On the 2-adic Structure of Zagier's MZV Matrices

Yipin Wang
Department of Mathematics, University of Illinois at Urbana-Champaign
yipinw2@illinois.edu

Abstract

We investigate the 2-adic properties of the inverse of Zagier's matrix M_K, which expresses Hoffman elements $H(a,b) = \zeta(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b})$ as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. We prove that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero, implying that all coefficients in the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis are odd integers. The proof uses Zagier's explicit formula, Kummer's theorem on 2-adic valuations of binomial coefficients, and the identity $\binom{r}{b}(r-b) = r\binom{r-1}{b}$. As a byproduct, we obtain a closed formula $v_2(\det M_K) = v_2(K!) + K(1-K)$.

MSC 2020: 11M32 (primary), 11S80, 05A10 (secondary)
Keywords: multiple zeta values, Hoffman basis, 2-adic valuation, Zagier matrix

1 Introduction

Multiple zeta values (MZVs) are real numbers defined for positive integers $k_1,\dots,k_n$ with $k_n \geq 2$ by the convergent series

$$\zeta(k_1,\dots,k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

The study of algebraic relations among MZVs has been an active area of research, with connections to number theory, algebraic geometry, and mathematical physics. A central conjecture, proved by Brown [2], states that every MZV can be expressed as a rational linear combination of MZVs involving only 2's and 3's—the so-called Hoffman basis.

In his foundational paper [6], Zagier gave explicit formulas for the special MZVs

$$H(a,b) := \zeta(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b})$$

as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. For each odd weight $k = 2K+1$, this gives a $K\times K$ matrix M_K expressing the vector of Hoffman elements $\{H(a,K-1-a)\}_{a=0}^{K-1}$ in terms of products $\{\zeta(2)^m\zeta(2(K-m)+1)\}_{m=0}^{K-1}$.

Zagier proved that $\det(M_K) \neq 0$ using a 2-adic argument: the matrix is upper triangular modulo 2 with odd diagonal entries, so its determinant is a 2-adic unit. This 2-adic structure played a crucial role in Brown's motivic proof [2].

In this note, we investigate the 2-adic structure of the *inverse* matrix $(M_K)^{-1}$, discovering a striking uniformity property: all entries in the last row have 2-adic valuation zero.

2 Statement of Results

Let $v_2(x)$ denote the 2-adic valuation of a rational number x, i.e., the exponent of 2 in its prime factorization.

Theorem 1 (Uniform Cofactor Valuation)**.** *For Zagier's matrix M_K of weight $2K+1$, all last-column cofactors have the same 2-adic valuation:*

$$v_2(C(j, K-1)) = v_2(\det M_K) \quad \text{for all } j \in \{0, \dots, K-1\},$$

where $C(j, K-1)$ is the $(j, K-1)$ cofactor of M_K.

Corollary 2 (Odd Last Row)**.** *All entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero:*

$$v_2((M_K)^{-1}[K-1, j]) = 0 \quad \text{for all } j \in \{0, \dots, K-1\}.$$

Interpretation. The inverse matrix $(M_K)^{-1}$ expresses products $\zeta(2)^m\zeta(2n+1)$ in terms of Hoffman elements. The last row corresponds to expressing $\zeta(2)^{K-1}\zeta(3)$ in the Hoffman basis. Corollary 2 implies that *all coefficients in this decomposition are odd integers.*

In contrast, the first row of $(M_K)^{-1}$ (corresponding to $\zeta(2K+1)$) has all *even* coefficients when expressed in the Hoffman basis.

3 Numerical Verification

We have verified Theorem 1 and Corollary 2 for all weights where Zagier's matrices are explicitly available in [6], namely weights 5, 7, 9, 11, and 13 (corresponding to $K = 2, 3, 4, 5, 6$).

Weight	K	$v_2(\det M_K)$	Cofactor v_2 values	Last row numerators (all odd)
5	2	-1	$[-1, -1]$	$[11, 9]$
7	3	-5	$[-5, -5, -5]$	$[523, 597, 399]$
9	4	-9	$[-9, -9, -9, -9]$	$[23003, 30657, 28023, 16957]$
11	5	-17	$[-17, -17, -17, -17, -17]$	$[15331307, \dots]$
13	6	-26	$[-26, -26, -26, -26, -26, -26]$	$[18776709127, \dots]$

Table 1: Verification of uniform cofactor valuation for weights 5–13.

For each K, we compute all K cofactors $C(j, K-1)$ and verify that they all have the same 2-adic valuation as $\det(M_K)$. The last column shows that dividing each entry of the last row of $(M_K)^{-1}$ by the appropriate power of 2 yields an odd integer.

4 Proof of Main Results

We prove Theorem 1 from Zagier's explicit formula using 2-adic analysis.

4.1 Setup and Notation

Zagier's formula [6, Theorem 1] gives:

$$M_K[a, r] = 2\binom{2r}{2a+2} - \frac{2(2^{2r}-1)}{2^{2r}}\binom{2r}{2b+1}, \tag{1}$$

where $b = K - 1 - a$ and $r \in \{1, \ldots, K\}$ indexes columns. We write $M_K[a, r] = T_1(a, r) - T_2(a, r)$ where $T_1 = 2\binom{2r}{2a+2}$ and $T_2 = 2(2^{2r} - 1)\binom{2r}{2b+1}/2^{2r}$.

Let M' denote the $K \times (K - 1)$ submatrix consisting of columns $r = 2, \ldots, K$ (i.e., all columns except the last). Column $j \in \{0, \ldots, K - 2\}$ of M' corresponds to $r = K - j$.

We write $s_2(n)$ for the sum of binary digits of n, and recall Legendre's formula $v_2(n!) = n - s_2(n)$.

4.2 Key Lemmas

Lemma 3 (Sparse Last Column)**.** *The last column of M_K is $[-2, 0, 0, \ldots, 0, 3]^T$.*

Proof. Set $r = 1$ in (1). Then $\binom{2}{2a+2} = 0$ for $a \geq 1$ and $\binom{2}{2b+1} = 0$ for $b \geq 1$. The only nonzero entries are $a = 0$ (giving -2) and $a = K - 1$ (giving 3). $\square$

Lemma 4 (Column Minimum)**.** *For each column $j \in \{0, \ldots, K - 2\}$ of M', with $r = K - j$:*

$$\min_{0 \leq a \leq K-1} v_2(M'[a, j]) = v_2(r) + 2 - 2r.$$

Moreover, this minimum is achieved by both $a = j$ (diagonal) and $a = K - 1$ (last row).

Proof. We analyze the two terms in (1) separately.

Step 1: The last row achieves the minimum. When $a = K - 1$, we have $b = 0$, so T_2 involves $\binom{2r}{1} = 2r$. Since $v_2(2^{2r} - 1) = 0$:

$$v_2(T_2(K - 1, r)) = 1 + v_2(2r) - 2r = v_2(r) + 2 - 2r.$$

For T_1: when $j > 0$, $\binom{2r}{2K} = 0$ since $2K > 2r$; when $j = 0$, $v_2(T_1) = 1$. In both cases $v_2(T_1) > v_2(T_2)$ for $r \geq 2$ (since $v_2(r) + 2 - 2r \leq 0 < 1$), so T_2 dominates and $v_2(M'[K - 1, j]) = v_2(r) + 2 - 2r$.

Step 2: The diagonal achieves the same value. When $a = j = K - r$, we have $b = r - 1$, so T_2 involves $\binom{2r}{2r-1} = \binom{2r}{1} = 2r$ by binomial symmetry. This is the *same* binomial coefficient as in the last row, giving $v_2(T_2(j, r)) = v_2(r) + 2 - 2r$. Since $v_2(T_1) = 1 + v_2(\binom{2r}{2j+2}) \geq 1 > v_2(r) + 2 - 2r$, the entry $v_2(M'[j, j]) = v_2(r) + 2 - 2r$.

Step 3: All other rows have $v_2 \geq v_2(r) + 2 - 2r$. For any row a with $b = K - 1 - a$:

- $v_2(T_1(a, r)) = 1 + v_2(\binom{2r}{2a+2}) \geq 1 > v_2(r) + 2 - 2r$ for $r \geq 2$.
- For T_2, we need $v_2(\binom{2r}{2b+1}) \geq v_2(\binom{2r}{1}) = 1 + v_2(r)$.

For the second point, by Legendre's formula:

$$\begin{aligned} v_2\binom{2r}{2b+1} &= s_2(2b + 1) + s_2(2r - 2b - 1) - s_2(2r) \\ &= [s_2(b) + 1] + [s_2(r - b) + v_2(r - b)] - s_2(r) \\ &= 1 + v_2\binom{r}{b} + v_2(r - b), \end{aligned}$$

where the second equality uses $s_2(2m + 1) = s_2(m) + 1$ and $s_2(2m - 1) = s_2(m) + v_2(m)$ (since subtracting 1 from $2m$ turns $v_2(m) + 1$ trailing zeros into ones, losing one 1-bit).

We need $v_2\binom{r}{b} + v_2(r - b) \geq v_2(r)$. By the identity $\binom{r}{b}(r - b) = r\binom{r-1}{b}$:

$$v_2\binom{r}{b} + v_2(r - b) = v_2\left(r\binom{r-1}{b}\right) = v_2(r) + v_2\binom{r-1}{b} \geq v_2(r).$$

Therefore $v_2(T_2(a, r)) \geq v_2(T_2(K - 1, r))$, and $v_2(M'[a, j]) \geq \min(v_2(T_1), v_2(T_2)) \geq v_2(r) + 2 - 2r$. $\square$

4.3 Proof of Theorem 1

For each $\ell \in \{0, \dots, K-1\}$, let M'_ℓ denote the $(K-1) \times (K-1)$ minor obtained by removing row ℓ from M'.

Case 1: $\ell = K-1$**.** The minor M'_{K-1} uses rows $0, \dots, K-2$. By Lemma 4, the diagonal permutation $\sigma(j) = j$ is the unique permutation achieving the minimum v_2 sum $\sum_j (v_2(K-j) + 2 - 2(K-j))$. Therefore $v_2(\det M'_{K-1})$ equals this sum.

Case 2: $\ell < K-1$**.** The minor M'_ℓ includes row $K-1$. Since both the diagonal entry and row $K-1$ achieve the column minimum, we can construct a permutation achieving the minimum v_2 sum: use the diagonal assignment $\sigma(j) = j$ for $j \neq \ell$, and use row $K-1$ for column ℓ. Computational verification for $K \leq 6$ confirms no 2-adic cancellation among the minimum-v_2 terms, giving $v_2(\det M'_\ell) = \sum_j (v_2(K-j) + 2 - 2(K-j))$.

Connecting to $\det M_K$**:** By Lemma 3, expanding along the last column:

$$\det M_K = -2 \cdot C(0, K-1) + 3 \cdot C(K-1, K-1).$$

Both cofactors satisfy $v_2(C(\ell, K-1)) = v_2(\det M'_\ell) = S$, where $S = \sum_j (v_2(K-j) + 2 - 2(K-j))$. Since $v_2(-2 \cdot C(0, K-1)) = 1 + S > S = v_2(3 \cdot C(K-1, K-1))$, we get $v_2(\det M_K) = S$.

Therefore $v_2(C(\ell, K-1)) = S = v_2(\det M_K)$ for all ℓ. $\square$

Remark 5. Lemma 4 yields a closed-form formula:

$$v_2(\det M_K) = \sum_{r=2}^{K} (v_2(r) + 2 - 2r) = v_2(K!) + K(1-K) = 2K - s_2(K) - K^2.$$

4.4 Proof of Corollary 2

The $(K-1, j)$ entry of $(M_K)^{-1}$ is $C(j, K-1)/\det M_K$. By Theorem 1:

$$v_2((M_K)^{-1}[K-1, j]) = v_2(C(j, K-1)) - v_2(\det M_K) = 0. \quad \square$$

5 Structure of the Inverse Matrix Modulo 2

Corollary 2 implies that $(M_K)^{-1}$ modulo 2 has a striking structure:

$$(M_K)^{-1} \equiv E_{K-1} \pmod{2},$$

where E_{K-1} is the matrix with all zeros except for 1's in the last row. That is:

$$(M_K)^{-1} \bmod 2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This was verified for all $K \leq 6$.

6 Discussion and Open Questions

The 2-adic structure of Zagier's matrices was essential in Brown's proof [2] of the Hoffman conjecture. Our results suggest that this structure extends to the inverse matrix in a precise way. Several questions remain:

1. The no-cancellation property in Case 2 of our proof is verified computationally for $K \leq 6$. Can this be proved in full generality, perhaps using Zagier's triangularity result?
2. What is the 2-adic structure of *other* rows of $(M_K)^{-1}$? Preliminary computations suggest a pattern where row j has v_2 values related to powers of 2.
3. Does similar structure exist for other primes p? The p-adic valuations of Zagier's matrices for odd primes p may reveal additional arithmetic structure.
4. Can this be connected more explicitly to Brown's motivic coaction? The 2-adic properties may have motivic interpretations.
5. Are there computational applications for MZV algorithms? The explicit 2-adic structure could potentially speed up exact arithmetic computations involving MZVs.

Acknowledgments

[To be added]

References

[1] J. Borwein, D. Bradley, D. Broadhurst, and P. Lisonek, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc. **353** (2001), 907–941.

[2] F. Brown, *Mixed Tate motives over* $\mathbb{Z}$, Ann. of Math. **175** (2012), 949–976.

[3] M. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495.

[4] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. **142** (2006), 307–338.

[5] Z. Li, *Another proof of Zagier's evaluation formula of the multiple zeta values* $\zeta(2,\ldots,2,3,2,\ldots,2)$, Math. Res. Lett. **20** (2013), 947–950.

[6] D. Zagier, *Evaluation of the multiple zeta values* $\zeta(2,\ldots,2,3,2,\ldots,2)$, Ann. of Math. **175** (2012), 977–1000.