Homework 3

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1 Ross 8.2

1.1 a

When $n > 0$, both the numerator and the denomator is greater than 0, so $a_n > 0 \forall n > 0$. Now consider $x_n = \frac{n}{n^2} = \frac{1}{n}$, the denomator is less than a_n, so $x_n > a_n \forall n > 0$.
Obviously 0 converges to 0, and we have proven in class that $\frac{1}{n}$ converges to 0 (pick $N = \frac{1}{\epsilon}$). $a_n \to 0$ by squeeze theorem.

1.2 c

This sequence converges to $\frac{4}{7}$. We will try to show that $x_n = |c_n - \frac{4}{7}| \to 0$.

$$\begin{aligned} |c_n - \frac{4}{7}| &= |\frac{28n+21}{49n-35} - \frac{4(7n-5)}{49n-35}| \\ &= \frac{28n+21-28n+20}{49n-35} \\ &= \frac{41}{49n-35} \\ \lim x_n &= 41 \times \lim(\frac{1}{49n-35}) \end{aligned}$$

Once again, we squeeze this sequence with 0 and $\frac{1}{n}$. Since the denomator is positive $x_n > 0 \forall n > 0$, and $49n - 35 > n \forall n > \frac{35}{48}$, so $x_n < \frac{1}{n}$ when n is large.
Therefore $\frac{1}{49n-35} \to 0$ by squeeze theorem.
Therefore $\lim x_n = 49 \times 0 = 0$, and $c_n \to \frac{4}{7}$

1.3 e

The sequence converges to 0. We define $x_n = \frac{1}{n}, y_n = \frac{-1}{n}$. Since $-1 \leq \sin n \leq 1$, $y_n \leq s_n \leq x_n$.
We have shown above that $\frac{1}{n} \to 0$. $\frac{-1}{n}$ can be shown with the same reasoning to converge $\forall n > \frac{1}{\epsilon}$. Therefore $s_n \to 0$ by squeeze theorem.

2 Q2

2.1 a

We will prove this via induction.
Base case: n = 1, $LHS = 1 + a; RHS = \frac{1-a^2}{1-a}$. $1 - a^2 = (1 + a)(1 - a)$, therefore LHS = RHS. Base case holds.
Inductive hypothesis, let the claim hold for an arbitrary $n \in \mathbb{N}$.

$$1 + a + ... + a^n + a^{n+1} = \frac{1 - a^{n+1}}{1 - a} + a^{n+1}$$

$$= \frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a}$$

$$= \frac{1 - a^{n+2}}{1 - a}$$

Thus we have proven the inductive hypothesis. Q.E.D.

2.2 b

Since $|s_{n+1} - s_n| < 1/2^n \forall n$, $|s_{n+2} - s_n| \leq 1/2^n + 1/2^{n+1}$ by the triangle inequality. We can further extend this so $|s_{n+k} - s_n| \leq 1/2^n + 1/2^{n+1} + ... + 1/2^{n+k-1}$
Let $\epsilon > 0$, we pick $N = \max\{\log_2(1/\epsilon), 1\}$. Pick $j, k > N$.

$$|s_j - s_k| \leq \sum_{i=j}^{j+k-1} \frac{1}{2^i} = \frac{1}{2^j}(1 + 1/2 + ... + 1/2^{k-1})$$

$$= \frac{1}{2^j}(2(1 - 1/2^k)) = \frac{1 - 1/2^k}{2^{j-1}} \leq \frac{1}{2^j}$$

Since $j > \log_2(1/\epsilon)$, $|s_j - s_k| \leq \frac{1}{2^j} < \epsilon$.
Therefore the sequence is cauchy. Q.E.D.

3 Ross 10.7

We essentially want to slowly pick items in S that is closer to $\sup S$. Consider $x_n = \sup S - 1/n$. Since $x_n < \sup S, \exists s_n \in S s.t. x_n \leq s_n < \sup S$. Both x_n and $\sup S$ converge to $\sup S$, therefore $s_n \to \sup S$ by squeeze theorem.

4 Q4

Since $s_n \to s$ by our assumption, let $\epsilon > 0, \exists N s.t. \forall n > N, |s_n - s| < \epsilon$.
Therefore $s - \epsilon < s_n < s + \epsilon$. Since s_n is between these two end points, and since the $\liminf$ is non-decreasing, we have $\liminf s_n \geq s - \epsilon$.
Since ϵ can be arbitrarily small, we must have $\liminf s_n \geq s$. Q.E.D.

5 Q5

5.1 a

We will attempt to prove this via induction.
Base case: $n = 1, s_n = 2, 2^2 - 2 > 0$
Inductive case: assume that $s_n^2 > 2$, then

$$s_{n+1}^2 - 2 = \frac{s_n^2}{4} + \frac{1}{s_n^2} + 2$$

. Since $s_n^2 > 2, \frac{s_n^2}{4} > 0, \frac{1}{s_n^2} > 0$. So $s_{n+1}^2 > 2$.
$s_n^2 - 2 > 0 \forall n \in \mathbb{N}$

5.2 b

We first prove that $s_n > 0 \forall n \in \mathbb{N}$. $n = 1, s_n = 2, 2 > 0$.
In the inductive case, assume $s_n > 0$, so $s_n/2 > 0, 1/s_n > 0$, therefore $s_n n + 1 > 0$. We have proven the inductive case.
To prove that it is monotone will again use induction.
Base case: $n = 1, s_n = 2, s_{n+1} = 2/2 + 1/2 = 3/2$, $s_{n+1} < s_n$. The base case holds.
Inductive case: assume that $s_n < s_{n-1}$. Consider $s_n - s_{n+1} = s_n - \frac{s_n}{2} - \frac{1}{s_n}$

$$\frac{s_n}{2} - \frac{1}{s_n} = \frac{s_n^2 - 2}{2s_n}$$

We have shown that $s_n^2 - 2 > 0$ and that $s_n > 0$, so this fraction is positive, therefore $s_n > s_{n+1}$. This sequence is monotonically decreasing.
Furthermore, since $0 < s_n < 2$, the sequence is bounded above and below. Therefore it converges. Q.E.D.

6 Q6

6.1 a

$\lim s_n = \sqrt{2}$.
We have shown that the sequence converges, so it is also cauchy. Let $\epsilon > 0$, there exists N such that for $i > N, |s_i - s_{i+1}| < \epsilon$.
Since ϵ can get arbitrarily small and that it converges, at very large values of n, s_n should be very close the the limit and to s_{n+1}. We should have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$. Therefore $\lim s_n$ should satisfy

$$\lim s_n = \frac{\lim s_n}{2} + \frac{1}{\lim s_n}$$

$$(\lim s_n)^2 = \frac{\lim s_n^2}{2} + 1$$

$$\frac{1}{2}(\lim s_n)^2 = 1$$

$$(\lim s_n)^2 = 2$$

$$\lim s_n = \sqrt{2}$$

6.2 b

The sequence s_n has $s_n \in \mathbb{Q}$ because each term is the sum of two rationals. Since it is convergent, the sequence is cauchy, but $s_n \to \sqrt{2}$, which is an irrational.

7 Q7

7.1 a

Since the sequence is -1 when n is odd and 1 otherwise, it oscillates ad infinitum. $-1 \leq s_n \leq 1$, so $\liminf s_n = -1, \limsup s_n = 1$
Let $x_n = \{s_n | n \bmod 2 \equiv 1\} = \{-1, -1, -1...\}$. Since $x_n = -1 \forall n \in \mathbb{N}$, $x_n \to -1 = \liminf$
Let $y_n = \{s_n | n \bmod 2 \equiv 0\} = \{1, 1, 1...\}$. Since $y_n = 1 \forall n \in \mathbb{N}$, $x_n \to 1 = \limsup$

7.2 b

We claim that $s_n \to 0$.
Let $\epsilon > 0$, choose $N = \log_2(1/\epsilon)$, $\forall n > N, |s_n - 0| = |s_n| = \frac{1}{2^n}$

$$\frac{1}{2^n} < \frac{1}{2^N} = \epsilon$$

Therefore the sequence converges to 0.
Since we have proven that a sequence converges iff its $\liminf = \limsup$, we have $\liminf s_n = \limsup s_n = 0$
Define $x_n = \{s_n | n \bmod 2 \equiv 1\}$. Since we are only taking odd n's from the set, $x_n < 0$, and since a subsequence of a convergent sequence also converges to its limit, $x_n \to \liminf$
By the same logic we define Define $y_n = \{s_n | n \bmod 2 \equiv 0\}$. We can see that $y_n > 0$, and by the convergence theorem stated above we have $y_n \to \limsup s_n = 0$

7.3 c

Using the limit theorem we can break this down into $\lim(-1)^n$ and $\lim 1/n$. From the work above and in class proofs we have $\liminf s_n = -1 + 0 = -1$ and $\limsup s_n = 1 + 0 = 1$
Similar to (a), we define $x_n = \{s_n | n \bmod 2 \equiv 1\} = (-1)^{2n-1} + 1/(2n-1)$. $\lim x_n = \lim(-1)^{2n-1} + \lim 1/(2n-1)$. The former is constant while the latter is a subsequence of $1/n$ so it converges to 0. Moreover, this sequence only decreasing due to $1/n$ so it is monotone. Finally we have $\lim x_n = \liminf s_n = -1$

Then we define $y_n = \{s_n | n \bmod 2 \equiv 0\} = (-1)^{2n} + 1/(2n) = 1 + 1/(2n)$. By the same logic as above the first term converges to 1 while the second converges to 0. It is also monotonically decreasing for the same reason. Finally we have $\lim y_n = \limsup s_n = 1$