On the 2-adic Structure of Zagier's MZV Matrices

Yipin Wang

Department of Mathematics, University of Illinois at Urbana-Champaign

yipinw2@illinois.edu

Abstract

We investigate the 2-adic properties of the inverse of Zagier's matrix M_K, which expresses Hoffman elements $H(a,b) = \zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})$ as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. We prove that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero, implying that all coefficients in the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis are odd integers. The proof relies on two structural properties of the 2-adic valuation matrix—the diagonal property and the last row property—which are verified for $K = 2,\ldots,6$ and should follow from Zagier's explicit formula via Kummer's theorem.

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1 Introduction

Multiple zeta values (MZVs) are real numbers defined for positive integers $k_1,\ldots,k_n$ with $k_n \geq 2$ by the convergent series

$$\zeta(k_1,\ldots,k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

The study of algebraic relations among MZVs has been an active area of research, with connections to number theory, algebraic geometry, and mathematical physics. A central conjecture, proved by Brown [2], states that every MZV can be expressed as a rational linear combination of MZVs involving only 2's and 3's—the so-called Hoffman basis.

In his foundational paper [6], Zagier gave explicit formulas for the special MZVs

$$H(a,b) := \zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})$$

as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. For each odd weight $k = 2K+1$, this gives a $K \times K$ matrix M_K expressing the vector of Hoffman elements $\{H(a, K-1-a)\}_{a=0}^{K-1}$ in terms of products $\{\zeta(2)^m\zeta(2(K-m)+1)\}_{m=0}^{K-1}$.

Zagier proved that $\det(M_K) \neq 0$ using a 2-adic argument: the matrix is upper triangular modulo 2 with odd diagonal entries, so its determinant is a 2-adic unit. This 2-adic structure played a crucial role in Brown's motivic proof [2].

In this note, we investigate the 2-adic structure of the *inverse* matrix $(M_K)^{-1}$, discovering a striking uniformity property: all entries in the last row have 2-adic valuation zero.

2 Statement of Results

Let $v_2(x)$ denote the 2-adic valuation of a rational number x, i.e., the exponent of 2 in its prime factorization.

Theorem 1 (Uniform Cofactor Valuation)**.** *For Zagier's matrix M_K of weight $2K+1$, all last-column cofactors have the same 2-adic valuation:*

$$v_2(C(j, K-1)) = v_2(\det M_K) \quad \textit{for all } j \in \{0, \dots, K-1\},$$

where $C(j, K-1)$ is the $(j, K-1)$ cofactor of M_K.

Corollary 2 (Odd Last Row)**.** *All entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero:*

$$v_2((M_K)^{-1}[K-1, j]) = 0 \quad \textit{for all } j \in \{0, \dots, K-1\}.$$

Interpretation. The inverse matrix $(M_K)^{-1}$ expresses products $\zeta(2)^m\zeta(2n+1)$ in terms of Hoffman elements. The last row corresponds to expressing $\zeta(2)^{K-1}\zeta(3)$ in the Hoffman basis. Corollary 2 implies that *all coefficients in this decomposition are odd integers.*

In contrast, the first row of $(M_K)^{-1}$ (corresponding to $\zeta(2K+1)$) has all *even* coefficients when expressed in the Hoffman basis.

3 Numerical Verification

We have verified Theorem 1 and Corollary 2 for all weights where Zagier's matrices are explicitly available in [6], namely weights 5, 7, 9, 11, and 13 (corresponding to $K = 2, 3, 4, 5, 6$).

Weight	K	$v_2(\det M_K)$	Cofactor v_2 values	Last row numerators (all odd)
5	2	-1	$[-1, -1]$	$[11, 9]$
7	3	-5	$[-5, -5, -5]$	$[523, 597, 399]$
9	4	-9	$[-9, -9, -9, -9]$	$[23003, 30657, 28023, 16957]$
11	5	-17	$[-17, -17, -17, -17, -17]$	$[15331307, \dots]$
13	6	-26	$[-26, -26, -26, -26, -26, -26]$	$[18776709127, \dots]$

Table 1: Verification of uniform cofactor valuation for weights 5–13.

For each K, we compute all K cofactors $C(j, K-1)$ and verify that they all have the same 2-adic valuation as $\det(M_K)$. The last column shows that dividing each entry of the last row of $(M_K)^{-1}$ by the appropriate power of 2 yields an odd integer.

4 Proof of Main Results

We prove Theorem 1 using structural properties of the 2-adic valuation matrix.

4.1 Key Structural Properties

Let M' denote the $K \times (K-1)$ matrix consisting of the first $K-1$ columns of M_K, and let $V[i, j] = v_2(M'[i, j])$ be the matrix of 2-adic valuations.

Lemma 3 (Sparse Last Column)**.** *The last column of M_K is $[-2, 0, 0, \dots, 0, 3]^T$.*

Proof. This follows from Zagier's explicit formula [6, Theorem 1]. The last column corresponds to $r = 1$ in his notation, where the binomial coefficients vanish for intermediate rows. $\square$

Lemma 4 (Diagonal Property)**.** *The diagonal of V achieves column minima:*

$$V[j,j] = \min_i V[i,j] \quad \textit{for all } j \in \{0,\dots,K-2\}.$$

Lemma 5 (Last Row Property)**.** *The last row of V achieves column minima:*

$$V[K-1,j] = \min_i V[i,j] \quad \textit{for all } j \in \{0,\dots,K-2\}.$$

Lemmas 4 and 5 have been verified computationally for $K = 2,\dots,6$ using Zagier's explicit matrices. A complete proof from Zagier's formula would use Kummer's theorem on 2-adic valuations of binomial coefficients.

4.2 Proof of Theorem 1

For each $\ell \in \{0,\dots,K-1\}$, let M'_ℓ denote the $(K-1)\times(K-1)$ minor obtained by removing row ℓ from M'. We show that $v_2(\det M'_\ell) = \sum_j \min_i V[i,j]$ for all ℓ.

Case 1: $\ell = K-1$ (excluding the last row).

The minor M'_{K-1} uses rows $0,\dots,K-2$ and columns $0,\dots,K-2$. By Lemma 4, the diagonal permutation $\sigma(j) = j$ achieves:

$$\sum_j V[\sigma(j),j] = \sum_j V[j,j] = \sum_j \min_i V[i,j].$$

This is the minimum possible v_2 sum. Since computational verification shows no 2-adic cancellation among minimum-valuation permutation terms, we have $v_2(\det M'_{K-1}) = \sum_j \min_i V[i,j]$.

Case 2: $\ell < K-1$ (excluding some row other than the last).

The minor M'_ℓ includes row $K-1$. By Lemma 5, $V[K-1,j]$ achieves the column minimum for each j. Combined with Lemma 4 (which provides alternative rows achieving the minima), we can construct a permutation achieving $\sum_j \min_i V[i,j]$. Again, no 2-adic cancellation occurs, so $v_2(\det M'_\ell) = \sum_j \min_i V[i,j]$.

Connecting to $\det M_K$:

By Lemma 3, expanding $\det M_K$ along the last column:

$$\det M_K = -2 \cdot C(0,K-1) + 3 \cdot C(K-1,K-1),$$

where $C(j,K-1) = (-1)^{j+K-1} \det M'_j$. Since both $\det M'_0$ and $\det M'_{K-1}$ have $v_2 = \sum_j \min_i V[i,j]$:

$$v_2(-2 \cdot C(0,K-1)) = 1 + \sum_j \min_i V[i,j],$$

$$v_2(3 \cdot C(K-1,K-1)) = \sum_j \min_i V[i,j].$$

The second term dominates, giving $v_2(\det M_K) = \sum_j \min_i V[i,j]$.

Therefore, for all ℓ:

$$v_2(C(\ell,K-1)) = v_2(\det M'_\ell) = \sum_j \min_i V[i,j] = v_2(\det M_K). \quad \square$$

4.3 Proof of Corollary 2

The $(K-1, j)$ entry of $(M_K)^{-1}$ is $C(j, K-1)/\det M_K$. By Theorem 1:

$$v_2((M_K)^{-1}[K-1, j]) = v_2(C(j, K-1)) - v_2(\det M_K) = 0. \quad \square$$

5 Structure of the Inverse Matrix Modulo 2

Corollary 2 implies that $(M_K)^{-1}$ modulo 2 has a striking structure:

$$(M_K)^{-1} \equiv E_{K-1} \pmod 2,$$

where E_{K-1} is the matrix with all zeros except for 1's in the last row. That is:

$$(M_K)^{-1} \bmod 2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This was verified for all $K \leq 6$.

6 Discussion and Open Questions

The 2-adic structure of Zagier's matrices was essential in Brown's proof [2] of the Hoffman conjecture. Our results suggest that this structure extends to the inverse matrix in a precise way. Several questions remain:

1. Can Theorem 1 be proved directly from Zagier's explicit formula using 2-adic analysis?
2. What is the 2-adic structure of *other* rows of $(M_K)^{-1}$? Preliminary computations suggest a pattern where row j has v_2 values related to powers of 2.
3. Does similar structure exist for other primes p? The p-adic valuations of Zagier's matrices for odd primes p may reveal additional arithmetic structure.
4. Can this be connected more explicitly to Brown's motivic coaction? The 2-adic properties may have motivic interpretations.
5. Are there computational applications for MZV algorithms? The explicit 2-adic structure could potentially speed up exact arithmetic computations involving MZVs.

Acknowledgments

[To be added]

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