Math 104, HW4

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1 Q1

1.1 a

Let x be an arbitrary point in $E = (0, 1)$. Choose $r = \min\{(1-x)/2, x/2\}$. Consider $S = \{d(s, x) < r\}$. Since $0 < x < 1$, $(1-x)/2$ and $x/2$ are both positive. Therefore $r > 0$, and since $x - x/2 > 0, x + (1-x)/2 < 1$, we have $S \subseteq E$
Therefore E is open.
Consider the complement of $E : E' = \mathbb{R} \setminus E$
Let $x' = 1, r' > 0$. Since E' is the complement of E, it is the union of $(-\infty, 0], [1, +\infty)$. If $r \geq 1$, we can see that $S' = \{d(s', x') < r'\}$ contains the point $1/2$ for instance, and $1/2 \notin E'$. Otherwise, let $a = x' - r'$, since $x' = 1, 0 < r' < 1, a \in S, a \notin E$. Therefore its complement is not open.
Thus we have shown that $(0, 1)$ is open and not closed.

1.2 b

Let $x = 1, r > 0, E = [0, 1]$. Consider $S = \{d(s, x) < r\}$. Since $r > 0, \exists s \in S s.t. s > x$. However since the interval only goes from 0 to 1, $s \notin E$. Therefore this interval is not open.
Consider $E' = \mathbb{R} \setminus E$.
Since E' is the complement of E, it is the union of $(-\infty, 0), (1, +\infty)$. If x is in the former, then pick $r' = -x/2$. Since $x < 0, -x > 0$, and $x + (-x/2) < 0$, so $S = \{d(s', x') < r'\} \subseteq E$.
If it is in the latter, pick $r' = (x-1)/2$. Since $x > 1$, and $x - (x-1)/2 > 1$, so $S = \{d(s', x') < r'\} \subseteq E$. Therefore its complement is open.
Thus we have shown that $[0, 1]$ is closed and not open.

1.3 c

Consider $x = 1$, let $r > 0$, we can consider this set to be a non-increasing series from 1 to 0. Let $S = \{d(s, x) < r\}$, now since $r > 0, \exists s' \in S s.t. 1/2 < s' < 1$, since this series is non-increasing, $s' \notin E$. Therefore the set is not open.
Consider the complement E'. Consider x'. If $x' > 1 or x' < 0$, we can choose r' exactly the same as part (b) of this question. We can see that the set with radius r' is a subset of E'.

If $0 < x' < 1$, we need to show that we can pick an r' small enough to have not let the "other" set in.
Since $x' \notin E$, and $0 < x' < 1$, then it must be "sanwiched" between two elements of E. Let the two around x' be $1/(n+1) < x' < 1/n$. Now we can apply the denseness of rationals theorem to show that $\exists q_1, q_2 s.t. 1/(n+1) < q_1 < x', x' < q_2 < 1/n$. Now let $r' = \min\{q_1, q_2\}$. We can see that all of the elements in this radius are in the set E'. Therefore its complement is open.
Thus we have shown that this set is closed and not open.

1.4 d

Let $x \in \mathbb{Q}, r > 0$, and $S = \{d(s, x) < r\}$. By the denseness of irrationals we know that $\exists a \notin \mathbb{Q} s.t. x < a < x + r$. Therefore $\mathbb{Q}$ is not open.
We can repeat the same argument but with irrationals. Let y be irrational, $r > 0$, and $S = \{d(s, y) < r\}$. By the denseness of ratioanls we know that $\exists b \in \mathbb{Q} s.t. y < b < y + r$. Therefore $\mathbb{Q}$'s complement is not open.
Therefore $\mathbb{Q}$ is neither open nor closed.

1.5 e

Let this set be E. Let $e \in E$ be an arbitrary point, and we choose $r = 1 - d(e, (0, 0))$. So we have our set $S = \{d(e, s) < r\}$. By the triangle inequality we have $d(s, (0, 0)) < 1 - r + r = 1$, so $s \in E \forall s \in S$. Therefore the set is open.
Let E's complement be called E', and let $x \in E'$ be a point such that $d(x, (0, 0)) = 1$. Let $r' > 0$, then consider the set $S' = \{d(x, s') < r'\}$. $\exists t \in S s.t. d(t, (0, 0)) < 1$. Then $t \in E$. Therefore its complement is not open.
Therefore this set is open and not closed.

2 Q2

2.1 a

Let $a \in U$ be an arbitrary point. Since U is a union of a collection of open sets, then it must belong to at least one element of this collection. Let that element be U_0.
Since U_0 is open, $\exists r > 0 s.t. S = \{s \in S | d(a, s) < r\} \subseteq U_0$. Therefore we have found an r that works for an arbitrary point in U. Thus U is open. Q.E.D.

2.2 b

Consider $V_0 = U_1 \cap U_2$.
From the intersection, we conclude that for all $v \in V_0, v \in U_1, v \in U_2$. Now consider an arbitrary point $w \in V$. Since it is in open sets U_1, U_2, $\exists r_1, r_2 s.t. \{d(w, v) < r_1\} \subset U_1, \{d(w, v) < r_2\} \subset U_2$
Now let $r = \min\{r_1, r_2\}$. Since r is the smaller of the two, $A = \{d(w, a) < r\} \subset V_0, \subset V_1$. Therefore $A \subset V_0$. Thus we have shown that V_0 is open.
We can then repeat this process finitely many times, taking the minimum of the radius each time. Finally we have that V is open. Q.E.D.

2.3 c

Consider $W = \cap_{n=0}^{\infty}(1/n, -1/n)$. Since $1/n \to 0$ and $-1/n \to 0$, $W = \{0\}$. This set has only 1 element and is therefore closed. Q.E.D.

3 Q3

Let $\epsilon > 0$, since $s_n \to s$, we have $\exists N s.t. \forall n > N, d(s_n, s) < \epsilon$. Now let $r = \epsilon$, we can see that $\exists n s.t. d(s_n, s) < r$.
Consider the complement of $E : F$. Consider the point s, since $s \notin E$, we have $s \in F$. Let $r' > 0$, define $Q = d(s, q) < r'$. Since we have shown above that $\exists n s.t. d(s_n, s) < r$ for $r > 0$, we know that Q will always overlap with E. Therefore we cannot find a radius small enough, and F is not open. Thus E is not closed. Q.E.D.

4 Q4

Since E is not closed, its complement F is not open. Let s be a boundary point in F: $s \in F s.t. \{p | d(s,p) < r\} \not\subset F \forall r > 0$
Now let e be an arbitrary point in E. Consider the sequence $s_n \in \{s \mid a \in E, d(a,s) < \frac{1}{n}\}$. We are attempting to draw "smaller and smaller" circles. Since we have shown above that $\exists p \in E s.t. d(s,p) < r \forall r > 0$, so we know that we can always pick an s_n that is closer to f. Now, since $1/n \to 0$, we know that $d(s_n, s) \to 0$, and therefore $s_n \to s, s \notin E$. Q.E.D.

5 Q5

Assume that there exists a sequence s_n that converges to $s \notin F$.
First, s must be in E. Since $F \subseteq E, \forall f \in F, f \in E$. Furthermore, E is sequentially compact, so every subsequence converges to an element in E. Therefore s_n cannot converge to an element outside of F, so for the sake of contradiction we assume that it converges to $s \in E$.
Let $\epsilon > 0$, then by our assumption there exists N such that $\forall n > N, d(s_n, s) < \epsilon$. Now since F is closed, we know that its complement is open. Let $F' = S \setminus F$. Since F is open, for any point $a \in F, \exists r > 0 s.t. \{b \mid d(a, b) < r\} \subseteq F'$. Now we let the region surrounding our convergent point s have value $r = k$, and we choose $\epsilon = k/2$. Since $r > 0, k/2 > 0$. Now for all $d(p, s) < k$, $p \in F'$. This is a contradiction since if our sequence converges to s, it must be able to get arbitrarily close, but we have just created a space where s_n cannot approach s. $\lightning$
Therefore our assumption is incorrect and $s_n \to s$ must have $s \in F$.

6 Q6

By the definition of lim sup we have

$$\lim_{N\to\infty}\{\sup\{\frac{a_{n+1}}{a_n} \mid n > N\} < C$$

Assume that the statement is not correct, so we have: for all $N \in \mathbb{N}, a_n \geq c^{n-N}a_N$
Let $k \in \mathbb{N}$, between $n \leq k \leq N$, there must be at least 1 value such that $(a_{k+1}/a_k) \geq C$ because otherwise, $a_n < c^{n-N}a_N$, and we have assumed that to be false. Now since we can show that there is at least 1 instance where $(a_{k+1}/a_k) \geq C$ for all possible intervals of $n, N \in N$, the limit superior cannot be less than C. $\lightning$
We have found a contradiction, therefore our assumption is not correct and there must exist some $N \in \mathbb{N}$ that satisfies $a_n < c^{n-N}a_N \forall n > N$. Q.E.D.

7 Q7

7.1 a

Lemma: $\limsup a + b \leq \limsup a + \limsup b$
Since $k^2 \neq 0 \forall k > 0$, $a_k \neq 0$. Consider a_{k+1}/a_k :

$$\frac{(k+1)^2}{3^{k+1}} \times \frac{3^k}{k^2} = \frac{(k+1)^2}{3k^2}$$

$(k+1)^2 < 2k^2$ for all $k > 3$, we have

$$\limsup \frac{(k+1)^2}{3k^2} < \limsup \frac{2k^2}{3k^2} = \frac{2}{3}$$

Therefore the ratio of the limit superior is less than 1, the sequence converges absolutely. Q.E.D.

7.2 b

$a_k = k^2/3^k$, define $b_k = 3^k/3^k$. Since $k^2 < 3^k$ for all $k > 2$, we have $a_k < b_k \forall k > 2$.
Now consider the nth root: taking the nth root does not change the order when both numbers are positive, so we have $\limsup a_k^{1/k} < \limsup b_k^{1/k} = 1$ since b_k is constant.
Thus we have $\limsup a_n < 1$. The sequence converges by root test. Q.E.D.

7.3 c

Consider $c_k = 2^k/3^k$. $2^k \geq k^2 \forall k \geq 2$. Furthermore, our sequence $a_k = k^2/3^k$ is positive. So we have $|a_k| \leq c_k$. c_k converges by geometric series with $r = 2/3 < 1$. Therefore a_k converges by limit comparison test.