Lecture 2: Banach Space

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Contents

1 Normed Linear Space

- **Definition** A ***metric space*** is a set M and a real-valued function $d(\cdot, \cdot) : M \times M \to \mathbb{R}$ which satisfies:

  1. (***Non-Negativity***) $d(x, y) \geq 0$
  2. (***Definiteness***) $d(x, y) = 0$ if and only if $x = y$
  3. (***Symmetric***) $d(x, y) = d(y, x)$
  4. (***Triangle Inequality***) $d(x, z) \leq d(x, y) + d(y, z)$

  The function d is called a ***metric*** on M. The metric space M equipped with metric d is denoted as (M, d).

- **Remark** Note that the definition of a *metric space* is only about the ***topology*** of the space. In the field of functional analysis, we are mostly concerned about ***the vector space***, i.e. a space that equipped with algebraic operations such as vector addition and scalar multiplications. In order to make the ***metric topological structure compatible*** with ***the algebraic structure** of **vector space***, we need to introduce additional function such as the ***norm***.

- **Definition** (***Normed Linear Space***)
  A ***normed linear space*** is a vector space, V, over $\mathbb{R}$ (or $\mathbb{C}$) and a function, $\|\cdot\| : V \to \mathbb{R}$ which satisfies:

  1. (***Non-Negativity***): $\|v\| \geq 0$ for all v in V;
  2. (***Positive Definiteness***): $\|v\| = 0$ if and only if $v = 0$;
  3. (***Absolute Homogeneity***) $\|\alpha v\| = |\alpha| \, \|v\|$ for all v in V and α in $\mathbb{R}$ (or $\mathbb{C}$)
  4. (***Subadditivity / Triangle Inequality***) $\|v + w\| \leq \|v\| + \|w\|$ for all v and w in V

  We denote the normed linear space as $(V, \|\cdot\|)$.

- **Remark** If the function $p : V \to \mathbb{R}$ only satisfies the condition 1, 3 and 4 (without *positive definiteness*), it is called a ***semi-norm***. The 1. *non-negativity* condition can be derived by the 3. *homogeneity* and 4. *subadditivity* conditions.

- **Remark** *A normed linear space* $(V, \|\cdot\|)$ is a ***metric space*** with *induced metric*

$$d(x, y) = \|x - y\| \,, \quad \text{for all } x, y \in V$$

- **Definition** (***Bounded Linear Operator***)
  A ***bounded linear transformation*** (or ***bounded operator***) is a mapping $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ from a normed linear space X to a normed linear space Y that satisfies

  1. (***Linearity***) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$
  2. (***Boundedness***) $\|Tx\|_Y \leq C \, \|x\|_X$ for small $C \geq 0$.

  The smallest such C is called *the **norm** of T*, written $\|T\|$ or $\|T\|_{X,Y}$. Thus

$$\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y$$

- **Remark** *A linear operator T* is a ***homomorphism*** *of a vector space (its domain) into another vector space*, that is, T ***preserves*** *the* ***two operations*** *of vector space.*

- **Proposition 1.1** *[Reed and Simon, 1980, Kreyszig, 1989]*
Let T be a linear transformation between two ***normed linear spaces****. The following are* ***equivalent****:*

  1. *T is* ***continuous*** *at* ***one*** *point.*
  2. *T is* ***continuous*** *at* ***all*** *points.*
  3. *T is* ***bounded****.*

- **Definition** A normed linear space $(V, \|\cdot\|)$ is ***complete*** if it is *complete* as a *metric space* in *the induced metric.*

- **Theorem 1.2** ***(The B.L.T. Theorem)*** *[Reed and Simon, 1980]*
Suppose T is a bounded linear transformation from a normed linear space $(V_1, \|\cdot\|_1)$ to a ***complete*** *normed linear space $(V_2, \|\cdot\|_2)$. Then T can be* ***uniquely extended*** *to a bounded linear transformation (with the same bound), $\widetilde{T}$, from the* ***completion*** *of V_1 to $(V_2, \|\cdot\|_2)$.*

2 Banach Space

2.1 Definition and Examples

- **Definition** A ***complete*** *normed linear space* is called a ***Banach space***.

- **Example** ($\mathcal{C}(X)$ and its subspace $\mathcal{C}_{\mathbb{R}}(X)$)
Let $\mathcal{C}(X)$ be the set of all *complex-valued* ***continuous functions*** on X and $\mathcal{C}_{\mathbb{R}}(X) \subseteq \mathcal{C}(X)$ be the set of all ***real-valued continuous functions*** on X. Also define $\mathcal{C}^b(X)$ as the set of all *complex-valued* ***bounded continuous functions*** on X. When X is ***a compact space***, $\mathcal{C}^b(X) = \mathcal{C}(X)$. Define the norm as

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \, .$$

  Then for ***compact Hausdorff space** X*, $\mathcal{C}(X)$ is a *(complex)* ***Banach space*** and $\mathcal{C}(X)$ is ***a (real) Banach space***.

- **Example** ($L^\infty(\mathbb{R})$ ***and its subspace*** $\mathcal{C}^0(\mathbb{R})$)
Let $L^\infty(\mathbb{R})$ be the set of (*equivalence classes of*) *complex-valued measurable functions* on $\mathbb{R}$ such that $|f(x)| \leq M$ a.e. with respect to Lebesgue measure for some $M < \infty$ ($f = g$ means $f(x) = g(x)$ a.e.). Let $\|f\|_\infty$ be ***the smallest such*** M. $L^\infty(\mathbb{R})$ is a ***Banach space*** with norm $\|\cdot\|_\infty$.

  The ***bounded continuous functions*** $\mathcal{C}^0(\mathbb{R})$ is a ***subspace*** of $L^\infty(\mathbb{R})$ and restricted to $\mathcal{C}^0(\mathbb{R})$ the $\|\cdot\|_\infty$-norm is just the usual ***supremum norm*** under which $\mathcal{C}^0(\mathbb{R})$ is ***complete*** (since *the uniform limit of continuous functions is continuous* See proof in chapter 1.). Thus, $\mathcal{C}^0(\mathbb{R})$ ***is a closed subspace of*** $L^\infty(\mathbb{R})$.

  Consider the set $\kappa(\mathbb{R})$ of ***continuous functions*** *with* ***compact support***, that is, the continuous functions that *vanish outside of some closed interval.* $\kappa(\mathbb{R})$ is a ***normed linear space*** under $\|\cdot\|_\infty$; but *is* ***not complete***, The ***completion*** of $\kappa(\mathbb{R})$ is ***not all*** *of* $\mathcal{C}^0(\mathbb{R})$; for example, if f is the function which is identically equal to one, then I *cannot be approximated by a function in* $\kappa(\mathbb{R})$ since $\|f - g\|_\infty \geq 1$ for all $g \in \kappa(\mathbb{R})$. The ***completion*** of $\kappa(\mathbb{R})$ is just $\mathcal{C}_\infty(\mathbb{R})$, *the continuous functions which* ***approach zero*** *at* ∞.

Some of the most powerful theorems in functional analysis (*Riesz-Markov*, *Stone-Weierstrass*) are generalizations of properties of $\mathcal{C}^0(\mathbb{R})$. $\blacksquare$

- **Example** (L^p ***spaces***)
  Let (X, μ) be a measure space and $p \geq 1$. We denote by $L^p(X, \mu)$ ***the set of equivalence classes*** of measurable functions which satisfy:

  $$\|f\|_p := \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} < \infty$$

  Two functions are *equivalent* if they differ only on *a set of measure zero.*

  The following theorem collects many of the standard facts about L^p spaces.

  Theorem 2.1 *Let* $1 \leq p < \infty$*, then*

  1. ***(The Minkowski Inequality)****: If* $f, g \in L^p(X, \mu)$*, then*

     $$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

  2. ***(Riesz-Fisher)****:* $L^p(X, \mu)$ *is* ***complete.***
  3. ***(The Hölder Inequality)*** *Let* $p, q,$ *and* r *be positive numbers satisfying* $p, q, r \geq 1$ *and* $p^{-1} + q^{-1} = r^{-1}$*. Suppose* $f \in L^p(X, \mu)$*,* $g \in L^q(X, \mu)$*. Then* $fg \in L^r(X, \mu)$ *and*

     $$\|fg\|_r \leq \|f\|_p \, \|g\|_q$$

  Remark *The Minkowski inequality* shows that $L^p(X, \mu)$ is a vector space and $\|\cdot\|_p$ satisfies the triangle inequality. This together with *Riesz-Fisher theorem* shows that $L^p(X, \mu)$ ***is a Banach space***.

- **Example** (***Sequence Spaces***)
  There is a nice class of spaces which is easy to describe and which we will often use to illustrate various concepts. In the following definitions,

  $$a := (a_n)_{n=1}^\infty$$

  always denotes a sequence of complex numbers.

  $$\ell^\infty := \left\{a : \|a\|_\infty := \sup_n |a_n| < \infty\right\}$$
  $$c_0 := \left\{a : \lim_{n \to \infty} a_n = 0\right\}$$
  $$\ell^p := \left\{a : \|a\|_p := \left(\sum_{n=1}^\infty |a_n|^p\right)^{\frac{1}{p}} < \infty\right\}$$
  $$s := \left\{a : \lim_{n \to \infty} n^p a_n = 0 \textit{ for all positive integers } p\right\}$$
  $$f := \{a : a_n = 0 \textit{ for all but a finite number of } n\}$$

It is clear that as sets $f \subseteq s \subseteq \ell^p \subseteq c_0 \subseteq \ell^\infty$.

The spaces ℓ^∞ and c_0 are Banach spaces with the $\|\cdot\|_\infty$ norm; ***ℓ^p is a Banach space with the $\|\cdot\|_p$ norm*** (note that $\ell^p = L^p(\mathbb{R}, \mu)$ where μ is the measure with *mass one at each positive integer* and *zero* everywhere else). It will turn out that s ***is a Frechet space***.

One of the reasons that these spaces are easy to handle is that *f is **dense in** ℓ^p (in $\|\cdot\|_p$; $p < \infty$* and ***f is dense in c_0 (in the $\|\cdot\|_\infty$ norm)***. Actually, the set of elements of f with only *rational entries* is also **dense** in ℓ^p and c_0. Since this set is ***countable***, ℓ^p and c_0 are ***separable***. ***ℓ^∞ is not separable***.

- **Example** (***The Bounded Operators***)
  In above we defined the concept of a *bounded linear transformation* or *bounded operator* from one normed linear space, X, to another Y; we will denote *the set of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$*. We can introduce a *norm* on $\mathcal{L}(X, Y)$ by defining

$$\|A\| := \sup_{x \neq 0,\, x \in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

  This norm is often called ***the operator norm***.

  We have the following proposition

  Proposition 2.2 *If Y is **complete**, $\mathcal{L}(X, Y)$ is a **Banach space**.*

- **Example** (***Hilbert Space***)
  All ***Hilbert spaces*** $(\mathcal{H}, \langle\cdot\,,\,\cdot\rangle)$ are ***Banach spaces*** with *induced norm* as

$$\|x\| = (\langle x\,,\,x\rangle)^{\frac{1}{2}}\,.$$

2.2 Isomorphism and Equivalence of Norms

- **Definition** (***Absolutely Summable***)
  A sequence of elements $(x_n)_{n=1}^\infty$ in a *normed linear space* X is called ***absolutely summable*** $\sum_{n=1}^\infty \|x_n\| < \infty$. It is called ***summable*** if $\sum_{n=1}^N x_n$ *converges* as $N \to \infty$ to an $x \in X$.

- **Proposition 2.3** ***(Criterion of Completeness for Normed Linear Space)*** *[Reed and Simon, 1980]*
  *A normed linear space is **complete** if and only if every **absolutely summable** sequence is **summable**.*

- **Definition** (***Isomorphism between Normed Linear Spaces***)
  A ***bounded linear operator*** from a normed linear space X to a normed linear space Y is called an ***isomorphism*** if it is a ***bijection*** which is ***continuous*** and which has ***a continuous inverse***.

  If it is ***norm preserving***, it is called ***an isometric isomorphism*** (any *norm preserving* map is called an ***isometry***).

- **Remark** The *isomorphism* is defined in above way is essentially a ***linear homemorphism***.

- **Definition** (***Norm Equivalence***)
  Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are called ***equivalent*** if there are

positive constants C and C' such that, for all $x \in X$,

$$C\,\|x\|_2 \le \|x\|_1 \le C'\,\|x\|_2$$

- **Remark** This concept is motivated by the following fact.

  Equivalent norms *on X define* ***the same topology*** *for X.*

- **Proposition 2.4** *The* ***completions*** *of the space in the two norms will be* ***isomorphic*** *if and only if* ***the norms are equivalent****.*

- **Proposition 2.5** *Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are* ***equivalent*** *if and only if* ***the identity map*** *is an* ***isomorphism****.*

- **Remark** An example is provided by *the sequence spaces.* The ***completion*** of f in the $\|\cdot\|_\infty$ norm is c_0 while the completion in the $\|\cdot\|_p$ norm is ℓ^p.

2.3 Subspace of a Banach Space

- **Definition** A ***subspace*** Y of a normed space X is a subspace of X considered as *a vector space*, with the *norm* obtained by ***restricting*** *the norm on X to the subset Y*. This norm on Y is said to be ***induced*** *by the norm on X*.

  If Y is closed in X, then Y is called ***a closed subspace*** of X.

- **Remark** A subspace Y of a ***Banach space*** X is a subspace of X considered as a normed space. Hence *we do not require Y to be complete.*

- **Proposition 2.6** *(**Subspace of a Banach space**). [Kreyszig, 1989]*
  A subspace Y of a Banach space X is ***complete*** *if and only if the set Y is* ***closed*** *in X.*

2.4 Basis and Separability

- **Definition** (***Basis of Normed Space***)
  If a normed space X contains *a sequence* (e_i) with the property that for *every* $x \in X$ there is a ***unique*** *sequence of scalars* (u^i) such that

$$\lim_{n\to\infty}\left\|x - \sum_{i=1}^{n} u^i\, e_i\right\| = 0, \tag{1}$$

  then (e_i) is called *a* ***Schauder basis (or basis)*** *for X*. The series $\sum_{i=1}^{\infty} u^i\, e_i$ which has the sum x is then called *the* ***expansion*** *of x with respect to* (e_i), and we write

$$x = \sum_{i=1}^{\infty} u^i\, e_i$$

- **Example** The (Schauder) basis of ℓ^p is (e_n) and

$$e_n := (\delta_{n,i}) = (0\,, \ldots, 0, 1, 0,\, \ldots)$$

  where the i-th component is 1 and the others are all zeros.

- **Proposition 2.7** *If a normed space X has a Schauder basis, then X is* ***separable****.*

- **Theorem 2.8** *(**Completion**). [Kreyszig, 1989]*
Let $X = (X, \|\cdot\|)$ be a normed space. Then there is a Banach space X and an isometry A from X onto a subspace W of X which is ***dense*** *in X. The space X is* ***unique****, except for isometries.*

2.5 Finite Dimensional Normed Spaces and Subspaces

- **Lemma 2.9** *(**Linear combinations**). [Kreyszig, 1989]*
Let $(x_1, \ldots, x_n)$ be a ***linearly independent*** *set of vectors in a normed space X (of any dimension). Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \ldots, \alpha_n$ we have*

$$\left\| \sum_{i=1}^{n} \alpha_i \, x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \,. \tag{2}$$

- **Theorem 2.10** *(**Completeness**). [Kreyszig, 1989]*
Every finite dimensional subspace Y of a normed space X is ***complete****. In particular, every* ***finite dimensional*** *normed space is* ***complete****.*

- **Remark** In other words, *every finite dimensional normed vector space* is a ***Banach space***.

- **Proposition 2.11** *(**Closedness**). [Kreyszig, 1989]*
Every finite dimensional subspace Y of a normed space X is ***closed*** *in X.*

- **Theorem 2.12** *(**Equivalent Norms**). [Kreyszig, 1989]*
If a vector space X is ***finite dimensional****,* ***all norms are equivalent****.*

- **Remark** This theorem is of considerable practical importance. For instance, it implies that ***convergence*** or *divergence* of a sequence in *a finite dimensional vector space* ***does not depend*** *on the particular* ***choice of a norm*** *on that space.* There is no ambiguity when we say $x_n \to x$ in *finite dimensional space.*

  In fact, ***there exists only one distinct norm topology for finite dimensional space***.

- **Definition** (***Compactness***).
A metric space X is said to be ***(sequentially) compact*** if *every sequence* in X has a ***convergent subsequence***. A subset M of X is said to be ***compact*** if M is *compact* considered as a subspace of X, that is, if every sequence in M has a convergent subsequence *whose limit is an element of M.*

- **Lemma 2.13** *(**Compactness**).*
A ***compact*** *subset M of a metric space is* ***closed*** *and* ***bounded****.*

- **Remark** *The* ***converse*** *of this lemma is in general* ***false****.* But for *finite dimensional space*, the converse is true:

- **Theorem 2.14** *(**Compactness**). [Kreyszig, 1989]*
In a ***finite*** *dimensional normed space X, any subset $M \subseteq X$ is* ***compact if and only if*** *M is* ***closed*** *and* ***bounded****.*

- **Remark** *In finite dimensional space*, the *compact* subsets are precisely *the closed and bounded* subsets, so that this property (***closedness*** and ***boundedness***) can be used for *defining* ***compactness***.

  However, *this can no longer be done* in the case of *an **infinite dimensional normed space***.

- **Lemma 2.15** ***(F. Riesz's Lemma)**. [Kreyszig, 1989]*
  *Let Y and Z be **subspaces** of a normed space X (of any dimension), and suppose that Y is **closed** and is a **proper subset** of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that*

$$\|z\| = 1, \quad \|z - y\| \geq \theta, \quad \text{for all } y \in Y.$$

- **Theorem 2.16** ***(Bounded Linear Operator)***
  *If a normed space X is finite dimensional, then every linear operator on X is **bounded**.*

- **Remark** (***Finite Dimensional Normed Space is Simple***)
  We summarizes the ***unique*** simple strcuture of finite dimensional normed space in terms of various concepts we discussed in this chapter:

  1. ***Completeness***: *Every finite dimensional normed vector space* is ***complete*** so it is a ***Banach space***;
  2. ***Norm Equivalence***: ***All norms*** *in a finite dimensional normed space are* ***equivalent***; therefore, ***convergence*** in one norm means convergence in all other norms.
  3. ***Topological Equivalence***: There exists ***only one distinct norm topology*** in a finite dimensional normed space;
  4. ***Compactness***: In a finite dimensional normed space, ***compactness*** is equivalent to ***closedness*** and ***boundedness***.
  5. ***Bounded Linear Operator***: ***Every linear operator*** between finite dimensional normed spaces is ***bounded***. Thus in finite dimensional space, every linear operator is ***continuous***.

2.6 Direct Sum of Banach Spaces

- **Definition** (***Direct Sum of Banach Spaces***)
  Let A be an index set (not necessarily countable), and suppose that for each $\alpha \in A$, X_α is a Banach space. Let

$$X := \left\{ (x_\alpha)_{\alpha \in A} : x_\alpha \in X_\alpha, \; \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha} < \infty \right\}.$$

  Then X with the norm

$$\|(x_\alpha)_{\alpha \in A}\|_X := \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha}$$

  is a Banach space. It is called ***the direct sum*** of the spaces X_α and is often written as $X = \bigoplus_{\alpha \in A} X_\alpha$.

- **Remark** (***Banach Spaces Direct Sum*** $\neq$ ***Hilbert Spaces Direct Sum***)
  Note that *the direct sum of Banach spaces* is ***not*** *necesssarily the direct sum of Hilbert spaces.*

  For instance, if we take countable numbers of copies of $\mathbb{C}$, *the Banach space direct sum* is ℓ_1, while *the Hilbert space direct sum* is ℓ_2.

  However, if only ***finite number*** *of Hilbert spaces* are involved, then both *Hilbert space direct sum* and *their Banach space direct sum* are *isomorphic* to each other.

2.7 Dual Space and Double Dual Space

- **Definition** (***Dual Space***)
  The space $\mathcal{L}(X, \mathbb{C})$ of all ***bounded linear functionals*** on *a normed linear space X* is called the ***dual space*** *of X*. This space $\mathcal{L}(X, \mathbb{C})$ is denoted as X^*.

  The dual space X^* is a Banach space *if X is a Banach space* (See Proposition 2.2). The ***norm*** of dual space is

  $$\|\lambda\| := \sup_{x \neq 0,\, \|x\| \leq 1} |\lambda(x)|\,,$$

  for all $\lambda \in X^*$.

- **Remark** By definition, we have the *dual norm inequality*

  $$|\lambda(x)| \leq \|\lambda\|_{X^*}\ \|x\|_X\,. \tag{3}$$

  In Hilbert space, since $\lambda(x) = \langle y_\lambda\,,\, x\rangle$ for some y_λ, it becomes *the Cauchy-Schwartz inequality.*

  $$|\langle y_\lambda\,,\, x\rangle| \leq \|y_\lambda\|\, \|x\|$$

- **Example** (***Hilbert Space***)
  Any ***Hilbert space*** $\mathcal{H}$ is ***isomorphic*** to its ***dual*** $\mathcal{H}^*$ according to *the Riesz Representation Theorem.* For instance $L^2(X, \mu) = (L^2(X, \mu))^*$.

- **Example** ($L^p(X, \mu)$ ***Spaces,*** $1 < p < \infty$)
  Suppose that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ then. according to *the Hölder inequality*, fg is in $L^1(X, \mu)$. Thus,

  $$\int_X f(x)\overline{g(x)} d\mu(x) < \infty$$

  makes sense, Let $g \in L^q(X, \mu)$ be fixed and define

  $$G(f) := \int_X f\overline{g} d\mu$$

  for each $f \in L^p(X, \mu)$. The Hölder inequality shows that $G(f)$ *is a* ***bounded linear functional*** on $L^p(X, \mu)$ with *norm* less than or equal to $\|g\|_q$; actually ***the norm*** $\|G\|$ ***is equal to*** $\|g\|_q$.

The *converse* of this statement is also *true*. That is, ***every bounded linear functional on*** L^p ***is of the form*** $G(f)$ ***for some*** $g \in L^q$. Furthermore, *different* ***functions*** *in* L^q *give rise to different* ***functionals*** *on* L^p. Thus, the mapping

$$L^q(M,\mu) \to (L^p(X,\mu))^*, \quad g \mapsto G_g(\cdot)$$

is a ***(conjugate linear) isometric isomorphism***.

In this sense, $L^q(M,\mu)$ ***is the dual of*** $L^p(X,\mu)$. Since the roles of p and q in the expression $p^{-1} + q^{-1} = 1$ are *symmetric*, it is clear that $L^p(X,\mu) = (L^q(X,\mu))^* = (L^p(X,\mu))^{**}$. That is, the ***dual*** of the ***dual*** of $L^p(X,\mu)$ is again $L^p(X,\mu)$. $\blacksquare$

- **Remark** Note that $L^\infty(X,\mu)$ space and $L^1(X,\mu)$ space are ***not dual*** spaces to each other. The dual space of $L^\infty(X,\mu)$ space is much larger than $L^1(X,\mu)$ space. In fact, $L^1(X,\mu)$ ***space is not dual to any Banach space***. This is different from ℓ^∞ and ℓ^1.

- **Example** ($\ell^\infty = (\ell^1)^*$, $\ell^1 = (c_0)^*$)
Suppose that $(\lambda_k)_{k=1}^\infty \in \ell^1$. Then for each $(a_k)_{k=1}^\infty \in c_0$,

$$\Lambda\left((a_k)_{k=1}^\infty\right) = \sum_{k=1}^\infty \lambda_k\, a_k$$

converges and $\Lambda(\cdot)$ is a ***continuous linear functional*** on c_0 with ***norm*** equal to $\sum_{k=1}^\infty |\lambda_k|$.

To see that *all continuous linear functionals on* c_0 *arise in this way*, we proceed as follows. Suppose $\lambda \in c_0^*$ and let e^k be the sequence in c_0 which has *all its terms equal to zero except for a one in the k-th place*. Define $\lambda_k = \lambda(e^k)$ and let $f^l = \sum_{k=1}^l (|\lambda_k|/\lambda_k)\, e^k$. If some λ_k is zero, we simply omit that term from the sum. Then for each l, $f^l \in c_0$ and $\|f^l\|_{c_0} = 1$. Since,

$$\lambda(f^l) = \sum_{k=1}^l |\lambda_k|, \quad \left|\lambda(f^l)\right| \le \|\lambda\|_{c_0^*}\, \|f^l\|_{c_0}$$

we have

$$\sum_{k=1}^l |\lambda_k| \le \|\lambda\|_{c_0^*}$$

Since this is true for all l, $\sum_{k=1}^\infty |\lambda_k| < \infty$ and

$$\Lambda\left((a_k)_{k=1}^\infty\right) = \sum_{k=1}^\infty \lambda_k\, a_k$$

is a *well-defined linear functional on* c_0. However, $\Lambda(\cdot)$ and $\lambda(\cdot)$ agree on *finite linear combinations* of the e^k. Because such *finite linear combinations are* ***dense*** *in* c_0 we conclude that $\lambda = \Lambda$. Thus every functional in c_0 arises from a sequence in ℓ^1, and the ***norms*** in ℓ^1 and c_0 coincide. Thus $\ell^1 = c_0^*$. A similar proof shows that $\ell^\infty = (\ell^1)^*$. $\blacksquare$

- **Remark** We see that $c_0 \subseteq (c_0)^{**} = (\ell^1)^* = \ell^\infty$.

- **Definition** (***Double Dual***)
Since the ***dual*** X^* of *a Banach space* is itself *a Banach space*, it also has *a* ***dual*** *space*, denoted by X^{**}. X^{**} is called ***the second dual***, ***the bidual***, or ***the double dual*** *of the space* X.

- **Proposition 2.17** *[Reed and Simon, 1980]*
  Let X be a Banach space. For each $x \in X$, let $\widetilde{x}(\cdot)$ be the linear functional on X^* which assigns to each $\lambda \in X^*$ the number $\lambda(x)$. Then the map $J : x \mapsto \widetilde{x}$ is an ***isometric isomorphism*** *of X onto a (possibly proper) subspace of X^{**}.*

- **Remark** From above proposition, we see that there exists an *embedding* from X to a subset of X^{**}

$$X \subseteq X^{**}, \quad X \hookrightarrow X^{**}$$

- **Definition** If the map $J : x \mapsto \widetilde{x}$ is ***surjective***, then X is said to be ***reflexive***. In other word, X is reflective if and only if $X = X^{**}$.

- **Example** $L^p(X, \mu)$ spaces are ***reflective*** for $1 < p < \infty$. Note that $L^p(X, \mu) = (L^q(X, \mu))^* = (L^p(X, \mu))^{**}$

- **Example** All Hilbert spaces $\mathcal{H}$ are ***reflective***.

- **Example** Since $c_0 \subseteq (c_0)^{**} = (\ell^1)^* = \ell^\infty$, c_0 is *not reflective.*

3 The Hahn-Banach Theorem

- **Remark** In dealing with Banach spaces, one often needs to *construct* ***linear functionals*** with ***certain properties***. This is usually done in two steps:

  1. one ***defines the linear functional*** on *a* ***subspace*** *of the Banach space* where it is easy to verify the desired properties;

  2. one appeals to (or proves) a general theorem which says that *any such functional* can be ***extended to the whole space*** while ***retaining the desired properties***.

  One of the basic tools of the second step is *the Hahn-Banach theorem.*

3.1 Extension Form of The Hahn-Banach Theorem

- **Definition** (***Sublinear Functional***)
  If X is a vector space, a ***sublinear functional*** on X is a map $p : X \to \mathbb{R}$ such that

  1. (***Homogeneity***): $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ and $x \in X$;

  2. (***Sublinearity***): $p(x + y) \leq p(x) + p(y)$,

- **Example** Every ***semi-norm*** is a *sublinear functional.* If p is a semi-norm, then the condition $f \leq p$ is equivalent to $|f| \leq p$.

- **Theorem 3.1** *(**The Hahn-Banach Theorem, Extension Form**) [Kreyszig, 1989, Reed and Simon, 1980, Luenberger, 1997, Folland, 2013]*
  Let X be a real normed linear space and p a ***sublinear functional*** *on X. Let f be a* ***linear functional*** *defined on a* ***subspace*** *M of X satisfying $f(x) \leq p(x)$ for all $x \in M$. Then there exists a* ***linear functional*** *F on X such that $F(x) \leq p(x)$ for all $x \in X$ and $F|_M = f$. (F is called an* ***extension*** *of f.)*

Proof: The ***idea*** of the proof is the following.

1. First we will show that if $x \in X$ but $x \notin M$, then we can ***extend*** f, to a functional g having the property $g(x) \leq p(x)$ on ***the space spanned by*** x ***and*** M (i.e. *the affine subspace* $M + \mathbb{R}\, x$).
2. We then use a ***Zorn's Lemma*** argument to show that *this process can be* ***continued to extend****), to the whole space* X.

(*Step 1*). If $y_1, y_2 \in M$, by linearity of functional f and $f \leq p$, we have

$$\begin{aligned} f(y_1) + f(y_2) &= f(y_1 + y_2) \\ &\leq p(y_1 + y_2) = p(y_1 - x + y_2 + x) \\ &\leq p(y_1 - x) + p(y_2 + x) \\ \Rightarrow f(y_1) - p(y_1 - x) &\leq p(y_2 + x) - f(y_2) \end{aligned}$$

Hence

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \inf_{y \in M} \{p(y + x) - f(y)\}$$

Let α be any number satisfying

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \alpha \leq \inf_{y \in M} \{p(y + x) - f(y)\}$$

and define $g : M + \mathbb{R}\, x \to \mathbb{R}$ by $g(y + \gamma x) := f(y) + \gamma\, \alpha$. Then g is *linear*, and $g|_M = f$, so that $g(y) \leq p(y)$ for $y \in M$. Moreover, if $\gamma > 0$ and $y \in M$,

$$\begin{aligned} g(y + \gamma x) &= \gamma(\gamma^{-1} f(y) + \alpha) \\ &= \gamma \left(f\left(\frac{y}{\gamma}\right) + \alpha \right) \\ &\leq \gamma \left[f\left(\frac{y}{\gamma}\right) + p\left(\frac{y}{\gamma} + x\right) - f\left(\frac{y}{\gamma}\right) \right] \quad \text{(upper bound of } \alpha) \\ &= \gamma p\left(\frac{y}{\gamma} + x\right) = p\left(y + \gamma\, x\right), \end{aligned}$$

where if $\gamma = -\mu < 0$,

$$\begin{aligned} g(y + \gamma x) &= \mu \left[f\left(\frac{y}{\gamma}\right) - \alpha \right] \\ &\leq \mu \left[f\left(\frac{y}{\gamma}\right) + p\left(\frac{y}{\gamma} - x\right) - f\left(\frac{y}{\gamma}\right) \right] \quad \text{(lower bound of } \alpha) \\ &\leq \mu p\left(\frac{y}{\gamma} - x\right) = p(y + \gamma x). \end{aligned}$$

Thus $g(z) \leq p(z)$ for all $z \in M + \mathbb{R}\, x$.

(*Step 2*). Apparently the same reasoning can be applied to *any linear functional* F *of* f *satisifying* $F \leq p$ *on its domain* and we shows that the *domain* of *a* ***maximal*** *linear extension* F satisfying $F \leq p$ must be the whole space X.

Define $\mathcal{F}$ as *the family of* ***all linear extensions*** F of f satisfying $F \leq p$. By construction in *Step 1* we see that $\mathcal{F}$ is ***partially ordered by inclusion*** where $F_1 \preceq F_2$ if F_2 is defined on a ***larger set*** than F_1 and $F_2(x) = F_1(x)$ where they are both defined.

Let $\{F_\alpha\}_{\alpha \in A}$ be a ***linearly ordered subset*** of $\mathcal{F}$; let X_α be the subspace on which F_α is defined. Define F on $\bigcup_{\alpha \in A} X_\alpha$ by setting $F(x) = F_\alpha(x)$ if $x \in X_\alpha$. Clearly $F_\alpha \preceq F$ so ***each linearly ordered subset of $\mathcal{F}$ has an upper bound***. By ***Zorn's lemma***, $\mathcal{F}$ has a ***maximal element*** Λ, defined on some set X', satisfying $\Lambda(x) \leq p(x)$ for $x \in X'$. But, X' *must be all of* X, since *otherwise* we could extend Λ to a $\widetilde{\Lambda}$ on a *larger* space by adding one dimension as above. Since this ***contradicts*** *the* ***maximality*** *of* Λ, we must have $X = X'$. Thus, the extension Λ is *everywhere defined.* ∎

- **Theorem 3.2** ***(The Complex Hahn-Banach Theorem, Extension Form)*** *[Kreyszig, 1989, Reed and Simon, 1980, Luenberger, 1997, Folland, 2013]*
  Let X be a complex normed linear space and p a ***semi-norm*** *on X. Let f be a* ***complex linear functional*** *defined on a* ***subspace*** *M of X satisfying $|f(x)| \leq |p(x)|$ for all $x \in M$. Then there exists a* ***complex linear functional*** *F on X such that $|F(x)| \leq |p(x)|$ for all $x \in X$ and $F|_M = f$. (F is called an* ***extension*** *of f.)*

- **Corollary 3.3** ***(The Existance of Minimum Norm Extension)***
  Let $f \in M^*$ be a bounded linear functional defined on a ***subspace*** *M of a real normed vector space X. Then there is a bounded linear functional $F \in X^*$ defined on X which is an* ***extension*** *of f satisfying $\|F\|_{X^*} = \|f\|_{M^*}$.*

  Note let $p(x) = \|f\|_{M^*} \|x\|$.

- **Corollary 3.4** *Let y be an element of a normed linear space X. Then there is a nonzero $F \in X^*$ such that $F(y) = \|F\|_{X^*} \|y\|_X$.*

- **Corollary 3.5** ***(The Existance of Distance Functional)***
  Let Z be a subspace of a normed linear space X and suppose that y is an element of X whose ***distance*** *from Z is $d = \inf_{z \in Z} \|y - z\|$. Then there exists a $F \in X^*$ so that $\|F\| \leq 1$, $F(y) = d$, and $F(z) = 0$* ***for all*** *z* ***in*** *Z.*

- **Remark** *The Hahn-Banach theorem*, particularly *Corollary 3.3*, is perhaps most profitably viewed as ***an existence theorem for a minimization problem***. Given an f on a subspace M of a normed space, it is not difficult to extend f to the whole space. ***An arbitrary extension***, however, will in general be ***unbounded*** or *have norm greater than the norm of f on M*. We therefore pose the problem of *selecting the extension of minimum norm.* The Hahn-Banach theorem both guarantees ***the existence of a minimum norm extension*** and tells us ***the norm of the best extension***.

- **Remark** (***Convex Analysis***)
  Note that any sublinear functional p is ***convex***, i.e. for all $x, y \in X$, $\alpha \in [0, 1]$

$$p(\alpha x + (1 - \alpha) y) \leq \alpha p(x) + (1 - \alpha) p(y).$$

  So the Hahn-Banach theorem can be stated as extention of linear functional $f(x) \leq p(x)$ bounded by a convex functional p. Then the inquality in the proof becomes

$$\frac{1}{\alpha} \sup_{y \in M} \{f(y) - p(y - \alpha\, x)\} \leq \frac{1}{\beta} \inf_{y \in M} \{p(y + \beta\, x) - f(y)\}$$

where the left-hand side in Hilbert space becomes

$$\sup_{y \in M} \left\{ \langle x_f \,,\, \alpha^{-1} y \rangle - p(\alpha^{-1} y - x) \right\}$$

This is ***the Legendre transformation*** *for convex function* $h(y) := p(y - x)$

$$h^*(x) := \sup_{y \in \alpha^{-1} M} \{ \langle x \,,\, y \rangle - h(y) \}$$

- **Proposition 3.6** *Let X be a Banach space. If X^* is* ***separable****, then X is* ***separable****.*

3.2 Geometric Form of The Hahn-Banach Theorem

- **Definition** The ***translation*** of a subspace is called *a* ***linear variety****.* It is written as $x + M$ where $x \in X$ is a fixed point and $M \subseteq X$ is a subspace of X.
- **Remark** *A linear variety* is also called *an* ***affine subspace****.*
- **Definition** A ***hyperplane*** H in a linear vector space X is a ***maximal*** *proper linear variety*, that is, a linear variety H such that $H \neq X$, and if V is any linear variety containing H, then either $V = X$ or $V = H$.
- **Remark** A *hyperplane* $H = x + M$ where M has ***codimension*** 1 in X, i.e.

$$X = \text{span}\{x, \text{basis of } M\}.$$

- **Proposition 3.7** *[Luenberger, 1997]*
Let H be a ***hyperplane*** *in a linear vector space X. Then there is a* ***linear functional*** *f on X and a constant c such that $H = \{x : f(x) = c\}$.* ***Conversely,*** *f is a nonzero linear functional on X, the set $\{x : f(x) = c\}$ is a hyperplane in X.*
- There exists an ***one-to-one correspondence*** between linear functional and hyperplane that does not passes the origin.

  Proposition 3.8 ***(Unique Linear Functional for Hyperplane)*** *[Luenberger, 1997]*
  Let H be a hyperplane in a linear vector space X. If H ***does not contain the origin****, there is a* ***unique*** *linear functional f on X such that $H = \{x : f(x) = 1\}$.*
- **Proposition 3.9** *Let f be a nonzero linear functional on a normed space X. Then the hyperplane $H = \{x : f(x) = c\}$ is* ***closed*** *for every c if and only if f is* ***continuous****.*
- **Remark** If f is a nonzero linear functional on a linear vector space X, we associate with the hyperplane $H = \{x : f(x) = c\}$ *the four sets*

$$\{x : f(x) \le c\}, \quad \{x : f(x) < c\}, \quad \{x : f(x) \ge c\}, \quad \{x : f(x) > c\}$$

  called ***half-spaces determined by*** H. The first two of these are referred to as ***negative half-spaces*** *determined by* f and the second two as ***positive half-spaces****.*

  If f is *continuous*, *the first and the third* half-spaces are ***closed*** and *the second and fourth* are ***open****.*

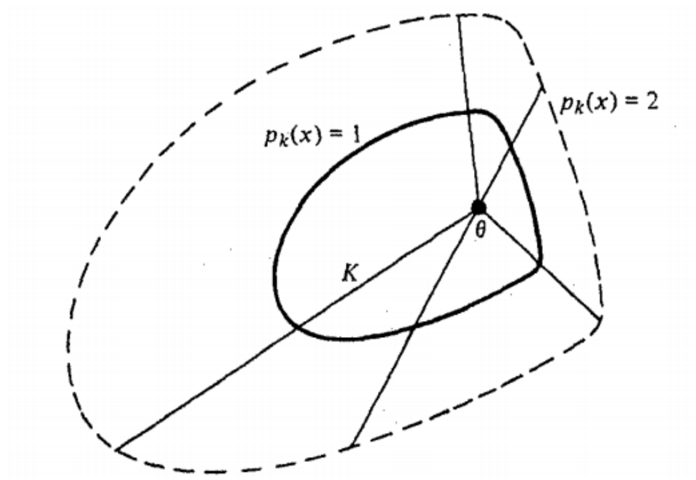

Figure 1: The Minkowski functional of a convex set [Luenberger, 1997]

- **Definition** (***The Minkowski Functional***) [Luenberger, 1997]
  Let K be ***a convex set*** in a *normed linear vector space* X and suppose 0 is *an **interior point** of* K. Then ***the Minkowski functional (or gauge)*** p of K is defined on X by

  $$p(x) := \inf\left\{r : \frac{x}{r} \in K, r > 0\right\} = [\sup\{t : t\,x \in X, t > 0\}]^{-1}.$$

  We note that for K *equal to the unit sphere in* X, *the Minkowski functional is* $\|x\|$. In the general case, $p(x)$ defines a kind of ***distance*** from *the origin to* x measured *with respect to* K; it is the *factor* by which K must be expanded so as to *include* x.

- **Lemma 3.10** *Let K be a convex set containing 0 as an interior point. Then **the Minkowski functional** p of K satisfies:*

  1. $0 \leq p(x) < \infty$ *for all* $x \in X$;
  2. *(**Homogeneity**):* $p(\lambda x) = \lambda p(x)$ *for all* $\lambda \geq 0$ *and* $x \in X$;
  3. *(**Sublinearity**):* $p(x + y) \leq p(x) + p(y)$,
  4. p *is **continuous**;*
  5. $\overline{K} = \{x : p(x) \leq 1\}$ *and* $\mathring{K} = \{x : p(x) < 1\}$.

  That is, *the Minkowski functional* is *a sublinear functional.*

- **Theorem 3.11** *(**Mazur's Theorem, Geometric Hahn-Banach Theorem**) [Luenberger, 1997]*
  *Let K be a **convex set** having a **nonempty interior** in a real normed linear vector space X. Suppose V is a **linear variety** in X containing no interior points of K. Then there is a **closed hyperplane** in X containing V but **containing no interior points** of K; i.e., there is an element $f \in X^*$ and a constant c such that $f(v) = c$ for all $v \in V$ and $f(k) < c$ for all $k \in K$.*

  Proof: By an appropriate translation we may assume that 0 is an interior point of K. Let M *be the subspace of* X *generated by* V. Then V is a ***hyperplane*** in M and does not contain 0; thus there is a ***linear functional*** f on M such that $V = \{x : f(x) = 1\}$.

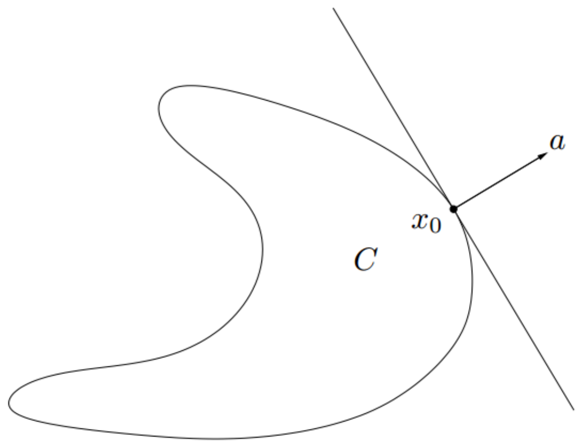

Figure 2: The supporting hyperplane of a convex set [Boyd et al., 2004]

We then show that $f(x) \leq p(x)$ for all $x \in M$, where $p(x)$ is ***the Minkowski functional*** (a sublinear functional). Since V contains no interior point of K, we have $f(x) = 1 \leq p(x)$ for $x \in V$. By homogeneity, $f(\alpha x) = \alpha \leq p(\alpha x)$ for $x \in V$ and $\alpha > 0$. While for $\alpha < 0$, $f(\alpha x) \leq 0 \leq p(\alpha x)$. Thus $f(x) \leq p(x)$ for all $x \in M$.

Then by *the Hahn-Banach Theorem*, there is an ***extension*** F of f from M to X with $F(x) \leq p(x)$. Let $H = \{x : F(x) = 1\}$. Since $F(x) \leq p(x)$ on X and since by Lemma 1 p is ***continuous***, F is ***continuous***, $F(x) < 1$ for $x \in \mathring{K}$, therefore, H is the desired closed hyperplane. $\blacksquare$

- **Remark** (***Geometric Interpretation of the Hahn-Banach theorem***)
  The ***geometric form*** *of the* **Hahn-Banach theorem**, in simplest form, says that given *a* ***convex set*** K containing ***an interior point***, and given a point x_0 *not in* $\mathring{K}$, there is a ***closed hyperplane containing*** x_0 but ***disjoint*** from $\mathring{K}$.

- **Definition** (***Supporting Hyperplane***)
  A ***closed*** *hyperplane* H in a normed space X is said to be ***a supporting hyperplane*** (or a ***support***) for the ***convex set*** K if K is *contained* in *one of the* **closed half-spaces** determined by H and H contains a point of $\overline{K}$.

- **Remark** Suppose $K \subseteq \mathbb{R}^n$, and x_0 is a point in its boundary ∂K, i.e.,
  $$x_0 \in \partial K = \overline{K} \setminus \mathring{K}.$$
  If $a \neq 0$ satisfies $\langle a\,,\, x\rangle \leq \langle a\,,\, x_0\rangle$ for all $x \in K$, then the hyperplane $\{x : \langle a\,,\, x\rangle = \langle a\,,\, x_0\rangle\}$ is called ***a supporting hyperplane*** to K at the point x_0.

- **Theorem 3.12** *(**Supporting Hyperplane Theorem**) [Luenberger, 1997, Rockafellar, 1970]*
  If x is ***not an interior point*** *of a convex set K which contains interior points, there is a* ***closed hyperplane*** *H containing x such that K lies on one side of H.*

- As a consequence of the above theorem, it follows that, for a convex set K with interior points, ***a supporting hyperplane*** *can be constructed containing* ***any boundary point of*** $\overline{K}$.

Theorem 3.13 *(**Eidelheit's Separation Theorem**) [Luenberger, 1997, Rockafellar, 1970]*
Let K_1 and K_2 be ***convex sets*** *in X such that K_1 has interior points and K_2* ***contains no***

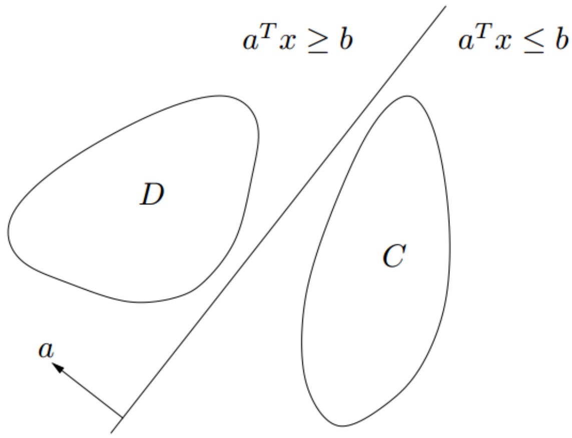

Figure 3: The existence of a separating hyperplane between two convex sets that does not overlap in interior set. [Boyd et al., 2004]

interior point of K_1*. Then there is a* ***closed hyperplane*** H ***separating*** K_1 *and* K_2*; i.e., there exists* $f \in X^*$ *such that*

$$\sup_{x \in K_1} f(x) \leq \inf_{x \in K_2} f(x) \tag{4}$$

In other words, K_1 *and* K_2 *lie in* ***opposite half-spaces*** *determined by* H*.*

Proof: Let $K = K_1 - K_2 = \{x_1 - x_2 : x_1 \in K_1, x_2 \in K_2\}$; then K contains an interior point and 0 not one of them. Also K is a convex set. By *The Supporting Hyperplane Theorem*, there is an $f \in X^*$, $f \neq 0$, such that $f(x) \leq 0$ for $x \in K$. Thus for $x_1 \in K_l$, $x_2 \in K_2$, $f(x_1) \leq f(x_2)$. Consequently, there is a real number c such that $\sup_{x \in K_1} f(x) \leq c \leq \inf_{x \in K_2} f(x)$. The desired hyperplane is $H = \{x : f(x) = c\}$. ∎

- **Corollary 3.14** *If* K *is a* ***closed convex*** *set and* $x \notin K$*, there is a* ***closed halfspace*** *that contains* K *but does not contain* x*.*

- **Theorem 3.15** *(**Dual Representation of Convex Set**)[Luenberger, 1997, Rockafellar, 1970]*
  If K *is a* ***closed convex*** *set in a normed space, then* K *is equal to the* ***intersection*** *of all the* ***closed half-spaces*** *that contain it.*

- **Remark** (***Duality for Convex Set***)
  Theorem above is often regarded as ***the geometric foundation*** *of* ***duality theory*** *for* ***convex sets***. By associating *closed hyperplanes* (or *half-spaces*) with elements of X^*, the theorem expresses ***a convex set in*** X ***as a collection of elements in*** X^*. See more in [Rockafellar, 1970].

- **Definition** Let K be a convex set in a real normed vector space X. The functional

$$h(f) := \sup_{x \in K} f(x)$$

  defined on X^* is called ***the support functional*** of K. $h \in X^{**}$.

- **Remark** The ***support functional*** of a *convex set K completely specifies the set (to within **closure**)*

$$\overline{K} = \bigcap_{f \in X^*} \{x : f(x) \le h(f)\}\,.$$

4 Linear Operators in Banach Space

4.1 Adjoints of Bounded Operator

- **Definition** (***Banach Space Adjoint***)
Let X and Y be *Banach spaces*, T *a* ***bounded linear operator*** from X to Y. The ***Banach space adjoint of*** T, denoted by $T^{'}$, is *the* ***bounded linear operator*** *from* Y^* *to* X^* defined by

$$\left(T'f\right)(x) = f\left(Tx\right)$$

for all $f \in Y^*$, $x \in X$.

- **Example** (***Adjoint of Right Shift Operator***)
Let $X = \ell^1 = Y$ and let be ***the right shift operator***

$$T\left(\xi_1, \xi_2, \ldots\right) = \left(0, \xi_1, \xi_2, \ldots\right)$$

Then $T' : \ell^\infty \to \ell^\infty$ is ***the left shift operator***

$$T'\left(\xi_1, \xi_2, \ldots\right) = \left(\xi_2, \xi_3, \ldots\right).$$

- **Proposition 4.1** *(**Isomorphism between Bounded Operator and its Adjoint**). [Reed and Simon, 1980]*
*Let X and Y be **Banach spaces**. The map $T \to T'$ is an **isometric isomorphism** of $\mathcal{L}(X, Y)$ into $\mathcal{L}(Y^*, X^*)$.*

- **Remark** (***Hilbert Space Adjoint***)
Let $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ be *the space of bounded linear operators* on $\mathcal{H}$. *The Banach space adjoint* of T^* then a mapping of $\mathcal{H}^*$ to $\mathcal{H}^*$. Let $C : \mathcal{H} \to \mathcal{H}^*$ be the map which assigns to each $y \in \mathcal{H}$, the bounded linear functional $\langle y\,,\,\cdot\rangle$ in $\mathcal{H}^*$. C is a ***conjugate linear isometry*** which is ***surjective*** by *the Riesz Representation theorem* (so it is ***unitary***). Now define a map $T^* : \mathcal{H} \to \mathcal{H}$ by

$$T^* = C^{-1} T' C$$

Then T^* satisfies

$$\langle x\,,\,Ty\rangle = (Cx)(Ty) = (T'Cx)(y) = \left\langle C^{-1}T'Cx\,,\,y\right\rangle = \langle T^*x\,,\,y\rangle\,,$$

T^* is called ***the Hilbert space adjoint of*** T, but usually we will just call it the adjoint and let the T^* distinguish it from T'. Notice that the map $T \to T^*$ is ***conjugate linear***, that is, $\alpha T \to \bar{\alpha} T^*$. This is because C is conjugate linear.

- **Proposition 4.2** *[Reed and Simon, 1980]*
*The map $T \to T^*$ is always **continuous** in the **weak** and **uniform operator topologies** but is only continuous in the **strong operator topology** if $\mathcal{H}$ is **finite dimensional**.*

4.2 Baire Category Theorem

- **Remark** (***Empty Interior = Complement is Dense***)
  Recall that if A is a subset of a space X, the ***interior*** of A is defined as *the union of all open sets of X that are contained in A.*

  To say that A has ***empty interior*** is to say then that A ***contains no open set of*** X *other than the empty set.* ***Equivalently***, A has ***empty interior*** if every point of A is *a* ***limit point*** *of the* ***complement*** *of* A, that is, if *the* ***complement*** *of A is* ***dense*** *in X*.

$$\mathring{A} = \emptyset \quad \Leftrightarrow \quad A^c \text{ is dense in } X$$

  In [Reed and Simon, 1980], if a subset $\overline{A}$ of X has *empty interior*, A is said to be ***nowhere dense*** in X.

- **Example** Some examples:

  1. The set $\mathbb{Q}$ of *rationals* has ***empty interior*** as a subset of $\mathbb{R}$
  2. The *interval* $[0, 1]$ has ***nonempty interior***.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Space***)
  A space X is said to be a ***Baire space*** if the following condition holds: Given ***any countable*** collection $\{A_n\}$ of ***closed*** sets of X each of which has ***empty interior*** in X, their ***union*** $\bigcup_{n=1}^{\infty} A_n$ also has ***empty interior*** in X.

- **Example** Some examples:

  1. The space $\mathbb{Q}$ of *rationals* is ***not*** *a* ***Baire space***. For each one-point set in $\mathbb{Q}$ is *closed and has empty interior* ***in*** $\mathbb{Q}$; and $\mathbb{Q}$ is *the countable union of its one-point subsets.*
  2. The space $\mathbb{Z}_+$, on the other hand, does form *a* ***Baire space***. Every subset of $\mathbb{Z}_+$ is *open*, so that there exist *no subsets* of $\mathbb{Z}_+$ having *empty interior*, except for the empty set. Therefore, $\mathbb{Z}_+$ satisfies the Baire condition vacuously.
  3. The *interval* $[0, 1] \times 0$ has ***empty interior*** as a *subset of the plane* $\mathbb{R}^2$, and so does the *subset* $\mathbb{Q} \times \mathbb{R}$.

- **Definition** (***Baire Category***)
  A subset A of a space X was said to be of ***the first category in*** X if it ***was contained*** *in the* ***union*** *of a* ***countable*** *collection of* ***closed*** *sets of X having* ***empty interiors*** *in X*; ***otherwise***, it was said to be of ***the second category in*** X.

- **Remark** *A space X is a* ***Baire space*** *if and only if every* ***nonempty open*** *set in X is of* ***the second category***.

- **Lemma 4.3** *(**Open Set Definition of Baire Space**) [Munkres, 2000]*
  X is a ***Baire space if and only if*** *given any* ***countable*** *collection $\{U_n\}$ of* ***open*** *sets in X, each of which is* ***dense*** *in X, their* ***intersection*** *$\bigcap_{n=1}^{\infty} U_n$ is also* ***dense*** *in X.*

- **Theorem 4.4** *(**Baire Category Theorem**). [Munkres, 2000]*
  If X is a ***compact Hausdorff*** *space or a* ***complete metric space***, *then X is a* ***Baire***

space.

- **Remark** In other word, neither ***compact Hausdorff*** space or a ***complete metric space*** is a *countable union of closed subsets with empty interior (that are nowhere dense).*

- **Lemma 4.5** *[Munkres, 2000]*
*Let $C_1 \supset C_2 \supset \dots$ be a **nested** sequence of **nonempty closed sets** in the **complete metric space** X. If diam $C_n \to 0$, then $\bigcap_n C_n = \emptyset$.*

- **Lemma 4.6** *[Munkres, 2000]*
*Any **open** subspace Y of a Baire space X is itself a Baire space.*

- **Theorem 4.7** *(**Discontinuity Point of Pointwise Convergence Function**) [Munkres, 2000]*
*Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a **Baire space**, the set of points at which f is **continuous** is **dense** in X.*

- **Remark** (***Use Baire Category Theorem as Proof by Contradition***)
The Baire category theorem is used to prove a certain subset C is ***dense*** in X by stating that X is a Baire space and C is countable intersection of dense open subsets in X (*C is a G_δ sets*).

  On the other hand, if $M = \bigcup_{n=1}^{\infty} A_n$ has ***nonempty interior***, then ***some*** of the sets $\bar{A}_n$ ***must have nonempty interior***. Otherwise, it contradicts with the Baire space definition.

4.3 Uniform Boundedness Theorem

- **Proposition 4.8** *[Reed and Simon, 1980]*
*Let X and Y be normed linear spaces. Then a linear map $: X \to Y$ is **bounded** if and only if*

$$T^{-1}\left\{y : \|y\|_Y \le 1\right\}$$

  *has a **nonempty interior**.*

  Proof: Suppose that T is given and the set in question contains the ball

$$\{x : \|x - x_0\|_X < \epsilon\}$$

  Then $\|x\| < \epsilon$ implies $x + x_0$ is in the ball of radius ϵ about x_0. Thus $\|T(x + x_0)\|_Y \le 1$ and

$$\|Tx\|_Y \le \|T(x + x_0)\|_Y + \|T(x_0)\|_Y \le 1 + \|T(x_0)\|_Y .$$

  Thus for *all* $x \in X$,

$$\|Tx\|_Y \le \epsilon^{-1}\left(1 + \|T(x_0)\|_Y\right)\|x\|_X$$

  so is *bounded*. The *converse* is easy. ∎

- **Theorem 4.9** *(**The Uniform Boundedness Theorem**). [Reed and Simon, 1980]*
*Let X be a **Banach space**. Let $\mathscr{F}$ be a family of **bounded** linear transformations from X to*

*some **normed linear space** Y. Suppose that for each $x \in X$, $\{\|Tx\|_Y : T \in \mathscr{F}\}$ is **bounded**, i.e.*

$$\sup_{T\in\mathscr{F}} \|Tx\|_Y < \infty.$$

*Then $\{\|T\| : T \in \mathscr{F}\}$ is **bounded**, i.e.*

$$\sup_{T\in\mathscr{F}} \|T\| < \infty.$$

Proof: Let

$$B_n := \{x : \|Tx\|_Y \le n, \ \forall\, T \in \mathscr{F}\}.$$

By the hypothesis each x is in some B_n, that is, $X = \bigcup_{n=1}^{\infty} B_n$. Moreover each B_n is ***closed*** since each T is continuous. By *the Baire category theorem*, some B_n has a ***nonempty interior***. By proposition 4.8, we conclude that the $\|T\|$'s are *uniformly bounded*. ∎

- **Corollary 4.10** ***(Separately Continuity of Bilinear Form on Banach Space = Joint Continuity)*** *[Reed and Simon, 1980]*
  *Let X and Y be Banach spaces and let $B(\cdot,\cdot)$ be a **separately continuous bilinear mapping** from $X \times Y$ to $\mathbb{C}$, that is, it is a **bounded** linear transformation if one of the two arguments is fixed. Then $B(\cdot,\cdot)$ is **jointly continuous**, that is, if $x_n \to 0$ and $y_n \to 0$ then $B(x_n, y_n) \to 0$.*

4.4 Open Mapping Theorem

- **Theorem 4.11** ***(Open Mapping Theorem)*** *[Reed and Simon, 1980]*
  *Let $T : X \to Y$ be a **surjective bounded linear transformation** of one **Banach** space **onto** another **Banach** space Y. Then if M is an **open** set in X, $T(M)$ is **open** in Y.*

  Proof: We need only show that, for every *neighborhood* N of x, $T(N)$ is a *neighborhood* of $T(x)$. Since $T(x+N) = T(x) + T(N)$, we need only show this for $x = 0$. Since neighborhoods contain balls it is sufficient to show that $T(B_X(0,r)) \supseteq B_Y(0,r')$, for some r' where

  $$B_X(0,r) = \{x \in X : \|x\|_X < r\}.$$

  However, since $T(B_X(0,r)) = r\,T(B_X(0,1))$, we need only show that $T(B_X(0,r))$ is a *neighborhood* of *zero* for some r. Finally, by the "*translation argument*" of the proposition, it is sufficient to show that $T(B_X(0,r))$ has a ***nonempty interior*** for ***some*** r.

  Since T is ***onto***,

  $$Y = \bigcup_{n=1}^{\infty} T(B_X(0,n))$$

  so some $\overline{T(B_X(0,n))}$ has a ***nonempty interior***.

  Now the hard work begins, since we want $T(B_X(0,n))$ to have a *nonempty interior*. By scaling and translating, we can suppose that $B_Y(0,\epsilon)$ is contained in $\overline{T(B_X(0,1))}$; we will show that $\overline{T(B_X(0,1))} \subset T(B_X(0,2))$ which will complete the proof.

Let $y \in \overline{T(B_X(0,1))}$. Pick $x_1 \in B_X(0,1)$ so $y - Tx_1 \in B_Y(0, \epsilon/2) \subset \overline{B_Y(0,1/2)}$. Now pick $x_2 \in B_X(0,1/2)$ so that

$$y - Tx_1 - Tx_2 \in B_Y(0, \epsilon/4)$$

By induction, we choose $x_n \in B_X(0, 2^{1-n})$ so that

$$y - \sum_{j=1}^{n} Tx_j \in B_Y(0, 2^{1-n}\epsilon)$$

Then $x = \sum_{j=1}^{\infty} x_j$ *exists* and is in $B_X(0,2)$ and

$$y = \sum_{j=1}^{\infty} Tx_j = Tx.$$

Thus $y \in T(B_X(0,2))$. ∎

- **Corollary 4.12** ***(Inverse Mapping Theorem)*** *[Reed and Simon, 1980]*
  A ***continuous bijection*** *of one Banach space onto another has a* ***continuous inverse****.*

- **Remark** Note T is an open map and $A = T^{-1}(T(A))$ for surjective map, then T^{-1} is *continuous.*

- **Theorem 4.13** ***(Banach-Schauder Theorem)*** *[Reed and Simon, 1980]*
  Let T be a ***continuous*** *linear map, $T : E \to F$, where E and F are Banach spaces. Then either $T(A)$ is* ***open*** *in $T(E)$ for* ***each open*** *$A \subseteq E$, or $T(E)$ is of* ***first category*** *in $\overline{T(E)}$.*

4.5 Closed Graph Theorem

- **Definition** (***Graph of Function***)
  Let be a mapping of a normed linear space X into a normed linear space Y. The ***graph of** T*, denoted by $\Gamma(T)$, is defined as

$$\Gamma(T) := \{(x, y) \in X \times Y : y = Tx\}.$$

- **Theorem 4.14** ***(Closed Graph Theorem)****[Reed and Simon, 1980]*
  Let X and Y be Banach spaces and T a linear map of X into Y. Then T is ***bounded*** *if and only if the* ***graph*** *of is* ***closed****.*

  Proof: Suppose that $\Gamma(T)$ is ***closed***. Then, since is linear, $\Gamma(T)$ is a ***subspace*** of ***the Banach space direct sum*** $X \oplus Y$. By assumption $\Gamma(T)$ is ***closed*** and thus is a ***Banach space*** in the norm

$$\|(x, Tx)\| = \|x\|_X + \|Tx\|_Y$$

  Consider the ***continuous*** *projection maps* π_1 and π_2,

$$\pi_1 : (x, Tx) \mapsto x, \quad \pi_2 : (x, Tx) \mapsto Tx.$$

  π_1 is a ***bijection*** so by ***the inverse mapping theorem*** π_1^{-1} is ***continuous***. But $T = \pi_2 \circ \pi_1^{-1}$, so T is ***continuous***. The *converse* is *trivial.* ∎

- **Remark** To avoid future confusion, we emphasize that the T in this theorem is implicitly assumed to be ***defined on all of*** X.

- **Remark** Consider the following statements:

  1. x_n converges to some element x;

  2. Tx_n converges to some element y;

  3. $Tx_n = y$.

  Usually to prove T is continuous, one need to show that given statement 1, the statement 2 and 3 are true. That is, we need to ***prove convergence*** of Tx_n and need to show ***identification*** of Tx and the limit of Tx_n.

  With ***close graph theorem***, we just need to show that given statement 1 ***and*** 2, statement 3 is true; that is, we just need to prove the identification part.

- **Corollary 4.15** *(**The Hellinger-Toeplitz Theorem**) [Reed and Simon, 1980]*
  *Let A be an **everywhere defined** linear operator on a **Hilbert space** $\mathcal{H}$ with*

  $$\langle x\,,\,Ay\rangle = \langle Ax\,,\,y\rangle$$

  *for all $x, y \in \mathcal{H}$; that is A is **self-adjoint**. Then A is **bounded**.*

  Proof: *We will prove that $\Gamma(A)$ is **closed**. Suppose that $\langle x_n\,,\,Ax_n\rangle \to \langle x\,,\,y\rangle$. We need only prove that $\langle x\,,\,y\rangle \in \Gamma(A)$, that is, that $y = Ax$. For any $z \in \mathcal{H}$,*

  $$\begin{aligned}\langle z\,,\,y\rangle &= \lim_{n\to\infty}\langle z\,,\,Ax_n\rangle = \lim_{n\to\infty}\langle Az\,,\,x_n\rangle \\ &= \langle Az\,,\,x\rangle = \langle z\,,\,Ax\rangle\end{aligned}$$

  *Thus $y = Ax$ and $\Gamma(A)$ is **closed**.* ∎

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