Homework 2

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1 Ross 4.15

For this we first prove that $\frac{1}{n} > 0 \forall n \in \mathbb{N}$. First we multiply both sides by n, since $n \in \mathbb{N}, n > 0$, the sign does not change.
We get $LHS = 1$ $RHS = 0$, since $1 > 0$ is an axiom, therefore we know that $1/n > 0$.
Now since we have proven $\frac{1}{n} \geq 0 \forall n \in \mathbb{N}$, by the ordered field axioms $a \leq b$.
Q.E.D.

2 Ross 4.16

Let the set in question be denoted by S. We will prove the claim via contradiction. We break its negative down into two cases: $\sup S < a$ or $\sup S > a$. For the former, let $x = \sup S | x \in \mathbb{R}$. By the denseness of rationals we see that there exists $q \in \mathbb{Q} s.t. x \leq q \leq a$. By the definition of this set we have $q \in S$. Therefore we have just found a member of this set that is greater than the supremum. This is a contradiction so x cannot be less than a.
For the latter, we once again let $x = \sup S | x \in \mathbb{R}$. Consider a. $a < x$ and by definition of S, $\forall s \in S, s < a$. We have found an upperbound that is less than our supremum. That is a contradiction so x cannot be greater than a.
$\sup S = a$ Q.E.D.

3 Ross 8.2

3.1 a

Claim: $a_n \to 0$
Proof: $a_n = \frac{n}{n^2+1} \leq \frac{n+1/n}{n^2+1} = \frac{1}{n}$ $(n \neq 0)$
Let $\epsilon > 0$, we select our $N = \frac{1}{\epsilon}$. $\forall n > N, |a_n - 0| < \frac{1}{n} < \epsilon$, thus the sequence converges.

3.2 c

Claim: $c_n \to \frac{4}{7}$
Proof: $|c_n - \frac{4}{7}| = |\frac{28n+21}{49n-35} - \frac{4(7n-5)}{49n-35}|$

$$= \frac{28n + 21 - 28n + 20}{49n - 35}$$

$$= \frac{41}{49n - 35} \leq \frac{41}{49n}$$

Let $\epsilon > 0$, we select our $N = \frac{41}{49\epsilon}$. For all $n > N, |c_n - \frac{4}{7}| \leq \frac{41}{49n} < \epsilon$

3.3 e

Claim: $s_n \to 0$
Proof: $|s_n - 0| = |\frac{1}{n} sinn| \leq \frac{1}{n}$.
Let $\epsilon > 0$, we select our $N = \frac{1}{\epsilon}$. For all $n > N, |s_n - 0| \leq \frac{1}{n} < \epsilon$

4 Ross 8.5

4.1 a

Let $\epsilon > 0$, since $a_n, b_n \to s$, $|a_n - s| < \epsilon \forall n > N_1$ and $|b_n - s| < \epsilon \forall n > N_2$. Let $k > \max\{N_1, N_2\}$, $a_k \leq s_k \leq b_k$. We subtract s from the expression and we have: $a_k - s \leq s_k - s \leq b_k - s$. Furthermore $|a_k - s| < \epsilon, |b_k - s| < \epsilon$. Since $s_k - s$ is "sandwiched" between two expressions whose absolute values are less than epsilon, then $|s_k - s| < \epsilon$.
$s_n \to 0$, Q.E.D.

4.2 b

Claim: $lims_n = 0$.
Proof: Since the absolute value s strictly non-negative, $t_n \geq 0$. Let $\epsilon > 0$, since $\lim t_n = 0$, $\exists N s.t. \forall k > N, t_k < \epsilon$. Then consider the seqence $d_n = 0$. Obviously $\lim d_n = 0$.

$$0 = |d_k - 0| \leq |s_k| = |s_k - 0| \leq t_k = |t_k - 0|$$

Therefore s_n converges to 0 by squeeze lemma.

5 Ross 8.7

5.1 a

Assume that this sequence a_n converges, let $a_n \to k$. By our assumption $\exists N \in \mathbb{R} s.t. |a_n - k| < \epsilon \forall n > N$. Since this is a cosine function it is cyclical, we can see that it goes 1, 0.5, -0.5, -1, -0.5, 0.5, ..., repeating ad infinitum.
Let $\epsilon = 0.1$. Select $t > N, t \bmod 6 \equiv 0$. By the pattern we observed above, we know that $a_t = 0$. Furthermore, we know that $a_{t+1} = 0.5$. By the definition of convergence we have $|a_t - k| < \epsilon$, $|a_{t+1} - k| < \epsilon$, subsituting the values we have calculated we have $|0 - k| < 0.1, |0.5 - k| < 0.1, |0 - k| + |0.5 - k| \leq 0.2$. However by the triangle property we know that $|0 - k| + |0.5 - k| \leq 0.5$. This is a contradiction, therefore our assuption is not correct.
a_n does not converge. Q.E.D.

5.2 b

For this problem we simply need to show that the sequence is not bounded. Assume that the sequence is bounded, and that there is a supremum k. By the Archimedean Principle $\exists n \in N s.t. n > k$. Consider s_n (if n is odd consider s_{n+1}), this term is greater than k. Therefore we have found a member in the set that is greater than the supremum. $\rightarrow\leftarrow$
The sequence is not bounded, therefore s_n cannot converge. Q.E.D.

5.3 c

The sequence here is very similar to that in section (a). The pattern is 0, 0.5, 1, 0.5, 0, -0.5, -1, -0.5, ... We can let $\epsilon = 0.1$ again and assume that it converges. So let $N \in \mathbb{R} s.t. |c_n - \lim c_n| < \epsilon \forall n > N$. Pick $i > N s.t. i \bmod 6 \equiv 0$. From the pattern that we observed, $c_{n+1} = 0.5$. By the triangle inequality we see that $\lim c_n$ cannot exist since we need the "two sides" (0.2) to be less than the other side (0.5).
We have found a contradiction, c_n does not converge.

6 Ross 8.10

Since $\lim s_n > a$, $\lim s_n - a > 0$. Let this value be d.
Consider $\epsilon = d$. Since the sequence converges we have $\exists N \in R s.t. \forall n > N, |s_n - \lim s_n| < \epsilon$. Since $\epsilon = \lim s_n - a$, we have

$$|s_n - \lim s_n| < \lim s_n - a$$

If $s_n \geq \lim s_n$, $s_n > a$ because $\lim s_n > a$.
Otherwise, $s_n < \lim s_n$. We can simplfy $|s_n - \lim s_n| < \lim s_n - a$ into $\lim s_n - s_n < \lim s_n - a$, and by algebraic manipulation we have $s_n > a$.
In both cases $s_n > a$. Q.E.D.

7 Q7

Claim: $\lim s_n = 1$
Let $\epsilon > 0$. Consider $a_n = 1$, $b_n = 1 - \frac{1}{n}$. Obviously a_n converges to 1.
For b_n, let $N = \frac{1}{\epsilon}$. $\forall k > N$, we have $|b_k - 1| = |1 - \frac{1}{k} - 1| = |-\frac{1}{k}| = \frac{1}{k} < \epsilon$.
Therefore $b_n \to 1$
Since $\frac{1}{n} > 0 \forall n \in \mathbb{N}$, $(1 - \frac{1}{n}) < 1$, so $\sqrt{(1 - \frac{1}{n})} > (1 - \frac{1}{n})$.
We have shown that $a_n \to 1$, $b_n \to 1$, and $b_n \leq s_n \leq a_n$. Therefore $s_n \to 1$ by squeeze theorem.
Q.E.D.