Lecture 7: Modes of Convergence

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1 Mode of Convergence

1.1 Convergence of Functions in Measure Space

- **Remark** (***Convergence of Functions vs. Convergence of Numbers and Vectors***)
  Convegence of numbers $a_n \to a$ and convergence of vector $\boldsymbol{v}_n \to \boldsymbol{v}$ are both ***unambiguous***:

  1. $a_n \to a$ means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n \geq N$, $|a_n - a| \leq \epsilon$;
  2. $\boldsymbol{v}_n \to \boldsymbol{v}$ means that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n \geq N$, $\|\boldsymbol{v}_n - \boldsymbol{v}\| \leq \epsilon$; Note that the chioce of norm in Euclidean space will not affect the convergence results: convergence in ℓ_p will implies convergence in ℓ_q norm.

  However, for functions $f_n : X \to \mathbb{C}$ and $f : X \to \mathbb{C}$, there can now be *many different ways* in which the sequence f_n may or may not converge to the limit f. Note that a_n can be thought as f_n with singular domain $X = \{1\}$ and $\boldsymbol{v}_n$ can be thought of f_n with finite set $X = \{1, \ldots, d\}$. On the other hand, once X *becomes* ***infinite***, the functions f_n acquire an ***infinite number of degrees of freedom***, and this allows them to approach f in any number of *inequivalent ways*.

- **Remark** (***Two Basic Modes of Convergence***) [Royden and Fitzpatrick, 1988, Tao, 2011]

  1. **Definition** (***Pointwise Convergence***)
     We say that f_n converges to f ***pointwise*** if, for any $x \in X$ and $\epsilon > 0$, there exists $N > 0$ (*that* ***depends*** *on* ϵ *and* x) such that for all $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$. Denoted as $f_n(x) \to f(x)$.
  2. **Definition** (***Uniform Convergence***)
     We say that f_n converges to f ***uniformly*** if, for any $\epsilon > 0$, there exists $N > 0$ (*that* ***depends*** *on* ϵ *only*) such that for all $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$ for every $x \in X$. Denoted as $f_n \to f$, *uniformly*.

     Unlike pointwise convergence, the time N at which $f_n(x)$ must be permanently ϵ-close to $f(x)$ is not permitted to depend on x, but must instead be chosen *uniformly* in x.

- **Remark** (***Uniform*** $\Rightarrow$ ***Pointwise, Not Vice Versa***)
  Uniform convergence implies *pointwise convergence*, but not conversely.

  Example The functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) := x/n$ converge ***pointwise*** to the zero function $f(x) := 0$, but ***not uniformly***.

- **Remark** (***Modes of Convergence of Measurable Functions***)
  When the domain X is equipped with the structure of a measure space $(X, \mathscr{B}, \mu)$, and the functions f_n (and their limit f) are measurable with respect to this space. In this context, we have some *additional modes of convergence*:

  1. **Definition** (***Pointwise Almost Everywhere Convergence***)
     We say that f_n converges to f ***pointwise almost everywhere*** if, for ***μ-almost everywhere*** $x \in X$, $f_n(x)$ converges to $f(x)$. It is denoted as $f_n \overset{a.e.}{\to} f$.

     In other words, there exists ***a null set*** E, $(\mu(E) = 0)$ such that for *any* $x \in X \setminus E$ and any $\epsilon > 0$, there exists $N > 0$ (*that* ***depends*** *on* ϵ *and* x) such that for all $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$.

2. **Definition** (***Uniformly Almost Everywhere Convergence***) [Tao, 2011]
   We say f_n converges to f ***uniformly almost everywhere***, ***essentially uniformly***, or ***in L^∞ norm*** if, for every $\epsilon > 0$, there exists N such that for every $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$, ***for μ-almost every $x \in X$***.

   That is, $f_n \to f$ *uniformly* in $x \in X \setminus E$, for some E with $\mu(E) = 0$.

   We can also formulate in terms of L^∞ ***norm*** as
   $$\|f_n(x) - f(x)\|_{L^\infty(X)} \stackrel{n\to\infty}{\longrightarrow} 0,$$
   where $\|f\|_{L^\infty(X)} = \operatorname{ess\,sup}_x |f(x)| \equiv \inf_{\{E:\mu(E)=0\}} \sup_{x \in X\setminus E} |f(x)|$ is the ***essential bound***. It is denoted as $f_n \stackrel{L^\infty}{\to} f$.

3. **Definition** (***Almost Uniform Convergence***) [Tao, 2011]
   We say that f_n converges to f ***almost uniformly*** if, for every $\epsilon > 0$, there exists an ***exceptional set*** $E \in \mathscr{B}$ of *measure* $\mu(E) \leq \epsilon$ such that f_n converges ***uniformly*** to f on the *complement* of E.

   That is, for arbitrary δ there exists some E with $\mu(E) \leq \delta$ such that $f_n \to f$ *uniformly* in $x \in X \setminus E$.

4. **Definition** (***Convergence in L^1 Norm***)
   We say that f_n converges to f ***in L^1 norm*** if the quantity
   $$\|f_n - f\|_{L^1(X)} = \int_X |f_n(x) - f(x)|\, d\mu \stackrel{n\to\infty}{\longrightarrow} 0.$$
   It is also called the convergence ***in mean***. Denoted as $f_n \stackrel{L^1}{\to} f$.

5. **Definition** (***Convergence in Measure***)
   We say that f_n converges to f ***in measure*** if, for every $\epsilon > 0$, the measures
   $$\mu\left(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}\right) \stackrel{n\to\infty}{\longrightarrow} 0.$$
   Denoted as $f_n \stackrel{\mu}{\to} f$.

- **Remark** The difference between the ***uniformly almost everywhere convergence*** vs. ***the almost uniformly convergence*** is that:

  1. the ***former*** corresponds to *uniform convergence* ***outside a null set***, and
  2. the ***latter*** corresponds to *uniform convergence* ***outside an arbitrary small measure set (but still not a null set)***.

- **Remark** Observe that each of *these five modes of convergence* is ***unaffected*** if one ***modifies*** f_n or f on ***a set of measure zero***. In contrast, the *pointwise* and *uniform* modes of convergence can be ***affected*** if one modifies f_n or f *even on a single point*.

- **Remark** In the context of *probability theory*, in which f_n and f are interpreted as *random variables*, [Billingsley, 2008, Folland, 2013]

convergence in L^1 *norm*	$\Leftrightarrow$	convergence in *mean*
pointwise convergence almost everywhere	$\Leftrightarrow$	***almost sure convergence***
convergence in ***measure***	$\Leftrightarrow$	convergence in ***probability***

- **Proposition 1.1** ***(Linearity of Convergence)***. *[Tao, 2011]*
  Let $(X, \mathscr{B}, \mu)$ be a measure space, let $f_n, g_n : X \to \mathbb{C}$ be sequences of measurable functions, and let $f, g : X \to \mathbb{C}$ be measurable functions.

  1. *Then f_n converges to f along one of the above seven modes of convergence* ***if and only if*** *$|f_n - f|$ converges to 0 along* ***the same mode****.*

  2. *If f_n converges to f along one of the above seven modes of convergence, and g_n converges to g along* ***the same mode****, then $f_n + g_n$ converges to $f + g$ along the same mode, and that $c\, f_n$ converges to $c\, f$ along the same mode for any $c \in \mathbb{C}$.*

  3. *(****Squeeze test****) If f_n converges to 0 along one of the above seven modes, and $|g_n| \le f_n$* ***pointwise*** *for each n, then g_n converges to 0 along* ***the same mode****.*

1.2 Modes of Convergence via Tail Support and Width

- **Remark** (***Tail Support and Width***)

  Definition Let $E_{n,m} := \{x \in X : |f_n(x) - f(x)| \ge 1/m\}$. Define ***the N-th tail support set***

$$T_{N,m} := \{x \in X : |f_n(x) - f(x)| \ge 1/m, \;\; \exists n \ge N\} = \bigcup_{n \ge N} E_{n,m}.$$

  Also let $\mu(E_{n,m})$ be the ***width*** of n-th event $\mathbb{1}\{E_{n,m}\}$. Note that $T_{N,m} \supseteq T_{N+1,m}$ is ***monotone nonincreasing*** and $T_{N,m} \subseteq T_{N,m+1}$ is ***monotone nondecreasing***.

  1. The ***pointwise convergence*** of f_n to f indicates that for every x, every $m \ge 1$, there exists some $N \equiv N(m, x) \ge 1$ such that $T_{N,m}^c \ni x$ or $T_{N,m} \not\ni x$. Equivalently, ***the tail support shrinks to emptyset***:

$$\bigcap_{N \in \mathbb{N}} T_{N,m} = \lim_{N \to \infty} T_{N,m} = \limsup_{n \to \infty} E_{n,m} = \emptyset, \quad \text{for all } m.$$

     Conversely, to prove ***not pointwise convergence***, we need to find a $x \in X$ and for an arbitrary fixed $m \ge 1$ such that

$$x \in \bigcap_{N \in \mathbb{N}} \bigcup_{n \ge N} \{x \in X : |f_n(x) - f(x)| \ge 1/m\} = \limsup_{n \to \infty} \{x \in X : |f_n(x) - f(x)| \ge 1/m\}.$$

  2. The ***pointwise almost everywhere convergence*** indicates that there exists ***a null set*** F with $\mu(F) = 0$ such that for every $x \in X \setminus F$ and any $m \ge 1$, there exists some $N \equiv N(m, x) \ge 1$ such that $(T_{N,m} \setminus F) \not\ni x$. Equivalently, ***the tail support shrinks to a nulll set***. Note that it makes no assumption on $(T_{N,m} \cap F)$.

$$\begin{aligned} & \lim_{N \to \infty} T_{N,m} \setminus F = \limsup_{n \to \infty} E_{n,m} \setminus F = \emptyset, \quad \text{for all } m. \\ \Leftrightarrow \quad & \bigcap_{N \in \mathbb{N}} T_{N,m} = \lim_{N \to \infty} T_{N,m} = F \\ \Leftrightarrow \quad & \mu\left(\lim_{N \to \infty} T_{N,m}\right) = \mu\left(\bigcap_{N \in \mathbb{N}} T_{N,m}\right) = 0 \end{aligned}$$

Conversely, to prove ***not pointwise almost convergence***, we need to find a $x \in X$ and for an arbitrary fixed $m \geq 1$ such that

$$x \in \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{x \in X : |f_n(x) - f(x)| \geq 1/m\} \setminus F = \limsup_{n \to \infty} \{x \in X \setminus F : |f_n(x) - f(x)| \geq 1/m\}.$$

3. The ***uniform convergence*** indicates that for each $m \geq 1$, there exists some $N(m) \geq 1$ (not depending on x) such that $T_{N,m} = \emptyset$. (i.e. $T_{N,m} \not\ni x$ for all $x \in X$.) So ***the tail support <u>is an empty set</u>***

4. The ***uniformly almost everywhere convergence*** indicates that there exists some null set F with $\mu(F) = 0$ such that for each $m \geq 1$, there exists some $N(m) \geq 1$ (not depending on x) such that $(T_{N,m} \setminus F) = \emptyset$. (i.e. $T_{N,m} \not\ni x$ for all $x \in X \setminus F$.) Equivalently, ***the tail support <u>is a null set</u>***:

$$\begin{aligned} & T_{N,m} = F \\ \Leftrightarrow \quad & \mu\left(T_{N,m}\right) = 0 \end{aligned}$$

5. The ***almost uniform convergence*** indicates that for every δ, there exists *some measurable set* F_δ with $\mu(F_\delta) < \delta$ such that for each $m \geq 1$ there exists some $N(m) \geq 1$ (not depending on x) such that $(T_{N,m} \setminus F_\delta) = \emptyset$. (i.e. $T_{N,m} \not\ni x$ for all $x \in X \setminus F_\delta$.) Equivalently, <u>***the measure of tail support shrinks to zero***</u>:

$$\begin{aligned} \mu\left(T_{N,m}\right) \leq \delta \quad & \Leftrightarrow \quad T_{N,m} = F_\delta \\ \lim_{N \to \infty} \mu\left(T_{N,m}\right) = 0 & \end{aligned}$$

6. The ***convergence in measure*** indicates that for any $m \geq 1$ and any $\delta > 0$, there exists $N \equiv N(m, \delta) \geq 1$ such that for all $n \geq N$, *the* <u>***width***</u> *of n-th event* <u>***shrinks to zero***</u>:

$$\begin{aligned} & \mu(E_{n,m}) \leq \delta \\ & \lim_{n \to \infty} \mu(E_{n,m}) := \lim_{n \to \infty} \mu\left(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}\right) = 0 \end{aligned}$$

- **Definition** Define the ***maximum variation*** between (f_n) and f as $\sup_{x \in X} |f_n(x) - f(x)|$. Note that

$$\sup_{x \in X} |f_n(x) - f(x)| \geq \sup_{x \in X \setminus F, \mu(F) = 0} |f_n(x) - f(x)|.$$

- **Remark** From *Borel-Cantelli Lemma*, we see that in order to show *the* ***pointwise almost everywhere*** *convergence*, i.e. $\mu\left(\bigcap_N T_{N,\epsilon}\right) = \mu(\limsup_{n \to \infty} E_{n,\epsilon}) = 0$ it suffice to show that ***the measure of the tail support is finite***, $\mu\left(T_{N,\epsilon}\right) = \sum_{n=N}^{\infty} \mu(E_{n,\epsilon}) < \infty$. Note that this condition implies that it not only converges ***in measure*** $\mu(E_{n,\epsilon}) \to 0$ but converge in ***an absolutely summable*** fashion.

1.3 Relationships between Different Modes of Convergence

- **Proposition 1.2** *[Tao, 2011]*
  Let $(X, \mathscr{F}, \mu)$ be a measure space, and let $f_n : X \to \mathbb{C}$ and $f : X \to \mathbb{C}$ be measurable functions

1. *If f_n converges to f **uniformly**, then f_n converges to f **pointwisely**.*

2. *If f_n converges to f **uniformly**, then f_n converges to f in L^∞ **norm**. **Conversely**, if f_n converges to f **in** L^∞ **norm**, then f_n converges to f **uniformly outside of a null set** (i.e. there exists a null set E such that the restriction $f_n|_{X/E}$ of f_n to the complement of E converges to the restriction $f|_{X/E}$ of f).*

3. *If f_n converges to f in L^∞ **norm**, then f_n converges to f **almost uniformly**.*

4. *If f_n converges to f **almost uniformly**, then f_n converges to f **pointwise almost everywhere**.*

5. *If f_n converges to f **pointwise**, then f_n converges to f **pointwise almost everywhere**.*

6. *If f_n converges to f in L^1 **norm**, then f_n converges to f **in measure**.*

7. *If f_n converges to f **almost uniformly**, then f_n converges to f **in measure**.*

Proof:

1. It is from the definition.

2. Note that for any $\epsilon > 0$, there exists $N(\epsilon) \geq 1$, such that for all $n \geq N(\epsilon)$, $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X$. Therefore, $\sup_{x\in X} |f_n(x) - f(x)| \leq \epsilon$. Then it holds that for any $x \in X \setminus E$, $\mu(E) = 0$, $\sup_{x\in X\setminus E} |f_n(x) - f(x)| \leq \epsilon$, so f_n converges to f in L^∞ norm.

   Since $f_n \overset{L^\infty}{\to} f$, then for any $\epsilon > 0$, there exists $N(\epsilon) \geq 1$, for all $n \geq N(\epsilon)$ such that the ***infimum*** of all essential upper bound M less than ϵ. In other words, let $d = \|f_n(x) - f(x)\|_{L^\infty} < \epsilon$, so given $\epsilon > 0$, there exists an upper bound $d + \epsilon > M > 0$ such that for any $x \in X \setminus E$ with some E such that $\mu(E) = 0$, and $\left| f_n|_{X\setminus E}(x) - f|_{X\setminus E}(x) \right| \leq M < 2\epsilon$. Therefore f_n converges to f uniformly outside a null set E.

3. This follows from the argument above.

4. Let (E_n) be a sequence of measurable sets in $\mathscr{B}$ such that for each $n \in \mathbb{N}$ we have $\mu(E_n) \leq 1/n$ and (f_n) converges uniformly to f on $X \setminus E_n$. Now pick an arbitrary $x \in X$. We have two cases.

   - In the first case $x \in \cap_{n\in\mathbb{N}} E_n$, in which case $\lim_{n\to\infty} f_n(x)$ is not necessarily $f(x)$. But this is not harmful, since $\mu(\cap_{n\in\mathbb{N}} E_n) = 0$.

   - In the second case $x \notin \cap_{n\in\mathbb{N}} E_n$, which implies $\lim_{n\to\infty} f_n(x) = f(x)$ from the uniform convergence.

5. This follows from the definition.

6. Apply *the Markov inequality*, then the result follows.

7. Pick an arbitrary $\delta > 0$, so that there exists an exceptional set E such that $\mu(E) \leq \delta$ and $f_n \to f$ uniformly on $X \setminus E$. That is, we can find $N \in \mathbb{N}$ such that for $n \geq N(\epsilon)$, $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X \setminus E$. For $n \geq N$, we have

$$\begin{aligned}
\mu\left(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\right) &= \mu\left(\{x \in E : |f_n(x) - f(x)| > \epsilon\}\right) \\
&\quad + \mu\left(\{x \in X \setminus E : |f_n(x) - f(x)| > \epsilon\}\right) \\
&\leq \mu(E) + \mu(\emptyset) = \delta + 0 = \delta \quad \blacksquare
\end{aligned}$$

- **Remark** This diagram shows the *relative strength* of different *modes of convergence*. The direction arrows $A \to B$ means "if A holds, then B holds".

$$\begin{tikzcd}
\textit{uniform} \arrow[r] \arrow[dd] \arrow[rr, bend left] & \textit{uniformly a.e.} \arrow[r, leftrightarrow] \arrow[d] & \textit{in } L^\infty \textit{ norm} \arrow[dl] \\
 & \textit{almost uniform} \arrow[d] \arrow[dr] & \textit{in } L^1 \textit{ norm} \arrow[d] \\
\textit{pointwise} \arrow[r] & \textit{pointwise a.e.} & \textit{in measure}
\end{tikzcd}$$

Moreover, here are some counter statements:

- $L^\infty \not\to L^1$: see the "*Escape to Width Infinity*" example below.
- ***uniform*** $\not\to L^1$: see the "*Escape to Width Infinity*" example below.
- $L^1 \not\to$ ***uniform*** : see the "*Typewriter Sequence*" example below.
- ***pointwise*** $\not\to L^1$: see the "*Escape to Horizontal Infinity*" example below.
- ***pointwise*** $\not\to$ ***uniform***: see the "$f_n = x/n$" example above.
- For finite measure space, ***pointwise a.e.*** $\to$ ***almost uniform***: see the Egorov's theorem.
- ***almost uniform*** $\not\to L^1$: see the "*Escape to Vertical Infinity*" example below.
- ***almost uniform*** $\not\to L^\infty$: see the "*Escape to Vertical Infinity*" example below. The *converse* is true, however.
- For bounded $f_n \le G, a.e.\ \forall n$, then ***pointwise a.e.*** $\to L^1$: see *Dominated Convergence Theorem.*
- $L^1 \not\to$ ***pointwise a.e.*** : see the "*Typewriter Sequence*" example below.
- ***in measure*** $\not\to$ ***pointwise a.e.*** : see the "*Typewriter Sequence*" example below.
- $L^1 \to$ ***convergence in integral***: by *triangle inequality.* Note that *the other modes of convergence does* **not directly** *lead to convergence in integral.*

1.4 Counter Examples

- **Example** (***Escape to Horizontal Infinity***).
  Let X be the real line with Lebesgue measure, and let
  $$f_n(x) \equiv \mathbb{1}\{x \in [n, n+1]\}.$$
  Note that the ***height*** and ***width do not shrink to zero***, but ***the tail set*** shrinks to ***the empty set***. We have the following statements on different modes of convergence:

  1. f_n ***converges pointwise*** *to* $f = 0$, (thus ***pointwise a.e.***)
  2. f_n ***does not*** *converges to* $f = 0$ ***uniformly***,

3. f_n ***does not*** *converges to* $f = 0$ in L^∞ ***norm***,
4. f_n ***does not*** *converges to* $f = 0$ ***almost uniformly***
5. f_n ***does not*** *converges to* $f = 0$ ***in measure***.
6. $\int_{\mathbb{R}} f_n dx = 1$ ***does not*** *converge* to $\int_{\mathbb{R}} f dx = 0$.
7. f_n ***does not*** *converges to* $f = 0$ ***in*** L^1 ***norm***.

Somehow, *all the* ***mass*** in the f_n has *escaped* by *moving off to infinity* in a ***horizontal direction***, leaving none behind for the pointwise limit f. In *frequency domain*, it corresponds to ***escaping to spatial infinity***.

- **Example** (***Escape to Width Infinity***).
  Let X be the real line with Lebesgue measure, and let
  $$f_n \equiv \frac{1}{n} \mathbb{1} \{x \in [0, n]\} .$$
  See that *the* ***height*** *goes to* ***zero***, but the ***width*** *(and* ***tail support****) go to* ***infinity***, causing the L^1 norm to stay ***bounded away from zero***. We have the following statements on different modes of convergence:

  1. f_n ***converges*** *to* $f = 0$ ***uniformly***. (Thus, ***pointwise***, ***pointwise a.e.***, ***uniformly a.e.***, ***almost uniformly***, ***in*** L^∞ ***norm*** and ***in measure***)
  2. $\int_{\mathbb{R}} f_n dx = 1$ ***does not converge*** *to* $\int_{\mathbb{R}} f dx = 0$. This is due to the ***increasingly wide*** *nature* of *the* ***support*** *of the* f_n. If all the f_n were supported *in a single set of finite measure*, this will not happen.
  3. f_n ***does not*** *converges to* $f = 0$ ***in*** L^1 ***norm***.

  In *frequency domain*, it corresponds to ***escaping to zero frequency***.

- **Example** (***Escape to Vertical Infinity***).
  Let X be the unit interval $[0, 1]$ with Lebesgue measure (restricted from $\mathbb{R}$), and let
  $$f_n = n \mathbb{1} \left\{x \in \left[n^{-1}, 2\, n^{-1}\right]\right\} .$$
  Note that the ***height*** *goes to* ***infinity***, but *the* ***width*** *(and* ***tail support****) go to* ***zero*** *(or* ***the empty set****)*, causing the L^1 *norm to stay* ***bounded away from zero***. We have the following statements on different modes of convergence:

  1. f_n ***converges pointwise*** *to* $f = 0$, (thus ***pointwise a.e.***)
  2. f_n ***converges*** *to* $f = 0$ ***almost uniformly***, (thus ***in measure***)
  3. f_n ***does not*** *converges to* $f = 0$ ***uniformly***,
  4. f_n ***does not*** *converges to* $f = 0$ in L^∞ ***norm***,
  5. $\int_{\mathbb{R}} f_n dx = 1$ ***does not*** *converge* to $\int_{\mathbb{R}} f dx = 0$.
  6. f_n ***does not*** *converges to* $f = 0$ ***in*** L^1 ***norm***.

  Note that we have finite measure on $X = [0, 1]$. This time, the mass has *escaped* ***vertically***, through *the* ***increasingly large*** *values of* f_n. In *frequency domain*, it corresponds to ***escaping to infinity frequency***.

- **Example** (***Typewriter Sequence***).
  Let f_n be defined by the formula

$$f_n \equiv \mathbb{1}\left\{x \in \left[\frac{n-2^k}{2^k}, \frac{n+1-2^k}{2^k}\right]\right\}$$

  whenever $k \geq 0$ and $2^k \leq n < 2^k + 1$. This is a sequence of indicator functions of ***intervals*** of ***decreasing length***, *marching across the unit interval* $[0, 1]$ ***over and over again***. See that *the **width** goes to **zero***, but *the **height** and the **tail support stay fixed*** (and thus ***bounded away from zero***). We have the following statements on different modes of convergence:

  1. f_n ***converges** to $f = 0$ **in** L^1 **norm***, (thus ***in measure***)
  2. f_n ***does not** converges to $f = 0$ **pointwise a.e.***, (thus ***not pointwise***, ***not almost uniformly***, ***not uniformly a.e.***, ***not uniformly***, ***not in*** L^∞ ***norm***)

1.5 Uniqueness

- **Proposition 1.3** *Let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions, and let $f, g : X \to \mathbb{C}$ be two additional measurable functions. Suppose that f_n converges to f **along one of the seven modes of convergence** defined above, and f_n converges to g **along another of the seven modes of convergence** (or perhaps the same mode of convergence as for f). Then f and g **agree almost everywhere**.*

- **Remark** It suffice to show that when f_n converges to f ***pointwise almost everywhere***, and f_n converges to g ***in measure***. We need to show that $f = g$ *almost everywhere.*

- **Remark** Even though the modes of convergence all *differ* from each other, they are all ***compatible*** in the sense that they ***never disagree*** about *which function f* a sequence of functions f_n ***converges to***, *outside of a set of measure zero.*

2 Modes of Convergence for Step Functions

2.1 Analysis

- **Remark** Consider the ***step function*** f_n as a constant multiple $f_n = A_n \mathbb{1}\{E_n\}$ of a measurable set E_n, which has a limit $f = 0$.

- **Definition** The modes of convergence for step function f_n is determined by the following quantities:

  1. the n-th ***width*** of f_n is $\mu(E_n)$;
  2. the n-th ***height*** of f_n is A_n;
  3. the N-th ***tail support*** $T_N \equiv \bigcup_{n \geq N} E_n$ of the sequence $f_1, f_2, f_3, \ldots$.

- **Remark** Assume the ***height*** A_n exhibit *one of two modes of behaviour*:

  1. $A_n \to 0$, ***converge to zero***;

2. (A_n) are ***bounded away from zero*** (i.e. there exists $c > 0$ such that $A_n \geq c$ for every n.)

- **Proposition 2.1** *The following regarding the seven modes of convergence of $f_n = A_n \mathbb{1}\{E_n\}$ to $f = 0$:*

  1. *f_n converges **uniformly** to zero if and only if $A_n \to 0$ as $n \to \infty$.*
  2. *f_n converges **in L^∞ norm** to zero if and only if $A_n \to 0$ as $n \to \infty$.*
  3. *f_n converges **almost uniformly** to zero if and only if $A_n \to 0$ as $n \to \infty$, or $\mu(T_N) \to 0$ as $N \to \infty$.*
  4. *f_n converges **pointwise** to zero if and only if $A_n \to 0$ as $n \to \infty$, or $\bigcap_{N=1}^{\infty} T_N = \emptyset$.*
  5. *f_n converges **pointwise almost everywhere** to zero if and only if $A_n \to 0$ as $n \to \infty$, or $\bigcap_{N=1}^{\infty} T_N$ is a null set.*
  6. *f_n converges **in measure** to zero if and only if $A_n \to 0$ as $n \to \infty$, or or $\mu(E_n) \to 0$ as $n \to \infty$.*
  7. *f_n converges **in L^1 norm** if and only if $A_n \mu(E_n) \to 0$ as $n \to \infty$.*

- **Remark** We summarize the above proposition:

  - When the ***height*** goes to ***zero***, then one has convergence to zero in ***all modes except possibly for L^1 convergence***, which requires that the ***product*** of the ***height*** and the ***width*** goes to zero.
  - If the ***height** is **bounded away from zero*** *(positive)* and the ***width*** is ***positive*** (finite support), then we ***never*** have ***uniform*** or L^1 convergence.
    * If ***the width goes to zero***, we have convergence in ***measure***.
    * If ***the measure of tail support goes to zero***, we have ***almost uniform*** *convergence.*
    * If ***the tail support shrinks** to a **null set***, we have ***pointwise almost everywhere*** *convergence.*
    * If ***the tail support shrinks** to **the empty set***, we have ***pointwise*** *convergence.*

- **Remark** Four counterexamples above are all step functions:

  1. In the ***escape to horizontal infinity*** scenario, *the height and width do not shrink to zero*, but *the tail set shrinks to the empty set (while remaining of infinite measure throughout)*
  2. In the ***escape to width infinity*** scenario, *the height goes to zero*, but *the width (and tail support) go to infinity*, causing the L^1 norm to *stay bounded away from zero.*
  3. In the ***escape to vertical infinity***, *the height goes to infinity*, but *the width (and tail support) go to zero (or the empty set)*, causing the L^1 *norm to stay bounded away from zero.*

2.2 Comparison

Table 1: Comparison of Modes of Convergence

	tail support	***width***	***maximum variation***	***subgraph***
definition	$T_{N,\epsilon} = \bigcup_{n \geq N} E_{n,\epsilon}$	$\mu(E_{n,\epsilon})$	$\sup_{x \in X}\{\lvert f_n(x) - f(x)\rvert\}$	$\Gamma(f_n) = \{(x,t) : 0 \leq t \leq f_n(x)\}$
pointwise	$\bigcap_{N=1}^{\infty} T_{N,\epsilon} = \emptyset$		*or*, $\to 0$ on X	
point-wise a.e.	$\mu\left(\bigcap_{N=1}^{\infty} T_{N,\epsilon}\right) = 0$		*or*, $\to 0$ on $X \setminus E$	
uniform	$T_{N,\epsilon} = \emptyset$		equivalently, $\to 0$ on X	
uniform a.e.	$\mu\left(T_{N,\epsilon}\right) = 0$		equivalently, $\to 0$ on $X \setminus E$	
L^{∞} ***norm***	$\mu\left(T_{N,\epsilon}\right) = 0$		equivalently, $\to 0$ on $X \setminus E$	
almost uniform	$\lim_{N\to\infty} \mu\left(T_{N,\epsilon}\right) = 0$		or, $\to 0$ on $X \setminus E$	
in measure		$\lim_{n\to\infty} \mu\left(E_{n,\epsilon}\right) = 0$	or, $\to 0$ on $X \setminus E$	
L^{1} ***norm***			$\to 0$ and support fixed or non-increasing	area of $\Gamma(f_n) = \mathcal{A}(\Gamma(f_n))$ $\lim_{n\to\infty} \mathcal{A}(\Gamma(\lvert f_n - f\rvert)) = 0$

3 Modes of Convergence With Additional Conditions

3.1 Finite Measure Space

- **Remark** If we assume that $(X, \mathscr{B}, \mu)$ has ***finite measure***, i.e. $\mu(X) < \infty$, we can shut down two of the four examples (namely, ***escape to horizontal infinity*** or ***escape to width infinity***) and creates a few more equivalences.

- **Example** *A probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ is a finite measure space since $\mathbb{P}(\Omega) = 1$.

- **Theorem 3.1** *(**Egorov's theorem**). [Royden and Fitzpatrick, 1988, Tao, 2011]*
*Let $(X, \mathscr{F}, \mu)$ be a **finite measure space**, that is, $\mu(X) < \infty$ and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions that converge **pointwise almost everywhere** to another function $f : X \to \mathbb{C}$, and let $\epsilon > 0$. Then there exists a μ-measurable set A of measure at most ϵ, such that f_n **converges uniformly** to f **outside** of A. That is, given finite measure space, convergence pointwise almost everywhere implies **converge almost uniformly**.*

- **Remark** *The* ***finite measure space*** condition allows us to use ***the downward convergence*** of measure without much concern.

- **Proposition 3.2** *Let X have **finite measure**, and let $f_n : X \to \mathbb{C}$ and $f : X \to \mathbb{C}$ be measurable functions. If f_n converges to f in L^∞ norm, then f_n also converges to f in L^1 norm.*

- **Remark** For finite measure space,

$$
\begin{tikzcd}
\textit{uniform} \arrow[r] \arrow[rr, bend left] \arrow[dd] & \textit{uniformly a.e.} \arrow[r, leftrightarrow] \arrow[d] & \textit{in } L^\infty \textit{ norm} \arrow[dl] \arrow[d, red] \\
 & \textit{almost uniform} \arrow[d] \arrow[dr] & \textit{in } L^1 \textit{ norm} \arrow[d] \\
\textit{pointwise} \arrow[r] & \textit{pointwise a.e.} \arrow[u, red, bend left] & \textit{in measure}
\end{tikzcd}
$$

3.2 Fast L^1 Convergence

- **Proposition 3.3** *(**Fast L^1 convergence**).*
*Suppose that $f_n, f : X \to \mathbb{C}$ are measurable functions such that $\sum_{n=1}^{\infty} \|f_n f\|_{L^1(\mu)} < \infty$; thus, not only do the quantities $\|f_n f\|_{L^1(\mu)}$ go to zero (which would mean L^1 convergence), but they converge in **an absolutely summable** fashion. Then*

  1. *f_n converges **pointwise almost everywhere** to f.*
  2. *f_n converges **almost uniformly** to f.*

- **Corollary 3.4** *(**Subsequence Convergence**). [Tao, 2011]*
*Suppose that $f_n : X \to \mathbb{C}$ are a sequence of measurable functions that converge in L^1 **norm** to a limit f. Then there exists a **subsequence** f_{n_j} that converges **almost uniformly** (and*

*hence, **pointwise almost everywhere**) to f (while remaining convergent in L^1 norm, of course).*

- **Corollary 3.5** *(**Subsequence Convergence in Measure**). [Tao, 2011]*
  *Suppose that $f_n : X \to \mathbb{C}$ are a sequence of measurable functions that **converge in measure** to a limit f. Then there exists a subsequence f_{n_j} that converges **almost uniformly** (and hence, **pointwise almost everywhere**) to f.*

- **Remark** It is instructive to see how this ***subsequence*** is extracted in the case of *the typewriter sequence*. In general, one can view the operation of passing to a subsequence as being able to ***eliminate*** "***typewriter***" situations in which *the tail support is much larger than the width*.

- **Exercise 3.6** *[Tao, 2011] Let $(X, \mathscr{B}, \mu)$ be a measure space, let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions converging **pointwise almost everywhere** as $n \to \infty$ to a measurable limit $f : X \to \mathbb{C}$, and for each n, let $f_{n,m} : X \to \mathbb{C}$ be a sequence of measurable functions converging **pointwise almost everywhere** as $m \to \infty$ (keeping n fixed) to f_n.*

  1. *If $\mu(X)$ is **finite**, show that there exists a sequence $m_1, m_2, \ldots$ such that f_{n,m_n} converges **pointwise almost everywhere** to f.*

  2. *Show the same claim is true if, instead of assuming that $\mu(X)$ is finite, we merely assume that X is **σ-finite**, i.e. it is the countable union of sets of finite measure.*

- **Exercise 3.7** *[Tao, 2011]*
  Let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions, and let $f : X \to \mathbb{C}$ be another measurable function. Show that the following are equivalent:

  1. *f_n converges **in measure** to f.*

  2. *Every **subsequence** f_{n_j} of the f_n has a **further subsequence** $f_{n_{j_i}}$ that converges **almost uniformly** to f.*

3.3 Domination and Uniform Integrability

- **Remark** Now we turn to the *reverse question*, of *whether **almost uniform** convergence, **pointwise almost everywhere** convergence, or convergence **in measure** can imply L^1 **convergence***. The *escape to vertical* and *width infinity* examples shows that without any further hypotheses, the answer to this question is ***no***.

- **Remark** [Tao, 2011] There are ***two major ways*** to shut down loss of mass via ***escape to infinity***.

  1. One is to enforce ***monotonicity***, which ***prevents each f_n from abandoning the location*** where the mass of the preceding $f_1, \ldots, f_{n-1}$ was concentrated and which thus shuts down the above three escape scenarios. More precisely, we have the monotone convergence theorem.

  2. The other major way is to ***dominate*** *all of the functions involved by an **absolutely convergent one***. This result is known as the dominated convergence theorem.

- **Definition** We say that a sequence $f_n : X \to \mathbb{C}$ is ***dominated*** if there exists an ***absolutely***

integrable function $g : X \to \mathbb{C}$ such that $|f_n(x)| \leq g(x)$ for all n and *almost every* x.

- **Definition** (***Uniform integrability***).
  A sequence $f_n : X \to \mathbb{C}$ of ***absolutely integrable*** functions is said to be ***uniformly integrable*** if the following three statements hold:

  1. (***Uniform bound on*** L^1 ***norm***) One has $\sup_n \|f_n\|_{L^1(\mu)} = \sup_n \int_X |f_n|\, d\mu < +\infty$.
  2. (***No escape to vertical infinity***) One has
  $$\lim_{M \to +\infty} \sup_n \int_{|f_n| \geq M} |f_n|\, d\mu \to 0.$$
  3. (***No escape to width infinity***) One has
  $$\lim_{\delta \to 0} \sup_n \int_{|f_n| \leq \delta} |f_n|\, d\mu \to 0.$$

- **Proposition 3.8** *(Property of Uniform Integrablility)*

  1. *If f is an* ***absolutely integrable*** *function, then the constant sequence $f_n = f$ is* ***uniformly integrable****. (Hint: use the monotone convergence theorem.)*
  2. *Every* ***dominated*** *sequence of measurable functions is* ***uniformly integrable****.*

- **Exercise 3.9** *Give an example of a sequence that is uniformly integrable but not dominated.*

- **Remark** In the case of a ***finite measure space***, there is *no escape to width infinity*, and the criterion for *uniform integrability* simplifies to just that of ***excluding vertical infinity***:

  Exercise 3.10 *Suppose that X has finite measure, and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions. Show that f_n is uniformly integrable* ***if and only if*** $\sup_n \int_{|f_n| \geq M} |f_n|\, d\mu \to 0$ *as $M \to \infty$.*

- **Exercise 3.11** *(****Uniform L^p bound on finite measure implies uniform integrability****).*
  Suppose that X have finite measure, let $1 < p < \infty$, and suppose that $f_n : X \to \mathbb{C}$ is a sequence of measurable functions such that $\sup_n \int_X |f_n|^p\, d\mu < +\infty$. Show that the sequence f_n is uniformly integrable.

- **Exercise 3.12** *Give an example of a sequence f_n of uniformly integrable functions that converge* ***pointwise almost everywhere*** *to zero, but do* ***not converge almost uniformly****,* ***in measure****, or* ***in*** L^1 ***norm****.*

- **Theorem 3.13** *(****Uniformly integrable convergence in measure****).*
  Let $f_n : X \to \mathbb{C}$ be a ***uniformly integrable*** *sequence of functions, and let $f : X \to \mathbb{C}$ be another function. Then f_n converges in L^1* ***norm*** *to f* ***if and only*** *if f_n converges to f* ***in measure****.*

- **Proposition 3.14** *Suppose that $f_n : X \to \mathbb{C}$ are a* ***dominated*** *sequence of measurable functions, and let $f : X \to \mathbb{C}$ be another measurable function. Show that f_n converges* ***pointwise almost everywhere*** *to f* ***if and only if*** *f_n converges in* ***almost uniformly*** *to f.*

4 Convergence in Distribution

- **Definition** (***Cumulative Distribution Function***) [Billingsley, 2008]
  Let $(X, \mathscr{F}, \mu)$ be a measure space. Given any real-valued measurable function $f : X \to \mathbb{R}$, we define the ***cumulative distribution function*** $F : \mathbb{R} \to [0, \infty]$ of f to be the function $F(\lambda) := \mu_f((-\infty, \lambda]) = \mu(\{x \in X : f(x) \le \lambda\})$ where $\mu_f = \mu \circ f^{-1}$ is a ***measure*** on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induced by function f.

- **Definition** (***Converge in Distribution***) [Van der Vaart, 2000]
  Let $(X, \mathscr{F}, \mu)$ be a measure space, $f_n : X \to \mathbb{R}$ be a sequence of real-valued *measurable functions*, and $f : X \to \mathbb{R}$ be another measurable function.

  We say that f_n ***converges in distribution*** to f if *the cumulative distribution function* $F_n(\lambda)$ of f_n converges ***pointwise*** to *the cumulative distribution function* $F(\lambda)$ of f at all $\lambda \in \mathbb{R}$ for which F is *continuous*. Denoted as $f_n \xrightarrow{d} f$ or $f_n \rightsquigarrow f$.

  Note that for the distribution $\mu_{f_n} \equiv \mu \circ f_n^{-1}$ is a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Thus $f_n \xrightarrow{d} f$ if and only if

$$\mu_{f_n}(A) \to \mu_f(A), \quad \forall A \in \mathscr{B}(\mathbb{R}).$$

- **Remark** (***Convergence of Measures Induced by Function***)
  Convergence in distribution is also called ***weak convergence*** in probability theory [Folland, 2013]. In general, it is actually ***not*** *a mode of* ***convergence of functions*** f_n ***itself*** but instead is the ***convergence of measures induced*** *by function f_n on $\mathscr{B}(\mathbb{R})$*.

  In functional analysis, however, ***weak convergence*** is actually reserved for a different mode of convergence, while ***the convergence in distribution*** is ***the weak* convergence***.

$$\begin{aligned} &\text{weak convergence} && \int f_n d\mu \to \int f d\mu, \quad \forall \mu \in \mathcal{M}(X), \\ &\text{convergence in distribution} && \int f d\mu_n \to \int f d\mu, \quad \forall f \in \mathcal{C}_0(X) \end{aligned}$$

  Definition (***Weak* Topology on Banach Space***)
  Let X be a *normed vector space* and X^* be its dual space. The ***weak* topology*** on X^* is *the weakest topology* on X^* so that $f(x)$ *is* ***continuous for all*** $x \in X$.

  The weak* topology on space of regular Borel measures $\mathcal{M}(X) \simeq (\mathcal{C}_0(X))^*$ on a ***compact Hausdorff*** space X, is often called ***the vague topology***. Note that $\mu_n \xrightarrow{w^*} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in \mathcal{C}_0(X)$.

- **Theorem 4.1** *(**The Portmanteau Theorem**). [Van der Vaart, 2000]*
  The following statements are equivalent.

  1. $X_n \rightsquigarrow X$.
  2. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *that are non-zero only on a* ***closed*** *and* ***bounded*** *set.*
  3. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***bounded continuous functions*** $h : \mathbb{R}^d \to \mathbb{R}$.

4. $\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X)]$ *for all* ***bounded measurable functions*** $h : \mathbb{R}^d \to \mathbb{R}$ *for which* $\mathbb{P}(X \in \{x : h \text{ is continuous at } x\}) = 1$.

- We can reformulate the definition of *convergence in distribution* as below:

  Definition [Wellner et al., 2013]
  Let (Ω, d) be a *metric space*, and $(\Omega, \mathscr{B})$ be *a measurable space*, where $\mathscr{B}$ is ***the Borel σ-field on*** Ω, the smallest σ-field containing *all the open balls* (as the basis of *metric topology* on Ω). Let $\{P_n\}$ and P be ***Borel probability measures*** on $(\Omega, \mathscr{B})$.

  Then the sequence P_n ***converges in distribution*** to P, which we write as $P_n \rightsquigarrow P$, if and only if

$$\int_{\Omega} f dP_n \to \int_{\Omega} f dP, \quad \text{for all } f \in \mathcal{C}_b(\Omega).$$

  Here $\mathcal{C}_b(\Omega)$ denotes the set of *all* ***bounded, continuous****, real functions on* Ω.

  We can see that ***the convergence in distribution*** is actually ***a weak* convergence***. That is, it is ***the weak convergence*** of ***bounded linear functionals*** $I_{\mathcal{P}_n} \xrightarrow{w^*} I_{\mathcal{P}}$ on *the space of all probability measures* $\mathcal{P}(\mathcal{X}) \simeq (\mathcal{C}_b(\mathcal{X}))^*$ on $(\mathcal{X}, \mathscr{B})$ where

$$I_{\mathcal{P}} : f \mapsto \int_{\Omega} f d\mathcal{P}.$$

  Note that the $I_{\mathcal{P}_n} \xrightarrow{w^*} I_{\mathcal{P}}$ is equivalent to $I_{\mathcal{P}_n}(f) \to I_{\mathcal{P}}(f)$ *for all* $f \in \mathcal{C}_b(\mathcal{X})$.

- **Remark** The density f_ξ of ξ is defined as for F_ξ uniformly continuous on $\mathbb{R}$

$$F_\xi(A) \equiv \mu \circ \xi^{-1}(A) \equiv \int_{\Omega} \mathbb{1}\left\{\xi^{-1}(A)\right\} d\mu \equiv \int_{\mathbb{R}} f_\xi \mathbb{1}\left\{A\right\} dx$$

  for all $A \in \mathscr{B}(\mathbb{R})$ and the integral is the Lebesgue integral with respect to Lebesgue measure.

- **Remark** Note that $\xi_n \to \xi$ and $\eta_n \to \eta$ in distribution, but it is possible $\xi_n + \eta_n \not\to \xi + \eta$.

- **Remark** It is related to following convergence

$$\sup_{A \in \mathscr{B}(\mathbb{R})} |F_n(A) - F(A)| \to 0$$
$$\text{or } F_n(A) \to F(A), \forall\, A \in \mathscr{B}(\mathbb{R})$$

- **Theorem 4.2** ***(Continuous Mapping Theorem)*** *[Van der Vaart, 2000]*
  Suppose that $f_n : X \to \mathbb{R}^k, n \geq 1$ *is a sequence of measureable functions and its limit* $f : X \to \mathbb{R}^k$ *is a measureable function. Let* $g : \mathbb{R}^k \to \mathbb{R}^m$ *be* ***continuous at every point*** *of a set* $C \subset \mathbb{R}^k$ *such that* $\mu(\{x : f(x) \in C\}) = \mu(X) = 1$. *Then*

  1. *If* $f_n \xrightarrow{a.e.} f$, *then* $g(f_n) \xrightarrow{a.e.} g(f)$;
  2. *If* $f_n \xrightarrow{\mu} f$, *then* $g(f_n) \xrightarrow{\mu} g(f)$;
  3. *If* $f_n \rightsquigarrow f$, *then* $g(f_n) \rightsquigarrow g(f)$.

  Proof: 1. Directly by the property of continuous map, since $g(\lim_{n\to\infty} y_{n,x}) = \lim_{n\to\infty} g(y_{n,x})$, where $y_{n,x} = f_n(x)$ for $x \in X/E$, $\mu(E) = 0$.

2. For any $\epsilon > 0$, there exists $\delta > 0$ such that the set

$$B_\delta \equiv \left\{ z \in \mathbb{R}^k \mid \exists y,\ \|z - y\| \le \delta,\ \|g(z) - g(y)\| > \epsilon \right\}.$$

Clearly, if $f(x) \notin B_\delta$ and $\|g(f_n(x)) - g(f(x))\| > \epsilon$, then $\|f_n(x) - f(x)\| > \delta$. So

$$\mu\left(\{x : \|g(f_n(x)) - g(f(x))\| > \epsilon\}\right) \le \mu\left(\{x : \|f_n(x) - f(x)\| > \delta\}\right) + \mu\left(\{x : f(x) \in B_\delta\}\right)$$

The first term on RHS converges to 0 as $n \to \infty$ for every fixed $\delta > 0$ due to the convergence in measure. Since $B_\delta \cap C \downarrow 0$, by continuity of g, the second term converges to 0 as $\delta \to 0$.

3. The event $\{x : g(f_n(x)) \in F\} \equiv \left\{x : f_n(x) \in g^{-1}(F)\right\}$ for any closed/open set F. Note that

$$g^{-1}(F) \subseteq \overline{g^{-1}(F)} \subset g^{-1}(F) \cup C^c$$

Thus there exists a sequence of $y_m \to y$ and $g(y_m) \in F$ for every closed F. If $y \in C$, then $g(y_m) \to g(y)$, which is in F, since F is closed. Otherwise, $y \in C^c$. By the portmanteau lemma, since f_n converges to f in distribution,

$$\begin{aligned}\limsup_{n\to\infty} \mu\left(\{x : g(f_n(x)) \in F\}\right) &\le \limsup_{n\to\infty} \mu\left(\left\{x : f_n(x) \in \overline{g^{-1}(F)}\right\}\right) \\ &\le \mu\left(\left\{x : f(x) \in \overline{g^{-1}(F)}\right\}\right)\end{aligned}$$

Since $\mu(X) = \mu(C) = 1$, so $\mu(C^c) = 0$. Thus the RHS

$$\begin{aligned}\mu\left(\left\{x : f(x) \in \overline{g^{-1}(F)}\right\}\right) &= \mu\left(\left\{x : f(x) \in g^{-1}(F)\right\}\right) \\ &= \mu\left(\{x : g(f(x)) \in F\}\right).\end{aligned}$$

Again by applying the portmanteau lemma, $g(f_n)$ converges to $g(f)$ in distribution. ∎

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