

HW4

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CSCI 5254 - Convex Optimization

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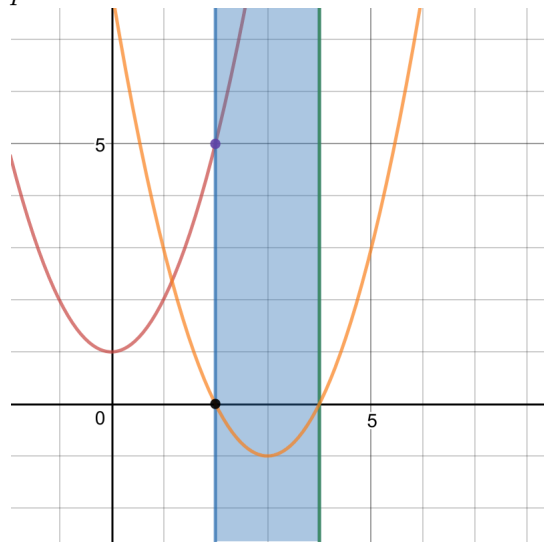
5.1)

a

Feasible set: $\{x \mid 2 \leq x \leq 4\}$

$$x^* = 2$$

$$p^* = 5$$



b

The Lagrangian:

$$\begin{aligned}\mathcal{L}(x, \lambda) &= x^2 + 1 + \lambda(x - 2)(x - 4) \\ &= (1 + \lambda)x^2 - 6\lambda x + (1 + 8\lambda)\end{aligned}$$

Gradient w.r.t. x :

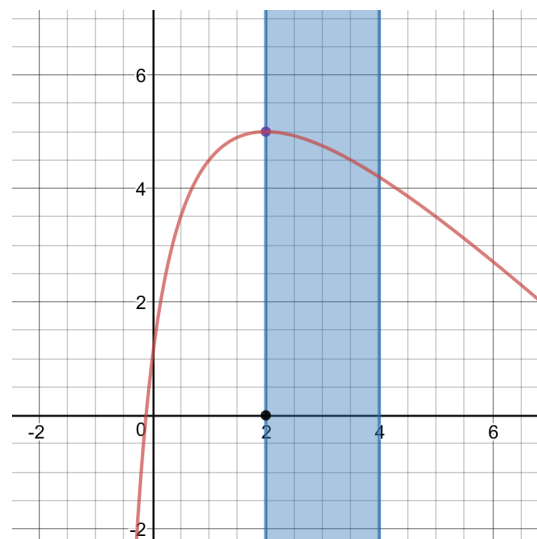
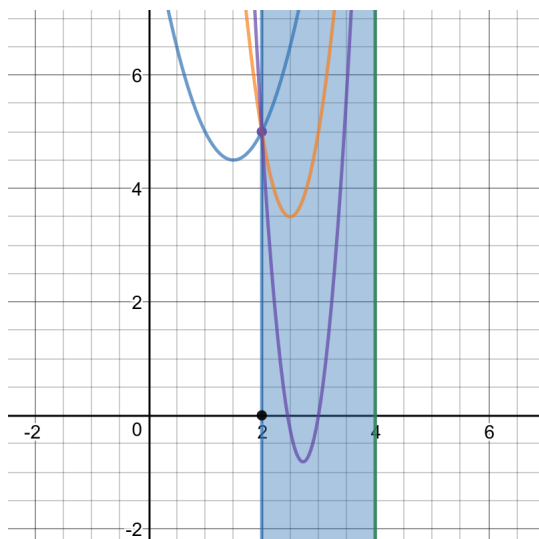
$$\nabla_x \mathcal{L}(x, \lambda) = 2x + 2\lambda x - 6\lambda = (1 + \lambda)x - 6\lambda$$

Set to zero, solve for x, plug back into lambda to get dual function:

$$\nabla_x \mathcal{L}(x, \lambda) = (1 + \lambda)x - 6\lambda = 0$$

$$x = \frac{3\lambda}{1 + \lambda}$$

$$g(\lambda) = \inf\left(\frac{-9\lambda^2}{1 - \lambda} + 8\lambda + 1\right)$$



c

The dual problem:

$$\begin{aligned} & \underset{\lambda}{\text{maximize}} && \frac{-9\lambda^2}{1 - \lambda} + 8\lambda + 1 \\ & \text{subject to} && \lambda \geq 0 \end{aligned}$$

Take the gradient of the dual w.r.t. λ , set to zero, to show strong duality:

$$\nabla_{\lambda} g(\lambda) = 0$$

$$\frac{-9\lambda(1 - \lambda) + 9\lambda^2}{(1 - \lambda)^2} + 8 = 0$$

$$-9\lambda(1 - \lambda) + 9\lambda^2 + 8(1 - \lambda)^2 = 0$$

$$\lambda^2 + 2\lambda - 8 = 0$$

$$(\lambda + 4)(\lambda - 2) = 0$$

λ must be greater than 2:

$$\lambda \neq -4, \lambda = 2$$

$$5 = p^* = g^* = g(2) = 5$$

5.11)

$$\begin{aligned} & \underset{\lambda}{\text{minimize}} && \sum_{i=1}^N \|A_i x + b_i\|_2 + \frac{1}{2} \|x - x_o\|_2^2 \\ & \text{subject to} && y_i = A_i x + b_i \end{aligned}$$

The Lagrangian:

$$\begin{aligned} \mathcal{L}(x, \lambda) &= \sum_{i=1}^N \|y_i\|_2 + \frac{1}{2} \|x - x_o\|_2^2 + \sum_{i=1}^N \lambda_i^T (y_i - A_i x + b_i) \\ &= \sum_{i=1}^N \|y_i\|_2 - \lambda_i^T y_i + \frac{1}{2} \|x - x_o\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x + b_i) \end{aligned}$$

Minimize w.r.t. y :

$$\begin{aligned} & \inf \left(\sum_{i=1}^N \|y_i\|_2 - \lambda_i^T y_i + \frac{1}{2} \|x - x_o\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x + b_i) \right) \\ &= \sum_{i=1}^N \inf \left(\|y_i\|_2 - \lambda_i^T y_i + \frac{1}{2} \|x - x_o\|_2^2 - \lambda_i^T (A_i x + b_i) \right) \end{aligned}$$

Based on Cauchy-Schwarz Inequality:

$$= \begin{cases} \frac{1}{2} \|x - x_o\|_2^2 - \lambda_i^T (A_i x + b_i) & \text{if } \|\lambda\|_* \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

Constraint on dual is

$$\|\lambda\|_* \leq 1$$

Minimize w.r.t. x via gradient and then set to zero:

$$\begin{aligned} \nabla_x \left(\frac{1}{2} \|x - x_o\|_2^2 - \sum_{i=1}^N \lambda_i^T (A_i x + b_i) \right) &= 0 \\ x - x_o - \sum_{i=1}^N \lambda_i^T A_i &= 0 \\ x &= x_o - \sum_{i=1}^N \lambda_i^T A_i \end{aligned}$$