Basic Rules of Dev

All of them are important!

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = c f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = n x^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\,g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$

Must know

should know

Winter 2010 Final

(a) (b) (c) also appears in Winter 2017

4 (a)

$$f(x) = \frac{2\sqrt{x^5} - x^{3/2}}{x^2} = 2x^{1/2} - x^{-1/2}$$
$$f'(x) = x^{-1/2} + 1/2x^{-3/2}$$

(b)

$$f(x) = \ln\left(\frac{x^4}{\sqrt{x-3}}\right) = 4\ln x - \frac{1}{2}\ln(x-3)$$
$$f'(x) = \frac{4}{x} - \frac{1}{2(x-3)}$$

simplify first using the log rules.

(c)

$$f(x) = e^3 + \arctan\left(e^x - e^{-x}\right)$$
$$f'(x) = 0 + \frac{1}{1 + (e^x - e^{-x})^2} \cdot \left(e^x + e^{-x}\right)$$

(d)

$$f(x) = \frac{3^x}{1 + \cos(x^2)}$$
$$f'(x) = \frac{\left(1 + \cos(x^2)\right) \cdot 3^x \cdot \ln 3 - 3^x \cdot \left(-\sin(x^2)\right) \cdot 2x}{\left(1 + \cos(x^2)\right)^2}$$

$\Rightarrow$ steps in details.

a)

$$f(x) = \frac{2x^{\frac{5}{2}} - x^{\frac{3}{2}}}{x^2} = \frac{2x^{\frac{5}{2}}}{x^2} - \frac{x^{\frac{3}{2}}}{x^2}$$

$$= 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$ ← before taking derivative, simplify first.

$$\therefore f'(x) = \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)'$$

$$= x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

b). $$f(x) = \ln\left(\frac{x^4}{\sqrt{x-3}}\right)$$ ← simplify first using the log rules

↓

$$\boxed{\begin{aligned} \ln\left(\tfrac{a}{b}\right) &= \ln a - \ln b \\ \ln(ab) &= \ln a + \ln b \\ \ln a^m &= m \ln a \end{aligned}}$$

$$= \ln x^4 - \ln(x-3)^{\frac{1}{2}}$$

$$= 4\ln x - \frac{1}{2}\ln(x-3)$$

$$\therefore f'(x)$$

$$= 4 \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x-3}$$

c) $f(x) = e^3 + \arctan(e^x - e^{-x})$

e^3 is just a number.

$(e^3)' = 0$

chain rule

$$f'(x) = 0 + \frac{1}{1+(e^x-e^{-x})^2} \cdot (e^x - e^{-x})'$$

$$= \frac{1}{1+(e^x-e^{-x})^2} \cdot (e^x + e^{-x}).$$

$(\arctan x)' = \frac{1}{1+x^2}$

d) quotient rule.

$(3^x)' = 3^x \ln 3.$

(e)

$$
\begin{aligned}
f(x) &= \left(1+x^2\right)^{2x} \\
\ln(f(x)) &= 2x\ln\left(1+x^2\right) \\
\frac{f'(x)}{f(x)} &= 2\ln\left(1+x^2\right) + \frac{2x}{1+x^2}\cdot 2x \\
&= 2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2} \\
f'(x) &= \left(1+x^2\right)^{2x}\left(2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2}\right)
\end{aligned}
$$

last question of question 4.

when variable is on the power.

take 'ln' from both sides.

April 2016 Q9.

$f'(x)$. $\uparrow$ $\downarrow$. Chart!

$f''(x)$ CU CD

[16] **9.** Given the function $f(x) = 2x^2 - x^4$.

(a) Find the domain of f and check for symmetry. Find asymptotes of f (if any).

(b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

a) domain: $(-\infty, \infty)$

symmetry: $\because f(x) = f(-x)$ $\therefore$ even function, symmetric to y-axis

no asymptotes.

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

b) $f'(x) = 4x - 4x^3$.

steps: ① find $f'(x)$

② set $f'(x) = 0$, Solve for x.

$$4x - 4x^3 = 0$$

$$x(4 - 4x^2) = 0$$

$x = 0$ or $4 - 4x^2 = 0$ ← $\boxed{4x^2 = 4,\ x = \pm 1}$

$x = 0$, $x = 1$, $x = -1$

③ draw a chart.

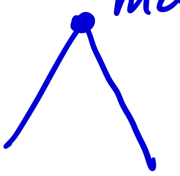
	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	↑	max	↓	min	↑	max	↓

④ Plug a random number that belongs to each interval into $f'(x)$, see $f'(x)$ is '+' or '−'

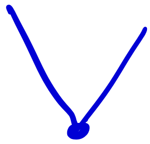
⑤ $f'(x) > 0$, $f(x) \uparrow$

$f'(x) < 0$, $f(x) \downarrow$.

max



$\uparrow$ max $\downarrow$



min.

$\downarrow$ min $\uparrow$

c) $f''(x) = 4 - 12x^2$

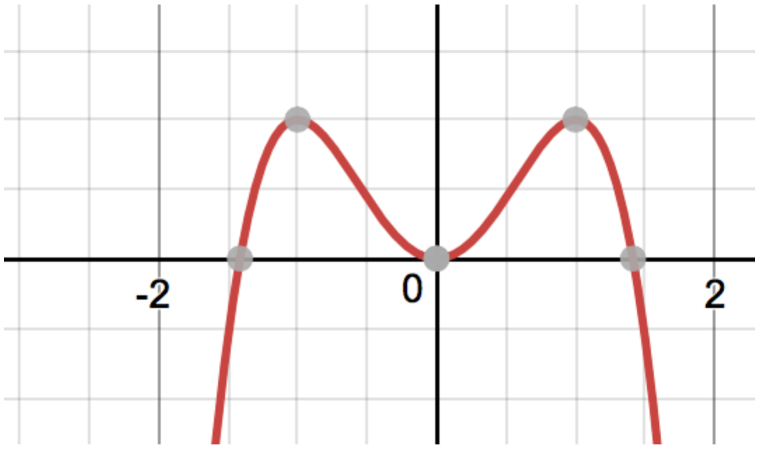
$f''(x) = 0$ $\qquad 4 - 12x^2 = 0$

$12x^2 = 4$

$x^2 = \frac{1}{3}$ $\qquad x = \pm\sqrt{\frac{1}{3}}$.

	$(-\infty, -\sqrt{\frac{1}{3}})$	$-\sqrt{\frac{1}{3}}$	$(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$	$\sqrt{\frac{1}{3}}$	$(\sqrt{\frac{1}{3}}, \infty)$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	CD		CU		CD

Inflection point: concavity changed, but increasing / decreasing didn't d)



How to sketch:

when $x = 0$, $y = ?$

$y = 0$, $x = ?$ (zero points)

$\uparrow$ $\downarrow$ CU CD.

max min values.

7. (11)

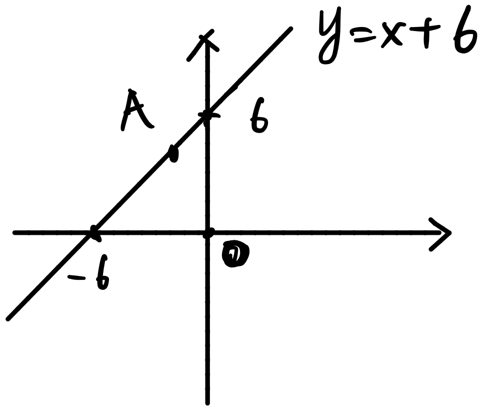
(a) Locate the point $\mathbf{A}(x_0, y_0)$ on the straight line $y = x + 6$ that is closest to the origin $\mathbf{O}(0,0)$.

(b) A rectangle is has its base on the $x-axis$ and its upper corners lie on the parabola $y = 12 - x^2$. What is the largest area of such rectangle.

Steps: ① find out a relation function. $f(x)$

② find $f'(x)$ ③ set $f'(x) = 0$

a)



A is on the function.

find distance OA that length OA is smallest.

since A is on $y = x+6$, $A(x, x+6)$

formula of distance between 2 points:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$\therefore$ distance between OA: $\sqrt{x^2 + (x+6)^2}$.

$f(x)$

$f'(x) = \quad 2x + 2(x+6) = 0$

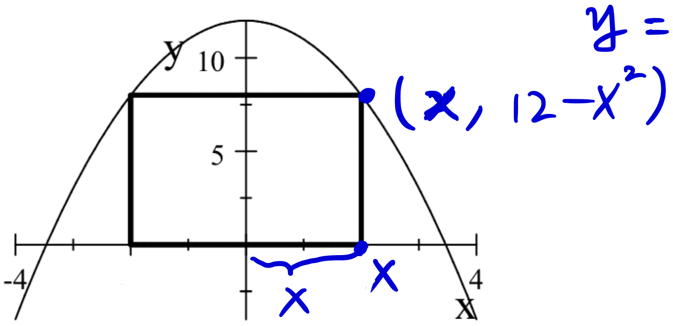
$x = -3, \quad y = 3.$

$\therefore \quad A(-3, 3)$

b)

(b) A rectangle is has its base $\overline{\mathbf{A}(-x,0)\,\mathbf{B}(x,0)}$ on the $x-axis$ and its upper corners lie on the parabola $y = 12 - x^2$. Then its area $A(x)$ is: $A(x) = 2xy = 2x\left(12 - x^2\right) = 24x - 2x^3 \rightarrow A'(x) = 24 - 6x^2 = 0 \rightarrow x = 2$ giving the largest area $A_{\max} = 16$

$y = 12 - x^2$



$A = \text{width} \cdot \text{height}$

$2x$ $\quad$ $(12 - x^2)$

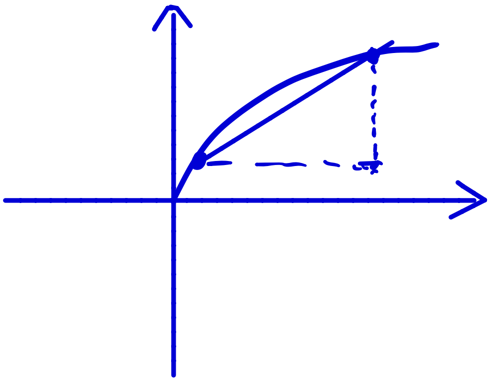
5. (7) Let $f(x) = x^3 - 2x + 3$.

(a) Calculate the slope m of the secant line joining the points $\mathbf{A}(-2, f(-2))$ and $\mathbf{B}(0, f(0))$.

(b) Locate the value $x = c$ (if any) on the interval $(-2, 0)$ such that $f'(c) = m$.

a) Steps: finding slope of secant.

$$\frac{y_2 - y_1}{x_2 - x_1} = m \leftarrow \text{slope}$$

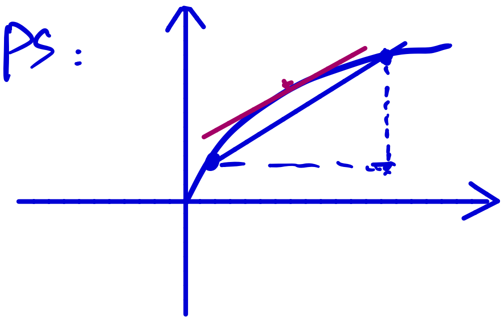


$\therefore$ a) $m = \dfrac{f(0) - f(-2)}{0 - (-2)} = \dfrac{3-(-1)}{2} = 2$

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

b) Steps:



① find $f'(x)$

② Plug in $c \Rightarrow f'(c)$

③ Set $f'(c) = m$.

Solve for c.

(mind the interval)

$\therefore$ b)

$f'(x) = 3x^2 - 2$.

$f'(c) = 3c^2 - 2$

$3c^2 - 2 = m = 2$

$3c^2 = 4$

$c = \pm\sqrt{\frac{4}{3}}$

$\because$ interval $(-2, 0)$ $\therefore c = -\sqrt{\frac{4}{3}}$

5. (12) Consider $y=\sqrt{25+x}$

(a) Use the definition of derivative to determine $\dfrac{dy}{dx}$

(b) Calculate the linearization $L(x)$ of $\sqrt{25+x}$ at $a=0$

(c) Use $L(x)$ to approximate $\sqrt{30}$.

Solution.

5. For $y=\sqrt{25+x}$

(a) Use the definition of derivative to determine $\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{\sqrt{25+x+h}-\sqrt{25+x}}{h}=$ $\lim\limits_{h\to 0}\dfrac{\sqrt{25+x+h}-\sqrt{25+x}}{h}\dfrac{\sqrt{25+x+h}+\sqrt{25+x}}{\sqrt{25+x+h}+\sqrt{25+x}}=\lim\limits_{h\to 0}\dfrac{(25+x)+h-(25+x)}{h\left(\sqrt{25+x+h}+\sqrt{25+x}\right)}=$ $\lim\limits_{h\to 0}\dfrac{1}{\sqrt{25+x+h}+\sqrt{25+x}}=\dfrac{1}{2\sqrt{x+25}}$

(b) The linearization of $\sqrt{25+x}$ at $a=0$ is $L(x)=f(0)+xf'(0)=$ $5+\dfrac{x}{2\sqrt{25}}=\dfrac{x}{10}+5$

(c) Use $L(x)$ to approximate $\sqrt{30}$: as $f(x)=\sqrt{25+x}=\sqrt{30}$ when $x=5\to L(5)=5+\dfrac{5}{2\sqrt{25}}=\dfrac{11}{2}=5.5\ (\approx 5.4772...=\sqrt{30})$

a) $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ $\leftarrow$ by def.

b) $L(x)=f(a)+f'(a)(x-a)$

c) $f(x)=\sqrt{25+x}=\sqrt{30}$ $\therefore$ $x=5$

plug 5 in $L(x)$

$L(5)=f(a)+f'(a)(x-5)$

$=\dfrac{5}{10}+5=5.5$.

6.

(a) Verify that the point $\mathbf{A}\,(2,1)$ lies on the curve $C: x^2+2y^2+2=x^3y^3$ and write an equation of the tangent line to C at $\mathbf{A}$.

(b) A spherical snowball is melting in such way that its diameter D is decreasing at the rate $\frac{dD}{dt} = -.01cm/\min$. At what rate is the volume $V = \frac{4\pi r^3}{3}$ (r is the radius of the snowball) of the snowball decreasing when $D = 9cm$?

(c) Use the l'Hospital's rule to calculate $\lim_{x\to 0} \frac{\sin^2(3x)}{1-\cos(2x)}$.

Solution.

(a) The point $\mathbf{A}\,(2,1)$ lies on the curve $C: x^2+2y^2+2=x^3y^3$ since $2^2+2+2=8=2^3$. Since the differentiation gives: $2x+4yy' = 3x^2y^3+3x^3y^2y' \overset{\text{At } \mathbf{A}}{\longrightarrow} 4+4y' = 12+24y' \to y' = -\frac{2}{5}$. The tangent line to C at $\mathbf{A}$ has equation: $y = y'(\mathbf{A})(x-2)+1 = -\frac{2}{5}(x-2)+1 \to y = \frac{9-2x}{5}$

(b) If a spherical snowball is melting in such way that its diameter $D=2r$ is decreasing at the rate $\frac{dD}{dt} \to \frac{dr}{dt} = \frac{-.01}{2} cm/\min$. Therefore, the rate $V' = 4\pi r^2 r'$ of the volume of the snowball when $D = 9cm \to r = 4.5cm$ is then $V' = 4\pi\left(\frac{81}{4}\right)\frac{-.01}{2} cm/\min = -0.405\,\pi cm/\min$.

(c) Use the l'Hospital's rule to calculate $\lim_{x\to 0}\frac{\sin^2(3x)}{1-\cos(2x)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0}\frac{6\sin(3x)\cos(3x)}{2\sin(2x)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0}\frac{18\left(\cos^2(3x)-\sin^2(3x)\right)}{4\cos(2x)} = \frac{9}{2}$.

(a) Verify that the point **A** $(2,1)$ lies on the curve $C : x^2+2y^2+2 = x^3y^3$ and write an equation of the tangent line to C at **A**.

Steps: ① verify. plug in the point

② implicit derivative.

$$2x + 4yy' = (x^3)'y^3 + x^3(y^3)'$$

$$2x + 4yy' = 3x^2y^3 + x^3 \cdot 3y^2 y'$$

$\rightarrow$ plug in $A(2,1)$ directly.

$$\Rightarrow \underline{y' = -\frac{2}{5}}$$

$\nwarrow$ this is the slope of tangent.

$$y - 1 = -\frac{2}{5}(x-2).$$

(b) A spherical snowball is melting in such way that its diameter D is decreasing at the rate $\frac{dD}{dt} = -.01cm/\min$. At what rate is the volume $V = \frac{4\pi r^3}{3}$ (r is the radius of the snowball) of the snowball decreasing when $D = 9cm$?

step ①: find derivative of $V = \frac{4\pi r^3}{3}$ in terms of t.

② $V' = \frac{4\pi}{3} \cdot 3r^2 r'$

$\because \frac{dD}{dt} = -0.1$, here we only have $\frac{dr}{dt}$

$D = 2r$ $\quad \therefore \frac{dr}{dt} = \frac{-0.1}{2}$

③ Plug in all values, get V'

Question pack:

a) implicit function derivative

b) Find dev on both sides in terms of t.

c) l'hospital

(c) Use the l'Hospital's rule to calculate $\lim_{x\to 0}\frac{\sin^2(3x)}{1-\cos(2x)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 0}\frac{6\sin(3x)\cos(3x)}{2\sin(2x)} \overset{\text{"}\frac{0}{0}\text{"}}{=}$
$\lim_{x\to 0}\frac{18\left(\cos^2(3x)-\sin^2(3x)\right)}{4\cos(2x)} = \frac{9}{2}.$

l'hospital 's rule.

$\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$\frac{\text{dev}}{\text{dev}}$ till end,

Plug in value.

1. (11)

(a) Solve: $2^{2x+2} = 3^{x-7}$

(b) Let $f(x) = \ln(x^2 - 1)$ and $g(x) = 1 - x$, calculate $(f \circ g)(x)$ and determine its domain.

(c) Calculate the inverse function $f^{-1}(x)$ for $f(x) = \ln(1 - 2x)$, and determine the range of both f and f^{-1}.

Solution

1. Then:

(a) $2^{2x+2} = 3^{x-7} \to (2x+2)\ln 2 = (x-7)\ln 3 \to x = \dfrac{2\ln 2 + 7\ln 3}{\ln 3 - 2\ln 2}$

(b) If $f(x) = \ln(x^2 - 1)$ and $g(x) = 1 - x$, then $(f \circ g)(x) = \ln\left((1-x)^2 - 1\right)$ and its domain is $(1-x)^2 - 1 = x^2 - 2x = x(x-2) > 0 \to (-\infty, 0) \cup (2, \infty)$.

(c) The inverse function $f^{-1}(x)$ for $f(x) = \ln(1 - 2x)$, and determine the range of both f and f^{-1}.

c) $y = \ln(1 - 2x)$

$x = \ln(1 - 2y)$

$e^x = e^{\ln(1-2y)}$

$e^x = 1 - 2y$

$2y = 1 - e^x$

$y = \dfrac{1 - e^x}{2}$

range of $f^{-1}(x)$ is domain of $f(x)$

$\downarrow$

$1 - 2x > 0$

$2x < 1$

$x < \frac{1}{2}$

$\therefore$ f^{-1} range: $y < \frac{1}{2}$

$f(x)$ range: $(-\infty, \infty)$

2. (7) Calculate the limit if it exists:

(a) $\lim_{x \to 2} \frac{|x-2|(x+3)}{x^2+x-6}$

(b) $\lim_{x \to 1} \frac{x-1}{3-\sqrt{x^2+8}}$

Solution

- Calculate the limits if it exists:

(a) $\lim_{x \to 2} \frac{|x-2|(x+3)}{x^2+x-6} = \lim_{x \to 2} \frac{|x-2|(x+3)}{(x+3)(x-2)} = \lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist as $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1 \neq 1 = \lim_{x \to 2^+} \frac{|x-2|}{x-2}$

(b) $\lim_{x \to 1} \frac{x-1}{3-\sqrt{x^2+8}} = \lim_{x \to 1} \frac{(x-1)\left(3+\sqrt{x^2+8}\right)}{9-(x^2+8)} = \lim_{x \to 1} \frac{(x-1)\left(3+\sqrt{x^2+8}\right)}{(1-x)(1+x)} = -\lim_{x \to 1} \frac{3+\sqrt{x^2+8}}{1+x} = -3$

limit exists when left approach = right approach

b) conjugate $(a-b)\underline{(a+b)}$

$$\frac{x-1}{3-\sqrt{x^2+8}} \cdot \frac{3+\sqrt{x^2+8}}{3+\sqrt{x^2+8}}$$

3. (6) Find all asymptotes for $y = \dfrac{\sqrt{9x^2+1}}{x^2-25}\dfrac{x^2+1}{x+5}$

Solution

. Find all asymptotes for $y = \dfrac{\sqrt{9x^2+1}}{x^2-25}\dfrac{x^2+1}{x+5}$

a) $x = \pm 5$ are **vertical** asymptotes as when $x \to \pm 5$ then $|y| \to \infty$

b) $$y = \frac{\pm x\sqrt{9+\frac{1}{x^2}}}{x^2\left(1-\frac{25}{x^2}\right)} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x\left(1+\frac{5}{x}\right)} = \frac{\pm\left(1+\frac{1}{x^2}\right)\sqrt{9+\frac{1}{x^2}}}{\left(1-\frac{25}{x^2}\right)\left(1+\frac{5}{x}\right)} \to y = \pm 3 \quad \frac{\pm 3}{1}$$

are **horizontal** asymptotes, as when $x \to \pm\infty$ then $y \to \pm 3$

VA: denominator $= 0$

HA: Short cut (use to verify)

$\dfrac{Ax^m}{Bx^n}$

$m > n$ $\quad \lim \to \infty$

$m < n$ $\quad \lim \to 0$

$m = n$ $\quad \lim \to \dfrac{A}{B}$

8. (14) Let $f(x) = 2x^3 - 21x^2 + 36x - 9$.

(a) Evaluate $f'(x)$ to determine intervals where $f(x)$ is increasing, intervals where it is decreasing, and all critical $x-$values to identify the local extrema.

(b) Evaluate $f''(x)$ to determine intervals where $f(x)$ is concave upwards, intervals where it is is concave downwards, and all critical $x-$values to identify the points of inflection.

(c) Use all the above information to sketch the graph $y = f(x)$.

Solution.

8. For $f(x) = 2x^3 - 21x^2 + 36x - 9$.

(a) Gives $f'(x) = 6x^2 - 42x + 36 = 6(x-1)(x-6) \rightarrow f(x)$ is increasing in $(-\infty, 1) \cup (6, \infty)$, decreasing in $(1, 6)$, and all critical $x-$values are $x_{\max} = 1 \rightarrow (1, 8)$, $x_{\min} = 6 \rightarrow (6, -117)$.

(b) Evaluate $f''(x) = 12x - 42 = 6(2x - 7) \rightarrow f(x)$ is concave upwards in $(3.5, \infty)$, it is concave downwards in $(-\infty, 3.5)$, and the critical $x-$value is $x = \frac{7}{2} \rightarrow$ point of inflection is $\left(\frac{7}{2}, -\frac{109}{2}\right)$.

(c) The graph $y = f(x)$ is therefore: