Hw10

Sunday, April 17, 2022 1:55 PM

PHIL 12A – Spring 2022

Problem Set 10 with Practice Solutions

67 points

1 Syntax and Semantics of Monadic Predicate Logic

1.1 Identity and Substitution

1.1.1 The Identity Predicate

1. **4 points** Write out a formula with no abbreviations, stating that "there are exactly 2 objects with property P, and at most two object with property Q".

2. **Practice** Exercise 4.38(2) of *Logic in Action*. Demonstrate the difference in meaning by providing a model in which the two formulas have different truth values.

Solution:

(1) Let $\mathcal{M}$ be a model such that $D = \{1,2,3,4,5\}$, $I(B) = \{1,3,5\}$, $I(G) = \{2,4\}$, and $I(L) = \{\langle 1,2\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 5,2\rangle\}$. Then $\mathcal{M} \vDash_g \exists! y \exists! x(B(x) \wedge G(y) \wedge L(x,y))$ (as 4 is the unique girl loved by exactly one boy), but $\mathcal{M} \nvDash_g \exists! x \exists! y(B(x) \wedge G(y) \wedge L(x,y))$ (as both boys 1 and 5 love exactly one girl). Thus, the formulas express different things.

(2) Let $\mathcal{M}$ be a model such that $D = \{1,2\}$, $I(A) = \{1\}$ and $I(B) = \{2\}$. Then $\mathcal{M} \vDash_g \exists! x A(x) \vee \exists! x B(x)$, but $\mathcal{M} \nvDash_g \exists! x(A(x) \vee B(x))$. Thus the formulas express different things.

2 Syntax and Semantics of Predicate Logic

3. These problems concern translating English sentences into the language of predicate logic:

   (a) **2 points** Assume the domain of discourse is all dogs. Translate the following sentences into predicate logic: (Use f for Fido, r for Rover, and L for loves)
   - **Practice**: Someone is loved by Rover. **Solution**: $\exists x L(r,x)$
   - Everyone who loves Rover loves Fido too.
   - Fido and Rover love each other.

   (b) **4 points** For each of the following, specify an appropriate domain of discourse, specify a translation key, and translate into predicate logic. (Note: you do not have to understand what a sentence means before you can attempt to translate it.)

   - **Practice** There is a greatest natural number.
     Solution: Let the domain be $\mathbb{N}$. Let $>$ be the greater-than relation.
     $$\exists x \forall y(x > y)$$
   - There is no largest number divisible by 3.
   - Every odd number is greater than some even number.
   - Dogs that love cats don't taunt **other** dogs.
   - Fido's friends are friends of Rover's friends.

   (c) **4 points** Translate the following sentences into predicate logical formulas. Assume the domain of discourse is cats and dogs.
   - Some dogs are not liked by all cats.
   - Every cat that is loved by all cats doesn't love any dog.
   - Every dog that only loves dogs is purred at by some cat who loves no dogs.
   - No cat loves a cat who is loved by a dog, except for the cats who love cats.

4. These problems concern describing situations using the language of predicate logic:

   (a) **Practice** Exercise 4.18 of *Logic in Action*.
   Solution: For example, $\forall x(G(x) \to \exists y(H(y) \wedge W(x,y)))$ ("every girl is wearing a hat") is true. By contrast, $\exists y(H(x) \wedge \forall x(G(x) \to W(x,y)))$ ("there is a hat such that every girl is wearing that hat") is false. There are many other acceptable answers.

   (b) **5 points** Exercise 4.22 of *Logic in Action*.

5. These problems concern describing binary relations:

   (a) **3 points** The formula $\forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \to R(x,z))$ expresses the *transitivity* of the relation R. Which of the following relations are transitive?
   - Being an ancestor of ... on the set of human beings.
   - Being a neighbor of ... on the set of human beings.
   - The *is divisible by* relation on the set of natural numbers. (Recall that a natural number n is divisible by a natural number m if there is some natural k such that $m \times k = n$.)

   (b) **Practice** Exercise 4.24 of *Logic in Action*.
   Solution:
   $$\exists x \exists y \exists z(R(x,y) \wedge R(y,z))$$

   (c) **Practice** Exercise 4.25 of *Logic in Action*.
   Solution: Place a reflexive loop on the middle point.

6. **12 points** The following natural language sentences are ambiguous: they can be interpreted in at least two different ways. For each one of them, provide two predicate logic formulas that correspond to two different readings (i.e., four formulas in total).

1. $\exists x,y\,(P(x) \wedge P(y) \wedge \neg x \doteq y) \wedge$
$\exists x,y,z\,[(Q(x) \wedge Q(y) \wedge Q(z)) \wedge (x \doteq y \vee x \doteq z \vee y \doteq z)]$

3. (a) · $\forall x\,(L(x,r) \to L(x,f))$
· $L(r,f) \wedge L(f,r)$

(b) · $\forall x\, \exists y\,((3|y) \wedge y > x)$
· $\forall x\,(2 \nmid x \to \exists y\,((2|y) \wedge x > y))$
· $\forall x\,(D(x) \wedge \exists y[L(x,y)] \to$
$\forall z\,(D(z) \wedge x \neq z) \to \neg T(x,z))$

· $\forall x\,(F(f,x) \to (\exists y\,[F(r,y) \wedge F(x,y)]))$

(c) · $\exists x\,(D(x) \wedge \exists y\, \neg L(y,x))$
· $\forall x\,(C(x) \wedge \forall y\,[C(y) \wedge L(x,y)] \to$
$\forall z\,(D(z) \to \neg L(x,z)))$
· $\forall x(D(x) \wedge \forall y[L(x,y) \to D(y)]$
$\to \exists z(C(z) \wedge P(z,x) \wedge \forall a(L(z,a) \to \neg D(a)))$

· $\forall x(C(x) \wedge \exists y(D(y) \wedge L(y,x))$
$\to \exists z(C(z) \wedge L(x,z)))$

4. (b) Let T be a constant and $g[T := \text{Tutankhamun}]$
$\forall x \forall y\,(M(x) \wedge \neg M(y) \wedge P(x,T) \wedge P(y,T) \to \exists a \exists b(M(a) \wedge P(a,x) \wedge P(a,y) \wedge \neg M(b)$
$\wedge P(b,x) \wedge P(b,y)))$

5. (a) i is transitive, family trees mean that it's all connected

$\wedge P(b,x) \wedge P(b,y)))$

5. (a) i is transitive, family trees mean that it's all connected

ii is not, ABC.

iii is transitive, $a = bn$, $b = cm$, $a = c \cdot mn$

(a) There is a cat who likes only dogs who like only dogs.

(b) Fido and Rover are liked by some cat.

Decide for each of the four predicate logic formulas that you have given whether they are true or false in the model in Figure 1. Explain your answers.

■: *dog* ●: *cat* ● → ■: ● *likes* ■ **f**:*Fido*, **r**:*Rover*, **t**:*Tibbles*, **l**:*Leo*

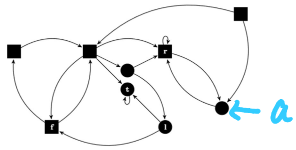

Figure 1: Model for Problem 6.

7. **10 points** Consider the model in Figure 2. Let the set of solid dots be the interpretation of the unary predicate symbol P. Let the edge relation of the graph be the interpretation of the binary predicate symbol R. What does

$$\forall x(P(x) \to (\exists y(R(x,y) \wedge \neg P(y) \wedge \forall z(R(y,z) \to P(z)))))$$

mean? Is this true in the model?

8. **8 points** Consider the four models in Figure 3. Let R be the binary predicate interpreted by the arrow in the diagram. Give for each model a predicate logic formula that is true in that model but not in the other three.

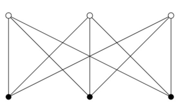

Figure 2: Model for Problem 7.

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6. (a) $\exists x (C(x) \wedge \exists y (L(x,y) \to D(y) \wedge \forall z (L(y,z) \to D(z))))$

False, no such cat who likes qualifying dogs and nothing else exists.

$\exists x (C(x) \wedge \exists y (L(x,y) \wedge D(y) \to \forall z (L(y,z) \to D(z))))$

True, Leo likes Fido, who only likes dogs.

(b) $\exists x (C(x) \wedge L(x,f) \wedge L(x,r))$

False, there is no cat who likes both Fido & Rover

$\exists x (C(x) \wedge L(x,f)) \wedge \exists x (C(x) \wedge L(x,r))$

True, Leo likes Fido and Cat a likes Rover, Leo and a are both cats.

7. For all solid points, it is connected to an empty point and all points connected to the empty point is a solid point. This statement is TRUE.

8. (a) $\exists x \exists y (x \neq y \wedge R(x,y) \wedge R(y,y))$

(b) $\exists x \exists y (x \neq y \wedge R(y,x) \wedge R(y,y))$

(c) $\exists x \exists y (x \neq y \wedge R(x,y) \wedge R(y,x))$

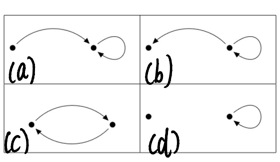

Figure 3: Models for Problem 8.

9. These problems concern validity and consequence in predicate logic:

(a) **6 points** Which of the following statements are true?

(i) $\vDash \forall x F(x) \vee \forall x \neg F(x)$;
(ii) $\vDash \exists x \neg F(x) \vee \forall x F(x)$;
(iii) $\vDash \forall x \forall y R(x,y) \to \forall y \forall x R(x,y)$;
(iv) $\vDash \exists x \forall y R(x,y) \leftrightarrow \forall x \exists y R(x,y)$;
(v) $\vDash \forall x R(x,x) \to \exists y \forall x R(y,x)$;
(vi) $\vDash \forall x R(x,x) \to \forall y \exists x R(y,x)$.

(b) **6 points** Which of the following statements are true? If the statement is false, provide a counterexample.

(i) $\vDash \forall x \forall y \forall z ((R(x,y) \wedge R(y,z)) \to R(x,z))$;
(ii) $\exists x R(x,x) \vDash \forall x \exists y R(y,y)$;
(iii) $\exists x \forall y R(x,y) \vDash \forall x \exists y R(x,y)$;
(iv) $\exists x \forall y R(x,y) \vDash \forall x \exists y R(y,x)$;
(v) $\exists x R(x,x) \vDash \exists x \exists y R(x,y)$;

(d) $\exists x \forall y (\neg R(x,y) \wedge \neg R(y,x))$

9. (a) (i) FALSE
(ii) TRUE
(iii) TRUE
(iv) FALSE
(v) FALSE
(vi) TRUE

(b)

(b) 6 points Which of the following statements are true? If the statement is false, provide a counterexample.

(i) $\vDash \forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \to R(x,z))$;
(ii) $\exists x R(x,x) \vDash \forall x \exists y R(y,y)$;
(iii) $\exists x \forall y R(x,y) \vDash \forall x \exists y R(x,y)$;
(iv) $\exists x \forall y R(x,y) \vDash \forall x \exists y R(y,x)$;
(v) $\exists x R(x,x) \vDash \exists x \exists y R(x,y)$;
(vi) $\forall x \exists x R(x,x) \vDash \forall x R(x,x)$.

(c) **4 points** Which of the following statements are true? If the statement is false, provide a counterexample.

(i) $\{\forall x \forall y (R(x,y) \to R(x,x))\} \vDash R(b,b)$;
(ii) $\{\forall x \forall y (R(x,y) \to R(y,x)), R(a,b)\} \vDash R(b,c)$;
(iii) $\{\forall x \exists y (R(x,y) \to R(y,x)), R(a,b)\} \vDash R(b,a)$;
(iv) $\{\forall x \forall y (R(x,y) \to R(y,x)), \neg R(a,b)\} \vDash \neg R(b,a)$.

10. **10 points Extra Credit** In an extra credit problem for Problem Set 8, we showed that if a formula of pure monadic predicate logic is satisfiable, then it is satisfiable in a model of size at most 2^k where k is the number of predicates occurring in the formula. (We stated this in terms of 'falsifiable', but φ is satisfiable iff $\neg\varphi$ is falsifiable, so we can state it either way.)

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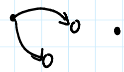
(vi) TRUE

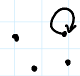
(b)
(i) FALSE, $R = a$ or b is TRUE, $x = \{0\}$, $y = \{1\}$, $z = \{0\}$

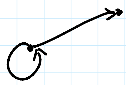
(ii) TRUE

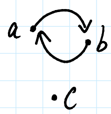
(iii) FALSE 

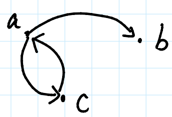
(iv) TRUE

(V) TRUE

(vi) FALSE 

(c)
(i) FALSE 

(ii) FALSE 

(iii) FALSE 

(iv) TRUE

Let us now consider a sentence with a binary predicate symbol:

$$\forall x \exists y R(x,y) \wedge \forall x \neg R(x,x) \wedge \forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \to R(x,z)).$$

Is this sentence satisfiable? If so, are there any *finite* models that make the sentence true? If so, give an example. If not, explain why not.

11. **8 points Extra Credit** A formula of predicate logic is in *prenex normal form* iff it is of the form

$$Q_1 x_1 \dots Q_n x_n \psi$$

where $Q_i \in \{\forall, \exists\}$ and ψ does not contain quantifiers. The following is an important fact about predicate logic.

Proposition 1. Every formula φ of predicate logic is equivalent to a formula in prenex normal form.

To get a feel for this, find prenex equivalents of the following formulas:

(a) $\exists x(\exists y R(y,x) \to P(x))$;
(b) $\forall x(P(x) \to \exists y R(x,y))$;
(c) $\exists x(P(x) \to \forall y R(x,y))$;
(d) $\exists x(\forall y R(y,x) \to P(x))$.

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10. Our model cannot have any disconnected point
$(\forall x \exists y R(x,y))$

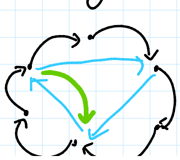
No reflexive point $(\forall x \neg R(x,x))$

Transitivity $\forall x \forall y \forall z[(R(x,y) \wedge R(y,z)) \to R(x,z)]$

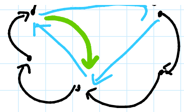
$\Rightarrow$ A model that satisfies the predicate does not contain , violates transitivity & no reflexivity

Assume that there is a finite model that satisfies the predicate. Let it have n elements.
By rule 1, we can form chains where $R(a,b)$, $R(b,c)$ ... exist, furthermore, since each point is connected to another point, this chain has no end.
$\Rightarrow \exists$ at least one cycle, let it have length m $(m \leq n)$

Let the elements be $\{\alpha_1 \dots \alpha_m\}$. By transitivity we can write $R(\alpha_1, \alpha_3)$, $R(\alpha_3, \alpha_5)$ ... Thus we reduce our model recursively until we have 2 elements, which is , and it is impossible.



$\Rightarrow$ It is impossible to be satisfied in any finite model.



$\Rightarrow$ It is impossible to be satisfied in any finite model.

11. (a) $\exists x(\exists y R(y,x) \to P(x))$

$= \exists x(\neg(\exists y R(y,x)) \vee P(x)) = \exists x \forall y(\neg R(y,x) \vee P(x))$

(b) $\forall x(P(x) \to \exists y R(x,y))$

$= \forall x(\neg P(x) \vee \exists y R(x,y)) = \forall x \exists y(\neg P(x) \vee R(x,y))$

(c) $\exists x(P(x) \to \forall y R(x,y))$

$= \exists x(\neg P(x) \vee \forall y R(x,y)) = \exists x \forall y(\neg P(x) \vee R(x,y))$

(d) $\exists x(\forall y R(y,x) \to P(x))$

$= \exists x(\neg(\forall y R(y,x)) \vee P(x)) = \exists x \exists y(\neg R(y,x) \vee P(x))$