Math 104, HW8

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1 Q1

For $x \neq 1$, $\frac{1}{x}$ is differentiable, and according to the inverse theorem, it is equal to $-\frac{1}{x^2}$
Since $\sin'(x) = \cos(x)$, and sin is well defined on all of $\mathbb{R}$ (the codomain of $\frac{1}{x} : \mathbb{R} \setminus 0 \to \mathbb{R}$), we can apply the chain rule to the second factor: $(\sin(\frac{1}{x}))' = \cos(\frac{1}{x})(-\frac{1}{x^2})$
Now since $x^2 : \mathbb{R} \to \mathbb{R}$ is continuous in all its domain, we attempt to differentiate at an arbitrary point x_0:

$$\lim_{x \to x_0} \frac{(x^2) - (x_0^2)}{x - x_0} = \lim_{x \to x_0} \frac{(x + x_0)(x - x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

Therefore $(x^2)' = 2x$
Finally we use the product rule and the derivative is

$$2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x})(-\frac{1}{x^2}) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

2 Q2

We claim that $f'(0)$ exists and is equal to 0.
Consider the definition of the derivative:

$$\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0}$$

Since this funciton is defined on every point but 0, we have $f' : \mathbb{R} \setminus 0 \to \mathbb{R}$
Now we can simplify to:

$$\lim_{x\to 0} x \sin(\frac{1}{x})$$

Now we apply the squeeze theorem with $-1 \leq \sin(s) \leq 1$, and

$$-x \leq x \sin(\frac{1}{x}) \leq x$$

Since $-x, x$ both converge to 0, our derivative also converges to 0.

3 Q3

We will use the fact that the derivative at $\mathbb{R} \setminus 0$ contains $\cos(\frac{1}{x})$, which fluctuates rapidly when x is close to 0 to show a contradiction.
Assume that the derivative funciton f' is continuous on $\mathbb{R}$. So let $\epsilon = 0.1, x_0 = 0$, then by our assumption there exists δ such that $|x - x_0| < \delta$ implies $|f'(x) - f'(0)| < \epsilon$
We have shown above that $f'(0) = 0$, so we expand $|f'(x) - f'(0)|$:

$$= |2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) - 0| \leq |2x\sin(\frac{1}{x})| + |\cos(\frac{1}{x})| \leq |\cos(\frac{1}{x})|$$

By the Archimedean Principle we know that there exists n such that $\frac{1}{n} < \delta$. Set $x = \frac{1}{2\pi n} < \frac{1}{n} < \delta$. Consider $f'(x) = |\cos(2\pi n)| = 1 > \epsilon$.
Therefore when $\epsilon = 0.1, x_0 = 0$, we have found an x such that for any $\delta > 0$, though $|x - x_0| < \delta$, $|f'(x) - f'(x_0)| > \epsilon$
Thus the function is not continuous. ■

4 Q4

4.1 a

Base case:
$(x^1)'$ By the definition of the derivative we know that $f'(a) = \lim_{x \to a} \frac{x-a}{x-a} = 1$ since the numerator and the denominator cancel.
Inductive step:
Assume that $(x^n)' = nx^{n-1}$ for some $n \in \mathbb{N}$ Consider x^{n+1}
We write it as $x^n x$, now we apply can the product rule. We have shown in the base case that $x' = 1$, so we have

$$(x^{n+1})' = nx^{n-1}x + 1x^n = (n+1)x^{n+1}$$

Thus we have shown the inductive step. ■

4.2 b

We rewrite $(\frac{f}{g})' = (f\frac{1}{g})'$ Now we attempt to apply the chain rule to $(\frac{1}{g})'$
Since g has codomain of $\mathbb{R}$ and is differentiable at a, we can use the chain rule.

$$(\frac{1}{g})' = -\frac{1}{(g(a))^2}g'(a)$$

Now we multiply $f(a)$ in with the product rule

$$(\frac{f}{g})' = f'(a)\frac{1}{g(a)} - f(a)\frac{1}{(g(a))^2}g'(a) = \frac{f'(a)g(a)}{(g(a))^2} - \frac{f(a)g'(a)}{(g(a))^2}$$

Thus it is proven.

5 Ross 29.5

My key observation to this problem is that when x, y are close to each other, $(x-y)^2$ becomes very small. So in order to be smaller than or equal to this, $|f(x) - f(y)|$ must also be able to get arbitrarily small.
Let $\epsilon > 0$, for any $a\mathbb{R}$, we can simply pick our delta to be $\min\{1, \sqrt(\epsilon/2)\}$
Let $|x-a| < \delta$, then $(x-a)^2$
Consider the derivative of this function: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$