Gate Complexity of the Algebraic Torus

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Abstract

We determine the gate complexity $t(p, q, n)$—the minimum number of compositions of affine maps $\mathbb{F}_q^n \to \mathbb{F}_q$ with arbitrary functions $\mathbb{F}_q \to \mathbb{F}_p$ needed to represent the indicator function of the algebraic torus $(\mathbb{F}_q^*)^n$ as an $\mathbb{F}_p$-linear combination—for all primes p and prime powers q with $\mathrm{char}(\mathbb{F}_q) \neq p$.

The answer exhibits a dichotomy governed by a single divisibility condition:

$$t(p,q,n) = \begin{cases} (q-1)^{n-1} & \text{if } p \mid (q-1), \\ \dfrac{q^n-1}{q-1} = |\mathbb{P}^{n-1}(\mathbb{F}_q)| & \text{if } p \nmid (q-1). \end{cases}$$

When $p \mid (q-1)$, the $\mathbb{F}_{p^k}$-Fourier transform of $\mathbf{1}_T$ is supported on the torus T, and the optimal construction uses $(q-1)^{n-1}$ gates indexed by $(\mathbb{F}_q^*)^{n-1}$. When $p \nmid (q-1)$, the Fourier transform has full support on $\mathbb{F}_q^n \setminus \{0\}$, and the optimal construction requires one gate per point of $\mathbb{P}^{n-1}(\mathbb{F}_q)$.

In both cases, the upper bound is a Fourier inversion identity and the lower bound is a Frobenius orbit counting argument. We give a cohomological interpretation: the gate complexity equals the Frobenius trace on compactly supported étale cohomology $H^*_{c,\text{ét}}(\mathbb{G}_m^{n-1}, \mathbb{F}_p)$.

We also show that depth-3 circuits escape the exponential barrier: for $n < p$, the torus indicator can be computed with $O(n)$ gates. Finally, we show that for $n = 2$, the spectral theory of the gate complexity model produces a *Steinberg polynomial* with an endoscopic decomposition governed by $\mathbb{Q}(\sqrt{-2})$ and a motivic factorization into CM abelian varieties over $\mathbb{F}_2$, connecting circuit complexity to arithmetic geometry.

1 Introduction

A central open problem in circuit complexity is to prove super-polynomial lower bounds for $\mathsf{AC}^0[6]$, the class of constant-depth circuits with AND, OR, NOT, and MOD-6 gates. The Razborov–Smolensky method [4, 5] gives exponential lower bounds for $\mathsf{AC}^0[p]$ when p is prime, but breaks down for composite moduli like $6 = 2 \times 3$.

The key difficulty is the interaction between different characteristics. MOD-6 gates can simulate both MOD-2 and MOD-3, combining information from $\mathbb{F}_2$ and $\mathbb{F}_3$ in a way that resists standard polynomial methods. In this paper we isolate this cross-characteristic interaction in its simplest form and study it through the lens of algebraic coding theory.

The model. The gate complexity model is inherently depth-2: a single layer of affine maps $\ell_i : \mathbb{F}_q^n \to \mathbb{F}_q$ composed with arbitrary functions $g_i : \mathbb{F}_q \to \mathbb{F}_p$, followed by an $\mathbb{F}_p$-linear combination. This corresponds to a depth-2 circuit with one layer of MOD-q gates feeding into a single MOD-p output gate.

We consider the gate complexity $t(p, q, n)$: the minimum number of (p, q)-gates needed to represent the indicator function $\mathbf{1}_T$ of the algebraic torus $T = (\mathbb{F}_q^*)^n$ as an $\mathbb{F}_p$-linear combination.

Here a (p,q)-gate is a composition $g \circ \ell$ where $\ell : \mathbb{F}_q^n \to \mathbb{F}_q$ is affine and $g : \mathbb{F}_q \to \mathbb{F}_p$ is arbitrary. The function $\mathbf{1}_T$ is the canonical "hard function" for this model: it is nonzero precisely on the torus, the complement of the union of coordinate hyperplanes.

Scope and limitations. Our exponential lower bound applies to this restricted depth-2 setting. A full $\mathsf{AC}^0[6]$ circuit has arbitrary constant depth, and the central open problem is precisely to understand how cross-characteristic interactions compose across multiple layers. We show in Section 10 that depth-3 already escapes the exponential barrier, reducing the gate count from exponential to linear.

1.1 Main Results

Our main result determines the gate complexity for all primes p and prime powers q with $\mathrm{char}(\mathbb{F}_q) \neq p$.

Theorem 1.1 (Main Theorem). *Let p be a prime and q a prime power with* $\mathrm{char}(\mathbb{F}_q) \neq p$. *Then*

$$t(p,q,n) = \begin{cases} (q-1)^{n-1} & \text{if } p \mid (q-1), \\ \dfrac{q^n - 1}{q-1} = |\mathbb{P}^{n-1}(\mathbb{F}_q)| & \text{if } p \nmid (q-1). \end{cases}$$

The dichotomy is governed by a single divisibility condition. Note that for $p = 2$, the condition $2 \mid (q-1)$ holds for all odd q, so the formula simplifies to $t(2,q,n) = (q-1)^{n-1}$. For $q = 2$, we have $q - 1 = 1$, so $p \nmid 1$ for all primes $p \geq 3$, giving $t(p,2,n) = 2^n - 1 = |\mathbb{P}^{n-1}(\mathbb{F}_2)|$.

Additional results.

(1) **Coding-theoretic framework (Section 2).** We reduce gate complexity to a minimum coset weight problem in a linear code over $\mathbb{F}_p$, with quotient dimension $\dim(C/C_0) = (q-1)^n$ in the cross-characteristic case.

(2) **Gate span completeness (Theorem 3.1).** Cross-characteristic gates span all functions $\mathbb{F}_q^n \to \mathbb{F}_p$. This fails in same characteristic, explaining the algebraic core of the $\mathsf{AC}^0[6]$ difficulty.

(3) **Fourier support dichotomy (Theorem 4.2).** Over $\mathbb{F}_{p^k}$, the Fourier transform $\widehat{\mathbf{1}_T}$ is supported on T when $p \mid (q-1)$ and on $\mathbb{F}_q^n \setminus \{0\}$ when $p \nmid (q-1)$.

(4) **Cohomological interpretation (Theorem 9.1).** Gate complexity equals the Frobenius trace on compactly supported étale cohomology: $t(p,q,n) = \mathrm{Tr}(\mathrm{Frob}_q \mid H^*_{c,\text{ét}}(\mathbb{G}_m^{n-1}, \mathbb{F}_p))$. The code quotient C/C_0 is isomorphic to the space of $\mathbb{F}_q^*$-orbit functions on T.

(5) **Depth-3 escape (Section 10).** For $n < p$, the torus indicator can be computed with $n+1$ gates at depth-3—an exponential improvement over depth-2.

(6) **Steinberg polynomial (Section 11).** For $n = 2$, the gate complexity model specializes to a random walk on $\mathbb{P}^1(\mathbb{F}_p)$ whose Steinberg polynomial $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ admits an endoscopic decomposition controlled by $\mathbb{Q}(\sqrt{-2})$ and a motivic factorization into CM abelian varieties over $\mathbb{F}_2$ [10].

1.2 Techniques

Upper bound. The construction is a Fourier inversion identity decomposed over projective lines. For each projective point $[a] \in \mathbb{P}^{n-1}(\mathbb{F}_q)$, we define a gate $g_{[a]} \circ \ell_a$ where $\ell_a(x) = a \cdot x$ and $g_{[a]}(v) = c_{[a]} \cdot \mathbf{1}_{v=0}$ with explicit coefficients $c_{[a]}$. The key observation is that the coefficients $c_{[a]}$ vanish in $\mathbb{F}_p$ precisely when $p \mid (q-1)$ and $a \notin T$, reducing the gate count from $|\mathbb{P}^{n-1}(\mathbb{F}_q)|$ to $(q-1)^{n-1}$ in this case.

Lower bound. The lower bound proceeds by a Frobenius orbit counting argument. The $\mathbb{F}_{p^k}$-Fourier transform (where $k = \mathrm{ord}_r(p)$ and $r = \mathrm{char}(\mathbb{F}_q)$) has the property that Fourier support is closed under the Frobenius action $\alpha \mapsto p\alpha$. We show:

- Each gate's Fourier support lies on a single $\mathbb{F}_q$-line through the origin.
- Each such line contains at most $(q-1)/k$ Frobenius orbits in its torus part.
- The Fourier support of $\mathbf{1}_T$ consists of all torus orbits (when $p \mid (q-1)$) or all nonzero orbits (when $p \nmid (q-1)$).
- Covering all required orbits forces $w \geq (q-1)^{n-1}$ or $w \geq (q^n-1)/(q-1)$ gates.

The factors of k cancel perfectly, so the final answer depends only on p, q, and n—not on the multiplicative order of p in $\mathbb{F}_r^*$.

Discussion. The conceptual message is a dichotomy: cross-characteristic gates always span the full function space, but doing so efficiently requires overcoming a Fourier-theoretic obstruction that grows exponentially in n. The formula reveals that the growth rate is controlled by either the torus dimension $|(\mathbb{F}_q^*)|^{n-1} = (q-1)^{n-1}$ or the projective space dimension $|\mathbb{P}^{n-1}(\mathbb{F}_q)| = (q^n-1)/(q-1)$, with the divisibility $p \mid (q-1)$ determining which regime applies.

1.3 Related Work

The polynomial method of Razborov [4] and Smolensky [5] gives exponential lower bounds for $\mathsf{AC}^0[p]$ for prime p, but fails for composite moduli. Barrington, Straubing, and Thérien [2] studied the algebraic structure of ACC^0 and showed connections to group theory. Viola [8] surveyed the state of small-depth computation and highlighted the $\mathsf{AC}^0[6]$ problem as a central challenge. Williams [9] proved nonuniform ACC^0 lower bounds via a different route (satisfiability algorithms), but the uniform case remains open.

The connection between gate complexity and coding theory parallels work on toric codes [3, 7], where code parameters are controlled by lattice geometry.

The building-theoretic perspective (Section 11) connects to work of Schneider–Stuhler [6] on representations and sheaves on Bruhat-Tits buildings, to Borel–Serre [1] on building cohomology, and to the depth-zero endoscopic theory of DeBacker–Reeder [11], Kaletha [12], and Kazhdan–Varshavsky [13]. The Steinberg polynomial and its arithmetic, developed in companion work [10], provides a concrete bridge between the gate complexity framework and the Langlands program.

1.4 Organization

Section 2 establishes the coding-theoretic framework. Section 3 proves gate span completeness. Section 4 develops the $\mathbb{F}_{p^k}$-Fourier transform and proves the support dichotomy. Section 5 proves the lower bound via orbit counting. Section 6 proves the upper bound via Fourier inversion. Section 7 analyzes the special case $q = 3$ in detail. Section 8 gives the alternative Vandermonde induction proof for $q = 3$ and shows why it fails for $q \geq 5$. Section 9 gives the cohomological interpretation connecting gate complexity to étale cohomology. Section 10 proves that depth-3 circuits escape the exponential barrier. Section 11 describes the connection to the Steinberg polynomial and the Langlands program. Section 12 discusses connections to $\mathsf{AC}^0[6]$ and future directions.

2 The Coding-Theoretic Framework

2.1 Setup and Notation

Throughout, p is a prime, q is a prime power with $\mathrm{char}(\mathbb{F}_q) = r \neq p$, and $n \geq 1$. Write $T = (\mathbb{F}_q^*)^n$ for the algebraic torus and $Z = \mathbb{F}_q^n \setminus T$ for the boundary.

Definition 2.1. *A (p, q)-gate on $\mathbb{F}_q^n$ is a function $g \circ \ell : \mathbb{F}_q^n \to \mathbb{F}_p$, where $\ell(u) = a \cdot u + b$ is affine ($a \in \mathbb{F}_q^n$, $b \in \mathbb{F}_q$) and $g : \mathbb{F}_q \to \mathbb{F}_p$ is arbitrary.*

Let G denote the set of all distinct gate evaluation vectors, with $|G| = G$, and form the gate evaluation matrix $M \in \mathbb{F}_p^{q^n \times G}$.

Definition 2.2. *The gate complexity is*

$$t(p, q, n) = \min\{\mathrm{wt}(c) : c \in \mathbb{F}_p^G, M_Z c = 0, M_T c = \mathbf{1}_T\}.$$

2.2 The Code and Its Quotient

Define linear codes over $\mathbb{F}_p$:

$$C = \ker(M_Z) = \{c \in \mathbb{F}_p^G : M_Z c = 0\},$$
$$C_0 = \ker(M) = \{c \in \mathbb{F}_p^G : Mc = 0\}.$$

The quotient C/C_0 maps isomorphically onto $\mathbb{F}_p^T$: every function $T \to \mathbb{F}_p$ is realizable. The target $\mathbf{1}_T$ determines a coset $c_0 + C_0$ inside C, and $t(p, q, n) = \min_{c \in c_0 + C_0} \mathrm{wt}(c)$.

3 Gate Span Completeness

Theorem 3.1. *Let p be a prime and q a prime power with $\mathrm{char}(\mathbb{F}_q) \neq p$. Then $\mathrm{span}_{\mathbb{F}_p}(G) = \mathbb{F}_p^{\mathbb{F}_q^n}$, and consequently $\dim(C/C_0) = (q - 1)^n$.*

Proof. We prove the contrapositive: any $\lambda : \mathbb{F}_q^n \to \mathbb{F}_p$ annihilating every gate must be zero.

Step 1. If $\sum_u \lambda(u)(g \circ \ell)(u) = 0$ for all gates, then choosing $g = \delta_v$ shows that each fiber sum $\sum_{\ell(u)=v} \lambda(u) = 0$ for all nonconstant ℓ and all v.

Step 2. Since $\mathrm{char}(\mathbb{F}_q) \neq p$, fix a nontrivial additive character $\psi : (\mathbb{F}_q, +) \to \mathbb{F}_p[\zeta]^*$. Multiplying fiber sums by $\psi(v)$ and summing gives $\widehat{\lambda}(\psi_a) = 0$ for all nonzero a.

Step 3. Since q^n is coprime to p, the DFT is invertible in $\mathbb{F}_p[\zeta]$. All Fourier coefficients vanishing implies $\lambda \equiv 0$.

The dimension formula follows: $\mathrm{rank}(M) = q^n$, $\mathrm{rank}(M_Z) = q^n - (q - 1)^n$, so $\dim(C/C_0) = (q - 1)^n$. □

Remark 3.2. *When $p = \mathrm{char}(\mathbb{F}_q)$, the DFT is not invertible and nontrivial annihilators exist. The quotient dimension collapses: for $p = q = 3$, $n = 2$, one has $\dim(C/C_0) = 1$ versus $(q - 1)^n = 4$ in the cross-characteristic case. This dichotomy is the algebraic core of the difficulty of $\mathsf{AC}^0[6]$.*

4 The $\mathbb{F}_{p^k}$-Fourier Transform

4.1 Setup

Let $r = \mathrm{char}(\mathbb{F}_q)$ and $k = \mathrm{ord}_r(p)$, the multiplicative order of p in $\mathbb{F}_r^*$. Since $r \mid p^k - 1$, the field $\mathbb{F}_{p^k}$ contains a primitive rth root of unity ζ.

Fix the nontrivial additive character $\chi : \mathbb{F}_q \to \mathbb{F}_{p^k}^*$ defined by $\chi(x) = \zeta^{\mathrm{Tr}(x)}$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_r$ is the field trace. (For q prime, this reduces to $\chi(x) = \zeta^x$.) The $\mathbb{F}_{p^k}$-Fourier transform of $f : \mathbb{F}_q^n \to \mathbb{F}_{p^k}$ is

$$\widehat{f}(\alpha) = \sum_{x \in \mathbb{F}_q^n} f(x)\chi(-\alpha \cdot x), \quad \alpha \in \mathbb{F}_q^n.$$

Since $\mathbb{F}_p \subset \mathbb{F}_{p^k}$, any function $f : \mathbb{F}_q^n \to \mathbb{F}_p$ has a well-defined $\mathbb{F}_{p^k}$-Fourier transform.

The Frobenius $\sigma : x \mapsto x^p$ acts on $\mathbb{F}_{p^k}$ with order k. Since Tr is $\mathbb{F}_r$-linear and $p \in \mathbb{F}_r$, we have $\sigma(\chi(v)) = \chi(v)^p = \zeta^{p\mathrm{Tr}(v)} = \zeta^{\mathrm{Tr}(pv)} = \chi(pv)$, so σ acts on $\mathbb{F}_q^n$ as $\alpha \mapsto p\alpha$ (scalar multiplication by $p \in \mathbb{F}_q$). For f taking values in $\mathbb{F}_p = \mathbb{F}_{p^k}^{\sigma}$:

$$\widehat{f}(p\alpha) = \widehat{f}(\alpha)^p, \tag{1}$$

so the Fourier support is a union of Frobenius orbits.

4.2 Fourier Support Dichotomy

Proposition 4.1. *Over $\mathbb{F}_{p^k}$, the Fourier transform of $\mathbf{1}_T$ is:*

$$\widehat{\mathbf{1}_T}(\alpha) = \prod_{j=1}^{n} S(\alpha_j), \quad S(a) = \sum_{c \in \mathbb{F}_q^*} \chi(-ac).$$

The per-coordinate factor satisfies:

$$S(a) = \begin{cases} q-1 & \text{if } a = 0, \\ -1 & \text{if } a \neq 0. \end{cases}$$

Proof. The torus indicator factorizes as $\mathbf{1}_T(x) = \prod_j \mathbf{1}_{x_j \neq 0}$, so the Fourier transform factorizes. For the sum $S(a) = \sum_{c \in \mathbb{F}_q^*} \chi(-ac)$: if $a = 0$, every term is 1 and $S(0) = q - 1$. If $a \neq 0$, the map $c \mapsto -ac$ is a bijection on $\mathbb{F}_q^*$, so $S(a) = \sum_{t \in \mathbb{F}_q^*} \chi(t) = \sum_{t \in \mathbb{F}_q} \chi(t) - 1 = 0 - 1 = -1$. □

Theorem 4.2 (Fourier Support Dichotomy). *Let $m(\alpha) = |\{j : \alpha_j = 0\}|$ for $\alpha \in \mathbb{F}_q^n$. Then in $\mathbb{F}_{p^k}$:*

$$\widehat{\mathbf{1}_T}(\alpha) = (-1)^{n-m(\alpha)}(q-1)^{m(\alpha)}.$$

Consequently:

(i) If $p \mid (q-1)$: $\widehat{\mathbf{1}_T}(\alpha) \neq 0 \iff \alpha \in T$. In particular, $\widehat{\mathbf{1}_T}(\alpha) = (-1)^n = \mathbf{1}_T(\alpha)$ for $p = 2$, recovering self-duality.

(ii) If $p \nmid (q-1)$: $\widehat{\mathbf{1}_T}(\alpha) \neq 0 \iff \alpha \neq 0$. The Fourier transform has full support on $\mathbb{F}_q^n \setminus \{0\}$.

Proof. By Proposition 4.1, $\widehat{\mathbf{1}_T}(\alpha) = \prod_j S(\alpha_j) = (-1)^{n-m(\alpha)}(q-1)^{m(\alpha)}$. This vanishes in $\mathbb{F}_{p^k}$ if and only if $m(\alpha) \geq 1$ and $q - 1 \equiv 0 \pmod{p}$. □

5 Lower Bound

Lemma 5.1 (Gate Fourier support)**.** *If $g \circ \ell$ is a gate with $\ell(x) = a \cdot x + b$, then* $\operatorname{supp}(\widehat{g \circ \ell}) \subseteq \mathbb{F}_q \cdot a$.

Proof. The Fourier transform of $g \circ \ell$ at α involves a sum over the affine hyperplane $\{x : a \cdot x + b = v\}$. This sum vanishes unless $\alpha \in (\ker a)^\perp = \mathbb{F}_q \cdot a$. □

Lemma 5.2 (Frobenius orbits)**.** *Let $k = \operatorname{ord}_r(p)$. The Frobenius $\alpha \mapsto p\alpha$ acts on $\mathbb{F}_q^n \setminus \{0\}$ with orbits of size dividing k. Each line $\mathbb{F}_q \cdot a$ through a nonzero a contains:*

(a) $(q-1)/k$ Frobenius orbits lying in $\mathbb{F}_q^* \cdot a$ (the torus part of the line), and

(b) one additional orbit $\{0\}$ (which has size 1).

For $a \in T$, the line $\mathbb{F}_q \cdot a$ meets T in exactly $(q-1)/k$ Frobenius orbits.

Proof. The orbits of $\mathbb{F}_q^*$ under multiplication by p have size $k = \operatorname{ord}_r(p)$, giving $(q-1)/k$ orbits. The line $\mathbb{F}_q \cdot a$ intersected with $\mathbb{F}_q^n \setminus \{0\}$ is $\mathbb{F}_q^* \cdot a$, which inherits the orbit decomposition. □

Theorem 5.3 (Lower bound)**.** *For all primes p and prime powers q with* $\operatorname{char}(\mathbb{F}_q) \neq p$*:*

$$t(p,q,n) \geq \begin{cases} (q-1)^{n-1} & \text{if } p \mid (q-1), \\ 1 + q + \cdots + q^{n-1} & \text{if } p \nmid (q-1). \end{cases}$$

Proof. Suppose $\mathbf{1}_T = \sum_{i=1}^{w} c_i (g_i \circ \ell_i)$ with $c_i \in \mathbb{F}_p^*$. Taking $\mathbb{F}_{p^k}$-Fourier transforms:

$$\widehat{\mathbf{1}_T} = \sum_{i=1}^{w} c_i \widehat{g_i \circ \ell_i}.$$

For any α with $\widehat{\mathbf{1}_T}(\alpha) \neq 0$, at least one gate must satisfy $\widehat{g_i \circ \ell_i}(\alpha) \neq 0$, placing α on the line $\mathbb{F}_q \cdot a_i$ by Lemma 5.1. Since the Fourier support is a union of Frobenius orbits by (1), each such orbit must be covered by some gate.

Case $p \mid (q-1)$**:** By Theorem 4.2(i), the Fourier support is T. The torus has $(q-1)^n/k$ Frobenius orbits, and each gate line covers at most $(q-1)/k$:

$$w \cdot \frac{q-1}{k} \geq \frac{(q-1)^n}{k} \implies w \geq (q-1)^{n-1}.$$

Case $p \nmid (q-1)$**:** By Theorem 4.2(ii), the Fourier support is $\mathbb{F}_q^n \setminus \{0\}$, which has $(q^n-1)/k$ Frobenius orbits. Each gate line covers at most $(q-1)/k$ orbits in $\mathbb{F}_q^n \setminus \{0\}$ (namely the orbits in $\mathbb{F}_q^* \cdot a_i$):

$$w \cdot \frac{q-1}{k} \geq \frac{q^n-1}{k} \implies w \geq \frac{q^n-1}{q-1} = |\mathbb{P}^{n-1}(\mathbb{F}_q)|.$$

□

Remark 5.4. *The factors of k cancel perfectly in the lower bound. This means the gate complexity depends only on q and n, not on the multiplicative order of p. The extension field $\mathbb{F}_{p^k}$ serves as an auxiliary tool but leaves no trace in the final answer.*

6 Upper Bound

Theorem 6.1 (Upper bound)**.** *For all primes p and prime powers q with* $\mathrm{char}(\mathbb{F}_q) \neq p$ *and* $n \geq 1$*:*

$$t(p,q,n) \leq \begin{cases} (q-1)^{n-1} & \text{if } p \mid (q-1), \\ 1 + q + \cdots + q^{n-1} & \text{if } p \nmid (q-1). \end{cases}$$

Proof. For each nonzero direction $a \in \mathbb{F}_q^n \setminus \{0\}$, define the homogeneous linear form $\ell_a(x) = a \cdot x$ and the gate function $g_a : \mathbb{F}_q \to \mathbb{F}_p$ by

$$g_a(v) = c_{[a]} \cdot \mathbf{1}_{v=0},$$

where $[a]$ denotes the projective class of a and

$$c_{[a]} = \frac{(-1)^{n-m(a)} \cdot (q-1)^{m(a)}}{q^{n-1}} \in \mathbb{F}_p, \tag{2}$$

with $m(a) = |\{j : a_j = 0\}|$ as before, and q^{n-1} is inverted in $\mathbb{F}_p$ (possible since $\mathrm{char}(\mathbb{F}_q) \neq p$). The coefficient $c_{[a]}$ depends only on the projective class $[a]$ since $m(ta) = m(a)$ for $t \in \mathbb{F}_q^*$.

Claim: The function

$$F(x) = \sum_{[a] \in \mathbb{P}^{n-1}(\mathbb{F}_q)} c_{[a]} \cdot \mathbf{1}_{a \cdot x = 0}$$

satisfies $F(x) = \mathbf{1}_T(x) + C$ for a constant $C \in \mathbb{F}_p$.

Proof of claim. Expand each indicator using the additive characters of $\mathbb{F}_q$:

$$\mathbf{1}_{a \cdot x = 0} = \frac{1}{q} \sum_{s \in \mathbb{F}_q} \chi(s \cdot a \cdot x) = \frac{1}{q} + \frac{1}{q} \sum_{s \in \mathbb{F}_q^*} \chi(s \cdot a \cdot x).$$

Substituting into F and using $\alpha = sa$ to parametrize $\mathbb{F}_q^n \setminus \{0\}$:

$$F(x) = C_0 + \frac{1}{q} \sum_{\alpha \in \mathbb{F}_q^n \setminus \{0\}} \frac{c_{[\alpha]}}{q-1} \chi(\alpha \cdot x),$$

where we used the fact that each $\alpha \neq 0$ is counted once for each $s \in \mathbb{F}_q^*$ in its projective class, and the factor $1/(q-1)$ compensates.

By Fourier inversion, $\mathbf{1}_T(x) = q^{-n} \sum_\alpha \widehat{\mathbf{1}_T}(\alpha) \chi(\alpha \cdot x)$. Matching coefficients shows $F(x) = \mathbf{1}_T(x) + C$ for some constant C.

Since a constant function can be absorbed into any single gate (by adjusting $g_a(v)$ for one gate), the number of gates equals the number of projective classes $[a]$ for which $c_{[a]} \neq 0$ in $\mathbb{F}_p$.

Counting nonzero gates. The coefficient $c_{[a]} = (-1)^{n-m(a)}(q-1)^{m(a)}/q^{n-1}$ vanishes in $\mathbb{F}_p$ if and only if $p \mid (q-1)$ and $m(a) \geq 1$ (since q^{n-1} is invertible and $(-1)^{n-m(a)}$ is a unit).

- If $p \mid (q-1)$: $c_{[a]} \neq 0$ only when $m(a) = 0$, i.e., $a \in T$. The number of such projective classes is $|T|/(q-1) = (q-1)^{n-1}$.
- If $p \nmid (q-1)$: $c_{[a]} \neq 0$ for all $[a] \in \mathbb{P}^{n-1}(\mathbb{F}_q)$, giving $(q^n-1)/(q-1)$ gates.

□

Proof of Theorem 1.1. Combine Theorem 5.3 and Theorem 6.1. □

7 The Special Case $q = 3$

For $q = 3$ and $p = 2$, the formula gives $t(2, 3, n) = 2^{n-1}$. This case admits a more detailed analysis.

7.1 Explicit Construction

The gates are indexed by $s \in (\mathbb{F}_3^*)^{n-1} = \{1, 2\}^{n-1}$. For each $s = (s_1, \dots, s_{n-1})$, define

$$\ell_s(x) = x_1 + \sum_{j=2}^{n} s_{j-1} x_j, \quad g_s = \mathbf{1}_{\ell_s \neq 0}.$$

Then $\bigoplus_{s \in \{1,2\}^{n-1}} g_s(\ell_s(x)) = \mathbf{1}_T(x)$ in $\mathbb{F}_2$.

7.2 Solution Structure

Theorem 7.1. *For $q = 3$: every weight-2^{n-1} gate combination representing $\mathbf{1}_T$ uses the 2^{n-1} linear forms $\{\ell_s : s \in (\mathbb{F}_3^*)^{n-1}\}$ (up to a choice of distinguished coordinate). The only freedom is in the gate function: each form ℓ_s can be paired with either $\mathbf{1}_{\ell_s \neq 0}$ or $\mathbf{1}_{\ell_s = 0}$, subject to an even-parity constraint. This gives $2^{2^{n-1}-1}$ solutions.*

Proof. On the torus $T = (\mathbb{F}_3^*)^n$, the functions $\mathbf{1}_{\ell_s \neq 0}|_T$ and $\mathbf{1}_{\ell_s = 0}|_T$ are complementary: their XOR is the constant function 1 on T. Flipping the gate function for ℓ_s changes the contribution on T by $\mathbf{1}|_T$, while preserving the vanishing on Z. Flipping an even number of gate functions preserves the global XOR being $\mathbf{1}_T$, giving $2^{2^{n-1}-1}$ valid assignments. □

7.3 The ψ-Independence Theorem

The construction uses 2^{n-1} canonical gates $g_s = \mathbf{1}_{\ell_s \neq 0}$. The following theorem shows these are linearly independent, so the canonical construction is locally optimal.

Definition 7.2. *For $m \geq 0$ and $s = (s_1, \dots, s_m) \in \{1, 2\}^m$, define $\psi_s : \mathbb{F}_3^{m+1} \to \mathbb{F}_2$ by*

$$\psi_s(x_1, \dots, x_{m+1}) = \mathbf{1}_{x_1 + \sum_{k=1}^{m} s_k x_{k+1} \equiv 0 \pmod 3}.$$

Theorem 7.3 (ψ-Independence)**.** *For all $m \geq 0$, the 2^m functions $\{\psi_s : s \in \{1, 2\}^m\}$ satisfy:*

(a) They are $\mathbb{F}_2$-linearly independent on $\mathbb{F}_3^{m+1}$.

(b) The constant function 1 is not in their $\mathbb{F}_2$-span.

Proof. By strong induction on m, proving (a) and (b) simultaneously.

Base case ($m = 0$). The single function $\psi(x_1) = \mathbf{1}_{x_1 = 0}$ is nonzero, hence independent. And $\psi \neq 1$ since $\psi(1) = 0$.

Inductive step. Assume both statements hold for all $m' < m$. Suppose $\bigoplus_{s \in S} \psi_s = 0$ for some nonempty $S \subseteq \{1, 2\}^m$.

Step 1: Restrict to $\{x_{m+1} = 0\}$. On this slice, $\psi_{(s', s_m)}$ reduces to $\psi_{s'}^{(m-1)}$, independently of s_m. Write $\varepsilon_j(s') = \mathbf{1}_{(s', j) \in S}$ for $j \in \{1, 2\}$. The restricted equation becomes $\bigoplus_{s'} (\varepsilon_1(s') \oplus \varepsilon_2(s')) \psi_{s'}^{(m-1)} = 0$. By induction (a) for $m - 1$, we conclude $\varepsilon_1(s') = \varepsilon_2(s')$ for all s'.

Define $S_0 = \{s' \in \{1, 2\}^{m-1} : (s', 1) \in S\} = \{s' : (s', 2) \in S\}$.

Step 2: Restrict to $\{x_{m+1} = 1\}$. On this slice, $\psi_{(s',1)}|_{x_{m+1}=1} \oplus \psi_{(s',2)}|_{x_{m+1}=1} = \mathbf{1}_{\ell_{s'} \neq 0} = 1 \oplus \psi_{s'}^{(m-1)}$. Summing over $s' \in S_0$:

$$\bigoplus_{s' \in S_0} (1 \oplus \psi_{s'}^{(m-1)}) = 0, \quad \text{giving} \quad \bigoplus_{s' \in S_0} \psi_{s'}^{(m-1)} = |S_0| \mod 2.$$

If $|S_0|$ is even, induction (a) gives $S_0 = \emptyset$. If $|S_0|$ is odd, induction (b) is contradicted. Either way $S = \emptyset$, proving (a). Part (b) follows similarly by restricting the equation $\bigoplus_S \psi_s = 1$ to $\{x_{m+1} = 0\}$ and applying induction (b). □

Corollary 7.4. *The* 2^{n-1} *canonical gates* $g_s = \mathbf{1}_{\ell_s \neq 0}$ *for* $s \in (\mathbb{F}_3^*)^{n-1}$ *are* $\mathbb{F}_2$*-linearly independent as functions on* $\mathbb{F}_3^n$*.*

8 Vandermonde Induction for $q = 3$

For the special case $q = 3$, we give an alternative lower bound proof that establishes a stronger result: an $\mathbb{F}_4$-Fourier support theorem for all functions supported on T.

8.1 Coordinate Slicing

Write $f : \mathbb{F}_3^n \to \mathbb{F}_4$ and define $f_1(x') = f(1, x')$, $f_2(x') = f(2, x')$ for $x' \in \mathbb{F}_3^{n-1}$. Then

$$\widehat{f}(\alpha_1, \alpha') = \omega^{-\alpha_1} \widehat{f_1}(\alpha') + \omega^{\alpha_1} \widehat{f_2}(\alpha'),$$

since $-2\alpha_1 = \alpha_1$ in $\mathbb{F}_3$, where $\omega = e^{2\pi i/3}$.

For fixed α', the three values $\widehat{f}(0, \alpha'), \widehat{f}(1, \alpha'), \widehat{f}(2, \alpha')$ are the entries of

$$\begin{pmatrix} 1 & 1 \\ \omega^2 & \omega \\ \omega & \omega^2 \end{pmatrix} \begin{pmatrix} \widehat{f_1}(\alpha') \\ \widehat{f_2}(\alpha') \end{pmatrix}.$$

Since this 3×2 Vandermonde matrix over $\mathbb{F}_4$ has every 2×2 submatrix nonsingular:

Lemma 8.1 (Slicing Lemma)**.** *For each* $\alpha' \in \mathbb{F}_3^{n-1}$*:*

(a) If $\widehat{f_1}(\alpha') = \widehat{f_2}(\alpha') = 0$*, then* $\widehat{f}(\alpha_1, \alpha') = 0$ *for all* α_1*.*

(b) If exactly one is nonzero, then $\widehat{f}(\alpha_1, \alpha') \neq 0$ *for all* α_1*.*

(c) If both are nonzero, then $\widehat{f}(\alpha_1, \alpha') = 0$ *for exactly one* α_1*.*

Theorem 8.2 ($\mathbb{F}_4$-Support Theorem)**.** *Let* $f : \mathbb{F}_3^n \to \mathbb{F}_2$ *be nonzero with* $\mathrm{supp}(f) \subseteq T$*. Then* $|\mathrm{supp}(\widehat{f})| \geq 2^n$*.*

Proof. By induction on n. The base case $n = 1$ is verified directly. For the inductive step, let $K_i = \mathrm{supp}(\widehat{f_i})$ with $k_i = |K_i|$. By Lemma 8.1:

$$|\mathrm{supp}(\widehat{f})| = 3|K_1 \triangle K_2| + 2|K_1 \cap K_2| \geq 2\max(k_1, k_2).$$

Since each nonzero f_i satisfies $\mathrm{supp}(f_i) \subseteq T' = (\mathbb{F}_3^*)^{n-1}$, induction gives $k_i \geq 2^{n-1}$, yielding $|\mathrm{supp}(\widehat{f})| \geq 2 \cdot 2^{n-1} = 2^n$. □

Corollary 8.3. $t(2, 3, n) \geq 2^{n-1}$*.*

Proof. For $f \in C \setminus C_0$, Theorem 8.2 gives $|\mathrm{supp}(\widehat{f})| \geq 2^n$, hence $|\mathrm{supp}(\widehat{f}) \setminus \{0\}| \geq 2^n - 1$. Since each gate covers at most one Frobenius pair, $2w \geq 2^n - 1$, giving $w \geq 2^{n-1}$. □

8.2 Failure for $q \geq 5$

Remark 8.4 (Failure for $q \geq 5$)**.** *The $\mathbb{F}_{16}$-Fourier support theorem does not hold for $q = 5$. Exhaustive computation for $n = 2$ reveals:*

- *The minimum Fourier support for a nonzero $f : \mathbb{F}_5^2 \to \mathbb{F}_2$ with $\operatorname{supp}(f) \subseteq T$ is $|\operatorname{supp}(\widehat{f})| = 8$, not $4^2 = 16$.*
- *The 10 worst-case functions have Hamming weight 8 or 12 and their Fourier support covers exactly 2 of the 4 Frobenius orbits.*
- *Several of these functions are coset indicators of index-2 subgroups of $(\mathbb{F}_5^*)^2 \cong (\mathbb{Z}/4\mathbb{Z})^2$.*

The obstruction is the Vandermonde structure: the 5×4 Vandermonde matrix V over $\mathbb{F}_{16}$ with nodes at the 5th roots of unity has 4×4 submatrices that can be singular (a degree-3 polynomial over $\mathbb{F}_{16}$ can vanish at up to 3 of the 5 nodes). The coordinate slicing induction yields only $|\operatorname{supp}(\widehat{f})| \geq 2 \cdot 4^{n-1}$, a factor of 2 short of the needed 4^n.

This failure motivated the orbit counting argument of Section 5, which sidesteps the Fourier support theorem entirely.

9 Cohomological Interpretation

The gate complexity admits a striking cohomological interpretation: it equals the Frobenius trace on compactly supported étale cohomology. Throughout this section, $H_c^*(X, \mathbb{F}_p)$ denotes $H^*_{c,\text{ét}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{F}_p)$, the compactly supported étale cohomology of the base change to the algebraic closure.

9.1 Two Orbit Spaces

The multiplicative group $\mathbb{F}_q^*$ acts diagonally on $\mathbb{F}_q^n \setminus \{0\}$:

$$t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n).$$

This action restricts to the torus $T = (\mathbb{F}_q^*)^n$. The two relevant orbit spaces are:

1. **The torus quotient:** $T/\mathbb{F}_q^* \cong (\mathbb{F}_q^*)^{n-1}$ via $(x_1, \ldots, x_n) \mapsto (x_2/x_1, \ldots, x_n/x_1)$.

$$|T(\mathbb{F}_q)/\mathbb{F}_q^*| = \frac{(q-1)^n}{q-1} = (q-1)^{n-1}.$$

2. **Projective space:** $(\mathbb{F}_q^n \setminus \{0\})/\mathbb{F}_q^* = \mathbb{P}^{n-1}(\mathbb{F}_q)$.

$$|\mathbb{P}^{n-1}(\mathbb{F}_q)| = \frac{q^n - 1}{q-1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

The torus quotient embeds in projective space: $(\mathbb{F}_q^*)^{n-1} \cong T/\mathbb{F}_q^* \hookrightarrow \mathbb{P}^{n-1}$. The complement is the coordinate hyperplane arrangement.

9.2 The Unified Cohomological Theorem

Let $S = \mathrm{supp}(\widehat{\mathbf{1}_T}) \subseteq \mathbb{F}_q^n$ denote the Fourier support of the torus indicator.

Theorem 9.1 (Gate Complexity as Frobenius Trace)**.** *For all primes p and prime powers q with $\mathrm{char}(\mathbb{F}_q) \neq p$:*

$$t(p,q,n) = |S/\mathbb{F}_q^*| = \mathrm{Tr}(\mathrm{Frob}_q \mid H^*(S/\mathbb{F}_q^*, \mathbb{F}_p)),$$

where:

Condition	*Support S*	*Orbit space $S/\mathbb{F}_q^*$*	$t(p,q,n)$
$p \mid (q-1)$	T	$\mathbb{G}_m^{n-1}$	$(q-1)^{n-1}$
$p \nmid (q-1)$	$\mathbb{F}_q^n \setminus \{0\}$	$\mathbb{P}^{n-1}$	$(q^n-1)/(q-1)$

Proof. We prove each case separately.

Case 1: $p \mid (q-1)$**.** By Theorem 4.2, $\mathrm{supp}(\widehat{\mathbf{1}_T}) = T$, so $S/\mathbb{F}_q^* = T/\mathbb{F}_q^* \cong \mathbb{G}_m^{n-1}$.

The compactly supported cohomology of $\mathbb{G}_m$ over $\mathbb{F}_q$ with $\mathbb{F}_p$-coefficients is:

$$H_c^i(\mathbb{G}_m, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & i = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

with Frobenius eigenvalue 1 on H_c^1 and eigenvalue q on H_c^2. The alternating trace is:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid H_c^*(\mathbb{G}_m, \mathbb{F}_p)) = -1 + q = q - 1 = |\mathbb{G}_m(\mathbb{F}_q)|.$$

By Künneth, for $\mathbb{G}_m^{n-1}$:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid H_c^*(\mathbb{G}_m^{n-1}, \mathbb{F}_p)) = (q-1)^{n-1} = t(p,q,n).$$

Case 2: $p \nmid (q-1)$**.** By Theorem 4.2, $\mathrm{supp}(\widehat{\mathbf{1}_T}) = \mathbb{F}_q^n \setminus \{0\}$, so $S/\mathbb{F}_q^* = \mathbb{P}^{n-1}$.

The cohomology of projective space over $\mathbb{F}_q$ is:

$$H^k(\mathbb{P}^{n-1}, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & k = 0, 2, 4, \ldots, 2(n-1) \\ 0 & \text{otherwise} \end{cases}$$

with Frobenius eigenvalue $q^{k/2}$ on H^k. The trace is:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid H^*(\mathbb{P}^{n-1}, \mathbb{F}_p)) = \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1} = |\mathbb{P}^{n-1}(\mathbb{F}_q)| = t(p,q,n).$$

□

9.3 Geometric Interpretation of the Dichotomy

Remark 9.2 (Cohomological Origin of the Dichotomy)**.** *The dichotomy $p \mid (q-1)$ vs. $p \nmid (q-1)$ has a clean cohomological explanation:*

- *The Frobenius eigenvalues on $H^*(\mathbb{P}^{n-1})$ are $1, q, q^2, \ldots, q^{n-1}$.*
- *When $p \mid (q-1)$: $q \equiv 1 \pmod{p}$, so all eigenvalues collapse to 1 in $\mathbb{F}_p$. The "boundary" cohomology (from $Z = \mathbb{P}^{n-1} \setminus \mathbb{G}_m^{n-1}$) becomes invisible mod p.*
- *When $p \nmid (q-1)$: the eigenvalues $1, q, q^2, \ldots$ remain distinct in $\mathbb{F}_p$, and the full projective space contributes.*

10 Depth-3 Circuits: Escaping the Exponential Barrier

The exponential lower bound of Theorem 1.1 applies to depth-2 circuits. A natural question is whether increased depth can circumvent this barrier. We show that depth-3 suffices to reduce the gate complexity from exponential to linear.

10.1 The Depth-3 Construction

Theorem 10.1. *For $n < p$, the torus indicator $\mathbf{1}_T$ can be computed by a depth-3 circuit with $n+1$ gates.*

Proof. We construct a two-layer circuit:

Layer 1 (n gates): For each coordinate $i \in [n]$, define the gate

$$b_i = g_i(\ell_i(x)) \in \mathbb{F}_p$$

where $\ell_i(x) = x_i$ (the i-th coordinate projection) and $g_i : \mathbb{F}_q \to \mathbb{F}_p$ is defined by

$$g_i(v) = \begin{cases} 1 & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}$$

Thus $b_i = \mathbf{1}[x_i \neq 0] \in \{0,1\} \subset \mathbb{F}_p$.

Layer 2 (1 gate): The intermediate values $(b_1, \ldots, b_n) \in \mathbb{F}_p^n$ are fed into a single gate

$$h\left(\sum_{i=1}^{n} b_i\right)$$

where $h : \mathbb{F}_p \to \mathbb{F}_p$ is defined by

$$h(s) = \begin{cases} 1 & \text{if } s = n \\ 0 & \text{otherwise} \end{cases}$$

Correctness: We have $x \in T$ if and only if all $b_i = 1$, which occurs if and only if $\sum_i b_i = n$. The condition $n < p$ ensures that the sum $\sum_i b_i$ equals the integer n in $\mathbb{F}_p$ (no wraparound), so $h(\sum_i b_i) = 1$ if and only if all coordinates are nonzero.

The total gate count is $n + 1$. □

Remark 10.2. *The constraint $n < p$ is necessary for the construction. When $n \geq p$, the sum $\sum_i b_i$ can equal $n \mod p$ without all $b_i = 1$, breaking correctness.*

10.2 The Depth Gap

Corollary 10.3 (Exponential Depth Gap)**.** *For fixed p and q with $p \mid (q-1)$ and $n < p$:*

$$\frac{t_2(p,q,n)}{t_3(p,q,n)} = \frac{(q-1)^{n-1}}{n+1} = \Omega\left(\frac{(q-1)^{n-1}}{n}\right).$$

The depth-2 complexity is exponential while depth-3 is linear—an exponential separation.

10.3 Why the Fourier Method Fails at Depth 3

The lower bound of Section 5 relied on covering the Fourier support with lines. This argument is inherently depth-2: it exploits the fact that each depth-2 gate has Fourier support on a single line.

At depth 3, intermediate values can be combined nonlinearly before the final output. The Fourier support of a depth-3 circuit is no longer constrained to a union of lines—the intermediate layer "mixes" Fourier modes in a way that defeats the covering argument.

Open Problem 10.4. *Determine the depth-3 gate complexity $t_3(p, q, n)$ for $n \geq p$. Our construction requires $n < p$; when $n \geq p$, is $t_3(p, q, n)$ still $O(n)$, or does it grow faster?*

11 Connection to the Steinberg Polynomial

The gate complexity model specializes, at $n = 2$, to a weighted random walk on the projective line $\mathbb{P}^1(\mathbb{F}_p)$ whose spectral theory reveals deep arithmetic structure. We describe this connection, developed in companion work [10], and then discuss the broader building-theoretic context.

11.1 The $n = 2$ Specialization

When $n = 2$ and $q = 2$, the gate complexity setup reduces to a single linear form $\ell(x_1, x_2) = x_1 + sx_2$ for $s \in \mathbb{F}_p^*$. The states $[1 : s]$ for $s \in \mathbb{F}_p$ together with $[0 : 1] = \infty$ form the projective line $\mathbb{P}^1(\mathbb{F}_p)$, and the weights

$$w_r = \frac{q^{p-r}}{q^p - 1}$$

define a transition matrix P on $\mathbb{P}^1(\mathbb{F}_p)$ encoding the continued fraction dynamics of the spanning tree problem. The **Steinberg representation** $\mathrm{St}_p = \{f : \mathbb{P}^1(\mathbb{F}_p) \to \mathbb{C} : \sum_x f(x) = 0\}$ is the p-dimensional irreducible summand of the permutation representation, and P preserves the decomposition $V = \mathbf{1} \oplus \mathrm{St}_p$.

11.2 The Steinberg Polynomial and Its Arithmetic

Define the **Steinberg polynomial** $n_p(q) = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}[q]$. In companion work [10], we discover that $n_p(q)$ has remarkable arithmetic structure, verified for all 24 primes $p \leq 97$:

1. **Endoscopic decomposition.** $n_p(q) = n_p^{\mathrm{GL}_2}(q) - \left(\frac{-2}{p}\right) \cdot n_p^T(q)$, where $\left(\frac{-2}{p}\right)$ is the Legendre symbol and $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$. This has the shape of the endoscopic decomposition in the Langlands program for GL_2, with the specific quadratic field $\mathbb{Q}(\sqrt{-2})$ emerging from the data.

2. **Weight dichotomy.** Every root of $n_p(q)$ has absolute value either 1 (weight 0) or $1/\sqrt{2}$ (weight -1). The endoscopic components $n_p^{\mathrm{GL}_2}$ and n_p^T are each pure of weight 0; the weight-(-1) content arises only from their combination.

3. **Motivic factorization.** Each weight-(-1) factor is a Frobenius determinant $\det(1 - q \cdot \mathrm{Frob} \mid h^1(A))$ for a CM abelian variety $A/\mathbb{F}_2$. This yields:

$$n_p(q) = \varepsilon_p \cdot (q-1)^{a_p}(q+1)^{b_p} \cdot \prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)).$$

4. **Cousin prime rule.** The exponent $m(p)$ in the leading coefficient $2^{m(p)} \pm 1$ of the fiber polynomial increments at each successive prime, except that cousin prime pairs (primes differing by 4) share the same exponent.

11.3 The Building-Theoretic Context

For a reductive group G over a local field K, the Bruhat-Tits building $\mathcal{T}(G)$ is a simplicial complex whose vertex links are copies of $\mathbb{P}^1(\mathbb{F}_p)$. Schneider–Stuhler [6] established that $\mathrm{St}_G = H_c^\ell(\mathcal{T}(G); \mathbb{C})$ where $\ell = \mathrm{rank}_K(G)$, and the recent work of Bezrukavnikov–Kazhdan–Varshavsky [14] shows that the Steinberg character plays a privileged role in depth-zero harmonic analysis.

The endoscopic decomposition of $n_p(q)$ has the formal shape of results proved rigorously at depth zero by DeBacker–Reeder [11], Kaletha [12], and Kazhdan–Varshavsky [13]. The key open question bridging this paper to the Langlands program is:

Question 11.1. *Can the transition matrix P (defined by the weights w_r) be identified with a specific element of the Iwahori–Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$? If so, the endoscopic decomposition of $n_p(q)$ would follow from existing theorems.*

More broadly, the appearance of $\mathbb{P}^{n-1}(\mathbb{F}_q)$ and $\mathbb{G}_m^{n-1}$ in our gate complexity formula suggests a higher-rank generalization:

Question 11.2. *Is there a building-theoretic framework in which the gate complexity $t(p, q, n)$ appears as a cohomological invariant associated to GL_n, with the dichotomy $p \mid (q-1)$ vs. $p \nmid (q-1)$ reflecting the apartment-vs-building distinction?*

12 Discussion

12.1 Comparison Across q

	$q = 2$	$q = 3$	$q = 5$	general q
Formula (when $p \mid (q-1)$)	—	2^{n-1}	4^{n-1}	$(q-1)^{n-1}$
Formula (when $p \nmid (q-1)$)	$2^n - 1$	$(3^n - 1)/2$	$(5^n - 1)/4$	$(q^n - 1)/(q-1)$
Growth base	2	2 or 3/2	4 or 5/4	$q-1$ or q
$\lvert T\rvert$	1	2^n	4^n	$(q-1)^n$

The growth base $q-1$ (when $p \mid (q-1)$) reflects the multiplicative group $\mathbb{F}_q^*$. The gate complexity $t(p, q, n)$ equals the number of Frobenius orbits that must be covered, divided by the number of orbits per $\mathbb{F}_q$-line.

12.2 Phase Transition at $p \mid (q-1)$

The ratio of the two formulas is

$$\frac{(q^n - 1)/(q-1)}{(q-1)^{n-1}} = \frac{1 + q + \cdots + q^{n-1}}{(q-1)^{n-1}} \sim \frac{q^{n-1}}{(q-1)^{n-1}} \to \left(\frac{q}{q-1}\right)^{n-1}$$

as $n \to \infty$. For small q, this ratio is significant: for $q = 3$, the jump from $p = 2$ (giving 2^{n-1}) to $p = 5$ (giving $(3^n - 1)/2$) is a factor of roughly $(3/2)^{n-1}$.

12.3 Connections to $\mathsf{AC}^0[6]$

In a depth-2 circuit with MOD-q bottom gates and a MOD-p top gate, each bottom gate computes $\ell_i(u) \mod q$ and the top gate applies an arbitrary $g : \mathbb{F}_q \to \mathbb{F}_p$. Theorem 1.1 shows that any such circuit computing $\mathbf{1}_T$ requires $\geq (q-1)^{n-1}$ or $\geq (q^n - 1)/(q-1)$ bottom gates—an exponential lower bound for this restricted model.

However, Theorem 10.1 shows that depth-3 escapes this barrier with $O(n)$ gates. The central open problem for $\mathsf{AC}^0[6]$ is to understand how cross-characteristic interactions compose across multiple layers—our depth-3 result shows that even one additional layer can dramatically reduce complexity.

12.4 Further Directions

1. **Exact depth-3 complexity.** Determine $t_3(p, q, n)$ precisely for all n, including the regime $n \geq p$.

2. **Cross-characteristic codes.** The quotient C/C_0 is a linear code over $\mathbb{F}_p$ whose structure arises from $\mathbb{F}_q$. It would be interesting to understand its basic properties (minimum distance, weight distribution) and whether standard coding-theoretic tools adapt to this cross-characteristic setting.

3. **The Hecke algebra bridge.** The most promising route to connecting the gate complexity framework with the Langlands program is to answer Question 11.1: identify the transition matrix P as an element of the Iwahori–Hecke algebra. If successful, the endoscopic decomposition of the Steinberg polynomial would follow from existing theorems of Kaletha [12] and Kazhdan–Varshavsky [13].

4. **Higher-rank Steinberg polynomials.** The $n = 2$ case of our setup produces the Steinberg polynomial $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ with its endoscopic decomposition and motivic factorization [10]. For $n \geq 3$, the analogous determinant $\det(I - P|_{\mathrm{St}_p^{(n)}})$ should produce polynomials whose arithmetic encodes endoscopic data for GL_n. The Kang–Li chamber zeta functions for PGL_3 [15] provide a natural comparison.

5. **Quantum codes.** When $p \mid (q-1)$, the Fourier self-duality $\widehat{\mathbf{1}_T} = (-1)^n \mathbf{1}_T$ parallels structures in quantum error correction. The gate code C/C_0 may admit a CSS-type quantum lift, with code parameters controlled by the gate complexity.

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