HW7

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CSCI 5254 - Convex Optimization

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8.16

Formulate the following asa CVX optimization problem:
Find the rectangle

$$R = \{\, x \in \mathbf{R}^n \mid l \preceq x \preceq u \,\}$$

of maximum volume enclosed in the polyhedron

$$P = \{\, x \mid Ax \preceq b \,\}$$

The volume can be expressed as:

Proposition 1.

$$v = \prod_{i=1}^{n} u_i - l_i \tag{1}$$

We want all the 2^n corners do be contained within the polyhedron. This every corner must meet the polyhedron constraint $Ac \preceq b$. Where c is the vector of corners. Each of these corners can be be more succinctly represented as the vector based on the upper and lower values of each edge:
If we express x_i as $u_i - l_i$ then this system becomes

$$\sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i$$

The problem can be expressed as:

$$\begin{aligned} \text{minimize} \quad & \prod_{i=1}^{n} u_i - l_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i \end{aligned}$$

The constraint is a posynomial as it is a summation of the monomial $a_{ij}(u_j - l_j)$.
To make the problem a non-linear geometric optimization problem, we take the log of the objective:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} log(u_i - l_i) \\ \text{subject to} & \sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i \end{array}$$

8.24

We make use of the Cauchy-Schwarz inequality and sustite p knowing that $||u||_2 \leq p$:

Proposition 1.

$$u^t x_i \leq ||u||_2 |x_i||_2 \tag{2}$$
$$||u||_2 ||x_i||_2 \leq p||x_i||_2 \tag{3}$$
$$u^t y_j \leq ||u||_2 |y_j||_2 \tag{4}$$
$$||u||_2 |y_j||_2 \leq p||y_j||_2 \tag{5}$$
$$-||u||_2 ||y_j||_2 \geq -p||y_j||_2 \tag{6}$$
$$\tag{7}$$

For x_i:

$$\begin{aligned} (a+u)^T x_i &\geq b \\ a^T x_i + u^T x_i &\geq b \\ a^T x_i + ||u||_2 |x_i||_2 &\geq b \\ a^T x_i + p|x_i||_2 &\geq b \\ a^T x_i - b &\geq -p|x_i||_2 \end{aligned}$$

For x_i:

$$\begin{aligned} (a+u)^T y_j &\leq b \\ a^T y_j + u^T y_j &\leq b \\ a^T y_j + ||u||_2 ||y_j||_2 &\leq b \\ a^T y_j + p||x_i||_2 &\leq b \\ a^T y_j - b &\leq -p||y_j||_2 \\ b - a^T y_j &\geq p||y_j||_2 \end{aligned}$$

The optimization problem:

$$\begin{array}{ll} \text{minimize} & p \\ \text{subject to} & b - a^T y_j \geq p||y_j||_2 \\ & a^T x_i - b \geq -p|x_i||_2 \\ & ||a||_2 \leq 1 \end{array}$$

Additional Exercises:

5.12

One heurisitc estiamte an initial $\hat{x}$ using the huber penalty function. We then use that $\hat{x}$ to estimate a $\hat{P}$ by aligning the indices of Ax and y to find a permutation matrix. Then using that same permutation we reoptimized for $\hat{x}$. We repeat this algorithm until the euclidean norm of the distance between the $\hat{x}_\tau$ and $\hat{x}_{\tau-1}$ is below some tolerance, τ being the current iteration step. **Code:**

```
above_tol = 1
tolerance = .00000001
% Seed our initial estimate of x using huber function
cvx_begin
variable x(n);
    minimize( sum(huber(A*x-y)) );
cvx_end
P_hat = eye(m)
x_prior = zeros(n)
while 1

    % Align the smallest indixes, find pi (the permutation index alignement)
    % and construct the permutation matrix P_hat accordingly:
    [Ax_values, Ax_idx] = sort(A*x);
    [y_values, y_idx] = sort(y);
    pi = [y_idx' ; Ax_idx'];
    P_temp = zeros(m, m);
    for i = 1 : m
       row = pi(1,i);
       col = pi(2,i);
       P_temp(row, col) = 1;
    end
    P_hat = P_temp;
    if P_hat*P_hat' ~= eye(m)
        "Invalid P_hat!"
        break
    end
    "Distance:"
    dist = norm(x - x_prior, 2)
    if dist <= tolerance
        break
    end
```

```
    x_prior = x;

    % Find x_hat
    cvx_begin
        variable x(n,1)
        minimize(norm(A*x-P_hat'*y, 2))
    cvx_end;
end
P_eye = eye(m);
cvx_begin
    variable x_eye(n,1)
    minimize(norm(A*x_eye-P_eye'*y, 2))
cvx_end;
"Distance x (P=I) and estimated x:"
norm(x_eye - x_true, 2)
"Distance x_true and estimated x:"
norm(x_true - x, 2)
```

Results:
"Distance estimated x (P=I) and x_true:"
ans = 3.4363
"Distance x_true and estimated x:"
ans = 0.0965

5.18

We can reformulate the problem as the original object being less than or equal to some value z:

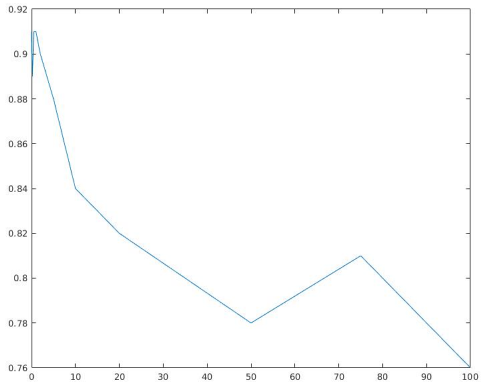
$$1 + max_{k \neq y_i} f_k(x_i) - f_{y_i}(x_i) \leq z_i, z_i \geq 0$$

This can be represented by the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_i z_i + \mu ||A||_F^2 \\ \text{subject to} & 1 + max_{k \neq y_i} f_k(x_i) - f_{y_i}(x_i) \leq z_i, \forall i \\ & 1^T b = 0, z \geq 0 \end{array}$$

This can be reexpressed using the individual inequality constraints:

$$\begin{array}{ll} \text{minimize} & \sum_i z_i + \mu ||A||_F^2 \\ \text{subject to} & 1^T b = 0, z \geq 0 \\ & 1 + a_k^T x_i + b - y_i \leq z_i, k = 1, 2, , y_i - 1, y_i + 1, , K, i = 1, 2, , m \end{array}$$



Code:

```
E = [];
U = [0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 50 75 100]
% This loop generates a new u value.
for u = 1:size(U,2)
    cvx_begin
        variable z(mTrain, 1)
        variable A(K, n)
        variable b(K, 1)
        minimize(sum(z) + U(u)*square_pos(norm(A,'fro')))
        subject to
        for i=1:mTrain
            for k=[1:y(i)-1 y(i)+1:K]
                1+(A(k,:)*x(:,i)+b(k))-(A(y(i),:)*x(:,i)+b(y(i))) <= z(i);
            end
            z(i) >= 0;
        end
        sum(b) == 0;
    cvx_end
    % Compute the predict predicted labels by computing the affine function
    % on xtest using the estimated optial A and b. Find the max in each
    % column (i.e. argmax for label) and round to get whole number value.
    correct = 0
    y_pred = zeros(1,mTest);

    for i=1:mTest
        [~, y_pred(i)] = max(A*xtest(:,i) + b);
        if (y_pred(i) == ytest(i))
            correct = correct + 1;
```

```
        end
    end
    percent_correct = correct/mTest
    E = [E ; percent_correct]
end
plot(U,E)
```

13.15

One heurisitic is to ensure that the 1-norm of w is minimized. We can then formulate and optimization problem subject to the following constraint:

Proposition 1.

$$E[(r-\bar{r})((r-\bar{r})] = \Sigma \tag{8}$$

$$E[rr^T] - \bar{r}\bar{r}^T = \Sigma \tag{9}$$

$$E[rr^T] = \Sigma + \bar{r}\bar{r}^T \tag{10}$$

Proposition 2.

$$E[z^T z] = c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} \tag{11}$$

$$\begin{aligned}
E[(z - w^T r)(z - w^T r)] &\leq .01E[z^2] \\
E[z^T z + r^T ww^T r - 2zw^T r] &\leq .01E[z^2] \\
E[z^T z] + E[r^T ww^T r] - 2E[zw^T r] &\leq .01E[z^2] \\
E[z^T z] + E[r^T ww^T r] - 2E[zw^T r] &\leq .01E[z^2] \\
E[z^T z] + E[w^T rr^T w] - 2E[(c^T r)^T w^T r] &\leq .01E[z^2] \\
c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2E[(c^T r)^T w^T r] &\leq .01E[z^2] \\
c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2w^T(\Sigma + \bar{r}\bar{r}^T)c &\leq .01E[z^2]
\end{aligned}$$

This becomes the optimization problem reflected in the matlab code:

$$\begin{array}{ll}
\text{minimize} & ||w||_1 \\
\text{subject to} & c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2w^T(\Sigma + \bar{r}\bar{r}^T)c \leq .01E[z^2]
\end{array}$$

Code:

```
ctc = mtimes(c', c);
rbarsq = dot(rbar, rbar');
zsqr = (ctc * rbarsq)
cvx_begin
  variable w(n)
  minimize norm(w,1)
```

```
  E_num = ( (rbar' * (c * c') * rbar)) + c'*diag(Sigma) + (w' * (Sigma + (rba
  subject to
      E_num <= 0.01 * (ctc * rbarsq);
      w <= c;
cvx_end
E_num/(ctc * rbarsq)
sum(abs(w > 0.01))
sum(abs(c > 0.01))
```

Results:
ans = 0.0100
ans = 108
ans = 500

16.5

See solution.

17.4

17.5

17.8

17.9