

Homework 3

Tianshuang (Ethan) Qiu

September 20, 2021

1 Ross 8.2

1.1 a

When $n > 0$, both the numerator and the denominator is greater than 0, so $a_n > 0 \forall n > 0$. Now consider $x_n = \frac{n}{n^2} = \frac{1}{n}$, the denominator is less than a_n , so $x_n > a_n \forall n > 0$.

Obviously 0 converges to 0, and we have proven in class that $\frac{1}{n}$ converges to 0 (pick $N = \frac{1}{\epsilon}$). $a_n \rightarrow 0$ by squeeze theorem.

1.2 c

This sequence converges to $\frac{4}{7}$. We will try to show that $x_n = |c_n - \frac{4}{7}| \rightarrow 0$.

$$\begin{aligned} |c_n - \frac{4}{7}| &= \left| \frac{28n + 21}{49n - 35} - \frac{4(7n - 5)}{49n - 35} \right| \\ &= \frac{28n + 21 - 28n + 20}{49n - 35} \\ &= \frac{41}{49n - 35} \end{aligned}$$

$$\lim x_n = 41 \times \lim \left(\frac{1}{49n - 35} \right)$$

Once again, we squeeze this sequence with 0 and $\frac{1}{n}$. Since the denominator is positive $x_n > 0 \forall n > 0$, and $49n - 35 > n \forall n > \frac{35}{48}$, so $x_n < \frac{1}{n}$ when n is large. Therefore $\frac{1}{49n - 35} \rightarrow 0$ by squeeze theorem.

Therefore $\lim x_n = 41 \times 0 = 0$, and $c_n \rightarrow \frac{4}{7}$

1.3 e

The sequence converges to 0. We define $x_n = \frac{1}{n}, y_n = \frac{-1}{n}$. Since $-1 \leq \sin n \leq 1$, $y_n \leq s_n \leq x_n$.

We have shown above that $\frac{1}{n} \rightarrow 0$. $\frac{-1}{n}$ can be shown with the same reasoning to converge $\forall n > \frac{1}{\epsilon}$. Therefore $s_n \rightarrow 0$ by squeeze theorem.

2 Q2

2.1 a

We will prove this via induction.

Base case: $n = 1$, $LHS = 1 + a$; $RHS = \frac{1-a^2}{1-a}$. $1 - a^2 = (1 + a)(1 - a)$, therefore $LHS = RHS$. Base case holds.

Inductive hypothesis, let the claim hold for an arbitrary $n \in \mathbb{N}$.

$$\begin{aligned} 1 + a + \dots + a^n + a^{n+1} &= \frac{1 - a^{n+1}}{1 - a} + a^{n+1} \\ &= \frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a} \\ &= \frac{1 - a^{n+2}}{1 - a} \end{aligned}$$

Thus we have proven the inductive hypothesis. Q.E.D.

2.2 b

Since $|s_{n+1} - s_n| < 1/2^n \forall n$, $|s_{n+2} - s_n| \leq 1/2^n + 1/2^{n+1}$ by the triangle inequality. We can further extend this so $|s_{n+k} - s_n| \leq 1/2^n + 1/2^{n+1} + \dots + 1/2^{n+k-1}$

Let $\epsilon > 0$, we pick $N = \max\{\log_2(1/\epsilon), 1\}$. Pick $j, k > N$.

$$\begin{aligned} |s_j - s_k| &\leq \sum_{i=j}^{j+k-1} \frac{1}{2^i} = \frac{1}{2^j} (1 + 1/2 + \dots + 1/2^{k-1}) \\ &= \frac{1}{2^j} (2(1 - 1/2^k)) = \frac{1 - 1/2^k}{2^{j-1}} \leq \frac{1}{2^j} \end{aligned}$$

Since $j > \log_2(1/\epsilon)$, $|s_j - s_k| \leq \frac{1}{2^j} < \epsilon$.

Therefore the sequence is cauchy. Q.E.D.

3 Ross 10.7

We essentially want to slowly pick items in S that is closer to $\sup S$.

Consider $x_n = \sup S - 1/n$. Since $x_n < \sup S$, $\exists s_n \in S$ s.t. $x_n \leq s_n \leq \sup S$. Both x_n and $\sup S$ converge to $\sup S$, therefore $s_n \rightarrow \sup S$ by squeeze theorem.

4 Q4

Since $s_n \rightarrow s$ by our assumption, let $\epsilon > 0, \exists N s.t. \forall n > N, |s_n - s| < \epsilon$.

Therefore $s - \epsilon < s_n < s + \epsilon$. Since s_n is between these two end points, and since the \liminf is non-decreasing, we have $\liminf s_n \geq s - \epsilon$.

Since ϵ can be arbitrarily small, we must have $\liminf s_n \geq s$. Q.E.D.

5 Q5

5.1 a

We will attempt to prove this via induction.

Base case: $n = 1, s_n = 2, 2^2 - 2 > 0$

Inductive case: assume that $s_n^2 > 2$, then

$$s_{n+1}^2 - 2 = \frac{s_n^2}{4} + \frac{1}{s_n^2} + 2$$

. Since $s_n^2 > 2, \frac{s_n^2}{4} > 0, \frac{1}{s_n^2} > 0$. So $s_{n+1}^2 > 2$.

$s_n^2 - 2 > 0 \forall n \in \mathbb{N}$

5.2 b

We first prove that $s_n > 0 \forall n \in \mathbb{N}$. $n = 1, s_n = 2, 2 > 0$.

In the inductive case, assume $s_n > 0$, so $s_n/2 > 0, 1/s_n > 0$, therefore $s_n n + 1 > 0$. We have proven the inductive case.

To prove that it is monotone will again use induction.

Base case: $n = 1, s_n = 2, s_{n+1} = 2/2 + 1/2 = 3/2, s_{n+1} < s_n$. The base case holds.

Inductive case: assume that $s_n < s_{n-1}$. Consider $s_n - s_{n+1} = s_n - \frac{s_n}{2} - \frac{1}{s_n}$

$$\frac{s_n}{2} - \frac{1}{s_n} = \frac{s_n^2 - 2}{2s_n}$$

We have shown that $s_n^2 - 2 > 0$ and that $s_n > 0$, so this fraction is positive, therefore $s_n > s_{n+1}$. This sequence is monotonically decreasing.

Furthermore, since $0 < s_n < 2$, the sequence is bounded above and below. Therefore it converges. Q.E.D.

6 Q6

6.1 a

$$\lim s_n = \sqrt{2}.$$

We have shown that the sequence converges, so it is also cauchy. Let $\epsilon > 0$, there exists N such that for $i > N$, $|s_i - s_{i+1}| < \epsilon$.

Since ϵ can get arbitrarily small and that it converges, we should have $\lim_{n \rightarrow \infty} s_n = \lim s_{n+1}$. Therefore $\lim s_n$ should satisfy

$$\lim s_n = \frac{\lim s_n}{2} + \frac{1}{\lim s_n}$$

$$(\lim s_n)^2 = \frac{\lim s_n^2}{2} + 1$$

$$\frac{1}{2}(\lim s_n)^2 = 1$$

$$(\lim s_n)^2 = 2$$

$$\lim s_n = \sqrt{2}$$

6.2 b

The sequence s_n has $s_n \in \mathbb{Q}$ because each term is the sum of two rationals. Since it is convergent, the sequence is cauchy, but $s_n \rightarrow \sqrt{2}$, which is an irrational.