Math 104, HW12

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1 Q1

1.1 a

First we know that $\sqrt{x^2} = |x|$. Since we know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$, we can show uniform convergence by the following.
Let $\epsilon > 0$, pick N such that for all $n > N$, $|\frac{1}{n}| < \epsilon^2$. Now since $|x| < 1$ and the domain of the square root being positive,

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2 + \frac{2}{\sqrt{n}}|x| + \frac{1}{n}} \leq \sqrt{(x + \frac{1}{\sqrt{n}})^2}$$

By our first statement the above expression is equal to $|(x + \frac{1}{\sqrt{n}})|$. By our definition of N,

$$||(x + \frac{1}{\sqrt{n}})| - |x|| \leq |x + \frac{1}{\sqrt{n}} - x| = |\frac{1}{\sqrt{n}}| < \epsilon$$

Thus we have $f_n \to |x|$ uniformly.

1.2 b

$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, and by the power rule we know that

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$$

1.3 c

Define $g : (-1, 1) \to \mathbb{R}$, $g(x) = -1$ for $x \in (-1, 0)$, $g(x) = 1$ for $x \in (0, 1)$, $g(0) = 0$
For any $x < 0$, let $a_n = \frac{1}{f_n'(x)} = \frac{\sqrt{x^2+\frac{1}{n}}}{x}$. By part a we know that $\sqrt{x^2 + \frac{1}{n}} \to |x|$ uniformly. Let $\epsilon > 0$ we pick N such that $|\sqrt{x^2 + \frac{1}{n}} - |x|| < \epsilon x - x$, therefore

$$|a_n - (-1)| = \frac{\sqrt{x^2 + \frac{1}{n}} + x}{x} = \frac{|\sqrt{x^2 + \frac{1}{n}} - |x||}{x} < \frac{\epsilon x}{x} = \epsilon$$

If $x > 0$, the same is true because the denominator is now positive and the absolute value sign should be flipped. Finally, if $x = 0$, $f_n'(x) = g(x) = 0$. Thus $f_n'(x) \to g(x)$ pointwise.
Since $f_n'(x)$ is a polynomial divided by a non-zero polynomial, it is continuous for all values of n and for all values of $x \in (-1, 1)$. However $g(x)$ is not continuous at 0. By our theorem about uniform convergence of continuous functions, we know that $f_n'(x) \not\to g(x)$ uniformly.

2 Q2

For any $x \in (-1, 1)$, shrink the domain of our power series to $[(x+1)/2, (1-x)/2]$. Now since both of these end points are within our radius of convergence, $\sum_{n=0}^{\infty} x^n \to \frac{1}{1-x}$ uniformly.
Then, we can take the derivative of both sides:

$$(\sum_{n=0}^{\infty} x^n \to \frac{1}{1-x}) = \sum_{n=1}^{\infty} nx^n$$

$$(\frac{1}{1-x})' = \frac{x}{(1-x)^2}$$

Thus $\sum_{n=1}^{\infty} nx^n \to \frac{x}{(1-x)^2}$

3 Q3

Let $x \in \mathbb{R}$, since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

$$(e^x)' = \sum_{j=1}^{\infty} \frac{jx^{j-1}}{j!}$$

However, since $\frac{jx^{j-1}}{j!} = \frac{x^{j-1}}{(j-1)!}$, since j begins at 1 and k at 0, each term can be matched bijectively to the first sum.
Thus they are the same sum. $\blacksquare$

4 Q4

Since $f(x) = e^{-x^2}$, by the last problem we know that it is equal to the sum of $\sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$
Now we integrate our power series term by term.

$$\int_0^y \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(1+2k)k!} y^{1+2k}$$

Let f_n be the above series, so $f_n' \to e^{x^2}$, therefore $f_n \to \int_0^x e^{x^2}$.

5 Q5

Since $f'(0)$ does not exist, the Taylor Polynomial of degree $n \geq 1$ does not exist. Thus the Taylor Series is 0, and we can let $\epsilon = 0.1$, at 0.5, $|f(x) - 0| = 0.4 > \epsilon$. Thus the Taylor Series does not converge to f, and therefore there is no power series that converge.

6 Q6

Base case: $n = 1$. We apply the chain rule:

$$(e^{\frac{1}{x^2}})' = e^{\frac{1}{x^2}}\frac{-2}{x^3}$$

Finally, let $a_{1,k} = 0$ for all $k \neq 3$, and $a_{1,3} = -2$, and our base case holds.
Inductive Case: let our formula hold for all $f^m(x)(m \leq n)$ for some $n \in \mathbb{N}$, then consider $f^{n+1}(x)$. We apply the product rule:

$$f^{n+1}(x) = (e^{\frac{1}{x^2}})'(\sum_{k=1}^{3n}\frac{a_{n,k}}{x^k}) + (e^{\frac{1}{x^2}})(\sum_{k=1}^{3n}\frac{a_{n,k}}{x^k})'$$

$$= e^{\frac{1}{x^2}}\frac{-2}{x^3}(\sum_{k=1}^{3n}\frac{a_{n,k}}{x^k}) + e^{\frac{1}{x^2}}(\sum_{k=1}^{3n}\frac{-ka_{n,k}}{x^{k+1}}) = e^{\frac{1}{x^2}}(\sum_{k=1}^{3n}\frac{-2a_{n,k}}{x^{k+3}}) + e^{\frac{1}{x^2}}(\sum_{k=1}^{3n}\frac{-ka_{n,k}}{x^{k+1}})$$

By factoring out $e^{\frac{1}{x^2}}$ we can see that the numerator contains constants, which is under the scope of $a_{n,k}$, the denominator has greatest possible degree of x^{k+3}. Since our sum can now go to $3(n+1) = 3n+3$, which is 3 more than n, the denominator can also be covered by our formula.
Thus we have proven the inductive case, and the proof is complete.

7 Q7

7.1 a

$\lim_{x\to 0} x^k = 0$, and $\lim_{x\to 0} \frac{1}{x^2} = \infty$ so $\lim_{x\to 0} e^{1/x^2} = 0$. Furthermore, both functions are differentiable at 0, thus we may apply l'hospital's rule to this problem. Let $y = \frac{1}{x}$, so if k is even,

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = \lim_{y\to\infty} e^{-y^2} y^k$$

Otherwise if k is odd, it is equal to $\lim_{y\to-\infty} e^{-y^2} y^k$
We consider the even case first: $\ln \lim_{y\to\infty} e^{-y^2} y^k$

$$= \lim_{y\to\infty} e^{-y^2} y^k = \lim_{y\to\infty} \frac{y^k}{e^{y^2}} = \lim_{y\to\infty} \frac{ky^{k-1}}{2ye^{y^2}}$$

We can repeat this process k times so that the numerator contains only constant terms, the denominator will at least have a e^{y^2} term, so the limit is 0.
For the odd case, the process is identical, after k iterations of L'Hospital's rule our expression contains only constants in the numerator, and the limit is 0.

7.2 b

In Q6, we have shown that $f^n(x) = e^{-y^2} \sum_{k=1}^{3n} \frac{a_{n,k}}{x^k}$, so we can break it down into each component of the sum. Now for each component consider $e^{\frac{1}{x^2}} \frac{a_{n,k}}{x^k} \frac{1}{x}$ where k is an integer between 1 and $3n$, then we can combine it into

$$e^{\frac{1}{x^2}} \frac{a_{n,k}}{x^{k+1}}$$

By part a and the fact that $a_{n,k}$ is a constant, the above expression converges to 0 as $x \to 0$. Therefore each term of the summation converges to 0 in our final expression, and $\lim_{x\to 0} \frac{f^n(x)}{x} = 0$

7.3 c

Base case: $f^0(0) = 0$. This is proven by our definition of f: $f(0) = 0$
Inductive case: assume that $f^n(0) = 0$ for some $n \geq 0$, then consider $f^{n+1}(0)$.

By the definition of the derivative:

$$f^{n+1}(0) = \lim_{x \to 0} \frac{f^n(x) - f^n(0)}{x - 0}$$

By our inductive hypothesis $f^n(0) = 0$, so the limit becomes $\lim_{x \to 0} \frac{f^n(x)}{x}$, which is known to be 0 from part b.

7.4 d

Since we know that the the derivative of f at 0 is 0 for all orders n, then the Taylor series for f centered at 0 is simply 0. It converges to f only at 0 since $f(0) = 0$ and the difference is 0.
For any other point $e^{\frac{1}{x^2}} \neq 0$, and we can pick $\epsilon = e^{\frac{1}{x^2}}/2$ to see that our Tyalor series does not converge to any $x \neq 0$