Math 74, Week 14

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1 Mon Lec, 3a

Let a be the length of this rectangle that is opposite the wall, and b be the length of the other side. So we have $a + 2b \leq 36$, and we try to maximize ab. We can simplify the first equation to $a = 36 - 2b$
By AM-GM, we have that $\frac{2a+b}{2} \geq \sqrt{2ab}$, now since we know that $a + 2b \leq 36$, we can subsitute that in.

$$18 \geq \frac{2a+b}{2} \geq \sqrt{2ab}$$

Thus the maximum the area ab can be is $18^2/2 = 162$. Now we try to find an a, b such that $ab = 162$. Let $a = 9, b = 18$, and $ab = 162$, achieving the maximum.

2 Mon Lec, 5a

We manipulate $2\sqrt{x} > 3 - \frac{1}{x}$, and it is equivalent to showing

$$2\sqrt{x} + \frac{1}{x} \geq 3$$

Then apply AM-GM to see that $\frac{\sqrt{x}+\sqrt{x}+1/x}{3} \geq \sqrt[3]{\sqrt{x}\sqrt{x}1/x} = 1$. Rearranging this gives

$$2\sqrt{x} + \frac{1}{x} \geq 3$$

Thus we have proven the statement.

3 Mon Lec, 6

We say two inequalities are equivalent when they are true and false at the same time.
In the first equation $(x-a)^2+1 > 0$, $(x-a)^2$ is non-negative, so the statement is always true.

$$4ax^2 + 4x + 1 > 0$$

$$(4a-4)x^2 + (2x+1)^2 > 0$$

This inequality holds true when $(4a - 4) > 0$, so the two inequalities are equivalent when $a > 1$

4 Mon Dis, 1a

By AM-GM, $\frac{a+b}{2} \geq \sqrt{ab}$, $\frac{b+c}{2} \geq \sqrt{bc}$, etc.
We can multiply these equations to get

$$\frac{(a+b)(b+c)...(e+a)}{2^5} \geq \sqrt{a^2b^2c^2d^2e^2}$$

$$(a+b)(b+c)(c+d)(d+e)(e+a) \geq 32abcde$$

5 Mon Dis, 1b

By AM-CM, $(\sum_{n=1}^{2021} n)/2021 \geq \sqrt[2021]{2021!}$. We can apply the arithmetic series sum to the lhs:

$$\frac{(2021+1)2021}{2 \times 2021} \geq \sqrt[2021]{2021!}$$

$$(\frac{2022}{2})^{2021} \geq 2021!$$

Thus it is proven.

6 Mon Dis, 3c

Let our box intersect the ellipsoid in octant 1 at (x, y, z). Since it is on the ellipsoid we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
By AM-GM inequality we have

$$(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2})/3 \geq \sqrt[3]{\frac{x^2}{a^2}\frac{y^2}{b^2}\frac{z^2}{c^2}}$$

$$\frac{1}{3} \geq \sqrt[3]{\frac{x^2y^2z^2}{a^2b^2c^2}}$$

$$\frac{1}{27} \geq \frac{x^2y^2z^2}{a^2b^2c^2}$$

Now the volume of our cube is simply $8xyz$ since each side is double the intersection point in octant 1.

$$\frac{1}{27a^2b^2c^2} \geq x^2y^2z^2$$

$$xyz \leq abc\sqrt{\frac{1}{27}}$$

$$8xyz \leq 8abc\sqrt{\frac{1}{27}}$$

Thus we have found the maximum value.