Math 104, HW5

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1 Ross 14.1

1.0.1 a

For this we will use the limit comparison test. $n^4 < (3/2)^n$ for large n, so let us only consider $n > 100$. Now we know that $\sum(\frac{1.5}{2})^n$ converges due to geometric series identity.
$\sum \frac{n^4}{2^n} < \sum(\frac{1.5}{2})^n$ for large n, and since tha latter converges, the former converges as well. Q.E.D.

1.1 b

For this we will use the limit comparison test. $3^n < n!$ for large n, so let us only consider $n > 10$. We know that $\sum(\frac{2}{3})^n$ converges due to geometric series identity.
$\sum \frac{2^n}{n!} < \sum(\frac{2}{3})^n$ for large n, since a larger denominator means a smaller value. Since tha latter converges, the former converges as well. Q.E.D.

1.2 c

For this we will use the limit comparison test. $n^2 < 2^n$ for $n > 2$. We know that $\sum(\frac{2}{3})^n$ converges due to geometric series identity.
$\sum \frac{n^2}{3^n} < \sum(\frac{2}{3})^n$ for large n. Since tha latter converges, the former converges as well. Q.E.D.

1.3 d

We apply the ratio test to $\sum \frac{n!}{n^4}$:

$$\limsup |\frac{(n+1)!}{(n+1)^4}\frac{n^4}{n!}| = n + 1 > 1$$

Therefore it does not converge by ratio test.

1.4 e

Since $cos^2 n \leq 1, \sum \frac{cos^2 n}{n^2} \leq \sum \frac{1}{n^2}$
Now we apply the root test to $\frac{1}{n^2}$:

$$|\frac{1}{n^2}|^{1/n} \leq 1^{1/n} = 1$$

Therefore $\sum \frac{1}{n^2}$ converges by root test, and $\sum \frac{cos^2 n}{n^2}$ converges by limit comparison.

1.5 f

We compare this to $\sum \frac{1}{n}$ where $n \geq 2$. $\log n < n$, so $1/\log n < 1/n$. Since we know that $\sum \frac{1}{n}$ is the harmonic series and does not converge, $\sum \frac{1}{\log n}$ also does not converge by limit comparison.

2 Ross 14.12

Since $\liminf |a_n| = 0$, for all $\epsilon > 0$, there exists $|a_n| \leq \epsilon$.
Pick our subsequence b_1 to be a_1, and b_2 to be $a_n s.t. |a_n| \leq \frac{|b-1|}{2}$. We can continue like this, defining b_n recursively:

$$b_n = a_k s.t. |a_k| < |b_{n-1}|/2$$

We know that we can always find such an a_k because the limit inferior is 0, so a_n must be able to get arbitrarily close to 0.
Since we have picked each element in our series to be less than or equal to half of the former, it is less than or equal to the geometric sum $a_1(\frac{1}{2})^n$. The geometric series converges, and so does our series by limit comparison test. Q.E.D.

3 Ross 14.14

Let the harmonic series be named h_n and the provided series be a_n.
From what was given, we can provide a formula for a_n:

$$a_n = \frac{1}{2^{\lfloor \frac{n}{2} \rfloor}}$$

Since $2^{\lfloor \frac{n}{2} \rfloor}$ will always be at least 1 greater than n, we have $h_n > a_n$
Now we attempt to show that a_n does not converge. We can the similar terms together, it is simple to see that it sums to $\frac{1}{2}$. We can further prove this using our formula for a_n: for any $a_n = \frac{1}{2^k}$, there are 2^{k-1} terms of such a_n, which sums to

$$2^{k-1}\frac{1}{2^k} = \frac{1}{2}$$

We see that sum any a_n with like terms to get $\frac{1}{2}$, so we state that $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{2}$
This series does not converge because the limit is not 0. Therefore $\frac{1}{n}$ also does not converge by limit comparison test.

4 Ross 15.3

If $p \leq 0$, we can simply apply the limit comparison test to see that it is greater than or equal to the harmonic series, which does not converge, implying that the series also does not converge.
Now for $p \geq 1$, we apply the integral test.

$$\int_2^\infty \frac{1}{n(\log n)^p}) \, dn = \lim_{n \to \infty} \int_2^n \frac{1}{n(\log n)^p} \, dn$$

(Assuming that log means the natural log) Let $u = \ln n, du = \frac{1}{n}$

$$= \lim_{n \to \infty} \int_{\ln 2}^{\ln n} \frac{1}{u^p} \, du$$

This integral can be evaluated to a finite value for $p > 1$ because we can use the power rule. When $p = 1$, the integral evaluates to $\ln u$, which is not finite when $u \to \infty$.
Q.E.D.

5 Q5

5.1 a

Let $\epsilon > 0, x_0 \in \mathbb{R}$. We pick $\delta = \min\{1, \frac{\epsilon}{1+2|x_0|}\}$
Let $|x - x_0| < \delta$, Consider $|f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0|$. By our definition of δ, the expression is strictly less than $|x + x_0|\delta$
Now we examine our definition of δ. Since it cannot be greater than 1, we have $|x - x_0| < 1$. By algebraic manipulation we can state that $|x + x_0| = |x - x_0 + 2x_0|$, and apply the triangle inequality: $|x + x_0| \leq |x - x_0| + |2x_0|$
We can now subsitute in 1 to get $|x + x_0| \leq 1 + |2x_0|$
Now we connect it all together:

$$|f(x) - f(x_0)| = |x^2 - x_0^2| < (1 + |2x_0|)\delta = \epsilon$$

Thus we have proven that it is continuous.
Q.E.D.

5.2 b

Let $\epsilon > 0, x_0 \in \mathbb{R}$. We pick $\delta = \min\{1, \frac{\epsilon}{(1+|x_0|)^2+|x_0+|x_0|^2|+|x_0^2|}\}$
Let $|x - x_0| < \delta$, Consider $|f(x) - f(x_0)| = |x^3 - x_0^3| = |x - x_0||x^2 + xx_0 + x_0^2|$. By our definition of δ, the expression is strictly less than $|x^2 + xx_0 + x_0^2|\delta \leq (|x|^2 + |xx_0| + |x_0^2|)\delta$
Now we examine our definition of δ. Since it cannot be greater than 1, we have $|x - x_0| < 1$. By the triangle inequality we can state that $|x| < 1 + |x_0|$.
We can now subsitute $|x|$:

$$|f(x) - f(x_0)| \leq (|x|^2 + |xx_0| + |x_0^2|)\delta < ((1 + |x_0|)^2 + |x_0 + |x_0|^2| + |x_0^2|)\delta = \epsilon$$

Thus we have proven that it is continuous.
Q.E.D.

6 Q6

6.1 If x = 0

Let $\epsilon > 0, x_0 = 0$. We pick $\delta = \epsilon^2$
Let $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = |\sqrt{x} - 0| = \sqrt{x}$ since $\sqrt{0} = 0$, and the result of a square root is always non-negative.
Since we have defined δ, $|x - x_0| < \delta \implies x < \delta \implies f(x) < \sqrt{\delta} = \epsilon$
Thus we have proved that the function is continuous at $x = 0$.

6.2 If x >0

Let $\epsilon > 0, x_0 \in \mathbb{R}^+$. We pick $\delta = \min\{1, \frac{|\sqrt{1+|x_0|}+\sqrt{x}_0|}{\epsilon}\}$
Let $|x - x_0| < \delta$, Consider $|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x}_0| = |\sqrt{x} - \sqrt{x}_0|\frac{|\sqrt{x}+\sqrt{x}_0|}{|\sqrt{x}+\sqrt{x}_0|} = |\frac{x-x_0}{|\sqrt{x}+\sqrt{x}_0|}| = |\frac{x-x_0}{|\sqrt{|x|}+\sqrt{x}_0|}|$
Now we examine our definition of δ. Since it cannot be greater than 1, we have $|x - x_0| < 1$. By the triangle inequality we can state that $|x| < 1 + |x_0|$. We can now subsitute $|x|, \delta$:

$$|f(x) - f(x_0)| = |\frac{x - x_0}{|\sqrt{|x|} + \sqrt{x}_0|}| < |\frac{\delta}{|\sqrt{1 + |x_0|} + \sqrt{x_0}|}| = \epsilon$$

Thus we have proven that it is continuous.
Q.E.D.

7 Ross 17.10

7.1 a

Let $x_0 = 0, \epsilon = 0.5$. Let $\delta > 0$, choose $x = \delta/2$. This fulfills the requirement that $|x - x_0| < \delta$. Now since $x > 0$, $f(x) = 1$. However we know that $f(x) = f(0) = 0$, $|f(x) - f(x_0)| = 1 > \epsilon$
Thus we have shown that when $\epsilon = 0.5$, we cannot find a small enough delta to fulfill the property of continuity, there will always be a "big jump" to the right. Therefore the function is not continuous.

7.2 b

We first observe the sine function, for any x, between $[x, x+2\pi]$, the function will yield a 1 and a -1 since it is cyclical.
Let $x_0 = 0, \epsilon = 0.01$. Let $\delta > 0$, choose $x = \min\{\delta/2, 1\}$. This fulfills the requirement that $|x - x_0| < \delta$. If $|f(x)| > 0.01$, the proof is complete since this violates the epsilon delta property.
Otherwise we attempt to make x smaller to violate the property. Since $0 < x < 1$, making x smaller will make $1/x$ bigger. Consider the interval

$$\frac{1}{x} < \frac{1}{x'} < \frac{1}{x} + 2\pi$$

$$0 < \frac{x}{1 + 2\pi x} < x' < x$$

This interval must yield a 1 since $1/x$ has changed by 2π, so let this value be denoted by $x', f(x') = 1$. Since x' is less than x, it is still within δ of 0. Therefore we have found a value inside our delta such that $|f(x') - f(x)| = 0.99 > \epsilon$
This violates the epsilon delta property, and the function is not continuous. Q.E.D.

7.3 c

Let $x_0 = 0, \epsilon = 0.5$. Let $\delta > 0$, choose $x = \delta/2$. This fulfills the requirement that $|x - x_0| < \delta$. Now since $x > 0$, $f(x) = 1$. However we know that $f(x) = f(0) = 0$, $|f(x) - f(x_0)| = 1 > \epsilon$
We can then repeat this process with $x' = -\delta/2$, with $f(x') = -1, |f(x') -$

$f(x)| = 1 > \epsilon$
Thus we have shown that when $\epsilon = 0.5$, we cannot find a small enough delta to fulfill the property of continuity, there will always be a "big jump" to the left and to the right. Therefore the function is not continuous.