MATH 203 Final Examination
April 2010

SOLUTIONS

MARKS

11

1 (a) $h(x) = g \circ f(x) = g\left((x-2)^2\right) = \sqrt{4-(x-2)^2} = \sqrt{4x-x^2}$. The domain of h is determined by the requirement

$$4x - x^2 \geq 0$$
$$x(4-x) \geq 0 \Rightarrow 0 \leq x \leq 4$$

So the domain is $[0,4]$. If we put $y = h(x)$ we see that $(x-2)^2 + y^2 = 4$ so $h(x)$ is the graph of the upper semi-circle of radius 2 centered at $(2,0)$. The range is therefore $[0,2]$.

1 (b) If $y = f(x) = e^{2x} + 2e^x$ we see that $y > 0$ for every x but $\lim_{x\to-\infty} f(x) = 0$,and $\lim_{x\to\infty} f(x) = \infty$ So the range is $(0,\infty)$. To find the inverse function, first solve for x. : We do this by letting $u = e^x$ as per the hint and we get

$$u^2 + 2u - y = 0$$
$$u = -1 \pm \sqrt{1+y}$$

We take the positive root because $u = e^x > 0$ so $e^x = \sqrt{1+y} - 1$. Now interchange x and y to get

$$y = \ln\left(\sqrt{1+x} - 1\right)$$

as the inverse function. It is defined for all $x > 0$ and has range $\mathbb{R}$.

10

2 (a)

$$\lim_{x\to-2} \frac{x^2 - x - 6}{4 - x^2} = \lim_{x\to-2} \frac{(x+2)(x-3)}{(2+x)(2-x)}$$
$$= \lim_{x\to-2} \frac{(x-3)}{(2-x)} = \frac{-5}{4} = -\frac{5}{4}$$

(b)

$$\lim_{x\to0} \frac{\sqrt{x+a^2} - a}{ax} = \lim_{x\to0} \frac{\sqrt{x+a^2} - a}{ax} \cdot \frac{\sqrt{x+a^2} + a}{\sqrt{x+a^2} + a}$$
$$= \lim_{x\to0} \frac{x}{ax} \frac{1}{\sqrt{x+a^2} + a}$$
$$= \frac{1}{a} \lim_{x\to0} \frac{1}{\sqrt{x+a^2} + a} = \frac{1}{a} \cdot \frac{1}{2a} = \frac{1}{2a^2}$$

6

3

$$f(x) = \frac{|x|\sqrt{4x^2+1} - 2x^2}{x^2-4} = \frac{|x|\sqrt{4x^2+1} - 2x^2}{(x-2)(x+2)}$$

and since $\lim_{x\to\pm 2}\left(|x|\sqrt{4x^2+1} - 2x^2\right) = 2\sqrt{17} - 8 \neq 0$, it follows that $x = \pm 2$ are vertical asymptotes. Also

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{\sqrt{4+1/x^2} - 2}{1-4/x^2}\right) = 0$$

so $y = 0$ is the only horizontal asymptote.

15

4 (a)

$$f(x) = \frac{2\sqrt{x^5} - x^{3/2}}{x^2} = 2x^{1/2} - x^{-1/2}$$
$$f'(x) = x^{-1/2} + 1/2x^{-3/2}$$

(b)

$$f(x) = \ln\left(\frac{x^4}{\sqrt{x-3}}\right) = 4\ln x - \frac{1}{2}\ln(x-3)$$
$$f'(x) = \frac{4}{x} - \frac{1}{2(x-3)}$$

(c)

$$f(x) = e^3 + \arctan\left(e^x - e^{-x}\right)$$
$$f'(x) = 0 + \frac{1}{1+(e^x - e^{-x})^2}\cdot\left(e^x + e^{-x}\right)$$

(d)

$$f(x) = \frac{3^x}{1+\cos(x^2)}$$
$$f'(x) = \frac{\left(1+\cos\left(x^2\right)\right)\cdot 3^x\cdot\ln 3 - 3^x\cdot\left(-\sin\left(x^2\right)\right)\cdot 2x}{\left(1+\cos\left(x^2\right)\right)^2}$$

(e)

$$\begin{aligned}
f(x) &= \left(1+x^2\right)^{2x} \\
\ln(f(x)) &= 2x\ln\left(1+x^2\right) \\
\frac{f'(x)}{f(x)} &= 2\ln\left(1+x^2\right) + \frac{2x}{1+x^2}\cdot 2x \\
&= 2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2} \\
f'(x) &= \left(1+x^2\right)^{2x}\left(2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2}\right)
\end{aligned}$$

15

5 (a) Substituting $(2,0)$ into $y + x\sqrt{1+y^2} + 2 = x^2$ gives $2+2=2^2$ which is correct. Differentiate implicitly:

$$y' + \sqrt{1+y^2} + \frac{xy}{\sqrt{1+y^2}}y' = 2x$$

now substitute before solving for y'

$$\begin{aligned}
y' + 1 &= 4 \\
y' &= 3
\end{aligned}$$

and an equation of the tangent line is $y = 3(x-2)$.

(b) We have $x^2 + y^2 = 25$. Differentiate with respect to time t to get

$$\begin{aligned}
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \\
&= -\left(\frac{-4}{3}\right)\cdot 15 \\
&= 20 \text{ m/s}
\end{aligned}$$

(c) Since the expression $\dfrac{e^x - x - 1}{x^2 + x^3}$ is an indeterminate form of the type $\dfrac{0}{0}$ at 0, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x - x - 1}{x^2 + x^3} &= (\mathrm{L'H})\ \lim_{x\to 0}\frac{e^x - 1}{2x + 3x^2} \\
&= (\mathrm{L'H})\ \lim_{x\to 0}\frac{e^x}{2 + 6x} \\
&= \frac{1}{2}
\end{aligned}$$

where we use L'Hôpital's rule a second time since the second expression is also an indeterminate form of the type $\dfrac{0}{0}$ at 0.

6

6 (a) $f(x) = \dfrac{x}{3x-1}$ so

$$\begin{aligned} m &= \frac{f(3) - f(1)}{3-1} = \frac{3/8 - 1/2}{2} \\ &= -\frac{1}{16} \end{aligned}$$

(b) $f'(x) = -\dfrac{1}{(3x-1)^2}$ so if $1 \le c \le 3$ and $f'(c) = m = -\dfrac{1}{16}$ we have

$$\begin{aligned} -\frac{1}{(3c-1)^2} &= -\frac{1}{16} \\ (3c-1)^2 &= 16 \\ 3c - 1 &= 4 \text{ (since } c > 0) \\ c &= \frac{5}{3} \end{aligned}$$

9

7 (a)

$$\begin{aligned} V(r) &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= \lim_{h\to 0} \frac{V(r+h) - V(r)}{h} = \frac{4}{3}\pi \lim_{h\to 0} \frac{(r+h)^3 - r^3}{h} \\ &= \frac{4}{3}\pi \lim_{h\to 0} \frac{r^3 + 3r^2h + 3rh^2 + h^3 - r^3}{h} \\ &= \frac{4}{3}\pi \lim_{h\to 0} \frac{h\left(3r^2 + 3rh + h^2\right)}{h} = \frac{4}{3}\pi \lim_{h\to 0}\left(3r^2 + 3rh + h^2\right) \\ &= \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2 \end{aligned}$$

(b) The formula is

$$\begin{aligned} L(r) &= f(a) + f'(a)(r-a) \\ &= \frac{4}{3}\pi a^3 + 4\pi a^2 (r-a) \end{aligned}$$

(c) We use the estimate $dV \approx \Delta V = L(r) - \dfrac{4}{3}\pi a^3$ and $\Delta r = r - a$ to get

$$\begin{aligned} L(r) - \frac{4}{3}\pi a^3 &= 4\pi a^2 (r-a) \\ dV &= 4\pi a^2 \Delta r \\ \Delta r &= \frac{dV}{4\pi a^2} = \frac{340}{4\pi \cdot (52)^2} \\ &= .01\text{cm} \end{aligned}$$

12

8 (a) $f(x) = xe^{-x^2}$ on $\left[-\frac{1}{2}, 1\right]$.

$$\begin{aligned} f'(x) &= e^{-x^2} + xe^{-x^2}(-2x) \\ &= e^{-x^2}\left(1-2x^2\right) \\ f'(x) &= 0 \Leftrightarrow 1-2x^2 = 0 \\ &\Leftrightarrow 2x^2 = 1 \end{aligned}$$

and so the critical numbers are $x = 1/\sqrt{2}$ and $x = -1/\sqrt{2}$ (but this one is not in the given interval). So we check: ; $f(-1/2) = -\frac{1}{2}e^{-\frac{1}{4}} \simeq -0.3894$; $f\left(1/\sqrt{2}\right) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \simeq 0.42888$; and $f(1) = e^{-1} \simeq 0.36788$. So the absolute minimum is $-\frac{1}{2}e^{-\frac{1}{4}} \simeq -0.3894$ at $-1/2$ and the absolute maximum is $\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \simeq 0.42888$ at $1/\sqrt{2}$.

(b) Let A be the area of the can. Given the radius is r and the height is h we see that the total area is the sum of the area of the base $\left(\pi r^2\right)$ plus the area of the cylindrical side $(2\pi rh)$

$$A = \pi r^2 + 2\pi rh; r > 0$$

Also the volume

$$\begin{aligned} V &= \pi r^2 h = 1000 \text{ so} \\ h &= \frac{1000}{\pi r^2} \text{ and} \\ A &= \pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = \pi r^2 + \frac{2000}{r} \\ A' &= 2\pi r - \frac{2000}{r^2} = 0 \Leftrightarrow r^3 = \frac{1000}{\pi} \text{ so} \\ r &= \frac{10}{\sqrt[3]{\pi}} = h \end{aligned}$$

Since

$$A'' = 2\pi + 4000r^{-3} > 0$$

we have a local minimum (and hence an absolute minimum).

16

9 (a) $f(x) = 2x^2 - x^4$ is defined for all x because it is a polynomial, so the domain is $\mathbb{R} = (-\infty, \infty)$. It is an even function since $f(-x) = f(x)$.

There are no vertical asymptotes because the domain is $\mathbb{R}$ (so there are no points a where the function can have limit ∞ or $-\infty$). There are no horizontal asymptotes because $\lim_{x\to\pm\infty} f(x) = -\infty$. So there are no asymptotes.

(b)

$$f'(x) = 4x - 4x^3 = 4x\left(1 - x^2\right)$$
$$= 4x(1-x)(1+x)$$

The critical numbers are $0, -1$ and 1. Constructing the table to see where it is increasing/decreasing we have

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$4x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$1 - x$	$+$	$+$	$+$	$+$	$+$	0	$-$
$1 + x$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
	$\nearrow$	Max.	$\searrow$	min.	$\nearrow$	Max.	$\searrow$

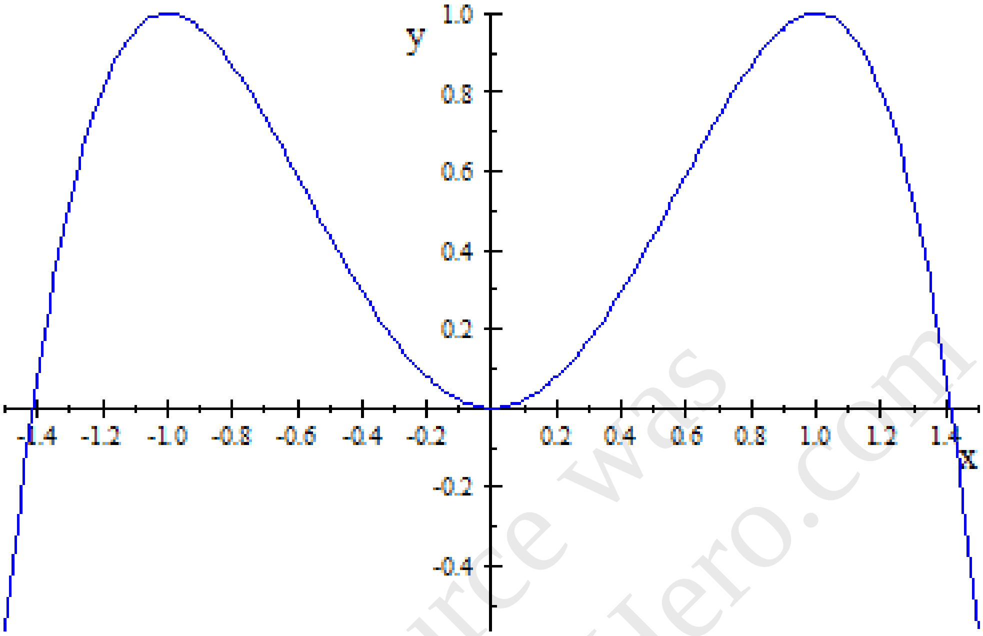
We can read off the intervals and the local extrema: local maxima at $(-1, 1)$ and $(1, 1)$ and a local minimum at $(0, 0)$.

(c)

$$f''(x) = 4 - 12x^2 = 4\left(1 - 3x^2\right)$$
$$= 0 \Longleftrightarrow x = \pm 1/\sqrt{3}$$

Here we will use the even symmetry and note that on $\left(0, 1/\sqrt{3}\right)$ $f''(x) > 0$ so the curve is concave up. By symmetry the curve is concave up on $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$. For $x > 1/\sqrt{3}$ (and $x < -1/\sqrt{3}$) it is concave down, so these points: $\left(\pm 1/\sqrt{3}, 1/3\right) = (\pm 0.577, 0.333)$ are inflection points.

(d) The graph:



5

Bonus

$$f(x) = \frac{\sin(ax)}{x-a}$$

$$\lim_{x\to a} f(x) = \frac{a\cos(ax)}{1} = a\cos\left(a^2\right) \text{ by L'Hôpital'srule}$$

If $a = 1$

$$\lim_{x\to 1} \frac{\sin(x)}{x-1} = \frac{\cos(x)}{1} = \cos(1) \text{ by L'Hôpital'srule}$$

What is wrong? L'Hôpital's rule doesn't apply if $\sin\left(a^2\right) \neq 0$ since the original expression $f(x) = \dfrac{\sin(ax)}{x-a}$ is not of the form $\dfrac{0}{0}$ near $x = a$: the numerator is near $\sin\left(a^2\right)$ while the denominator is near 0. If $a = 0$ we get $\dfrac{0}{x} = 0$ which has limit 0 anyway, without L'Hôpital's rule. If not, we can only use L'Hôpital's rule when $\sin\left(a^2\right) = 0$, i.e. when $a^2 = \pi n$ or $a = \sqrt{n\pi}$; $n = 1, 2, 3, \cdots$.