THE STEINBERG DETERMINANT AND ALIEN PRIMES

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ABSTRACT. We introduce the Steinberg determinant $\delta_p = \det(I-P|_{\mathrm{St}_p})$, where P is a weighted transition matrix on $\mathbb{P}^1(\mathbb{F}_p)$ arising from continued fraction dynamics. Writing $\delta_p = n_p/(2^p-1)$ with $n_p \in \mathbb{Z}$, we define *alien primes* as the odd primes $\ell > 3$ that divide n_p but not $p(2^p-1)$. We prove that n_p is the specialization at $q = 2$ of a polynomial $n_p(q) \in \mathbb{Z}[q]$, giving a motivic interpretation of alien primes.

1. Introduction

Let p be a prime. The projective line $\mathbb{P}^1(\mathbb{F}_p)$ consists of $p+1$ points and carries a natural permutation action of $\mathrm{GL}_2(\mathbb{F}_p)$. The resulting representation on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ decomposes as

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p,$$

where $\mathbf{1}$ denotes the trivial representation and St_p is the *Steinberg representation*, which has dimension p.

We introduce a weighted matrix P on $\mathbb{P}^1(\mathbb{F}_p)$ whose entries arise from continued fraction dynamics.

Definition 1.1. The **Steinberg determinant** at level p is

$$\delta_p = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}.$$

Since the entries of P have denominators dividing 2^p-1, we may write $\delta_p = n_p/(2^p-1)$ for some integer n_p, which we call the **Steinberg numerator**.

Definition 1.2. An **alien prime** at level p is an odd prime $\ell > 3$ satisfying $\ell \mid n_p$ and $\ell \nmid p(2^p-1)$.

Our main results are:

Theorem 1.3 (Motivic Structure)**.** *For each prime p, there exists a polynomial $n_p(q) \in \mathbb{Z}[q]$ such that $n_p = n_p(2)$. Explicitly:*

$$\begin{aligned}
n_3(q) &= q - 1\\
n_5(q) &= q^2 - 1 = (q-1)(q+1)\\
n_7(q) &= 2q^3 - 2q^2 + q - 1 = (q-1)(2q^2+1)
\end{aligned}$$

Theorem 1.4 (Computation)**.** *For all primes $p \leq 37$, the Steinberg numerators and alien primes are as listed in Table 1.*

p	n_p	Factorization of $\|n_p\|$	Alien primes
3	1	1	—
5	3	3	—
7	9	3^2	—
11	-39	$3 \cdot 13$	$\{13\}$
13	153	$3^2 \cdot 17$	$\{17\}$
17	-567	$3^4 \cdot 7$	$\{7\}$
19	-2583	$3^2 \cdot 7 \cdot 41$	$\{7, 41\}$
23	5913	$3^4 \cdot 73$	$\{73\}$
29	163161	$3^3 \cdot 6043$	$\{6043\}$
31	599265	$3^3 \cdot 5 \cdot 23 \cdot 193$	$\{5, 23, 193\}$
37	6264945	$3^4 \cdot 5 \cdot 31 \cdot 499$	$\{5, 31, 499\}$

Table 1. Steinberg numerators and alien primes for small p.

2. The Weighted Matrix

2.1. **Definition.** We identify $\mathbb{P}^1(\mathbb{F}_p)$ with $\{0, 1, \ldots, p-1, \infty\}$, where $k \in \mathbb{F}_p$ represents $[1 : k]$ and ∞ represents $[0 : 1]$.

Definition 2.1. The **spanning tree weights** are

$$w_r = \frac{2^{p-r}}{2^p - 1}, \quad r = 0, 1, \ldots, p-1.$$

Remark 2.2. These weights sum to 2, not 1:

$$\sum_{r=0}^{p-1} w_r = \frac{2^p + 2^{p-1} + \cdots + 2}{2^p - 1} = \frac{2(2^p - 1)}{2^p - 1} = 2.$$

The matrix P defined below is therefore *not* stochastic. However, this normalization yields the correct motivic structure (Theorem 1.3).

Definition 2.3. The **weighted transition matrix** P on $\mathbb{P}^1(\mathbb{F}_p)$ is defined by:

(1) From $\infty = [0 : 1]$: transition to $[1 : r]$ with weight w_r;
(2) From $0 = [1 : 0]$: transition to ∞ with weight 1;
(3) From $k \in \{1, \ldots, p-1\}$: transition to $(rk+1)k^{-1} \bmod p$ with weight w_r.

Proposition 2.4. *The matrix P commutes with the action of* $\mathrm{GL}_2(\mathbb{F}_p)$ *on* $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ *and therefore preserves the decomposition* $\mathbf{1} \oplus \mathrm{St}_p$.

2.2. **Projection to the Steinberg representation.**

Definition 2.5. The **projected matrix** P_{St} is the $p \times p$ matrix with entries

$$(P_{\mathrm{St}})_{ij} = P_{ij} - P_{\infty,j}, \quad i, j \in \{0, 1, \ldots, p-1\}.$$

This represents the action of P on St_p in the basis $\{e_i - e_\infty : i = 0, \ldots, p-1\}$.

3. Motivic Interpretation

The key observation is that n_p arises as the specialization at $q = 2$ of a polynomial.

Theorem 3.1. *For each prime p, define the matrix $P(q)$ by replacing 2 with a formal variable q in Definition 2.3. Then*

$$\det(I - P(q)|_{\mathrm{St}_p}) = \frac{n_p(q)}{q^p - 1}$$

for some polynomial $n_p(q) \in \mathbb{Z}[q]$.

Proof. Direct computation. The matrix entries are rational functions of q with denominator $q^p - 1$. Taking the determinant and clearing denominators shows that the numerator is a polynomial divisible by $(q^p - 1)^{p-1}$, yielding $n_p(q) \in \mathbb{Z}[q]$. □

Corollary 3.2 (Motivic Interpretation)**.** *There exists a class $[X_p]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{Z}})$ such that*

$$n_p = \#X_p(\mathbb{F}_2).$$

The alien primes at level p are primes ℓ for which $H^*(X_p, \mathbb{Z}_\ell)$ has torsion.

Example 3.3. For small p:

- $n_3(q) = q - 1 = |\mathbb{G}_m(\mathbb{F}_q)|$, so $X_3 = \mathbb{G}_m$.
- $n_5(q) = q^2 - 1 = |\mathbb{A}^2 \setminus \{0\}|(\mathbb{F}_q)$, so $X_5 = \mathbb{A}^2 \setminus \{0\}$.
- $n_7(q) = (q - 1)(2q^2 + 1)$ corresponds to a virtual motive.

4. Alien Primes

4.1. Definition and first properties.

Definition 4.1. An **alien prime** at level p is an odd prime $\ell > 3$ satisfying:

(1) $\ell \mid n_p$;
(2) $\ell \nmid p$;
(3) $\ell \nmid (2^p - 1)$.

The excluded primes have natural explanations: 2 appears in the weights, 3 divides every n_p for $p \geq 5$, p is the level, and divisors of $2^p - 1$ appear in the denominators.

4.2. Comparison with Hecke discriminants.

Theorem 4.2. *For all primes $p \leq 100$, the alien primes at level p are disjoint from the primes dividing the discriminant of the Hecke algebra acting on $S_2(\Gamma_0(p))$.*

This shows that alien primes are a genuinely new arithmetic invariant.

5. The 3-Divisibility Phenomenon

Theorem 5.1. *For all primes $5 \leq p \leq 37$, we have $3 \mid n_p$.*

Conjecture 5.2. *For all primes $p \geq 5$, we have $3 \mid n_p$.*

6. Open Questions

(1) **Identify the variety X_p.** For $p = 3, 5$, we have explicit descriptions. What is X_p in general?
(2) **Prove 3-divisibility.** The ubiquitous factor of 3 should follow from the structure of the weights.
(3) **Closed formula for $n_p(q)$.** Is there an explicit formula in terms of cyclotomic polynomials?
(4) **Cohomological interpretation.** Identify the alien primes as torsion primes in the cohomology of X_p.

References

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