Math 104, HW7

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1 Q1

We first show that the limits exist $\implies$ uniformly continuous. We extend the definition of $f : [a, b]$ where $f(a) = \lim_{x \to a}, f(b) = \lim_{x \to b}$. Since the limit exists, we know that f can get arbitrily close it. More precisely, we know that there exists δ such that $|x - a| < \delta \implies |f(x) - \lim_{x \to a}| < \epsilon$ for any $\epsilon > 0$. This also demonstrates continuity at a. We can repeat this logic to show that it is continuous at b. Then, since f is continuous on the closed inverval and continuous, it is uniformly continuous.
Now we show that uniformly continuous $\implies$ limits exist. Let $a_n \in (a, b)$ be an arbitrary sequence that converges to a. Then since it is convergent, we know that it is cauchy. Now since f is uniformly continuous, we know that $f(a_n)$ is also cauchy, and it is therefore convergent. Therefore the limit exists as $x \to a$. We can repeat this argument for b to show that $\lim_{x \to b} f(x)$ exists.
Thus it is proven.

2 Q2

2.1 Forward statement

Let $f : S \to S^*$ be continuous, let $E \subset S^*$ be a closed set, then $E' = S^* \setminus E$ is open and $f^{-1}(E')$ is also open by continuity.
Consider $f^{-1}(E')$, since f is defined on all S, each $\exists f(s) \forall s \in S$. Therefore all $s \in S$ either has an image in E or in E', and $f^{-1}(E') = S \setminus f^{-1}(E)$. Since we know that both S and $f^{-1}(E')$ are open, $f^{-1}(E)$ must be closed.

2.2 Converse

Now assume that for any closed $F \subset S^*$, $f^{-1}(F)$ is also closed. We essentially construct the same proof for open sets but with one more step. For any point $s_0 \in S$, $\epsilon > 0$, construct open set $G = \{d^*(f(s_0), s^*) < \epsilon\}$. Now take its complement $G' = S^* \setminus G$. Since S^* and G are both open, G' is closed.
By our assuption $H' = f^{-1}(G')$ is closed, then its complement $H = S \setminus H'$ is open. Since $f(s_0) \in G$, we have $s_0 \in H$. Then we can find a "ball" inside this set such that for some $\delta > 0$, $\{d(s - s_0) < \delta\} \subseteq H$
Thus it follows that $d(s - s_0) < \delta$ implies $d^*(f(s_0), s^*) < \epsilon$ and we have shown that f is continuous at s_0. Since s_0 is an arbitrary point in S, f is continuous.

3 Q3

Let $C = \{(1/n, \infty)...\}$ for all $n \in \mathbb{N}$. We claim that C is a cover for $(0, \infty)$. To see that it is true, let $x \in (0, \infty)$, if $x > 1$, it is in every subset. Otherwise, by the Archimedean Principle there exists $m \in \mathbb{N}$ such that $1/m < x$. Therefore $x \in (1/m, \infty) \in C$.

Now we take $C \cup (-\infty, 0]$. Let $x \in \mathbb{R}$, if $x \leq 0$, then it is in the second set, otherwise it is in the first.

Let C' be a finite subset of C, then we must have $C' = \{(1/a, \infty), ..., (1/b, \infty)\}$ with natural numbers $a \leq b$. Now consider $r = 1/(2a)$. Since $0 < r < 1/a$, $r \notin C'$, and $r \notin (-\infty, 0]$

We can find a such r for any finite subset of C, therefore there is no finite subcover of $\mathbb{R}$. Thus $\mathbb{R}$ is not compact.

4 Q4

Consider $X = S \setminus F$. Since F is closed, its complement X is open. Now let C be any arbitrary open cover of F. We take the union of the above two sets: $C' = X \cup C$. Since C covers F and A covers the rest of the metric space, and since $E \subset S$, C' is an open cover of E.
Now we apply the definition of compactness, so there exists a finite subcover in C'. Let this subcover be Y. If $X \notin Y$, then our proof is complete since $F \subseteq E$. Any cover that covers E must also cover F.
Otherwise, we remove A: $Y \setminus X$. Since we have removed what we have added, $(Y \setminus X) \subseteq C$. This set is a subcover for F, since $X \cup F = \emptyset$, we are not removing any point that is inside F, thus we have found a subcover that covers F.

5 Q5

Any set with the interval property can be written as $(a, b), [a, b], (a, b], [a, b)$, where a is the infimum of the set and b the supremum. Square brackets mean that the infimum/supremum is in the set and round brackets mean that they are not.

5.1 a

If I is closed, then $R \setminus I$ is open. Let I be $[x, y]$. Then its complement is $(-\infty, x) \cup (y, \infty)$. For any point i_0 in this complement, we pick $r = \min\{|i_0 - x|/2, |i_0 - y|/2\}$, and thus $\{|i - i_0| < r\} \subset (\mathbb{R} \setminus I)$. Having square brackets on both ends ensures that I's complement is open, therefore I is closed.
Otherwise consider I with a round bracket. Then its complement will contain that point. Let that point be i_1. At i_1, for any $r > 0$, $\{|i_1 - i| < r\} \cap I \neq \emptyset$. There will always be some part of that set that is in our original I. Therefore an interval with any round brackets is not closed.

5.2 b

If I is open, then let it be (x, y). For any point i_0 in this complement, we pick $r = \min\{|i_0 - x|/2, |i_0 - y|/2\}$, and thus $\{|i - i_0| < r\} \subseteq I$. Having round brackets on both ends ensures that I is open.
Otherwise consider I with a square bracket. Then it contains this point. Let this point be i_1 At i_1, for any $r > 0$, $\exists j \in \{|i_1 - i| < r\} \notin I$
No matter how small we make the radius, this set will contain points that are not in I. Therefore having any square brackets make the interval not open.

6 Q6

We first assume that the set $G = E \cup F$ is disconnected. Then it follows that there exists two open sets A, B such that $G = (G \cap A) \cup (G \cap B)$, $(G \cap A) \cap (G \cap B) = \emptyset$, and $A \cap G \neq \emptyset$, $B \cap G \neq \emptyset$
We now attempt to show that these two sets cannot exist.
Let $x \in E \cap F$, then without loss of generality assume that $x \in A$. Now take $y \in B$. Now one of the following must be true: $y \in E$ or $y \in F$. To make this simpler we name the subset that y is in: H. Since $x \in E \cap F$, $x \in H$. Now we have $x \in A \cap H, y \in B \cap H$.
Since E, F are connected, H must also be connected. Since we know that $x \in A \cap H, y \in B \cap H$, we know that $A \cap H \neq \emptyset$, $B \cap H \neq \emptyset$. According to our assumption, $G \subseteq A \cup B$, then $H \subset A \cup B$. However, this is not possible because H is connected and per the defintion of connectedness, it cannot be split by two open sets that satisfy the above constraints.
Therefore as shown above the sets A, B cannot exist. We have reached a contradiction, and G is therefore connected.

7 Q7

Since the equation we are given is a unit circle, we know from trigonometry that we can paramatrize the unit circle into $x = \cos(x), y = \sin(x), 0 \leq x \leq 2\pi$

Now since sine and cosine are both continuous, their sum is also continuous. Since $[0, 2\pi]$ is an interval, it is connected. We apply the theorem that continuous functions map connected sets to connected sets to see that the image (the unit circle) is connected as well.