Math 104, HW11

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November 27, 2021

1 Q1

Assume that $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = 0$, then let $\epsilon > 0$. By our assumption there exists $N \in \mathbb{R}$ such that $\sup\{|f_n(x) - f(x)|\} < \epsilon$ for all $x \in S, n > N$. Since the supremum is less than ϵ, every member in that set must also be less than epsilon, which is the definition for uniform convergence. Therefore $f_n \to f$ uniformly.

Now assume that $f_n \to f$ uniformly. Assume that $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = k > 0$, then let $\epsilon = k/2$. Since $f_n \to f$ uniformly for any $\epsilon > 0$ we can find $N \in \mathbb{R}$ such that $|f_m(x) - f(x)| < \epsilon$ for all $x \in S, m > N$. Now consider this m. It implies that the supremum of the set $\{|f_m(x) - f(x)|\}$ is less than or equal to $k/2$, which is less than the supremum of the set as n approaches infinity. Since we know that $\limsup y_n \leq \sup y_n$, we have reached a contradiction, our assumption is incorrect and $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = 0$. Thus we have proven the converse.

2 Q2

We claim that $f_n \to f : [0,1] \to \mathbb{R}$, $f(x) = x^2$ uniformly.
Let $\epsilon > 0$, choose $N = \max\{1, \frac{4}{\epsilon}\}$, consider $n > N$. Take $x \in [0,1]$, $|f_n(x) - f(x)| = |(x - \frac{1}{n})^2 - x^2|$

$$= |x^2 + \frac{1}{n^2} - \frac{2x}{n} - x^2| = |\frac{1}{n^2} - \frac{2x}{n}|$$

Since we know that $n \geq 1$ and $x \in [0,1]$, we know that $\frac{1}{n^2} < \frac{2x}{n}$, so their difference is (nonstrictly) less than $\frac{2x}{n}$
Now since $n > N$, $\frac{2x}{n} \leq \frac{2}{n} < \frac{\epsilon}{2} < \epsilon$. Therefore $|f_n(x) - f(x)| < \epsilon$, and we have proven that $f_n \to f$ uniformly.

3 Q3

3.1 Pointwise Convergent

Let $\epsilon > 0$ and $x \in [0,1]$, choose $N = \frac{\epsilon}{2(1-x)}$. For all $n > N$, consider $|f_n(x) - f(x)| = |nx^n(1-x) - 0| \leq |Nx^n(1-x)| \leq |\frac{\epsilon}{2}x^n| \leq |\epsilon/2| < \epsilon$
Thus given any point in $[0,1]$ and ϵ, we can find a N such that $|f_n(x) - f(x)| < \epsilon$

3.2 Not Unif. Convergent

Choose $\epsilon = \frac{1}{10e}$, let $N \in \mathbb{R}$. Consider $m > N$. Since $f_m(x)$ is a polynomial we know that it is differentiable. Furthermore, since $f_m(0) = 0$, $f_m(1) = 0$, the maximum of $f_m(x)$ in $[0,1]$ must appear in $(0,1)$, then its derivative must be 0. $f_m(x) = mx^m(1-x) = mx^m - mx^{m+1}$.

$$f'_m(x) = m^2x^{m-1} - m(m+1)x^m = 0$$

$$m^2x^{m-1} = m^2x^m + mx^m$$

$$m^2 = m^2x + mx$$

$$x = \frac{m}{m+1}$$

We find that $f'_m(x) = 0$ when $x = \frac{m}{m+1}$, now we attempt to find $f_m(x)$ at this x. $f_m(x) = (\frac{m}{m+1})^m \frac{m}{m}$
Since we are intrested in the behavior of this value as $N \to \infty$, we know that the limit of the maximum of this function is $\lim_{m\to\infty}(\frac{m}{m+1})^m = \frac{1}{e}$. Recall that our $\epsilon = \frac{1}{10e}$ which is less than the maximum of the function $\lim_{m\to\infty} f_m$, therefore there is no N that can satisfy the requirements for uniform convergence. Thus it is not uniformly convergent. ■

4 Ross 25.7

Since $|\cos(k)| \leq 1$, $|\frac{1}{n^2}\cos(nx)| \leq \frac{1}{n^2}$, and since $\sum \frac{1}{n^2}$ converges by the p series test, $\sum \frac{1}{n^2}\cos(nx)$ converges uniformly as well by the M-test.
Furthermore, since $\frac{1}{n^2}\cos(nx)$ is continuous for all $n \in \mathbb{N}$, and it converges uniformly, it must converge to a continuous function.
$\blacksquare$

5 Q5

5.1 Pointwise Convergent

Let $x \in (0,1), \epsilon > 0$, pick $N = \log_x((1-x)\epsilon) = \frac{\ln((1-x)\epsilon)}{\ln(x)}$, consider an arbitrary $n > N$. $f_n(x) = \sum_{k=0}^n g_k = \sum_{k=0}^n x_k$

$$|f_n(x) - f_x| = |(\sum_{k=0}^{n} x_k) - \frac{1}{1-x}|$$

We apply the formula for a geometric sum to the former, since $x < 1$, $(1-x^n)$ is positive

$$= |\frac{1-x^n}{1-x} - \frac{1}{1-x}| = |\frac{-x^n}{1-x}| = \frac{x^n}{1-x}$$

Now recall our N, we know that the difference is strictly less than $\frac{x^N}{1-x} = \frac{\epsilon(1-x)}{1-x} = \epsilon$
Thus we have proven that $\sum_k = 0^\infty g_k \to f$ pointwise.

5.2 Not Uniformly Convergent

We will use the fact that finite sums of finite numbers cannot be infinite to create a contradiction.
First for our function f, as $x \to 1$, $(1-x) \to 0$, and thus $\frac{1}{1-x} \to infty$.
Assume that $f_n \to f$ uniformly. Then choose $\epsilon = 1$, by our assumption there exists $N \in \mathbb{R}$ such that for all $m > n$, $|f_m(x) - f(x)| < \epsilon$ for all $x \in (0,1)$.
Since f_m is the sum of $m+1$ terms starting from x^0, and $0 < x < 1$, we know that $\lim_{x\to 1} f_m \leq (m+1) < \infty$. Now we just need to find this point.
We are looking for a value such that the difference is equal to our epislon: $f(x) = m + 1 + 1 = m + 2$. Now we try to find this point in $(0,1)$:

$$\frac{1}{1-x} = m+2$$
$$1 = (1-x)(m+2) = m + 2 - xm - 2x$$
$$x(m+2) = m+1$$
$$x = \frac{m+1}{m+2}$$

Since $m > 0$, $x \in (0,1)$, and at this point, $|f_m(x) - f(x)| = \epsilon$, and we have a contradiction. Therefore our assumption is incorrect and $\sum_k = 0^\infty g_k \not\to f$ uniformly.

6 Ross 25.3

6.1 a

We claim that the series of functions converge go $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}$.
Consider $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = \lim_{n\to\infty} \sup\{|\frac{n+\cos x}{2n+\sin^2 x} - \frac{1}{2}|\}$
Since both cosine and sine are non-strictly between 0 and 1, the greatest $\frac{n+\cos x}{2n+\sin^2 x}$ can be is $\frac{n+1}{2n}$, and the smallest it can be is $\frac{n}{2n+1}$, therefore we have the inequality:

$$|\frac{n + \cos x}{2n + \sin^2 x} - \frac{1}{2}| \leq \max\{\frac{n+1}{2n}, \frac{n}{2n+1}\} - \frac{1}{2}$$

Now consider $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$ since the difference $|\frac{n+1}{2n} - \frac{1}{2}| = |\frac{1}{2n}|$ approaches 0 as $n \to \infty$. Similarly $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$
Therefore we know that $\max\{\frac{n+1}{2n}, \frac{n}{2n+1}\} - \frac{1}{2} = 0$, thus

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\} = 0$$

From our theorem in Q1 we know that $f_n \to f$ uniformly.

6.2 b

By the theorem that if f_n are continuous and $f_n \to f$ uniformly, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$, we know that we can subsitute the integral for

$$\int_2^7 \frac{1}{2}dx = \frac{7}{2} - \frac{2}{2} = 2.5$$

7 Ross 23.1

7.1 a

We compute $\limsup(a_n)^{\frac{1}{n}} = \limsup(n^2)^{\frac{1}{n}} = \limsup(n^{\frac{1}{n}})^2 = 1$ Therefore $R = 1/1 = 1$
Now we check -1: $\sum(-1)^n n^2$ does not converge by the alternating series test.
For $x = 1$, $\lim_{n\to\infty} n^2 \not\to 0$, therefore it diverges.
Finally, the intereval of convergence is found to be $(-1, 1)$

7.2 c

We compute $\limsup(a_n)^{\frac{1}{n}} = \limsup(\frac{2^n}{n^2})^{\frac{1}{n}} = \limsup \frac{2^{n\frac{1}{n}}}{1} = 2$ Therefore $R = 1/2 = \frac{1}{2}$
Now we check $-\frac{1}{2}$: $\sum(-1)^n \frac{1}{n^2}$ converges by the alternating series test.
For $x = \frac{1}{2}$, $\sum \frac{1}{n^2}$ converges by p-series test.
Finally, the intereval of convergence is found to be $[-\frac{1}{2}, \frac{1}{2}]$

7.3 e

We compute $\limsup(a_n)^{\frac{1}{n}} = \limsup(\frac{2^n}{n!})^{\frac{1}{n}} = \limsup \frac{2}{n!^{\frac{1}{n}}} = 0$ Therefore $R = \infty$
The interval of convergence is defined to be $(-\infty, \infty)$

7.4 g

We compute $\limsup(a_n)^{\frac{1}{n}} = \limsup(\frac{3^n}{n4^n})^{\frac{1}{n}} = \limsup(\frac{3}{4})^{n\frac{1}{n}}\frac{1}{1} = \frac{3}{4}$ Therefore $R = 1/(3/4) = \frac{4}{3}$
Now we check $-\frac{4}{3}$: $\sum(-1)^n \frac{1}{n}$ converges by the alternating series test.
For $x = \frac{4}{3}$, $\sum \frac{1}{n}$ does not converge by p-series test.
Finally, the intereval of convergence is found to be $[-\frac{4}{3}, \frac{4}{3})$