

# HW5

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## 6.1

**Proposition 1.**

$$\log(x + 1) \leq x \quad (1)$$

$$-\log(x + 1) \geq x \quad (2)$$

$$x > -1 \quad (3)$$

**Proposition 2.**

$$-\log(1 - x) \text{ is convex} \quad (4)$$

**Proposition 3.**

$$\phi(\|u\|_\infty) = -a^2 \log\left(1 - \frac{\|u\|_\infty}{a}\right) \quad (5)$$

Left inequality working backwards:

$$\begin{aligned} \|u\|_2^2 &\leq -a^2 \sum_{i=1}^m \log\left(1 - \frac{u_i^2}{a^2}\right) \\ \sum_{i=1}^m \frac{|u_i|^2}{a^2} &\leq - \sum_{i=1}^m \log\left(1 - \frac{u_i^2}{a^2}\right) \end{aligned}$$

Given proposition 1:

$$\sum_{i=1}^m \frac{|u_i|^2}{a^2} \leq - \sum_{i=1}^m \log\left(1 - \frac{u_i^2}{a^2}\right)$$

true when:

$$-\frac{u_i^2}{a^2} \geq -1$$

Right inequality:

$$\begin{aligned}
& \text{Given: } u_i^2 \leq \|u_i\|_\infty^2 \\
& \sum_{i=1}^m -\log(1 - \frac{u_i^2}{a^2}) \leq \sum_{i=1}^m -\log(1 - \frac{\|u_i\|_\infty^2}{a^2}) \text{ given proposition 1} \\
& \sum_{i=1}^m -\log(1 - \frac{u_i^2}{a^2}) \leq \frac{u_i^2}{\|u\|_2^\infty} \sum_{i=1}^m -\log(1 - \frac{\|u_i\|_\infty^2}{a^2}) \text{ given } \frac{u_i^2}{\|u\|_2^\infty} \geq 1 \\
& -a^2 \sum_{i=1}^m \log(1 - \frac{u_i^2}{a^2}) \leq -a^2 \frac{u_i^2}{\|u\|_2^\infty} \sum_{i=1}^m \log(1 - \frac{\|u_i\|_\infty^2}{a^2}) \\
& -a^2 \sum_{i=1}^m \log(1 - \frac{u_i^2}{a^2}) \leq \frac{u_i^2}{\|u\|_2^\infty} \phi(\|u\|_\infty)
\end{aligned}$$

## 6.9

To show convexity, the following level set must be convex:

$$S_\alpha = \{ t_i \mid \max_{i=1, \dots, k} |\frac{p(t_i)}{q(t_i)} - y_i| \leq \alpha \}$$

Due to absolute value, following inequalities must hold:

$$-\alpha q(t_i) \leq y_i q(t_i) - p(t_i) \leq \alpha q(t_i)$$

This is represent two inequalities that define a polyhedron and is therefore convex. Since the level set is convex, the original minimization problem is at least quasiconvex.

## 7.3

**Proposition 1.**

$$P(x|y=1) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}} dz \quad (6)$$

$$P(x|y=0) = 1 - \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}} dz \quad (7)$$

Ordering probability terms in order of  $y=1$  and  $y=0$ , our total probability is:

$$p(a, b) = \prod_{i=1}^q P_i(a^T u_i + b | y=1) \prod_{i=q+1}^m (1 - P_i(a^T u_i + b | y=0))$$

The negative log likelihood:

$$l(a, b) = \sum_{i=1}^q -\log(P_i(a^T u_i + b | y=1)) + \sum_{i=q+1}^m -\log(1 - P_i(a^T u_i + b | y=0))$$

The negative log likelihood is a convex function, so minimizing this function is a convex optimization problem.

## 7.4

**Proposition 1.**

$$\text{Sample mean: } u = \frac{1}{N} \sum_{k=1}^N y_k \quad (8)$$

$$\text{Covariance: } Y = \frac{1}{N} \sum_{k=1}^N (y_k - u)(y_k - u)^T \quad (9)$$

$$\begin{aligned} & -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}R^{-1} \sum_{k=1}^N (y_k - a)(y_k - a)^T \\ &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}R^{-1} \sum_{k=1}^N (y_k y_k^T - a y_k^T - y_k a^T + a a^T) \\ &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}R^{-1} \left( \sum_{k=1}^N y_k y_k^T - \sum_{k=1}^N a y_k^T - \sum_{k=1}^N y_k a^T + \sum_{k=1}^N a a^T \right) \\ &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}R^{-1} \left( \sum_{k=1}^N y_k y_k^T - \sum_{k=1}^N a y_k^T - \sum_{k=1}^N y_k a^T + N a a^T \right) \end{aligned}$$

Substitute sample mean:

$$\begin{aligned} &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}R^{-1} \sum_{k=1}^N y_k y_k^T - N a y^T - N u a^T + N a a^T \\ &= R^{-1} \sum_{k=1}^N (y_k - a)(y_k - a)^T - R^{-1} N(a - u)(a - u)^T \\ &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}(N R^{-1} Y + R^{-1} N(a - u)(a - u)^T) \\ &= -\frac{N}{2}n\log(2\pi) - \frac{N}{2}\log(\det(R)) - \frac{1}{2}(N \text{tr}(R^{-1} Y) + N(a - u)R^{-1}(a - u)^T) \end{aligned}$$