

Math 104, HW9

Tianshuang (Ethan) Qiu

November 8, 2021

1 Q1

1.1 a

To show that the inverse g as defined in the problem is a function, we simply need to show that each unique input has a single output. This implies that our function f must be injective on \mathbb{R} , which is to say $f(a) = f(b) \iff a = b$.

We can assume that there exists $x_0, x_1 \in I$ such that $f(x_0) = f(x_1)$. Then by the Mean Value Theorem we know that there is a point $y \in (x_0, x_1)$ where $f'(y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = 0$. However the problem specifies that $f'(x) \neq 0$, therefore our assumption is incorrect and f must be injective to \mathbb{R} . Thus its inverse g exists.

1.2 b

We claim that f is monotone. Since $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists, we know that it satisfies the epsilon-delta property at a . Since f is differentiable, its derivative is defined on all I . Therefore each point in I also must satisfy the epsilon-delta property, and f' is therefore continuous. Now since it is continuous, if $f'(b) > 0$ and $f'(c) < 0$ for some $b, c \in I$, then by Intermediate Value Theorem there must be $d \in (b, c)$ such that $f'(d) = 0$. However we know that to be false, therefore f is monotone.

Let $\epsilon > 0, y_0 \in f(I), x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$. Without loss of generality assume that f is monotonically increasing. Therefore $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Now we can simply take $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + \epsilon) - f(x_0)\}$. Then for any $|y_1 - y_0| < \delta$, let $f(x_1) = y_1$, then $x_0 - \epsilon < x_1 < x_0 + \epsilon$ by monotonicity. Thus g is continuous.

If f is monotonically decreasing we can simply repeat the above argument but with the signs flipped since if $a > b$, now we will have $f(b) > f(a)$

2 Ross 30.2

2.1 a

$\sin(0) - 0 = 0$, so we attempt to use l'hospital's rule. Assume that the limit exists, then it must be equal to

$$\lim_{x \rightarrow 0} \frac{x^2}{\cos x} = \lim_{x \rightarrow 0} \frac{0}{1} = 0$$

Therefore the limit is 0

2.2 b

For this problem we need to use l'hoospital's rule 3 times

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \tan(x) \sec^2(x)}{6x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x (\sec^2(x) + 2 \tan^2(x))}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

We used the chain rule for the second step, both the chain rule and the product rule for the third step.

2.3 c

We combine these fractions, then apply l'hospital's rule twice:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x + \sin x - x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{0}{1} = 0 \end{aligned}$$

2.4 d

We know that if $\lim_{x \rightarrow a} f(x) = b$, $\lim_{x \rightarrow b} g(x) = c$, then $\lim_{x \rightarrow a} g(f(x)) = g(\lim_{x \rightarrow a} f(x))$. Assume that the limit does exist for our expression, and since the natural log is continuous, we can apply this theorem.

$$\ln(\lim_{x \rightarrow 0} \cos x^{1/x^2}) = \lim_{x \rightarrow 0} \ln(\cos x^{1/x^2}) = \lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2}$$

Now we can use l'hospital's Theorem

$$= \lim_{x \rightarrow 0} \frac{-\sin x / \cos x}{2x} = \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

Now to find $\lim_{x \rightarrow 0} \cos x^{1/x^2}$, we simply apply the inverse of the natural log:

$$e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}$$

3 Q3

3.1 a

Since $x_n \rightarrow \infty$, x_n get can arbitrarily large. More rigorously, for any $r \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that if $m > n$, $x_m > r$

Consider $y_n = \frac{1}{x_n}$. Let $\epsilon > 0$, let $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $n \in \mathbb{N}$ such that $x_n > \epsilon_0$, which we know exists as we have shown above. Now since $\epsilon_0 > 0$, we know that x_n, y_n are positive, so we have $|y_n| = |\frac{1}{x_n}| < |\frac{1}{\epsilon_0}| \leq \epsilon$

Thus we have shown that $|y_n|$ can get arbitrarily small, therefore $y_n \rightarrow 0$

■

3.2 b

Since $\lim_{x \rightarrow a} f(x) = \infty$, then for any $r \in \mathbb{R}$, there exists a $\delta > 0$ such that $|x - a| < \delta \implies f(x) > r$

Let $g(x) = \frac{1}{f(x)}$. We know that g is well defined since $f(x) \neq 0$ for $x \in (a, b)$.

Let $\epsilon > 0$, take $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $\delta > 0$ such that $f(a + \delta) > \epsilon_0$, which we know exists as we have shown above.

$|g(a + \delta)| < \frac{1}{\epsilon_0} \leq \epsilon$ We have shown that $|g(x)|$ gets arbitrarily small when x is close to a , therefore $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$

■

4 Q4

Let P be a partition such that $P = \{t_1 = a < t_2 < \dots < t_n = b\}$, furthermore let P be evenly spaced such that $t_i - t_{i-1}$ is equal for all $1 < i < n$. Let $M(s)$ denote the supremum of f in a set s , and $m(s)$ the infimum.

We find the upper and lower Darboux Sum. Since $f(x) = x$, if $x_0 > x_1$, $f(x_0) > f(x_1)$, so the infimum is at the lower bound of the interval and the supremum the upper bound.

$$U(f, P) = \sum_{i=1}^n M(s)(t_i - t_{i-1}) = \sum_{i=1}^n (t_i)(t_i - t_{i-1})$$

$$L(f, P) = \sum_{i=1}^n m(s)(t_i - t_{i-1}) = \sum_{i=1}^n (t_{i-1})(t_i - t_{i-1})$$

Since P is evenly spaced, we know that $t_i = a + (b - a)i/n$

Let $\epsilon > 0$, consider $U(f, P) - L(f, P)$, we can combine the sums to get

$$U(f, P) - L(f, P) = \sum_{i=1}^n (t_i - t_{i-1})(t_i - t_{i-1}) = \frac{(b - a)^2}{n}$$

Now for any $\epsilon > 0$ we simply need to choose $n > (b - a)^2/\epsilon$, which implies that $U(f, P) - L(f, P) < \epsilon$. Since it is less than any positive number and $U(f, P) \geq L(f, P)$, we have $U(f) = L(f)$ and the function is integrable.

In order to find $U(f)$ we need to substitute $U(f, P) = \sum_{i=1}^n (t_i)(t_i - t_{i-1}) = \sum_{i=1}^n (a + (b - a)i/n)((b - a)/n) = (b - a)/n \left(\frac{(a + (b - a)/n) + b}{2} n \right)$

The last step is using the sum of an arithmetic sequence, now we can tidy up to see that $U(f, P) = \frac{(b - a)(a + (b - a)/n) + b}{2}$. Since $U(f)$ is the infimum of the set $U(f, P)$ where P is a partition, and we know that this sum is minimized as $n \rightarrow \infty$ since the term $(b - a)/n \rightarrow 0$ as proven in the last problem. Thus we have found

$$U(f) = \lim_{n \rightarrow \infty} \frac{(b - a)(a + (b - a)/n) + b}{2} = \frac{(b - a)(b + a)}{2}$$

5 Ross 32.6

Since f is bounded, we know that U_n, L_n are finite. Therefore we have $\lim U_n - \lim L_n = 0$. Since we know that $U(f, P)$ where P is a partition has an infimum at $U(f)$ and similarly $L(f, P)$ has a supremum at $L(f)$. We have $\lim U_n = \lim L_n$, and that $U_n \geq L_n$ with the equivalence happening if and only if they are the infimum and the supremums and the function is integrable. We must have $\lim L_n = L(f)$ and $\lim U_n = U(f)$. Since $L(f) = U(f)$, the function is integrable. Thus we have

$$\int_a^n f = L(f) = U(f) = \lim L_n = \lim U_n$$

5.1 Q6

We essentially have to show that since this is a finite subset, we can create finitely many “thin rectangles” around each of these points. These rectangles have negligible width and converges to 0.

For any $s_n \in S$, for ease of notation let $f(s_n) = y_n$. Now we consider the upper and lower Darboux sums of this function.

We define our partition P as such: $P = \{a, s_n - \delta, s_n + \delta, b\}$ for each $s_n \in S$ with $\delta > 0$ (if $a \in S$ or $b \in S$, our first partitions become $\{a, a + \delta, \dots\}$ or our last partitions become $\{\dots, b - \delta, b\}$). Since S is finite, we can iterate through all the points in it with the above expression.

For the lower sum, since each partition contains some points that are not in S , the infimum for every partition is 0, and we have $U(f, P) = 0$

For the upper sum, each partition that doesn't contain a point in S has a supremum of 0, a partition $[s_n - \delta, s_n + \delta]$ has a supremum at s_n with a value of y_n . So we have

$$U(f, P) \leq 2\delta \sum_{i=1}^n y_i$$

If a or b is in S , the above inequality still holds since our interval would only be δ instead of 2δ and $\delta < 2\delta$

Let $z = \max\{y_1, y_1, \dots, y_n\}$. For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{4nz}$. Now consider our inequality:

$$U(f, P) \leq 2\delta \sum_{i=1}^n y_i \leq 2\delta nz = \frac{\epsilon}{2} < \epsilon$$

Thus we have shown that $U(f, P)$ can be less than any positive number, and we also know that $U(f, P) \geq L(f, P) = 0$ so we have $U(f, P) = L(f, P) = 0$. Thus the function is integrable.

Since this equivalency can only happen if $U(f, P) = U(f)$ and $L(f, P) = L(f)$, we have $\int_a^b f = 0$

6 Q7

Let P be an equi-distant partition of $[0, \pi/2]$. We compute the upper and lower Darboux sums:

$$U(f, P) = \sum_{i=1}^n M(s)(t_i - t_{i-1})$$

$$L(f, P) = \sum_{i=1}^n m(s)(t_i - t_{i-1})$$

Since we know that $\sin x$ is monotonically increasing in our domain, so $M([a, b]) = \sin(b)$, $m([a, b]) = \sin(a)$.

Since P is equi-distant, we have $t_i = a + (b - a)i/n$, so when we take $U(f, P) - L(f, P)$, we can get a clean telescopic series:

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n \sin(a + (b - a)\frac{i}{n})(b - a)/n - \sin(a + (b - a)\frac{i-1}{n})(b - a)/n \\ &= \frac{b - a}{n}(\sin b - \sin a) \end{aligned}$$

We know that $a = 0, b = \pi/2$, so we have $U(f, P) - L(f, P) = \frac{\pi}{2n}1$. For any $\epsilon > 0$, pick $n > \max\{\frac{\pi}{2\epsilon}, 1\}$. After partitioning $[0, \pi/2]$ into n equal partitions, the difference between our upper and lower sum is $\frac{\pi}{2n} < \frac{\pi}{\epsilon\pi} = \epsilon$. Thus we have shown that the difference between $U(f, P)$ and $L(f, P)$ can get arbitrarily small. Therefore the two converges to the same number, and $\sin x$ is integrable on our interval.