Lecture 7: Bounded Operators

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Contents

1 Topologies of Bounded Operators

- **Definition** (***Uniform Operator Topology***)
  Let $\mathcal{L}(X,Y)$ be *the space of bounded linear operators* from *Banach space* X to *Banach space* Y. $\mathcal{L}(X,Y)$ is a *Banach space* with *norm*

  $$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

  The induced topology on $\mathcal{L}(X,Y)$ is called ***the uniform operator topology*** (or ***norm topology***).

- **Definition** (***Strong Operator Topology***)
  The strong operator topology is *the* ***weakest topology*** on $\mathcal{L}(X,Y)$ such that *the evaluation maps*

  $$E_x : \mathcal{L}(X,Y) \to Y$$

  given by $E_x(T) = Tx$ are ***continuous for all*** $x \in X$.

- **Remark** (***Multiplication Map***)
  Consider the ***multiplication map*** $\mathcal{L}(X,Y) \times \mathcal{L}(Y,Z) \to \mathcal{L}(X,Z)$

  $$(A,B) \mapsto BA$$

  1. *In* ***uniform operator topology***, the map is ***jointly continuous***.
  2. *In* ***strong operator topology***, the map is ***separately*** but ***not jointly continuous*** if X, Y, and Z are *infinite-dimensional*.

- **Definition** (***Weak Operator Topology***)
  The weak operator topology on $\mathcal{L}(X,Y)$ is *the* ***weakest topology*** such that *the evaluation maps*

  $$E_{x,f} : \mathcal{L}(X,Y) \to \mathbb{C}$$

  given by $E_{x,f}(T) = f(Tx)$ are all ***continuous for all*** $x \in X$***,*** $f \in Y^*$.

- **Remark** (***Neighborhood in the Bounded Operator Topologies***)
  1. In ***uniform operator topology***: *A* ***neighborhood basis*** at the *origin* is given by sets of the form

     $$\{S \in \mathcal{L}(X,Y) : \ \|S\| < \epsilon\}.$$

  2. In ***strong operator topology***: *A* ***neighborhood basis*** at the *origin* is given by sets of the form

     $$\{S \in \mathcal{L}(X,Y) : \ \|Sx_i\|_Y < \epsilon, \ i = 1, \ldots, n\}$$

     where $\{x_i\}_{i=1}^n$ is a ***finite*** *collection of elements* of X and ϵ is positive.

3. In ***weak operator topology***: *A* ***neighborhood basis*** at the *origin* is given by sets of the form

$$\{S \in \mathcal{L}(X,Y) : \ |f_j\,(Sx_i)| < \epsilon, \ i = 1, \ldots, n, \ j = 1, \ldots, m\}$$

where $\{x_i\}_{i=1}^{n}$ and $\{f_j\}_{j=1}^{m}$ are ***finite*** *families of elements of* X *and* Y^*, respectively.

- **Remark** (***Convergence in the Bounded Operator Topologies***)
Let $\mathcal{L}(X,Y)$ be *the space of bounded linear operators* from *Banach space* X to *Banach space* Y. $\{T_\alpha\}$ is a net of operators in $\mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X,Y)$.

1. $\{T_\alpha\}$ ***converges*** to T in ***uniform operator topology*** (i.e. ***norm topology***) if and only if

$$\|T_\alpha - T\| \to 0.$$

That is, $T_\alpha \to T$ in ***norm***.

2. $\{T_\alpha\}$ ***converges*** to an operator T in ***strong operator topology*** if and only if

$$\|T_\alpha x - Tx\|_Y \to 0, \quad \forall x \in X.$$

That is, $T_\alpha \xrightarrow{s} T$ or $(T_\alpha x)$ converges ***strongly in*** Y for ***every*** $x \in X$.

3. $\{T_\alpha\}$ ***converges*** to an operator T in ***weak operator topology*** if and only if

$$|f\,(T_\alpha x) - f\,(Tx)| \to 0, \quad \forall x \in X, \forall f \in Y^*.$$

That is, $T_\alpha \xrightarrow{w} T$ or $(T_\alpha x)$ converges ***weakly in*** Y for ***every*** $x \in X$.

- **Remark**

uniformly operator converg $\Rightarrow$ ***strongly operator converg*** $\Rightarrow$ ***weakly operator converg***

- **Remark** (***Weak Operator Topology*** *vs.* ***Weak Topology*** *on* $\mathcal{L}(X,Y)$)
We compare the weak operator topology and the weak topology on $\mathcal{L}(X,Y)$ where $\mathcal{L}(X,Y)$ is treated as Banach space:

1. ***The weak operator topology*** on $\mathcal{L}(X,Y)$ is *the weakest topology* such that

$$f\,(Tx) \ \text{ is } \textbf{\textit{continous}} \text{ w.r.t. } T, \ \textbf{\textit{for all}} \ x \in X, \ f \in Y^*$$

2. ***The weak topology*** on $\mathcal{L}(X,Y)$ is *the weakest topology* such that

$$F\,(T) \ \text{ is } \textbf{\textit{continous}} \text{ w.r.t. } T, \ \textbf{\textit{for all}} \ F \in (\mathcal{L}(X,Y))^*$$

- **Remark** In general, *the* ***weak*** *and* ***strong operator topologies*** on $\mathcal{L}(X,Y)$ will ***not be first-countable*** so that questions of *compactness*, *net convergence*, and *sequential convergence* are complicated.

- **Proposition 1.1** *(**Weakly Operator Convergence in Hilbert Space**) [Reed and Simon, 1980]*
*Let $\mathcal{L}(\mathcal{H})$ denote the bounded operators on a Hilbert space $\mathcal{H}$. Let T_n be a **sequence** of bounded operators and suppose that $\langle T_n x\,,\,y\rangle$ converges as $n \to \infty$ for each $x, y \in \mathcal{H}$. Then there exists $T \in \mathcal{L}(\mathcal{H})$ such that $T_n \xrightarrow{w} T$.*

- **Remark** (***Stronly Operator Convergence in Hilbert Space***) [Reed and Simon, 1980]
  If $T_n x$ converges for each $x \in \mathcal{H}$, then there exists $T \in \mathcal{L}(\mathcal{H})$ such that $T_n \xrightarrow{s} T$.

- **Definition** (***Kernel*** *and* ***Range*** *of Linear Operator*)
  Let $T \in \mathcal{L}(X, Y)$. The set of vectors $x \in X$ so that $Tx = 0$ is called the ***kernel* of T**; that is,

  $$Ker(T) := \{x \in X : Tx = 0\}.$$

  Note that $\text{Ker}(T) \subseteq X$ is a ***closed subspace*** of X.

  The set of vectors $y \in Y$ so that $y = Tx$ for some $x \in X$ is called the ***range* of T**; that is,

  $$Ran(T) := \{y \in Y : y = Tx\}.$$

  Note that $\text{Ran}(T) \subseteq Y$ is a ***subspace*** of Y, and Ran T may *not be closed*.

- **Example** Consider the bounded operators on ℓ^2.

  1. Let T_n be defined by

     $$T_n(\xi_1, \xi_2, \ldots) = \left(\frac{1}{n}\xi_1, \frac{1}{n}\xi_2, \ldots\right).$$

     Then $T_n \to 0$ ***uniformly***.

  2. Let S_n be defined by

     $$S_n(\xi_1, \xi_2, \ldots) = (\underbrace{0, \ldots, 0}_{n}, \xi_{n+1}, \xi_{n+2}, \ldots).$$

     Then $S_n \to 0$ ***strongly*** but ***not uniformly***.

  3. Let W_n be defined by

     $$W_n(\xi_1, \xi_2, \ldots) = (\underbrace{0, \ldots, 0}_{n}, \xi_1, \xi_2, \ldots).$$

     Then $W_n \to 0$ in the ***weak operator*** *topology* but ***not*** *in the* ***strong*** *or* ***uniform*** *topologies*.

2 The Spectrum

2.1 Finite Dimensional Case

- **Remark** (***Eigenvalues*** *of Linear Transformation in* ***Finite Dimensional Space***)
  If is a linear transformation on $\mathbb{C}^n$, then the ***eigenvalues*** of are the complex numbers λ such that the ***determinant*** (called ***the characteristic determinant***)

  $$\det(\lambda I - T) = 0.$$

  The set of such λ is called ***the spectrum of*** T. It can consist of ***at most*** *n points*, since $\det(\lambda I - T)$ is *a* ***polynomial*** *of degree n*, called ***the characteristic polynomial*** *of T*.

- **Remark** If λ is ***not an eigenvalue***, then $\lambda I - T$ ***has an inverse*** since

$$\det(\lambda I - T) \neq 0.$$

- **Proposition 2.1** *(**Invariance of Eigenvalue under Change of Basis**) [Kreyszig, 1989]*
*All matrices representing a given linear operator $T : X \to X$ on a **finite dimensional normed space** X relative to various bases for X have the **same eigenvalues**.*

- **Theorem 2.2** *(**The Existence of Eigenvalues**). [Kreyszig, 1989]*
*A linear operator on a **finite dimensional** complex normed space $X \neq \{0\}$ has **at least one eigenvalue**.*

2.2 Infinite Dimensional Case

- **Definition** (***Resolvent*** *and* ***Spectrum***)
Let $T \in \mathcal{L}(X)$. A complex number λ is said to be in ***the resolvent set*** $\rho(T)$ *of* T if

$$\lambda I - T$$

is a ***bijection*** with a ***bounded inverse***.

$$R_\lambda(T) := (\lambda I - T)^{-1}$$

is called ***the resolvent*** *of* T *at* λ. Note that $R_\lambda(T)$ is defined on $\mathrm{Ran}\,(\lambda I - T)$.

If $\lambda \notin \rho(T)$, then λ is said to be in the ***spectrum*** $\sigma(T)$ ***of*** T.

- **Remark** The name "***resolvent***" is appropriate, since $R_\lambda(T)$ helps to solve the equation $(\lambda I - T)\, x = y$. Thus, $x = (\lambda I - T)^{-1}\, y = R_\lambda(T) y$ provided $R_\lambda(T)$ exists.

- **Definition** (***Point Spectrum, Continuous Spectrum*** *and* ***Residual Spectrum***)
Let $T \in \mathcal{L}(X)$

  1. ***Point Spectrum***: An $x \neq 0$ which satisfies

  $$\begin{aligned} &Tx = \lambda x \\ \text{or }\; &(\lambda I - T)\, x = 0, \quad \text{for some } \lambda \in \mathbb{C} \end{aligned}$$

  is called an ***eigenvector*** *of* T; λ is called ***the corresponding eigenvalue***.

  If λ is an *eigenvalue*, then $(\lambda I - T)$ is ***not injective*** (i.e. $\mathrm{Ker}\,(\lambda I - T) \neq \{0\}$) so λ is *in the spectrum of* T. ***The set of all eigenvalues*** is called ***the point spectrum of*** T. It is denoted as $\sigma_p(T)$.

  2. ***Continuous Spectrum***: If λ is ***not an eigenvalue*** and if $\mathrm{Ran}\,(\lambda I - T)$ is ***dense*** but the resolvent $R_\lambda(T)$ is ***unbounded***, then λ is said to be in ***the continuous spectrum***. It is denoted as $\sigma_c(T)$.

  3. ***Residual Spectrum***: If λ is ***not an eigenvalue*** and if $\mathrm{Ran}\,(\lambda I - T)$ is ***not dense***, then λ is said to be in ***the residual spectrum***. It is denoted as $\sigma_r(T)$.

- **Remark** (***Pure Point Spectrum for Finite Dimensional Case***)
If X is ***finite dimensional*** *normed linear space*, $T \in \mathcal{L}(X)$ then $\sigma_c(T) = \sigma_r(T) = \emptyset$.

Table 1: Comparison between different subset of spectrums and resolvent set

	point spectrum $\sigma_p(T)$	***continuous spectrum*** $\sigma_c(T)$	***residual spectrum*** $\sigma_r(T)$	***resolvent set*** $\rho(T)$
$R_\lambda(T)$ ***exists***	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$R_\lambda(T)$ *is* ***bounded***	$\times$	$\times$	$-$	$\checkmark$
$R_\lambda(T)$ *is defined in a* ***dense*** *subset of* Y	$\times$	$\checkmark$	$\times$	$\checkmark$

- **Remark** (***Partition*** *of Complex Space* $\mathbb{C}$)
  All four sets above are disjoint and they forms a partition of $\mathbb{C}$

$$\begin{aligned}\mathbb{C} &= \rho(T) \cup \sigma(T) \\ &= \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).\end{aligned}$$

  We will prove this later.

- **Remark** (***Some Special Case***)
  1. If X ***finite dimensional***, $\mathbb{C} = \rho(T) \cup \sigma_p(T)$ since $\sigma_c(T) = \sigma_r(T) = \emptyset$.
  2. If $T \in \mathcal{L}(\mathcal{H})$ and T is ***self-adjoint***, $\mathbb{C} = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T)$ since $\sigma_r(T) = \emptyset$.
  3. If $T \in \mathcal{L}(\mathcal{H})$ and T is ***self-adjoint and compact***, $\mathbb{C} = \rho(T) \cup \sigma_p(T)$

- **Remark** If X is a function space, the *eigenvectors* of *linear operator* T is called the ***eigenfunctions*** of T.

- **Definition** (***Eigenspace of Linear Operator***)
  The subspace of domain $D(T)$ consisting of $\{0\}$ and ***all eigenvectors*** of T corresponding to *an eigenvalue* λ of T is called ***the eigenspace of*** T corresponding to that eigenvalue λ.

2.3 Spectrum of Bounded Linear Operator in Banach Space

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- **Definition** (***Spectral Radius of Linear Operator***)
  Let

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

  $r(T)$ is called ***the spectral radius of*** T.

- **Proposition 2.3** *(**Spectral Radius Calculation**) [Reed and Simon, 1980]*
  *Let X be a **Banach space**, $T \in \mathcal{L}(X)$. Then*

$$\lim_{n\to\infty} \|T^n\|^{1/n}$$

  exists and is equal to $r(T)$.

- **Theorem 2.4** *(**Spectrum and Resolvent of Adjoint**) (**Phillips**) [Reed and Simon, 1980]*
  *Let X be a **Banach space**, $T \in \mathcal{L}(X)$. Then*

$$\sigma(T) = \sigma(T') \;\; \textit{and} \;\; R_\lambda(T') = (R_\lambda(T))'.$$

- **Proposition 2.5** *(**Spectrum of Adjoint**) [Reed and Simon, 1980]*
  Let X be a Banach space and $T \in \mathcal{L}(X)$. Then,

  1. *If λ is in the **residual spectrum** of T, then λ is in the **point spectrum** of T'.*
  2. *If λ is in the **point spectrum** of T, then λ is in **either** the **point** or the **residual spectrum** of T'.*

2.4 Spectrum of Self-Adjoint Operator in Hilbert Space

- **Proposition 2.6** *(**Spectral Radius Calculation**) [Reed and Simon, 1980]*
  *Let X be a **Hilbert space**, $T \in \mathcal{L}(X)$ and T is **self-adjoint**. Then*

$$r(T) = \|T\|$$

- **Theorem 2.7** *(**Spectrum and Resolvent of Adjoint**) (**Phillips**) [Reed and Simon, 1980]*
  *If X is a **Hilbert space** and $T \in \mathcal{L}(X)$, then*

$$\sigma(T) = \sigma(T^*) \;\; \textit{and} \;\; R_\lambda(T^*) = (R_\lambda(T))^*.$$

- **Proposition 2.8** *(**Spectrum of Self-Adjoint Operator**) [Reed and Simon, 1980]*
  *Let be a **self-adjoint operator** on a **Hilbert space** $\mathcal{H}$. Then,*

  1. *T has **no residual spectrum**, i.e. $\sigma_r(T) = \emptyset$.*
  2. *$\sigma(T)$ is a subset of $\mathbb{R}$.*
  3. ***Eigenvectors** corresponding to **distinct eigenvalues** of T are **orthogonal**.*

- **Remark** (***Resemblence to Symmetric or Hermitian Matrix***)
  This property is the same as the *spectrum* for *symmetric* real matrix or *Hermitian matrix* in *finite dimensional case*. That is,

  1. *the **eigenvalues** of* symmetric real matrices *or* Hermitian matrices *are all **real-valued***;
  2. the ***eigenspaces*** corresponds to ***distinct eigenvalue***s are ***orthorgonal*** to each other.

2.5 Positive Semidefinite Operators and the Polar Decomposition

- **Definition** (***Positive-Semidefinite Operator***)
  Let $\mathcal{H}$ be a ***Hilbert space***. An operator $B \in \mathcal{L}(\mathcal{H})$ is called ***positive-semidefinite*** if

$$\langle Bx\,,\,x\rangle \geq 0 \text{ for all } x \in \mathcal{H}.$$

  We write $B \succeq 0$ if is *positive-semidefinite* and $B \succeq A$ if $(B - A) \succeq 0$.

Similarly, B is called ***positive-definite*** if

$$\langle Bx\,,\,x\rangle > 0 \text{ for all } x \neq 0 \in \mathcal{H}.$$

The *positive semidefinite operator* is sometimes called ***positive*** *operator.*

- **Proposition 2.9** *(**Positive Semi-Definiteness $\Rightarrow$ Self-Adjoint**) [Reed and Simon, 1980]*
Every (bounded) ***positive semidefinite*** *operator on a* ***complex Hilbert space*** *is* ***self-adjoint****.*

  Proof: Notice that $\langle Ax\,,\,x\rangle$ takes only real value, so

  $$\langle Ax\,,\,x\rangle = \overline{\langle Ax\,,\,x\rangle} = \langle x\,,\,Ax\rangle$$

  By *the polarization identity*,

  $$\langle Ax\,,\,y\rangle = \langle x\,,\,Ay\rangle$$

  if $\langle Ax\,,\,x\rangle = \langle x\,,\,Ax\rangle$ for all x. Thus, if A is positive, it is self-adjoint. ∎

- **Remark** (***Square Root of Positive Semidefinite Operator***)
For any $A \in \mathcal{L}(\mathcal{H})$ notice that ***the normal operator*** *is* ***positive semi-definite***

  $$A^*A \succeq 0$$

  since

  $$\langle A^*Ax\,,\,x\rangle = \|Ax\|^2 \geq 0.$$

  Just as $|z| = \sqrt{\bar{z}z}$, we want to find the modulus of a linear operator as

  $$|A| := \sqrt{A^*A}$$

  To show the square root of positive semidefinite operator makes sense, we have the following lemma

  Lemma 2.10 *The power series for $\sqrt{1-z}$ about zero converges* ***absolutely*** *for all complex numbers z satisfying $|z| \leq 1$.*

  Theorem 2.11 *(**Square Root Lemma**) [Reed and Simon, 1980]*
  Let $A \in \mathcal{L}(\mathcal{H})$ and $A \succeq 0$. Then there is a ***unique*** *$B \in \mathcal{L}(\mathcal{H})$ with $B \succeq 0$ and $B^2 = A$. Furthermore, B* ***commutes*** *with every bounded operator which commutes with A.*

- **Definition** For $A \in \mathcal{L}(\mathcal{H})$, we can define ***absolute value*** *of* A as the square root of its normal operation

  $$|A| := \sqrt{A^*A}$$

- **Remark** For $|\cdot|$ operation on linear operator A:

  1. $|\lambda A| = |\lambda|\,|A|$
  2. $|\cdot|$ is ***norm continuous*** on $\mathcal{L}(\mathcal{H})$

3. in general the following equations ***do not hold***

$$|AB| = |A|\,|B|\,, \quad |A| = |A^*|$$

- **Definition** (***Partial Isometry***)
An operator $U \in \mathcal{L}(\mathcal{H})$ is called an ***isometry*** if

$$\|Ux\| = \|x\|\,, \quad \text{all } x \in \mathcal{H}.$$

U is called a ***partial isometry*** if U is an *isometry* when ***restricted*** to the *closed subspace* $(\mathrm{Ker}(U))^{\perp}$.

- **Remark** (***Partial Isometry*** = ***Unitary*** (***Ker***$(U))^{\perp} \to$ ***Ran***(U))
If U is a ***partial isometry***, $\mathcal{H}$ can be written as

$$\mathcal{H} = (\mathrm{Ker}(U)) \oplus (\mathrm{Ker}(U))^{\perp}, \quad \mathcal{H} = (\mathrm{Ran}(U)) \oplus (\mathrm{Ran}(U))^{\perp}$$

and U is a ***unitary operator*** between $(\mathrm{Ker}(U))^{\perp}$, *the* ***initial subspace*** *of* U, and $\mathrm{Ran}(U)$, *the* ***final subspace*** of U.

Moreover, its *adjoint* is its *inverse*, $U^* = (U_{(\mathrm{Ker}(U))^{\perp}})^{-1} : \mathrm{Ran}(U) \to (\mathrm{Ker}(U))^{\perp}$.

- **Proposition 2.12** *(**Projection Operators by Partial Isometry**) [Reed and Simon, 1980]*
*Let U be a **partial isometry**. Then $P_i = U^*U$ and $P_f = UU^*$ are respectively the **projections** onto the **initial** and **final subspaces** of U, i.e.*

$$P_i := U^*U = P_{(Ker(U))^{\perp}}, \quad P_f := UU^* = P_{Ran(U)},$$

*Conversely, if $U \in \mathcal{L}(\mathcal{H})$ with U^*U and UU^* **projections**, then U is a **partial isometry**.*

- **Theorem 2.13** *(**Polar Decomposition**) [Reed and Simon, 1980]*
*Let A be a bounded linear operator on a **Hilbert space**. Then there is a **partial isometry** U such that*

$$A = U\,|A|$$

*U is **uniquely** determined by the condition that Ker(U) = Ker(A). Moreover, Ran(U) = $\overline{\text{Ran}(A)}$.*

3 Compact Operators

3.1 Definitions and Basic Properties

- **Definition** (***Kernel of Integral Operator***)
Consider the simple operator T_K, defined in $\mathcal{C}[0,1]$ by

$$(T_K f)(x) = \int_0^1 K(x,y) f(y) dy,$$

where the function $K(x,y)$ is *continuous* on the square $0 \le x, y \le 1$. T_K is called an ***integral kernel operator*** and $K(x,y)$ is called *the* ***kernel*** *of the integral operator* T_K.

- **Remark** (***Properties of Integral Kernel Operator***)
  We summary some important property of the integral kernel operator T_K:

  1. T_K is ***bounded linear operator*** on $\mathcal{C}[0,1]$.

  $$|(T_K f)(x)| \leq \left(\sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)|\right)\left(\sup_{y\in[0,1]} |f(y)|\right)$$
  $$\Rightarrow \|T_K f\|_\infty \leq \left(\sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)|\right)\|f\|_\infty$$

  2. For $K^*(x,y) := \overline{K(y,x)}$,

  $$(T_K)^* = T_{K^*}$$

  3. Let B_M denote the functions f in $\mathcal{C}[0,1]$ such that $\|f\|_\infty \leq M$, i.e. closed $\|\|_\infty$-ball in $\mathcal{C}[0,1]$

  $$B_M := \{f \in \mathcal{C}[0,1] : \|f\|_\infty \leq M\}$$

  The set of functions $T_K(B_M) := \{T_K f : f \in B_M\}$ is ***equicontinuous***.

  Proof: Since $K(x,y)$ is *continuous* on the *compact* set $[0,1]\times[0,1]$, $K(x,y)$ is *uniformly continuous*. Thus, given an $\epsilon > 0$, we can find $\delta > 0$ such that $|x - x'| < \delta$ implies $|K(x,y) - K(x',y)| < \epsilon$ for all $y \in [0,1]$. Thus, for all $f \in B_M$

  $$\left|(T_K f)(x) - (T_K f)(x')\right| \leq \left(\sup_{(x,y)\in[0,1]\times[0,1]} \left|K(x,y) - K(x',y)\right|\right)\|f\|_\infty$$
  $$\leq \epsilon M. \quad \blacksquare$$

  4. Moreover, $\underline{T_K(B_M) := \{T_K f : f \in B_M\}}$ <u>is ***precompact*** in $\mathcal{C}[0,1]$</u>, i.e. its closure $\overline{T_K(B_M)}$ is ***compact***. In other word, for every sequence $f_n \in B_M$, the *sequence* $T_K f_n$ has a ***convergent subsequence***.

  This follows from the fact that $T_K(B_M)$ is *equicontinuous* and *uniformly bounded* by $\|T_K\|\, M$. So by *the Ascoli's theorem*, we have the result.

  5. The *operator norm* of T_K is *bounded above* by the L^2 *norm* of kernel function K

  $$\|T_K\| \leq \|K\|_{L^2}$$

  6. The eigenfunctions of T_K $\{\varphi_n\}_{n=1}^\infty$ forms a complete orthonormal basis in $L^2(M,\mu)$.

  $$K(x,y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x)\overline{\varphi_n(y)}$$

  where λ_n is the eigenvalue corresponding to eigenfunction φ_n.

- **Definition** (***Compact Operator***)
  Let X and Y be *Banach spaces*. An operator $T \in \mathcal{L}(X, Y)$ is called ***compact*** (or ***completely continuous***) if T takes ***bounded sets*** in X into ***precompact sets*** in Y.

  Equivalently, T is ***compact*** if and only if for every ***bounded*** *sequence* $\{x_n\} \subseteq X$, $\{Tx_n\}$ has a ***subsequence convergent*** in Y.

- **Example** (***Finite Rank Operators***)
  Suppose that *the **range** of T is **finite dimensional**.* That is, every vector in the range of T can be written

$$Tx = \sum_{i=1}^{n} \alpha_i y_i,$$

  for some fixed family $\{y_i\}_{i=1}^{n}$ in Y. If x_n is any *bounded sequence* in X, the corresponding $\alpha_i^{(n)}$ are *bounded* since T is *bounded*. The usual subsequence trick allows one to extract a *convergent subsequence* from $\{Tx_n\}$ which proves that T is *compact*. $\blacksquare$

- An important property of the compact operator is

  Theorem 3.1 *(**Weakly Convergent** + **Compact Operator** = **Uniformly Convergent**)* *[Reed and Simon, 1980]*
  *A **compact** operator maps **weakly convergent** sequences into **norm convergent** sequences; i.e. if $T \in \mathcal{L}(X)$ is compact, then*

$$x_n \xrightarrow{w} x \quad \Rightarrow \quad Tx_n \xrightarrow{norm} Tx.$$

  *The converse holds true if X is **reflective**.*

- **Proposition 3.2** *[Reed and Simon, 1980]*
  *Let X and Y be **Banach spaces**, $T \in \mathcal{L}(X, Y)$.*

  1. *If $\{T_n\}$ are **compact** and $T_n \to T$ in the **norm topology**, then T is **compact**.*
  2. *T is **compact** if and only if T' is **compact**.*
  3. *If $S \in \mathcal{L}(Y, Z)$ with Z a Banach space and if T **or** S is **compact**, then ST is **compact**.*

- The proposition above shows that the space of compact operators on $\mathcal{H}$ is a ***closed subspace*** of $\mathcal{L}(\mathcal{H})$, thus it is *a Banach space too.*

  Definition (***Space of Compact Operators***)
  Now assume that $\mathcal{H}$ is a ***separable Hilbert space***. We denote *the Banach space of **compact operators*** on *a separable Hilbert space* by $\text{Com}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$.

- **Theorem 3.3** *(**Compact Operator Approximated by Finite Rank Operator**)[Reed and Simon, 1980]*
  *Let $\mathcal{H}$ be a **separable Hilbert space**. Then every **compact operator** on $\mathcal{H}$ is the **norm limit** of a sequence of operators of **finite rank**.*

3.2 The Spectrum of Compact Operator

- **Remark** (***Fredholm Alternative***)
  The basic principle which makes compact operators important is ***the Fredholm alternative***:

If A is ***compact***, then ***exactly one*** *of the following two statements holds true*:

1. $$A\varphi = \varphi \text{ has a solution;}$$

2. $$(I - A)^{-1} \text{ exists.}$$

From *the Fredhold alternative*, we see that if ***for any*** φ there is ***at most one*** ψ (***uniqueness*** *statement*) such that

$$(I - A)\,\psi = \varphi$$

then there is ***always exactly one*** (*i.e.* ***existence*** *statement*). That is, ***compactness*** *and* ***uniqueness*** *together imply* ***existence***.

- **Theorem 3.4** *(**Analytic Fredholm Theorem**) [Reed and Simon, 1980]*
  *Let D be an **open connected** subset of $\mathbb{C}$. Let $f : D \to \mathcal{L}(\mathcal{H})$ be an **analytic operator-valued function** such that $f(z)$ is **compact** for each $z \in D$. Then, either*

  1. *$(I - f(z))^{-1}$ exists for **no** $z \in D$; or*
  2. *$(I - f(z))^{-1}$ exists for **all** $z \in D \setminus S$ where S is a **discrete** subset of D (i.e. S is a set which has no limit points in D.) In this case, $(I - f(z))^{-1}$ is **meromorphic** in D, **analytic** in $D \setminus S$, the **residues** at the poles are **finite rank operators**, and if $z \in S$ then*

     $$f(z)\varphi = \varphi$$

     *has a **nonzero solution** in $\mathcal{H}$*

- **Corollary 3.5** *(**The Fredholm Alternative**) [Reed and Simon, 1980]*
  *If A is a **compact** operator on $\mathcal{H}$, then **either** $(I - A)^{-1}$ exists **or** $\varphi = \varphi$ has a solution.*

- **Theorem 3.6** *(**Riesz-Schauder Theorem**) [Reed and Simon, 1980]*
  *Let A be a **compact** operator on $\mathcal{H}$, then $\sigma(A)$ is a **discrete set** having **no limit points except** perhaps $\lambda = 0$.*

  *Further, any **nonzero** $\lambda \in \sigma(A)$ is an **eigenvalue** of **finite multiplicity** (i.e. the corresponding space of eigenvectors is **finite dimensional**).*

- **Remark** (***Compact*** *Operator has only* ***Nonzero Point Spectrum*** *with* ***Finite Dimensional Eigenspace***)
  Riesz-Schauder Theorem states that *the* ***spectrum*** *for* ***compact*** *operator on* ***Hilbert*** *space* consists of *only* the point spectrum besides $\lambda = 0$.

  Moreover, *the* ***eigenspace*** corresponding to *each* ***nonzero eigenvalue*** is *finite dimensional.*

- **Theorem 3.7** *(**The Hilbert-Schmidt Theorem**) [Reed and Simon, 1980]*
  *Let A be a **self-adjoint compact operator** on $\mathcal{H}$. Then, there is a **complete orthonormal basis**, $\{\phi_n\}_{n=1}^{\infty}$, for $\mathcal{H}$ so that*

  $$A\phi_n = \lambda_n \phi_n$$

  and $\lambda_n \to 0$ as $n \to \infty$.

- **Remark** (***Eigendecomposition*** *of Hilbert Space based on* ***Self-Adjoint Compact Operator***)
  In other word, given a self-adjoint compact operator A on $\mathcal{H}$, *the HIlbert space $\mathcal{H}$ is the direct sum of eigenspaces of A.*

$$\mathcal{H} = \bigoplus_{\lambda_n \in \sigma(A) \subset \mathbb{R}} \mathrm{Ker}\,(\lambda_n I - A)$$

  A ***self-adjoint compact operator*** on $\mathcal{H}$ is the closest counterpart of ***Hermitian matrix / Symmetric Real matrix*** in infinite dimensional space.

- **Theorem 3.8** ***(Canonical Form for Compact Operators)*** *[Reed and Simon, 1980]*
  Let A be a ***compact*** *operator on $\mathcal{H}$. Then there exist* ***(not necessarily complete) orthonormal sets*** *$\{\psi_n\}_{n=1}^{N}$ and $\{\phi_n\}_{n=1}^{N}$ and* ***positive real numbers*** *$\{\lambda_n\}_{n=1}^{N}$ with $\lambda_n \to 0$ so that*

$$A = \sum_{n=1}^{N} \lambda_n \langle \psi_n\,,\,\cdot \rangle\, \phi_n \tag{1}$$

  The sum in (1), which may be finite or infinite, ***converges in norm****. The numbers, $\{\lambda_n\}_{n=1}^{N}$, are called the* ***singular values of*** *A.*

- **Remark** (***SVD for Compact Operator***)
  Recall for *finite dimensional case*, *the* ***singular value decomposition (SVD)***

$$A = \sum_{n=1}^{N} \lambda_n \phi_n \psi_n^T.$$

  The *singular value decomposition* is a generalization for *the spectral decomposition* for *self-adjoint operator*. But it only exists for ***compact operator***.

3.3 The Trace Class

- We generalize the definition of *trace* of linear operator from finite dimensional space to infinite dimensional space:

  Definition (***Trace of Positive Semi-Definite Operator***)
  Let $\mathcal{H}$ be a ***separable Hilbert space***, $\{\phi_n\}_{n=1}^{\infty}$ an ***orthonormal basis*** Then for any ***positive semi-definite*** operator $A \in \mathcal{L}(\mathcal{H})$, we define

$$\mathrm{tr}\,(A) = \sum_{n=1}^{\infty} \langle A\phi_n\,,\,\phi_n \rangle$$

  The number $\mathrm{tr}\,(A)$ is called ***the trace of*** A.

- **Proposition 3.9** ***(Properties*** *of* ***Trace)*** *[Reed and Simon, 1980]*
  Let $\mathcal{H}$ be a separable Hilbert space, $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis. Then for any ***positive semi-definite*** *operator $A \in \mathcal{L}(\mathcal{H})$, its trace $\mathrm{tr}\,(A)$ as defined above is* ***independent*** *of the orthonormal basis chosen. The trace has the following properties:*

1. ***(Linearity):*** $\mathrm{tr}\,(A+B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)$.

2. ***(Positive Homogeneity):*** $\mathrm{tr}\,(\lambda A) = \lambda \mathrm{tr}\,(A)$ *for all* $\lambda \geq 0$.

3. ***(Unitary Invariance):*** $\mathrm{tr}\left(U\,A\,U^{-1}\right) = \mathrm{tr}\,(A)$ *for any* ***unitary*** *operator* U.

4. ***(Monotonicity):*** *if* $B \succeq A \succeq 0$, *then* $\mathrm{tr}\,(B) \geq \mathrm{tr}\,(A)$

- **Remark** (***Trace of General Linear Operator***)
Let $A \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator on separable Hilbert space. Instead of considering *the trace of* A, we consider *the trace of modulus of* A,

$$\mathrm{tr}\,(|A|) = \mathrm{tr}\left(\sqrt{A^*A}\right).$$

- **Definition** (***Trace Class***)
An operator $A \in \mathcal{L}(\mathcal{H})$ is called ***trace class*** if and only if

$$\mathrm{tr}\,(|A|) = \mathrm{tr}\left(\sqrt{A^*A}\right) < \infty.$$

The family of all trace class operators is denoted by $\mathcal{B}_1(\mathcal{H})$.

- The following lemma is used in proof of part 2 in next proposition

Lemma 3.10 *Every* $B \in \mathcal{L}(\mathcal{H})$ *can be written as a linear combination of* ***four unitary operators****.*

- **Proposition 3.11** *(**Space of Trace Class Operator**) [Reed and Simon, 1980]*
The family of all trace class operators $\mathcal{B}_1(\mathcal{H})$ *is a* ***$*$-ideal*** *in* $\mathcal{L}(\mathcal{H})$*, that is,*

1. $\mathcal{B}_1(\mathcal{H})$ *is a* ***vector space****.*

2. *(**Operator Multiplication**) If* $A \in \mathcal{B}_1(\mathcal{H})$ *and* $B \in \mathcal{L}(\mathcal{H})$*, then* $AB \in \mathcal{B}_1(\mathcal{H})$ *and* $BA \in \mathcal{B}_1(\mathcal{H})$*.*

3. *(**Adjoint**) If* $A \in \mathcal{B}_1(\mathcal{H})$ *then* $A^* \in \mathcal{B}_1(\mathcal{H})$*.*

- **Remark Definition** (***$*$-Algebra***)
An ***algebra*** $\mathcal{A}$ *over field* K is a K***-vector space*** together with a ***binary product*** $(a, b) \mapsto ab$ satisfying

1. $a(bc) = (ab)c$,

2. $\lambda(ab) = (\lambda a)b = a(\lambda b)$,

3. $a(b+c) = ab + ac$,

4. $(a+b)c = ac + bc$,

for all $a, b, c \in \mathcal{A}$ and $\lambda \in K$.

A ***$*$-algebra*** $\mathcal{A}$ is a ***algebra*** over $\mathbb{C}$ with a unary ***involution*** $* : a \mapsto a^*$ such that

1. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$,

2. $(ab)^* = b^* a^*$,

3. $(a^*)^* = a$,

for all $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$.

Example (***Hilbert Adjoint as $*$-Operation***)
For $\mathcal{L}(\mathcal{H})$, let the $*$-operation be *the **Hilbert adjoint***, i.e. $\langle Tx\,,\, y\rangle = \langle x\,,\, T^{*}y\rangle$ so $\mathcal{L}(\mathcal{H})$ *is a **$*$-algebra*** with *operator addition* and *operator multiplication*.

Definition (***Left Ideal***)
For an arbitrary ***ring*** $(R, +, \cdot)$, let $(R, +)$ be its ***additive group***. A *subset* I is called a ***left ideal*** of R if it is an *additive subgroup* of R that "*absorbs multiplication from the left by elements of* R"; that is, I is a *left ideal* if it satisfies the following two conditions:

1. $(I, +)$ is a *subgroup* of $(R, +)$,
2. For every $r \in R$ and every $x \in I$, the *product* rx is in I.

- **Proposition 3.12** *(**Norm of Trace Class**) [Reed and Simon, 1980]*
  Let $\|\cdot\|_1$ *be defined in* $\mathcal{B}_1(\mathcal{H})$ *by*

$$\|A\|_1 = \mathrm{tr}\left(|A|\right).$$

  Then $\mathcal{B}_1(\mathcal{H})$ *is a **Banach space** with norm* $\|\cdot\|_1$ *and*

$$\|A\| \leq \|A\|_1$$

- **Remark** $\mathcal{B}_1(\mathcal{H})$ is ***not closed*** under *the operator norm* $\|\cdot\|$ in $\mathcal{L}(\mathcal{H})$.

- **Proposition 3.13** *(**Compactness**) [Reed and Simon, 1980]*
  *Every $A \in \mathcal{B}_1(\mathcal{H})$ is **compact**. A **compact** operator A is in $\mathcal{B}_1(\mathcal{H})$ **if and only if***

$$\sum_{n=1}^{\infty} \lambda_n < \infty$$

  where $\{\lambda_n\}$ *are the **singular values** of* A.

- **Corollary 3.14** *(**Finite Rank Approximation**) [Reed and Simon, 1980]*
  The finite rank operators are $\|\cdot\|_1$***-dense*** *in* $\mathcal{B}_1(\mathcal{H})$.

- **Proposition 3.15** *[Reed and Simon, 1980]*
  If $A \in \mathcal{B}_1(\mathcal{H})$ *and* $\{\varphi_n\}_{n=1}^{\infty}$ *is **any** orthonormal basis, then*

$$\sum_{n=1}^{\infty} \langle A\phi_n\,,\, \phi_n\rangle$$

  *converges **absolutely** and the limit is **independent** of the choice of basis.*

3.4 Hilbert-Schmidt Operator

- **Definition** (***Hilbert-Schmidt Operator***)
  An operator $T \in \mathcal{L}(\mathcal{H})$ is called ***Hilbert-Schmidt*** if and only if

$$\mathrm{tr}\left(T^{*}T\right) < \infty.$$

  The family of all Hilbert-Schmidt operators is denoted by $\mathcal{B}_2(\mathcal{H})$ or $\mathcal{B}_{HS}(\mathcal{H})$.

- **Proposition 3.16** ***(Space of Hilbert-Schmidt Operator)*** *[Reed and Simon, 1980]*

  1. *The space of all Hilbert-Schmidt operators $\mathcal{B}_2(\mathcal{H})$ is a* ***$*$-ideal** in $\mathcal{L}(\mathcal{H})$,*

  2. ***(Inner Product)**: If $A, B \in \mathcal{B}_2(\mathcal{H})$, then for **any orthonormal basis** $\{\varphi_n\}_{n=1}^{\infty}$,*

  $$\sum_{n=1}^{\infty} \langle A^* B \varphi_n \,,\, \varphi_n \rangle$$

  *is **absolutely summable**, and its **limit**, denoted by $\langle A \,,\, B \rangle_{HS}$, is **independent** of the orthonormal basis chosen, i.e.*

  $$\langle A \,,\, B \rangle_{HS} = \mathrm{tr}\,(A^* B)$$

  3. *$\mathcal{B}_2(\mathcal{H})$ with inner product $\langle \cdot \,,\, \cdot \rangle_{HS}$ is a **Hilbert space**.*

  4. ***(Norm)**: Let $\|\cdot\|_2$ be defined in $\mathcal{B}_2(\mathcal{H})$ by*

  $$\|A\|_2 := \sqrt{\langle A \,,\, A \rangle}_{HS} = \sqrt{\mathrm{tr}\,(A^* A)}.$$

  Then

  $$\|A\| \leq \|A\|_2 \leq \|A\|_1\,, \quad \textit{and} \quad \|A\|_2 = \|A^*\|_2$$

  5. ***(Compactness)** Every $A \in \mathcal{B}_2(\mathcal{H})$ is **compact** and a **compact operator**, A, is in $\mathcal{B}_2(\mathcal{H})$ **if and only if***

  $$\sum_{n=1}^{\infty} \lambda_n^2 < \infty$$

  *where $\{\lambda_n\}$ are the **singular values** of A.*

  6. ***(Finite Rank Approximation)** The **finite rank operators** are $\|\cdot\|_2$**-dense** in $\mathcal{B}_2(\mathcal{H})$.*

  7. *$A \in \mathcal{B}_2(\mathcal{H})$ **if and only if***

  $$\{\|A\varphi_n\|\}_{n=1}^{\infty} \in \ell^2$$

  *for **some** orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$.*

  8. *$A \in \mathcal{B}_1(\mathcal{H})$ if and only if $A = BC$ with $B, C \in \mathcal{B}_2(\mathcal{H})$.*

  9. *$\mathcal{B}_2(\mathcal{H})$ is not $\|\cdot\|$-closed in $\mathcal{L}(\mathcal{H})$.*

- **Theorem 3.17** ***(Hilbert-Schmidt Operator of L^2 Space)*** *[Reed and Simon, 1980]*
  *Let (M, μ) be a **measure space** and $\mathcal{H} = L^2(M, \mu)$. Then $T \in \mathcal{L}(\mathcal{H})$ is **Hilbert-Schmidt if and only if** there is a function*

  $$K \in L^2(M \times M, \mu \otimes \mu)$$

  with

  $$(Tf)(x) = \int_M K(x, y) f(y) d\mu(y),$$

  Moreover,

  $$\|T\|_2^2 = \int_{M \times M} |K(x, y)|^2 \, d\mu(x) d\mu(y).$$

Proof: Let $K \in L^2(M \times M, \mu \otimes \mu)$ and let T_K be the associated integral operator. It is easy to see (Problem 25) that T_K is a well-defined operator on $\mathcal{H}$ and that

$$\|T_K\| \leq \|K\|_{L^2}$$

Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2(M, \mu)$. Then $\{\varphi_n(x)\overline{\varphi_m(y)}\}_{n,m=1}^{\infty}$ is an orthonormal base for $L^2(M \times M, \mu \otimes \mu)$ so

$$K = \sum_{n,m=1}^{\infty} \lambda_{n,m} \varphi_n(x) \overline{\varphi_m(y)}$$

Let

$$K_N = \sum_{n,m=1}^{N} \lambda_{n,m} \varphi_n(x) \overline{\varphi_m(y)}$$

Then each K_N is *the integral kernel* of *a finite rank operator*. In fact,

$$T_{K_N} = \sum_{n,m=1}^{N} \lambda_{n,m} \langle \varphi_m \,, \cdot \rangle \, \varphi_n.$$

Since $\|K_N - K\|_{L^2} \to 0$ as $N \to \infty$, by inequality above, we have $\|T_{K_N} - T_K\| \to 0$. Thus T_K is *compact*. In fact,

$$\mathrm{tr}\,(T_K^* T_K) = \sum_{n=1}^{\infty} \|T_K \varphi_n\|^2 = \sum_{n,m=1}^{\infty} |\lambda_{n,m}|^2 = \|K\|_{L^2}^2$$

Thus $T_K \in \mathcal{B}_2(\mathcal{H})$ and $\|T_K\|_2 = \|K\|_{L^2}$.

We have shown that the map $\mapsto A_K$ is an ***isometry*** of $L^2(M \times M, \mu \otimes \mu)$ into $\mathcal{B}_2(\mathcal{H})$, so its *range is* ***closed***. But ***the finite rank operators*** clearly come from *kernels* and since they are ***dense*** in $\mathcal{B}_2(\mathcal{H})$ the range of $\mapsto A_K$ is all of $\mathcal{B}_2(\mathcal{H})$. ∎

- **Remark** A ***Hilbert-Schmidt*** operator T on a ***square integrable*** *space* $L^2(M, \mu)$ is a ***integral kernel operator***.

  In other word, for $T \in \mathcal{L}(\mathcal{H})$, if $\mathrm{tr}\,(T^*T) < \infty$, then T is a ***compact operator***. If, in particular, $\mathcal{H} = L^2(M, \mu)$, then T can be written as the *integral kernel operator*

$$(Tf)(x) = \int_M K(x, y) f(y) d\mu(y),$$

- **Theorem 3.18** ***(Mercer's Theorem)*** *[Borthwick, 2020].*
  Suppose Ω is a ***compact domain*** *and T is a* ***positive Hilbert-Schmidt operator*** *on $L^2(\Omega)$. If the integral kernel $K(\cdot, \cdot)$ is* ***continuous*** *on $\Omega \times \Omega$, then the* ***eigenfunction*** *φ_k is* ***continuous*** *on Ω if $\lambda_k > 0$, and the expansion*

$$K(x, y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$$

  converges ***uniformly*** *on* ***compact*** *sets.*

3.5 Trace of Linear Operator

- **Definition** (***Trace***)
  The map $\text{tr} : \mathcal{B}_1(\mathcal{H}) \to \mathbb{C}$ given by

$$\text{tr}\,(A) = \sum_{n=1}^{\infty} \langle A\phi_n\,,\,\phi_n\rangle$$

  where $\{\phi_n\}_{n=1}^{\infty}$ is *any orthonormal basis* in $\mathcal{H}$ is called ***the trace***.

- **Remark** For $A \in \mathcal{B}_1(\mathcal{H})$, $\sum_{n=1}^{\infty} |\langle A\phi_n\,,\,\phi_n\rangle| < \infty$ for *any orthonormal basis* $\{\phi_n\}_{n=1}^{\infty}$.

- **Remark** (***Decomposition of Self-Adjoint operator***)
  For any $A \in \mathcal{L}(\mathcal{H})$ and A being self-adjoint,

$$A = A_+ - A_-$$

  where both A_+ and A_- are ***positive*** and $A_+A_- = 0$.

  Not surprisingly, $A \in \mathcal{B}_1(\mathcal{H})$ if and only if

$$\text{tr}\,(A_+) < \infty, \;\; \text{tr}\,(A_-) < \infty,$$

  and

$$\text{tr}\,(A) = \text{tr}\,(A_+) - \text{tr}\,(A_-)\,.$$

- Finally, we collect the property of trace for linear operators:

  Proposition 3.19 *(**Properties of Trace**) [Reed and Simon, 1980]*

  1. $\text{tr}\,(\cdot)$ *is linear.*
  2. $\text{tr}\,(A^*) = \overline{\text{tr}\,(A)}$.
  3. $\text{tr}\,(AB) = \text{tr}\,(BA)$ *if* $A \in \mathcal{B}_1(\mathcal{H})$ *and* $B \in \mathcal{L}(\mathcal{H})$.

- **Remark** If $A \in \mathcal{B}_1(\mathcal{H})$, the map

$$B \mapsto \text{tr}\,(AB)$$

  is a ***linear functional*** on $\mathcal{L}(\mathcal{H})$. We can also hold $B \in \mathcal{L}(\mathcal{H})$ *fixed* and obtain a ***linear functional*** on $\mathcal{B}_1(\mathcal{H})$ given by the map

$$A \mapsto \text{tr}\,(BA)\,.$$

  The set of these functionals is just ***the dual of*** $\mathcal{B}_1(\mathcal{H})$.

- **Proposition 3.20** *(**Dual Space of Compact Operators**) [Reed and Simon, 1980]*

  1. $\mathcal{B}_1(\mathcal{H}) = (Com(\mathcal{H}))^*$. *That is, the map* $A \mapsto \text{tr}\,(A\cdot)$ *is an* ***isometric isomorphism*** *of* $\mathcal{B}_1(\mathcal{H})$ *onto* $(Com(\mathcal{H}))^*$.
  2. $\mathcal{L}(\mathcal{H}) = (\mathcal{B}_1(\mathcal{H}))^*$. *That is, the map* $B \mapsto \text{tr}\,(B\cdot)$ *is an* ***isometric isomorphism*** *of* $\mathcal{L}(\mathcal{H})$ *onto* $(\mathcal{B}_1(\mathcal{H}))^*$.

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