Lecture 3: The Boolean Algebra, σ-Algebra and Limits in Set Theory

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Contents

1 Set Theory Basics

1.1 Set, Function and Axiom of Choice

- **Definition** Given a set X, the collection of all subsets of X, denoted as 2^X, is defined as
$$2^X := \{E : E \subseteq X\}$$
- **Remark** The followings are basic operation on 2^X: For $A, B \in 2^X$,
  1. ***Inclusion***: $A \subseteq B$ if and only if $\forall x \in A$, $x \in B$.
  2. ***Union***: $A \cup B = \{x : x \in A \vee x \in B\}$.
  3. ***Intersection***: $A \cap B = \{x : x \in A \wedge x \in B\}$.
  4. ***Difference***: $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
  5. ***Complement***: $A^c = X \setminus A = \{x : x \in X \wedge x \notin A\}$.
  6. ***Symmetric Difference***: $A \Delta B = (A \setminus B) \cup (B \setminus A) = \{x \in X : x \notin A \vee x \notin B\}$.

  We have ***deMorgan's laws***:
$$\left(\bigcup_{a \in A} U_a\right)^c = \bigcap_{a \in A} U_a^c, \quad \left(\bigcap_{a \in A} U_a\right)^c = \bigcup_{a \in A} U_a^c$$
- **Remark** Note that the following equality is useful:
$$A \Delta B = (A \cup B) \setminus (A \cap B)$$
- **Definition** ***An equivalence relation*** on X is a relation R on X such that
  1. (***Reflexivity***): xRx for all $x \in X$;
  2. (***Symmetry***): xRy if and only if yRx for all $x, y \in X$;
  3. (***Transitivity***): xRy and yRz then xRz for all $x, y, z \in X$.

  The equivalence class of an element x is denoted as $[x] := \{y \in X : xRy\}$. We usually denote the equivalence relation R as $\sim$. The set of equivalence classes provides ***a partition of the set*** X in that every $z \in X$ can must belong to *only one equivalence class* $[x]$. That is $[x] \cap [y] = \emptyset$ if $x \not\sim y$ and $X = \bigcup_{x \in X} [x]$.

  The set of all equivalence classes of X by $\sim$, denoted $X/\sim := \{[x] : x \in X\}$, is ***the quotient set*** of X by $\sim$. $X = \bigcup_{C \in X/\sim} C$.
- **Definition** $f : X \to Y$ is a ***function*** if for each $x \in X$, there exists a unique $y = f(x) \in Y$. X is called the ***domain*** of f and Y is called the ***codomain*** of f. $f(X) = \{y \in Y : y = f(x)\}$ is called the ***range*** of f

  The ***pre-image*** of f is defined as
$$f^{-1}(E) = \{x \in X : f(x) \in E\}.$$

- **Remark** The pre-image operation ***commutes*** with ***all basic set operations***:

$$
\begin{aligned}
A \subseteq B &\Rightarrow f^{-1}(A) \subseteq f^{-1}(B) \\
f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) &= \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \\
f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) &= \bigcap_{\alpha\in A} f^{-1}(E_\alpha) \\
f^{-1}(A \setminus B) &= f^{-1}(A) \setminus f^{-1}(B) \\
f^{-1}(E^c) &= \left(f^{-1}(E)\right)^c
\end{aligned}
$$

- **Remark** The image operation ***commutes*** with only ***inclusion*** *and* ***union*** *operations*:

$$
\begin{aligned}
A \subseteq B &\Rightarrow f(A) \subseteq f(B) \\
f\left(\bigcup_{\alpha\in A} E_\alpha\right) &= \bigcup_{\alpha\in A} f(E_\alpha)
\end{aligned}
$$

  For the other operations:

$$
\begin{aligned}
f\left(\bigcap_{\alpha\in A} E_\alpha\right) &\subseteq \bigcap_{\alpha\in A} f(E_\alpha) \\
f(A \setminus B) &\supseteq f(A) \setminus f(B)
\end{aligned}
$$

- **Definition** A map $f : X \to Y$ is ***surjective, or, onto***, if for every $y \in Y$, there exists a $x \in X$ such that $y = f(x)$. In set theory notation:

$$f : X \to Y \text{ is surjective } \Leftrightarrow f^{-1}(Y) \subseteq X.$$

  A map $f : X \to Y$ is ***injective***, if for every $x_1 \neq x_2 \in X$, their map $f(x_1) \neq f(x_2)$, or equivalently, $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

  If a map $f : X \to Y$ is both *surjective* and *injective*, we say f is a ***bijective***, or there exists an ***one-to-one correspondence*** between X and Y. Thus $Y = f(X)$.

- **Remark**

$$
\begin{aligned}
f^{-1}(f(B)) &\supseteq B, \quad \forall B \subseteq X \\
f(f^{-1}(E)) &\subseteq E, \quad \forall E \subseteq Y \\
f : X \to Y \text{ is surjective } &\Leftrightarrow f^{-1}(Y) \subseteq X. \\
&\Rightarrow f(f^{-1}(E)) = E. \\
f : X \to Y \text{ is injective } &\Rightarrow f^{-1}(f(B)) = B \\
&\Rightarrow f\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f(E_\alpha) \\
&\Rightarrow f(A \setminus B) = f(A) \setminus f(B)
\end{aligned}
$$

- **Proposition 1.1** *The following statements for composite functions are true:*

  1. *If f, g are both injective, then $g \circ f$ is injective.*
  2. *If f, g are both surjective, then $g \circ f$ is surjective.*
  3. *Every **injective** map $f : X \to Y$ can be writen as $f = \iota \circ f_R$ where $f_R : X \to f(X)$ is a **bijective** map and ι is the **inclusion map**.*
  4. *Every **surjective** map $f : X \to Y$ can be writen as $f = f_p \circ \pi$ where $\pi : X \to (X/\sim)$ is **a quotient map** (projection $x \mapsto [x]$) for the equivalent relation $x \sim y \Leftrightarrow f(x) = f(y)$ and $f_p : (X/\sim) \to Y$ is defined as $f_p([x]) = f(x)$ **constant** in each coset $[x]$.*
  5. *If $g \circ f$ is **injective**, then f is **injective**.*
  6. *If $g \circ f$ is **surjective**, then g is **surjective**.*

- **Principle 1.2** *(**The Axiom of Choice**).*
  If $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets, then $\prod_{\alpha \in A} X_\alpha$ is non-empty.

- **Corollary 1.3** *If $\{X_\alpha\}_{\alpha \in A}$ is a **disjoint** collection of nonempty sets, there is a set $Y \subset \bigcup_{\alpha \in A} X_\alpha$ such that $Y \cap X_\alpha$ contains **precisely one element** for each $\alpha \in A$.*

1.2 The Limits of Sets

- **Definition** A ***nested*** sequence of sets $E_1, E_2, \ldots$ is ***nondecreasing*** if $E_i \subseteq E_{i+1}$, and it is ***nonincreasing*** if $E_i \supseteq E_{i+1}$.

- **Definition** The $\underline{\textit{infimum}}$ and the $\underline{\textit{supremum}}$ of a collection of sets $\{E_n\}_{n \geq k}$ is given by

$$\inf_{n \geq k} E_n = \bigcap_{n=k}^{\infty} E_n, \quad \sup_{n \geq k} E_n = \bigcup_{n=k}^{\infty} E_n,$$

  respectively.

- **Remark** Note that

  1. $\inf\limits_{n \geq 1} E_n, \ldots, \inf\limits_{n \geq k} E_n, \ldots$ is ***monotone increasing*** as k increases since

$$\inf_{n \geq k} E_n \subseteq \inf_{n \geq k+1} E_n.$$

     *The **more*** sets that are involved in the ***intersection***, *the **less** cardinality of* the intersection will be. As k increases, *less* sets are involved in the intersection.

  2. $\sup\limits_{n \geq 1} E_n, \ldots, \sup\limits_{n \geq k} E_n, \ldots$ is ***monotone decreasing***. as k increases since

$$\sup_{n \geq k} E_n \supseteq \sup_{n \geq k+1} E_n.$$

     *The **more*** sets that are involved in the ***union***, *the **more** cardinality of* the union will be. As k increases, *less* sets are involved in the union.

- **Definition** [Resnick, 2013]
  The ***limit infimum*** and ***limit supremum*** is defined as

$$\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n, \quad \limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n, \tag{1}$$

  respectively.

- **Remark** It is clear that for ***nested sequence*** $\{E_n\}_{n\geq 1}$ that is ***nondecreasing***,

$$\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = \limsup_{n\to\infty} E_n$$

  so define *the* ***limit*** *of monotone increasing nested sets* as $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.

  Similarly, for ***nonincreasing nested sets*** $\{E_n\}_{n\geq 1}$,

$$\liminf_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n = \limsup_{n\to\infty} E_n$$

  so define *the* ***limit*** *of monotone decreasing nested sets* as $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$.

- **Remark** (***Limit Infimum and Limit Supremum of a Sequence***)
  Note that the notion

$$\liminf_{n\to\infty} a_n \equiv \lim_{k\to\infty}\inf_{n\geq k} a_n = \sup_{k\geq 1}\inf_{n\geq k} a_n$$

  and

$$\limsup_{n\to\infty} a_n \equiv \lim_{k\to\infty}\sup_{n\geq k} a_n = \inf_{k\geq 1}\sup_{n\geq k} a_n.$$

  It is ***the limit infimum*** *and* ***limit supremum*** among all the ***accumulation points*** of a sequence (a_n), respectively.

  Proposition 1.4 *The following properties hold*

  1. $\inf_{n\geq 1} a_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup_{n\geq 1} a_n$, *if the total infimum and total supremum exists.*

  2. $$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n,$$
     $$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

  3. ***A lower bound*** *on* $\liminf a_n \geq c$ *means that the sequence* a_n *will "****no smaller than the case*** *... " and* c *is a* ***lower bound*** *for* ***all possible sub-sequence*** (a_{k_n}).

  4. ***An upper bound*** *on* $\limsup a_n \leq b$ *means that the sequence* a_n *will "****no greater than the case*** *... " and* b *is a* ***upper bound*** *for* ***all possible sub-sequence*** (a_{k_n}).

Unlike *the limit operation*, which *may not exists* for some sequence (a_n), ***the limit infimum and limit supremum are always exists***, provided that the sequence lies in any ***partially ordered set***, where the suprema and infima exist, such as in a complete lattice. The *limit* exists ***if and only if*** the *limit infimum* and *limit supremum* are equal: $\lim_{n\to} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

- **Remark** Under complement operation, we have

$$\left(\liminf_{n\to\infty} E_n\right)^c = \limsup_{n\to\infty} E_n$$

and *vice versa*.

- **Proposition 1.5** *The* ***interpretation*** *of limit infimum and limit supremum*

$$\liminf_{n\to\infty} E_n = \{x:\ x \in E_n,\ \textit{for}\ \textbf{all but finite}\ n\} = \{x:\ \exists k,\, \forall n \ge k,\, x \in E_n\}$$
$$\limsup_{n\to\infty} E_n = \{x:\ x \in E_n,\ \textit{for}\ \textbf{infinitely many}\ n\} = \{x:\ \exists k,\, \forall n \ge k,\, x \in E_n\}$$

Proof: Define the indicator function of set $\mathbb{1}_C(x) = \mathbb{1}\{x \in C\} = 1$, if $x \in C$; $= 0$, o.w. Then

$$x \in \limsup_{n\to\infty} E_n = \bigcap_{k\ge 1}\bigcup_{n\ge k} E_n$$

indicates that for $k \ge 1$, there exists some $n_k > k$ such that $x \in E_{n_k} \Leftrightarrow \mathbb{1}_{E_{n_k}}(x) = 1$. Therefore

$$\sum_{n=1}^{\infty} \mathbb{1}_{E_n}(x) \ge \sum_{k=1}^{\infty} \mathbb{1}_{E_{n_k}}(x) = \infty,$$

and

$$\limsup_{n\to\infty} E_n \subseteq \left\{x \;\middle|\; \sum_{n=1}^{\infty} \mathbb{1}_{E_n}(x) = \infty\right\}.$$

For the converse part, see that for any $x \in \left\{x \;\middle|\; \sum_{n=1}^{\infty} \mathbb{1}_{E_n}(x) = \infty\right\}$, it indicates that there exists an infinite sub-sequence with indices $\{n_k\} \to \infty$ such that $x \in E_{n_k}$, so, by definition, for any $k > 1$, there exists some $n \ge k$, such that $x \in E_n$, or $x \in \bigcup_{n\ge k} E_n$. Clearly, $x \in \limsup_{n\to\infty} E_n \Rightarrow \limsup_{n\to\infty} E_n \supseteq \left\{x \;\middle|\; \sum_{n=1}^{\infty} \mathbb{1}_{E_n}(x) = \infty\right\}$. This completes the proof for limit supremum.

For limit infimum, consider the following set

$$\left\{x \;\middle|\; \sum_{n=1}^{\infty} \mathbb{1}_{E_n^c}(x) < \infty\right\}.$$

To show $\liminf_{n\to\infty} E_n \subseteq \left\{x \;\middle|\; \sum_{n=1}^{\infty} \mathbb{1}_{E_n^c}(x) < \infty\right\}$, we see that $x \in \liminf_{n\to\infty} E_n$, iff for some $k \ge 1$, $x \in E_n \Rightarrow \mathbb{1}_{E_n}(x) = 1$; or $\mathbb{1}_{E_n^c}(x) = 0$ holds for all $n \ge k \Rightarrow \sum_{n\ge k} \mathbb{1}_{E_n^c}(x) = 0$.

Choose one such k, the following decomposition holds

$$\sum_{n=1}^{\infty} \mathbb{1}_{E_n^c}(x) = \sum_{n=1}^{k} \mathbb{1}_{E_n^c}(x) + \sum_{n\geq k} \mathbb{1}_{E_n^c}(x)$$
$$\leq k < \infty,$$

which prove the inclusion part.

To show the converse, see that $x \in \left\{x \,\middle|\, \sum_{n=1}^{\infty} \mathbb{1}_{E_n^c}(x) < \infty\right\}$, means that it is possible to find $k \geq 1$ such that $\sum_{n\geq k} \mathbb{1}_{E_n^c}(x) = 0$, which means that $x \in E_n, \forall n \geq k$, therefore $x \in \liminf_{n\to\infty} E_n \Rightarrow \liminf_{n\to\infty} E_n \supset \left\{x \,\middle|\, \sum_{n=1}^{\infty} \mathbb{1}_{E_n^c}(x) < \infty\right\}$. ∎

- **Remark** Note

  1. $\liminf E_n$ is "***lower bound***" for the event $\{x \in E_n\}$, since $x \in \liminf E_n$ indicates only ***finitely many*** of n that x is ***not*** in E_n; In other words, (a_n) will "***finally***" lies in E_n., or "***with a few exceptions, ...*** "

     It is an ***assertion*** even in the ***worst*** case.

  2. $\limsup$ is "***upper bound***" for the event $\{x \in E_n\}$, as it indicates there ***exists*** a ***infinite sub-sequence***, k_n, such that $x \in E_{k_n}$ for every k_n.

     It is an ***assertion*** for the ***infinitely often occurrence*** of a event.

2 Development of σ-Algebra

2.1 Boolean Algebra

- **Definition** [Tao, 2011]
  Let X be a set. *A (concrete)* ***Boolean algebra (Boolean field)*** on X is a *collection of subsets* $\mathscr{B}$ of X which obeys the following properties:

  1. (***Empty set***) $\emptyset \in \mathscr{B}$;

  2. (***Complements***) For any $E \in \mathscr{B}$, then $E^c \equiv (X \setminus E) \in \mathscr{B}$;

  3. (***Finite unions***) For any $E, F \subset \mathscr{B}$, $E \cup F \in \mathscr{B}$.

  We sometimes say that E is $\mathscr{B}$***-measurable***, or ***measurable with respect to*** $\mathscr{B}$, if $E \in \mathscr{B}$.

- **Remark** Note that ***the finite difference*** $A - B$, $A\Delta B$ and ***intersections*** $A \cap B$ are also ***closed*** under the Boolean algebra.

- **Definition** A ***field (algebra)*** is a *non-empty collection of subsets* in X that is ***closed*** under ***finite union*** and ***complements***.

  It is just a *subset (sub-algebra)* of *Boolean field* $(X, \subset, \cup, \cdot^c)$.

- **Definition** Given two Boolean algebras $\mathscr{B}, (\mathscr{B})'$ on X, we say that $(\mathscr{B})'$ is ***finer*** than, a ***sub-algebra*** of, or a ***refinement*** of $\mathscr{B}$, or that $\mathscr{B}$ is ***coarser*** than or a ***coarsening*** of $(\mathscr{B})'$, if $\mathscr{B} \subset (\mathscr{B})'$.

- **Remark** In ***abstract Boolean algebra***, $\cup$ is replaced by *join* operation $\vee$ and $\cap$ is replaced by *meet* operation $\wedge$.

- **Remark** The definition of Boolean algebra ***does not requires*** X to have a ***topology***. It focus on a collection of subsets that is ***closed*** under *the set union operation* $\cup$ and the set complement $\cdot^c$. In other words, the concerns is the ***set-algebraic property*** not the topological property. Note that the set intersection operation $\cap$ can be obtained from composite of set union and set complement operations.

- **Definition** [Tao, 2011]
  Let X be partitioned into *a union* $X = \bigcup_{\alpha \in I} A_\alpha$ *of* **disjoint sets** A_α, which we refer to as ***atoms***. Then this partition *generates* ***a Boolean algebra*** $\mathscr{A}((A_\alpha)_{\alpha \in I})$, defined as *the collection of all the sets E of the form* $E = \bigcup_{\alpha \in J} A_\alpha$ for some $J \subseteq I$, i.e. $\mathscr{A}((A_\alpha)_{\alpha \in I})$ is *the collection of all sets that can be represented as* ***the union of one or more atoms***. Then $\mathscr{A}((A_\alpha)_{\alpha \in I})$ is ***a Boolean algebra***, and we refer to it as the ***atomic algebra*** with *atoms* $(A_\alpha)_{\alpha \in I}$.

- **Definition** A Boolean algebra is ***finite*** if it only consists of *finite many of subsets* (i.e., its *cardinality* is finite) [Tao, 2011].

- **Remark** The definition of ***atomic algebra*** as *generated* by ***atoms*** resembles the definition of ***topology*** *generated* by ***basis***.

  - In both cases, a subset in the collection of *atomic algebra / topology* is seen as the ***union*** of some subsets in the *atoms / basis*.

  - On the other hand, ***atoms*** *are all* ***disjoint***, while *sets in* ***basis*** are ***not necessarily disjoint***. In fact, by definition, for any two sets in basis that have nonempty intersection, there must exists a third set in basis that is a subset of the intersection.

- **Example** The followings are examples of *Boolean algebra*:

  1. ***The trivial algebra*** $\{X, \emptyset\}$ is *atomic algebra* with atoms $\{X\}$.

  2. ***The discrete algebra*** 2^X is *atomic algebra* generated by collection of ***singletons*** $\{x\}$.

- **Remark** The *non-empty atoms* of an *atomic algebra* are determined up to ***relabeling***. More precisely, if $X = \bigcup_{\alpha \in I} A_\alpha = \bigcup_{\alpha' \in I'} A'_{\alpha'}$ are two partitions of X into non-empty atoms A_α, $A'_{\alpha'}$, then $\bigcup_{\alpha \in I} A_\alpha = \bigcup_{\alpha' \in I'} A'_{\alpha'}$ if and only if exists a ***bijection*** $\phi : \alpha \to \alpha'$ such that $A'_{\phi(\alpha)} = A_\alpha$ for all $\alpha \in I$. [Tao, 2011]

- **Remark** There is a ***one-to-one correspondence*** between ***finite Boolean algebras*** on X and ***finite partitions*** of X into non-empty sets. (its cardinality is 2^m, for some m). [Tao, 2011]

- **Definition** [Tao, 2011]
  Let n be an integer. The ***dyadic algebra*** $\mathscr{D}_n$ at scale 2^{-n} in $\mathbb{R}^d$ is defined to be the atomic algebra generated by the *half-open dyadic cubes*

  $$\left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right) \times \cdots \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right)$$

  of length 2^{-n}. Note that $\mathscr{D}_n \subset \mathscr{D}_{n+1}$.

- **Example** Here are some more examples for Boolean algebra [Tao, 2011]

1. The collection $\overline{\mathcal{E}[\mathbb{R}^d]}$ of ***elementary sets*** (boxes and its finite union and intersections) and co-elementary sets (its complements is elementary) in $\mathbb{R}^d$ forms a Boolean algebra.

2. The collection $\overline{\mathcal{J}[\mathbb{R}^d]}$ of ***Jordan measureable set*** (contained in finite union of elementary sets) and co-Jordan measureable sets in $\mathbb{R}^d$ forms a Boolean algebra.

3. The collection $\mathcal{L}[\mathbb{R}^d]$ of ***Lebesgue measureable set*** (contained in countable union of elementary sets) in $\mathbb{R}^d$ forms a Boolean algebra.

4. The collection $\mathcal{N}[\mathbb{R}^d]$ of ***Lebesgue null sets*** and ***co-null sets*** (its complement is null set) in $\mathbb{R}^d$ forms a Boolean algebra. we refer to it as ***the null algebra*** on $\mathbb{R}^d$.

5. Given $Y \subset X$, and $\mathscr{B}$ is a Boolean algebra on X, then the ***restriction*** of algebra on Y is $\mathscr{B}|_Y = \mathscr{B} \cap 2^Y = \{E \cap Y : E \in \mathscr{B}\}$, which is a *sub-algebra*.

6. The ***dyadic algebra*** $\mathscr{D}_n$ at ***scale*** 2^{-n} in $\mathbb{R}^d$ is defined to be ***the atomic algebra*** generated by the *half-open dyadic cubes* of length 2^{-n}.

7. Note that $\left\{\emptyset, \mathbb{R}^d\right\} \subset \mathscr{D}_n \subset \overline{\mathcal{E}[\mathbb{R}^d]} = \bigcup_{n\ge 1} \mathscr{D}_n \subset \overline{J[\mathbb{R}^d]} \subset L[\mathbb{R}^d] \subset 2^{\mathbb{R}^d}$. $N[\mathbb{R}^d] \subset L[\mathbb{R}^d]$. Although $\mathscr{D}_n$ for given n is atomic algebra, $\overline{\mathcal{E}[\mathbb{R}^d]}$ and all its predecessors are ***non-atomic***, since they do not have finite cardinality.

8. $\bigwedge_{\alpha\in I} \mathscr{B}_\alpha \equiv \bigcap_{\alpha\in I} \mathscr{B}_\alpha$ for all $\alpha \in I$ is a Boolean algebra (I is arbitrary), which is ***the finest algebra*** that is ***coarser*** than any $\mathscr{B}_\alpha$.

- **Example** (***Boolean Algebra Generated by*** $\mathcal{F}$)

Definition Given a collection of sets $\mathcal{F}$, then $\langle \mathcal{F} \rangle_{bool}$ is ***the Boolean algebra generated*** by $\mathcal{F}$, i.e. the ***intersection*** of all the Boolean algebras that contain $\mathcal{F}$.

$$\langle \mathcal{F} \rangle_{bool} = \bigwedge_{\mathscr{B}_\alpha \supseteq \mathcal{F}} \mathscr{B}_\alpha.$$

Proposition 2.1 *We have the following results regarding* $\langle \mathcal{F} \rangle_{bool}$

1. $\langle \mathcal{F} \rangle_{bool}$ *is the* ***coarest*** *Boolean algebra that contains* $\mathcal{F}$.

2. *Note that* $\mathcal{F}$ *is a Boolean algebra if and only if* $\mathcal{F} = \langle \mathcal{F} \rangle_{bool}$.

3. *If* $\mathcal{F}$ *is collection of* n ***sets****, then* $\langle \mathcal{F} \rangle_{bool}$ *is* ***a finite Boolean algebra*** *with cardinality* 2^{2^n}.

Exercise 2.2 *(**Recursive description of a generated Boolean algebra**). [Tao, 2011] Let* $\mathcal{F}$ *be a collection of sets in a set* X. *Define the sets* $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ ***recursively*** *as follows:*

1. $\mathcal{F}_0 := \mathcal{F}$.

2. *For each* $n \ge 1$*, we define* $\mathcal{F}_n$ *to be the collection of all sets that* ***either*** *the* ***union*** *of a* ***finite number*** *of sets in* $\mathcal{F}_{n-1}$ *(including the empty union* $\emptyset$*), or the* ***complement*** *of such a union.*

Show that $\langle \mathcal{F} \rangle_{bool} = \bigcup_{n=0}^{\infty} \mathcal{F}_n$.

2.2 σ-Algebra

- **Definition** Given space X, a ***σ-field*** (or, ***σ-algebra***) $\mathscr{F}$ is a non-empty collection of *subsets* in X such that
  1. $\emptyset \in \mathscr{F}$; $X \in \mathscr{F}$;
  2. ***Complements***: For any $B \in \mathscr{F}$, then $B^c \equiv (X - B) \in \mathscr{F}$;
  3. ***Countable union***: for any sub-collection $\{B_k\}_{k=1}^{\infty} \subset \mathscr{F}$,
  $$\bigcup_{k=1}^{\infty} B_k \in \mathscr{F};$$
  Also, ***Countable intersection***: $\bigcap_{k=1}^{\infty} B_k \in \mathscr{F}$, ***de Morgan's law***.

  We refer to the pair $(X, \mathscr{F})$ of a set X together with a σ-algebra on that set as ***a measurable space***.

- **Remark** The prefix σ usually denotes "***countable union***". Other instances of this prefix include a ***σ-compact topological space*** (*a countable union of compact sets*), a ***σ-finite measure space*** (*a countable union of sets of finite measure*), or F_{σ} ***set*** (*a countable union of closed sets*) for other instances of this prefix.

- **Remark** A σ-algebra can be ***equivalently*** defined as an algebra that is closed under ***countable disjoint union***. Using the following transformation, for given $\{E_j\}$,
$$F_j = E_j - \bigcup_{i=1}^{j-1} E_i, \forall\, j \in \mathbb{N}.$$
Then $F_i \cap F_j = \emptyset, i \neq j$ and $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j$.

- **Remark** A field (algebra) may not be a σ-field since it *may not be* ***closure*** *under* ***countable union***.

- **Remark** ***The closure under countable union*** for σ-algebra is the key property to make sure that it is a proper *domain* to define ***measure***, since a desired property for a *measure* μ is the ***countably additive over disjoint sets***: $\mu\left(\sum_{t=1}^{\infty} A_t\right) = \sum_{t=1}^{\infty} \mu\left(A_t\right)$. Thus $\sum_{t=1}^{\infty} A_t$ need to be included in the domain of a proper measure.

- **Remark** ***(σ-Algebra vs. Boolean Algebra)***
  1. **Proposition 2.3** *Any σ-algebra is Boolean-algebra.*
  2. **Proposition 2.4** *Any* ***atomic algebra*** *is σ-algebra.*
  3. **Proposition 2.5** *An algebra of* ***finite*** *set X is a σ-algebra of X and it is* ***the power set*** 2^X *itself.*

- **Example** Here are some more examples for σ-algebra [Tao, 2011]
  1. ***The trivial algebra*** $\{X, \emptyset\}$ is *σ-algebra* since it is an atomic algebra.
  2. ***The discrete algebra*** 2^X is *σ-algebra* since it is an atomic algebra.
  3. All the ***finite Boolean algebra*** is σ-algebra.

4. The ***dyadic algebra*** $\mathscr{D}_n$ at ***scale*** 2^{-n} in $\mathbb{R}^d$ is a ***σ-algebra*** since it is an atomic algebra.
5. The collection $\mathcal{L}[\mathbb{R}^d]$ of ***Lebesgue measureable set*** (contained in countable union of elementary sets) in $\mathbb{R}^d$ forms a Boolean algebra.
6. The collection $\mathcal{N}[\mathbb{R}^d]$ of ***Lebesgue null sets*** and ***co-null sets*** (its complement is null set) in $\mathbb{R}^d$ forms a Boolean algebra. we refer to it as ***the null algebra*** on $\mathbb{R}^d$.
7. Given $Y \subset X$ as a subspace of X, and $\mathscr{B}$ is a σ-algebra on X, then the ***restriction*** of algebra on Y is $\mathscr{B}|_Y = \mathscr{B} \cap 2^Y = \{E \cap Y : E \in \mathscr{B}\}$, which is a ***σ-algebra on subspace*** Y.
8. Note that both the collections of *elementary sets* $\mathcal{E}[\mathbb{R}^d]$ and ***the Jordan measurable sets*** $\mathcal{J}[\mathbb{R}^d]$ ***do not form a σ-algebra***.
9. If $\{\mathscr{B}_\alpha\}$ are σ-algebras, then $\bigwedge_{\alpha \in I} \mathscr{B}_\alpha \equiv \bigcap_{\alpha \in I} \mathscr{B}_\alpha$ for all $\alpha \in I$ is a σ-algebra (I is arbitrary), which is ***the finest σ-algebra*** that is ***coarser*** than any $\mathscr{B}_\alpha$.

- **Example** (***σ-Algebra Generated by*** $\mathcal{F}$)

Definition Denote $\sigma(\mathcal{F}) := \langle \mathcal{F} \rangle$ as ***the σ-algebra generated by*** $\mathcal{F}$, given by

$$\sigma(\mathcal{F}) = \langle \mathcal{F} \rangle = \bigwedge_{\mathscr{B}_\alpha \supseteq \mathcal{F}} \mathscr{B}_\alpha.$$

It is the ***coarsest*** σ-algebra containing $\mathcal{F}$, for any σ-algebra that contains $\mathcal{F}$.

It is easy to see that

$$\langle \mathcal{F} \rangle_{bool} \subseteq \langle \mathcal{F} \rangle$$

The equality holds if and only if $\langle \mathcal{F} \rangle_{bool}$ is a σ-algebra.

Proposition 2.6 *(**Recursive description of a generated σ-algebra**). [Tao, 2011]* *$\sigma(\mathcal{F})$ is generated according to the following procedure:*

1. *For every set $A \in \mathcal{F}$, $A \in \sigma(\mathcal{F})$; $\mathcal{F} \subset \sigma(\mathcal{F})$;*
2. *Take the **finite union** and **finite intersection** of any **subcollections** $\{A_k\} \subset \mathcal{F}$, put $\bigcup_{k=1}^{n} A_k \in \sigma(\mathcal{F}), n \geq 1$ and $\bigcap_{k=1}^{n} A_k \in \sigma(\mathcal{F}), n \geq 1$;*
3. *Put the **countably infinite union** and **intersections** of any **subcollections** $\{A_k\} \subset \mathcal{F}$, put $\bigcup_{k=1}^{\infty} A_k \in \sigma(\mathcal{F})$ and $\bigcap_{k=1}^{\infty} A_k \in \sigma(\mathcal{F})$;*
4. *Put the **complements** $A^c \in \sigma(\mathcal{F}), \forall A \in \sigma(\mathcal{F})$;*

Finally we have the ***monotonicity***:

1. **Proposition 2.7** *If $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\sigma(\mathcal{F}_1) \subset \sigma(\mathcal{F}_2)$.*
2. **Proposition 2.8** *If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \sigma(\mathcal{F}_1)$, then $\sigma(\mathcal{F}_2) = \sigma(\mathcal{F}_1)$.*
3. **Proposition 2.9** *Let $\mathscr{F}$ be a σ-algebra on a set X. Let $S \subset X$ be a subset of X.*

   Then

   $$\sigma(\mathscr{F} \cup \{S\}) = \{(E_1 \cap S) \cup (E_2 \cap S^c) : E_1, E_2 \in \mathscr{F}\}$$

   *where σ denotes **generated σ-algebra**.*

- **Remark** Note that $\mathscr{F}_1 \cup \mathscr{F}_2$ is usually not a σ-algebra.

2.3 Borel σ-Algebra

- **Definition** (***Borel σ-algebra***). [Tao, 2011]
  Let X be a ***metric space***, or more generally ***a topological space***. The ***Borel σ-algebra*** $\mathcal{B}[X]$ of X is defined to be the σ-algebra *generated by the* ***open subsets*** *of* X.

  Elements of $\mathcal{B}[X]$ will be called ***Borel measurable***.

- **Example** The followings are examples of *Borel measurable subsets* in X:

  1. Any ***the open set*** and ***the closed set*** (which are *complements* of open sets), including ***The arbitrary union*** of *open sets*, and ***arbitrary intersection*** of *closed set*.
  2. The ***countable unions*** of ***closed sets*** (known as F_σ sets),
  3. The ***countable intersections*** of ***open sets*** (known as G_δ sets),
  4. The ***countable intersections*** of F_σ sets, and so forth.

- **Exercise 2.10** *Show that the Borel σ-algebra $\mathcal{B}[\mathbb{R}^d]$ of a Euclidean set is generated by any of the following collections of sets:*

  1. *The open subsets of $\mathbb{R}^d$.*
  2. *The closed subsets of $\mathbb{R}^d$.*
  3. *The compact subsets of $\mathbb{R}^d$.*
  4. *The open balls of $\mathbb{R}^d$.*
  5. *The boxes in $\mathbb{R}^d$.*
  6. *The elementary sets in $\mathbb{R}^d$.*

  (Hint: To show that two families $\mathcal{F}, \mathcal{F}'$ of sets generate the same σ-algebra, it suffices to show that every σ-algebra that contains $\mathcal{F}$, contains $\mathcal{F}'$ also, and conversely.)

- **Remark** $\mathcal{B}[X] \subset \mathcal{L}[X]$, i.e. the Borel σ-algebra is ***coarser*** than the Lebesgue σ-algebra.

- **Remark** There exist ***Jordan measurable*** (and hence Lebesgue measurable) subsets of $\mathbb{R}^d$ which are ***not Borel measurable***. [Tao, 2011]

- **Remark** Despite this demonstration that ***not all Lebesgue measurable subsets are Borel measurable***, it is *remarkably* ***difficult*** *(though not impossible)* to exhibit a specific set that is not Borel measurable. Indeed, a large majority of the explicitly constructible sets that one actually encounters in practice tend to be Borel measurable, and one can view the property of Borel measurability intuitively as a kind of "*constructibility*" property. A Borel σ-algebra is large enough to contain all subsets in X that is of "practical use" in computing measures and integrations within $(0, 1]$.

- **Exercise 2.11** *Show that the Lebesgue σ-algebra on $\mathbb{R}^d$ is generated by the union of the Borel σ-algebra and the null σ-algebra.*

3 Topology, σ-algebra and Borel σ-algebra

3.1 Definition

- **Definition** [Munkres, 2000]
  Given space X, a collection of subsets $\mathscr{T}$ is called a ***topology*** on X, if the following conditions holds
  1. $\emptyset \in \mathscr{T}$; $X \in \mathscr{T}$;
  2. ***Aribitray Union property***: for any sub-collection $\{U_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{T}$,
     $$\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathscr{T};$$
     Note that Λ could be uncountable.
  3. ***Finite intersection***: for any finite sub-collection $\{U_k\}_{1\le k\le n} \subset \mathscr{T}$,
     $$\bigcap_{k=1}^{n} U_k \in \mathscr{T}.$$

  $U \in \mathscr{T}$ is called an open set in topology $\mathscr{T}$ on X.
- **Definition** [Royden and Fitzpatrick, 1988, Billingsley, 2008, Folland, 2013, Resnick, 2013]
  Given space X, a ***σ-field*** (or, ***σ-algebra***) $\mathscr{F}$ is a non-empty collection of subsets in X such that
  1. $\emptyset \in \mathscr{F}$; $X \in \mathscr{F}$;
  2. ***Complements***: For any $B \in \mathscr{F}$, then $B^c \equiv (X - B) \in \mathscr{F}$;
  3. *Finite union*: for any $A, B \subset \mathscr{F}$,
     $$A \cup B \in \mathscr{F};$$
  4. ***Countable union***: for any sub-collection $\{B_k\}_{k=1}^{\infty} \subset \mathscr{F}$,
     $$\bigcup_{k=1}^{\infty} B_k \in \mathscr{F};$$
     Also, Countable intersection: $\bigcap_{k=1}^{\infty} B_k \in \mathscr{F}$, ***de Morgan's law***.
- **Definition** Given a ***topological space*** $(X, \mathscr{T})$, a ***Borel σ-field*** (or, ***Borel σ-algebra***) $\mathscr{B}$ is *the σ-algebra generated* from ***open sets*** *(or closed sets) in* $\mathscr{T}$. Note that this σ-algebra is not, in general, the whole power set.

  The *Borel σ-algebra* on X is the ***smallest*** σ-algebra containing *all open sets* (or, equivalently, all closed sets).
- **Remark** We compare the (open-set) topoloy with σ-algebra:

- ***The open-set topology*** on X is ***closed*** under ***any union***, or ***finite intersection*** *operation*. It does ***not*** *consider* the ***complements*** as the complements defines *a* ***closed set*** *not in open-set topology*. It contains the open sets as *the basic environment* in investigating the ***infinitesimal behavior*** of functions in ***analysis***.
- ***A*** σ***-algebra*** concerns more about the ***closure*** under a set of ***operations*** on X: *countable union*, *countable intersection*, ***complementation***. It has nothing to do with *the open set*, *closed set*, or the *continuity*.
- The ***analysis*** replies on ***topology*** on space X; while the ***modern algebra*** replies on ***the closure of operation*** on a space X. A σ-algebra is a collection of subsets in X that endows a ***algebraic structure***.

- **Remark** The ***Borel*** σ***-algebra*** lies in between, which concerns both ***algebraic*** and ***analytical structure***.
  - A ***open set*** U is a *Borel set* in $\mathscr{B}$; also a ***closed set*** $C \equiv U^c$ is a *Borel set* in $\mathscr{B}$.
  - Any ***countable union*** *of* ***closed set***, denoted as "F_σ set", $F_{\sigma,\Lambda} = \bigcup_{\lambda\in\Lambda} C_\lambda \in \mathscr{B}$
  - Any ***countable intersection*** *of* ***open sets***, denoted as "G_δ set", $G_{\delta,\Lambda} = \bigcap_{\lambda\in\Lambda} U_\lambda \in \mathscr{B}$.
  - Note that a F_σ set is ***not closed*** (but could be open) and a G_δ set is ***not open*** *(but could be closed)*.

  The Borel σ-algebra contains *open sets*, *closed sets*, *G_δ sets*, *F_σ sets*, and their further *countable union and intersections*, according to the topology.

- **Example** On the Euclidean space $\mathbb{R}^n$, another σ-algebra is of importance: *the collection of all* ***Lebesgue measurable sets***. This σ-algebra contains ***more sets*** than the Borel σ-algebra on $\mathbb{R}^n$ and is preferred in integration theory, as it gives a complete measure space.
- **Remark** Note that not all Lebesgue measurable subsets are Borel measurable [Tao, 2011].
- **Remark** The ***Lebesgue*** σ***-algebra*** on $\mathbb{R}^d$ is generated by the ***union*** of ***the Borel*** σ***-algebra*** and ***the null*** σ***-algebra***. [Tao, 2011]

  Thus, The Lebesgue σ-algebra on $\mathbb{R}^d$ is a ***completion*** of the Borel σ-algebra [Tao, 2011].
- **Example** The σ-algebra $\mathscr{F}$ is the domain where *a* ***probability measure*** $\mathbb{P} : \mathscr{F} \to \mathbb{R}$ is defined, whereas when considering a measure on $\mathbb{R}$, *a* ***Borel*** σ***-algebra*** $\mathcal{B}$ generated by ***order topology*** $\{(a, b], a < b, \forall a, b \in \mathbb{R}\}$ is of primarily concern.
- **Remark** ***A common problem*** is to find a good notion of a ***measure*** on a ***topological space*** that is ***compatible*** with the ***topology*** in some sense.
  - One way to do this is to define a measure on the ***Borel sets*** of *the topological space*.
  - In general, however, the ***algebraic structure*** of the σ-**algebra**: ***closure under*** complements, finite intersections and countably unions, instead of its ***geometric*** structure or ***topology***, are ***crucial*** to define a proper measure that ***mimic the length, area and volume*** in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$, respectively.

    Also there are several problems with this: for example, such a measure may not have a well defined support.

3.2 Comparison

Table 1: Comparison between σ-algebra and topology

	Boolean Algebra	***σ-Algebra***	***Borel σ-Algebra***	***Topology***
compatibility		$\Leftarrow \checkmark \Rightarrow$	*σ-algebra generated* by ***open subsets***	*no relation*
collection of subsets	✓	✓	✓	✓
include emptyset	✓	✓	✓	✓
include fullset	✓	✓	✓	✓
finite union	✓	✓	✓	✓
countable union		✓	✓	✓
arbitrary union				✓
finite intersection	✓	✓	✓	✓
countable intersection		✓	✓	
complements	✓	✓	✓	
structure	***analytical***	***analytical***	***analytical*** & ***topological***	***topological***
related ***measure***	✓	✓	✓	
set ***in*** *collection*	***elementary sets***; ***Jordan measurable sets***; ***atomic algebra***; *dyadic algebra*; ***finite union*** of measurable sets; etc.	*Boolean measurable set*; ***Lebesgue measurable sets***, Lebesgue null sets; ***the countable union*** and complements etc.	open sets, ***closed sets***, ***compact sets***, *elementary sets*, G_δ and F_σ sets etc.	***open sets***
set ***not in*** *collection*	some ***Lebesgue measurable sets***	some ***non-measurable sets***	some Jordan measurable set but not Borel measurable	***closed set***, G_δ and F_σ sets
function	***Boolean measurable function***; *Rieman integrable function*,	***Lebesgue measurable function***, *σ-finite function*, *continuous function*	***Borel measurable function***, *continuous function*	***continuous function***

4 Example

- **Example** 1. For finite set X, the power set 2^X is both algebra and σ-algebra of X.
  2. In particular, all finite Boolean algebra (atomic algebra) is a σ-algeba [Tao, 2011];
  3. All Lebesgue measuerable sets form a σ-algebra; All null sets and co-null sets form a σ-algebra [Tao, 2011].
  4. The elementary algebra $\overline{\mathcal{E}[\mathbb{R}^d]}$ and Jordan algebra $\overline{J[\mathbb{R}^d]}$ are *not* σ-algebra.
  5. the Lebesgue σ-algebra on $\mathbb{R}^d$ is generated by the union of the Borel σ-algebra and the null σ-algebra.
  6. An *algebra* of X can be defined as the collection of all *finite* and *cofinite* (i.e., its complement is finite) subsets in X. It is *not* a σ-algebra if X is infinite [Billingsley, 2008].
  7. A σ-algebra $\mathscr{F}$ of X can be defined as the collection of all *countable* and *co-countable* (i.e., its complement is countable) subsets in X. There exists subset A of X that is uncountable with uncountable complement. Thus $A \notin \mathscr{F}$, by definition, but $A \in 2^X$, and $\mathscr{F} \subsetneq 2^X$ [Billingsley, 2008].
  8. Use the σ-algebra $\mathscr{F}$ of X as defined above: note that the uncountable union A of singleton sets is uncountable and if A has uncountable complement, $A \notin \mathscr{F}$, although each singleton set is in $\mathscr{F}$. It shows that *arbitrary* union of sets may not in $\mathscr{F}$ [Billingsley, 2008].
  9. The restriction of σ-algebra $\mathscr{F}$ on subset Y, i.e. $\mathscr{F}|_Y$ is a σ-algebra.

- **Example** [Tao, 2011] The generation of σ-algebra, given a collection of sets $\mathcal{F}$. First, define the *ordinal* with ω_1 being the first uncountable ordinal. Define the sets $\mathcal{F}_\alpha$ for every countable ordinal $\alpha \in \omega_1$

  1. $\mathscr{F}_\alpha \equiv \mathcal{F}$
  2. For each countable successor ordinal $\alpha = \beta + 1$, we define F_α to be the collection of all sets that either the union of an *at most countable* number of sets in $\mathcal{F}_{n-1}$ (including the empty union ;), or the complement of such a union;
  3. For each countable limit ordinal $\alpha = sup_{\beta<\alpha}\beta$, we define $\mathcal{F}_\alpha \equiv \bigcup_{\beta<\alpha} \mathcal{F}_\beta$

  Then $\sigma(\mathcal{F}) = \bigcup_{\alpha\in\omega_1} \mathcal{F}_\alpha$ is the σ-algebra generated by $\mathcal{F}$.

- **Example** 1. If $\mathscr{C} = \{\emptyset\}$ or $\mathscr{C} = \{X\}$, then $\sigma(\mathscr{C}) = \{\emptyset, X\}$. It is a *Borel σ-algebra* over indiscrete topology on X.
  2. If $\mathscr{C} = \{\{a\}, a \in X\}$ the collection of all singletons, then $\sigma(\mathscr{C})$ is the collection of all *countable* and *co-countable* (i.e., its complement is countable) subsets in X. (called σ-algebra of countable and co-countable sets.)
  3. If $\mathscr{C} = \{(a, b], 0 \le a < b \le 1\}$ the collection of all subintervals in $(0, 1]$, then $\sigma(\mathscr{C})$ is the *Borel σ-algebra* on $(0, 1]$, denote as $\mathcal{B}((0, 1])$.

- **Example** Find the σ-algebra generated from the subintervals of $(0, 1]$.

Solution: We take the following sets

- Define a collection $\mathscr{B}_0$, where $\emptyset \in \mathscr{B}_0$ and $(0,1] \in \mathscr{B}_0$;
- Find the collection $\mathscr{C}$ of all *disjoint* subintervals $\mathscr{C} = \{(a_i, b_i], 0 \leq \cdots \leq a_i < b_i \leq a_{i+1} \cdots \leq 1\}$. And let $\mathscr{C} \subset \mathscr{B}_0$.
- Suppose $A = \bigcup_{i=1}^n (a_i, a_i'], n \in \mathbb{N}$ with $a_1 < a_1' \leq a_2 \cdots \leq a_n' \leq 1$, then $A^c = (0, a_1] \cup \left(\bigcup_{i=1}^{n-1} (a_i', a_{i+1}]\right) \cup (a_n', 1]$. Let $A, A^c \in \mathscr{B}_0$.
- Take the intersection btw A, B, as $A \cap B = \bigcup_{i=1}^n \bigcup_{j=1}^m ((a_i, a_i'] \cap (b_j, b_j'])$. Note that $A \cap B$ is union of disjoint subintervals, or intervals, or emptyset. So $A \cap B \in \mathscr{B}_0$.
- Repeated the above procedures until all finite union of disjoint subintervals in $\mathscr{C}$ is in $\mathscr{B}_0$.
- $\mathscr{B}_0$ is an algebra but not σ-algebra. It does not contain $\{b_i\} = \bigcup_{i \in \mathbb{N}} (b_i - \frac{1}{n}, b_i]$. The Borel σ-algebra is $\sigma(\mathscr{B}_0)$, including all the countable union and intersections of elements in $\mathscr{B}_0$.

- **Example** [Resnick, 2013, Billingsley, 2008]
  - $A_n = \left(\frac{1}{n}, 1\right]$, then $A_{n+1} \supset A_n$, the limits of sequence $\{A_n, n \geq 1\}$ is given as

$$\lim_{n\to\infty} A_n = \bigcup_{n\geq 1} \left(\frac{1}{n}, 1\right] = (0,1]$$

.

  - $A_n = \left[\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right]$, then

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \lim_{k\to\infty} \inf_{n\geq k} \left[\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right] \\
&= \bigcup_{k=1}^{\infty} \bigcap_{n \geq 2k-1} \left[\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right] \\
&= \bigcup_{k=1}^{\infty} \left[\frac{1}{2k}, 1 - \frac{1}{2k}\right] = (0,1)
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} A_n &= \lim_{k\to\infty} \sup_{n\geq k} \left[\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right] \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n \geq 2k-1} \left[\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right] \\
&= \bigcap_{k=1}^{\infty} \left[-\frac{1}{2k-1}, 1 + \frac{1}{2k-1}\right] = [0,1]
\end{aligned}$$

Thus $\lim_{n\to} A_n$ does not exists, although the end points are convergent. ∎

- **Example** [Resnick, 2013] Suppose $A_n = \left\{\frac{m}{n}, m \in \mathbb{N}\right\}, n \in \mathbb{N}$. What is $\liminf_{n\to} A_n$ and $\limsup_{n\to} A_n$?

Solution: Since $\frac{m_0}{n_0}$ for given (m_0, n_0), if $\frac{m_0}{n_0} \in \liminf_{n\to} A_n$, then $\exists k$ such that $\frac{m_0}{n_0} \in \left\{\frac{m}{n}, m \in \mathbb{N}\right\}$ for all $n \geq k$. It is impossible for $\frac{m_0}{n_0} \notin \mathbb{N}$. Therefore,

$$\begin{aligned}\liminf_{n\to} A_n &= \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\left\{\frac{m}{n}, m \in \mathbb{N}\right\} \\ &= \{0, 1, 2, \ldots, \} = \mathbb{N}.\end{aligned}$$

Since $\frac{m_0}{n_0}$ for given (m_0, n_0), for any $k \geq 1$, there always exists $n = k\,n_0 \geq k$ such that $\frac{m_0}{n_0} = \frac{n\,m_0}{n\,n_0} \in \left\{\frac{n_0\,m}{n\,n_0}, m \in \mathbb{N}\right\} \equiv \left\{\frac{m}{n}, m \in \mathbb{N}\right\}$

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{\frac{m}{n}, m \in \mathbb{N}\right\} \\ &= \left\{\frac{m}{n} | n, m \in \mathbb{N}\right\} = \mathbb{Q}. \quad \blacksquare\end{aligned}$$

- **Example** [Resnick, 2013] Show that

$$\liminf_{n\to\infty} A_n = \left\{x \mid \lim_{n\to\infty} \mathbb{1}\left\{x \in A_n\right\} = 1\right\}.$$

- **Example** [Resnick, 2013]

1. Suppose $\mathcal{C}$ is a finite partition of Ω, that is

$$\mathcal{C} = \{A_1, \ldots, A_k\}, \Omega = \bigcup_{i=1}^{k} A_i,\ A_i \cap A_j = \emptyset,\ \forall i \neq j.$$

Show that the minimal algebra $\mathcal{A}(\mathcal{C})$ generated by $\mathcal{C}$ is the class of unions of subfamilies of $\mathcal{C}$; that is,

$$\mathcal{A}(\mathcal{C}) = \left\{\bigcup_{I} A_j : I \subset \{1, \ldots, k\}\right\}.$$

(This includes the empty set.)

2. What is the σ-algebra generated from $\mathcal{C}$?

3. If $\mathcal{C} = \{A_1, \ldots\}$ is a countable partition of Ω, what is the induced σ-algebra?

4. If $\mathcal{A}$ is an algebra of subsets of Ω, we say $A \in \mathcal{A}$ is an atom of $\mathcal{A}$; if $A \neq \emptyset$ and for $\emptyset \neq B \in \mathcal{A}$, if $B \subset A$, then $B = A$. Thus A cannot be split into smaller nonempty set that is in $\mathcal{A}$.

   Example: $\Omega = \mathbb{R}$, and $\mathcal{A}$ is the algebra generated by intervals with integer end points $(a, b], a, b \in \mathbb{Z}$. What is the atoms in $\mathcal{A}$?

5. As converse to (1), prove that if $\mathcal{A}$ is a finite algebra of subsets of Ω, then the atoms of $\mathcal{A}$ constitute a finite partition of Ω that generates $\mathcal{A}$.

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