From Elliptic Functions to Modular forms

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Elliptic functions

Definition(Elliptic funtions)

If f is meromorphic on $\mathbb{C}$, and $\exists \omega_1, \omega_2 \in \mathbb{C}^*$, such that $\forall z \in \mathbb{C}, f(z+\omega_1) = f(z), f(z+\omega_2) = f(z)$, then we say that f if an elliptic function.(double period mero.)

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- Normalization: Let $\tau = \frac{\omega_1}{\omega_2}$, suppose Im $\tau > 0$.(If not, exchange ω_1 and ω_2), then the function $F(z) = f(\omega_2 z)$ has period 1 and τ, where $\tau \in \mathcal{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$(the upper complex plane)

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Example

$f(z) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(z+m+n\tau)^3}$($\tau$ is fixed and $\tau \in \mathcal{H}$). Note that this function is well defined since we can verify that the series is absolutely convergent and uniformly convergent.

Elliptic functions

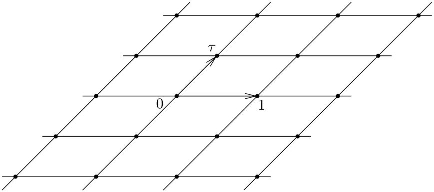

Figure 1. The lattice Λ generated by 1 and τ

Elliptic functions

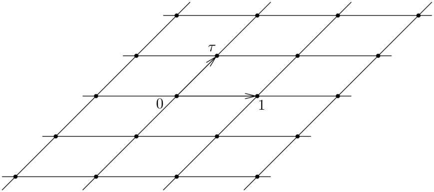

Figure 1. The lattice Λ generated by 1 and τ

Definition

(1) (Lattices) For a $z \in \mathbb{C}$, define a point set $\Lambda = \{m + n\tau : m, n \in \mathbb{Z}\}$, say a lattice.
(2) We define an equivalence relation: we say $z_1 \sim z_2$ iff $z_1 - z_2 \in \Lambda$.

Definition (Fundamental parallelogram)

For a $\tau \in \mathbb{C}$, define a point set $P_0 = \{z \in \mathbb{C} : z = a + b\tau$,where $0 \leq a < 1, 0 \leq b < 1\}$.

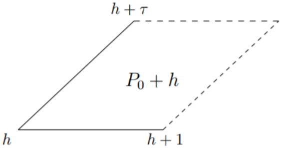

Figure 2. A period parallelogram

Remark

(1) $\forall z \in \mathbb{C}, \exists! \omega \in P_0$, such that $z \sim \omega$
(2) If $\frac{\omega_1}{\omega_2} \in \mathbb{R}$, then when $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$, f will be simple period, when $\frac{\omega_1}{\omega_2} \in \mathbb{R} - \mathbb{Q}$, f will be constant. We often suppose $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$.

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Definition

We say that two lattice are similar if $\exists m \in \mathbb{C}, m\Lambda = \Lambda'$, denoted $\Lambda \approx \Lambda'$.

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Definition

We say that two lattice are similar if $\exists m \in \mathbb{C}, m\Lambda = \Lambda'$, denoted $\Lambda \approx \Lambda'$.

Lemma

(1) For two lattices: $\Lambda = \omega_1\mathbb{Z} \bigoplus \omega_2\mathbb{Z}, \Lambda' = \omega_1'\mathbb{Z} \bigoplus \omega_2'\mathbb{Z}$, $\Lambda = \Lambda'$(as two point sets) $\iff$ $\binom{\omega_1}{\omega_2} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\binom{\omega_1'}{\omega_2'}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$.
(2) For two lattices: $\Lambda = \mathbb{Z} \bigoplus \tau\mathbb{Z}, \Lambda' = \mathbb{Z} \bigoplus \tau'\mathbb{Z}$, $\Lambda = \Lambda'$(as two point sets) $\iff$ $\tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau'$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$(Fractional linear transformation).

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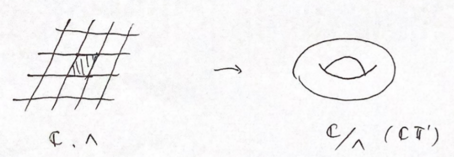
Complex tori

Definition(Complex tori)

For a lattice Λ, $\mathbb{C}/\Lambda := \{z + \Lambda : z \in \mathbb{C}\}$(equivalent class)



Definition(Complex tori)

For a lattice Λ, $\mathbb{C}/\Lambda := \{z + \Lambda : z \in \mathbb{C}\}$(equivalent class)

Remark(structures on complex tori)

(1) A complex torus is an Abelian group(add.).(induced by ($\mathbb{C}$,+))
(2) A complex torus is a compact Riemann surface.

Complex tori

Remark

Elliptic functions are holomorphic functions: $\mathbb{CT}^1 \to \mathbb{CP}^1$

Complex tori

Remark

Elliptic functions are holomorphic functions: $\mathbb{CT}^1 \to \mathbb{CP}^1$

Recall a theorem about Riemann surface:

Remark

Every holomorphic function on a compact Riemann surface is constant.

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Properties of elliptic functions

Theorem

Holomorphic elliptic functions are constant.

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For every elliptic function, $\#\{\text{zeros}\}=\#\{\text{poles}\}$ $(z \in P_0)$.

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Holomorphic elliptic functions are constant.

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Theorem

For every elliptic function, $\#\{\text{zeros}\}=\#\{\text{poles}\}$ $(z \in P_0)$.

Proof.

Compute the integral on $\partial\overline{P_0}$,

$$\int_{\partial\overline{P_0}} \frac{f'(z)}{f(z)} dz = N(f, \partial\overline{P_0}) - P(f, \partial\overline{P_0}) = 0.$$

Properties of elliptic functions

Corollary

If f is an elliptic function, then the solution numbers of $f(z) = c$ $(z \in P_0)$ are the same for every $c \in \mathbb{CP}^1$.In other words,
$\#\{z \in P_0 : f(z) = c_1\} = \#\{z \in P_0 : f(z) = c_2\}$

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Proof.

$$
\begin{aligned}
\#\{z \in P_0 : f(z) = c_1\} &= \#\{\text{zeros of } f(z) - c_1\} \\
&= \#\{\text{poles of } f(z) - c_1\} \\
&= \#\{\text{poles of } f(z) - c_2\} \\
&= \#\{\text{zeros of } f(z) - c_2\} \\
&= \#\{z \in P_0 : f(z) = c_2\}
\end{aligned}
$$

We call this number the order of f.

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Weierstrass p function

Definition(Weierstrass p function)

For a lattice Λ, let

$$p(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} [\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}]$$

Notice that when $|\omega| \to +\infty$, $\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = O(\frac{1}{\omega^3})$, thus, p(z) is well defined.

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Theorem

p is elliptic.

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Theorem

p is elliptic.

Proof.

Notice that $p'(z) = -2\sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^3}$ is elliptic, whose periods are 1 and τ.

Weierstrass p function

Proof.

Thus,

$$p'(z+1) - p'(z) = 0 = p'(z+\tau) - p'(z) (\forall z \in \mathbb{C}).$$

It follows that, for some a,b $\in \mathbb{C}$,

$$p(z+1) - p(z) = a, p(z+\tau) - p(z) = b (\forall z \in \mathbb{C}).$$

However, p is even. $p(\frac{1}{2}) = p(-\frac{1}{2}) \Rightarrow a = 0$, $p(\frac{\tau}{2}) = p(-\frac{\tau}{2}) \Rightarrow b = 0$.
The theorem holds.

Weierstrass p function

We know that p' is odd and elliptic, which implies that $p'(\frac{1}{2}) = -p'(-\frac{1}{2}) = -p'(\frac{1}{2}) = 0$. Similarly, $p'(\frac{\tau}{2}) = p'(\frac{1+\tau}{2}) = 0 = p'(\frac{1}{2})$. On the other hand, p' has a unique pole at 0, which is of order 3. Therefore, $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the only zeros of p', which are of order 1. Denote $F(z) = (p(z) - p(\frac{1}{2}))(p(z) - p(\frac{\tau}{2}))(p(z) - p(\frac{1+\tau}{2}))$. Then F has only three zeres, $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, which are of order 2. So does p'^2. So we have obtained that $\frac{F}{p'^2}$ is holomorphic and elliptic, thus, it is constant, say $\frac{F}{p'^2} = c$. In order to find c, we take the Laurent expansion for p and p', $p(z) = \frac{1}{z^2} + ..., p'(z) = -\frac{2}{z^3} + ...$ We conclude that c=$\frac{1}{4}$. Denote that $p(\frac{1}{2}) = e_1, p(\frac{\tau}{2}) = e_2, p(\frac{1+\tau}{2}) = e_3$, we have the following theorem.

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Theorem

p'$^2 = 4(p - e_1)(p - e_2)(p - e_3)$.(*Elliptic curves*)

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Modular forms

Definition (Modular forms) A first glance

We say $f : \mathcal{H} \to \mathbb{C}$ is a modular form if
(1) f is holomorphic,
(2) $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)(\forall \tau \in \mathcal{H}, \gamma \in SL_2(\mathbb{Z}))$,
(3) f is holomorphic at ∞.
Then we say that f is a modular form, denoted $f \in \mathcal{M}(SL_2(\mathbb{Z}))$.

Modular forms

Consideration

Given a lattice, which is generated by ω_1 *and* ω_2, we know that the lattice corresponds with a p function, and p *and* p' satisfies an elliptic curve.
Our goal is to put all these elliptic curves into a certain space, viewing every elliptic curves as a point in that space, and study the functions on that space, say F.
By the above thinking, since every elliptic curve is determined by two complex number $\omega_1\ \omega_2$. It is reasonable to consider $\mathbb{C}^2$ to be that space.
However, we know that if (ω_1, ω_2) and (ω_1', ω_2') generates the same lattice, the elliptic curves should be the same as well. So we should let the function F satisfies $F(\omega_1, \omega_2) = F(\omega_1', \omega_2')$.
Furthermore, if we want to take a normalization by substituting (ω_1, ω_2) with τ,$\tau = \frac{\omega_1}{\omega_2}$, we should suppose $f(\tau) = F(\tau, 1)$. Now we wonder the invariant for f exactly.

Modular forms

$GL_2(\mathbb{Z}) \curvearrowright \mathbb{C}^2 \ni (\omega_1, \omega_2)$, *invariant*

$\downarrow$ up to similarity

$GL_2(\mathbb{Z}) \curvearrowright \mathbb{CP}^1 \ni [\omega_1, \omega_2]$, *invariant*

$\downarrow$ we don't consider elliptic functions for $\dfrac{\omega_1}{\omega_2} \in \mathbb{R}$ or $= \infty$

$GL_2(\mathbb{Z}) \curvearrowright \mathbb{CP}^1 - \mathbb{RP}^1 \ = \mathcal{H}^+ \sqcup \mathcal{H}^-,$ *invariant*

$\downarrow$ delete $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which exchanges $\mathcal{H}^+ and\ \mathcal{H}^-$.

$SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}^+, invariant.$

Modular forms

Consideration

For two similar lattices $\Lambda = m\Lambda'$, the value of the elliptic functions will be multiplied by a constant m^k(k is the order of the elliptic function). We want a function $F(\omega_1, \omega_2) : \mathbb{C}^2 \to \mathbb{C}$ to satisfies $F(\gamma \cdot (\omega_1, \omega_2))$ and $F(m\omega_1, m\omega_2) = m^k \cdot F(\omega_1, \omega_2)$ However, we suppose another function $f(\tau) = F(\tau, 1)$ to take values for lattice class up to similarity. We wonder the invariant for $f(\tau)$. Compute that

$$\begin{aligned} f(\gamma \cdot \tau) &= f(\frac{a\tau + b}{c\tau + d}) = F(\frac{a\tau + b}{c\tau + d}, 1) = \frac{1}{(c\tau + d)^k} F(a\tau + b, c\tau + d) \\ &= \frac{1}{(c\tau + d)^k} F(\gamma \cdot (\tau, 1)) = \frac{1}{(c\tau + d)^k} f(\tau). \end{aligned}$$

Since the function f take values for lattice class, it is reasonable to suppose $\tau \in \mathcal{H}$. We have the following definition.

Modular forms

Definition (Modular forms)

We say $f : \mathcal{H} \to \mathbb{C}$ is a modular form if
(1) f is holomorphic,
(2) $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)(\forall \tau \in \mathcal{H}, \gamma \in SL_2(\mathbb{Z}))$,
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Definition(Automorphic forms and cusp forms)

[1] Instead of holomorphic, if f is meromorphic, and satisfies (2) and (3), we say f an automorphic form. Denoted $f \in \mathcal{A}_k(SL_2(\mathbb{Z}))$.
[2] If f is a modular form, take its Fourier expansion, if the constant term $= 0$, namely $f = \sum_{n=1}^{+\infty} a_n q^n (q = e^{2\pi i \tau})$, say f is a cusp form. $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$.

References

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[2]Fred Diamond, Jerry Shurman. GTM-228: A First Course in Modular Forms.