

Math 104, HW4

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1 Q1

1.1 a

Let x be an arbitrary point in $E = (0, 1)$. Choose $r = \min\{(1 - x)/2, x/2\}$. Consider $S = \{d(s, x) < r\}$. Since $0 < x < 1$, $(1 - x)/2$ and $x/2$ are both positive. Therefore $r > 0$, and since $x - x/2 > 0, x + (1 - x)/2 < 1$, we have $S \subseteq E$.

Therefore E is open.

Consider the complement of $E : E' = \mathbb{R} \setminus E$

Let $x' = 1, r' > 0$. Since E' is the complement of E , it is the union of $(-\infty, 0], [1, +\infty)$. If $r \geq 1$, we can see that $S' = \{d(s', x') < r'\}$ contains the point $1/2$ for instance, and $1/2 \notin E'$. Otherwise, let $a = x' - r'$, since $x' = 1, 0 < r' < 1, a \in S, a \notin E$. Therefore its complement is not open.

Thus we have shown that $(0, 1)$ is open and not closed.

1.2 b

Let $x = 1, r > 0, E = [0, 1]$. Consider $S = \{d(s, x) < r\}$. Since $r > 0, \exists s \in S \text{ s.t. } s > x$. However since the interval only goes from 0 to 1, $s \notin E$. Therefore this interval is not open.

Consider $E' = \mathbb{R} \setminus E$.

Since E' is the complement of E , it is the union of $(-\infty, 0), (1, +\infty)$. If x is in the former, then pick $r' = -x/2$. Since $x < 0, -x > 0$, and $x + (-x/2) < 0$, so $S = \{d(s', x') < r'\} \subseteq E$.

If it is in the latter, pick $r' = (x - 1)/2$. Since $x > 1$, and $x - (x - 1)/2 > 1$, so $S = \{d(s', x') < r'\} \subseteq E$. Therefore its complement is open.

Thus we have shown that $[0, 1]$ is closed and not open.

1.3 c

Consider $x = 1$, let $r > 0$, we can consider this set to be a non-increasing series from 1 to 0. Let $S = \{d(s, x) < r\}$, now since $r > 0, \exists s' \in S \text{ s.t. } 1/2 < s' < 1$, since this series is non-increasing, $s' \notin E$. Therefore the set is not open.

Consider the complement E' . Consider x' . If $x' > 1$ or $x' < 0$, we can choose r' exactly the same as part (b) of this question. We can see that the set with radius r' is a subset of E' .

If $0 < x' < 1$, we need to show that we can pick an r' small enough to have not let the “other” set in.

Since $x' \notin E$, and $0 < x' < 1$, then it must be “sanwiched” between two elements of E . Let the two around x' be $1/(n+1) < x' < 1/n$. Now we can apply the denseness of rationals theorem to show that $\exists q_1, q_2$ s.t. $1/(n+1) < q_1 < x', x' < q_2 < 1/n$. Now let $r' = \min\{q_1, q_2\}$. We can see that all of the elements in this radius are in the set E' . Therefore its complement is open. Thus we have shown that this set is closed and not open.

1.4 d

Let $x \in \mathbb{Q}, r > 0$, and $S = \{d(s, x) < r\}$. By the denseness of irrationals we know that $\exists a \notin \mathbb{Q}$ s.t. $x < a < x + r$. Therefore \mathbb{Q} is not open.

We can repeat the same argument but with irrationals. Let y be irrational, $r > 0$, and $S = \{d(s, y) < r\}$. By the denseness of rationals we know that $\exists b \in \mathbb{Q}$ s.t. $y < b < y + r$. Therefore \mathbb{Q} 's complement is not open.

Therefore \mathbb{Q} is neither open nor closed.

1.5 e

Let this set be E . Let $e \in E$ be an arbitrary point, and we choose $r = 1 - d(e, (0, 0))$. So we have our set $S = \{d(e, s) < r\}$. By the triangle inequality we have $d(s, (0, 0)) < 1 - r + r = 1$, so $s \in E \forall s \in S$. Therefore the set is open.

Let E 's complement be called E' , and let $x \in E'$ be a point such that $d(x, (0, 0)) = 1$. Let $r' > 0$, then consider the set $S' = \{d(x, s') < r'\}$. $\exists t \in S$ s.t. $d(t, (0, 0)) < 1$. Then $t \in E$. Therefore its complement is not open. Therefore this set is open and not closed.

2 Q2

2.1 a

Let $a \in U$ be an arbitrary point. Since U is a union of a collection of open sets, then it must belong to at least one element of this collection. Let that element be U_0 .

Since U_0 is open, $\exists r > 0$ s.t. $S = \{s \in S \mid d(a, s) < r\} \subseteq U_0$. Therefore we have found an r that works for an arbitrary point in U . Thus U is open. Q.E.D.

2.2 b

Consider $V_0 = U_1 \cup U_2$.

From the intersection, we conclude that for all $v \in V_0, v \in U_1, v \in U_2$.

Now consider an arbitrary point $w \in V$. Since it is in open sets U_1, U_2 , $\exists r_1, r_2$ s.t. $\{d(w, v) < r_1\} \subset U_1, \{d(w, v) < r_2\} \subset U_2$

Now let $A = \{\}$