Math 104, HW12

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1 Q1

1.1 a

First we know that $\sqrt{x^2} = |x|$. Since we know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$, we can show uniform convergence by the following.
Let $\epsilon > 0$, pick N such that for all $n > N$, $|\frac{1}{n}| < \epsilon^2$. Now since $|x| < 1$ and the domain of the square root being positive,

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2 + \frac{2}{\sqrt{n}}|x| + \frac{1}{n}} \leq \sqrt{(x + \frac{1}{\sqrt{n}})^2}$$

By our first statement the above expression is equal to $|(x + \frac{1}{\sqrt{n}})|$. By our definition of N,

$$||(x + \frac{1}{\sqrt{n}})| - |x|| \leq |x + \frac{1}{\sqrt{n}} - x| = |\frac{1}{\sqrt{n}}| < \epsilon$$

Thus we have $f_n \to |x|$ uniformly.

1.2 b

$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, and by the power rule we know that

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$$

1.3 c

Define $g : (-1, 1) \to \mathbb{R}$, $g(x) = -1$ for $x \in (-1, 0)$, $g(x) = 1$ for $x \in (0, 1)$, $g(0) = 0$
For any $x < 0$, let $a_n = \frac{1}{f_n'(x)} = \frac{\sqrt{x^2+\frac{1}{n}}}{x}$. By part a we know that $\sqrt{x^2 + \frac{1}{n}} \to |x|$ uniformly. Let $\epsilon > 0$ we pick N such that $|\sqrt{x^2 + \frac{1}{n}} - |x|| < \epsilon x - x$, therefore

$$|a_n - (-1)| = \frac{\sqrt{x^2 + \frac{1}{n}} + x}{x} = \frac{|\sqrt{x^2 + \frac{1}{n}} - |x||}{x} < \frac{\epsilon x}{x} = \epsilon$$

If $x > 0$, the same is true because the denominator is now positive and the absolute value sign should be flipped. Finally, if $x = 0$, $f_n'(x) = g(x) = 0$. Thus $f_n'(x) \to g(x)$ pointwise.

Since $f_n'(x)$ is a polynomial divided by a non-zero polynomial, it is continuous for all values of n and for all values of $x \in (-1, 1)$. However $g(x)$ is not continuous at 0. By our theorem about uniform convergence of continuous functions, we know that $f_n'(x) \not\to g(x)$ uniformly.

2 Q2