HW5 HW6

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CSCI 5254 - Convex Optimization

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6.1

Proposition 1.

$$log(x+1) \leq x \tag{1}$$
$$-log(x+1) \geq x \tag{2}$$
$$x > -1 \tag{3}$$

Proposition 2.

$$-log(1-x) \text{ is convex} \tag{4}$$

Proposition 3.

$$\phi(||u||_\infty) = -a^2 log(1 - \frac{||u||_\infty}{a^2}) \tag{5}$$

Left inequality working backwards:

$$||u||_2^2 \leq -a^2 \sum_{i=1}^{m} log(1 - \frac{u_i^2}{a^2})$$
$$\sum_{i=1}^{m} \frac{|u_i|^2}{a^2} \leq -\sum_{i=1}^{m} log(1 - \frac{u_i^2}{a^2})$$

Given proposition 1:

$$\sum_{i=1}^{m} \frac{|u_i|^2}{a^2} \leq -\sum_{i=1}^{m} log(1 - \frac{u_i^2}{a^2})$$

true when:

$$-\frac{u_i^2}{a^2} \geq -1$$

Right inequality:

$$\text{Given: } u_i^2 \leq ||u_i||_\infty^2$$

$$\sum_{i=1}^{m} -log(1-\frac{u_i^2}{a^2}) \leq \sum_{i=1}^{m} -log(1-\frac{||u_i||_\infty^2}{a^2}) \text{ given proposition 1}$$

$$\sum_{i=1}^{m} -log(1-\frac{u_i^2}{a^2}) \leq \frac{u_i^2}{||u||_2^\infty} \sum_{i=1}^{m} -log(1-\frac{||u_i||_\infty^2}{a^2}) \text{ given } \frac{u_i^2}{||u||_2^\infty} \geq 1$$

$$-a^2 \sum_{i=1}^{m} log(1-\frac{u_i^2}{a^2}) \leq -a^2 \frac{u_i^2}{||u||_2^\infty} \sum_{i=1}^{m} log(1-\frac{||u_i||_\infty^2}{a^2})$$

$$-a^2 \sum_{i=1}^{m} log(1-\frac{u_i^2}{a^2}) \leq \frac{u_i^2}{||u||_2^\infty} \phi(||u||_\infty)$$

6.9

To show convexity, the following level set must be convex:

$$S_\alpha = \{\, t_i \mid \max_{i=1,\ldots,k} |\frac{p(t_i)}{q(t_i)} - y_i| \leq \alpha \,\}$$

Due to absolute value, following inequalities must hold:

$$-\alpha q(t_i) \leq y_i q(t_i) - p(t_i) \leq \alpha q(t_i)$$

This is represent two inequalities that define a polyhedron and is therefore convex. Since the level set is convex, the original minimzation problem is at least quasiconvex.

7.3

Proposition 1.

$$P(x|y=1) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-z^2}{2}} dz \tag{6}$$

$$P(x|y=0) = 1 - \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-z^2}{2}} dz \tag{7}$$

Ordering probability terms in order of $y = 1$ and $y = 0$, our total probability is:

$$p(a,b) = \prod_{i=1}^{q} P_i(a^T u_i + b|y=1) \prod_{i=q+1}^{m} (1 - P_i(a^T u_i + b|y=0))$$

The negative log likelihood:

$$l(a,b) = \sum_{i=1}^{q} -log(P_i(a^T u_i + b|y=1)) + \sum_{i=q+1}^{m} -log(1 - P_i(a^T u_i + b|y=0))$$

The negative log likelihood is a convex function, so minimizing this function is a convex optimization problem.

7.4

a)

Proposition 1.

$$\text{Sample mean: } u = \frac{1}{N}\sum_{k=1}^{N} y_k \tag{8}$$

$$\text{Covariance: } Y = \frac{1}{N}\sum_{k=1}^{N}(y_k - u)(y_k - u)^T \tag{9}$$

$$-\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}R^{-1}\sum_{k=1}^{N}(y_k - a)(y_k - a)^T$$

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}R^{-1}\sum_{k=1}^{N}(y_k y_k^T - a y_k^T - y_k a^T + aa^T)$$

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}R^{-1}(\sum_{k=1}^{N} y_k y_k^T - \sum_{k=1}^{N} a y_k^T - \sum_{k=1}^{N} y_k a^T + \sum_{k=1}^{N} aa^T))$$

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}R^{-1}(\sum_{k=1}^{N} y_k y_k^T - \sum_{k=1}^{N} a y_k^T - \sum_{k=1}^{N} y_k a^T + Naa^T)$$

Substitute sample mean:

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}R^{-1}\sum_{k=1}^{N} y_k y_k^T - Nay^T - Nua^T + Naa^T$$

$$= R^{-1}\sum_{k=1}^{N}(y_k - a)(y_k - a)^T - R^{-1}N(a-u)(a-u)^T$$

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}(NR^{-1}Y + R^{-1}N(a-u)(a-u)^T)$$

$$= -\frac{N}{2}nlog(2\pi) - \frac{N}{2}log(det(R)) - \frac{1}{2}(Ntr(R^{-1}Y) + N(a-u)R^{-1}(a-u)^T)$$

Set the gradient to zero to see a and R optimal values.

$$\nabla_a l(R,a) = -2R^{-1}(a-u) = 0$$

$$\therefore a = u$$

$$\nabla_R l(R,a) = -R^{-1} + R^{-1}(Y - (a-u)(a-u)^T)R^{-1} = 0$$

$$R = Y + (a-u)(a-u)^T$$

$$R = Y + (0)(0)^T$$

$$\therefore R = Y$$

7.8

Express sign function as a probability where we order values with $y > 1$ followed by $y < 0$:

$$\prod_{i=1}^{k} prob(a_i^T x + b_i + v_i > 0) \prod_{i=k+1}^{m} prob(a_i^T x + b_i + v_i < 0)$$

Since a_i and b_i are known values, the only RV is the noise term. We can epxress v_i as an expression of $a_i^T x + b_i$. P represents the cumulative density function of v_i. We can represent the probability as follows:

$$\prod_{i=1}^{k} P(-a_i^T x - b_i) \prod_{i=k+1}^{m} 1 - P(-a_i^T x - b_i)$$

Log likelihood below is concave so if we maximize, we obtain a convex problem:

$$l(x) = \sum_{i=1}^{k} log(P(-a_i^T x - b_i)) + \sum_{i=k+1}^{m} log(1 - P(-a_i^T x - b_i))$$

7.9

Given

$$y_i = f(a_i^T x + b_i + v_i), i = 1, \ldots, m$$

We know that a_i and b_i are knowns, so lets expression the random variable v_i as an expression of all other terms. We assume that f is an invertible function.

$$v_i = f^{-1}(y_i) - a_i^T x - b_i$$

The probability of observing $y_i, \ldots, y_m$ is:

$$\prod_{i=1}^{m} prob(f^{-1}(y_i) - a_i^T x - b_i)$$

$$l(x, f) = \sum_{i=1}^{m} log(prob(f^{-1}(y_i) - a_i^T x - b_i))$$

This log probability is concave w.r.t x and f. Thus maximizing generates a convex optimization problem.

Additional Exercises:

3.9

a)

Given:

$$z = [\Re x, \Im x]$$

Setup a system of equations using the vector breakdown of x for its $\Re$ and $\Im$ components:

$$||x||_2^2 = ||z||_2^2$$
$$\begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \begin{bmatrix} \Re x \\ \Im x \end{bmatrix} = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix}$$

This becomes the optimization problem:

$$\begin{array}{ll} \underset{z}{\text{minimize}} & ||z||_2 \\ \text{subject to} & \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \begin{bmatrix} \Re x \\ \Im x \end{bmatrix} = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix} \end{array}$$

b)

Define the second order cone:

$$K_i = \{\, (z, t) \mid ||z||_2 \leq t \,\}$$

The SOCP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & |z||_2 \\ & \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \begin{bmatrix} \Re x \\ \Im x \end{bmatrix} = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix} \end{array}$$

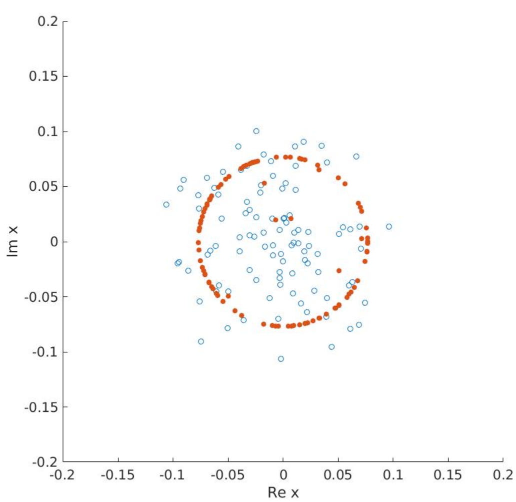
c)

Code:

```
randn('state',0);
m = 30; n = 100;
Are = randn(m,n); Aim = randn(m,n);
bre = randn(m,1); bim = randn(m,1);
A = Are + i*Aim;
b = bre + i*bim;

Atot = [Are -Aim; Aim Are];
btot = [bre; bim];
z_2 = Atot'*inv(Atot*Atot')*btot;
x_2 = z_2(1:100) + i*z_2(101:200);
```

```
cvx_begin
    variable x(n) complex
    minimize( norm(x) )
    subject to
    A*x == b;
cvx_end

cvx_begin
    variable xinf(n) complex
    minimize( norm(xinf,Inf) )
    subject to
    A*xinf == b;
cvx_end

figure(1)
scatter(real(x),imag(x)), hold on,
scatter(real(xinf),imag(xinf),[],'filled'), hold off,
axis([-0.2 0.2 -0.2 0.2]), axis square,
xlabel('Re x'); ylabel('Im x');
```

Results: The red dots represent the infinity norm.



4.1

a)

Code:

```
M = [1 -1/2; -1/2 2];
m = [-1 0]';
A = [1 2; 1 -4; 5 76];
```

```
b = [-2 -3 1]';
delta = .1

cvx_begin
    variable x(2)
    dual variable y
    minimize(quad_form(x, M)+m'*x)
    subject to
        y: A*x <= b;
cvx_end
p_star = cvx_optval
y
x
```

Results:
p_star = 8.2222
y =
1.8994
3.4684
0.0931
x =
-2.3333
0.1667

<u>KKT Conditions</u>
Primal:

$$x_1^* + 2x_2^* \leq u_1$$
$$x_1^* + -4x_2^* \leq u_2$$
$$5x_1^* + 76x_2^* \leq 1$$

Dual:

$$\lambda_1^*, \lambda_2^*, \lambda_3^* \geq 0$$

Complementary Slackness:

$$\lambda_1^*(x_1^* + 2x_2^* - u_1) = 0$$
$$\lambda_2^*(x_1^* + -4x_2^* - u_2) = 0$$
$$\lambda_3^*(5x_1^* + 76x_2^* - 1) = 0$$

First Order Conditions:

$$4x_2^* - x_1^* + 2\lambda_1^* - 4\lambda_2^* + 76\lambda_3^* = 0$$
$$2x_1^* - x_2^* - 1 + \lambda_1^* + \lambda_2^* + 5\lambda_2^* = 0$$

b)

Code:

```
M = [1 -1/2; -1/2 2];
m = [-1 0]';
A = [1 2; 1 -4; 5 76];
b = [-2 -3 1]';

cvx_begin
    variable x(2)
    dual variable y
    minimize(quad_form(x, M)+m'*x)
    subject to
        y: A*x <= b;
cvx_end
p_star = cvx_optval

array = [0 -1 1];
table = [];
delta = 0.1;

for i = array
    for j = array
        p_pred = p_star - [y(1) y(2)]*[i; j]*delta;
        cvx_begin
            variable x(2)
            minimize(quad_form(x,M)+m'*x)
            subject to
                A*x <= b+[i;j;0]*delta
        cvx_end
        p_exact = cvx_optval;
        table = [table; i*delta j*delta p_pred p_exact]
    end
end
```

Results:

d_1	d_2	p^*_{pred}	p^*_{exact}
0	0	8.2222	8.2222
0	-0.1000	8.5691	8.7064
0	0.1000	7.8754	7.9800
-0.1000	0	8.4122	8.5650
-0.1000	-0.1000	8.7590	8.8156
-0.1000	0.1000	8.0653	8.3189
0.1000	0	8.0323	8.2222
0.1000	-0.1000	8.3791	8.7064
0.1000	0.1000	7.6854	7.7515

We can see that $p^*_{pred} \leq p^*_{exact}$ for all pertubations.

5.2

The objective function $\max\limits_{i=1,\ldots,k} |f(t_i) - y_i|$ is not convex, however it is quasiconvex:

$$\{\, t, y, \alpha \mid \max_{i=1,\ldots,k} |f(t_i) - y_i| \leq \alpha \,\}$$

as it is a linear inequality.

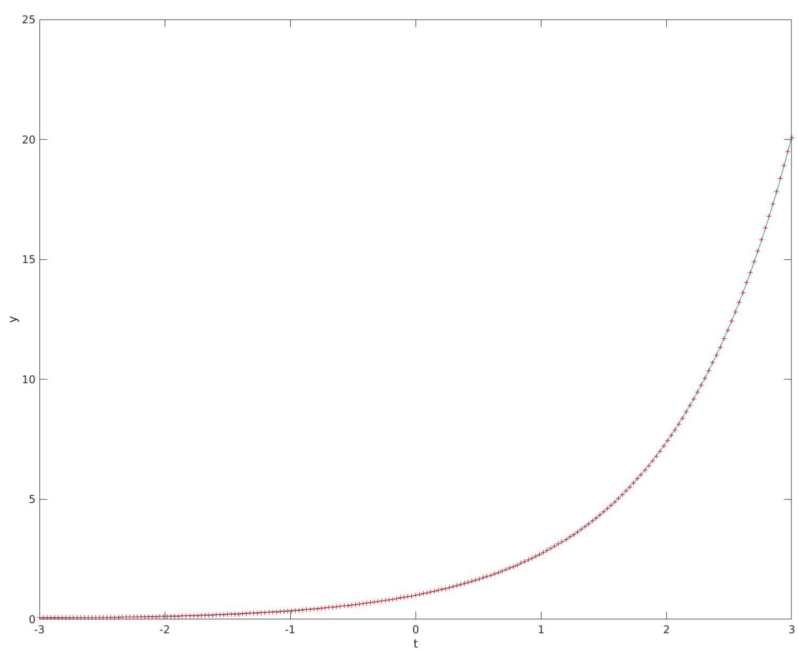


Figure 1: Data and optimal function fit.

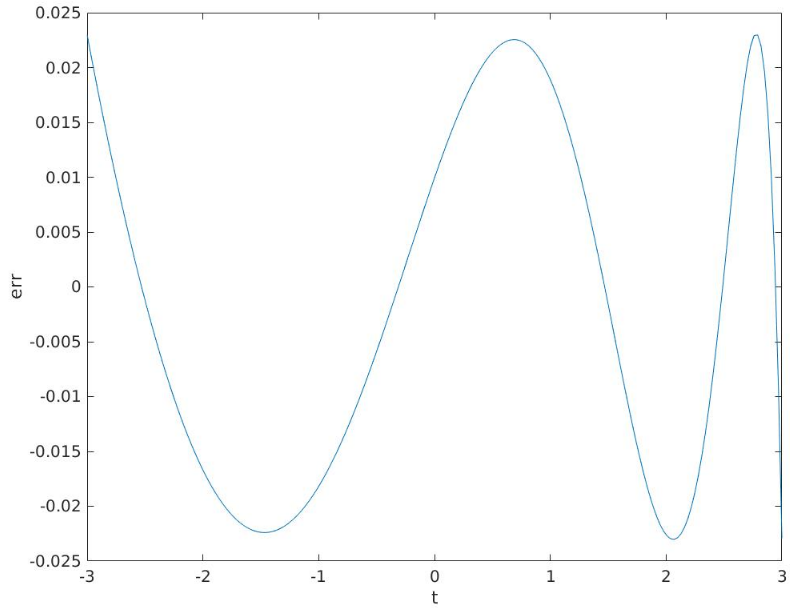


Figure 2: Error for the given t value.

To solve we can use the bisection method:

Code:

```
upper = exp(5);
lower = 0;
tolerance = .001
k = 201
t=(-3:6/(k-1):3)';
y=exp(t);
% 1 + t + t^2
T=[ones(k,1) t t.^2];

while upper - lower >= tolerance
    midpoint = (lower + upper)/2
    cvx_begin
    % a_0, a_1, a_2
    variable a(3)
    % b_0, b_1
    variable b(2)
    subject to
        abs(T*a-y.*(T*[1;b])) <= midpoint*T*[1;b]
    cvx_end
    if strcmp(cvx_status,'Solved')
        a_star = a;
        b_star = b;
```

```
            upper = midpoint;
            value = midpoint;
        else
            lower = midpoint
        end
end

y_star = T*a_star./(T*[1;b_star]);
y_star
a_star
b_star

figure(1);
plot(t,y,'g', t,y_star,'r');
xlabel('t');
ylabel('y');

figure(2);
plot(t, y_star-y);
xlabel('t');
ylabel('err');
```

Results:
$a_star =$
1.0099
0.6115
0.1133
$b\_star =$
-0.4147
0.0485

5.6

Note: I do not deserve full credit for the below code. It was infered from the given code and solutions online.

Code:

```
% tv_img_interp.m
% Total variation image interpolation.
% Defines m, n, Uorig, Known.
% Load original image.
pwd()
Uorig = double(imread('/home/carl/CUBoulder/coursework/5254/HW5/tv_img_interp
[m, n] = size(Uorig);
% Create 50% mask of known pixels.
```

```
rand('state', 1029);
Known = rand(m,n) > 0.5;
%%%%% Put your solution code here
% Calculate and define Ul2 and Utv.
% Placeholder:
cvx_begin
variable Ul2(m, n);
Ul2(Known) == Uorig(Known);
Ux = Ul2(2:end,2:end) - Ul2(2:end,1:end-1);
Uy = Ul2(2:end,2:end) - Ul2(1:end-1,2:end);
% Squared / l2 norm
minimize(norm([Ux(:); Uy(:)], 2));
cvx_end
cvx_begin
variable Utv(m, n);
Utv(Known) == Uorig(Known);
Ux = Utv(2:end,2:end) - Utv(2:end,1:end-1);
Uy = Utv(2:end,2:end) - Utv(1:end-1,2:end);
% abs or l1 norm
minimize(norm([Ux(:); Uy(:)], 1)); % tv roughness measure
cvx_end
%%%%%
% Graph everything.
figure(1); cla;
colormap gray;
subplot(221);
imagesc(Uorig)
title('Original image');
axis image;
subplot(222);
imagesc(Known.*Uorig + 256-150*Known);
title('Obscured image');
axis image;
subplot(223);
imagesc(Ul2);
title('l_2 reconstructed image');
axis image;
subplot(224);
imagesc(Utv);
title('Total variation reconstructed image');
axis image;
```

Results:

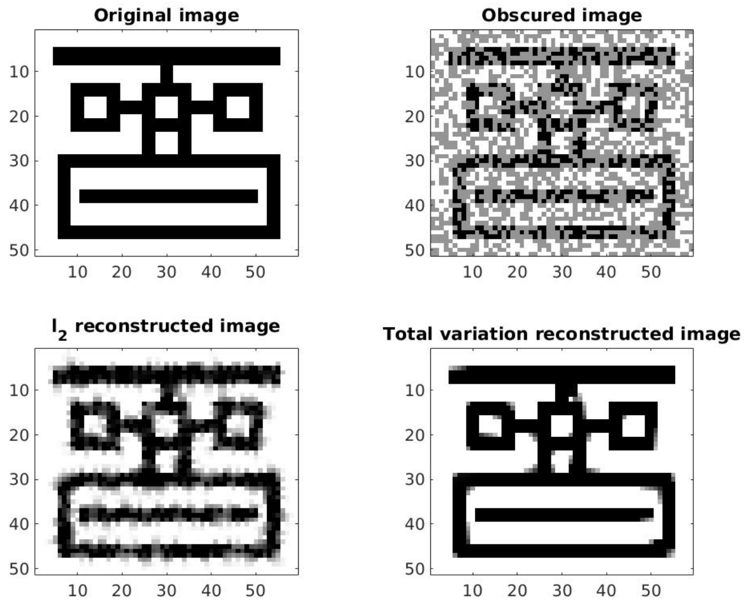


Figure 3: Interpolation results.

5.13

a)

We constrain the problem such that $c^T x_i$ for all censored data points ($i = M+1, \ldots, K$) must be greater than the lower bound D while minimizing the uncensored data $i = 1, \ldots, M$.

$$\begin{array}{ll} \underset{c}{\text{minimize}} & \sum_{i=1}^{M} (y_i - c^T x_i)^2 \\ \text{subject to} & c^T x_i \geq D, \text{ for } i = M+1, \ldots, K \end{array}$$

b)

Code:

```
% data for censored fitting problem.
randn('state',0);
n = 20; % dimension of x's
M = 25; % number of non-censored data points
K = 100; % total number of points
c_true = randn(n,1);
X = randn(n,K);
y = X'*c_true + 0.1*(sqrt(n))*randn(K,1);
% Reorder measurements, then censor
[y, sort_ind] = sort(y);
sort_ind
```

```
X = X(:,sort_ind);
D = (y(M)+y(M+1))/2;
y = y(1:M);
X_uncen = X(:,1:M)
X_cen = X(:,M+1:K)
cvx_begin
    variable c(n)
    minimize(sum_square(y - X_uncen'*c))
    subject to
        X_cen'*c >= D
cvx_end
cvx_begin
    variable c_ls(n)
    minimize(sum_square(y - X_uncen'*c_ls))
cvx_end

norm(c - c_true, 2) / norm(c_true,2)

norm(c_ls - c_true, 2) / norm(c_true,2)
```

Results:
Errors:
$\hat{c} = 0.1538$
$c_{ls} = 0.3907$

5.15

a)

We can optimize the following:

$$\begin{array}{ll} \underset{P}{\text{minimize}} & \frac{1}{N}\sum_{i=1}^{N}(d_i - (x_i - y_i)^T P(x_i - y_i))^2 \\ \text{subject to} & P \succeq 0 \end{array}$$

Another approach would be to maxmize $i = 1, \ldots, M$ dissimilar points for the P-metric while keep $i = M+1, \ldots, N$ similar points less then some arbitrarily small value α:

$$\begin{array}{ll} \underset{P}{\text{maxmize}} & \sum_{i=1}^{M}((x_i - y_i)^T P(x_i - y_i))^{\frac{1}{2}} \\ \text{subject to} & P \succeq 0 \\ & \sum_{i=M+1}^{N}(x_i - y_i)^T P(x_i - y_i) \leq \alpha \end{array}$$

b)

Code:

```
%% data for learning a quadratic metric
% provides X, Y, d, X_test , Y_test , d_test
rand('seed',0);
randn('seed',0);
n = 5; % dimension
N = 100; % number of distance samples
N_test = 10;
X = randn(n,N);
Y = randn(n,N);
X_test = randn(n,N_test);
Y_test = randn(n,N_test);
P =randn(n,n);
P = P*P'+eye(n);
sqrtP = sqrtm(P);
d = norms(sqrtP*(X-Y)); % exact distances
d = pos(d+randn(1,N)); % add noise and make nonnegative
d_test = norms(sqrtP*(X_test-Y_test));
d_test = pos(d_test+randn(1,N_test));
P
alpha = 5;
[d_test, sort_ind] = sort(d_test);
X_test = X_test(:,sort_ind);
Y_test = Y_test(:,sort_ind);
diff = X_test-Y_test

clear P sqrtP;
cvx_begin
    variable P(n,n)
    minimize((1/N_test)*pow_pos(sum(d_test' - sqrt(diag(diff'*P*diff))),2)),
    subject to
    P>0
cvx_end
```

Result: Mean Squared Error = +1.24901e-10

6.4

a)

Since the noise determans the stoachsticity, the cumulative distribution function for the normal distrubtion can be defined as follows:

$$\Phi(\frac{x-u}{\sigma})$$

where

$$x = y_i(a_{i,j} - a_{i,k})$$

This gives:

$$\Phi(\frac{y_i(a_i - a_j)}{\sigma})$$

Thus the total probability of outcomes y given abilities a is:

$$p(y|a) = \prod_{i=1}^{n} \Phi(\frac{y_i(a_i - a_j)}{\sigma})$$

The log likelihood is:

$$l(a) = \sum_{i}^{n} log(\Phi(\frac{y_i(a_i - a_j)}{\sigma}))$$

This is concave so we can minimie the negative log likelihood:

$$\begin{array}{ll} \underset{a}{\text{minimize}} & \sum_{i}^{n} log(\Phi(\frac{y_i(a_i - a_j)}{\sigma})) \\ \text{subject to} & 0 \preceq a \preceq 1 \end{array}$$

The constraint is a relaxation of the binary constraint:

$$a_i \in 0, 1$$

b,c)

Code:

```
n = 10;
m = 45;
m_test = 45;
sigma= 0.250;
test = [...]
train = [...]

A1 = sparse(1:m,train(:,1),train(:,3),m,n);
A2 = sparse(1:m,train(:,2),-train(:,3),m,n);
A = A1+A2;

cvx_begin
    variable a_hat(n)
    minimize(-sum(log_normcdf(A*a_hat/sigma)))
    subject to
    a_hat >= 0
    a_hat <= 1
```

```
cvx_end
a_hat

res = sign(a_hat(test(:,1))-a_hat(test(:,2)));
Pml = 1-length(find(res-test(:,3)))/m_test
```

Results b):
a_hat =
1.0000
0.0000
0.6829
0.3696
0.7946
0.5779
0.3795
0.0895
0.6736
0.5779
Results c):
$P_{ml} = 0.8667$
About 86% of the time time, the ML prediction is correct.

6.6

a)

Our noise is I.I.D from a gaussian distribution we can minimize the sum of squares likelihood, expressing $v(t)$ in terms of the other components:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \sum_{t=2}^{N+2}(y(t) - \sum_{\tau=1}^{k} h(\tau)x(t-\tau))^2 \\ \text{subject to} & x(N) \geq x(N-1) \geq ... \geq x(1) \geq 0 \\ & x(t) = 0, t \leq 0 \end{array}$$

Since x monotonically increases with t, we can minimize.
Code:

```
clear all; close all;
% create problem data
randn('state',0);
N = 100;
% create an increasing input signal
xtrue = zeros(N,1);
xtrue(1:40) = 0.1;
xtrue(50) = 2;
xtrue(70:80) = 0.15;
```

```
xtrue(80) = 1;
xtrue = cumsum(xtrue);
% pass the increasing input through a moving-average filter
% and add Gaussian noise
h = [1 -0.85 0.7 -0.3]; k = length(h);
yhat = conv(h,xtrue);
y = yhat(1:end-3) + randn(N,1);


cvx_begin
    variable x(N-1)
    minimize(pow_pos((y - conv(h,x)),2))
    subject to
        x >= 0
cvx_end
```

Results:
Cannot figure out how to express the above problem in CVX.

12.4

We can formulate this as a SOCP for quasiconvex optimization.

Formulating the level set:

$$\{()\} S_{ij}, t| \frac{\alpha p_j}{|||x_i - x_j||^2} \leq t | S_{ij} \geq \beta R_{ij})$$

$$\begin{array}{ll} \underset{t}{\text{minimize}} & t \\ \text{subject to} & t|||x_i - x_j||^2 \leq -\alpha p_j \\ & \beta > 0, R_{ij} \geq 0 \end{array}$$

15.3

a)

We want to maxmize the given logarithm network utility as it is a concave function.

$$\begin{array}{ll} \underset{t}{\text{maxmize}} & \sum_{j=1}^{n} log(f_j) \\ \text{subject to} & Rf \preceq c, f \succeq 0 \end{array}$$

b)

Latency is the sum of link delays when the link traffic t_i is zero.

$$d_i = \frac{1}{c_i}$$

resulting in zero flow. The link delay vector can be repesented as:

$$(\frac{1}{c_1}, \dots, \frac{1}{c_m})$$

To some these delays, wil multply by R^T and find the maximum element to get L^{min}:

$$L^{min} = max(R^T(\frac{1}{c_1}, \dots, \frac{1}{c_m})$$

This imples that the minimum latency is the maximum of the flow latency.

c)

We still want to maximize the logirthm network utility, however we can ensure that the latency is minimial, which can be expressed as an additional constraint:

$$\begin{array}{ll} \underset{t}{\text{maxmize}} & \sum_{j=1}^{n} log(f_j) \\ \text{subject to} & Rf \preceq c, f \succeq 0 \\ & \sum_{i=1}^{m} \frac{R_{ij}}{c_i - r_i^T f} \leq L, j = 1, \dots, n \end{array}$$

The new constraint $\sum_{i=1}^{m} \frac{R_{ij}}{c_i - r_i^T f} \leq L$ implies that The network flow from i to j devided by the delay cannot be greater than the minimized latency.

d)

Note I do not deserve full credit for this. Heavily inspired from a solution online:

Code:

```
% max utility
cvx_begin
    variable f(n)
    maximize geo_mean(f)
    R*f <= c
cvx_end
Umax=sum(log(f));

% min latency
Lmin = max(R'*(1./c));

N = 20;
ds = 1.10*Lmin*logspace(0,1,N);
Uopt = [];
for d = ds
```

```
    cvx_begin
        variable f(n)
        maximize geo_mean(f);
        R'*inv_pos(c-R*f) <= d*ones(n,1)
    cvx_end
    Uopt = [Uopt n*log(cvx_optval)];
end
semilogx(ds,Uopt,'k-',[Lmin,ds],[Umax,ones(1,N)*Umax],...
'k--',[1,1]*Lmin,[Uopt(1),Umax],'k--')

xlabel('L'); ylabel('U');
```

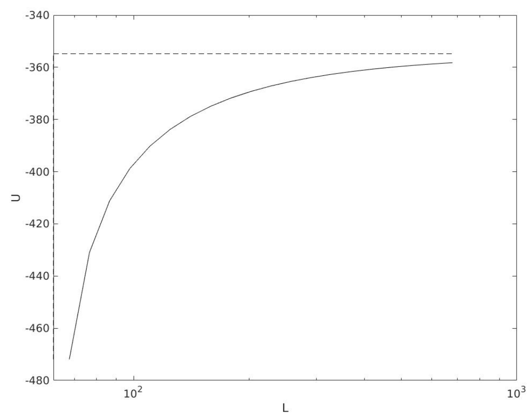


Figure 4: Utilite vs latency tradeoff.