Math 104, HW9

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1 Q1

1.1 a

To show that the inverse g as defined in the problem is a function, we simply need to show that each unique input has a single output. This implies that our function f must be injective on to $\mathbb{R}$, which is to say $f(a) = f(b) \iff a = b$.
We can assume that there exists $x_0, x_1 \in I$ such that $f(x_0) = f(x_1)$. Then by the Mean Value Theorem we know that there is a point $y \in (x_0, x_1)$ where $f'(y) = \frac{f(x_0)-f(x_1)}{x_0-x_1} = 0$ However the problem specifies that $f'(x) \neq 0$, therefore our assumption is incorrect and f must be injective to $\mathbb{R}$. Thus its inverse g exists.

1.2 b

We claim that f is monotone. Since $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, we know that it satisfies the epsilon-delta property. Since f is differentiable, its derivative is defined on all I. Therefore each point in I also must satisfy the epsilon-delta property, and f' is therefore continuous. Now since it is continuous, if $f'(b) > 0$ and $f'(c) < 0$ for some $b, c \in I$, then by Intermediate Value THeorem there must be $d \in (b, c)$ such that $f(d) = 0$. However we know that to be false, therefore f is monotone.
Let $\epsilon > 0, y_0 \in f(I), x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$. Without loss of generality assume that f is monotonically increasing. Therefore $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Now we can simply take $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + s\epsilon) + f(x_0)\}$. For any $|y_1 - y_0| < \delta$, let $f(x_1) = y_1$, then $x_0 - \epsilon < x_1 < x_0 + \epsilon$ by monoticity. Thus g is continuous.

2 Ross 30.2

2.1 a

$\sin(0) - 0 = 0$, so we attempt to use l'hospital's rule. Assume that the limit exists, then it must be equal to

$$\lim_{x\to 0} \frac{x^2}{\cos x} = \lim_{x\to 0} \frac{0}{1} = 0$$

Therefore the limit is 0

2.2 b

For this problem we need to use l'hoospital's rule 3 times

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\tan(x)\sec^2(x)}{6x}$$
$$= \lim_{x\to 0} \frac{2\sec^2 x(\sec^2(x) + 2\tan^2(x))}{6} = \frac{2}{6} = \frac{1}{3}$$

We used the chain rule for the second step, both the chain rule and the product rule for the third step.

2.3 c

We combine these fractions, then apply l'hospital's rule twice:

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \sin x - x\sin x}$$
$$= \lim_{x\to 0} \frac{0}{1} = 0$$

2.4 d

We know that if $\lim_{x\to a} f(x) = b, \lim_{x\to b} g(x) = c$, then $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$. Assume that the limit does exist for our expression, and since the natural log is continuous, we can apply this theorem.

$$\ln(\lim_{x\to 0} \cos x^{1/x^2}) = \lim_{x\to 0} \ln(\cos x^{1/x^2}) = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$$

Now we can use l'hospital's Theorem

$$= \lim_{x\to 0} \frac{-\sin x/\cos x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

Now to find $\lim_{x\to 0} \cos x^{1/x^2}$, we simply apply the inverse of the natural log:

$$e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}$$

3 Q3

3.1 a

Since $x_n \to \infty$, x_n get can arbitrarily large. More rigorously, for any $r \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that if $m > n$, $x_m > r$

Consider $y_n = \frac{1}{x_n}$. Let $\epsilon > 0$, let $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $n \in \mathbb{N}$ such that $x_n > \epsilon 0$, which we know exists as we have shown above. Now since $\epsilon_0 > 0$, we know that x_n, y_n are positive, so we have $|y_n| = |\frac{1}{x_n}| < |\frac{1}{\epsilon_0}| \leq \epsilon$

Thus we have shown that $|y_n|$ can get arbitrarily small, therefore $y_n \to 0$

$\blacksquare$

3.2 b

Since $\ln_{x \to a} f(x) = \infty$, then for any $r \in \mathbb{R}$, there exists a $\delta > 0$ such that $|x - a| < \delta \implies f(x) > r$

Let $g(x) = \frac{1}{f(x)}$. We know that g is well defined since $f(x) \neq 0$ for $x \in (a, b)$. Let $\epsilon > 0$, take $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $\delta > 0$ such that $f(a + \delta) > \epsilon_0$, which we know exists as we have shown above.

$|g(a + \delta)| < \frac{1}{\epsilon_0} \leq \epsilon$ We have shown thata $|g(x)|$ gets arbitrarily small when x is close to a, therefore $\lim_{x \to a} \frac{1}{f(x)} = 0$

$\blacksquare$

4 Q4

Let P be a partition such that $P = \{t_0 = a < t_1 < ... < t_n = b\}$. Let $M(s)$ denote the supremum of f in a set s, and $m(s)$ the infimum.
We find the upper and lower Darboux Sum:

$$U(f, P) = \sum_{i=1}^{n} M(s)(t_i - t_{i-1})$$

$$L(f, P) = \sum_{i=1}^{n} m(s)(t_i - t_{i-1})$$

Since $f(x) = x$, if $x_0 > x_1, f(x_0) > f(x_1)$, so the infimum is at the lower bound of the interval and the supremum the upper bound/ Now we can rewrite

$$U(f, P) = \sum_{i=1}^{n} (t_i)(t_i - t_{i-1}) = t_1 t_1 - t_1 t_0 + t_2 t_2 - t_2 t_1 + ... + t_n t_n - t_n t_{n-1}$$

$$L(f, P) = \sum_{i=1}^{n} (t_{i-1})(t_i - t_{i-1})$$

Let $\epsilon > 0$, consider $U(f, P) - L(f, P)$, we can combine the sums to get

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (t_i - t_{i-1})(t_i - t_{i-1})$$