Math 104, HW9

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November 7, 2021

1 Q1

1.1 a

To show that the inverse g as defined in the problem is a function, we simply need to show that each unique input has a single output. This implies that our function f must be injective on to $\mathbb{R}$, which is to say $f(a) = f(b) \iff a = b$.
We can assume that there exists $x_0, x_1 \in I$ such that $f(x_0) = f(x_1)$. Then by the Mean Value Theorem we know that there is a point $y \in (x_0, x_1)$ where $f'(y) = \frac{f(x_0)-f(x_1)}{x_0-x_1} = 0$ However the problem specifies that $f'(x) \neq 0$, therefore our assumption is incorrect and f must be injective to $\mathbb{R}$. Thus its inverse g exists.

1.2 b

We claim that f is monotone. Since $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, we know that it satisfies the epsilon-delta property. Since f is differentiable, its derivative is defined on all I. Therefore each point in I also must satisfy the epsilon-delta property, and f' is therefore continuous. Now since it is continuous, if $f'(b) > 0$ and $f'(c) < 0$ for some $b, c \in I$, then by Intermediate Value THeorem there must be $d \in (b, c)$ such that $f(d) = 0$. However we know that to be false, therefore f is monotone.
Let $\epsilon > 0, y_0 \in f(I), x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$. Without loss of generality assume that f is monotonically increasing. Therefore $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Now we can simply take $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + s\epsilon) + f(x_0)\}$. For any $|y_1 - y_0| < \delta$, let $f(x_1) = y_1$, then $x_0 - \epsilon < x_1 < x_0 + \epsilon$ by monoticity. Thus g is continuous.
If f is monotonically decreasing we can simply repeat the above argument but with the signs flipped since if $a > b$, now we will have $f(b) > f(a)$

2 Ross 30.2

2.1 a

$\sin(0) - 0 = 0$, so we attempt to use l'hospital's rule. Assume that the limit exists, then it must be equal to

$$\lim_{x\to 0} \frac{x^2}{\cos x} = \lim_{x\to 0} \frac{0}{1} = 0$$

Therefore the limit is 0

2.2 b

For this problem we need to use l'hoospital's rule 3 times

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\tan(x)\sec^2(x)}{6x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x(\sec^2(x) + 2\tan^2(x))}{6} = \frac{2}{6} = \frac{1}{3}$$

We used the chain rule for the second step, both the chain rule and the product rule for the third step.

2.3 c

We combine these fractions, then apply l'hospital's rule twice:

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \sin x - x \sin x}$$

$$= \lim_{x\to 0} \frac{0}{1} = 0$$

2.4 d

We know that if $\lim_{x\to a} f(x) = b, \lim_{x\to b} g(x) = c$, then $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$. Assume that the limit does exist for our expression, and since the natural log is continuous, we can apply this theorem.

$$\ln(\lim_{x\to 0} \cos x^{1/x^2}) = \lim_{x\to 0} \ln(\cos x^{1/x^2}) = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$$

Now we can use l'hospital's Theorem

$$= \lim_{x \to 0} \frac{-\sin x / \cos x}{2x} = \lim_{x \to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

Now to find $\lim_{x \to 0} \cos x^{1/x^2}$, we simply apply the inverse of the natural log:

$$e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}$$

3 Q3

3.1 a

Since $x_n \to \infty$, x_n get can arbitrarily large. More rigorously, for any $r \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that if $m > n$, $x_m > r$

Consider $y_n = \frac{1}{x_n}$. Let $\epsilon > 0$, let $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $n \in \mathbb{N}$ such that $x_n > \epsilon 0$, which we know exists as we have shown above. Now since $\epsilon_0 > 0$, we know that x_n, y_n are positive, so we have $|y_n| = |\frac{1}{x_n}| < |\frac{1}{\epsilon_0}| \leq \epsilon$

Thus we have shown that $|y_n|$ can get arbitrarily small, therefore $y_n \to 0$

$\blacksquare$

3.2 b

Since $\ln_{x \to a} f(x) = \infty$, then for any $r \in \mathbb{R}$, there exists a $\delta > 0$ such that $|x - a| < \delta \implies f(x) > r$

Let $g(x) = \frac{1}{f(x)}$. We know that g is well defined since $f(x) \neq 0$ for $x \in (a, b)$. Let $\epsilon > 0$, take $\epsilon_0 = \max\{\frac{1}{\epsilon}, 1\}$. Find $\delta > 0$ such that $f(a + \delta) > \epsilon_0$, which we know exists as we have shown above.

$|g(a + \delta)| < \frac{1}{\epsilon_0} \leq \epsilon$ We have shown thata $|g(x)|$ gets arbitrarily small when x is close to a, therefore $\lim_{x \to a} \frac{1}{f(x)} = 0$

$\blacksquare$

4 Q4

Let P be a partition such that $P = \{t_1 = a < t_2 < ... < t_n = b\}$, furthermore let P be evenly spaced such that $t_i - t_{i-1}$ is equal for all $1 < i < n$. Let $M(s)$ denote the supremum of f in a set s, and $m(s)$ the infimum.
We find the upper and lower Darboux Sum. Since $f(x) = x$, if $x_0 > x_1, f(x_0) > f(x_1)$, so the infimum is at the lower bound of the interval and the supremum the upper bound.

$$U(f, P) = \sum_{i=1}^{n} M(s)(t_i - t_{i-1}) = \sum_{i=1}^{n}(t_i)(t_i - t_{i-1})$$

$$L(f, P) = \sum_{i=1}^{n} m(s)(t_i - t_{i-1}) = \sum_{i=1}^{n}(t_{i-1})(t_i - t_{i-1})$$

Since P is evenly spaced, we know that $t_i = a + (b - a)i/n$
Let $\epsilon > 0$, consider $U(f, P) - L(f, P)$, we can combine the sums to get

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(t_i - t_{i-1})(t_i - t_{i-1}) = \frac{(b-a)^2}{n}$$

Now for any $\epsilon > 0$ we simply need to choose $n > (b - a)^2/\epsilon$, which implies that $U(f, P) - L(f, P) < \epsilon$. Since it is less than any positive number and $U(f, P) \geq L(f, P)$, we have $U(f) = L(f)$ and the function is integrable.
In order to find U(f) we need to subsitute $U(f, P) = \sum_{i=1}^{n}(t_i)(t_i - t_{i-1}) = \sum_{i=1}^{n}(a + (b - a)i/n)((b - a)/n) = (b - a)/n(\frac{((a+(b-a)/n)+b)n}{2})$
The last step is using the sum of an arithmetic sequence, now we can tidy up to see that $U(f, P) = \frac{(b-a)(a+(b-a)/n)+b}{2}$. Since $U(f)$ is the infimum of the set $U(f, P)$ where P is a partition, and we know that this sum is minimized as $n \to \infty$ since the term $(b - a)/n \to 0$ as proven in the last problem. Thus we have found

$$U(f) = \lim_{n\to\infty} \frac{(b-a)(a + (b-a)/n) + b}{2} = \frac{(b-a)(b+a)}{2}$$

5 Ross 32.6

Since f is bounded, we know that U_n, L_n are finite. Therefore we have $\lim U_n - \lim L_n = 0$. Since we know that $U(f, P)$ where P is a partition has an infimum at $U(f)$ and similarly $L(f, P)$ has a supremum at $L(f)$. We have $\lim U_n = \lim L_n$, and that $U_n \geq L_n$ with the equivalence happening if and only if they are the infimum and the supremums and the function is integrable. We must have $\lim L_n = L(f)$ and $\lim U_n = U(f)$. Since $L(f) = U(f)$, the function is integrable. Thus we have

$$\int_a^n f = L(f) = U(f) = \lim L_n = \lim U_n$$

5.1 Q6

We essentially have to show that since this is a finite subset, we can create finitely many "thin rectangles" around each of these points. These rectangles have negligible width and converges to 0.

For any $s_n \in S$, for ease of notation let $f(s_n) = y_n$. Now we consider the upper and lower Darboux sums of this function.

We define our partition P as such: $P = \{a, s_n - \delta, s_n + \delta, b\}$ for each $s_n \in S$ with $\delta > 0$(if $a \in S$ or $b \in S$, our first partitions become $\{a, a + \delta, ...\}$ or our last partitions become $\{..., b - \delta, b\}$). Since S is finite, we can iterate through all the points in it with the above expression.

For the lower sum, since each partiton contains some points that are not in S, the infimum for every partiton is 0, and we have $U(f, P) = 0$

For the upper sum, each partition that doesn't contain a point in S has a supremum of 0, a partiton $[s_n - \delta, s_n + \delta]$ has a supremum at s_n with a value of y_n. So we have

$$U(f, P) \leq 2\delta \sum_{i=1}^{n} y_i$$

If a or b is in S, the above inequality still holds since our interval would only be δ instead of 2δ and $\delta < 2\delta$

Let $z = \max\{y_1, y_1, ..., y_n\}$. For any $\epsilon > 0$, choose $\delta < \frac{z}{2n}$. Now consider our inequality:

$$U(f, P) \leq 2\delta \sum_{i=1}^{n} y_i \leq 2\delta z < \epsilon$$

Thus we have shown that $U(f, P)$ can be less than any positive number, and we also know that $U(f, P) \geq L(f, P) = 0$ so we have $U(f, P) = L(f, P) = 0$. Thus the function is integrable.

Since this equivalency can only happen if $U(f, P) = U(f)$ and $L(f, P) = L(f)$, we have $\int_a^b f = 0$

6 Q7

Let P be an equi-distant partition of $[0, \pi/2]$. We compute the upper and lower Darboux sums:

$$U(f,P) = \sum_{i=1}^{n} M(s)(t_i - t_{i-1})$$

$$L(f,P) = \sum_{i=1}^{n} m(s)(t_i - t_{i-1})$$

Since we know that $\sin x$ is monotonically increasing in our domain, so $M([a,b]) = \sin(b), m([a,b]) = \sin(a)$.
Since P is equi-distant, we have $t_i = a + (b-a)i/n$, so when we take $U(f,P) - L(f,P)$, we can get a clean telescopic series:

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \sin(a+(b-a)\frac{1}{n})(b-a)/n - \sin(a+(b-a)\frac{i-1}{n})(b-a)/n)$$

$$= \frac{b-a}{n}(\sin b - \sin a)$$

We know that $a = 0, b = \pi/2$, so we have $U(f,P) - L(f,P) = \frac{\pi}{2n}1$
For any $\epsilon > 0$, pick $n > \max\{\frac{\pi}{2\epsilon}1\}$. After partitioning $[0, \pi/2]$ into n equal partitions, the difference between our upper and lower sum is $\frac{\pi}{2n} < \frac{\pi}{\epsilon\pi} = \epsilon$
Thus we have shown that the difference between $U(f,P)$ and $L(f,P)$ can get arbitrarily small. Therefore the two converges to the same number, and $\sin x$ is integrable on our interval.