Homework 4

Due Tuesday, September 28 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. For a-d, the metric space is $\mathbb{R}$. For e the metric space is $\mathbb{R}^2$. Prove that
   a) $(0,1)$ is open and not closed.
   b) $[0,1]$ is closed and not open.
   c) $\{0\} \cup \{1, 1/2, 1/3, ..., 1/n, ...\}$ is closed and not open.
   d) $\mathbb{Q}$ is neither open nor closed.
   e) $\{x \in \mathbb{R}^2 | d(x,(0,0)) < 1\} \subset \mathbb{R}^2$ is open and not closed.
2. Let S be a metric space with distance function d.
   a) Let U be the union of a collection of open sets. Prove that U is open.
   (in more detail: Let A be some set, finite or infinite. Suppose that for each $\alpha \in A$, there exists an open subset of S called U_α. Show that
   $$U = \bigcup_{\alpha \in A} U_\alpha = \{s \in S | s \in U_\alpha \text{ for some } \alpha \in A\}$$
   is open.)
   b) Let $U_1, ..., U_k$ be a finite collection of open sets. Prove that $V = U_1 \cap U_2 \cap ... \cap U_k$ is open.
   c) Find an infinite collection of sets $U_1, U_2, ..., U_n, ...$, each $U_n \subset \mathbb{R}$, such that the intersection of all of the U_n is NOT open.

   (In Problems 3 and 4, you will prove the following theorem: a set $E \subset S$ is closed if and only if every convergent sequence in E converges to an element of E. Therefore, do not use that theorem in problems 3 and 4. Use only the definition of a closed set as the complement of an open set.)
3. Let S be a metric space with distance function d. Let $E \subset S$ and let (s_n) be a sequence in E. Assume that $s_n \to s$ and $s \notin E$. Prove that E is not closed.
   (Hint: Show that for every $r > 0$, there exists an n such that $d(s_n, s) < r$.)
4. Let S be a metric space with distance function d. Assume that $E \subseteq S$ is not closed. Prove that there exists a sequence (s_n) in E such that $s_n \to s$ and $s \notin E$.
   (Hint: Prove that there exists $s \in S \setminus E$ which violates the definition of an open set. Use that violation to construct a sequence converging to s.)
5. Let S be a metric space with distance function d. Let E be a sequentially compact subset of S and let F be a closed subset of S such that $F \subseteq E$. Prove that F is sequentially compact.
6. Let $a_n \in \mathbb{R}$, $a_n > 0$ for all $n \in \mathbb{N}$. Consider the sequence $\left(\frac{a_{n+1}}{a_n}\right)$. Let $C \in \mathbb{R}$ such that
   $$\limsup \frac{a_{n+1}}{a_n} < C.$$
   Prove that there exists $N \in \mathbb{N}$ such that for $n > N$,
   $$a_n < C^{n-N} a_N.$$

   (see next page for problem 7)

7. Consider the series $\sum \frac{k^2}{3^k}$.

a) Use the ratio test to show the series converges.

b) Use the root test to show the series converges. (You can use the fact that $\lim n^{1/n} = 1$ without proof. See Ross Theorem 9.7)

c) Use the limit comparison test to show the series converges (Hint: Compare to a geometric series. You might need to consider only large k.)