

Math 104, HW12

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1 Q1

1.1 a

First we know that $\sqrt{x^2} = |x|$. Since we know that $\frac{1}{n} \rightarrow 0$ and $\frac{1}{n^2} \rightarrow 0$, we can show uniform convergence by the following.

Let $\epsilon > 0$, pick N such that for all $n > N$, $|\frac{1}{n}| < \epsilon^2$. Now since $|x| < 1$ and the domain of the square root being positive,

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2 + \frac{2}{\sqrt{n}}|x| + \frac{1}{n}} \leq \sqrt{(x + \frac{1}{\sqrt{n}})^2}$$

By our first statement the above expression is equal to $|(x + \frac{1}{\sqrt{n}})|$. By our definition of N ,

$$|(x + \frac{1}{\sqrt{n}})| - |x| \leq |x + \frac{1}{\sqrt{n}} - x| = |\frac{1}{\sqrt{n}}| < \epsilon$$

Thus we have $f_n \rightarrow |x|$ uniformly.

1.2 b

$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, and by the power rule we know that

$$f'_n(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$$

1.3 c

Define $g : (-1, 1) \rightarrow \mathbb{R}$, $g(x) = -1$ for $x \in (-1, 0)$, $g(x) = 1$ for $x \in (0, 1)$, $g(0) = 0$

For any $x < 0$, let $a_n = \frac{1}{f'_n(x)} = \frac{\sqrt{x^2 + \frac{1}{n}}}{x}$. By part a we know that $\sqrt{x^2 + \frac{1}{n}} \rightarrow |x|$ uniformly. Let $\epsilon > 0$ we pick N such that $|\sqrt{x^2 + \frac{1}{n}} - |x|| < \epsilon x - x$, therefore

$$|a_n - (-1)| = \frac{\sqrt{x^2 + \frac{1}{n}} + x}{x} = \frac{|\sqrt{x^2 + \frac{1}{n}} - |x||}{x} < \frac{\epsilon x}{x} = \epsilon$$

If $x > 0$, the same is true because the denominator is now positive and the absolute value sign should be flipped. Finally, if $x = 0$, $f'_n(x) = g(x) = 0$. Thus $f'_n(x) \rightarrow g(x)$ pointwise.

Since $f'_n(x)$ is a polynomial divided by a non-zero polynomial, it is continuous for all values of n and for all values of $x \in (-1, 1)$. However $g(x)$ is not continuous at 0. By our theorem about uniform convergence of continuous functions, we know that $f'_n(x) \not\rightarrow g(x)$ uniformly.

2 Q2

For any $x \in (-1, 1)$, shrink the domain of our power series to $[(x+1)/2, (1-x)/2]$. Now since both of these end points are within our radius of convergence, $\sum_{n=0}^{\infty} x^n \rightarrow \frac{1}{1-x}$ uniformly.

Then, we can take the derivative of both sides:

$$\left(\sum_{n=0}^{\infty} x^n \rightarrow \frac{1}{1-x}\right) = \sum_{n=1}^{\infty} nx^n$$

$$\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

Thus $\sum_{n=1}^{\infty} nx^n \rightarrow \frac{x}{(1-x)^2}$

3 Q3

Let $x \in \mathbb{R}$, since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

$$(e^x)' = \sum_{j=1}^{\infty} \frac{jx^{j-1}}{j!}$$

However, since $\frac{jx^{j-1}}{j!} = \frac{x^{j-1}}{(j-1)!}$, since j begins at 1 and k at 0, each term can be matched bijectively to the first sum.

Thus they are the same sum. ■

4 Q4

Since $f(x) = e^{-x^2}$, by the last problem we know that it is equal to the sum of $\sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$.
Now we integrate our power series term by term.

$$\int_0^y \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(1+2k)k!} y^{1+2k}$$

Let f_n be the above series, so $f'_n \rightarrow e^{x^2}$, therefore $f_n \rightarrow \int_0^x e^{x^2}$.

5 Q5

Since $f'(0)$ does not exist, the Taylor Polynomial of degree $n \geq 1$ does not exist. Thus the Taylor Series is 0, and we can let $\epsilon = 0.1$, at 0.5, $|f(x) - 0| = 0.4 > \epsilon$. Thus the Taylor Series does not converge to f , and therefore there is no power series that converge.

6 Q6

Base case: $n = 1$. We apply the chain rule:

$$(e^{\frac{1}{x^2}})' = e^{\frac{1}{x^2}} \frac{-2}{x^3}$$

Finally, let $a_{1,k} = 0$ for all $k \neq 3$, and $a_{1,3} = -2$, and our base case holds.

Inductive Case: let our formula hold for all $f^m(x)$ ($m \leq n$) for some $n \in \mathbb{N}$, then consider $f^{n+1}(x)$. We apply the product rule:

$$\begin{aligned} f^{n+1}(x) &= (e^{\frac{1}{x^2}})' \left(\sum_{k=1}^{3n} \frac{a_{n,k}}{x^k} \right) + (e^{\frac{1}{x^2}}) \left(\sum_{k=1}^{3n} \frac{a_{n,k}}{x^k} \right)' \\ &= e^{\frac{1}{x^2}} \frac{-2}{x^3} \left(\sum_{k=1}^{3n} \frac{a_{n,k}}{x^k} \right) + e^{\frac{1}{x^2}} \left(\sum_{k=1}^{3n} \frac{-k a_{n,k}}{x^{k+1}} \right) = e^{\frac{1}{x^2}} \left(\sum_{k=1}^{3n} \frac{-2a_{n,k}}{x^{k+3}} \right) \end{aligned}$$