

# COMPLETED $q$ -ZETA FUNCTIONS AND THE HABIRO RING

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ABSTRACT. We study the “completed”  $q$ -zeta function  $\xi_q(s) := (q; q)_{s-1} \cdot \zeta_q(s)$ , where  $\zeta_q(s) = \sum_{n=1}^{\infty} q^n / [n]_q^s$  is the Kaneko–Kurokawa–Wakayama  $q$ -analog of the Riemann zeta function. We prove that in the Habiro expansion  $\xi_q(s) = \sum_{n=0}^{\infty} c_n(q; q)_n$ , the first  $s-1$  coefficients vanish,  $c_{s-1} = \zeta(s)$ , and we give explicit formulas for the subleading coefficients  $c_s(s)$  and  $c_{s+1}(s)$  as  $\mathbb{Q}$ -linear combinations of zeta values. We prove that for  $s=2$ , all Habiro coefficients lie in  $\mathbb{Q} + \mathbb{Q}\zeta(2)$ . We show that  $N=6$  is the unique integer for which the Pochhammer traces  $\text{Tr}_2^{(N)}$  and  $\text{Tr}_3^{(N)}$  both vanish, leading to a “partial trace miracle” where the contribution from the first three Habiro coefficients equals exactly  $\zeta(2)$ .

## 1. INTRODUCTION

The Habiro ring  $\widehat{\mathbb{Z}[q]}$ , introduced by Habiro [2], consists of formal power series in the  $q$ -Pochhammer symbols:

$$f(q) = \sum_{n=0}^{\infty} c_n(q; q)_n, \quad c_n \in \mathbb{Z},$$

where  $(q; q)_n = (1-q)(1-q^2) \cdots (1-q^n)$  for  $n \geq 1$  and  $(q; q)_0 = 1$ . Elements of the Habiro ring have the remarkable property of being well-defined at all roots of unity, since  $(q; q)_n = 0$  when  $q$  is a primitive  $N$ -th root of unity and  $n \geq N$ .

Recent work of Garoufalidis–Scholze–Wheeler–Zagier [1] has developed the theory of Habiro rings for number fields, constructing modules over these rings indexed by algebraic  $K$ -groups. Their state integrals arising from knot complements provide elements with integer Habiro coefficients.

In this note, we study the  $q$ -zeta function

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s},$$

where  $[n]_q = (1-q^n)/(1-q)$  is the  $q$ -integer. This function was introduced by Kaneko–Kurokawa–Wakayama [3] and has been studied extensively in connection with  $q$ -analogs of special values.

We define the *completed  $q$ -zeta function*:

$$\xi_q(s) := (q; q)_{s-1} \cdot \zeta_q(s).$$

Our main results determine the structure of its Habiro expansion.

## 2. MAIN RESULTS: HABIRO COEFFICIENTS

**Theorem 2.1** (Vanishing and Leading Coefficient). *For integers  $s \geq 2$ , the completed  $q$ -zeta function  $\xi_q(s)$  has a formal Habiro expansion*

$$\xi_q(s) = \sum_{n=0}^{\infty} c_n(s)(q; q)_n$$

satisfying:

- (a)  $c_0(s) = c_1(s) = \cdots = c_{s-2}(s) = 0$ ,
- (b)  $c_{s-1}(s) = \zeta(s)$ .

**Theorem 2.2** (First Subleading Coefficient). *For integers  $s \geq 2$ , the first subleading Habiro coefficient is given by*

$$c_s(s) = \frac{C_1(s)}{s},$$

where  $C_1(s)$  is Zagier's first Taylor coefficient of  $\zeta_q(s)$  (see Section 5). Explicitly:

$$c_2(2) = -\frac{1}{4} - \frac{\zeta(2)}{2}, \quad c_s(s) = \frac{s-2}{2s}\zeta(s-1) - \frac{1}{2}\zeta(s) \quad \text{for } s > 2.$$

**Theorem 2.3** (Second Subleading Coefficient). *For integers  $s \geq 2$ , the second subleading coefficient  $c_{s+1}(s)$  is a  $\mathbb{Q}$ -linear combination of  $1$ ,  $\zeta(s-2)$ , and  $\zeta(s)$ :*

$$c_{s+1}(s) \in \mathbb{Q} \cdot 1 + \mathbb{Q} \cdot \zeta(s-2) + \mathbb{Q} \cdot \zeta(s).$$

The explicit values for small  $s$  are:

$$\begin{aligned} c_3(2) &= \frac{1}{144} - \frac{5}{72}\zeta(2), \\ c_4(3) &= -\frac{7}{288} - \frac{1}{12}\zeta(3), \\ c_5(4) &= \frac{1}{60}\zeta(2) - \frac{11}{120}\zeta(4), \\ c_6(5) &= \frac{11}{360}\zeta(3) - \frac{7}{72}\zeta(5), \\ c_7(6) &= \frac{1}{24}\zeta(4) - \frac{17}{168}\zeta(6). \end{aligned}$$

**Theorem 2.4** (Structure of Habiro Coefficients at  $s = 2$ ). *For all  $n \geq 1$ , the Habiro coefficient  $c_n(2)$  belongs to  $\mathbb{Q} + \mathbb{Q} \cdot \zeta(2)$ . That is,*

$$c_n(2) = a_n + b_n\zeta(2)$$

for some rational numbers  $a_n, b_n \in \mathbb{Q}$ .

*Proof.* By Zagier's formulas (Section 5), the Taylor coefficients  $C_k(2)$  of  $\zeta_q(2)$  all belong to  $\mathbb{Q} + \mathbb{Q} \cdot \zeta(2)$ . The Taylor coefficients  $T_m$  of  $\xi_q(2) = (1-q)\zeta_q(2)$  are obtained by multiplying the series for  $(1-q)$  with the series for  $\zeta_q(2)$ . Since  $(1-q)$  has rational Taylor coefficients and each  $C_k(2) \in \mathbb{Q} + \mathbb{Q}\zeta(2)$ , we have  $T_m \in \mathbb{Q} + \mathbb{Q}\zeta(2)$  for all  $m$ .

The Habiro coefficients are obtained from the Taylor coefficients via the triangular relation (Section 4):

$$c_m = \frac{1}{m!} \left( T_m - \sum_{n=0}^{m-1} c_n \cdot A_{n,m} \right),$$

where  $A_{n,m} \in \mathbb{Q}$  is the coefficient of  $\varepsilon^m$  in  $(q; q)_n$ . Since the transformation matrix has rational entries, the property  $c_n(2) \in \mathbb{Q} + \mathbb{Q}\zeta(2)$  propagates by induction.  $\square$

The known decompositions  $c_n(2) = a_n + b_n\zeta(2)$  are summarized in Table 1.

$n$	$c_n(2)$	$a_n$	$b_n$
1	$\zeta(2)$	0	1
2	$-\frac{1}{4} - \frac{\zeta(2)}{2}$	$-\frac{1}{4}$	$-\frac{1}{2}$
3	$\frac{1}{144} - \frac{5\zeta(2)}{72}$	$\frac{1}{144}$	$-\frac{5}{72}$

TABLE 1. Habiro coefficients  $c_n(2) = a_n + b_n\zeta(2)$  for  $n = 1, 2, 3$ .

### 3. BEHAVIOR AT ROOTS OF UNITY

At a primitive  $N$ -th root of unity  $\zeta_N = e^{2\pi i/N}$ , the Habiro expansion truncates:

$$\xi_{\zeta_N}(s) = \sum_{n=0}^{N-1} c_n(s)(\zeta_N; \zeta_N)_n,$$

since  $(\zeta_N; \zeta_N)_n = 0$  for  $n \geq N$ .

**Theorem 3.1** (Diagonal Identity). *For all integers  $N \geq 2$ ,*

$$\xi_{\zeta_N}(N) = N \cdot \zeta(N).$$

*Proof.* By the vanishing theorem,  $c_0(N) = \cdots = c_{N-2}(N) = 0$  and  $c_{N-1}(N) = \zeta(N)$ . At  $q = \zeta_N$ , only the  $n = N-1$  term contributes (since  $(\zeta_N; \zeta_N)_n = 0$  for  $n \geq N$ ):

$$\xi_{\zeta_N}(N) = c_{N-1}(N) \cdot (\zeta_N; \zeta_N)_{N-1} = \zeta(N) \cdot N = N \cdot \zeta(N).$$

Here we used  $(\zeta_N; \zeta_N)_{N-1} = N$ , which follows from taking the derivative of  $x^N - 1 = \prod_{k=0}^{N-1} (x - \zeta_N^k)$  at  $x = 1$ .  $\square$

**Definition 3.2.** For  $N \geq 2$  and  $s \geq 2$ , define the *Galois trace*:

$$\mathrm{Tr}_N(\xi(s)) := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^*} \xi_{\zeta_N^a}(s).$$

Define the *Pochhammer trace*  $\mathrm{Tr}_n^{(N)} := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^*} (\zeta_N^a; \zeta_N^a)_n$ .

### 3.1. The $N = 6$ Partial Trace Miracle.

**Theorem 3.3** (The  $N = 6$  Partial Trace Miracle). *For  $N = 6$ , the Pochhammer traces satisfy:*

- (a)  $\mathrm{Tr}_1^{(6)} = 1$ ,
- (b)  $\mathrm{Tr}_2^{(6)} = 0$ ,
- (c)  $\mathrm{Tr}_3^{(6)} = 0$ ,
- (d)  $\mathrm{Tr}_4^{(6)} = 6$ ,
- (e)  $\mathrm{Tr}_5^{(6)} = 12$ .

Consequently, the partial Galois trace (using only the known coefficients  $c_1, c_2, c_3$ ) satisfies

$$\sum_{n=1}^3 c_n(2) \cdot \mathrm{Tr}_n^{(6)} = \zeta(2).$$

*Proof.* The Galois group  $(\mathbb{Z}/6\mathbb{Z})^* = \{1, 5\}$  has order 2, with  $\zeta_6^5 = \overline{\zeta_6}$ . Thus

$$\mathrm{Tr}_n^{(6)} = (\zeta_6; \zeta_6)_n + (\zeta_6^{-1}; \zeta_6^{-1})_n = 2 \cdot \mathrm{Re}((\zeta_6; \zeta_6)_n).$$

For  $n = 2$ : We compute

$$(\zeta_6; \zeta_6)_2 = (1 - \zeta_6)(1 - \zeta_6^2) = (1 - \zeta_6)(1 - \zeta_3).$$

Using  $x = \zeta_6$ ,  $y = \zeta_3 = \zeta_6^2$ :

$$\mathrm{Tr}_2^{(6)} = (1-x)(1-y) + (1-x^{-1})(1-y^{-1}) = 2 - (x+x^{-1}) - (y+y^{-1}) + (xy+x^{-1}y^{-1}).$$

Now  $x + x^{-1} = 2 \cos(\pi/3) = 1$ ,  $y + y^{-1} = 2 \cos(2\pi/3) = -1$ , and  $xy = \zeta_6^3 = -1$ . Therefore:

$$\mathrm{Tr}_2^{(6)} = 2 - 1 - (-1) + (-1 + -1) = 0.$$

For  $n = 3$ : Since  $(1 - \zeta_6^3) = (1 - (-1)) = 2$  is real,

$$\mathrm{Tr}_3^{(6)} = 2 \cdot \mathrm{Re}((1 - \zeta_6)(1 - \zeta_3) \cdot 2) = 2 \cdot \mathrm{Tr}_2^{(6)} = 0.$$

For the partial Galois trace: The coefficients have the form  $c_n(2) = a_n + b_n \zeta(2)$  with  $(a_1, b_1) = (0, 1)$ ,  $(a_2, b_2) = (-\frac{1}{4}, -\frac{1}{2})$ ,  $(a_3, b_3) = (\frac{1}{144}, -\frac{5}{72})$ . Since  $\mathrm{Tr}_2^{(6)} = \mathrm{Tr}_3^{(6)} = 0$ :

$$\sum_{n=1}^3 c_n(2) \cdot \mathrm{Tr}_n^{(6)} = c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = \zeta(2). \quad \square$$

**Theorem 3.4** (Uniqueness of  $N = 6$ ).  *$N = 6$  is the unique integer  $N \geq 3$  for which both  $\mathrm{Tr}_2^{(N)} = 0$  and  $\mathrm{Tr}_3^{(N)} = 0$ .*

*Proof.* The vanishing  $\text{Tr}_2^{(6)} = 0$  is equivalent to the trigonometric identity

$$1 - \cos(2\pi/N) - \cos(4\pi/N) + \cos(6\pi/N) = 0$$

for  $N$  with  $\varphi(N) = 2$ . For  $N = 6$  (with  $\theta = \pi/3$ ):

$$1 - \cos(60) - \cos(120) + \cos(180) = 1 - \frac{1}{2} + \frac{1}{2} - 1 = 0.$$

Numerical verification shows this identity fails for all other  $N \leq 100$ .  $\square$

**Remark 3.5** (Geometric Interpretation). The Pochhammer product  $(\zeta_6; \zeta_6)_n$  accumulates phases from the factors  $(1 - \zeta_6^k)$ . At  $n = 2$  and  $n = 3$ , this accumulated phase equals  $-90$ , placing the product on the imaginary axis. Since  $\text{Tr}_n^{(6)} = 2 \cdot \text{Re}((\zeta_6; \zeta_6)_n)$ , this gives zero. The special arithmetic of  $6 = 2 \times 3$ —combining the properties of  $\zeta_2 = -1$  and  $\zeta_3$ —creates this perfect cancellation.

**Remark 3.6** (On the Full Galois Trace). The *full* Galois trace  $\text{Tr}_6(\xi(2)) = \sum_{n=1}^5 c_n(2) \cdot \text{Tr}_n^{(6)}$  involves the unknown coefficients  $c_4(2)$  and  $c_5(2)$ . While the partial trace using  $c_1, c_2, c_3$  equals  $\zeta(2)$  exactly (Theorem 3.3), the full trace receives additional contributions:

$$\text{Tr}_6(\xi(2)) = \zeta(2) + c_4(2) \cdot 6 + c_5(2) \cdot 12.$$

Determining the exact value of  $\text{Tr}_6(\xi(2))$  requires computing  $c_4(2)$  and  $c_5(2)$ , which in turn requires Zagier's formula for  $C_3(2)$ .

Note that direct computation of  $\xi_{\zeta_N}(s)$  via the series  $\zeta_q(s) = \sum q^n/[n]_q^s$  is problematic at roots of unity, since  $[n]_q = 0$  when  $N \mid n$ , creating poles. The Habiro expansion provides the correct interpretation of  $\xi_{\zeta_N}(s)$  as a finite sum.

#### 4. TAYLOR-HABIRO TRANSFORMATION

Write  $q = e^{-\varepsilon}$  with  $\varepsilon \rightarrow 0^+$ . Both the Taylor basis  $\{\varepsilon^k\}$  and the Habiro basis  $\{(q; q)_n\}$  expand functions near  $q = 1$ :

$$\xi_q(s) = \sum_{m=0}^{\infty} T_m \varepsilon^m = \sum_{n=0}^{\infty} c_n(q; q)_n.$$

The  $q$ -Pochhammer symbol has the expansion

$$(q; q)_n = n! \cdot \varepsilon^n \cdot \left( 1 - \frac{n(n+1)}{4} \varepsilon + O(\varepsilon^2) \right).$$

This gives a triangular relationship:  $(q; q)_n$  starts at order  $\varepsilon^n$ , so  $c_n$  can be determined recursively from  $T_0, T_1, \dots, T_n$ .

Explicitly, if  $A_{n,m}$  denotes the coefficient of  $\varepsilon^m$  in  $(q; q)_n$ , then:

$$c_m = \frac{1}{m!} \left( T_m - \sum_{n=0}^{m-1} c_n \cdot A_{n,m} \right).$$

The Taylor coefficients  $T_m$  of  $\xi_q(s) = (q; q)_{s-1} \cdot \zeta_q(s)$  are computed from Zagier's Taylor coefficients  $C_k(s)$  of  $\zeta_q(s)$ .

## 5. ZAGIER'S FORMULAS

D. Zagier (personal communication) provided explicit formulas for the Taylor coefficients of  $\zeta_q(s)$ . Writing

$$\zeta_q(s) = \sum_{k=0}^{\infty} C_k(s) \varepsilon^k \quad \text{where } q = e^{-\varepsilon},$$

we have:

$$C_0(s). \quad C_0(s) = \zeta(s).$$

$$C_1(s).$$

$$C_1(s) = \begin{cases} -\zeta(2) - \frac{1}{2} & \text{if } s = 2, \\ \frac{s-2}{2}\zeta(s-1) - \frac{s}{2}\zeta(s) & \text{if } s > 2. \end{cases}$$

$$C_2(s).$$

$$C_2(s) = \begin{cases} \frac{13}{24} + \frac{7}{12}\zeta(2) & \text{if } s = 2, \\ -\frac{7}{24} - \frac{3}{4}\zeta(2) + \frac{5}{4}\zeta(3) & \text{if } s = 3, \\ \frac{(s-3)(3s-4)}{24}\zeta(s-2) - \frac{s(s-2)}{4}\zeta(s-1) + \frac{s(3s+1)}{24}\zeta(s) & \text{if } s > 3. \end{cases}$$

## 6. NUMERICAL VERIFICATION

We verify our exact formulas numerically:

$s$	$c_s(s)$ (exact)	Numerical
2	$-\frac{1}{4} - \frac{\zeta(2)}{2}$	-1.0725
3	$\frac{\zeta(2)}{6} - \frac{\zeta(3)}{2}$	-0.3269
4	$\frac{\zeta(3)}{4} - \frac{\zeta(4)}{2}$	-0.2406
5	$\frac{3\zeta(4)}{10} - \frac{\zeta(5)}{2}$	-0.1938

  

$s$	$c_{s+1}(s)$ (exact)	Numerical
2	$\frac{1}{144} - \frac{5\zeta(2)}{72}$	-0.1073
3	$-\frac{7}{288} - \frac{\zeta(3)}{12}$	-0.1245
4	$\frac{\zeta(2)}{60} - \frac{11\zeta(4)}{120}$	-0.0718
5	$\frac{11\zeta(3)}{360} - \frac{7\zeta(5)}{72}$	-0.0641

The diagonal identity  $\xi_{\zeta_N}(N) = N \cdot \zeta(N)$  is verified numerically:

$N$	$\xi_{\zeta_N}(N)$ (computed)	$N \cdot \zeta(N)$
2	3.28987	3.28987
3	3.60617	3.60617
4	4.32929	4.32929
5	5.18464	5.18464
6	6.10406	6.10406

## 7. THE SKIPPING PHENOMENON

**Remark 7.1** (The Skipping Pattern). The Habiro coefficients exhibit a remarkable “skipping” phenomenon:

- $c_s(s) \in \mathbb{Q} \cdot \zeta(s-1) + \mathbb{Q} \cdot \zeta(s)$  (plus  $\mathbb{Q}$  for  $s = 2$ )
- $c_{s+1}(s) \in \mathbb{Q} \cdot \zeta(s-2) + \mathbb{Q} \cdot \zeta(s)$  (plus  $\mathbb{Q}$  for  $s = 2, 3$ )—skips  $\zeta(s-1)$

## 8. REMARKS AND QUESTIONS

**Remark 8.1.** Conjecturally, the space  $\mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3) + \cdots + \mathbb{Q}\zeta(s)$  is the space of periods of mixed Tate motives over  $\mathbb{Z}$  of weight  $\leq s$ . The appearance of this space in our formulas suggests a motivic interpretation of the Habiro coefficients.

**Remark 8.2.** The non-integrality of these coefficients contrasts with the GSWZ framework [1], where state integrals from knot complements yield integer Habiro coefficients. This suggests that  $\xi_q(s)$  does not arise from a GSWZ-type construction.

**Question 8.3.** Is there a closed formula for all  $c_n(s)$  as  $\mathbb{Q}$ -linear combinations of  $\{1, \zeta(2), \dots, \zeta(s)\}$ ?

**Question 8.4.** Can  $c_4(2)$  and  $c_5(2)$  be determined individually? This would require either Zagier’s formula for  $C_3(2)$  or another computational approach.

**Question 8.5.** What is the exact value of the full Galois trace  $\text{Tr}_N(\xi(2))$  for various  $N$ ? Is  $\text{Tr}_N(\xi(2))$  always a rational number?

**Question 8.6.** Is there a conceptual explanation for the “partial trace miracle” at  $N = 6$ ? Does it arise from properties of cyclotomic fields, the Habiro ring structure, or motivic considerations?

**Question 8.7.** Does the structural result  $c_n(s) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \cdots + \mathbb{Q}\zeta(s)$  hold for all  $s \geq 2$ ?

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