$$1. \qquad \mathrm{A} \oplus \mathrm{B} = \mathrm{B} \oplus \mathrm{A}$$

Prove:

$$A = \{a \mid for\ some\ a \in A\}$$

$$B = \{b \mid for\ some\ b \in B\}$$

$$\mathrm{A} \oplus \mathrm{B} = \{p \mid p = a + b\ for\ some\ a \in A\ and\ b \in B\}$$

$$\mathrm{B} \oplus \mathrm{A} = \{p \mid p = b + a\ for\ some\ a \in A\ and\ b \in B\}$$

Therefore,

$$\mathrm{A} \oplus \mathrm{B} = \mathrm{B} \oplus \mathrm{A}$$

$$2. \qquad \mathrm{A} \oplus \mathrm{B_x} = (\mathrm{A} \oplus \mathrm{B})_{\mathrm{x}}$$

Prove:

$$B_x = \{b_x \mid b_x = b + x\ for\ some\ b \in B\}$$

$$\mathrm{A} \oplus \mathrm{B_x} = \{p \mid p = a + b_x = a + b + x\ for\ some\ a \in A\ and\ b \in B\}$$

$$(\mathrm{A} \oplus \mathrm{B})_{\mathrm{x}} = \{p \mid p = a + b + x\ for\ some\ a \in A\ and\ b \in B\}$$

Therefore,

$$\mathrm{A} \oplus \mathrm{B_x} = (\mathrm{A} \oplus \mathrm{B})_{\mathrm{x}}$$

$$3. \qquad \mathrm{A} \ominus \mathrm{B}_x = (\mathrm{A} \ominus \mathrm{B})_{-\mathrm{x}}$$

Prove:

$$\mathrm{A} \ominus \mathrm{B}_x = \{p \mid p + b + x \in A, \forall b + x \in B\}$$

$$\mathrm{A} \ominus \mathrm{B} = \{p' \mid p' + b \in A, \forall b \in B\}$$

$$(\mathrm{A} \ominus \mathrm{B})_{-\mathrm{x}} = \{p \mid p = p' - x \; for\ some\ p' \in \mathrm{A} \ominus \mathrm{B}\}$$

$$\because p' = p + x$$

$$\therefore (\mathrm{A} \ominus \mathrm{B})_{-\mathrm{x}} = \{p \mid p + x + b \in A, \forall b + x \in B\}$$

Therefore,

$$\mathrm{A} \ominus \mathrm{B}_x = (\mathrm{A} \ominus \mathrm{B})_{-\mathrm{x}}$$

4. $\mathrm{A_x} \ominus \mathrm{B} = (\mathrm{A} \ominus \mathrm{B})_\mathrm{x}$

Prove:

$$A_x = \{a_x \mid a_x = a + x \; for\ some\ a \in A\}$$

$$\mathrm{A_x} \ominus \mathrm{B} = \{p \mid p + b \in A_x, \forall b \in B\}$$

$$\mathrm{A_x} \ominus \mathrm{B} = \{p \mid p + b - x \in A, \forall b \in B\}$$

$$\mathrm{A} \ominus \mathrm{B} = \{p' \mid p' + b \in A, \forall b \in B\}$$

$$(\mathrm{A} \ominus \mathrm{B})_\mathrm{x} = \{p \mid p = p' + x \; for\ some\ p' \in \mathrm{A} \ominus \mathrm{B}\}$$

$$\because p' = p - x$$

$$\therefore (\mathrm{A} \ominus \mathrm{B})_\mathrm{x} = \{p \mid p - x + b \in A, \forall b \in B\}$$

Therefore,

$$\mathrm{A_x} \ominus \mathrm{B} = (\mathrm{A} \ominus \mathrm{B})_\mathrm{x}$$

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