HW4

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CSCI 5254 - Convex Optimization

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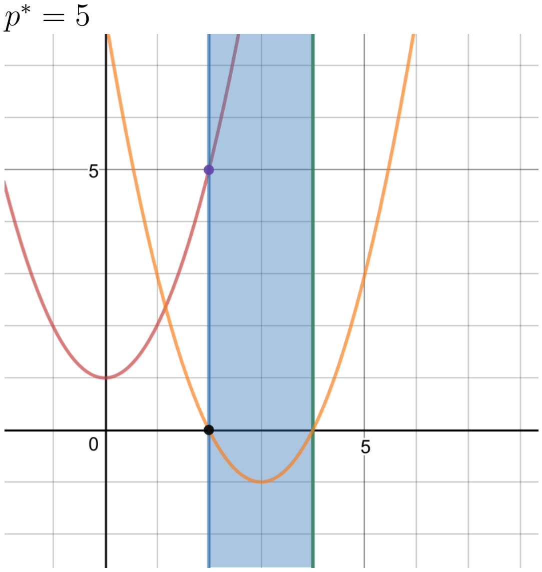
5.1)

a

Feasible set: $\{\, x \mid 2 \leq x \leq 4 \,\}$
$x^* = 2$
$p^* = 5$



b

The Lagrangian:

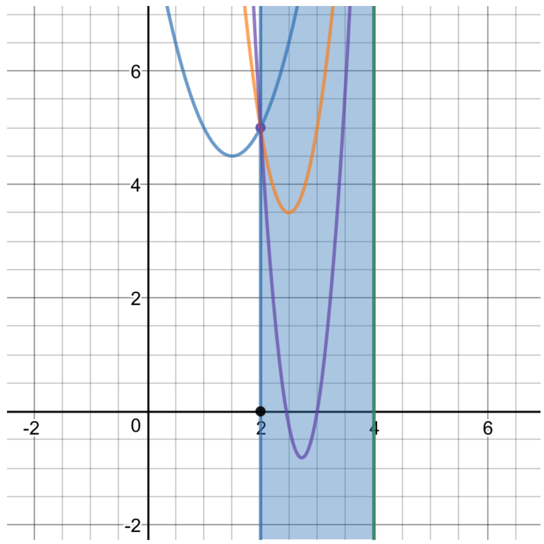
$$\begin{aligned}\mathscr{L}(x,\lambda) &= x^2 + 1 + \lambda(x-2)(x-4) \\ &= (1+\lambda)x^2 - 6\lambda x + (1+8\lambda)\end{aligned}$$

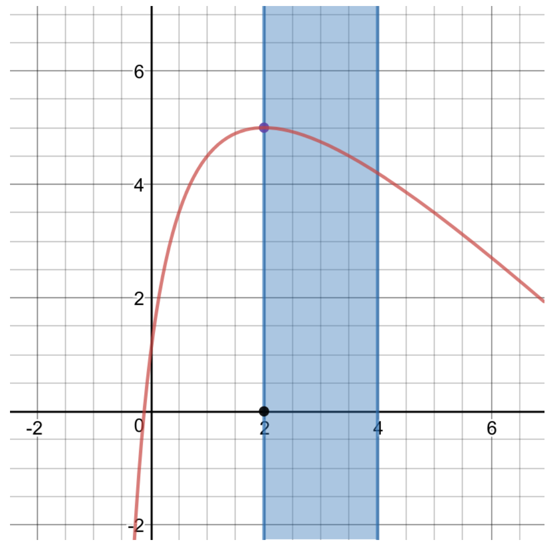
Gradient w.r.t. x:

$$\nabla_x \mathscr{L}(x,\lambda) \quad = 2x + 2\lambda x - 6\lambda = (1+\lambda)x - 6\lambda$$

Set to zero, solve for x, plug back into lambda to get dual function:

$$\nabla_x \mathscr{L}(x, \lambda) = (1 + \lambda)x - 6\lambda = 0$$
$$x = \frac{3\lambda}{1 + \lambda}$$
$$g(\lambda) = inf(\frac{-9\lambda^2}{1 - \lambda} + 8\lambda + 1)$$





c

The dual problem:

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & \frac{-9\lambda^2}{1 - \lambda} + 8\lambda + 1 \\ \text{subject to} \quad & \lambda \geq 0 \end{aligned}$$

Take the gradient of the dual w.r.t. λ, set to zero, to show strong duality:

$$\nabla_x g(\lambda) = 0$$
$$\frac{-9\lambda(1 - \lambda) + 9\lambda^2}{(1 + \lambda)^2} + 8 = 0$$
$$-9\lambda(1 - \lambda) + 9\lambda^2 + 8(1 + \lambda)^2 = 0$$
$$\lambda^2 + 2\lambda - 8 = 0$$
$$(\lambda + 4)(\lambda - 2) = 0$$
$$\lambda \text{ must be greater than 2:}$$
$$\lambda \neq -4, \lambda = 2$$
$$5 = p^* = g^* = g(2) = 5$$

5.11)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & \sum_{i=1}^{N} ||A_i x + b_i||_2 + \frac{1}{2}||x - x_o||_2^2 \\ \text{subject to} & y_i = A_i x + b_i \end{array}$$

The Lagrangian:

$$\mathscr{L}(x, y, \lambda_1 \dots \lambda_N) = \sum_{i=1}^{N} ||y_i||_2 + \frac{1}{2}||x - x_o||_2^2 + \sum_{i=1}^{N} \lambda_i^T (y_i - A_i x + b_i)$$

$$= \sum_{i=1}^{N} ||y_i||_2 - \lambda_i^T y_i + \frac{1}{2}||x - x_o||_2^2 - \sum_{i=1}^{N} \lambda_i^T (A_i x + b_i)$$

Minimize w.r.t. y:

$$inf(\sum_{i=1}^{N} ||y_i||_2 - \lambda_i^T y_i + \frac{1}{2}||x - x_o||_2^2 - \sum_{i=1}^{N} \lambda_i^T (A_i x + b_i))$$

$$= \sum_{i=1}^{N} inf(||y_i||_2 - \lambda_i^T y_i + \frac{1}{2}||x - x_o||_2^2 - \lambda_i^T (A_i x + b_i))$$

Based on Cauchy-Schwarz Inequality:

$$= \begin{cases} \frac{1}{2}||x - x_o||_2^2 - \lambda_i^T (A_i x + b_i) & \text{if } ||\lambda||_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

Constraint on dual is

$$||\lambda||_* \leq 1$$

Minimize w.r.t. x via gradient and then set to zero:

$$\nabla_x(|\frac{1}{2}||x - x_o||_2^2 - \sum_{i=1}^{N} \lambda_i^T (A_i x + b_i)) = 0$$

$$x - x_o - \sum_{i=1}^{N} \lambda_i^T A_i = 0$$

$$x = x_o - \sum_{i=1}^{N} \lambda_i^T A_i$$

Plug into Lagrangian to get dual function:

$$g(\lambda_1, \dots, \lambda_N) = \sum_{i=1}^{N} (A_i x_o + b_1) - \frac{1}{2}|| \sum_{i=1}^{N} A_i^T \lambda_i||^2$$

The Dual Problem:

$$\begin{array}{ll} \underset{\lambda}{\text{Maximize}} & \sum_{i=1}^{N} ||A_i x + b_i||_2 + \frac{1}{2}||x - x_o||_2^2 \\ \text{subject to} & y_i = A_i x + b_i \end{array}$$

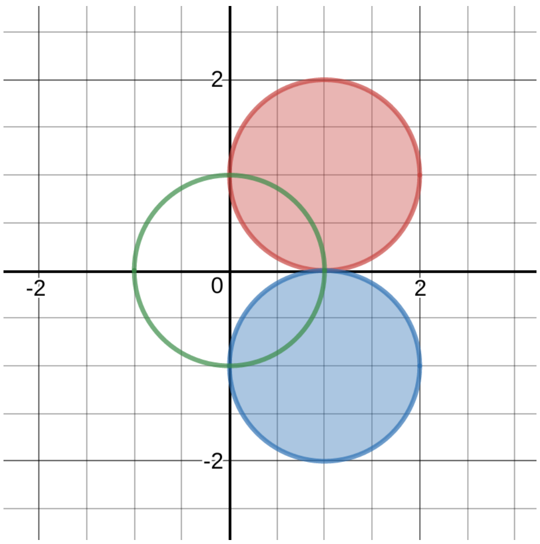
5.13)

See attached paper.

5.26)

See attached paper.

a)



5.27)

See attached paper.

5.39)

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