Math 104, HW10

Tianshuang (Ethan) Qiu

November 14, 2021

1 Q1: Ross 33.7

1.1 a

Let P be an arbitrary partition such that $P = \{a = t_1 < t_2 < ... < t_n = b\}$.
Now consider

$$U(f^2, P) - L(f^2, P) = (M[t_1, t_2] - m[t_1, t_2])(t_2 - t_1) + ... + (M[t_{n-1}, t_n] - m[t_{n-1}, t_n])(t_n - t_{n-1})$$

Since RHS and LHS has the same partition P, each t_k is also the same on the RHS. We consider just $P_1 = [t_1, t_2]$:
Let the $M(f^2, P_1) = f(x_0)^2, m(f^2, P_1) = f(x_1)^2$

$$U(f^2, P_1) - L(f^2, P_1) = (f(x_0)^2 - f(x_1)^2)(t_2 - t_1) = (f(x_0) + f(x_1))(f(x_0) - f(x_1))(t_2 - t_1)$$

Now consider the same partition for f:

$$U(f, P_1) - L(f, P_1) = (M(f, P_1) - m(f, P_1))(t_2 - t_1)$$

Since $B \geq f(x)$ for all $x \in [a, b]$, we have $2B \geq (f(x_0) + f(x_1))$. Since $M(f, P_1)$ is the maximum of the function over this interval and $m(f, P_1)$ is the minimum, their difference is greater than any other differences in this interval P_1, namely $M(f, P_1) - m(f, P_1) \geq (f(x_0) - f(x_1))$.
Therefore we have $U(f^2, P_1) - L(f^2, P_1) \leq 2B(U(f, P_1) - L(f, P_1))$. Now we can repeat this with all intervals of $[t_k, t_k - 1]$ where $2 \leq k \leq n$, thus we have shown that

$$U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P))$$

for any partition P

1.2 b

Since f is integrable, for any $\epsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$. Now for any $\epsilon > 0$, choose $\epsilon_0 = \epsilon \times 4B$ where B is the absolute bound for f, since f is integrable we find P_0 that the difference between the Darboux sums is less than ϵ_0
Now consider $U(f^2, P_0) - L(f^2, P_0)$, from part(a) we know that $U(f^2, P_0) - L(f^2, P_0) \leq 2B(U(f, P_0) - L(f, P_0)) \leq \frac{\epsilon}{2} < \epsilon$
Therefore f^2 is integrable.

2 Q1, Ross 33.8

By our theorem we know that the sum(difference) of two integrable functions is integrable. Therefore we know that $(f+g)$ and $(f-g)$ are integrable. By 33.7 we know that $(f+g)^2$, $(f-g)^2$ are integrable. Now we simply take the difference: $(f+g)^2-(f-g)^2 = 4fg$. We apply the integrability theorem again and we know that this is integrable as well. Thus fg must be integrable.

3 Q2

3.1 a

The function is continuous at decreasing intervals as $|x|$ decreases, it is also continuous at $x = 0$. Since $-1 \leq \text{sgn}(x) \leq 1$, $-x \leq f(x) \leq x$ therefore the function converges to 0 at $x = 0$ by squeeze theorem.
Now for any other point, the continuity breaks when $\sin \frac{1}{x}$ is 0 since the whole expression which was not 0 before suddenly "drops" or "rises" to 0. More rigorously, if $\sin \frac{1}{x} \neq 0$, then $\text{sgn}(x) = 1$ or -1, then $f(x) = x$ or $-x$, which does not have the value 0 unless $x = 0$. Therefore the function is discontinuous at all points where $\sin(\frac{1}{x}) = 0$, or $\frac{1}{x} = n\pi$ where n is an integer. Since it is continuous everywhere else, we have that the function is continuous on $[-1, -\frac{1}{\pi}), (-\frac{1}{\pi}, -\frac{1}{2\pi})...(\frac{1}{2\pi}, \frac{1}{\pi}), (\frac{1}{\pi}, 1]$

3.2 b

Even though f is not piecewise continuous on all of $[-1, 1]$, the discontinuities increase near 0. We claim that it is piecewise continuous on $[-1, 0)$ and $(0, 1]$. Let a_n be the sequence of postive discontinuous points: $a_n = \frac{1}{n\pi}$, and let $b_n = -a_n$. Since $0 < a_n < 1/n$ we know that it converges to 0, by similar logic so does b_n. Let $0 < x_0 \leq 1$, we know that between each a_n the function is either x or $-x$ which is uniformly continuous. Moreover, we know that since $a_n \to 0$, there is $n \in \mathbb{N}$ such that $a_n < x_0$. Let n_0 be the smallest such n. Let the first partition be $[x_0, a_{n_0-1}]$, and the last $[a_1, 1]$. The same is true if $-1 \leq x_1 < 0$, let n_1 be the smallest $n \in \mathbb{N}$ such that $a_n > x_1$. We define our first partiton as $[-1, a_1]$, and the last $[a_n, x_1]$, between each closed interval the function is uniformly continuous.
Now for any $\epsilon > 0$, choose $u = \sqrt{\epsilon}/4, v = -u$. As we have shown above the function is integrable in $[-1, v]$ and $[u, 1]$. Therefore we only need to consider the interval $[v, u]$. In this interval the greatest value f can take is u when $\text{sgn}(\sin(\frac{1}{x})) = 1$ and $f(x) = x$ Similarly the least value it can take is v when $f(x) = -x$

$$U(f, [v, u]) = u(u - v)$$

$$L(f, [v, u]) = v(u - v)$$

Thus $U(f, [v, u]) - L(f, [v, u]) = (u - v)^2 = \frac{\epsilon}{16} < \epsilon$. Therefore the function is integrable.

4 Ross 34.2

4.1 a

We first assumes that the function e^{t^2} is the derivative of a function $F(t)$, so by the Fundamental Theorem of Calculus, we simplify the expression into $\lim_{x\to 0} \frac{F(x)-F(0)}{x}$. Since the denominator approaches 0, we apply L'Hospital's rule and the limit is equal to $\lim_{x\to 0} \frac{F'(x)-F'(0)}{1} = \frac{e^{x^2}(x')-e^0(0')}{1} = 1$
In the last step, we needed to apply the chain rule and take the derivative of the function inputs, ending with $1-0=1$

4.2 b

We first assumes that the function e^{t^2} is the derivative of a function $F(t)$, so by the Fundamental Theorem of Calculus, we simplify the expression into $\lim_{x\to 0} \frac{F(3+h)-F(3)}{h}$. Since the denominator approaches 0, we apply L'Hospital's rule and the limit is equal to $\lim_{x\to 0} \frac{F'(3+h)-F'(3)}{1} = \frac{e^{(3+h)^2}(3+h)'-e^3(0')}{1} = e^9$

5 Q4

5.1 a

For $x < 0$, $F(x) = 0$ since $f(x) = 0$
For $0 \leq x \leq 1$, $F(x) = \frac{1}{2}x^2$ since we have proven the power rule and $\frac{1}{x}x^2$ has a derivative of x.
For $x > 1$, $F(x) = \frac{1}{2} + 4(x-1)$. The one half comes from $F(1) - F(0)$, and since the function is constant, the upper and lower Darboux sums will be the same for any partition.

5.2 b

F is continuous everywhere. In the intervals $x < 0, 0 < x < 1, x > 1$ we know F is continuous since their functions are continuous. At $x = 0$, let $\epsilon_0 > 0$, pick $\delta_0 = \sqrt{\epsilon_0}$ Let $|x - 0| < \delta$, then if $x < 0$, $|F(x) - F(0)| = 0$, otherwise $|F(x) - F(0)| < \epsilon_0/2 - 0 < \epsilon$, therefore F is continuous at 0
At $x = 1$, let $\epsilon_1 > 0$, pick $\delta_1 = \sqrt{\epsilon_0}$. If $x > 1$

5.3 c

F is differentiable at $(-\infty, 1)$. $F'(x)$ for $x < 0$ is 0 since it is constant. $F'(x)$ on $[0, 1]$ is x and since $F'(x) = 0$ on both negative and positive sides of $x = 0$, therefore F is differentiable at 0
For $x > 1$, $F'(x) = 4$. Therefore it is also differentiable at $(1, \infty)$. It is not differentiable at 1 since on the negative side it is 1, but on the positive it is 4

6 Ross 34.5

For each $x \in \mathbb{R}$, limit both F, f to $[x-1, x+2]$. Since f is continuous on this interval and F is its integral, F is differentiable at x by the Fundamental Theorem of Calculus.
Since the upper bound of the integral computed in F has an upper bound of $x+1$, by the same theorem we know that it is equal to $f(x+1)$.

7 Ross 34.7

Let J be the interval $(-\infty, \infty)$, $u : J \to \mathbb{R}$ is defined as $u = x^2$ and $u' = 2x$, and $f : \mathbb{R} \to \mathbb{R}$, $f = -\frac{2}{3}(1-a)^{3/2}$, by the power rule $f' = \sqrt{1-a}$. Now since $U(J) \subseteq \mathbb{R}$, we can apply u-subsitution here.
Let the integral equal I, we know that $2I = \int_0^1 2x\sqrt{1-x^2}dx = \int_0^1 u'(f' \circ u) = \int_{u(0)}^{u(1)} f' = f(1) - f(0) = \frac{2}{3}$
Now we have $2I = \frac{2}{3}$, so our original integral $I = \frac{1}{3}$

8 Ross 34.12

Assume that there exists some f where it is not 0 everywhere that satisfies this integrala. Since g is an arbitrary continuous function and f is continuous, let $g = f$. Now our integral becomes $\int_a^b f^2(x)dx = 0$ Now since $f^2(x) \geq 0$, we know that $f^2(x) = 0$ everywhere in our interval.
To see that it is true, we let $f^2(x_0) > 0$, then by continuity there exists $\delta > 0$ such that $|f^2(x) - f^2(x_0)| < f^2(x_0)/2$, then in this interval the integral is at least $\delta f^2(x_0)/2$, thus the whole integral must be greater than or equal to it. Since the expression is positive, there is no way for the integral to be 0. Thus x_0 does not exist and $f^2(x) = 0$ in our interval.
From our conclusion above it is simple to see that $f(x) = 0$ everywhere for $f^2(x) = 0$, therefore we have reached a contradiction, and $f(x) = 0$ for all $x \in [a, b]$