HW3

Carl Mueller
CSCI 5254 - Convex Optimization

February 27, 2018

4.1)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x_1, x_2) \\ \text{subject to} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Make a sketch of the feasible set:

a)

$$f_0(x_1, x_2) = x_1 + x_2$$
$$p^* = inf \{ x_1 + x_2 \mid 2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \}$$
$$p^* = 3/5$$
$$x^* \in \{ (2/5, 1/5) \}$$

b)

$$f_0(x_1, x_2) = -x_1 - x_2$$
$$p^* = inf \{ -x_1 - x_1 \mid 2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \}$$

No lowerbound as $x \to \infty$ & $x_2 \to \infty$ is in the feasible set then $-x_1 - x_2 \to -\infty$

c)

$$
\begin{aligned}
&f_0(x_1, x_2) = x_1 \\
&p^* = inf\,\{\, x_1 \mid 2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \,\} \\
&p^* = 0 \\
&x^* \in \{\, (0, x_2) \,\}
\end{aligned}
$$

d)

$$
\begin{aligned}
&f_0(x_1, x_2) = max(x_1, x_2) \\
&p^* = inf\,\{\, max(x_1, x_2) \mid 2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \,\} \\
&\text{Say } x_1 = x_2 \\
&2x_1 = 1 - x_1 \\
&x_1 = 1/3 \\
&\therefore \\
&p^* = 1/3 \\
&x^* \in \{\, (1/3, 1/3) \,\}
\end{aligned}
$$

e)

$$
\begin{aligned}
&f_0(x_1, x_2) = x_1^2 + 9x_2^2 \\
&p^* = inf\,\{\, max(x_1, x_2) \mid 2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \,\} \\
&\text{Let } 2x_1 + x_2 = 1, x_1 + 3x_2 = 1 \\
&2x_1 + x_2 = 1 \\
&x = (1/3, 1/3) \\
&x_1 + 3x_2 = 1 \\
&x = (1/2, 1/6) \\
&\text{ This gives the smallest } p^* \text{ and satisfies all constraints} \\
&\therefore \\
&p^* = 1/2 \\
&x^* \in \{\, (1/2, 1/6) \,\}
\end{aligned}
$$

4.3)

We use the optimality criterion:

$$
\nabla_{f_0}(x^*)^T(y - x) \geq 0, \;\; \forall y \in x, x \in \text{feasible set}
$$

$$\begin{aligned}
&\nabla f_0(x^*) \\
&= ([1, 1/2, -1] \cdot \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13.0 \end{bmatrix}) \cdot \begin{bmatrix} y_1 - 1 \\ y_2 - 1/2 \\ y_3 + 1 \end{bmatrix} \\
&= [-1, 0, 2] \cdot \begin{bmatrix} y_1 - 1 \\ y_2 - 1/2 \\ y_3 + 1 \end{bmatrix} \\
&= -1(y_1 - 1) + 2(y_2 + 1) \geq 0
\end{aligned}$$

This statistfies the optimality condition.

4.7)

a)

$$\begin{aligned}
&f_0(x) \text{ is convex} \\
&\text{Show } fracf_0(x)c^T x + d \text{ is quasiconvex.} \\
&\{\, x \mid fracf_0(x)c^T x + d \leq \alpha \,\} \\
&\{\, x \mid f_0(x) \leq \alpha(c^T x + d) \,\} \\
&\{\, x \mid f_0(x) \leq \hat{\alpha} \,\} \\
&\text{Since } f_0(x) \text{ is convex, all its level sets are convex} \\
&\therefore \\
&\frac{f_0(x)}{c^T x + d} \text{ is quasiconvex.}
\end{aligned}$$

b)

$$\begin{aligned}
&\text{Let } t = \frac{1}{c^T x + d} \;\&\; y = \frac{x}{c^T x + d} \\
&g(y, t) = \frac{f_0(x)}{c^T x + d} \\
&g_i \text{ is convex since the perspective of a convex function is convex} \\
&\text{For the constraints, the perspective still holds:} \\
&\text{Since } f_i \text{ is convex: } g_i(y, t) \leq 0,\ \ i = 1, \ldots, m \\
&\frac{x}{t} = y \\
&\frac{Ay}{t} = b \\
&Ay = bt \\
&t = \frac{1}{c^T x + d} \\
&tc^T(\frac{y}{t}) + dt = 1 \\
&c^T y + dt = 1
\end{aligned}$$

4.8

a)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \end{aligned}$$

Three scenarios:
1)

$$\begin{aligned} & Ax = b \text{ is infeasible, then} \\ & p^* = \infty \end{aligned}$$

2)

$$\begin{aligned} & Ax = b \text{ is feasible and } c \perp Null(A) \\ & p^* = c^T x^* = c^T y^* \end{aligned}$$

3)

$$\begin{aligned} & Ax = b \text{ is feasible and } c \not\perp Null(A) \\ & \text{Problem is unbounded and:} \\ & p^* = -\infty \end{aligned}$$

c)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & l \preceq 1 \preceq u \end{aligned}$$

We multiple c into the constraint to get the values for p^*:

$$c^T l \preceq c^T x \preceq c^T u$$

$$p^* = \begin{cases} c^T l; & x = l, c > 0 \\ c^T u; & x = u, c > 0 \\ c^T x; & x \in [l, u], c = 0 \end{cases}$$

4.11

b)

$$\text{minimize} \quad ||Ax - b||_1$$

$$\begin{aligned} & \sum_{i=1}^{n} |y_i - f(x_i)| \\ & = t \cdot 1 \\ & \text{For the norm given:} \\ & S = t \cdot 1 = \sum_{1}^{n} |Ax - b| \end{aligned}$$

Equivalent to the linear program:

$$\begin{aligned} &\text{minimize} && t \cdot 1 \\ &\text{subject to} && -t \preceq Ax - b \preceq t \end{aligned}$$

The optimal solution is when the k^{th} value is:

$$|a_i^T x - b| = t_i$$

b)

$$\begin{aligned} &\text{minimize} && ||Ax - b||_1 \\ &\text{subject to} && ||x||_\infty \leq 1 \end{aligned}$$

This is equivalent to the linear program:

$$\begin{aligned} &\text{minimize} && t \cdot 1 \\ &\text{subject to} && -t \preceq Ax - b \preceq t \\ & && ||x||_\infty \leq 1 \end{aligned}$$

The last constraint is equivalent to:

$$\begin{aligned} &max(|x_1|, |x_2|, \ldots, |x_n|) \leq 1 \\ &-\vec{1} \leq x \leq \vec{1} \end{aligned}$$

4.12)

We want to minimuze the given cost:

$$C = \sum_{i,j=1}^{n} c_{ij} x_{ij}$$

We want to the net flow to be conserved at each node so that

$$b_i + \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0, \ i = 1, \ldots, n$$

Flow links are bounded as well:

$$l_{ij} \leq x_{ij} \leq u_{ij}$$

4.15)

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \preceq b \\ & && x \in \{0, 1\}, i = 1, \ldots, n \end{aligned}$$

Relaxation method:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \preceq b \\ & && 0 \leq x_i \leq 1, i = 1, \ldots, n \end{aligned}$$

a)

The feasible set of the relaxed problem is a superset of the feasible set of the unrelaxed problem:

$$\{ x \mid Ax \preceq b, x \in \{0,1\}, i = 1, \ldots, n \} \supseteq \{ x \mid Ax \preceq b, 0 \leq x_i \leq 1, i = 1, \ldots, n \}$$

This means that there exists a x^* in relaxed feasible such that:

$$f(x^*_{relaxed}) = p^*_{relaxed} \leq f(x^*_{unrelaxed}) = p^*_{unrelaxed}$$

b)

If the solution to the relaxtion method is such that $x^* \in 0, 1$ then it is a solution to the boolean L.P.

4.23)

$$\text{minimize} \quad ||Ax - b||_4$$

Note that the norm is defined as:

$$(\sum_{1}^{n} (a_i^T x - b_1)^4)^{\frac{1}{4}}$$

We solve via a change of variable metho to convert to a quadratic program:

$$\begin{aligned} &\text{Define: } y_i = a_i^T x - b_i \\ &\text{Defintion} z_i = y_i^2 \end{aligned}$$

The new minimization problem is formulated as:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} z^2 \\ \text{subject to} \quad & y = Ax - b \\ & y^2 = z \end{aligned}$$

4.40)

c)

$$\text{minimize} \quad (Ax + b)^T F(x)^{-1} (Ax + b)$$

We defined this via an epigraph by definiing the above in terms of t-level sets: The new minimization problem is formulated as:

$$(Ax + b)^T F(x)^{-1} (Ax + b) \leq t$$

For which we minimize t.
The above is equivalent to the below block matrix through Shur's complement:

$$\begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \end{array}$$

4.43)

a)

Using a property $_i(x) \leq t$ for some t iff $A(x) \preceq tI$.

$$\begin{array}{ll} \text{minimize} & \lambda_1(x) \\ \text{subject to} & A(x) \preceq tI \end{array}$$

b)

Using a property $_i(x) \leq t$ for some t iff $A(x) \preceq tI$ and $_i(x) \geq \gamma$ for some t iff $A(x) \succeq I$

$$\begin{array}{ll} \text{minimize} & \lambda_1(x) \\ \text{subject to} & \gamma I \preceq A(x) \preceq tI \end{array}$$