Math 74, Week 5

Tianshuang (Ethan) Qiu

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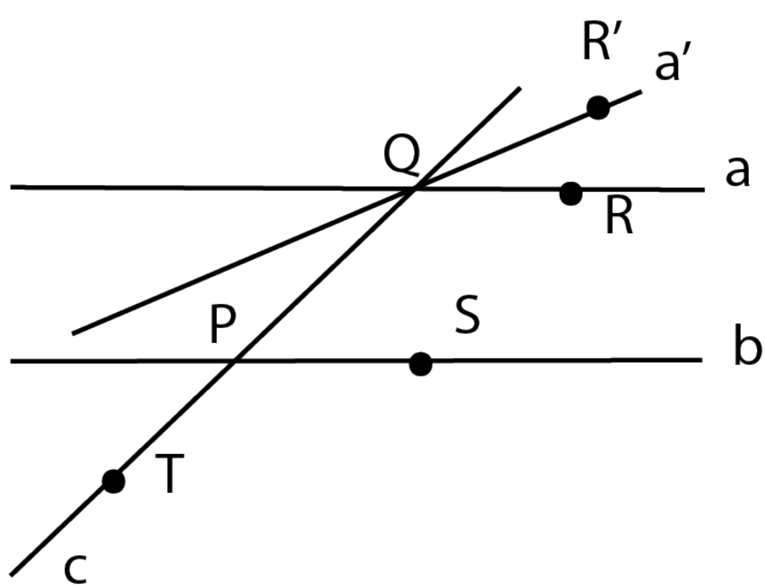
1 Mon Lec, 2a

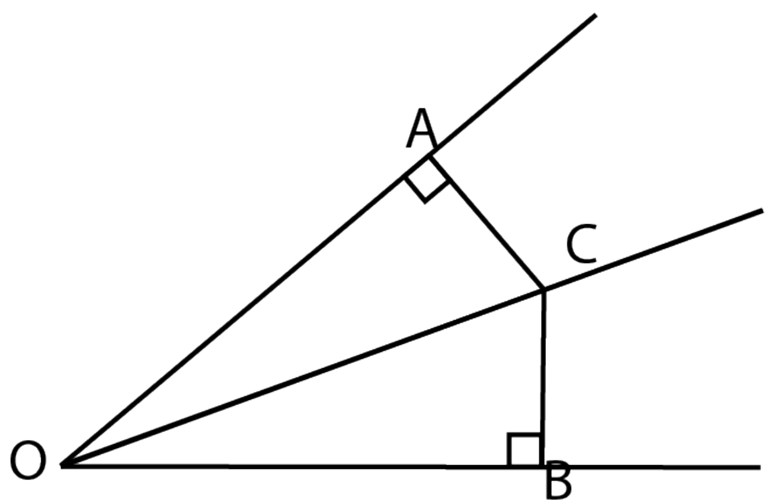
Let the statement: "There is at most one parallel line to a given line l through a given point P." be statement A;
"If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other." be statement B.



We first prove that $A \implies B$. Let $a \parallel b$, and c intersects a at point Q. Assume that statement B is false so c does not intersect b. Since it does not intersect and b and c are coplanar, we have $b \parallel c$. $a \parallel b$, and a, c interesect

at Q. However by A we know that there can be at most one line parallel to b at point Q. ↯. Our assumption is incorrect, $A \implies B$
Then we show that $B \implies A$. Let $a \parallel b$, and c intersects b at point P. Assume that A is incorrect, so we construct a' to also be parallel to b. We know that b must intersect a by B. We name this point P. Then since we have assumed that $a \parallel a'$, our line must also intersect a' at P. Consider $\angle SPT, \angle RQT$, and they must be equal because they $a \parallel b$, and corresponding angles are equal when the two lines are parallel. By the same logic we have $\angle SPT = \angle R'QT$.
Using the transitive property we can get $\angle RQT = \angle R'QT$. However this cannot be true because if the two angles are equal, a, a' overlap and they become the same line. ↯
Our assumption is incrroect and $B \implies A$. Therefore $A \iff B$. Q.E.D.

2 Mon Lec, 3a



2.1 Bisector $\implies$ equal distance from legs

Let OC bisect $\angle AOB$, choose A, B such that $CA \perp OA, CB \perp OB$
Consider $\triangle AOC, BOC$, since $CA \perp OA, CB \perp OB$, we can write the fol-

lowing using the inner sum of triangles:

$$\angle ACO + \angle COA + 90^\circ = 190^\circ$$

$$\angle BCO + \angle COB + 90^\circ = 190^\circ$$

Since OC bisect $\angle AOB$, we have $\angle COA = \angle COB$, so $\angle ACO = \angle BCO$. Finally, since $\triangle AOC, BOC$ share OC, we have $\triangle AOC \cong \triangle BOC$ (ASA congruency). Therefore $CA = CB$. Q.E.D.

2.2 Equal distance from legs $\implies$ angle bisection

Let OC be a ray from O, choose point C and draw $CA \perp OA, CB \perp OB$.
$AC = BC$
Consider $\triangle AOC, BOC$, since $CA \perp OA, CB \perp OB$, they are both right triangles. Using $AC = BC$, we have $\triangle AOC \cong \triangle BOC$ (HL right triangle congruency). Therefore $\angle COA = \angle COB$. Q.E.D.

3 Mon Dis, 1b (second bullet)

$$\prod_{1}^{n} = (1 - \frac{1}{n^2})$$

We examine $1 - 1/k^2$ and factor it into $\frac{k^2-1}{k^2} = \frac{(k+1)(k-1)}{k^2}$. Since k is incrementing by 1 in our series, we can cancel the majority of terms out since it is telescoping. We can expand our series into

$$\frac{1 \times 3}{2^2} \times \frac{2 \times 4}{3^2} \times ... \times \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{1}{2} \times \frac{n+1}{n} = \frac{n+1}{2n}$$

4 Mon Dis, 1d

(Assuming that)

4.1 $4^n + 15n - 1$

The largest common divisor for these expressions is 1.

4.2 $n^3 - n$

The largest common divisor for these expressions is 6.

4.3 $2^{n+2} + 7n$

The largest common divisor for these expressions is 5.

5 Wed Lec, 3a

Base case: $n = 1, 1 = 1^2$. Base case holds.
Inductive hypothesis: assume that for some $n \geq 1$, $1+3+5+...+(2n-1) = n^2$.
Inductive proof: consider $n+1$, $1+3+5+...+(2n-1)+(2n+1)$, using our inductive hypothesis, we can susbsitute everything but the last term: $n^2+2n+1 = (n+1)^2$
Thus we have proven the inductive step. Q.E.D.

6 Wed Lec, 3c

Base case: $n = 1, 1/(4 \times 1^2 - 1) = 1/3 = 1/(2 \times 1 + 1)$. Base case holds.
Inductive hypothesis: assume that for some $n \geq 1$, $\frac{1}{4\times 1^2-1} + \frac{1}{4\times 2^2-1} + ... + \frac{1}{4\times n^2-1} = \frac{n}{2n+1}$.
Inductive proof: consider $n+1$,

$$\frac{1}{4\times 1^2-1} + \frac{1}{4\times 2^2-1} + ... + \frac{1}{4\times n^2-1} + \frac{1}{4\times (n+1)^2-1}$$

Using our inductive hypothesis, we can susbsitute everything but the last term:

$$\frac{n}{2n+1} + \frac{1}{4\times(n+1)^2-1} = \frac{n(4(n+1)^2-1)}{(2n+1)(4(n+1)^2-1)} + \frac{2n+1}{(2n+1)(4(n+1)^2-1)}$$

$$= \frac{n(4n^2+3+8n)+2n+1}{(2n+1)(4n^2+3+8n)} = \frac{4n^3+3n+8n^2+2n+1}{8n^3+6n+16n^2+4n^2+3+8n} = \frac{4n^3+8n^2+5n+1}{8n^3+20n^2+14n+3}$$

We apply long division by $2n+3$ to the denominator.

$$\frac{8n^3+20n^2+14n+3}{2n+3} = 4n^2+4n+1$$

Now we apply long division by $n+1$ to the numerator.

$$\frac{4n^3+8n^2+5n+1}{n+1} = 4n^2+4n+1$$

Therefore we can factor the expression into

$$\frac{(n+1)(4n^2+4n+1)}{(2n+3)(4n^2+4n+1)} = \frac{n+1}{2n+3}$$

Thus we have proven the inductive step. Q.E.D.

7 Wed Lec, 6b