i$ for $i = 1, \ldots, k$. Let $\mathcal{R}$ be the ***subspace*** of $\mathcal{F}(V_1 \times \ldots \times V_k)$ *spanned* by all elements of the following forms:

$$\begin{aligned} &(v_1, \ldots, a\, v_i, \ldots, v_k) - a\,(v_1, \ldots, v_i, \ldots, v_k) \\ &(v_1, \ldots, v_i + v_i', \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k) \end{aligned} \tag{4}$$

  with $v_j, v_j' \in V_j$, $i \in \{1, \ldots, k\}$, and $a \in \mathbb{R}$.

Define ***the tensor product of the spaces*** $V_1, \ldots, V_k$, denoted by $V_1 \otimes \ldots \otimes V_k$, to be the following ***quotient vector space***:

$$V_1 \otimes \ldots \otimes V_k = \mathcal{F}(V_1 \times \ldots \times V_k)/\mathcal{R}$$

and let $\Pi : \mathcal{F}(V_1 \times \ldots \times V_k) \to V_1 \otimes \ldots \otimes V_k$ be ***the natural projection***. The ***equivalence class*** of an element $(v_1, \ldots, v_k)$ in $V_1 \otimes \ldots \otimes V_k$ is denoted by

$$v_1 \otimes \ldots \otimes v_k = \Pi(v_1, \ldots, v_k) \tag{5}$$

and is called ***the (abstract) tensor product of*** $(v_1, \ldots, v_k)$.

It follows from the definition that abstract tensor products satisfy

$$\begin{aligned} v_1 \otimes \ldots \otimes (a\, v_i) \otimes \ldots \otimes v_k &= a(v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_k), \\ v_1 \otimes \ldots \otimes (v_i + v_i') \otimes \ldots \otimes v_k &= (v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_k) + (v_1 \otimes \ldots \otimes v_i' \otimes \ldots \otimes v_k) \end{aligned}$$

- **Proposition 1.2** *(**Characteristic Property of the Tensor Product Space**).*
*Let $V_1, \ldots, V_k$ be finite-dimensional real vector spaces. If $A : V_1 \times \ldots \times V_k \to X$ is **any multilinear map** into a vector space X, then there is a **unique linear map** $\widetilde{A} : V_1 \otimes \ldots \otimes V_k \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} V_1 \times \ldots \times V_k \arrow[r, "A"] \arrow[d, "\pi"'] & X \\ V_1 \otimes \ldots \otimes V_k \arrow[ur, dashed, "\widetilde{A}"'] & \end{tikzcd} , \tag{6}$$

where π is the map $\pi(v_1, \ldots, v_k) = v_1 \otimes \ldots \otimes v_k$.

- **Remark** The *characteristic property of the tensor product space* states that *any mulilinear function* $\tau : V_1 \times \ldots \times V_k \to \mathbb{R}$ descends into a *linear map* $\widetilde{\tau} : V_1 \otimes \ldots \otimes V_k \to \mathbb{R}$ so that any linear combinations of tensor products $v_{i_1} \otimes \ldots \otimes v_{i_k}$ is expressed as

$$\widetilde{\tau}\left(a^{i_1 \ldots i_k}\, v_{i_1} \otimes \ldots \otimes v_{i_k}\right) = a^{i_1 \ldots i_k}\, \tau(v_{i_1}, \ldots, v_{i_k})$$

- **Proposition 1.3** *(**A Basis for the Tensor Product Space**).*
*Suppose $V_1, \ldots, V_k$ are real vector spaces of dimensions $n_1 \ldots n_k$, respectively. For each $j = 1, \ldots, k$, suppose $(E_1^{(j)}, \ldots, E_{n_j}^{(j)})$ is a **basis** for V_j. Then the set*

$$\mathfrak{C} = \left\{ E_{i_1}^{(1)} \otimes \ldots \otimes E_{i_k}^{(k)} :\ 1 \le i_j \le n_j, j = 1, \ldots, k \right\}$$

***is a basis** for $V_1 \otimes \ldots \otimes V_k$, which therefore has **dimension equal to** $n_1 \ldots n_k$.*

- **Proposition 1.4** *(**Abstract vs. Concrete Tensor Products**).[Lee, 2003.]*
*If $V_1, \ldots, V_k$ are finite-dimensional vector spaces, there is a **canonical isomorphism***

$$V_1^* \otimes \ldots \otimes V_k^* \simeq L(V_1, \ldots, V_k; \mathbb{R}) \tag{7}$$

*under which the **abstract tensor product** defined by (5) corresponds to the **tensor product of covectors** defined by (2).*

- **Remark** Since we are assuming the vector spaces are all finite-dimensional, we can also identify each V_j with its second dual space V_j^{**}, and thereby obtain ***another canonical identification***

$$V_1 \otimes \ldots \otimes V_k \simeq L(V_1^*, \ldots, V_k^*; \mathbb{R}) \tag{8}$$

- **Remark** (**Kronecker Product vs. Tensor Product**)
The two notions represent operations on different objects: Kronecker product on matrices; tensor product on linear maps between vector spaces. But there is a connection: Given two matrices, we can think of them as representing linear maps between vector spaces equipped with a chosen basis. ***The Kronecker product of the two matrices then represents the tensor product of the two linear maps***. (This claim makes sense because the tensor product of two vector spaces with distinguished bases comes with a distinguish basis.)

- **Remark** As we see, the space of tensor product defines a set of ***parallel linear systems***. All ***sub-systems*** are ***independent***. Each sub-system has its own ***basis***, its own ***linear operations*** and its own ***representation***. The tensor product operation *group* these independent linear systems together and *consider them as a whole*.

  For ***the whole system perspective***, its representations are collected locally and then concatenated together. The linear map on ***the concatenated representation*** is essentially the same as ***applying linear map*** in each sub-system and ***multiplying*** them together. This is the same as computing the joint distribution by product of marginal distributions. ***The multiplication principle*** is applied when counting the size of the whole system.

  The space of tensor product $V_1 \otimes \ldots \otimes V_k$ reflect a ***divide-and-conquer strategy*** and a ***local-global strategy*** to study the complex functions such as *multilinear functionals* $\alpha(v_1, \ldots, v_k)$. In the former, we study it by ***perturbing*** *the input of each sub-system*. In the latter, we think of it as ***a linear map*** on the k-tensors $v_1 \otimes \ldots \otimes v_k$.

1.2 Covariant and Contravariant Tensors on a Vector Space

- **Definition** Let V be a finite-dimensional real vector space. If k is a positive integer, ***a covariant k-tensor*** on V is an element of *the k-fold tensor product* $V^* \otimes \ldots \otimes V^*$, which we typically think of as ***a real-valued multilinear function of k elements of V***:

$$\alpha : \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R}$$

  The number k is called ***the rank of*** α. A 0-tensor is, by convention, just a real number (a real-valued function depending multilinearly on *no vectors*!).

  We denote ***the vector space*** of ***all covariant k-tensors on V*** by the shorthand notation

$$T^k V^* = \underbrace{V^* \otimes \ldots \otimes V^*}_{k}$$

- **Example** (***Covariant Tensors***).
Let V be a finite-dimensional vector space.

1. *Every linear functional* $\omega : V \to \mathbb{R}$ is multilinear, so ***a covariant*** 1***-tensor is just a covector***. Thus, $T^1(V^*)$ is equal to V^*.

2. *A covariant* 2*-tensor* on V is *a real-valued* ***bilinear function*** of two vectors, also called ***a bilinear form***. One example is *the dot product* on $\mathbb{R}^n$. More generally, ***every inner product is a covariant*** 2***-tensor***.

3. *The determinant*, thought of as a function of n vectors, is ***a covariant*** n***-tensor on*** $\mathbb{R}^n$.

- **Definition** For any finite-dimensional real vector space V, we define *the space of* ***contravariant tensors*** *on* V of ***rank*** k to be the vector space

$$T^k V = \underbrace{V \otimes \ldots \otimes V}_{k}$$

In particular, $T^1(V) = V$, and by convention $T^0(V) = \mathbb{R}$. Because we are assuming that V is finite-dimensional, it is possible to identify this space with *the set of* ***multilinear functionals of*** k ***covectors***:

$$T^k V = \left\{ \text{multilinear functionals } \alpha : \underbrace{V^* \times \ldots \times V^*}_{k} \to \mathbb{R} \right\}$$

- **Definition** Even more generally, for any nonnegative integers k, l, we define *the space of* ***mixed tensors on*** V ***of type*** (k, l) as

$$T^{(k,l)} V = \underbrace{V \otimes \ldots \otimes V}_{k} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l}$$

- **Corollary 1.5** *Let V be an n-dimensional real vector space. Suppose (E_i) is any basis for V and (ϵ^j) is the dual basis for V^*. Then the following sets constitute bases for the tensor spaces over V:*

$$\left\{\epsilon^{i_1} \otimes \ldots \otimes \epsilon^{i_k} :\ 1 \le i_s \le n, s = 1, \ldots, k\right\} \text{ is basis for } T^k V^*;$$
$$\left\{E_{i_1} \otimes \ldots \otimes E_{i_k} :\ 1 \le i_s \le n, s = 1, \ldots, k\right\} \text{ is basis for } T^k V;$$
$$\left\{E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_l} :\ 1 \le i_1, \ldots i_k, j_1, \ldots, j_l \le n\right\} \text{ is basis for } T^{(k,l)} V; \quad (9)$$

Therefore, dim $T^k V^* =$ dim $T^k V = n^k$ and dim $T^{(k,l)} V = n^{k+l}$

- **Remark** (***Coordinate Representation of Covariant*** k***-Tensor***)
In particular, once a basis is chosen for V, every ***covariant*** k***-tensor*** $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_{i_1, i_2, \ldots, i_k} \epsilon^{i_1} \otimes \ldots \otimes \epsilon^{i_k} \tag{10}$$

where the n^k coefficients $\alpha_{i_1, i_2, \ldots, i_k}$ are determined by

$$\alpha_{i_1, i_2, \ldots, i_k} = \alpha\left(E_{i_1}, \ldots, E_{i_k}\right) \tag{11}$$

For instance, covariant 2-tensor is bilinear form. Every *bilinear form* can be written as $\beta = \beta_{i,j} \epsilon^1 \otimes \epsilon^2$, for some uniquely determined $n \times n$ matrix $(\beta_{i,j})$.

1.3 Symmetric and Alternating Tensors

1.3.1 Symmetric Tensors

- **Definition** Let V be a finite-dimensional vector space. A *covariant k-tensor* α on V is said to be ***symmetric*** if its value is *unchanged* by ***interchanging*** any pair of arguments:

  $$\alpha\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k\right) = \alpha\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k\right)$$

  whenever $i \leq i < j \leq k$.

- **Definition** The set of ***symmetric covariant k-tensors*** is a linear subspace of the space $T^k(V^*)$ of all covariant k-tensors on V; we denote this subspace by $\underline{\Sigma^k(V^*)}$

  There is a ***natural projection*** from $T^k(V^*)$ to $\Sigma^k(V^*)$ defined as follows. First, let S_k denote ***the symmetric group on k elements***, that is, the group of ***permutations*** of the set $\{1, \ldots, k\}$. Given a k-tensor α and a permutation $\sigma \in S_k$, we define a new k-tensor ${}^{\sigma}\alpha$ by

  $${}^{\sigma}\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

  Note that ${}^{\tau}({}^{\sigma}\alpha) ={}^{\tau\sigma} \alpha$ where $\tau\sigma$ represents the composition of τ and σ, that is, $\tau\sigma(i) = \tau(\sigma(i))$.(*This is the reason for putting σ before α in the notation ${}^{\sigma}\alpha$ instead of after it.*)

  We define a ***projection*** $\text{Sym} : T^k(V^*) \rightarrow \Sigma^k(V^*)$ called ***<u>symmetrization</u>*** by

  $$\text{Sym } \alpha = \frac{1}{k!} \sum_{\sigma \in S_k} {}^{\sigma}\alpha$$

  More explicitly, this means that

  $$\text{Sym } \alpha(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

- **Proposition 1.6** ***(Properties of Symmetrization).***
  Let α be a covariant tensor on a finite-dimensional vector space.

  1. *Sym α is symmetric.*
  2. *Sym $\alpha = \alpha$ if and only if α is symmetric.*

- **Definition** If $\alpha \in \Sigma^k(V^*)$ and $\beta \in \Sigma^k(V^*)$, we define their ***symmetric product*** to be the $(k + l)$-tensor $\alpha\,\beta$(denoted by juxtaposition with no intervening product symbol) given by

  $$\alpha\,\beta = \text{Sym }(\alpha \otimes \beta)$$

  More explicitly, the action of $\alpha\,\beta$ on vectors $v_1, \ldots, v_{k+l}$ is given by

  $$\alpha\,\beta(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

- **Proposition 1.7** ***(Properties of the Symmetric Product).***

1. *The symmetric product is **symmetric** and **bilinear**: for all symmetric tensors α, β, γ and all $a, b \in \mathbb{R}$,*

$$\alpha\,\beta = \beta\,\alpha$$
$$(a\,\alpha + b\,\beta)\,\gamma = a\,\alpha\,\gamma + b\,\beta\,\gamma = \gamma\,(a\,\alpha + b\,\beta)$$

2. *If α and β are **covectors**, then*

$$\alpha\,\beta = \frac{1}{2}\,(\alpha \otimes \beta + \beta \otimes \alpha)\,.$$

1.3.2 Alternating Tensors

- **Definition** Assume that V is a finite-dimensional real vector space. ***A covariant k-tensor*** α on V is said to be ***alternating*** (or ***antisymmetric*** or ***skew-symmetric***) if it ***changes sign*** whenever two of its arguments are *interchanged.* This means that for all vectors $v_1, \ldots, v_k \in V$ and every pair of distinct indices i, j it satisfies

$$\alpha\,(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha\,(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

  Alternating covariant k-tensors are also variously called ***exterior forms***, ***multicovectors***, or ***k-covectors***.

  The subspace of ***all alternating covariant*** k***-tensors*** on V is denoted by $\Lambda^k(V^*) \subseteq T^k(V^*)$.

- **Definition** Recall that for any permutation $\sigma \in S_k$, ***the sign of*** σ, denoted by sgn σ, is equal to $+1$ if σ *is **even*** (i.e., can be written as *a composition of an **even** number of transpositions*), and -1 if σ is ***odd***.

- **Lemma 1.8** *The following statements are equivalent for a covariant k-tensor α:*

  1. *α is **alternating**;*

  2. *For any vectors $v_1, \ldots, v_k \in V$, and **any permutation** $\sigma \in S_k$*

  $$\alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (sgn\ \sigma)\alpha\,(v_1, \ldots, v_k)$$

  3. *With respect to any basis, the components $\alpha_{i_1,\ldots,i_k}$ of α change sign whenever two indices are interchanged.*

- **Lemma 1.9** *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

  1. *α is **alternating**.*

  2. *$\alpha(v_1, \ldots, v_k) = 0$ whenever the k-tuple $(v_1, \ldots, v_k)$ is **linearly dependent**.*

  3. *α gives the value zero whenever **two of its arguments** are **equal**:*

  $$\alpha(v_1, \ldots, w, \ldots, w, v_k) = 0.$$

- **Definition** We define a projection Alt $: T^k(V^*) \to \Lambda^k(V^*)$, called ***alternation***, as follows:

$$\text{Alt}\ \alpha = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\,\{\sigma\})\,({}^{\sigma}\alpha)$$

where S_k is *the symmetric group on k elements.* More explicitly, this means

$$\text{Alt } \alpha(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sign}\{\sigma\})\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

- **Proposition 1.10** ***(Properties of Alternation).***
  Let α be a covariant tensor on a finite-dimensional vector space.

  1. *Alt α is alternating.*

  2. *Alt $\alpha = \alpha$ if and only if α is alternating.*

1.4 Tensor Fields and Tensor Bundle

- **Definition** Now let M be a smooth manifold with or without boundary. We define the ***bundle of covariant k-tensors*** on M by

$$T^k T^* M = \bigsqcup_{p \in M} T^k \left(T_p^* M\right)$$

  Analogously, we define ***the bundle of contravariant k-tensors*** by

$$T^k TM = \bigsqcup_{p \in M} T^k \left(T_p M\right)$$

  and ***the bundle of mixed tensors of type (k, l)*** by

$$T^{(k,l)} TM = \bigsqcup_{p \in M} T^{(k,l)} \left(T_p M\right)$$

- **Remark** There are natural identifications

$$\begin{aligned}
&T^{(0,0)}TM = T^0 T^* M = T^0 TM = M \times \mathbb{R};\\
&T^{(0,1)}TM = T^1 T^* M = T^* M;\\
&T^{(1,0)}TM = T^1 TM = TM;\\
&T^{(0,k)}TM = T^k T^* M;\\
&T^{(k,0)}TM = T^k TM.
\end{aligned}$$

  Any one of these bundles is called ***a tensor bundle over*** M. (Thus, the tangent and cotangent bundles are special cases of tensor bundles.)

- **Definition** *A **section** of a tensor bundle* is called *a **(covariant, contravariant, or mixed) tensor field** on M.* ***A smooth tensor field*** is a section that is smooth in the usual sense of smooth sections of vector bundles.

- **Remark** The ***spaces of smooth sections*** of these tensor bundles, $\Gamma\left(T^k T^* M\right)$, $\Gamma\left(T^k TM\right)$, and $\Gamma(T^{(k,l)}TM)$, are ***infinite-dimensional vector spaces** over* $\mathbb{R}$, and ***modules*** over $\mathcal{C}^\infty(M)$. We also denote the *space of smooth covariant tensor fields* as

$$\mathcal{T}^k(M) = \Gamma\left(T^k T^* M\right).$$

- **Remark** (***Coordinate Representation of Tensor Fields***)
  In any smooth local coordinates (x^i), sections of these bundles can be written (using the summation convention) as

$$A = \begin{cases} A_{i_1,\ldots,i_k}\, dx^{i_1} \otimes \ldots \otimes dx^{i_k}, & A \in \Gamma\left(T^k T^* M\right); \\ A^{i_1,\ldots,i_k}\, \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}}, & A \in \Gamma\left(T^k T M\right); \\ A^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}\, \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_k} & A \in \Gamma\left(T^{(k,l)} T M\right); \end{cases} \tag{12}$$

  The functions $A_{i_1,\ldots,i_k}$, $A^{i_1,\ldots,i_k}$, or $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_l}$ are called the ***component functions*** of A in the chosen coordinates.

- **Proposition 1.11** *(**Smoothness Criteria for Tensor Fields**).*
  Let M be a smooth manifold with or without boundary, and let $A : M \to T^k T^* M$ be a rough section. The following are equivalent.

  1. *A is smooth.*
  2. *In **every** smooth coordinate chart, the **component functions** of A are smooth.*
  3. *Each point of M is contained in **some** coordinate chart in which A has **smooth component functions**.*
  4. *If $X_1, \ldots, X_k \in \mathfrak{X}(M)$, then the function $A(X_1, \ldots, X_k) : M \to \mathbb{R}$, defined by*

     $$A(X_1, \ldots, X_k)(p) = A_p\left(X_1\big|_p, \ldots, X_k\big|_p\right) \tag{13}$$

     is smooth
  5. *Whenever $X_1, \ldots, X_k$ are smooth vector fields defined on **some open subset** $U \subseteq M$, the function $A(X_1, \ldots, X_k)$ is smooth on U.*

- **Lemma 1.12** *(**Tensor Characterization Lemma**).[Lee, 2003.]*
  A map

  $$\mathcal{A} : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k} \to \mathcal{C}^\infty(M). \tag{14}$$

  *is **induced** by a **smooth covariant k-tensor field** A as in (13) if and only if it is **multilinear** over $\mathcal{C}^\infty(M)$.*

- **Definition** For symmetric and alternating tensor field, we have the following definition:

  1. ***A symmetric tensor field*** on a manifold (with or without boundary) is simply *a covariant tensor field* whose value at each point is a *symmetric tensor*.

     The ***symmetric product*** of two or more tensor fields is defined pointwise, just like the tensor product. Thus, for example, if A and B are *smooth covector fields*, their symmetric product is ***the smooth 2-tensor field*** AB, which is given by

     $$AB = \frac{1}{2}(A \otimes B) + \frac{1}{2}(B \otimes A).$$

  2. ***Alternating tensor fields*** are called ***differential forms***;

1.5 Pullbasks of Tensor Fields

- **Definition** Suppose $F : M \to N$ is a smooth map. For any point $p \in M$ and any *k-tensor* $\alpha \in T^k(T^*_{F(p)}N)$, we define a tensor $dF^*_p(\alpha) \in T^k\left(T^*_p M\right)$, called ***the pointwise pullback*** of α by F at p, by

$$dF^*_p(\alpha)(v_1, \ldots, v_k) = \alpha\left(dF_p(v_1), \ldots, dF_p(v_k)\right)$$

for any $v_1, \ldots, v_k \in T_p M$.

- **Definition** If A is *a covariant k-tensor field on N*, we define a rough***k-tensor field*** F^*A ***on*** M; called ***the pullback of A by F***, by

$$(F^*A)_p = dF^*_p(A_{F(p)}).$$

This tensor field acts on vectors $v_1, \ldots, v_k \in T_p M$ by

$$(F^*A)_p(v_1, \ldots, v_k) = A_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_k)\right).$$

- **Proposition 1.13** ***(Properties of Tensor Pullbacks)***.
Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.

  1. $F^*(fB) = (f \circ F)\, F^*(B)$
  2. $F^*(A \otimes B) = F^*A \otimes F^*(B)$
  3. $F^*(A + B) = F^*A + F^*(B)$
  4. *$F^*(B)$ is a **(continuous) tensor field**, and is **smooth** if B is **smooth**.*
  5. $(G \circ F)^*B = F^*(G^*B)$.
  6. $(Id_N)^*B = B$.

- **Corollary 1.14** ***(Coordinate Representation of Pullback Tensor Fields)***
Let $F : M \to N$ be smooth, and let B be a covariant k-tensor field on N. If $p \in M$ and (y^i) are smooth coordinates for N on a neighborhood of $F(p)$, then F^*B has the following expression in a neighborhood of p:

$$F^*\left(B_{i_1,\ldots,i_k}\, dy^{i_1} \otimes \ldots \otimes dy^{i_k}\right) = \left(B_{i_1,\ldots,i_k} \circ F\right) d\left(y^{i_1} \circ F\right) \otimes \ldots \otimes \left(y^{i_k} \circ F\right).$$

- **Remark** F^*B is computed as follows: whereaver you see y^i in the expression for B, just substitute the ith component function of F and expand.

1.6 Contraction

- **Proposition 1.15** *Let V be a finite-dimensional vector space. There is a natural (basis-independent) **isomorphism** between $T^{(k+1,l)}V$ and the space of **multilinear** maps*

$$\underbrace{V^* \times \ldots \times V^*}_{k} \times \underbrace{V \times \ldots \times V}_{l} \to V$$

- **Definition** We can use the result of Proposition 1.15 to define a natural operation called ***trace*** or ***contraction***, which *lowers the rank of a tensor by* 2.

  For $F = v \otimes \omega \in T^{(1,1)}V$. Define the operator $\text{tr} : T^{(1,1)}V \to \mathbb{R}$ is just ***the trace of*** F for i.e. the sum of the diagonal entries of any matrix representation of F. More generally, we define $\text{tr} \; : T^{(k+1,l+1)}V \to T^{(k,l)}V$ by letting $\text{tr}\, F(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l)$ be the ***trace*** of the $(1,1)$***-tensor***

$$F(\omega^1, \ldots, \omega^k, \cdot, v_1, \ldots, v_l, \cdot) \in T^{(1,1)}V$$

  In terms of a basis, the ***components*** of $\text{tr}\, F$ are

$$(\text{tr}\; F)^{i_1, \ldots, i_k}_{j_1, \ldots, j_l} = F^{i_1, \ldots, i_k, m}_{j_1, \ldots, j_l, m}.$$

  In other words, just ***set the last upper and lower indices equal*** and ***sum***.

- **Remark** We consider a $(1,1)$-tensor $F = v \otimes \omega$. Under standard basis, $v = v^i E_i$ and $\omega = \omega_j\, \epsilon^j$, F has representation

$$\begin{aligned} F &= v \otimes \omega \\ &= (v^i E_i) \otimes (\omega_j\, \epsilon^j) \\ &= (\omega_j\, v^i) E_i \otimes \epsilon^j := F^i_j\; E_i \otimes \epsilon^j \end{aligned}$$

  There is an isomorphism $T^{(1,1)}V \to L(V; V)$ as $F \mapsto [F^i_j]_{j,i}$. Then the ***trace*** of F is

$$\begin{aligned} \text{tr}\; (v \otimes \omega) &= \omega(v) \\ &= \omega_i\, v^i \\ &= \text{tr} \left(\begin{bmatrix} \omega_1\, v^1 & \ldots & \omega_1\, v^n \\ \vdots & \ddots & \vdots \\ \omega_n\, v^1 & \ldots & \omega_n\, v^n \end{bmatrix} \right) = \text{tr}\; [F^i_j]_{j,i} \end{aligned}$$

- **Remark** We have the formula for a (k,l)-tensor field F

$$F(\omega^1, \ldots, \omega^k, V_1, \ldots, V_l) = \underbrace{\text{tr} \circ \ldots \circ \text{tr}}_{k+l} \Big(F \otimes \omega^1 \otimes \ldots \otimes \omega^k \otimes V_1 \otimes \ldots \otimes V_l \Big), \qquad (15)$$

  where each trace operator acts on an upper index of F and the lower index of the corresponding 1-form, or a lower index of F and the upper index of the corresponding vector field.

  For instance, for covariant 2-tensor field $g = \omega^1 \otimes \omega^2$:

$$\begin{aligned} g(X, Y) &= \text{tr} \left(\text{tr}(\omega^1 \otimes \omega^2 \otimes X \otimes Y) \right) \\ &= \text{tr} \left(\text{tr}(\omega^2 \otimes Y)\, \omega^1 \otimes X \right) \\ &= \text{tr} \left((\omega^2(Y))\, \omega^1 \otimes X \right) \\ &= (\omega^2(Y))\, \text{tr} \left(\omega^1 \otimes X \right) \\ &= (\omega^2(Y))\, (\omega^1(X)) \end{aligned}$$

2 Differential Forms

2.1 Elementary k-covectors

- **Definition** Given a positive integer k, ***an ordered k-tuple*** $I = (i_1, \dots, i_k)$ of *positive integers* is called *a **multi-index** of length k.* If I is such a multi-index and $\sigma \in S_k$ is a permutation of $\{1, \dots, k\}$, we write I for the following multi-index:

$$I_\sigma = \left(i_{\sigma(1)}, \dots, i_{\sigma(k)}\right).$$

Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$.

- **Definition** Let V be an n-dimensional vector space, and suppose $(\epsilon^1, \dots, \epsilon^n)$ is any *basis* for V^*. We now define a collection of *k-covectors* on V that generalize ***the determinant function*** on $\mathbb{R}^n$.

For each multi-index $I = (i_1, \dots, i_k)$ of length k such that $1 \le i_1 \le \dots \le i_k \le n$, define ***a covariant k-tensor*** $\epsilon^I = \epsilon^{i_1, \dots, i_k}$ by

$$\epsilon^I(v_1, \dots, v_k) = \det \begin{bmatrix} \epsilon^{i_1}(v_1) & \dots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \dots & \epsilon^{i_k}(v_k) \end{bmatrix} = \det \begin{bmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \dots & v_k^{i_k} \end{bmatrix}. \tag{16}$$

In other words, if $\boldsymbol{V}$ denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \dots, v_k$ with respect to the basis (E_i) dual to (ϵ^i), then $\epsilon^I(v_1, \dots, v_k)$ is the ***determinant of the $k \times k$ submatrix*** consisting of rows $i_1, \dots, i_k$ of $\boldsymbol{V}$. Because the determinant changes sign whenever two columns are interchanged, it is clear that ϵ^I is *an alternating k-tensor*. We call ϵ^I ***an elementary alternating tensor*** or ***elementary k-covector***.

- **Definition** If I and J are multiindices of length k, we define the Kronecker delta function:

$$\delta_J^I = \det \begin{bmatrix} v_{j_1}^{i_1} & \dots & v_{j_k}^{i_1} \\ \vdots & \ddots & \vdots \\ v_{j_1}^{i_k} & \dots & v_{j_k}^{i_k} \end{bmatrix}$$

(I represent the row number, J represent the column number.)

- **Remark** The following is the property of Kronecker delta

$$\delta_J^I = \begin{cases} \text{sign}\,\{\sigma\} & \text{if neither } I \text{ nor } J \text{ has a repeated index, } J = I_\sigma,\ \sigma \in S_k \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I \end{cases}$$

- **Lemma 2.1** ***(Properties of Elementary k-Covectors).***
Let (E_i) be a basis for V, let (ϵ^i) be the dual basis for V^*, and let ϵ^I be as defined above.

  1. *If I has a repeated index, then $\epsilon^I = 0$.*
  2. *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\epsilon^I = sign\{\sigma\}\, \epsilon^J$.*
  3. *The result of evaluating ϵ^I on a sequence of basis vectors is*

$$\epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I.$$

- **Definition** A multi-index $I = (i_1, \dots, i_k)$ is said to be ***increasing*** if $i_1 < \dots < i_k$. It is useful to use a primed summation sign to denote a sum over *only increasing multi-indices*

$$\sum_I{}' a_I \epsilon^I = \sum_{\{I: i_1 < \dots < i_k\}} a_I \epsilon^I.$$

- **Proposition 2.2** *(**A Basis for** $\Lambda^k(V^*)$)*
Let V be an n-dimensional vector space. If (ϵ^i) is any basis for V^*, then for each positive integer $k \le n$, the collection of k-covectors

$$\underline{\mathcal{E} = \left\{\epsilon^I : I \text{ is an increasing multi-index of length } k\right\}}$$

*is **a basis for** $\Lambda^k(V^*)$. Therefore,*

$$dim\ \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If $k > n$, then dim $\Lambda^k(V^*) = 0$.

- **Remark** In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V^*)$ is 1-***dimensional*** and is spanned by $\epsilon^{1,\dots,n}$.

- **Proposition 2.3** *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T : V \to V$ is any **linear map** and $v_1, \dots, v_n$ are arbitrary vectors in V, then*

$$\omega\left(Tv_1, \dots, Tv_n\right) = (\det T)\, \omega\left(v_1, \dots, v_n\right). \tag{17}$$

2.2 Wedge Product

- **Definition** Let V be a finite-dimensional real vector space. Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their ***wedge product*** or ***exterior product*** to be the following $(k+l)$-*covector*:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\mathrm{Alt}\left(\omega \otimes \eta\right) = \frac{1}{k!\,l!}\sum_{\sigma \in S_{k+l}} \left(\mathrm{sign}\left\{\sigma\right\}\right)\left({}^{\sigma}\left(\omega \otimes \eta\right)\right) \tag{18}$$

- The coefficients come from the following lemma:

  Lemma 2.4 *Let V be an n-dimensional vector space and let $(\epsilon^1, \dots, \epsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$,*

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ} \tag{19}$$

  *where $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$ is obtained by **concatenating** I and J.*

- **Proposition 2.5** *(**Properties of the Wedge Product**).*
*Suppose $\omega, \omega', \eta, \eta'$ and ξ are **multicovectors** on a finite-dimensional vector space V.*

  1. *(**Bilinearity**): For $a, a; \in \mathbb{R}$,*

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

2. ***(Associativity):***

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$$

3. ***(Anticommutativity):*** *For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega \tag{20}$$

4. *If (ϵ^i) is any basis for V^* and $I = (i_1, \ldots, i_k)$ is any multi-index, then*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^I \tag{21}$$

5. *For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$,*

$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \ldots, v_k) = \det\left(\omega^j(v_i)\right) \tag{22}$$

- **Remark** Because of part (4) of this lemma, henceforth we generally use the notations ϵ^I and $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ ***interchangeably***

- **Definition** A k-covector η is said to be ***decomposable*** if it can be expressed in the form $\eta = \omega^1 \wedge \ldots \wedge \omega^k$, where $\omega^1, \ldots, \omega^k$ are *covectors.*

- **Remark** It is important to be aware that not every k-covector is decomposable when $k > 1$; however, it follows from Proposition 2.2 and above Proposition 2.5 (4) that ***every k-covector*** *can be written as a* ***linear combination*** *of* ***decomposable ones***.

- **Definition** For any n-dimensional vector space V, define **a vector space** $\Lambda(V^*)$ by

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

  It follows from Proposition 2.2 that $\dim \Lambda(V^*) = 2^n$. The wedge product turns $\Lambda(V^*)$ into an ***associative algebra***, called ***the exterior algebra*** (or ***Grassmann algebra***) of V.

- **Remark** For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$, ***the exterior product*** is considered as the ***determinant function*** *of a $k \times k$ submatrix*

$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \ldots, v_k) = \det \begin{bmatrix} \omega^1(v_1) & \ldots & \omega^1(v_k) \\ \vdots & \ddots & \vdots \\ \omega^k(v_1) & \ldots & \omega^k(v_k) \end{bmatrix}$$

  where ***vectors*** $v_1, \ldots, v_k$ *forms* ***column vector***, and ***covectors*** $\omega^1, \ldots, \omega^k$ *form the* ***row vector***.

  In other words, we can think of ***exterior product of covectors*** as ***an abstraction of determinant operation***.

2.3 Interior Product

- **Definition** Let V be a finite-dimensional vector space. For each $v \in V$, we define a *linear map* $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$, called ***interior multiplication (interior product)*** by v, as follows:

$$(\iota_v \omega)(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1}).$$

In other words, $(\iota_v \omega)$ is obtained from ω by ***inserting*** v ***into the first slot***. By convention, we interpret $(\iota_v \omega)$ to be ***zero*** when ω is a 0-***covector*** (i.e., a ***number***). Another common notation is

$$v \lrcorner \omega = (\iota_v \omega).$$

This is often read "v ***into*** ω."

- **Proposition 2.6** *Let V be a finite-dimensional vector space and $v \in V$.*

  1. $\iota_v \circ \iota_v = 0$.
  2. *If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\iota_v(\omega \wedge \eta) = \iota_v(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_v(\eta) \tag{23}$$

- **Remark** It is easy to verify the following form

$$\iota_v\left(\omega^1 \wedge \ldots \wedge \omega^k\right) = v \lrcorner \left(\omega^1 \wedge \ldots \wedge \omega^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \left(\omega^1 \wedge \ldots \wedge \widehat{\omega}^i \wedge \ldots \wedge \omega^k\right) \tag{24}$$

$$\Leftrightarrow \left(\omega^1 \wedge \ldots \wedge \omega^k\right)(v, v_2, \ldots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \left(\omega^1 \wedge \ldots \wedge \widehat{\omega}^i \wedge \ldots \wedge \omega^k\right)(v_2, \ldots, v_k)$$

where the hat indicates that ω^i is ***omitted***. In *determiant form*, it can be written as

$$\det \boldsymbol{V} = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \det \boldsymbol{V}_1^i \tag{25}$$

where $\boldsymbol{V}_j^i$ denote the $(k-1) \times (k-1)$ submatrix of $\boldsymbol{V}$ obtained by ***deleting*** the i-th row and j-th column. This is just ***the expansion of*** $\det \boldsymbol{V}$ ***by minors*** along the first column, and therefore is equal to $\det \boldsymbol{v}$.

- **Remark** The *exterior product* ***increase*** the *rank* of tensor, while the *interior product* ***decrease*** the *rank* of tensor by 1.

2.4 Differential Forms on Manifolds

- **Definition** Let $T^k T^* M$ be the *bundle* of all covariant k-tensors on M. The subset of $T^k T^* M$ consisting of ***alternating tensors*** is denoted by $\Lambda^k(T^* M)$:

$$\Lambda^k(T^* M) = \bigsqcup_{p \in M} \Lambda^k(T_p^* M).$$

$\Lambda^k(T^* M)$ is a *smooth subbundle* of $T^k T^* M$, so it is a *smooth vector bundle of rank* $\binom{n}{k}$.

- **Remark** $\Lambda^k(T^*M)$ is ***the bundle of all alternating covariant k-tensors (exterior forms, k-covectors)*** on M.

- **Definition** A ***section*** of $\Lambda^k(T^*M)$ is called ***a differential k-form***, or just a ***k-form***; this is a *(continuous) tensor field* whose value at each point is an *alternating tensor*. The integer k is called *the* ***degree*** *of the form*. We denote the *vector space* of ***smooth k-forms*** by

$$\Omega^k(M) = \Gamma\left(\Lambda^k(T^*M)\right).$$

- **Remark** *A k-form is just an alternating covariant k-tensor fields.*

- **Remark** The ***wedge product*** of *two differential forms* is defined ***pointwise***: $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Thus, ***the wedge product of a k-form with an l-form is a $(k+l)$-form***. If f is a 0-form (i.e. a smooth function) and ω is a k-form, we interpret the wedge product $f \wedge \omega$ to mean the ordinary product $f\omega$.

- **Remark** The direct sum of all vector spaces of smooth k-forms for $k \leq n$ is

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M). \tag{26}$$

  Then the wedge product turns $\Omega^*(M)$ into *an* ***associative, anticommutative graded algebra***.

- **Remark** (***Duality of Basis***)
  The basis of differential k-forms $(dx^{i_1} \wedge \ldots \wedge dx^{i_k})$ in $\Gamma\left(\Lambda^k(T^*M)\right)$ acts on the local coordinate frames $(\partial/\partial x^i)$ in TM

$$\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right)\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J$$

- **Remark** (***Coordinate Representation of k-Forms***)
  In any smooth chart, a k-form ω can be written locally as

$$\omega = \sum_I{}' \omega_I dx^I := \sum_I{}' \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

  where the coefficients ω^I are ***continuous functions*** defined on the coordinate domain, and we use dx^I as an abbreviation for $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (not to be mistaken for the differential of a real-valued function x^I). Also $\sum_I' \epsilon^I$ means that sum with increasing multi-indices. ***The component function*** ω_I is computed as

$$\omega_I = \omega\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right).$$

  Note that ω_I is ***the determinant of a $k \times k$ principal sub-matrix (i.e. principal minors)*** whose *rows and columns* are indexed by increasing multi-index I.

- **Example** The followings are some basic differential k-forms:

  1. Any smooth function $f \in \mathcal{C}^\infty(M)$ is a 0-***form***;

2. ***A differential* 1*-form*** is ***the covariant vector field*** df

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

3. ***A differential* 2*-form*** is written as

$$\omega = \sum_{i<j} \omega_{i,j}\, dx^i \wedge dx^j$$

- **Definition** If $F : M \to N$ is a smooth map and ω is ***a differential form*** on N, *the* ***pullback*** F^* *is a* ***differential form on*** M; defined as for any covariant tensor field:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_p\left(dF_p(v_1), \ldots, dF_p(v_k)\right).$$

- **Lemma 2.7** *Suppose $F : M \to N$ is smooth.*

  1. $F^* : \Omega^k(N) \to \Omega^k(M)$ *is* **linear** *over* $\mathbb{R}$.
  2. $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.
  3. *In any smooth chart,*

$$F^*\left(\sum_I{}' \omega_I\, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\; d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F) \tag{27}$$

- **Proposition 2.8** *(**Pullback Formula for Top-Degree Forms**).*
Let $F : M \to N$ be a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^*\left(u\, dy^1 \wedge \ldots \wedge dy^n\right) = (u \circ F)\,(\det(DF))\, dx^1 \wedge \ldots dx^n \tag{28}$$

where DF represents ***the Jacobian matrix of*** F *in these coordinates.*

Note that $d(y^i \circ F) = dF^i = \det(DF)^i_j dx^j$

- **Corollary 2.9** *(**Change of Coordinates for Differential Forms**)*
If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \widetilde{U}$:

$$d\widetilde{x}^1 \wedge \ldots \wedge d\widetilde{x}^n = \det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n. \tag{29}$$

- **Remark** The equation (28) provides a computational formula for pullback of differential forms under coordinate systems for domain and codomain. And the equation (29) provides the fomula for change of variables of differential forms.

- **Definition** ***Interior multiplication*** also extends naturally to ***vector fields*** and ***differential forms***, simply by letting it act *pointwise*: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$-form $X \lrcorner\, \omega = \iota_X \omega$ by

$$(X \lrcorner\, \omega)_p = X_p \lrcorner\, \omega_p.$$

- **Proposition 2.10** *Let X be a smooth vector field on M.*
  1. *If ω is a smooth differential form, then $\iota_X \omega$ is smooth.*
  2. *$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is* **linear** *over $\mathcal{C}^\infty(M)$ and therefore corresponds to a* ***smooth bundle homomorphism*** *$\iota_X : \Lambda^k(T^*M) \to \Lambda^{k-1}(T^*M)$.*

2.5 Exterior Derivatives of Differential Forms

- **Remark** For each smooth manifold M with or without boundary, we will show that there is ***a differential operator*** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω. Thus, it will follow that *a necessary condition* for a smooth k-form ω to be equal to $d\eta$ for some $(k-1)$-form η is that $d\omega = 0$.

- **Definition** If $\omega = \sum'_J \omega_J dx^J$ is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, we define its ***exterior derivative*** $d\omega$ to be the following $(k+1)$-form:
$$d\omega := d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J, \tag{30}$$
where $d\omega_J$ is the differential of the function ω_J. In somewhat more detail, this is
$$d\omega := d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}. \tag{31}$$

- **Remark** The *exterior derivatives* of a k-form is ***a linear combination*** of $(k+1)$*-forms*. It component function is ***the principal minior*** *of* ***Jacobian matrix of component functions*** $(\frac{\partial \omega_j}{\partial x^i})$.

- **Remark** When ω is a 1-form, this becomes
$$\begin{aligned} d\omega = d\left(\sum_j \omega_j dx^j\right) &= \sum_j d\omega_j \wedge dx^j \\ &= \sum_j \sum_i \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j. \end{aligned}$$
Note that the component is the determinant of a 2×2 sub-matrix of Jacobian $(\frac{\partial \omega_j}{\partial x^i})$.

- **Remark** The ***exterior differentiation*** defines the differential of k-form. It is an ***extension*** of ***differentiation*** to ***determinant function***.

- **Definition** If $A = \oplus_k A^k$ is a *graded algebra*, a *linear map* $T : A \to A$ is said to be a map ***of degree*** m if $T(A^k) \subseteq A^{k+m}$ for each k. It is said to be an ***antiderivation*** if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x \in A^k$ and $y \in A^l$.

- **Remark** (***The Exterior Differentiation vs. The Interior Multiplication***)
  1. The ***exterior differentiation*** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an ***antiderivation*** of *degree* $+1$ whose ***square is zero***.
  2. On the other hand, the ***interior multiplication*** $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is an ***antiderivation*** of *degree* -1 whose ***square is zero***, where $X \in \mathfrak{X}(M)$.

- Another important feature of the exterior derivative is that *it **commutes** with all pullbacks.*

  Proposition 2.11 *(**Naturality of the Exterior Derivative**).*
  *If $F : M \to N$ is a smooth map, then for each k **the pullback map** $F^* : \Omega^k(N) \to \Omega^k(M)$ **commutes** with d: for all $\omega \in \Omega^k(N)$,*

$$F^*(d\omega) = d\,(F^*\omega)\,. \tag{32}$$

2.6 An Invariant Formula for the Exterior Derivative

- **Proposition 2.12** *(**Exterior Derivative of a** 1**-Form**).*
  For any smooth 1-form ω and smooth vector fields X and Y,

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{33}$$

  Proof: Since any smooth 1-form can be expressed locally as a sum of terms of the form $u\,dv$ for smooth functions u and v, it suffices to consider that case. Suppose $\omega = u\,dv$, and X, Y are smooth vector fields. The LHS of (33)

$$\begin{aligned} d\,(u\,dv)\,(X, Y) &= (du \wedge dv)(X, Y) = du(X)dv(Y) - du(Y)dv(X) \\ &= X(u)Y(v) - X(v)Y(u) \end{aligned}$$

  The RHS is

$$\begin{aligned} &= X(u\,dv(Y)) - Y(u\,dv(X)) - u\,dv([X, Y]) \\ &= X(u\,Y(v)) - Y(u\,X(v)) - u\,[X, Y](v) \\ &= X(u)Y(v) + u\,XY(v) - Y(u)X(v) - u\,YX(v) - u\,((XY - YX)v) \\ &= X(u)Y(v) - Y(u)X(v) + u\,(XY(v) - YX(v)) - u\,(XY(v) - YX(v)) \\ &= X(u)Y(v) - Y(u)X(v). \end{aligned}$$

  Thus (33) holds. $\blacksquare$.

- **Proposition 2.13** *Let M be a smooth n-manifold with or without boundary, let (E_i) be a smooth local frame for M, and let (ϵ^i) be the dual coframe. For each i, let $b^i_{j,k}$ denote the **component functions** of **the exterior derivative** of ϵ^i in this frame, and for each j, k, let $c^i_{j,k}$ be the **component functions** of the **Lie bracket** $[E_j, E_k]$:*

$$d\epsilon^i = \sum_{j<k} b^i_{j,k}\epsilon^j \wedge \epsilon^k; \quad [E_j, E_k] = c^i_{j,k}E_i$$

  Then $b^i_{j,k} = -c^i_{j,k}$.

- **Proposition 2.14** ***(Invariant Formula for the Exterior Derivative).***
Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields $X_1, \ldots, X_{k+1}$ on M,

$$\begin{aligned} d\omega\left(X_1, \ldots, X_{k+1}\right) = &\sum_{1 \le i \le k+1} (-1)^i X_i \left(\omega\left(X_1, \ldots, \widehat{X}_i \ldots, X_{k+1}\right)\right) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}\right), \end{aligned} \tag{34}$$

where the hats indicate ***omitted*** *arguments.*

- **Remark** The proof of formula (34) and (33) is only based on the definition of k-form and vector fields, and it does not involve any specific coordinate system. Thus it can be used to give an ***invariant definition*** of ***the exterior differentiation*** d.

3 Directional Derivatives of Vector Fields

3.1 Connections

- **Remark** There are *two alternatives* for the definition of ***geodesics***:
  - Geodesics is the *"**shortest**" path* that connects two points on the surface; This definition is hard since the definition of manifold is abstract.
  - Geodesics is the curve on the surface that has ***zero tangential acceleration***. This is the motivation to introduce the concept of *connections*.

- **Remark** Although the ***velocity*** of a curve γ in a manifold M is well defined, the ***acceleration*** of the curve on M is **not** since it requires comparison between tangent vectors in two different tangent spaces $T_{\gamma(t)}M$ and $T_{\gamma(t+\Delta)}M$.

- **Remark** To do so, we need a way to ***compare values of the vector field at different points***, or intuitively, to "***connect***" ***nearby tangent spaces***. This is where a connection comes in: it will be an additional piece of data on a manifold, *a **rule*** for *computing directional derivatives of vector fields.*

- **Remark** ***A connection*** is a ***coordinate-independent*** set of rules for taking ***directional derivatives of vector fields***.

- **Definition** Let $\pi : E \to M$ be a *smooth vector bundle* over *a smooth manifold* M with or without boundary, and let $\Gamma(E)$ denote the space of *smooth sections* of E. A ***connection*** in E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

1. $\nabla_X Y$ is ***linear*** *over* $\mathcal{C}^\infty(M)$ ***in*** X: for $f_1, f_2 \in \mathcal{C}^\infty(M)$ and $X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{(f_1 X_1 + f_2 X_2)} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$$

2. $\nabla_X Y$ is ***linear*** *over* $\mathbb{R}$ ***in*** Y: for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \Gamma(E)$,

$$\nabla_X(a_1\, Y_1 + a_2\, Y_2) = a_1\, \nabla_X Y_1 + a_2\, \nabla_X Y_2$$

3. ∇ satisfies the following ***product rule***: for $f \in \mathcal{C}^\infty(M)$,

$$\nabla_X(fY) = f\, \nabla_X Y + (Xf)\, Y$$

The symbol ∇ is read "***del***" or "***nabla***," and $\nabla_X Y$ is called ***the covariant derivative of*** Y ***in the direction*** X.

- **Remark** There is *a variety of types of connections* that are useful in different circumstances. The type of connection we have defined here is sometimes called ***a Koszul connection*** to distinguish it from other types.

- **Lemma 3.1** *(**Locality**). [Lee, 2018]*
*Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$, and $p \in M$, the covariant derivative $\nabla_X Y|_p$ depends **only** on the values of X and Y in an arbitrarily **small neighborhood** of p. More precisely, if $X = \widetilde{X}$ and $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.*

- **Proposition 3.2** *(**Restriction of a Connection**).[Lee, 2018]*
*Suppose ∇ is a connection in a smooth vector bundle $E \to M$. For every open subset $U \subseteq M$, there is a **unique connection** ∇^U on the **restricted bundle** $E|_U$ that satisfies the following relation for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$:*

$$\nabla^U_{(X|_U)}(Y|_U) = (\nabla_X Y)|_U. \tag{35}$$

- **Proposition 3.3** *Under the hypotheses of Lemma 3.1, $\nabla_X Y|_p$ depends **only** on the **values** of Y in a **neighborhood** of p and the **value** of X **at** p.*

- **Remark** In the situation of these two propositions, we typically just refer to the *restricted connection* as ∇ instead of ∇^U; the proposition guarantees that there is no ambiguity in doing so. Thus if X is a vector field defined in a neighorhood of p,

$$\nabla_v Y = \nabla_X Y|_p, \quad \text{for } v = X_p.$$

3.2 Connections in the Tangent Bundle

- We focus on the connection in tangent bundle.

Definition Suppose M is a smooth manifold with or without boundary. By the definition we just gave, *a connection in TM* is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

satisfying properties (1)-(3) above. *A connection in the tangent bundle TM* is often called simply ***a connection on M***. (The terms ***affine connection*** and ***linear connection*** are also sometimes used in this context.)

- **Definition** For computations, we need to examine how a connection appears in terms of *a local frame*. Let (E_i) be a *smooth local frame* for TM on an open subset $U \subseteq M$. For every choice of the indices i and j, we can expand the vector field $\nabla_{E_i} E_j$ in terms of this same frame:

$$\nabla_{E_i} E_j = \Gamma_{i,j}^{k}\, E_k. \tag{36}$$

As i, j, and k range from 1 to $n = \dim M$, this defines n^3 smooth functions $\Gamma_{i,j}^{k} : U \to \mathbb{R}$, called ***the connection coefficients*** *of* ∇ ***with respect to the given frame***.

- The following proposition shows that the connection is completely determined in U by its connection coefficients.

Proposition 3.4 ***(Coordinate Representation of Connection)*** *[Lee, 2018]*
Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose (E_i) is a smooth local frame over an open subset $U \subseteq M$, and let $\left\{\Gamma_{i,j}^{k}\right\}$ be the connection coefficients of ∇ with respect to this frame. For smooth vector fields $X, Y \in \mathfrak{X}(M)$, written in terms of the frame as $X = X^i\, E_i$, $Y = Y^j\, E_j$, one has

$$\nabla_X Y = \left(X(Y^k) + X^i Y^j\, \Gamma_{i,j}^{k}\right)\, E_k. \tag{37}$$

- **Remark** The n^3 functions $\{\Gamma_{i,j}^{k}\}$ are called ***the Christoffel symbols*** under *the metric connections*. [do Carmo Valero, 1976]

- **Remark** The smooth function $\Gamma_{i,j}^{k} \in \mathcal{C}^{\infty}(M)$ has three indices: *two lower indices* (i, j) cooresponds to the index of ***component*** X^i ***for the directional vector field*** X, and the index of ***component*** Y^j ***for the differentiated vector field*** Y in $\nabla_X Y$; *the one upper index* k correponds to the index of the ***basis*** vector field $\partial/\partial x^k$ which spans the space of vector fields.

- **Remark** The *first term* of (37) accounts for *the* ***change of position relative*** *to the local frame* when moving Y from one tangent space to another along the direction of X. The second term accounts for the ***additional "rotation" of frames***. For Euclidean space, the basis is fixed when moving along the tangent direction (i.e. no *rotation* just *translation*).

- **Proposition 3.5** ***(Transformation Law for Connection Coefficients)***. *[Lee, 2018]*
Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose we are given two smooth local frames (E_i) and $(\widetilde{E}_j)$ for TM on an open subset $U \subseteq M$, related by $\widetilde{E}_i = A_i^j E_j$ for some matrix of functions (A_i^j). Let $\Gamma_{i,j}^{k}$ and $\widetilde{\Gamma}_{i,j}^{k}$ denote the connection coefficients of ∇ with respect to these two frames. Then

$$\widetilde{\Gamma}_{i,j}^{k} = (A^{-1})_t^k A_i^r A_j^s \Gamma_{r,s}^{t} + (A^{-1})_t^k A_i^s E_s(A_j^t) \tag{38}$$

- **Lemma 3.6** *Suppose M is a smooth n-manifold with or without boundary, and M admits* ***a global frame*** *(E_i). Formula (37) gives a* ***one-to-one correspondence*** *between connections in TM and choices of n^3 smooth real-valued functions $\{\Gamma_{i,j}^{k}\}$ on M.*

- **Proposition 3.7** *The tangent bundle of every smooth manifold with or without boundary admits a connection.*

- **Proposition 3.8** ***(The Difference Tensor)*.**
Let M be a smooth manifold with or without boundary. For any two connections ∇^0 and ∇^1 in TM, define a map $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by
$$D(X,Y) = \nabla^0_X Y - \nabla^1_X Y.$$
*Then D is **bilinear** over $\mathcal{C}^\infty(M)$, and thus defines a $(1,2)$**-tensor field** called **the difference tensor between ∇^0 and ∇^1**.*

- **Theorem 3.9** *Let M be a smooth manifold with or without boundary, and let ∇^0 be any connection in TM. Then **the set $\mathcal{A}(TM)$ of all connections** in TM is equal to the following **affine space**:*
$$\mathcal{A}(TM) = \left\{\nabla^0 + D : \; D \in \Gamma(T^{(1,2)}TM)\right\},$$
where $D \in \Gamma(T^{(1,2)}TM)$ is interpreted as a map from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$, and $\nabla^0 + D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by
$$(\nabla^0 + D)(X,Y) = \nabla^0_X Y + D(X,Y).$$

- **Remark** Finally we can define ***the covariant derivative of every** 1**-form*** ω based on connection on TM. In particular, ***the connection on** 1**-form*** $\nabla : \mathfrak{X}(M) \times \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ can be defined as
$$\begin{aligned} \langle \nabla_X \omega \,,\, Y\rangle &= \nabla_X \langle \omega \,,\, Y\rangle - \langle \omega \,,\, \nabla_X Y\rangle \\ \Rightarrow (\nabla_X \omega)(Y) &= X(\omega(Y)) - \omega(\nabla_X Y). \end{aligned} \tag{39}$$
where $\langle \omega \,,\, Y\rangle = \omega(Y)$ is *a natural pairing*. The coordinate representation of connection on 1-form is
$$\nabla_X \omega = \left(X(\omega_k) - X^j \omega_i \Gamma^i_{j,k}\right) \epsilon^k \tag{40}$$
where (ϵ^i) are coframes and $\omega = \omega_k\, \epsilon^k$, $X = X^i\, E_i$.

- **Remark** For a covariant 2-tensor field $g = g_{i,j} dx^i \otimes dx^j$, the covariant derivative of g in direction of Z is
$$(\nabla_{(Z)} g)(X,Y) = Z\left(g(X,Y)\right) - g\left(\nabla_Z X, Y\right) - g\left(X, \nabla_Z Y\right)$$

3.3 Total Covariant Derivatives

- **Proposition 3.10** ***(The Total Covariant Derivative)*.** *[Lee, 2018]*
Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every $F \in \Gamma(T^{(k,l)}TM)$, the map
$$\nabla F : \underbrace{\Omega^1(M) \times \ldots \times \Omega^1(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{l+1} \to \mathcal{C}^\infty(M)$$
given by
$$\nabla F\left(\omega_1, \ldots, \omega_k, Y_1, \ldots, Y_l, X\right) = \left(\nabla_X F\right)\left(\omega_1, \ldots, \omega_k, Y_1, \ldots, Y_l\right) \tag{41}$$
*defines a **smooth** $(k, l{+}1)$**-tensor field** on M called **the total covariant derivative of** F.*

- **Remark** The total covariant derivative of $Y \in \mathfrak{X}(M) := \Gamma(T^{(1,0)}TM)$ is a $(1,1)$-***tensor field***

$$\nabla Y(\omega, X) = (\nabla_X Y)(\omega) = \omega\left(\nabla_X Y\right).$$

Similarly, the total covariant derivative of $\omega \in \mathfrak{X}^*(M) = \Omega^1(M) = \Gamma(T^{(0,1)}TM)$ is a $(0,2)$-***tensor field***

$$\nabla\omega(Y, X) = (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega\left(\nabla_X Y\right)$$

- **Remark** It can be verified that the following formula for total covariant derivative holds

$$\nabla_Y F = \operatorname{tr}\left(\nabla F \otimes Y\right) \tag{42}$$

- **Definition** Given vector fields $X, Y \in \mathfrak{X}(M)$, let us introduce the notation $\nabla^2_{X,Y} F$ for the (k,l)-tensor field obtained by inserting X, Y in the last two slots of $\nabla^2 F = \nabla(\nabla F)$:

$$\nabla^2_{X,Y} F(\ \ldots\ ) = \nabla^2 F(\ldots, Y, X)$$

- **Proposition 3.11** *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth vector field or tensor field F,*

$$\nabla^2_{X,Y} F = \nabla_X\left(\nabla_Y F\right) - \nabla_{(\nabla_X Y)} F. \tag{43}$$

- **Example** (***The Covariant Hessian***).
Let u be a smooth function on M.
  - The ***total covariant derivative of a smooth function is equal to its*** 1***-form*** $\nabla u = du \in \Omega^1(M) = \Gamma(T^{(0,1)}TM)$ since

$$\nabla u(X) = \nabla_X u = Xu = du(X)$$

  - The 2-tensor $\nabla^2 u = \nabla(du)$ is called ***the covariant Hessian of*** u. Its action on smooth vector fields X, Y can be computed by the following formula:

$$\nabla^2 u(Y, X) = \nabla^2_{X,Y} u = \nabla_X \nabla_Y u - \nabla_{(\nabla_X Y)} u = X(Yu) - (\nabla_X Y)(u) \tag{44}$$

In any local coordinates, it is

$$\nabla^2 u = u_{\,;i,j}\, dx^i \otimes dx^j$$

where

$$u_{\,;i,j} = \frac{\partial}{\partial x^j}\frac{\partial u}{\partial x^i} - \Gamma^k_{j,i}\frac{\partial u}{\partial x^k}$$

3.4 Vector and Tensor Fields Along Curves

- **Definition** Let M be a smooth manifold with or without boundary. Given a smooth curve $\gamma : I \to M$, ***a vector field along*** γ is a *continuous* map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$; it is ***a smooth vector field along*** γ if it is *smooth* as a map from I to TM.

  We let $\mathfrak{X}(\gamma)$ denote ***the set of all smooth vector fields along*** γ. It is a *real vector space* under pointwise vector addition and multiplication by constants, and it is a module over $\mathcal{C}^\infty(I)$ with multiplication defined pointwise:

$$(fX)(t) = f(t)X(t).$$

- **Remark** (***Construction of A Smooth Vector Field Along the Curve***)
  Suppose $\gamma : I \to M$ is a smooth curve and $\widetilde{V} \in \mathfrak{X}(M)$ is a smooth vector field on an open subset of M containing the image of γ. The smooth vector field along the curve γ, $V = \widetilde{V} \circ \gamma$:

$$V(t) = \widetilde{V}_{\gamma(t)} \in T_{\gamma(t)}M.$$

  A smooth vector field along γ is said to be ***extendible*** if there *exists* a smooth vector field $\widetilde{V}$ on a neighborhood of the image of γ that is related to V in this way.

  Not every vector field along a curve need be extendible; for example, if $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$, then γ' is not extendible.

- **Definition** More generally, ***a tensor field along*** γ is a *continuous* map σ from I to some tensor bundle $T^{(k,l)}TM$ such that $\sigma(t) \in T^{(k,l)}T_{\gamma(t)}M$ for each $t \in I$.

  It is a ***smooth tensor field along*** γ if it is *smooth* as a map from I to $T^{(k,l)}TM$, and it is ***extendible*** if there is a smooth tensor field $\widetilde{\sigma}$ on a neighborhood of $\gamma(I)$ such that $\widetilde{\sigma} = \sigma \circ \gamma$.

- **Theorem 3.12** *(**Covariant Derivative Along a Curve**).*
  *Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For each smooth curve $\gamma : I \to M$, the **connection** determines **a unique operator***

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

  *called **the covariant derivative along** γ, satisfying the following properties:*

  1. *(**Linearity over** $\mathbb{R}$):*

$$D_t\,(a\,V + b\,W) = a\,D_t(V) + b\,D_t(W), \quad \textit{for } a, b \in \mathbb{R}.$$

  2. *(**Product Rule**):*

$$D_t(f\,V) = f'\,V + f D_t(V), \quad \textit{for } f \in \mathcal{C}^\infty(I).$$

  3. *If $V \in \mathfrak{X}(\gamma)$ is **extendible**, then for every extension $\widetilde{V}$ of V,*

$$D_t(V(t)) = \nabla_{\gamma'(t)}\widetilde{V}.$$

There is ***an analogous operator*** *on the space of* ***smooth tensor fields*** *of any type along* γ.

- **Remark** (***Coordinate Representation for Covariant Derivatives Along a Curve***)
Choose smooth coordinates (x^i) for M in a neighborhood of $\gamma(t_0)$, and write

$$V(t) = V^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

for t near t_0, where $V^1, \ldots, V^n$ are *smooth real-valued functions* defined on some neighborhood of t_0 in I. By the properties of D_t, since each $\frac{\partial}{\partial x^i}$ is extendible,

$$\begin{aligned} D_t\left(V_t\right) &= \dot{V}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + V^i(t)\,\nabla_{\gamma'(t)}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \\ &= \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\,\Gamma^k_{i,j}(\gamma(t))\right)\frac{\partial}{\partial x^k}\Big|_{\gamma(t)} \end{aligned} \tag{45}$$

3.5 Geodesics

- **Definition** Let M be a smooth manifold with or without boundary and let ∇ be a connection in TM. For every smooth curve $\gamma : I \to M$, we define the ***acceleration*** of γ to be ***the vector field*** $D_t(\gamma')$ ***along*** γ.

- **Definition** A smooth curve γ is called a ***geodesic*** (***with respect to*** ∇) if *its acceleration is zero*: $D_t(\gamma'(t)) = 0$.

- **Remark** ***Geodesic*** is the curve whose ***tangential acceleration*** is ***zero***. From *the connection* ∇ *point of view*, it specify both the directional vector field and the target vector field equal to $\gamma'(t)$. That is, the *tangential acceleration along a curve* γ is

$$\nabla_{\gamma'(t)}\gamma'(t).$$

- **Remark** (***The Ordinary Differential Equations for the Geodesic***)
In terms of smooth coordinates (x^i), if we write the component functions of γ as $\gamma(t) = (x^1(t), \ldots, x^n(t))$. From (45) and $D_t\left(\gamma'(t)\right)$, we have a set of ordinary differential equations called ***the geodesic equations***:

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\,\Gamma^k_{i,j}(x(t)) = 0, \quad k = 1, \ldots, n. \tag{46}$$

where $x(t) := (x^1(t), \ldots, x^n(t))$. A (parameterized) curve γ is a geodesic ***if and only if*** its component functions satiesfy *the geodesic equations*. This is a set of ***second-order nonlinear ODEs***.

- **Theorem 3.13** *(**Existence and Uniqueness of Geodesics**). [Lee, 2018]*
Let M be a smooth manifold and ∇ a connection in TM. For every $p \in M$, $w \in T_pM$, and $t_0 \in \mathbb{R}$, there ***exist*** *an open interval $I \subseteq \mathbb{R}$ containing t_0 and a* ***geodesic*** *$\gamma : I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = w$. Any two such geodesics* ***agree*** *on their common domain.*

- **Remark** From the geodesic equation, we see that ***the only parameters*** *of the ODE that determines the geodesic is* **the cooefficients of the connection** $\{\Gamma^k_{i,j}\}$. That is, *the geodesic* is solely determined by *the connection* ∇. Thus we also call it a ∇-***geodesic***.

- **Remark** The ***geodesic equation*** *under the initial boundary condition* can be written in the following form:

$$\dot{x}^k(t) = v^k(t) \tag{47}$$
$$\dot{v}^k(t) = -v^i(t)v^j(t)\,\Gamma_{i,j}^k(x(t)) \tag{48}$$

Treating $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ as coordinates on $U \times \mathbb{R}^n$, we can recognize (48) as the equations for the ***flow*** of ***the vector field*** $G \in \mathfrak{X}(U \times \mathbb{R}^n)$ given by

$$G_{(x,v)} = v^k \frac{\partial}{\partial x^k}\Big|_{(x,v)} - v^i v^j\,\Gamma_{i,j}^k(x)\frac{\partial}{\partial v^k}\Big|_{(x,v)}. \tag{49}$$

The importance of G stems from the fact that it actually defines ***a global vector field on the total space of*** TM, called ***the geodesic vector field***. It can be verified that the components of G under a change of coordinates *take the same form* in *every coordinate chart*.

Note that G acts on a function $f \in \mathcal{C}^\infty(U \times \mathbb{R}^n)$ as

$$Gf(p, v) = \frac{d}{dt}\Big|_{t=0} f(\gamma_v(t), \gamma_v'(t)). \tag{50}$$

- **Definition** A geodesic $\gamma : I \to M$ is said to be ***maximal*** if it *cannot be extended* to a geodesic on a *larger interval*, that is, if there does not exist a geodesic $\widetilde{\gamma} : \widetilde{I} \to M$ defined on an interval $\widetilde{I}$ properly containing I and satisfying $\widetilde{\gamma}|_I = \gamma$.

  A ***geodesic segment*** is a geodesic whose domain is a ***compact interval***.

- **Corollary 3.14** *Let M be a smooth manifold and let ∇ be a connection in TM. For each $p \in M$ and $v \in T_pM$, there is a* ***unique maximal geodesic*** *$\gamma : I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing* 0.

- **Definition** The ***unique maximal geodesic*** γ with $\gamma(0) = p$ and $\gamma'(0) = v$ is often called simply ***the geodesic with initial point*** p ***and initial velocity*** v, and is denoted by γ_v. (Note that we can always find $p = \pi(v)$ where $\pi : TM \to M$ is the natural projection.)

3.6 Parallel Transport

- **Definition** Let M be a smooth manifold and let ∇ be a connection in TM. *A smooth vector or tensor field V along a smooth curve γ* is said to be ***parallel along*** γ *(with respect to ∇)* if $D_t(V) \equiv 0$.

- **Remark** A *geodesic* can be characterized as a curve whose ***velocity vector field is parallel along the curve***.

- **Remark** (***Coordinate Representation of Vector Field Parallel Along a Curve***) Given a smooth curve γ with a local coordinate representation $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, formula (45) shows that a vector field V is parallel along γ if and only if

$$\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\,\Gamma_{i,j}^k(\gamma(t)) = 0, \quad k = 1, , \ldots, n \tag{51}$$

This is a set of ***linear ordinary differential equations*** with respect to $(V^1(t), \ldots, V^n(t))$.

- **Theorem 3.15** ***(Existence and Uniqueness of Parallel Transport).***
Suppose M is a smooth manifold with or without boundary, and ∇ is a connection in TM. Given a smooth curve $\gamma : I \to M$, $t_0 \in I$, and a vector $v \in T_{\gamma(t_0)}M$ or tensor $v \in T^{(k,l)}T_{\gamma(t_0)}M$, there exists a ***unique parallel vector or tensor field*** *V along γ such that $V(t_0) = v$.*

- **Remark** *The vector or tensor field* whose existence and uniqueness are proved in Theorem aboive is called ***the parallel transport of v along γ***.

- **Definition** For each $t_0, t_1 \in I$, we define a map
$$P^{\gamma}_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M, \tag{52}$$
called ***the parallel transport map***, by setting
$$P^{\gamma}_{t_0,t_1}(v) = V(t_1), \quad \forall v \in T_{\gamma(t_0)}M$$
where V *is the parallel transport of v along γ.*

  This map is ***linear***, because *the equation of parallelism is linear.* It is in fact an ***isomorphism***, because $P^{\gamma}_{t_1,t_0}$ is an ***inverse*** for it.

- **Remark** (***Parallel Frames Along a Curve***)
Given any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)}M$, we can ***parallel transport the vectors b_i along*** γ, thus obtaining an n-tuple of *parallel vector fields* $(E_1, \ldots, E_n)$ along γ. Because each parallel transport map is an *isomorphism*, ***the vectors*** $(E_i(t))$ ***form a basis for*** $T_{\gamma(t)}M$ ***at each point*** $\gamma(t)$. Such an n-tuple of vector fields along γ is called ***a parallel frame along*** γ.

  Every smooth (or piecewise smooth) vector field along γ can be expressed in terms of such a frame as
$$V(t) = V^i(t)\, E_i(t),$$
and then the properties of covariant derivatives along curves, together with the fact that the E_i's are parallel, imply
$$D_t(V_t) = \dot{V}^i(t)\, E_i(t) \tag{53}$$
wherever V and γ are smooth. This means that *a vector field is* ***parallel*** *along γ if and only if* ***its component functions with respect to the frame*** (E_i) ***are constants***.

- **Theorem 3.16** ***(Parallel Transport Determines Covariant Differentiation).*** *[Lee, 2018]*
Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose $\gamma : I \to M$ is a smooth curve and V is a smooth vector field along γ. For each $t_0 \in I$,
$$D_t V(t_0) = \lim_{\Delta t \to 0} \frac{P^{\gamma}_{(t_0+\Delta t),t_0}(V(t_0 + \Delta t)) - V(t_0)}{\Delta t} \tag{54}$$

- **Corollary 3.17** ***(Parallel Transport Determines the Connection).*** *[Lee, 2018]*
Let M be a smooth manifold with or without boundary, and let ∇ be a connection in TM. Suppose X and Y are smooth vector fields on M. For every $p \in M$,
$$\nabla_X Y\big|_p = \lim_{t \to 0} \frac{P^{\gamma}_{t,0}(Y_{\gamma(t)}) - Y_p}{t}, \tag{55}$$
where $\gamma : I \to M$ is any smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$.

- **Remark** See similarity between (55) and the definition of Lie derivatives:

$$(\mathscr{L}_X Y)_p = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)} \left(Y_{\theta_t(p)}\right) - Y_p}{t},$$

where θ is the ***flow of*** X in the neighborhood of p such that $\theta_0(p) = p$, $(\theta^{(p)})'(0) = X_p$.

- **Remark** A smooth vector or tensor field on M is said to be ***parallel*** *(with respect to* ∇*)* if it is *parallel along* ***every smooth curve*** *in* M.

- **Proposition 3.18** *Suppose M is a smooth manifold with or without boundary, ∇ is a connection in TM, and A is a* ***smooth vector or tensor field*** *on M. Then A is parallel on M if and only if $\nabla A \equiv 0$.*

- **Remark** It is always possible to extend a vector at a point to a parallel vector field along any given curve. However, it may not be possible in general to extend it to a ***parallel vector field*** on an open subset of the manifold. The impossibility of finding such extensions is intimately connected with the phenomenon of ***curvature***.

- **Remark** We see that both the concept of ***connections*** and the concept of ***parallel transport along a curve*** can be derived from each other.

$$\nabla \rightleftharpoons P^{\gamma}_{t_0,t_1}$$

They both define a way that "*connects*" the tangent space T_pM at $p = \gamma(t_0)$ and the tangent space T_qM at $q = \gamma(t_0 + \Delta t)$ in p's close neighborhood. The former begins with ***a set of rules*** *for a mapping* and the latter begins with ***covariant derivatives along a curve***.

4 Lie Derivatives

- **Definition** Suppose M is a smooth manifold, V is a *smooth vector field* on M; and θ is the ***flow of*** V. For any smooth vector field W on M, define ***a rough vector field*** on M, *denoted by* $\mathscr{L}_V W$ and called the <u>***Lie derivative*** *of W with respect to V*</u>, by

$$\begin{aligned}(\mathscr{L}_V W)_p &= \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right) - W_p}{t} \\ &= \left.\frac{d}{dt}\right|_{t=0} d\left(\theta_{-t}\right)_{\theta_t(p)} \left(W_{\theta_t(p)}\right),\end{aligned} \tag{56}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighborhood of p, and θ_{-t} is the inverse of θ_t, so both $d(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right)$ and W_p are elements of T_pM.

- **Remark** If M has nonempty boundary, this definition of $\mathscr{L}_V W$ makes sense as long as V is tangent to ∂M so that its flow exists.

- **Lemma 4.1** *Suppose M is a smooth manifold with or without boundary, and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume in addition that V is tangent to ∂M. Then $(\mathscr{L}_V W)_p$ exists for every $p \in M$, and $\mathscr{L}_V W$ is a* ***smooth vector field***.

- The following theorem is critical to understand the ***Lie derivatives*** and ***Lie bracket***.

Theorem 4.2 *If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathscr{L}_V W = [V, W]$.*

- **Remark** This theorem allows us to extend the definition of the ***Lie derivative*** to arbitrary *smooth vector fields* on a smooth manifold M with boundary. Given $V, W \in \mathfrak{X}(M)$ we define $(\mathscr{L}_V W)_p$ for $p \in \partial M$ by embedding M in a smooth manifold $\widetilde{M}$ without boundary (such as the double of M), extending V and W to smooth vector fields on $\widetilde{M}$, and computing the Lie derivative there. By virtue of the preceding theorem, $(\mathscr{L}_V W)_p = [V, W]_p$ is independent of the choice of extension.

- **Remark** This thoerem also gives us a ***geometric interpretation*** of *the Lie bracket of two vector fields*: it is the ***directional derivative*** *of the second vector field* along *the* ***flow*** *of the first*.

- **Corollary 4.3** *Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.*

  1. *(**Anti-symmetric**) $\mathscr{L}_V W = -\mathscr{L}_W V$.*
  2. *(**Product Rule**) $\mathscr{L}_V [W, X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$.*
  3. *$\mathscr{L}_{[V,W]} X = \mathscr{L}_V \mathscr{L}_W X - \mathscr{L}_W \mathscr{L}_V X$.*
  4. *If $g \in \mathcal{C}^\infty(M)$, then $\mathscr{L}_V (gW) = (Vg) W + g \mathscr{L}_V W$.*
  5. *(**Pushforward**) If $F : M \to N$ is a **diffeomorphism**, then $F_*(\mathscr{L}_V X) = \mathscr{L}_{F_* V} F_* X$.*

- **Remark** Note that the Lie derivative is ***not linear over $\mathcal{C}^\infty(M)$ in** V*, i.e.

$$\mathscr{L}_{fV} W \neq f \mathscr{L}_V W$$

- **Remark** If V and W are vector fields on M and θ is the flow of V, the Lie derivative $(\mathscr{L}_V W)_p$, by definition, expresses the t-derivative of *the* ***time-dependent vector*** $d\,(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right) \in T_pM$ at $t = 0$. The next proposition shows how it can also be used to compute the derivative of this expression at other times.

- **Proposition 4.4** *Suppose M is a smooth manifold with or without boundary and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \emptyset$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any (t_0, p) in the domain of θ,*

$$\left.\frac{d}{dt}\right|_{t=t_0} d\,(\theta_{-t})_{\theta_t(p)} \left(W_{\theta_t(p)}\right) = d(\theta_{-t_0}) \left((\mathscr{L}_V W)_{\theta_{t_0}(p)}\right). \tag{57}$$

5 Lie Group and Lie Algebra

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