

HW1

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3.1)

Proposition . Suppose $f : R \rightarrow R$ is convex, and $a, b \in \text{dom} f$ with $a < b$.

a)

Show that:

$$f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

Proposition 1. Jensen's inequality: $f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta)f(b)$

Proof.

Let $x \in [a, b]$

$$b-x+x-a=b-a$$

$$\frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$$

$$\text{Let } \theta = \frac{b-x}{b-a} \text{ then } 1-\theta = \frac{x-a}{b-a}$$

$$\text{Let } f(x) = f(\theta a + (1-\theta)b)$$

Using Jensen's inequality:

$$f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta)f(b)$$

$$f(x) = f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta)f(b) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

□

b)

Show that: $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$

Proof.

Using inequality in a), we subtract $f(a)$ from both sides and simplify:

$$\begin{aligned}f(x) - f(a) &\leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) - f(a) \\ \frac{f(x) - f(a)}{x-a} &\leq \frac{(1 - \frac{x-a}{b-a})}{x-a} - \frac{f(a)}{x-a} + \frac{x-a}{(x-a)(b-a)}f(b) \\ \frac{f(x) - f(a)}{x-a} &\leq \frac{f(a)}{x-a} - \frac{f(a)}{b-a} - \frac{f(a)}{x-a} + \frac{f(b)}{b-a} \\ \frac{f(x) - f(a)}{x-a} &\leq \frac{f(b) - f(a)}{b-a}\end{aligned}$$

The same process can be done by subtracting $f(b)$ from both sides to get:

$$\frac{f(b)}{b-a} \leq \frac{f(b) - f(x)}{b-x}$$

Combine inequalities:

$$\frac{f(x) - f(a)}{x-a} \leq \frac{f(b) - f(a)}{b-a} \leq \frac{f(b) - f(x)}{b-x}$$

□

c)

Show that: $f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$

Proposition 1. $f'(a) = \lim_{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$ $f'(b) = \lim_{x \rightarrow b} \frac{f(x)-f(b)}{x-b}$

Proof.

Let:

$$f(b) \geq f(a) + f'(a)(b-a)$$

$$\frac{f(b) - f(a)}{b-a} \geq f'(a) \text{ where } b > a$$

and:

$$f(a) \geq f(b) + f'(b)(a-b)$$

$$\frac{f(a) - f(b)}{a-b} \geq f'(b)$$

$$\frac{f(b) - f(a)}{b-a} \leq f'(b)$$

$$f'(a) \leq \frac{f(b) - f(a)}{b-a} \leq f'(b)$$

□

d)

Show that: Suppose f is twice differentiable. Use c) to show that

$$\frac{f'(b) - f'(a)}{b - a} \geq 0$$

Proof.

Differentiate both sides of result in part b):

$$f''(a) \leq \frac{f'(b) - f'(a)}{b - a} \leq f''(b)$$

Since $b - a \geq 0$ it follows that :

$$0 \leq \lim_{b \rightarrow a} f'(a) = \lim_{b \rightarrow a} \frac{f'(b) - f'(a)}{b - a}$$

□

3.15)

For $0 \leq \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$

We also define $u_0(x) = \ln(x)$

a)

Show that for $x > 0$, $u_0(x) = \lim_{\alpha \rightarrow 0} u_\alpha(x)$

Proof.

$$\lim_{\alpha \rightarrow 0} u_\alpha(x)$$

L'Hopital's Rule:

$$\lim_{\alpha \rightarrow 0} x^\alpha \ln(x)$$

$$= \ln(x) = u_0(x) \text{ by given definition.}$$

□

b)

Show that $u_\alpha(x)$ is concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$

Proof. Monotonicity and Concavity

Show that: $\nabla^2 u_\alpha(x) \leq 0$

$$u'_\alpha(x) = x^{\alpha-1} \geq 0 \text{ therefore monotonic increasing.}$$

$$u''_\alpha(x) = (\alpha - 1)x^{\alpha-2} \geq 0$$

$\alpha - 1$ is always negative and $x^{\alpha-2}$ is always positive, therefore:

$$u''_\alpha(x) \leq 0 \text{ which implies concavity.}$$

□

Proof. $u_\alpha(1) = 0$

$$\frac{1^\alpha - 1}{\alpha} = 0 \quad \forall \alpha, \alpha \in (0, 1)$$

□

3.16)

Show if concave, convex, quasiconcave, quasiconvex.

b)

$$f(x) = x_1 x_2$$

Proof. Convexity and Concavity

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \nabla^2 - f(x_1, x_2) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Neither of which are positive semidefinite or negative semidefinite, respectively, therefore not convex nor

□

Proposition . Let g be convex function

$$g(x_1) = \frac{\alpha}{x_1}$$

Proof. Quasiconvexity & Quasiconcavity

$$g'(x_1) = -\frac{\alpha}{x_1^2}$$

$$g'(x_1) = \frac{3\alpha}{x_1^3} \geq 0 \quad \forall x_1 \in R_{++}$$

$$\text{epi}(g(x_1)) = \{ (x_1, x_2) \mid \frac{\alpha}{x_1} \leq x_2 \}$$

$$= \{ (x_1, x_2) \mid x_1 x_2 \geq \alpha \} \text{ which is the definition of the superlevel set of } f(x_1, x_2)$$

Since g is convex, we know its epigraph is convex.

Since f 's superlevel set is the epigraph of a convex function, we know it is convex

Thus f is quasiconvex

□

c)

$$f(x) = 1/x_1x_2 \text{ on } R_{++}^2$$

Proof. Convexity and Concavity

Calculate the Hessian:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_1^3x_2} & \frac{1}{x_1^2x_2^2} \\ \frac{1}{x_1^2x_2^2} & \frac{2}{x_1x_2^3} \end{bmatrix}$$

□

e)

$$f(x) = x_1^2/x_2 \text{ on } R \times R_{++}$$

Proof. Convexity and Concavity

Calculate the Hessian:

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

Using Sylvester's Criterion:

$$\frac{4x_1^2}{x_2^5} - \frac{4x_1^2}{x_2^5} > 0$$

Positive semidefinite matrix therefore convex!

□

3.18)

Use the proof of the log determinant to prove the following:

a)

$f(X) = \text{tr}(X^{-1})$ is convex on $\text{dom } f = S_{++}^n$.

Proof.

Using a linear affine function substitution:

$$g(t) = f(Z + tV) \text{ where } Z + tV \succ 0$$

$$f(Z + tV) = \text{tr}((Z + tV)^{-1})$$

$$= \text{tr}((Z^{1/2}(I + tZ^{1/2}V^{-1/2})Z^{-1/2})^{-1})$$

$$= \text{tr}((Z^{1/2})^{-1}(I + tZ^{1/2}V^{-1/2})Z^{-1/2})^{-1})$$

$$\text{Using Spectral Decomposition: } \hat{V} = Z^{1/2}VZ^{-1/2} = Q\Lambda Q^T$$

$$= \text{tr}((Z)^{-1}(I + tQ\Lambda Q^T)^{-1})$$

$$= \text{tr}(Z^{-1}(Q^T)^{-1}(I + t\Lambda)^{-1}Q^{-1})$$

$$= \text{tr}(Z^{-1}Q(I + t\Lambda)^{-1}Q^T)$$

$$= \text{tr}(Q^T Z^{-1}Q(I + t\Lambda)^{-1})$$

$$= \sum_{i,i} (Q^T Z^{-1})_{ii} \left(\frac{1}{1 + t\lambda_i} \right)$$

$$(Q^T Z^{-1})_{ii} \text{ is always positive}$$

$$\left(\frac{1}{1 + t\lambda_i} \right) \text{ is a sum of convex functions}$$

Therefore $f(X)$ is convex.

□

b)

$$f(X) = (\det(X))^{1/n} \text{ is concave on } \text{dom} f = S_{++}^n.$$

Proof.

Using a linear affine function substitution:

$$g(t) = f(Z + tV) \text{ where } Z + tV \succ 0$$

$$f(g(t)) = (\det(Z + tV))^{1/n}$$

$$f(g(t)) = (\det^{1/2}(I + tZ^{1/2}V^{-1/2})Z^{-1/2})^{1/n}$$

$$f(g(t)) = (\det(Z(I + tQ\Lambda Q^T)))^{1/n}$$

$$f(g(t)) = (\det(Z)\det(I + tQ\Lambda Q^T))^{1/n}$$

$$f(g(t)) = \det(Z)^{1/n} \det(Q^T Q(I + t\Lambda))^{1/n}$$

$$f(g(t)) = \det(Z)^{1/n} \prod_i (I + t\lambda_i)^{1/n}$$

$$\prod_i (I + t\lambda_i)^{1/n} \text{ is the gemotric mean which is concae thus } f \text{ is concave.}$$

□

3.19)

a)

Show $f(x) = \sum \alpha_i x_i$ is convex where $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$. and x_i demnnotes the i'th largest component. Hint: $f(x) = \sum_{i=1}^k x_i$ is convex on R^n

Proof.

$$f(x) = \alpha_r(x_1 + x_2 + \dots + x_r) + (\alpha_{r-1} - \alpha_r)(x_1 + x_2 + \dots + x_{r-1}) + \dots$$

$(x_1 + x_2 + \dots + x_r)$ is convex based of the hint.

$\alpha_{r-1} - \alpha_r$ is positive.

Therefore f is a sum on nonnegative convex functions

□

3.22)

Show the following functions are convex.

b)

$$f(x, u, v) = -\sqrt{uv - x^T x} \text{ on } \text{dom} f = \{ (x, u, v) \mid uv > x^T x, u, v > 0 \}$$

Proposition . $\frac{x^T x}{u}$ and $-\sqrt{x_1 x_2}$ is convex on R_{++}^2 .

Proof.

$$\text{Let } h(x_1, x_2) = -\sqrt{x_1 x_2}$$

$$\text{Let } g_1(x, u, v) = u \rightarrow \text{concave.}$$

$$\text{Let } g_2(x, u, v) = v - \frac{x^T x}{u} \rightarrow \text{concave.}$$

$h(g_1, g_2) = f$ is a concave if h is concave \hat{h} is nonincreasing and g_1, g_2 are both concave.

Thus f is convex.

□

c)

$$f(x, u, v) = -\log(uv - x^T x) \text{ on } \text{dom} f = \{ (x, u, v) \mid uv > x^T x, u, v > 0 \}$$

Proposition . $\frac{x^T x}{u}$ and $-\sqrt{x_1 x_2}$ is convex on R_{++}^2 .

Proof.

Let $h(x_1, x_2) = -\log(x_1 x_2) \rightarrow$ convex and nonincreasing.

Let $g_1(x, u, v) = u \rightarrow$ concave.

Let $g_2(x, u, v) = v - \frac{x^T x}{u} \rightarrow$ concave.

$h(g_1, g_2) = f$ is a convex if h is concave \hat{h} is nonincreasing and g_1, g_2 are both concave.

Thus f is concave.

□

3.24)

Determine if the following are xconvex, concave, quasiconvex, or quasiconcave.

c)

$\{a_1, \dots, a_n\}$ where $a_1 < a_2 < \dots < a_n$ with $\text{prob}(x = a_i) = p_i$.

$$\text{prob}(\alpha \leq x \leq \beta)$$

Proof.

$$\text{prob}(\alpha \leq x \leq \beta) = \sum_{i=j}^k p_i$$

This is a sum of linear function in the probability space, each of which are convex and nonnegative. As a

□

h)

$$\{\beta - \alpha \mid \text{prob}(\alpha \leq x \leq \beta) \geq 0.9\}$$

Proposition . Let $f(p)$ be a function that returns an interval for the given probabability.

Then $f(\sum_{k=i}^j) = a_j - a_i$ and $f(.9) = \gamma$

Proof.

$$\{\beta - \alpha \mid \text{prob}(\alpha \leq x \leq \beta) \geq 0.9\}$$

$$\{\beta - \alpha \mid \sum_{i=j}^k p_i \geq 0.9\}$$

We need to find a $f(p) \geq \gamma$

Thus quasiconcave iff all intervals of width less than γ have $p < .9$

$$\sum_{k=i}^j p_k < .9$$

□

3.26)

a

Show $\sum_{i=1}^k i(x)$ is convex on S^n where

$$\sum_{i=1}^k i(x) = \sup \{ \text{tr}(v^T x v) \mid v \in R^{n \times k}, v^T v = I \}$$

Proof.

$\text{tr}(v^T x v)$ is a linear function

$\sup \text{tr}(v^T x v)$ is a pointwise supremum of linear functions which is convex thus the function is convex.

□

3.35)

$$S_C(y) = \sup \{ y^T x \mid x \in C \}$$

a

Show that $S_B = S_{\text{conv} B}$.

I.e. that:

$$\sup \{ y^T u \mid u \in C \} = \sup \{ y^T v \mid v \in \text{conv}(C) \}$$

Proof. Disprove the negative:

$$\sup \{ y^T u \mid u \in C \} < \sup \{ y^T v \mid v \in \text{conv}(C) \}$$

$$y^T u < y^T v$$

$$y^T u < y^T \sum_i \theta_i u_i$$

Thus can't be true for all i since there must be one point where $u = v$.

Thus we must conclude equality since the same contradiction holds for $>$.

□

b

Let B be close and convex. Show that $A \subseteq B \Leftrightarrow S_A(y) \leq S_B(y)$.

Proof. $A \subseteq B \Rightarrow \sup \{ y^T u \mid u \in A \} \leq \sup \{ y^T v \mid v \in B \}$

Show less than:

$y^T u < y^T v$; because there exists $\exists v \in B$ where $v < u$ since $A \subseteq B$ for some y

Show equality:

$y^T u = y^T v$ because $\exists v \in B$ where $v = u$ since $A \subseteq B$

Show greater than is not true.

$y^T u > y^T v$ but because $\exists v \in B$ where $v > u$ since $A \subseteq B$

this contradicts the strict $>$ inequality.

□

Proof. $A \not\subseteq B \Rightarrow \sup \{ y^T u \mid u \in A \} \geq \sup \{ y^T v \mid v \in B \}$

If $x \in A$ and $x \notin B$, then there exists a support hyperplane that separates B from x .

This means that $y^T x > y^T y \forall y \in B$.

□

3.36)

Derive the conjugates for the following functions.

a)

$$f(x) = \max_{i=1} x_i \text{ on } \mathbb{R}^n$$

$$f^*(y) = f^*(y) = \sup(y^T x - \max_{i=1} x_i)$$

Proof. $y \prec 0$

$$ty_k \rightarrow \infty \text{ if } t \rightarrow \infty$$

Thus $\sup(y^T x - \max_{i=1} x_i)$ DNE so $y \preceq 0$ not in the domain.

□

Proof. $y \succeq 0$

$$f^*(y) = \sup(y^T x - \max_{i=1} x_i)$$

Say $x = t \vec{1}$

Substitute into f :

$$t \vec{1}^T y - \max_{i=1} t \vec{1}$$

$$t \vec{1} - t$$

Where $\vec{1}^T y$ must be $= 1$ to be bounded since $t \in (-\infty, \infty)$

Thus $y \succeq 0$ in the domain

□

a)

Proof. $y \succeq 0$

$$f^*(y) = \sup(y^T x - \max_{i=1} x_i)$$

Say $x = t \vec{1}$

Substitute into f :

$$t \vec{1}^T y - \max_{i=1} t \vec{1}$$

$$t \vec{1} - t$$

Where $\vec{1}^T y$ must be $= 1$ to be bounded since $t \in (-\infty, \infty)$

Thus $y \succeq 0$ in the domain

□