Math 104, HW10

Tianshuang (Ethan) Qiu

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1 Q1: Ross 33.7

1.1 a

Let P be an arbitrary partition such that $P = \{a = t_1 < t_2 < ... < t_n = b\}$.
Now consider

$$U(f^2, P) - L(f^2, P) = (M[t_1, t_2] - m[t_1, t_2])(t_2 - t_1) + ... + (M[t_{n-1}, t_n] - m[t_{n-1}, t_n])(t_n - t_{n-1})$$

Since RHS and LHS has the same partition P, each t_k is also the same on the RHS. We consider just $P_1 = [t_1, t_2]$:
Let the $M(f^2, P_1) = f(x_0)^2, m(f^2, P_1) = f(x_1)^2$

$$U(f^2, P_1) - L(f^2, P_1) = (f(x_0)^2 - f(x_1)^2)(t_2 - t_1) = (f(x_0) + f(x_1))(f(x_0) - f(x_1))(t_2 - t_1)$$

Now consider the same partition for f:

$$U(f, P_1) - L(f, P_1) = (M(f, P_1) - m(f, P_1))(t_2 - t_1)$$

Since $B \geq f(x)$ for all $x \in [a, b]$, we have $2B \geq (f(x_0) + f(x_1))$. Since $M(f, P_1)$ is the maximum of the function over this interval and $m(f, P_1)$ is the minimum, their difference is greater than any other differences in this interval P_1, namely $M(f, P_1) - m(f, P_1) \geq (f(x_0) - f(x_1))$.
Therefore we have $U(f^2, P_1) - L(f^2, P_1) \leq 2B(U(f, P_1) - L(f, P_1))$. Now we can repeat this with all intervals of $[t_k, t_k - 1]$ where $2 \leq k \leq n$, thus we have shown that

$$U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P))$$

for any partition P

1.2 b

Since f is integrable, for any $\epsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \epsilon$. Now for any $\epsilon > 0$, choose $\epsilon_0 = \epsilon \times 4B$ where B is the absolute bound for f, since f is integrable we find P_0 that the difference between the Darboux sums is less than ϵ_0
Now consider $U(f^2, P_0) - L(f^2, P_0)$, from part(a) we know that $U(f^2, P_0) - L(f^2, P_0) \leq 2B(U(f, P_0) - L(f, P_0)) \leq \frac{\epsilon}{2} < \epsilon$
Therefore f^2 is integrable.

2 Q1, Ross 33.8

By our theorem we know that the sum(difference) of two integrable functions is integrable. Therefore we know that $(f+g)$ and $(f-g)$ are integrable. By 33.7 we know that $(f+g)^2$, $(f-g)^2$ are integrable. Now we simply take the difference: $(f+g)^2-(f-g)^2 = 4fg$. We apply the integrability theorem again and we know that this is integrable as well. Thus fg must be integrable/

3 Q2

3.1 a

The function is only continuous at $x = 0$. Let $\epsilon > 0$, pick $\delta = /4$. When