Étale Cohomology

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Abstract

Étale cohomology is an important tool in the theory of modern arithmetic geometry. In this article, we review the definitions and basic properties of étale morphisms and étale sheaves in order to define étale cohomology and compare étale cohomology with Zariski cohomology.

Contents

Chapter 1

Introduction

In the theory of algebraic geometry, let X be a scheme, we have a topological space $\mathrm{sp}(X)$, called Zariski topology in this article. Then for any presheaf $\mathscr{P}$ on X, called Zariski presheaf in this article, we define its Čech cohomology $\check{H}^n(\mathfrak{U},\mathscr{P})$ with respect to the open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ to be the cohomology of the Čech complex

$$C^n(\mathfrak{U},\mathscr{P}) = \prod_{i_0,\cdots,i_n\in I} \mathscr{P}(U_{i_0\cdots i_n})$$

where $U_{i_0\cdots i_n} = U_{i_0}\cap\cdots\cap U_{i_n}$. Besides, for any sheaf $\mathscr{F}$ on X, called Zariski sheaf in this article, we define the cohomology of this sheaf $H^n(X,\mathscr{F})$ to be $R^n\Gamma(X,\mathscr{F})$. However, the Zariski topology is so coarse that methods in algebraic topology can't be applied in the Zariski theory of sheaf cohomology and Čech cohomology.

In the 1940s, Weil observed that some of his results on the numbers of points on certain varieties over finite fields could be explained by the existence of a certain cohomology theory in which the Lefschetz fixed point formula holds. About 1958, Grothendieck define a finer "topology" called étale topology and construct the theory of étale cohomology modelling the ordinary theory of sheaf cohomology. In this cohomology theory, we have useful tools like Poincaré duality, Lefschetz fixed point formula, Leray spectral sequence, etc.. In 1974, Deligne published the paper [4], gave the first proof of the Weil conjectures mainly by using Lefschetz trace formula of étale cohomology, which is the most successful application of étale cohomology theory.

In order to construct a finer "topology" on which the cohomology theory behaves well, we define firstly the étale morphisms of schemes to be flat and unramified morphisms and study their basic properties in chapter 2. Then in section 3.1 we define a category $X_{ét}$ of which the objects are étale morphisms of X, called the étale topology. It is clear that étale topology is not a topological space but a category, while étale morphisms play roles as open sets in topological spaces since they are actually open mappings (2.2.5). Note that any open immersion is an étale morphism by 2.2.2 (i), hence the étale topology for a scheme has more "open sets" than its Zariski topology. In other words, étale topology is truly a "finer topology".

Modelling the definition of open coverings in algebraic geometry, for any object U in $X_{ét}$, we define an étale covering of U to be a set $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha\in I}$ of morphisms if U is the union of the images of U_α. This definition helps us to define étale sheaves and Čech cohomology later.

Let X be a scheme, we define the étale presheaves in chapter 3.1 to be contravariant functors on its étale topology $X_{ét}$. And we say that an étale presheaf $\mathscr{F}$ is an étale sheaf if the following sequence is exact for any étale covering $\{U_\alpha \to U\}_{\alpha\in I}$

$$0 \to \mathscr{F}(U) \to \prod_{\alpha\in I} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta\in I} \mathscr{F}(U_\alpha \times_U U_\beta) \tag{1.1}$$

Note that the definitions of étale presheaves and étale sheaves are similar to those in algebraic geometry since presheaves can also be considered as a contravariant functor [3, P61] and a presheaf is a sheaf if and only if we have the similar exact sequence [2, P2].

Then in 3.2, we aim to define the stalks of étale sheaves. Moreover, we will find that it is just the strictly henselization of Zariski stalks, *i.e.* $\mathscr{O}_{X_{ét},s} \cong \tilde{\mathscr{O}}_{X,s}$. After introducing the stalks of étale sheaves, it will be convenient to study basic properties of the functors f_*, f^*, $f_!$, and $f^!$ in section 3.2 and section 3.3, which will be useful for the calculation of étale cohomology.

Similar to Čech cohomology in algebraic geometry, in section 4.1, we want to define Čech cohomology for an étale sheaf with respect to an étale covering. In fact, just let $U_{\alpha_0,\cdots\alpha_n} = U_{\alpha_0} \times_U \cdots \times_U U_{\alpha_n}$ and define the same Čech complex. Note that when $U_{\alpha_0} \to X$ and $U_{\alpha_1} \to X$ are open immersions, $U_{\alpha_0} \times_U U_{\alpha_1} = U_{\alpha_0} \cap U_{\alpha_1}$, which is compatible with the ordinary Čech cohomology in algebraic geometry.

Similar to sheaf cohomology in algebraic geometry, we define the étale cohomology $H^i_{ét}(U, \mathscr{F})$ of an étale sheaf $\mathscr{F}$ to be $R^i\Gamma(U, \cdot)$. Then by using spectral sequence to study étale cohomology, we will see that the étale cohomology of an étale sheaf originating from a quasi-coherent (Zariski) sheaf is identical with the Zariski cohomology, namely $H^q_{ét}(X, \mathscr{M}_{ét}) \cong H^q_{Zar}(X, \mathscr{M})$.

Chapter 2

Étale Morphisms

2.1 Unramified Morphisms

Definition 2.1.1. *(Unramified morphism). Let $f : X \to Y$ be a morphism locally of finite type, x a point in X, and $y = f(x)$. We say f is unramified at x if $\mathfrak{m}_y \mathscr{O}_{X,x} = \mathfrak{m}_x$ and the residue field $k(x)$ is a finite separable extension of the residue field $k(y)$. If f is unramified at every point in X, we say that f is unramified.*

Proposition 2.1.2. *[1, Prop.2.2.1] Suppose $f : X \to Y$ is a morphism locally of finite type, $x \in X$ and $y = f(x)$. Then the conditions below are equivalent*

(i) f is unramified at x.

(ii) $(\Omega_{X/Y})_x = 0$.

(iii) The diagonal morphism $\Delta : X \to X \times_Y X$ is an open immersion in a neighborhood of x.

Proof. (i) $\Rightarrow$ (ii).

Let $W = \operatorname{Spec} B$ be a neighbourhood of x in X, and $V = \operatorname{Spec} A$ a neighbourhood of y in Y such that $f(W) \subset V$

$$(\Omega_{X/Y})_x \cong (\Omega_{X/Y}|_W)_x \cong (\tilde{\Omega}_{B/A})_x \cong \Omega_{\mathscr{O}_{X,x}/A} \cong \Omega_{\mathscr{O}_{X,x}/\mathscr{O}_{Y,y}}.$$

In fact, the last isomorphism is obtained from the following exact sequence

$$\Omega_{\mathscr{O}_{Y,y}/A} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{X,x} \to \Omega_{\mathscr{O}_{X,x}/A} \to \Omega_{\mathscr{O}_{X,x}/\mathscr{O}_{Y,y}} \to 0,$$

where $\Omega_{\mathscr{O}_{Y,y}/A} \cong (\Omega_{A/A})_y = 0$. Then by [1, Prop.2.1.2] we have

$$(\Omega_{X/Y})_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}/\mathfrak{m}_y \mathscr{O}_{X,x} \cong \Omega_{k(x)/k(y)} = 0.$$

The assumption that f is locally of finite type yields that $(\Omega_{X/Y})_x$ is a finitely generated $\mathscr{O}_{X,x}$-module by [3, Chapter 2, Cor.8.5]. Use Nakayama's lemma, we have $(\Omega_{X/Y})_x = 0$.

(ii) $\Rightarrow$ (iii).

Take apart the diagonal morphism Δ into a closed immersion $X \to V$ and an open immersion $V \to X \times_Y X$. It suffices to show that the ideal sheaf $\mathscr{I}$ of the closed immersion vanishes in a neighbourhood of $\Delta(x)$. Notice that

$$\mathscr{I}_{\Delta(x)} \otimes_{\mathscr{O}_{X\times_Y X,\Delta(x)}} \mathscr{O}_{X\times_Y X,\Delta(x)}/\mathscr{I}_{\Delta(x)} \cong (\mathscr{I}/\mathscr{I}^2)_{\Delta(x)} \cong (\Omega_{X/Y})_x = 0$$

where the last isomorphism comes from the definition of $\Omega_{X/Y}$. Since f is locally of finite type, use [1, Prop.1.10.5 (iv)], we know that $\mathscr{I}_{\Delta(x)}$ is a finitely generated $\mathscr{O}_{X\times_Y X,\Delta(x)}$-mod. Note that $\mathscr{O}_{X\times_Y X,\Delta(x)}$ is a local ring, so $\mathscr{I}_{\Delta(x)} \subset \mathfrak{m}_{\Delta(x)}$. Hence we can apply Nakayama's lemma, and obtain that $\mathscr{I}_{\Delta(x)} = 0$. Then $\mathscr{I}_{\Delta(x)}$ vanishes in a neighbourhood of $\Delta(x)$, so Δ is an isomorphism in a neighbourhood of x.

(iii) $\Rightarrow$ (i).

By replacing X by a neighbourhood of x, we are reduced to the case where Δ is an open immersion. With loss of generality, we can also assume that $Y = \operatorname{Spec} k$ where $k = k(y)$.

Indeed, consider f by a base change $\operatorname{Spec} k(y) \to Y$, denote $X_y = X \times_Y \operatorname{Spec} k(y)$, we have a Cartesian diagram

$$\begin{tikzcd} X_y \times_{\operatorname{Spec} k} X_y \arrow[r] \arrow[d, "\Delta_{X_y}"] & X \times_Y X \arrow[d, "\Delta"] \\ X_y \arrow[r] & X. \end{tikzcd}$$

Thus, Δ is an open immersion implies that Δ_{X_y} is an open immersion. In the diagram:

$$\begin{tikzcd} X_y \arrow[r] \arrow[d, "f_y"] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd}$$

one can show that $\mathscr{O}_{X_y,x} \cong \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$, then the residue field of X_y at x is isomorphic to $k(x)$. f_y is unramified implies that (0) is a maximal ideal in $\mathscr{O}_{X_y,x}$, then $\mathscr{O}_{X_y,x}$ is a field. It follows that $\mathfrak{m}_y\mathscr{O}_{X,x} = \mathfrak{m}_x$. Then f is unramified.

To prove our assertion under the condition that $Y = \operatorname{Spec} k$, let $\bar{k}$ be an algebraic closure of k. Then for any closed point $t \in X$, we have a k-morphism $t : \operatorname{Spec} \bar{k} \to X$ with image t. Denote its graph map by Γ_t. Then we have a Cartesian diagram

$$\begin{tikzcd} \operatorname{Spec} \bar{k} \arrow[r, "\Gamma_t"] \arrow[d, "t"] & X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_{\operatorname{Spec} k} X \end{tikzcd}$$

By our assumption, Δ is an open immersion, then Γ_t is also an open immersion. Therefore, we know that every closed point in $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is isolated and $\Gamma(t, \mathscr{O}_{X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}}) = \bar{k}$. This implies that $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} =_{i=1}^{n} \operatorname{Spec} \bar{k}$since f is locally of finite type. Hence X is a finite set. Then we can assume that X consists of only one point. Indeed, if not, we replace X by a neighbourhood of x. Use the property of Artin rings [1, Lem.1.9.2], $X = \operatorname{Spec} A$ for some finite k-algebra A. Then we have $A \otimes_k \bar{k} \cong \prod_{i=1}^{n} \bar{k}$. Thus, $A \otimes_k \bar{k}$ is reduced then A is reduced. Since X consists of only one point, A must be a field. Note that $A \otimes_k \bar{k}$ is reduced also implies that the field extension A over k is separable. Then (i) follows. $\square$

Proposition 2.1.3. *[1, Prop.2.2.2] In the following, morphisms are assumed to be locally of finite presentation.*
(i) Unramified morphisms are locally quasi-finite,
(ii) Immersions are unramified.
(iii) Composites of unramified morphisms are unramified.
(iv) Base change of unramified morphisms are unramified.
(v) If the composite $X \to Y \to Z$ is unramified, then so is $X \to Y$.

Proof. For (i), just use the definition of unramified morphism.

For (ii), the statement holds obviously for open immersion, since open immersions are local isomorphism. For closed immersions, since they can be viewed as a surjection between rings stalk-locally [2, 42], the differential sheaf must be null.

For (iii) and (v), use the exact sequence,

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y},$$

where $X \xrightarrow{f} Y \to Z$.

For (iv), use $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$, where A' and B are A-algebras, and $B' = B \otimes_A A'$. $\square$

2.2 Étale Morphisms

Definition 2.2.1. *(Étale morphism). If f is a morphism locally of finite presentation, we say f is étale at x if f is flat at x and unramified at x. If f is étale at every point, we say f is étale.*

Proposition 2.2.2. *[1, Prop.2.3.1] In the following statement, morphisms are assumed to be locally of finite presentation.*
(i) Composites of étale morphisms are étale.
(ii) Open immersions are étale.
(iii) Base change of étale morphisms are étale.

(iv) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that gf is étale and g is unramified. Then f is étale.

(v) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that gf is flat at $x \in X$ and $g : Y \to Z$ is unramified, Then f is flat at x.

Proof. (i) and (iii) are trivial. (ii) just need to notice that open immersions are local isomorphism then flat.

For (iv), by 2.2.4 in [1], the graph $\Gamma_f : X \to X \times_Z Y$ of f is an open immersion, which is étale by (ii). Since étale is stable under base change, the projection $p_2 : X \times_Z Y \to Y$ is étale. By (1), $f = p_2\Gamma_f$ is étale. And (v) is completely similar to (iv).

$\square$

Example 2.2.3. *[1, Prop.2.3.3] Suppose A is a ring, $F(t)$ is a monic polynomial in $A[t]$, and $B = A[t]/(F(t))$. Then the canonical morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is étale at a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ if and only if $\mathfrak{q}$ doesn't contain the image of $F'(t)$. Thus, f is étale if and only if $F(t)$ and $F'(t)$ are relatively prime in $A[t]$*

Proof. The assumption that $F(t)$ is monic implies that B over A is free and then flat. Notice that $A \to A[t] \to A[t]/(F(t))$ induces an exact sequence

$$(F(t))/(F^2(t)) \xrightarrow{\delta} \Omega_{A[t]/A} \otimes_{A[t]} B \to \Omega_{B/A} \to 0,$$

where $\delta(F(t)) = dF(t) \otimes 1 = F'(t)dt \otimes 1$. It follows that

$$\Omega_{B/A} \cong \Omega_{A[t]/A} \otimes_{A[t]} B/im\,\delta \cong A[t]/(F(t), F'(t)).$$

Therefore, $(\Omega_{B/A})_{\mathfrak{q}} = 0$ if and only if $\mathfrak{q}$ doesn't contain the image of $F'(t)$ in B. $\square$

Theorem 2.2.4. *[1, Theorem.2.3.5] (Chevalley). Suppose $f : X \to Y$ is a morphism locally of finite presentation, x is a point in X. Denote $A = \mathscr{O}_{Y,f(x)}$, then f is étale at x if and only if $\mathscr{O}_{X,x}$ is A-isomorphic to a localization of $A[t]/(F(t))$ at a maximal ideal, where $F(t)$ is a monic polynomial in $A[t]$ and $\mathfrak{n}$ does not contain the image of $F'(t)$ in $A[t]/(F(t))$.*

Proof. The sufficiency is just 2.2.3. For the necessity, we may assume that $Y = \operatorname{Spec} A$. Indeed, we can take the base change by $\operatorname{Spec} A \to Y$ since the problem is local. Denote by U an affine open neighbourhood of x in X such that $f|_U : U \to \operatorname{Spec} A$ is unramified. We approach A by finitely generated $\mathbb{Z}$-modules which are A-subalgebras, say $A = \varinjlim_\lambda A_\lambda$. By [1, Lem.1.10.4 (ii), Prop.2.2.3], we can find an unramified affine morphism $f_\lambda : U_\lambda \to \operatorname{Spec} A_\lambda$ of finite type such that $f|_U$ is a base change of f_λ. Apply the Zariski Main Theorem [1, Theorem.1.10.13], we have a factorization of f_λ which can be taken a base change to a factorization of $f|_U$:

$$U \overset{j}{\hookrightarrow} \text{Spec } B' \xrightarrow{\bar{f}} \text{Spec } A$$

where j is an open immersion and $\bar{f}$ is finite. Suppose $\mathfrak{n}'$ is the prime ideal of B' which corresponds to the image of x in Spec B'. Since it is above the maximal ideal $\mathfrak{m}$ of A, $\mathfrak{n}'$ is a maximal ideal of B'. Notice that $\mathscr{O}_{X,x}$ is A-isomorphic to $B'_{\mathfrak{n}'}$. Denote by $\mathfrak{n}'_1,...,\mathfrak{n}'_r$ all the maximal ideals of B' distinct from $\mathfrak{n}'$. Since $B'/\mathfrak{n}'$ is a finite separable extension of $A/\mathfrak{m}$, it is generated by a single element. By the Chinese remainder theorem, we can find $u \in \mathfrak{n}'_1 \cap \cdots \cap \mathfrak{n}'_r$ such that the image of u in $B'/\mathfrak{n}'$ is nonzero and generates $B'/\mathfrak{n}'$ over $A/\mathfrak{m}$. Let $B = A[u]$ and let $\mathfrak{n} = \mathfrak{n}' \cap B$. By [1, Prop.2.3.4], $\mathscr{O}_{X,x}$ is A-isomorphic to $B_\mathfrak{n}$. Let $\bar{u}$ be the image of u in $B/\mathfrak{m}B$, let $f(t) \in (A/\mathfrak{m})[t]$ be the minimal polynomial of $\bar{u}$ over $A/\mathfrak{m}$, and let $d = \deg f$. We have

$$B/\mathfrak{m}B \cong (A/\mathfrak{m})[t]/(f(t)).$$

Apply Nakayama's lemma, B is generated by $1, u, ..., u^{d-1}$ as an A-module. Hence there exists a monic polynomial $F(t) \in A[t]$ of degree d such that $F(u) = 0$. The image of $F(t)$ in $(A/\mathfrak{m})[t]$ is necessarily $f(t)$. Since $A/\mathfrak{m}$ is a field, Spec $B/\mathfrak{m}B \to$ Spec $A/\mathfrak{m}$ is étale at $\mathfrak{n}/\mathfrak{m}B$. By 2.2.3, we have $f'(\bar{u}) \notin \mathfrak{n}/\mathfrak{m}B$. Hence we have $F'(u) \notin \mathfrak{n}$.

Since f is also flat at x. Then $B_\mathfrak{n}$ is flat over A. Denote by $\mathfrak{n}_0$ the inverse image of $\mathfrak{n}$ under the epimorphism

$$A[t]/(F(t)) \to B, \quad t \mapsto u$$

of A-algebras. Then $\mathfrak{n}_0$ does not contain the image of $F'(t)$ in $A[t]/(F(t))$. By 2.2.3, Spec $A[t]/(F(t)) \to$ Spec A is étale at $\mathfrak{n}_0$, and in particular, unramified in a neighbourhood of $\mathfrak{n}_0$. Using 2.2.2 (v) to the composite

$$\text{Spec } B \to \text{Spec } A[t]/(F(t)) \to \text{Spec } A,$$

we know that the left morphism is flat at $\mathfrak{n}$. Thus, $(A[t]/(F(t)))_{\mathfrak{n}_0} \to B_\mathfrak{n}$ is faithfully flat and then injective. However, $A[t]/(F(t)) \to B$ is surjective, so we have $(A[t]/(F(t)))_{\mathfrak{n}_0} \cong B_\mathfrak{n}$ Therefore, $(A[t]/(F(t)))_{\mathfrak{n}_0}$ is A-isomorphic to $\mathscr{O}_{X,x}$. □

Proposition 2.2.5. *[1, Prop.2.3.8] Étale morphisms are open mappings.*

Proof. Since the problem is local, we only need to prove for the case where f : Spec $B \to$ Spec A is an étale morphism between affine schemes. Apply [1, Lem.1.10.4(ii), Prop.2.3.7], we can assume that f is the base change of the morphism f_0 : Spec $B_0 \to$ Spec A_0 where A_0 is a subalgebra of A which is finitely generated over $\mathbb{Z}$ and B_0 is an étale finitely generated A_0-algebra. Denote by

$\{A_\lambda\}$ the family of subalgebras of A being finitely generated over A_0. We have the following Cartesian diagrams:

$$\begin{array}{ccccc}
\operatorname{Spec} B & \xrightarrow{v_\lambda} & \operatorname{Spec} B_\lambda & \longrightarrow & \operatorname{Spec} B_0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\lambda} & & \downarrow{\scriptstyle f_0} \\
\operatorname{Spec} A & \xrightarrow{u_\lambda} & \operatorname{Spec} A_\lambda & \longrightarrow & \operatorname{Spec} A_0
\end{array}$$

where $B_\lambda = B_0 \otimes_{A_0} A_\lambda$. Note that every f_λ is flat, and now A_λ, B_λ are noetherian by the finite type conditions. Then apply [1, Theorem 1.5.2], we know that f_λ is an open mapping. By [1, Prop.1.10.1(ii)], any quasi-compact open subset U of $\operatorname{Spec} B$ can be written by the form $U = v_\lambda^{-1}(U_\lambda)$ for some λ and some open subset U_λ of $\operatorname{Spec} B_\lambda$. Since $f_\lambda(U_\lambda)$ is an open set and

$$f(u) = f v_\lambda^{-1}(U_\lambda) = u_\lambda^{-1} f_\lambda(U_\lambda),$$

$f(U)$ is also an open set. $\square$

Definition 2.2.6. *(Radical morphism). If f is universally injective, that is, any base change of f is injective, we say that f is radical.*

Proposition 2.2.7. *[1, Prop.2.3.9] A morphism is an open immersion if and only if it is étale and radical.*

Proof. Denote the morphism by $f : X \to Y$. The necessity is obvious. We are to show the sufficiency. Notice that f is injective, we just need to show that f is locally an open immersion. Hence we can assume that f is the base change of $f_0 : \operatorname{Spec} B_0 \to \operatorname{Spec} A_0$ by $Y \to \operatorname{Spec} A_0$ where A_0 is a finitely generated $\mathbb{Z}$-algebra and B_0 is a finitely generated A_0-algebra. Since f is radical, f is clearly separated and quasi-finite. Then use the Zariski Main Theorem [1, Theorem.1.10.13], we can factorize f_0 as the composite of an open immersion and a finite morphism. After taking a base change, we have a factorization of f:

$$X \overset{j}{\hookrightarrow} \bar{X} \overset{\bar{f}}{\to} Y.$$

where j is an open immersion, and $\bar{f}$ is a finite morphism. From 2.2.5, we know that $f(X)$ is an open set. So we can assume that f is surjective. Otherwise, replace Y by $f(X)$ and $\bar{X}$ by $\bar{f}^{-1}(f(X))$. Since f is étale, it is universally open again by 2.2.5. At the same time, f is universally injective, hence it is universally an homeomorphism. It implies that f is proper and integral. Since $\bar{f}$ is separated, by [3, Ch.2 Cor.4.8(e)], j is proper. Then j is an open and closed immersion, which implies that it is finite. Thus, f is finite. Now in order to prove that f is an isomorphism, we are reduced to the case where $Y = \operatorname{Spec} A$, and $X = \operatorname{Spec} B$

such that B is a finite A-algebra. Since $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is radical and integral, $B_{\mathfrak{p}}$ is a local ring for any $\mathfrak{p} \in \mathrm{Spec}\, A$. However, f is also étale, the field extension $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is both purely inseparable (by [1, Prop.1.7.1 (iii)]) and separable, then the two fields are isomorphic. Apply Nakayama's lemma, $f_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective. However, $f_{\mathfrak{p}}$ is faithfully flat since it is flat and $\mathrm{Spec}\, B_{\mathfrak{p}} \to \mathrm{Spec}\, A_{\mathfrak{p}}$ is surjective. It follows that $f_{\mathfrak{p}}$ is injective. Then we have $A_{\mathfrak{p}} \cong B_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}\, A$. This implies that $A \cong B$.

□

Proposition 2.2.8. *[1, Prop.2.3.10] Let $f : X \to Y$ be a morphism, if f is unramified and separated and Y is connected, then there is a bijection between the set of sections of f and the set of connected components of X on which f induces an isomorphism.*

Proof. Suppose that $s : Y \to X$ is a section of f, then s is a closed immersion using the fact that sections of separated morphisms are closed immersions. Furthermore, $fs = id$ yields that s is radical and that s is étale by 2.2.2 (iv). Thus, s is an open immersion by 2.2.7 which induces an isomorphism between Y and a connected component of X. □

2.3 Henselization

Definition 2.3.1. *(Henselian rings). Let $(R, \mathfrak{m})$ be a local ring, $k = R/\mathfrak{m}$ its residue field. Then if the following statement holds for $(R, \mathfrak{m})$, we say that R is Henselian.*

For any monic polynomial $f(t) \in R[t]$ and any factorization $\bar{f}(t) = \bar{g}(t)\bar{h}(t)$, where $\bar{f}(t)$ is the image of $f(t)$ in $k[t]$ and $\bar{g}(t), \bar{h}(t)$ are two relatively prime monic polynomials in $k[t]$, there exist uniquely two relatively prime polynomials $g(t)$ and $h(t)$ in $R[t]$ such that

(i) $f(t) = g(t)h(t)$,

(ii) $\bar{g}(t)$ and $\bar{h}(t)$ are the images of $g(t)$ and $h(t)$ in $k[t]$ respectively.

Moreover, if k is separably closed, we say that R is strictly henselian.

Proposition 2.3.2. *[1, Prop.2.8.3] R is henselian if and only if for any étale morphism $g : X \to \mathrm{Spec}\, R$, any section of $g_s : X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ is induced by a section of g.*

Proof. "Necessity".

For an section of g_s, suppose x is its image. Use 2.2.4, we know that there exists a monic polynomial $f(t)$ in $R[t]$ and a maximal ideal $\mathfrak{n}$ of $B = R[t]/(f(t))$ which doesn't contain the image of $f'(t)$ in B such that $\mathscr{O}_{X,x}$ is R-isomorphic to $B_{\mathfrak{n}}$.

Let $\bar{f}(t)$ be the image of $f(t)$ in $k[t]$. Then the section g_s induces a k-morphism Spec $k \to$ Spec $\mathscr{O}_{X,x}/\mathfrak{m}\mathscr{O}_{X,x}$. We compose this k-morphism with the canonical morphisms

$$\text{Spec } \mathscr{O}_{X,x}/\mathfrak{m}\mathscr{O}_{X,x} \cong \text{Spec } B_{\mathfrak{n}}/\mathfrak{m}B_{\mathfrak{n}} \to \text{Spec } B/\mathfrak{m}B \cong \text{Spec } k[t]/(\bar{f}(t)),$$

we obtain a k-morphism

$$\text{Spec } k \to \text{Spec } k[t]/(\bar{f}(t)),$$

which corresponds to a k-homomorphism

$$k[t]/(\bar{f}(t)) \to k.$$

Denote by $\bar{a}$ the image of t under this homomorphism, then $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$ in k. Hence $\bar{a}$ is a simply root of $\bar{f}(t)$, and we have the factorization

$$\bar{f}(t) = (t - \bar{a})\bar{h}(t)$$

where $\bar{h}(t)$ is a monic polynomial in $k[t]$ being relatively prime to $t - \bar{a}$. The definition of henselian yields a lifted factorization

$$f(t) = (t - a)h(t)$$

for some $a \in R$ which lifts $\bar{a}$ and $h(t) \in R[t]$ which lifts $\bar{h}(t)$, where $(t - a)$ and $h(t)$ are relatively prime. The R-homomorphism

$$R(t)/(f(t)) \to R[t]/(t - a) \cong R,\ t \mapsto a$$

induces an R-homomorphism

$$B_{\mathfrak{n}} = (R[t]/(f(t)))_{\mathfrak{n}} \to R$$

then a Spec R-morphism

$$\text{Spec } R \to \text{Spec } B_{\mathfrak{n}} \cong \text{Spec } \mathscr{O}_{X,x}.$$

Compose it with the canonical morphism Spec $\mathscr{O}_{X,x}$, we obtain a section of $g : X \to \text{Spec } R$ which induces g_s.

"Sufficiency".

Denote by $\text{Idem}(A)$ the set of idempotent elements in A for any ring A.

Claim. Given the assumption, we know that for any $A = R[t]/(f(t))$ where $f(t)$ is a monic polynomial in $R[t]$, we have a bijection $p : \text{Idem}(A) \to \text{Idem}(A/\mathfrak{m}A)$ induced by the canonical homomorphism $A \to A/\mathfrak{m}$. If the claim holds, use [1, Lem.2.8.1 (i), (ii)], we will have a bijection between the set of pairs $(g(t), h(t))$ of

A and the set of pairs $(\bar{g}(t), \bar{h}(t))$ of $A/\mathfrak{m}A$, where $f(t) = g(t)h(t)$, $g(t)$ and $h(t)$ are relatively prime in $A[t]$, and $\bar{f}(t) = \bar{g}(t)\bar{h}(t)$, $\bar{g}(t)$ and $\bar{h}(t)$ are relatively prime in $k[t]$. As a sequence, R is henselian.

Now we prove the claim. Note firstly that p is an injection for any local ring A. Indeed, suppose that $e_1, e_2 \in A$ such that $e_1 \equiv e_2 \mod \mathfrak{m}A$. We have

$$(e_1 - e_2)^3 = e_1^3 - 3e_1^2 e_2 + 3e_1 e_2^2 - e_2^3 = e_1 - 3e_1 e_2 + 3e_1 e_2 - e_2 = e_1 - e_2$$

Then

$$(e_1 - e_2)((e_1 - e_2)^2 - 1) = 0.$$

Notice that $e_1 - e_2 \in \mathfrak{m}A$, then $(e_1 - e_2)^2 - 1 \in A^\times$. Hence $e_1 - e_2 = 0$. Thus, p is injective. To prove that p is surjective, we use [1, Lem.2.8.2], let B represents the functor $\mathrm{Idem}(\,\cdot\, \otimes_R A)$. Just need to show that the map

$$\mathrm{Hom}_R(B, R) \to \mathrm{Hom}_R(B, R/\mathfrak{m})$$

is surjective. Note that elements in $\mathrm{Hom}_R(B, R)$ and $\mathrm{Hom}_R(B, R/\mathfrak{m})$ can be respectively identified with sections of $g : X \to \operatorname{Spec} R$ and sections of $g_s : X \times_{\operatorname{Spec} R} \operatorname{Spec} k \to \operatorname{Spec} k$. And this is implied by our assumption since g is étale. $\square$

Definition 2.3.3. *(Essentially étale morphisms). Let $A \to A'$ be a local homomorphism of local rings. We say A' is essentially étale over A if there exists an étale A-algebra B such that A' is A-isomorphic to $B_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of B lying over the maximal ideal of A. Furthermore, we say this homomorphism is strictly essentially étale if it induces an isomorphism on the residue field besides.*

Definition 2.3.4. *(Henselization). Let A be a local ring.*

Suppose $\mathscr{S}$ is the category whose objects are strictly essentially étale local A-algebras, and whose morphisms are A-homomorphisms. By [1, Lem. 2.8.6], "upper bound" for objects and morphisms can be obtained in $\mathscr{S}$[1], then we can take direct limits in $\mathscr{S}$. We define the henselization A^h of A to be

$$A^h = \varinjlim_{A' \in ob\,\mathscr{S}} A'.$$

Let Ω be a separable closure of k. Define $\mathscr{T}$ as follows. Objects in $\mathscr{T}$ are pairs $(A', \beta_{A'})$ where A' is an essentially étale local A-algebras and $\beta_{A'} : k(A') \to \Omega$ are k-homomorphisms. A morphism in $\mathscr{T}$ from (A_1, β_{A_1}) to (A_2, β_{A_2}) is an A-algebra homomorphism $\phi : A_1 \to A_2$ such that $\beta_{A_1} = \beta_{A_2}\bar{\phi}$, where $\bar{\phi}$ is the homomorphism

[1]For a precise description, confer [1, Section 2.7 (I1),(I2)]. Our aim is to take direct limit on a category instead of a directed set, then the "upper bound" condition for the category is similar to require the set to be directed

induced by ϕ on the residue field. By [1, Lem. 2.8.16], "upper bound" for objects and morphisms can be obtained in $\mathscr{T}$. Then we can define the strict henselization of A relative to the inclusion $i : k \to \Omega$ to be

$$\tilde{A} = \varinjlim_{(A',\beta_{A'})\in ob\,\mathscr{T}} A',$$

Chapter 3

Étale Sheaves

3.1 Étale Topology and Étale Sheaves

Definition 3.1.1. *(Grothendieck topology).*

Let $\mathscr{C}$ be a category on which fiber products exist. For each object U in $\mathscr{C}$, suppose we specify a family $\mathscr{T}_U$ whose elements are sets of morphisms in C of the form $\{U_\alpha \to U\}_{\alpha\in I}$. We call the elements in $\mathscr{T}_U$ the coverings of U. We say that $\mathscr{T}$ is a Grothendieck topology on $\mathscr{C}$ if the following axioms hold:
(GT1) For any isomorphism $V \to U$ in $\mathscr{C}$, the set $\{V \to U\}$ is a covering in $\mathscr{T}_U$.
(GT2) Let $\{U_\alpha \to U\}_{\alpha\in I}$ be a covering in $\mathscr{T}_U$. For any morphism $V \to U$ in C, the set $\{U_\alpha \times_U V \to V\}_{\alpha\in I}$ defined by base change is a covering in $\mathscr{T}_U$.
(GT3) Let $\{U_\alpha \to U\}_{\alpha\in I}$ be a covering in $\mathscr{T}_U$, and for each $\alpha \in I$, let $\{U_{\alpha\beta} \to U_\alpha\}_{\beta\in I_\alpha}$ be a covering in $\mathscr{T}_{U_\alpha}$. Then the set $\{U_{\alpha\beta} \to U\}_{\alpha\in I,\beta\in I_\alpha}$ defined by taking composites is a covering in $\mathscr{T}_U$.

An étale presheaf $\mathscr{F}$ on $\mathscr{C}$ is called an étale sheaf if for any $U \in \mathrm{ob}\,\mathscr{C}$ and any covering $\{U_\alpha \to U\}_{\alpha\in I}$ in $\mathscr{T}_U$, the sequence

$$0 \to \mathscr{F}(U) \to \prod_{\alpha\in I} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta\in I} \mathscr{F}(U_\alpha \times_U U_\beta) \tag{3.1}$$

is exact, where the homomorphisms in this sequence are defined by

$$\mathscr{F}(U) \to \prod_{\alpha\in I} \mathscr{F}(U_\alpha), \quad s \mapsto (s|_{U_\alpha})$$
$$\prod_{\alpha\in I} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta\in I} \mathscr{F}(U_\alpha \times_U U_\beta), \quad (s_\alpha) \mapsto (s_\beta|_{U_\alpha\times_U U_\beta} - s_\alpha|_{U_\alpha\times_U U_\beta}).$$

Example 3.1.2. *Let X be a scheme, then the Zariski topology together with ordinary open coverings forms a Grothendieck topology naturally.*

Definition 3.1.3. *(Étale topology). Let X be a scheme, we define a category $X_{\acute{e}t}$: Objects in $X_{\acute{e}t}$ are étale morphisms $U \to X$ and arrows are X-morphisms $U \to V$ (U and V are étale X-schemes). Note that morphisms in $X_{\acute{e}t}$ are étale and $X_{\acute{e}t}$ admits fiber product. For each U, a set $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ of morphisms in $X_{\acute{e}t}$ is called an étale covering of U if U is the union of images of U_α. Then, $X_{\acute{e}t}$ together with the étale coverings satisfies (GT1-GT3) and forms a Grothendieck topology, called the étale topology on X. Let $f : X' \to X$ be a morphism of schemes, f induces a functor*

$$f_{\acute{e}t} : X_{\acute{e}t} \to X'_{\acute{e}t}, \quad U \mapsto U \times_X X'.$$

Definition 3.1.4. *(Étale sheaves). An étale presheaf of sets on X is defined to be a contravariant functor from $X_{\acute{e}t}$ to the category of set. Morphisms of étale presheaves are defined to be natural transformations of functors. We use the notations of restriction maps as those in ordinary sheaf theory below. When the above exact sequence 3.1 holds for an étale presheaf, we say that it is an étale sheaf. Denote the category of étale presheaves on X by $\mathscr{P}_X$ and the category of étale sheaves on X by $\mathscr{S}_X$*

Definition 3.1.5. *Let $f : X' \to X$ be a morphism of schemes.*
(i) For any étale presheaf $\mathscr{F}'$ on X', we define an étale presheaf $f_{\mathscr{P}}\mathscr{F}'$ on X by

$$(f_{\mathscr{P}}\mathscr{F}')(U) := \mathscr{F}'(U \times_X X').$$

(ii) For any étale presheaf $\mathscr{F}$ on X, we define an étale presheaf $f^{\mathscr{P}}\mathscr{F}$ on X' as follows.
For any object U' in $X'_{\acute{e}t}$, let $I_{U'}$ be the category whose objects are pairs (U, ϕ), where U is an object in $X_{\acute{e}t}$ and $\phi : U' \to U$ is an X-morphism, and morphisms in $I_{U'}$ from (U_1, ϕ_1) to (U_2, ϕ_2) is an X-morphism $\xi : (U_1, \phi_1) \to (U_2, \phi_2)$ satisfying $\xi\phi_1 = \phi_2$. Define

$$(f^{\mathscr{P}}\mathscr{F})(U') := \varinjlim_{(U,\phi) \in ob I^{\circ}_{U'}} \mathscr{F}(U)$$

for each $U' \in ob X'_{\acute{e}t}$[1]*.*

Proposition 3.1.6. *These two functors are adjoint, i.e.:*

$$\mathrm{Hom}(f^{\mathscr{P}}\mathscr{F}, \mathscr{F}') \cong \mathrm{Hom}(\mathscr{F}, f_{\mathscr{P}}\mathscr{F}')$$

Definition 3.1.7. *(Sheafification of étale presheaves). Suppose $\mathscr{G}$ is an étale presheaf on X, if there exists a pair $(\mathscr{G}^{\#}, \theta)$, where $\mathscr{G}^{\#}$ is an étale sheaf on X and $\theta : \mathscr{G} \to \mathscr{G}^{\#}$ is a morphism of étale presheaves such that for any étale sheaf $\mathscr{F}$ on*

[1] Actually, in the category $I^{\circ}_{U'}$, "upper bound" for any two morphisms and for any two objects can be obtained. Direct limit can then be well defined on it. For a precise proof, see [1, 177]

X and any morphism of étale presheaves $\phi : \mathscr{G} \to \mathscr{F}$, there exists one and only one morphism $\psi : \mathscr{G}^{\#} \to \mathscr{F}$ such that $\psi\theta = \phi$. Then we say $\mathscr{G}^{\#}$ is a sheafification of $\mathscr{G}$.

Proposition 3.1.8. *[1, Prop.5.2.1, Lem.5.2.2, Lem.5.2.3] For any étale presheaf $\mathscr{G}$ on X, there exists one and only one sheafification of $\mathscr{G}$ up to an isomorphism. In other words, we have the adjunction,*

$$\mathrm{Hom}_{sh}(\mathscr{G}^{\#}, \mathscr{F}) \cong \mathrm{Hom}_{pre}(\mathscr{G}, i\mathscr{F})$$

where $\# : \mathscr{P}_X \to \mathscr{S}_X$ and $i : \mathscr{S}_X \to \mathscr{P}_X$ the inclusion functor. For convenience, we often omit i.

Proof. (Sketch). Define firstly a functor $+ : \mathscr{P}_X \to \mathscr{P}_X$. For every object U in $X_{\acute{e}t}$, let J_U be the category whose objects are étale coverings of U, whose morphisms are the refinement of étale coverings[2]. Note that every refinement induces a homomorphism between the Čech cohomology[3] $\check{H}^0(\mathfrak{U}', \mathscr{G}) \to \check{H}^0(\mathfrak{U}, \mathscr{G})$. Notice that J_U is a directed family then we define

$$\mathscr{H}^0(\mathscr{F})^+(U) := \varinjlim_{\mathfrak{U} \in ob J_U} \check{H}^0(\mathfrak{U}, \mathscr{H}^i(\mathscr{F}))$$

Claim.

(i) For any étale presheaf $\mathscr{G}$, $\mathscr{G}^+$ satisfies the identity condition of étale sheaves, which means that the first entry of 3.1 is exact.

(ii) For any étale presheaf $\mathscr{G}$ satisfying the identity condition, $\mathscr{G}^+$ is an étale sheaf.

(iii) For any étale presheaf $\mathscr{G}$ and any étale sheaf $\mathscr{F}$ on X, the canonical morphism $\mathscr{G} \to \mathscr{G}^+$ induces a bijection

$$\Phi : \mathrm{Hom}(\mathscr{G}^+, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{G}, \mathscr{F})$$

We now use (GT3) in 3.1.1 to show (i). To check the identity condition, let $\mathfrak{U} = \{U_\alpha \to U\}$ be an étale covering. For two elements $\xi_i\,(i = 1, 2)$ in $\mathscr{G}^+(U)$, we choose an étale covering $\mathfrak{B}$ to represent them. We take the composite covering, denoted simply by $\mathfrak{U} \times_U \mathfrak{B}$. Since the images of ξ_i in $\check{H}^0(\mathfrak{B}, \mathscr{G}) \to \check{H}^0(\mathfrak{B} \times_U U_\alpha, \mathscr{G}) \to \mathscr{G}^+(U_\alpha)$ are the same by the assumption, there exists a refinement $\mathfrak{M}_\alpha$ of $\mathfrak{B} \times_U U_\alpha$ for each α, which is an étale covering of U_α such that the images of ξ_i in $\check{H}^0(\mathfrak{M}_\alpha, \mathscr{G})$ are the same. However, $\bigcup_\alpha \mathfrak{M}_\alpha$ is a refinement of $\mathfrak{B}$, which is also an étale covering of U. Then ξ_i have the same image in $\check{H}^0(\bigcup_\alpha \mathfrak{M}_\alpha, \mathscr{G})$.

[2] The definition of refinements of étale coverings is similar to that in algebraic geometry. For a precise definition, confer[1, Section 5.1 P75]

[3] The definition of Čech cohomology is in 4.1.1

Hence ξ_i have the same image in $\mathscr{G}^+(U)$, *i.e.* they represent the same element. This proves the exactness of the first entry.

We can also prove (ii) and (iii) mainly by applying the axioms of Grothendieck topology similarly, confer [1, Lem.5.3.2, Lem.5.2.2]. We define the sheafification $\mathscr{G}^{\#}$ to be $\mathscr{G}^{++}$, which satisfies the desired property. $\square$

Remark 3.1.9. *$+$ is left exact, $\# = + \circ +$ is also left exact. Furthermore, the adjunction implies that $\#$ is also right exact, and the inclusion $i : \mathscr{S}_X \to \mathscr{I}_X$ is a left exact functor.*

Definition 3.1.10. *Suppose $f : X' \to X$ is a morphism of schemes, $\mathscr{F}'$ is an étale sheaf on X', it is clear that $f_{\mathscr{P}}\mathscr{F}$ is also an étale sheaf, denoted by $f_*\mathscr{F}$, called the direct image of $\mathscr{F}'$. Suppose $\mathscr{F}$ is an étale sheaf on X, we define $f^*\mathscr{F} = (f^{\mathscr{P}}\mathscr{F})^{\#}$, called the inverse image of $\mathscr{F}$ on X'. One can show that*

$$\mathrm{Hom}(f^*\mathscr{F}, \mathscr{F}') \cong \mathrm{Hom}(\mathscr{F}, f_*\mathscr{F}')$$

which is a functorial bijection.

3.2 Stalks of Étale Sheaves

Definition 3.2.1. *(Geometric point). Let X be a scheme. A geometric point of X is a morphism $\gamma : s \to X$ such that s is the spectrum of a separably closed field. We often write $k(s)$ for the separably closed field such that $s = \mathrm{Spec}\, k(s)$. (Giving a geometric point is equivalent to giving a separably closed extension $k(s)$ of the residue field $k(x)$). Let $f : X \to X'$ be a morphism. Then $f\gamma : s \to X'$ is a geometric point of X'.*

Definition 3.2.2. *(Stalk of étale sheaves). Suppose X is a scheme, $\mathscr{F}$ is an étale sheaf on X. Let $s \to X$ be a geometric point of X, where $s = \mathrm{Spec}\, k(s)$ with $k(s)$ separably closed. For convenience, we denote the geometric point simply by s. We define the stalk of $\mathscr{F}$ at s to be $\Gamma(s, \mathscr{F}|_s)$, where $\mathscr{F}|_s$ is the inverse image of $\mathscr{F}$ on s, denoted by $\mathscr{F}_s$. Note that*

$$\mathscr{F}_s \cong \varinjlim_{(U,s_U)\in ob I_s^\circ} \mathscr{F}(U).$$

Proposition 3.2.3. *[1, Prop.5.3.1] Let X be a scheme and let $s \to X$ be a geometric point of X. For any étale presheaf $\mathscr{F}$ on X, we have*

$$\mathscr{F}_s^\sharp \cong \varinjlim_{(U,s_U)\in ob I_s^\circ} \mathscr{F}(U).$$

Proof. Note firstly that

$$\Gamma(s, \mathscr{F}) \cong \Gamma(s, \mathscr{F}^{\sharp}).$$

Denote $\gamma : s \to X$, then we have

$$\mathscr{F}_s^{\sharp} = \Gamma(s, \gamma^*(\mathscr{F}^{\sharp})) \cong \Gamma(s, (\gamma^{\mathscr{P}} \mathscr{F})^{\sharp}) \cong \Gamma(s, \gamma^{\mathscr{P}} \mathscr{F}) = \varinjlim_{(U, s_U) \in ob I_s^{\circ}} \mathscr{F}(U).$$

$\square$

Notations as above, an object (U, s_U) in I_s is called an étale neighborhood of s in X.

Lemma 3.2.4. *[1, Lem.5.3.2, Prop.5.3.3] Let X be a scheme.*

(i) Let f be a morphism between two étale sheaves. Then f is an isomorphism if and only if it induces an isomorphism on stalks of every point in X.

(ii) Two morphisms of étale sheaves are identical if and only if they induce the same morphism on the stalk of every point of X.

(iii) A sequence

$$\mathscr{F} \to \mathscr{G} \to \mathscr{H}$$

of étale sheaves on X is exact if and only if for any geometric point s of X, the sequence

$$\mathscr{F}_s \to \mathscr{G}_s \to \mathscr{H}_s$$

is exact.

The proof is completely similar to that in algebraic geometry.

Definition 3.2.5. *Suppose X is a scheme and $s \to X$ is a geometric point of X. We define an étale sheaf $\mathscr{O}_{X_{ét}}$ on X to be $\mathscr{O}_{X_{ét}}(U) = \mathscr{O}_U(U)$ for any object U in $X_{ét}$.*

Denote the image of the geometric point in U by ξ, we have

$$\begin{aligned}
\mathscr{O}_{X_{ét}, s} &\cong \varinjlim_{(U, s_U) \in I_s^{\circ}} \mathscr{O}_U(U) \\
&\cong \varinjlim_{(U, s_U) \in I_s^{\circ}} \varinjlim_{\xi \in V \subseteq U} \mathscr{O}_U(V) \\
&\cong \varinjlim_{(U, s_U) \in I_s^{\circ}} \mathscr{O}_{U, \xi} \\
&\cong \tilde{\mathscr{O}}_{X, s},
\end{aligned}$$

which is the strictly henselization of the Zariski stalk $\mathscr{O}_{X,x}$.

Proposition 3.2.6. *[1, Prop.5.3.4]*

Suppose X is a scheme, then for any geometric point s' in X and any étale X-scheme U, the canonical map

$$\mathrm{Hom}_X(\widetilde{X}_{s'}, U) \to \mathrm{Hom}_X(s', U)$$

is bijective.

Proof. Taking the graphs of X-morphisms, we get the following one-to-one correspondences

$$\mathrm{Hom}_X(\widetilde{X}_{s'}, U) \cong \mathrm{Hom}_{\widetilde{X}_{s'}}(\widetilde{X}_{s'}, \widetilde{X}_{s'} \times_X U),$$
$$\mathrm{Hom}_X(s', U) \cong \mathrm{Hom}_{s'}(s', s' \times_X U).$$

Hence, by substituting $\widetilde{X}_{s'} \times_X U$ with W, it suffices to show that for any étale $\widetilde{X}_{s'}$-scheme W, the canonical map

$$\mathrm{Hom}_{\widetilde{X}_{s'}}(\widetilde{X}_{s'}, W) \to \mathrm{Hom}_{s'}(s', s' \times_{\widetilde{X}_{s'}} W)$$

is bijective. This follows from 2.2.8 and 2.3.2. □

Proposition 3.2.7. *[1, Prop.5.3.7] Suppose f is a finite morphism, $y \in Y$, $\overline{k(y)}$ is a separable closure of $k(y)$ and $\mathscr{F}$ is an étale sheaf on X. Then we have*

$$(f_*\mathscr{F})_{\bar{y}} \cong \bigoplus_{x \in X \times_Y \mathrm{Spec}\, \overline{k(y)}} \mathscr{F}_{\bar{x}}.$$

Proof. Note that we have the Cartesian diagrams

$$\begin{tikzcd}
X \times_Y \widetilde{Y}_{\bar{y}} \arrow[r, "f_{\bar{y}}"] \arrow[d] & \widetilde{Y}_{\bar{y}} \arrow[d] \\
X \times_Y \mathrm{Spec}\, \mathscr{O}_{Y,y} \arrow[r] \arrow[d] & \mathrm{Spec}\, \mathscr{O}_{Y,y} \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

In order to study $X \times_Y \widetilde{Y}_{\bar{y}}$ and $X \times_Y \mathrm{Spec}\, \mathscr{O}_{Y,y}$, we can replace Y by an affine neighbourhood of y, and X by the inverse image of that affine neighbourhood. Then $X \times_Y \mathrm{Spec}\, \mathscr{O}_{Y,y}$ is affine, denoted by $\mathrm{Spec}\, B$. Let A be $\mathscr{O}_{Y,y}$, use [1, Prop.2.8.20] we have

$$(B \otimes_A \widetilde{A})_n \cong \mathrm{Spec}\, \widetilde{B_{\mathfrak{n}}} \cong \mathrm{Spec}\, \widetilde{\mathscr{O}_{X,x}} \cong \widetilde{X}_{\bar{x}}.$$

Since $f_{\bar{y}}$ is finite then integral, we have the following bijections between sets

$$\mathrm{Spec}_{max}\,(B\otimes_A \widetilde{A}) \leftrightarrow f_{\bar{y}}^{-1}(\bar{y}) \leftrightarrow X\times_Y \mathrm{Spec}\,\overline{k(y)}.$$

Use the proof of [1, Prop.2.8.3 (iii)⇒(i)], we have

$$X\times_Y \widetilde{Y}_{\bar{y}} \cong_{\mathfrak{n}\in \mathrm{Spec}_{max}\,(B\otimes_A \widetilde{A})} (B\otimes_A \widetilde{A})_{\mathfrak{n}} \cong_{x_i\in X\times_Y \mathrm{Spec}\,\overline{k(y)}} \widetilde{X}_{\bar{x}_i}.$$

Then use the proof of [1, 2.8.20], we know that there exists an étale morphism $U\to Y$ such that U is affine, $X\times_Y U \cong_{i=1}^{n} V_i$ for some finite U-schemes V_i, $\widetilde{Y}_{\bar{y}}$ factors through U, and $V_i\times_U \widetilde{Y}_{\bar{y}} \cong \widetilde{X}_{\bar{x}_i}$.

Replacing Y by U, and X by each V_i, we are reduced to the case where Y is affine, $X\times_Y \mathrm{Spec}\,\overline{k(y)}$ contains only one point x and $X\times_Y \widetilde{Y}_{\bar{y}} \cong \widetilde{X}_{\bar{x}}$.

Fix an affine open neighbourhood U_0 of y in Y. By 3.2.6, for any affine étale neighbourhood $(V,\bar{x}_V)$ of $\bar{x}$ such that the image of V in Y is contained in U_0, the X-morphism $\bar{x}_V : \mathrm{Spec}\,\overline{k(x)} \to V$ defines an X-morphism

$$g : \widetilde{X}_{\bar{x}} \to V$$

such that the composite of g with the morphism $\mathrm{Spec}\,\overline{k(x)} \to \widetilde{X}_{\bar{x}}$ coincides with $\bar{x}_V$. Denote by $J_{\bar{y}}$ the category of affine étale neighbourhood of $\bar{y}$ in U_0. Then we have

$$\widetilde{Y}_{\bar{y}} \cong \mathrm{Spec}\,(\varinjlim_{(U,s_U)\in ob J_{\bar{y}}^{\circ}} \mathscr{O}_U(U)).$$

Apply [1, 1.10.9], there exist an affine étale neighbourhood (U,s_U) of $\bar{y}$ in Y and an X-morphism $g_U : X\times_Y U \to V$ such that the composite

$$\widetilde{X}_{\bar{x}} \cong X\times_Y \widetilde{Y}_{\bar{y}} \to X\times_Y U \xrightarrow{g_U} V$$

coincides with $g : \widetilde{X}_{\bar{x}} \to V$. The composite

$$\mathrm{Spec}\,\overline{k(x)} \to \widetilde{X}_{\bar{x}} \cong X\times_Y \widetilde{Y}_{\bar{y}} \to X\times_Y U$$

makes $X\times_Y U$ an étale neighbourhood of $\bar{x}$ in X which dominates the étale neighbourhood $(V,\bar{x}_V)$. It follows that

$$\mathscr{F}_{\bar{x}} \cong \varinjlim_{(U,s_U)\in ob J_{\bar{y}}^{\circ}} \mathscr{F}(X\times_Y U).$$

However, we have

$$(f_*\mathscr{F})_{\bar{y}} \cong \varinjlim_{(U,s_U)\in ob J_{\bar{y}}^{\circ}} (f_*\mathscr{F})(U) \cong \varinjlim_{(U,s_U)\in ob J_{\bar{y}}^{\circ}} \mathscr{F}(X\times_Y U).$$

Hence $(f_*\mathscr{F})_{\bar{y}} \cong \mathscr{F}_{\bar{x}}$. Our assertion follows. $\square$

Corollary 3.2.8. *Following the above condition, the functor f_* is exact and faithful.*

Corollary 3.2.9. *[1, Cor.5.3.8] Suppose $f : X \to Y$ is an immersion. Then for any étale sheaf $\mathscr{F}$ on X, the canonical morphism $f^* f_* \mathscr{F} \to \mathscr{F}$ is an isomorphism.*

Proof. For open immersions, the assertion holds obviously. If f is a closed immersion, for any point x in X,

$$(f^* f_* \mathscr{F})_{\bar{x}} \cong (f_* \mathscr{F})_{f(\bar{x})} \cong \mathscr{F}_{\bar{x}}.$$

Indeed, $X \times_Y \operatorname{Spec} \overline{k(y)}$ consists of only one point in its underlying topological space. □

Corollary 3.2.10. *[1, Cor.5.3.9] Given a Cartesian diagram*

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

If f is finite, then for any étale sheaf $\mathscr{F}$ on X, we have a canonical isomorphism

$$g^* f_* \mathscr{F} \xrightarrow{\cong} f'_* g'^* \mathscr{F}.$$

Proof. Apply f_* to the canonical morphism , we have

$$f_* \mathscr{F} \to f_* g'_* g'^* \mathscr{F} \cong g_* f'_* g'^* \mathscr{F}.$$

Use the adjunction of g^* and g_*, we have the morphism

$$g^* f_* \mathscr{F} \to f'_* g'^* \mathscr{F}.$$

View on stalks and using 3.2.7, we have

$$(g^* f_* \mathscr{F})_{y'} \cong \bigoplus_{x \in X \times_Y \operatorname{Spec} \overline{k(g(y'))}} \mathscr{F}_x,$$

and

$$(f'_* g'^* \mathscr{F})_{y'} \cong \bigoplus_{x' \in X \times_Y \operatorname{Spec} \overline{k(y')}} \mathscr{F}_{g(y')},$$

which are actually isomorphic. This shows that the morphism is an isomorphism. □

Corollary 3.2.11. *[1, Cor.5.3.10] Suppose $f : X \to Y$ is a finite surjective radical morphism. Then for any étale sheaf $\mathscr{F}$ on X and any étale sheaf on Y, the canonical morphisms $f^* f_* \mathscr{F} \to \mathscr{F}$ and $\mathscr{G} \to f_* f^* \mathscr{G}$ are isomorphisms.*

Proof. Notice that $X \times_Y \operatorname{Spec} \overline{k(y)} \to \operatorname{Spec} \overline{k(y)}$ is a base change of f. Thus, it is surjective and injective, which implies that $X \times_Y \operatorname{Spec} \overline{k(y)}$ consists of only one point. Following 3.2.7, we complete the proof. $\square$

Remark 3.2.12. *Furthermore, under the condition above, the two functors f_* and f^* define an equivalence between the category of étale sheaves on X and the category of étale sheaves on Y.*

Proposition 3.2.13. *[1, Prop.5.4.1] Suppose $i : Y \to X$ is a closed immersion, and $j : X \backslash Y \hookrightarrow Y$ be the complement. Then we have*

(i) The functor $i_* : \mathscr{S}_Y \to \mathscr{S}_X$ is a fully faithful functor.

(ii) An étale sheaf $\mathscr{F}$ on X is isomorphic to an étale sheaf $i_* \mathscr{G}$ for some étale sheaf G on Y if and only if $j^* \mathscr{F} = 0$.

Proof. For (i), just need to notice that

$$\operatorname{Hom}(\mathscr{G}_1, \mathscr{G}_2) \to \operatorname{Hom}(i_* \mathscr{G}_, i_* \mathscr{G}_2) \to \operatorname{Hom}(i^* i_* \mathscr{G}_1, i^* i_* \mathscr{G}_2) \cong \operatorname{Hom}(\mathscr{G}_1, \mathscr{G}_2),$$

where $\mathscr{G}_i (i = 1, 2)$ are two étale sheaves on Y. One can show that i_* and i^* are inverse to each other.

For (ii) The necessity is clear. For the sufficiency, suppose $\mathscr{F}$ is an étale sheaf on X with $j^* \mathscr{F} = 0$, then $\mathscr{F} \to i_* i^* \mathscr{F}$ induces isomorphisms on stalks since $\mathscr{F}_x = 0$ for all $x \notin Y$. Let $\mathscr{G} = i^* \mathscr{F}$. $\square$

3.3 The Functor $f_!$

Definition 3.3.1. *(Functors $i_{\mathscr{P}}$ and $i^{\mathscr{P}}$). Suppose $i : \mathscr{C} \to \mathscr{C}'$ is a covariant functor. Then it induces two covariant functors as below.*

$$i^{\mathscr{P}} : \mathscr{P}_{\mathscr{C}'} \to \mathscr{P}_{\mathscr{C}}, \quad i^{\mathscr{P}}(\mathscr{F}')(U) = \mathscr{F}'(i(U))$$

for any $\mathscr{F}' \in ob\ \mathscr{P}_{\mathscr{C}'}$ and $U \in ob\ \mathscr{C}$.

For any U' in $\mathscr{C}'$, define a category $I_{U'}$ whose objects are pairs (U, ϕ) such that each U is an object in $\mathscr{C}$, each $\phi : U' \to i(U)$ is a morphism in $\mathscr{C}'$. For two objects $(U_i, \phi_i)\,(i = 1, 2)$ in $I_{U'}$, a morphism $\xi : (U_1, \phi_1) \to (U_2, \phi_2)$ is a morphism $\xi : U_1 \to U_2$ in $\mathscr{C}$ such that $i(\xi)\phi_1 = \phi_2$. We define

$$(i_{\mathscr{P}} \mathscr{F})(U') = \varinjlim_{(U,\phi) \in ob I_{U'}^{\circ}} \mathscr{F}(U)$$

for any étale sheaf $\mathscr{F}$ on $\mathscr{C}$ and any $U' \in ob\mathscr{C}$. One can show that $(i_{\mathscr{P}}, i^{\mathscr{P}})$ is an adjoint pair.

Example 3.3.2. *(i) Suppose $f : X' \to X$ is a morphism of schemes, then for the functor $f_{ét}$ defined in 3.1.3, we have $(f_{ét})^{\mathscr{P}} = f_{\mathscr{P}}$ and $(f_{ét})_{\mathscr{P}} = f^{\mathscr{P}}$, where the right hand sides are defined in 3.1.5.*

(ii) If $f : X' \to X$ is an étale morphism. We can define a functor $i : X'_{ét} \to X_{ét}$ by viewing étale X'-schemes as étale X-schemes. Then we have $i^{\mathscr{P}}\mathscr{F} \cong f^*\mathscr{F}$ for any étale sheaf $\mathscr{F}$ on X, where the right hand side has been defined in 3.1.10. We also have its left adjoint functor $i_{\mathscr{P}}$.

Definition 3.3.3. *(The functor $f_!$). Suppose $f : X' \to X$ is an étale morphism, $\mathscr{F}'$ is an étale sheaf on X', we define $f_!\mathscr{F}' = (i_{\mathscr{P}}\mathscr{F}')^{\#}$, where $i_{\mathscr{P}}$ is the functor defined above.*

Definition 3.3.4. *(Support). Suppose $\mathscr{F}$ is an étale sheaf on X, then the smallest closed subset F of X such that $\mathscr{F}|_{X\backslash F} = 0$ is called the support of $\mathscr{F}$. Suppose U is an object in $X_{ét}$, and s is a section in $\mathscr{F}(U)$, then the support of s is defined to be the smallest closed subset F of U such that $s|_{U\backslash F} = 0$.*

Proposition 3.3.5. *[1, Prop.5.4.2] Suppose $i : Y \to X$ is a closed immersion, and $j : X\backslash Y \hookrightarrow X$ is its complement. Then*

(i) we have the adjoint pairs (i^*, i_*), $(i_*, i^!)$, (j^*, j_*), $(j_!, j^*)$,

(ii) we know that i_*, i^*, j^*, $j_!$ are exact, j_*, $i^!$ are left exact,

(iii) we have $i^!i_*$, i^*i_*, $j^*j_!$, j^*j_* all isomorphic to id, and $i^*j_!$, $i^!j_!$, $i^!j_*$, j^*i_* are all isomorphic to 0,

(iv) for any étale sheaf $\mathscr{F}$ on X, the following sequences are exact

$$0 \to i_*i^!\mathscr{F} \to \mathscr{F} \to j_*j^*\mathscr{F},$$
$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to i_*i^*\mathscr{F} \to 0.$$

Proof. Suppose $\mathscr{K}$ is kernel of the canonical morphism $\mathscr{F} \to j_*j^*\mathscr{F}$. Since j^* is exact, $j^*\mathscr{K}$ is the kernel of the morphism $j^*\mathscr{F} \to j^*(j_*j^*\mathscr{F})$, which is actually an isomorphism. Hence $j^*\mathscr{F} = 0$. Use 3.2.13, there exists an étale sheaf $i^!\mathscr{F}$ on Y such that $i_*i^!\mathscr{F} \cong \mathscr{F}$. In other words, we have the exact sequence

$$0 \to i_*i^!\mathscr{F} \to \mathscr{F} \to j_*j^*\mathscr{F}.$$

For any étale sheaf $\mathscr{G}$ on Y, we use the above exact sequence by replacing $\mathscr{F}$ with $i^*\mathscr{G}$. Notice that $j_*j^*i_*\mathscr{G} = 0$, use the exactness of i_*, we obtain $i^!i_*\mathscr{G} \cong \mathscr{G}$. For any morphism $\phi : i_*\mathscr{G} \to \mathscr{F}$, we have a commutative diagram

$$\begin{array}{ccc} i_*\mathscr{G} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ j_*j^*i_*\mathscr{G} & \longrightarrow & j_*j^*\mathscr{F} \end{array}$$

Notice that $j_* j^* i_* \mathscr{G} = 0$, the image of ϕ must be contained in $\ker(\mathscr{F} \to j_* j^* \mathscr{F}) = i_* i^! \mathscr{F}$. Therefore, we have

$$\mathrm{Hom}(i_* \mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}(i_* \mathscr{G}, i_* i^! \mathscr{F}) \cong \mathrm{Hom}(\mathscr{G}, i^! \mathscr{F})$$

since i_* is exact and faithful. We have obtained the adjoint pair $(i_*, i^!)$.

Suppose $\mathscr{H}$ is an étale sheaf on $X \backslash Y$. Denote by $\iota : (X \backslash Y)_{\acute{e}t} \to X_{\acute{e}t}$ the functor induced by j as in 3.3.2(ii). One can show that for an object V in $X_{\acute{e}t}$, $\iota_{\mathscr{P}} \mathscr{H}(V) = \mathscr{H}(V)$ if the image of V is contained in $X \backslash Y$, and $\iota_{\mathscr{P}} \mathscr{H}(V) = 0$ otherwise. Then by the definition of $j_!$, we have $j^* j_! \mathscr{H} \cong \mathscr{H}$. For any étale sheaf on X, we have

$$\mathrm{Hom}(j_! \mathscr{H}, \mathscr{F}) \cong \mathrm{Hom}(\iota_{\mathscr{P}} \mathscr{H}, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{H}, \iota^{\mathscr{P}} \mathscr{F}) \cong \mathrm{Hom}(\mathscr{H}, j^* \mathscr{F}).$$

Hence $(j_!, j^*)$ is an adjoint pair. $\square$

Proposition 3.3.6. *[1, Prop.5.5.1] Suppose $f : X' \to X$ is an étale morphism.*

(i) We have the adjoint pair $(f_!, f^*)$.

(ii) Given a Cartesian diagram

$$\begin{array}{ccc} X' \times_X Y & \xrightarrow{g'} & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X. \end{array}$$

For any étale sheaf $\mathscr{F}'$ on X', we have a canonical isomorphism

$$f'_! g'^* \mathscr{F}' \xrightarrow{\cong} g^* f_! \mathscr{F}'.$$

(iii) Suppose $h : X'' \to X'$ is another étale morphism. We have $(fh)_! \cong f_! h_!$.

(iv) For any point x in X, let $\overline{k(x)}$ be a separable closure of $k(x)$. We have

$$(f_! \mathscr{F}')_{\bar{x}} \cong \bigoplus_{x' \in X' \times_X \mathrm{Spec}\, \overline{k(x)}} \mathscr{F}'_{\bar{x}'}.$$

Then the functor $f_!$ is exact and faithful.

Proof. (i) Completely similar to the end of the proof above the above proposition.

(ii) Apply $\mathrm{Hom}(\cdot, \mathscr{G})$, use 3.2.10 and the adjunction.

(iii) It follows from $(fh)^* \cong h^* f^*$ and the adjunction.

(iv) Use (ii) by taking g to be the morphism $\mathrm{Spec}\,\overline{k(x)} \to X$. Denote $Y = \mathrm{Spec}\,\overline{k(x)}$ We have

$$\begin{aligned} f'_! g'^* \mathscr{F}'(Y) &\cong \bigoplus_{\psi \in \mathrm{Hom}_Y(Y, X' \times_X Y)} g'^* \mathscr{F}'(U_\psi) \\ &\cong \bigoplus_{x' \in X' \times_X \mathrm{Spec}\,\overline{k(x)}} \mathscr{F}'_{\bar{x}'}. \end{aligned}$$

□

Chapter 4

Étale Cohomology

4.1 Čech Cohomology

Definition 4.1.1. *(Čech cohomology group). Suppose U is an object in $X_{ét}$ and $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ an étale covering. Denote*

$$U_{\alpha_0,\dots,\alpha_n} = U_{\alpha_0} \times_U \cdots \times_U U_{\alpha_n}.$$

For any étale presheaf $\mathscr{F}$, let

$$C^n(\mathfrak{U}, \mathscr{F}) = \prod_{\alpha_0,\dots,\alpha_n} \mathscr{F}(U_{\alpha_0\dots\alpha_n}).$$

Define a homomorphism

$$d : C^n(\mathfrak{U}, \mathscr{F}) \to C^{n+1}(\mathfrak{U}, \mathscr{F})$$

where

$$(ds)_{\alpha_0\dots\alpha_{n+1}} = \sum_{i=1}^{n+1} (-1)^i s_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{n+1}}|_{U_{\alpha_0\dots\alpha_{n+1}}}$$

One can check that $(C^\cdot(\mathfrak{U}, \mathscr{F}), d)$ forms a complex, called Čech complex, its cohomology groups are called Čech cohomology groups of $\mathscr{F}$ with respect to $\mathfrak{U}$, denoted by $\check{H}^\cdot(\mathfrak{U}, \mathscr{F})$.

Remark 4.1.2. *We don't have the alternative cochain which can be applied to calculate the Čech cohomology like ordinary Čech cohomology. For example, if we take $\mathscr{C}$ to be the category of open subsets of a topological space X, $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ an open covering, let $\mathfrak{U}' = \{_{\alpha \in I} U_\alpha \to U\}$. Suppose $\mathscr{F}$ is an étale sheaf on X, then one can show that*

$$C^\cdot(\mathfrak{U}, \mathscr{F}) \cong C^\cdot(\mathfrak{U}', \mathscr{F}).$$

Since $\mathfrak{U}'$ has only one element, the alternative n-cochains for $\mathfrak{U}'$ of $\mathscr{F}$ are trivial, whereas $\check{H}^{\cdot}(\mathfrak{U}, \mathscr{F})$ is not trivial in general.

However, we have similar morphisms of complexes for refinement, and we have the homotopy invariance.

Definition 4.1.3. *Suppose X is an scheme, U is an object in $X_{ét}$, then define*

$$\check{H}^i(U, \cdot) = \varinjlim_{\mathfrak{U} \in ob\, J_U} \check{H}^i(\mathfrak{U}, \cdot),$$

where J_U is the same notion as in the proof of 3.1.8.

4.2 Basic Properties of Étale Cohomology

The categories $\mathscr{P}_X$ and $\mathscr{S}_X$ defined in 3.1.4 are abelian categories with enough injective objects, and $\check{H}^n(\mathfrak{U}, \mathscr{I}) = 0$ for any injective étale presheaf and any $n \geq 1$.[1, Prop.5.1.1, Prop.5.2.4]

Definition 4.2.1. *(Étale Cohomology). Let X be a scheme, denote $\mathscr{S}_X$ the category of étale sheaves of abelian groups on X. For any object U in $X_{ét}$, the functor $\Gamma(U, \cdot) : \mathscr{S}_X \to \mathscr{A}$ is left exact, where $\mathscr{A}$ is the category of abelian groups. By the above lemma, we can define the right derived functors $R^i\Gamma(U, \cdot)$ $(i \geqslant 0)$ of $\Gamma(U, \cdot)$, denoted also by $H^i_{ét}(U, \cdot)$ or $H^i(U, \cdot)$. For an étale sheaf $\mathscr{F}$ on X, we called $H^i(U, \mathscr{F})$ the étale cohomology groups of $\mathscr{F}$. Define an étale presheaf $\mathscr{H}^i(\mathscr{F})$ on X to be $\mathscr{H}^i(\mathscr{F})(U) = H^i(U, \mathscr{F})$ for any object U in $X_{ét}$.*

Lemma 4.2.2. *[1, Lem.5.6] For any étale sheaf $\mathscr{F}$ on a scheme X, $\mathscr{H}^i(\mathscr{F})^+ = 0$ for all $i \geq 1$.*

Proof. Suppose $\mathscr{I}^{\cdot}$ is an injective resolution of $\mathscr{F}$. Then $\mathscr{H}^i(\mathscr{F})(U)$ is the i-th cohomology group of $\mathscr{I}^{\cdot}(U)$. So $\mathscr{H}^i(\mathscr{F})^\#$ is the i-th étale cohomology sheaf of $\mathscr{I}^{\cdot}$ which is an exact sequence. Thus, $\mathscr{H}^i(\mathscr{F})^\# = 0$ for all $i \geq 1$. Then we have $\mathscr{H}^i(\mathscr{F})^+ = 0$ since $\mathscr{H}^i(\mathscr{F})^+$ is an étale subsheaf of $\mathscr{H}^i(\mathscr{F})^\#$. □

Proposition 4.2.3. *[1, Prop.5.6.2] Suppose X is a scheme and $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ is an étale covering in $X_{ét}$. Then we have the following biregular spectral sequences for any $\mathscr{F}$ on X*

$$E_2^{ij} = \check{H}^i(\mathfrak{U}, \mathscr{H}^j(\mathscr{F})) \Rightarrow H^{i+j}(U, \mathscr{F}),$$
$$E_2^{ij} = \check{H}^i(U, \mathscr{H}^j(\mathscr{F})) \Rightarrow H^{i+j}(U, \mathscr{F}).$$

Proof. For any injective étale sheaf $\mathscr{I}$, since it is also injective in the category of étale presheaves, we have $\check{H}^j(\mathfrak{U},\mathscr{I})=0$ for any $j\geq 1$ by [1, 5.2.4 (i)]. Suppose $\mathscr{I}^{\cdot}$ is an injective resolution of $\mathscr{F}$. We have a bicomplex $K^{pq}=C^p(\mathfrak{U},\mathscr{I}^q)$. Noticing that $\check{H}^j(\mathfrak{U},\mathscr{I}^i)=\mathscr{I}^i(U)$ when $j=0$, and $\check{H}^j(\mathfrak{U},\mathscr{I}^i)=0$ when $j\geq 1$, we can compute the second spectral sequence of the bicomplex:

$$E_{II,2}^{ij}=H_{II}^iH_I^j(K^{\cdot\cdot})=H^i(\mathscr{H}^j(\mathfrak{U},\mathscr{I}^{\cdot}))=\begin{cases}H^i(U,\mathscr{F}) & \text{if } j=0,\\ 0 & \text{if } j\geq 1.\end{cases}$$

So the second spectral sequence of this bicomplex degenerates, then by [2, Prop.2.2.2],

$$H^n(K^{\cdot\cdot})=E_{II,2}^{n,0}=H^n(U,\mathscr{F}).$$

Then, compute the first spectral sequence, we have

$$E_{I,2}^{ij}=H_I^iH_{II}^j(K^{\cdot\cdot})=\check{H}^i(\mathfrak{U},\mathscr{H}^j(\mathscr{F}))\Rightarrow H^{i+j}(U,\mathscr{F}).$$

Take the direct limit, we have

$$\check{H}^i(U,\mathscr{H}^j(\mathscr{F}))\Rightarrow H^{i+j}(U,\mathscr{F}).$$

□

Corollary 4.2.4. *[1, Cor.5.6.3] Suppose $\mathscr{F}$ is an étale sheaf on a scheme X, U is an object in $X_{ét}$, then we have the canonical isomorphism $\check{H}^1(U,\mathscr{F})\cong H^1(U,\mathscr{F})$ and the canonical injection $\check{H}^2(U,\mathscr{F})\hookrightarrow H^2(U,\mathscr{F})$*

Proof. Apply [2, Prop.2.2.3] to the spectral sequence in the above proposition, we have

$$0\to\check{H}^1(U,\mathscr{F})\to H^1(U,\mathscr{F})\to\check{H}^0(U,\mathscr{H}^1(\mathscr{F}))\to\check{H}^2(U,\mathscr{F})\to H^2(U,\mathscr{F}).$$

By 4.2.2,

$$\check{H}^0(U,\mathscr{H}^1(\mathscr{F}))\cong\mathscr{H}^j(\mathscr{F})^+(U)=0.$$

We obtain the desired isomorphism and the monomorphism. □

Definition 4.2.5. *An étale sheaf $\mathscr{F}$ on a scheme X is called flasque if for any étale covering $\mathfrak{U}=\{U_\alpha\to U\}_{\alpha\in I}$ in $X_{ét}$, we have $H^1(\mathfrak{U},\mathscr{F})=0$. Note that use the above corollary, we have $H^i(U,\mathscr{F})=\check{H}^i(U,\mathscr{F})=0$ $(\forall i\geq 1)$ for any flasque étale sheaf $\mathscr{F}$ on X and any U in $obX_{ét}$.*

Proposition 4.2.6. *[1, Prop.5.6.5]*

(i) Injective étale sheaves are flasque.

(ii) Suppose $f : X' \to X$ is a morphism of schemes. If $\mathscr{F}'$ is a flasque étale sheaf on X', then $f_*\mathscr{F}$ is a flasque étale sheaf on X.

Proof. (i) It has been proved in the proof of 4.2.3.

(ii) Suppose $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ is an étale covering in $X_{\acute{e}t}$. Denote $f^*\mathfrak{U} = \{U_\alpha \times_X X' \to U \times_X X'\}_{\alpha \in I}$, it is an étale covering in $X'_{\acute{e}t}$. By definition, we have

$$\check{H}^i(\mathfrak{U}, f_*\mathscr{F}') = \check{H}^i(f^*\mathfrak{U}, \mathscr{F}').$$

Since $\mathscr{F}'$ is flasque, we have $\check{H}^i(f^*\mathfrak{U}, \mathscr{F}') = 0$ for any $i \geq 1$. It follows that $f_*\mathscr{F}'$ is flasque.

□

Proposition 4.2.7. *[1, Prop.5.6.9] Suppose $f : X' \to X$ is a morphism of schemes and $\mathscr{F}'$ is an étale sheaf on X'. Then we have a biregular spectral sequence*

$$E_2^{ij} = H^i(X, R^j f_*\mathscr{F}') \Rightarrow H^{i+j}(X', \mathscr{F}').$$

Proof. After using [2, Prop.2.2.10], we will obtain the desired spectral sequence. It suffices to show that $f_*\mathscr{I}$ is acyclic for any injective étale sheaf $\mathscr{I}$ in the sense of having null i-th étale cohomology for all $i \geq 1$. Use 4.2.6, we complete the proof.

□

4.3 Calculations of Étale Cohomology

Proposition 4.3.1. *[1, Prop.5.7.1] Suppose f is a finite surjective radical morphism. Then for any $\mathscr{F}$ on Y, we have $H^i(Y, \mathscr{F}) \cong H^i(X, f^*\mathscr{F})$.*

Proof. Apply 3.2.12 we obtain the result. □

Corollary 4.3.2. *[1, Cor.5.7.2] Suppose X is a scheme over a field K and L is a finite purely inseparable extension of K. Then $H^i(X, \mathscr{F}) \cong H^i(X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L, \mathscr{F}|_{X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L})$ for any étale sheaf $\mathscr{F}$ on X.*

Proof. Note that $\mathrm{Spec}\, L \to \mathrm{Spec}\, K$ is radical, then it is clear that its base change $X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$ is also radical. Use 4.3.1. □

Proposition 4.3.3. *[1, Prop.5.7.3] Suppose A is a strictly henselian ring. Then we have $H^i(\mathrm{Spec}\, A, \mathscr{F}) = 0$ for any étale sheaf $\mathscr{F}$ on $\mathrm{Spec}\, A$ and any $i \geq 1$.*

Proof. Denote by s the closed point of $\operatorname{Spec} A$, then $\Gamma(\operatorname{Spec}, \mathscr{F}) = \mathscr{F}_s$. It follows that $\Gamma(\operatorname{Spec} A, \cdot)$ is an exact functor. The desired assertion follows. $\square$

Proposition 4.3.4. *[1, Prop.5.7.4] Suppose $f : X \to Y$ is a finite morphism and $\mathscr{F}$ is an étale sheaf on X. Then we have $R^i f_* \mathscr{F} = 0$ and $H^i(Y, f_*\mathscr{F}) \cong H^i(X, \mathscr{F})$ for any $i \geq 1$.*

Proof. By 3.2.8, f_* is an exact functor, then we have $R^i f_* \mathscr{F} = 0$. From the spectral sequence in 4.2.7, we obtain the isomorphism of the cohomology groups. $\square$

Suppose X is a scheme. For any étale sheaf on X, we define a Zariski sheaf $i_*\mathscr{F}$ to be

$$(i_*\mathscr{F})(V) = \mathscr{F}(V)$$

for any open subset V of X. For any Zariski presheaf $\mathscr{G}$ on X, we define an étale presheaf $i^{\mathscr{P}}\mathscr{G}$ to be

$$(i^{\mathscr{P}}\mathscr{G})(U) = \mathscr{G}(\operatorname{im}(U \to X))$$

for any object U in $X_{ét}$. Let $i^*\mathscr{G} = (i^{\mathscr{P}}\mathscr{G})^\#$ be the étale sheaf associated to $i^{\mathscr{P}}\mathscr{G}$. Then one can show that (i^*, i_*) is an adjoint pair.

Proposition 4.3.5. *[1, Lem.5.7.4] Suppose X is a scheme, i_* the functor defined above. Then for any étale sheaf $\mathscr{F}$ on X, we have a biregular spectral sequence*

$$E_2^{pq} = H^p_{Zar}(X, R^q i_* \mathscr{F}) \Rightarrow H^{p+q}_{ét}(X, \mathscr{F}).$$

Proof. Similar to 4.2.7, we want to apply [2, Prop.2.2.10] for

$$\mathscr{S}_X \xrightarrow{i_*} \mathscr{S}_{X_{Zar}} \xrightarrow{\Gamma(X,\cdot)} \mathfrak{Ab}$$

where $\mathscr{S}_{X_{Zar}}$ is the category of Zariski sheaves of X. It suffices to show that $i_*\mathscr{I}$ is $\Gamma(X, \cdot)$-acyclic for any injective étale sheaf $\mathscr{I}$. Notice that the functor $\operatorname{Hom}(\,\cdot\,, i_*\mathscr{I}) \cong \operatorname{Hom}(i^*\cdot, \mathscr{I})$ is exact, hence the Zariski sheaf $i_*\mathscr{I}$ is injective then $\Gamma(X, \cdot)$-acyclic. $\square$

Proposition 4.3.6. *[1, Prop.5.7.5] Suppose X is a scheme, $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{M}_{ét}$ is the étale sheaf defined by $U \mapsto \Gamma(U, \pi^*\mathscr{M})$ for any object $\pi : U \to X$ in $X_{ét}$. Then we have $H^q_{Zar}(X, \mathscr{M}) \cong H^q_{ét}(X, \mathscr{M}_{ét})$ for all q.*

Proof. By using 4.3.5 and [2, Prop.2.2.2], it suffices to show that $R^q i_* \mathscr{M}_{ét} = 0$ for any $q \geq 1$. Note that $R^q i_* \mathscr{M}_{ét}$ is the Zariski sheaf associated to the Zariski presheaf defined by $U \mapsto H^q_{ét}(U, \mathscr{M}_{ét})$ for open subsets U of X. Hence it suffices to show that $H^q_{ét}(U, \mathscr{M}_{ét}) = 0$ for any $q \geq 1$ any affine scheme U, and any quasi-coherent $\mathscr{O}_U$-module $\mathscr{M}$. Then for any étale covering $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$, where

I is finite, each U_α and U are affine, suppose $\mathfrak{U}' = \{_{\alpha\in I}U_\alpha \to U\}$. By 4.1.2 we have

$$C^{\cdot}(\mathfrak{U}, \mathscr{M}_{ét}) \cong C^{\cdot}(\mathfrak{U}', \mathscr{M}_{ét}) \cong \mathscr{M}_{ét}((_{\alpha\in I}U_\alpha) \times_U \cdots \times_U (_{\alpha\in I}U_\alpha)).$$

Note that $_{\alpha\in I}U_\alpha$ is affine, we denote it by Spec B, and denote U by Spec A. Then $C^{\cdot}(\mathfrak{U}, \mathscr{M}_{ét}) \cong M \otimes_A B \otimes_A B \otimes \cdots \otimes_A B$ for some M since $\mathscr{M}$ is quasi-coherent. Note that the induced map $A \to B$ is flat and surjective on spectrums, so in order to show that $\check{H}^q(\mathfrak{U}, \mathscr{M}_{ét}) = 0$ for any $q \geq 1$, we just need to prove the following claim.

Claim. Suppose $A \to B$ is a faithfully flat homomorphism of rings and M is an A-module. Then the following sequence is exact

$$0 \to M \overset{d^{-1}}{\to} M \otimes_A B \overset{d_0}{\to} M \otimes_A B \otimes_A B \overset{d_1}{\to} M \otimes_A B \otimes_A B \otimes_A B \overset{d_2}{\to} \cdots,$$

where

$$d_n(x \otimes b_0 \otimes \cdots \otimes b_n) = \sum_{i=1}^{n+1} (-1)^i x \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_n.$$

Proof of the claim:

Since B is faithfully flat over A, it suffices to show that after tensoring B the sequence is exact, *i.e*.

$$0 \to M \otimes_A B \overset{d_{-1}}{\to} M \otimes_A B \otimes_A B \overset{d_0}{\to} M \otimes_A B \otimes_A B \otimes_A B \overset{d_1}{\to} \cdots,$$

where

$$d_n(x \otimes b_0 \otimes \cdots \otimes b_n \otimes b) = \sum_{i=1}^{n+1} (-1)^i x \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_n \otimes b.$$

We define a chain homotopy for the sequence itself from id to 0 to show that it is exact. Let

$$\begin{aligned} D_n : \quad & M \otimes_A (B^{\otimes(n+2)}) \to M \otimes_A (B^{\otimes(n+1)}) \\ & x \otimes b_0 \otimes \cdots \otimes b_n \otimes b \mapsto x \otimes b_1 \otimes \cdots \otimes b_n \otimes b_0 b. \end{aligned}$$

Then we have

$$d_{n-1}D_n + D_{n+1}d_n = id.$$

Our claim follows.

Now we have $\check{H}^q(\mathfrak{U}, \mathscr{M}_{ét}) = 0$ for any $q \geq 1$.Then

$$\check{H}^q(U, \mathscr{M}_{ét}) \cong \varinjlim_{\mathfrak{U}} \check{H}^q(\mathfrak{U}, \mathscr{M}_{ét}) = 0$$

for any $q \geq 1$. Indeed, note that U is affine then quasi-compact. For any étale covering, considering the images of the étale morphisms, we can choose a finite étale covering to be a refinement of it.

Furthermore, use 4.2.4, we have $H^1_{ét}(U, \mathscr{M}_{ét}) \cong \check{H}^1(U, \mathscr{M}_{ét}) = 0$. Apply induction, suppose we have shown that $H^q_{ét}(U, \mathscr{M}_{ét}) = 0$ for any $1 \leq q \leq n$ and any object U in $X_{ét}$. Then we use the biregular spectral sequence

$$E_2^{pq} = \check{H}^p(U, \mathscr{H}^q(\mathscr{M}_{ét})) \Rightarrow H^{p+q}_{ét}(U, \mathscr{M}_{ét})$$

in 4.2.3. We have

$$\check{H}^p(U, \mathscr{H}^q(\mathscr{M}_{ét})) = \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{M}_{ét}))$$

where $\mathfrak{U}$ runs over étale coverings $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I}$ of U with I being finite and each U_α being affine. Notice that each U_α is affine, by the induction hypothesis we have

$$\mathscr{H}^q(\mathscr{M}_{ét})(U_{\alpha_0 \ldots \alpha_n}) \cong H^q_{ét}(U_{\alpha_0 \ldots \alpha_n}, \mathscr{M}_{ét}) = 0$$

for any $1 \leq q \leq n$. Hence $\check{H}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{M}_{ét})) = 0$ then $E_2^{pq} = 0$ for any p and $1 \leq q \leq n$. Apply [2, Prop.2.2.3], we have an exact sequence

$$0 \to E_2^{n+1,0} \to H^{n+1}_{ét}(U, \mathscr{M}_{ét}) \to E_2^{0,n+1}.$$

Moreover,

$$E_2^{0,n+1} = \check{H}^0(U, \mathscr{H}^{n+1}(\mathscr{M}_{ét})) = (\mathscr{H}^{n+1}(\mathscr{M}_{ét}))^+(U) = 0$$

and

$$E_2^{n+1,0} = \check{H}^{n+1}(U, \mathscr{M}_{ét}) = 0.$$

It follows that $H^{n+1}_{ét}(U, \mathscr{M}_{ét}) = 0$. Our assertion follows. $\square$

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