Wed Lec 4d

mod 3

$\equiv 0$	$\equiv 1$	$\equiv 2$
0, 3, 6, 9	1, 4, 7, 10	2, 5, 8, 11
$\Rightarrow 3n \ (n \in \mathbb{Z})$	$\Rightarrow 3n+1$	$\Rightarrow 3n+2$

mod 6

0	1	2	3	4	5
0, 6, 12, 18		2, 8, 14, 20		4, 10, 16, 22	
	1, 7, 13, 19		3, 9, 15, 21		5, 11, 17, 21
$\Rightarrow 6n$	$\Rightarrow 6n+1$	$\Rightarrow 6n+2$	$\Rightarrow 6n+3$	$\Rightarrow 6n+4$	$\Rightarrow 6n+5$

All the numbers that $\equiv 0$ mod 6 is $\equiv 0$ mod 3, because being divisible by 6 means that it has a factor of 3.

Wed Dis 6

mod 5

	1	2	3	4
1	1	2	3	4
2	2	4	1	2
3	3	1	4	2
4	4	3	2	1

mod 6

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

There are 0's in 6 but not in 5. It may be because 5 is prime.

Math 74, Week 6

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1 Mon Lec, 1a

Base case: $n = 1, n = 2, a_1 = 3, a_2 = 5$, base cases hold.
Assume that for some $n \in \mathbb{N}$, all $m \in \mathbb{N}1, 1 \leq m \leq n$ satisfies this identity,
we have

$$a_{n+1} = 3a_n - 2a_{n-1} = 3(2^n + 1) - 2(2^{n-1} + 1) = 3 \times 2^n + 3 - 2 \times 2^{n-1} - 2$$

$$= 2^{n-1}(6 - 2) + 1 = 2^{n-1}(2^2) + 1 = 2^{n+1} + 1$$

Thus we have proven the inductive case. Q.E.D.

2 Mon Lec, 3c

Lemma: a sequence given by $f_n = 1/\sqrt{5}(\phi^n - \bar{\phi}^n)$ where $\phi = (1 + \sqrt{5})/2, \bar{\phi} = (1 - \sqrt{5})/2$ has the property $f_n + f_n + 1 = f_n + 2$ for $n \geq 0$
Base case: $n = 0, f_0 = 0, f_1 = 1, f_2 = 1$, base case holds. Assume that for some n, the statement hodls for all $m \leq n$.
Consider f_{n+1},

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right)$$

We get use this property to get a similar result for f_{n-1}. Now we add them
together:

$$f_n + f_{n-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n + \left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-1}\right)$$

$$= \frac{1}{\sqrt{5}}(((\frac{1+\sqrt{5}}{2})^{n-1}(3+\sqrt{5})/2) - (\frac{1-\sqrt{5}}{2})^{n-1}(3+\sqrt{5})/2)$$

Since $(3+\sqrt{5}) = (1+\sqrt{5})^2/2$, the expression simplifies to

$$= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1})$$

. Thus we have proven the fibbonacci property from the closed form.

Base case: $n = 0, m = 0, f_n = 1, f_m = 1, f_{n+m+1} = 2$, base case holds.

For any $n, m \in \mathbb{N}$, assume that the statement is true for all $n' \le n, m' \le m$.

Now consider $n + 1$ and f_{n+m+2}.

Using our inductive hypothesis, we know that $f_{n+m+1} = f_m f_n + f_{m+1} f_{n+1}$

Now consider $f_m f_{n+1} + f_{m+1} f_{n+2}$, use our lemma to get that it is equal to $f_m(f_n + f_{n-1}) + f_{m+1}(f_n + f_{n+1})$
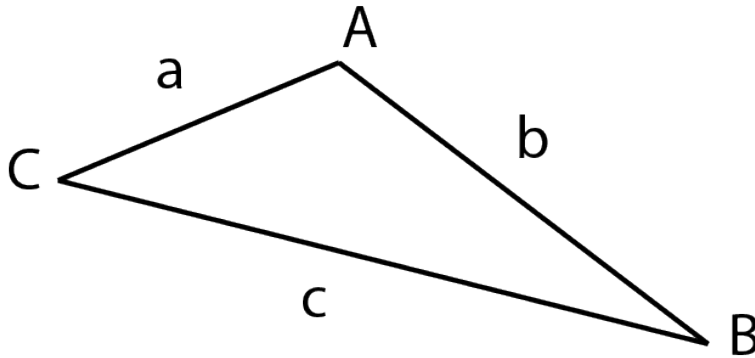
$$= f_m f_n + f_m f_{n-1} + f_{m+1} f_n + f_{m+1} f_{n+1}$$

Now we apply IH,

$$= f_{m+n} + f_{m+n+1} = f_{m+n+2}$$

The last step is using our lemma (the fibbonacci identity). Thus we have proven the inductive step. Q.E.D.

3   Mon Lec, 4a

Base case: let there be three people, they form a triangle. Let the sides have length $a < b < c$. We can see that A and C will exchange pies, and B

will throw at A, leaving him the survivor. Therefore base case holds.

Assume that for some odd n, the statement holds for all odd $m \leq n$. Now consider $n + 2$.

We first remove the two people who are closest to each other. We call this the alternate game. Then the remaining n people satisfies the inductive hypothesis, and there will be one survivor. Now consider the original game, since these two removed people are closest to each other, they will throw the pie at each other.

Then we break down the remaining players into two scenarios when we add the two back: the two new players are closer than their original targets, or they are further than their original targets.

In either case they will not throw their pies at the survivor in the alternate game. Therefore he will still be the survivor in the original game.

Thus we have proven the inductive case. Q.E.D.

4   Mon Dis, 4

Base case: $n = 1, S = \{1, 2\}, 1 \mid 2$, base case holds.
Assume that for $n \in \mathbb{N}$, the statement holds for $m \leq n$. Consdier $n + 1$ and
a set $S = \{a_1, a_2, ..., a_{n+2}\}$. We break it down into two cases.

If $a_{n+1} \leq 2n$ Then we can simply take $S' = S \setminus a_{n+2}$, now we have a set
that has $n + 1$ elements, and the largest one is less than $2n$, so we can apply
the inductive hypothesis and conclude that there is an integer that divides
another in S'.

If $a_{n+1} > 2n$ Now if $a_{n+1} > 2n$, and the maximum of the set cannot exceed
$2n + 22$, we must have $a_{n+1} = 2n + 1.a_{n+2} = 2n + 2$. Now if $(n + 1) \in S$, we
will have $n + 1 \mid 2n + 2$. Otherwise, take the set $\{a_1, a_2, ..., a_n, n + 1\}$, all of
these are less than $2n$ since we know what a_{n+1} and a_{n+2} are. Now we can
apply the inductive hypothesis to this set since it has $n + 1$ items. Therefore
there must be two integers that divides each other.
Thus we have proven the inductive case. Q.E.D.

5   Mon Dis, 7

Base case: $n = 1$, we can simply take a block out, the remaining 3 form an
L shape. Base case holds
Assume that for some $n \in \mathbb{N}$, the statement holds for all $m \leq n$, now consider
$2^{n+1} \times 2^{n+1}$. We first divide the square into four squares of $2^n \times 2^n$. Take
the right most square and remove a single tile. By the inductive hypothesis
this square is tilable with L's since it has $2^n \times 2^n$ tiles. Now for the remaing
three squares, we remove one tile at the center like this:
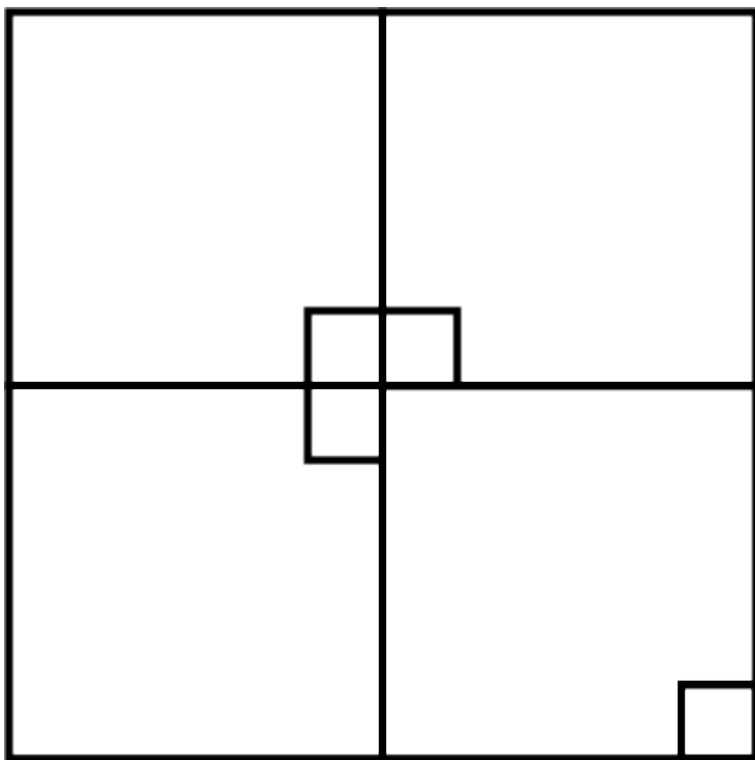
Now the remaining 3 tiles can be filled with a single L. Thus we have proven the inductive step. Q.E.D.

6 Wed Lec, 5

Let our 9 digit number be represented by $abcde2021$. Using the subtraction theorem we know that $abcde0000/2021$ must be an integer.

We can factor $2021 = 43 \times 47$, so it shares no factors with 10000. So $2021 \mid abcde$. Now the amount that we are seeking is equal to the amount of 5 digit numbers that is divisible by 2021

$$\left\lfloor \frac{99999}{2021} \right\rfloor - \left\lfloor \frac{9999}{2021} \right\rfloor = 49 - 4 = 45$$

7 Wed Dis, 6

$$(y - \frac{1}{y})^3 = x^3$$

$$y^3 - 3\frac{1}{y} + 3y - \frac{1}{y^3} = x^3$$

$$y^3 - \frac{1}{y^3} + 3(y - \frac{1}{y}) = x^3$$

$$y^3 - \frac{1}{y^3} + 3x = x^3$$

$$y^3 - \frac{1}{y^3} = x^3 - 3x$$

8 Fri Lec, 3b

Since $a \equiv b \mod 6$, we have $a = 6n + k$, $b = 6m + k$, where $m, n, k \in \mathbb{Z}$.
Let $h \in \mathbb{Z}$, we multiply a, b by h: $ah = 6nh + kh$, $bh = 6mh + kh$.
Since mutiples of 6 will always be equivalent to 0 when modded by 6, the only things that are left on both sides are kh, and $kh \equiv kh \mod 6$.
Q.E.D.

9 Fri Lec, 7

We factor 12 into 3×2^2, and 15 into 3×5. Since our perfect square needs to have these as factors, xy^2 must have 3 as a factor, and xy must have 5 as a factor.
Furthermore, since xy has 5 as a factor, xy^2 must have 25 as a factor since the $12xy^2$ is a perfect square and 12 cannot be divided by 5. Now we attempt to construct one such number. let $y = 5$, $x = 3$, $12xy^2 = 900 = 30^2$, $15xy = 225 = 15^2$. $x + y = 8$