Homework 3

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1 Ross 8.2

1.1 a

When $n > 0$, both the numerator and the denomator is greater than 0, so $a_n > 0 \forall n > 0$. Now consider $x_n = \frac{n}{n^2} = \frac{1}{n}$, the denomator is less than a_n, so $x_n > a_n \forall n > 0$.
Obviously 0 converges to 0, and we have proven in class that $\frac{1}{n}$ converges to 0 (pick $N = \frac{1}{\epsilon}$). $a_n \to 0$ by squeeze theorem.

1.2 c

This sequence converges to $\frac{4}{7}$. We will try to show that $x_n = |c_n - \frac{4}{7}| \to 0$.

$$\begin{aligned}
|c_n - \frac{4}{7}| &= |\frac{28n+21}{49n-35} - \frac{4(7n-5)}{49n-35}| \\
&= \frac{28n+21-28n+20}{49n-35} \\
&= \frac{41}{49n-35} \\
\lim x_n &= 41 \times \lim(\frac{1}{49n-35})
\end{aligned}$$

Once again, we squeeze this sequence with 0 and $\frac{1}{n}$. Since the denomator is positive $x_n > 0 \forall n > 0$, and $49n - 35 > n \forall n > \frac{35}{48}$, so $x_n < \frac{1}{n}$ when n is large.
Therefore $\frac{1}{49n-35} \to 0$ by squeeze theorem.
Therefore $\lim x_n = 49 \times 0 = 0$, and $c_n \to \frac{4}{7}$

1.3 e

The sequence converges to 0. We define $x_n = \frac{1}{n}, y_n = \frac{-1}{n}$. Since $-1 \leq \sin n \leq 1$, $y_n \leq s_n \leq x_n$.
We have shown above that $\frac{1}{n} \to 0$. $\frac{-1}{n}$ can be shown with the same reasoning to converge $\forall n > \frac{1}{\epsilon}$. Therefore $s_n \to 0$ by squeeze theorem.

2 Q2

2.1 a

We will prove this via induction.
Base case: n = 1, $LHS = 1 + a; RHS = \frac{1-a^2}{1-a}$. $1 - a^2 = (1 + a)(1 - a)$, therefore LHS = RHS. Base case holds.
Inductive hypothesis, let the claim hold for an arbitrary $n \in \mathbb{N}$.

$$1 + a + ... + a^n + a^{n+1} = \frac{1 - a^{n+1}}{1 - a} + a^{n+1}$$

$$= \frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a}$$

$$= \frac{1 - a^{n+2}}{1 - a}$$

Thus we have proven the inductive hypothesis. Q.E.D.

2.2 b

Since $|s_{n+1} - s_n| < 1/2^n \forall n$, $|s_{n+2} - s_n| \leq 1/2^n + 1/2^{n+1}$ by the triangle inequality. We can further extend this so $|s_{n+k} - s_n| \leq 1/2^n + 1/2^{n+1} + ... + 1/2^{n+k-1}$
Let $\epsilon > 0$, we pick $N > \log_2(1/\epsilon)$. Pick $j, k > N$.

$$|s_j - s_k| \leq \sum_{i=j}^{j+k-1} \frac{1}{2^i} = \frac{1}{2^j}(1 + 1/2 + ... + 1/2^{k-1})$$

$$= \frac{1}{2^j}(2(1 - 1/2^k)) = \frac{1 - 1/2^k}{2^{j-1}} \leq \frac{1}{2^j}$$

Since $j > \log_2(1/\epsilon)$, $|s_j - s_k| \leq \frac{1}{2^j} < \epsilon$.
Therefore the sequence is cauchy. Q.E.D.