Math 104, HW6

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1 Q1

Let $x_0 \in \mathbb{R}, \epsilon > 0$. We first define $\log_a a^k = k$. Choose $\delta = \log_a(\frac{\epsilon}{a^{x_0}} + 1)$
Let $|x - x_0| < \delta$, consider $|f(x) - f(x_0)| = |a^x - a^{x_0}| \leq |a^{|x|} - a^{|x_0|}|$. Since we know that $|x - x_0| < \delta$, we have $|x| < \delta + |x_0|$
Now we subsitute our inequality in:

$$|a^{|x|} - a^{|x_0|}| < |a^{\delta + |x_0|} - a^{|x_0|}| = |a^{x_0}(a^{\delta} - 1)| = |a^{x_0}(\frac{\epsilon}{a^{x_0}} + 1 - 1)| = |a^{x_0}\frac{\epsilon}{a^{x_0}}| = \epsilon$$

Therefore we have shown the epsilon delta property. ■

2 Ross 17.12

2.1 a

Assume that there is a point $x_0 \in (a, b)$ where $f(x_0) \neq 0$. We now try to show that this contradicts the epsilon-delta property of continuity.
Now let $f(x_0) = k, \epsilon = |k|/2$. By continuity there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Now let this arbitrary delta be δ_0. By the density of rationals we know that there exists $q \in \mathbb{Q} s.t. x_0 - \delta < q < x_0 + \delta$. By the definition of this function $f(q) = 0$, and $|f(x_0) - f(q)| = k > \epsilon$. ↯
We have found a contradiction of the continuity principle, so our assumption must be wrong and $f(x) = 0$ in the domain.

2.2 b

We can simply repeat the argument above but replace 0 with $g(r)$. We need to argue a bit more since g is now a function and not a constant.
Assume that there is a point $x_0 \in (a, b)$ where $f(x_0) \neq g(x_0)$. We now try to show that this contradicts the epsilon-delta property of continuity.
Now let $f(x_0) = k, \epsilon = |k - g(x_0)|/2$. By continuity there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Now let this arbitrary delta be δ_0. By the density of rationals we know that there exists $q \in \mathbb{Q} s.t. x_0 - \delta < q < x_0 + \delta$. By the definition of this function $f(q) = g(q)$, and $|f(x_0) - f(q)| = |k - g(x_0)| > \epsilon$. ↯
We have found a contradiction of the continuity principle, so our assumption must be wrong and $f(x) = 0$ in the domain.

3 Ross 17.13

3.1 a

Let $x_0 \in \mathbb{Q}, \epsilon_0 = 0.5$. We assume that the function is continuous at x_0, then there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Let this delta be δ_0. By the density of irrationals we know that there exists $a \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_0 - \delta < a < x_0 + \delta$. Then by the definition of this function we have $f(a) = 0$. $|a - x_0| < \delta$ but $|f(a) - f(x_0)| > \epsilon$ ↯
Therefore this function is not continuous at any rational x.
For any irrational x_1, we can repeat the above reasoning and use the density of rationals to find a rational that is within delta of x_1 but further than epsilon.

3.2 b

Let $x_0 = 0$, we choose $\delta = \epsilon/2$. If $|x - x_0| < \delta$ we can split the cases into two: if x is rational or if x is irrational. If $x \in \mathbb{Q}$ we have $f(x) = x$, then we can use our definiiton of delta $|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon/2 < \epsilon$. Otherwise if x is irrational we have $f(x) = 0$ and $|0 - 0| = 0 < \epsilon$. Therefore $h(x)$ is continuous at 0.

Let $x_0 \neq 0$. If x_0 is rational then $f(x_0) = x_0$, then we pick $\epsilon = x_0/2$, and we can see that for any $\delta > 0$, by the density of irrationals there exists an irrational $x' s.t. x_0 - \delta < x' < x_0 + \delta$, and $f(x') = 0$, so $|f(x') - f(x_0)| = x_0 > \epsilon$, therefore the function is not continuous at any non-zero rational.
Now let x_0 be an irrational, we can pick $\epsilon = |x_0|/2$, and for any $\delta > 0$, by the density of rationals there exists an rational $x' s.t. x_0 - \delta < x' < x_0 + \delta$ and $|x'| > |x_0|$. $f(x') = x'$ Then by our definiiton of epsilon we have $|f(x') - f(x_0)| = |x'| > \epsilon$
Thus the function is not continuous when $x \neq 0$

4 Q4

Let $\epsilon = 1$, assume that there exists some $\delta > 0$ such that for any x_0, x_1, $|x_0 - x_1| < \delta \implies |f(x_0) - f(x_1)| < \epsilon$

Let $\delta > 0, \delta_0 = \min\{\delta, 1\}$. Let $x_0 = \max\{100, 1/\delta_0\}, x_1 = x_0 + \delta_0/2$. We expand

$$x_0 = 1/\delta_0^2, x_1 = (1/\delta_0^2 + \delta_0/2)^2 = (\frac{2+\delta_0^3}{2\delta_0^2})^2 = \frac{4 + \delta_0^6 + 4delta_0^3}{4\delta_0^4} > \frac{4}{4\delta_0^4}$$

Since $\delta_0 \leq 1$, we can say that the above fraction is greater than or equal to 1, which is epsilon.

Therefore we have $|x_0 - x_1| = \delta_0/2 < \delta$, $|f(x_0) - f(x_1)| > \frac{4}{4\delta_0^4} \geq 1 = \epsilon$. Therefore we have shown that for $\epsilon = 1$, we cannot possibly find a delta to satisfy the epsilon-delta property. Therefore it is not uniformly continuous.

5 Q5

Since the sum of continuous functions are continuous, and we know that all powers of x is continuous, $f(x)$ is continuous everywhere.
We will attempt to use the intermediate value theorem, since if we can find a positive value and a negative value, and since the function is continuous, there must be a point where $f(x) = 0$ in between it.
First we know that $lim_{x\to\infty} x = \infty$. If a_1, a_2 are non-zero, $|a_1 x| < |a_2 x^2|$ when $|x| > |a_1/\sqrt{a_2}|$; likewise, $|a_2 x^2| < |a_3 x^3|$ when $|x| > |a_2^2/\sqrt[3]{a_3}|$
Now we can see that since the absolute value of $a_3 x^3$ will be greater than the rest of the terms, its behavior as $|x|$ becomes large dictates the polynomial's behavior.
Therefore if $a_3 > 0, f(x) \to \infty$ as $x \to infty$, and $f(x) \to -\infty$ as $x \to -\infty$ otherwise $f(x) \to -\infty$ as $x \to \infty, f(x) \to \infty$ as $x \to -\infty$. In all cases one is positive and the other is negative, so there must be a 0 in between.
■

6 Q6

Since f is continuous and $x_n \in \mathbb{R}$, we know that x_n converges. By the definition of continuity we know that $f(x_n)$ also converges, and since the codomain: $\mathbb{R}$ is complete, we know that $f(x_n)$ is cauchy.
Q.E.D.

7 Q7

Define $g : [0, 1] \to \mathbb{R}, g(x) = f(x)$. Now g is continuous since f is continuous. Since the domain of g is closed and bounded, g is uniformly continuous. Therefore for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in 0, 1$, $|x - y| < \delta \implies |g(x) - g(y)| < \epsilon$
We can see that this function has a cyclical nature to it, with each period having length 1. For any $x \in \mathbb{R}$, let $a = 1$, by the Archimedean Principle we know that there exists an integer N such that $Na > x$. Let S be the set of such integers. Since it is bounded below, we can pick the smallest one n. Since $n - 1$ is not in the set, $n - 1 \leq x < n$
Now we can simply subtract $n - 1$ from x to get $x' \in [0, 1]$. By the property of this function $f(x) = f(x')$.
For any $\epsilon > 0$ we can find $\delta > 0$ that satisfies the epsilon-delta property in g. Then since this function is cyclical. We know that the neighborhood near x is exactly the neighborhood near x', therefore our δ that works on g will also work with f.
Thus we have found a delta for any epsilon that satisfies uniform continuity. Q.E.D.