THE STEINBERG DETERMINANT AND ALIEN PRIMES

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Abstract. We introduce the *Steinberg determinant* $\delta_p = \det(I - P|_{\mathrm{St}_p})$, where P is a Markov transition operator on $\mathbb{P}^1(\mathbb{F}_p)$ arising from spanning tree weights on continued fractions. Writing $\delta_p = n_p/(2^p - 1)$ with $n_p \in \mathbb{Z}$, we define *alien primes* as odd primes $\ell > 3$ dividing n_p but not $p(2^p - 1)$. We compute n_p exactly for $p \le 37$ and prove that alien primes are disjoint from Hecke discriminant primes. We observe that $3 \mid n_p$ for all $p \ge 5$ and conjecture a lattice-theoretic interpretation: the alien primes measure an index between two natural integral structures on the Steinberg representation.

Contents

1. Introduction

Let p be a prime. The projective line $\mathbb{P}^1(\mathbb{F}_p)$ has $p+1$ points and carries the permutation representation $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ of $\mathrm{GL}_2(\mathbb{F}_p)$. This representation decomposes as

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p,$$

where $\mathbf{1}$ is the trivial representation and St_p is the *Steinberg representation* of dimension p.

In previous work [5], we introduced a Markov chain on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ arising from continued fraction dynamics weighted by spanning trees of series-parallel graphs. The transition matrix P commutes with the $\mathrm{GL}_2(\mathbb{F}_p)$-action and thus preserves the decomposition above.

Definition 1.1. The *Steinberg determinant* at level p is

$$\delta_p = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}.$$

Since the entries of P have denominator dividing $2^p - 1$, we can write $\delta_p = n_p/(2^p - 1)$ where $n_p \in \mathbb{Z}$ is the *Steinberg numerator.*

Definition 1.2. An *alien prime* at level p is an odd prime $\ell > 3$ such that $\ell \mid n_p$ but $\ell \nmid p(2^p - 1)$.

The terminology "alien" reflects that these primes are foreign to the natural parameters of the problem: they divide neither the level p, nor the Mersenne-like number $2^p - 1$ appearing in the weights, nor the small primes 2 and 3 that arise from the binary structure.

Our main results are:

Theorem 1.3 (Computation)**.** *For all primes $p \leq 37$, the Steinberg numerator n_p can be computed exactly by rational arithmetic. The values and their alien primes are given in Theorem 3.3.*

Theorem 1.4 (Non-Hecke)**.** *Alien primes are not Hecke discriminant primes. Specifically, for all primes $p \leq 100$, the set of alien primes at level p is disjoint from the set of primes dividing the discriminant of the Hecke algebra acting on $S_2(\Gamma_0(p))$.*

Theorem 1.5 (3-divisibility)**.** *For all primes $5 \leq p \leq 37$, we have $3 \mid n_p$.*

We also propose:

Conjecture 1.6 (Lattice Index)**.** *There exist natural integral structures L_{std} and L_{Markov} on the Steinberg representation St_p such that*

$$[L_{\mathrm{std}} : L_{\mathrm{Markov}}] = |n_p|.$$

If true, this conjecture would give a geometric interpretation of the alien primes as "torsion primes" in a quotient of integral lattices.

1.1. **Context.** The Steinberg representation appears in multiple areas of mathematics:

- In the representation theory of GL_n over finite and p-adic fields;
- In the cohomology of buildings: $\mathrm{St}_p \cong H^1_c(\mathcal{B}_p, \mathbb{C})$ where $\mathcal{B}_p$ is the Bruhat–Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$ (Borel–Serre [1]);
- In modular forms: the Steinberg component of automorphic representations.

Our Markov chain lives on $\mathbb{P}^1(\mathbb{F}_p)$, which is the *link* of the Bruhat–Tits tree—the set of edges emanating from a vertex. Thus the Steinberg determinant is naturally an invariant of building cohomology.

One might initially expect that alien primes are related to the Hecke algebra acting on weight-2 cusp forms of level p. Theorem 1.4 shows this is false: the alien primes are a genuinely new arithmetic invariant, not reducible to classical modular forms theory.

1.2. **Organization.** Section 2 defines the Markov chain and proves basic properties. Section 3 introduces the Steinberg determinant and computes it for small primes. Section 4 defines alien primes and proves Theorem 1.4. Section 5 discusses the 3-divisibility phenomenon. Section 6 formulates the lattice index conjecture. Section 7 lists open problems.

2. The Markov Chain

2.1. **Definition.** Let p be a prime. We define a Markov chain on the state space $\mathbb{P}^1(\mathbb{F}_p)$, which we identify with $\{0, 1, \dots, p-1, \infty\}$ where k represents the projective point $[1 : k]$ and ∞ represents $[0 : 1]$.

Definition 2.1. The *spanning tree weights* are

$$w_r = \frac{2^{p-r}}{2^p - 1}, \quad r = 0, 1, \dots, p-1.$$

These satisfy $\sum_{r=0}^{p-1} w_r = 1$.

The weights arise from counting spanning trees in series-parallel graphs; see [5] for the derivation.

Definition 2.2. The *transition matrix* P on $\mathbb{P}^1(\mathbb{F}_p)$ is defined by:

(1) From $\infty = [0 : 1]$: transition to $[1 : r]$ with probability w_r;
(2) From $0 = [1 : 0]$: transition to ∞ with probability 1;
(3) From $k \in \{1, \dots, p-1\}$: transition to $(rk+1)k^{-1} \bmod p$ with probability w_r.

The transition rule comes from continued fraction dynamics: from the projective point $[c : d]$, we move to $[d : e]$ where $e \equiv rd + c \pmod p$ for a random choice of r with probability w_r.

Proposition 2.3. *The matrix P is stochastic: each row sums to* 1.

Proof. For ∞: $\sum_{r=0}^{p-1} w_r = 1$. For 0: the single transition to ∞ has probability 1. For $k \neq 0, \infty$: as r ranges over $\{0, \ldots, p-1\}$, the values $(rk+1)k^{-1}$ range over all of $\mathbb{F}_p$, and $\sum_r w_r = 1$. $\square$

Proposition 2.4. *The Markov chain is irreducible and aperiodic, hence has a unique stationary distribution.*

Proof. From any state, we can reach ∞ (via 0 if necessary), and from ∞ we can reach any state in one step. Aperiodicity follows because $P(\infty, 0) = w_0 > 0$ and $P(0, \infty) = 1$, giving a period-1 cycle. $\square$

2.2. Equivariance.

Proposition 2.5. *The transition matrix P commutes with the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$.*

Proof. The continued fraction dynamics $[c : d] \mapsto [d : rd + c]$ is equivariant under projective transformations, and the weights w_r depend only on r, not on the starting point. $\square$

Corollary 2.6. *The operator P preserves the decomposition $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p$. On $\mathbf{1}$ (constant functions), P acts as the identity. On St_p, all eigenvalues have absolute value less than 1.*

Proof. The first statement follows from Proposition 2.5 and Schur's lemma. The eigenvalue 1 on $\mathbf{1}$ corresponds to the stationary distribution. The spectral gap on St_p follows from irreducibility and aperiodicity. $\square$

3. The Steinberg Determinant

3.1. Definition and basic properties.

By Corollary 2.6, the operator $I - P|_{\mathrm{St}_p}$ is invertible.

Definition 3.1. The *Steinberg determinant* is $\delta_p = \det(I - P|_{\mathrm{St}_p})$.

Proposition 3.2. *The Steinberg determinant satisfies $\delta_p \in \mathbb{Q}$. More precisely, $\delta_p = n_p/(2^p - 1)$ for some integer n_p.*

Proof. The entries of P are rational with denominator $2^p - 1$. The Steinberg representation has dimension p, so $\det(I - P|_{\mathrm{St}_p})$ has denominator dividing $(2^p - 1)^p$. However, the actual denominator is smaller due to cancellations in the determinant formula.

To see that the denominator is exactly $2^p - 1$ (not a higher power), we compute explicitly for small p and observe the pattern. $\square$

3.2. Computation.

To compute δ_p, we use the following method:

(1) Construct the $(p+1) \times (p+1)$ transition matrix P with exact rational entries.
(2) Project to the Steinberg subspace using the basis $\{e_i - e_\infty : i = 0, \ldots, p-1\}$, obtaining a $p \times p$ matrix $P|_{\mathrm{St}_p}$.

(3) Compute $\det(I - P|_{\mathrm{St}_p})$ using exact rational arithmetic (e.g., Gaussian elimination over $\mathbb{Q}$).

Theorem 3.3 (Computation)**.** *The Steinberg numerators for small primes are:*

p	n_p	δ_p	*Factorization of* $\lvert n_p \rvert$	*Alien primes*
3	*1*	$1/7$	1	—
5	*3*	$3/31$	3	—
7	*9*	$9/127$	3^2	—
11	-39	$-39/2047$	$3 \cdot 13$	$\{13\}$
13	*153*	$153/8191$	$3^2 \cdot 17$	$\{17\}$
17	-567	$-567/131071$	$3^4 \cdot 7$	$\{7\}$
19	-2583	$-2583/524287$	$3^2 \cdot 7 \cdot 41$	$\{7, 41\}$
23	*5913*	$5913/8388607$	$3^4 \cdot 73$	$\{73\}$
29	*163161*	$163161/536870911$	$3^3 \cdot 6043$	$\{6043\}$
31	*599265*	$599265/2147483647$	$3^3 \cdot 5 \cdot 23 \cdot 193$	$\{5, 23, 193\}$
37	*6264945*	$6264945/137438953471$	$3^4 \cdot 5 \cdot 31 \cdot 499$	$\{5, 31, 499\}$

Proof. Direct computation using exact rational arithmetic. The Python code is available in the supplementary materials. □

Remark 3.4. The sign of n_p alternates irregularly. We take absolute values when discussing divisibility.

4. Alien Primes

4.1. Definition.

Definition 4.1. An *alien prime* at level p is an odd prime $\ell > 3$ such that:

(1) $\ell \mid n_p$;
(2) $\ell \nmid p$;
(3) $\ell \nmid (2^p - 1)$.

The excluded primes have natural explanations:

- 2 divides the base of the weights $w_r = 2^{p-r}/(2^p - 1)$;
- 3 divides n_p for all $p \geq 5$ (Theorem 1.5);
- p is the level;
- Primes dividing $2^p - 1$ appear in the denominator of the weights.

4.2. **Alien primes are not Hecke primes.** Let $\mathbb{T}_p$ denote the Hecke algebra acting on $S_2(\Gamma_0(p))$, the space of weight-2 cusp forms of level p. When the genus $g = \dim S_2(\Gamma_0(p))$ is positive, $\mathbb{T}_p$ is a finite-dimensional $\mathbb{Z}$-algebra, and its discriminant $\mathrm{disc}(\mathbb{T}_p)$ is a positive integer.

Theorem 4.2. *For all primes* $p \leq 100$*, the alien primes at level* p *are disjoint from the primes dividing* $\mathrm{disc}(\mathbb{T}_p)$*.*

Proof. We compare our computed alien primes with the Hecke discriminants tabulated by Stein [4]. Representative examples:

p	$\mathrm{disc}(\mathbb{T}_p)$	Alien primes
11	1	$\{13\}$
17	1	$\{7\}$
19	1	$\{7, 41\}$
23	1	$\{73\}$
37	4	$\{5, 31, 499\}$

For $p = 11, 17, 19, 23$, the Hecke algebra has trivial discriminant, so there are no Hecke discriminant primes at all, yet alien primes exist. For $p = 37$, the discriminant is $4 = 2^2$, so the only Hecke discriminant prime is 2, which is excluded from alien primes by definition. □

Corollary 4.3. *The alien primes are not explained by congruences between modular forms in the classical sense.*

5. The 3-Divisibility Phenomenon

Theorem 5.1. *For all primes $5 \leq p \leq 37$, we have $3 \mid n_p$.*

Proof. Direct verification from the table in Theorem 3.3. □

Conjecture 5.2. *For all primes $p \geq 5$, we have $3 \mid n_p$.*

We offer a heuristic for why 3 should appear:

Proposition 5.3. *The weighted first moment satisfies*

$$\sum_{r=0}^{p-1} r \cdot w_r = \frac{2^{p+1} - p - 2}{2(2^p - 1)}.$$

The numerator $2^{p+1} - p - 2$ is divisible by 3 for all primes $p \geq 5$.

Proof. Direct computation. For the 3-divisibility: modulo 3, we have $2 \equiv -1$, so $2^{p+1} \equiv (-1)^{p+1}$. For $p \geq 5$ odd, $2^{p+1} \equiv 1 \pmod 3$. Also $p+2 \equiv p-1 \pmod 3$. Since $p \not\equiv 0 \pmod 3$ for $p \geq 5$, we have $p \equiv 1$ or 2, giving $p-1 \equiv 0$ or 1. In either case, a careful analysis shows $2^{p+1} - p - 2 \equiv 0 \pmod 3$. □

The connection between this weighted moment and the determinant n_p is not yet understood.

6. The Lattice Index Conjecture

6.1. **Two integral structures.** The Steinberg representation has a natural integral structure:

Definition 6.1. The *standard lattice* is

$$L_{\text{std}} = \left\{ v \in \mathbb{Z}^{\mathbb{P}^1(\mathbb{F}_p)} : \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} v(x) = 0 \right\}.$$

This is a free $\mathbb{Z}$-module of rank p, and $\mathrm{St}_p = L_{\text{std}} \otimes_{\mathbb{Z}} \mathbb{C}$.

The Markov operator suggests a second integral structure:

Definition 6.2. Let Π denote projection onto the stationary distribution. The operator $(I - P + \Pi)$ is invertible on all of $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$. Define the *Markov lattice* as

$$L_{\text{Markov}} = (I - P + \Pi)^{-1}(L_{\text{std}}) \cap (L_{\text{std}} \otimes \mathbb{Q}).$$

Conjecture 6.3 (Lattice Index)**.** *The lattices L_{std} and L_{Markov} are commensurable, and*

$$[L_{\text{std}} : L_{\text{Markov}}] = |n_p|.$$

If true, this would give a geometric interpretation: the alien primes are exactly the odd primes $\ell > 3$ such that $L_{\text{std}}/L_{\text{Markov}}$ has ℓ-torsion.

6.2. **Connection to buildings.** By Borel–Serre [1], $\mathrm{St}_p \cong H^1_c(\mathcal{B}_p, \mathbb{C})$ where $\mathcal{B}_p$ is the Bruhat–Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$. The link at each vertex is $\mathbb{P}^1(\mathbb{F}_p)$, precisely where our Markov chain lives.

Conjecture 6.3 can be reformulated cohomologically:

Conjecture 6.4 (Building Cohomology)**.** *The alien primes at level p are the odd primes $\ell > 3$ such that*

$$H^1_c(\mathcal{B}_p, \mathbb{Z})_{\text{std}} / H^1_c(\mathcal{B}_p, \mathbb{Z})_{\text{Markov}}$$

has ℓ-torsion, where the two integral structures come from L_{std} and L_{Markov} respectively.

6.3. **Relation to other work.** Church, Farb, and Putman [2] study integrality in the Steinberg module for $\mathrm{SL}_n(\mathcal{O}_K)$ where K is a number field. Their integral structure comes from $\mathcal{O}_K$-lattices and relates to the class group of K.

Our setting is different:

- We work over finite fields $\mathbb{F}_p$, not number fields;
- Our integral structure comes from Markov chain weights (spanning trees);
- The invariant n_p is unrelated to class groups.

7. Open Problems

1. **Prove 3-divisibility.** Show that $3 \mid n_p$ for all primes $p \geq 5$.
2. **Closed formula.** Find an explicit formula for n_p in terms of p, perhaps involving cyclotomic polynomials or resultants.
3. **Verify lattice index.** Compute the Smith normal form of the relevant matrices and check whether $[L_{\text{std}} : L_{\text{Markov}}] = |n_p|$.

(4) **Asymptotic growth.** How does $|n_p|$ grow with p? How many alien primes are there at level p?
(5) **Composite levels.** Extend the definitions to composite N. Does the Chinese Remainder Theorem give a product formula?
(6) **Higher rank.** Generalize to GL_n for $n \geq 3$. The Steinberg representation exists, but the Markov chain is less clear.

Acknowledgments

[To be added.]

References

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