Hw9

Saturday, April 9, 2022 5:51 PM

PHIL 12A - Spring 2022

Problem Set 9

94 points.

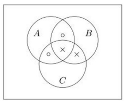
5 Syntax and Semantics of Monadic Predicate Logic

5.1 Syllogistic Logic

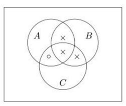
1. **practice** Exercise 3.2, 3.3, and 3.4 in *Logic in Action*.

   Solution:

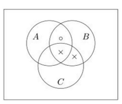
   Exercise 3.2 on page 3-15: The syllogistic pattern is not valid because it is possible to build a counter-example in which the premises are both true but the conclusion is false. This can be seen in the following diagram, where A = 'philosopher', B = 'barbarian' and C = 'greek':



   Exercise 3.3 on page 3-16: The syllogistic pattern is not valid because it is possible to build a counter-example in which the premises are both true but the conclusion is false. This can be seen in the following diagram, where A = 'philosopher', B = 'barbarian' and C = 'greek':



   Exercise 3.4 on page 3-16: The syllogistic pattern is valid because after we update with the information given in the premises it is impossible for the conclusion to be false. This can be illustrated by the following diagram, where A = 'philosopher', B = 'barbarian' and C = 'greek':



2. **6 points** Consider the following syllogisms:

   (1) No B is A / Some A are C / No B is C.

   (2) Not all A are B / Not all B are C / Not all A are C

   (3) All A are C / Not all B are C / Not all B are A

   (a) Which ones of the syllogisms are valid and which ones are invalid?

   (b) Use the diagram method to show the validity of the syllogisms you claimed are valid. (Check chapter 3 of *Logic in Action* for the diagram method).

   (c) Use diagrams and show the invalidity of the syllogisms you claimed are invalid.

5.2 Pure Monadic Predicate Logic

5.2.1 Pure Monadic Predicate Logic I

3. **practice** Exercise 4.6 in *Logic in Action*.

   Solution:

   $\forall x(B(x) \wedge W(x))$ expresses "Everything is a boy and walks."

   $\exists x(B(x) \rightarrow W(x))$ expresses "There is something (someone) that walks if it's a boy."

4. **3 points** Formalize the syllogistic argument number (1) in question 2 in monadic predicate logic.

5. **8 points total; each part 2 points** Give a recursive definition of the set of *free variables* of a formula φ, i.e., those variables with at least one free occurrence in φ, as described in the slides. To do so, fill in the question marks in the following template (note that we do not assume that the metavariables 'x' and 'y' refer to distinct variables, so you need to analyze the case where $x = y$ and the case where $x \neq y$):

   (a) x is a free variable of $P(y)$ iff ?

   (b) x is a free variable of $\neg\varphi$ iff ?

   (c) for $\# \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, x is a free variable of $(\varphi \# \psi)$ iff ?

   (d) for $Q \in \{\forall, \exists\}$, x is a free variable of $Qy\varphi$ iff ?

6. **10 points** Given a model $\mathcal{M} = (D, I)$, a subset $A \subseteq D$ is said to be *definable* in the language of pure monadic predicate logic iff there is a pure monadic formula φ with exactly one free variable x such that for some variable assignment g,

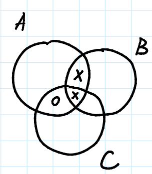
$$A = \{d \in D \mid \mathcal{M} \vDash_{g[x:=d]} \varphi\},$$

   i.e., A is exactly the set of objects d such that φ is true under the variable assignment that maps x to d.

   For example, in the model in Figure 1, the set $\{1, 2\}$ is defined by the formula $\neg\texttt{Sophomore}(x)$. For **every subset** of the domain of that model, say whether it is definable by a formula or not. If it is definable, give a formula that defines it.

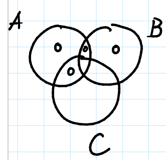
2.

(1)



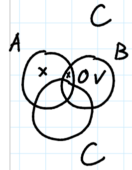
INVALID

(2)



INVALID

(3)



VALID

4.

$$\frac{[\forall x\ (B(x) \rightarrow \neg A(x))] \quad [\exists x\ (A(x) \wedge C(x))]}{\forall x\ (B(x) \rightarrow \neg C(x))}$$

5.

(a) $x = y$

(b) x is a free var of φ

(c) x is a free var in φ or in ψ

(d) x is a free var in φ, $x \neq y$

$$A = \{d \in D \mid \mathcal{M} \vDash_{g[x:=d]} \varphi\},$$

i.e., A is exactly the set of objects d such that φ is true under the variable assignment that maps x to d.

For example, in the model in Figure 1, the set $\{1, 2\}$ is defined by the formula $\neg$`Sophomore`(x). For **every subset** of the domain of that model, say whether it is definable by a formula or not. If it is definable, give a formula that defines it.

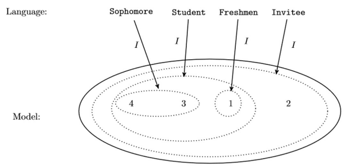


Figure 1: Model for Problem 6.

5.2.2 Pure Monadic Predicate Logic II

7. **practice** Exercise 4.14 of *Logic in Action*.

   Solution:

   (1) $\neg\forall x\neg(A(x) \wedge B(x))$;

   (2) $\neg\forall x(A(x) \rightarrow B(x))$;

   (3) $\forall x(A(x) \rightarrow \neg B(x))$

8. **3 points** Translate the following syllogistic statements into monadic predicate logic, withough using universal quantifiers:
   (a) Not all A are B.
   (b) No A is B.
   (c) All non-A are B.

9. **16 points** Determine whether each of the following formulas are true in the model in Figure 2, where R stands for red, G stands for green, B stands for blue, P stands for purple, S stands for square, and C stands for circle:
   (a) **practice** $\exists x(P(x) \wedge C(x))$;
   Solution True.
   (b) **practice** $\forall x(C(x) \vee S(x))$;
   Solution True.
   (c) $\exists x P(x) \wedge \exists x G(x)$;
   (d) $\forall x C(x) \vee \forall x S(x)$;
   (e) $\forall x(\neg R(x) \rightarrow S(x))$
   (f) $\forall x(G(x) \rightarrow P(x))$;
   (g) $\exists x(P(x) \rightarrow \neg C(x))$;
   (h) $\exists x(\neg P(x) \vee \neg C(x))$
   (i) $\exists x G(x) \vee \neg\exists x\neg R(x)$;
   (j) $\exists x(G(x) \vee (P(x) \wedge \neg C(x)))$.
   (k) $\forall x(\neg B(x) \rightarrow (\neg S(x) \vee G(x)))$

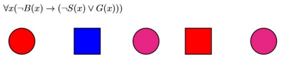


Figure 2:

10. **practice** Exercise 4.17 of *Logic in Action*.

    Solution: The book states that $\exists x\forall y\varphi \vDash \forall y\exists x\varphi$ and $\forall y\forall x\varphi \vDash \forall x\forall y\varphi$, while $\forall x\exists y\varphi \nvDash \exists y\forall x\varphi$. The remaining possible implication is $\exists x\exists y\varphi \vDash \exists y\exists x\varphi$, which indeed holds.

11. **10 points** For each of the following sentences, say whether or not it is valid. If it is not valid, present a model in which it is false.
    (a) **practice** $\forall x(P(x) \wedge Q(x)) \rightarrow (\forall x P(x) \wedge \forall x Q(x))$;
    Solution: Valid.

    (b) $\exists x(P(x) \wedge Q(x)) \leftrightarrow (\exists x P(x) \wedge \exists x Q(x))$;
    (c) $(\forall x P(x) \vee \forall x Q(x)) \rightarrow \forall x(P(x) \vee Q(x))$;
    (d) $\forall x(P(x) \vee Q(x)) \rightarrow \forall x\forall y(P(x) \vee Q(y))$;
    (e) $\forall x(\exists x P(x) \rightarrow P(x))$;
    (f) $\exists x(P(x) \rightarrow \forall x P(x))$.

12. **4 points** An important fact about pure monadic predicate logic is the following.

    Proposition 1. Each sentence φ of pure monadic predicate logic is equivalent to a sentence $\varphi^\star$ containing the same predicate symbols as φ *and only one variable.*

    To get a feel for this, **explain** the following equivalences by appealing to the semantics of monadic predicate logic:
    (a) $\forall x\exists y(P(x) \wedge Q(y))$ is equivalent to $\forall x P(x) \wedge \exists y Q(y)$;
    (b) $\forall x P(x) \wedge \exists y Q(y)$ is equivalent to $\forall z P(z) \wedge \exists z Q(z)$.

13. **5 points** *Extra Credit.* In the slides, we mentioned the important lemma that if a formula φ of pure monadic predicate logic is *not valid*, then it is falsified in a model on the domain $D = \{1, \ldots, 2^k\}$ where k is the number of predicate symbols appearing in φ. In this extra credit problem, we will show that if φ is not valid, then it is falsified in a model where D has *at most* 2^k elements (from which the lemma easily follows).

    Suppose $\mathcal{M} = (D, I)$ is a model and g an assignment such that $\mathcal{M} \nvDash_g \varphi$. We will shrink $\mathcal{M}$ to a model with no more than 2^k elements that also falsifies φ.

    Let $Pred(\varphi)$ be the set of all unary predicates that occur in φ. For example, if φ is $\forall x(P_1(x) \rightarrow P_3(x))$, then $Pred(\varphi) = \{P_1, P_3\}$.

    We assume that $Pred(\varphi)$ has k elements. It follows that $Pred(\varphi)$ has 2^k *subsets.*

    For each $d \in D$, let

    $$\widehat{d} = \{d' \in D \mid \text{ for all } P \in Pred(\varphi) : d \in I(P) \text{ iff } d' \in I(P)\}.$$

    That is, $\widehat{d}$ is the *set* of all objects that are "indistinguishable" from d in $\mathcal{M}$ using predicates that appear in φ. The idea behind the proof is that such indistinguishable objects should be *collapsed* into a single object $\widehat{d}$. (For example, in the model in Figure 6, the objects 1 and 3 cannot be distinguished from each other by any of the four predicates, so they can be collapsed into a single object $\widehat{1} = \widehat{3}$.) There can be at most 2^k distinct sets $\widehat{d}$ because each distinct $\widehat{d}$ defines a distinct subset of $Pred(\varphi)$, namely the subset $\{P \in Pred(\varphi) \mid d \in I(P)\}$, and there are only 2^k subsets of $Pred(\varphi)$. This leads us to the definition of our collapsed model $\widehat{\mathcal{M}} = (\widehat{D}, \widehat{I})$:
    - $\widehat{D} = \{\widehat{d} \mid d \in D\}$;
    - for each predicate $P \in Pred(\varphi)$, $\widehat{d} \in \widehat{I}(P)$ iff $d \in I(P)$;
    - for each predicate $P \notin Pred(\varphi)$, $\widehat{I}(P) = \varnothing$.

(d) x is a free var in φ, $x \neq y$

6. $\{1\}$: Freshman(x)
   $\{2\}$: $\neg$ Student(x)
   $\{1,2\}$: $\neg$ Sophomore(x)
   $\{3,4\}$: Sophomore(x)
   $\{2,3,4\}$: $\neg$ Freshman(x)
   $\{1,3,4\}$: Student(x)
   $\{1,2,3,4\}$: Invitee(x)
   $\{\}$: $\neg$ Invitee(x)

8. (a) $\exists x\ A(x) \wedge \neg B(x)$
   (b) $\forall x\ A(x) \rightarrow \neg B(x)$
   (c) $\forall x\ \neg A(x) \rightarrow B(x)$

9. (c) False
   (d) False
   (e) False
   (f) True
   (g) True
   (h) True
   (i) False
   (j) False
   (k) False

11. (b) Invalid   P(x): a, b   Q(x): c, d
    Second is true but first is false
    (c) Valid
    (d) Valid
    (e) Invalid   P(x): a, b   c
    (f) Valid

12. (a) $\forall x \exists y (P(x) \wedge Q(y))$
    for all $d \in D$, $\mathcal{M} \vDash g(x:=d)$ [for some $e \in D$, $\mathcal{M} \vDash g(y:=e)$ $P(x) \wedge Q(y)$]
    $= \mathcal{M} \vDash g(x:=d, y:=e)$ $P(x) \wedge Q(y)$
    $= \mathcal{M} \vDash g(x:=d, y:=e)$ $P(x)$ and $\mathcal{M} \vDash g(x:=d, y:=e)$ $Q(y)$
    True iff $d \in I(P)$ and $e \in I(Q)$

    $\forall x P(x) \wedge \exists y\ Q(y)$
    for all $d \in D$, $\mathcal{M} \vDash g(x:=d)$ $P(x)$ and for some $e \in D$, $\mathcal{M} \vDash g(y:=e)$ $Q(y)$
    $\mathcal{M} \vDash g(x:=d)$ $P(x)$ $\wedge$ $\mathcal{M} \vDash g(y:=e)$ $Q(y)$
    True iff $d \in I(P)$ and $e \in I(Q)$

- $D = \{d \mid d \in D\}$;
- for each predicate $P \in Pred(\varphi)$, $\widehat{d} \in \widehat{I}(P)$ iff $d \in I(P)$;
- for each predicate $P \notin Pred(\varphi)$, $\widehat{I}(P) = \varnothing$.

$d \in I(P)$ and $e \in I(Q)$

(b) $\forall x P(x) \wedge \exists y Q(y)$
for all $d \in D$, $\mathcal{M} \models_{g(x:=d)} P(x)$ AND
for some $e \in D$, $\mathcal{M} \models_{g(y:=e)} Q(y)$
True iff
$d \in I(P)$ and $e \in I(Q)$

(b) $\forall z P(z) \wedge \exists z Q(z)$
for all $d \in D$, $\mathcal{M} \models_{g(z:=d)} P(z)$ AND
for some $e \in D$, $\mathcal{M} \models_{g(z:=e)} Q(z)$
True iff
$d \in I(P)$ and $e \in I(Q)$

13. Base Case: ψ is an atomic formula, it is the form $P(x)$, if false ANYWHERE in D,
take it in $\hat{D}$, otherwise take any $d \in D$ to be $\hat{D}$. $\widetilde{M} \models_{\tilde{g}} P(x) \leftrightarrow M \models_g P(x)$.

Inductive case: Assume ψ is of the form $\exists x \alpha$. Then it's true iff for some $d \in D$, $\mathcal{M} \models_{g[x:=d]} \alpha$.
Let there be k preds in α. There can be at most 2^k distinct subsets.
Define $\widetilde{\mathcal{M}} = (\widetilde{D}, \widetilde{I})$. $\widetilde{\mathcal{M}} \models_{\tilde{g}} \alpha = \widetilde{\mathcal{M}} \models_{\tilde{g}[x:=\tilde{d}]}$.
$\tilde{d} = d$ iff $d \in I(P)$ for $P \in Pred(\alpha)$
$\therefore$ $\tilde{d}$ is equivalent to $d \in D$
$\therefore$ $\widetilde{\mathcal{M}} \models_{\tilde{g}} \alpha \leftrightarrow \widetilde{\mathcal{M}} \models_g \alpha$

Given any variable assignment g for $\mathcal{M}$, define the variable assignment $\widehat{g}$ for $\widehat{\mathcal{M}}$ by

$$\widehat{g}(x_i) = \widehat{g(x_i)}.$$

It follows that for any variable x and $d \in D$,

$$\widehat{g[x := d]} = \widehat{g}[x := \widehat{d}].$$

It also follows that for any subformula ψ of φ and variable assignment g,

$$\widehat{\mathcal{M}} \models_{\widehat{g}} \psi \text{ iff } \mathcal{M} \models_g \psi.$$

Prove this 'iff' by induction on ψ. Give only the base case where ψ is an atomic formula of the form $P(x)$ and the inductive step where ψ is of the form $\exists x\alpha$. Use the equations given above (plus the inductive hypothesis for the $\exists$ case).

From what you proved and our initial assumption that $\mathcal{M} \not\models_g \varphi$, it follows that $\widehat{\mathcal{M}} \not\models_{\widehat{g}} \varphi$, which shows that φ is indeed falsified in a model with at most 2^k elements!

5.3 Constants and Functions

5.3.1 Constants

14. **10 points** Let φ be a formula in which the constant c appears but the variable x does not appear. **Prove** that if φ is valid, then $\forall x \varphi^c_x$ is valid, where φ^c_x is the formula obtained from φ by replacing all occurrences of c with x. Intuitively: "if φ holds of an arbitrary element, then it holds of everything." In your argument you may assume (without proof) the fact that for any model $\mathcal{M} = (D, I)$ and variable assignment g for $\mathcal{M}$:

$$\mathcal{M} \models_g \varphi \text{ iff } \mathcal{M} \models_{g[x:=[\![c]\!]^g_I]} \varphi^c_x.$$

You may also assume that if two models $\mathcal{M}$ and $\mathcal{N}$ differ only on the interpretation of a constant that does not appear in a formula ψ, then $\mathcal{M} \models_g \psi$ iff $\mathcal{N} \models_g \psi$.

5.3.2 Function Symbols

15. **5 points** *Extra Credit* In the slides, we mentioned the fact that there is an algorithm that converts any formula φ of monadic predicate logic with unary function symbols into a formula φ' of monadic predicate logic *without function symbols* such that φ is valid iff φ' is valid.

To give you a feel for how this could be true, let ψ be a formula of monadic predicate logic, P a unary predicate, and f a unary function symbol not occurring in ψ. **Prove** that the formula

$$\forall z(\psi(z) \to P(f(z)))$$

is valid if and only if the formula

$$\forall x \exists y(P(y) \leftrightarrow Q(x)) \to \forall z(\psi(z) \to Q(z))$$

14. We will prove the contrapositive. Assume that $\forall x \varphi^c_x$ is not valid. Then $\mathcal{M} \not\models_g \forall x \varphi^c_x$, and there are some $d \in D$ where $\mathcal{M} \not\models_{g[x:=d]} \varphi^c_x$.
Consider $\mathcal{M}' = (D, I')$, where $I'(c) = d$,
then $\mathcal{M}' \models_g \varphi \leftrightarrow \mathcal{M} \models_{g[x:=d]} \varphi^c_x$
thus $\mathcal{M}' \not\models_g \varphi$

is valid, where Q is a new predicate symbol that does not appear in ψ.

Suggestion: suppose $\forall z(\psi(z) \to P(f(z)))$ is *not* valid, so it is falsified in some model $\mathcal{M}$. Then define a model $\mathcal{M}'$ that differs from $\mathcal{M}$ at most in the interpretation of the predicate Q (i.e., just define $I^{\mathcal{M}'}(Q)$ for us), such that $\mathcal{M}'$ falsifies $\forall x \exists y(P(y) \leftrightarrow Q(x)) \to \forall z(\psi(z) \to Q(z))$ (prove it). In the other direction, suppose $\forall x \exists y(P(y) \leftrightarrow Q(x)) \to \forall z(\psi(z) \to Q(z))$ is not valid, so it is falsified in some model $\mathcal{N}$. Then define a model $\mathcal{N}'$ that differs from $\mathcal{N}$ at most in the interpretation of the function symbol f (i.e., just define $I^{\mathcal{N}'}(f)$ for us) such that $\mathcal{N}'$ falsifies $\forall z(\psi(z) \to P(f(z)))$ (prove it).

5.4 Identity and Substitution

5.4.1 The Identity Predicate

16. **practice** Exercise 4.36 of *Logic in Action*. Write out your formula with no abbreviations.

    Solution: $\exists x \exists y \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge \forall v(P(v) \leftrightarrow (v = y \vee v = x \vee v = z)))$

17. **practice** Exercise 4.38 (2) of *Logic in Action*. Demonstrate the difference in meaning by providing a model in which the two formulas have different truth values.

    Solution:

    (1) Let $\mathcal{M}$ be a model such that $D = \{1,2,3,4,5\}$, $I(B) = \{1,3,5\}$, $I(G) = \{2,4\}$, and $I(L) = \{\langle 1,2\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 5,2\rangle\}$. Then $\mathcal{M} \models_g \exists! y \exists! x(B(x) \wedge G(y) \wedge L(x,y))$ (as 4 is the unique girl loved by exactly one boy), but $\mathcal{M} \not\models_g \exists! x \exists! y(B(x) \wedge G(y) \wedge L(x,y))$ (as both boys 1 and 5 love exactly one girl). Thus, the formulas express different things.

    (2) Let $\mathcal{M}$ be a model such that $D = \{1,2\}$, $I(A) = \{1\}$ and $I(B) = \{2\}$. Then $\mathcal{M} \models_g \exists! x A(x) \vee \exists! x B(x)$, but $\mathcal{M} \not\models_g \exists! x(A(x) \vee B(x))$. Thus the formulas express different things.

5.4.2 Substitution

18. **6 points** Give the results of the following substitutions:

    (a) **practice** $(P(x))^x_c$;

    Solution: $(P(x))^x_c = P(x^x_c) = P(c)$

    (b) $(P(x) \to (Q(x) \to R(y)))^x_y$;

    (c) $(P(x) \to \forall x P(x))^x_y$;

    (d) $(\exists x(P(x) \vee Q(y)))^x_y$;

    (e) $(\exists x(P(x) \vee Q(y)))^y_x$;

    (f) $(P(f(x)) \to \exists x P(f(x)))^x_{f(x)}$;

    (g) $\exists x(P(x) \vee Q(f(y)))^y_{f(x)}$.

18. (b) $P(y) \to (Q(y) \to R(y))$

(c) $P(y) \to \forall x P(x)$

(d) $\exists x(P(x) \vee Q(y))$

(e) $\exists x(P(x) \vee Q(y))$

(f) $P(f(f(x))) \to \exists x P(f(x))$

(g) $\exists x(P(x) \vee Q(f(y)))$

19. **8 points total; 2 points each part** Give a recursive definition of the set of terms t that are *substitutable* for a variable x in a formula φ. That is, fill in the question marks in the following template (note that we do not assume that the metavariables 'x' and 'y' refer to distinct variables, so you need to analyze the case where $x = y$ and the case where $x \neq y$ in part (d)):

    (a) t is always substitutable for x in an atomic formula $P(u)$ or $s \doteq t$ (because there are no quantifiers in an atomic formula to capture a variable in t);

    (b) t is substitutable for x in $\neg\varphi$ iff ?

    (c) for $\# \in \{\wedge, \vee, \to, \leftrightarrow\}$, t is substitutable for x in $(\varphi \# \psi)$ iff ?

    (d) for $Q \in \{\forall, \exists\}$, t is substitutable for x in $Qy\varphi$ iff ?

19.

(a) True

(b) t is substitutable in φ

(c) t is substitutable in φ and in ψ

(d) t is a free and unbound variable in φ