# ON THE 2-ADIC STRUCTURE OF ZAGIER'S MZV MATRICES

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Abstract. We investigate the 2-adic properties of the inverse of Zagier's matrix $M_K$, which expresses Hoffman elements $H(a,b) = \zeta(2,\ldots,2,3,2,\ldots,2)$ as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. We prove that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero (Theorem 1), implying that all coefficients in the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis are odd integers. As a companion result, we establish a closed-form inverse for the binomial core matrix $B_N[a,i] = \binom{2i}{2a}$ (Theorem 6): its inverse is given explicitly in terms of the Euler–secant numbers $E_{2n}$ and the hyperbolic secant function, with the exact 2-adic valuation of every entry governed by binary carry counting via Kummer's theorem. As byproducts, we obtain the closed formula $v_2(\det M_K) = 2K - s_2(K) - K^2$ and the congruence $E_{2n} \equiv 1 \pmod 4$ for all $n \geq 0$.

## 1. Introduction

Multiple zeta values (MZVs) are real numbers defined for positive integers $k_1,\ldots,k_n$ with $k_n \geq 2$ by the convergent series

$$\zeta(k_1,\ldots,k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

A central result, proved by Brown [1], states that every MZV is a $\mathbb{Q}$-linear combination of the *Hoffman basis* elements $\zeta(k_1,\ldots,k_n)$ where each $k_i \in \{2,3\}$.

Zagier [7] gave explicit formulas for the special MZVs

$$H(a,b) := \zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})$$

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as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. For each odd weight $w = 2K+1$, this gives a $K \times K$ matrix $M_K$ expressing the vector of Hoffman elements $\{H(a, K-1-a)\}_{a=0}^{K-1}$ in terms of products $\{\zeta(2)^m\zeta(2(K-m)+1)\}_{m=0}^{K-1}$.

Zagier proved that $\det(M_K) \neq 0$ using a 2-adic argument: the matrix is upper triangular modulo 2 with odd diagonal entries. This 2-adic structure played a crucial role in Brown's motivic proof [1].

In this paper, we investigate the 2-adic structure of the inverse matrix $(M_K)^{-1}$ and of a related binomial matrix. Our first main result (Theorem 1) establishes that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero, implying that the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis has exclusively odd coefficients. Our second main result (Theorem 6) gives a closed-form inverse for the *binomial core matrix* $B_N[a,i] = \binom{2i}{2a}$, expressed in terms of the Euler–secant numbers and the function $\operatorname{sech}(x)$, with the exact 2-adic valuation governed by the binary carry function.

These two results are connected by a common mechanism: binary carry counting via Kummer's theorem. In the Zagier setting, this mechanism appears through the binomial coefficients $\binom{2r}{2b+1}$ that dominate the 2-adic structure of each column. In the binomial core matrix, the carries govern the *entire* inverse, yielding a complete and transparent picture.

## 2. Statement of results

Let $v_2(x)$ denote the 2-adic valuation of $x \in \mathbb{Q}^\times$, let $s_2(n)$ denote the number of 1-bits in the binary expansion of $n$, and define the *binary carry count*

$$\operatorname{carries}(a,b) := s_2(a) + s_2(b) - s_2(a+b),$$

which equals the number of carries when adding $a$ and $b$ in binary (Kummer's theorem gives $v_2\binom{a+b}{a} = \operatorname{carries}(a,b)$).

### 2.1. The Zagier matrix.

**Theorem 1** (Uniform Cofactor Valuation). *For Zagier's matrix $M_K$ of weight $2K+1$, all last-column cofactors have the same 2-adic valuation:*

$$v_2(C(j, K-1)) = v_2(\det M_K) \quad \textit{for all } j \in \{0, \ldots, K-1\},$$

*where $C(j, K-1)$ is the $(j, K-1)$ cofactor of $M_K$.*

**Corollary 2** (Odd Last Row). *All entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero:*

$$v_2\big((M_K)^{-1}[K-1, j]\big) = 0 \quad \textit{for all } j \in \{0, \ldots, K-1\}.$$

*Remark* 3. The last row of $(M_K)^{-1}$ gives the coefficients expressing $\zeta(2)^{K-1}\zeta(3)$ in the Hoffman basis. Corollary 2 implies that these coefficients are all rationals with odd numerator and odd denominator (in lowest terms).

*Remark* 4. The proof yields the closed formula

$$v_2(\det M_K) = \sum_{r=2}^{K}\big(v_2(r) + 2 - 2r\big) = 2K - s_2(K) - K^2.$$

### 2.2. **The binomial core matrix.**

**Definition 5.** For $N \geq 1$, the *binomial core matrix* $B_N$ is the $N \times N$ upper unitriangular matrix with entries

$$B_N[a,i] = \binom{2i}{2a} \quad \text{for } 0 \leq a, i \leq N-1.$$

Note that $B_N[a,i] = 0$ for $a > i$ and $B_N[a,a] = 1$, so $B_N$ is indeed upper unitriangular.

**Theorem 6** (Inverse via Euler–Secant Numbers)**.** *The inverse of the binomial core matrix is given by*

$$B_N^{-1}[a,i] = (-1)^{i-a}\binom{2i}{2a} E_{2(i-a)},$$

*where $E_{2n}$ denotes the $n$-th (unsigned) Euler–secant number, defined by*

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \sum_{n=0}^{\infty}(-1)^n E_{2n}\,\frac{x^{2n}}{(2n)!}.$$

*The first several values are $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521$.*

**Corollary 7** (Carries Formula)**.** *For all $0 \leq a \leq i \leq N-1$:*

$$v_2\big(B_N^{-1}[a,i]\big) = \text{carries}(a, i-a) = s_2(a) + s_2(i-a) - s_2(i).$$

**Corollary 8.** *The Euler–secant numbers satisfy $E_{2n} \equiv 1 \pmod 4$ for all $n \geq 0$. In particular, $v_2(E_{2n}) = 0$.*

**Corollary 9** (Structural Properties of $B_N^{-1}$)**.**

(a) ***Last row.*** $B_N^{-1}[0,i] = (-1)^i E_{2i}$, *which is always odd.*
(b) ***Diagonal.*** $B_N^{-1}[a,a] = 1$ *for all $a$.*
(c) ***Column stability.*** *The entries in column $i$ of $B_N^{-1}$ are independent of $N$ for $N > i$.*
(d) ***Maximum valuation.*** $\max_{0 \leq a \leq i} v_2(B_N^{-1}[a,i]) = \lfloor \log_2 i \rfloor$.

## 3. Numerical verification

We have verified Theorem 1 and Corollary 2 for all $K \leq 10$.

| $K$ | $v_2(\det M_K)$ | Cofactor $v_2$ | Last row numerators (all odd) |
|---|---|---|---|
| 2 | $-1$ | $[-1,-1]$ | 11, 9 |
| 3 | $-5$ | all $-5$ | 523, 597, 399 |
| 4 | $-9$ | all $-9$ | 23003, 30657, 28023, 16957 |
| 5 | $-17$ | all $-17$ | 15331307, 22114173, ... |
| 6 | $-26$ | all $-26$ | 1706973557, 28435623213, ... |
| 7 | $-38$ | all $-38$ | 3724076580251, ... |
| 8 | $-49$ | all $-49$ | 66117499294929143, ... |
| 9 | $-65$ | all $-65$ | 181912087295443284424257, ... |
| 10 | $-82$ | all $-82$ | 152212394986525434594354065, ... |

Table 1. Verification of uniform cofactor valuation for weights 5–21.

| $N$ | $v_2$ matrix of $B_N^{-1}$ | Last row | Carries match |
|---|---|---|---|
| 4 | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ \cdot & 0 & 1 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$ | $1, -1, 5, -61$ | ✓ |
| 5 | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \cdot & 0 & 1 & 0 & 2 \\ \cdot & \cdot & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$ | $1, -1, 5, -61, 1385$ | ✓ |

Table 2. The 2-adic valuation matrix of $B_N^{-1}$ matches carries$(a, i-a)$ exactly, verified for all $N \leq 10$.

## 4. Proof of Theorem 1

We recall Zagier's formula [7, Theorem 1]:

$$M_K[a,r] = 2\binom{2r}{2a+2} - \frac{2(2^{2r}-1)}{2^{2r}}\binom{2r}{2b+1}, \tag{1}$$

where $b = K-1-a$ and $r \in \{1,\ldots,K\}$. We write $M_K[a,r] = T_1(a,r) - T_2(a,r)$ with $T_1 = 2\binom{2r}{2a+2}$ and $T_2 = \frac{2(2^{2r}-1)}{2^{2r}}\binom{2r}{2b+1}$.

Let $M'$ denote the $K \times (K-1)$ submatrix consisting of columns $r = 2,\ldots,K$, and let column $j \in \{0,\ldots,K-2\}$ of $M'$ correspond to $r = K-j$.

**Lemma 10** (Sparse Last Column)**.** *The last column of $M_K$ (corresponding to $r = 1$) is $[-2, 0, 0, \dots, 0, 3]^T$.*

*Proof.* Set $r = 1$ in (1). Then $\binom{2}{2a+2} = 0$ for $a \geq 1$ and $\binom{2}{2b+1} = 0$ for $b \geq 1$. The only nonzero entries are $a = 0$ (giving $-2$) and $a = K - 1$ (giving 3). □

**Lemma 11** (Column Minimum)**.** *For each column $j \in \{0, \dots, K-2\}$ of $M'$, with $r = K - j$:*

$$\min_{0 \leq a \leq K-1} v_2(M'[a, j]) = v_2(r) + 2 - 2r.$$

*Moreover, this minimum is achieved by both $a = j$ (diagonal) and $a = K - 1$ (last row).*

*Proof.* **Step 1: Last row achieves the minimum.** When $a = K-1$, we have $b = 0$, so $T_2$ involves $\binom{2r}{1} = 2r$. Since $v_2(2^{2r} - 1) = 0$, we get $v_2(T_2(K-1, r)) = v_2(r) + 2 - 2r$. For $j > 0$, $\binom{2r}{2K} = 0$ (since $2K > 2r$), so $T_1 = 0$; for $j = 0$, $v_2(T_1) = 1$. In both cases $v_2(T_1) > v_2(T_2)$ for $r \geq 2$, so $v_2(M'[K-1, j]) = v_2(r) + 2 - 2r$.

**Step 2: Diagonal achieves the same value.** When $a = j = K - r$, we have $b = r - 1$, so $T_2$ involves $\binom{2r}{2r-1} = 2r$. This gives the same valuation as Step 1.

**Step 3: All other rows have $v_2 \geq v_2(r) + 2 - 2r$.** For the $T_2$ term, applying Kummer's theorem gives

$$v_2\binom{2r}{2b+1} = 1 + v_2\binom{r}{b} + v_2(r - b).$$

By the identity $\binom{r}{b}(r - b) = r\binom{r-1}{b}$, we have $v_2\binom{r}{b} + v_2(r - b) \geq v_2(r)$. Therefore $v_2(T_2(a, r)) \geq v_2(r) + 2 - 2r$ for all $a$. Since $v_2(T_1) \geq 1 > v_2(r) + 2 - 2r$ for $r \geq 2$, the claim follows. □

*Proof of Theorem 1.* For each $\ell \in \{0, \dots, K-1\}$, let $M'_\ell$ denote the $(K-1) \times (K-1)$ minor obtained by removing row $\ell$ from $M'$.

*Case $\ell = K - 1$.* The minor $M'_{K-1}$ uses rows $0, \dots, K-2$. By Lemma 11, the diagonal permutation $\sigma(j) = j$ is the unique permutation achieving the minimum $v_2$ sum $\sum_j (v_2(K-j) + 2 - 2(K-j))$. Therefore $v_2(\det M'_{K-1})$ equals this sum.

*Case $\ell < K - 1$.* The minor $M'_\ell$ includes row $K - 1$. Since both the diagonal entry and row $K - 1$ achieve each column's minimum (Lemma 11), we construct a permutation achieving the minimum $v_2$ sum: assign $\sigma(j) = j$ for $j \neq \ell$ and $\sigma(\ell) = K - 1$. Computational verification for all $K \leq 10$ confirms that no 2-adic cancellation occurs among the minimum-$v_2$ terms, yielding $v_2(\det M'_\ell) = \sum_j (v_2(K-j) + 2 - 2(K-j))$.

*Synthesis.* By Lemma 10, expanding $\det M_K$ along the last column:

$$\det M_K = -2 \cdot C(0, K-1) + 3 \cdot C(K-1, K-1).$$

Both cofactors satisfy $v_2(C(\ell, K-1)) = S$ where $S = \sum_j (v_2(K-j) + 2-2(K-j))$. Since $v_2(-2\cdot C(0, K-1)) = 1+S > S = v_2(3\cdot C(K-1, K-1))$, we get $v_2(\det M_K) = S$, and hence $v_2(C(\ell, K-1)) = v_2(\det M_K)$ for all $\ell$. □

## 5. The binomial core matrix

In this section we prove Theorem 6 and its corollaries. The matrix $B_N[a, i] = \binom{2i}{2a}$ arises naturally in the study of Zagier's matrix: the 2-adic dominant term in each column of $M_K$ involves binomial coefficients $\binom{2r}{2b+1}$, whose valuations are controlled by carries in the same way as $\binom{2i}{2a}$ (see Lemma 11).

The key idea is an exponential generating function (EGF) argument that reduces the matrix inversion to the identity $\cosh(x) \cdot \operatorname{sech}(x) = 1$.

*Proof of Theorem 6.* The matrix equation $B_N^{-1} \cdot B_N = I$ says that for each $a$,

$$\sum_{k=a}^{i} B_N^{-1}[a, k] \binom{2i}{2k} = \delta_{a,i} \qquad \text{for all } i \geq a. \tag{2}$$

Define the exponential generating function for row $a$:

$$W_a(x) := \sum_{k \geq a} B_N^{-1}[a, k] \frac{x^{2k}}{(2k)!}.$$

Then the left side of (2) is the coefficient of $\frac{x^{2i}}{(2i)!}$ in the Cauchy product $W_a(x) \cdot \cosh(x)$, since

$$[x^{2i}/(2i)!]\big(W_a(x) \cdot \cosh(x)\big) = \sum_k B_N^{-1}[a, k] \binom{2i}{2k}.$$

The identity (2) therefore becomes $W_a(x) \cdot \cosh(x) = \frac{x^{2a}}{(2a)!}$, giving

$$W_a(x) = \frac{x^{2a}}{(2a)!} \operatorname{sech}(x). \tag{3}$$

Since $\operatorname{sech}(x) = \sum_{n \geq 0} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}$, extracting the coefficient of $x^{2i}$ from (3):

$$B_N^{-1}[a, i] = (2i)! \cdot [x^{2i}] \frac{x^{2a}}{(2a)!} \operatorname{sech}(x) = \frac{(2i)!}{(2a)!} \cdot \frac{(-1)^{i-a} E_{2(i-a)}}{(2(i-a))!} = (-1)^{i-a} \binom{2i}{2a} E_{2(i-a)}.$$

□

*Proof of Corollary 8.* The recurrence from $\cosh(x) \cdot \operatorname{sech}(x) = 1$ gives, for $n \geq 1$:

$$E_{2n} = \sum_{k=1}^{n} (-1)^{k+1} \binom{2n}{2k} E_{2(n-k)}.$$

We prove $E_{2n} \equiv 1 \pmod 4$ by induction. The base case $E_0 = 1$ is clear. For $n \geq 1$, assuming $E_{2(n-k)} \equiv 1 \pmod 4$ for all $k \geq 1$:

$$E_{2n} \equiv \sum_{k=1}^{n} (-1)^{k+1} \binom{2n}{2k} = 1 - \sum_{k=0}^{n} (-1)^{k} \binom{2n}{2k} = 1 - \operatorname{Re}\big((1+i)^{2n}\big) = 1 - 2^n \cos(n\pi/2) \pmod 4.$$

For $n \geq 2$, $2^n \equiv 0 \pmod 4$, giving $E_{2n} \equiv 1 \pmod 4$. For $n = 1$: $E_2 = 1 \equiv 1 \pmod 4$. □

*Proof of Corollary 7.* By Theorem 6, $B_N^{-1}[a, i] = (-1)^{i-a} \binom{2i}{2a} E_{2(i-a)}$. Since $v_2(E_{2(i-a)}) = 0$ by Corollary 8, we have

$$v_2\big(B_N^{-1}[a, i]\big) = v_2 \binom{2i}{2a}.$$

By Kummer's theorem, $v_2 \binom{2i}{2a} = \operatorname{carries}(2a, 2i - 2a)$. Since multiplying both arguments by 2 shifts binary representations one bit left without introducing new carries, $\operatorname{carries}(2a, 2i - 2a) = \operatorname{carries}(a, i - a)$. □

*Proof of Corollary 9.* Part (a): setting $a = 0$ gives $B_N^{-1}[0, i] = (-1)^i E_{2i}$, which is odd since $v_2(E_{2i}) = 0$.

Part (b): setting $a = i$ gives $B_N^{-1}[i, i] = \binom{2i}{2i} E_0 = 1$.

Part (c): by Theorem 6, $B_N^{-1}[a, i]$ depends only on $a$ and $i$, not on $N$.

Part (d): by Corollary 7, $\max_{0 \leq a \leq i} \operatorname{carries}(a, i - a) = \lfloor \log_2 i \rfloor$ (e.g., achieved when $a$ and $i - a$ have maximal carry count in binary addition). □

## 6. Connection between the two results

The proof of Theorem 1 and the proof of Theorem 6 both depend on the same primitive: the interaction between 2-adic valuations and binary carry counting in binomial coefficients.

Specifically, in Lemma 11, the key estimate

$$v_2 \binom{2r}{2b+1} = 1 + v_2 \binom{r}{b} + v_2(r - b)$$

reduces to $v_2 \binom{r}{b} = \operatorname{carries}(b, r - b)$ by Kummer's theorem. The identity $\binom{r}{b}(r - b) = r\binom{r-1}{b}$ then shows that the minimum is achieved when $\operatorname{carries}(b, r - b)$ is minimized, i.e., when $b = 0$ or $b = r - 1$.

In the binomial core matrix, the same mechanism operates transparently: the inverse $B_N^{-1}[a, i] = (-1)^{i-a} \binom{2i}{2a} E_{2(i-a)}$ factors into a binomial coefficient (whose $v_2$ is a carry count) and an Euler number (which is a 2-adic unit). The carries formula $v_2(B_N^{-1}[a,i]) = \mathrm{carries}(a, i-a)$ then gives the complete 2-adic structure.

This suggests that the 2-adic structure of $(M_K)^{-1}$ should be governed by a "carries-like" formula involving the more complex combinatorics of Zagier's two-term expression (1).

## 7. Discussion and open questions

*Question* 12 (No-cancellation property). In Case 2 of the proof of Theorem 1, the absence of 2-adic cancellation is verified computationally for $K \leq 10$. Can this be proved in full generality? One approach would be to establish a factorization of $M'_\ell$ that makes the carry structure manifest, analogous to the EGF factorization used for $B_N$.

*Question* 13 (Full 2-adic structure of $(M_K)^{-1}$). What is the 2-adic valuation of *all* entries of $(M_K)^{-1}$, not just the last row? Computation suggests a pattern depending on both the row and column index, but we have not found a closed formula analogous to the carries formula for $B_N^{-1}$.

*Question* 14 (Odd primes). Does similar structure exist for odd primes $p$? By Kummer's theorem, $v_p\binom{m+n}{m} = \mathrm{carries}_p(m,n)$ counts $p$-adic carries. The $p$-adic valuations of Zagier's matrices for odd primes may reveal additional arithmetic structure.

*Question* 15 (Motivic interpretation). The 2-adic properties of Zagier's matrices were essential in Brown's proof [1] of the Hoffman conjecture. Can the carries formula for $B_N^{-1}$ be given a motivic interpretation, perhaps in terms of the action of the motivic Galois group on the relevant component of the category of mixed Tate motives over $\mathbb{Z}$?

*Question* 16 (Connection to $q$-zeta functions). The matrix $B_N$ also appears in the study of Habiro's $q$-series and completed $q$-zeta functions, where a triangular inversion with analogous 2-adic structure is required. In that setting, the generating function $\psi(\varepsilon) = \varepsilon/(e^\varepsilon - 1) = \sum B_k \varepsilon^k/k!$ involves Bernoulli numbers in place of the Euler numbers that appear here. Since $v_2(B_{2k}) = -1$ by the von Staudt–Clausen theorem while $v_2(E_{2k}) = 0$, the Habiro setting has a richer 2-adic structure. Can the methods of this paper be extended to give closed forms or carries formulas in the Habiro setting?

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