Étale Covering

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Connection between Galois Theory and Algebraic topology

Theorem.(Fundamental Theorem of Galois Theory)

If F is a finite dimensional Galois extension of K , then there is a bijection:

$$\{\text{Intermediate fields of F/K}\}$$

$$\updownarrow$$

$$\{\text{Subgroups of } Gal(F/K)\}.$$

Besides,

$$|Gal(F/K)| = [F : K].$$

Connection between Galois Theory and Algebraic topology

Review of algebraic topology.

Definition.(Covering space)

We firstly define the covering map. Let $p : X \to S$ satisfy the following condition:
Each point $s \in S$ has an open neighborhood U in S such that $p^{-1}(U)$ is a union of disjoint open sets in X , each of which is mapped homeomorphically onto U by p.
A covering space of a space S is a space X together with a covering map.

Connection between Galois Theory and Algebraic topology

Review of algebraic topology.

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A covering space of a space S is a space X together with a covering map.

Definition.

$\forall s \in S$, say $p^{-1}(s)$ the fiber of s, and say $|p^{-1}(s)|$ the sheet of the covering which is not depended by the choice of s. Then we have a group action $\pi_1(S, s) \curvearrowright p^{-1}(s)$.

Proposition.

The induced map of the fundamental groups

$$p_\pi : \pi_1(X, x) \to \pi_1(S, s)$$

is injective. Define $H_e := p_\pi(\pi_1(X, x))$.

Connection between Galois Theory and Algebraic topology

Proposition.

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Proposition.

$[\pi_1(S, s) : H_e]$=the sheet of p.

Recall the fact in Galois theory:

$$|Gal(F/K)| = [F : K].$$

Connection between Galois Theory and Algebraic topology

Theorem.

Let S be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection :

$$\{\text{Path-connected covering space}\}$$

$$\updownarrow$$

$$\{\text{Subgroup of } \pi_1(S,s)\}.$$

Recall the Fundamental Theorem of Galois theory.

Connection between Galois Theory and Algebraic topology

Definition.(Universal covering)

A simply-connected covering space of S is called a universal cover, say $\tilde{S}$. It is unique up to isomorphism.

In Galois theory, we have "Algebraic Closure".

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A simply-connected covering space of S is called a universal cover, say $\tilde{S}$. It is unique up to isomorphism.

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Proposition.

$Aut(\tilde{S}/S) \cong \pi_1(S, s)$. Hence $Aut(\tilde{S}/S)$ can be viewed as a second definition of the fundamental group of S.

In Galois theory, we have "absolute Galois group".

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Definition.(Sheaf)

Let X be a topological space. A sheaf $\mathscr{F}$ of abelian groups on X consists of the data:

(a) for every open subset U $\subseteq X$, an abelian group $\mathscr{F}(U)$,

(b) for every inclusion V $\subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

(0) $\mathscr{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,

(1) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and

(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

(3) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_1} = 0$ for all i, then $s = 0$;

(4) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

Definition.(Stalk)

If $\mathscr{F}$ is a sheaf on X, and if P is a point of X, we define the stalk $\mathscr{F}_P$ of $\mathscr{F}$ at P to be the direct limit of the groups $\mathscr{F}(U)$ for all open sets U containing P, via the restriction maps ρ. Namely

$$\varinjlim_{P\in U} \mathscr{F}(U).$$

Definition.(Morphisms between sheaves on X)

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X , a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U , such that whenever $V \subseteq U$ is an inclusion, and the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X.

Definition.(Spectrum)

(1) As a set, we define Spec A to be the set of all prime ideals of A.
(2) As a topological space. If $\mathfrak{a}$ is any ideal of A , we define the subset $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ to be the set of all prime ideals which contain $\mathfrak{a}$. Let the subsets of the form $V(\mathfrak{a})$ to be the closed subsets. Note that $V(A) = \varnothing$; $V((0)) =$Spec A.
(3) A sheaf of rings $\mathscr{O}$ on Spec A. For each prime ideal $\mathfrak{p} \subseteq A$, let A_p be the localization of A at $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathscr{O}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} A_{\mathfrak{p}}$, such that $s(\mathfrak{p}) \in A_{\mathrm{p}}$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A : to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of p , contained in U , and elements a,$f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in A_{q}.
Let A be a ring. The spectrum of A is the pair consisting of the topological space Spec A together with the sheaf of rings $\mathscr{O}$ defined above.

Definition.(Ringed space)

A ringed space is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X. A morphism of ringed spaces from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a pair $\left(f, f^{\#}\right)$ of a continuous map $f : X \to Y$ and a map $f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X$ of sheaves of rings on Y. The ringed space $(X, \mathscr{O}_X)$ is a locally ringed space if for each point $P \in X$, the stalk $\mathscr{O}_{X,P}$ is a local ring.

Definition.(Affine scheme)

An affine scheme is a locally ringed space which is isomorphic to spectrum of some ring.

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Definition.(Scheme)

A scheme is a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathscr{O}_X|_U$, is an affine scheme.
We call X the underlying topological space of the scheme $(X, \mathscr{O}_X)$, and $\mathscr{O}_X$ its structure sheaf.
A morphism of schemes is a morphism as locally ringed space.

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A morphism of schemes is a morphism as locally ringed space.

Example.

If k is a field, Spec k is an affine scheme whose topological space consists of one point, and whose structure sheaf consists of the field k.

Definition.(Connected scheme)

A scheme is connected if its underlying topological space is connected.

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Definition.(Finite morphism)

A morphism $f : X \to Y$ is a finite morphism if there exists a covering of Y by open affine subsets V_i =Spec B_i , such that for each i , $f^{-1}(V_i)$ is affine, equal to Spec A_i , where A_i is a B_i-algebra which is a finitely generated B_i-module.

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Theorem.

Finite morphisms are closed maps.

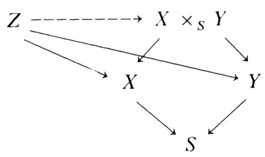
Definition.(Open immersion)

An open subscheme of a scheme X is a scheme U, whose topological space is an open subset of X, and whose structure sheaf $\mathscr{O}_U$ is isomorphic to the restriction $\mathscr{O}_X|_U$ of the structure sheaf of X. An open immersion is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Algebraic Geometry

Definition.(Fibred product)

Let S be a scheme, and let X, Y be schemes over S, i.e., schemes with morphisms to S. We define the fibred product of X and Y over S, denoted $X \times_S Y$, to be a scheme, together with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, such that given any scheme Z over S, and given morphisms $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$, and $g = p_2 \circ \theta$. The morphisms p_1 and p_2 are called the projection morphisms of the fibred product onto its factors.



Algebraic Geometry

Theorem.

For any two schemes X and Y over a scheme S , the fibred product $X \times_S Y$ exists, and is unique up to unique isomorphism.

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Definition.(Base change)

Let S be a fixed scheme which we think of as a base scheme, meaning that we are interested in the category of schemes over S. For example, think of S$=$ Spec k, where k is a field. If S' is another base scheme, and if $S' \to S$ is a morphism, then for any scheme X over S, we let $X' = X \times {}_S S'$, which will be a scheme over S'. We say that X' is obtained from X by making a base change $S' \to S$.

If a property of morphisms is also held for its base change, we sat that this property is stable under base change.

Definition.(Sheaf of modules)

Let $(X, \mathscr{O}_X)$ be a ringed space. A sheaf of $\mathscr{O}_X$-modules (or simply an $\mathscr{O}_X$-module) is a sheaf $\mathscr{F}$ on X, such that for each open set U $\subseteq X$, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module, and for each inclusion of open sets V $\subseteq U$, the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$.
A morphism $\mathscr{F} \to \mathscr{G}$ of sheaves of $\mathscr{O}_X$-modules is a morphism of sheaves, such that for each open set U $\subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules.

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Definition.(Flat module)

Let A be a ring. An A-module M is called flat if the functor $N \mapsto M \otimes_A N$ on the category of A-modules is exact.

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Definition.(Flat morphism)

A morphism $f : X \to Y$ of schemes is called flat if $\forall x \in X, \mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{Y,f(x)}$.

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Definition.(Unramified morphism)

Let $f : X \to Y$ be a morphism locally of finite type, x a point in X, and $y = f(x)$. We say f is unramified if $\mathfrak{m}_y \mathscr{O}_{X,x} = \mathfrak{m}_x$ and the residue field $k(x)$ is a finite separable extension of the residue field $k(y)$.(Consider algebraic number theory)

Étale Covering

Definition.(Étale morphism)

If f is a morphism locally of finite presentation, we say f is étale if f is flat and unramified.

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Theorem.

The following properties of morphisms are stable under base change.
(1) Finite.
(2) Affine.
(3) Flat.
(4) Étale.
(5) Locally free of finite rank.

Étale Covering

Proposition.

In the following, morphisms are assumed to be locally of finite presentation.

(1) Composites of étale morphisms are étale.

(2) Open immersions are étale.

(3) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that gf is étale and g is étale. Then f is étale.

(4) Étale morphisms are open mappings.

Étale Covering

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(4) Étale morphisms are open mappings.

Proof.

Let us prove (3) as an example. By 2.2.4 in [1], the graph $\Gamma_f : X \to X \times_Z Y$ of f is an open immersion, which is étale by (2). Since étale is stable under base change, the projection $p_2 : X \times_Z Y \to Y$ is étale. By (1), $f = p_2\Gamma_f$ is étale.

[1]Fu Lei. Etale Cohomology Theory[M].WORLD SCIENTIFIC:2011-01.

Definition.(X/G)

Let X be a scheme on which a finite group G acts on the right. If X=Spec A, this is equivalent to saying G acts on A on the left. Let $\mathrm{Hom}(X,Z)^G$ be the subset of $\mathrm{Hom}(X,Z)$ consisting of morphisms invariant under G. A morphism is called the quotient of X by G if it is a morphism $X \to Y$ is invariant under the action of G, such that the canonical map

$$\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)^G$$

is bijective for any scheme Z, that is, Y represents the functor $Z \mapsto \mathrm{Hom}(X,Z)^G$. We often denote Y by X/G.

Étale Covering

Definition.(Étale Covering)

An étale covering space of a scheme S is a finite étale morphism $X \to S$.

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The group $\mathrm{Aut}(X/S)$ acts on X on the left. Let $G = \mathrm{Aut}(X/S)^{\circ}$ be the opposite group, then G acts on X on the right. If $S = X/G$, we say $X \to S$ is a Galois étale covering space with Galois group G.

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Remark.

If X is a variety over an algebraically closed field and $\pi : Y \to X$ is finite and étale, then each fibre of π has exactly the same number of points. Moreover, each $x \in X$ has an étale neighborhood $(U, u) \to (X, x)$ such that $Y \times_X U$ is a disjoint union of open subvarieties (or subschemes) U_i each of which is mapped isomorphically onto U by $\pi \times 1$. Thus, a finite étale map is the natural analogue of a finite covering space in the theory of algebraic topology.

Definition.(Geometric point)

Let X be a scheme. A geometric point of X is a morphism $\gamma : s \to X$ such that s is the spectrum of a separably closed field. We often write k(s) for the separably closed field such that $s = \operatorname{Spec} k(s)$. (Giving a geometric point is equivalent to giving a separably closed extension k(s) of the residue field k(x)). Let $f : X \to X'$ be a morphism. Then $f\gamma : s \to X'$ is a geometric point of X'.

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Definition.(Pointed scheme)

A pointed scheme is a pair (X, γ) such that X is a scheme, and $\gamma : s \to X$ is a geometric point of X. A morphism $f : (X, \gamma) \to (X', \gamma')$ between pointed schemes is a morphism $f : X \to X'$ such that $f\gamma = \gamma'$.

Étale Covering

Theorem.1

Let (S, γ) be a pointed scheme, and let $(X_i, \alpha_i) \to (S, \gamma)(i = 1, 2)$ be two morphisms of pointed schemes.
(1) If X_1 is connected and X_2 is unramified and separated over S , then there exists at most one S-morphism from $(X_1, \alpha_1)\, to \,(X_2, \alpha_2)$.
(2) Suppose X_1 and X_2 are étale covering spaces of S. There exists a connected pointed étale covering space (X_3, α_3) of (S, γ) dominating $(X_i, \alpha_i)\,(i = 1, 2)$, that is, there exist S-morphisms from (X_3, α_3) to (X_i, α_i).

Étale Covering

Proof.

(1) An S-morphism f from X_1 to X_2 is completely determined by its graph $\Gamma_f : X_1 \to X_1 \times_S X_2$, which is a section of the projection $\pi_1 : X_1 \times_S X_2 \to X_1$. If $f(\alpha_1) = \alpha_2$, then $\Gamma_f(\alpha_1) = (\alpha_1, \alpha_2)$. Such Γ_f is unique if it exists by [1]2.3.10(i) which is a property of étale morphism.

(2) α_1 and α_2 define a point $\alpha_3 = (\alpha_1, \alpha_2)$ of $X_1 \times_S X_2$. Let X_3 be the connected component of $X_1 \times_S X_2$ containing the image of α_3. Then (X_3, α_3) dominates $(X_i, \alpha_i)\,(i = 1, 2)$.

Theorem.2

Let (S, γ) be a pointed connected noetherian scheme, X_1 and X_2 two étale covering spaces of S, $u : X_1 \to X_2$ an S-morphism, and $X_i(\gamma)(i = 1, 2)$ the sets of geometric points of X_i lying above γ. If the map $X_1(\gamma) \to X_2(\gamma)$ induced by u is bijective, then u is an isomorphism.

Theorem.2

Let (S, γ) be a pointed connected noetherian scheme, X_1 and X_2 two étale covering spaces of S, $u : X_1 \to X_2$ an S-morphism, and $X_i(\gamma)(i = 1, 2)$ the sets of geometric points of X_i lying above γ. If the map $X_1(\gamma) \to X_2(\gamma)$ induced by u is bijective, then u is an isomorphism.

Sketch of proof.

The image of any connected component of X_2 in S is both open and closed. Since S is connected, the image is S. Replacing X_2 by its connected components and X_1 by the inverse images of these components, we are reduced to the case where X_2 is connected. Note that u is finite and étale. So $u_*\mathscr{O}_{X_1}$ is a locally free $\mathscr{O}_{X_2}$-module of constant finite rank.

Let $s \in S$ be the image of γ and let $x_2 \in X_2$ be a point above s. Then

$$X_1 \times_{X_2} \text{ Spec } \mathscr{O}_{X_2,x_2} \cong \text{Spec } A$$

for some $\mathscr{O}_{X_2,x_2}$-algebra A which is free of finite rank as an $\mathscr{O}_{X_2,x_2}$-module. Since $X_1(\gamma) \to X_2(\gamma)$ is bijective, there is one and only one point x_1 in X_1 lying above x_2. So A is a local ring. Since u is étale, $\mathfrak{m}_{x_2}$ A is the maximal ideal of A and $A/\mathfrak{m}_{x_2}A$ is finite separable over $\mathscr{O}_{X_2}, x_2/\mathfrak{m}_{x_2}$. Again because $X_1(\gamma) \to X_2(\gamma)$ is bijective, we must have $\mathscr{O}_{X_2,x_2}/\mathfrak{m}_{x_2} \cong A/\mathfrak{m}_2 A$. It follows that the rank of $u_*\mathscr{O}_{X_1}$ is 1. Let x_2' be an arbitrary point of X_2 and let A' be an $\mathscr{O}_{X_2,x_2'}$-algebra such that

$$X_1 \times_{X_2} \text{Spec } \mathscr{O}_{X_2,x_2'} \cong \text{Spec } A'.$$

Since $\text{rank}\,(u_*\mathscr{O}_{X_1}) = 1$, A' is a free $\mathscr{O}_{X_2}, x_2'-$module of rank 1.

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The homomorphism $\mathscr{O}_{X_2,x_2'}/\mathfrak{m}_{x_2'} \to A'/\mathfrak{m}_{x_2'}A$ is a nonzero homomorphism of one dimensional vectors spaces. It is necessarily surjective. By Nakayama's lemma, the homomorphism $\mathscr{O}_{X_2,x_2'} \to A'$ is also surjective. It is injective since it is faithfully flat. So we have $\mathscr{O}_{X_2,x_2'} \cong A'$. Hence $\mathscr{O}_{X_2} \cong u_*\mathscr{O}_{X_1}$, and u is an isomorphism. The proof is complete.

Étale Covering

Theorem.3

Let (S, γ) be a pointed connected noetherian scheme, X a connected étale covering space of S, $X(\gamma)$ the set of geometric points in X lying above γ , and $G = \mathrm{Aut}(X/S)^{\circ}$. The following conditions are equivalent:
(1) $X/G \cong S$, that is, X is a Galois covering of S.
(2) G acts transitively on $X(\gamma)$.

For the proof, see [1]3.2.8.

Étale Covering

Theorem.4

let S be a connected noetherian scheme and let X be a connected étale covering space of S. Then any S-morphism $u : X \to X$ is an isomorphism.

Étale Covering

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let S be a connected noetherian scheme and let X be a connected étale covering space of S. Then any S-morphism $u : X \to X$ is an isomorphism.

Proof.

Note that u is necessarily étale and finite. So u(X) is both open and closed. As X is connected, we have u(X)=X. Let γ be a geometric point of S , let α be a geometric point of X lying above γ, and let x be the image of α. Then there exists $x' \in X$ such that $u(x') = x$. Moreover the residue field $k(x')$ is a finite separable extension of the residue field k(x). So there exists a geometric point α' of X with image x' such that $u\alpha' = \alpha$. Thus u induces a surjective map $X(\gamma) \to X(\gamma)$. As $X(\gamma)$ is finite. This map is bijective. By Theorem 2, u is an isomorphism.

Étale Covering

Theorem.5

Let (S, γ) be a pointed connected noetherian scheme, and let (Y, β) be a pointed étale covering space of (S, γ). There exists a pointed connected Galois étale covering space (X, α) of (S, γ) dominating (Y, β), that is, there exists an S-morphism from (X, α) to (Y, β).

Proof.

Let $\beta_1, \dots, \beta_k$ be all the distinct geometric points of Y lying above γ. They define a geometric point α of $Y^k = Y \times_S \cdots \times_S Y$ such that $p_i\alpha = \beta_i (i = 1, \dots, k)$, where $p_i : Y^k \to Y$ are the projections. Let X be the connected component of Y^k containing the image of α. It suffices to show that X is a Galois covering of S. By Theorem 3 and Theorem 4 , it suffices to show that for any geometric point α' of X lying above γ, there exists an S-morphism $u : X \to X$ such that $u\alpha = \alpha'$.

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Let $j : X \hookrightarrow Y^k$ be the open immersion. Since $p_i j\alpha = \beta_i (i = 1, \dots, k)$ are distinct, $p_i j\alpha'$ are distinct by Theorem 1(1). Let σ be a permutation of $\{1, \dots, k\}$ such that $p_i j\alpha' = p_{\sigma(i)} j\alpha$, and let $v : Y^k \to Y^k$ be the S-morphism with the property $p_i v = p_{\sigma(i)}$. We have

$$p_i v j\alpha = p_{\sigma(i)} j\alpha = p_i j\alpha'.$$

It follows that $vj\alpha = j\alpha'$. Since X is the connected component of Y^k containing the images of α and α', v induces a morphism $u : X \to X$ with the property $u(\alpha) = \alpha'$. This proves our assertion.

Recall some homological algebra.

Proposition.

Consider the following conditions of the category I :
(I1) Given two morphisms $i \rightrightarrows j$ in I , there exists a morphism $j \to k$ so that its composite with these two morphisms are the same.
(I2) Given two objects i and j in I , there exists an object k in I admitting morphisms $i \to k$ and $j \to k$.
Suppose objects in I form a set and I satisfies (I1) and (I2). For any covariant functor

$$F : I \to \mathscr{C}, \varinjlim_{i} F(i)$$

exists.

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Remark.

Let (S, γ) be a pointed connected noetherian scheme. Let I be the opposite category of the category of pointed connected Galois étale covering spaces of (S, γ). For any $i \in obI$, denote by (X_i, α_i) the corresponding pointed connected Galois étale covering space of (S, γ). By Theorem 1(2) and Theorem 5, I satisfies the conditions (I1) and (I2), so I admits direct limit.

Definition.(Étale fundamental group)

We define the fundamental group of (S, γ) to be

$$\pi_1(S, \gamma) = \varprojlim_{i} \operatorname{Aut}(X_i/S)^{\circ}$$

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Definition.(Étale fundamental group)

We define the fundamental group of (S, γ) to be

$$\pi_1(S, \gamma) = \varprojlim_i \mathrm{Aut}\,(X_i/S)^\circ$$

Recall the second definition of the fundamental group in algebraic topology.
Note that $\mathrm{Aut}\,(X_i/S)^\circ$ acts on the left on the set $\mathrm{Hom}\,(X_i, X)$ for any object X in $\mathrm{Et}(S)$. So $\pi_1(S, \gamma)$ acts on the left on the sets

$$\varinjlim_i \mathrm{Hom}_S\,(X_i, X) \cong F(X).$$

Then the action of $\pi_1(S, \gamma)$ on F(X) factors through the finite quotient $\mathrm{Aut}\,(X_i/S)^\circ$. Put the discrete topology on $\mathrm{Aut}\,(X_i/S)^\circ$, and put the product topology on $\pi_1(S, \gamma) = \varprojlim_i \mathrm{Aut}\,(X_i/S)^\circ$. Then $\pi_1(S, \gamma)$ acts continuously on the discrete finite set $F(X)$.

Étale Covering

Theorem.

Let K be a field, Ω a separably closed field containing K, γ : Spec Ω $\rightarrow \operatorname{Spec} K$ the corresponding geometric point, and K_s the separable closure of K contained in Ω. Then we have a canonical isomorphism

$$\pi_1(\operatorname{Spec} K, \gamma) \cong \operatorname{Gal}(K_s/K).$$

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Proof.

Let $\{K_i\}$ be the family of finite Galois extensions of K contained in Ω and let α_i : Spec $\Omega \to SpecK_i$ be the corresponding geometric points. Then $\{(\operatorname{Spec} K_i, \alpha_i)\}$ is cofinal in the category of pointed connected Galois étale covering spaces of $(\operatorname{Spec} K, \gamma)$. We haveAut $(\operatorname{Spec} K_i / \operatorname{Spec} K)^\circ = \operatorname{Gal}(K_i/K)$.So we have

$$\pi_1(\operatorname{Spec} K, \gamma) \cong \varprojlim_i \operatorname{Gal}(K_i/K) \cong \operatorname{Gal}(K_s/K).$$

Example.

(1) $\pi_1(\mathbb{A}^1(\mathbb{C}) - \{0\}) \cong \varprojlim_i \mathbb{Z}/i\mathbb{Z} \cong \prod_p \mathbb{Z}_p = \hat{\mathbb{Z}}$.

(2) $\pi_1(Spec(\mathbb{F}_p)) \cong \hat{\mathbb{Z}}$.

Étale Covering

Theorem.

Let S be a normal connected noetherian scheme, K its function field, Ω a separably closed field containing K , and K_s the separable closure of K in Ω. Denote by γ both the geometric point $\mathrm{Spec}\,\Omega \to \mathrm{Spec}\,K$ and the geometric point defined by the composite $\mathrm{Spec}\ \Omega \to \mathrm{Spec}\ K \to S$. Denote K_{ur} is the subfield of K_s generated by all finite separable extensions of K contained in K_s that are unramified over S. In particular, we have

$$\pi_1(S,\gamma) \cong \mathrm{Gal}\,(K_{ur}/K)\,.$$

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