Math 104, HW9

Tianshuang (Ethan) Qiu

November 7, 2021

1 Q1

1.1 a

To show that the inverse g as defined in the problem is a function, we simply need to show that each unique input has a single output. This implies that our function f must be injective on to $\mathbb{R}$, which is to say $f(a) = f(b) \iff a = b$.
We can assume that there exists $x_0, x_1 \in I$ such that $f(x_0) = f(x_1)$. Then by the Mean Value Theorem we know that there is a point $y \in (x_0, x_1)$ where $f'(y) = \frac{f(x_0)-f(x_1)}{x_0-x_1} = 0$ However the problem specifies that $f'(x) \neq 0$, therefore our assumption is incorrect and f must be injective to $\mathbb{R}$. Thus its inverse g exists.

1.2 b

We claim that f is monotone. Since $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, we know that it satisfies the epsilon-delta property. Since f is differentiable, its derivative is defined on all I. Therefore each point in I also must satisfy the epsilon-delta property, and f' is therefore continuous. Now since it is continuous, if $f'(b) > 0$ and $f'(c) < 0$ for some $b, c \in I$, then by Intermediate Value THeorem there must be $d \in (b, c)$ such that $f(d) = 0$. However we know that to be false, therefore f is monotone.
Let $\epsilon > 0, y_0 \in f(I), x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$. Without loss of generality assume that f is monotonically increasing. Therefore $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Now we can simply take $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + s\epsilon) + f(x_0)\}$. For any $|y_1 - y_0| < \delta$, let $f(x_1) = y_1$, then $x_0 - \epsilon < x_1 < x_0 + \epsilon$ by monoticity. Thus g is continuous.

2 Ross 30.2

2.1 a

$\sin(0) - 0 = 0$, so we attempt to use l'hospital's rule. Assume that the limit exists, then it must be equal to

$$\lim_{x \to 0} \frac{x^2}{\cos x} = \lim_{x \to 0} \frac{0}{1} = 0$$

Therefore the limit is 0

2.2 b

For this problem we need to use l'hoospital's rule 3 times

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \tan \sec^2(x)}{6x}$$
$$= \lim_{x \to 0} \frac{2 \sec^2(x)(\sec^2(x) + 2 \tan^2(x))}{6} = \frac{2}{6} = \frac{1}{3}$$

We used the chain rule for