Lecture 7: Complete Metric Spaces and Function Spaces

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Contents

1 Complete Metric Space

- **Definition** (***Cauchy Net in Topological Vector Space***)
  A *net* $\{x_\alpha\}_{\alpha \in I}$ in ***toplogocial vector space*** X is called ***Cauchy*** if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta)\in I\times I}$ ***converges*** *to zero*. (Here $I \times I$ is ***directed*** in the usual way: $(\alpha, \beta) \prec (\alpha', \beta')$ if and only if $\alpha \prec \alpha'$ and $\beta \prec \beta'$.)

- **Definition** (***Completeness***)
  A toplogocial vector space X is ***complete*** if every Cauchy net converges.

- **Proposition 1.1** *(**Complete First Countable Topological Vector Space**)*
  *If X is a **first-countable topological vector space** and every **Cauchy sequence** in X converges, then every **Cauchy net** in X converges.*

- **Proposition 1.2** *(**Completeness of Euclidean Space**) [Munkres, 2000]*
  *Euclidean space $\mathbb{R}^k$ is **complete** in either of its usual **metrics**, the **euclidean metric** d or the **square metric** ρ.*

- **Lemma 1.3** *(**Convergence in Product Space is Weak Convergence**) [Munkres, 2000]*
  Let X be the product space $X = \prod_\alpha X_\alpha$; let x_n be a sequence of points of X. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α.

- **Proposition 1.4** *(**Completeness of Countable Product Space**) [Munkres, 2000]*
  *There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is **complete**.*

- **Definition** (***Uniform Metric in Function Space***)
  Let (Y, d) be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the ***standard bounded metric*** on Y derived from d. If $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha\in J}$ are points of the cartesian product Y^J, let

  $$\bar{\rho}(x, y) = \sup\left\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\right\}.$$

  It is easy to check that $\bar{\rho}$ is a metric; it is called ***the uniform metric*** on Y^J corresponding to the metric d on Y.

  Note that ***the space of all functions*** $f : J \to Y$, ***denoted*** as Y^J, is a subset of the product space $J \times Y$. We can define uniform metric in the function space: if f, $g : J \to Y$, then

  $$\bar{\rho}(f, g) = \sup\left\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\right\}.$$

- **Proposition 1.5** *(**Completeness of Function Space Under Uniform Metric**) [Munkres, 2000]*
  *If the space Y is **complete** in the metric d, then the space Y^J is **complete** in the **uniform metric** $\bar{\rho}$ corresponding to d.*

- **Definition** (***Space of Continuous Functions and Bounded Functions***)
  Let Y^X be *the space of all functions* $f : X \to Y$, where X is a *topological space* and Y is a *metric space with metric* d. Denote the ***subspace*** of Y^X consisting of all ***continuous functions*** f as $\mathcal{C}(X, Y)$.

  Also denote *the set of all **bounded functions*** $f : X \to Y$ as $\mathcal{B}(X, Y)$. (A function f is said to be ***bounded*** if its image $f(X)$ is a ***bounded subset*** of *the metric space* (Y, d).)

- **Proposition 1.6** ***(Completeness of*** $\mathcal{C}(X,Y)$ ***and*** $\mathcal{B}(X,Y)$ ***Under Uniform Metric)*** *[Munkres, 2000]*
  *Let X be a topological space and let (Y,d) be a metric space. The set $\mathcal{C}(X,Y)$ of **continuous** functions is **closed** in Y^X under the **uniform metric**. So is the set $\mathcal{B}(X,Y)$ of **bounded functions**. Therefore, if Y is **complete**, these spaces are **complete** in the **uniform metric**.*

- **Definition** (***Sup Metric on Bounded Functions***)
  If (Y,d) is a metric space, one can define another metric *on the set $\mathcal{B}(X,Y)$ of **bounded functions*** from X to Y by the equation

$$\rho(x,y) = \sup\{d(f(x),g(x)) : x \in X\}\,.$$

  It is easy to see that ρ is well-defined, for the set $f(X) \cup g(X)$ is ***bounded*** if both $f(X)$ and $g(X)$ are. The metric ρ is called ***the sup metric***.

- **Theorem 1.7** ***(Existence of Completion)*** *[Munkres, 2000]*
  *Let (X,d) be a metric space. There is an **isometric embedding** of X into a **complete** metric space.*

- **Definition** (***Completion***)
  Let X be a *metric space*. If $h: X \to Y$ is an ***isometric embedding*** of X into a ***complete*** *metric space* Y, then the ***subspace*** $h(X)$ of Y is a *complete metric space*. It is called ***the completion of*** X.

- **Definition** (***Topological Complete***)
  A space X is said to be ***topologically complete*** if there *exists* a metric for the *topology* of X relative to which X is *complete*.

- **Proposition 1.8** ***(Properties of Topological Complete)*** *[Munkres, 2000]*
  The followings are properties of topological completeness:

  1. *A **closed** subspace of a topologically complete space is topologically complete.*
  2. *A **countable product** of topologically complete spaces is topologically complete (in the **product topology**).*
  3. *An **open** subspace of a topologically complete space is topologically complete.*
  4. *A G_δ **set** in a topologically complete space is topologically complete.*

2 Compactness in Metric Spaces

2.1 Total Boundedness and Equicontinuous

- **Remark** (*Relate* ***Compactness*** *to* ***Completeness***)
  How is ***compactness*** of a metric space X related to ***completeness*** of X?

  The followings is from *the sequential compactness* and definition of *completeness*:

  Proposition 2.1 *Every **compact** metric space is **complete**.*

  The *converse* does not hold – ***a complete metric space need not be compact***. It is reasonable to ask what ***extra condition*** one needs to impose on a complete space to be assured of its compactness. Such a condition is the one called *total boundedness*.

- **Definition** (***Total Boundedness***)
  A metric space (X, d) is said to be ***totally bounded*** if for every $\epsilon > 0$, there is a ***finite covering*** *of* X *by* ϵ***-balls***.

- **Theorem 2.2** *(**Total Boundedness** + **Completeness** = **Compactness**)[Munkres, 2000]*
  A metric space (X, d) *is* ***compact*** ***if and only if*** *it is* ***complete*** *and* ***totally bounded***.

- **Remark** We now apply this result to find ***the compact subspaces*** of the space $\mathcal{C}(X, \mathbb{R}^n)$, *in the* ***uniform topology***. We know that a subspace of $\mathbb{R}^n$ is compact if and only if it is ***closed*** and ***bounded***.

  One might hope that an analogous result holds for $\mathcal{C}(X, \mathbb{R}^n)$. ***But*** it does not, even if X is *compact*. One needs to assume that the subspace of $\mathcal{C}(X, \mathbb{R}^n)$ satisfies *an* ***additional condition***, called ***equicontinuity***.

- **Definition** (***Equicontinuity***) [Reed and Simon, 1980, Munkres, 2000]
  Let (Y, d) be a *metric space*. Let $\mathscr{F}$ be a *subset* of the function space $\mathcal{C}(X, Y)$ (i.e. $f \in \mathscr{F}$ is continuous). If $x_0 \in X$, *the set* $\mathscr{F}$ *of functions* is said to be ***equicontinuous at*** x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that *for all* $x \in U$ and ***all*** $f \in \mathscr{F}$,

  $$d(f(x), f(x_0)) < \epsilon.$$

  If the set $\mathscr{F}$ is *equicontinuous* at x_0 for each $x_0 \in X$, it is said simply to be ***equicontinuous*** or $\mathscr{F}$ is an ***equicontinuous family***.

  We say $\mathscr{F}$ is a ***uniformly equicontinuous family*** if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ whenever $p(x, x') < \delta$ for all $x, x' \in X$ and ***every*** $f \in \mathscr{F}$.

- **Remark** An *equicontinuous family* of functions is *a family of continuous functions*.

- **Remark** ***Continuity*** of the function f at x_0 means that ***given*** f and given $\epsilon > 0$, there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for $x \in U$. ***Equicontinuity*** **of** $\mathscr{F}$ **means that** ***a single neighborhood*** U **can be chosen that will work for all the functions** f in the collection $\mathscr{F}$.

- **Lemma 2.3** *(**Total Boundedness** $\Rightarrow$ **Equicontinuous**) [Munkres, 2000]*
  Let X *be a* ***space****; let* (Y, d) *be a* ***metric*** *space. If the subset* $\mathscr{F}$ *of* $\mathcal{C}(X, Y)$ *is* ***totally bounded*** *under the* ***uniform metric*** *corresponding to d, then* $\mathscr{F}$ *is* ***equicontinuous*** *under d.*

- **Lemma 2.4** *(**Equicontinuous** + **Compactness** $\Rightarrow$ **Total Boundedness**) [Munkres, 2000]*
  Let X *be a space; let* (Y, d) *be a metric space; assume* X *and* Y *are* ***compact****. If the subset* $\mathscr{F}$ *of* $\mathcal{C}(X, Y)$ *is* ***equicontinuous*** *under d, then* $\mathscr{F}$ *is* ***totally bounded*** *under the* ***uniform*** *and* ***sup*** *metrics corresponding to d.*

- **Definition** (***Pointwise Bounded***)
  If (Y, d) is a *metric space*, a *subset* $\mathscr{F}$ of $\mathcal{C}(X, Y)$ is said to be ***pointwise bounded*** under d if for each $x \in X$, the subset

  $$F_a = \{f(a) : f \in \mathscr{F}\}$$

  of Y is ***bounded*** under d.

- **Theorem 2.5** ***(Ascoli's Theorem, Classical Version).*** *[Munkres, 2000]*
  Let X be a ***compact*** *space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding* ***uniform topology****. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has* ***compact closure if and only if*** *$\mathscr{F}$ is* ***equicontinuous*** *and* ***pointwise bounded*** *under d.*

- **Corollary 2.6** *[Munkres, 2000]*
  Let X be ***compact****; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding* ***uniform topology****. A subspace $\mathscr{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is* ***compact if and only if*** *it is* ***closed, bounded*** *under the* ***sup metric*** *ρ, and* ***equicontinuous*** *under d.*

- **Corollary 2.7** ***(Ascoli's Theorem, Sequence Version)*** *[Reed and Simon, 1980]*
  Let $\{f_n\}$ be a family of ***uniformly bounded equicontinuous functions*** *on $[0, 1]$. Then* ***some subsequence*** *$\{f_{n,m}\}$ converges* ***uniformly*** *on $[0, 1]$.*

- **Definition** (***Continuous Functions that Vanish At Infinity*** $\mathcal{C}_0(X, \mathbb{R})$)
  Let X be a space. A subset $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R})$ is said to ***vanish uniformly at infinity*** if given $\epsilon > 0$, there is a ***compact subspace*** C of X such that $|f(x)| < \epsilon$ for $x \in X \setminus C$ and $f \in \mathcal{F}$.

  If $\mathcal{F}$ consists of a single function f, we say simply that f ***vanishes at infinity***. Let $\mathcal{C}_0(X, \mathbb{R})$ denote *the set of continuous functions $f : X \to \mathbb{R}$ that* ***vanish at infinity***.

- **Corollary 2.8** *[Munkres, 2000]*
  Let X be ***locally compact Hausdorff****; give $\mathcal{C}_0(X, \mathbb{R})$ the uniform topology. A subset $\mathcal{F}$ of $\mathcal{C}_0(X, \mathbb{R})$ has* ***compact closure if and only if*** *it is* ***pointwise bounded****,* ***equicontinuous****, and* ***vanishes uniformly at infinity****.*

2.2 Pointwise and Compact Convergence

- **Definition** (***Topology of Pointwise Convergence / Point-Open Topology***)
  Given a point x of the set X and an open set U of the space Y, let

  $$S(x, U) = \left\{ f : f \in Y^X \text{ and } f(x) \in U \right\}.$$

  The sets $S(x, U)$ are a ***subbasis*** for *topology* on Y^X, which is called ***the topology*** of ***pointwise convergence*** (or ***the point-open topology***)

- **Remark** (***Basis of Point-Open Topology***)
  The general *basis element* for this topology is a *finite intersection* of subbasis elements $S(x, U)$. Thus a typical ***basis element*** about the function f *consists of all functions g that are* ***"close"*** *to f* ***at finitely many points***. Such a *neighborhood* is illustrated in Figure 1; it consists of all functions g whose graphs *intersect the three vertical intervals* pictured.

- **Remark** ***The topology of pointwise convergence on Y^X is the product topology***.

  If we replace X by J and denote the general element of J by α to make it look more familiar, then the set $S(\alpha, U)$ of all functions $x : J \to Y$ such that $x(\alpha) \in U$ is just the subset $\pi_\alpha^{-1}(U)$ of Y^J, which is *the standard subbasis element* for the product topology.

- **Proposition 2.9** ***(Pointwise Convergence Topology)****[Munkres, 2000]*
  A sequence f_n of functions ***converges*** *to the function f in the* ***topology of pointwise***

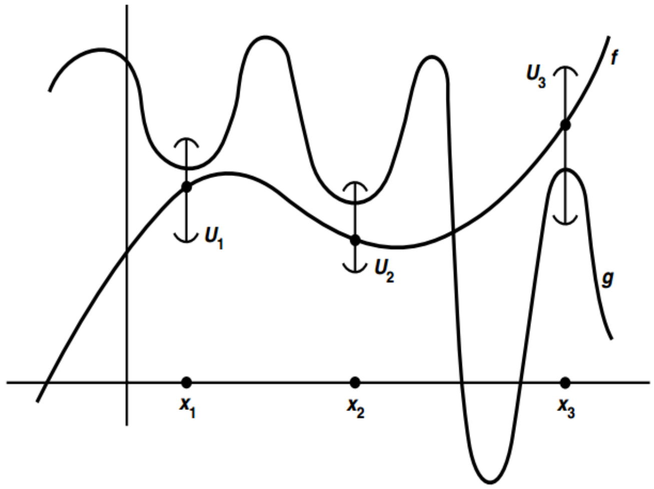

Figure 1: The function g in neighborhood of f in topology of pointwise convergence. [Munkres, 2000]

convergence if and only if *for* ***each*** x *in* X*, the sequence* $f_n(x)$ *of* ***points of*** Y *converges to the point* $f(x)$*.*

- **Remark** Compare the *subbasis* of *the point-open topology* on function space Y^X and *the weak topology* on space X

$$\begin{aligned} S(x,U) &= \left\{f : f \in Y^X \text{ and } f(x) \in U\right\} \quad \textit{point-open topology}. \\ B(f,U) &= \{x : x \in X \text{ and } f(x) \in U\} \quad \textit{weak topology}. \end{aligned}$$

- **Example** (***Pointwise Convergence Does Not Preserve Continuity***)
Consider the space $\mathbb{R}^I$, where $I = [0, 1]$. The sequence (f_n) of continuous functions given by $f_n(x) = x^n$ *converges* in the ***topology of pointwise convergence*** to the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases},$$

This example shows that the subspace $\mathcal{C}(I, \mathbb{R})$ of continuous functions is ***not closed*** in $\mathbb{R}^I$ *in the topology of pointwise convergence*. Note that $\mathcal{C}(I, \mathbb{R})$ is ***closed*** *in* $\mathbb{R}^I$ *under* ***uniform topology*** due to *Uniform Limit theorem.*

- **Definition** (***Topology of Compact Convergence***)
Let (Y, d) be a *metric space*; let X be a *topological space*. Given an element f of Y^X, a ***compact subspace*** C of X, and a number $\epsilon > 0$, let $B_C(f, \epsilon)$ denote the set of all those elements g of Y^X for which

$$\sup\{d(f(x), g(x)) : x \in C\} < \epsilon.$$

The sets $B_C(f, \epsilon)$ form a ***basis*** for a topology on Y^X. It is called the ***topology of compact convergence*** (or sometimes the "***topology of uniform convergence on compact sets***").

- **Proposition 2.10** *(**Topology of Uniform Convergence in Compact Sets**) [Munkres, 2000]*

A sequence $f_n : X \to Y$ of functions converges to the function f in the ***topology of compact convergence*** *if and only if for* ***each compact subspace*** *C of X, the sequence $f_n|_C$ converges* ***uniformly*** *to $f|_C$.*

- **Definition** (***Compactly Generated Space***)
  A space X is said to be ***compactly generated*** if it satisfies the following condition: A set A is ***open (or closed)*** in X if $A \cap C$ is ***open (or closed)*** in C for each ***compact subspace*** C of X.

- **Lemma 2.11** *[Munkres, 2000]*
  If X is ***locally compact****, or if X satisfies* ***the first countability axiom****, then X is* ***compactly generated.***

- The crucial fact about compactly generated spaces is the following:

  Lemma 2.12 ***(Continuous Extension on Compact Generated Space)*** *[Munkres, 2000]*
  If X is compactly generated, then a function $f : X \to Y$ is ***continuous*** *if for each* ***compact subspace*** *C of X, the restricted function $f|_C$ is* ***continuous.***

  Proof: Let V be an *open* subset of Y; we show that $f^{-1}(V)$ is *open* in X. Given any subspace C of X,

  $$f^{-1}(V) \cap C = (f|_C)^{-1}(V).$$

  If C is *compact*, this set is *open* in C because $f|_C$ is *continuous.* Since X is *compactly generated*, it follows that $f^{-1}(V)$ is *open* in X. ∎

- **Theorem 2.13** *($\mathcal{C}(X,Y)$* ***on Compact Generated Space****)* *[Munkres, 2000]*
  Let X be a ***compactly generated space****: let (Y,d) be a metric space. Then $\mathcal{C}(X,Y)$ is* ***closed*** *in Y^X in the* ***topology of compact convergence****.*

  Proof: Let $f \in Y^X$ be a *limit point* of $\mathcal{C}(X,Y)$; we wish to show f is *continuous.*

  It suffices to show that $f|_C$ is *continuous* for each *compact subspace* C of X, since by lemma above, we can extend f on entire space. For each n, consider the *neighborhood* $B_C(f, 1/n)$ of f; it *intersects* $\mathcal{C}(X,Y)$, so we can choose a function $f_n \in \mathcal{C}(X,Y)$ lying in this neighborhood. The sequence of functions $f_n|_C : C \to Y$ *converges uniformly* to the function $f|_C$, so that by *the uniform limit theorem*, $f|_C$ is *continuous.* ∎

- **Corollary 2.14** ***(Compact Convergence Limit)*** *[Munkres, 2000]*
  Let X be a ***compactly generated space****; let (Y,d) be a* ***metric*** *space. If a sequence of* ***continuous*** *functions $f_n : X \to Y$ converges to f in the* ***topology of compact convergence****, then f is* ***continuous****.*

- **Remark** (***Useful Topologies on*** Y^X)

  1. ***Uniform Topology***: generated by the ***basis***

     $$B_U(f,\epsilon) = \left\{ g \in Y^X : \sup_{x \in X} \bar{d}(f(x), g(x)) < \epsilon \right\}$$

     It corresponds to ***the uniform convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *uniform topology*, following *the Uniform Limit Theorem.*

2. ***Topology of Pointwise Convergence***: generated by the ***basis***

$$\begin{aligned} B_{U_1,\ldots,U_n}(x_1,\ldots,x_n,\epsilon) &= \bigcap_{i=1}^{n} S(x_i,U_i) \\ &= \left\{ f \in Y^X : f(x_1) \in U_1,\ldots,f(x_n) \in U_n \right\}, \quad 1 \le n < \infty. \end{aligned}$$

It corresponds to ***the pointwise convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***not closed*** in Y^X under the *topology of pointwise convergence*

3. ***Topology of Compact Convergence***: generated by the ***basis***

$$B_C(f,\epsilon) = \left\{ g \in Y^X : \sup_{x \in C} d(f(x),g(x)) < \epsilon \right\}.$$

It corresponds to ***the uniform convergence*** of f_n to f in Y^X for $x \in C$. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *topology of compact convergence* ***if X is compactly generated***.

- **Theorem 2.15** ***(Relationship between Topologies on*** Y^X***)*** *[Munkres, 2000]*
Let X be a space; let (Y,d) be a metric space. For the function space Y^X , one has the following ***inclusions of topologies:***

$$\textbf{(uniform)} \supseteq \textbf{(compact convergence)} \supseteq \textbf{(pointwise convergence)}.$$

If X is ***compact****, the* ***first two*** *coincide, and if X is* ***discrete****, the* ***second two*** *coincide.*

- **Remark** Note that both *uniform topology* and *topology of compact convergence* rmade specific use of the metric d for the space Y, i.e. it can only be defined when the image of function Y is a metric space.

But ***the topology of pointwise convergence*** does *not use the definition of metric d* in Y. In fact, ***it is defined for any image space*** Y.

- **Definition** (***Compact-Open Topology on Continuous Function Space***)
Let X and Y be topological spaces. If C is a ***compact subspace*** of X and U is an *open* subset of Y, define

$$S(C,U) = \{ f \in \mathcal{C}(X,Y) : f(C) \subseteq U \}.$$

The sets $S(C,U)$ form a ***subbasis*** for a *topology* on $\mathcal{C}(X,Y)$ that is called ***the compact-open topology***.

- **Proposition 2.16** ***(Compact-Open on*** $\mathcal{C}(X,Y)$ ***= Compact Convergence)*** *[Munkres, 2000]*
Let X be a space and let (Y,d) be a metric space. On the set $\mathcal{C}(X,Y)$, the ***compact-open topology*** *and the* ***topology of compact convergence coincide****.*

- **Corollary 2.17** ***(Compact Convergence on*** $\mathcal{C}(X,Y)$ ***Need Not*** d***)*** *[Munkres, 2000]*
Let Y be a metric space. The ***compact convergence topology*** *on $\mathcal{C}(X,Y)$ does* ***not*** *depend on the* ***metric*** *of Y. Therefore if X is* ***compact****, the* ***uniform topology*** *on $\mathcal{C}(X,Y)$ does not depend on the metric of Y.*

- **Remark** The fact that the definition of ***the compact-open topology*** does not involve a ***metric*** is just one of its useful features.

Another is the fact that it satisfies the requirement of "***joint continuity***. Roughly speaking, this means that the expression $f(x)$ is *continuous* not only in the *single*"variable x, but is ***continuous jointly*** *in* ***both*** *the x and f* .

- **Theorem 2.18** ***(Compact-Open Topology $\Rightarrow$ Joint Continuity for x and f)***
Let X be ***locally compact Hausdorff****; let $\mathcal{C}(X, Y)$ have the* ***compact-open topology****. Then the map*

$$e : X \times \mathcal{C}(X, Y) \to Y$$

defined by the equation

$$e(x, f) = f(x)$$

is ***continuous****. The map e is called* ***the evaluation map****.*

- **Definition** Given a function $f : X \times Z \to Y$, there is a corresponding function $F : Z \to \mathcal{C}(X, Y)$, defined by the equation

$$(F(z))(x) = f(x, z).$$

Conversely, given $F : Z \to \mathcal{C}(X, Y)$, this equation defines a corresponding function $f : X \times Z \to Y$. We say that F is the map of Z into $\mathcal{C}(X, Y)$ that is induced by f.

- **Proposition 2.19** *Let X and Y be spaces; give $\mathcal{C}(X, Y)$ the* ***compact-open topology****. If $f : X \times Z \to Y$ is* ***continuous****, then* ***so is*** *the induced function $F : Z \to \mathcal{C}(X, Y)$. The* ***converse*** *holds if X is* ***locally compact Hausdorff****.*

2.3 Ascoli's Theorem

- **Theorem 2.20** ***(Ascoli's Theorem, General Version)****. [Munkres, 2000]*
Let X be a space and let (Y, d) be a ***metric*** *space. Give $\mathcal{C}(X, Y)$ the* ***topology of compact convergence****; let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$.*

  1. *If $\mathcal{F}$ is* ***equicontinuous*** *under d and the set*

  $$F_a = \{f(a) : f \in \mathcal{F}\}$$

  has ***compact closure*** *for each $a \in X$, then $\mathcal{F}$ is* ***contained*** *in a* ***compact subspace*** *of $\mathcal{C}(X, Y)$.*

  2. *The* ***converse*** *holds if X is* ***locally compact Hausdorff****.*

- **Remark** Compare with classical version, we see generalizations:

  1. X need not to be ***compact***; $\Rightarrow$ does not even need X to be topological. $\Leftarrow$ holds when X is ***locally compact Hausdorff*** .

  2. $\mathcal{C}(X, Y)$ is under ***compact-open topology*** which is ***weaker*** than ***uniform topology***, i.e. we does not require convergence of sequence *uniformly* but only *uniformly in a compact subset*.

3. $\mathcal{F}$ does not need to be ***pointwise bounded*** under d. In other word, the set

$$F_a = \{f(a) : f \in \mathcal{F}\}$$

need not to be ***bounded*** but need to have ***compact closure*** for each $a \in X$. Note that for metric space Y, if Y is finite dimensional, it is the same requirement as boundness. But compact closure is stronger than bounded.

- **Proposition 2.21** *(**Equicontinuity** + **Pointwise Convergence** $\Rightarrow$ **Compact Convergence**) [Munkres, 2000]*
*Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of **continuous** functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f in the **topology of pointwise convergence**. If $\{f_n\}$ is **equicontinuous**, then f is **continuous** and f_n converges to f in the **topology of compact convergence**.*

3 Baire Category Theorem

- **Remark** (***Empty Interior*** = ***Complement is Dense***)
Recall that if A is a subset of a space X, the ***interior*** of A is defined as *the union of all open sets of X that are contained in A.*

  To say that A has ***empty interior*** is to say then that A ***contains no open set of*** X *other than the empty set.* ***Equivalently***, A has ***empty interior*** if every point of A is *a **limit point** of the **complement** of A*, that is, if *the **complement** of A is **dense** in X*.

$$\mathring{A} = \emptyset \quad \Leftrightarrow \quad A^c \text{ is dense in } X$$

  In [Reed and Simon, 1980], if a subset $\overline{A}$ of X has *empty interior*, A is said to be ***nowhere dense*** in X.

- **Example** Some examples:

  1. The set $\mathbb{Q}$ of *rationals* has ***empty interior*** as a subset of $\mathbb{R}$
  2. The *interval* $[0, 1]$ has ***nonempty interior***.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Space***)
A space X is said to be a ***Baire space*** if the following condition holds: Given ***any countable*** collection $\{A_n\}$ of ***closed*** sets of X each of which has ***empty interior*** in X, their ***union*** $\bigcup_{n=1}^{\infty} A_n$ also has ***empty interior*** in X.

- **Example** Some examples:

  1. The space $\mathbb{Q}$ of *rationals* is ***not** a **Baire space***. For each one-point set in $\mathbb{Q}$ is *closed and has empty interior **in*** $\mathbb{Q}$; and $\mathbb{Q}$ is *the countable union of its one-point subsets.*
  2. The space $\mathbb{Z}_+$, on the other hand, does form *a **Baire space***. Every subset of $\mathbb{Z}_+$ is *open*, so that there exist *no subsets* of $\mathbb{Z}_+$ having *empty interior*, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.

3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Category***)
  A subset A of a space X was said to be of ***the first category in*** X if it ***was contained*** *in the* ***union*** *of a* ***countable*** *collection of* ***closed*** *sets of* X *having* ***empty interiors*** *in* X; ***otherwise***, it was said to be of ***the second category in*** X.

- **Remark** *A space X is a* ***Baire space*** *if and only if every* ***nonempty open*** *set in X is of* ***the second category***.

- **Lemma 3.1** *(**Open Set Definition of Baire Space**) [Munkres, 2000]*
  X is a ***Baire space if and only if*** *given any* ***countable*** *collection $\{U_n\}$ of* ***open*** *sets in X, each of which is* ***dense*** *in X, their* ***intersection*** $\bigcap_{n=1}^{\infty} U_n$ *is also* ***dense*** *in X.*

- **Theorem 3.2** *(**Baire Category Theorem**). [Munkres, 2000]*
  If X is a ***compact Hausdorff*** *space or a* ***complete metric space***, *then X is a* ***Baire space***.

- **Remark** In other word, neither ***compact Hausdorff*** space or a ***complete metric space*** is a *countable union of closed subsets with empty interior (that are nowhere dense).*

- **Lemma 3.3** *[Munkres, 2000]*
  Let $C_1 \supset C_2 \supset \dots$ be a ***nested*** *sequence of* ***nonempty closed sets*** *in the* ***complete metric space*** *X. If diam $C_n \to 0$, then $\bigcap_n C_n = \emptyset$.*

- **Lemma 3.4** *[Munkres, 2000]*
  Any ***open*** *subspace Y of a Baire space X is itself a Baire space.*

- **Theorem 3.5** *(**Discontinuity Point of Pointwise Convergence Function**) [Munkres, 2000]*
  Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a ***Baire space***, *the set of points at which f is* ***continuous*** *is* ***dense*** *in X.*

- **Remark** (***Use Baire Category Theorem as Proof by Contradition***)
  The Baire category theorem is used to prove a certain subset C is ***dense*** in X by stating that X is a Baire space and C is countable intersection of dense open subsets in X (*C is a G_δ sets*).

  On the other hand, if $M = \bigcup_{n=1}^{\infty} A_n$ has ***nonempty interior***, then ***some*** of the sets $\bar{A}_n$ ***must have nonempty interior***. Otherwise, it contradicts with the Baire space definition.

References

James R Munkres. *Topology, 2nd*. Prentice Hall, 2000.

Michael Reed and Barry Simon. *Methods of modern mathematical physics: Functional analysis*, volume 1. Gulf Professional Publishing, 1980.