

A Switching Lemma for DNFs over \mathbb{F}_p : the Canonical Decision Tree Approach

Abstract

We prove a switching lemma for DNFs over \mathbb{F}_p via the canonical decision tree (CDT) method of Razborov, obtaining the M -independent bound

$$\Pr[\text{DT}(f|_\rho) \geq s] \leq \left(\frac{2pwq}{1-q} \right)^s$$

for any width- w DNF f with *any number of terms* M , where ρ is a random restriction keeping each variable alive with probability q . The key observation is that the p -ary branching of the CDT over \mathbb{F}_p is effectively binary: all $p-1$ nonzero branches lead to the same continuation, so the encoding requires only $2w$ (not pw) bits per step. For $p=2$ this recovers the classical bound $(4wq/(1-q))^s$. As a corollary, depth- d circuits over \mathbb{F}_p computing generalized parity require size $\exp(\Omega_p(n^{1/(d-1)}))$.

1 Introduction

Håstad's switching lemma [1] is the cornerstone of AC^0 lower bounds. For a width- w CNF or DNF f on $\{0,1\}^n$ and a random restriction ρ_q keeping each variable alive with probability q :

$$\Pr[\text{DT}(f|_\rho) \geq s] \leq (Cwq)^s,$$

where C is an absolute constant and DT denotes decision tree depth. Two features are essential: the bound controls DT depth (not merely certificate complexity), and it is *independent* of the number of clauses M .

Extending this to \mathbb{F}_p is nontrivial. Over \mathbb{Z}_p^n (for a prime p), a literal is $\ell_i(x) = \mathbf{1}[x_i \neq 0]$, and random restrictions assign each variable to: alive with probability q , or dead with a uniform value in $\{0, \dots, p-1\}$ with probability $(1-q)/p$ each. The recent work [6] proved the single-gate switching lemma $(C_p q K/s)^s$ for fan-in K gates, which already yields tight AC^0 lower bounds.

In this note we prove the full M -independent DNF/CNF switching lemma over \mathbb{F}_p , adapting Razborov's canonical decision tree proof [2, 3].

Theorem 1 (Main). *Let $f = T_1 \vee \dots \vee T_M$ be a width- w DNF over \mathbb{F}_p^n (each term $T_j = \bigwedge_{i \in S_j} \ell_i$ with $|S_j| \leq w$). For $\rho \sim \rho_q$:*

$$\Pr[\text{DT}(f|_\rho) \geq s] \leq \left(\frac{2pwq}{1-q} \right)^s.$$

An identical bound holds for width- w CNFs by duality.

For $p=2$ and small q , this gives $(4wq)^s$, recovering the classical bound.

Remark 2 (Comparison with Boolean). In the Boolean case ($p=2$), the constant in the switching lemma is $C \approx 4$ – 7 depending on the proof method. Our proof gives $C_p = 2p$; for $p=2$ this is $C_2 = 4$, matching Razborov's original argument. Whether C_p can be improved to $O(1)$ independent of p is an interesting open question.

2 Setup

2.1 Literals and formulas over \mathbb{F}_p

We work over \mathbb{Z}_p^n for a prime p . A *literal* is $\ell_i(x) = \mathbf{1}[x_i \neq 0]$. A *width- w DNF* is $f = T_1 \vee \dots \vee T_M$ where each *term* is $T_j = \bigwedge_{i \in S_j} \ell_i$ with $|S_j| \leq w$. Term T_j evaluates to 1 iff all variables in S_j are nonzero.

2.2 Random restrictions

A *random restriction* $\rho \sim \rho_q$ independently sets each variable x_i to:

- alive (denoted $*$) with probability q ;
- dead with value $v \in \{0, \dots, p-1\}$ with probability $(1-q)/p$ each.

We write $A(\rho)$ for the set of alive variables and $\sigma(\rho)$ for the dead assignment.

2.3 Effect on terms

Under restriction ρ , a term T_j is:

- *falsified*: some variable in S_j is dead with value 0;
- *fully satisfied*: every variable in S_j is dead with a nonzero value;
- *active*: no variable in S_j is dead-0, and at least one variable is alive.

The restricted function $f|_\rho$ depends only on the alive variables.

3 The canonical decision tree over \mathbb{F}_p

Definition 3 (CDT for DNFs). *Given a width- w DNF $f = T_1 \vee \dots \vee T_M$ and a restriction ρ , the canonical decision tree $\mathcal{T}(f|_\rho)$ is defined recursively:*

1. *If some term T_j is fully satisfied under ρ : output 1 (leaf, depth 0).*
2. *Otherwise, let T_j be the first active term (smallest j). Query the smallest-index alive variable $x_i \in S_j$.*
 - **Branch** $x_i = 0$: *term T_j is falsified. Recurse on $f|_{\rho'}$ where ρ' extends ρ by setting $x_i = 0$.*
 - **Branch** $x_i \neq 0$: *literal ℓ_i is satisfied. Recurse on $f|_{\rho'}$ where ρ' extends ρ by setting $x_i = 1$ (any nonzero value).*
3. *If no active term exists (all terms falsified): output 0 (leaf, depth 0).*

The depth $\text{DT}(f|_\rho)$ is the depth of $\mathcal{T}(f|_\rho)$.

Proposition 4 (Binary effective branching). *In the CDT, the subtrees rooted at branches $x_i = v$ for $v = 1, 2, \dots, p-1$ are all identical. Hence the CDT is effectively a binary tree.*

Proof. The CDT state depends only on which terms are active and which variables are alive. For a variable x_i in active term T_j :

- Value 0: T_j becomes falsified.
- Any value $v \neq 0$: literal $\ell_i = \mathbf{1}[x_i \neq 0]$ becomes 1. Term T_j either becomes satisfied (if x_i was the last alive variable) or continues with one fewer alive variable. The effect is the same for all $v \neq 0$. \square

4 Proof of Theorem 1

We adapt Razborov’s encoding argument [2, 3].

Bad restrictions. Let $\text{BAD}_s = \{\rho : \text{DT}(f|_\rho) \geq s\}$. For $\rho \in \text{BAD}_s$, there exists a root-to-node path in $\mathcal{T}(f|_\rho)$ of depth s .

Extracting a path. Fix $\rho \in \text{BAD}_s$. By Proposition 4, the CDT is an effectively binary tree. Choose the *lexicographically first* (left-first) path of depth s in the CDT. This path queries variables x_{v_1}, \dots, x_{v_s} in terms $T_{\alpha_1}, \dots, T_{\alpha_s}$, with branch indicators $b_1, \dots, b_s \in \{0, 1\}$ (where 0 = “value is 0” and 1 = “value is nonzero”).

Injection. Define $\Phi(\rho) = (\tilde{\rho}, \tau)$ where:

- $\tilde{\rho}$ agrees with ρ except each x_{v_i} is set to *dead with value 1* (instead of alive).
- The *encoding* is $\tau = (k_1, b_1, \dots, k_s, b_s)$, where $k_i = \text{pos}(x_{v_i}, S_{\alpha_i}) \in \{0, \dots, w - 1\}$ is the position of x_{v_i} within term T_{α_i} (in sorted order).

Preservation of falsified terms. We kill to value 1 (nonzero). This does not falsify any term, since falsification requires a dead variable with value 0. Formally: for any term T_j , if T_j is falsified under ρ (has some dead variable with value 0), then T_j is still falsified under $\tilde{\rho}$ (the same dead-0 variable persists). Conversely, if T_j is not falsified under ρ , the only change is that some alive variables become dead-1, which does not create any dead-0 variable.

Hence: *the set of falsified terms is identical under ρ and $\tilde{\rho}$.*

Injectivity. Given $(\tilde{\rho}, \tau)$, we reconstruct ρ iteratively:

1. Initialize $\rho := \tilde{\rho}$. Let the active terms be those not falsified (same as under the original ρ).
2. For $i = 1, \dots, s$:
 - (a) *Determine the current term.* If $i = 1$, let T be the first non-falsified term. If $b_{i-1} = 0$ (previous variable falsified a term), advance to the next non-falsified term. If $b_{i-1} = 1$, stay at the same term.
 - (b) *Identify the variable.* The k_i -th variable (sorted order) in the current term T is x_{v_i} . This variable is currently dead with value 1 in ρ .
 - (c) *Revive.* Set $\rho(x_{v_i}) := *$ (alive).
3. Output ρ .

This procedure is well-defined because:

- *Step 2(a):* the non-falsified terms are the same under $\tilde{\rho}$ and ρ (at any intermediate stage), since we only change dead-1 to alive (never affecting dead-0 variables).

- *Step 2(b)*: the variable at position k_i in the current term is dead-1 (it was set to dead-1 by the forward map and has not been revived in a previous step, since the CDT queries each variable at most once).

Since the procedure uniquely determines ρ from $(\tilde{\rho}, \tau)$, the map Φ is injective on each fiber $\text{BAD}_\tau := \{\rho \in \text{BAD}_s : \Phi(\rho) \text{ has encoding } \tau\}$.

Encoding count.

$$|\{\tau\}| \leq (2w)^s,$$

since each step contributes w choices for the position k_i and 2 choices for the branch b_i . Crucially, b_i records only “zero vs. nonzero” (not the specific nonzero value), which suffices by Proposition 4.

Probability ratio. Each x_{v_i} changes from alive (probability q) to dead with value 1 (probability $(1 - q)/p$):

$$\frac{\Pr[\rho]}{\Pr[\tilde{\rho}]} = \prod_{i=1}^s \frac{q}{(1 - q)/p} = \left(\frac{pq}{1 - q} \right)^s.$$

Combining.

$$\begin{aligned} \Pr[\text{BAD}_s] &= \sum_{\tau} \Pr[\text{BAD}_\tau] = \sum_{\tau} \sum_{\rho \in \text{BAD}_\tau} \Pr[\rho] \\ &= \sum_{\tau} \sum_{\rho \in \text{BAD}_\tau} \Pr[\tilde{\rho}] \cdot \left(\frac{pq}{1 - q} \right)^s \\ &\leq \sum_{\tau} 1 \cdot \left(\frac{pq}{1 - q} \right)^s = (2w)^s \cdot \left(\frac{pq}{1 - q} \right)^s = \left(\frac{2pwq}{1 - q} \right)^s. \end{aligned}$$

5 AC⁰ lower bound

Corollary 5. *For any prime p and constant depth $d \geq 2$, any depth- d circuit over \mathbb{F}_p computing $\text{PAR}_p(x) = \mathbf{1}[\sum_i x_i \not\equiv 0 \pmod{p}]$ has size $\exp(\Omega_p(n^{1/(d-1)}))$.*

Proof. The argument is standard. Let C be a depth- d circuit of size S with bottom fan-in w . Apply $d - 1$ rounds of random restrictions with $q = c/(pw)$ for a small constant c .

At each round, every bottom gate (a width- w DNF or CNF) satisfies

$$\Pr[\text{DT} \geq s] \leq (2pwq/(1 - q))^s = (2c/(1 - q))^s \leq (3c)^s$$

for small q . Setting $s = \lceil C' \ln S \rceil$ for large C' , this is $< 1/S^2$. By a union bound over S gates, all gates simplify to DT depth $\leq s$ with positive probability.

Each gate is replaced by its depth- s decision tree expansion, reducing the circuit depth by 1 and setting the new bottom fan-in to $s = O(\ln S)$. After $d - 1$ rounds:

$$n' = n \cdot \prod_{i=1}^{d-1} q_i \geq \frac{n}{(Cp)^{d-1} \cdot w \cdot (\ln S)^{d-2}}$$

alive variables survive. For PAR_p to remain non-constant, we need $n' \geq 1$, giving $w \cdot (\ln S)^{d-2} \leq n/(Cp)^{d-1}$. Since $w \leq S$, this forces $S \geq \exp(\Omega_p(n^{1/(d-1)}))$. \square

6 Discussion

6.1 The key observation: binary effective branching

The CDT for a DNF over \mathbb{F}_p branches p ways at each query, but $p-1$ of these branches are identical (all nonzero values have the same effect on clause satisfaction). This is the crucial observation: the encoding needs only 1 bit per step to specify the branch (zero vs. nonzero), rather than $\log_2 p$ bits.

In contrast, for more general “ \mathbb{F}_p -valued” gate types (e.g., functions that distinguish between different nonzero values), the p -ary branching would be genuinely p -ary, and the encoding would require $\log_2 p$ bits per step. This would give a bound of $(p^2 w q / (1 - q))^s$.

6.2 Killing to value 1

The choice to kill staircase variables to dead-1 (any fixed nonzero value) is essential. Killing to value 0 (as in the injection of [7]) preserves the set of *surviving* clauses, but the CDT structure changes because the active terms differ. Killing to a nonzero value preserves the set of *falsified* terms (since falsification requires a dead-0 variable), which is what the CDT-based inverse needs.

6.3 Open problems

1. **Optimal constant.** Our bound gives $C_p = 2p$. For $p = 2$, this is $C_2 = 4$, matching Razborov’s proof. Can C_p be made $O(1)$? In the Boolean case, Razborov’s bound of 4 is not tight; improved bounds give $C_2 \leq 7/2$ [4]. The *correct* constant over \mathbb{F}_p is likely $O(p)$ since the probability ratio $pq/(1-q)$ inherently involves p .
2. **Projections and depth hierarchy.** RST [5] extended the switching lemma to random *projections* to prove average-case depth hierarchy for the Boolean PH relative to a random oracle. Extending this to \mathbb{F}_p would give analogous results for \mathbb{F}_p -valued computation models.
3. **ACC⁰.** Our switching lemma handles AND/OR gates over \mathbb{F}_p . The frontier of AC⁰-style lower bounds is ACC⁰ (circuits with MOD _{m} gates). Connecting the \mathbb{F}_p framework to ACC⁰ lower bounds remains a major open problem.

References

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