HW2

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3.1)

Proposition . Suppose $f : R \rightarrow R$ is convex, and $a, b \epsilon dom f$ with $a < b$.

a)

Show that:

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

Proposition 1. Jensen's inequality: $f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta) f(b)$

Proof.

$$\begin{aligned}
&\text{Let } x \in [a, b] \\
&b - x + x - a = b - a \\
&\frac{b-x}{b-a} + \frac{x-a}{b-a} = 1 \\
&\text{Let } \theta = \frac{b-x}{b-a} \text{ then } 1 - \theta = \frac{x-a}{b-a} \\
&\text{Let } f(x) = f(\theta a + (1-\theta)b) \\
&\text{Using Jenson's inequality:} \\
&f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta) f(b) \\
&f(x) = f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta) f(b) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)
\end{aligned}$$

□

b)

Show that: $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$

Proof.

Using inequality in a), we subtract $f(a)$ from both sides and simplify:

$$f(x) - f(a) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) - f(a)$$

$$\frac{f(x)-f(a)}{x-a} \leq \frac{(1-\frac{x-a}{b-a})}{x-a} - \frac{f(a)}{x-a} + \frac{x-a}{(x-a)(b-a)} f(b)$$

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(a)}{x-a} - \frac{f(a)}{b-a} - \frac{f(a)}{x-a} + \frac{f(b)}{b-a}$$

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$$

The same process can be done by subtracting $f(b)$ from both sides to get:

$$\frac{f(b)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

Combine inequalities:

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

□

c)

Show that: $f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$

Proposition 1. $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ $f'(b) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b}$

Proof.

Let:

$$f(b) \geq f(a) + f'(a)(b-a)$$

$$\frac{f(b)-f(a)}{b-a} \geq f'(a) \text{ where } b > a$$

and:

$$f(a) \geq f(b) + f'(b)(a-b)$$

$$\frac{f(a)-f(b)}{a-b} \geq f'(b)$$

$$\frac{f(b)-f(a)}{b-a} \leq f'(b)$$

$$f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$$

□

d)

Show that: Suppose f is twice differentiable. Use c) to show that

$$\frac{f'(b) - f'(a)}{b - a} \geq 0$$

Proof.

Differentiate boths sides of result in part b):

$$f''(a) \leq \frac{f'(b) - f'(a)}{b - a} \leq f''(b)$$

Since $b - a \geq 0$ it follows that :

$$0 \leq \lim_{b \to a} f'(a) = \lim_{b \to a} \frac{f'(b) - f'(a)}{b - a}$$

$\square$

3.15)

For $0 \leq \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$

We also define $u_0(x) = ln(x)$

a)

Show that for $x > 0, u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$

Proof.

$$\lim_{\alpha \to 0} u_\alpha(x)$$

L'Hopital's Rule:

$$\lim_{\alpha \to 0} x^\alpha ln(x)$$

$= ln(x) = u_0(x)$ by given definition.

$\square$

b)

Show that $u_\alpha(x)$ is concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$

Proof. Monotonicity and Concavity

Show that: $\nabla^2 u_\alpha(x) \leq 0$

$u'_\alpha(x) = x^{\alpha-1} \geq 0$ therefore monotonic increasing.

$u''_\alpha(x) = (\alpha - 1)x^{\alpha-2} \geq 0$

$\alpha - 1$ is always negative and $x^{\alpha-2}$ is always postive, therefore:

$u''_\alpha(x) \leq 0$ which implies concavity.

□

Proof. $u_\alpha(1) = 0$

$$\frac{1^\alpha - 1}{\alpha} = 0 \ \forall \alpha, \alpha \in (0,1)$$

□

3.16)

Show if concave, convex, quasiconcave, quasiconvex.

b)

$$f(x) = x_1 x_2$$

Proof. Convexity and Concavity

$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \nabla^2 - f(x_1, x_2) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Neither of which are positive semidefinite or negative semidefinite, respectively, therefore not convex nor c

□

Proposition . Let g be convex function

$$g(x_1) = \frac{\alpha}{x_1}$$

Proof. Quasiconvecity & Quasiconcavity

$g'(x_1) = -\frac{\alpha}{x_1^2}$
$g'(x_1) = \frac{3\alpha}{x_1^3} \geq 0 \ \forall x_1 \in R_{++}$
$epi(g(x_1)) = \{\, (x_1, x_2) \mid frac\alpha x_1 \leq x_2 \,\}$
$= \{\, (x_1, x_2) \mid x_1 x_2 \geq \alpha \,\}$ which is the definition of the superlevel set of $f(x_1, x_2)$
Since g is convex, we know its epigraph is convex.
Since f's superlevel set is the epigraph of a convex function, we know it is convex
Thus f is quasiconvex

□

c)

$$f(x) = 1/x_1x_2 \text{ on } R^2_{++}$$

Proof. Convexity and Concavity

Calculate the Hessian:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$$

□

e)

$$f(x) = x_1^2/x_2 \text{ on } RxR_{++}$$

Proof. Convexity and Concavity

Calculate the Hessian:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

Using Sylvester's Criterion:

$$\frac{4x_1^2}{x_2^5} - \frac{4x_1^2}{x_2^r} > 0$$

Positve semidefinite matrix therefore convex!

□

3.18)

Use the proof of the log determinant to prove the following:

a)

$f(X) = tr(X^{-1})$ is convex on $dom f = S^n_{++}$.

Proof.

Using a linear affine funcion substitution:
$g(t) = f(Z + tV)$ where $Z + tV \succ 0$
$f(Z + tV) = tr((Z + tV)^{-1})$
$= tr((Z^{1/2}(I + tZ^{1/2}V^{-1/2})Z^{-1/2})^{-1})$
$= tr((Z^{1/2})^{-1}(I + tZ^{1/2}V^{-1/2})Z^{-1/2})^{-1})$
Using Spectral Decomposition: $\hat{V} = Z^{1/2}VZ^{-1/2} = Q\Lambda Q^T$
$= tr((Z)^{-1}(I + tQ\Lambda Q^T)^{-1})$
$= tr(Z^{-1}(Q^T)^{-1}(I + t\Lambda)^{-1}Q^{-1})$
$= tr(Z^{-1}Q(I + t\Lambda)^{-1}Q^T)$
$= tr(Q^T Z^{-1}Q(I + t\Lambda)^{-1})$
$= \sum_{i,i}(Q^T Z^{-1})(\frac{1}{1 + t\lambda_i})$
$(Q^T Z^{-1})_{ii}$ is always postive
$(\frac{1}{1 + t\lambda_i})$ is a sum of convex functions
Therefore $f(X)$ is convex.

□

b)

$f(X) = (det(X))^{1/n}$ is concave on $domf = S^n_{++}$.

Proof.

Using a linear affine funcion substitution:
$g(t) = f(Z + tV)$ where $Z + tV \succ 0$
$f(g(t)) = (det(Z + tV))^{1/n}$
$f(g(t)) = (det(^{1/2}(I + tZ^{1/2}V^{-1/2})Z^{-1/2}))^{1/n}$
$f(g(t)) = (det(Z(I + tQ\Lambda Q^T))^{1/n}$
$f(g(t)) = (det(Z)det(I + tQ\Lambda Q^T))^{1/n}$
$f(g(t)) = det(Z)^{1/n}det(Q^T Q(I + t\Lambda))^{1/n}$
$f(g(t)) = det(Z)^{1/n}\prod_i (I + t\lambda_i)^{1/n}$
$\prod_i (I + t\lambda_i)^{1/n}$ is the gemotric mean which is concae thus f is concave.

□

3.19)

a)

Show $f(x) = \sum \alpha_i x_i$ is convex where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$. and x_i demnnotes the i'th largest component. Hint: $f(x) = \sum_{i=1}^{k} x_i$ is convex on R^n

Proof.

$f(x) = \alpha_r(x_1 + x_2 + \cdots + x_r) + (\alpha_{r-1} - \alpha_r)(x_1 + x_2 + \cdots + x_{r-1}) + \ldots$
$(x_1 + x_2 + \cdots + x_r)$ is convex based of the hint.
$\alpha_{r-1} - \alpha_r$ is positive.
Therefore f is a sum on nonnegative convex functions

□

3.22)

Show the following functions are convex.

b)

$f(x, u, v) = -\sqrt{uv - x^T x}$ on $dom f = \{ (x, u, v) \mid uv > x^T x, u, v > 0 \}$

Proposition . $\frac{x^T x}{u}$ and $-\sqrt{x_1 x_2}$ is convex on R^2_{++}.

Proof.

Let $h(x_1, x_2) = -\sqrt{x_1 x_2}$
Let $g_1(x, u, v) = u \rightarrow concave.$
Let $g_2(x, u, v) = v - \dfrac{x^T x}{u} \rightarrow concave.$
$h(g_1, g_2) = f$ is a concave if h is concave $\hat{h}$ is nonincreasing and g_1, g_2 are both concave.
Thus f is convex.

□

c)

$f(x, u, v) = -log(uv - x^T x)$ on $dom f = \{ (x, u, v) \mid uv > x^T x, u, v > 0 \}$

Proposition . $\frac{x^T x}{u}$ and $-\sqrt{x_1 x_2}$ is convex on R^2_{++}.

Proof.

Let $h(x_1, x_2) = -log(x_1x_2) \rightarrow$ convex and nonincreasing.
Let $g_1(x, u, v) = u \rightarrow$ concave.
Let $g_2(x, u, v) = v - \dfrac{x^T x}{u} \rightarrow$ concave.
$h(g_1, g_2) = f$ is a convex if h is concave $\hat{h}$ is nonincreasing and g_1, g_2 are both concave.
Thus f is concave.

□

3.24)

Determine if the following are xcconvex, concave, quasiconvex, or quasiconcave.

c)

$\{ a_1, \dots, a_n \}$ where $a_1 < a_2 < \cdots < a_n$ with $prob(x = a_i) = p_i$.

$$prob(\alpha \leq x \leq \beta)$$

Proof.

$$prob(\alpha \leq x \leq \beta) = \sum_{i=j}^{k} p_i$$

This is a sum of linear function in the probability space, each of which are convex and nonegative. As a

□

h)

$\{ \beta - \alpha \mid prob(\alpha \leq x \leq \beta) \geq 0.9 \}$

Proposition . Let $f(p)$ be a function that returns an interval for the given propbability. Then $f(\sum_{k=i}^{j}) = a_j - a_i$ and $f(.9) = \gamma$

Proof.

$$\{ \beta - \alpha \mid prob(\alpha \leq x \leq \beta) \geq 0.9 \}$$

$$\{ \beta - \alpha \mid \sum_{i=j}^{k} p_i \geq 0.9 \}$$

We need to find a $f(p) \geq \gamma$
Thus quasiconcanv iff all invervals of width less that γ have $p < .9$

$$\sum_{k=i}^{j} p_k < .9$$

□

3.26)

a

Show $\sum_{i=1}^{k} {}_i(x)$ is convex on S^n where

$$\sum_{i=1}^{k} {}_i(x) = sup \{ tr(v^T x v) \mid v \in R^{n \times k}, v^T v = I \}$$

Proof.

$tr(v^T xv)$ is a linear function
$suptr(v^T xv)$is a pointwise supremum of linear functions which is convex thus the function is convex.

□

3.35)

$S_C(y) = sup \{ y^T x \mid x \in C \}$

a

Show that $S_B = S_{convB}$.
I.e. that:

$$sup \{ y^T u \mid u \in C \} = sup \{ y^T v \mid v \in conv(C) \}$$

Proof. Disprove the negative:

$$sup \{ y^T u \mid u \in C \} < sup \{ y^T v \mid v \in conv(C) \}$$
$$y^T u < y^T v$$
$$y^T u < y^T \sum_i \theta_i u_i$$

Thus can't be true for all i since there must be one point where $u = v$.
Thus we must conclude equality since the same contradition holds for $>$.

□

b

Let B be close and convex. Show that $A \subseteq B \Leftrightarrow S_A(y) \leq S_B(y)$.

Proof. $A \subseteq B \Rightarrow sup\{\, y^T u \mid u \in A \,\} \leq sup\{\, y^T v \mid v \in B \,\}$

Show less than:
$y^T u < y^T v$; bebcause there exists$\exists v \in B$ where $v < u$ since $A \subseteq B$ for some y
Show equality:
$y^T u = y^T v$ because $\exists v \in B$ where $v = u$ since $A \subseteq B$
Show greater than is not true.
$y^t u > y^T v$ but because $\exists v \in B$ where $v > u$ since $A \subseteq B$
this contradicts the strict $>$ inequality.

□

Proof. $A \not\subseteq B \Rightarrow sup\{\, y^T u \mid u \in A \,\} \geq sup\{\, y^T v \mid v \in B \,\}$

If $x \in A$ and $x \notin B$, then there exists a support hyperplane that separates B from x.
This means that $y^T x > y^T y \ \forall y \in B$.

□

3.36)

Derive the conjugates for the following functions.

a)

$f(x) = max_{i=1} x_i$ on R^n

$$f^*(y) = f^*(y) = sup(y^T x - max_{i=1} x_i)$$

Proof. $y \prec 0$

$ty_k \rightarrow \infty$ if $t \rightarrow \infty$
Thus $sup(y^T x - max_{i=1} x_i)$ DNE so $y \preceq 0$ not in the domain.

□

Proof. $y \succeq 0$

$f^*(y) = sup(y^T x - max_{i=1} x_i)$
Say $x = t\overrightarrow{1}$
Substitute into f:
$t\overrightarrow{1}^T y - max_{i=1} t\overrightarrow{1}$
$t\overrightarrow{1} - t$
Where $\overrightarrow{1}^T y$ must be $= 1$ to be bounded since $t \in (-\infty, \infty)$
Thus $y \succeq 0$ in the domain

□

d)

$f(x) = x^p$ on R_{++}, where $p > 1$ or $p < 0$.

$$f^*(y) = f^*(y) = sup(xy - x^p)$$

Proof. $y < 0$

$$y = x^{p-1} \text{ where } x \neq 1 \text{ is bounded.}$$

□

Proof. $y > 0$

Find the point where the rate of change of the difference is 0 and thus bounded:
$\frac{d}{dx} yx = \frac{d}{dx} x^p$
$y = px^{p-1}$
$(\frac{y}{p})^{\frac{1}{p-1}} = x$
The function value for that value of x is:
$f^*(y) = (p-1)(\frac{y}{p})^{\frac{p}{p-1}}$

□

3.39)

d)

Show that conjugate of the conjugate of a close convex function is itsel: $f = f^{**}$

Proof.

Fenchel's Inequality:
$f^{**}(x) \leq f(x) \forall x$
Since $epi(f) \subseteq epi f^{**}$
Since f and f^{**} are closed, convex, then Fenchel's inequality becomes the string equality: $epi(f) = epi(f^{*}$
Using a contradiction, suppose:
$epi(f) \neq epi(f^{**})$
Say point $(x, f^{**}(x)) \notin epi(f) \rightarrow$ support hjyperplan separating $epi(f)$ from point
But since $epi(f) \subseteq epi(f^{**})$this is a contradictions and therefore:
$epi(f) = epi(f^{**}) \Rightarrow f = f^{**}$

□

Proof. $y > 0$

Find the point where the rate of change of the difference is 0 and thus bounded:
$\frac{d}{dx} yx = \frac{d}{dx} x^p$
$y = px^{p-1}$
$(\frac{y}{p})^{\frac{1}{p-1}} = x$
The function value for that value of x is:
$f^*(y) = (p-1)(\frac{y}{p})^{\frac{p}{p-1}}$

□