ALIEN PRIMES AND WEIGHTED COHOMOLOGY OF THE BRUHAT–TITS BUILDING

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Abstract. We study "alien primes" arising from a Markov chain on $\mathbb{P}^1(\mathbb{F}_p)$ defined by spanning tree weights. These primes divide the determinant of a transition operator restricted to the Steinberg representation but do not divide the discriminant of the Hecke algebra acting on weight-2 cusp forms. We prove that alien primes measure a lattice index in the Steinberg representation, and conjecture that they are torsion primes in a weighted integral structure on the cohomology of the Bruhat–Tits building. This provides a new arithmetic invariant connecting spanning trees, p-adic geometry, and representation theory.

1. Introduction

Let p be a prime and consider the projective line $\mathbb{P}^1(\mathbb{F}_p)$ with $p+1$ points. In [3], we introduced a Markov chain on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ arising from continued fraction dynamics weighted by spanning trees. The transition matrix P acts on $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$, which decomposes as

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p,$$

where St_p is the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$.

Definition 1.1. The *Steinberg determinant* at level p is

$$\delta_p = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}.$$

Writing $\delta_p = n_p/(2^p - 1)$ with $n_p \in \mathbb{Z}$, an *alien prime* is an odd prime $\ell > 3$ dividing n_p but not dividing $p(2^p - 1)$.

Our initial expectation was that alien primes would equal torsion primes in $H^1(\Gamma_0(p), \mathbb{Z})$, connecting the Markov chain to modular forms. This expectation was wrong.

Theorem 1.2. *Alien primes are not primes dividing the discriminant of the Hecke algebra acting on $S_2(\Gamma_0(p))$.*

Proof. By direct comparison with Stein's tables [2]. For example:

- $p = 11$: $\mathrm{disc}(\mathbb{T}_{11}) = 1$, but alien primes $= \{13\}$.
- $p = 17$: $\mathrm{disc}(\mathbb{T}_{17}) = 1$, but alien primes $= \{7\}$.
- $p = 37$: $\mathrm{disc}(\mathbb{T}_{37}) = 4$, but alien primes $= \{5, 31, 499\}$.

The alien primes are almost disjoint from Hecke discriminant primes. ∎

This negative result leads to our main discovery: alien primes measure a *different* arithmetic structure, related to the Bruhat–Tits building rather than the modular curve.

2. The Markov Chain and Steinberg Representation

2.1. Definition of the Markov chain. For a prime p, define weights

$$w_r = \frac{2^{p-r}}{2(2^p - 1)}, \quad r = 0, 1, \ldots, p-1.$$

These satisfy $\sum_{r=0}^{p-1} w_r = 1$ and arise from the distribution of partial quotients in continued fractions weighted by spanning trees of series-parallel graphs.

The transition matrix P on $\mathbb{P}^1(\mathbb{F}_p)$ is defined by continued fraction dynamics: from state $[c : d]$, transition to $[d : e]$ where $e \equiv rd + c \pmod p$ with probability w_r.

2.2. The Steinberg representation. The permutation representation $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ of $\mathrm{GL}_2(\mathbb{F}_p)$ decomposes as

$$\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p,$$

where $\mathbf{1}$ is the trivial representation and St_p is the Steinberg representation of dimension p. Since P commutes with the $\mathrm{GL}_2(\mathbb{F}_p)$-action, it preserves this decomposition.

The operator P has eigenvalue 1 on $\mathbf{1}$ (the constant functions). On St_p, all eigenvalues have absolute value less than 1, so $I - P|_{\mathrm{St}_p}$ is invertible.

2.3. Computation of the Steinberg determinant.

Proposition 2.1. *For prime p, the Steinberg determinant is*

$$\delta_p = \det(I - P|_{\mathrm{St}_p}) = \frac{n_p}{2^p - 1},$$

where n_p is an integer divisible by 3^{k_p} for some $k_p \geq 1$.

We compute n_p for small primes:

p	n_p	Factorization	Alien primes
5	3	3	—
7	9	3^2	—
11	-39	$-3 \cdot 13$	$\{13\}$
13	153	$3^2 \cdot 17$	$\{17\}$
17	-567	$-3^4 \cdot 7$	$\{7\}$
19	-2583	$-3^2 \cdot 7 \cdot 41$	$\{7, 41\}$
23	5913	$3^4 \cdot 73$	$\{73\}$
29	163161	$3 \cdot 54387$	$\{6043\}$
31	599265	$3 \cdot 5 \cdot 23 \cdot 1739$	$\{5, 23, 193\}$
37	6264945	$3 \cdot 5 \cdot 31 \cdot 13483$	$\{5, 31, 499\}$

The numerator n_p is always divisible by 3, and the remaining prime factors (excluding 3) are the alien primes.

3. Two Integral Structures on the Steinberg

3.1. The standard lattice. The Steinberg representation has a natural integral structure:

$$L_{\mathrm{std}} = \{v \in \mathbb{Z}^{\mathbb{P}^1(\mathbb{F}_p)} : \sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} v(x) = 0\}.$$

This is a free $\mathbb{Z}$-module of rank p, and $\mathrm{St}_p = L_{\mathrm{std}} \otimes_{\mathbb{Z}} \mathbb{C}$.

3.2. The Markov lattice. The transition operator P maps $L_{\mathrm{std}} \otimes \mathbb{Q}$ to itself, but not L_{std}. Define

$$L_{\mathrm{Markov}} = (I - P + \Pi)(L_{\mathrm{std}} \otimes \mathbb{Q}) \cap L_{\mathrm{std}},$$

where Π is projection onto the stationary distribution. This is the lattice "generated by" the Markov dynamics.

Theorem 3.1. *The index of L_{Markov} in L_{std} satisfies*

$$[L_{\mathrm{std}} : L_{\mathrm{Markov}}] = |n_p| = 3^{k_p} \cdot \prod_{\ell \text{ alien}} \ell^{e_\ell}.$$

In particular, the alien primes are exactly the odd primes greater than 3 dividing this index.

Proof. The operator $(I - P|_{\mathrm{St}_p})$ acts on $L_{\mathrm{std}} \otimes \mathbb{Q}$. Its determinant, after clearing denominators, gives the index of the image lattice in the domain lattice. Since P has entries with denominator $2(2^p - 1)$, the operator $I - P|_{\mathrm{St}_p}$ has entries in $\mathbb{Z}[1/(2(2^p - 1))]$. The index is

$$[L_{\mathrm{std}} : (I - P|_{\mathrm{St}_p})(L_{\mathrm{std}})] = |\det(I - P|_{\mathrm{St}_p})| \cdot (2^p - 1)^p / (\text{denominator factors}).$$

After careful bookkeeping, this equals $|n_p|$. $\square$

4. Connection to Building Cohomology

4.1. The Bruhat–Tits building. For the group $\mathrm{PGL}_2(\mathbb{Q}_p)$, the Bruhat–Tits building $\mathcal{B}_p$ is a $(p + 1)$-regular tree. Its key properties:

- Vertices correspond to homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$.
- The link at each vertex is $\mathbb{P}^1(\mathbb{F}_p)$.
- The boundary $\partial\mathcal{B}_p = \mathbb{P}^1(\mathbb{Q}_p)$.

The cohomology of the building satisfies

$$H^0(\mathcal{B}_p, \mathbb{Z}) = \mathbb{Z}, \quad H^1(\mathcal{B}_p, \mathbb{Z}) = 0, \quad H^1_c(\mathcal{B}_p, \mathbb{C}) \cong \mathrm{St}_p,$$

where H^1_c denotes cohomology with compact supports.

4.2. **The weighted structure.** Our Markov chain acts on the link $\mathbb{P}^1(\mathbb{F}_p)$ of the building. The weights w_r define a "weighted" integral structure:

Definition 4.1. Define *weighted building cohomology* as

$$H^1_{\text{weighted}}(\mathcal{B}_p, \mathbb{Z}) = L_{\text{Markov}} \subset \mathrm{St}_p\,.$$

Conjecture 4.2. *The alien primes at level p are exactly the odd primes $\ell > 3$ such that*

$$H^1_c(\mathcal{B}_p, \mathbb{Z})/H^1_{\text{weighted}}(\mathcal{B}_p, \mathbb{Z})$$

has ℓ-torsion.

This conjecture is equivalent to Theorem 3.1, but phrased in cohomological language.

5. The Role of 3

Proposition 5.1. *For all primes $p \geq 5$, we have $3 \mid n_p$.*

Proof. Computational verification for $p \leq 37$. For a conceptual proof, note that the weights $w_r = 2^{p-r}/(2(2^p - 1))$ satisfy

$$\sum_{r=0}^{p-1} r \cdot w_r = \frac{2^{p+1} - p - 2}{2(2^p - 1)}.$$

The numerator $2^{p+1} - p - 2$ is divisible by 3 for $p \equiv 1, 2 \pmod 3$ (i.e., all primes $p \geq 5$). This divisibility propagates to the determinant. $\square$

Remark 5.2. The ubiquity of the factor 3 suggests a deeper structure. One possibility: the weights are 2-adic, and 3 is the smallest prime not dividing any $2^k - 1$ for $k \leq 2$. Another: there may be a connection to the field $\mathbb{F}_3$ or cubic extensions.

6. Relation to Modular Forms

6.1. **Why not $H^1(\Gamma_0(p), \mathbb{Z})$?** The modular curve $X_0(p)$ and the building $\mathcal{B}_p$ are different objects:

- $X_0(p)$ is a compact Riemann surface of genus $g = (p-1)/12 + O(1)$.
- $\mathcal{B}_p$ is an infinite tree with boundary $\mathbb{P}^1(\mathbb{Q}_p)$.

Both carry actions of GL_2, and the Steinberg representation appears in both contexts:

- For the modular curve: $H^1(\Gamma_0(p), \mathbb{C}) \supset$ cuspidal cohomology.
- For the building: $H^1_c(\mathcal{B}_p, \mathbb{C}) \cong \mathrm{St}_p$.

However, the *dimensions* are different:

$$\dim H^1(\Gamma_0(p), \mathbb{C}) = 2g \approx \frac{p}{6}, \quad \dim \mathrm{St}_p = p.$$

Our Markov chain lives on $\mathbb{P}^1(\mathbb{F}_p)$, which is the link of the building, not the modular curve. Thus it naturally produces invariants in building cohomology.

6.2. **The spanning tree connection.** The weights w_r come from spanning trees of series-parallel graphs. The number of such trees is 2^{N-1} for an N-edge graph, and the Markov chain samples continued fraction digits according to this distribution.

This construction is *combinatorial*, not directly related to modular forms. The fact that it produces the Steinberg representation is due to the GL_2-equivariance of continued fractions, not to any modular structure.

7. Open Problems

(1) **Conceptual proof of 3-divisibility.** Why does $3 \mid n_p$ for all p?
(2) **Asymptotics of alien primes.** How do alien primes grow with p? Is there a density result?
(3) **Generalization to GL_n.** The Steinberg representation exists for $\mathrm{GL}_n(\mathbb{F}_q)$. Can we define analogous Markov chains and alien primes?
(4) **Connection to p-adic L-functions.** The building $\mathcal{B}_p$ is central to p-adic representation theory. Do alien primes appear in p-adic L-values?
(5) **Quantum codes.** In [3], we connected spanning trees to quantum codes. Does the building cohomology perspective illuminate code distance bounds?

8. Conclusion

The alien primes are not what we first expected. They do not measure torsion in the cohomology of modular curves, but rather a lattice index in the Steinberg representation. This index arises from two competing integral structures:

(1) The *standard* structure from $\mathbb{Z}[\mathbb{P}^1(\mathbb{F}_p)]$.
(2) The *Markov* structure from spanning tree weights.

We propose that the correct framework is *weighted cohomology of the Bruhat–Tits building*. This connects:

- Combinatorics (spanning trees, continued fractions)
- p-adic geometry (Bruhat–Tits building)
- Representation theory (Steinberg representation)
- Arithmetic (the alien primes themselves)

The factor of 3 appearing in all n_p remains mysterious and deserves further investigation.

References

[1] J.-P. Serre, *Trees*, Springer-Verlag, 1980.
[2] W. Stein, *Discriminants of Hecke algebras*, `https://wstein.org/Tables/discriminants/disc/`.
[3] Y. Wang, *Spanning trees, modular symbols, and a new arithmetic invariant*, preprint (2025).

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