Homework 1

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1 Ross 1.4

1.1 a

n = 1, 1 = 1
n = 2, 1 + 3 = 4
n = 3, 1 + 3 + 5 = 9
n = 4, 1 + 3 + 5 + 7 = 16
Conjecture: $\sum_{i=1}^{n} 2i - 1 = n^2$

1.2 b

Base case: n = 1. $\sum_{i=1}^{1} 2i - 1 = 1$, base case holds.
Inductive hypothesis: let n $\in \mathbb{N}$, and $\sum_{i=1}^{n} 2i - 1 = n^2$
Inductive proof: consider n+1, $\sum_{i=1}^{n+1} 2i - 1 = (\sum_{i=1}^{n} 2i - 1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2$.
Thus we have proven that for n+1, the sum of the sequence is also $(n+1)^2$, and our conjecture holds.
Q.E.D.

2 Ross 1.12

2.1 a

$(a+b) = a+b$
$(a+b)^2 = a^2+2ab+b^2$
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
The theorem holds for all the above cases.

2.2 b

We will prove this with a combinatorics proof.
$\binom{n}{k}$ calculates the number of ways to choose k elements from a set with n. Likewise, $\binom{n}{k-1}$ gives the ways to choose $k-1$ from n.
When we evaluate $\binom{n+1}{k}$, we can line up the $n+1$ elements in a row. To choose the k elements, we can either choose the first element or not. If we choose the first element, there are now n elements left and $k-1$ elements to choose; if we do not, there are n elements left and still $k-1$ to choose. It is precisely the two elements above: $\binom{n}{k}+\binom{n}{k-1}$.
Q.E.D.

2.3 c

Base case: let $n=1$, $(a+b)^1 = a+b$, base case holds.
Inductive hypothesis: assume that the hypothesis holds for some $n \in \mathbb{N}$
Inductive proof: $(a+b)^{n+1} = (a+b)(a+b)^n$, therefore by the inductive hypothesis, it is equal to

$$(a+b)(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n)$$

$$= a(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n) + b(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ...\binom{n}{n}b^n)$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^nb + ... + \binom{n}{n}ab^n + \binom{n}{0}a^nb + \binom{n}{1}a^{n-1}b^2 + ... + \binom{n}{n}b^{n+1}$$

Now we group like terms:

$$= \binom{n}{0}a^{n+1} + (\binom{n}{1} + \binom{n}{0})a^nb + ... + (\binom{n}{n} + \binom{n}{n-1})ab^n + \binom{n}{n}b^{n+1}$$

Using part(b) and that $\binom{n}{0} = \binom{n+1}{0} = 1$ and $\binom{n}{n} = \binom{n+1}{n+1} = 1$:

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i$$

Thus we have proven the inductive case.
Q.E.D.

3 Q3

For this problem we pick intermediary ab.
$\because a \leq b, b \geq 0, \therefore ab \leq bb = b^2$. By the same logic, we can multiply both sides by a and get $a^2 = aa \leq ab$
Since $a^2 \leq ab \leq b^2$, $a^2 \leq b^2$ must hold according to the order field axiom.
To prove the converse, we will take its contrapositive: $a > b \implies a^2 > b^2$. We can repeat the logic as above, using ab as the intermediary. We arrive at the conclusion that $a^2 > ab > b^2$, proving our statement.
Thus we have proved the statement and its converse. $a \leq b \iff a^2 \leq b^2$ $(a, b \geq 0)$.
Q.E.D.

4 Ross 4.7

4.1 a

Since the infimum is a member of the set of lowerbounds for any set, and the supremum is a member of the set of upperbounds, we have $\inf S \leq s(\forall s \in S)$, and $\sup S \geq s(\forall s \in S)$. Therefore by the ordered field axiom, $\inf S \leq \sup S$ for any set.
Thus we have proven $\inf T \leq \sup T$ and $\inf S \leq \sup S$. Now we show that $\inf T \leq \inf S$.
Suppose that the above statement is false, so $\inf S > \inf T$. Therefore $\min s|s \in \mathbb{R}, s \leq x(\forall x \in S) > \min t|t \in \mathbb{R}, s \leq y \forall y \in T$. Then consider $k = (\inf S + \inf T)/2$. Since $\inf S > \inf T$, $k < \inf S$. From the definition of infimum, $k < s(\forall s \in S)$. Furthermore, since $k > \inf T$, $\exists t \in T$such that $t > k$. However, $S \subseteq T$, so every element of S is in T. It is impossible for an element to exist in T but not in S. Therefore our assumption is incorrect.
We conclude that $\inf T \leq \inf S$.
We can repeat the same argument symetrically for the supremum and show that $\sup T \geq \sup S$. Then by the ordered field axioms we can arrive at the conclusion $\inf T \leq \inf S \leq \sup S \leq \sup T$.
Q.E.D.

4.2 b

Let $x = \sup S, y = \sup T, Z = S \cup T$. Furthermore, let $x \geq y$ (switch S, T if $x < y$). To show that $x = \sup Z$, we need to show that $\forall z \in Z, x \geq z$, and that x is the minimum of all upperbounds of Z.
To show the first part, we assume that the statement is false. $\exists z \in Z s.t. z > x$. Since Z is the union of S and T, all the elements inside must be from S or T. $x = \sup S$, $y = \sup T$, and since $x \geq y$, x is an upperbound for both S and T. This is a contradiction to our assumption that $\exists z \in Z s.t. z > x$. Therefore our assumption is incorrect and x must be an upperbound for Z.
To second part, we also assume that it is false. $\exists a < x, s.t. a \geq z(\forall z \in Z)$. Consider $b_0 = \frac{a+x}{2}$. Firstly we can see that $b_0 > a$ since $a < x$. Secondly, if $b_0 \in S$, our proof is complete since $b_0 > a$ and $b_0 \in S$, therefore $b_0 \in Z$. We have found an element in the superset that is greater than our assumed supremum. This is a contradiction.
If $b_0 \notin S$

5 Ross 4.8