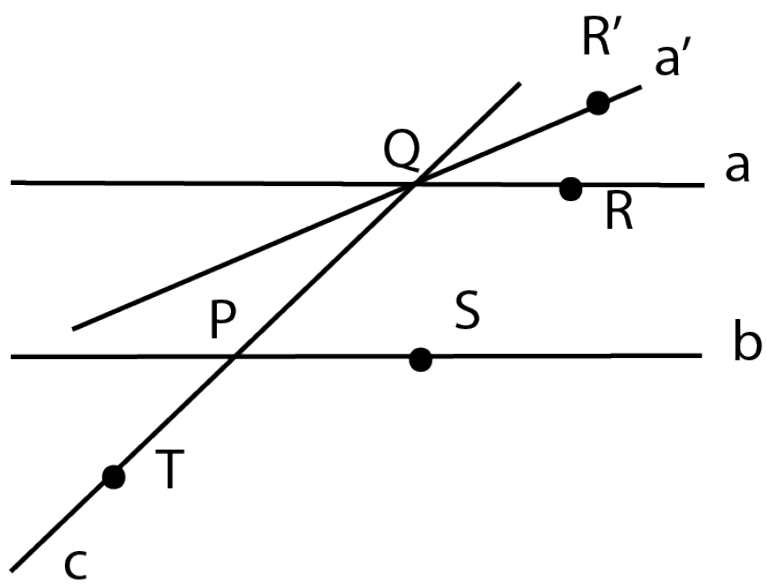
Math 74, Week 5

Tianshuang (Ethan) Qiu

September 27, 2021

1 Mon Lec, 2a

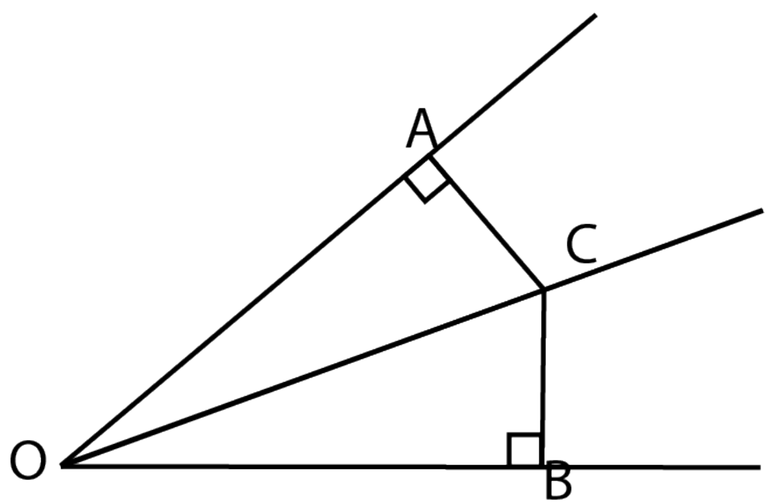
Let the statement: "There is at most one parallel line to a given line l through a given point P." be statement A;
"If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other." be statement B.



We first prove that $A \implies B$. Let $a \parallel b$, and c intersects a at point Q. Assume that statement B is false so c does not intersect b. Since it does not intersect and b and c are coplanar, we have $b \parallel c$, $a \parallel b$, and a, c interesect at

Q. However by A we know that there can be at most one line parallel to b at point Q. ↯. Our assumption is incorrect, $A \implies B$
Then we show that $B \implies A$. Let $a \parallel b$, and c intersects b at point P. Assume that A is incorrect, so we construct a' to also be parallel to b. We know that b must intersect a by B. We name this point P. Then since we have assumed that $a \parallel a'$, our line must also intersect a' at P. Consider $\angle SPT, \angle RQT$, and they must be equal because they $a \parallel b$, and corresponding angles are equal when the two lines are parallel. By the same logic we have $\angle SPT = \angle R'QT$.
Using the transitive property we can get $\angle RQT = \angle R'QT$. However this cannot be true because if the two angles are equal, a, a' overlap and they become the same line. ↯
Our assumption is incrroect and $B \implies A$. Therefore $A \iff B$. Q.E.D.

2 Mon Lec, 3a



2.1 Bisector $\implies$ equal distance from legs

Let OC bisect $\angle AOB$, choose A, B such that $CA \perp OA, CB \perp OB$
Consider $\triangle AOC, BOC$, since $CA \perp OA, CB \perp OB$, we can write the fol-

lowing using the inner sum of triangles:

$$\angle ACO + \angle COA + 90^\circ = 190^\circ$$

$$\angle BCO + \angle COB + 90^\circ = 190^\circ$$

Since OC bisect $\angle AOB$, we have $\angle COA = \angle COB$, so $\angle ACO = \angle BCO$. Finally, since $\triangle AOC, BOC$ share OC, we have $\triangle AOC \cong \triangle BOC$ (ASA congruency). Therefore $CA = CB$. Q.E.D.

2.2 Equal distance from legs $\implies$ angle bisection

Let OC be a ray from O, choose point C and draw $CA \perp OA, CB \perp OB$.
$AC = BC$
Consider $\triangle AOC, BOC$, since $CA \perp OA, CB \perp OB$, they are both right triangles. Using $AC = BC$, we have $\triangle AOC \cong \triangle BOC$ (HL right triangle congruency). Therefore $\angle COA = \angle COB$. Q.E.D.

3 Mon Dis, 1b (second bullet)

$$\prod_{1}^{n} = (1 - \frac{1}{n^2})$$

We examine $1 - 1/k^2$ and factor it into $\frac{k^2-1}{k^2} = \frac{(k+1)(k-1)}{k^2}$. Since k is incrementing by 1 in our series, we can cancel the majority of terms out since it is telescoping. We can expand our series into

$$\frac{1 \times 3}{2^2} \times \frac{2 \times 4}{3^2} \times ... \times \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{1}{2} \times \frac{n+1}{n} = \frac{n+1}{2n}$$

4 Mon Dis, 1d

4.1 $4^n + 15n - 1$

The largest common divisor for these expressions is 3.

4.2 $n^3 - n$

The largest common divisor for these expressions is 6.

4.3 $2^{n+2} + 7n$

The largest common divisor for these expressions is 5.

5 Wed Lec, 3a

Base case: $n = 1, 1 = 1^2$. Base case holds.
Inductive hypothesis: assume that for some $n \geq 1$, $1+3+5+...+(2n-1) = n^2$.
Inductive proof: consider $n+1$, $1+3+5+...+(2n-1)+(2n+1)$, using our inductive hypothesis, we can susbsitute everything but the last term: $n^2+2n+1 = (n+1)^2$
Thus we have proven the inductive step. Q.E.D.

6 Wed Lec, 3c

Base case: $n = 1, 1/(4 \times 1^2 - 1) = 1/3 = 1/(2 \times 1 + 1)$. Base case holds.
Inductive hypothesis: assume that for some $n \geq 1$, $\frac{1}{4\times1^2-1} + \frac{1}{4\times2^2-1} + ... + \frac{1}{4\times n^2-1} = \frac{n}{2n+1}$.
Inductive proof: consider $n+1$,

$$\frac{1}{4\times1^2-1} + \frac{1}{4\times2^2-1} + ... + \frac{1}{4\times n^2-1} + \frac{1}{4\times(n+1)^2-1}$$

Using our inductive hypothesis, we can susbsitute everything but the last term:

$$\frac{n}{2n+1} + \frac{1}{4(n+1)^2-1} = \frac{n(4(n+1)^2-1)}{(2n+1)(4(n+1)^2-1)} + \frac{2n+1}{(2n+1)(4(n+1)^2-1)}$$

$$= \frac{n(4n^2+3+8n)+2n+1}{(2n+1)(4n^2+3+8n)} = \frac{4n^3+3n+8n^2+2n+1}{8n^3+6n+16n^2+4n^2+3+8n} = \frac{4n^3+8n^2+5n+1}{8n^3+20n^2+14n+3}$$

We apply long division by $2n+3$ to the denominator.

$$\frac{8n^3+20n^2+14n+3}{2n+3} = 4n^2+4n+1$$

Now we apply long division by $n+1$ to the numerator.

$$\frac{4n^3+8n^2+5n+1}{n+1} = 4n^2+4n+1$$

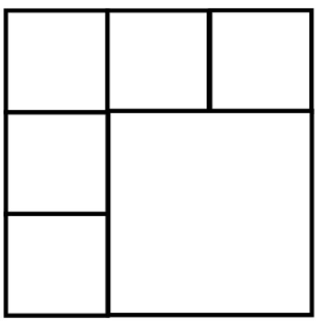
Therefore we can factor the expression into

$$\frac{(n+1)(4n^2+4n+1)}{(2n+3)(4n^2+4n+1)} = \frac{n+1}{2n+3}$$

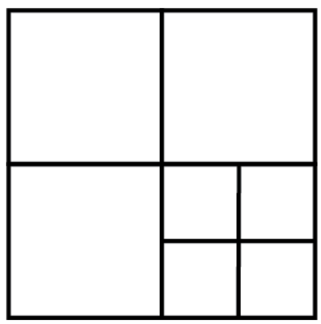
Thus we have proven the inductive step. Q.E.D.

7 Wed Lec, 6b

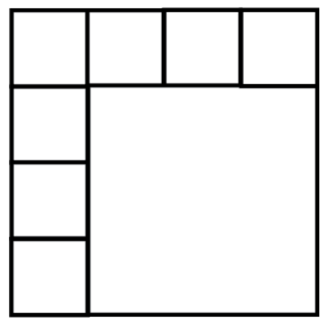
We claim that we can divide a single square into all $n \in \mathbb{N}, n \geq 6$ squares.
First we define a process of "opening a window": by connecting midpoints of sides that are across each other, we draw a cross on a square, dividing it into 4 smaller squares. The total gain of this process is $4 - 1 = 3$ squares.
We split the cases into the following: $n = 3k, n = 3k + 1, n = 3k + 2 (k \geq 2)$, and we will prove by induction each case.
Base cases: $n = 6, 7, 8$, we can divide it up like the following:



(a) $n = 6$



(b) $n = 7$



(c) $n = 8$

Inductive case: Assume that for some $k \geq 2$, we can divide a square into $3k, 3k + 1, 3k + 2$ smaller squares, then in order to get $3(k + 1)$ squares, we can simply "open a window" in any of the $3k$ sub squares. This yields a total of $3k + 3 = 3(k + 1)$ squares.
We can prove the cases for $3(k + 1) + 1, 3(k + 1) + 2$ by also "opening a window" using a subsquare in squares that has been divided into $3k + 1$ and $3k + 2$ parts, yielding $3k + 4$ and $3k + 5$ squares.
Thus we have proven the inductive case. Q.E.D.

8 Wed Dis, 3a

Base case: $n = 1$, $1^3 = (1 \times 2/2)^2 = 1$, base case holds.
Inductive case: Assume that

$$1^3 + 2^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$$

for some $n \geq 1$, consider $1^3 + 2^3 + ... + n^3 + (n+1)^3$, we can apply our inductive hypothesis to get

$$= (\frac{n(n+1)}{2})^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2 + 4(n+1))}{4} = \frac{(n+1)^2(n^2 + 4n + 4)}{4} = (\frac{(n+1)^2(n+2)^2}{2^2})$$

The expression above is exactly what we would expect if we subsitute $(n+1)$ for n in $(\frac{n(n+1)}{2})^2$. Therefore we have proven the inductive case. Q.E.D.

9 Wed Dis, 4a

Base case: $n = 0, 1 = (x-1)/(x-1) = 1$ since $x \neq 1$. Base case holds.
Inductive case: Assume that the statement is true for some $n \geq 0$. Consider $n+1$, we have: $1 + x + x^2 + ... + x^n + x^{n+1}$. Now we apply our inductive hypothesis:

$$= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} = \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x-1} = \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x-1} = \frac{x^{n+2} - 1}{x - 1}$$

The expression above is exactly what we would expect if we subsitute $(n+1)$ for n in $\frac{x^{n+1}-1}{x-1}$. Thus we have proven the inductive case. Q.E.D.

10 Fri Lec, 2b

$n! < n^n \forall n \geq 2$
Base case: $n = 2, n! = 2, n^n = 4, n! < n^n$, base case holds.
Inductive case: assume that $n! < n^n$ for some $n \geq 2$, consider $(n+1)!, (n+1)^{n+1}$. By the definition of factorials we have $(n+1)! = (n+1)n! < (n+1)n^n$
Now, consider $(n+1)n^n, (n+1)^{n+1}$. Since the latter is $n+1$ multiplied by itself $n+1$ times, and the former only has $n+1$ once and n n times, and since $n < n+1$, we have $(n+1)n^n < (n+1)^{n+1}$.
Now we can put it all together: $(n+1)! = (n+1)n! < (n+1)n^n < (n+1)^{n+1}$.
Thus we have proven the inductive case. Q.E.D.

11 Fri Lec, 3a

Base case: $n = 0, n^3 - n = 0, n \bmod 6 \equiv 0$ Base case holds.
Inductive case: Assume that $(n^3 - n) \mid 6$ for some $n \geq 0$, consider $(n+1)^3 - (n+1)$.

$$= n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$$

Integers that are divisible by 2 are spaced such that there is one every other, and those divisible by 3 are spaced such that there is one every third. Here we have a product of 3 consecutive integers, so there must be at least 1 integer between $n, n+1, n+2$ that is divisible by 2, and at least 1 that is divisible by 3. Since $6 = 2 \times 3$, we have $n(n+1)(n+2) \mid 6$.
We have proven the inductive case. Q.E.D.

12 Fri Lec, 3b

Base case: $n = 0, 2^2 + 7^0 = 5, 5 \bmod 5 \equiv 0$ Base case holds.
Inductive case: Assume that $2^{n+2} + 7^n \mid 5$ for some $n \geq 0$, consider $2^{n+3} + 7^{n+1}$.
We add then subtract $2^{n+2} \times 7$ from the latter, this does not change the value but will alow us to manipulate the terms:

$$7 \times 2^{n+2} - 7 \times 2^{n+2} + 2^{n+3} + 7^{n+1} = 7(2^{n+2} + 7^n) + 2^{k+2}(2 - 7)$$

$$= 7(2^{n+2} + 7^n) - 5 \times 2^{k+2}$$

By the inductive hypothesis $2^{n+2} + 7^n \mid 5$, and since $-5 \mid 5$, both terms are divisible by 5, therefore the sum must also be divisible by 5.
We have proven the inductive case. Q.E.D.