# HW7

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CSCI 5254 - Convex Optimization

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## 8.16

Formulate the following asa CVX optimization problem:
Find the rectangle

$$R = \{ x \in \mathbf{R}^n \mid l \preceq x \preceq u \}$$

of maximum volume enclosed in the polyhedron

$$P = \{ x \mid Ax \preceq b \}$$

The volume can be expressed as:

**Proposition 1.**

$$v = \prod_{i=1}^{n} u_i - l_i \tag{1}$$

We want all the $2^n$ corners do be contained within the polyhedron. This every corner must meet the polyhedron constraint $Ac \preceq b$. Where $c$ is the vector of corners. Each of these corners can be be more succinctly represented as the vector based on the upper and lower values of each edge:
If we express $x_i$ as $u_i - l_i$ then this system becomes

$$\sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i$$

The problem can be expressed as:

$$\begin{aligned} \text{minimize} \quad & \prod_{i=1}^{n} u_i - l_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i \end{aligned}$$

The constraint is a posynomial as it is a summation of the monomial $a_{ij}(u_j - l_j)$.
To make the problem a non-linear geometric optimization problem, we take the log of the objective:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} log(u_i - l_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ij}(u_j - l_j) \leq b_i \end{aligned}$$

## 8.24

We make use of the Cauchy-Schwarz inequality and sustite $p$ knowing that $||u||_2 \leq p$:

**Proposition 1.**

$$u^t x_i \leq ||u||_2 |x_i||_2 \tag{2}$$
$$||u||_2||x_i||_2 \leq p||x_i||_2 \tag{3}$$
$$u^t y_j \leq ||u||_2 |y_j||_2 \tag{4}$$
$$||u||_2 |y_j||_2 \leq p||y_j||_2 \tag{5}$$
$$-||u||_2||y_j||_2 \geq -p||y_j||_2 \tag{6}$$
$$\tag{7}$$

For $x_i$:

$$\begin{aligned} (a+u)^T x_i &\geq b \\ a^T x_i + u^T x_i &\geq b \\ a^T x_i + ||u||_2 |x_i||_2 &\geq b \\ a^T x_i + p|x_i||_2 &\geq b \\ a^T x_i - b &\geq -p|x_i||_2 \end{aligned}$$

For $x_i$:

$$\begin{aligned} (a+u)^T y_j &\leq b \\ a^T y_j + u^T y_j &\leq b \\ a^T y_j + ||u||_2||y_j||_2 &\leq b \\ a^T y_j + p||x_i||_2 &\leq b \\ a^T y_j - b &\leq -p||y_j||_2 \\ b - a^T y_j &\geq p||y_j||_2 \end{aligned}$$

The optimization problem:

$$\begin{aligned} \text{minimize} \quad & p \\ \text{subject to} \quad & b - a^T y_j \geq p||y_j||_2 \\ & a^T x_i - b \geq -p|x_i||_2 \\ & ||a||_2 \leq 1 \end{aligned}$$

**Additional Exercises:**

## 5.12

One heurisitc estiamte an initial $\hat{x}$ using the huber penalty function. We then use that $\hat{x}$ to estimate a $\hat{P}$ by aligning the indices of $Ax$ and $y$ to find a permutation matrix. Then using that same permutation we reoptimized for $\hat{x}$. We repeat this algorithm until the euclidean norm of the distance between the $\hat{x}_\tau$ and $\hat{x}_{\tau-1}$ is below some tolerance, $\tau$ being the current iteration step. **Code:**

```
above_tol = 1
tolerance = .00000001
% Seed our initial estimate of x using huber function
cvx_begin
variable x(n);
    minimize( sum(huber(A*x-y)) );
cvx_end
P_hat = eye(m)
x_prior = zeros(n)
while 1

    % Align the smallest indixes, find pi (the permutation index alignement)
    % and construct the permutation matrix P_hat accordingly:
    [Ax_values, Ax_idx] = sort(A*x);
    [y_values, y_idx] = sort(y);
    pi = [y_idx' ; Ax_idx'];
    P_temp = zeros(m, m);
    for i = 1 : m
       row = pi(1,i);
       col = pi(2,i);
       P_temp(row, col) = 1;
    end
    P_hat = P_temp;
    if P_hat*P_hat' ~= eye(m)
        "Invalid P_hat!"
        break
    end
    "Distance:"
    dist = norm(x - x_prior, 2)
    if dist <= tolerance
        break
    end
```

```
    x_prior = x;

    % Find x_hat
    cvx_begin
        variable x(n,1)
        minimize(norm(A*x-P_hat'*y, 2))
    cvx_end;
end
P_eye = eye(m);
cvx_begin
    variable x_eye(n,1)
    minimize(norm(A*x_eye-P_eye'*y, 2))
cvx_end;
"Distance x (P=I) and estimated x:"
norm(x_eye - x_true, 2)
"Distance x_true and estimated x:"
norm(x_true - x, 2)
```

**Results:**
"Distance estimated x (P=I) and x_true:"
ans = 3.4363
"Distance x_true and estimated x:"
ans = 0.0965

## 5.18

We can reformulate the problem as the original object being less than or equal to some value $z$:

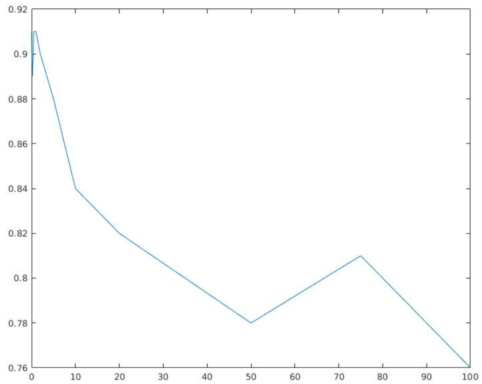
$$1 + max_{k \neq y_i} f_k(x_i) - f_{y_i}(x_i) \leq z_i, z_i \geq 0$$

This can be represented by the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_i z_i + \mu ||A||_F^2 \\ \text{subject to} & 1 + max_{k \neq y_i} f_k(x_i) - f_{y_i}(x_i) \leq z_i, \forall i \\ & 1^T b = 0, z \geq 0 \end{array}$$

This can be reexpressed using the individual inequality constraints:

$$\begin{array}{ll} \text{minimize} & \sum_i z_i + \mu ||A||_F^2 \\ \text{subject to} & 1^T b = 0, z \geq 0 \\ & 1 + a_k^T x_i + b - y_i \leq z_i, k = 1, 2, , y_i - 1, y_i + 1, , K, i = 1, 2, , m \end{array}$$




**Code:**

```
E = [];
U = [0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 50 75 100]
% This loop generates a new u value.
 for u = 1:size(U,2)
     cvx_begin
         variable z(mTrain, 1)
         variable A(K, n)
         variable b(K, 1)
         minimize(sum(z) + U(u)*square_pos(norm(A,'fro')))
         subject to
         for i=1:mTrain
             for k=[1:y(i)-1 y(i)+1:K]
                 1+(A(k,:)*x(:,i)+b(k))-(A(y(i),:)*x(:,i)+b(y(i))) <= z(i);
             end
             z(i) >= 0;
         end
         sum(b) == 0;
     cvx_end
     % Compute the predict predicted labels by computing the affine function
     % on xtest using the estimated optial A and b. Find the max in each
     % column (i.e. argmax for label) and round to get whole number value.
     correct = 0
     y_pred = zeros(1,mTest);

     for i=1:mTest
         [~, y_pred(i)] = max(A*xtest(:,i) + b);
         if (y_pred(i) == ytest(i))
             correct = correct + 1;
```

```
        end
    end
    percent_correct = correct/mTest
    E = [E ; percent_correct]
end
plot(U,E)
```

## 13.15

One heurisitic is to ensure that the 1-norm of $w$ is minimized. We can then formulate and optimization problem subject to the following constraint:

**Proposition 1.**

$$E[(r-\bar{r})((r-\bar{r})] = \Sigma \quad (8)$$
$$E[rr^T] - \bar{r}\bar{r}^T = \Sigma \quad (9)$$
$$E[rr^T] = \Sigma + \bar{r}\bar{r}^T \quad (10)$$

**Proposition 2.**

$$E[z^Tz] = c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} \quad (11)$$

$$\begin{aligned}
E[(z-w^Tr)(z-w^Tr)] &\leq .01E[z^2] \\
E[z^Tz + r^Tww^Tr - 2zw^Tr] &\leq .01E[z^2] \\
E[z^Tz] + E[r^Tww^Tr] - 2E[zw^Tr] &\leq .01E[z^2] \\
E[z^Tz] + E[r^Tww^Tr] - 2E[zw^Tr] &\leq .01E[z^2] \\
E[z^Tz] + E[w^Trr^Tw] - 2E[(c^Tr)^Tw^Tr] &\leq .01E[z^2] \\
c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2E[(c^Tr)^Tw^Tr] &\leq .01E[z^2] \\
c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2w^T(\Sigma + \bar{r}\bar{r}^T)c &\leq .01E[z^2]
\end{aligned}$$

This becomes the optimization problem reflected in the matlab code:

$$\begin{array}{ll}
\text{minimize} & ||w||_1 \\
\text{subject to} & c^T diag(\Sigma) + \bar{r}^T cc^T \bar{r} + w^T(\Sigma + \bar{r}\bar{r}^T)w - 2w^T(\Sigma + \bar{r}\bar{r}^T)c \leq .01E[z^2]
\end{array}$$

**Code:**

```
ctc = mtimes(c', c);
rbarsq = dot(rbar, rbar');
zsqr = (ctc * rbarsq)
cvx_begin
  variable w(n)
  minimize norm(w,1)
```

```
  E_num = ( (rbar' * (c * c') * rbar)) + c'*diag(Sigma) + (w' * (Sigma + (rba
  subject to
      E_num <= 0.01 * (ctc * rbarsq);
      w <= c;
cvx_end
E_num/(ctc * rbarsq)
sum(abs(w > 0.01))
sum(abs(c > 0.01))
```

**Results:**
ans = 0.0100
ans = 108
ans = 500

## 16.5

We begin by stating that $X = \theta s$. The optmization problem can be formulated as:

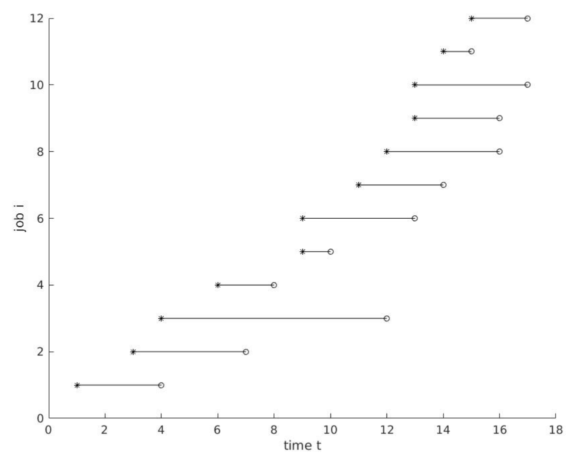
$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{t=1}^{T} \phi(s_t) \\
\text{subject to} & S_{min} \preceq s \preceq S_{max} \\
& |s_{t+1} - s_t| \leq R \\
& s = X1, X^T 1 \succeq W, X \succeq 0 \\
& X_{ti} = 0, t = 1, \ldots, A_i - 1, 1 = i, \ldots, n \\
& X_{ti} = 0, t = D_i + 1, \ldots, T, 1 = i, \ldots, n
\end{array}
$$

The top two constraitns are the processor speed limits and slew rates. Each row coefficients $\theta_{ti}$ for $1 = 1, \ldots, n$ must sum to one therefore $X1 = s$. The last two constraints express that each component of $X_{ti}$ is 0 for any values outside the range $[A_i, D_i]$ for the given job $i$. **Code:**

```
cvx_begin
    variable X(T,n)
    s = sum(X');
    minimize(sum(alpha+beta*s+gamma*square(s)))
    subject to
        X >= 0;
        Smin <= s <= Smax;
        abs(s(2:end)-s(1:end-1))<=R; % slew rate constraint
        % Timing constraints for each job
        for i=1:n
            for t=1:A(i)-1
                X(t,i)==0;
            end
```

```
                for  t=D(i)+1:T
                     X(t,i)==0;
                end
          end
          sum(X)>=W';
cvx_end
theta = X./(s'*ones(1,n));
figure;
bar((s'*ones(1,n)).*theta,1,'stacked');
xlabel('Time: tt');
ylabel('Stacked speed: s_t');
```

**Results:**



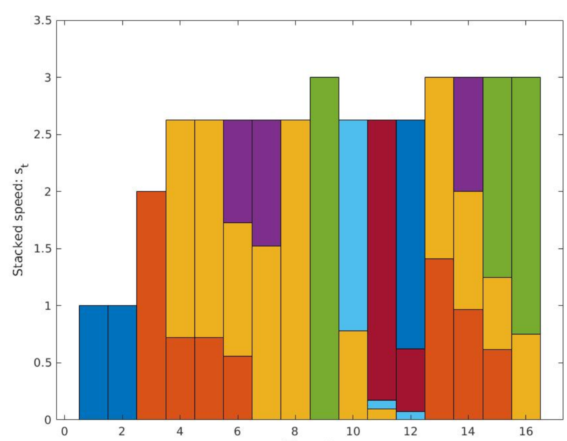




Figure 1: Stacked processor speed, each stack representing portion of speed devoted to allocated jobs.

### 17.4

We must maximize the net revenue minus the penalty to maximize the net profit:

$$\begin{array}{ll} \text{maximize} & trace(R^T N) - p^T s \\ \text{subject to} & s = q - A^T N T \\ & N \succeq 0 \\ & \sum_{t=1}^{T} N_{it} = I_i \end{array}$$

**Code:**

```
cvx_begin
    variable N(n,T)
    % we need s positive to ensure no negative penalties.
    s =pos(q-diag(Acontr'*N*Tcontr))
    maximize(sum(diag(R'*N())) - p'*s)
    subject to
        N >= 0;
        sum(N)== I';
cvx_end

net_profit = cvx_optval
revenue = sum(diag(R'*N))
payment = p'*s

% Highest ad revenue %
cvx_begin
    variable N(n,T)
    maximize (sum(diag(R'*N)))
    subject to
        N >= 0;
        ones(1,n)*N == I';
cvx_end
hi_ad_revenue = cvx_optval;
s = pos(q-diag(Acontr'*N*Tcontr))
hi_ad_payment = p'*s
hi_ad_net_profit = hi_ad_revenue-hi_ad_payment
```

**Results:**
Full ad porfolio:
net_profit = 230.5660
revenue = 268.2319
payment = 37.6659

Highest earning ad portfolio:
hi_ad_revenue = 305.1017
hi_ad_payment = 232.2602
hi_ad_net_profit = 72.8415

## 17.5

For the first condition: $\phi(v) = v^2$ which will penalize higher values. For the second condition: $\phi(v) = huber(v)$ which is less sensitive to outliers and thus will allow a few large preference violations. The minimization problem for the squared error becomes becomes:

$$\begin{array}{ll} \text{minimize} & \sum \phi(V) \\ \text{subject to} & V = max(r_j + 1 - r_i, 0) \end{array}$$

For the huber penalty function, since CVX requires huber be passed an affine function, we must reformulate:

$$\begin{array}{ll} \text{minimize} & \sum \phi(V) \\ \text{subject to} & r_j + 1 - r_i, 0 \geq 0 \end{array}$$

**Code:**

```
cvx_begin
variable R(50)
V = max(R(preferences(:,2))+1-R(preferences(:,1)),0)
minimize(sum(square_phi(V)))
cvx_end
sum(V>0.001)
histogram(V)

cvx_begin
variable R(50)
V = R(preferences(:,2))+1-R(preferences(:,1))
minimize(sum(huber_phi(V)))
subject to
    R(preferences(:,2))+1-R(preferences(:,1)) >= 0
cvx_end
sum(V>0.001)
histogram(V)

function square_penalty = square_phi( x )
square_penalty = pow_pos(x, 2)
end
```

```
function huber_penalty = huber_phi( x )
huber_penalty = huber(x)
end
```

**Results:**
Squared penalty:
sum(V¿0.001) = 781
Huber Penalty:
sum(V¿0.001) = 900

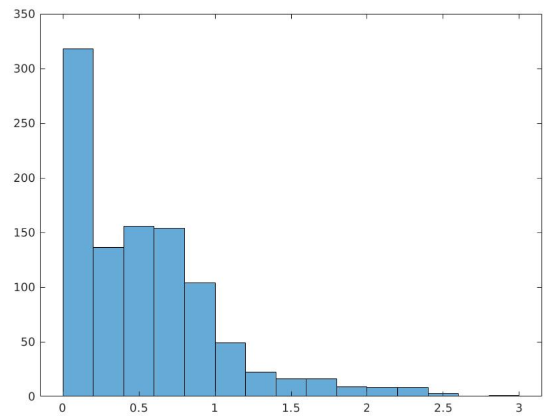



Figure 2: Histogram for square penalty.

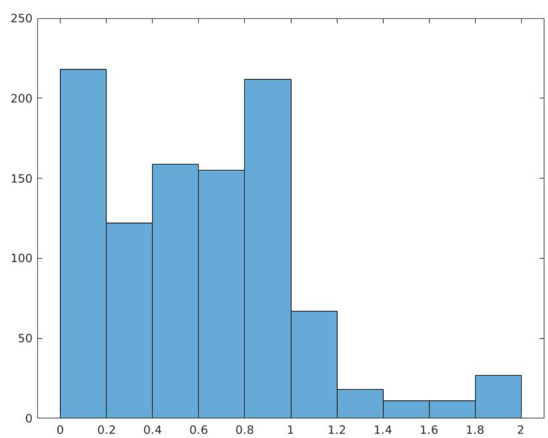



Figure 3: Histogram for huber penalty.

## 17.8

## 17.9