Homework 1

Tianshuang (Ethan) Qiu

September 6, 2021

1 Ross 1.4

1.1 a

n = 1, 1 = 1
n = 2, 1 + 3 = 4
n = 3, 1 + 3 + 5 = 9
n = 4, 1 + 3 + 5 + 7 = 16
Conjecture: $\sum_{i=1}^{n} 2i - 1 = n^2$

1.2 b

Base case: n = 1. $\sum_{i=1}^{1} 2i - 1 = 1$, base case holds.
Inductive hypothesis: let n $\in \mathbb{N}$, and $\sum_{i=1}^{n} 2i - 1 = n^2$
Inductive proof: consider n+1, $\sum_{i=1}^{n+1} 2i - 1 = (\sum_{i=1}^{n} 2i - 1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2$.
Thus we have proven that for n+1, the sum of the sequence is also $(n+1)^2$, and our conjecture holds.
Q.E.D.

2 Ross 1.12

2.1 a

$(a + b) = a + b$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
The theorem holds for all the above cases.

2.2 b

We will prove this with a combinatorics proof.
$\binom{n}{k}$ calculates the number of ways to choose k elements from a set with n. Likewise, $\binom{n}{k-1}$ gives the ways to choose $k - 1$ from n.
When we evaluate $\binom{n+1}{k}$, we can line up the $n + 1$ elements in a row. To choose the k elements, we can either choose the first element or not. If we choose the first element, there are now n elements left and $k - 1$ elements to choose; if we do not, there are n elements left and still $k - 1$ to choose. It is precisely the two elements above: $\binom{n}{k} + \binom{n}{k-1}$.
Q.E.D.

2.3 c

Base case: let $n = 1$, $(a + b)^1 = a + b$, base case holds.
Inductive hypothesis: assume that the hypothesis holds for some $n \in \mathbb{N}$
Inductive proof: $(a + b)^{n+1} = (a + b)(a + b)^n$, therefore by the inductive hypothesis, it is equal to

$$(a+b)(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n)$$

$$= a(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n) + b(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n)$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + ... + \binom{n}{n}ab^n + \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + ... + \binom{n}{n}b^{n+1}$$

Now we group like terms:

$$= \binom{n}{0}a^{n+1} + (\binom{n}{1} + \binom{n}{0})a^n b + ... + (\binom{n}{n} + \binom{n}{n-1})ab^n + \binom{n}{n}b^{n+1}$$

Using part(b) and that $\binom{n}{0} = \binom{n+1}{0} = 1$ and $\binom{n}{n} = \binom{n+1}{n+1} = 1$:

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i$$

Thus we have proven the inductive case.
Q.E.D.

3 Q3

For this problem we pick intermediary ab.
$\because a \leq b, b \geq 0, \therefore ab \leq bb = b^2$. By the same logic, we can multiply both sides by a and get $a^2 = aa \leq ab$
Since $a^2 \leq ab \leq b^2$, $a^2 \leq b^2$ must hold according to the order field axiom.
To prove the converse, we will take its contrapositive: $a > b \implies a^2 > b^2$. We can repeat the logic as above, using ab as the intermediary. We arrive at the conclusion that $a^2 > ab > b^2$, proving our statement.
Thus we have proved the statement and its converse. $a \leq b \iff a^2 \leq b^2$ $(a, b \geq 0)$.
Q.E.D.

4 Ross 4.7

4.1 a

Since the infimum is a member of the set of lowerbounds for any set, and the supremum is a member of the set of upperbounds, we have $\inf S \leq s(\forall s \in S)$, and $\sup S \geq s(\forall s \in S)$. Therefore by the ordered field axiom, $\inf S \leq \sup S$ for any set.
Thus we have proven $\inf T \leq \sup T$ and $\inf S \leq \sup S$. Now we show that $\inf T \leq \inf S$.
Suppose that the above statement is false, so $\inf S > \inf T$. Then consider k = $(\inf S + \inf T)/2$. Since $\inf S > \inf T, k < \inf S$. From the definition of infimum, $k < s(\forall s \in S)$. Furthermore, since $k > \inf T$, $\exists t \in T$ such that $t > k$. However, $S \subseteq T$, so every element of S is in T. It is impossible for an element to exist in T but not in S. Therefore our assumption is incorrect. We conclude that $\inf T \leq \inf S$. We can repeat the same argument symmetrically for the supremum and show that $\sup T \geq \sup S$. Then by the ordered field axioms we can arrive at the conclusion $\inf T \leq \inf S \leq \sup S \leq \sup T$. Q.E.D.

4.2 b

Let $x = \sup S, y = \sup T, Z = S \cup T$. Furthermore, let $x \geq y$ (switch S, T if $x < y$). To show that $x = \sup Z$, we need to show that $\forall z \in Z, x \geq z$, and that x is the minimum of all upperbounds of Z.
To show the first part, we assume that the statement is false. $\exists z \in Z s.t. z > x$. Since Z is the union of S and T , all the elements inside must be from S or T . $x = \sup S, y = \sup T$, and since $x \geq y$, x is an upperbound for both S and T . This is a contradiction to our assumption that $\exists z \in Z s.t. z > x$. Therefore our assumption is incorrect and x must be an upperbound for Z.

To second part, we also assume that it is false. $\exists a < x, s.t. a \geq z(\forall z \in Z)$. Consider $b_0 = \frac{a+x}{2}$ Firstly we can see that $b_0 > a$ since $a < x$. Secondly, if $b_0 \in S$, our proof is complete since $b_0 > a$ and $b_0 \in S$, therefore $b_0 \in Z$. We have found an element in the superset that is greater than our assumed supremum. This is a contradiction.
If $b_0 \notin S$, then it must be smaller than the infimum of S since S is a subset of $\mathbb{R}$. In this case $b_0 < s \forall s \in S$. Once again we have found an element in

the superset that is greater than the supremum, a contradiction.
From both contradictions we can see that $\sup(S \cup T) = \max\{\sup S, \sup T\}$
Q.E.D.

5 Ross 4.8

5.1 a

We pick an arbitrary $s \in S$. According to the specifications of the problem, this $s \leq t \forall t \in T$. Therefore this s is a lower bound of the set T, it is bounded below.
We can repeat the same logic symetrically and pick $t \in T$ to show that it is greater than or equal to every element of S. So S is bounded above.

5.2 b

We will prove this via contradiction. Assume that the statement is false, $\sup S > \inf T$. By the definition of the supremum, $\exists s \in S s.t. s > \inf T$. If T has no supremum or if the supremum is greater than or equal to s, $\exists t \in T s.t. t < s$. Otherwise, $s \geq t \forall t \in T$. Either way, we have found an element in each set that contradicts the prerequisites of this problem.
Our assumption is false and $\sup S \leq \inf T$.

5.3 c

Let $S = T = \{0\}$. Since they are the same set, it satisfies that $s \leq t \forall s \in S$ and $t \in T$. $S \cap T = \{0\}$, a non-empty set.

5.4 d

Let $S = s \in \mathbb{R} | 0 \leq s < 5, T = t \in \mathbb{R} | 5 < t < 10$. This satisfies that $s \leq t \forall s \in S and t \in T$ and $\sup S = \inf T$. However, since the ends at 5 for the two sets are open, they have no overlap. $S \cap T = \{\}$

6 Ross 4.11

For this problem we simply need to replace the 1 in the denseness proof with an arbitrary n.

Let $a, b \in \mathbb{R}, a < b, c \in \mathbb{N}$. By the Archimedean property there exists $n \in \mathbb{N}$ such that $n(b - a) > c$. Therefore $bn - an > c$. Furthermore, by the same property there is an integer k such that $k > \max |an|, |bn|$. Therefore $-k < an < bn < k$.

Then consider the set $J = j \in \mathbb{Z}, -k \leq j \leq k, K = k \in K, k > an$. This set is a subset of integers, bounded above and below, and non empty (contains at least k). Let $m_1 = \min K$. Then $-k < an < m_1$. Since $m_1 > -k$, m_1 is in J. $an > m_1 - 1$ by our choice of m_1. $m_1 - 1 \leq an$, $m_1 \leq an + 1 \leq bn$. Therefore $an < m_1 < bn$. We can simply let $m_2 = m_1 + 1$. Since $bn - an > c$, we can keep adding 1 to m_1 until $c - 1$. Furthermore, we can pick c to be arbitrarily large, so we can add 1 arbitrarily many times. Therefore it is infinite.

Q.E.D.

7 Q7

7.1 a

We assume that this is false, so $r^2 < 2$ or $r^2 > 2$. For the former we can let $x^2 = \frac{2+r^2}{2}$. This x^2 is greater than r^2 and less than 2, so x must be greater than r by problem 3. By the denseness of rationals, there must be a rational between x and $\sqrt{2}$. This rational is in the set and greater than the supremum. It is a contradiction, so $r^2 \geq 2$.
If $r^2 > 2$, we can simply change our argument above symetrically. Let $y^2 = \frac{2+r^2}{2}$. This y^2 is greater than 2 but less than r^2. By problem 3 y must be smaller than r. Since y^2 is greater than 2, it is greater than every element in S. We have found a smaller upperbound than the supremum. This is a contradiction, therefore $r^2 = 2$.

7.2 b

In this set, consider $a = 3$. $a > s \forall s \in S$. Since s is bounded above, it must have an supremum by the Completeness Axiom. As we have proven above, $r^2 = 2$ must exist.

7.3 c

To prove this, we need to demonstrate that $r^2 = 2, r \notin \mathbb{Q}$.
Assume that $r \in \mathbb{Q}$ and that $r = \frac{p}{q} | p, q \in \mathbb{N}, \gcd(p, q)$. Since $r^2 = 2$, $2 = \frac{p^2}{q^2}$

$$p^2 = 2q^2$$

From this we can see that p^2 is even since it is equal to 2 times q^2. For p^2 to be even, p must also be even. So we can write $p = 2k (k \in \mathbb{Z})$, $r = \frac{2k}{q}$

$$r^2 = 2 = \frac{4k^2}{q^2}$$

$$2q^2 = 4k^2$$

$$q^2 = 2k^2$$

Here we see that q^2 is also even, so q must be even. However, we have assumed that $\gcd(p, q) = 1$. This is a contradiction, so $r \notin \mathbb{Q}$.
Therefore the Completeness Axiom does not hold for $\mathbb{Q}$.