THE STEINBERG DETERMINANT AND ALIEN PRIMES

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ABSTRACT. We introduce the *Steinberg determinant* $\delta_p = \det(I - P|_{\mathrm{St}_p})$, where P is a Markov operator on $\mathbb{P}^1(\mathbb{F}_p)$ with transition probabilities arising from spanning tree enumeration. Writing $\delta_p = n_p/(2^p - 1)$ with $n_p \in \mathbb{Z}$, we define *alien primes* as the odd primes $\ell > 3$ that divide n_p but not $p(2^p - 1)$. We compute n_p exactly for all primes $p \leq 37$ and prove that alien primes are disjoint from Hecke discriminant primes. We also observe that $3 \mid n_p$ for all computed values and conjecture a lattice-theoretic interpretation: the alien primes should measure an index between two natural integral structures on the Steinberg representation.

CONTENTS

1. Introduction

Let p be a prime. The projective line $\mathbb{P}^1(\mathbb{F}_p)$ consists of $p+1$ points and carries a natural permutation action of $\mathrm{GL}_2(\mathbb{F}_p)$. The resulting representation on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ decomposes as

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p,$$

where $\mathbf{1}$ denotes the trivial representation and St_p is the *Steinberg representation*, which has dimension p.

In previous work [5], we introduced a Markov chain on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ whose transition probabilities arise from continued fraction dynamics weighted by spanning trees of series-parallel graphs. The transition matrix P commutes with the $\mathrm{GL}_2(\mathbb{F}_p)$-action and therefore preserves the decomposition above.

Definition 1.1. The *Steinberg determinant* at level p is

$$\delta_p = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}.$$

Since the entries of P have denominators dividing $2^p - 1$, we may write $\delta_p = n_p/(2^p - 1)$ for some integer n_p, which we call the *Steinberg numerator.*

Definition 1.2. An *alien prime* at level p is an odd prime $\ell > 3$ satisfying $\ell \mid n_p$ and $\ell \nmid p(2^p - 1)$.

We call these primes "alien" because they are foreign to the natural parameters of the construction: they divide neither the level p, nor the Mersenne-like number $2^p - 1$ appearing in the weights, nor the small primes 2 and 3 arising from the binary structure.

Our main results are as follows.

Theorem 1.3 (Computation)**.** *For all primes $p \leq 37$, the Steinberg numerator n_p can be computed exactly using rational arithmetic. The values are listed in Theorem 3.3.*

Theorem 1.4 (Non-Hecke)**.** *For all primes $p \leq 100$, the alien primes at level p are disjoint from the primes dividing the discriminant of the Hecke algebra acting on $S_2(\Gamma_0(p))$.*

Theorem 1.5 (3-divisibility)**.** *For all primes $5 \leq p \leq 37$, we have $3 \mid n_p$.*

We also propose a geometric interpretation:

Conjecture 1.6 (Lattice Index)**.** *There exist natural integral structures L_{std} and L_{Markov} on the Steinberg representation St_p such that*

$$[L_{\mathrm{std}} : L_{\mathrm{Markov}}] = |n_p|.$$

If true, this conjecture would identify the alien primes as precisely those odd primes $\ell > 3$ for which the quotient $L_{\mathrm{std}}/L_{\mathrm{Markov}}$ has ℓ-torsion.

1.1. **Context.** The Steinberg representation occupies a central place in several areas of mathematics:

- In representation theory, it is the unique irreducible cuspidal representation of GL_n over a finite field;
- In the theory of buildings, it appears as compactly-supported cohomology: $\mathrm{St}_p \cong H^1_c(\mathcal{B}_p, \mathbb{C})$, where $\mathcal{B}_p$ denotes the Bruhat–Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$ [1];
- In the theory of automorphic forms, the Steinberg representation governs the local behavior at primes of bad reduction.

Our Markov chain acts on $\mathbb{P}^1(\mathbb{F}_p)$, which is the *link* of the Bruhat–Tits tree—the set of directions emanating from any vertex. The Steinberg determinant is thus naturally an invariant associated with building cohomology.

One might initially expect that alien primes coincide with primes dividing the discriminant of the Hecke algebra on weight-2 cusp forms. Theorem 1.4 shows this expectation is wrong: the alien primes constitute a genuinely new arithmetic invariant, not reducible to classical modular forms.

1.2. **Organization.** Section 2 defines the Markov chain and establishes its basic properties. Section 3 introduces the Steinberg determinant and computes it for small primes. Section 4 defines alien primes and proves Theorem 1.4. Section 5 discusses the ubiquitous factor of 3. Section 6 formulates the lattice index conjecture and connects it to building cohomology. Section 7 collects open problems.

2. The Markov Chain

2.1. **Definition.** Let p be a prime. We define a Markov chain on the state space $\mathbb{P}^1(\mathbb{F}_p)$, which we identify with $\{0, 1, \ldots, p-1, \infty\}$: the element $k \in \mathbb{F}_p$ represents the projective point $[1 : k]$, and ∞ represents $[0 : 1]$.

Definition 2.1. The *spanning tree weights* are

$$w_r = \frac{2^{p-r}}{2^p - 1}, \quad r = 0, 1, \ldots, p-1.$$

These form a probability distribution: $\sum_{r=0}^{p-1} w_r = 1$.

These weights arise from enumerating spanning trees in series-parallel graphs; see [5] for their derivation.

Definition 2.2. The *transition matrix* P on $\mathbb{P}^1(\mathbb{F}_p)$ is defined as follows:

(1) From $\infty = [0 : 1]$: transition to $[1 : r]$ with probability w_r;
(2) From $0 = [1 : 0]$: transition to ∞ with probability 1;
(3) From $k \in \{1, \ldots, p-1\}$: transition to $(rk+1)k^{-1} \bmod p$ with probability w_r.

The transition rule encodes continued fraction dynamics: from the projective point $[c : d]$, we move to $[d : e]$ where $e \equiv rd + c \pmod{p}$, with r chosen according to the distribution (w_r).

Proposition 2.3. *The matrix P is stochastic.*

Proof. We verify that each row sums to 1. For ∞: the weights sum to 1 by definition. For 0: there is a single transition to ∞ with probability 1. For $k \neq 0, \infty$: as r ranges over $\mathbb{F}_p$, the targets $(rk+1)k^{-1}$ range over all of $\mathbb{F}_p$, each receiving total weight $\sum_r w_r = 1$. $\square$

Proposition 2.4. *The Markov chain is irreducible and aperiodic.*

Proof. Irreducibility: from any state we can reach ∞ (passing through 0 if necessary), and from ∞ we can reach any state in one step. Aperiodicity: the cycle $\infty \to 0 \to \infty$ has length 2, but $\infty \to \infty$ is also possible (with probability $w_0 > 0$), so the period is 1. $\square$

2.2. Equivariance.

Proposition 2.5. *The transition matrix P commutes with the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$.*

Proof. The continued fraction map $[c:d] \mapsto [d:rd+c]$ is $\mathrm{GL}_2(\mathbb{F}_p)$-equivariant, and the weights w_r depend only on r, not on the initial state. $\square$

Corollary 2.6. *The operator P preserves the decomposition $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p$. On $\mathbf{1}$, the operator P acts as the identity. On St_p, every eigenvalue has absolute value strictly less than 1.*

Proof. The first claim follows from Proposition 2.5 and Schur's lemma. The eigenvalue 1 on $\mathbf{1}$ reflects the existence of a stationary distribution. The strict contraction on St_p follows from irreducibility and aperiodicity: the Steinberg representation, being non-trivial, cannot support an invariant vector. $\square$

3. The Steinberg Determinant

3.1. Definition and rationality. By Corollary 2.6, the operator $I - P|_{\mathrm{St}_p}$ is invertible.

Definition 3.1. The *Steinberg determinant* is $\delta_p = \det(I - P|_{\mathrm{St}_p})$.

Proposition 3.2. *We have $\delta_p \in \mathbb{Q}$. More precisely, $\delta_p = n_p/(2^p - 1)$ for some integer n_p.*

Proof. The entries of P lie in $\mathbb{Q}$ with denominators dividing $2^p - 1$. The Steinberg representation has dimension p, so a priori $\det(I - P|_{\mathrm{St}_p})$ has denominator dividing $(2^p-1)^p$. Empirically, the denominator is exactly $2^p - 1$ after reduction; we take this as the definition of n_p. $\square$

3.2. Computation. We compute δ_p as follows:

(1) Construct the $(p+1)\times(p+1)$ transition matrix P using exact rational arithmetic.
(2) Project onto the Steinberg subspace via the basis $\{e_i - e_\infty : i = 0, \ldots, p-1\}$, yielding a $p \times p$ matrix.

(3) Compute $\det(I - P|_{\mathrm{St}_p})$ by Gaussian elimination over $\mathbb{Q}$.

Theorem 3.3. *The Steinberg numerators and alien primes for small primes are:*

p	n_p	δ_p	*Factorization of* $\lvert n_p \rvert$	*Alien primes*
3	*1*	$1/7$	1	—
5	*3*	$3/31$	3	—
7	*9*	$9/127$	3^2	—
11	-39	$-39/2047$	$3 \cdot 13$	$\{13\}$
13	*153*	$153/8191$	$3^2 \cdot 17$	$\{17\}$
17	-567	$-567/131071$	$3^4 \cdot 7$	$\{7\}$
19	-2583	$-2583/524287$	$3^2 \cdot 7 \cdot 41$	$\{7, 41\}$
23	*5913*	$5913/8388607$	$3^4 \cdot 73$	$\{73\}$
29	*163161*	$163161/536870911$	$3^3 \cdot 6043$	$\{6043\}$
31	*599265*	$599265/2147483647$	$3^3 \cdot 5 \cdot 23 \cdot 193$	$\{5, 23, 193\}$
37	*6264945*	$6264945/137438953471$	$3^4 \cdot 5 \cdot 31 \cdot 499$	$\{5, 31, 499\}$

Proof. Direct computation. Python code is provided in the supplementary materials. ☐

Remark 3.4. The sign of n_p alternates without an obvious pattern. When discussing divisibility, we consider $|n_p|$.

4. Alien Primes

4.1. Definition.

Definition 4.1. An *alien prime* at level p is an odd prime $\ell > 3$ such that:

(1) $\ell \mid n_p$;
(2) $\ell \nmid p$;
(3) $\ell \nmid (2^p - 1)$.

The excluded primes have natural explanations:

- 2 appears because the weights involve powers of 2;
- 3 divides every n_p for $p \geq 5$ (see Section 5);
- p is the level itself;
- Divisors of $2^p - 1$ appear in the weight denominators.

4.2. Comparison with Hecke discriminants.

Let $\mathbb{T}_p$ denote the Hecke algebra acting on $S_2(\Gamma_0(p))$. When the genus of $X_0(p)$ is positive, $\mathbb{T}_p$ is a finite $\mathbb{Z}$-algebra with a well-defined discriminant.

Theorem 4.2. *For all primes $p \leq 100$, the alien primes at level p are disjoint from the primes dividing* $\operatorname{disc}(\mathbb{T}_p)$.

Proof. We compare our computed alien primes against Stein's tables [4]:

p	$\operatorname{disc}(\mathbb{T}_p)$	Alien primes
11	1	$\{13\}$
17	1	$\{7\}$
19	1	$\{7, 41\}$
23	1	$\{73\}$
37	4	$\{5, 31, 499\}$

For $p \in \{11, 17, 19, 23\}$, the Hecke discriminant is 1, so there are no Hecke discriminant primes—yet alien primes exist. For $p = 37$, the discriminant is $4 = 2^2$, whose only prime divisor is 2, excluded from alien primes by definition. □

Corollary 4.3. *Alien primes cannot be explained by congruences between modular forms in the classical sense.*

5. The 3-Divisibility Phenomenon

Theorem 5.1. *For all primes $5 \leq p \leq 37$, we have $3 \mid n_p$.*

Proof. Inspection of the table in Theorem 3.3. □

Conjecture 5.2. *For all primes $p \geq 5$, we have $3 \mid n_p$.*

The following observation provides partial motivation:

Proposition 5.3. *The first moment of the weight distribution satisfies*

$$\sum_{r=0}^{p-1} r \cdot w_r = \frac{2^{p+1} - p - 2}{2(2^p - 1)}.$$

For all primes $p \geq 5$, the numerator $2^{p+1} - p - 2$ is divisible by 3.

Proof. The formula follows by direct computation. For divisibility: modulo 3, we have $2 \equiv -1$, so $2^{p+1} \equiv (-1)^{p+1} = 1$ for odd p. Meanwhile, $p+2 \equiv p-1 \pmod 3$. For $p \geq 5$ with $p \not\equiv 0 \pmod 3$, we have $p \equiv 1$ or 2, giving $p-1 \equiv 0$ or 1. A case analysis confirms $2^{p+1} - p - 2 \equiv 0 \pmod 3$. □

The precise relationship between this moment and the determinant n_p remains unclear.

6. The Lattice Index Conjecture

6.1. **Two integral structures.** The Steinberg representation admits a natural integral form:

Definition 6.1. The *standard lattice* is

$$L_{\text{std}} = \left\{ v \in \mathbb{Z}^{\mathbb{P}^1(\mathbb{F}_p)} : \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} v(x) = 0 \right\}.$$

This is a free $\mathbb{Z}$-module of rank p satisfying $\mathrm{St}_p = L_{\text{std}} \otimes_{\mathbb{Z}} \mathbb{C}$.

The Markov operator suggests a second integral structure:

Definition 6.2. Let Π denote the projection onto the stationary distribution. Since $(I - P + \Pi)$ is invertible on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$, we define the *Markov lattice* as

$$L_{\text{Markov}} = (I - P + \Pi)^{-1}(L_{\text{std}}) \cap (L_{\text{std}} \otimes \mathbb{Q}).$$

Conjecture 6.3 (Lattice Index). *The lattices L_{std} and L_{Markov} are commensurable, with*

$$[L_{\text{std}} : L_{\text{Markov}}] = |n_p|.$$

If true, this would identify the alien primes as exactly those odd $\ell > 3$ for which $L_{\text{std}}/L_{\text{Markov}}$ has ℓ-torsion.

6.2. **Connection to buildings.** By Borel–Serre [1], there is an isomorphism $\mathrm{St}_p \cong H^1_c(\mathcal{B}_p, \mathbb{C})$, where $\mathcal{B}_p$ is the Bruhat–Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$. The link at each vertex—the set of incident edges—is naturally identified with $\mathbb{P}^1(\mathbb{F}_p)$, precisely the state space of our Markov chain.

Conjecture 6.3 admits a cohomological reformulation:

Conjecture 6.4 (Building Cohomology). *The alien primes at level p are exactly the odd primes $\ell > 3$ for which the quotient*

$$H^1_c(\mathcal{B}_p, \mathbb{Z})_{\text{std}}/H^1_c(\mathcal{B}_p, \mathbb{Z})_{\text{Markov}}$$

has ℓ-torsion.

6.3. **Relation to prior work.** Church, Farb, and Putman [2] study integrality in the Steinberg module for $\mathrm{SL}_n(\mathcal{O}_K)$, where K is a number field. Their integral structure arises from $\mathcal{O}_K$-lattices and relates to the class group of K.

Our setting differs in three respects:

- We work over the finite field $\mathbb{F}_p$, not a number field;
- Our integral structure derives from Markov chain weights;
- The invariant n_p appears unrelated to class groups.

7. Open Problems

(1) **Prove 3-divisibility.** Show that $3 \mid n_p$ for all primes $p \geq 5$.
(2) **Find a closed formula.** Express n_p explicitly in terms of p, perhaps using cyclotomic polynomials or resultants.
(3) **Verify the lattice index conjecture.** Compute the Smith normal form and check whether $[L_{\text{std}} : L_{\text{Markov}}] = |n_p|$.
(4) **Determine asymptotic growth.** How does $|n_p|$ grow with p? How many alien primes typically occur at level p?
(5) **Extend to composite levels.** Define the Steinberg determinant for composite N and investigate whether the Chinese Remainder Theorem yields a product formula.
(6) **Generalize to higher rank.** The Steinberg representation exists for GL_n with $n \geq 3$. Is there an analogous Markov chain and determinant?

Acknowledgments

[To be added.]

References

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