HW1

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2.11 a)

Show that the hyperbolic set $\{x \in R^2_+ \mid \prod_{i=1}^n x_i \geq 1\}$ is convex for $n = 2$.

Proposition 1. Consider the convex combination of two points $x, y \in R^2_+$ called z.

$$z = \theta x + (1-\theta)y$$

where the components of z are:

$$z_1 = \theta x_1 + (1-\theta)y_1$$
$$z_2 = \theta x_2 + (1-\theta)y_2$$

Proposition 2. If $x \succeq y$ the $z \succeq y$ and $z_1 z_2 \geq y_1 y_2 \geq 1$

Proposition 3. If $y \succeq x$ the $z \succeq x$ and $z_1 z_2 \geq x_1 x_2 \geq 1$

Proposition 4. If $x \not\succeq y$ then either $(y_1 - x_1)$ or $(y_2 - x_2)$ is negative. Therefore $(y_1 - x_1)(y_2 - x_2) < 0$.

Proof. We show that $z_1 z_2 \geq 1$:

$$(\theta x_1 + (1-\theta)y_1)(\theta x_2 + (1-\theta)y_2) \geq 1$$
$$= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta)x_1 y_2 + \theta(1-\theta)x_2 y_1$$

After some ugly algebra:

$$= \theta x_1 x_2 + (1-\theta)y_1 y_2 - \theta(1-\theta)(y_1 - x_1)(y_2 - x_2)$$

$$\theta x_1 x_2 + (1-\theta) y_1 y_2 \geq 1$$

Based on proposition 4, the following is also true:

$$-\theta(1-\theta)(y_1 - x_1)(y_2 - x_2) \geq 0$$

Therefore:

$$\theta x_1 x_2 + (1-\theta) y_1 y_2 - \theta(1-\theta)(y_1 - x_1)(y_2 - x_2) \geq 1$$

Thus proving convexity. □

2.11 b)

Show that the hyperbolic set $\{x \in R^2_+ \mid \prod_{i=1}^n x_i \geq 1\}$ is convex in the general case.

Proposition 1. If $a\ , b \geq 0$ and $0 \leq \theta \leq 1$ then $a^\theta b^{1-\theta} \leq \theta a + (1 - \theta b)$

Proposition 2. $\prod_i x_i \geq 1$

Proposition 3. $\prod_i y_i \geq 1$

Proof. Given z as a convex combination: $z = \theta\ x + (1-\theta)\ y$, show that

$$\prod_i \theta\ x_i + (1-\theta)\ y_i \geq 1$$

Given proposition 1:

$$\prod_i x_i^\theta y_i^\theta \leq \prod_i \theta x_i + (1-\theta) y_i = \prod_i z_i$$

$$\prod_i x_i^\theta y_i^{(1-\theta)}$$

$$= (\prod_i x_i)^\theta (\prod_i y_i)^{(1-\theta)}$$

Given propositions 2 and 3:

$$(\prod_i x_i)^\theta (\prod_i y_i)^{(1-\theta)} \geq 1$$

Since either term $\prod_i x_i$ and $\prod_i y_i$ is $\geq\ 1$. Thus:

$$1 \leq \prod_i x_i^\theta y_i^\theta \leq \prod_i \theta x_i + (1-\theta) y_i = \prod_i z_i$$

Which therefore proves convexity in the general. □

2.12 c)

The set $\{x\ R^n \mid a_1^T x \ \leq\ b_1,\ a_2^T b_2 \ \leq\ b_2\}$ is the intersection of two halfspaces:

$$a_1^T x \leq b_1$$
$$a_2^T x \leq b_2$$

Halfspaces are convex therefore a wedge is a intersection of convex sets and therefore is convex.

2.12 e)

The set

$$\{x \mid dist(x, S) \ \leq\ dist(x, T)\}$$

is not generally convex. For example, consider the set $S \ =\ \{x \mid x = (0, 0)\}$ (the origin) and the complement set of a ball $T \ =\ \{x \mid U \backslash B(x, r), x = [0, 0]\}$ centered at the origin. The original set creates a washer between the ball's complement which is not convex.

2.12 f)

The set $\{x \mid x \ +\ S_2 \ \subseteq\ S_1\}$ where S_1 is convex. If $y \ \in\ S_2$ then $x \ +\ y \ \in\ S_1$. Then the original set:

$$\{x \mid x \ +\ S_2 \ \subseteq\ S_1\} \ = \cap \{x \mid x \ +\ y \ \subseteq\ S_1\} \forall y \in S_2$$

Thus the intersection of all sets of S_1 shifted by $-y$ each of which are convex.

2.12 g)

The set: $\{\, x \mid\ ||x - a||_2 \leq \theta ||x - b||_2 \,\}$:

$$\{x \mid ||x - a||_2 \ \leq\ \theta ||x - b||_2\}$$
$$= \{x \mid ||x - a||_2^2 \ \leq\ \theta^2 ||x - b||_2^2\}$$

Using definition of a ball:

$$= \{x \mid (x - a)T(x - a) \ \leq\ \theta^2 (x - b)^T (x - b)\}$$
$$= \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x \ +\ a^T a \ -\ \theta^2 b^T b \leq\ 0\}$$

If $\theta = 1$ then the remaining inequality indicates a halfspace. For all other values, we're left with a ball. All cases of which are convex.

2.14 a)

Given $S \subseteq R^n$, let $||.||$ be a norm in R^n, $a \geq 0$, and the set:

$$S_a = \{x \mid dist(x, S) \leq\ a\} \text{ where } dist(x, S) = \mathbf{inf}_{y \in S} ||x - y||$$

Show if S is convex then S_a is convex.

Proof.

$$S_a = \{ x \mid dist(x, S) \leq a \} \text{ where } dist(x, S) = \mathbf{inf}_{y \in S}||x - y||$$
$$\text{Substitute line segment between } x_1 \text{ \& } x_2 \text{ into the constraint and show } \leq a:$$
$$\mathbf{inf}_{y \in S}||\theta x_1 + (1-\theta)x_2 - y_o||$$
$$\text{Create convex combination representing } y_o \text{ and substitute:}$$
$$\mathbf{inf}_{y \in S}||\theta x_1 + (1-\theta)x_2 - \theta y_1 + (1-\theta)y_2||$$
$$= \mathbf{inf}_{y \in S}||\theta(x_1 - y_1) + (1-\theta)(x_2 - y_2)||$$
$$\text{Triangle inequality and scalability of norm:}$$
$$\leq (\theta)\mathbf{inf}_{y \in S}||(x_1 - y_1)|| + (1-\theta)\mathbf{inf}_{y \in S}||x_2 - y_2||$$
$$\leq a$$

Thus the set is convex. □

2.14 b)

Given $S \subseteq R^n$, let $||.||$ be a norm in R^n, $a \geq 0$, and the set:

$$S_{-a} = \{ x \mid B(x, a) \subseteq S \}$$

where $B(x, a)$ is the ball in the norm $||.||$ with center x and radius a. Show if S is convex. S_{-a} is convex.

$$S_{-a} = \{ x \mid ||y - x|| \leq a \}$$
$$\text{Let } x_o = \theta x_1 + (1-\theta)x_2 \ \in S_{-a}$$
$$\text{Let } y_o = \theta y_1 + (1-\theta)y_2 \ \in S$$
$$S_{-a} = \{ x \mid ||\theta y_1 + (1-\theta)y_2 - \theta x_1 + (1-\theta)x_2|| \leq a \}$$
$$\text{Triangle inequality and scalability of norm:}$$
$$= \theta||(y_1 - x_1)|| + (1-\theta)||(y_2 - x_2)|| \leq a$$

Thus the set is convex.

2.15 a)

Given $\alpha \leq E[f(x)] \leq \beta$ where $\mathbf{prob}(x_i = a_i) = p_i$ and $E[f(x)] = \sum_{i=1}^{n} p_i f(a_i)$. Also the set $P = \{ p \mid \mathbf{1}^T P = 1.p_i > 0 \}$ is the intersection of a set of halfsapces ($p_i > 0$) and a hyperplane($\mathbf{1}^T P = 1$) creating a polyhedron and thefore is convex.

$$\alpha \leq E[f(x)] \leq \beta$$
$$\alpha \leq E[f(x)] \ \cap \ E[f(x)] \leq \beta$$
$$\alpha \leq \sum_{i=1}^{n} p_i f(a_i) \ \cap \ \sum_{i=1}^{n} p_i f(a_i) \leq \beta$$

Each of the above inequalities are inequalities in P since the $E[]$ function lies within the space P. Therfore both these inqualities are halfspaces in P, and are convex
Thus

$$\alpha \leq \sum_{i=1}^{n} p_i f(a_i) \ \cap \ \sum_{i=1}^{n} p_i f(a_i) \leq \beta$$

is an intersection of halspaces and is therefore convex.

2.15 b)

Given $\alpha \leq E[f(x)] \leq \beta$ where $\mathbf{prob}(x_i \geq \alpha)$. Again the set $P = \{\, p \mid \mathbf{1}^T P = 1, p_i > 0 \,\}$ is the intersection of a set of halfsapces ($p_i > 0$) and a hyperplane ($\mathbf{1}^T P = 1$) creating a polyhedron and thefore is convex.

$$\mathbf{prob}(x_i \geq \alpha) = \sum_{x_i \geq \alpha} p_i \leq \beta$$

This is another inquality in the space of P. Since P is convex, therefore the halspace in P is convex and thus the original probability set is convex.

2.15 f)

Given $\mathbf{var}(x) \leq \alpha$.
Let $a = [.75, .25]$, $p = [.5, .5]$, $\alpha = .05$.

$$\mathbf{var}(x) \leq \alpha$$
$$\sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \alpha$$
$$\text{Plug in values } a,\ p\ \&\ \alpha\text{:}$$
$$(.5 * .75^2 + .5 * .25^2) - (.5 * .75 + .5 * .25)^2$$
$$= (.3125) - (.25)$$
$$= .0625 \nleq .05$$

Thus not convex.

2.15 g)

Given $\mathbf{var}(x) \geq \alpha$.

$$\begin{aligned}\mathbf{var}(x) &\leq \alpha \\ \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 &\geq \alpha \\ p^T(a^2) - (p^T a)^2 &\geq \alpha \\ p^T(a^2) - (p^T a a^T p) &\geq \alpha\end{aligned}$$

Since aa^T makes a positive semi-definite matrix, the above defines a convex set.

2.19 a)

Let

$$f : R^m -> R^n$$

$$f(x) = \frac{Ax + b}{c^T x + d}$$

$$\mathbf{dom} f = \{\, x \mid c^T x + d > 0 \,\}$$

$$C = \{\, y \mid g^T y \leq h \,\}, \text{ where C is convex.}$$

$$f^{-1}(C) = \{\, x \in \mathbf{dom} f \;\mid\; f(x) \in C \,\}$$

Describe $f^{-1}(C)$:

Proof.

$$\begin{aligned}f^{-1}(C) = \{\, x \;\mid\; g^T f(x) \in C \,\} \\ \text{Apply the affine function:} \\ = \{\, x \mid\; g^T \frac{Ax + b}{c^T x + d} \leq h, c^T x + d > 0 \,\} \\ = \{\, x \mid\; g^T A x + g^T b \leq h c^T x + h d, c^T x + d > 0 \,\}\end{aligned}$$

This is the intersection of two half spaces:

$$\{\, x \mid c^T x + d > 0 \,\} \cap \{\, x \mid\; g^T A x + g^T b \leq h c^T x + h d, c^T x + d > 0 \,\}$$

$\square$

2.19 b)

Let

$$f : R^m - > R^n$$
$$f(x) = \frac{Ax+b}{c^T x + d}$$
$$\mathbf{dom} f = \{ x \mid c^T x + d > 0 \}$$
$$C = \{ y \mid Gy \preceq h \}, \text{ where C is convex.}$$
$$f^{-1}(C) = \{ x \in \mathbf{dom} f \ \mid \ f(x) \in C \}$$

Describe $f^{-1}(C)$:

Proof.

$$\begin{aligned} f^{-1}(C) = \{ y \mid Gy \preceq h \} \\ \text{Apply the affine function:} \\ = \{ x \in \mathbf{dom} f \mid Gf(x) \preceq h \} \\ = \{ x \mid G\frac{Ax+b}{c^T x + d} \leq h, c^T x + d > 0 \} \\ = \{ x \mid GAx + Gb \leq hc^T x + hd, c^T x + d > 0 \} \end{aligned}$$

We subtract Gb and $hc^T x$ from both sides to have the "matrix $\preceq$ vector" form.

$$f^{-1}(C) = \{ x \mid (GA - hc^T)x \preceq hd - Gb, c^T x + d > 0 \}$$

This is again a polyhedron intersecting the domain of f which is a half space. □

2.33 a)

Given

$$K_{m+} = \{ x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq ... \geq x_n \geq 0 \}$$

Show K_{m+} is proper:

Convexity: K_{m+} is a system of inequalities $x_1 \geq x_2 \geq ... \geq x_n \geq 0$ which constitutes a polyhedron.

Closed: Since the system of inequalities are not strict, the polyhedron is closed.

Pointed: A cone is pointed if $x \in K_{m+}$ & $-x \in K_{m+}$ if $x = 0$. Say $x = [1, 0]$ and $-x = [-1, 0]$. $-x$ violates inequality constraints. Only if $x, -x = 0$ are the constraints not violated.

2.33 b)

Find the dual cone K^*_{m+}

Given the definition of the dual cone:

$$K^* = \{\, y \mid x^T y \geq 0 \ \forall x \in K \,\}$$

Show that

$$\sum_{i=1}^{n} x_i y_i \geq 0 \ \forall x \in K_{m+}$$

:

$$\begin{aligned} \sum_{i=0}^{n} x_i y_i = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots + (x_{n_1} - x_n)(y_i + \cdots + y_{n+1}) \\ \text{we know } (x_{n_1} - x_n) \geq 0 \text{ based on the constraints on } K_{m+} \\ \therefore x_T y \geq 0 \iff y_1,\ y_1 + y_2,\ y_1 + \cdots + y_n \geq 0 \\ \therefore K^*_{m+} = \{\, y \mid \sum_{i=1}^{n} y_i \geq 0, k = 1, \ldots, n \,\} \end{aligned}$$

2.3)

If we can recursively show that the midpoint of the midpoint and the end of a line segment is contained by the set for all line segments, then midpoint convexity implies convexity. For example:

$$\frac{x_1 + \dfrac{x_1 + x_2}{2}}{2} = \frac{x_1}{2} + \frac{x_1}{4} + \frac{x_2}{4} = 3/4x_1 + 1/4x_2 \in C$$

And the next recursion:

$$\frac{x_1 + \dfrac{x_1 + \dfrac{x_1 + x_2}{2}}{2}}{2} = \frac{x_1}{2} + \frac{x_1}{4} + \frac{x_1}{8} + \frac{x_2}{8} = 7/8x_1 + 1/8x_2 \in C$$

The coefficients always add to one, meeting the requirement of convexity for all line segments and the recursively reassigned midpoints. The general pattern shows a consistent convex combination of which the points is always contained in C:

$$(\frac{2^k - 1}{2^k})x_1 + (1 - \frac{2^k - 1}{2^k})x_2 \in C$$

As $k \to \infty$:

$$lim_{k \to \infty}(\frac{2^k - 1}{2^k})x_1 + (1 - \frac{2^k - 1}{2^k})x_2 = x_1 \in C$$