Lecture 6: Sard's Theorem

Tianpei Xie

Oct. 17th., 2022

Contents

1 Sard's Theorem

1.1 Sets of Measure Zero

- **Definition** A set $A \subseteq \mathbb{R}^n$ to have ***measure zero*** if for any $\delta > 0$, A can be ***covered*** by ***a countable collection of open rectangles***, the ***sum*** of whose ***volumes*** is less than δ (Fig.).

- **Lemma 1.1** *Suppose $A \subseteq \mathbb{R}^n$ is a **compact** subset whose intersection with $\{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero for **every** $c \in \mathbb{R}$. Then A has n-dimensional measure zero.*

- **Proposition 1.2** *Suppose A is an open or closed subset of $\mathbb{R}^{n-1}$ or $\mathbb{H}^{n-1}$, and $f : A \to \mathbb{R}$ is a **continuous** function. Then the **graph** of f has measure zero in $\mathbb{R}^n$.*

- **Corollary 1.3** *Every **proper affine subspace** of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$.*

- **Proposition 1.4** *Suppose $A \subseteq \mathbb{R}^n$ has **measure zero** and $F : A \to \mathbb{R}^n$ is a **smooth** map. Then $F(A)$ has measure zero.*

- **Definition** Let M be a smooth n-manifold with or without boundary. A subset $A \subseteq M$ has ***measure zero*** in M if for ***every smooth chart*** (U, φ) for M, the *subset* $\varphi(A \cap U) \subseteq \mathbb{R}^n$ has n***-dimensional measure zero***.

- The following lemma shows that we need only check this condition for *a single collection of smooth charts* whose *domains cover* A.

  Lemma 1.5 *Let M be a smooth n-manifold with or without boundary and $A \subseteq M$. Suppose that for **some collection** $\{(U_\alpha, \varphi_\alpha)\}$ of smooth charts whose domains **cover** A, $\varphi_\alpha(A \cap U_\alpha)$ has **measure zero** in $\mathbb{R}^n$ for each α. Then A has measure zero in M.*

- **Proposition 1.6** *Suppose M is a smooth manifold with or without boundary and $A \subseteq M$ has **measure zero** in M. Then $M \setminus A$ is **dense** in M.*

- **Theorem 1.7** *Suppose M and N are smooth n-manifolds with or without boundary, $F : M \to N$ is a **smooth** map, and $A \subseteq M$ is a subset of **measure zero**. Then $F(A)$ has measure zero in N.*

1.2 Proof of Sard's Theorem

- *The Sard's theorem* underlies all of our results about embedding, approximation, and transversality.

- **Theorem 1.8** *(**Sard's Theorem**).*
  *Suppose M and N are smooth manifolds with or without boundary and $F : M \to N$ is a **smooth map**. Then the set of **critical values** of F has **measure zero** in N.*

  Proof:

1.3 Corollaries

- **Corollary 1.9** *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a* ***smooth map****. If $\dim M < \dim N$, then $F(M)$ has measure zero in N.*

- **Remark** It is important to be aware tha Corollary above is false if F is merely assumed to be **continuous**. For example, there is a continuous map $F : [0, 1] \to \mathbb{R}^2$ whose image is the entire unit square $[0, 1] \times [0, 1]$. (Such a map is called ***a space-filling curve***).

- **Corollary 1.10** *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an* ***immersed submanifold*** *with or without boundary. If $\dim S < \dim M$, then S has measure zero in M.*

2 The Whitney Embedding Theorem

- Our first application of *Sard's theorem* is to show that ***every smooth manifold can be embedded into a Euclidean space***. In fact, we will show that every smooth n-manifold with or without boundary is *diffeomorphic* to a *properly embedded submanifold* (with or without boundary) of $\mathbb{R}^{2n+1}$.

- **Theorem 2.1** ***(Whitney Embedding Theorem).***
Every smooth n-manifold with or without boundary admits a proper ***smooth embedding*** *into $\mathbb{R}^{2n+1}$.*

- **Theorem 2.2** ***(Whitney Immersion Theorem).***
Every smooth n-manifold with or without boundary admits a ***smooth immersion*** *into $\mathbb{R}^{2n}$.*

- **Theorem 2.3** ***(Strong Whitney Embedding Theorem).***
If $n > 0$, every smooth n-manifold admits a ***smooth embedding*** *into $\mathbb{R}^{2n}$.*

- **Theorem 2.4** ***(Strong Whitney Immersion Theorem).***
If $n > 1$, every smooth n-manifold admits a ***smooth immersion*** *into $\mathbb{R}^{2n-1}$.*

  Because of these results, the first two theorems are sometimes called the *easy* or *weak Whitney embedding and immersion theorems.*

3 The Whitney Approximation Theorems

3.1 Whitney Approximation Theorem for Functions

- We begin with a theorem about ***smoothly approximating functions*** *into Euclidean spaces*. Our first theorem shows, in particular, that *any continuous function* from a smooth manifold M into $\mathbb{R}^k$ can be *uniformly approximated* by a smooth function.

- **Theorem 3.1** ***(Whitney Approximation Theorem for Functions).***
Suppose M is a smooth manifold with or without boundary, and $F : M \to \mathbb{R}^k$ is a ***continuous*** *function. Given any* ***positive continuous*** *function $\delta : M \to \mathbb{R}$, there exists a* ***smooth*** *function $\widetilde{F} : M \to \mathbb{R}^k$ that is δ-close to F. If F is smooth on a closed subset $A \subseteq M$, then $\widetilde{F}$*

can be chosen to be equal to F on A.

3.2 Tubular Neighborhoods

3.3 Smooth Approximation of Maps Between Manifolds

- Now we can extend *the Whitney approximation theorem* to *maps between manifolds*. This extension will have important applications to *line integrals*.

- **Theorem 3.2** *(**Whitney Approximation Theorem**).*
  *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold (without boundary), and $F : N \to M$ is a **continuous** map. Then F is **homotopic** to a smooth map. If F is already smooth on a closed subset $A \subseteq N$, then the **homotopy** can be taken to be relative to A.*

4 Transversality

- As our final application of *Sard's theorem*, we show how *submanifolds* can be *perturbed* so that ***they intersect "nicely."*** To explain what this means, we introduce the concept of ***transversality***.

- **Definition** Suppose M is a smooth manifold. Two *embedded submanifolds* $S, S' \subseteq M$ are said to ***intersect transversely*** if for each $p \in S \cap S'$, the tangent spaces $\underline{T_pS}$ and $\underline{T_pS'}$ *together* ***span*** $\underline{T_pM}$ (where we consider T_pS and T_pS' as subspaces of T_pM).

- **Definition** If $F : N \to M$ is a smooth map and $S \subseteq M$ is an *embedded submanifold*, we say that ***F is transverse to S*** if for every $x \in F^{-1}(S)$, the spaces $\underline{T_{F(x)}S}$ and $\underline{dF_x(T_xN)}$ *together* ***span*** $\underline{T_{F(x)}M}$.