Math 104, HW7

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1 Q1

We first show that the limits exist $\implies$ uniformly continuous. We extend the definition of $f : [a, b]$ where $f(a) = \lim_{x\to a}, f(b) = \lim_{x\to b}$. Since the limit exists, we know that f can get arbitrily close it. More precisely, we know that there exists δ such that $|x - a| < \delta \implies |f(x) - \lim_{x\to a}| < \epsilon$ for any $\epsilon > 0$. This also demonstrates continuity at a. We can repeat this logic to show that it is continuous at b. Then, since f is continuous on the closed inverval and continuous, it is uniformly compact.
Now we show that uniformly continuous $\implies$ limits exist. Let $a_n \in (a, b)$ be an arbitrary sequence that converges to a. Then since it is convergent, we know that it is cauchy. Now since f is uniformly continuous, we know that $f(a_n)$ is also cauchy, and it is therefore convergent. Therefore the limit exists as $x \to a$. We can repeat this argument for b to show that $\lim_{x\to b} f(x)$ exists.
Thus it is proven.

2 Q2

forward statement Let $f : S \to S^*$ be continuous, let $E \subset S^*$ be a closed set, then $E' = S^* \setminus E$ is open and $f^{-1}(E')$ is also open by continuity.
Consider $f^{-1}(E')$, since f is defined on all S, each $\exists f(s) \forall s \in S$. Therefore all $s \in S$ either has an image in E or in E', and $f^{-1}(E') = S \setminus f^{-1}(E)$. Since we know that both S and $f^{-1}(E')$ are open, $f^{-1}(E)$ must be closed.

converse Now assume that for any closed $F \subset S^*$, $f^{-1}(F)$ is also closed. We essentially construct the same proof for open sets but with one more step. For any point $s_0 \in S$, $\epsilon > 0$, construct open set $G = \{d^*(f(s_0), s^*) < \epsilon\}$. Now take its complement $G' = S^* \setminus G$. Since S^* and G are both open, G' is closed.
By our assuption $H' = f^{-1}(G')$ is closed, then its complement $H = S \setminus H'$ is open. Since $f(s_0) \in G$, we have $s_0 \in H$. Then we can find a "ball" inside this set such that for some $\delta > 0$, $\{d(s - s_0) < \delta\} \subseteq H$
Thus it follows that $d(s - s_0) < \delta$ implies $d^*(f(s_0), s^*) < \epsilon$ and we have shown that f is continuous.

3 Q3

Let $C = \{(1/n, \infty)...\}$ for all $n \in \mathbb{N}$. We claim that C is a cover for $(0, \infty)$. To see that it is true, let $x \in (0, \infty)$, if $x > 1$, it is in every subset. Otherwise, by the Archimedean Principle there exists $m \in \mathbb{N}$ such that $1/m < x$. Therefore $x \in (1/m, \infty) \in C$.
Now we take $C \cup (-\infty, 0]$. Let $x \in \mathbb{R}$, if $x \leq 0$, then it is in the second set, otherwise it is in the first.
Let C' be a finite subset of C, then we must have $C' = \{(1/a, \infty), ..., (1/b, \infty)\}$ with natural numbers $a \leq b$. Now consider $r = 1/(2a)$. Since $0 < r < 1/a$, $r \notin C'$, and $r \notin (-\infty, 0]$
We can find a such r for any finite subset of C, therefore there is no finite subcover of $\mathbb{R}$. Thus $\mathbb{R}$ is not compact.

4 Q4

Consider $X = S \setminus F$. Since F is closed, its complement X is open. Now let C be any arbitrary open cover of F. We take the union of the above two sets: $C' = X \cup C$. Since C covers F and A covers the rest of the metric space, and since $E \subset S$, C' is an open cover of E.
Now we apply the definition of compactness, so there exists a finite subcover in C'. Let this subcover be Y. If $X \notin Y$, then our proof is complete since $F \subseteq E$. Any cover that covers E must also cover F.
Otherwise, we remove A: $Y \setminus X$. Since we have removed what we have added, $(Y \setminus X) \subseteq C$. This set is a subcover for F, since $X \cup F = \emptyset$, we are not removing any point that is inside F, thus we have found a subcover that covers F.

5 Q5