Math 74, Week 9

Tianshuang (Ethan) Qiu

October 24, 2021

1 Mon Lec, 2

1.1 a

Our monovariant is the sum of all the numbers. Consider $a = ka', b = kb', \gcd(a, b) = k, lcm(a, b) = ka'b'$ Then $a + b = k(a' + b'), \gcd(a, b) + lcm(a, b) = k(a'b' + 1)$. $a'b' + 1$ will always be greater than or equal to $a' + b'$ for $a', b' > 0$, so it is nondecreasing.

1.2 b

The upperbound is the nL, where n is the amount of numbers, and L the product of all the numbers. Since L is the largest number that can appear by this process, multiplying it by the amount of numbers gives an upperbound.

1.3 c

Our monovariant changes discretely, and it can change by at least 1 per turn. Since it is nondecreasing, it must stabilize at a value in this process.

1.4 d

When the monovariant stops changing, we have $a + b = \gcd(a, b) + lcm(a, b)$ therefore $a = \gcd(a, b), b = lcm(a, b)$ or vice versa. Therefore the game also stops.

2 Mon Lec, 6

Let our monovariant be the largest of all the numbers. Since we are replacing them with the absolute value of the differences, the sum can only get smaller. Furthermore, since it can only take discrete changes, it must stabilize at some point.
At this point, all the number must be $\mathcal{M}$ or 0. Otherwise, the game will take the difference and $\mathcal{M}$ will now be smaller. Now we rescale this to only have 1 and 0.
We now describe a new game: instead of taking the absolute value of the difference, we add the two adjacent numbers. Even though this appears to violate our monovariant, we notice that our new circle is, in fact, equivalent to the original game if we take every number modulo 2. This is true because $a + b = a - b \bmod 2$ Therefore we can directly compute the original game's state with our new game.
Now we notice that the new game has numbers around it according to Pascal's triangle. Since an even number at a position mean that the original game has a 0 at the same position, we just need to find a row in Pascal's triangle that has 7 even numbers in the middle (the 8th comes from the edge 1's summing to 2). We find it at $8, 28, 56, 70, 56, 28, 8, 2$.
Note that this reasoning will apply to all 2^k initial settings, since the nth row of Pascal's triangle is $\binom{2^n}{k}$, which is even for $1 < k < 2^n$, and the ends sum to 2.

3 Mon Dis, 1c

We have shown in class that there can be at most 7 moves for each square. Therefore we construct the following square.

$$\begin{bmatrix} -2 & -1 \\ -3 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \to \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

4 Mon Dis, 5

4.1 a

Since $\phi > 1$, $\lim_{m\to\infty} \phi^m = \infty$, thus the series diverges to ∞

4.2 b

Since $\phi > 1$, we can apply the geometric series sum with r $= 1/phi$:

$$\frac{1}{1-1/\phi} = \frac{1}{1-2/(1+\sqrt{5})} = \frac{1}{(-1+\sqrt{5})/(1+\sqrt{5})} = \frac{1+\sqrt{5}}{\sqrt{5}-1} = \frac{(1+\sqrt{5})(1+\sqrt{5})}{(\sqrt{5}-1)(1+\sqrt{5})}$$

$$= \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

Now consider ϕ^2:

$$(\frac{1+\sqrt{5}}{2})^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

Thus we can see that they are equal.

5 Wed Lec, 5f

The first boy has n choices, and the second $n-1$, and so on. The last boy has 1 choice since there is only one girl left. This gives a total of $n!$ possible configurations.

6 Fri Lec, 4d

6.1 $\sin(2\alpha)$

From $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$, we know that

$$\sin(\alpha) = \frac{e^{\alpha i} - e^{-\alpha i}}{2i}$$

$$\cos(\alpha) = \frac{e^{\alpha i} + e^{-\alpha i}}{2}$$

Now we multiply these to get:

$$\sin(\alpha)\cos(\alpha) = \frac{e^{2\alpha i} - e^{-2\alpha i}}{4i}$$

On the other side:

$$\sin(2\alpha) = \frac{e^{2\alpha i} - e^{-2\alpha i}}{2i}$$

Thus we can see that it is equal to $2\sin(\alpha)\cos(\alpha)$

6.2 $\cos(2\alpha)$

$$\cos(2\alpha) = \frac{e^{2\alpha i} + e^{-2\alpha i}}{2}$$

$$\cos^2(\alpha) = (\frac{e^{\alpha i} + e^{-\alpha i}}{2})^2 = \frac{e^{2\alpha i} + e^{-2\alpha i} + 2}{4}$$

$$\sin^2(\alpha) = (\frac{e^{\alpha i} - e^{-\alpha i}}{2i})^2 = \frac{e^{2\alpha i} + e^{2-\alpha i} - 2}{-4}$$

$\cos^2(\alpha) - \sin^2(\alpha)$ removes the negative in -4, leading to $\frac{2(e^{2\alpha i} + e^{-2\alpha i})}{4}$, which simplifies to $\cos(2\alpha)$

7 Fri Lec, 6

7.1 a

Since we know that our nonagon is inscribed inside a unit circle, all complex points on our shape has length 1. Now using polar angles we know that each side has angle $\frac{2\pi}{9}$. Now we simply write each point as $\cos(2\pi j/9) + i\sin(2\pi j/9)$, using our equations, we know that it is equal to

$$\frac{e^{(2\pi j/9)i} + e^{(-2\pi j/9)i}}{2} + \frac{e^{(2\pi j/9)i} - e^{(-2\pi j/9)i}}{2} = e^{(2\pi j/9)i}$$

Thus we have $A_j = e^{2\pi j i/9}$

7.2 b

We calculate $z_1 = e^{2\pi i/9}$, so $z_1^5 = e^{10\pi i/9}$ Since we know that $e^{2\pi i}$ gets us back to z_0, this ends up at the same spot: z_1, since we can write $z_1^5 \equiv e^{1\pi i/9}$ in our system (essentially modding 9).
Similarly, for z_1^9, we know that it is equal to $e^{2\pi i} = 0$, so we are at A_0.