Math 74, Week 4

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1 Lec Mon, 1c

1.1 a

Since each term is the product of x^a, y^b, z^c, and $a + b + c = 2020$, we can simplify this problem into dogs and biscuits with 2020 biscuits and 3 dogs: $\binom{2020+3-1}{3-1} = \binom{2022}{2}$

1.2 b

Before combining, we expand each term by picking one variable from each of the 2020 $(x + y + z)$ multiplied together. So we have 3^{2020}.

1.3 c

This is a "no dogs go hungry" problem. We can just simply pick x, y, z from the first three brackets, feeding each dog a biscuit. Now we have 2017 biscuits with 3 dogs $\binom{2017+3-1}{3-1} = \binom{2019}{2}$.

1.3.1 Alternate solution

We can reach the same result by subtracting the amount where there is only x, or only y, or only z, or xy, xz, yz.
For the first three, there is only 1 way for that to happen since that variable has to be raised to 2020. For xz, we have $a + b = 2020$, feeding 2018 biscuits to 2 dogs. We need to subtract 2 since we have already counted having only one term. Therefore $\binom{2018+2-1}{2-1} = 2019$.

Adding them together we have $1 \times 3 + 2019 \times 3 = 6060$. Now we subtract it from what we had in part (a): $\binom{2022}{2} - 6060$ which is the same as $\binom{2019}{2}$

2 Dis Mon, 1a

LHS is the amount of ways to choose a team with k people and a captain from a group with n people. It chooses the team first: $\binom{n}{k}$. Then from that team we choose a captain with k ways to do it.
RHS calculates the amount of ways to choose a captain first: n, then the rest of the team: $\binom{n-1}{k-1}$. Both sides calculate the same thing. Therefore LHS = RHS.
Q.E.D.

3 Dis Mon, 4

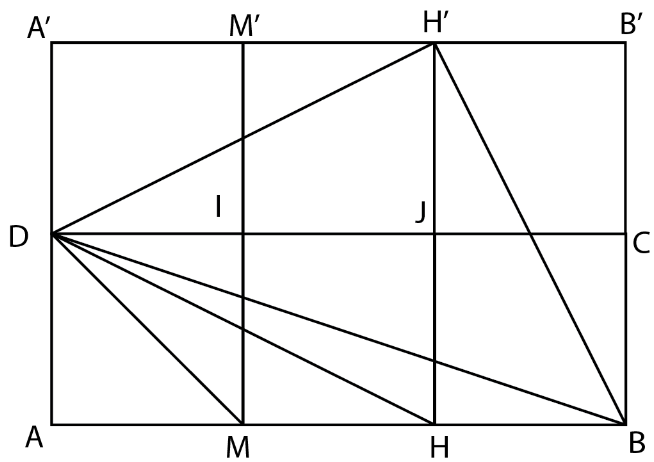
$$x + \frac{1}{x} = 7$$

$$(x + \frac{1}{x})^2 = 49$$

$$x^2 + \frac{1}{x^2} + 2 = 49$$

$$x^2 + \frac{1}{x^2} = 47$$

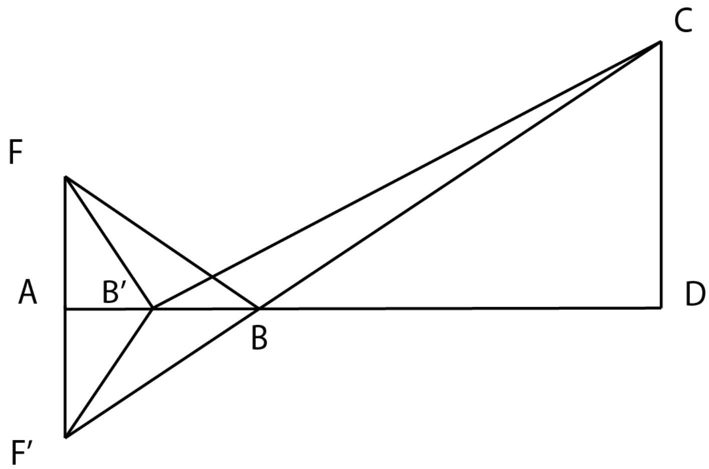
4 Lec Wed, 1a



The sum of these angles is 90°.

Proof: We attempt to move all three angles to $\angle HBC$, since a square has 4 right angles, we can see that $\angle HBC = 90°$.
We construct a row of 3 identical squares with $A'M' = AM, M'H' = MH, etc$, and connect $DH', H'B$
Consider $\triangle DAM$, it is a right isosceles triangle, since it is in a square ($DA = AM, DA \perp AM$). Now consider $\triangle DA'H', \triangle H'B'B$. Since all the squares have the same side length, we have $DA' = H'B', A'H' = B'B$. Furthermore, since these are all squares, we have $\angle DA'H' = \angle H'B'B = 90°$. Therefore $\triangle DA'H' \cong \triangle H'B'B$ by SAS property. $DH' = H'B, \angle H'B'B = \angle DH'A$. Since the sum of the three angles in a triangle is $180°$ and $\angle H'B'B = DA'H' = 90°$, we can see that $\angle B'H'B + \angle AH'D = 90°$. Finally, since M'H'B' forms a line $\angle M'H'B' = 180°$, we have $\angle DH'B = \angle DAM = 90°$. Now we have $\triangle DH'B \sim \triangle DAM$ by RAR property for similarity, therefore $\angle AMD = \angle H'BD$.
By essentially the same logic as $\triangle DA'H' \cong \triangle H'B'B$, we can prove that $\triangle DAH \cong \triangle H'B'B$, so $\angle DHA = \angle B'BH$.
We have proven that $\angle AMD = \angle H'BD$, $\angle DHA = \angle B'BH$. Since the sum these two and $\angle DBA$ is a right angle, we have proven our claim.

5 Lec Wed, 1b



We reflect F over AD to F', and connect $F'C$. Now we first show that B is the optimal point.

By the definiton of reflection, we can see that if we were to reflect F' back over AD it will overlap with our original point. This "overlap" causes all three vertices of $\triangle ABF$ to overlap with those of $\triangle ABF'$. By Euclid's definition they are congruent.

Consider a point on AD that is not $B : B'$. By similar reasoning we can also show that $\triangle AB'F \cong \triangle AB'F'$.

From these congruencies we see that $B'F = B'F', BF = BF'$. The farmers route can then be converted into $F'B + B'C$ and $F'B + BC$. Furthermore, since F'B'C forms a triangle, we have $F'B+B'C > F'B+BC$ by the triangle inequality, thus proving that B gives the shortest route to the cow.

Since $FF' \perp AD$ and $CD \perp AD$, we have $FF' \parallel CD$. By the property of alternate interior angles, $\angle FF'C = \angle F'CD$.

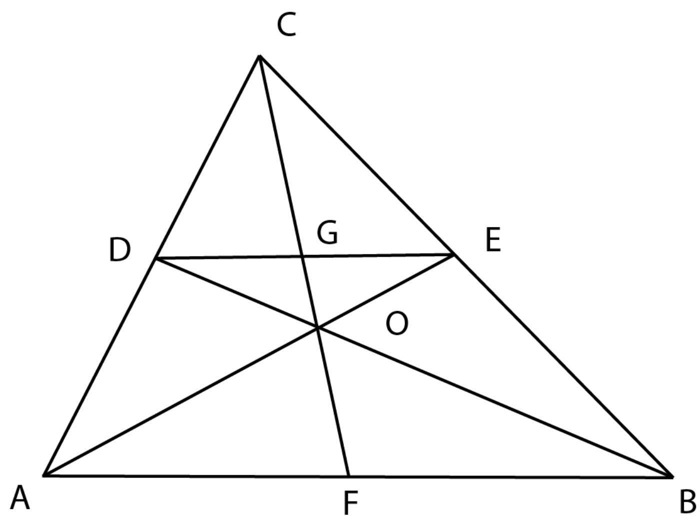
$\angle ABF' + \angle ABC = 180°$ since F'C is a line. By the same logic, $\angle CBD + \angle ABC = 180°$. Therefore $\angle ABF' = DBC$. We have now shown $\triangle ABF' \sim \triangle CBD$ by AAA property.

$AF = AF' = 2$. Let $AB = x$, BD would then equal $4 - x$. By property of similarity we have $\frac{CD}{F'A} = \frac{DB}{AB}$

$$\frac{6}{2} = \frac{4-x}{x}$$

Solving the above equation yields $x = 1$.

6 Lec Wed, 2b

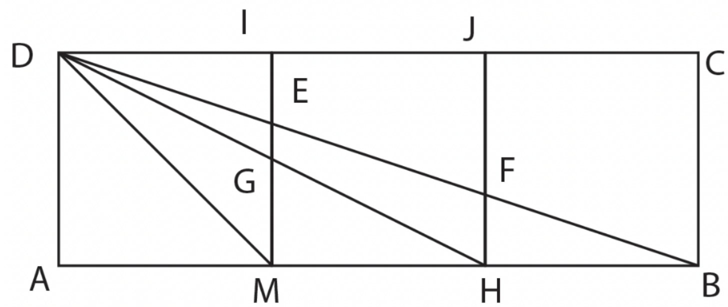


Let $AD = CD, CE = EB$, the two segments intersect at O. Connect BD, AE, DE, CO, and extend CO to intersect AB at F.
This assumes that 2a ($DE \parallel AB$, $DE = 0.5AB$) has already been proven.
Since $DE \parallel AB$, the alternate interior angles are equal, $\angle EDB = \angle DBA$, $\angle DEA = EAB$. Since the sum of all the angles of a triangle is 180°, the third angle must also be the same. Therefore $\triangle DEO \sim \triangle BAO$ by AAA similarity.
Because $DE = 0.5AB$, $DO = 0.5BO, EO = 0.5AO$ by the principles of similarity.
Using $DE \parallel AB$ again, we can see that $\angle EGF = \angle AFG$. By the same reasoning as above we can see that $\triangle GOE \sim \triangle FOA$ by AAA similarity. Therefore

$$\frac{GE}{AF} = \frac{EO}{OA} = \frac{1}{2}$$

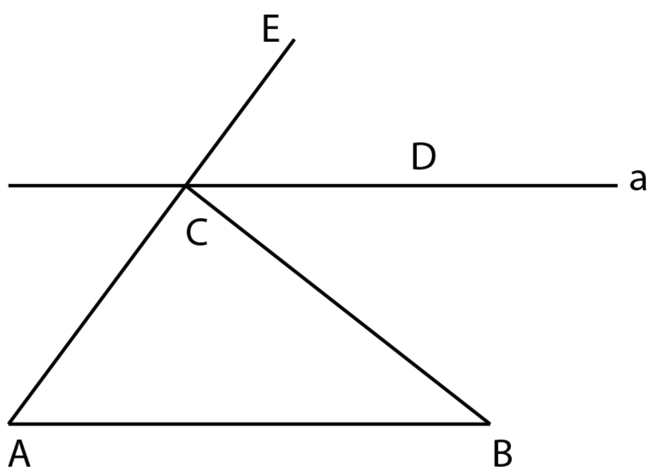
Since $DE \parallel AB$, we can show that $\angle CGE = \angle CFB$, $\angle CEG = \angle CBF$ due to the corresponding angles being equal. $\triangle CGE \sim \triangle CFB$ by AAA similarity. Since E is the midpoint of BC, $CE = 0.5CB$, then by the principle of similarity, $GE = 0.5FB$
Since $GE = \frac{1}{2}FB = \frac{1}{2}AF$, $AF = FB$. Therefore all three medians intersect at point O.

7 Dis Wed, 1a



Since these are all squares we have $DA = AM = BC = CJ$. And since DM connects the diagonal of a square, $\angle DMA = \angle ADM = 45°$.
Let DA have length x, then DM would have length $\sqrt{2}x$. Since $\angle DMA = 45°$, its compliment $\angle DMH = 135°$. MB has length $2x$, and DM has length x.
$\frac{MH}{DM} = \frac{DM}{MB} = \frac{1}{\sqrt{2}}$, and since $\triangle DMH, \triangle BMD$ both have angle DMH, $\triangle DMH \sim \triangle BMD$ (RAR similarity), and $\angle DHM = \angle BDM$.
Now we prove a congruency between two right triangles. From the squares we have $AD = CB = x$, $CD = AB = 3x$, and $\angle DAB = \angle BCD = 90°$, so $\triangle DAB \cong \triangle BCD$ by SAS congruency. Therefore $\angle ABD = \angle BDC$
Thus we have moved all three angles into the right angle $\angle ADC$, proving that the sum of these three angles is 90°. Q.E.D.

8 Dis Wed, 3b



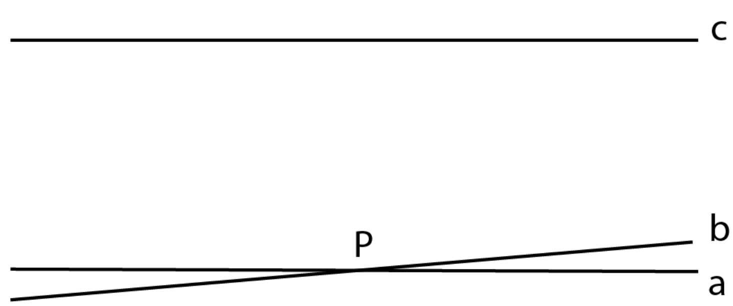
Let $\triangle ABC$ be an arbitrary triangle, extend AC to E, construct $a \parallel AB$. For this proof I need the axiom that a straight line is 180°, and that the corresponding angles and alternate interior angles between two parallel lines are equal.
Since $a \parallel AB$, we have $\angle BAC = \angle DCE$, and $\angle ABC = \angle BCD$. Since the three angles form $\angle ACE$, which is a straight line, the 3 angles add up to 180°.

9 Lec Fri, 3

If the 5th postulate was indeed redundant (provable from others), then the 4 postulates alone must always define the Euclidean space, and that there should be no other spaces that could follow the first 4 postulates but not the fifth.
Hyperbolic geometry satisfies the first 4 with its own definitions of lines, circles, and right angles. The resulting space does not follow Euclid's 5th postulate: two lines whose sum of the inner angles on one side is less than $180°$ can still curve away from each other and never intersect.
This shows that we need Euclid's 5th postulate to properly define a Euclidean space.

10 Lec Fri, 4b

Let us label "There is at most 1 parallel line to a given line l through a given point P" "Statement A", and "two lines that are parallel to the same line are also parallel to eachother" "Statement B".
For this problem we need to prove that $A \iff B$. We will begin by proving $A \implies B$.



1 Assume that statement B is false, so the two lines are not parallel to each other. Then let $a \parallel c, b \parallel c$, and a, c intersect at point P. Then at P, there are two different lines that are both parallel to c. ↯
Therefore our initial assumption is incorrect, statement B is true.

2 Now we try to prove that $B \implies A$. Assume that statement A is false, so there are at least 2 different lines that go through that point. Then let $a \parallel c, b \parallel c$, a, b go through the same point P. Then by statement B, $a \parallel b$, but a, b both contain P, so they have to be the same line. ↯
Therefore there exists at most one line through P parallel to c.