Assignment 1 Zeshun Zong Risk & Prf. Management. zz1509@nyu.edu

1. Let X_1 be the r.v. represent the number on the first dice
X_2 be the r.v. — — second dice.

$\mathbb{E}[X_1+X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{7}{2}+\frac{7}{2} = 7$

$Var(X_1+X_2) = Var(X_1) + Var(X_2) = 2.92+2.92 = 5.84$ ~~stdev~~ $\sigma = \sqrt{5.84} = 2.42$

2. $\mathbb{E}\left[\sum_{n=1}^{10^6} X_n\right] = \sum_{n=1}^{10^6} \mathbb{E}X_n = 10^6 \times 3.5 = 3500000.$

$Var\left(\sum_{n=1}^{10^6} X_n\right) = \sum_{n=1}^{10^6} Var(X_n) = 10^6 \times 2.92$, $\sigma = \sqrt{10^6 \cdot 2.92} = 1708.$

3. $\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{N} \mathbb{E}[X_n] = \frac{N(k+1)}{2}$, $Var\left(\sum_{n=1}^{N} X_n\right) = \sum_{n=1}^{N} Var(X_n)$, $\sigma = \sqrt{\sum_{n=1}^{N} Var(X_n)} = \sqrt{\frac{N(k^2-1)}{12}}$

where $\mathbb{E}X_n = \frac{1}{k}[1+2+\cdots+k] = \frac{k+1}{2}$ and $Var(X_n) = \frac{1}{k}[1+4+\cdots+k^2] - \left(\frac{k+1}{2}\right)^2 = \frac{k^2-1}{12}$.

4. $\mathbb{E}[X^2] - (\mathbb{E}X)^2 = \mathbb{E}[(X-\mathbb{E}X)^2]$

proof. $RHS = \mathbb{E}[X^2 + (\mathbb{E}X)^2 - 2X\mathbb{E}X] = \mathbb{E}X^2 + (\mathbb{E}X)^2 - 2(\mathbb{E}X)\cdot(\mathbb{E}X)$

$= \mathbb{E}X^2 + (\mathbb{E}X)^2 - 2(\mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = LHS.$ ∎

5. Let X_i be the r.v. represent the number of rolls needed for the i^{th} distinct number to come up since the appearence of the $(i-1)^{th}$ distinct number (excluded the $(i-1)^{th}$ appearance).

We seek to find $\mathbb{E}[X_1+X_2+\cdots+X_6]$.

Observe that each X_i follow a geometric distribution with the probability of success $p_i = 1-\frac{i-1}{6}$, and $P(X_i=n) = (1-p_i)^{n-1}p_i$.

We know that $\mathbb{E}[X_i] = \frac{1}{p_i}$.

Hence. $\mathbb{E}[X_1+X_2+\cdots+X_6] = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_6 = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$

$= 14.7$

6. $P = \frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \cdots + \underbrace{\frac{5}{6}\times\cdots\times\frac{5}{6}}_{2n}\times\frac{1}{6} + \cdots$

$$= \frac{1}{6}\cdot\left[1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots + \left(\frac{25}{36}\right)^n + \cdots\right]$$

$$= \frac{1}{6}\cdot\frac{1}{1-\frac{25}{36}} = \frac{1}{6}\times\frac{36}{11} = \frac{6}{11}.$$

7. Annual interest rate is 4%.

$$PV = \sum_{i=1}^{30} 40\cdot(1.04)^{-i} = 40\cdot\sum_{i=1}^{30}(1.04)^{-i} = 40\cdot\frac{\frac{1}{1.04}\left[1-\left(\frac{1}{1.04}\right)^{30}\right]}{1-\frac{1}{1.04}}$$

$$= 691.68$$

8. $PV = \frac{1000}{(1.04)^{30}} = 308.32$

9. Sum is 1000.

This is equivalent to depositing \$1000 in the bank, Earning a yearly interest of 4% × 1000 = 40 dollars and getting your \$1000 back at the end.

10. $S = \sum_{i=1}^{30} 40\cdot(1+r)^{-i} + \frac{1000}{(1+r)^{30}} = 40\sum_{i=1}^{30}(1+r)^{-i} + 1000\cdot(1+r)^{-30}.$

$$\frac{\partial S}{\partial r} = 40\cdot\sum_{i=1}^{30}(-i)(1+r)^{-i-1} + 1000\cdot(-30)\cdot(1+r)^{-31}.$$

$$= -17292.03.$$

11. $S = 40 \cdot \sum_{i=1}^{30} (1+r)^{-i} + 1000 \cdot (1+r)^{-30} = 1100.$

$$4 \cdot \frac{\frac{1}{1+r}\left[1-\left(\frac{1}{1+r}\right)^{30}\right]}{1-\frac{1}{1+r}} + 100 \cdot (1+r)^{-30} = 110.$$

Solve for r we get

$$r = 3.46\%.$$

12. Let R be the r.v. represent the interest rate.

$$PV = 1000 \cdot \sum_{i=1}^{10} (1+R)^{-i} = 1000 \cdot \frac{\frac{1}{1+R}\left[1-\left(\frac{1}{1+R}\right)^{10}\right]}{1-\frac{1}{1+R}}.$$

Let $Q = \frac{1}{1+R}$. then $PV = 1000 \cdot \frac{Q-Q^{11}}{1-Q}$.

$$\mathbb{E}[PV] = \mathbb{E}\left[1000 \cdot \frac{Q-Q^{11}}{1-Q}\right]$$

$$= 1000 \cdot \mathbb{E}\left[\frac{Q-Q^{11}}{1-Q}\right] = 1000 \cdot \mathbb{E}\left[\frac{\frac{1}{1+R}-\left(\frac{1}{1+R}\right)^{11}}{1-\frac{1}{1+R}}\right]$$

$$= 1000 \cdot \left[\frac{\frac{1}{1.01}-\left(\frac{1}{1.01}\right)^{11}}{1-\frac{1}{1.01}} \times \frac{1}{6} + \frac{\frac{1}{1.02}-\left(\frac{1}{1.02}\right)^{11}}{1-\frac{1}{1.02}} \times \frac{1}{6} + \cdots + \frac{\frac{1}{1.06}-\left(\frac{1}{1.06}\right)^{11}}{1-\frac{1}{1.06}} \times \frac{1}{6}\right]$$

$$= 1000 \times 8.36280 = 8362.80.$$

12. Let R be the r.v. represent the percentage interest rate.

Then $PV = \dfrac{1000}{(1+R)^{10}}$

$$\mathbb{E}[PV] = \mathbb{E}\left[\frac{1000}{(1+R)^{10}}\right] = 1000 \cdot \mathbb{E}\left[\frac{1}{(1+R)^{10}}\right]$$

$$= 1000 \cdot \left[\frac{1}{6} \times \frac{1}{1.01^{10}} + \frac{1}{6} \times \frac{1}{1.02^{10}} + \cdots + \frac{1}{6} \times \frac{1}{1.06^{10}}\right]$$

$$= 719.60$$

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

14. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

15. $BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

16. $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

17. Suppose use eggplant x liters, olive oil y liters, tahini z liters.

$$50x + 5000y + 3000z = 300$$
$$4x + 5000y + 2000z = 200$$
$$15x + 0 + 10z = 25$$

$\Rightarrow \begin{bmatrix} 50 & 5000 & 3000 \\ 4 & 5000 & 2000 \\ 15 & 0 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 200 \\ 25 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.65 \\ 0.01 \\ 0.02 \end{bmatrix}$

4.5.

18. $u(p) = 1 - e^{-\frac{p}{1000}}$.

$$\mathbb{E}U = \frac{1}{6} \times u(4) + \frac{1}{6} \times u(4^2) + \cdots + \frac{1}{6} \cdot u(4^6)$$

$$= \frac{1}{6} \times \left[1 - e^{-\frac{4}{1000}} + 1 - e^{-\frac{16}{1000}} + 1 - e^{-\frac{64}{1000}} + \cdots + 1 - e^{-\frac{4^6}{1000}} \right]$$

$$= 0.322$$

19. $1 - e^{-\frac{p}{1000}} = 0.322 \Rightarrow e^{-\frac{p}{1000}} = 0.678$

$$\Rightarrow -\frac{p}{1000} = -0.389 \quad \Rightarrow p = 389$$

20. Certainty Equivalent of net utility is \$441.

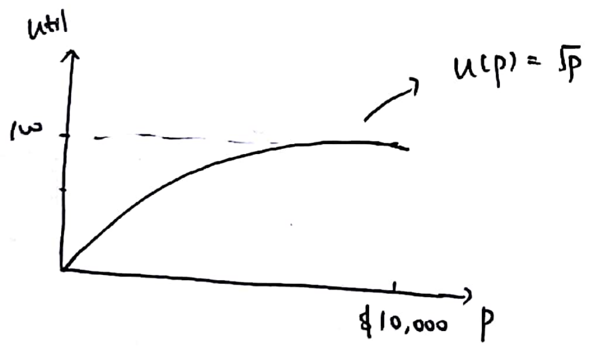
$\Rightarrow$ Certainty Equivalent of $u(p) = 1 - e^{-\frac{p}{n}}$ is also \$441.

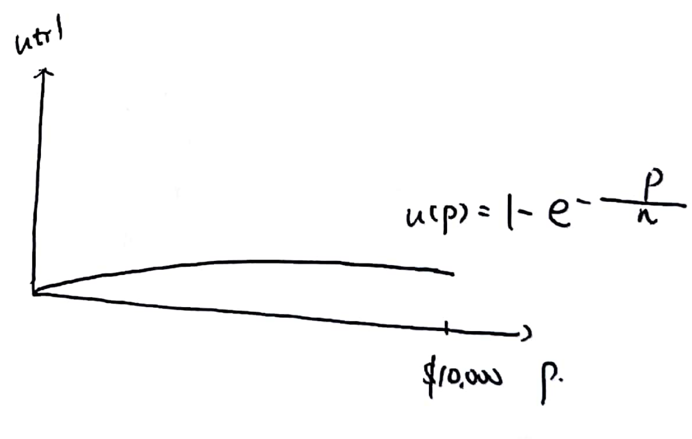
i.e. $\mathbb{E}U = 1 - e^{-\frac{441}{n}}$.

$$\Rightarrow \frac{1}{6} \times \left[1 - e^{-\frac{4}{n}} + 1 - e^{-\frac{16}{n}} + \cdots + 1 - e^{-\frac{4^6}{n}} \right] = 1 - e^{-\frac{441}{n}}.$$

$$\Rightarrow e^{-\frac{4}{n}} + e^{-\frac{16}{n}} + \cdots + e^{-\frac{4^6}{n}} = 6 \cdot e^{-\frac{441}{n}}. \quad \Rightarrow \underline{n = 1268.48}.$$

21.





由 扫描全能王 扫描创建

22.

$p_0 = 100$ ⟨ $p_{up} = 110$, $p_{down} = 90$

$S = \$105$.

Ptf 1: one put option. Payoff is 0 if stock goes up

price v. 15 if stock goes down

where v is the value of the option.

We replicate ptf 1 with ptf 2: x shares of stock + $\$y$ in money market account,

such that they yield the same payoff on the second day.

i.e.

$$\begin{cases} 110x + y = 0 \\ 90x + y = 15 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{4} \\ y = 82.5. \end{cases}$$

In the absence of arbitrage, the two portfolios must have the same value at the beginning.

$$\Rightarrow \quad V = 100 \times \left(-\tfrac{3}{4}\right) + 82.5 = 7.5$$

23.

Call option strike S.

		payoff
p_0	p_{up}	$(p_{up} - S)_+$
	p_{down}	$(p_{down} - S)_+$.

where $(\lambda)_+ = \begin{cases} \lambda & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \leq 0. \end{cases}$

Replicate portfolio: x shares of stock + $\$y$ in money market account.

$$\begin{cases} x \cdot p_{up} + y = (p_{up} - S)_+ \\ x \cdot p_{down} + y = (p_{down} - S)_+ \end{cases} \Rightarrow \begin{cases} x = \dfrac{(p_{up}-S)_+ - (p_{down}-S)_+}{p_{up} - p_{down}} \\ y = (p_{up}-S)_+ - p_{up} \cdot \dfrac{(p_{up}-S)_+ - (p_{down}-S)_+}{p_{up} - p_{down}}. \end{cases}$$

$$V = p_0 x + y.$$

Now suppose we assume that $P_{up} > S > P_{down}$.

The system is reduced to $\begin{cases} x = \dfrac{P_{up} - S}{P_{up} - P_{down}} \\ y = P_{down} \cdot \dfrac{P_{up} - S}{P_{up} - P_{down}} \end{cases}$.

$$v = x \cdot P_0 - y = (P_0 - P_{down}) \frac{P_{up} - S}{P_{up} - P_{down}}$$

24. $$\frac{dv}{dS} = \frac{P_{down} - P_0}{P_{up} - P_{down}}$$

$$\frac{dv}{dP_0} = \frac{P_{up} - S}{P_{up} - P_{down}}$$

$$\frac{dv}{dP_{up}} = (P_0 - P_{down}) \cdot \frac{(P_{up} - P_{down}) - (P_{up} - S)}{(P_{up} - P_{down})^2} = \frac{(S - P_{down})(P_0 - P_{down})}{P_{up} - P_{down}}.$$

$$\frac{dv}{dP_{down}} = \frac{S - P_{up}}{P_{up} - P_{down}} + (P_0 - P_{down}) \frac{(P_{up} - S)}{(P_{up} - P_{down})^2}$$

$$= \frac{(S - P_{up})(P_{up} - P_{down}) + (P_0 - P_{down})(P_{up} - S)}{(P_{up} - P_{down})^2}$$

$$= \frac{(P_{up} - S)(P_0 - P_{up})}{(P_{up} - P_{down})^2}$$