Summary: essential in differential geometry

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Contents

1 Concepts

1.1 Curves and Surfaces

- The definition of *regular parameterized curve* $\alpha(s)$ and definition of arc length $L(\alpha)$.
- The concept of *curvature* k, *torsion* τ with the tangent vector $\boldsymbol{t}$, *normal vector* $\boldsymbol{n}$, *binormal vector* $\boldsymbol{b} = \boldsymbol{t} \wedge \boldsymbol{n}$.
- The *Frenet trihedron* $(\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s))$ and a system of differential equations (*Frenet formula*)
- The concept of *osculating plane* $(\boldsymbol{t}, \boldsymbol{n})$, *normal plane* $(\boldsymbol{n}, \boldsymbol{b})$ and *rectifying plane* $(\boldsymbol{t}, \boldsymbol{b})$.

- The definition of a *regular* surface $\mathcal{S}$ in $\mathbb{R}^3$, *parameterization*, system of coordinates, coordinate neighborhood.
- The change of coordinates, Jacobian determinant and *diffeomorphism*.
- The definition of tangent space $T_p\mathcal{S}$ via a family of embedded curves $\{\alpha(s)\} \subset \mathcal{S}$ and the basis of tangent space $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ under the parameterization $\boldsymbol{x}(u, v)$ of p.
- The definition of the *differential* of $f : \mathcal{S}_1 \to \mathcal{S}_2$ on surface $\mathcal{S}_1$ as linear mapping $df : T_p\mathcal{S}_1 \to T_p\mathcal{S}_2$.
- The preimage of a map, *regular value*, *critical point*, critical value, regular point.

1.2 Vector fields

- The definition of (tangent) vector field on the surface and the differentiable vector field;
- The definition of trajectory of vector field and the solution of the system of first-order differential equations;
- The uniqueness and existence of the trajectory, given initial condition;
- The definition of the first integral of the vector field;
- The derivative of function relative to vector field; the vector field as an operator on space of smooth functions $\mathbb{C}^\infty(U)$

$$
\begin{aligned}
\boldsymbol{w}(u, v) &= a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v} \\
\boldsymbol{w}(f) &= \frac{d}{dt}(f \circ \alpha)\bigg|_{t=0} = a(u, v)\frac{\partial f}{\partial u} + b(u, v)\frac{\partial f}{\partial v}
\end{aligned}
$$

- The existence of local (orthogonal) parameterization given a set of independent vector fields or fields of directions
- The field of directions as the equivalence class of vector fields; the integral curve of a field of directions;
- The Lie bracket $[\boldsymbol{w}, \boldsymbol{v}] \equiv \boldsymbol{w}\,\boldsymbol{v} - \boldsymbol{v}\,\boldsymbol{w}$ as operator for $f\,g$ functions.

1.3 Local and Intrinsic geometry of surface

- The *Gauss map* $N : \mathcal{S} \to \mathbb{S}^2$ associates each point $p \in \mathcal{S}$, the *unit* normal vector $N(p)$ to the tangent plane $T_p\mathcal{S}$ as $N(p) = \boldsymbol{x}_u \wedge \boldsymbol{x}_v / \|\boldsymbol{x}_u \wedge \boldsymbol{x}_v\|$.
- The *shape operator* on $T_p\mathcal{S}$ as the differential of Gauss map $dN_p : T_p\mathcal{S} \to T_p\mathcal{S}$. Motivated as the rate of change of the unit normal vectors $N(p)$ along a given curve α. The shape operator dN_p is *self-adjoint*.

- The inner product $\langle \cdot \, , \, \cdot \rangle_p$ on tangent plane $T_p\mathcal{S}$.
- The concept of *isometry* as the property of transformation that preserving the inner product (length and angle).
- The *first fundamental form* of a regular surface $\mathcal{S} \subset \mathbb{R}^3$ at $p \in \mathcal{S}$ is defined as a quadratic form, $I_p : T_pS \to \mathbb{R}$ given by
$$I_p(\boldsymbol{w}) = \langle w \, , \, w \rangle_p = \|w\|_2^2 \geq 0 \;\; \boldsymbol{w} \in T_pS.$$
- The quadratic form Π_p defined in T_pS by $\Pi_p(\boldsymbol{v}) = -\langle dN_p(\boldsymbol{v}) \, , \, \boldsymbol{v} \rangle$ is called the *second fundamental form* of $\mathcal{S}$ at p, where dN_p is the differential of Gauss map at p, referred as the shape operator.
- The *Christoffel symbols* $\Gamma_{i,j}^k, \; i, j, k = 1, 2$ as the linear coefficients in representing the tangential component of the second-order derivatives of parameterization $\boldsymbol{x}_{uu}, \boldsymbol{x}_{uv}, \boldsymbol{x}_{vv}$ under the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ in $T_p\mathcal{S}$.
- The *Gaussian curvature* $\boldsymbol{K} = \det(dN_p)$ as the determinant of the shape operator.
- The quantity that only depends on the first fundamental form is *invariant* under isometry, or that is *intrinsic* to the local *geometry* of the surface.

- The intuition behind the coefficients of first and second fundamental form and the *Christoffel symbols*
  1. E, F, G are quantities related to the *first-order derivatives* of the coordinate curve (metric term in *unit* velocity field);
  2. e, f, g determines the *normal component of the second-order derivatives* of the coordinate curve along $\boldsymbol{N}$;
  3. The Christoffel symbols $\Gamma_{i,j}^k$ determines the projection of the second-order derivatives of the coordinate curve, or the derivative of the tangent vector field along each basis of the tangent space; that is, they determine the *tangential component of the second-order derivatives* of the coordinate curve. It is a function of E, F, G and its first derivatives.
  4. The Gaussian curvature by Gaussian formula is related to the third-order derivatives of the coordinate curve (i.e. the differential of the Christoffel symbol).
- The Christoffel symbols can be obtained via the coefficients of the first fundamental form and its derivatives.
- The Gauss Theorem states that the Gaussian curvature can be determined by the Christoffel

symbols and their derivatives, which is invariant under isometries.

- The concept of normal curvature k_n as the geometric interpretation of the second fundamental form.
- The concept of the principal curvature $k_{\max}, k_{\min}$ and principal directions and the notion like the curve of curvature.
- The classification of the local geometry at a point via the Gaussian curvature
  - Elliptic, if $\boldsymbol{K} = \det(dN_p) > 0$; e.g. sphere, All curves passing through an $\sim$ point have their normal vector pointing towards the same side of the tangent plane.
  - Hyperbolic, if $\boldsymbol{K} = \det(dN_p) < 0$; $\sim$ normal vector pointing towards the opposite side of the tangent plane.
  - Parabolic, if $\boldsymbol{K} = \det(dN_p) = 0$ but $dN_p \neq 0$; e.g. the cylinder, one of the principal curvature is nonzero.
  - Planar, if $dN_p = 0$. Note: may not be in a plane.

- The quantities that depend only on the intrinsic geometry of the surface
  1. The coefficients of the first and second fundamental form, E, F, G, e, f, g;
  2. The Christoffel symbols $\Gamma^k_{i,j}$;
  3. The Gaussian curvature $\boldsymbol{K}$;
  4. The covariant derivative $\nabla_{\boldsymbol{z}} \boldsymbol{w}$ and the notion of affine connection ∇;
  5. The trajectory of the geodesic γ via $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

1.4 Parallel transport and geodesic

- The definition of differentiable *vector field* $\boldsymbol{w}(p) = \sum_k w_k (\boldsymbol{e}_k)_p$ for $\boldsymbol{e}_k = \boldsymbol{x}_{\xi_k}$ in the *tangent bundle* $T\mathcal{S} = \bigcup_{p \in \mathcal{S}} \{p\} \times T_p\mathcal{S}$.
- In regular surface, the *covariant derivative* $\frac{D\boldsymbol{w}}{dt} \equiv \nabla_{\boldsymbol{v}} \boldsymbol{w}$ is seen as *tangential projection* of the Euclidean derivative of the field $\boldsymbol{w}$ along a curve α with $\boldsymbol{v} = \dot{\alpha}(0)$. The operator ∇ is a differential operator on the space of vector fields.
- An important characteristic when tangent space rolls on the surface: the origin (the contact point) will move above the surface, so the *basis vector* $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is a *smooth function* of parameterization.
- The covariant derivative satisfies a set of four properties for affine connections, invariant of inner product and symmetry of Christoffel symbols.
- The intuition: make a connection of vector field $\boldsymbol{w}$ in tangent space $T_p\mathcal{S}$ to $T_q\mathcal{S}$ of different points along a curve.

- The notion of *parallel vector field* along a curve, $\nabla_{\dot{\alpha}(0)}\boldsymbol{w} = 0$.
- The definition of *parallel transport* of vector field $\boldsymbol{w}(p)$ in $T_p\mathcal{S}$ to $\boldsymbol{w}(q)$ in $T_q\mathcal{S}$.
- Given a parallel unit vector field $\boldsymbol{v}$ along a curve, we can compute the infinitesimal parallel transport of the other unit vector field $\boldsymbol{w}$, $\nabla_{\dot{\alpha}(0)}\boldsymbol{w}$ via the differential of the angle from $\boldsymbol{v}$ to $\boldsymbol{w}$ along the curve.
- On surface, the covariant derivative equivalently define an affine connection and is the infinitesimal parallel transport.
- Given a Riemannian metric, one can determine a natural connection called *Levi-Civita connection.*

- Define $k_g = \langle d\dot{\alpha}(t)/dt\,,\, \boldsymbol{N} \wedge \dot{\alpha}\rangle$ is the *geodesic curvature* and $k^2 = k_n^2 + k_g^2$.
- The curve γ on $\mathcal{S}$ is said to be *geodesic* at $t \in I$ if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, its velocity field is parallel along the curve.
- The trajectory of geodesic is *uniquely* determined via a set of *second-order ordinary differential equations* with coefficients as the Christoffel symbols.
- The curve that *minimize* the arc length joins two point is the geodesic.

- Given a point p and the direction $\boldsymbol{v} = \dot{\alpha}$, at local neighborhood $U \ni p$, there exists a unique geodesic from p to any point $q \in U$ satisfies $\gamma(0) = p, \gamma(t_1) = q, \dot{\gamma}(0) = \boldsymbol{v}$.
- The definition of *exponential map* at p as $\exp_p : T_p\mathcal{S} \to \mathcal{S}$ as $\exp_p(\boldsymbol{v}) = \gamma(1, \boldsymbol{v})$ and $\exp_p(0) = p$, where $\gamma(1, \boldsymbol{v}) = \gamma(\|\boldsymbol{v}\|, \boldsymbol{v}/\|v\|)$ is the point $\gamma(1)$ for the geodesic with initial value $\gamma(0) = p$ and $\dot{\gamma}(0) = \boldsymbol{v}$.
- The exponential map is a *diffeomorphism* in $B_\epsilon \subset T_p\mathcal{S}$, $(d\exp_p)_0(\boldsymbol{v}) = \boldsymbol{v}, \forall \boldsymbol{v} \in B_\epsilon$, and thus defines a parameterization on $\mathcal{S}$.
- The *geodesic polar coordinate* $\exp_p(\rho_0, \theta_0) = q$ with p as the origin is given by intersection of the *geodesic circle* $\exp_p(\rho_0, \theta(t))$ and the *radical geodesic* $\exp_p(\rho(t), \theta_0)$.
- The geodesic polar coordinate system is orthogonal with $E = 1, F = 0$, $\lim_{\rho\to 0} G = 0$, $\lim_{\rho\to 0}(\sqrt{G})_\rho = 1$.

1.5 The Gauss-Bonnet Theorem and the non-Euclidean geometry

- **Theorem 1.1** ***(Gauss-Bonnet Theorem (Local))*** *(**?**)*
  *Let $\boldsymbol{x} : U \to \mathcal{S}$ be an **isothermal parametrization** (i.e., $F = 0, E = G = \lambda^2(u, v)$) of an oriented surface $\mathcal{S}$, where $U \subset \mathcal{R}^2$ is **homeomorphic** to an **open disk** and $\boldsymbol{x}$ is compatible with the orientation of $\mathcal{S}$. Let $\mathcal{R} \subset \boldsymbol{x}(U)$ be a **simple region** of $\mathcal{S}$ and let $\alpha : I \to \mathcal{S}$ be such that $\partial\mathcal{R} = \alpha(I)$. Assume that α is **positively oriented**, parametrized by arc length s, and let $\alpha(s_0), \ldots, \alpha(s_k)$ and $\theta_0, \ldots, \theta_k$ be, respectively, the vertices and the **external angles** of*

α. Then

$$\sum_{i=1}^{k}\int_{s_i}^{s_{i+1}} k_g(s)ds + \iint_{\mathcal{R}} \boldsymbol{K}d\sigma + \sum_{i=1}^{k}\theta_i = 2\pi \tag{1}$$

*where $k_g(s)$ is the **geodesic curvature** of the regular arcs of α and $\boldsymbol{K}$ is the **Gaussian curvature** of $\mathcal{S}$.*

- **Remark** It is seen that the techniques used in the proof of this theorem may also be used to give an interpretation of the **Gaussian curvature** in terms of ***parallelism***.

  Let $\boldsymbol{x} : U \to \mathcal{S}$ be an **isothermal parametrization** (i.e., $F = 0, E = G = \lambda^2(u,v)$) at point $p \in \mathcal{S}$ and let $\mathcal{R} \subset \boldsymbol{x}(U)$ be a *simple* region *without vertices*, containing p in its interior. Let $\alpha : [0,1] \to \boldsymbol{x}(U)$ be a curve parametrized by arc length s such that the trace of α is the boundary of $\mathcal{R}$. Let $\boldsymbol{w}_0$ be a unit vector ***tangent*** to $\mathcal{S}$ at $\alpha(0)$ and let $\boldsymbol{w}(s)$, $s \in [0,1]$, be the ***parallel transport*** of $\boldsymbol{w}_0$ *along* α. By using representation of algebraic value in terms of E, F, G and the Gauss-Green theorem in the uv plane, we obtain

$$\begin{aligned} 0 &= \int_0^1 \frac{D\boldsymbol{w}}{ds}\,ds \\ &= \int_0^1 \frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right\}ds + \int_0^1 \frac{d\varphi}{ds}ds \\ &= -\iint_{\mathcal{R}} \boldsymbol{K}d\sigma + \varphi(1) - \varphi(0) \end{aligned}$$

  where $\varphi = \varphi(s)$ is a differentiable *determination* of the ***angle*** from $\boldsymbol{x}_u$ to $\boldsymbol{w}(s)$.

  It follows that $\varphi(1) - \varphi(0) = \Delta\varphi$ is given by

$$\Delta\varphi = \iint_{\mathcal{R}} \boldsymbol{K}d\sigma \tag{2}$$

  Now, $\Delta\varphi$ does not depend on the choice of $\boldsymbol{w}_0$, and it follows from the expression above that $\Delta\varphi$ does not depend on the choice of $\alpha(0)$ either. By taking the limit

$$\lim_{\mathcal{R}\to p}\frac{\Delta\varphi}{A(\mathcal{R})} = \boldsymbol{K}(p),$$

  where $A(\mathcal{R})$ denotes the ***area*** of the region $\mathcal{R}$, we obtain the desired interpretation of $\boldsymbol{K}$.

- Given a triangulation J of a regular region $\mathcal{R} \subset \mathcal{S}$ of a surface $\mathcal{S}$, we shall denote by F the *number of triangles (**faces**)*, by E *the number of sides (**edges**)*, and by V *the number of **vertices** of the triangulation.* The number

$$F - E + V = \chi$$

  is called ***the Euler-Poincaré characteristic of the triangulation***.

- **Proposition 1.2** *If $\mathcal{R} \subset \mathcal{S}$ is a regular region of a surface $\mathcal{S}$, the **the Euler-Poincaré characteristic** does **not** depend on the **triangulation** of $\mathcal{R}$. It is convenient, therefore, to denote it by $\chi(\mathcal{R})$*

- **Proposition 1.3** ***Every regular region** of a regular surface admits a **triangulation**.*

- **Theorem 1.4** *(**Gauss-Bonnet Theorem (Global)**) (?)*
  *Let $\mathcal{R} \subset \mathcal{S}$ be a **regular region** of an oriented surface and let $C_1, \ldots, C_n$ be the closed, simple, piece-wise regular curves which form the **boundary** $\partial\mathcal{R}$ of $\mathcal{R}$. Suppose that each C_i is positively oriented and let $\theta_0, \ldots, \theta_p$ be the set of **all external angles of the curves** $C_1, \ldots, C_n$. Then*

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) ds + \iint_{\mathcal{R}} \boldsymbol{K} d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi\, \chi(\mathcal{R}) \tag{3}$$

  where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i.

- **Proposition 1.5** *A **compact** surface of **positive curvature** is **homeomorphic** to a sphere.*

- **Proposition 1.6** *Let $\mathcal{S}$ be an orientable surface of **negative or zero curvature**. Then two geodesics γ_1 and γ_2 which start from a point $p \in \mathcal{S}$ **cannot meet again** at a point $q \in \mathcal{S}$ in such a way that the **traces** of γ_1 and γ_2 constitute the **boundary** of a simple region $\mathcal{R}$ of $\mathcal{S}$.*

- **Proposition 1.7** *If there exist two simple **closed geodesics** Γ_1 and Γ_2 on a **compact**, connected surface $\mathcal{S}$ of **positive curvature**, then Γ_1 and Γ_2 **intersect**.*

- **Remark** (***Non-Euclidean Geometry***)
  Let T be a **geodesic triangle** (that is, the sides of T are **geodesics**) in an oriented surface $\mathcal{S}$. Assume that Gauss curvature $\boldsymbol{K}$ does not change sign in T. Let $\theta_1, \theta_2, \theta_3$ be the **external angles** of T and let $\varphi_1 = \pi - \theta_1$, $\varphi_2 = \pi - \theta_2$, $\varphi_3 = \pi - \theta_3$ be its **interior angles**. By the Gauss-Bonnet theorem,

$$\iint_{T} \boldsymbol{K} d\sigma + \sum_{i=1}^{3} \theta_i = 2\pi$$

  Thus,

$$\iint_{T} \boldsymbol{K} d\sigma = 2\pi - \sum_{i=1}^{3} \theta_i = -\pi + \sum_{i=1}^{3} \varphi_i \tag{4}$$

  It follows that the sum of the interior angles, $\sum_{i=1}^{3} \varphi_i$, of a geodesic triangle is

  1. Equal to π if $\boldsymbol{K} = 0$. (i.e. **plane**)
  2. Greater than π if $\boldsymbol{K} > 0$. (i.e. **elliptic**)
  3. Smaller than π if $\boldsymbol{K} < 0$. (i.e. **hyperbolic**)

  Furthermore, the difference $\sum_{i=1}^{3} \varphi_i - \pi$ (the ***excess*** of T) is given precisely by $\iint_T \boldsymbol{K} d\sigma$. If $\boldsymbol{K} \neq 0$ on T, this is the ***area of image*** $N(T)$ of T by the ***Gauss map*** $N : \mathcal{S} \to \mathbb{S}^2$. This was the form in which Gauss himself stated his theorem:

  ***The excess of a geodesic triangle** T **is equal to the area of its spherical image** $N(T)$.*

  The above fact is related to a historical controversy about the possibility of proving Euclid's fifth axiom (the axiom of the parallels), from which it follows that the sum of the interior angles of any triangle is equal to π.

2 Formula

- The cross product (vector product) of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ under the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is denoted as $\boldsymbol{u} \wedge \boldsymbol{v}$ and computed as

$$\langle \boldsymbol{u} \wedge \boldsymbol{v}\,,\, \boldsymbol{w}\rangle = \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \equiv \det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \tag{5}$$

  and

$$\boldsymbol{u} \wedge \boldsymbol{v} \equiv \boldsymbol{u} \times \boldsymbol{v} \equiv \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \boldsymbol{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{e}_3 \tag{6}$$

- The Frenet trihedron $(\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s))$ and a system of differential equations (Frenet formula)

$$\begin{aligned} \boldsymbol{t}' &= k\,\boldsymbol{n} \\ \boldsymbol{n}' &= -k\,\boldsymbol{t} - \tau\,\boldsymbol{b} \\ \boldsymbol{b}' &= \tau\,\boldsymbol{n} \end{aligned} \tag{7}$$

- (Tangent vector via basis)
  For $\dot\alpha(0) \equiv \boldsymbol{w} \in T_pS$, for some $\alpha = \boldsymbol{x} \circ \boldsymbol{\beta}$, where $\boldsymbol{\beta}(t) = (u(t), v(t))$, with $\boldsymbol{\beta}(0) = q = \boldsymbol{x}^{-1}(p)$. Then

$$\dot\alpha(0) = \frac{d}{dt}(\boldsymbol{x} \circ \boldsymbol{\beta})(0) \tag{8}$$

$$= \boldsymbol{x}_u \dot u(0) + \boldsymbol{x}_v \dot v(0) \tag{9}$$

  Thus under the basis $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ of T_pS, the coordinate of $\boldsymbol{w}$ in T_pS is $(\dot u(0), \dot v(0))$, and $\boldsymbol{w}$ is the velocity of the curve α is represented as $(u(t), v(t))$ in parameterization $\boldsymbol{x}$ at $t = 0$.

- (Differential of map via basis)
  If $\boldsymbol{w} = (\dot u(0), \dot v(0))$ in $T_p(S_1)$, and $f(u,v) = (f_1(u,v), f_2(u,v))$, with $\alpha(t) = (u(t), v(t))$, then the tangent of $\beta = f \circ \alpha$ at $f(p)$ is given as

$$\dot\beta(0) = df_p(\boldsymbol{w}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{bmatrix} \begin{bmatrix} \dot u(0) \\ \dot v(0) \end{bmatrix} \tag{10}$$

  Thus df_p as a linear mapping under coordinates $(\boldsymbol{x}_u, \boldsymbol{x}_v)$ in T_pS is given as the matrix

$$df_p = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}.$$

- The inner product of $\boldsymbol{w} = (\dot u(0), \dot v(0)) \in T_p\mathcal{S}$ with itself gives

$$\begin{aligned} I_p(\boldsymbol{w}) &= \langle w\,,\, w\rangle_p = \|w\|_2^2 \\ &= E\,(\dot u(0))^2 + 2\,F\,(\dot u(0)\dot v(0)) + G\,(\dot v(0))^2 \end{aligned}$$

- The coefficients for the first and second fundamental form

$$
\begin{aligned}
E(u,v) &= \langle \boldsymbol{x}_u \,,\, \boldsymbol{x}_u \rangle \\
F(u,v) &= \langle \boldsymbol{x}_u \,,\, \boldsymbol{x}_v \rangle \\
G(u,v) &= \langle \boldsymbol{x}_v \,,\, \boldsymbol{x}_v \rangle \\
e(u,v) &= -\langle N_u \,,\, \boldsymbol{x}_u \rangle = \langle N \,,\, \boldsymbol{x}_{uu} \rangle \\
f(u,v) &= -\langle N_u \,,\, \boldsymbol{x}_v \rangle = \langle N \,,\, \boldsymbol{x}_{vu} \rangle = \langle N \,,\, \boldsymbol{x}_{uv} \rangle = -\langle N_v \,,\, \boldsymbol{x}_u \rangle \\
g(u,v) &= -\langle N_v \,,\, \boldsymbol{x}_v \rangle = \langle N \,,\, \boldsymbol{x}_{vv} \rangle
\end{aligned} \tag{11}
$$

- The equation for Gaussian curvature

$$
\boldsymbol{K} = \frac{eg - f^2}{EG - F^2} \tag{12}
$$

- The local basis of $\mathbb{R}^3$ given by the trihedron $(\boldsymbol{x}_u, \boldsymbol{x}_v, \boldsymbol{N})$

$$
\begin{aligned}
\frac{\partial \boldsymbol{x}_u}{\partial u} &= \boldsymbol{x}_{uu} = \Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\,N \\
\frac{\partial \boldsymbol{x}_u}{\partial v} &= \boldsymbol{x}_{uv} = \Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v + f\,N \\
\frac{\partial \boldsymbol{x}_v}{\partial u} &= \boldsymbol{x}_{vu} = \Gamma^1_{21}\boldsymbol{x}_u + \Gamma^2_{21}\boldsymbol{x}_v + f\,N \\
\frac{\partial \boldsymbol{x}_v}{\partial v} &= \boldsymbol{x}_{vv} = \Gamma^1_{22}\boldsymbol{x}_u + \Gamma^2_{22}\boldsymbol{x}_v + g\,N \\
\frac{\partial N}{\partial u} &= N_u = a_{11}\boldsymbol{x}_u + a_{21}\boldsymbol{x}_v \\
\frac{\partial N}{\partial v} &= N_v = a_{12}\boldsymbol{x}_u + a_{22}\boldsymbol{x}_v
\end{aligned} \tag{13}
$$

- (The equations of Christoffel symbols via first fundamental form)

$$
\begin{aligned}
&\begin{cases} \Gamma^1_{11}E + \Gamma^2_{11}F & = \langle \boldsymbol{x}_{uu} \,,\, \boldsymbol{x}_u \rangle \quad = \frac{1}{2}E_u \\ \Gamma^1_{11}F + \Gamma^2_{11}G & = \langle \boldsymbol{x}_{uu} \,,\, \boldsymbol{x}_v \rangle \quad = F_u - \frac{1}{2}E_v \end{cases} \\
&\begin{cases} \Gamma^1_{12}E + \Gamma^2_{12}F & = \langle \boldsymbol{x}_{uv} \,,\, \boldsymbol{x}_u \rangle \quad = \frac{1}{2}E_v \\ \Gamma^1_{12}F + \Gamma^2_{12}G & = \langle \boldsymbol{x}_{uv} \,,\, \boldsymbol{x}_v \rangle \quad = \frac{1}{2}G_u \end{cases} \\
&\begin{cases} \Gamma^1_{22}E + \Gamma^2_{22}F & = \langle \boldsymbol{x}_{vv} \,,\, \boldsymbol{x}_u \rangle \quad = F_v - \frac{1}{2}G_u \\ \Gamma^1_{22}F + \Gamma^2_{22}G & = \langle \boldsymbol{x}_{vv} \,,\, \boldsymbol{x}_v \rangle \quad = \frac{1}{2}G_v \end{cases}
\end{aligned} \tag{14}
$$

- By solving the equations $(\boldsymbol{x}_{vv})_u = (\boldsymbol{x}_{uv})_v$, $(\boldsymbol{x}_{uu})_v = (\boldsymbol{x}_{uv})_u$ and $\boldsymbol{N}_{uv} = \boldsymbol{N}_{vu}$, one obtain the following equations

$$
\begin{aligned}
\left(\Gamma^2_{12}\right)_u - \left(\Gamma^2_{11}\right)_v - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{12}\Gamma^2_{11} + \left(\Gamma^2_{12}\right)^2 &= -\boldsymbol{K}E \\
\left(\Gamma^1_{12}\right)_u - \left(\Gamma^1_{11}\right)_v - \Gamma^2_{11}\Gamma^1_{22} + \Gamma^2_{12}\Gamma^1_{12} &= \boldsymbol{K}F \\
e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} &= e_v - f_u \\
e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} &= f_v - g_u,
\end{aligned} \tag{15}
$$

where $\boldsymbol{K}$ is the Gaussian curvature shown in Gaussian theorem. The first two equations are called the *Gauss formula* and the last two equations are called the *Mainardi-Codazzi equations*. These four equations are known as the *compatibility equations of the theory of surfaces*.

- In surface in $\mathbb{R}^3$, it is an *affine connection*, satisfies the following properties: for $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{y}, \boldsymbol{z}$ the vector field in $U \subset \mathcal{S}$ and $f : U \to \mathbb{R}$ is a differentiable function in $\mathcal{S}$; $\nabla_{\boldsymbol{y}}(f)$ is the directional derivative of f in the direction of $\boldsymbol{y}$, λ, μ are real numbers,

  1. The affine property for vector field
  $$\nabla_{\boldsymbol{y}}(\lambda \boldsymbol{w} + \mu \boldsymbol{v}) = \lambda \nabla_{\boldsymbol{y}}(\boldsymbol{w}) + \mu \nabla_{\boldsymbol{y}}(\boldsymbol{v});$$
  $$\nabla_{\lambda \boldsymbol{y} + \mu \boldsymbol{z}}(\boldsymbol{w}) = \lambda \nabla_{\boldsymbol{y}}(\boldsymbol{w}) + \mu \nabla_{\boldsymbol{z}}(\boldsymbol{w})$$
  2. The Leibniz rule
  $$\nabla_{\boldsymbol{y}}(f\boldsymbol{w}) = \nabla_{\boldsymbol{y}}(f)\, \boldsymbol{w} + f \nabla_{\boldsymbol{y}}(\boldsymbol{w});$$
  $$\nabla_{f\boldsymbol{y}}(\boldsymbol{v}) = f \nabla_{\boldsymbol{y}}(\boldsymbol{v});$$
  3. The metric-preserving property
  $$\nabla_{\boldsymbol{y}}(\langle \boldsymbol{w}\,,\, \boldsymbol{v} \rangle) = \langle \nabla_{\boldsymbol{y}}(\boldsymbol{w})\,,\, \boldsymbol{v} \rangle + \langle \boldsymbol{w}\,,\, \nabla_{\boldsymbol{y}}(\boldsymbol{v}) \rangle;$$
  4. The symmetry property
  $$\nabla_{\boldsymbol{e}_i}(\boldsymbol{e}_j) = \nabla_{\boldsymbol{e}_j}(\boldsymbol{e}_i), \quad \boldsymbol{e}_i = \boldsymbol{x}_{\xi_i} \text{ for parameterization } \boldsymbol{x}(\xi_1, \ldots, \xi_m).$$

  The first two properties defines the *affine connection* in $U \subset \mathcal{S}$. The last two properties associate the connection with the Riemannian metric and guarantee that the Christoffel symbols are symmetric w.r.t. lower indices. These four properties defines the *unique* connections or covariant derivatives, and parallel transport, geodesic on the surface.

- For $\partial_k = \frac{\partial}{\partial \xi_k}$ as a differential operator and a basis vector field,
$$\nabla_i(\partial_j) \equiv \nabla_{\partial_i}(\partial_j) = \Gamma_{i,j}^k \partial_k$$

- Let $\boldsymbol{w} = \sum_k w_k \boldsymbol{e}_k$ and $\boldsymbol{v} = \sum_k v_k \boldsymbol{e}_k$ in $T_p\mathcal{S}$, then
$$\nabla_{\boldsymbol{v}} \boldsymbol{w} = \sum_k \left(\sum_i v_i (\partial_i w_k) + \sum_{i,j} v_i \Gamma_{i,j}^k w_j \right) \boldsymbol{e}_k$$
or in each component ($\partial_k = \frac{\partial}{\partial \xi_k} \equiv \boldsymbol{e}_k$)
$$(\nabla_{\boldsymbol{v}} \boldsymbol{w}) = v_i \left\{ (\partial_i w_k) + \Gamma_{i,j}^k w_j \right\} \partial_k,$$
where we ignore the summation over common indices i, j, k.

- Let $\boldsymbol{w}$ be a differentiable vector field of *unit* vectors along a parameterized curve $\alpha : I \to \mathcal{S}$ on an oriented surface $\mathcal{S}$. Since $\boldsymbol{w}(t), t \in I$ is a unit vector field, $d\boldsymbol{w}(t)/dt$ is normal to $\boldsymbol{w}(t)$, and therefore,

$$\frac{D\boldsymbol{w}}{dt} = \lambda \left(\boldsymbol{N} \wedge \boldsymbol{w}(t) \right),$$

where $\lambda = \lambda(t)$ denoted as $[D\boldsymbol{w}/dt]$, is called *the algebraic value* of the covariant derivative of $\boldsymbol{w}$ at p.

Note that $\lambda = [D\boldsymbol{w}/dt] = \langle d\boldsymbol{w}(t)/dt \, , \, \boldsymbol{N} \wedge \boldsymbol{w} \rangle$ and its sign depends on the orientation of the surface.

- Given the Riemannian metric as $\left\langle \boldsymbol{x}_{\xi_i} \, , \, \boldsymbol{x}_{\xi_j} \right\rangle = \boldsymbol{J}_{i,j}$, one can solve the Christoffel symbols as

$$\Gamma^k_{i,j} = \frac{1}{2} \sum_m \boldsymbol{J}^{k,m} \left\{ \frac{\partial \boldsymbol{J}_{j,m}}{\partial \xi_i} + \frac{\partial \boldsymbol{J}_{m,i}}{\partial \xi_j} - \frac{\partial \boldsymbol{J}_{i,j}}{\partial \xi_m} \right\}, \quad i, j, k = 1, 2 \tag{16}$$

where $\sum_m \boldsymbol{J}^{k,m} \boldsymbol{J}_{m,j} = \delta_k(j)$. Therefore, given the Riemannian metric in $\mathcal{S}$, there exist unique covariant derivative or connection in $\mathcal{S}$, which is called the *Levi-Civita connection* of *the Riemannian structure.*

- To find the trajectory of the geodesic γ under parameterization $\boldsymbol{x}(\xi_1(t), \ldots, \xi_n(t))$,

$$\frac{d^2 \xi^k}{dt^2} + \sum_{i,j \in \{1,\ldots,n\}} \Gamma^k_{i,j} \frac{d\xi^i}{dt} \frac{d\xi^j}{dt} = 0, \quad k = 1, \ldots, n. \tag{17}$$

Note that $\Gamma^k_{i,j}, \; i, j, k = 1, \ldots, n$ are functions of intrinsic coordinate functions $(\xi_1(t), \ldots, \xi_n(t))$.

- Consider the Gaussian curvature $\boldsymbol{K}(\rho, \theta)$ in a polar system. Since $E(\rho, \theta) = 1, F(\rho, \theta) = 0$, it means that the following *Gauss-Jacobi equation* is satisfied

$$\boldsymbol{K} = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}} \tag{18}$$

In other words, this is the differential equation for $\sqrt{G}(\rho, \theta)$ given the curvature $\boldsymbol{K}(\rho, \theta)$.

$$(\sqrt{G})_{\rho\rho} + \boldsymbol{K}\sqrt{G} = 0 \tag{19}$$

For constant $\boldsymbol{K}$, the equation (18) a linear differential equation of the second order with constant coefficients.

- The second covariant derivative $\nabla^2 f = \nabla \left(\nabla f \right)$ for a smooth function $f : \mathcal{S} \to \mathbb{R}$, and it gives as

$$\nabla^2 f = \left(\partial_i \partial_j f - \Gamma^k_{i,j} \partial_k f \right) \boldsymbol{e}_i \wedge \boldsymbol{e}_j \tag{20}$$

3 Important theorems

- **Proposition 3.1** *The differential $dN_p : T_pS \to T_pS$ of the Gauss map is a self-adjoint linear map, i.e. $\langle dN_p(\boldsymbol{w}_1)\,,\,\boldsymbol{w}_2\rangle = \langle \boldsymbol{w}_1\,,\,dN_p(\boldsymbol{w}_2)\rangle$ for $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ any two vectors in T_pS.*

- **Theorem 3.2** *(Meusnier)*
  All curves lying on a surface $\mathcal{S}$ and having at a given point $p \in \mathcal{S}$ the same tangent line have at this point the same normal curvatures.

- **Theorem 3.3** *(THEOREMA EGREGIUM, Gauss)*
  The Gaussian curvature $\boldsymbol{K}$ of a surface is invariant by local isometries.

- **Proposition 3.4** *Let $\boldsymbol{w}, \boldsymbol{v}$ be parallel vector fields along $\alpha : I \to \mathcal{S}$. Then $\langle \boldsymbol{w}(t)\,,\,\boldsymbol{v}(t)\rangle$ is constant. In particular, $|\boldsymbol{w}(t)|$ and $|\boldsymbol{v}(t)|$ are constant, and the angle between $\boldsymbol{w}$, and $\boldsymbol{v}$ is constant.*

- **Proposition 3.5** *Let $\alpha : I \to \mathcal{S}$ be a parameterized curve in $\mathcal{S}$ and let $\boldsymbol{w}_0 \in T_{\alpha(t_0)}S$, $t_0 \in I$. There exists a unique parallel vector field $\boldsymbol{w}(t)$ along $\alpha(t)$, with $\boldsymbol{w}(t_0) = \boldsymbol{w}_0$.*

- **Proposition 3.6** *Given a point $p \in \mathcal{S}$ and a vector $\boldsymbol{w} \in T_p(\mathcal{S}), \boldsymbol{w} \neq 0$, there exists an $\epsilon > 0$ and a unique parameterized geodesic $\gamma : (-\epsilon, \epsilon) \to \mathcal{S}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \boldsymbol{w}$.*

- **Lemma 3.7** *Let $\boldsymbol{v}, \boldsymbol{w}$ be two differentiable vector fields along the curve $\alpha : I \to \mathcal{S}$, with $|\boldsymbol{w}(t)| = |\boldsymbol{v}(t)| = 1$, $t \in I$. Then*
  $$\left[\frac{D\boldsymbol{w}}{dt}\right] - \left[\frac{D\boldsymbol{v}}{dt}\right] = \frac{d\phi}{dt},$$
  where ϕ is one of the differentiable determination of the angle from $\boldsymbol{v}$ to $\boldsymbol{w}$.

- **Proposition 3.8** *Let $\boldsymbol{x}(u, v)$ be an orthogonal parameterization ($F = 0$) of a neighborhood of an oriented surface $\mathcal{S}$, and $\boldsymbol{w}(t)$ be a differentiable vector field of unit vectors along the curve $\boldsymbol{x}(u(t), v(t))$. Then*
  $$\left[\frac{D\boldsymbol{w}}{dt}\right] = \frac{1}{2\sqrt{EG}}\left\{G_u \frac{dv}{dt} - E_v \frac{du}{dt}\right\} + \frac{d\phi}{dt},$$
  where $\phi(t)$ is the angle from $\boldsymbol{x}_u$ to $\boldsymbol{w}(t)$ in the given orientation.

- **Theorem 3.9** *(The dependence of the geodesic on its initial conditions). Given $p \in \mathcal{S}$, there exists an $\epsilon_1 > 0, \epsilon_2 > 0$ and a differentiable map*
  $$\gamma : (-\epsilon_2, \epsilon_2) \times B_{\epsilon_1} \to \mathcal{S}, \quad B_{\epsilon_1} \subset T_p\mathcal{S}$$
  such that for $\boldsymbol{v} \in B_{\epsilon_1}, \boldsymbol{v} \neq 0$, $t \in (-\epsilon_2, \epsilon_2)$, the curve $t \mapsto \gamma(t, \boldsymbol{v})$ is the geodesic of $\mathcal{S}$ with $\gamma(0, \boldsymbol{v}) = p$ and $\dot{\gamma}(0, \boldsymbol{v}) = \boldsymbol{v}$ and for $\boldsymbol{v} = 0$, then $\gamma(t, 0) = p$.

- **Proposition 3.10** *The exponential map $\exp_p : B_\epsilon \subset T_p\mathcal{S} \to \mathcal{S}$ is a diffeomorphism in a neighborhood $U \subset B_\epsilon$ of the origin 0 of $T_p\mathcal{S}$.*

- **Proposition 3.11** *Let* $\boldsymbol{x} : (U - \ell) \to (V - L)$ *be a system of geodesic polar coordinates* (ρ, θ)*. Then the coefficients* $E \equiv E(\rho, \theta)$*,* $F \equiv F(\rho, \theta)$ *and* $G \equiv G(\rho, \theta)$ *of the first fundamental form satisfies the conditions*

$$E = 1, \qquad F = 0, \qquad \lim_{\rho \to 0} G = 0, \qquad \lim_{\rho \to 0} (\sqrt{G})_\rho = 1.$$

- **Theorem 3.12** *(Minding). Any two regular surfaces with the same constant Gaussian curvature are locally isometric. More precisely, let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be two regular surfaces with the same constant curvature* $\boldsymbol{K}$*. Choose points* $p_1 \in \mathcal{S}_1$*,* $p_2 \in \mathcal{S}_2$*, and orthonormal basis* $\{\boldsymbol{e}_1, \boldsymbol{e}_2\} \in T_{p_1}\mathcal{S}_1$ *and* $\{\boldsymbol{f}_1, \boldsymbol{f}_2\} \in T_{p_2}\mathcal{S}_2$*. Then there exists neighborhood* V_1 *of* p_1*,* V_2 *of* p_2 *and an isometry* $\psi : V_1 \to V_2$ *such that* $d\psi(\boldsymbol{e}_1) = \boldsymbol{f}_1$ *and* $d\psi(\boldsymbol{e}_2) = \boldsymbol{f}_2$*.*

- **Proposition 3.13** *(The minimal arc length regular curve joins two points is the geodesic). Let* $\alpha : [0, t_1] \to \mathcal{S}$ *is a parameterized regular curve with parameter as the arc length. Suppose that the arc length of* α *between any two points* $t, \tau \in I$ *is smaller than or equal to the arc length of any regular parameterized curve joining* $\alpha(t), \alpha(\tau)$*. Then* α *is a geodesic.*

- **Proposition 3.14** *(The geodesic joins two points has the minimal arc length). Let* p *be a point on a surface* $\mathcal{S}$*. Then there exists a neighborhood* $W \subset \mathcal{S}$ *of* p *such that if* $\gamma : I \to W$ *is a parameterized geodesic with* $\gamma(0) = p$*,* $\gamma(t_1) = q$*,* $t_1 \in I$*, and* $\alpha : [0, t_1] \to \mathcal{S}$ *is a parameterized regular curve joining* p *and* q*, we have*

$$\ell_\gamma \leq \ell_\alpha,$$

  where ℓ_α *denotes the arc length of the curve* α*. Moreover, if* $\ell_\gamma = \ell_\alpha$*, then the trace of* γ *coincides with the trace of* α *between* p *and* q*.*

  Note the above proposition holds only locally. It is seen that two nonantipodal points of a sphere may be connected by two meridians of unequal lengths, and only the smallest one satisfies the property. That is, the geodesics, if sufficiently extended, may not be the shortest path between its end points.