PHIL 12A – Spring 2022
Problem Set 3

80 points.

1 Syntax and Semantics of Propositional Logic

1.1 Semantics of the Propositional Language

1.1.1 Semantics of the Propositional Language III

1. **[10 POINTS]** The following LSAT problem is taken verbatim from the Princeton Review's *Cracking the LSAT with 3 Practice Tests*, 2014 Edition.
   "A store is creating a window display featuring four hats and three scarves. The only hats being considered are A, B, C, D, E, and F, and the only scarves being considered are J, K, L, M, and N.

   If A is displayed, then neither B nor L can be displayed.

   B is displayed only if D is displayed.

   C cannot be displayed unless J is displayed.

   D can be displayed only if K is displayed.

   If L is displayed, then M must be displayed.

   F cannot be displayed unless D is not displayed.

   1. Which one of the following is a possible display of hats in the window?

   (A) A, B, C, F

   (B) A, C, D, E

   (C) A, D, E, F

   (D) B, C, D, F

   (E) B, C, E, F

   2. If F is displayed, which one of the following must be true?

   (A) A is not displayed.

   (B) B is not displayed.

   (C) K is not displayed.

(D) L is displayed.

(E) M is displayed."

To do: Formalize this LSAT problem using propositional logic. First, translate the six conditions beginning with 'If A is displayed...' into formulas of propositional logic.

Question 1 asks whether the six conditions are *satisfiable* together with some extra formulas. Use your knowledge of propositional logic to answer these satisfiability questions.

Question 2 asks whether the six conditions together with the proposition symbol F have various *logical consequences*. Use your knowledge of propositional logic to answer these consequence questions.

2. **[6 POINTS]** On the notion of equivalence: Exercises 2.12 of Logic in Action

3. **[8 POINTS]** On the relationship between logical consequence, validity and satisfiability. Let φ and ψ be two formulas such that ψ is a logical consequence of φ. Explain why the following statements are true:

   (a) **[PRACTICE]** If φ is valid, then ψ is valid.

   (b) If φ is satisfiable, then ψ is satisfiable.

4. **[10 POINTS extra credit]** On the duality between $\wedge$ and $\vee$. Given a formula φ whose only connectives are $\neg$, $\wedge$, and $\vee$, let $\varphi^\star$ be the result of interchanging $\wedge$ and $\vee$ and replacing each sentence symbol by its negation. More formally, we recursively define a function $(\cdot)^\star$ on the set of formulas whose only connectives are $\neg$, $\wedge$, and $\vee$:

   - $p^\star := \neg p$;
   - $(\neg\varphi)^* := \neg\varphi^*$;
   - $(\varphi \wedge \psi)^\star := (\varphi^\star \vee \psi^\star)$;
   - $(\varphi \vee \psi)^\star := (\varphi^\star \wedge \psi^\star)$.

   Show that for any formula φ, $\varphi^\star$ is equivalent to $\neg\varphi$. Prove this by induction. For the **base case** of a sentence symbol p, obviously $p^\star$ is equivalent to $\neg p$, because $p^\star$ *is* $\neg p$. For the **inductive hypothesis**, assume that $\varphi^\star$ is equivalent to $\neg\varphi$ and $\psi^\star$ is equivalent to $\neg\psi$. Then using the definition of $(\cdot)^\star$ and our semantics, prove that:

   (a) **[PRACTICE]** $(\neg\varphi)^*$ is equivalent to $\neg\neg\phi$;

   (b) $(\varphi \wedge \psi)^\star$ is equivalent to $\neg(\varphi \wedge \psi)$;

   (c) $(\varphi \vee \psi)^\star$ is equivalent to $\neg(\varphi \vee \psi)$.

5. **[8 POINTS]** On tautologies: Which of the following are tautologies?

   (a) $(p \to q) \vee (q \to p)$;

   (b) $(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$;

   (c) $(p \to (q \to p)) \to p$;

(d) $\neg p \to (p \to q)$.

6. **[8 POINTS]** On testing satisfiability: Which of the following are satisfiable?

   (a) $p \wedge (q \vee \neg p) \wedge \neg(p \wedge q)$;

   (b) $p \wedge \neg(q \to (q \to \neg p))$;

   (c) $\neg\neg\neg\neg p \wedge \neg\neg\neg\neg\neg\neg p$;

   (d) $(\neg p \vee q \vee (p \wedge r)) \wedge \neg(p \to q) \wedge \neg(r \to q)$.

2 Basic Theory of Propositional Logic

2.1 Economy of Language

2.1.1 Economy of Language I & II

7. **[12 POINTS]**

   (a) Give a formula equivalent to $\neg(p \vee (q \to r))$ in which the only connectives are $\neg$ and $\wedge$.

   (b) Give a formula equivalent to $p \to (q \wedge (p \to r))$ in which the only connectives are $\neg$ and $\vee$.

   (c) Give a formula equivalent to $(p \wedge q) \leftrightarrow (q \to r)$ in which the only connectives are $\neg$ and $\to$.

8. **[8 POINTS]** Write out part of the inductive proof that every formula is equivalent to one in which the only connectives are $\neg$ and $\vee$. In particular, assume as the **inductive hypothesis** that φ and ψ equivalent to $S(\varphi)$ and $S(\psi)$, respectively, for the function S defined in the Economy of Language II slides. Then prove that $\varphi \wedge \psi$ is equivalent to $S(\varphi \wedge \psi)$ (don't forget to appeal to the **inductive hypothesis** at some point in your proof).

9. **[10 POINTS]** Suppose we extend our language with a binary connective $\downarrow$ (for 'neither. . . nor') with the following truth table:

$\widehat{V}(\varphi)$	$\widehat{V}(\psi)$	$\widehat{V}(\varphi\downarrow\psi)$
1	1	0
1	0	0
0	1	0
0	0	1

Show how every formula of our language can be translated into an equivalent formula in which the only connective is $\downarrow$. (You do not need to give the full inductive proof. Just give the correct translation.)

2.2 Truth Functions

2.2.1 Truth Functions I

10. **[10 POINTS]** Consider the following ternary truth function min^3 (for 'minority'):

$$min^3(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Show that min^3 is definable by a formula of our language.

2.2.2 Truth Functions II

11. **[5 POINTS extra credit]** Let $\mathcal{L}_{\wedge,\vee}(\{p\})$ be the set of formulas of $\mathcal{L}(\{p\})$ in which the only connectives are $\wedge$ and $\vee$. How many *equivalence classes* of formulas of $\mathcal{L}_{\wedge,\vee}(\{p\})$ are there? Justify your answer.