Math 74, Week 3

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1 Wed Lec, 1a

Prove that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

1.1 algebraic

We first expand the expression:

$$LHS = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!}$$

Then we simplify:

$$LHS = \frac{n!}{(n-k)(n-k-1)!k!} + \frac{n!}{(k+1)(n-k-1)!k!}$$

$$LHS = \frac{(n!(k+1)) + (n!(n-k))}{(n-k)(k+1)(n-k-1)!k!}$$

$$LHS = \frac{n!(n+1)}{(k+1)!(n-k)!}$$

$$LHS = \frac{(n+1)!}{(k+1)!(n-k)!}$$

Now we expand the right side:

$$RHS = \frac{(n+1)!}{(k+1)!(n-k)!}$$

We see thatt RHS = LHS.
Q.E.D.

1.2 Combinatorial

The right hand side calculates the number of bitstrings of length $n+1$ with $k+1$ 0's. Since each bit can either end in 0 or 1, we can split them into different cases.
If the the last is 0, there are k 0's left in the substring of length n. So to calculate this, we use $\binom{n}{k}$.
If the last bit is 1, there are still $k+1$ 0's left in the substring before it. Using the formula, we get $\binom{n}{k+1}$
Adding them together equals the right hand side. They are evaluating the same thing.
Q.E.D.

2 Wed Lec, 4b

Let people be the pigeons and each letter is a different hole.
Number of pigeons = number of names = 33
Number of holes = number of letters = 30
Since the amount of pigeons is greater than the amount of holes, some hole must have at least 2 pigeons by PHP. Therefore some ending letter must be shared by at least 2 names.

3 Wed Lec, 5

3.1 a

We can form a segment from any two points. So the total amount of segments is $\binom{50}{2} = \frac{50!}{2!48!} = 1225$

3.2 b

Since no three points are co-linear, we can choose any 3 points to form a triangle. So the total amount is $\binom{50}{3} = \frac{50!}{3!47!} = 19600$

3.3 c

Similar to part (b), we choose any 4 points to form a quadrilateral. The total amount is $\binom{50}{4} = \frac{50!}{46!4!} = 230300$.

4 Wed Dis, 2

Since the three books needs to be next to each other in the specified order, we can think of it as if the three are "glued together" into one book. So essentially we are looking for the amount of ways to arrange 5 books.
There are 5 ways to choose the first one, 4 for the second, ..., for a total of $5! = 120$ ways to arrange it.

5 Wed Dis, 4

We are looking for the amount of ways to rearrange "GAUSS". Since there are 5 letters, we have $5! = 120$. Then we remove the ways that we over count the repeat "S". $\frac{5!}{2!} = 60$
Similarly, for "RAMANUJAN", we take all possible permutations and divide by repeating letters "A" and "N": $\frac{9!}{3!2!} = 30240$

6 Fri Lec, 1b

6.1 Combinatorial

RHS is the number of ways to choose a team and a captain out of n people: choose the captain first, then choose the rest of the team (each member can either be chosen or not): $n \times 2^{n-1}$
LHS chooses the team first, then chooses the captain. There are $\binom{n}{k}$ ways to choose k people from n to form a team. Then, from that team there are k ways to pick a captain. Finally we sum up all possible number of team size (from 0 to n): $\sum_{k=0}^{n} k\binom{n}{k}$
RHS = LHS, Q.E.D.

7 Fri Lec, 4

Let the n students be our pigeons, and the friends they have be f, so f would be the holes. $0 \leq f \leq n-1$
Case 1: if someone has 0 friends, then there cannot be anyone with $n-1$ friends. If there is a person who is not friends with anyone, and since friendships are mutual, then there cannot be anyone who is friends with everyone. In this case $0 \leq f < n-1$. There are $n-1$ holes, $n > n-1$. By PHP there must be 2 people with the same amount of friends.
Case 2: otherwise, $0 < f \leq n-1$. Once again there are $n-1$ holes, by PHP there must be 2 people with the same amount of friends.
Q.E.D.