Lecture 5: Measure Theory on Compact Space

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Contents

1 Subspaces of Continuous Functions

- **Remark** (***Useful Topologies on*** Y^X)

  1. ***Uniform Topology***: generated by the ***basis***

     $$B_U(f,\epsilon) = \left\{ g \in Y^X : \sup_{x\in X} \bar{d}(f(x), g(x)) < \epsilon \right\}$$

     It corresponds to ***the uniform convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *uniform topology*, following *the Uniform Limit Theorem*.

  2. ***Topology of Pointwise Convergence***: generated by the ***basis***

     $$\begin{aligned} B_{U_1,\ldots,U_n}(x_1,\ldots,x_n,\epsilon) &= \bigcap_{i=1}^{n} S(x_i, U_i) \\ &= \left\{ f \in Y^X : f(x_1) \in U_1, \ldots, f(x_n) \in U_n \right\}, \quad 1 \le n < \infty. \end{aligned}$$

     It corresponds to ***the pointwise convergence*** of f_n to f in Y^X. $\mathcal{C}(X,Y)$ is ***not closed*** in Y^X under the *topology of pointwise convergence*. Note that *the topology of poinwise convergence* is *the* ***product topology*** *of* Y^X.

  3. ***Topology of Compact Convergence***: generated by the ***basis***

     $$B_C(f,\epsilon) = \left\{ g \in Y^X : \sup_{x\in C} d(f(x), g(x)) < \epsilon \right\}, \; C \text{ is compact set.}$$

     It corresponds to ***the uniform convergence*** of f_n to f in Y^X for $x \in C$. $\mathcal{C}(X,Y)$ is ***closed*** in Y^X under the *topology of compact convergence* ***if*** X ***is compactly generated***.

     On $\mathcal{C}(X)$, the topology of compact convergence is equal to the compact-open topology:

     Definition (***Compact-Open Topology on Continuous Function Space***)
     Let X and Y be topological spaces. If C is a ***compact subspace*** of X and U is an *open* subset of Y, define

     $$S(C,U) = \{ f \in \mathcal{C}(X,Y) : f(C) \subseteq U \}.$$

     The sets $S(C,U)$ form a ***subbasis*** for a *topology* on $\mathcal{C}(X,Y)$ that is called ***the compact-open topology***.

  We see that the *uniform topology* is the *finest* among them all and the *topology of pointwise convergence* is the *coarest*.

  $$\textbf{\textit{(uniform)}} \supseteq \textbf{\textit{(compact convergence)}} \supseteq \textbf{\textit{(pointwise convergence)}}.$$

- **Definition** (***Subspace of Continuous Functions***)
  Let $\mathcal{C}(X) := \mathcal{C}(X,\mathbb{R})$ be *the space of* ***continuous*** *real-valued functions* on topological space X and $\mathcal{B}(X) := \mathcal{B}(X,\mathbb{R})$ be *the space of* ***bounded*** *real-valued functions* on X.

  1. The intersection of $\mathcal{B}(X)$ and $\mathcal{C}(X)$ is the space of all ***bounded continuous*** *functions*

     $$\mathcal{BC}(X) := \mathcal{BC}(X,\mathbb{R}) = \mathcal{B}(X,\mathbb{R}) \cap \mathcal{C}(X,\mathbb{R})$$

     Note that $\mathcal{BC}(X) \subseteq \mathcal{B}(X)$ is a ***closed subspace***.

2. Define the ***support*** of a function f, $\mathrm{supp}(f)$ as *the* ***smallest closed set*** *outside of which f vanishes.* The subset $\mathcal{C}_c(X) \subseteq \mathcal{C}(X)$ is the space of all *continuous functions* with ***compact support***

$$\mathcal{C}_c(X) = \{f \in \mathcal{C}(X, \mathbb{R}) : \mathrm{supp}\ (f)\ \textit{is compact}\}.$$

Note that by *Tietze Extension Theorem, the locally compact Hausdorff space X* has a rich supply of continuous functions that vanishes outside a compact set.

3. Recall also that $\mathcal{C}_0(X)$ is the space of *continuous functions* on X that ***vanishes at infinity***, i.e. for all $\epsilon > 0$, $|f(x)| < \epsilon$ if $x \in X \setminus C$ for some ***compact subset*** $C \subseteq X$.

$$\mathcal{C}_0(X) = \{f \in \mathcal{C}(X, \mathbb{R}) : f\ \textit{vanishes at infinity}\}.$$

Note that

$$\mathcal{C}_c(X) \subseteq \mathcal{C}_0(X) \subseteq \mathcal{BC}(X) \subseteq \mathcal{C}(X)$$

- Recall that

  Proposition 1.1 *If X is a* ***locally compact Hausdorf*** *space, $\mathcal{C}(X)$ is a* ***closed*** *subspace of $\mathbb{R}^X$ in the* ***topology of compact convergence****.*

- **Proposition 1.2** *[Folland, 2013]*
  If X is a topological space, $\mathcal{BC}(X)$ is a ***closed*** *subspace of $\mathcal{B}(X)$ in the* ***uniform metric****; in particular, $\mathcal{BC}(X)$ is* ***complete****.*

- **Proposition 1.3** *[Folland, 2013]*
  If X is a ***locally compact Hausdorf*** *space, $\mathcal{C}_0(X)$ is a* ***closure*** *of $\mathcal{C}_c(X)$ in the* ***uniform metric****.*

- **Remark** Note that $\mathcal{C}_0(X) = \overline{\mathcal{C}_c(X)}$ is the ***completion*** of $\mathcal{C}_c(X)$ *under uniform metrc.*

2 Measures on Locally Compact Hausdorff Space

2.1 Baire σ-algebra

- **Definition** ***A G_δ set*** is a set which is a ***countable intersection*** of *open sets.*

- **Proposition 2.1** *[Reed and Simon, 1980]*
  Let I be a ***compact Hausdorff*** *space and let $f \in \mathcal{C}(X)$. Then $f^{-1}([a, \infty))$ is a* ***compact G_δ set****.*

- **Definition** (***Baire σ-algebra***)
  The Baire σ-algebra is the *σ-algebra $\mathscr{C}$ generated* by *the* ***compact G_δ*** *in a* ***compact space*** X. Each *measurable* set in Baire σ-algebra is called a ***Baire set***

- **Definition** (***Baire σ-algebra on Locally Compact Hausdorff Space***)
  In general, for *a* ***locally compact Hausdorff*** X, the ***Baire σ-algebra*** is generated as

$$\sigma\left(\left\{f^{-1}(U) : f \in \mathcal{C}_c(X),\ U \in \mathscr{B}(\mathbb{R})\right\}\right)$$

That is, *the Baire sets* of a *locally compact Hausdorff* space form ***the smallest σ-algebra*** such that all *compactly supported continuous functions in* $\mathcal{C}_c(X)$ are ***measurable***.

- **Remark** Every *Baire set* is *regular Borel measurable* if X is *second-countable locally compact Hausdorff*. Baire sets avoid some pathological properties of Borel sets on spaces ***without a countable basis (second-countable)*** *for the topology.*

- **Definition** (***Baire Measure***)
Given *a measurable space* $(X, \mathscr{C})$, where X is a ***compact space***, and $\mathscr{C}$ is the *Baire σ-algebra* generated by all *compact G_δ sets* in X, ***the Baire measure*** is a *nonegative* function $\mu : \mathscr{C} \to [0, +\infty)$ that obeys the following axioms:

  1. (***Finiteness***) $\mu(X) < \infty$.
  2. (***Empty set***) $\mu(\emptyset) = 0$.
  3. (***Countable additivity***) Whenever $E_1, E_2, \ldots \in \mathscr{B}$ are a ***countable sequence*** of ***disjoint*** *measurable sets*, then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

- **Remark** Baire measure is Borel measure. In practice, the use of *Baire measures* on *Baire sets* can often be replaced by the use of ***regular Borel measures*** on ***Borel sets***.

- **Definition** (***Baire Measurable Function***)
The functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) *measurable relative to the Baire σ-algebra* are called ***Baire measurable functions***.

- **Theorem 2.2** *(**Continuous Functions are Absolutely Integrable under Baire Measure**)[Reed and Simon, 1980]*
*If μ is a **Baire measure**, then $\mathcal{C}(X) \subseteq L^p(X, \mu)$ for all $1 \le p < \infty$ and $\mathcal{C}(X)$ is **dense** in $L^1(X, \mu)$ or any L^p space.*

- **Remark** $\mathcal{C}(X) = L^\infty(X, \mu)$ under uniform metric by definition of L^∞ norm.

2.2 Radon Measure

- **Remark** Despite the fact that Baire sets are all that are needed, the reader no doubt wants to *repress G_δ* and consider *all **Borel** sets*, i.e. the σ-algebra $\mathscr{B}$ generated by all open sets.

- **Definition** (***Outer Regularity***) [Folland, 2013]
Let μ be a ***Borel*** measure on X and E a *Borel subset* of X. The measure μ is called ***outer regular*** on E if
$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ is open}\}$$

- **Definition** (***Inner Regularity***) [Folland, 2013]
Let μ be a ***Borel*** measure on X and E a *Borel subset* of X. The measure μ is called ***inner regular*** on E if
$$\mu(E) = \sup\{\mu(C) : C \subseteq E, C \text{ is compact}\}$$

- **Definition** If μ is *outer* and *inner regular* on *all Borel sets*, μ is called ***regular***.
- **Remark** ***Baire measure*** is equivalent to a ***regular Borel measure (Randon measure)*** in the context of ***compact space*** X.
- **Definition** (***Radon Measure***) [Folland, 2013]
  A ***Radon measure*** μ on X is a *Borel measure* that is
  1. ***finite*** on *all* ***compact*** *sets*; i.e. for any ***compact subset*** $K \subseteq X$,
  $$\mu(K) < \infty.$$
  2. ***outer regular*** on *all Borel sets*; i.e. for any ***Borel set*** E
  $$\mu(E) = \inf \{\mu(U) : U \supseteq E, U \text{ is open}\}.$$
  3. ***inner regular*** on all *open sets*; i.e. for any ***open set*** U
  $$\mu(U) = \sup \{\mu(K) : K \subseteq U, K \text{ is compact and Borel}\}.$$
- **Remark** ***Baire measure*** is a ***Radon measure***.

  Randon measure is called ***regular Borel measure*** in [Reed and Simon, 1980].

2.3 Positive Linear Functionals on $\mathcal{C}_c(X)$

- **Definition** (***Positive Linear Functional***)
  Let $\mathcal{C}(X)$ be *the space of* ***continuous*** *functions* on X. A ***positive linear functional*** on $\mathcal{C}(X)$ is a (not necessarily a priori continuous) *linear functiona* I with $I(f) > 0$ for all f with $f(x) \geq 0$ pointwise.
- **Lemma 2.3** *(**Bounded by Unit Ball in Uniform Metric**) [Folland, 2013]*
  If I is a positive linear functional on $\mathcal{C}_c(X)$, for each compact $C \subseteq X$ there is a constant κ_C such that $|I(f)| < \kappa_C \|f\|_u$ for all $f \in \mathcal{C}_c(X)$ such that supp$(f) \subset K$.
- **Remark** If μ is a *Borel measure* on X such that $\mu(C) < \infty$ for every compact subset $C \subseteq X$, then $\mathcal{C}_c(X) \subseteq L^1(X, \mu)$. Therefore, $f \mapsto \int f d\mu$ is a ***positive linear functional*** on $\mathcal{C}_c(X)$.

  The following theorem shows that the ***every positive linear functionals*** on $\mathcal{C}_c(X)$ can be ***represented*** as the *integral with respect to* ***some Radon measure*** μ.
- **Theorem 2.4** *(**The Riesz-Markov Representation Theorem**). [Folland, 2013]*
  *Let X be a **locally compact Hausdorff** space, if I is a **positive linear functional** on $\mathcal{C}_c(X)$, there is a **unique Radon measure** μ on X such that*
  $$I(f) = \int f d\mu$$
  for all $f \in \mathcal{C}_c(X)$. Moreover, μ satisfies the following conditions:
  1. *for all **open** sets $U \subseteq X$,*
  $$\mu(U) = \sup \{I(f) : f \in \mathcal{C}_c(X), supp(f) \subseteq U,\ 0 \leq f \leq 1\}. \tag{1}$$

2. for all ***compact*** *sets* $K \subseteq X$

$$\mu(K) = \inf \{I(f) : f \in \mathcal{C}_c(X), f \geq \mathbb{1}_K\}. \tag{2}$$

Proof: Let us begin by establishing ***uniqueness***. If μ is a ***Radon measure*** such that $I(f) = \int f d\mu$ for all $f \in \mathcal{C}_c(X)$, and $U \subset X$ is open, then clearly $I(f) \leq \mu(U)$ whenever $\mathrm{supp}(f) \subseteq U$ and $0 \leq f \leq 1$ (denoted as $f \prec U$). On the other hand, if $K \subset U$ is *compact*, by ***Urysohn's lemma*** there is an $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subseteq U$ and $f = 1$ on K, whence

$$\mu(K) \leq \int f d\mu = I(f) \leq \mu(U).$$

Since μ is ***inner regular*** on U, i.e. $\mu(U) = \sup_{K \subset U, K \text{ compact}} \mu(K)$ it follows that (1) is satisfied. Thus μ is ***determined*** by I according to (1) on ***open sets***, and hence on ***all Borel sets*** because of ***outer regularity***.

This argument proves the uniqueness of μ and also suggests how to go about proving ***existence***. We begin by defining *a set function* $\mu : 2^X \to \mathbb{R}_+$ as

$$\mu(U) = \sup\{I(f) : f \in \mathcal{C}_c(X),\ \mathrm{supp}(f) \subseteq U,\ 0 \leq f \leq 1\}$$

for U ***open***, and we then define $\mu^*(E)$ for an arbitrary $E \subset X$ by

$$\mu^*(E) = \inf \{\mu(U) : U \supset E, U \text{ open}\}.$$

Clearly $\mu(U) \leq \mu(V)$ if $U \subseteq V$, and hence $\mu^*(U) = \mu(U)$ if U is *open*. The outline of the proof is now as follows.

1. First we shall establish that

   (a) μ^* is an ***outer measure***. (i.e. satisfying *monotonicity*, *countable subadditivity*)

   (b) Every *open set* is ***μ^*-measurable***.

   At this point it follows from ***Carathéodory's theorem*** that every *Borel set* is μ^*-*measurable* and that $\mu = \mu^*|_{\mathcal{B}(X)}$ is a ***Borel measure***. (The notation is *consistent* because $\mu^*(U) = \mu(U)$ for U open.) The measure μ is ***outer regular*** and satisfies (1) by definition.

2. We next *show* that μ ***satisfies*** (2). This clearly implies that μ is ***finite*** on ***compact sets***, and ***inner regularity*** on ***open sets*** also follows easily. Indeed, if U is *open* and for any α such that $\alpha < \mu(U)$, there exists an $f \in \mathcal{C}_c(X)$ such that $\mathrm{supp}(f) \subseteq U,\ 0 \leq f \leq 1$ and $I(f) > \alpha$, and let $K = \mathrm{supp}(f)$. If $g \in \mathcal{C}_c(X)$ and $g \geq \mathbb{1}_K$, then $g - f \geq 0$ and hence $I(g) \geq I(f) > \alpha$. But then $\mu(K) > \alpha$ by (2), so $\mu(U) = \sup \mu(K)$, i.e. μ is ***inner regular*** on U.

3. Finally, we prove that

   $$I(f) = \int f d\mu$$

   for all $f \in \mathcal{C}_c(X)$. With this, the proof of the theorem will be complete.

We start the proof.

1. We *construct* a *Borel measure* μ and prove *its outer regularity* first.

   (a) $\mu(\emptyset) = 0$ and μ is monotone as shown above. By definition, for any $E \subset X$,

   $$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \mu(U_j) : U_j \text{ is open, and } E \subseteq \bigcup_{j=1}^{\infty} U_j \right\}.$$

   The RHS is *an outer measure* by proposition (See [Folland, 2013]). So it suffice to show that

   $$\mu(\bigcup_{j=1}^{\infty} U_j) \leq \sum_{j=1}^{\infty} \mu(U_j)$$

   for U_j open sets. Let $U := \bigcup_{j=1}^{\infty} U_j$ be an open subset, $f \in \mathcal{C}_c(X)$, $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subset U$. Denote the *compact set* $K = \mathrm{supp}(f)$.

   Given that X is a *locally compact Hausdorff space* and K is its *compact subset* with *open cover* $\bigcup_{j=1}^{\infty} U_j$, there is a finite sub-cover $K \subset \bigcup_{j=1}^{n} U_j$ for some n. Then there exists *a* ***partition of unity*** on K subordinate to $\{U_j\}_{j=1}^{n}$ consisting of *compactly supported functions* $g_1, \ldots, g_n \in \mathcal{C}_c(X)$ with $\mathrm{supp}(g_j) \subseteq U_j$, $0 \leq g_j \leq 1$, and $\sum_{j=1}^{n} g_j(x) = 1$ for $x \in K$. Thus $f(x) = \sum_{j=1}^{n} f(x) g_j(x)$ for $x \in K$ and $\mathrm{supp}(f g_j) \subseteq U_j$, $0 \leq f g_j \leq 1$. The linear functional

   $$I(f) = I\left(\sum_{j=1}^{n} f g_j\right) = \sum_{j=1}^{n} I(f g_j) \leq \sum_{j=1}^{n} \mu(U_j) \leq \sum_{j=1}^{\infty} \mu(U_j).$$

   Since this is true for all $f \in \mathcal{C}_c(X)$, $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subset U$, thus $\mu(U) = \sup\{I(f) : f \prec U\} \leq \sum_{j=1}^{\infty} \mu(U_j)$.

   (b) To show that every *open set* is μ^*-measurable, let $U \subset X$ be any open set and $E \subset X$ be a subset so that $\mu^*(E) < \infty$, then we need to show that

   $$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U).$$

   It suffice to show that $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$ since the other side holds by *subadditivity*.

   First suppose that E is *open*. Then $E \cap U$ is *open*, so given $\epsilon > 0$ we can find $f \in \mathcal{C}_c(X)$ such that $f \prec E \cap U$ and

   $$I(f) > \mu(E \cap U) - \epsilon.$$

   Also, $E \setminus (\mathrm{supp}(f))$ is *open*, so we can find $g \in \mathcal{C}_c(X)$ such that $g \prec E \setminus (\mathrm{supp}(f))$ and

   $$I(g) > \mu(E \setminus (\mathrm{supp}(f))) - \epsilon.$$

   But then $f + g \prec E$, so

   $$\begin{aligned} \mu(E) \geq I(f) + I(g) &> \mu(E \cap U) - \mu(E \setminus (\mathrm{supp}(f))) - 2\epsilon \\ &\geq \mu(E \cap U) - \mu(E \setminus U) - 2\epsilon \end{aligned}$$

Letting $\epsilon \to 0$, we obtain the desired inequality.

For *the general case*, if $\mu^*(E) < \infty$, we can find an *open* $V \supset E$ such that $\mu(V) < \mu^*(E) + \epsilon$, and hence

$$\begin{aligned}\mu^*(E) + \epsilon > \mu(V) &\geq \mu(V \cap U) - \mu(V \setminus U) \\ &\geq \mu(E \cap U) - \mu(E \setminus U).\end{aligned}$$

Letting $\epsilon \to 0$, we are done.

2. If K is *compact*, $f \in \mathcal{C}_c(X)$, and $f \geq \mathbb{1}_K$, let

$$U_\epsilon := \{x : f(x) > 1 - \epsilon\}.$$

Then U_ϵ is *open*, and if $g \prec U_\epsilon$, we have $(1-\epsilon)^{-1} f - g \geq 0$ and so $I(g) \leq (1-\epsilon)^{-1} I(f)$. Thus

$$\mu(K) \leq \mu(U_\epsilon) \leq (1-\epsilon)^{-1} I(f),$$

and letting $\epsilon \to 0$ we see that $\mu(K) \leq I(f)$. On the other hand, for any *open* $U \supset K$, by ***Urysohns lemma***, there exists $f \in \mathcal{C}_c(X)$ such that $f \geq \mathbb{1}_K$ and $f \prec U$, whence

$$I(f) \leq \mu(U).$$

Since μ is *outer regular* on K, (2) follows.

3. It suffices to show that $I(f) = \int f d\mu$ if $f \in \mathcal{C}(X, [0,1])$, as $\mathcal{C}(X)$ is the *linear span* of the latter set. Given $N \in \mathbb{N}$, for $1 \leq j \leq N$ let

$$K_j := \left\{x : f(x) \geq \frac{j}{N}\right\}$$

and let $K_0 = \operatorname{supp}(f)$. Note that $K_{j-1} \supseteq K_j$. Also, define $f_1, \ldots, f_N \in \mathcal{C}_c(X)$ by

$$\begin{aligned} f_j(x) &= 0 && \text{if } x \notin K_{j-1}, \quad \text{i.e. } f_j \in \left[0, \frac{j-1}{N}\right); \\ f_j(x) &= f(x) - \frac{j-1}{N} && \text{if } x \in K_{j-1} \setminus K_j, \quad \text{i.e. } f_j \in \left[\frac{j-1}{N}, \frac{j}{N}\right); \\ f_j(x) &= \frac{1}{N} && \text{if } x \in K_j, \quad \text{i.e. } f_j \in \left[\frac{j}{N}, 1\right]. \end{aligned}$$

In other words,

$$\begin{aligned} & f_j(x) = \min\left\{\max\left\{f(x) - \frac{j-1}{N}, 0\right\}, \frac{1}{N}\right\}. \\ \Rightarrow & \frac{1}{N}\mathbb{1}_{K_{j-1}} \geq f_j \geq \frac{1}{N}\mathbb{1}_{K_j} \\ \Rightarrow & \frac{1}{N}\mu\left(K_{j-1}\right) \geq \int f_j d\mu \geq \frac{1}{N}\mu\left(K_j\right). \end{aligned}$$

Also, if U is an *open* set containing K_{j-1} we have $N f_j \prec U$ and so $I(f_j) \leq N^{-1}\mu(U)$. Hence, by (2) and *outer regularity*,

$$\frac{1}{N}\mu\left(K_{j-1}\right) \geq I(f_j) \geq \frac{1}{N}\mu\left(K_j\right)$$

Moreover, $f = \sum_{j=1}^{N} f_j$, so that

$$\frac{1}{N}\sum_{j=0}^{N-1}\mu\left(K_j\right) \geq \int f d\mu \geq \frac{1}{N}\sum_{j=1}^{N}\mu\left(K_j\right),$$
$$\text{and } \frac{1}{N}\sum_{j=0}^{N-1}\mu\left(K_j\right) \geq I(f) \geq \frac{1}{N}\sum_{j=1}^{N}\mu\left(K_j\right).$$

It follows that

$$\left|I(f) - \int f d\mu\right| \leq \frac{\mu(K_0) - \mu(K_N)}{N} \leq \frac{\mu\left(\mathrm{supp}(f)\right)}{N}.$$

As $N \to \infty$, since $\mu\left(\mathrm{supp}(f)\right) < \infty$, $I(f) = \int f d\mu$. ∎

- **Remark** Following *the Riesz-Markov Theorem*

$$\mu(X) = \sup\left\{\int_X f d\mu : f \in \mathcal{C}_c(X),\ 0 \leq f \leq 1\right\}.$$

- The following theorem is another version of *the Riesz representation theorem*:

Theorem 2.5 ***(The Riesz-Markov Theorem)*** *[Reed and Simon, 1980]*
*Let X be a **compact Hausdorff** space. For any **positive linear functional** I on $\mathcal{C}(X)$, there is a **unique Baire measure** μ on X with*

$$I(f) = \int f d\mu$$

- **Remark** (***Radon Measures*** $\Leftrightarrow$ ***Positive Linear Functionals on*** $\mathcal{C}_c(X)$)
The Riesz-Markov theorem relates ***linear functionals*** on spaces of ***continuous*** *functions on a* ***locally compact*** *space* to ***measures*** in ***measure theory***.

- **Remark** ***Not to be confused*** with *another Riesz representation theorem*, which related *linear functions on Hilbert space* as inner product with some element in Hilbert space

$$I(f) = \langle f\,,\, g_I\rangle$$

for some $g_I \in \mathcal{H}$.

- **Remark** (***Duality between*** $\mathcal{C}_0(X)$ ***and*** $\mathcal{M}(X)$)
The Riesz representation theorem establishes the ***foundation*** of the ***the duality*** between *the space of compactly supported continuous functions* and *the space of all Radon* ***measures*** on X.

In particular, for *locally compact Hausdorff* X,

$$\{\mu : \mu \textit{ is a Radon measure on } X\} \simeq \{I \in \mathcal{C}_0(X)^* : I \text{ is positive}\}$$

2.4 Dual Space of $\mathcal{C}_0(X)$

- **Theorem 2.6** *(**Monotone Convergence Theorem for Nets**) [Reed and Simon, 1980]*
*Let μ be a **regular Borel** measure on a **compact Hausdorff** space X. Let $\{f_\alpha\}_{\alpha \in J}$ be an **increasing net** of continuous functions. Then*

$$f_\alpha \to f \in L^1(X, \mu), \quad a.e.$$

if and only if $\sup_\alpha \|f_\alpha\|_1 < \infty$ *and in that case*

$$\|f_\alpha - f\|_1 \to 0.$$

- **Lemma 2.7** *[Reed and Simon, 1980]*
Let $f, g \in \mathcal{C}(X)$ with $f, g \geq 0$. Suppose $h \in \mathcal{C}(X)$ and $0 \leq h \leq f + g$. Then, we can write $h = h_1 + h_2$ with $0 \leq h_1 \leq f$, $0 \leq h_2 \leq g$, $h_1, h_2 \in \mathcal{C}(X)$.

- **Theorem 2.8** *(**Decomposition of Real Linear Functional**) [Reed and Simon, 1980, Folland, 2013]*
*Let X be a **compact** space, $I \in (\mathcal{C}(X))^*$ be any continuous linear functional on $\mathcal{C}(X)$. Then I can be written*

$$I = I_+ - I_-$$

*with I_+ and I_- **positive linear functionals**. Moreover,*

$$I_+ + I_- = \|I\|$$

*and this **uniquely determines** I_+ and I_- .*

- **Definition** (***Complex Radon Measure***)
A ***signed Radon measure*** is a ***signed Borel measure*** whose ***positive*** and ***negative variations*** are ***Radon***, and a ***complex Radon measure*** is a ***complex Borel measure*** whose *real and imaginary parts* are *signed Radon measures.*

- **Remark** In [Reed and Simon, 1980], one defines ***the complex Baire measure*** as a *finite linear complex combination* of *Baire measures.*

- **Definition** (***Space of Complex Radon Measures***)
On *locally compact Hausdorff space X*, We denote *the space of complex Radon measures on X* by $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$ we define

$$\|\mu\| = |\mu|\,(X),$$

where $|\mu|$ is the ***total variation*** of μ.

- **Proposition 2.9** *($\mathcal{M}(X)$ **is Normed Linear Space**) [Folland, 2013]*
*If μ is a **complex Borel measure**, then μ is **Radon** if and only if $|\mu|$ is **Radon**. Moreover, $\mathcal{M}(X)$ is a vector space and $\mu \mapsto \|\mu\|$ is a **norm** on it.*

- **Theorem 2.10** *(**The Riesz-Markov Theorem, Locally Compact Version**) [Reed and Simon, 1980, Folland, 2013]*
*Let X be a **locally compact Hausdorff** space. For any continuous linear functional I*

on $\mathcal{C}_0(X)$, (the space of continuous functions *on X that vanishes at infinity), there is a* ***unique regular countably additive complex Borel measure*** *μ on X such that*

$$I(f) = \int_X f d\mu, \quad \textit{for all } f \in \mathcal{C}_0(X).$$

The ***norm*** *of I as a linear functional is* ***the total variation*** *of μ, that is*

$$\|I\| = |\mu|(X).$$

Finally, I is ***positive*** *if and only if the measure μ is* ***non-negative****.*

- **Remark** In other word, the map $\mu \mapsto I_\mu$, is an ***isometric isomorphism*** from $\mathcal{M}(X)$ to $(\mathcal{C}_0(X))^*$, or

$$\mathcal{M}(X) \simeq (\mathcal{C}_0(X))^*.$$

- **Corollary 2.11** *[Reed and Simon, 1980, Folland, 2013]*
Let X be a ***compact Hausdorff*** *space. Then the* ***dual space*** *$\mathcal{C}(X)^*$ is* ***isometric isomorphism*** *to $\mathcal{M}(X)$.*

- **Definition** Given $\mathcal{M}(X) \simeq (\mathcal{C}_0(X))^*$, we define subspaces of $\mathcal{M}$:

$$\mathcal{M}_+(X) = \{I \in \mathcal{M}(X) : I \textit{ is a positive linear functional}\},$$
$$\mathcal{M}_{+,1}(X) = \{I \in \mathcal{M}(X) : \|I\| = 1\}.$$

Thus $\mathcal{M}_+(X)$ is *identified* with ***the space of all positive Randon measures on*** X.

- **Remark** (***Isometric Embedding of*** $L^1(\mu)$ ***into*** $M(X)$)
Let μ be a fixed *positive Radon measure* on X. If $f \in L^1(\mu)$, *the complex measure*

$$d\nu_f = f d\mu$$

is easily seen to be ***Radon***, and $\|\nu\| = \int |f|\, d\mu = \|f\|_1$. Thus $f \mapsto \nu_f$ is an ***isometric embedding*** of $L^1(\mu)$ into $M(X)$ whose range consists precisely of those $\nu \in \mathcal{M}(X)$ such that $\nu \ll \mu$.

- **Remark** (***Two Perspectives of Measures***)
For *regular Borel measure* μ or in general, *Radon measures* on ***locally compact*** space X, there are two perspectives:

  1. ***Nonegative set function on the σ-algebra $\mathscr{A}$***: as a ***measure of the volume*** of a *subset* in X;

  2. ***Positive linear functional on*** $\mathcal{C}_0(X)$: as a ***integral*** *of compactly supported continuous functions with respect to* ***given measure****.*

In some cases, it is important to think of ***measures*** not merely as individual objects but instead as *elements of* $(\mathcal{C}_0(X))^*$, so that we can employ *geometric* ideas.

- **Remark** (***Weak* Topology on*** $\mathcal{M}(X)$)
The weak* topology on $\mathcal{M}(X)$, X a ***compact Hausdorff*** space, is often called ***the vague topology***. Note that $\mu_n \xrightarrow{w^*} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in \mathcal{C}_0(X)$.

It can be shown that *the linear combinations of point masses* are ***weak* dense*** in $\mathcal{M}(X)$. That is, for given $\mu \in \mathcal{M}(X)$, $f_1, \ldots, f_n \in \mathcal{C}(X)$ and $\epsilon > 0$, that we can find $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and $x_1, \ldots, x_m \in X$ so that

$$\left|\mu(f_i) - \sum_{j=1}^{m} \alpha_j f_i(x_j)\right| < \epsilon, \quad \forall\, i = 1, \ldots, n,$$

i.e. $\sum_{j=1}^{m} \alpha_j \delta_{x_j} \to \mu$ where $\delta_x(f) = f(x)$ is the ***evaluation map*** and $\delta_x(\cdot) \mapsto \delta_x$ is identified with the ***point mass***.

- **Proposition 2.12** *(**Criterion for Weak* (Vague) Convergence on** $\mathcal{M}(X)$) [Folland, 2013]*
  Suppose $\mu_1, \mu_2, \ldots \in \mathcal{M}(\mathbb{R})$, and let $F_n(x) = \mu_n((-\infty, x])$ and $F(x) = \mu((-\infty, x])$.

  1. *If $\sup_n \|\mu_n\| < \infty$ and $F_n(x) \to F(x)$ for **every x** at which F is **continuous**, then $\mu_n \to \mu$ **vaguely**.*
  2. *If $\mu_n \to \mu$ **vaguely**, then $\sup_n \|\mu_n\| < \infty$. If, in addition, the μ_ns are **positive**, then $F_n(x) \to F(x)$ at **every** x at which F is **continuous**.*

- Finally, we tends to the geometrical properties of subspace of $\mathcal{M}(X)$

  Definition (***Convex Cone***)
  A set A in a vector space Y is called ***convex*** if x and $y \in A$ and $0 \le t \le 1$ implies $tx + (1-t)y \in A$. Thus A is *convex* if the ***line segment*** between x and y is in A whenever x and y are in A. A is called a ***cone*** if $x \in A$ implies $tx \in A$ for all $t > 0$. If A is *convex* and a *cone*, it is called a ***convex cone***.

- **Proposition 2.13** *(**Geometry of** $\mathcal{M}_+(X)$ **and** $\mathcal{M}_{+,1}(X)$) [Reed and Simon, 1980]*
  *Let X be a **compact Hausdorff** space. Then $\mathcal{M}_{+,1}(X)$ is **convex** and $\mathcal{M}_+(X)$ is a **convex cone**.*

References

Gerald B Folland. *Real analysis: modern techniques and their applications.* John Wiley & Sons, 2013.

Michael Reed and Barry Simon. *Methods of modern mathematical physics: Functional analysis*, volume 1. Gulf Professional Publishing, 1980.