Lecture 20: Curvature

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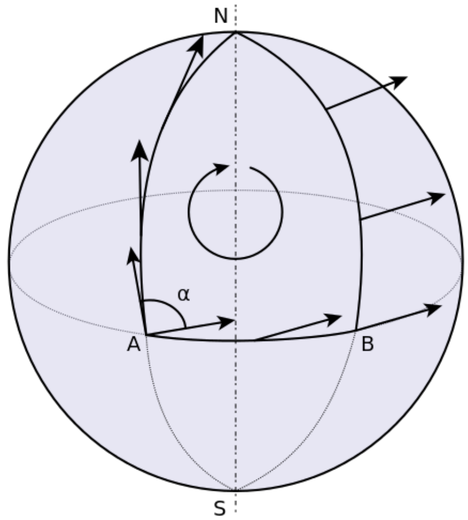

Figure 1: Result of parallel transport along the x^1-axis and the x^2-coordinate lines [Lee, 2018]

1 Local Invariants

- **Remark** For any geometric structure defined on smooth manifolds, it is of great interest to address ***the local equivalence question***: Are all examples of the structure locally equivalent to each other (under an appropriate notion of local equivalence)?

  The most important technique for proving that *two geometric structures* are ***not locally equivalent*** is to find ***local invariants***, which are quantities that must be preserved by local equivalences. In order to address the general problem of local equivalence of Riemannian or pseudo-Riemannian metrics, we will define a local invariant for all such metrics called ***curvature***.

  Initially, its definition will have nothing to do with ***the curvature of curves***, but later we will see that the two concepts are intimately related.

- **Remark** The *sphere* and the *plane* are *not locally isometric*. The key idea is that *every tangent vector* in the plane can be extended to a *parallel vector field*, so every Riemannian manifold that is *locally isometric to* $\mathbb{R}^2$ must have the same property locally.

- **Remark** Given a Riemannian 2-manifold M, here is one way to attempt to construct ***a parallel extension*** of a vector $z \in T_pM$ working in any smooth local coordinates (x^1, x^2) centered at p:

  1. first ***parallel transport*** z ***along the*** x^1***-axis***;

  2. then ***parallel transport*** the *resulting vectors* along the coordinate lines ***parallel to the*** x^2***-axis*** (Fig. 1).

  By construction, the resulting *vector field* Z *is* <u>***parallel along every*** x^2***-coordinate line***</u> *and* ***along*** *the* x^1*-axis.*

  The question is *whether this vector field is* <u>**parallel along** x^1**-coordinate lines**</u> *other than the* x^1*-axis*, or in other words, whether $\nabla_{\partial_1} Z \equiv 0$. Observe that $\nabla_{\partial_1} Z$ ***vanishes*** when $x^2 = 0$.

If we could show that

$$\nabla_{\partial_2}\nabla_{\partial_1}Z = 0 \tag{1}$$

then it would follow that $\nabla_{\partial_1}Z \equiv 0$, because ***the zero vector field*** is *the* ***unique*** *parallel transport of* ***zero*** along the x^2-curves. If we knew that

$$\nabla_{\partial_2}\nabla_{\partial_1}Z = \nabla_{\partial_1}\nabla_{\partial_2}Z \tag{2}$$

then (1) would follow immediately, because $\nabla_{\partial_2}Z \equiv 0$ everywhere by construction.

- **Remark** Let us look more closely at the quantity $\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z$ when X, Y, and Z are smooth vector fields.

$$\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_Y Z &= \overline{\nabla}_X(Y(Z^k)\partial_k) = X\left(Y^j\partial_j(Z^k)\right)\partial_k = XY(Z^k)\partial_k \\
\overline{\nabla}_Y\overline{\nabla}_X Z &= YX(Z^k)\partial_k \\
\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z &= (XY - YX)(Z^k)\partial_k = [X,Y](Z^k)\partial_k = \overline{\nabla}_{[X,Y]}Z \\
\Rightarrow \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z &= \overline{\nabla}_{[X,Y]}Z.
\end{aligned}$$

Recall that a Riemannian manifold is said to be ***flat*** if it is ***locally isometric*** to a ***Euclidean space***, that is, if every point has a neighborhood that is ***isometric*** to an open set in $\mathbb{R}^n$ with its ***Euclidean metric***.

We say that a ***connection*** ∇ on a smooth manifold M satisfies ***the flatness criterion*** if whenever X, Y, Z are smooth vector fields defined on an open subset of M, the following identity holds:

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = \nabla_{[X,Y]}Z \tag{3}$$

- **Remark** The geometric interpretation of the term $\nabla_X\nabla_Y Z$ is the *two-step process*:

  1. First, ***parallel transport*** of Z along *the* ***flow*** *of vector field* Y;
  2. Then, ***parallel transport*** of Z along *the* ***flow*** *of vector field* X

  Then the resulting vector field is $\nabla_X\nabla_Y Z$.

- **Proposition 1.1** *If (M,g) is a* ***flat*** *Riemannian or pseudo-Riemannian manifold, then its* ***Levi-Civita connection*** *satisfies* ***the flatness criterion****.*

2 The Curvature Tensor

2.1 Definitions

- **Definition** Let (M,g) be a Riemannian or pseudo-Riemannian manifold, and define a map $R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{4}$$

- The following proposition make sure this multilinear map defines a $(1,3)$-tensor field

Proposition 2.1 *The map R defined above is* ***multilinear*** *over $\mathcal{C}^{\infty}(M)$, and thus defines a $(1,3)$-**tensor field** on M.*

- **Definition** For each pair of vector fields $X, Y \in \mathfrak{X}(M)$, the map $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $Z \mapsto R(X,Y)Z$ is a ***smooth bundle endomorphism*** of TM, called ***the curvature endomorphism determined by X and Y***.

  The ***tensor field*** R itself is called ***the (Riemann) curvature endomorphism*** or the $(1,3)$-***curvature tensor***.

- **Remark** (***Coordinate Representation of the $(1,3)$-Curvature Tensor***)
  We adopt the convention that ***the last index is the contravariant (upper) one***. This is contrary to our default assumption that *covector arguments come first*. Thus, for example, *the curvature endomorphism* can be written in terms of local coordinates (x^i) as

$$R = R^{l}_{i,j,k}\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l},$$

  where the coefficients $R^{l}_{i,j,k}$ are defined by

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^{l}_{i,j,k}\,\frac{\partial}{\partial x^l}.$$

- **Remark** (***Understanding the Geometric Meaning of the $(1,3)$-Curvature Tensor***)
  The $(1,3)$-tensor $R(X,Y)Z$ describes *the* ***difference*** *of resulting* ***vector fields*** after ***parallel transporting*** vector field Z through ***two different routes***:

  1. First ***parallel transporting*** along ***the flow of*** Y, then ***parallel transporting*** along ***the flow of*** X, the resulting vector field is $\nabla_X \nabla_Y Z$;

  2. First ***parallel transporting*** along ***the flow of*** X, then ***parallel transporting*** along ***the flow of*** Y, the resulting vector field is $\nabla_Y \nabla_X Z$;

  The last term $\nabla_{[X,Y]}Z$ provides additional ***correction*** if X and Y are ***not orthorgonal***.

  Thus $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is ***close related to*** *the* ***angle*** *of these* ***two resulting vector fields***. If the surface is ***flat***, this angle should be ***zero*** since *the vector field* ***does not rotate*** during the transport and it is ***regardless of the path it takes***. On the other hand, if ***the surface bends***, then the vector field *will rotate* during the parallel transport and thus traversing through different paths will cause the vector field *points to different directions* in final destination, i.e. the angle is not zero.

- **Proposition 2.2** ***(The Riemann Curvature via Coefficients of Connection)*** *[Lee, 2018]*
  Let (M,g) be a Riemannian or pseudo-Riemannian manifold. In terms of any smooth local coordinates, the components of the $(1,3)$-curvature tensor are given by

$$R^{l}_{i,j,k} = \partial_i \Gamma^{l}_{j,k} - \partial_j \Gamma^{l}_{i,k} + \Gamma^{m}_{j,k}\Gamma^{l}_{i,m} - \Gamma^{m}_{i,k}\Gamma^{l}_{j,m}. \tag{5}$$

- **Remark** The curvature endomorphism also measures ***the failure*** *of* ***second covariant derivatives*** *along* ***families*** *of curves to* ***commute***. Given a smooth one-parameter *family of curves* $\Gamma : J \times I \to M$, recall that the velocity fields $\partial_t \Gamma(s,t) = (\Gamma_s)'(t)$ and $\partial_s \Gamma(s,t) = (\Gamma^{(t)})'(s)$ are smooth vector fields along Γ.

Proposition 2.3 *Suppose (M, g) is a smooth Riemannian or pseudo-Riemannian manifold and $\Gamma : J \times I \to M$ is a smooth one-parameter **family** of curves in M. Then for every smooth vector field V along Γ,*

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \partial_t \Gamma)V \tag{6}$$

- **Definition** We define the <u>***(Riemann) curvature tensor***</u> to be the <u>$(0, 4)$-*tensor field*</u> $Rm = R^\flat$ (also denoted by *Riem* by some authors) obtained from the $(1, 3)$-curvature tensor R by ***lowering its last index***. Its *action* on vector fields is given by

$$Rm(X, Y, Z, W) := \langle R(X, Y)Z \, , \, W \rangle_g \tag{7}$$

  This quanitity measures the angle between $R(X, Y)Z$ and W.

- **Remark** (***Coordinate Representation of the Riemann Curvature Tensor***)
  In terms of any smooth local coordinates, it is written

$$Rm = R_{i,j,k,l} \, dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

  where $R_{i,j,k,l} = g_{l,m} R^m_{i,j,k}$. We also see that

$$R_{i,j,k,l} = g_{l,m} \left(\partial_i \Gamma^m_{j,k} - \partial_j \Gamma^m_{i,k} + \Gamma^p_{j,k} \Gamma^m_{i,p} - \Gamma^p_{i,k} \Gamma^m_{j,p} \right). \tag{8}$$

- **Proposition 2.4** *The **curvature tensor** is a <u>**local isometry invariant**</u>: if (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian or pseudo-Riemannian manifolds and $\varphi : M \to \widetilde{M}$ is a local isometry, then $\varphi^* \widetilde{Rm} = Rm$.*

2.2 Flat Manifolds

2.3 Symmetries of the Curvature Tensor

3 Ricci and Scalar Curvatures

3.1 The Ricci Identities

3.2 Ricci and Scalar Curvatures

4 The Weyl Tensor

5 Curvatures of Conformally Related Metrics

References

John M Lee. *Introduction to Riemannian manifolds*, volume 176. Springer, 2018.