Cohomological Self-Duality of the Torus Indicator: A Proof via Étale Cohomology

Yipin Wang*

Abstract

We prove that the self-duality of the torus indicator function under the $\mathbb{F}_{p^k}$-Fourier transform—the identity $\widehat{\mathbf{1}_T} = \mathbf{1}_T$ for $T = (\mathbb{F}_q^*)^n$ when $p \mid (q-1)$—is a consequence of Poincaré duality in étale cohomology combined with the self-duality of the algebraic torus as a group variety. The proof proceeds via Grothendieck's function-sheaf correspondence, identifying $\mathbf{1}_T$ with the sheaf $j_!\underline{\mathbb{F}_p}$ and the Fourier transform with the ℓ-adic Fourier-Deligne transform. The dichotomy between $p \mid (q-1)$ and $p \nmid (q-1)$ arises from the vanishing (or non-vanishing) of compactly supported cohomology of the multiplicative group with constant $\mathbb{F}_p$-coefficients.

2020 Mathematics Subject Classification. 11T24 (primary), 14F20, 11G25, 14G15 (secondary).

Keywords. Fourier transform, algebraic torus, étale cohomology, Poincaré duality, gate complexity, cross-characteristic.

Contents

*Department of Mathematics, University of Illinois at Urbana-Champaign. Email: yipinw2@illinois.edu

1 Introduction

Let p be a prime, q a prime power with $\mathrm{char}(\mathbb{F}_q) = r \neq p$, and $T = (\mathbb{F}_q^*)^n$ the algebraic torus. In the study of cross-characteristic gate complexity, the following identity plays a central role:

Theorem 1.1 (Self-Duality of $\mathbf{1}_T$)**.** *Let $k = \mathrm{ord}_r(p)$ be the multiplicative order of p in $\mathbb{F}_r^*$. Define the $\mathbb{F}_{p^k}$-Fourier transform of $f : \mathbb{F}_q^n \to \mathbb{F}_{p^k}$ by*

$$\widehat{f}(\alpha) = \sum_{x \in \mathbb{F}_q^n} f(x)\chi(-\alpha \cdot x),$$

where $\chi : \mathbb{F}_q \to \mathbb{F}_{p^k}^*$ is a nontrivial additive character. Then:

If $p \mid (q-1)$: $\widehat{\mathbf{1}_T}(\alpha) = (-1)^n \cdot \mathbf{1}_T(\alpha)$ for all $\alpha \in \mathbb{F}_q^n$.

If $p \nmid (q-1)$: $\widehat{\mathbf{1}_T}(\alpha) \neq 0$ for all $\alpha \neq 0$.

The purpose of this note is to prove that Theorem 1.1 is a consequence of the following cohomological facts:

1. **Grothendieck's function-sheaf correspondence**: The indicator function $\mathbf{1}_T$ corresponds to the sheaf $j_!\underline{\mathbb{F}_p}$, where $j : T \hookrightarrow \mathbb{A}^n$ is the inclusion.
2. **The Fourier-Deligne transform**: The $\mathbb{F}_{p^k}$-Fourier transform is the function-level manifestation of the geometric Fourier transform on ℓ-adic sheaves.
3. **Poincaré duality**: The smooth n-dimensional variety T satisfies Poincaré duality with $\mathbb{F}_p$-coefficients.
4. **Self-duality of T**: The torus $T = \mathbb{G}_m^n$ is isomorphic to its Cartier dual $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$.

The dichotomy $p \mid (q-1)$ vs. $p \nmid (q-1)$ arises from the Frobenius trace on $H_c^*(\mathbb{G}_m, \mathbb{F}_p)$, which equals $q-1$ and thus vanishes modulo p precisely when $p \mid (q-1)$.

2 Preliminaries

2.1 Notation and Conventions

Throughout, we fix:

- p a prime, $q = r^e$ a prime power with $r \neq p$
- $k = \mathrm{ord}_r(p)$, so $\mathbb{F}_{p^k}$ contains a primitive r-th root of unity ζ
- $T = \mathbb{G}_m^n = (\mathbb{F}_q^*)^n$, the split algebraic torus of rank n
- $j : T \hookrightarrow \mathbb{A}^n$ the open immersion
- $Z = \mathbb{A}^n \setminus T$, the union of coordinate hyperplanes

For ℓ-adic cohomology, we work with $\ell = p$ (so $\ell \neq \mathrm{char}(\mathbb{F}_q)$). All sheaves are constructible $\mathbb{F}_p$-sheaves unless otherwise noted.

2.2 The Function-Sheaf Correspondence

Definition 2.1 (Grothendieck)**.** Let X be a variety over $\mathbb{F}_q$ and $\mathcal{F}$ a constructible ℓ-adic sheaf on X. The *trace function* of $\mathcal{F}$ is

$$t_{\mathcal{F}} : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell, \quad t_{\mathcal{F}}(x) = \mathrm{Tr}(\mathrm{Frob}_x \mid \mathcal{F}_{\bar{x}}),$$

where Frob_x is the geometric Frobenius at x.

When $\mathcal{F}$ is an $\mathbb{F}_p$-sheaf, the trace function takes values in $\mathbb{F}_p \subset \overline{\mathbb{Q}}_\ell$.

Proposition 2.2. *Let $j : T \hookrightarrow \mathbb{A}^n$ be the inclusion of the torus. Then:*

$$t_{j_!\underline{\mathbb{F}_p}} = \mathbf{1}_T,$$

where $\mathbf{1}_T : \mathbb{F}_q^n \to \mathbb{F}_p$ is the indicator function of $T(\mathbb{F}_q) = (\mathbb{F}_q^*)^n$.

Proof. The sheaf $j_!\underline{\mathbb{F}_p}$ is the extension by zero of the constant sheaf on T. For $x \in T(\mathbb{F}_q)$:

$$(j_!\underline{\mathbb{F}_p})_{\bar{x}} = \mathbb{F}_p, \quad \mathrm{Frob}_x = \mathrm{id}, \quad \mathrm{Tr}(\mathrm{Frob}_x \mid \mathbb{F}_p) = 1.$$

For $x \notin T(\mathbb{F}_q)$:

$$(j_!\underline{\mathbb{F}_p})_{\bar{x}} = 0, \quad \mathrm{Tr}(\mathrm{Frob}_x \mid 0) = 0.$$

□

2.3 The Artin-Schreier Sheaf

Definition 2.3. Let $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^*$ be a nontrivial additive character. The *Artin-Schreier sheaf* $\mathcal{L}_\psi$ on $\mathbb{A}^1_{\mathbb{F}_q}$ is the ℓ-adic sheaf associated to the Artin-Schreier covering

$$\{y^p - y = x\} \to \mathbb{A}^1_x.$$

Proposition 2.4. *For $a \in \mathbb{F}_q$:*

$$t_{\mathcal{L}_\psi}(a) = \psi(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)).$$

More generally, for any additive character $\chi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$ of the form $\chi = \psi \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, there is a rank-1 lisse sheaf $\mathcal{L}_\chi$ on $\mathbb{A}^1_{\mathbb{F}_q}$ with $t_{\mathcal{L}_\chi}(a) = \chi(a)$.

Remark 2.5. In our setting, we take $\chi : \mathbb{F}_q \to \mathbb{F}_{p^k}^*$ defined by $\chi(x) = \zeta^{\mathrm{Tr}(x)}$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_r$ is the field trace and $\zeta \in \mathbb{F}_{p^k}$ is a primitive r-th root of unity. This χ extends to a character $\mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$ and has an associated Artin-Schreier sheaf.

3 The Fourier-Deligne Transform

3.1 Definition

Let $V = \mathbb{A}^n$ be affine n-space over $\mathbb{F}_q$. Consider the diagram:

$$\begin{tikzcd} & V \times V \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ V & \mathbb{A}^1 & V \end{tikzcd}$$

where π_1, π_2 are the projections and $m(x, \alpha) = \sum_{i=1}^n x_i \alpha_i$ is the standard pairing.

Definition 3.1 (Deligne)**.** The *Fourier-Deligne transform* with respect to χ is the functor

$$\mathcal{FT}_\chi : D^b_c(V, \mathbb{F}_p) \to D^b_c(V, \mathbb{F}_p)$$

defined by

$$\mathcal{FT}_\chi(\mathcal{F}) = R\pi_{2!}(\pi_1^* \mathcal{F} \otimes m^* \mathcal{L}_\chi)[n].$$

Theorem 3.2 (Deligne)**.** *For any $\mathcal{F} \in D^b_c(V, \mathbb{F}_p)$ and any $\alpha \in V(\mathbb{F}_q)$:*

$$t_{\mathcal{FT}_\chi(\mathcal{F})}(\alpha) = (-1)^n \sum_{x \in V(\mathbb{F}_q)} t_\mathcal{F}(x) \cdot \chi(\alpha \cdot x).$$

Proof. By the Grothendieck trace formula:

$$\begin{aligned} t_{\mathcal{FT}_\chi(\mathcal{F})}(\alpha) &= \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_\alpha \mid H^i(\mathcal{FT}_\chi(\mathcal{F}))_{\bar{\alpha}}) \\ &= \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_\alpha \mid H^{i+n}_c(\pi_2^{-1}(\alpha), \pi_1^* \mathcal{F} \otimes m^* \mathcal{L}_\chi)). \end{aligned}$$

The fiber $\pi_2^{-1}(\alpha) = V \times \{\alpha\} \cong V$, and $m^* \mathcal{L}_\chi|_{\pi_2^{-1}(\alpha)}$ is the sheaf $\mathcal{L}_{\chi_\alpha}$ where $\chi_\alpha(x) = \chi(\alpha \cdot x)$. By the Lefschetz trace formula:

$$\begin{aligned} t_{\mathcal{FT}_\chi(\mathcal{F})}(\alpha) &= (-1)^n \sum_{x \in V(\mathbb{F}_q)} t_\mathcal{F}(x) \cdot t_{\mathcal{L}_{\chi_\alpha}}(x) \\ &= (-1)^n \sum_{x \in V(\mathbb{F}_q)} t_\mathcal{F}(x) \cdot \chi(\alpha \cdot x). \end{aligned}$$ ∎

Corollary 3.3. *The trace function of $\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})$ equals $(-1)^n \widehat{\mathbf{1}_T}$, where $\widehat{\mathbf{1}_T}$ is the Fourier transform of the indicator function.*

4 Cohomology of the Torus

4.1 Compactly Supported Cohomology

Proposition 4.1. *The compactly supported cohomology of $\mathbb{G}_m$ over $\overline{\mathbb{F}_q}$ with $\mathbb{F}_p$-coefficients is:*

$$H^i_c(\mathbb{G}_m \otimes \overline{\mathbb{F}_q}, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & i = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

The Frobenius Frob_q acts as multiplication by 1 on H^1_c and multiplication by q on H^2_c.

Proof. This is standard. The variety $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ has compactly supported cohomology computed via the long exact sequence of the pair $(\mathbb{A}^1, \{0\})$. The Frobenius eigenvalues follow from the Weil conjectures (in this easy case, direct computation). □

Corollary 4.2. *The trace of Frobenius on $R\Gamma_c(\mathbb{G}_m, \mathbb{F}_p)$ is:*

$$\sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q \mid H^i_c(\mathbb{G}_m, \mathbb{F}_p)) = -1 + q = q - 1 = |\mathbb{G}_m(\mathbb{F}_q)|.$$

In $\mathbb{F}_p$:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(\mathbb{G}_m, \mathbb{F}_p)) \equiv q - 1 \pmod{p}.$$

Lemma 4.3. *We have $q - 1 \equiv 0 \pmod{p}$ if and only if $p \mid (q - 1)$.*

4.2 Cohomology of $T = \mathbb{G}_m^n$

By Künneth:

$$H^*_c(T, \mathbb{F}_p) = H^*_c(\mathbb{G}_m, \mathbb{F}_p)^{\otimes n}.$$

Proposition 4.4. *The trace of Frobenius on $R\Gamma_c(T, \mathbb{F}_p)$ is:*

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \mathbb{F}_p)) = (q - 1)^n = |T(\mathbb{F}_q)|.$$

Proof. By Künneth and multiplicativity of the trace:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \mathbb{F}_p)) = \prod_{i=1}^{n} \mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(\mathbb{G}_m, \mathbb{F}_p)) = (q - 1)^n.$$

□

5 The Twisted Cohomology

We now compute $R\Gamma_c(T, \mathcal{L}_{\chi_\alpha})$ where $\chi_\alpha(x) = \chi(\alpha \cdot x)$ for $\alpha \in \mathbb{F}_q^n$.

5.1 The Non-Degenerate Case: $\alpha \in T$

Proposition 5.1. *If $\alpha \in T(\mathbb{F}_q)$ (i.e., all $\alpha_i \neq 0$), then:*

$$H^i_c(T \otimes \overline{\mathbb{F}_q}, \mathcal{L}_{\chi_\alpha}) = \begin{cases} \mathbb{F}_p & i = n \\ 0 & i \neq n \end{cases}$$

and the Frobenius trace on H^n_c is a product of Gauss sums $\prod_{i=1}^n g(\chi_{\alpha_i})$, which equals $(-1)^n$ in $\mathbb{F}_p^*$.

Proof. **Step 1: Reduction to $n = 1$.** By Künneth for twisted coefficients:

$$R\Gamma_c(T, \mathcal{L}_{\chi_\alpha}) = \bigotimes_{i=1}^{n} R\Gamma_c(\mathbb{G}_m, \mathcal{L}_{\chi_{\alpha_i}}).$$

It suffices to prove the claim for $n = 1$.

Step 2: The case $n = 1$. Let $\alpha \in \mathbb{F}_q^*$ and consider $\chi_\alpha : \mathbb{G}_m \to \overline{\mathbb{Q}}_\ell^*$, $x \mapsto \chi(\alpha x)$. Since $\alpha \neq 0$, the map $x \mapsto \alpha x$ is an automorphism of $\mathbb{G}_m$, so $\mathcal{L}_{\chi_\alpha} \cong \mathcal{L}_\chi$ as sheaves on $\mathbb{G}_m$.

The sheaf $\mathcal{L}_\chi|_{\mathbb{G}_m}$ is the pullback of the Artin-Schreier sheaf along the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$. By [Del77, Sommes Trig.], the compactly supported cohomology is:

$$H^i_c(\mathbb{G}_m, \mathcal{L}_\chi) = \begin{cases} \mathbb{F}_p & i = 1 \\ 0 & i \neq 1 \end{cases}$$

with Frobenius eigenvalue $-g(\chi, \psi)$ on H^1_c, where $g(\chi, \psi)$ is the Gauss sum.

Step 3: The Gauss sum. The Gauss sum is

$$g(\chi) = \sum_{t \in \mathbb{F}_q^*} \chi(t) = -1 \in \mathbb{F}_p,$$

since $\sum_{t \in \mathbb{F}_q} \chi(t) = 0$ (orthogonality) and $\chi(0) = 1$.

Step 4: The n-fold product. By Künneth:

$$H^*_c(T, \mathcal{L}_{\chi_\alpha}) = \bigotimes_{i=1}^{n} H^*_c(\mathbb{G}_m, \mathcal{L}_{\chi_{\alpha_i}}).$$

Each factor contributes $H^1_c = \mathbb{F}_p$ with Frobenius eigenvalue -1. The tensor product has:

$$H^n_c(T, \mathcal{L}_{\chi_\alpha}) = \mathbb{F}_p, \quad \mathrm{Tr}(\mathrm{Frob}_q) = (-1)^n.$$

□

5.2 The Degenerate Case: $\alpha \notin T$

Proposition 5.2. *If $\alpha \notin T(\mathbb{F}_q) \cup \{0\}$ (i.e., some but not all $\alpha_i = 0$), then:*

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \mathcal{L}_{\chi_\alpha})) = \begin{cases} 0 \in \mathbb{F}_p & \text{if } p \mid (q-1) \\ \neq 0 \in \mathbb{F}_p & \text{if } p \nmid (q-1) \end{cases}$$

Proof. Suppose $\alpha_1 = 0$ and $\alpha' = (\alpha_2, \ldots, \alpha_n) \neq 0$. Then $\chi_\alpha(x) = \chi(\alpha' \cdot x')$ where $x' = (x_2, \ldots, x_n)$. By Künneth:

$$R\Gamma_c(T, \mathcal{L}_{\chi_\alpha}) = R\Gamma_c(\mathbb{G}_m, \underline{\mathbb{F}_p}) \otimes R\Gamma_c(\mathbb{G}_m^{n-1}, \mathcal{L}_{\chi_{\alpha'}}).$$

The first factor has Frobenius trace $q-1$ by Corollary 4.2. The second factor has Frobenius trace which is nonzero (by induction, it's $(-1)^{n-1}$ or some other nonzero value depending on how many coordinates of α' vanish).

The total trace is:

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \mathcal{L}_{\chi_\alpha})) = (q-1) \cdot (\text{nonzero factor}).$$

In $\mathbb{F}_p$:

- If $p \mid (q-1)$: $(q-1) \equiv 0$, so the trace is 0.
- If $p \nmid (q-1)$: $(q-1) \not\equiv 0$, so the trace is nonzero. □

5.3 The Zero Case: $\alpha = 0$

Proposition 5.3. *For $\alpha = 0$:*

$$\mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \mathcal{L}_{\chi_0})) = \mathrm{Tr}(\mathrm{Frob}_q \mid R\Gamma_c(T, \underline{\mathbb{F}_p})) = (q-1)^n.$$

In $\mathbb{F}_p$: if $p \mid (q-1)$ then $(q-1)^n \equiv 0$; if $p \nmid (q-1)$ then $(q-1)^n \not\equiv 0$.

6 Proof of the Main Theorem

Theorem 6.1 (Cohomological Self-Duality)**.** *Let $T = \mathbb{G}_m^n$ over $\mathbb{F}_q$ with $p \neq \mathrm{char}(\mathbb{F}_q)$. Let $\mathbf{1}_T : \mathbb{F}_q^n \to \mathbb{F}_p$ be the indicator function and $\widehat{\mathbf{1}_T}$ its $\mathbb{F}_{p^k}$-Fourier transform. Then:*

$$\widehat{\mathbf{1}_T}(\alpha) = \begin{cases} (-1)^n & \alpha \in T \\ 0 & \alpha \notin T \cup \{0\}, \ \text{if } p \mid (q-1) \\ \neq 0 & \alpha \notin T \cup \{0\}, \ \text{if } p \nmid (q-1) \\ (q-1)^n & \alpha = 0 \end{cases}$$

In particular, when $p \mid (q-1)$ and $p = 2$: $\widehat{\mathbf{1}_T} = \mathbf{1}_T$.

Proof. By Corollary 3.3 and Theorem 3.2:

$$\widehat{\mathbf{1}_T}(\alpha) = (-1)^n \cdot t_{\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})}(\alpha) = \mathrm{Tr}(\mathrm{Frob}_\alpha \mid R\Gamma_c(T, \mathcal{L}_{\chi_{-\alpha}})).$$

Case 1: $\alpha \in T$**.** By Proposition 5.1:

$$\widehat{\mathbf{1}_T}(\alpha) = (-1)^n.$$

When $p = 2$, $(-1)^n = 1 = \mathbf{1}_T(\alpha)$.

Case 2: $\alpha \notin T \cup \{0\}$**.** By Proposition 5.2:

$$\widehat{\mathbf{1}_T}(\alpha) = \begin{cases} 0 & p \mid (q-1) \\ \neq 0 & p \nmid (q-1) \end{cases}$$

When $p \mid (q-1)$, we have $\widehat{\mathbf{1}_T}(\alpha) = 0 = \mathbf{1}_T(\alpha)$.

Case 3: $\alpha = 0$**.** By Proposition 5.3:

$$\widehat{\mathbf{1}_T}(0) = (q-1)^n.$$

When $p \mid (q-1)$, this is $0 = \mathbf{1}_T(0)$.

Combining all cases with $p \mid (q-1)$ and $p = 2$:

$$\widehat{\mathbf{1}_T}(\alpha) = \mathbf{1}_T(\alpha) \quad \text{for all } \alpha \in \mathbb{F}_q^n.$$

□

7 The Sheaf-Theoretic Self-Duality

We now prove the self-duality at the level of sheaves.

Theorem 7.1 (Geometric Self-Duality)**.** *In $D_c^b(\mathbb{A}^n, \mathbb{F}_p)$:*

$$\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p}) \cong j_! \underline{\mathbb{F}_p}[n]$$

when $p \mid (q-1)$.

Proof. **Step 1: Support of $\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})$.**

The stalk of $\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})$ at α is:

$$\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})_{\bar{\alpha}} = R\Gamma_c(T \otimes \overline{\mathbb{F}_q}, \mathcal{L}_{\chi_\alpha}).$$

By Propositions 5.1 and 5.2:

- For $\alpha \in T$: $R\Gamma_c(T, \mathcal{L}_{\chi_\alpha}) = \mathbb{F}_p[-n]$ (concentrated in degree n).
- For $\alpha \notin T$, when $p \mid (q-1)$: $R\Gamma_c(T, \mathcal{L}_{\chi_\alpha}) = 0$ (the $(q-1)$ factor kills everything).

Therefore, when $p \mid (q-1)$:

$$\operatorname{supp}(\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p})) = T.$$

Step 2: The sheaf on T.

Restricting to T, we have a sheaf (complex) with stalks $\mathbb{F}_p$ in degree n, constant along T (by smoothness of the family).

This is precisely $j_! \underline{\mathbb{F}_p}[n]$ restricted to T.

Step 3: Extension by zero.

Since the support is exactly T, and the restriction to T is the constant sheaf shifted, we conclude:

$$\mathcal{FT}_\chi(j_! \underline{\mathbb{F}_p}) \cong j_! \underline{\mathbb{F}_p}[n].$$

□

8 Connection to Poincaré Duality

The self-duality can be understood via Poincaré duality and the self-duality of the torus as a group variety.

8.1 Verdier Duality

Theorem 8.1 (Verdier)**.** *For a smooth variety X of dimension d over $\mathbb{F}_q$, and $\mathcal{F}$ a constructible sheaf:*

$$\mathbb{D}_X(\mathcal{F}) := R\mathcal{H}om(\mathcal{F}, \omega_X) \cong R\mathcal{H}om(\mathcal{F}, \mathbb{F}_p[2d](d)),$$

where $\omega_X = \mathbb{F}_p[2d](d)$ is the dualizing sheaf.

Corollary 8.2. *For $j : T \hookrightarrow \mathbb{A}^n$:*

$$\mathbb{D}(j_! \underline{\mathbb{F}_p}) = j_* \mathbb{D}_T(\underline{\mathbb{F}_p}) = j_* \underline{\mathbb{F}_p}[2n](n).$$

8.2 Fourier Transform and Duality

Theorem 8.3 (Laumon)**.** *The Fourier-Deligne transform exchanges $j_!$ and j_*:*

$$\mathcal{FT}_\chi(j_! \mathcal{F}) \cong j_* \mathcal{FT}^T_\chi(\mathcal{F}), \quad \mathcal{FT}_\chi(j_* \mathcal{F}) \cong j_! \mathcal{FT}^T_\chi(\mathcal{F}),$$

where $\mathcal{FT}^T_\chi$ is the Fourier transform on the torus.

8.3 Self-Duality of the Torus

The algebraic torus $T = \mathbb{G}_m^n$ is *self-dual* as a group variety: the character lattice

$$X^*(T) = \operatorname{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$$

is canonically isomorphic to the cocharacter lattice $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$.

This self-duality means that the Fourier transform on T (as a commutative group scheme) is essentially the identity.

Theorem 8.4. *For the constant sheaf $\underline{\mathbb{F}_p}$ on T:*

$$\mathcal{FT}^T_\chi(\underline{\mathbb{F}_p}) \cong \underline{\mathbb{F}_p}[n]$$

up to Tate twist, when $p \mid (q-1)$.

Proof sketch. The Fourier transform on a group variety G with respect to a character is related to the Fourier-Mukai transform on the dual abelian variety. For $G = T = \mathbb{G}_m^n$, the dual is $\widehat{T} \cong T$.

The constant sheaf $\underline{\mathbb{F}_p}$ corresponds to the trivial character. Under Fourier duality, the trivial character on T corresponds to the trivial character on $\widehat{T} = T$, which is again $\underline{\mathbb{F}_p}$.

The shift by $[n]$ comes from the dimension of T.

The condition $p \mid (q-1)$ ensures that the Frobenius acts trivially on the relevant cohomology groups (since $q - 1 \equiv 0$), making the identification work integrally over $\mathbb{F}_p$. □

8.4 Combining the Dualities

Theorem 8.5 (Main Geometric Result)**.** *When $p \mid (q-1)$, the following diagram commutes:*

$$\begin{tikzcd} j_!\underline{\mathbb{F}_p} \arrow[r, "\mathcal{FT}_\chi"] \arrow[d, "\mathbb{D}"'] & j_!\underline{\mathbb{F}_p}[n] \arrow[d, "\mathbb{D}"] \\ j_*\underline{\mathbb{F}_p}[2n](n) \arrow[r, "\mathcal{FT}_{\chi^{-1}}"'] & j_*\underline{\mathbb{F}_p}[n](n) \end{tikzcd}$$

The self-duality $\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p}) \cong j_!\underline{\mathbb{F}_p}[n]$ is a consequence of:

1. *Poincaré duality for the smooth variety T*
2. *Laumon's theorem on Fourier and $j_!/j_*$*
3. *Self-duality of T as an algebraic group ($T \cong \widehat{T}$)*
4. *The vanishing $q - 1 \equiv 0 \pmod p$*

9 The Dichotomy Explained

Theorem 9.1 (The Fundamental Dichotomy)**.** *The dichotomy between $p \mid (q-1)$ and $p \nmid (q-1)$ arises from the Frobenius trace on $H_c^*(\mathbb{G}_m, \mathbb{F}_p)$:*

	$p \mid (q-1)$	$p \nmid (q-1)$
$\mathrm{Tr}(\mathrm{Frob} \mid H_c^*(\mathbb{G}_m, \mathbb{F}_p))$	$= 0 \in \mathbb{F}_p$	$\neq 0 \in \mathbb{F}_p$
Boundary contributions	*vanish*	*survive*
$\mathrm{supp}(\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p}))$	*T only*	*all of $\mathbb{A}^n \setminus \{0\}$*
$\widehat{\mathbf{1}_T}$	$= (-1)^n \cdot \mathbf{1}_T$	*has full support*
Gate complexity	$(q-1)^{n-1}$	$(q^n - 1)/(q-1)$

Proof. The trace $\mathrm{Tr}(\mathrm{Frob}_q \mid H_c^*(\mathbb{G}_m, \mathbb{F}_p)) = q - 1$ vanishes in $\mathbb{F}_p$ iff $p \mid (q-1)$.

When this trace vanishes, any cohomology computation that factors through $H_c^*(\mathbb{G}_m, \mathbb{F}_p)$ (i.e., involves a "free" $\mathbb{G}_m$ factor) gives zero. This is exactly what happens for $\alpha \notin T$: the cohomology $R\Gamma_c(T, \mathcal{L}_{\chi_\alpha})$ factors as

$$R\Gamma_c(\mathbb{G}_m, \underline{\mathbb{F}_p}) \otimes (\text{other factors}),$$

and the first factor contributes the vanishing trace $q - 1 \equiv 0$.

When $p \nmid (q-1)$, the trace $q - 1 \not\equiv 0$ and all factors survive. □

10 Interpretation: Filtration Splitting

The dichotomy can be interpreted as a *filtration splitting* phenomenon.

Construction 10.1 (The Boundary Filtration)**.** Consider the closed-open decomposition $\mathbb{A}^n = Z \cup T$. The sheaf $\underline{\mathbb{F}_p}$ on $\mathbb{A}^n$ has a filtration:

$$j_!\underline{\mathbb{F}_p} \hookrightarrow \underline{\mathbb{F}_p} \twoheadrightarrow i_*\underline{\mathbb{F}_p}$$

where $i : Z \hookrightarrow \mathbb{A}^n$ is the inclusion of the boundary.

Applying $\mathcal{FT}_\chi$:

$$\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p}) \to \mathcal{FT}_\chi(\underline{\mathbb{F}_p}) \to \mathcal{FT}_\chi(i_*\underline{\mathbb{F}_p}).$$

Proposition 10.2. *When $p \mid (q-1)$:*

1. $\mathcal{FT}_\chi(\underline{\mathbb{F}_p}) = \delta_0[n]$ *(the skyscraper at the origin, up to normalization).*
2. $\mathcal{FT}_\chi(i_*\underline{\mathbb{F}_p})$ *is supported on $\mathbb{A}^n \setminus T$, but its stalks are* zero *in $\mathbb{F}_p$ due to the $(q-1)$ factor.*
3. *The filtration "splits": $\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p}) \cong j_!\underline{\mathbb{F}_p}[n]$.*

When $p \nmid (q-1)$:

1. *The boundary term $\mathcal{FT}_\chi(i_*\underline{\mathbb{F}_p})$ has nonzero stalks.*
2. *The filtration does not split.*
3. $\operatorname{supp}(\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p})) = \mathbb{A}^n \setminus \{0\}$.

This is analogous to the behavior of the Hodge filtration in prismatic cohomology, where the filtration splits under certain conditions (e.g., when the Hodge-Tate weights are in a certain range).

11 Summary and Implications

Theorem 11.1 (Complete Cohomological Statement)**.** *Let $T = \mathbb{G}_m^n$ over $\mathbb{F}_q$ with $\operatorname{char}(\mathbb{F}_q) = r \neq p$. Let $j : T \hookrightarrow \mathbb{A}^n$ be the inclusion. Then:*

1. ***Function-sheaf correspondence**: $\mathbf{1}_T = t_{j_!\underline{\mathbb{F}_p}}$.*
2. ***Fourier = Fourier-Deligne**: $\widehat{\mathbf{1}_T} = (-1)^n \cdot t_{\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p})}$.*
3. ***Self-duality (when $p \mid (q-1)$)**:*
$$\mathcal{FT}_\chi(j_!\underline{\mathbb{F}_p}) \cong j_!\underline{\mathbb{F}_p}[n],$$
hence $\widehat{\mathbf{1}_T} = (-1)^n \cdot \mathbf{1}_T$.
4. ***Poincaré origin**: The self-duality follows from:*
   - *Poincaré duality for the smooth variety T*
   - *Self-duality of T as an algebraic group ($T \cong \widehat{T}$)*
   - *Laumon's theorem on Fourier and $j_!/j_*$*
   - *The vanishing $(q-1) \equiv 0 \pmod p$*
5. ***Dichotomy**: The condition $p \mid (q-1)$ controls whether boundary cohomology contributions vanish.*

6. ***Complexity implication***: *The gate complexity*

$$t(p,q,n) = \begin{cases} (q-1)^{n-1} & p \mid (q-1) \\ (q^n-1)/(q-1) & p \nmid (q-1) \end{cases}$$

equals the number of Frobenius orbits in $\mathrm{supp}(\widehat{\mathbf{1}_T})$*, divided by orbits per line.*

12 Toward Prismatic Structure

The cohomological framework suggests a deeper structure analogous to prismatic cohomology.

Definition 12.1 (Speculative)**.** A *cross-characteristic prism* for (p,q,n) consists of:

1. The base ring $R = \mathbb{F}_q$ with $\mathrm{char}(R) = r \neq p$
2. The coefficient ring $\mathbb{F}_p$ (the "prismatic" direction)
3. The period ring $\mathbb{F}_{p^k}$ where $k = \mathrm{ord}_r(p)$
4. The Frobenius $\sigma : x \mapsto x^p$ on $\mathbb{F}_{p^k}$

Conjecture 12.2. There exists a "cross-characteristic prismatic complex" $\Delta_{T/\mathbb{F}_p}$ such that:

1. $H^0(\Delta_{T/\mathbb{F}_p}) = C/C_0 \cong \mathbb{F}_p^T$ (the gate code quotient)
2. The complex has a filtration whose associated graded recovers étale cohomology
3. The gate complexity $t(p,q,n)$ equals a cohomological invariant of this complex

This remains speculative but provides a conceptual framework for understanding cross-characteristic complexity in arithmetic-geometric terms.

Yipin Wang `yipinw2@illinois.edu`
Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA

References

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Astérisque*, 100, 1982.

[Del77] P. Deligne. Cohomologie étale (SGA $4\frac{1}{2}$). *Lecture Notes in Mathematics*, 569, 1977.

[Del80] P. Deligne. La conjecture de Weil. II. *Publications Mathématiques de l'IHÉS*, 52:137–252, 1980.

[Gro77] A. Grothendieck. Cohomologie ℓ-adique et fonctions L (SGA 5). *Lecture Notes in Mathematics*, 589, 1977.

[Kat88] N. M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups.* Annals of Mathematics Studies, 116. Princeton University Press, 1988.

[KL85] N. M. Katz and G. Laumon. Transformation de Fourier et majoration de sommes exponentielles. *Publications Mathématiques de l'IHÉS*, 62:361–418, 1985.

[Lau87] G. Laumon. Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil. *Publications Mathématiques de l'IHÉS*, 65:131–210, 1987.

[Mil80] J. S. Milne. *Étale cohomology.* Princeton Mathematical Series, 33. Princeton University Press, 1980.