Math 104

Tianshuang (Ethan) Qiu

September 8, 2021

1 Wed Lec, 1a

Prove that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

1.1 algebraic

We first expand the expression:

$$LHS = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!}$$

Then we simplify:

$$LHS = \frac{n!}{(n-k)(n-k-1)!k!} + \frac{n!}{(k+1)(n-k-1)!k!}$$

$$LHS = \frac{(n!(k+1)) + (n!(n-k))}{(n-k)(k+1)(n-k-1)!k!}$$

$$LHS = \frac{n!(n+1)}{(k+1)!(n-k)!}$$

$$LHS = \frac{(n+1)!}{(k+1)!(n-k)!}$$

Now we expand the right side:

$$RHS = \frac{(n+1)!}{(k+1)!(n-k)!}$$

We see thatt RHS = LHS.
Q.E.D.

1.2 Combinatorial

The right hand side calculates the number of bitstrings of length $n+1$ with $k+1$ 0's. Since each bit can either end in 0 or 1, we can split them into different cases.
If the the last is 0, there are k 0's left in the substring of length n. So to calculate this, we use $\binom{n}{k}$.
If the last bit is 1, there are still $k+1$ 0's left in the substring before it. Using the formula, we get $\binom{n}{k+1}$
Adding them together equals the right hand side. They are evaluating the same thing.
Q.E.D.