Lecture 5: Concentration of Measure and Isoperimetry

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1 The Classic Isoperimetry Inequalities

1.1 Brunn-Minkowski Inequality

- **Theorem 1.1** *(**Brunn-Minkowski Inequality**) [Boucheron et al., 2013, Vershynin, 2018, Wainwright, 2019]*
  *Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0, 1]$,*
  $$Vol\left(\lambda A + (1-\lambda) B\right)^{\frac{1}{n}} \geq \lambda\, Vol(A)^{\frac{1}{n}} + (1-\lambda)\, Vol(B)^{\frac{1}{n}}. \tag{1}$$
  Note: a convex body in $\mathbb{R}^n$ is closed and compact set.

- **Theorem 1.2** *(**The Prékopa-Leindler Inequality**). [Boucheron et al., 2013, Wainwright, 2019]*
  *Let $\lambda \in (0, 1)$, and let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be **non-negative measurable functions** such that for all $x, y \in \mathbb{R}^n$,*
  $$h\left(\lambda x + (1-\lambda) y\right) \geq f(x)^{\lambda} g(y)^{1-\lambda}.$$
  Then
  $$\int_{\mathbb{R}^n} h(x) dx \geq \left(\int_{\mathbb{R}^n} f(x) dx\right)^{\lambda} \left(\int_{\mathbb{R}^n} g(x) dx\right)^{1-\lambda}. \tag{2}$$

- **Corollary 1.3** *(**Weaker Brunn-Minkowski Inequality**) [Boucheron et al., 2013, Wainwright, 2019]*
  *Let $A, B \subset \mathbb{R}^n$ be **non-empty compact sets**. Then for all $\lambda \in [0, 1]$,*
  $$Vol\left(\lambda A + (1-\lambda) B\right) \geq Vol(A)^{\lambda}\, Vol(B)^{1-\lambda}. \tag{3}$$

1.2 Blow-Up of Sets

1.3 The Classical Isoperimetry Theorem

2 Concentration via Isoperimetry

2.1 Levy's Inequalities and Concentration Function

2.2 Isoperimetric Inequalities on the Unit Sphere

- **Remark** (***Volume Ratio of Unit Balls and its Interior***) [Vershynin, 2018]
  Let $B(0, 1) := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the unit ball in $\mathbb{R}^n$. The volume ratio between $B(0, 1)$ and its ϵ-interior $B(0, 1-\epsilon)$ is
  $$\frac{\mathrm{Vol}(B(0, 1-\epsilon))}{\mathrm{Vol}(B(0, 1))} = (1-\epsilon)^n \leq \exp\left(-n\epsilon\right)$$
  The inequality is due to $1 - x \leq e^{-x}$.

As $n \to \infty$, the above ratio goes to 0. In other words, most of volume in $B(0, 1)$ is ***concentrated*** in the ***boundary*** $\partial B = \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. This phenomenon is called "***the curse of dimensionality***".

- **Definition**

2.3 Gaussian Isoperimetric Inequalities and Concentration of Gaussian Measure

2.4 Edge Isoperimetric Inequality on the Binary Hypercube

2.5 Vertex Isoperimetric Inequality on the Binary Hypercube

2.6 Convex Distance Inequality

References

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Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.

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