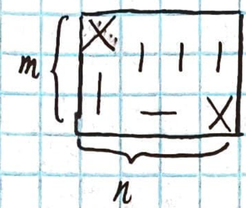
LEC 3

3.

(d) An $m \times n$ board w/ diagonal corners removed can be tiled iff EXACTLY ONE of m, n is odd.

(e) We first prove that when one is odd the claim holds
Let m be odd and n be even



① Fill the first column w/ vertical tiles. This is possible due to the corner is removed, leaving $n-1$ spaces, which is even.

② Fill the last row w/ horizontal tiles. There are $m-2$ empty tiles from the corner and column, so it's possible.

③ We are left w/ a $(m-1) \times (n-1)$ rectangle. Since $m-1$ is even, we can simply fill every column w/ $\frac{m}{2}$ verticals.

We have filled all spaces.

Let both m, n be even.
We color the board in chessboard pattern such that no adjacent tiles share the same color.
$\therefore$ a domino MUST occupy 1 dark and 1 light tile
Now we observe the board and find that diagonal corners have the same color (m, n are even, so it will fall to the same color on both row & column)
$\therefore$ There are now $\frac{mn}{2}$ light ~~blocks~~ & $\frac{mn}{2}-2$ dark tiles (or vice versa)
Since # light tiles must equal tiles # dark tiles, this configuration cannot be filled w/ dominoes.

Let m, n both be odd.

$\because$ m, n are odd

$\therefore$ mn is odd

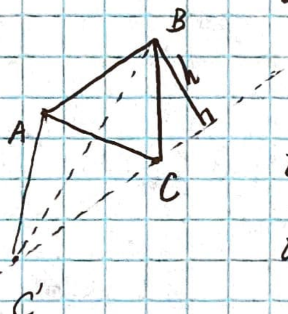
$\therefore$ $mn-2$ is odd.

However, a domino takes up 2 tiles and $2n$ is even

$\therefore$ It is impossible to fill this configuration w/ dominoes

Q.E.D.

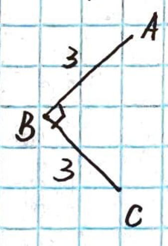
5. In this configuration, $S_{\triangle ABC}$ remains constant



Let $\triangle ABC$ be an arbitrary configuration of the witches. $S_{\triangle ABC} = AB \cdot h \cdot \frac{1}{2}$, $S_{\triangle ABC'} = AB \cdot h \cdot \frac{1}{2}$, where C' is an arbitrary point on the line $\parallel AB$.

We can repeat the argument above for points A, B, reaching the same conclusion.

This problem states that the witches are in the configuration below



Using our lemma above, we can see that all transformations must have $S_{\triangle AB'C'} = \frac{9}{2}$

However the goal configuration has $S_{\triangle AB'C'} = 4$. $4 \neq \frac{9}{2}$, therefore it is impossible.

Q.E.D.

6. Vieta's formula states: let x_1, x_2 be roots to $ax^2+bx+C=0$.
Then we have: $x_1 + x_2 = -\frac{b}{a}$, $x_1 x_2 = \frac{c}{a}$

$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$\therefore (x_1 + x_2)^2 - 4x_1x_2 > 12$

$\frac{b^2}{a^2} - \frac{4c}{a} > 12$, in this case $a=1$, $b=-m-5$, $c=2m+3$

$\therefore (m+5)^2 - (8m+12) > 12$

$m^2 + 25 + 10m - 8m - 12 > 12$

$m^2 - 8m + 1 > 0$

$m_1, m_2 = \frac{8 \pm \sqrt{64-4}}{2} = 4 \pm \frac{\sqrt{60}}{2} = 4 \pm \sqrt{15}$

$\therefore$ m must satisfy $m > 4+\sqrt{15}$ or $m < 4-\sqrt{15}$

DIS 2

1. (c) Let $m, n \in \mathbb{N}$ $(m<n)$ s.t. mn is even. It is possible for n people to each shake m other people's hands.

Since each handshake requires 2 people, the total hands shaken from k handshakes is $2k$, which is even. If mn was odd, there is no solution for n people to shake m other hands.

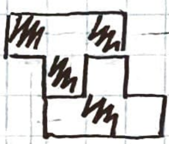
If mn is even, we label the people $1, 2, \dots, n$.

If m is even, we can simply form a circle w/ n people. Each person shakes hands w/ $\frac{m}{2}$ people on his left and $\frac{m}{2}$ people on his right. This accounts for a total of $m \cdot n$ shakes.

If n is even, (and $m < \frac{n}{2}$) divide n into 2 equal lines. Each person shakes the hand of the person after the one infront of ~~him~~. (Person 1 from line 1 shakes w/ person 2 from line 2). He then goes down line 2, shaking hands w/ m people, wrapping around to the first if he reaches the end. This leads to a total of $m \cdot n$ shakes.

If $m > \frac{n}{2}$, simply run the above algorithm w/ $(m \leftarrow n-m)$, and then each person can shake hands w/ everyone they did NOT shake hands w/ in the algorithm.

2. When $n \bmod 4 = 0$, the board can be tiled.



Proof: if $n \bmod 4 = 1$ or 3, n^2 is odd and since each tetromino has 4 squares, $4k$ is even. $\therefore$ It is impossible.

If $n \bmod 4 = 2$, we color the board using checkerboard pattern. As illustrated above, a tile can either have $1B+3W$ or $3W+1B$. Let the amount of $1B+3W$ tiles be x, so the amount of $3W+1B$ is also x since the total num of B & W squares in an even square is even. Using the total number of squares we have $2x \cdot 4 = n^2$

$$(B+3W)x + (3B+W)x = \frac{n^2}{2}(B+W)$$

$$4Bx + 4Wx = \frac{n^2}{2}(B+W)$$

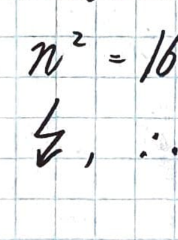
$$4x = \frac{n^2}{2}$$

$$n^2 = 8x$$

If $n^2 = 8x$ and n is an integer, $x \bmod 2$ must be 0. Otherwise n will have $\sqrt{2}$ in its factors. If $x \bmod 2 = 0$, $n^2 = 16k$, $n \bmod 4 = 0 \neq 2$.

$\lightning$, $\therefore$ it is impossible.

If $n \bmod 4 = 0$, we can split it into k smaller 4×4 squares. Then we tile each square as below.



Q.E.D.

LEC 4.

5. 11 ways to choose a captain
from the 10: one can either be in or out
$\Rightarrow 11 \cdot 2^{10}$

A: both are valid but one is faster

6. _ _ _

Case 1: last digit is 0, 9 ways to pick hundreds, 8 to pick tens.

$1 \times 9 \times 8 = 72$

Case 2: last digit is 2, 4, 6, 8, 8 ways to pick hundreds, 8 to pick tens

$4 \times 8 \times 8 = 256$

$256 + 72 = 328$

A: applies to disjoint cases. They apply when they are disjoint.

7. Some get diff pie = All pie config \ everyone has the same pie

$\because \exists$ 3 flavors

$\therefore$ 3 ways to give everyone the same pie

$\therefore$ Some 2 get diff pie = $3^7 - 3$

~~DEF~~

LEC Fri

3b. We line the people up, 1 is married to 2, 3 to 4, 5-6, 7-8, 9-10. Now we only need to find how many ways there are to line them up.

$\Rightarrow 10!$

4. $\dfrac{11!}{4! \cdot 4! \cdot 2!}$

DIS Wed

3.

(a) $26^6 + 26^7 + 26^8 + 26^9$

(b) $26^5 + 26^6 + 26^7 + 26^8$

7.

(b) By inclusion exclusion, $\left\lfloor \frac{150}{4} \right\rfloor + \left\lfloor \frac{150}{6} \right\rfloor + \left\lfloor \frac{150}{7} \right\rfloor - \left\lfloor \frac{150}{28} \right\rfloor - \left\lfloor \frac{150}{12} \right\rfloor - \left\lfloor \frac{150}{42} \right\rfloor$

$+ \left\lfloor \frac{150}{84} \right\rfloor = 37 + 25 + 21 - 5 - 12 - 3 + 1 = 62 + 16 - 15 + 1 = 64$