ECO325: Lecture Notes
Advanced Economic Theory: Macro

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1 Lecture 1. September 6. 2018

Definition 1.1. A **growth miracle** are episodes where thee growth in a country far exceeds the world average over a extended period of time. <u>Result</u> the country experiencing the miracle moves up the wold income distribution.

Definition 1.2. A **growth disaster** is an episodes where the growth in a country falls short at the world average for an extended period of time. <u>Result</u> the country moves down in the world income distribution.

Facts Data from the 20th century suggest that

1. Real output grows at a (more or less) constant rate.

2. Stock of real capital grows at a (more or less) constant rate (but it grows faster than labor input).

3. Growth rates of real output and the stock of capital are about the same.

4. The rate of growth of output per capita varies greatly across countries.

1.1 Solow Growth Model (continuous time version)

Solow growth model decomposes the growth in output per capita into portions accounted for by increase in inputs and the portion contributed to increases in productivity.

In the baseline model we denote K as capital, L as labor and A as technology.

1.1.1 Production Function

Remark 1.1. Harrod-neutral technology here, refer to Uzawa's theorem.

Definition 1.3. The **effective labor input** is defined as $A(t)L(t)$

Definition 1.4. The production function is defined as

$$Y(t) = F(K(t), A(t)L(t))$$

Typically Cobb-Douglas form is taken

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}, \ \alpha \in (0,1)$$

Property 1.1. Properties on production function.

1. <u>CRS in K and AL</u>: $Y(cK, cAL) = cY(K, AL), \ \forall c \geq 0$ implies

   - All gains from specialization have been exhausted.
   - Inputs other than K and AL are unimportant.

Definition 1.5. Define $c := \frac{1}{AL}$, the **intensive form** of production function is

$$y = \frac{Y}{AL} = f(k)$$

where $y := \frac{Y}{AL}$ denotes the **output per unit of effective labor** and $k := \frac{K}{AL}$ denote the capital stock per unit of effective labor.

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2.1 Dynamic Transitions

Remark 2.1. For the dynamic transition function of capital per unit of effective labor,

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t)$$

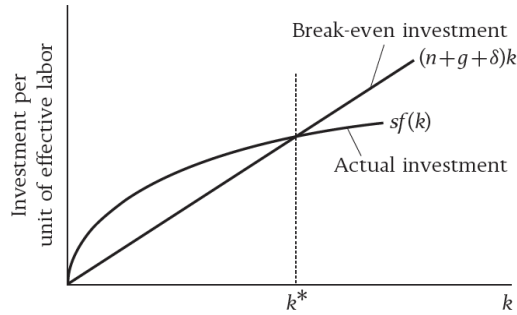And dynamic transition and phase diagram can be expressed as

Figure 1: Dynamic Transition of Capital Per Unit of Effective Labor

Definition 2.1. Steady level of capital per unit of effective labor(k^*) is defined as the level of capital per unit of effective labor that equates break-even investment per unit of effective labor and actual investment per unit of effective labor. [1]

$$k^* := \{k \in \mathbb{R}_+ : sf(k) = (n + g + \delta)k\}$$

Remark 2.2. The values of other endogenous variables at steady state are also derived from k^*.

Example 2.1.

$$y^* = f(k^*) \tag{1}$$
$$i^* = sf(k^*) = (n + g + \delta)k^* \tag{2}$$
$$c^* = y^* - i^* = f(k^*) - (n + g + \delta)k^* = (1 - s)f(k^*) \tag{3}$$

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[1] The definition can also be expressed as $k^* := \{k \in \mathbb{R}_+ : \hat{k} = 0\}$
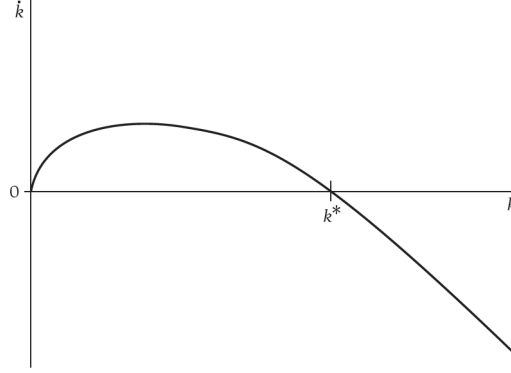
Figure 2: Phase Diagram of Capital Per Unit of Effective Labor

For the growth rate of each endogenous variable (per unit of effective labor).

$$\frac{\partial k(t)}{\partial t}\Big|_{k=k^*} = 0 \tag{4}$$

$$\frac{\partial c(t)}{\partial t}\Big|_{k=k^*} = 0 \tag{5}$$

$$\frac{\partial y(t)}{\partial t}\Big|_{k=k^*} = 0 \tag{6}$$

above relations are equivalent to

$$\text{On steady state} \begin{cases} \dot{c}(t) = 0 \\ \dot{k}(t) = 0 \\ \dot{y}(t) = 0 \end{cases} \tag{7}$$

Proof. By definition of consumption per unit of effective labor,
$c(\cdot) = (1 - s)f(k(t))$
$\implies \dot{c}(t) := \frac{\partial c(\cdot)}{\partial t} = (1 - s)f'(k(t))\dot{k}(t)$ by chain rule
Since $\dot{k}|_{k=k^*} = 0$ and $(1 - s)f'(k(t)) < \infty$
Thus $\dot{c}(t)|_{k=k^*} = 0$ ∎

2.2 Balanced Growth Path

Definition 2.2. A **balanced growth path** is a situation where each variable in the model are all growing at a constant rate. [2] [3]

[2] Variables are not required to grow at the same rate by this definition.
[3] Variables remaining fixed are also considered as growing at a constant rate ($g = 0$).

2.2.1 Growth Rates on Balanced Growth Path

Population and Technology By definition of population and technological progress,

$$g_A := \frac{\dot{A}}{A} = g \tag{8}$$

$$g_L := \frac{\dot{L}}{L} = n \tag{9}$$

Capital per person Since $\frac{K(t)}{L(t)} = \frac{k(t)A(t)L(t)}{L(t)} = k(t)A(t)$, and the growth rate of $x(t)$ can be found as $\frac{\partial \ln x(t)}{\partial t}$. Then

Solution.

$$\frac{\partial \ln \frac{K(t)}{L(t)}}{\partial t} = \frac{\partial k(t)A(t)}{\partial t}$$
$$= \frac{\partial \ln k(t)}{\partial t} + \frac{\partial \ln A(t)}{\partial t}$$
$$= \frac{\dot{k}(t)}{k(t)} + g$$

And at the steady state, by definition, $\dot{k}(t)|_{k-k^*} = 0$, therefore

$$g^*_{\frac{K}{L}} = g \tag{10}$$

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Output and Consumption per person Similarly,

Solution.

$$\frac{Y(t)}{L(t)} = y(t)A(t)$$
$$g_{\frac{Y}{L}} = \frac{\partial \ln y(t) + \ln A(t)}{\partial t}$$
$$= \frac{\partial \ln y}{\partial t} + \frac{\partial \ln A(t)}{\partial t}$$
$$= g + \frac{\dot{y}}{y}$$

and for consumption per person,

$$\frac{C(t)}{L(t)} = c(t)A(t)$$
$$g_{\frac{C}{L}} = g + \frac{\dot{c}}{c}$$

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Thus, on the balanced growth path, [4]

$$g_{\frac{Y}{L}}^* = g \tag{11}$$

$$g_{\frac{C}{L}}^* = g \tag{12}$$

$$\blacksquare$$

Proposition 2.1. Along the balanced growth path, consumption and output per person also growth at rate g.

Proposition 2.2. Along the balanced growth path, aggregate variables, $Y(t), I(t), C(t)$ are all growing at a rate $n + g$.

$$g_Y^* = g_C^* = g_I^* = n + g \tag{13}$$

Proof.

$$\begin{aligned}
g_K &= \frac{\partial \ln K(t)}{\partial t} \\
&= \frac{\partial \ln A(t)L(t)k(t)}{\partial t} \\
&= \frac{\partial \ln A(t)}{\partial t} + \frac{\partial \ln L(t)}{\partial t} + \frac{\partial \ln k(t)}{\partial t} \\
&= g + n + \frac{\dot{k}}{k}
\end{aligned}$$

and at balanced growth path, $\frac{\dot{k}}{k}|_{k=k^*} = 0$, therefore

$$g_K^* = n + g$$

and proof for $C(t)$ and $I(t)$ follows the same path. $\blacksquare$

Definition 2.3. The **golden rule level of capital per unit of effective labor** (k_G) is the steady state level of capital per unit of effective labor that maximizes steady state consumption per unit of effective labor.

$$k_G = argmax_{k^* \in \mathbf{k}^*(\Theta)} \{c^* = f(k^*) - (n + g + \delta)k^*\}$$

First Order Necessary Condition. Solve

$$\begin{aligned}
\frac{\partial c^*(k^*)}{\partial k^*} &= 0 \\
\implies \frac{\partial f(k^*) - (n + g + \delta)k^*}{\partial k^*} &= 0 \\
\implies f'(k^*) &= (n + g + \delta)
\end{aligned}$$

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[4]g_X^* denotes the growth rate of variable X on the balanced growth path.

Thus, golden rule level of capital stock per unit of effective labor k_G can be expressed as [5]

$$k_G = \{k \in \mathbb{R}_+ : f'(k) = (n + g + \delta)\} \tag{14}$$

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2.3 Experiment

2.3.1 Impact of Change in the Saving Rate $(s_1 > s_0)$

Suppose at time t_0, the saving rate parameter increases discretely: $s_0 \to s_1$.

Remark 2.3. The relation of c_0^* and c_1^* depends on the relative position of s_1 and the golden rule level of saving rate s_G.

2.3.2 Derive the Effect of Change in s Mathematically

Goal Find $\frac{\partial k^*}{\partial s}$. And notice that $k^*(n + g + \delta) = sf(k^*)$ for any steady state capital level k^*. And the steady state level of capital per unit of effective labor can be written as a function of parameters, as $k^*(n, g, \delta, s)$.

Impact on k^*

Solution. At any steady state level, k^* satisfies

$$sf(k^*(n, g, \delta, s)) = (n + g + \delta)k^*(n, g, \delta, s)$$

Differentiate both sides with respect to s,
We have

$$sf'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n + g + \delta)\frac{\partial k^*}{\partial s}$$

Rearrange and get

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n + g + \delta) - sf'(k^*)}$$

Notice that the slope of break-even investment is greater than the slope of the actual investment at the steady state, therefore

$$\frac{\partial k^*}{\partial s} > 0$$

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[5]Notice that the zero solution, $k^* = 0$ is a trivial steady state and we ignore this case during this course.
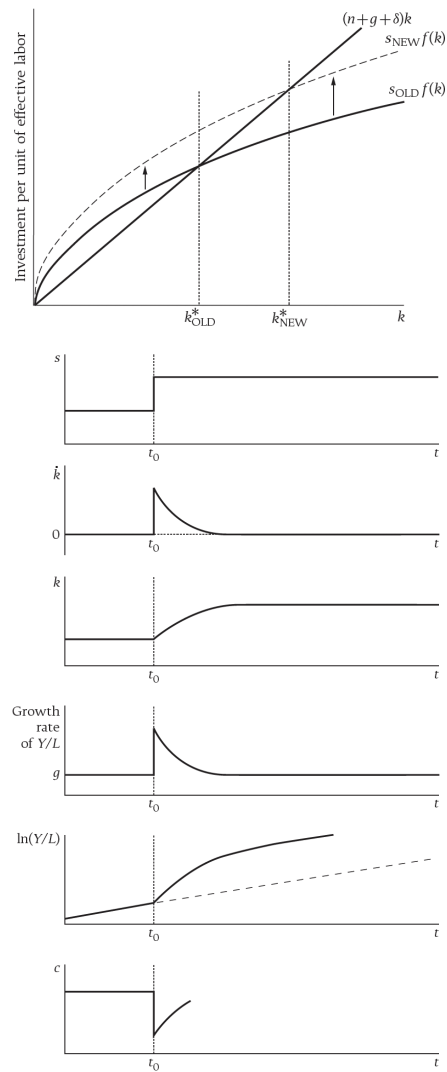
Figure 3: Effect of an Increase in Saving Rate.

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Impact on y^*

Solution. Using chain rule we have

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s}$$

$$= \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial s} > 0, \ \forall k^* \in \mathbf{k}^*(\Theta)$$

∎

To get a sense on how much y^* changes wit respect to change in s, we could look at the <u>elasticity</u>.

$$\eta = \frac{\partial y^*}{\partial s}\frac{s}{y^*} = f'(k^*)\frac{\partial k^*}{\partial s}\frac{s}{f(k^*)} = \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)}$$

Recall that $(n+g+\delta) = \frac{sf(k^*)}{k^*}$ and rearrange the elasticity

$$\eta = \frac{\partial y^*}{\partial s}\frac{s}{y^*}$$

$$= \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)}$$

$$= \frac{sf'(k^*)}{\frac{sf(k^*)}{k^*} - sf'(k^*)}$$

$$= \frac{f'(k^*)}{\frac{f(k^*)}{k^*} - f'(k^*)}$$

$$= \frac{f'(k^*)\frac{k^*}{f(k^*)}}{1 - f'(k^*)\frac{k^*}{f(k^*)}}$$

$$= \frac{\alpha_K}{1 - \alpha_K}$$

Remark 2.4. α_K denotes the elasticity of output per unit of effective unit labor with respect to capital stock per unit of effective labor, along the balanced growth path. And

$$\alpha_K \approx \frac{1}{3}$$

Remark 2.5. If the production function is in the Cobb-Douglas form, then $\alpha_K = \alpha$.

Example 2.2. If $\alpha_K \approx \frac{1}{3}$ then $\frac{\partial y^*}{\partial s}sy^* \approx \frac{1}{2}$.

Impact on c^*   Notice that on the balanced growth path $c^* = y^* - i^*$.

$$c^* = f(k^*) - (n + g + \delta)k^*$$

and differentiate with respect to s

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n + g + \delta)]\frac{\partial k^*}{\partial s}$$

And notice that the sign of $\frac{\partial c^*}{\partial s}$ depends on the relative slope of production function and break-even investment. By the first order condition of golden rule level of capital per unit of effective labor, $(n + g + \delta) = f'(k_G)$

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - f'(k_G)]\frac{\partial k^*}{\partial s}$$

And

$$\begin{cases} k^* = k_G \implies f'(k^*) = f'(k_G) \implies \frac{\partial c^*}{\partial s} = 0 \\ k^* < k_G \implies f'(k^*) > f'(k_G) \implies \frac{\partial c^*}{\partial s} > 0 \\ k^* > k_G \implies f'(k^*) < f'(k_G) \implies \frac{\partial c^*}{\partial s} < 0 \end{cases}$$