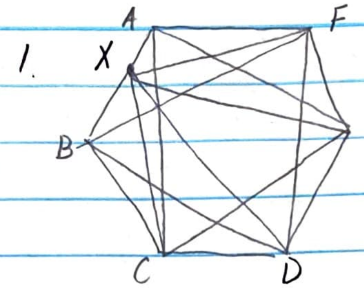
Wed Lec

1.

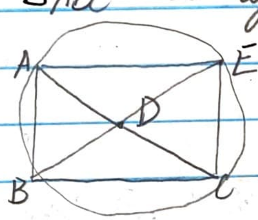


Since $ABCDEF$ is regular, all the sides and interior angles are the same. So we have $\triangle AFE \cong \triangle FED \cong \triangle EDC \cong \triangle DCB$ (SAS)

Now we know that ~~AE~~ $AE = FD = EC = DB$, and $\triangle ACE$ is equilateral, so is $\triangle BFD$. So by our theorem proven in class, $XA + XC = XE$, $XB + XF = XD$

We combine the equations: $XD + XE = XA + XC + XB + XF$

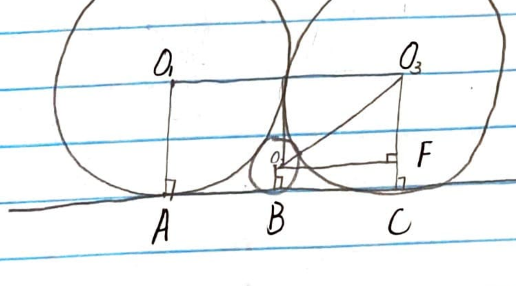
2. Let $\triangle ABC$ be right, rotate $\triangle ABC$ around the midpoint of AC. Now we have a rectangle $ABCE$.



Since $AD = DC$, then $BD = DE$. And since $\triangle CEA$ is rotated from $\triangle ABC$, $AC = BE$. Therefore we can inscribe $ABCE$ in a circle centered at D with radius AD.

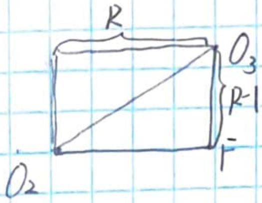
According to Ptolemy's Thm, $AC \cdot BE = AE \cdot BC + AB \cdot EC$, but using rotation congruency, $AC^2 = AB^2 + BC^2$ which is the Pythagorean Thm.

3.



Let O_1, O_2, O_3 be the ~~radius~~ centers, since AB is tangent to the circles, $O_1A \perp AB$, $O_2B \perp AB$, $O_3C \perp AC$. Let $CF = 1$, thus O_2FCB is a rectangle, and $O_2F \perp FC$.

Thus $O_2O_3 = R+1$, $O_3F = R-1$, $O_2F = R$



$\therefore$ in Rt$\triangle O_2FO_3$, $(R+1)^2 = (R-1)^2 + R^2$

$R^2 + 2R + 1 = R^2 + 1 - 2R + R^2$

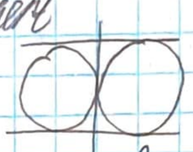
$0 = R^2 - 4R$

$R = \pm 4$, since length > 0, $R = 4$.

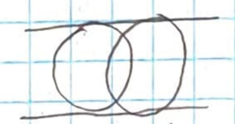
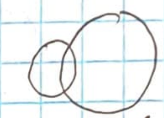
Wed Dis

1. When the two circles have 0 pts in common  

$\Rightarrow$ 0 common tangent $\Rightarrow 2$ lines

1 pts in common  or 

$\Rightarrow$ 3 lines $\Rightarrow$ 1 line

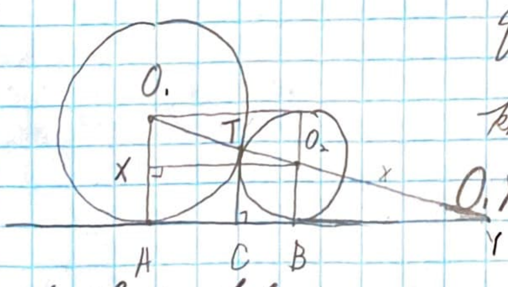
2. pts in common  

$\Rightarrow$ 2 lines $\Rightarrow$ 0 lines

All pts in common  $\Rightarrow \infty$ lines

$\therefore$ Two circles can have 0, 1, 2, 3 or infinite common tangent lines

2.d



We know $O_1A = 7$, $O_2B = 5$, ~~AA~~ furthermore we know that $O_1O_2 = 12$. Connect $O_2X \perp O_1A$, so $O_1X = O_1A - O_2B = 2$. Extend O_1O_2 to Y, and use the fact that $OA \parallel TC \parallel O_2B$ to construct similar triangles $\triangle O_1AY$, $\triangle O_2BY$, $\triangle TCY$.

$$\frac{7}{12+x} = \frac{5}{x} \qquad 7x = 60 + 5x \qquad 2x = 60 \qquad x = 30, \qquad TY = x + 5 = 35$$

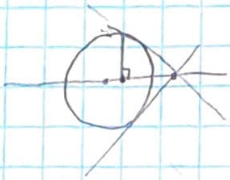
$$\frac{TC}{TY} = \frac{1}{6} = \frac{35/6}{35}$$

$\therefore \; TC = 35/6$

Fri Lec

3.

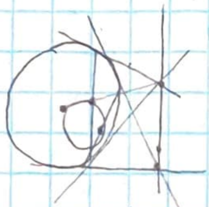
(a)



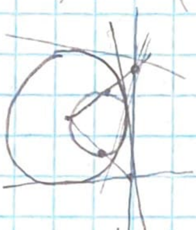
points inside ~~are~~ are mapped to the outside on the same line and vice versa

$\therefore$ lines thru center are mapped to itself

(b)



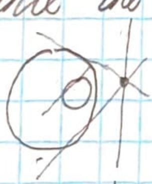
lines outside the circle becomes a circle inside our big circle



lines tangent becomes a circle that shares 1 point w/ the larger

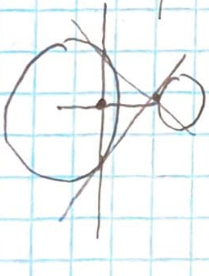
lines that go thru the circle becomes a circle that intersects our circle in 2 points

(c) since the inversion is reversible:



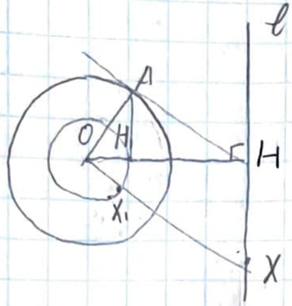
circles inside our circle becomes a line outside the circle

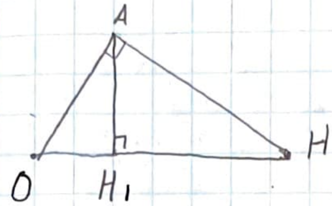
circles that have 1 pt overlap becomes a tangent line



circles that are outside becomes a line that ~~intersection~~ intersects at 2 pts.

4.





Since AH is tangent to O, $OA \perp AH$, by property of inversion $AH_1 \perp OH$.

$\angle AHH_1 = \angle H_1HA$, $\angle AOH_1 = \angle AOH_1$ since the other angles contain a Rt$\angle$, $\triangle AOH_1 \sim \triangle HOA \sim \triangle HAH_1$ (AAA) Therefore $\frac{OH_1}{OA} = \frac{OA}{OH} \Rightarrow OH \cdot OH_1 = OA^2 = r^2$

We can repeat the above proof for x ~~but~~ if we swap H, H_1 w/ X, X_1 since they are both on a ray from the center.

Q.E.D.

Math 74, Week 10

Tianshuang (Ethan) Qiu

October 31, 2021

1 Mon Lec, 6c

$$z^{n-1} - 1 = \prod_{k=0}^{n-1}(z - \omega_k)$$

As shown in class, the complex roots are evenly spaced across the unit circle, $2\pi/n$ apart. So we have

$$\omega_k = e^{2\pi i \frac{k}{n}}$$

2 Mon Lec, 6f

$(z-1)(z^{n-1}+z^{n-2}+...+z^2+z+1) = (z^n+z^{n-1}+...+z^2+z)-(z^{n-1}+z^{n-2}+...+z+1) = z^n-1$

Therefore we can switch our statement to $\frac{z^n-1}{z-1}$
Now we subsitute our answer from the previous question in, and since $\omega_0 = 1$, it cancels out with the first term.
We can factor the expression into

$$\prod_{k=1}^{n-1}(z - \omega_k)$$

where

$$\omega_k = e^{2\pi i \frac{k}{n}}$$

Essentially the same as 6c but with $z = 1$ removed.

3 Mon Lec, 7b

3.1 6c

Sum: $-\frac{0}{1} = 0$
Product: $(-1)^n \frac{1}{1} = (-1)^n$

3.2 6f

Sum: $-\frac{1}{1} = -1$
Product: $(-1)^n \frac{0}{1} = 0$

4 Mon Dis, 1a

$$|A_0A1|...|A_0A_8| = \prod_{k=0}^{8} |1-\omega_k| = |\prod_{k=0}^{8}(1-\omega_k)|$$

The last equivalency is due the fact that multiplication of the modulus is equal to the modulus of the product.
We have proven above that $(z-1)(z^{n-1}+z^{n-2}+...+z^2+z+1) = z^n - 1$, so consider

$$\frac{z^n-1}{z-1} = \frac{(z-1)(z^{n-1}+z^{n-2}+...+z^2+z+1)}{z-1} = z^{n-1}+z^{n-2}+...+z^2+z+1$$

Now using the roots of the polynomial we know that $z^9 - 1 = (z-1)(z-\omega)...(z-\omega^8)$, in this case we have divided out $z-1$, so we have

$$z^8 + z^7 + ... + 1 = (z-\omega)...(z-\omega^8)$$

Let $z = 1$, and we have $9 = (z-\omega)...(z-\omega^8)$, since $|9| = 9$, we have shown that $|A_0A1|...|A_0A_8| = 9$

5 Mon Dis, 3f

Let $x = y - 2$, so $x^3 = y^3 - 6y^2 + 12y - 8$. Now we plug y back

$$y^3 - 6y^2 + 12y - 8 = -6(y-2)^2 - 12y + 24 - 6$$

$$y^3 - 6y^2 + 12y - 8 = -6y^2 + 12y - 6$$

$$y^3 = 2$$

Now since we know that $2^3 = 8$, we can directly solve: $y_1 = \sqrt[3]{2}, x_1 = \sqrt[3]{2} + 2$
Then we factor $(y^3)/(y - \sqrt[3]{2}) = y^2 + \sqrt[3]{2}y + \sqrt[3]{4}$ Now we apply the quadratic formula to get

$$y_2 = \frac{-1-\sqrt{3}i}{\sqrt[3]{2}}, y_3 = \frac{-1+\sqrt{3}i}{\sqrt[3]{2}}$$

So we have $x_2 = \frac{-1-\sqrt{3}i}{\sqrt[3]{2}} - 2, x_3 = \frac{-1+\sqrt{3}i}{\sqrt[3]{2}} - 2$