

Math 104, HW8

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October 30, 2021

1 Q1

For $x \neq 1$, $\frac{1}{x}$ is differentiable, and according to the inverse theorem, it is equal to $-\frac{1}{x^2}$

Since $\sin'(\frac{1}{x}) = \cos(x)$, and \sin is well defined on all of \mathbb{R} (the codomain of $\frac{1}{x} : \mathbb{R} \setminus 0 \rightarrow \mathbb{R}$), we can apply the chain rule to the second factor: $(\sin(\frac{1}{x}))' = \cos(\frac{1}{x})(-\frac{1}{x^2})$

Now since $x^2 : \mathbb{R} \rightarrow \mathbb{R}$ is continuous in all its domain, we attempt to differentiate at an arbitrary point x_0 :

$$\lim_{x \rightarrow x_0} \frac{(x^2) - (x_0^2)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x + x_0)(x - x_0)}{x - x_0} = \lim_{x \rightarrow x_0} (x + x_0) = 2x_0$$

Therefore $(x^2)' = 2x$

Finally we use the product rule and the derivative is

$$2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x})(-\frac{1}{x^2}) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

2 Q2

We claim that $f'(0)$ exists and is equal to 0.

Consider the definition of the derivative:

$$\lim_{x \rightarrow 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0}$$

Since this function is defined on every point but 0, we have $f' : \mathbb{R} \setminus 0 \rightarrow \mathbb{R}$

Now we can simplify to:

$$\lim_{x \rightarrow 0} x \sin(\frac{1}{x})$$

Now we apply the squeeze theorem with $-1 \leq \sin(s) \leq 1$, and

$$-x \leq x \sin(\frac{1}{x}) \leq x$$

Since $-x, x$ both converge to 0, our derivative also converges to 0.

3 Q3

We will use the fact that the derivative at $\mathbb{R} \setminus 0$ contains $\cos(\frac{1}{x})$, which fluctuates rapidly when x is close to 0 to show a contradiction.

Assume that the derivative function f' is continuous on \mathbb{R} . So let $\epsilon = 0.1, x_0 = 0$, then by our assumption there exists δ such that $|x - x_0| < \delta$ implies $|f'(x) - f'(0)| < \epsilon$

We have shown above that $f'(0) = 0$, so we expand $|f'(x) - f'(0)|$:

$$= |2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) - 0| \leq |2x \sin(\frac{1}{x})| + |\cos(\frac{1}{x})| \leq |\cos(\frac{1}{x})|$$

By the Archimedean Principle we know that there exists n such that $\frac{1}{n} < \delta$. Set $x = \frac{1}{2\pi n} < \frac{1}{n} < \delta$. Consider $f'(x) = |\cos(2\pi n)| = 1 > \epsilon$.

Therefore when $\epsilon = 0.1, x_0 = 0$, we have found an x such that for any $\delta > 0$, though $|x - x_0| < \delta$, $|f'(x) - f'(x_0)| > \epsilon$

Thus the function is not continuous. ■

4 Q4

4.1 a

Base case:

$(x^1)'$ By the definition of the derivative we know that $f'(a) = \lim_{x \rightarrow a} \frac{x-a}{x-a} = 1$ since the numerator and the denominator cancel.

Inductive step:

Assume that $(x^n)' = nx^{n-1}$ for some $n \in \mathbb{N}$ Consider x^{n+1}

We write it as $x^n x$, now we apply can the product rule. We have shown in the base case that $x' = 1$, so we have

$$(x^{n+1})' = nx^{n-1}x + 1x^n = (n+1)x^{n+1}$$

Thus we have shown the inductive step. ■

4.2 b

We rewrite $(\frac{f}{g})' = (f\frac{1}{g})'$ Now we attempt to apply the chain rule to $(\frac{1}{g})'$ Since g has codomain of \mathbb{R} and is differentiable at a , we can use the chain rule.

$$(\frac{1}{g})' = -\frac{1}{(g(a))^2}g'(a)$$

Now we multiply $f(a)$ in with the product rule

$$(\frac{f}{g})' = f'(a)\frac{1}{g(a)} - f(a)\frac{1}{(g(a))^2}g'(a) = \frac{f'(a)g(a)}{(g(a))^2} - \frac{f(a)g'(a)}{(g(a))^2}$$

Thus it is proven.

5 Ross 29.5

My key observation to this problem is that when x, y are close to each other, $(x - y)^2$ becomes very small. So in order to be smaller than or equal to this, $|f(x) - f(y)|$ must also be able to get arbitrarily small.

Let $\epsilon > 0$, for any $a \in \mathbb{R}$, we can simply pick our delta to be $\min\{1, \sqrt{\epsilon/2}\}$. Let $|x - a| < \delta$, then $(x - a)^2 < \delta^2 = \epsilon/2$. Now by the statement of the problem $|f(x) - f(a)| \leq (x - a)^2 < \epsilon/2$. Therefore f is continuous everywhere.

Now consider the absolute value of the derivative of this function:

$$|f'(a)| = \lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|}$$

We can slightly modify the specification of the problem. Since $x^2 = (-x)^2$, we can rewrite the function assumption to be $|f(x) - f(y)| \leq |x - y|^2$.

Now we have $|f'(a)| \leq \lim_{x \rightarrow a} |x - a| = 0$. Since $|f'(a)| \geq 0$, we have $|f'(a)| = 0$. Thus we have shown that the function has a derivative of 0 everywhere, and it is therefore constant. ■

6 Ross 29.13

Consider $h(x) = g(x) - f(x)$. $h(0) = g(0) - f(0) = 0$, $h'(x) = g'(x) - f'(x)$.

By the specification of the problem we know that $h'(x) \geq 0$ when $x \geq 0$

Now consider $h(y)$ with $y \geq 0$. Assume that there is a point $z \geq 0$ such that $g(z) < f(z)$, so $h(z) = g(z) - f(z) < 0$. Since $h(0) = 0$, and h is differentiable, by the Mean Value Theorem we know that there exists a point $p \in (0, z)$ where $h'(p) = \frac{h(z) - 0}{z - 0} < 0$. However by our assumption we know that $h'(z) \geq 0 \forall z \geq 0$. We have reached a contradiction.

Therefore our assumption is incorrect, $g(x) \geq f(x)$ for all $x \geq 0$.

7 Ross 29.17

7.1 h is differentiable implies equal derivative and limit

Let h be differentiable at a . Then by the definition of the derivative we have $\lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$ exists, which implies that $\lim_{x \rightarrow a} h(x)$ exists. Since the limit exists, for any $\epsilon > 0$, $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|h(x) - h(a)| < \epsilon$. $h(a) = g(a)$ When x is less than a , $|f(x) - g(a)| < \epsilon$. Since the difference is less than any positive number, $f(a)$ must equal $g(a)$.

Now consider the fact that $\lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$ exists. Then we must have

$$\lim_{x \rightarrow a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{h(x) - h(a)}{x - a}$$

By the definition of h we can now break the above statement down into:

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{g(x) - g(a)}{x - a}$$

Since $f'(a), g'(a)$ exist, then the above equation must imply $\lim_{x \rightarrow a^+} f'(a) = \lim_{x \rightarrow a^-} g'(a)$, thus $f'(a) = g'(a)$

7.2 Converse

Let $f(a) = g(a), f'(a) = g'(a)$. Since $f(a) = g(a)$, and f, g are differentiable and therefore continuous, the function h is continuous at a . Now consider the derivative: $\lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$.

We break it down into two cases: x converging from the negative and the positive

$$\begin{aligned} \lim_{x \rightarrow a^+} \frac{h(x) - h(a)}{x - a} &= \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = f'(a) \\ \lim_{x \rightarrow a^-} \frac{h(x) - h(a)}{x - a} &= \lim_{x \rightarrow a^-} \frac{g(x) - g(a)}{x - a} = g'(a) \end{aligned}$$

Now since $f'(a) = g'(a)$, we know that $\lim_{x \rightarrow a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \rightarrow a^-} \frac{h(x) - h(a)}{x - a}$. Thus the limit exists, and therefore h is differentiable at a

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