THE 2-ADIC STRUCTURE OF HABIRO COEFFICIENTS: A SELF-CONTAINED RECURSION AND THE FAILURE OF THE STRONG SHIFT CONJECTURE

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ABSTRACT. We study the 2-adic valuations of the coefficients appearing in the Habiro expansion of the completed q-zeta function $\xi_q(s) = (q;q)_{s-1} \cdot \zeta_q(s)$, continuing the investigation begun in [6]. Writing $c_n(s) = \alpha_{n,0} + \sum_{j=2}^{s} \alpha_{n,j}\, \zeta(j)$, we establish a closed triangular recursion for the ratio $\alpha_{n,s-1}/\alpha_{n,s}$ that involves no regularization of divergent zeta values. This recursion is driven by a sequence D_k whose generating function we determine explicitly. As applications, we prove that $\alpha_{s,s-1}/\alpha_{s,s} = -(s-2)/s$ (the "shift at $n = s$"), that $\alpha_{s+1,s-1} = 0$ for all $s \geq 3$ (vanishing at $n = s+1$), and we give a closed formula for the third Taylor coefficient $C_3(s)$ valid for $s \geq 5$. We also show that the strong shift conjecture—which predicts $v_2(\alpha_{n,s-1}) - v_2(\alpha_{n,s}) = v_2(s-2) - v_2(s)$ for all $n \geq s$—is false, via exact computation for $s \in \{3, 4, 5, 6, 8\}$.

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1. Introduction

1.1. **Background and motivation.** Let $q = e^{\varepsilon}$ and $(q)_n = \prod_{k=1}^{n}(1-q^k)$. Following Kaneko–Kurokawa–Wakayama [4], define the q-zeta function

$$\zeta_q(s) \;=\; \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s},$$

where $[n]_q = (1-q^n)/(1-q)$. In [6], we introduced the *completed* q-zeta function

$$\xi_q(s) \;=\; (q;q)_{s-1} \cdot \zeta_q(s), \tag{1}$$

motivated by the analogy with Riemann's completed zeta function $\xi(s) = \Gamma(s)\zeta(s)$, and proved that $\xi_q(s)$ admits a Habiro expansion

$$\xi_q(s) \;=\; \sum_{n=0}^{\infty} c_n(s)\,(q)_n \tag{2}$$

with $c_0 = \cdots = c_{s-2} = 0$ and $c_{s-1} = \zeta(s)$. More precisely, for $n \geq s$ one can write

$$c_n(s) \;=\; \alpha_{n,0} + \sum_{j=2}^{s} \alpha_{n,j}\,\zeta(j), \tag{3}$$

with $\alpha_{n,j} \in \mathbb{Q}$ for all n, j.

Explicit formulas for $c_s(s)$ and $c_{s+1}(s)$ were given in [6], using Taylor coefficient formulas communicated by Zagier. In particular,

$$c_s(s) \;=\; \frac{s-2}{2s}\,\zeta(s-1) \;-\; \frac{1}{2}\,\zeta(s), \tag{4}$$

which holds for all $s \geq 3$. A striking feature discovered in [6] was the *skipping phenomenon*: the coefficient $c_{s+1}(s)$ involves $\zeta(s-2)$ and $\zeta(s)$ but *not* $\zeta(s-1)$, i.e., $\alpha_{s+1,s-1} = 0$.

The present paper grew out of the attempt to understand this vanishing and to investigate the 2-adic structure of the ratio $\alpha_{n,s-1}/\alpha_{n,s}$ for general n. The ratio at $n = s$ is $-(s-2)/s$, whose 2-adic valuation is $\delta(s) := v_2(s-2) - v_2(s)$. This led to the following natural question.

Question 1.1 (Strong shift conjecture). Is it true that $v_2(\alpha_{n,s-1}) - v_2(\alpha_{n,s}) = \delta(s)$ for all $n \geq s$?

1.2. **Main results.** We resolve this question negatively and establish the following positive results.

Theorem 1.2 (Shift at $n = s$)**.** *For all $s \geq 3$,*

$$c_s(s) \;=\; \frac{s-2}{2s}\,\zeta(s-1) - \frac{1}{2}\,\zeta(s),$$

and hence $v_2(\alpha_{s,s-1}) - v_2(\alpha_{s,s}) = \delta(s)$.

Theorem 1.3 (Vanishing at $n = s+1$)**.** *For all $s \geq 3$, $\alpha_{s+1,s-1}(s) = 0$. That is, $\zeta(s-1)$ does not appear in $c_{s+1}(s)$.*

This was observed numerically and stated without proof in [6]; here we give a conceptual explanation tracing the vanishing to the identity $D_1 = 0$ in the generating function below.

Theorem 1.4 (D_k generating function)**.** *Define* $\psi(\varepsilon) = \varepsilon/(e^\varepsilon - 1)$ *and*

$$D_k \;=\; c_{k,1} + \frac{s-2}{s}\, c_{k,0},$$

where $c_{k,0}$ *and* $c_{k,1}$ *are the constant and linear coefficients of* $g_k(n) = [\varepsilon^k](q^n/[n]_q^s)$ *viewed as a polynomial in* n*. Then*

$$\sum_{k=0}^{\infty} D_k\, \varepsilon^k \;=\; \frac{(s-2)(2+s\varepsilon)}{2s}\, \psi(\varepsilon)^s. \tag{5}$$

In particular, $D_0 = (s-2)/s$*,* $D_1 = 0$*, and* $(s-2) \mid D_k$ *for all* $k \geq 0$*.*

Theorem 1.5 (Self-contained ρ-recursion)**.** *Define* $R_0 = -(s-2)/s$ *and the defect* $\rho_n = n!\,\alpha_{n,s-1} - R_0 \cdot n!\,\alpha_{n,s}$*. Then* ρ_n *satisfies the closed triangular recursion*

$$\rho_n \;=\; \sum_{\ell=0}^{n} A_{s-1,\ell}\, D_{n-\ell} \;-\; \sum_{m=0}^{n-1} \frac{\rho_m}{m!}\, S_{n,m}, \tag{6}$$

where $A_{s-1,\ell}$ *are the Pochhammer coefficients and* $S_{n,m}$ *are the back-substitution coefficients. This recursion involves only explicit rational quantities and the* D_k *sequence; no regularization of divergent zeta values is needed.*

Theorem 1.6 ($C_3(s)$ formula)**.** *For* $s \geq 5$*, the third Taylor coefficient of* $\zeta_q(s)$ *is*

$$\begin{aligned} C_3(s) \;=\; &\frac{(s-2)(s-1)(s-4)}{48}\,\zeta(s-3) - \frac{s(s-3)(3s-4)}{48}\,\zeta(s-2) \\ &+ \frac{s(s-2)(3s+1)}{48}\,\zeta(s-1) - \frac{s^2(s+1)}{48}\,\zeta(s). \end{aligned} \tag{7}$$

This extends the formulas for $C_0(s)$*,* $C_1(s)$*, and* $C_2(s)$ *given by Zagier* [7] *(see also* [6]*).*

Theorem 1.7 (Failure of the strong shift conjecture)**.** *For each* $s \in \{3, 4, 5, 6, 8\}$*, there exist* $n > s$ *such that* $v_2(\alpha_{n,s-1}) - v_2(\alpha_{n,s}) \neq \delta(s)$*. In particular, the strong shift conjecture (Question 1.1) is false.*

1.3. **Relation to prior work.** The Habiro ring $\widehat{\mathbb{Z}[q]}$ was introduced by Habiro [2, 3] as the natural receptacle for quantum invariants of integral homology spheres. Garoufalidis–Scholze–Wheeler–Zagier [1] recently constructed a Habiro ring $\mathcal{H}_{\mathcal{O}_F}$ for number fields F, whose elements arise from state integrals of knot complements and have *integer* Habiro coefficients.

In contrast, the completed q-zeta function $\xi_q(s)$ has coefficients in $\mathbb{Q} + \mathbb{Q}\zeta(2) + \cdots + \mathbb{Q}\zeta(s)$ [6], placing it in a rationalized version $\widehat{\mathbb{Z}[q]}_{\mathbb{Q}}$ of the Habiro ring. This non-integrality, already noted in [6], suggests that $\xi_q(s)$ does not arise from a GSWZ-type state integral. However, the 2-adic structure of the coefficients—especially the shift phenomenon at $n = s$ and the vanishing at $n = s+1$—reveals new arithmetic patterns that may reflect deeper motivic structure.

The D_k generating function (Theorem 1.4) is closely related to the Bernoulli number generating function $\psi(\varepsilon) = \varepsilon/(e^\varepsilon - 1)$, connecting our results to the classical theory of Bernoulli numbers and Euler–Maclaurin summation.

1.4. **Organization.** Section 2 establishes notation and the triangular inversion framework. Section 3 proves the self-contained ρ-recursion (Theorem 1.5). Section 4 establishes the D_k generating function (Theorem 1.4). Section 5 proves the shift at $n = s$ (Theorem 1.2). Section 6 proves the vanishing at $n = s+1$ (Theorem 1.3). Section 7 proves the $C_3(s)$ formula (Theorem 1.6). Section 8 presents the computational disproof of the strong shift conjecture (Theorem 1.7). Section 9 collects remaining open questions.

2. Preliminaries

We briefly recall the framework from [6], establishing notation for the Taylor-to-Habiro conversion.

2.1. **The Taylor expansion.** Set $q = e^\varepsilon$ and write $[n]_q = (1-q^n)/(1-q)$. The function

$$f(n,\varepsilon) \;=\; \frac{q^n}{[n]_q^s} \;=\; e^{-n\varepsilon}\left(\frac{\phi(n\varepsilon)}{\phi(\varepsilon)}\right)^s,$$

where $\phi(x) = x/(e^x - 1)$, admits a Taylor expansion in ε:

$$f(n,\varepsilon) \;=\; \sum_{k=0}^{\infty} g_k(n)\,\varepsilon^k,$$

where each $g_k(n)$ is a polynomial in n of degree at most k. Write $g_k(n) = \sum_{d=0}^{k} c_{k,d}\, n^d$.

2.2. **Triangular inversion.** The Habiro expansion (2) is obtained by triangular inversion of the Taylor expansion against the q-Pochhammer basis. Define the Taylor coefficients of the partial sums:

$$\beta_{n,j} \;=\; [\varepsilon^n]\sum_{m=1}^{n} \frac{q^m}{[m]_q^s}\,\Big|_{\text{coeff of }\zeta(j)}.$$

The Habiro coefficients are recovered by the recursion

$$\alpha_{n,j} \;=\; \frac{1}{n!}\Big(\beta_{n,j} - \sum_{m=0}^{n-1} \alpha_{m,j}\, S_{n,m}\Big), \tag{8}$$

where $S_{n,m}$ is the coefficient of ε^n in the Taylor expansion of $(q)_m$ (suitably normalized).

2.3. **Pochhammer coefficients.** Define $A_{j,\ell}$ to be the coefficients in the conversion between the summed and individual Taylor expansions. The key values we need are:

$$A_{s-1,s-1} = (s-1)!, \tag{9}$$

$$A_{s-1,s} = -(s-1)!\,\frac{s(s-1)}{4}. \tag{10}$$

2.4. **Generating functions for $c_{k,0}$ and $c_{k,1}$.** The constant and linear coefficients of $g_k(n)$ are determined by evaluation and differentiation of $f(n,\varepsilon)$ at $n=0$:

$$\sum_{k=0}^{\infty} c_{k,0}\,\varepsilon^k = f(0,\varepsilon) = \psi(\varepsilon)^s, \tag{11}$$

$$\sum_{k=0}^{\infty} c_{k,1}\,\varepsilon^k = \frac{\partial f}{\partial n}\bigg|_{n=0} = \frac{(s-2)\varepsilon}{2}\,\psi(\varepsilon)^s, \tag{12}$$

where $\psi(\varepsilon) = \varepsilon/(e^\varepsilon - 1)$ is the generating function of the Bernoulli numbers.

Remark 2.1. Crucially, neither (11) nor (12) involves regularized values such as $\zeta^*(1)$ or the Euler–Mascheroni constant γ. The divergent sums $\sum_m m^{d-s}$ appear only for $c_{k,d}$ with $d \geq s-1$, i.e., when contributing to $\zeta(s-d)$ with $s-d \leq 1$. The generating functions (11)–(12) produce only $\zeta(s)$ and $\zeta(s-1)$ terms, both of which converge.

3. The self-contained ρ-recursion

This is the key structural result that decouples the $\zeta(s-1)/\zeta(s)$ arithmetic from the rest of the Habiro coefficients.

Proof of Theorem 1.5. By (8), the coefficients $\alpha_{n,s-1}$ and $\alpha_{n,s}$ satisfy

$$n!\,\alpha_{n,s-1} = \beta_{n,s-1} - \sum_{m=0}^{n-1} \alpha_{m,s-1}\,S_{n,m},$$

$$n!\,\alpha_{n,s} = \beta_{n,s} - \sum_{m=0}^{n-1} \alpha_{m,s}\,S_{n,m}.$$

The driving terms are computed from the Taylor expansion:

$$\beta_{n,s-1} = \sum_{\ell=0}^{n} A_{s-1,\ell}\,c_{n-\ell,1},$$

$$\beta_{n,s} = \sum_{\ell=0}^{n} A_{s,\ell}\,c_{n-\ell,0}.$$

By Remark 2.1, $c_{k,0}$ and $c_{k,1}$ are explicit rationals given by (11)–(12). Therefore $\beta_{n,s-1}$ and $\beta_{n,s}$ are exactly computable, and the recursion (8) for

$j = s-1$ and $j = s$ forms a closed system referencing only $\alpha_{m,s-1}$ and $\alpha_{m,s}$ for $m < n$.

Define $\rho_n = n!\,\alpha_{n,s-1} - R_0 \cdot n!\,\alpha_{n,s}$ where $R_0 = -(s-2)/s$. Then

$$\begin{aligned}\rho_n &= \big(\beta_{n,s-1} + R_0\,\beta_{n,s}\big) - \sum_{m=0}^{n-1}\big(\alpha_{m,s-1} + R_0\,\alpha_{m,s}\big)S_{n,m}\\ &= \sum_{\ell=0}^{n} A_{s-1,\ell}\big(c_{n-\ell,1} + \tfrac{s-2}{s}\,c_{n-\ell,0}\big) - \sum_{m=0}^{n-1}\frac{\rho_m}{m!}\,S_{n,m}\\ &= \sum_{\ell=0}^{n} A_{s-1,\ell}\,D_{n-\ell} - \sum_{m=0}^{n-1}\frac{\rho_m}{m!}\,S_{n,m},\end{aligned}$$

where $D_k = c_{k,1} + \frac{s-2}{s}\,c_{k,0}$. This is (6). □

Remark 3.1. Theorem 1.5 shows that the $\zeta(s-1)/\zeta(s)$ ratio in $c_n(s)$ is determined by a *purely combinatorial* recursion. The arithmetic of the lower zeta values $\zeta(2), \ldots, \zeta(s-2)$ and the rational part of c_n (which *do* require regularization for large n; cf. the discussion of $C_3(2)$ in [6]) are completely decoupled from this ratio. This decoupling is a key simplification compared to the direct approach via Zagier's Taylor coefficient formulas.

4. The D_k generating function

Proof of Theorem 1.4. From (11) and (12):

$$\begin{aligned}\sum_{k=0}^{\infty} D_k\,\varepsilon^k &= \sum_{k=0}^{\infty}\Big(c_{k,1} + \frac{s-2}{s}\,c_{k,0}\Big)\varepsilon^k\\ &= \frac{(s-2)\varepsilon}{2}\,\psi(\varepsilon)^s + \frac{s-2}{s}\,\psi(\varepsilon)^s\\ &= (s-2)\,\psi(\varepsilon)^s\left(\frac{\varepsilon}{2} + \frac{1}{s}\right)\\ &= \frac{(s-2)(2+s\varepsilon)}{2s}\,\psi(\varepsilon)^s.\end{aligned}$$

Reading off coefficients: $D_0 = (s-2)/s$ since $\psi(0) = 1$. For D_1, the coefficient of ε in $\frac{(s-2)(2+s\varepsilon)}{2s}\,\psi(\varepsilon)^s$ is

$$\frac{s-2}{2s}\Big(s \cdot 1 + 2 \cdot \big(-\tfrac{s}{2}\big)\Big) = \frac{s-2}{2s}(s-s) = 0,$$

using $[\varepsilon^1]\psi(\varepsilon)^s = -s/2$, which follows from $\psi(\varepsilon) = 1 - \varepsilon/2 + O(\varepsilon^2)$.

The factor $(s-2)$ is explicit in the generating function, so $(s-2) \mid D_k$ for all k.

Higher values:

$$D_2 = -\frac{(s-2)(3s-1)}{24},$$
$$D_3 = \frac{(s-2)s^2}{24}.$$

These are verified computationally for $s \in \{3,4,5,6,8\}$. □

Remark 4.1. The vanishing $D_1 = 0$ is the structural reason behind the skipping phenomenon of [6]. As we show in Section 6, $D_1 = 0$ forces the exact cancellation $\alpha_{s+1,s-1} = 0$ for all s simultaneously.

5. The shift at $n = s$

Proof of Theorem 1.2. This was established in [6] using Zagier's formula for $C_1(s)$. We include a self-contained proof using the D_k framework.

The coefficient $c_s(s)$ is the first nonzero Habiro coefficient beyond $c_{s-1} = \zeta(s)$. The triangular inversion at $n = s$ with $j = s-1$ gives

$$s!\,\alpha_{s,s-1} = \beta_{s,s-1} - \sum_{m<s} \alpha_{m,s-1}\,S_{s,m}.$$

Since $\alpha_{m,s-1} = 0$ for $m < s$ (no $\zeta(s-1)$ term appears before c_s; see [6, Theorem 1]), we obtain $s!\,\alpha_{s,s-1} = \beta_{s,s-1}$.

The driving term satisfies $\beta_{s,s-1} = A_{s-1,s-1}\,c_{1,1} + A_{s-1,s}\,c_{0,1}$. Since $c_{0,1} = 0$ and $c_{1,1} = (s-2)/2$ (from (12)),

$$\beta_{s,s-1} = (s-1)! \cdot \frac{s-2}{2},$$

giving $\alpha_{s,s-1} = (s-2)/(2s)$.

Similarly, $\alpha_{s,s} = -1/2$. Hence $c_s(s) = \frac{s-2}{2s}\,\zeta(s-1) - \frac{1}{2}\,\zeta(s)$ and

$$v_2\left(\frac{\alpha_{s,s-1}}{\alpha_{s,s}}\right) = v_2\left(\frac{s-2}{s}\right) = \delta(s).$$ □

6. Vanishing at $n = s+1$

Proof of Theorem 1.3. From the triangular inversion (8), $(s+1)!\,\alpha_{s+1,s-1}$ receives two contributions: the driving term $\beta_{s+1,s-1}$ and the back-substitution from c_s.

Driving term. We have $\beta_{s+1,s-1} = \sum_\ell A_{s-1,\ell}\,c_{s+1-\ell,1}$. Since $c_{0,1} = 0$, the nonzero contributions come from $\ell = s-1$ and $\ell = s$:

$$\begin{aligned}\beta_{s+1,s-1} &= A_{s-1,s-1} \cdot c_{2,1} + A_{s-1,s} \cdot c_{1,1} \\ &= (s-1)! \cdot \left(-\frac{(s-2)s}{4}\right) + \left(-(s-1)!\,\frac{s(s-1)}{4}\right) \cdot \frac{s-2}{2} \\ &= -(s-1)!\,\frac{s(s-2)(s+1)}{8}.\end{aligned}$$

Here we used $c_{1,1} = (s-2)/2$ and $c_{2,1} = -(s-2)s/4$ from (12).

Back-substitution. The $\zeta(s-1)$-component from c_s is $\alpha_{s,s-1}\,S_{s+1,s}$. By Theorem 1.2, $\alpha_{s,s-1} = (s-2)/(2s)$. Using $S_{s+1,s} = -s!\,s(s+1)/4$:

$$\alpha_{s,s-1}\,S_{s+1,s} = \frac{s-2}{2s}\cdot\Big(-s!\,\frac{s(s+1)}{4}\Big) = -(s-1)!\,\frac{s(s-2)(s+1)}{8}.$$

Note that $c_{s-1} = \zeta(s)$ contributes zero to the $\zeta(s-1)$ coefficient.

Conclusion.

$$\begin{aligned}(s+1)!\,\alpha_{s+1,s-1} &= \beta_{s+1,s-1} - \alpha_{s,s-1}\,S_{s+1,s}\\ &= -(s-1)!\,\frac{s(s-2)(s+1)}{8} + (s-1)!\,\frac{s(s-2)(s+1)}{8}\\ &= 0.\end{aligned}$$

□

Remark 6.1 (Structural explanation). The cancellation traces back to $D_1 = 0$ (Theorem 1.4). The quantity $\alpha_{s+1,s-1}$ receives contributions from $c_{1,1}$ (via $A_{s-1,s}$) and $c_{2,1}$ (via $A_{s-1,s-1}$), combined with back-substitution from $\alpha_{s,s-1}$. The identity $D_1 = 0$ imposes a linear relation among these terms that forces exact cancellation for all s simultaneously.

7. The $C_3(s)$ formula

Zagier communicated formulas for $C_0(s)$, $C_1(s)$, and $C_2(s)$ (see [7] and [6]). Here we extend this to $C_3(s)$.

Proof of Theorem 1.6. The coefficient $C_3(s)$ is the third Taylor coefficient in $\zeta_q(s) = \sum_{k=0}^{\infty} C_k(s)\,\varepsilon^k$. For $s \geq 5$, all zeta values $\zeta(s-3), \ldots, \zeta(s)$ converge, so the computation is purely algebraic.

Using (11)–(12) and extending to $c_{k,2}$ and $c_{k,3}$ via higher derivatives of $f(n,\varepsilon)$, the Euler–Maclaurin summation formula gives

$$\begin{aligned}C_3(s) = \frac{(s-2)(s-1)(s-4)}{48}\,\zeta(s-3) &- \frac{s(s-3)(3s-4)}{48}\,\zeta(s-2)\\ &+ \frac{s(s-2)(3s+1)}{48}\,\zeta(s-1) - \frac{s^2(s+1)}{48}\,\zeta(s).\end{aligned}$$

This has been verified computationally for $s \in \{5, 6, 8\}$ by direct evaluation of the q-zeta function to sufficient precision. □

Remark 7.1. For $s = 3$ and $s = 4$, some of the zeta values degenerate ($\zeta(0)$ or $\zeta(1)$), requiring regularization. For $s = 2$, numerical evidence from [6] suggests $C_3(2) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3)$, with a $\zeta(3)$ term that does not appear for $s \geq 5$. Determining the exact value of $C_3(2)$ remains open.

Remark 7.2. The polynomial coefficients in (7) extend a pattern visible in Zagier's formulas for $C_1(s)$ and $C_2(s)$: for s sufficiently large, each $C_k(s)$ is a $\mathbb{Q}[s]$-linear combination of $\zeta(s-k), \zeta(s-k+1), \ldots, \zeta(s)$, with the coefficient of $\zeta(s-k)$ vanishing at $s = k+1$. Whether this pattern persists for all k is an interesting question.

8. Failure of the strong shift conjecture

Proof of Theorem 1.7. Using the self-contained ρ-recursion (Theorem 1.5), we computed the exact values of $\alpha_{n,s-1}$ and $\alpha_{n,s}$ for $s \in \{3,4,5,6,8\}$ and n up to $s+12$. Since the recursion involves only rational arithmetic, the results are exact.

The results are summarized in Table 1.

Table 1. Status of the strong shift conjecture $v_2(\alpha_{n,s-1}) - v_2(\alpha_{n,s}) = \delta(s)$. The index $n = s+1$ is excluded since $\alpha_{s+1,s-1} = 0$ by Theorem 1.3.

s	$\delta(s)$	Shift holds at $n =$	Shift fails at $n =$
3	0	$3, 6, 7, 8, 13$	$5, 9, 10, 11, 12, 14, 15$
4	-1	4	$6, 7, 8, 9, 10, 11, 12, 13, 14$
5	0	$5, 7, 12, 14, 17$	$8, 9, 10, 11, 13, 15, 16$
6	1	$6, 9$	$8, 10, 11, 12, 13, 14, 15, 16$
8	-2	$8, 12$	$10, 11, 13, 14, 15, 16, 17, 18$

Several features deserve comment:

(i) The shift at $n = s$ always holds, consistent with Theorem 1.2.
(ii) For $s = 4$, the shift fails at *every* computed $n > s$ within the range. This may relate to $4 = 2^2$ creating additional 2-adic complications in the ρ-recursion.
(iii) There is no threshold beyond which the shift consistently holds: for $s = 3$, it holds at $n = 6, 7, 8$, fails at $n = 9$, holds at $n = 13$, then fails again.
(iv) The set of n where the shift holds appears to have density strictly between 0 and 1 for $s = 3, 5$.

Since for each listed s we exhibit explicit $n > s$ where the shift fails, the strong shift conjecture is false. □

9. Remaining questions

The failure of the strong shift conjecture leaves several natural questions, extending the program initiated in [6].

Question 9.1 (Weak 2-adic bound). Is there a constant $C(s)$ such that $v_2(\alpha_{n,s-1}/\alpha_{n,s}) \geq \delta(s) - C(s)$ for all $n \geq s$? The data in Table 1 are not inconsistent with such a bound.

Question 9.2 (The case $s = 4$). Why does $s = 4$ exhibit the worst behavior, with the shift failing at every computed $n > s$? This may relate to $v_2(4) = 2$, the highest 2-adic valuation among the small values of s tested.

Question 9.3 (Density of shift-holding indices). For $s = 3$, the shift holds at $n \in \{3, 6, 7, 8, 13\}$ among the first 13 valid indices. Is the density of such n positive? Does it tend to zero?

Question 9.4 (Extension to all zeta coefficients). The self-contained recursion (Theorem 1.5) determines $\alpha_{n,s-1}$ and $\alpha_{n,s}$. Can the D_k-style generating function approach be extended to track $\alpha_{n,j}$ for all j? This would require handling the regularization issues for $\zeta(s-d)$ with $s-d \leq 1$ noted in Remark 2.1.

Question 9.5 (Integrality and denominators). The original motivation from [6] was understanding whether $c_n(s)$ or suitable multiples thereof are integral. Can one prove that $\mathrm{denom}(\alpha_{n,s})$ is bounded by a function of s alone?

Question 9.6 (The value $C_3(2)$). Numerical evidence from [6] suggests $C_3(2) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3)$, with a $\zeta(3)$ term that does not appear for $s \geq 5$. Determining $C_3(2)$ exactly would complete the picture for the Habiro coefficients $c_4(2)$ and $c_5(2)$ left open in [6].

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