Homework 3

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1 Ross 8.2

1.1 a

Since $n^2+1 > n \forall n \in \mathbb{R}$, the fraction $\frac{n}{n^2+1} < 1$. Furthermore since $n \geq 0$ and $n^2 > 0$ the fraction is positive. So we have $0 < \frac{n}{n^2+1} < 1$. It is bounded.
Now we need to show that it is monotonically decreasing, and we will do this via induction.
Base case: $a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_2 < a_1$
Inductive case: assume that $a_n < a_{n-1}$, consider

$$a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

We take $a_{n+1} - a_n$

$$= \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$$
$$= \frac{(n+1)(n^2+1) - n((n+1)^2+1)}{(n^2+1)((n+1)^2+1)}$$
$$= \frac{n^3+n+n^2+n-n(n+1)^2-n}{(n^2+1)((n+1)^2+1)}$$
$$= \frac{n^3+n+n^2+n-n^3-n-2n^2-n}{(n^2+1)((n+1)^2+1)}$$
$$= \frac{-n^2}{(n^2+1)((n+1)^2+1)}$$

For $n \geq 1$, $-n^2 < 0, (n^2+1)((n+1)^2+1) > 0$, so the fraction is negative. $a_n > a_{n+1}$. We have proven the inductive case.
We have proven that the sequence is bounded and monotonically decreasing, therefore it converges. Q.E.D.

1.2 c

This sequence converges to $\frac{4}{7}$. We will try to show that $|c_n - \frac{4}{7}|$ is bounded and monotonically decreasing.
We first solve for $4n+3 = 7n-5$, and get $n = \frac{8}{3}$. This means that for $n > 8/3$, $7n-5 > 4n+3$, so $c_n < 1$. We have found an upperbound for this sequence.

Now let $n > \frac{5}{7}$. We can see that both the numerator and the denomator is greater than 0. So $c_n > 0$, we have found a lower bound for the sequence. Now we will show that it is monotonically decresing through induction. Let our base case be $n = 3$, $c_n = \frac{15}{16}, c_{n+1} = \frac{19}{23}, c_{n+1} < c_n$. Base case holds. Now we assume that the sequence is monotonically decreasing for all $k < n$, and let $c_n = \frac{4n+3}{7n-5}$. $c_{n+1} = \frac{4(n+1)+3}{7(n+1)-5}$

$$c_{n+1} - c_n = \frac{(4n+7)(7n-5) - (4n+3)(7n+2)}{(7n-5)(7n+2)} = \frac{28n^2 + 29n - 35 - 28n^2 - 29n - 6}{(7n-5)(7n+2)}$$

$$= \frac{-41}{(7n-5)(7n+2)}$$

1.3 e

We will try to show that this sequence is cauchy.