Lecture 8: Vector Fields

Tianpei Xie

Oct. 16th., 2022

Contents

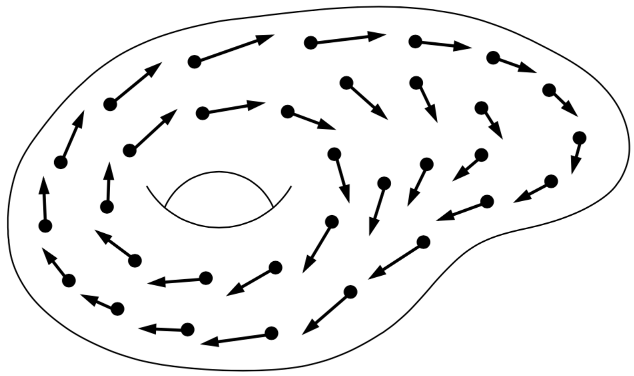

Figure 1: A vector field [Lee, 2003.]

1 Vector Fields on Manifolds

1.1 Definitions

- **Definition** If M is a smooth manifold with or without boundary, a ***vector field*** on M is a ***section*** of the map $\pi : TM \to M$. More concretely, a *vector field* is a ***continuous*** map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \mathrm{Id}_M, \tag{1}$$

or *equivalently*, $X_p \in T_pM$ for each $p \in M$.

- **Remark** We write the ***value of*** X ***at*** p as X_p instead of $X(p)$ to be consistent with our notation for elements of the tangent bundle, as well as to avoid conflict with the notation $v(f)$ for the action of a vector on a function.

- **Remark** You should visualize a vector field on M in the same way as you visualize vector fields in Euclidean space: as an arrow attached to each point of M, chosen to be tangent to M and to ***vary continuously from point to point*** (Fig. 1).

- **Remark** We can compare several related concepts:
  - ***Tangent vector*** $v \in T_pM$ is both a *geometric tangent vector*, i.e. the tangent direction for some curve on M passing p, and a *derivation* at p defined on $\mathcal{C}^\infty(M)$ that satisfies the product rule. For the latter case, v is a *linear functional* that act on functions $f \in \mathcal{C}^\infty(M)$. $v(f)$ induces the *directional derivatives* of f along v and *derivation at* p.
  - The ***differential*** of $F : M \to N$ at p, dF_p, is a *linear operator* from T_pM to T_pN. dF_p maps a *tangent vector* at p in M to a *tangent vector* at $F(p)$ in N. Since tangent vectors are functions, dF_p is also a linear functional if $F : M \to \mathbb{R}$. $dF_p(v) \in T_{F(p)}N$ can act on a function on N to have $dF_p(v)(f)$.
  - The ***vector field*** X is a *continuous* map from a point $p \in M$ to a *tangent vector* $X_p = v \in T_pM$. Thus for smooth function f on M, X_pf is a real-value, i.e. the directional derivative of f along $v = X_p$. A vector field X is also ***a derivation operator*** on $\mathcal{C}^\infty(M)$. It maps a *smooth function* f to a *new smooth function* Xf which at each point is X_pf. A vector field is a *global generalization* of a tangent vector.

- **Definition** When the map $X : M \to TM$ is *smooth* and the tangent bundle TM is given a *smooth manifold structure*, X is a ***smooth vector field***.

  In addition, for some purposes it is useful to consider maps from M to TM that would be vector fields except that they *might not be continuous*. A ***rough vector field*** on M is a (*not necessarily continuous*) map $X : M \to TM$ satisfying (1).

- **Definition** Just as for functions, if X is a vector field on M, the ***support of*** X is defined to be *the closure of the set* $\{p \in M : X_p \neq 0\}$. A vector field is said to be ***compactly supported*** if its support is a *compact set*.

- **Remark** (***Coordinate Representation of Vector Field At a Point***)
  Suppose M is a smooth n-manifold (with or without boundary). If $X : M \to TM$ is a *rough vector field* and $(U, (x^i))$ is any *smooth coordinate chart* for M, we can write the ***value** of* X *at any point* $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \, \frac{\partial}{\partial x^i}\bigg|_p. \tag{2}$$

  This defines n ***functions*** $X^i : U \to \mathbb{R}$, called the ***component functions*** of X in the given chart.

- Note that a *component function* is a real-value function on neighborhood $U \subseteq M$. It is necessary to distinguish (2) from the coordinate representation of tangent vector $v \in T_pM$

$$v = v^i \, \frac{\partial}{\partial x^i}\bigg|_p,$$

  where $v^i \in \mathbb{R}$ is a fixed constant.

- **Proposition 1.1** ***(Smoothness Criterion for Vector Fields)*** *[Lee, 2003.]*
  *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the **restriction** of X to U is **smooth** if and only if its **component functions** with respect to this chart are **smooth**.*

- **Definition** If M is a smooth manifold with or without boundary and $A \subseteq M$ is an arbitrary subset, ***a vector field along*** A is a *continuous* map $X : A \to TM$ satisfying $\pi \circ X = \mathrm{Id}_A$ (or in other words $X_p \in TpM$ for each $p \in A$).

  We call it ***a smooth vector field along*** A if for each $p \in A$, there is a *neighborhood* V of p in M and a *smooth vector field* $\widetilde{X}$ on V that agrees with X on $V \cap A$.

- **Lemma 1.2** ***(Extension Lemma for Vector Fields)****. [Lee, 2003.]*
  *Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a **smooth global vector field** $\widetilde{X}$ on M such that $\widetilde{X}|_A = X$ and $\mathrm{supp}\,\widetilde{X} \subseteq U$.*

- As an important special case, ***any vector*** *at a point can be* ***extended*** *to a* ***smooth vector field*** *on the* ***entire manifold***.

  Proposition 1.3 *Let M be a smooth manifold with or without boundary. Given $p \in M$ and $v \in T_pM$, there is a smooth global vector field X on M such that $X_p = v$.*

- **Remark** (***The Space of all Vector Fields on a Manifold is a Vector Space***)
  If M is a smooth manifold with or without boundary, it is standard to use the notation $\mathfrak{X}(M)$ to denote ***the set of all smooth vector fields on M***.

$\mathfrak{X}(M)$ *is a* ***vector space*** under pointwise addition and scalar multiplication:

1. For any $a, b \in \mathbb{R}$ and any $X, Y \in \mathfrak{X}(M)$,
$$(a\,X + b\,Y)_p = a\,X_p + b\,Y_p.$$

2. The *zero element* of this vector space is the ***zero vector field***, whose value at each $p \in M$ is $0 \in T_pM$.

In addition, *smooth vector fields* can be multiplied by *smooth real-valued functions*: if $f \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX : M \to TM$ by
$$(fX)_p = f(p)\,X_p.$$

- **Proposition 1.4** *Let M be a smooth manifold with or without boundary.*

  1. *If X and Y are smooth vector fields on M and $f, g \in \mathcal{C}^\infty(M)$, then $fX + gY$ is a smooth vector field.*

  2. *$\mathfrak{X}(M)$ is a* ***module*** *over the* ***ring*** *$\mathcal{C}^\infty(M)$.*

- **Remark** (***Coordinate Representation of Vector Field***)
We can generalize the formula (2) as the coordinate representation of the vector field X
$$X = X^i \frac{\partial}{\partial x^i}. \tag{3}$$
where $(\frac{\partial}{\partial x^i})$ are *the coordinate vector fields*, which are ***basis*** for $\mathfrak{X}(M)$ and X^i is the i-th component function of X in the given coordinates.

  In partial differential equations (PDEs), we usually write (3) in *dot-product form*
$$X = \boldsymbol{X} \cdot \nabla = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \tag{4}$$
$$\text{where } \boldsymbol{X} = [X^1, \ldots, X^n], \quad \nabla := \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right).$$
∇ (*the nabla symbol*) is also called ***gradient operator***.

- Note that we need to distinguish the difference between the ***coordinate vector*** $\frac{\partial}{\partial x^i}|_p$ in tangent space T_pM and the ***coordinate vector fields*** $\frac{\partial}{\partial x^i}$ as latter is a smooth function on M.

1.2 Examples of Smooth Vector Fields

- **Example** (***Coordinate Vector Fields***).
If $(U, (x^i))$ is any *smooth chart* on M, the assignment
$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$
determines a *vector field* on U, called ***the i-th coordinate vector field*** and denoted by $\frac{\partial}{\partial x^i}$ (without p since it is now a function of p). It is ***smooth*** because its component functions are *constants*.

- **Example** (***The Euler Vector Field***).
  The *vector field* V on $\mathbb{R}^n$ whose value at $x \in \mathbb{R}^n$ is

  $$V_x = x^1 \frac{\partial}{\partial x^1}\bigg|_x + \ldots + x^n \frac{\partial}{\partial x^n}\bigg|_x.$$

  is ***smooth*** because its coordinate functions are *linear*. It *vanishes* at the origin, and points *radially* ***outward*** *everywhere else*. It is called ***the Euler vector field*** because of its appearance in *Eulers homogeneous function theorem.*

- **Example** (***The Angle Coordinate Vector Field on the Circle***).
  Let θ be any *angle coordinate* on a proper open subset $U \subseteq \mathbb{S}^1$, and let $\frac{d}{d\theta}$ denote the corresponding coordinate vector field. Because any other angle coordinate $\widetilde{\theta}$ differs from θ by an additive constant in a neighborhood of each point, the transformation law for coordinate vector fields (i.e. the change of coordinate law) shows that $\frac{d}{d\theta} = \frac{d}{d\widetilde{\theta}}$ on their common domain.

  For this reason, there is a ***globally defined vector field*** on $\mathbb{S}^1$ whose coordinate representation is $\frac{d}{d\theta}$ with respect to any angle coordinate. It is a ***smooth*** vector field because its component function is *constant* in *any such chart.* We denote this global vector field by $\frac{d}{d\theta}$, even though, strictly speaking, it cannot be considered as a coordinate vector field on the entire circle at once.

- **Example** (***Angle Coordinate Vector Fields on Tori***).
  On the n-dimensional torus $\mathbb{T}^n$, choosing an angle function θ^i for the i-th circle factor, $i = 1, \ldots, n$, yields local coordinates $(\theta^1, \ldots, \theta^n)$ for $\mathbb{T}^n$. An analysis similar to that of the previous example shows that the coordinate vector fields $\frac{d}{d\theta^1}, \ldots, \frac{d}{d\theta^n}$ are ***smooth*** and ***globally defined*** on $\mathbb{T}^n$.

- **Example** (***Restriction of Vector Field on Open Submanifold***)
  If $U \subseteq M$ is open, the fact that T_pU is *naturally identified* with T_pM for each $p \in U$ (See Chapter 3) allows us to identify TU with the open subset $\pi^{-1}(U) \subseteq TM$. Therefore, a vector field on U can be thought of either as a map from U to TU or as a map from U to TM whichever is more convenient.

  If X is a *vector field* on M, *its restriction $X|_U$ is a vector field on U*, which is *smooth if X is.*

1.3 Local and Global Frames

- Coordinate vector fields in a smooth chart provide a convenient way of representing vector fields, because their values form a basis for the tangent space at each point. However, they are not the only choices.

- **Definition** Suppose M is a smooth n-manifold with or without boundary. An *ordered k-tuple* $(X_1, \ldots, X_k)$ of ***vector fields*** defined on some subset $A \subseteq M$ is said to be ***linearly independent*** if $(X_1|_p, \ldots, X_k|_p)$ is a *linearly independent* k-tuple in T_pM for each $p \in A$, and is said to ***span the tangent bundle*** if the k-tuple $(X_1|_p, \ldots, X_k|_p)$ spans T_pM at each $p \in A$.

- **Definition** A ***local frame*** for M is *an ordered n-tuple of* ***vector fields*** $(E_1, \ldots, E_n)$ defined on an **open subset** $U \subseteq M$ that is ***linearly independent*** and ***spans the tangent bundle***; thus the vectors $(E_1|_p, \ldots, E_n|_p)$ form a basis for T_pM at each $p \in U$.

$(E_1, \ldots, E_n)$ is called a ***global frame*** if $U = M$, and a ***smooth frame*** if each of the vector fields E_i is *smooth*.

We often use the shorthand notation (E_i) to denote a frame $(E_1, \ldots, E_n)$.

- If M has dimension n, then to check that an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ is a local frame, it suffices to check either that *it is linearly independent* **or** that *it spans the tangent bundle*.

- **Example** (***Local and Global Frames***).
  - The ***standard coordinate vector fields*** $(\frac{\partial}{\partial x^i})$ form a ***smooth global frame*** for $\mathbb{R}^n$.
  - If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M (possibly with boundary), then the *coordinate vector fields* form a ***smooth local frame*** $(\frac{\partial}{\partial x^i})$ on U, called a ***coordinate frame***. *Every point of M is in the domain of such a local frame.*
  - The ***angle coordinate vector field*** $\frac{d}{d\theta}$ constitutes a *smooth global frame* for the circle $\mathbb{S}^1$.
  - The n-tuple of vector fields $(\frac{d}{d\theta^i})$ is a *smooth global frame* for the n-torus $\mathbb{T}^n$.

- **Proposition 1.5** ***(Completion of Local Frames).***
  Let M be a smooth n-manifold with or without boundary.
  1. *If $(X_1, \ldots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an **open subset** $U \subseteq M$, with $1 \leq k < n$, then for each $p \in U$ there exist smooth vector fields $X_{k+1}, \ldots, X_n$ in a **neighborhood** V of p such that $(X_1, \ldots, X_n)$ is a smooth local frame for M on $U \cap V$.*
  2. *If $(v_1, \ldots, v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p \in M$, with $1 \leq k \leq n$, then there exists a **smooth local frame** (X_i) on a **neighborhood** of p such that $X_i|_p = v_i$ for $i = 1, \ldots, k$.*
  3. *If $(X_1, \ldots, X_n)$ is a linearly independent n-tuple of smooth vector fields along a **closed subset** $A \subseteq M$, then there exists a **smooth local frame** $(\widetilde{X}_1, \ldots, \widetilde{X}_n)$ on some neighborhood of A such that $\widetilde{X}_i|_A = X_i$ for $i = 1, \ldots, n$.*

- **Definition** A k-tuple of vector fields $(E_1, \ldots, E_k)$ defined on some subset $A \subseteq \mathbb{R}^n$ is said to be ***orthonormal*** if for each $p \in A$, the vectors $(E_1|_p, \ldots, E_k|_p)$ are ***orthonormal*** with respect to the Euclidean dot product (where we identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ in the usual way).

  A *(local or global) frame* consisting of *orthonormal vector fields* is called an ***orthonormal frame***.

- **Lemma 1.6** ***(Gram-Schmidt Algorithm for Frames).***
  Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over an open subset $U \subseteq \mathbb{R}^n$. Then there is a smooth orthonormal frame (E_j) over U such that span$(E_1|_p, \ldots, E_j|_p) =$ span$(X_1|_p, \ldots, X_j|_p)$ for each $j = 1, \ldots, n$ and each $p \in U$.

- Although *smooth local frames* are plentiful, *global ones* are not.

  Definition A smooth manifold with or without boundary is said to be ***parallelizable*** if it admits a **smooth global frame**.

- **Example** These are some examples of parallizable or non-parallizable manifolds:
  - $\mathbb{R}^n$, $\mathbb{S}^1$ and $\mathbb{T}^n$ are all *parallelizable manifold*.
  - *All **Lie groups*** are *parallelizable*.
  - Most smooth manifolds are *not parallelizable*. The simplest example of a *nonparallelizable manifold* is $\mathbb{S}^2$. (In fact, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the ***only*** spheres that are parallelizable.)

1.4 Vector Fields as Derivations of $\mathcal{C}^\infty(M)$

- An essential property of vector fields is that they define ***operators*** on the space of smooth real-valued functions.

  Definition If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf : U \to \mathbb{R}$, defined by
  $$(X\,f)_p = X_p\,f$$
  Note that $v\,f \equiv v(f)$ as we omit the parenthesis.

- **Remark** Be careful not to confuse the notations fX and Xf:
  - the former fX is ***the smooth vector field*** on U obtained by multiplying X by f,
  - while the latter Xf is ***the real-valued functio***n on U obtained by *applying* the vector field X to the smooth function f.

- **Remark** Because the action of a tangent vector on a function is determined by the values of the function in an arbitrarily small neighborhood, it follows that Xf is ***locally determined***. In particular, for any open subset $V \subseteq U$,
  $$(X\,f)\big|_V = X\left(f\big|_V\right). \tag{5}$$

- **Proposition 1.7** *Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a **rough vector field**. The following are equivalent:*
  1. *X is smooth.*
  2. *For **every** $f \in \mathcal{C}^\infty(M)$, the function Xf is smooth on M.*
  3. *For **every open subset** $U \subseteq M$ and every $f \in \mathcal{C}^\infty(U)$, the function Xf is smooth on U.*

- **Definition** Define a map $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is called a ***derivation*** (as distinct from a *derivation at p*, defined in Chapter 3) if it is **linear** over $\mathbb{R}$ and satisfies the *Leibnitz rule*
  $$X(fg) = f\,X(g) + g\,X(f), \qquad \forall\; f, g \in \mathcal{C}^\infty(M) \tag{6}$$

- **Remark** As the tangent vector itself is a ***linear functional*** on $\mathcal{C}^\infty(M)$, the vector field X is a ***linear operator*** that maps a function to another function on $\mathcal{C}^\infty(M)$. The ***value*** *of function Xf* at p is *the directional derivative* of f *along with* $X(p)$ at point p.

- The next proposition shows that *derivations* of $\mathcal{C}^\infty(M)$ can be identified with smooth vector fields:

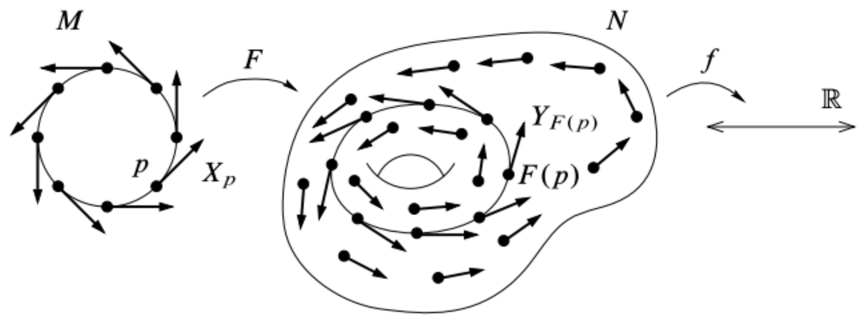

Figure 2: F**-related vector fields [Lee, 2003.]**

Proposition 1.8 *Let M be a smooth manifold with or without boundary. A map $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a **derivation if and only** if it is of the form $Df = Xf$ for **some smooth vector field** $X \in \mathfrak{X}(M)$.*

- **Remark** Because of this result, we sometimes ***identify*** *smooth vector fields* on M with *derivations* of $\mathcal{C}^\infty(M)$, using the same letter for both *the vector field* (thought of as a smooth map from M to TM) and the *derivation* (thought of as a linear map from $\mathcal{C}^\infty(M)$ to itself)

2 Vector Fields and Smooth Maps

2.1 Smooth Maps on Vector Fields

- **Remark** If $F : M \to N$ is a *smooth map* and X is a *vector field* on M, then for each point $p \in M$, we obtain a *vector* $dF_p(X_p) \in T_{F(p)}N$ by applying the differential of F to X_p. Can we map a vector field to a vector field via differential dF_p ? Unfortunately, this does not in general true.

- **Definition** Suppose $F : M \to N$ is *smooth* and X is a *vector field* on M, and suppose there happens to be a *vector field* Y on N with the property that for each $p \in M$,

$$dF_p(X_p) = Y_{F(p)}.$$

In this case, we say *the **vector fields** X and Y are **F-related*** (see Fig. 2).

- **Remark** The ***differential*** dF_p is defined *locally*, and it ***does not guarantee to map a vector field (a global concept) to a vector field***. For example, if F is *not surjective*, there is no way to decide what vector to assign to a point $q \in N \setminus F(M)$. If F is *not injective*, then for some points of N there may be several different vectors obtained by applying dF to X at different points of M.

- **Proposition 2.1** *Suppose $F : M \to N$ is a smooth map between manifolds with or without boundary, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Then X and Y are F-**related if and only if** for **every smooth real-valued function** f defined on an open subset of N,*

$$X(f \circ F) = (Yf) \circ F \tag{7}$$

Proof: For any $p \in M$ and any smooth real-valued f defined in a neighborhood of $F(p)$,

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)(f),$$

while

$$((Yf) \circ F)(p) = (Yf)(F(p)) = Y_{F(p)}(f).$$

Thus, (7) is true for all f if and only if $dF_p(X_p) = Y_{F(p)}$ for all p, i.e., if and only if X and Y are F-related. ∎

- **Proposition 2.2** *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a* ***diffeomorphism****. For every $X \in \mathfrak{X}(M)$, there is a* ***unique*** *smooth vector field on N that is F-related to X.*

  Proof: Note that for $Y \in \mathfrak{X}(N)$ to be F-related to X means that $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$. If F is a *diffeomorphism*, therefore, we define Y by

  $$Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}), \quad \forall q \in N.$$

  We can show that Y is F-related to X. Note that $Y : N \to TN$ is the *composition* of the following smooth maps:

  $$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN.$$

  It follows that Y is smooth. ∎

- **Definition** Suppose M and N are *smooth manifolds with or without boundary*, and $F : M \to N$ is a ***diffeomorphism***. For every $X \in \mathfrak{X}(M)$, there is a ***unique*** *smooth vector field* Y on N that is F-related to X. We denote the ***unique vector field*** that is ***F-related*** to X by $\underline{F_* X}$, and call it the <u>***pushforward of X by F***</u>. And $F_* X$ is defined explicitly by the formula

  $$(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}), \quad \forall q \in N. \tag{8}$$

  As long as the inverse map F^{-1} can be computed explicitly, the ***pushforward*** of a vector field can be computed directly from this formula. Note that sometimes the *pushforward* of X is denoted as $F_{\#} X$.

- **Corollary 2.3** *Suppose $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. For any $f \in \mathcal{C}^\infty(N)$,*

  $$(F_* X\, f) \circ F = X(f \circ F)$$

2.2 Vector Fields and Submanifolds

- **Remark** If $S \subseteq M$ is an immersed or embedded submanifold (with or without boundary), *a vector field X on M* does ***not necessarily*** *restrict to a vector field on S*, because X_p may not lie in the *subspace* $T_pS \subseteq T_pM$ at a point $p \in S$.

- **Definition** Given a point $p \in S$, a vector field X on M is said to <u>***be tangent to***</u> S at p if $X_p \in T_pS \subseteq T_pM$. It ***is tangent to*** S if it is tangent to S at every point of S.

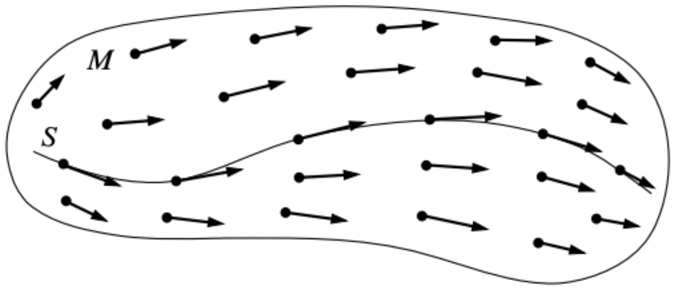

Figure 3: A vector field tangent to a submanifold. [Lee, 2003.]

- **Proposition 2.4** *Let M be a smooth manifold, $S \subseteq M$ be an* ***embedded submanifold*** *with or without boundary, and X be a smooth vector field on M. Then X is* ***tangent*** *to S if and only if $(Xf)|_S = 0$ for* ***every*** $\underline{f \in \mathcal{C}^\infty(M)}$ ***such that*** $\underline{f|_S \equiv 0}$.

- **Remark** Suppose $S \subseteq M$ is an ***immersed submanifold*** with or without boundary, and Y is a smooth vector field on M. If there is a vector field $X \in \mathfrak{X}(S)$ that is ***ι-related to Y***, where $\iota : S \hookrightarrow M$ is the inclusion map, then clearly Y ***is tangent to*** S, because $Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$.

  The converse is true as well.

  Proposition 2.5 *(**Restricting Vector Fields to Submanifolds**). [Lee, 2003.]*
  Let M be a smooth manifold, let $S \subseteq M$ be an ***immersed submanifold*** *with or without boundary, and let $\iota : S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathfrak{X}(M)$ is* ***tangent to*** *S, then there is a* ***unique smooth vector field*** *on S, denoted by $Y|_S$, that is* ***ι-related to*** *Y.*

3 Lie Brackets

- In this section we introduce an important way of ***combining*** *two smooth vector fields* to obtain *another vector field.*

- **Definition** Let X and Y be smooth vector fields on a smooth manifold M and $f \in \mathcal{C}^\infty(M)$ is smooth function on M. Define an ***operator*** $[X, Y] : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, called ***the Lie bracket*** of X and Y, defined by

$$[X, Y]\, f = XYf - YXf. \tag{9}$$

- **Remark** A vector field maps a smooth function on M to another smooth function on M. Thus it is valid to define $XYf = Xg$ where $g = Yf$ is the derivation of f under Y. $[X, Y]_p f$ is ***a second-order (directional) derivatives*** of f at p along two directions Y_p and X_p.

- **Remark** Note that XY itself is not a vector field since it does not necessarily satisfy the Leibnitz rule. For example, $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. Let $f(x, y) = x$ and $g(x, y) = y$. Then direct computation shows that $XY(fg) = 2x$, while $fXYg + gXYf = x$, so XY is not a derivation of $\mathcal{C}^\infty\mathbb{R}^2$.

- **Lemma 3.1** *The* ***Lie bracket*** *of* ***any pair*** *of smooth vector fields is a smooth vector field.*

  Proof: It suffices to show that $[X, Y]$ is a *derivation* of $\mathcal{C}^\infty(M)$. For arbitrary $f, g \in \mathcal{C}^\infty(M)$, we compute

$$\begin{aligned}
[X,Y](fg) &= XY(fg) - YX(fg) \\
&= X(f\,Y(g) + g\,Y(f)) - Y(f\,X(g) + g\,X(f)) \\
&= f\,XY(g) + g\,XY(f) - f\,YX(g) - g\,YX(f) \\
&= f\,(XY - YX)(g) + g\,(XY - YX)(f) \\
&= f\,[X,Y](g) + g\,[X,Y](f). \qquad \blacksquare
\end{aligned}$$

- **Remark** The ***value*** of the vector field $[X,Y]$ at a point $p \in M$ is the *derivation at* p given by the formula

$$[X,Y]_p f = X_p(Yf) - Y_p(Xf). \tag{10}$$

However, this formula is of limited usefulness for computations, because it requires one to compute terms involving *second derivatives* of f that will always cancel each other out.

- **Proposition 3.2** ***(Coordinate Formula for the Lie Bracket)****. [Lee, 2003.]*
Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M. Then $[X,Y]$ has the following coordinate expression:

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}, \tag{11}$$

or more concisely,

$$[X,Y] = \left(XY^j - YX^j \right) \frac{\partial}{\partial x^j}. \tag{12}$$

- **Remark** One trivial application of (12) is to compute the Lie brackets of the coordinate vector fields $\frac{\partial}{\partial x^i}$ in any smooth chart: because *the component functions* of *the coordinate vector fields* are *all constants*, it follows that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] \equiv 0, \quad \forall\, i,j. \tag{13}$$

This also follows from the definition of the Lie bracket, and is essentially a restatement of the fact that ***mixed partial derivatives of smooth functions commute***.

- **Proposition 3.3** ***(Properties of the Lie Bracket)****.*
*The **Lie bracket** satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:*

  1. ***Bilinearity****: For $a, b \in \mathbb{R}$,*

$$\begin{aligned}
[aX + bY, Z] &= a[X,Z] + b[Y,Z], \\
[Z, aX + bY] &= a[Z,X] + b[Z,Y].
\end{aligned}$$

  2. ***Antisymmetry****:*

$$[X,Y] = -[Y,X]$$

3. ***Jacobi Identity***:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

4. *For $f,g\in\mathcal{C}^\infty(M)$,*

$$[fX,gY]=fg[X,Y]+(fXg)\,Y-(gYf)\,X \tag{14}$$

The significance of part 4 of this proposition might not be evident at this point, but it will become clearer in the next chapter, where we will see that it expresses the fact that the ***Lie bracket*** satisfies *product rules* with respect to ***both of its arguments***.

- **Proposition 3.4** *(**Naturality of the Lie Bracket**). [Lee, 2003.]*
*Let $F:M\to N$ be a smooth map between manifolds with or without boundary, and let $X_1,X_2\in\mathfrak{X}(M)$ and $Y_1,Y_2\in\mathfrak{X}(N)$ be **vector fields** such that X_i is F**-related to** Y_i for $i=1,2$. Then $[X_1,X_2]$ is F**-related to** $[Y_1,Y_2]$.*

  Proof: Using the fact that X_i and Y_i are F-related, for $f\in\mathcal{C}^\infty(M)$, $X_i(f\circ F)=(Y_if)\circ F$ for $i=1,2$. Thus

$$\begin{aligned}[X_1,X_2](f\circ F)&=X_1X_2(f\circ F)-X_2X_1(f\circ F)\\&=X_1((Y_2f)\circ F)-X_2((Y_1f)\circ F)\\&=Y_1Y_2f\circ F-Y_2Y_1f\circ F\\&=([Y_1,Y_2]f)\circ F.\end{aligned}$$

  So $[X_1,X_2]$ is F**-related to** $[Y_1,Y_2]$. ∎

- **Corollary 3.5** *(**Pushforwards of Lie Brackets**).*
*Suppose $F:M\to N$ is a **diffeomorphism** and $X_1,X_2\in\mathfrak{X}(M)$. Then*

$$F_*[X_1,X_2]=[F_*X_1,F_*X_2].$$

- **Corollary 3.6** *(**Brackets of Vector Fields Tangent to Submanifolds**).*
*Let M be a smooth manifold and let S be an **immersed submanifold** with or without boundary in M. If Y_1 and Y_2 are smooth vector fields on M that are **tangent** to S, then $[Y_1,Y_2]$ is also **tangent** to S.*

4 The Lie Algebra of a Lie Group

4.1 Lie Algebra

4.2 Induced Lie Algebra Homomorphisms

4.3 The Lie Algebra of a Lie Subgroup

References

John Marshall Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 2003. ISBN 0-387-95448-1.