Mathematical Induction (Lecture)

Worksheet 5: MI^s on Sequences, Pascal's Triangle, and Pie Fights[1]

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MATH 74: Transition to Upper-Division Mathematics
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Read: *Session 6: Mathematical Induction* (vol. I)

- §4. Strong Induction, up to Exercise 14 (pp. 115-119)

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Recurrences)** Prove the direct formulas by MI^s.
   (a) If $a_1 = 3$, $a_2 = 5$, and $a_{n+1} = 3a_n - 2a_{n-1}$ for all $n \geq 2$, then $a_n = 2^n + 1$ for all $n \in \mathbb{N}$.
   (b) **(Fibonacci*)** If $f_0 = 0$, $f_1 = 1$, and $\forall n \geq 1$ $f_{n+1} = f_n + f_{n-1}$, then $\forall n \geq 0$:
   $$f_n = \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n) \text{ for } \phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$
   Is there a non-MI way to arrive at these formulas? (*Hint*: Use some linear algebra: eigenvalues?)

2. **(Sequence Cycling)** Let $a_{n+1} = a_n - a_{n-1}$ for any $n \geq 2$, where $a_1 = 1$ and $a_2 = 2$.
   (a) Find a pattern for $\{a_n\}$ and prove it.
   (b) Prove that $a_n = -2\cos\frac{(n+1)\pi}{3}$ for all $n \geq 1$.
   (c) **($\mathbb{C}$-preview*)** How could we have arrived at the direct formula in (b) without guessing? (*Die-Hards Hint*: If $x^2 = x - 1$ has roots $r_{1,2}$, set $a_n = Ar_1^n + Br_2^n$. With $n = 0, 1$, create a linear system to calculate that $A = -r_1$ and $B = -r_2$. Finally, use de Moivre's formula to rewrite r_1^{n+1} and r_2^{n+1} and simplify the direct formula for a_n.)

3. Each statement below depends on two variables, or has a double index. Decide on what you will induct, and using MI^s, prove the statement.
   (a) **(Pascal's Triangle)** The number in the nth row and kth diagonal of Pascal's Triangle equals the binomial coefficient $\binom{n}{k}$ for $n, k \geq 0$.
   (b) **(Hockey-Stick)** For $n \geq k \geq 0$:
   $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$.
   (c) **(Double Fibonacci)** For all $n, m \geq 0$:
   $$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}.$$
   Re-prove this, using the direct formula for f_n.

4. **(Pie-Fight*)** There is an *odd* number of people (at least 3) and the distances between any two of them are all *different*. Every person throws a pie into the face of his/her closest neighbor.
   (a) Using MI, prove that there will be at least one "survivor" (a person not "pied" by another).
   (b) Why should the number of people be *odd*; e.g., what is wrong with having 100 people making the pie fight? What if some distances were the *same*? Does this change the problem?

5. **(Finite Induction)** Recall Problem 2, IMO '88: If the ratio $\frac{a^2+b^2}{1+ab} \in \mathbb{Z}$ for some $a, b \in \mathbb{N}$, then this ratio is a perfect square. In the solution, we created a chain of pairs (a_n, b_n) of non-negative numbers that were solutions for the same ratio k, and which steadily got "smaller" until a pair with a 0 was hit: $(0, b_n)$ or $(a_n, 0)$.
   (a) What quantity exactly "got smaller"? (This is called a decreasing *monovariant*.)
   (b) What ensures that we reach a pair with a 0? Can't we go on forever? Was MI involved?

6. **(True/False Jeopardy)** Supply convincing reasoning for your answer.
   (a) T F For a recurrence sequence with $a_{n+1} = 3a_n - 4a_{n-1} + 2a_{n-2}$, to prove a conjectured equation for a_n using MI^s, we will have 3 base cases.
   (b) T F MI helps us come up with direct formulas for sequences.
   (c) T F A bank that has an unlimited supply of 4-peso and 10-peso bills can pay any even number of pesos greater than 6.
   (d) T F Sending two or more newly-wed couples on honeymoons to different planets proves that the Pie-Throwing Problem is false for an even number of people.