Lecture 0: Summary of Topology (Part 2)

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Contents

1 Connectedness and Compactness

- **Remark** ***Connectedness*** and ***compactness*** are basic ***topological properties***. Both of them are defined based on a collection of open subsets.

  1. ***Connectedness*** is a ***global topological property***: a topological space is *connected* if it cannot be partitioned by two *disjoint nonempty open* subsets. *Connectedness* reveals the information of ***entire space*** *not just within a neighborhood*. Connectedness is ***compatible*** with the ***continuity*** of functions as it implies ***the intermediate value theorem***, which in turn, can be used to construct *inverse function*. Moreover, *connectedness* defines ***an equivalence relationship*** which allows a ***partition*** of the space into ***components***.

  2. ***Connectedness*** is a ***local-to-global topological property***: a topological space is *compact* if every open cover have a finite sub-cover. Using ***finite sub-cover***, ***local properties*** defined *within each neighborhood* can be ***generalized globally*** to entire space. Concept of functions that are closely related to compactness is ***the uniformly continuity*** and ***the maximum value theorem***. The compactness allows us to drop dependency on each individual point x.

  Compared to *connectedness*, ***compactness*** is usually a ***strong condition*** on the topological space.

1.1 Connected Spaces

1.1.1 Definitions

- **Definition** (***Separation*** *and* ***Connectedness***)
  Let X be a topological space. A ***separation*** of X is a pair U, V of ***disjoint nonempty open*** *subset*s of X whose union is X.

  The space X is said to be <u>***connected***</u> if there *does not exist a separation* of X.

- **Definition** Equivalently, X is ***connected*** if and only if the only subsets of X that are ***both open and closed*** are $\emptyset$ and X itself.

- **Remark** (***Proof of Connectedness***)
  As the definition suggests, the proof of connectedness is done ***by contradition***. One first assume that the set X has a ***seperation***; it can be separated into two ***disjoint nonempty open*** sets such that $X = A \cup B$. Then we proof by contradiction using ***existing connectedness conditions*** and the ***property of open subsets (basis, continuity etc.)***.

- **Remark** ***Connectedness*** is obviously a ***topological property***, since it is formulated entirely in terms of *the collection of open sets* of X.

  Said differently, if X is ***connected***, so is any space ***homeomorphic*** to X.

- **Lemma 1.1** *(Separation and Connected Subspace) [Munkres, 2000]*
  If Y is a ***subspace*** *of X, a* ***separation*** *of Y is a pair of disjoint nonempty sets A and B whose union is Y,* ***neither*** *of which contains a* ***limit point*** *of the other. The space Y is connected if there exists no separation of Y.*

- **Example** (***Indiscrete Topology is Connected***)
  Let X denote a two-point space in ***the indiscrete topology***. Obviously there is *no separation* of X, so X is *connected*.

- **Example** ($\mathbb{Q}$ ***is Not Connected***)
  The *rationals* $\mathbb{Q}$ are ***not connected***. Indeed, *the only connected subspaces* of $\mathbb{Q}$ are the *one-point sets*: If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose *an irrational number* a lying between p and q, and write Y as the union of the open sets

$$Y \cap (-\infty, a) \text{ and } Y \cap (a, +\infty).$$

- **Lemma 1.2** *If the sets C and D form a **separation** of X, and if Y is a **connected** subspace of X, then Y lies **entirely within** either C or D.*

- **Proposition 1.3** *(**Connectedness by Union**) [Munkres, 2000]*
  *The **union** of a collection of connected subspaces of X that **have a point in common** is connected.*

- **Proposition 1.4** *(**Connectedness by Closure**)[Munkres, 2000]*
  Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$, then B is also connected.

- **Remark** If B is formed by *adjoining* to the ***connected*** *subspace A some or all of its **limit points***, then B is connected.

- **Proposition 1.5** *(**Connectedness by Continuity**) [Munkres, 2000]*
  *The **image** of a connected space under a **continuous** map is connected.*

- **Proposition 1.6** *(**Connectedness by Finite Product**) [Munkres, 2000]*
  *A **finite** cartesian product of connected spaces is connected.*

- **Remark** Countable infinite product of connected spaces ***may not be connected***. It depends on the ***topology*** of the product space.

- **Example** ($\mathbb{R}^\omega$ ***is Not Connected under Box Topology***)
  Consider the cartesian product $\mathbb{R}^\omega$ in ***the box topology***. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of *all **bounded** sequences of real numbers*, and the set B of *all **unbounded** sequences*. These sets are ***disjoint***, and each is ***open*** in the box topology.

- **Example** ($\mathbb{R}^\omega$ ***is Connected under Product Topology***)
  Consider the cartesian product $\mathbb{R}^\omega$ in ***the product topology***. Let $\widetilde{\mathbb{R}}^n$ denote the ***subspace*** of $\mathbb{R}^\omega$ consisting of all sequences $x = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. The space $\widetilde{\mathbb{R}}^n$ is clearly ***homeomorphic*** to $\mathbb{R}^n$, so that it is ***connected***. It follows that the space $\mathbb{R}^\infty$ that is the ***union*** of the spaces $\widetilde{\mathbb{R}}^n$ is ***connected***, for these spaces have the point $0 = (0, 0, \ldots)$ in common. We show that the ***closure*** of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is ***connected*** as well.

1.1.2 Connected Subspaces of Real Line

- **Definition** (***Linear Continuum***)
  A ***simply ordered set*** L having *more than one element* is called a ***linear continuum*** if the following hold:

1. L has the ***least upper bound property***.

2. If $x < y$, there exists z such that $x < z < y$.

- **Proposition 1.7** *(**Linear Continuum is Connected**) [Munkres, 2000]*
*If L is a **linear continuum** in the **order topology**, then L is **connected**, and so are **intervals** and **rays** in L.*

- **Corollary 1.8** *(**$\mathbb{R}$ is Connected**)*
*The real line $\mathbb{R}$ is **connected** and so are **intervals** and **rays** in $\mathbb{R}$.*

- **Theorem 1.9** *(**Intermediate Value Theorem**). [Munkres, 2000]*
*Let $f : X \to Y$ be a **continuous** map, where X is a **connected** space and Y is an ordered set in the **order topology**. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there **exists** a point c of X such that $f(c) = r$.*

- **Definition** (***Path Connectedness***)
Given points x and y of the space X, a ***path*** in X from x to y is a *continuous map* $f : [a, b] \to X$ of some ***closed interval*** in the real line into X, such that $f(a) = x$ and $f(b) = y$.

  A space X is said to be ***path connected*** if ***every pair*** of points of X can be ***joined by a path*** in X.

- **Remark** It is easy to see that ***a path-connected space X is connected*** since $X = f([a, b])$ is the image of connected space under continuous function f. The *converse* is *not true*, i.e. connected $\not\Rightarrow$ path-connected.

- **Example** (***Punctured Euclidean Space $\mathbb{R}^n \setminus \{0\}$ is Path Connected***)
Define ***punctured euclidean space*** to be the space $\mathbb{R}^n \setminus \{0\}$, where 0 is the origin in $\mathbb{R}^n$. If $n > 1$, this space is ***path connected***: Given x and y different from 0, we can join x and y by the *straight-line path* between them if that path does not go through the origin. Otherwise, we can choose a point z *not on the line joining x and y*, and take the *broken-line path* from x to z, and then from z to y.

- **Example** (*Common Path-Connected Spaces*)
The following spaces are ***path-connected***:

  1. ***The unit ball*** $\mathbb{B}^n = \{x : \|x\| \le 1\}$ is *path-connected*;

  2. ***The unit sphere*** $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ by the equation $\mathbb{S}^{n-1} = \{x : \|x\| = 1\}$ is *path connected*. For the map $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ defined by $g(x) = x/\|x\|$ is *continuous* and *surjective*; and the continuous image of path connected space is path connected.

- **Example** ***The ordered square*** I_o^2 (i.e. $I \times I$ under *dictionary order topology*) is ***connected*** but ***not path connected***.

- **Example** ***The topologist's sine curve*** is defined as the ***closure*** $\bar{S}$ of the set
$$S = \{(x, \sin(1/x)) : 0 < x \le 1\}.$$
$\bar{S}$ is ***connected*** but ***not path-connected***.

- **Remark** Recall that a topological space X is

  - ***connected*** if there do not exist two *disjoint*, *nonempty*, *open* subsets of X whose union is X;

- ***path-connected*** if every pair of points in X can be ***joined by a path*** in X, and
- ***locally (path-)connected*** if X has a ***basis*** of ***(path-)connected open subsets***.

We have ***path-connected*** $\Rightarrow$ ***connected*** but ***connected*** $\not\Rightarrow$ ***locally-connected***.

1.1.3 Components and Local Connectedness

- Given an arbitrary space X, there is a natural way to ***break*** *it up into pieces* that are connected (or path connected).

  Definition (***Connected Component*** *as* ***Equivalence Class***)
  Given X, define an *equivalence relation* on X by setting $x \sim y$ if there is *a* ***connected subspace*** of X containing *both* x and y. The *equivalence classes* are called ***the components*** (or the ***connected components***) of X.

- **Proposition 1.10** *(Characterization of Connected Components)*
  The components of X are ***connected disjoint subspaces*** *of X whose union is X, such that each nonempty* ***connected*** *subspace of X* ***intersects only one*** *of them.*

- **Definition** (***Path Component***)
  We define another *equivalence relation* on the space X by defining $x \sim y$ if there is a *path* in X from x to y. The *equivalence classes* are called ***the path components*** *of X*.

- **Proposition 1.11** *(Characterization of Path Components)*
  The path components of X are ***path-connected disjoint subspaces*** *of X whose union is X, such that each nonempty* ***path-connected*** *subspace of X* ***intersects only one*** *of them.*

- **Example** Each connected component of $\mathbb{Q}$ in $\mathbb{R}$ consists of *a single point*. ***None*** of the components of $\mathbb{Q}$ are ***open*** in $\mathbb{Q}$.

- **Example** The "***topologists sine curve*** $\bar{S}$ of the preceding section is a space that has ***a single component*** (since it is *connected*) and ***two path components***. One path component is the curve S and the other is *the vertical interval* $V = 0 \times [-1, 1]$. Note that S is ***open*** in $\bar{S}$ but ***not closed***, while V is ***closed*** but ***not open***.

  If one forms a space from $\bar{S}$ by *deleting* all points of V having *rational second coordinate*, one obtains a space that has ***only one component*** but ***uncountably many path components***.

- **Remark** From the example of topologist's sine curve, we see that the *connectedness does not imply the path-connectedness* since ***neither*** *of* ***two path components*** *are* ***both*** *open and closed*. Note that the vertical line is the set of ***limit points*** of the curve $\sin(1/x)$ but not every sequence approaches to the vertical curve is convergent.

- **Example** See some of examples below:

  1. The intervals and rays in $\mathbb{R}$ are ***both connected and locally connected***.
  2. The subspace $[1, 0) \cup (0, 1]$ of $\mathbb{R}$ is ***not connected***, but it is ***locally connected***.
  3. The rationals $\mathbb{Q}$ are ***neither connected nor locally connected***.
  4. *The topologists sine curve* is ***connected*** but ***not locally connected***.

- **Proposition 1.12** *(**Characterization of Locally Connectedness**) [Munkres, 2000]*
  *A space X is locally connected **if and only if** for every **open** set U of X, each **component** of U is **open** in X.*

- **Proposition 1.13** *(**Characterization of Locally Path-Connectedness**) [Munkres, 2000]*
  *A space X is locally path connected **if and only if** for every **open** set U of X, each **path component** of U is **open** in X.*

- **Proposition 1.14** *(**Relationship between Components and Path Components**)*
  *If X is a topological space, each **path component** of X lies in a **component** of X. If X is **locally path connected**, then the **components** and **the path components** of X are the same.*

1.2 Compact Spaces

Remark (***Metric Space*** *and* ***Compact Hausdorff Space***)
Two of the most well-behaved classes of spaces to deal with in mathematics are ***the metrizable spaces*** and ***the compact Hausdorff spaces***.

1. ***Metrizable space*** (X, d):
   - ***subspace*** of *metrizable* space is *meterizable*;
   - ***compact subspace*** of *metric* space is ***bounded*** in that metric and is ***closed***;
   - *every metrizable space* is ***normal*** (T_4);
   - ***compactness*** = ***sequential compactness*** = ***limit point compactness***;
   - ***sequence lemma***: for $A \subset X$, $x \in \bar{A}$ if and only if there exists a squence of points in A that converges to x. ($\Rightarrow$ need X being metric space);
   - f is ***continuous*** at x if and only if $x_n \to x$ leads to $f(x_n) \to f(x)$ ($\Leftarrow$ part holds for metric space)
   - ***unform limit theorem***: If the *range* of f_n is a *metric space* and f_n are *continuous*, then $f_n \to f$ *uniformly* means that f is a *continuous* function.
   - ***unform continuity theorem***: if f is a *countinous* map between two *metric spaces*, and the domain is ***compact***, then f is ***uniformly continuous***.
   - every metric space is ***first-countable***.
2. ***Compact Hausdorff Space***:
   - ***subspace*** of *compact Hausdorff space* is *compact Hausdorff* if and only if it is ***closed***.
   - ***closed subspace*** of *compact* space is ***compact***;
   - ***compact subspace*** of *Hausdorff* space is ***closed***;
   - *compact Hausdorff space* X is ***normal*** (T_4), thus it is ***completely regular***;
   - ***arbitrary product*** of *compact (Hausdorff)* space is *compact (Hausdorff)*;
   - ***compactness*** $\Rightarrow$ ***sequential compactness***;

- ***compactness*** = ***net compactness***, i.e. every *net* has a convergence *subnet*;
- ***image*** of *compact* space under continuous map f is *compact*;
- ***continuous bijection*** between two *compact Hausdorff* spaces is a ***homemorphism*** (and is a ***closed map***);
- ***closed graph theorem***: f is ***continuous*** if and only if its ***graph*** is ***closed***;
- ***uncountability***: for *compact Hausdorff space*, if the space has *no isolated points*, then it is *uncountable*;
- if compact Hausdorff space is ***second-countable***, then it is ***metrizable***.

1.2.1 Definitions

- **Definition** (***Covering of Set*** *and* ***Open Covering of Topological Set***)
  A collection $\mathscr{A}$ *of subsets* of a space X is said to ***cover*** X, or to be *a* ***covering*** *of* X, if the union of the elements of $\mathscr{A}$ is equal to X.

  It is called an ***open covering of*** X if its elements are *open subsets* of X.
- **Definition** (***Compactness***)
  A topological space X is said to be ***compact*** if *every open covering* $\mathscr{A}$ of X contains a ***finite*** *subcollection* that also *covers* X.
- **Example** (*Compactness is a strong condition*)
  Consider the following examples that are *connected* by *not compact*:
  1. The ***real line*** $\mathbb{R}$ is ***not compact*** since the open covering $\mathscr{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ has no finite sub-covering.
  2. The ***half interval*** $(0, 1]$ is ***not compact*** since the open covering $\mathscr{A} = \{(1/n, 1] : n \in \mathbb{Z}_+\}$ has no finite sub-covering.
- **Example** (***Finite Set is Compact***)
  Any space X containing only ***finitely*** *many points* is necessarily ***compact***, because in this case *every open covering of* X *is finite.*
- **Example** The following *subspace* of $\mathbb{R}$ is ***compact*** but *not connected*:
  $$X = \{0\} \cup \{1/n : n \in \mathbb{Z}_+\}.$$
- **Definition** If Y is a subspace of X, a collection $\mathscr{A}$ of *subsets of* X is said to ***cover*** Y if the *union* of its elements *contains* Y.
- **Lemma 1.15** ***(Subspace Compactness)*** *[Munkres, 2000]*
  Let Y *be a subspace of* X*. Then* Y *is compact if and only if every covering of* Y *by sets* ***open in*** X *contains* a finite subcollection covering Y.
- **Remark** A ***compact subset*** of a topological space is one that is a compact space in the ***subspace topology***.
- **Proposition 1.16** ***(Compactness by Closed Subspace)*** *[Munkres, 2000]*
  Every ***closed subspace*** *of a compact space is compact.*

- **Proposition 1.17** ***(Compact Subspace + Hausdorff ⇒ Closedness)*** *[Munkres, 2000]*
  Every ***compact*** *subspace of a* ***Hausdorff*** *space is* ***closed****.*

- **Remark** (***Compactness ⇒ Closedness***)
  Since *the Hausdorff condition is mild*, we can safely say that being *compact* implies that being *closed*.

- **Exercise 1.18** ***(Compact Subspace in Metric Space)***
  Show that every ***compact subspace*** *of* ***a metric space*** *is* ***bounded*** *in that* ***metric*** *and is* ***closed****. Find a metric space in which not every closed bounded subspace is compact.*

- **Proposition 1.19** *If Y is a* ***compact subspace*** *of the* ***Hausdorff*** *space X and x_0 is not in Y, then there exist* ***disjoint open*** *sets U and V of X containing x_0 and Y, respectively.*

- **Remark** To prove *the compact subspace* is *closed*, one need the *Hausdorff condition.*

- **Proposition 1.20** ***(Compactness by Continuity)*** *[Munkres, 2000]*
  The ***image*** *of a* ***compact*** *space under a* ***continuous*** *map is compact.*

- **Theorem 1.21** ***(Closed Graph Theorem)*** *[Reed and Simon, 1980, Munkres, 2000]*
  Let $f : X \to Y$; let Y be ***<u>compact Hausdorff</u>****. Then f is* ***<u>continuous if and only if</u>*** *the* ***<u>graph</u>*** *of f,*

  $$G(f) = \{(x, f(x)) : x \in X\},$$

  is ***<u>closed</u>*** *in $X \times Y$.*

- **Theorem 1.22** ***(Homemorphism by Domain Compactness)****[Munkres, 2000]*
  Let $f : X \to Y$ be a ***bijective continuous*** *function. If X is* ***compact*** *and Y is* ***Hausdorff****, then f is a* ***homeomorphism****.*

- **Proposition 1.23** ***(Compactness by Finite Product)*** *[Munkres, 2000]*
  The product of ***finitely*** *many compact spaces is compact.*

  Lemma 1.24 ***(The Tube Lemma)****.[Munkres, 2000]*
  Consider the product space $X \times Y$, where Y is ***compact****. If N is an open set of $X \times Y$ containing the* ***slice*** *$x_0 \times Y$ of $X \times Y$, then N contains some* ***tube*** *$W \times Y$ about $x_0 \times Y$, where W is a* ***neighborhood*** *of x_0 in X.*

- **Remark** (***Compactness by Infinite Product***)
  Unlike *the connectedness property*, which *may not hold* for infinite product space, *the infinite product of compact space is indeed compact.* This is called ***the Tychonoff theorem***,

- To prove *compactness*, the following property is useful:

  Definition (***Finite Intersection Property***)
  A collection $\mathscr{C}$ of subsets of X is said to have ***<u>the finite intersection property</u>*** if for *every finite subcollection*

  $$\{C_1, \ldots, C_n\}$$

  of $\mathscr{C}$, the ***intersection*** $C_1 \cap \ldots \cap C_n$ is ***nonempty****.*

- **Proposition 1.25** ***(Equivalent Definition of Compactness)*** *[Munkres, 2000]*
  Let X be a topological space. Then X is ***compact if and only if*** *for every collection $\mathscr{C}$ of*

closed *sets in X having* ***the finite intersection property****, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is* ***nonempty****.*

Proof: Given a collection $\mathscr{A}$ of subsets of X, let

$$\mathscr{C} = \{X \setminus A : A \in \mathscr{A}\}$$

be the collection of their *complements*. Then the following statements hold:

1. $\mathscr{A}$ is a collection of open sets if and only if $\mathscr{C}$ is a collection of closed sets.
2. The collection $\mathscr{A}$ covers X if and only if the *intersection* $\bigcap_{C \in \mathscr{C}} C$ of all the elements of $\mathscr{C}$ is ***empty***.
3. The ***finite subcollection*** $\{A_1, \ldots, A_n\}$ of $\mathscr{A}$ covers X if and only if the ***intersection*** of the corresponding elements $C_i = X \setminus A_i$ of $\mathscr{C}$ is ***empty***.

The proof of the theorem now proceeds in two easy steps: taking the ***contrapositive*** (of the theorem), and then the ***complement*** (of the sets)!

There are two equivalent statements regarding the compactness of set:

1. "*Given any collection $\mathscr{A}$ of open subsets of X, if $\mathscr{A}$ covers X, then some finite subcollection of $\mathscr{A}$ covers X.*"
2. "*Given any collection $\mathscr{A}$ of open sets, if* ***no finite subcollection*** *of $\mathscr{A}$ covers X, then A* ***does not cover*** *X.*"
3. $\Rightarrow$ "*Given any collection $\mathscr{C}$ of* ***closed sets****, if* ***every finite intersection*** *of elements of $\mathscr{C}$ is* ***nonempty****, then the* ***intersection of all the elements*** *of $\mathscr{C}$ is* ***nonempty***"
   $\blacksquare$

- **Remark** (***Nested Sequence of Closed Sets in Compact Space***)
  A special case of this proposition occurs when we have a ***nested*** *sequence* $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ *of* ***closed sets*** in *a* ***compact space*** X.

  If each of the sets C_n is nonempty, then the collection $\mathscr{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ automatically has ***the finite intersection property***. Then the intersection

  $$\bigcap_{n \in \mathbb{Z}_+} C_n$$

  is nonempty.

1.2.2 Compact Subspaces of the Real Line

- **Theorem 1.26** *[Munkres, 2000]*
  Let X be a ***simply ordered*** *set having the* ***least upper bound property****. In the order topology, each* ***closed interval*** *in X is* ***compact****.*
- **Corollary 1.27** ***(Closed Interval in Real Line is Compact)****[Munkres, 2000]*
  Every ***closed interval*** *in $\mathbb{R}$ is* ***compact****.*
- **Proposition 1.28** ***(Closed and Bounded in Euclidean Metric = Compact)****[Munkres, 2000]*

A subspace A of $\mathbb{R}^n$ is ***compact*** *if and only if it is* ***closed** and is **bounded*** *in the* ***euclidean metric*** *d or the* ***square metric*** *ρ*

- **Theorem 1.29** ***(Extreme Value Theorem).*** *[Munkres, 2000]*
  Let $f : X \to Y$ be ***continuous****, where Y is an* ***ordered set*** *in the order topology. If X is* ***compact****, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.*

- **Definition** (***Distance to Subset***)
  Let (X, d) be a ***metric space***; let A be a nonempty subset of X. For each $x \in X$, we define ***the distance*** *from x to A* by the equation

$$d(x, A) = \inf \{d(x, a) : a \in A\}.$$

- **Remark** The distance to subset $d(x : A)$ is a ***continuous*** function with respect to the first argument.

- **Remark** Recall that the ***diameter*** of a *bounded subset A* of a *metric space* (X, d) is the number

$$\sup \{d(a_1, a_2) : a_1, a_2 \in A\}.$$

- **Lemma 1.30** ***(The Lebesgue Number Lemma).*** *[Munkres, 2000]*
  Let $\mathscr{A}$ be an ***open covering*** *of the* ***metric space*** *(X, d). If X is* ***compact****, there is a $\delta > 0$ such that for each subset of X having* ***diameter less than*** *δ, there exists an element of $\mathscr{A}$ containing it.*

  The number δ is called a ***Lebesgue number*** *for the covering $\mathscr{A}$.*

- **Remark** ***The Lebesgue number*** is a ***threshold*** *on* ***diameter of subset*** so that all of subsets with diameter less than this threshold is fully contained in one of the open sets in the covering of X. The *existance* of this number relies on the ***compactness*** of domain X.

  This number is used in ϵ-δ ***condition*** to prove *the uniform continuity.*

- **Definition** (***Uniform Continuity***)
  A function $f : (X, d_X) \to (Y, d_Y)$ is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \quad \Rightarrow \quad d_Y(f(x_0), f(x_1)) < \epsilon.$$

- **Theorem 1.31** ***(Uniform Continuity Theorem).*** *[Munkres, 2000]*
  Let $f : X \to Y$ be a ***continuous*** *map of the* ***compact*** *metric space (X, d_X) to the metric space (Y, d_Y). Then f is* ***uniformly continuous****.*

- **Remark**

$$f \text{ continous } + \text{ compact domain } \Rightarrow f \text{ uniformly continous}$$

- **Definition** If X is a space, a point x of X is said to be ***an isolated point*** of X if *the one-point set* $\{x\}$ is ***open*** in X.

- **Theorem 1.32** ***(Uncountability in Compact Hausdorff Space)*** *[Munkres, 2000]*
  Let X be a nonempty ***compact Hausdorff*** *space. If X has* ***no isolated points****, then X is* ***uncountable****.*

- **Corollary 1.33** *[Munkres, 2000]*
  Every ***closed interval*** *in $\mathbb{R}$ is* ***uncountable****.*

- **Exercise 1.34** ***(Cantor Set)*** *[Munkres, 2000]*
  Let A_0 be the ***closed interval*** *$[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by* ***deleting*** *its "****middle third*** *$(1/3, 2/3)$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, define A_n by the equation*

  $$A_n = A_{n-1} \setminus \left(\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^k}, \frac{2+3k}{3^k} \right) \right).$$

  The ***intersection***

  $$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

  is called ***the Cantor set****; it is a* ***subspace*** *of $[0, 1]$.*

  1. *Show that C is* ***totally disconnected****.*
  2. *Show that C is* ***compact****.*
  3. *Show that each set A_n is a* ***union*** *of* ***finitely*** *many disjoint* ***closed intervals*** *of length $1/3^n$; and show that the* ***end points*** *of these intervals lie in C.*
  4. *Show that C has* ***no isolated points****.*
  5. *Conclude that C is* ***uncountable****.*

1.2.3 Limit Point Compactness

- **Definition** (***Limit Point Compactness***)
  A space X is said to be ***limit point compact*** if every infinite subset of X has a ***limit point***.

- **Proposition 1.35** ***(Compactness $\Rightarrow$ Limit Point Compactness)*** *[Munkres, 2000]*
  Compactness *implies* ***limit point compactness****, but* ***not conversely****.*

- **Example** (**Limit Point Compactness $\not\Rightarrow$ Compactness**)
  Let Y consist of ***two points***; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is ***limit point compact***, for *every nonempty subset of X has a* ***limit point***. It is ***not compact***, for the *covering* of X by the open sets $U_n = \{n\} \times Y$ has *no finite subcollection covering X.* ∎

- **Definition** (***Sequential Compactness***)
  Let X be a topological space. If (x_n) is a *sequence* of points of X, and if

  $$n_1 < n_2 < \ldots < n_i < \ldots$$

  is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n).

The space X is said to be ***sequentially compact*** if *every sequence of points of X has a* ***convergent subsequence****.*

- **Theorem 1.36** ***(Equivalent Definitions of Compactness in Metric Space)*** *[Munkres, 2000]*
  Let X be a ***metrizable space****. Then the following are* ***equivalent:***
  1. *X is* ***compact****.*
  2. *X is* ***limit point compact****.*
  3. *X is* ***sequentially compact****.*

1.2.4 Local Compactness

- **Definition** (***Local Compactness***)
  A space X is said to be ***locally compact at x*** if there is some ***compact subspace*** C of X that contains a ***neighborhood*** of x.

  If X is *locally compact* at *each of its points*, X is said simply to be ***locally compact***.

- **Example** For the one-dimensional space:
  1. The real line $\mathbb{R}$ is ***locally compact***. The point x lies in some interval (a, b), which in turn is *contained* in *the compact subspace* $[a, b]$.
  2. The subspace $\mathbb{Q}$ of rational numbers is ***not locally compact***.

- **Example** For product space of $\mathbb{R}$:
  1. The ***finite dimensional space*** $\mathbb{R}^n$ is ***locally compact***; the point x lies in some basis element $(a_1, b_1) \times \ldots \times (a_n, b_n)$, which in turn lies in *the compact subspace* $[a_1, b_1] \times \ldots \times [a_n, b_n]$.
  2. The ***countable infinite dimensional space*** $\mathbb{R}^\omega$ is ***not locally compact***; *none of its basis elements are contained in compact subspaces.* For if

     $$B = (a_1, b_1) \times \ldots \times (a_n, b_n) \times \mathbb{R} \times \ldots \times \mathbb{R} \times \ldots$$

     were contained in a *compact subspace*, then its ***closure***

     $$\bar{B} = [a_1, b_1] \times \ldots \times [a_n, b_n] \times \mathbb{R} \times \ldots \times \mathbb{R} \times \ldots$$

     would be ***compact***, which it is not.

- **Example** (***Simply Ordered Set with Least Upper Bound Property***)
  Every ***simply ordered*** *set* X *having* ***the least upper bound property*** is ***locally compact***: Given a basis element for X, it is contained in a *closed interval* in X, which is compact.

- **Example** (***Manifold***) [Lee, 2018]
  Every ***topological manifold*** is ***locally compact Hausdorff***.

  Thus ***every smooth manifold*** is ***locally compact Hausdorff***.

- **Definition** (***Precompactness***)
  A subset of X is said to be ***precompact*** in X if its ***closure*** in X is ***compact***.

- If X is *not a compact Hausdorff space*, then *under what conditions* is X *homeomorphic* with *a* ***subspace*** *of a compact Hausdorff space* ?

  Theorem 1.37 ***(Unique One-Point Compactification)*** *[Munkres, 2000]*
  Let X be a space. Then X is ***locally compact Hausdorff*** *if and only if there exists a space Y satisfying the following conditions:*

  1. *X is a subspace of Y.*
  2. *The set $Y \setminus X$ consists of* ***a single point*** *(which is the limit point of X).*
  3. *Y is a* ***compact Hausdorff*** *space.*

  If Y and Y' are two spaces satisfying these conditions, then there is a ***homeomorphism*** *of Y with Y' that equals* ***the identity map*** *on X.*

- **Definition** (***One-Point Compactification***)
  If Y is a ***compact Hausdorff*** space and X is a proper *subspace* of Y whose ***closure*** equals Y, then Y is said to be a ***compactification*** of X.

  If $Y \setminus X$ equals *a single point*, then Y is called ***the one-point compactification*** of X.

- **Remark** (***Locally Compact Hausdorff*** = *Existance of Unique One-Point Compactification*)
  X has a ***one-point compactification*** Y if and only if X is a ***locally compact Hausdorff space*** that is *not itself compact*.

  We speak of Y as "***the***" *one-point compactification* because Y is ***uniquely*** determined up to a *homeomorphism*.

- **Example** ***The one-point compactification*** of the real line $\mathbb{R}$ is ***homeomorphic*** with the ***circle*** $\mathbb{S}^1$.

  Similarly, ***the one-point compactification*** of $\mathbb{R}^2$ is ***homeomorphic*** to the ***sphere*** $\mathbb{S}^2$.

- **Proposition 1.38** ***(Locally Compact Hausdorff = Precompact Basis)*** *[Munkres, 2000]*
  Let X be a ***Hausdorff*** *space. Then X is* ***locally compact if and only if*** *given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is* ***compact*** *and $\bar{V} \subseteq U$.*

- **Corollary 1.39** ***(Closed or Open Subspace)*** *[Munkres, 2000]*
  Let X be locally compact Hausdorff; let A be a subspace of X. If A is ***closed*** *in X or* ***open*** *in X, then A is locally compact.*

- **Corollary 1.40** *[Munkres, 2000]*
  A space X is ***homeomorphic*** *to an* ***open*** *subspace of a* ***compact Hausdorff*** *space* ***if and only if*** *X is* ***locally compact Hausdorff***.

- **Remark** *Locally Compact Hausdorff = Open Subspace of Compact Hausorff*

- **Theorem 1.41** *[Treves, 2016]*
  Every locally compact Hausdorff topological vector space is ***finite-dimensional***.

- **Remark** (***Equivalent Definition of Locally Compact Hausdorff Space***)
  For a ***Hausdorff space*** X, the following are ***equivalent***:

1. X is ***locally compact***.

2. Each point of X has a ***precompact*** neighborhood.

3. X has a basis of ***precompact*** open subsets.

1.3 The Tychonoff Theorem

- **Lemma 1.42** ***(Existance of Maximal Collection with Finite Intersection Property)*** *[Munkres, 2000]*
Let X be a set; let $\mathscr{A}$ be a collection of subsets of X having the ***finite intersection property****. Then there is a collection $\mathscr{D}$ of subsets of X such that $\mathscr{D}$* ***contains*** *$\mathscr{A}$, and $\mathscr{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathscr{D}$ has this property.*

  [Hint: apply Zorn's Lemma to *the collection of collections of subsets* with finite intersection property]

- **Definition** We often say that a collection $\mathscr{D}$ satisfying the conclusion of this theorem is ***maximal*** *with respect to* ***the finite intersection property***.

- **Lemma 1.43** ***(Elements of Maximal Collection with Finite Intersection Property)*** *[Munkres, 2000]*
Let X be a set; let $\mathscr{D}$ be a collection of subsets of X that is ***maximal*** *with respect to* ***the finite intersection property****. Then:*

  1. *Any* ***finite intersection of elements*** *of $\mathscr{D}$ is an* ***element*** *of $\mathscr{D}$.*

  2. *If A is a subset of X that* ***intersects every element*** *of $\mathscr{D}$, then A is an element of $\mathscr{D}$.*

- **Theorem 1.44** ***(Tychonoff Theorem)****. [Munkres, 2000]*
An ***arbitrary product*** *of* ***compact*** *spaces is* ***compact*** *in the* ***product topology****.*

1.4 Nets and Convergence in Topological Space

- **Definition** (***Directed System of Index Set***)
A ***directed system*** is *an index set I* together with an ***ordering*** $\prec$ which satisfies:

  1. If $\alpha, \beta \in l$ then there exists $\gamma \in I$ so that $\gamma \succ \alpha$ and $\gamma \succ \beta$.

  2. $\prec$ is a ***partial ordering***.

- **Definition** A subset K of I is said to be ***cofinal*** in I if for each $\alpha \in I$, there exists $\beta \in K$ such that $\alpha \preceq \beta$.

- **Proposition 1.45** *If I is a directed system, and K is cofinal in I, then K is a directed system.*

- **Definition** (***Net***)
A ***net*** in a topological space X is a mapping from a ***directed system*** I to X; we denote it by $\{x_\alpha\}_{\alpha \in I}$

- **Remark** (***Net*** *vs.* ***Sequence***)

Net is a generalization and abstraction of ***sequence***. The directed system I is ***not necessarily countable***. So $\{x_\alpha\}_{\alpha \in I}$ may not be a countable sequence. *A sequence is a net with countable index set* $I \subseteq \mathbb{N}$. The directed system can be any set e.g. a graph.

- **Definition** If $P(\alpha)$ is a ***proposition*** depending on an ***index*** α in a *directed set* I we say $P(\alpha)$ ***is eventually true*** if there is a β in I with $P(\alpha)$ *true* if *for all* $\alpha \succ \beta$.

  We say $P(\alpha)$ ***is frequently true*** if it is ***not eventually false***, that is, if for any β *there exists* an $\alpha \succ \beta$ with $P(\alpha)$ *true.*

- **Definition** (***Convergence***)
  A ***net*** $\{x_\alpha\}_{\alpha \in I}$ in a topological space X is said to ***converge*** to a point $x \in X$ (written $x_\alpha \to x$) if for ***any neighborhood*** N of x, ***there exists*** a $\beta \in l$ so that $x_\alpha \in N$ if $\alpha \succeq \beta$. The point x that being converged to is called ***the limit point*** of x_α.

  Note that if $x_\alpha \to x$, then x_α is ***eventually in all neighborhoods of*** x. If x_α is ***frequently in any neighborhood of*** x, we say that x is a ***cluster point*** of x_α.

- **Remark** This definition *generalizes* the ϵ-δ language for convergence in metric space. Notice that the notions of *limit* and *cluster point* generalize the same notions for sequences in a metric space..

- **Proposition 1.46** *(**Net Lemma**) [Reed and Simon, 1980]*
  *Let A be a set in a topological space X. Then, a point $x \in \bar{A}$, the **closure** of A **if and only if** there is a net $\{x_\alpha\}_{\alpha \in I}$ with $x_\alpha \in A$, So that $x_\alpha \to x$.*

- **Proposition 1.47** *[Munkres, 2000]*

  1. *(**Continuous Function**): A function f from a topological space X to a topological space Y is **continuous** if and only if for **every convergent net** $\{x_\alpha\}_{\alpha \in I}$ **in** X, with $x_\alpha \to x$, the net $\{f(x_\alpha)\}_{\alpha \in I}$ **converges in** Y to $f(x)$.*

  2. *(**Uniqueness of Limit Point for Hausdorff Space**): Let X be a **Hausdorff space**. Then a net $\{x_\alpha\}_{\alpha \in I}$ in X can have **at most one limit**; that is, if $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.*

- **Definition** A net $\{x_\alpha\}_{\alpha \in I}$ is a ***subnet*** of a net $\{y_\beta\}_{\beta \in J}$ if and only if there is a function $F : I \to J$ such that

  1. $x_\alpha = y_{F(\alpha)}$ for each $\alpha \in I$.

  2. For all $\beta' \in J$, there is an $\alpha' \in I$ such that $\alpha \succ \alpha'$ implies $F(\alpha) \succ \beta'$ (that is, $F(\alpha)$ is ***eventually larger*** *than* ***any fixed*** $\beta \in J \Rightarrow F(I)$ is ***cofinal*** in J).

- **Proposition 1.48** *A point x in a topological space X is a **cluster point** of a **net** $\{x_\alpha\}_{\alpha \in I}$ if and only if **some subnet** of $\{x_\alpha\}_{\alpha \in I}$ **converges** to x.*

- **Theorem 1.49** *(**The Bolzano-Weierstrass Theorem**) [Reed and Simon, 1980, Munkres, 2000]*
  *A space X is **compact if and only if every net** in X **has a convergent subnet**.*

  Proof: To prove the implication $\Rightarrow$, let $B_\alpha = \{x_\beta : \alpha \preceq \beta\}$ and show that $\{B_\alpha\}$ has ***the finite intersection property***.

  To prove $\Leftarrow$, let $\mathscr{A}$ be a collection of ***closed sets*** having *the finite intersection property*, and let $\mathscr{B}$ be the collection of *all finite intersections* of elements of $\mathscr{A}$, ***partially ordered*** by *reverse inclusion.*

- **Remark** (***Compactness*** *via* ***Generalized Sequential Compactness***)
  By *generalization* of ***squences*** $\Rightarrow$ ***nets***, we obtain a *generalization* of the result of ***sequential compactnesss*** *in* ***metric space*** to ***compactness*** in *general topological space.*

  If the ***first countability axiom*** is satisfied, we can use *subsequence* and *sequence* in place of *subnet* and *net.*

2 Countability and The Separation Axioms

- **Remark** (***Countability***)
  A topological space X is said to be

  1. ***first-countable*** if there is a ***countable neighborhood basis*** at each point,

  2. ***second-countable*** if there is ***a countable basis*** for its topology.

- **Remark** (***The Separation Axioms***)
  A topological space is called a

  1. T_1 ***space***: every pair of ***disjoint one-point sets*** can be ***separated*** *by* ***one open set***, *which* ***contains only one*** *of the singular pair.*

     It is equivalent to say that *every* ***one point set*** *is* ***closed.***

  2. ***Hausdorff*** (or T_2): every pair of ***disjoint one-point sets*** can be ***separated*** *by* ***two disjoint open sets***, *each containing one of the singular sets, respectively.*

  3. ***regular*** (or T_3): it is T_1 and every pair of ***disjoint one-point set and closed set*** can be ***separated*** *by* ***two disjoint open sets***, *each containing one of the pair (singular set and closed set), respectively.*

     It is equivalent to say that each point has ***closed neighborhood basis***.

  4. ***normal*** (or T_4) if and only if it is T_1 and every pair of ***disjoint closed sets*** can be ***separated*** *by* ***two disjoint open sets***, *each containing one of the closed sets, respectively.*

- **Remark** The ***connectedness*** and ***compactness*** are both ***global topological properties*** of space;

  On the other hand, ***the countability axioms*** and ***the separation axioms*** describes ***the local topological properties*** of the space.

- **Remark** Both *the countability axioms* and *the separation axioms* arise from deeper study of topology itself.

  1. ***first-countable*** $\Rightarrow$ if ***convergent sequence*** is adequate to ***detect limit points*** of a set.

  2. ***second-countable*** $\Rightarrow$ ***separability*** (i.e. existance of ***countable dense set***); ***Lindelöf space*** (existance of *countable open subcovering*); ***topological manifolds***;

3. ***Hausdorff** (T_2)* $\Rightarrow$ if ***convergent sequence has at most one limit point***

4. ***regular** (T_3)* $\Rightarrow$ ***Urysohn metrization theorem:*** if + ***second-countable*** then ***metrizable***

5. ***normal** (T_4)* $\Rightarrow$ ***Urysohn lemma:*** if every pair of ***disjoint closed sets*** in X can be *separated by* ***disjoint open sets***, then each such pair can be ***separated by a continuous function***.

   $\Rightarrow$ ***Urysohn metrization theorem:*** since ***regular*** + ***second-countable*** $\Rightarrow$ ***normal***.

   $\Rightarrow$ ***Tietze extension theorem***: any ***real-valued continous function*** on ***closed subspace*** of *normal space* can be extended to the entire space.

   $\Rightarrow$ ***Existence of finite partitions of unity***:

2.1 The Countability Axioms

- **Definition** (***First-Countable***)
  A space X is said to have a ***<u>countable basis at x</u>*** if there is a ***countable*** *collection* $\mathscr{B}$ of ***neighborhoods*** of x such that *each neighborhood* of x ***contains at least one*** of the elements of $\mathscr{B}$.

  A space that has *a* ***countable basis*** *at each of its points* is said to satisfy ***<u>the first countability axiom</u>***, or to be ***<u>first-countable</u>***.

- **Remark** *Every metric space is first-countable.*

- **Proposition 2.1** *(**Limit Point Detected by Convergent Sequence**) [Munkres, 2000]*
  Let X be a topological space.

  1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the **converse** holds if X is **first-countable**.*

  2. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The **converse** holds if X is **first-countable**.*

- **Definition** (***Second-Countable***)
  If a space X has ***a countable basis*** *for its topology*, then X is said to satisfy ***the second countability axiom***, or to be ***<u>second-countable</u>***.

- **Example** ($\mathbb{R}$)
  The real line $\mathbb{R}$ has a ***countable basis***, which is the collection of all *open intervals* (a, b) with ***rational end points***.

- **Example** ($\mathbb{R}^n$ and $\mathbb{R}^\omega$ under *product topology*)

  1. *The finite dimensional space* $\mathbb{R}^n$ has a ***countable basis***, which is the collection of all *product of intervals* with ***rational end points***.

  2. *The countable infinite dimensional space* $\mathbb{R}^\omega$ has a ***countable basis***, which is the collection of all products $\prod_{n\in\mathbb{Z}_+} U_n$, where U_n is an *open interval with* ***rational end points*** *for* ***finitely many*** *values* of n, and $U_n = \mathbb{R}$ for *all other values* of n.

- **Example** ($\mathbb{R}^\omega$ *under* ***Uniform Topology Not Second-Countable***)
  In the ***uniform topology***, $\mathbb{R}^\omega$ satisfies *the first countability axiom (being metrizable)*. However, it *does* ***not satisfy the second***.

- **Example** (***Topological Manifolds***)

  Definition Suppose M is a ***topological space***. We say that M is a ***topological manifold*** of *dimension* n or a ***topological*** n***-manifold*** if it has the following properties:

  1. M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.

  2. M is ***second-countable***: there exists a ***countable basis*** for the topology of M.

  3. M is ***locally Euclidean of dimension*** n: each point of M has a neighborhood that is ***homeomorphic*** to an open subset of $\mathbb{R}^n$.

- Both countability axioms are well behaved with respect to the operations of taking subspaces or countable products:

  Proposition 2.2 *(**Subspaces and Countable Product**) [Munkres, 2000]*
  *A **subspace** of ______*

  1. *a first-countable space is first-countable;*

  2. *a second-countable space is second-countable.*

  *And a **countable product** of ______*

  1. *first-countable spaces is first-countable;*

  2. *second-countable spaces is second-countable.*

- **Definition** (***Dense Subset***)
  A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$. (That is, *every point in X is a limit point of A.*)

- **Definition** (***Separability***)
  A topological space X is called ***separable*** if and only if it has a ***countable dense set***.

- **Definition** (***Lindelöf Space***)
  A space for which *every open covering* contains *a* ***countable*** *subcovering* is called a ***Lindelöf space***.

- **Proposition 2.3** *(**Properties of Second-Countability**) [Munkres, 2000]*
  *Suppose that X has a **countable basis**. Then:*

  1. *Every **open covering** of X contains a **countable** subcollection covering X. (X is **Lindelöf space**)*

  2. *There exists a **countable** subset of X that is **dense** in X. (X is **separable**)*

- **Proposition 2.4** *(**Metric Space Equivalence**) [Munkres, 2000]*
  *Suppose that X is a **metrizable spcae**. The following statements are equivalent:*

  1. *X has a **countable basis** (**second-countable**).*

  2. *X has a **countable dense subset** (**separable**).*

3. *Every **open covering** of X contains a **countable** subcollection covering X. (**Lindelöf space**).*

- **Example** (***The Product of two Lindelöf Spaces Need Not be Lindelöf***)
  The space $\mathbb{R}_\ell$ is *Lindelöf*, but the product space $\mathbb{R}_\ell^2$ is not. $\mathbb{R}_\ell^2$ is called ***the Sorgenfrey plane***.

  The space $\mathbb{R}_\ell^2$ has as basis all sets of the form $[a, b) \times [c, d)$. To show it is not *Lindelöf*, consider the subspace

  $$L = \{(x, -x) : x \in \mathbb{R}_\ell\}.$$

  It is easy to check that L is ***closed*** in $\mathbb{R}_\ell^2$. Let us cover $\mathbb{R}_\ell^2$ by ***the open set*** $\mathbb{R}_\ell^2 \setminus L$ and by all *basis elements* of the form

  $$[a, b) \times [-a, d).$$

  Each of these open sets intersects L in ***at most one point***. Since L is ***uncountable***, *no countable subcollection covers* $\mathbb{R}_\ell^2$. ∎

- **Example** (***The Subspace of Lindelöf Space Need Not be Lindelöf***)
  The ***ordered square*** I_o^2 is ***compact***; therefore it is *Lindelöf*, trivially. However, the *subspace* $A = I \times (0, 1)$ is ***not Lindelöf***. For A is the union of the disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in A. This collection of sets is ***uncountable***, and ***no proper subcollection covers*** A. ∎

- **Proposition 2.5** *(**Compact Metrizable Space**) [Munkres, 2000]*
  *Every **compact metrizable** space X has a countable basis (i.e. **second-countable**).*

  [Hint: Let $\mathscr{A}_n$ be a finite covering of X by $1/n$-balls.]

- **Proposition 2.6** *(**Preservation by Continuity**) [Munkres, 2000]*
  Let $f : X \to Y$ be continuous.

  1. *If X is **Lindelöf** , then $f(X)$ is **Lindelöf** ;*

  2. *if X has a **countable dense subset**, then $f(X)$ satisfies the same condition.*

- **Proposition 2.7** *(**Preservation by Product**) [Munkres, 2000]*
  *If X is a **countable product** of spaces having countable dense subsets (**separable**), then X has a countable dense subset (**separable**).*

- **Proposition 2.8** *(**Preservation by Continuous Open Map**) [Munkres, 2000]*
  *Let $f : X \to Y$ be **continuous** <u>**open map**</u>. If X satisfies **the first** or **the second countability axiom**, then $f(X)$ satisfies the same axiom.*

2.2 The Separation Axioms

2.2.1 Definitions and Properties

- **Definition** (***The Separation Axioms***)

  1. A topological space is called a <u>T_1 ***space***</u> if and only if for all x and y, $x \neq y$, there is an ***open set*** U with $y \in U$, $x \notin U$.

     Equivalently, a space is T_1 *if and only if* $\{x\}$ is ***closed*** for each x.

2. A topological space is called ***Hausdorff*** (or T_2) if and only if for all all x and y, $x \neq y$, there are ***open sets*** U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

3. A topological space is called ***regular*** (or T_3) if and only if it is T_1 and for all x and C, ***closed***, with $x \notin C$, there are ***open sets*** U, V such that $x \in U$, $C \subset V$, and $U \cap V = \emptyset$.

   Equivalently, a space is T_3 if *the* ***closed neighborhoods*** *of any point are a* ***neighborhood base***.

4. A topological space is called ***normal*** (or T_4) if and only if it is T_1 and for all C_1, C_2, ***closed***, with $C_1 \cap C_2 = \emptyset$, there are ***open sets*** U, V with $C_1 \subset U$, $C_2 \subset V$, and $U \cap V = \emptyset$.

- **Proposition 2.9**

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$$

- **Remark** (***Separation axioms*** $\neq$ ***Discounnected Space***)
These axioms are called ***separation axioms*** for the reason that they involve "*separating* certain kinds of *sets from one another* by ***disjoint open sets***.

  We have used the word "*separation*" before, of course, when we studied *connected spaces*. But in that case, we were trying to find *disjoint open sets* ***whose union was the entire space***.

- **Lemma 2.10** *Let X be a topological space. Let one-point sets in X be closed.*

  1. *X is* ***regular*** *if and only if given a point x of X and a neighborhood U of x, there is a* ***neighborhood*** *V of x such that $\bar{V} \subseteq U$.*

  2. *X is* ***normal*** *if and only if given a* ***closed*** *set A and an open set U containing A, there is an* ***open set*** *V containing A such that $\bar{V} \subseteq U$.*

- **Remark** X is ***regular*** $\Leftrightarrow$ Each point of X has a ***closed neighborhood***

  Note that X is ***locally compact Hausdorff*** $\Leftrightarrow$ Each point of X has a ***precompact neighborhood*** i.e. it has a closed neighborhood and *the closure is compact*.

- **Proposition 2.11** *(**Simply Ordered Set is Hausdorff**) [Munkres, 2000]*
Every ***simply ordered set*** *is a* ***Hausdorff*** *space in the* ***order topology***.

- **Proposition 2.12** *(**Order Topology is Regular**) [Munkres, 2000]*
Every ***order topology*** *is a* ***regular***.

- **Remark** It can be shown actually that every ***order topology*** is a ***normal***, which includes all of these two previous results.

- **Proposition 2.13** *(**Preservation of Hausdorff and Regular Axioms**)*

  1. *The* ***product*** *of two Hausdorff/regular spaces is a Hausdorff/regular space.*

  2. *A* ***subspace*** *of a Hausdorff/regular space is a Hausdorff/regular space.*

- **Example** ($\mathbb{R}_K$ ***is Hausdorff but Not Regular***)
The space $\mathbb{R}_K$ is ***Hausdorff*** but ***not regular***. Recall that $\mathbb{R}_K$ denotes the reals in the topology having as *basis all open intervals* (a, b) *and all sets of the form* $(a, b) \setminus K$, where

$K = \{1/n : n \in \mathbb{Z}_+\}$. This space is *Hausdorff*, because any two distinct points have *disjoint open intervals* containing them. But it is ***not regular***. The set K is ***closed*** in $\mathbb{R}_K$, and it does *not contain the point* 0.

But it is ***not regular***. The set K is ***closed*** in $\mathbb{R}_K$, and it does *not contain the point* 0. Suppose that there exist *disjoint open sets* U and V containing 0 and K, respectively. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) \setminus K$, since each basis element of the form (a, b) containing 0 intersects K. Choose n large enough that $1/n \in (a, b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d). Finally, choose z so that $z < 1/n$ and $z > \max\{c, 1/(n + 1)\}$. Then z belongs to both U and V, so they are not disjoint. $\blacksquare$

- **Example** (***$\mathbb{R}_\ell$ is Normal***)
The space $\mathbb{R}_\ell$ is ***normal***. Recall that $\mathbb{R}_\ell$ is $\mathbb{R}$ with ***lower limit topology***. (i.e. the basis element is *the half-interval* $[a, b)$.) It is immediate that *one-point sets are closed* in $\mathbb{R}_\ell$, since the topology of $\mathbb{R}_\ell$ is *finer* than that of $\mathbb{R}$.

  To check ***normality***, suppose that A and B are *disjoint closed sets* in $\mathbb{R}_\ell$. For each point a of A choose a basis element $[a, x_a)$ *not intersecting* B; and for each point b of B choose a basis element $[b, x_b)$ *not intersecting* A. The open sets

$$U = \bigcup_{a\in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b\in B} [b, x_b)$$

  are ***disjoint open sets*** about A and B, respectively.

- **Example** (***The Sorgenfrey plane $\mathbb{R}_\ell^2$ is Not Normal***)
The space $\mathbb{R}_\ell$ is regular, so the product space $\mathbb{R}_\ell^2$ is regular. Thus this example serves *two purposes*. It shows that ***a regular space need not be normal***, and it shows that ***the product of two normal spaces need not be normal***.

- **Definition** (***Perfect Map***)
A ***closed continuous surjective*** *map* $p : X \to Y$ is called a ***perfect map*** if $p^{-1}(\{y\})$ is ***compact*** for each $y \in Y$.

- **Remark** *A perfect map* is a *quotient map.*

- **Proposition 2.14** *(**Preservation Properties of Perfect Map**) [Munkres, 2000]*
*Let $p : X \to Y$ be a **perfect map**, i.e. it is a **closed continuous surjective** map who preimage of one point set is **compact**. Then*

  1. *If X is **Hausdorff**, then so is Y.*
  2. *If X is **regular**, then so is Y.*
  3. *If X is **locally compact**, then so is Y.*
  4. *If X is **second-countable**, then so is Y.*

- **Theorem 2.15** *(**Preservation Properties of Orbit Space**) [Munkres, 2000]*
*Let G be a **compact topological group**; let X be a topological space; let α be an **action** of G on X. The orbit space X/G is the quotient space under equivalence relationship $x \sim \alpha(x)$.*

  1. *If X is **Hausdorff**, then so is X/G.*

2. *If X is **regular**, then so is X/G.*

3. *If X is **normal**, then so is X/G.*

4. *If X is **locally compact**, then so is X/G.*

5. *If X is **second-countable**, then so is X/G.*

- **Definition** If X and Y are topological spaces, a map $F : X \to Y$ (continuous or not) is said to be ***proper*** if for every ***compact*** set $K \subseteq Y$, the ***preimage*** $F^{-1}(K)$ is ***compact***.

2.2.2 Normal Space

- **Remark** As we have seen, unlike its name suggested, normal spaces are *not as well-behaved* as one might wish. On the other hand, ***most of the spaces*** with which we are familiar do *satisfy this axiom*, as we shall see.

  Its ***importance*** comes from the fact that the results one can prove ***under the hypothesis of normality*** are central to much of topology. The ***Urysohn metrization theorem*** and ***the Tietze extension theorem*** are two such results

- **Proposition 2.16** *[Munkres, 2000]*
  *Every **locally compact Hausdorff** space is **regular**.*

- **Theorem 2.17** ***(Regular + Second-Countable ⇒ Normal)****[Munkres, 2000]*
  *Every **regular** space with a **countable basis** is **normal**.*

- **Proposition 2.18** ***(Regular + Lindelöf ⇒ Normal)****[Munkres, 2000]*
  *Every **regular Lindelöf** space is **normal**.*

- **Theorem 2.19** *[Munkres, 2000]*
  *Every **metrizable** space is **normal**.*

- **Theorem 2.20** *[Munkres, 2000, Reed and Simon, 1980]*
  *Every **compact Hausdorff** space X is **normal**.*

- **Theorem 2.21** *[Munkres, 2000]*
  *Every **well-ordered** set X is **normal** in the **order topology**.*

  In fact, a stronger theorem holds:

  Theorem 2.22 *Every **order** topology is normal*

- **Example** (***The Uncountable Product of Normal Spaces Need Not be Normal***)
  If J is ***uncountable***, the product space $\mathbb{R}^J$ is ***not normal***.

  This example serves *three purposes*. It shows that ***a regular space $\mathbb{R}^J$ need not be normal***. It shows that ***a subspace of a normal space need not be normal***, for $\mathbb{R}^J$ is *homeomorphic* to the subspace $(0,1)^J$ of $[0,1]^J$, which (assuming *the Tychonoff theorem*) is ***compact Hausdorff*** and therefore *normal*. And it shows that ***an uncountable product of normal spaces need not be normal***. It leaves unsettled the question as to whether *a finite or a countable product of normal spaces might be normal*.

- **Example** (***The Finite Product of Normal Spaces Need Not be Normal***).

Table 1: Comparison the Urysohn Lemma and Geometric Hahn-Banach Theorem

	Urysohn's Lemma	*Geometric Hahn-Banach Theorem*
space	***normal*** *topological space* T_4	***normed linear*** *space*
weaker space	***completely regular*** *topological space*	***locally convex*** *space*
objects	*two* ***closed*** *subsets* A, B	*two* ***convex*** *subsets* A, B
separation pre-condition	***closed*** *subsets are* ***disjoint***	***convex*** *sets are* ***disjoint***
separating function	***continuous function*** $f : X \to [0, 1]$	***a hyperplane defined by linear functional*** $\ell(x) = a$
conclusion	*two* ***closed sets*** *can be* ***separated*** *by* f	*two* ***convex sets*** *can be* ***separated*** *by* ***hyperplane***
conclusion in math	$f(A) = \{0\}$ and $f(B) = \{1\}$	$\sup_{a\in A} \ell(a) \le a \le \inf_{b\in B} \ell(b)$

Recall $S_\Omega = \{x : x \in X \text{ and } x < \Omega\}$ is the ***uncountable section*** of a ***well-ordered set*** X by Ω where Ω is the ***largest element*** of X (called ***the minimal uncountable well-ordered set***).

Consider ***the well-ordered set*** $\bar{S}_\Omega$, *in the order topology*, and consider the subset S_Ω, *in the subspace topology* (which is the same as the order topology). *Both spaces are* ***normal***, but the product space $S_\Omega \times \bar{S}_\Omega$ is ***not normal***.

his example serves *three purposes*. First, it shows that ***a regular space need not be normal***, for $S_\Omega \times \bar{S}_\Omega$ is a *product of regular spaces* and therefore regular. Second, it shows that ***a subspace of a normal space need not be normal***, for $S_\Omega \times \bar{S}_\Omega$ is a *subspace* of $\bar{S}_\Omega \times \bar{S}_\Omega$, which is a ***compact Hausdorff*** *space* and therefore ***normal***. Third, it shows that ***the product of two normal spaces need not be normal***.

2.3 Important Theorems

2.3.1 The Urysohn Lemma

- **Theorem 2.23** *(**Urysohn Lemma**). [Munkres, 2000]*
  *Let X be a **normal** space; let A and B be **disjoint closed subsets** of X. Let $[a, b]$ be a **closed interval** in the real line. Then there exists a **continuous** map*
  $$f : X \to [a, b]$$
  *such that $f(x) = a$ for **every** x in A, and $f(x) = b$ for **every** x in B.*

- **Corollary 2.24** *(**Urysohn Lemma for** G_δ). [Munkres, 2000]*
  *Let X be a **normal** space. Then there exists a **continuous** map*
  $$f : X \to [0, 1]$$
  *such that $f(x) = 0$ for **every** $x \in A$, and $f(x) > 0$ for **every** $x \notin A$ **if and only if** A is a G_δ set, i.e. it equal to a countable intersection of open sets in X.*

- **Theorem 2.25** ***(Strong Form of Urysohn Lemma)**. [Munkres, 2000]*
*Let X be a **normal** space. Then there exists a **continuous** map*

$$f : X \to [0,1]$$

*such that $f(x) = 0$ for $x \in A$, **and** $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ **otherwise if and only if** A and B are disjoint closed G_δ set in X.*

- **Definition** (***Separation by Continuous Function***)
If A and B are two subsets of the topological space X, and if there is a **continuous** function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be ***separated by a continuous function***.

- **Remark** *The Urysohn lemma* says that if *every pair of* ***disjoint closed sets*** in X can be *separated by disjoint open sets*, then each such pair can be ***separated by a continuous function***. The ***converse*** is trivial, for if $f : X \to [0,1]$ is the function, then $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ are *disjoint open sets* containing A and B, respectively.

- **Remark** (***Separation by Continuous Function*** *vs* ***Separation by Linear Function***)
We can compare *the Urysohn lemma* with *the geometric Hahn-Banach theorem* which seperate two ***convex sets*** with *linear functional*. See Table 1. The *geometric Hahn-Banach theorem* can be seen as a generalization of *the Urysohn lemma* in ***normed linear space***.

- **Remark** (***Continuous Function in Compact Hausdorff Space***) [Reed and Simon, 1980]
The Urysohn lemma suggests that there are ***a lot of continuous functions*** in *normal space*. *The space of all real-valued continuous functions $\mathcal{C}_{\mathbb{R}}(X)$* on a ***compact Hausdorff space*** X (which is normal space) has a ***dense subset*** since any real-valued continuous functions on $[0,1]$ is a *uniform limit* of ***polynomials***.

- **Definition** (***Completely Regular***)
A space X is ***completely regular*** if *one-point sets* are *closed* in X and if for each point x_0 and each ***closed*** set A not containing x_0, there is a ***continuous function*** $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

- **Remark**

$$\text{normal} \;\Rightarrow\; \text{completely regular} \;\Rightarrow \text{regular}$$

Proposition 2.26 *A **subspace** of a completely regular space is completely regular.*

*A **product** of completely regular spaces is completely regular.*

- **Example** ($S_\Omega \times \bar{S}_\Omega$ ***is Completely Regular but Not Normal***).
$S_\Omega \times \bar{S}_\Omega$ is ***not normal*** but it is the product space of two completely regular spaces.

- **Theorem 2.27** ***(Urysohn Lemma, Locally Compact Version)**. [Folland, 2013]*
*Let X be a **locally compact Hausdorff** space and $K \subseteq U \subseteq X$ where K is **compact** and U is **open**. Then there exists a **continuous** map*

$$f : X \to [0,1]$$

*such that $f(x) = 1$ for **every** $x \in K$, and $f(x) = 0$ for x outside a **compact subset** of U.*

- **Corollary 2.28** *[Folland, 2013]*
  Every ***locally compact Hausdorff*** *space is* ***completely regular****.*

- **Remark** (***Dual Space of*** $\mathcal{C}_c(X)$ ***on Locally Compact Hausdorff Space***) [Reed and Simon, 1980, Folland, 2013]
  The famous ***Riesz-Markov theorem*** shows that the ***dual space*** of $\mathcal{C}_c(X)$, the space of compactly supported continuous function on *locally compact Hausdorff space* X is isomorphic to *the space of* ***signed regular Borel measures*** on X, i.e. $(\mathcal{C}_c(X))^* \simeq \mathcal{M}(X)$. The proof of *Riesz-Markov theorem* is based on ***the Urysohn lemma*** *for locally compact space.*

2.3.2 The Urysohn Metrization Theorem

- **Theorem 2.29** ***(Urysohn Metrization Theorem)****. [Munkres, 2000]*
  Every ***regular*** *space X with a* ***countable basis*** *is* ***metrizable****.*

- **Theorem 2.30** ***(Embedding Theorem)****. [Munkres, 2000]*
  Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of ***continuous*** *functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is* ***positive*** *at x_0 and* ***vanishes outside*** *U. Then the function $F : X \to \mathbb{R}^J$ defined by*

  $$F(x) = (f_\alpha(x))_{\alpha \in J}$$

  is a ***topological embedding*** *of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$ for each α then F* ***embeds*** *X in $[0,1]^J$.*

- **Definition** (***Separation*** *of* ***Points*** *From* ***Closed*** *Set by* ***Continuous*** *Functions*)
  A family of continuous functions that satisfies the hypotheses of *the embedding theorem above* is said to ***separate points from closed sets in*** X.

  The existence of such a family is readily seen to be *equivalent*, for a space X in which one-point sets are *closed*, to the requirement that X be *completely regular.*

- **Corollary 2.31** ***(Embedding Equivalent Definition*** *of* ***Completely Regular)*** *[Munkres, 2000]*
  A space X is ***completely regular if and only if*** *it is* ***homeomorphic*** *to a subspace of $[0,1]^J$ for some J.*

2.3.3 The Tietze Extension Theorem

- **Theorem 2.32** ***(Tietze Extension Theorem)*** *[Munkres, 2000, Reed and Simon, 1980]*
  Let X be a ***normal space****; let A be a* ***closed subspace*** *of X.*

  1. *Any* ***continuous*** *map of A into the* ***closed interval*** *$[a, b]$ of $\mathbb{R}$ may be* ***extended*** *to a* ***continuous*** *map of* ***all of*** *X into $[a, b]$.*
  2. *Any* ***continuous*** *map of A into $\mathbb{R}$ may be* ***extended*** *to a* ***continuous*** *map of* ***all of*** *X into $\mathbb{R}$.*

- **Theorem 2.33** ***(Tietze Extension Theorem, Locally Compact Version)*** *[Folland, 2013]*

Table 2: Comparison Tietze Extension Theorem and Hahn-Banach Theorem

	Tietze Extension Theorem	*Hahn-Banach Theorem*
space	***normal*** *topological space* T_4	***normed linear*** *space*
subspace	***topological*** *subspace*	***linear*** *subspace*
function to be extended	***real-valued continuous function***	***linear functional***
additional constraint	*the subspace is* ***closed***	*the functional* ***bounded above by a sublinear functional***
conclusion	*the domain of continuous function can be* ***extended to entire space***	*the domain of linear functional can be* ***extended to entire space***

Let X be a ***locally compact Hausdorff space****; let K be a* ***compact subspace*** *of X. If $f \in \mathcal{C}(K)$ is a* ***continuous*** *map of K into $\mathbb{R}$, there exists a* ***continuous*** *extension $F \in \mathcal{C}(X)$ of* ***all of*** *X into $\mathbb{R}$ such that $F|_K = f$. Moreover, F may be taken to* ***vanish outside a compact set****.*

- **Remark** (***Extension of Continuous Function vs. Extension of Linear Functional***)
  We can compare *the Tietze extension theorem* with *the Hahn-Banach theorem* in normed linear space. See from Table 2 that the Hahn-Banach theorem generalize the Tietze extension theorem from normal topological space to the normed linear space (which is metrizable so normal).

2.4 The Stone-Ĉech Compactification

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2.5 Embeddings of Manifolds

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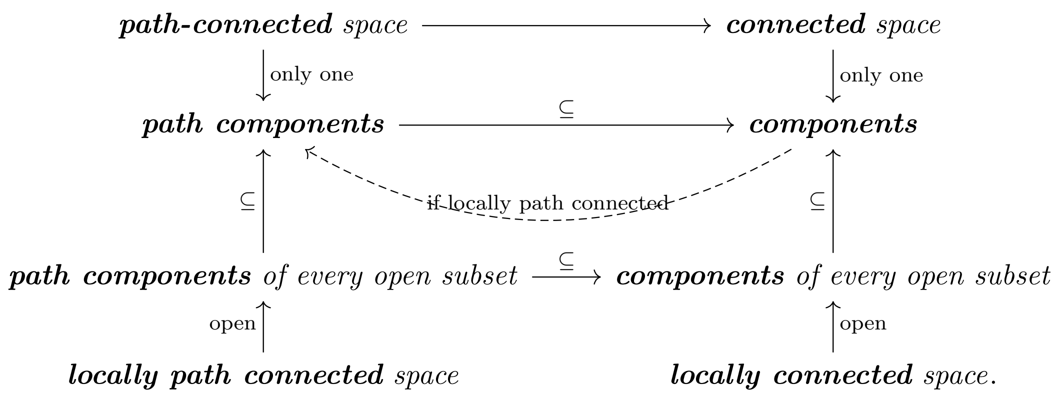
3 Summary of Preservation of Topological Properties

Table 3: Summary of Preservation of Topological Properties Under Transformations

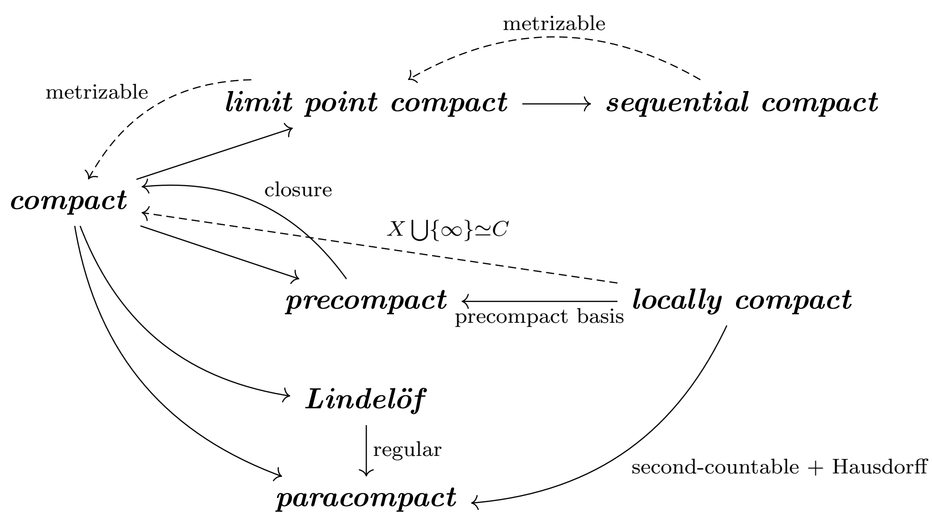
	subspace	***product space***	***image of continuous function***
connected	✓	✓ under ***product topology***	✓
locally connected	if ***open and connected*** subspace, ✓	if ***all but finitely many*** *of spaces are* ***connected***, ✓	in general ×
compact	if ***closed*** subspace, ✓;	for ***arbitrary*** product, ✓	✓
locally compact	if ***closed*** *or* ***open*** subspace and Hausdorff, ✓	if ***finite*** product, ✓; if ***infinite*** product ×	if f is a ***perfect map***, then ✓; in general ×
first-countable	✓	if ***countable*** product, ✓	if f is a ***open map***, then ✓; in general ×
second-countable	✓	if ***countable*** product, ✓	if f is a ***open map or perfect map***, then ✓; in general ×
separable	if metrizable, then ✓; in general ×	if ***countable*** product, ✓	✓
Lindelöf	if metrizable, then ✓; in general ×	×	✓
T_1 ***axiom***	✓	for ***arbitrary*** product, ✓	in general ×
Hausdorff T_2	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓; in general ×
regular T_3	✓	for ***arbitrary*** product, ✓	if f is a ***perfect map***, then ✓ ; in general ×
completely regular	✓	for ***arbitrary*** product, ✓	in general ×
normal T_4	×	×	×
paracompact	if ***closed*** subspace, ✓;	×	×
topologically complete	for ***closed*** *or* ***open*** subspace, ✓	if ***countable*** product, ✓	×

4 Summary of Relationships between Topological Properties

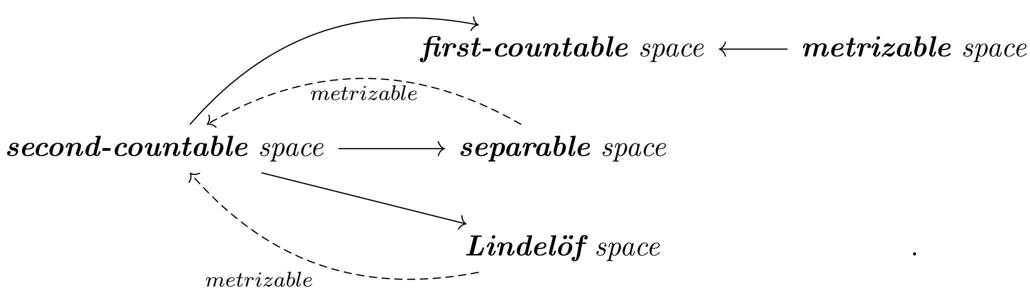
- ***Connectedness***



- ***Compactness***



- ***Countablity Axioms***



- ***Separation Axioms***

$$
\begin{tikzcd}
T_1 \text{ space} & & & \textbf{\textit{metrizable}} \text{ space} \arrow[d] \arrow[dlll] \arrow[dddd, bend left] \\
\textbf{\textit{Hausdorff}} \text{ space} \arrow[u] & \textbf{\textit{regular}} \text{ space} \arrow[l] \arrow[urr, bend left, "+ \text{ countable locally finite basis}"] \arrow[dddrr, bend right, "\text{Lindelöf}"] & & \textbf{\textit{normal}} \text{ space} \arrow[ll] \arrow[dll, "\text{Urysohn lemma}"] \\
 & \textbf{\textit{completely regular}} \text{ space} \arrow[u] & & \\
 & & & \textbf{\textit{compact Hausdorff}} \text{ space} \arrow[uu] \arrow[d] \\
 & \textbf{\textit{locally compact Hausdorff}} \text{ space} \arrow[uu] \arrow[urr, dashed, "\textit{compactification}"'] & & \textbf{\textit{paracompact Hausdorff}} \text{ space} \arrow[uuu, bend right]
\end{tikzcd}
$$

5 Summary of Counterexamples for Topological Properties

Table 4: Summary of Counterexamples for Topological Properties

	$\mathbb{R}^\omega$ $\mathscr{T}_{prod}$	$\mathbb{R}^\omega$ $\mathscr{T}_{box}$	$\mathbb{R}^\omega$ $\mathscr{T}_{unif}$	$\mathbb{R}_K$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell^2$	I_o^2	S_Ω	$\bar{S}_\Omega$	$S_\Omega \times \bar{S}_\Omega$	$(x, \sin(1/x))$
connected	✓	$\times$	$\times$	✓	$\times$	$\times$	✓	$\times$	$\times$	$\times$	✓
path connected	✓	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
locally connected	✓	$\times$	✓	$\times$	$\times$	$\times$	✓	$\times$	$\times$	$\times$	$\times$
locally path connected	✓	$\times$	✓	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	✓	$\times$	✓	$\times$	✓
limit point compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	✓	✓	✓		✓
sequentially compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	✓	✓	✓		✓
locally compact	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	✓	✓	✓	✓	✓
paracompact	✓	✓	✓	$\times$	✓	$\times$	✓	$\times$	✓	$\times$	✓
first-countable	✓	$\times$	✓	✓	✓	✓	✓	✓	$\times$	$\times$	
second-countable	✓	$\times$	$\times$	✓	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	
separable	✓	$\times$	$\times$	✓	✓	✓	$\times$	$\times$	$\times$	$\times$	
Lindelöf	✓	$\times$	$\times$	✓	✓	$\times$	✓	$\times$	✓	$\times$	✓
T_1 ***axiom***	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Hausdorff T_2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
regular T_3	✓	✓	✓	$\times$	✓	✓	✓	✓	✓	✓	
completely regular	✓	✓	✓	$\times$	✓	✓	✓	✓	✓	✓	
normal T_4	✓	✓	✓	$\times$	✓	$\times$	✓	✓	✓	$\times$	
locally metrizable	✓	$\times$	✓	$\times$			$\times$	✓	$\times$	$\times$	
metrizable	✓	$\times$	✓	$\times$	$\times$	$\times$	✓	$\times$	$\times$	$\times$	$\times$

1. $(\mathbb{R}^\omega, \mathscr{T}_{prod})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***product topology***. Note that under product topology, the ***basis*** is of form $\prod_{n\in\mathbb{Z}_+} U_n$ where there exists some N so that for all $n \geq N$, $U_n = \mathbb{R}$.

2. $(\mathbb{R}^\omega, \mathscr{T}_{box})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***box topology***. Note that under box topology, the ***basis*** is of form $\prod_{n\in\mathbb{Z}_+} U_n$ where $U_n \neq \mathbb{R}$ for all n.

3. $(\mathbb{R}^\omega, \mathscr{T}_{unif})$: space of ***countable infinite*** real sequence $(a_n)_{n\in\mathbb{Z}}$ equipped with ***uniform topology***. Note that the uniform topology is induced by ***the uniform metric*** $\bar{\rho}$ on $\mathbb{R}^\omega$, which is defined by the equation

$$\bar{\rho}((x_n)_{n\in\mathbb{Z}_+}, (y_n)_{n\in\mathbb{Z}_+}) = \sup\left\{\bar{d}(x_n, y_n) : n \in \mathbb{Z}_+\right\},$$

   where $\bar{d}$ is ***the standard bounded metric*** on $\mathbb{R}$.

4. $\mathbb{R}_K$: the real line $\mathbb{R}$ equipped with the ***K-topology***. The K-topology is ***generated*** by *all open intervals* (a, b) **and** *all sets* ***of the form***

$$(a, b) \setminus K \text{ where } K = \{1/n : n \in \mathbb{Z}_+\}.$$

5. $\mathbb{R}_\ell$: the real line $\mathbb{R}$ equipped with the ***lower limit topology***. The basis of lower limit topology is the collection of all ***half-open intervals*** of the form

$$[a, b) = \{x : a \leq x < b\},$$

   where $a < b$. $\mathbb{R}_\ell$ is also called ***the Sorgenfrey line***.

6. $\mathbb{R}_\ell^2 = \mathbb{R}_\ell \times \mathbb{R}_\ell$: is called ***the Sorgenfrey plane***.

7. I_o^2: is called ***ordered square*** where $I = [0, 1]$. It is the set $[0, 1] \times [0, 1]$ in ***the dictionary order topology***. In dictionary order relationship, $(x_1, x_2) < (y_1, y_2)$ if and only if $x_1 < y_1$ or $(x_1 = y_1) \wedge (x_2 < y_2)$. In dictionary order topology, open intervals are of the form

$$\{(x_1, x_2) : x_1 \in (a, b) \text{ or } (x_1 = c) \wedge (x_2 \in (d, e))\} = ((a, b) \times I) \cup (c \times (d, e)).$$

8. S_Ω: is ***the uncountable ordinal space***. If A is a ***well-ordered set*** then A itself contains a ***smallest element*** which we will denote by a_0. For each element x in a *well-ordered set* A, ***the section at*** x is defined to be the subset

$$S_x = (-\infty, x) = [a_0, x) = \{y \in A : y < x\}.$$

   The uncountable ordinal space S_Ω *is an* ***uncountable well-ordered set*** in which *each section* S_x *is* ***countable***. This description of S_Ω is justified by the following:

   Lemma 5.1 *There exists an uncountable well-ordered set A such that S_x is countable for each $x \in A$, and any two uncountable well-ordered sets satisfying this property are* ***order isomorphic*** *(that is, they have the same order type).*

9. $\bar{S}_\Omega$: is ***the closed uncountable ordinal space***. It is defined by $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ with ***the well-ordering*** given by: (a) if $x, y \in S_\Omega$ then $x < y$ in $\bar{S}_\Omega$ iff $x < y$ in S_Ω, and (b) if $x \in S_\Omega$ then $x < \Omega$. Notice that Ω is a ***maximal element*** in $\bar{S}_\Omega$ (but S_Ω *does not have a maximal element*). S_Ω is the section of Ω in $\bar{S}_\Omega$.

10. $S_\Omega \times \bar{S}_\Omega$

11. $\bar{S}$: is called ***the topologist's sine curve***. It is the closure of the graph

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

That is $\bar{S} = S \cup \{(x, y) : x = 0\}$.

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