MAT395 Independent Reading in Mathematical Economics

Individual Decision Making, Market Equilibrium, Market Failure, and Other Topics.

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May 28, 2019

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- GitHub: `https://github.com/TianyuDu/Spikey_UofT_Notes`
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1 Chapter 1. Preference and Choice

1.1 Preference Relations

Definition 1.1.

(i) The **strict preference** relation, $\succ$, is defined by

$$x \succ y \iff x \succsim y \wedge \neg(y \succsim x) \tag{1.1}$$

(ii) The **indifference** relation, $\sim$, is defined by

$$x \sim y \iff x \succsim y \wedge y \succsim x \tag{1.2}$$

Definition 1.2 (1.B.1)**.** The preference relation $\succsim$ is **rational** if it possesses the following two properties

(i) *Completeness*
$$\forall x, y \in X,\ x \succsim y \vee y \succsim x \tag{1.3}$$

(ii) *Transitivity*
$$\forall x, y, z \in X,\ x \succsim y \wedge y \succsim z \implies x \succsim z \tag{1.4}$$

Proposition 1.1 (1.B.1)**.** If $\succsim$ is rational, then

(i) $\succ$ is both **reflexive** ($\neg\ x \succ x$) and **transitive** ($x \succ y \wedge y \succ z \implies x \succ z$);

(ii) $\sim$ is both **reflexive** and **transitive**;

(iii) $x \succ y \succsim z \implies x \succ z$.

Example 1.1. Typical scenarios when transitivity of preference is violated:

(i) *Just perceptible differences*;

(ii) *Framing problem*;

(iii) *Observed preference might from the result of the interaction of several more primitive rational preferences (Condorcet paradox)*;

(iv) *Change of tastes.*

Definition 1.3 (1.B.2)**.** A function $u : X \to \mathbb{R}$ is a **utility function representing preference relation** $\succsim$ if
$$\forall x, y \in X,\ x \succsim y \iff u(x) \geq u(y) \tag{1.5}$$

Proposition 1.2 (1.B.2)**.** If a preference relation $\succsim$ can be represented by a utility function, then $\succsim$ is rational.

1.2 Choice Rules

Definition 1.4. A **choice structure**, $(\mathscr{B}, C(\cdot))$, is a tuple consists of

(i) The collection of **budget sets** $\mathscr{B}$, which is a set of nonempty subsets of X.

(ii) The **choice rule**, $C(B) \subset B$, is a *correspondence* for every $B \subset \mathscr{B}$ denotes the individual's choice from among the alternatives in B. If $C(B)$ is not a singleton, it can be interpreted as the *acceptable alternatives* in B, which the individual would actually chosen if the decision-making process is run repeatedly.

Definition 1.5 (1.C.1)**.** The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the **weak axiom of revealed preference** if

$$\Big(\underbrace{\exists B \in \mathscr{B} \text{ s.t. } x, y \in B \wedge x \in C(B)}_{x \succsim *y \text{ revealed.}}\Big) \implies \Big(\forall B' \in \mathscr{B} \text{ s.t. } x, y \in B',\ y \in C(B') \implies x \in C(B')\Big) \tag{1.6}$$

Definition 1.6. Given a choice structure $(\mathscr{B}, C(\cdot))$, the **revealed preference relation** $\succsim^*$ is defined as

$$x \succsim^* y \iff \exists B \in \mathscr{B} \text{ s.t. } x, y \in B \wedge x \in C(B) \tag{1.7}$$

Remark 1.1 (Interpretation on the definition of WARP)**.** If x is *revealed* at least as good as y, then y cannot be revealed preferred to x.

1.3 The Relationship between Preference Relations and Choice Rules

Definition 1.7. Given rational preference relation $\succsim$ on X, the **preference-maximizing choice rule** is defined as

$$C^*(B, \succsim) := \{x \in B : x \succsim y \ \forall y \in B\} \ \forall B \in \mathscr{B} \tag{1.8}$$

We say the rational preference relation **generates** the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

Assumption 1.1. Assume $C^*(B, \succsim) \neq \varnothing$ for all $B \in \mathscr{B}$.

Proposition 1.3 (1.D.1 (Rational $\rightarrow$ WARP))**.** Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom.

Definition 1.8 (1.D.1)**.** Given choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ **rationalizes** $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim) \ \forall B \in \mathscr{B} \tag{1.9}$$

That is, $\succsim$ *generates the choice structure* $(\mathscr{B}, C(\cdot))$.

Remark 1.2. In general, for a given choice structure $(\mathscr{B}, C(\cdot))$, there may be more than one rational preference relation $\succsim$ rationalizing it.

Proposition 1.4 (1.D.2 (WARP $\rightarrow$ Rational))**.** If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) The weak axiom is satisfied;

(ii) $\mathscr{B}$ includes all subsets of X up to three elements.

Then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$.

2 Chapter 2. Consumer Choice

2.1 Commodities

Definition 2.1. Assume the number of **commodities** is finite and equal to L. In general, a **commodity vector** or **commodity bundle** is an element in a **commodity space**, typically $\mathbb{R}^L$.

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L \tag{2.1}$$

Remark 2.1 (Time Aggregation)**.** The time/location of commodity matters in some scenarios, and can be built into the definition of a commodity.

Remark 2.2. We should also note that in some contexts it becomes convenient, and even necessary, to expand the set of commodities to include goods and services that may potentially be available for purchase but are not actually so and even some that may be available by means other than market exchange.

2.2 The Consumption Set

Definition 2.2. The **consumption set** is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by his environment.

Assumption 2.1. For simplicity, we assume the consumption set to be $\mathbb{R}^L_+$, which is *convex*.

$$X := \mathbb{R}^L_+ = \{\mathbf{x} \in \mathbb{R}^L : x_\ell \geq 0,\ \forall \ell \in [L]\} \tag{2.2}$$

2.3 Competitive Budgets

Definition 2.3. A **price vector** is defined as

$$\mathbf{p} := \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L \tag{2.3}$$

For simplicity, here we always assume

(i) *Positive price*: $\mathbf{p} \gg \mathbf{0}$;

(ii) *Price-taking assumption*: $\mathbf{p}$ is beyond the influence of the consumer.

Definition 2.4 (2.D.1)**.** The **Walrasian**, or **competitive budget set** is defined as

$$B_{\mathbf{p},w} := \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} \leq w\} \tag{2.4}$$

where w is the *wealth* of consumer, and assumed to be positive.

Definition 2.5. The **consumer's problem** is choosing a consumption bundle $\mathbf{x} \in B_{\mathbf{p},w}$, for each given $(\mathbf{p}, w) \in \mathbb{R}^L_{++}$.

Definition 2.6. The set $\{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} = w\}$ is called the **budget hyperplane**.

Proposition 2.1. The price vector $\mathbf{p}$ is orthogonal to the budget hyperplane.

Proposition 2.2. The Walrasian budget set $B_{\mathbf{p},w}$ is a *convex* set.

2.4 Demand Functions and Comparative Statics

Definition 2.7. The consumer's **Walrasian demand correspondence** $x(\mathbf{p}, w) : \mathbb{R}^{L+1}_{++} \rightrightarrows \mathbb{R}^L_+$ assigns a *set* of chosen consumption bundles for each price-wealth pair $(\mathbf{p}, w)$. When $x(\mathbf{p}, w)$ is single-valued, we refer to it as a **demand function**

$$\mathbf{x}(\mathbf{p}, w) = \begin{bmatrix} x_1(\mathbf{p}, w) \\ x_2(\mathbf{p}, w) \\ \vdots \\ x_L(\mathbf{p}, w) \end{bmatrix} \tag{2.5}$$

Definition 2.8 (2.E.1)**.** The Walrasian demand correspondence $x(\mathbf{p}, w) : \mathbb{R}^{L+1}_{++} \rightrightarrows \mathbb{R}^L_+$ is **homogenous of degree zero** if

$$x(\alpha\mathbf{p}, \alpha w) = x(\mathbf{p}, w)\ \forall (\mathbf{p}, w, \alpha) \in \mathbb{R}^{L+2}_{++} \tag{2.6}$$

Also note that

$$B_{\mathbf{p},w} = B_{\alpha\mathbf{p},\alpha w}\ \forall (\mathbf{p}, w, \alpha) \in \mathbb{R}^{L+2}_{++} \tag{2.7}$$

Definition 2.9 (2.E.2)**.** The Walrasian demand correspondence $x(\mathbf{p}, w)$ satisfies **Walras' law** if

$$\forall (\mathbf{p}, w) \gg \mathbf{0},\ \forall \mathbf{x} \in x(\mathbf{p}, w),\ \mathbf{p} \cdot \mathbf{x} = w \tag{2.8}$$

Assumption 2.2. For simplicity, we assume $x(\mathbf{p}, w)$ is always *single-valued, continuous and differentiable.*

Proposition 2.3. The **family of Walrasian budget sets** defined as

$$\mathscr{B}^{\mathscr{W}} := \{B_{\mathbf{p},w} : \mathbf{p}, w \gg \mathbf{0}\} \tag{2.9}$$

altogether with Walrasian demand homogeneous to degree zero forms a *choice structure*

$$(\mathscr{B}^{\mathscr{W}}, x(\cdot)) \tag{2.10}$$

Definition 2.10. For fixed prices $\overline{\mathbf{p}} \in \mathbb{R}^L_{++}$, the function of wealth $\mathbf{x}(\overline{\mathbf{p}}, w)$ is called consumer's **Engel function**. Its image in $\mathbb{R}^L_+$,

$$E_{\overline{\mathbf{p}}} := \{\mathbf{x}(\overline{\mathbf{p}}, w) : w \in \mathbb{R}_{++}\} \subset \mathbb{R}^L_+ \tag{2.11}$$

is defined as the **wealth expansion path**.

Definition 2.11. Given $(\mathbf{p}, w)$, the **wealth effect** is defined as

$$D_w\mathbf{x}(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial w} \\ \frac{\partial x_2(\mathbf{p},w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(\mathbf{p},w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L \tag{2.12}$$

For the ℓ-th commodity, it's called **normal** at $(\mathbf{p}, w)$ if $\frac{\partial x_\ell(\mathbf{p},w)}{\partial w} \geq 0$, and **inferior** otherwise. And the ℓ-th commodity is normal/inferior if its normal/inferior every where in $\mathbb{R}^{L+1}_{++}$.

Definition 2.12. The **offer curve** is defined as the locus

$$\{\mathbf{x}(\mathbf{p}, w) : p_j > 0\} \tag{2.13}$$

for any chosen j.

Definition 2.13. Good ℓ is said to be a **Giffen good** at $(\mathbf{p}, w)$ if

$$\frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_\ell} > 0 \tag{2.14}$$

Definition 2.14. The **price effects** at $(\mathbf{p}, w)$ is defined as

$$D_{\mathbf{p}}(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_1(\mathbf{p},w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_L(\mathbf{p},w)}{\partial p_L} \end{bmatrix} \tag{2.15}$$

Proposition 2.4 (2.E.1)**.** If the Walrasian demand function $x(\mathbf{p}, w)$ is homogenous of degree zero, then for all $\mathbf{p}$ and w, then

$$\sum_{k=1}^{L} \frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_k} p_k + \frac{\partial x_\ell(\mathbf{p}, w)}{\partial w} w = 0 \text{ for } \ell = 1, \ldots, L \tag{2.16}$$

Equivalently,

$$D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ \mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ w = \mathbf{0} \tag{2.17}$$

Proof. Apply *Euler's theorem* on homogenous functions to each component x_ℓ.

$$\underbrace{D_{(\mathbf{p},w)}\mathbf{x}(\mathbf{p}, w)}_{L\times(L+1)} \cdot \underbrace{(\mathbf{p}, w)}_{(L+1)\times 1} = 0\ \mathbf{x}(\mathbf{p}, w) = \mathbf{0} \tag{2.18}$$

$$\implies [\underbrace{D_{\mathbf{p}}(\mathbf{p}, w)}_{L\times L} | \underbrace{D_w\mathbf{x}(\mathbf{p}, w)}_{L\times 1}] \cdot (\mathbf{p}, w) = D_{\mathbf{p}}(\mathbf{p}, w)\ \mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ w = \mathbf{0} \tag{2.19}$$

∎

Definition 2.15. The **elasticities of demand ℓ with respect to price k and wealth** is defined as

$$\varepsilon_{\ell,k}(\mathbf{p},w) := \frac{\partial x_\ell(\mathbf{p},w)}{\partial p_k}\frac{p_l}{x_\ell(\mathbf{p},w)} \tag{2.20}$$

$$\varepsilon_{\ell,w}(\mathbf{p},w) := \frac{\partial x_\ell(\mathbf{p},w)}{\partial w}\frac{w}{x_\ell(\mathbf{p},w)} \tag{2.21}$$

Corollary 2.1. Dividing both sides of the equality in proposition (2.E.1) by x_ℓ gives

$$\sum_{k=1}^{L}\varepsilon_{\ell,k}(\mathbf{p},w) + \varepsilon_{\ell,w}(\mathbf{p},w) = 0 \ \forall \ell \in \{1,\ldots,L\} \tag{2.22}$$

Proposition 2.5 (2.E.2 Cournot Aggregation)**.** If the Walrasian demand function $x(\mathbf{p},w)$ satisfies *Walras' law*, then for every $(\mathbf{p},w)$,

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(\mathbf{p},w)}{\partial p_k} + x_k(\mathbf{p},w) = 0 \quad \text{for } k = 1,\ldots,L \tag{2.23}$$

Equivalently,

$$\mathbf{p}^T D_{\mathbf{p}}\mathbf{x}(\mathbf{p},w) + \mathbf{x}(\mathbf{p},w)^T = \mathbf{0}^T \tag{2.24}$$

Proof. Differentiate both sides of Walras' law identity $\mathbf{p}^T\mathbf{x} = w$ with respect to $\mathbf{p}$. ■

Proposition 2.6 (2.E.3. Engel Aggregation)**.** If the Walrasian demand function $x(\mathbf{p},w)$ satisfies *Walras' law*, then for every $(\mathbf{p},w)$,

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(\mathbf{p},w)}{\partial w} = 1 \tag{2.25}$$

or equivalently

$$\mathbf{p}\cdot D_w x(\mathbf{p},w) = 1 \tag{2.26}$$

Proof. Differentiate both sides of Walras' law identity $\mathbf{p}^T\mathbf{x} = w$ with respect to w. ■

Proposition 2.7 (Exer. 2.E.2)**.**

$$\sum_{\ell=1}^{L} b_\ell(\mathbf{p},w)\varepsilon_{\ell k}(\mathbf{p},w) + b_k(\mathbf{p},w) = 0 \tag{2.27}$$

and

$$\sum_{\ell=1}^{L} b_\ell(\mathbf{p},w)\varepsilon_{\ell w}(\mathbf{p},w) = 1 \tag{2.28}$$

where $b_\ell := \frac{x_\ell p_\ell}{w}$ is defined to be the portion of wealth spent on commodity ℓ.

2.5 The Weak Axiom of Revealed Preference and the Law of Demand

Assumption 2.3. In the section, we assume $\mathbf{x}(\mathbf{p},w)$ is

(i) Single-valued;

(ii) homogeneous to degree zero;

(iii) satisfies Walras' law.

Definition 2.16 (2.F.1)**.** The Walrasian demand function $\mathbf{x}(\mathbf{p}, w)$ satisfies the **weak axiom of revealed preference** if for every two $(\mathbf{p}, w), (\mathbf{p}', w') \in \mathbb{R}^{L+1}_{++}$,

$$\underbrace{\mathbf{p} \cdot \mathbf{x}(\mathbf{p}', w') \leq w \wedge \mathbf{x}(\mathbf{p}, w) \neq \mathbf{x}(\mathbf{p}', w')}_{\text{revealed: } \mathbf{x}(\mathbf{p},w) \succ^* \mathbf{x}(\mathbf{p}',w')} \implies \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w) > w' \tag{2.29}$$

Equivalently,

$$\mathbf{x}(\mathbf{p}', w') \in B_{\mathbf{p},w} \wedge \mathbf{x}(\mathbf{p}', w') \notin C(B_{\mathbf{p},w}) \implies \mathbf{x}(\mathbf{p}, w) \notin C(B_{\mathbf{p}',w'}) \tag{2.30}$$

Corollary 2.2 (Equivalent Definition)**.** The weak axiom says, given our assumptions and $\mathbf{x}(\mathbf{p}_1, w_1) \neq \mathbf{x}(\mathbf{p}_2, w_2)$, we <u>cannot</u> have both

$$\mathbf{x}(\mathbf{p}_1, w_1) \in B_{\mathbf{p}_2,w_2} \wedge \mathbf{x}(\mathbf{p}_2, w_2) \in B_{\mathbf{p}_1,w_1} \tag{2.31}$$

Definition 2.17. A price change $\Delta\mathbf{p}$ is a **Slutsky compensated price change** if the consumer is given a **Slutsky wealth compensation** with amount

$$\Delta w = \Delta\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, w) \tag{2.32}$$

such that the consumer's initial consumption is just affordable at the new price.

Proposition 2.8 (2.F.1)**.** Suppose that the Walrasian demand function $\mathbf{x}(\mathbf{p}', w')$ is homogenous of degree zero and satisfies Walras' law. Then $\mathbf{x}(\mathbf{p}', w')$ satisfies the weak axiom <u>if and only if</u> the following property holds:
For any *compensated price change* from $(\mathbf{p}, w)$ to $(\mathbf{p}', w' := \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w))$,

$$\Delta\mathbf{p} \cdot \Delta\mathbf{x} \leq 0 \tag{2.33}$$

with strict inequality whenever $\mathbf{x}(\mathbf{p}, w) \neq \mathbf{x}(\mathbf{p}', w')$.

Corollary 2.3 (Compensated Law of Demand)**.** $\Delta\mathbf{p} \cdot \Delta\mathbf{x} \leq 0$ says demand and price move in opposite directions, *under Slutsky compensation.*

Definition 2.18. At infinitesimal price change, the Slutsky compensation can be written as

$$dw = \mathbf{x}(\mathbf{p}, w) \cdot d\mathbf{p} \tag{2.34}$$

and the compensated law of demand becomes

$$d\mathbf{p} \cdot d\mathbf{x} \leq 0 \tag{2.35}$$

Then the total derivative of $\mathbf{x}$ is

$$\begin{aligned}
d\mathbf{x} &= D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ d\mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ dw && (2.36)\\
&= D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ d\mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ [\mathbf{x}(\mathbf{p}, w) \cdot d\mathbf{p}] && (2.37)\\
&= [\underbrace{D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)}_{L\times L} + \underbrace{D_w\mathbf{x}(\mathbf{p}, w)}_{L\times 1}\underbrace{\mathbf{x}(\mathbf{p}, w)^T}_{1\times L}]d\mathbf{p} && (2.38)\\
&\implies d\mathbf{p}^T \underbrace{[D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w) + D_w\mathbf{x}(\mathbf{p}, w)\mathbf{x}(\mathbf{p}, w)^T]}_{L\times L} d\mathbf{p} \leq 0 && (2.39)
\end{aligned}$$

and the **Slutsky/substitution matrix** is defined as

$$\begin{aligned}
S(\mathbf{p}, w) &:= [D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w) + D_w\mathbf{x}(\mathbf{p}, w)\mathbf{x}(\mathbf{p}, w)^T] && (2.40)\\
s_{\ell k} &= \underbrace{\frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_k}}_{\text{total effect}} + \underbrace{\frac{\partial x_\ell(\mathbf{p}, w)}{\partial w} x_k(p, w)}_{\text{wealth effect}} && (2.41)
\end{aligned}$$

where $s_{\ell k}$ is the **substitution effect**.

Remark 2.3. The above identity (Slutsky equation) suggests the total impact of price change in p_k on demand for x_ℓ can be decomposed into two portions, substitution effect and income effect.

Corollary 2.4 (Slutsky Equation)**.**

$$\frac{\partial x_i(\mathbf{p}, w)}{\partial p_j} = \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} - \frac{\partial x_i(\mathbf{p}, w)}{\partial w} x_j(\mathbf{p}, w) \tag{2.42}$$

Remark 2.4. Consider the scenario when only p_k changes, with Slutsky compensation, consumer's wealth changes by $dw = x_k(\mathbf{p}, w)dp_k$. So the wealth effect on x_ℓ is $\frac{\partial x_\ell}{\partial w}dw = \frac{\partial x_\ell}{\partial w}x_k(\mathbf{p}, w)dp_k$.

Proposition 2.9 (2.F.2)**.** If a differentiable Walrasian demand function $\mathbf{x}(\mathbf{p}, w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any $(\mathbf{p}, w)$, the Slutsky matrix $S(\mathbf{p}, w)$ is negative semi-definite.

Corollary 2.5. Given $S(\mathbf{p}, w)$ is negative semi-definite, we have

$$\mathbf{e}_\ell^T S(\mathbf{p}, w)\mathbf{e}_\ell \leq 0 \ \forall \ell \in \{1, \dots, L\} \tag{2.43}$$

$$\implies s_{\ell\ell} \leq 0 \ \forall \ell \in \{1, \dots, L\} \tag{2.44}$$

which suggests the *substitution effect of good ℓ with respect to its own price is always negative.*

Remark 2.5. Proposition 2.F.2 does *not* imply, in general, that the matrix $S(\mathbf{p}, w)$ is symmetric.

Proposition 2.10 (2.F.3)**.** Suppose that the Walrasian demand function $\mathbf{x}(\mathbf{p}, w)$ is differentiable, homogenous of degree zero, and satisfies Walras' law. Then for every $(\mathbf{p}, w)$

$$\mathbf{p}^T S(\mathbf{p}, w) = \mathbf{0} \wedge S(\mathbf{p}, w)\mathbf{p} = \mathbf{0} \tag{2.45}$$

Proof. By propositions 2.E.1 to 2.E.3. ■

3 Chapter 3. Classical Demand Theory

3.1 Preference Relations: Basic Properties

Definition 3.1 (3.B.1)**.** The preference relation $\succsim$ on X is **rational** if it possesses the following two properties

(i) *Completeness.* $\forall \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \succsim \mathbf{y} \vee \mathbf{y} \succsim \mathbf{x}$;

(ii) *Transitivity.* $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \mathbf{x} \succsim \mathbf{y} \wedge \mathbf{y} \succsim \mathbf{z} \implies \mathbf{x} \succsim \mathbf{z}$.

3.1.1 Desirability Assumptions

Definition 3.2 (3.B.2)**.** The preference relation $\succsim$ on X is **monotone** if $\mathbf{x} \in X$ and $\mathbf{y} \gg \mathbf{x} \implies \mathbf{y} \succ \mathbf{x}$; It is **strongly monotone** if $\mathbf{y} \geq \mathbf{x} \wedge \mathbf{x} \neq \mathbf{y}$

Remark 3.1. If $\succsim$ is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities.

Definition 3.3 (3.B.3)**.** A preference relation $\succsim$ on X is **locally nonsatiated** if

$$\forall \mathbf{x} \in X, \varepsilon > 0,\ \exists \mathbf{y} \in \overline{\mathcal{B}}(\mathbf{x}, \varepsilon) \cap X \ s.t.\ \mathbf{y} \succ \mathbf{x} \tag{3.1}$$

Remark 3.2. Local nonsatiation rules out the extreme situation in which all commodities are bads, since in that case no consumption at all (the point $\mathbf{x} = \mathbf{0}$) would be a satiation point.

Proposition 3.1 (Exercise 3.B.1)**.**

$$\text{Strongly Monotone} \implies \text{Monotone} \implies \text{Locally Non-satiation} \tag{3.2}$$

Definition 3.4. The **indifference set** containing point $\mathbf{x}$ is defined as $\{\mathbf{y} \in X : \mathbf{x} \sim \mathbf{y}\}$. The **upper contour set** of bundle $\mathbf{x}$ is $\{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$. The **lower contour set** of $\mathbf{x}$ is defined as $\{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$.

Remark 3.3 (Implication of Local Nonsatiation)**.** One implication of local nonsatiation (and, hence, of monotonicity) is that it rules out "thick" indifference sets.

3.1.2 Convexity Assumptions

Definition 3.5 (3.B.4)**.** The preference relation $\succsim$ on X is **convex** if for every $\mathbf{x} \in X$, the upper contour set if $\mathbf{x}$ is convex.

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \mathbf{y} \succsim \mathbf{x} \wedge \mathbf{z} \succsim \mathbf{x} \implies \alpha \mathbf{y} + (1-\alpha)\mathbf{z} \succsim \mathbf{x}\ \forall \alpha \in [0,1] \tag{3.3}$$

Remark 3.4 (Implication of Convexity)**.** Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification.

Remark 3.5. The convex assumption can hold only if X is convex.

Definition 3.6 (3.B.5)**.** The preference relation $\succsim$ on X is **strictly convex** if

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \mathbf{y} \succ \mathbf{x} \wedge \mathbf{z} \succ \mathbf{x} \wedge \mathbf{y} \neq \mathbf{z} \implies \alpha \mathbf{y} + (1-\alpha)\mathbf{z} \succ \mathbf{x}\ \forall \alpha \in (0,1) \tag{3.4}$$

Definition 3.7 (3.B.6)**.** A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is **homothetic** if

$$\forall \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \sim \mathbf{y} \implies \alpha \mathbf{x} \sim \alpha \mathbf{y},\ \forall \alpha \in \mathbb{R}_+ \tag{3.5}$$

Definition 3.8 (3.B.7)**.** The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is **quasilinear** with respect to commodity 1 (the **numeraire** commodity) if $\forall \mathbf{x}, \mathbf{y} \in X$

(i) $\mathbf{x} \sim \mathbf{y} \implies \mathbf{x} + \alpha \mathbf{e}_1 \sim \mathbf{y} + \alpha \mathbf{e}_1\ \forall \alpha \in \mathbb{R}$;

(ii) *Good 1 is desirable*: $\forall \mathbf{x} \in X, \alpha \in \mathbb{R}_{++},\ \mathbf{x} + \alpha \mathbf{e}_1 \succ \mathbf{x}$.

3.2 Preference and Utility

Definition 3.9 (Example 3.C.1)**.** The **lexicographic preference relation** on $X = \mathbb{R}^2_+$ defines $x \succsim y$ if either $x_1 > y_1$ or $x_1 = y_1 \wedge x_2 \geq y_2$.

Definition 3.10 (3.C.1)**.** The preference relation $\succsim$ on X is **continuous** if it is *preserved under limits*. That's

$$\forall ((x^n, y^n)_n)_{n=1}^\infty\ s.t.\ x := \lim_{n\to\infty} x^n,\ y := \lim_{n\to\infty} y^n, \quad x^n \succsim y^n\ \forall n \implies x \succsim y \tag{3.6}$$

Proposition 3.2 (Equivalent Definition)**.** A preference relation $\succsim$ is continuous if and only if for all $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ and lower contour set $\{y \in X : x \succsim y\}$ are closed.

Proof. Suppose $\succsim$ is continuous, fix $x \in X$. Then for any sequence in the upper contour set of x, the limit point is also in the upper contour set of x. As a result, for every $x \in X$, U_x contains all limit points, so it is closed. ∎

Proposition 3.3. Lexicographic preference relation is *not* continuous.

Proof.

$$x^n := (1/n, 0) \text{ and } y^n := (0, 1) \tag{3.7}$$

∎

Proposition 3.4.][3.C.1] Let $\succsim$ be a continuous preference relation on X, there is a continuous utility function $u : X \to \mathbb{R}$ representing $\succsim$.

Proof. Construction of utility function:

(i) For each $x \in X$, by monotonicity and continuity of $\succsim$, there exists an unique $\alpha(x)$ such that

$$\alpha(x)e \sim x \tag{3.8}$$

(ii) Take $\alpha(x)$ as the utility function.

■

Remark 3.6. Above proposition guarantees the existence of continuous utility function for any continuous $\succsim$. But, not all utility functions representing $\succsim$ are continuous. We can construct discontinuous utility function by compositing a continuous utility function with a discontinuous but strictly increasing transformation.

Remark 3.7. It is possible for continuous preferences *not* to be representable by a differentiable (but still continuous) utility function (*Leontief*).

Lemma 3.1. The upper contour set of a quasi-concave function is convex.

Proposition 3.5. [?] is this bi-conditional? If $\succsim$ is (strictly) convex, then $u(\cdot)$ representing $\succsim$ is (strictly) quasi-concave.

Proposition 3.6. A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if and only if it admits a utility function u homogeneous of degree one.

Proposition 3.7. A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is *quasilinear* with respect to the first commodity (numeraire) if and only if it admits a utility function u in the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

Remark 3.8. Increasingness and quasi-concavity are ordinal properties of u; they are preserved for any arbitrary increasing transformation of the utility index. In contrast, the special forms of the utility representations in above propositions are not preserved; they are cardinal properties that are simply convenient choices for a utility representation.

3.3 The Utility Maximization Problem

Definition 3.11. Suppose a consumer chooses her most *preferred consumption bundle* given prices $p \gg 0$ and wealth level $w > 0$, then the **utility maximization problem**(UMP) of this consumer is

$$\max_{x \geq 0} u(x) \ s.t. \ p \cdot x \leq w \tag{3.9}$$

Proposition 3.8 (3.D.1)**.** If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proof. Note $B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$ is compact. The proposition is an immediate consequence of the extreme value theorem. ■

3.3.1 The Walrasian Demand Correspondence/Function

Definition 3.12. The **Walrasian demand correspondence**, $x(p, w)$, is the set of solutions to consumer's UMP. When the solution is unique, it is referred to as the walrasian demand function.

Proposition 3.9 (3.D.2)**.** Suppose u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on $X := \mathbb{R}^L_+$. The the Walrasian demand correspondence, $x(p, w)$, satisfies

(i) *Homogeneous of degree zero* in (p, w);

(ii) *Walras' law*: $p \cdot x = w$;

(iii) *Convexity* if $\succsim$ is convex (i.e. u is quasi-concave), then $x(p, w)$ is convex;

(iv) *Uniqueness* if $\succsim$ is strictly convex (i.e. u is strictly quasi-concave), then $x(p, w)$ is a singleton.[1]

[1] A singleton set is trivially convex.

Proposition 3.10 (Kuhn-Tucker Necessary Condition)**.** Let $x^* \in x(p,w)$, then there exists a *Lagrangian multiplier* $\lambda \geq 0$ such that

$$\nabla u(x^*) \leq \lambda p \tag{3.10}$$
$$x^* \cdot [\nabla u(x^*) - \lambda p] = 0 \text{ (complementary slackness)} \tag{3.11}$$

As a result, for any interior optimum ($x^* \gg 0$),

$$\nabla u(x^*) = \lambda p \tag{3.12}$$

Corollary 3.1. If $\nabla u(x^*) \gg 0$, then the first order necessary condition for an interior optimum to UMP is equivalent to

$$\frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} = \frac{p_\ell}{p_k} \tag{3.13}$$

for every ℓ, k.

Definition 3.13. The left hand side of above equality is the **marginal rate of substitution** of good ℓ for good k at x^*, $MRS_{\ell k}$ at (x^*). It tells us the amount of good k that the consumer must be given to compensate her for a one-unit marginal reduction in her consumption of good ℓ ($\frac{dx_k}{dx_\ell}$).

Proposition 3.11 (Interpretation of λ)**.** The Lagrangian multiplier λ gives the **shadow price** of relaxing the wealth constraint in UMP. Therefore it equals the *marginal utility value of wealth* at the optimum.

Proof. This is an immediate consequence of the envelope theorem. ■

Proposition 3.12. If u is quasi-concave and monotone, and has $\nabla u(x) \neq 0\ \forall x \in \mathbb{R}^L_+$, then the Kuhn-Tucker conditions are indeed sufficient.

Proposition 3.13. Indeed, if preferences are continuous, strictly convex, and locally nonsatiated on the consumption set $\mathbb{R}^L_+$,then $x(p,w)$ (which is then a function) is always continuous at all $(p,w) \gg 0$.

3.3.2 The Indirect Utility Function

Definition 3.14. The value function of consumer's UMP, $v(p,w) := u(x^*(p,w))$, is called the **indirect utility function**.

Proposition 3.14 (3.D.3)**.** Suppose u is a continuous utility function representing a locally nonsatiated $\succsim$ on $\mathbb{R}^L_+$, then $v(p,w)$ satisfies

(i) Homogeneous of degree zero;

(ii) Strictly increasing in w and non-increasing in p_ℓ for every ℓ;

(iii) Quasi-convex (i.e. its lower contour set is convex);

(iv) Continuous in (p,w).

Proof. Show quasi-convexity of $v(p,w)$. Let $\overline{v} \in \mathbb{R}$ be an attainable utility level, the corresponding lower contour is $L := \{(p,w) : v(p,w) \leq \overline{v}\}$. Let $(p,w),(p',w') \in L$, $\alpha \in [0,1]$. Show $(p'',w'') := \alpha(p,w) + (1-\alpha)(p',w') \in L$ by showing $u(x) \leq \overline{v}$ for every $p'' \cdot x \leq w''$. Suppose $p'' \cdot x \leq w''$, then

$$\alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w' \tag{3.14}$$
$$\implies p \cdot x \leq w \vee p' \cdot x \leq w' \tag{3.15}$$

which implies either $u(x) \leq v(p,w)$ or $u(x) \leq v(p',w')$, by the definition of value function of maximization problems. Since both $v(p,w), v(p',w') \leq \overline{v}$, then $u(x) \leq \overline{v}$. Therefore $v(p'',w'') \leq \overline{v}$. So $(p'',w'') \in L$, and L is convex. ■

Proposition 3.15 (Transformation on v)**.** [?] Does this require f to be strictly increasing? Note that the indirect utility function depends on the utility representation chosen. In particular, if $v(p,w)$ is the indirect utility function when the consumer's utility function is u, then the indirect utility function corresponding to utility representation $\tilde{u}(x) = f \circ u(x)$ is $\tilde{v}(p,w) = f \circ v(p,w)$.

Proof. the maximizer of such an optimization problem is invariant under such a monotonically increasing transformation f. ■

3.4 The Expenditure Minimization Problem

Definition 3.15. Suppose a consumer chooses her most *preferred consumption bundle* given prices $p \gg 0$ and wealth level $u > u(0)$, then the **expenditure minimization problem**(EMP) of this consumer is

$$\min_{x \geq 0} p \cdot x \ s.t. \ u(x) \geq u \tag{3.16}$$

Definition 3.16. The value function of above optimization problem is called the **expenditure function**, denoted as $e(p, u)$.

Assumption 3.1. We assume that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$.

Proposition 3.16 (3.E.1, the Duality)**.** Suppose u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$, and $p \gg 0$. Then,

(i) If x^* is optimal in the UMP when wealth $w > 0$, then x^* is the in the EMP with utility level $u(x^*)$, and the minimal expenditure is w;

(ii) If x^* is optimal in the EMP with utility level $u > u(0)$, then x^* is optimal in the UMP with wealth level $p \cdot x^*$, and the attained maximal utility is u.

Proof. Contradiction. ■

Corollary 3.2. For any $p \gg 0$, $w > 0$, and $u > u(0)$,

$$e(p, v(p, w)) = w \tag{3.17}$$

$$v(p, e(p, u)) = u \tag{3.18}$$

Corollary 3.3.

$$h(p, u) = x(p, e(p, u)) \tag{3.19}$$

$$x(p, w) = h(p, v(p, w)) \tag{3.20}$$

Proposition 3.17 (3.E.2)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$. Then the expenditure function $e(p, u)$ possesses the following properties

(i) Homogeneous of degree one <u>in p</u>;

(ii) Strictly increasing in u and nondecreasing in p_ℓ for every ℓ;

(iii) Concave in p;

(iv) Continuous in (p, u).

Proof. Show the concavity of e, let $p, p' \gg 0$, $\alpha \in [0, 1]$, and $\overline{u} > u(0)$. Define $p'' := \alpha p + (1 - \alpha)p'$, then

$$e\left(p'', \overline{u}\right) = p'' \cdot x'' \tag{3.21}$$

$$= \alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \tag{3.22}$$

$$\geq \alpha e(p, \overline{u}) + (1 - \alpha)e\left(p', \overline{u}\right) \tag{3.23}$$

■

3.4.1 The Hicksian (or Compensated) Demand Function

Definition 3.17. The set of solutions to EMP, $h(p,u) \subseteq \mathbb{R}^L_+$, is known as the **Hicksian, or compensated, demand correspondence, or function** if single-valued.

Proposition 3.18 (3.E.3)**.** Suppose that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand $h(p,u)$ possess the following properties

(i) *Homogeneous of degree zero* in p;

(ii) *No excess utility*: $\forall x \in h(p,u),\ u(x) = u$;

(iii) *Convexity*: if $\succsim$ is convex, then $h(p,u)$ is convex;

(iv) *Uniqueness*: if $\succsim$ is strictly convex, then $h(p,u)$ is a singleton.

Definition 3.18. As prices vary, $h(p,u)$ gives precisely the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u. The amount of wealth compensated to ensure the original utility level attainable is referred to as the **Hicksian wealth compensation**.

$$\Delta w_{\text{ Hicks }} = e\left(p', u\right) - w \tag{3.24}$$

3.4.2 Hicksian Demand and the Compensated Law of Demand

Proposition 3.19 (3.E.4, the Compensated Law of Demand)**.** Suppose that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. And suppose $h(p,u)$ is single-valued everywhere, then for all p' and p'',

$$\left(p'' - p'\right) \cdot \left[h\left(p'', u\right) - h\left(p', u\right)\right] \leq 0 \tag{3.25}$$

That's, *Demand and price move in opposite directions for price changes that are accompanied by Hicksian wealth compensation.*

Proof. Suppose $u > u(0)$, and the price changes from p' tp p'', then

$$p'' \cdot h(p'', u) \leq p'' \cdot h(p', u) \tag{3.26}$$

$$p' \cdot h(p'', u) \geq p' \cdot h(p', u) \tag{3.27}$$

subtracting above inequalities gives the desired result. ∎

3.5 Duality: A Mathematical Introduction

Definition 3.19. A **half-space** is a set of the form $\left\{x \in \mathbb{R}^L : p \cdot x \geq c\right\}$ for some $p \in \mathbb{R}^L$, $p \neq 0$ is called the **normal vector** to the half-space. Its boundary $\left\{x \in \mathbb{R}^L : p \cdot x = c\right\}$ is called a **hyperplane**.

Definition 3.20 (3.F.1)**.** For any nonempty closed set $K \subset \mathbb{R}^L$, the **support function** of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\} \tag{3.28}$$

Proposition 3.20 (3.F.1, The Duality Theorem)**.** Let K be a nonempty closed set, and let μ_K be its support function. Then there is a unique $\mathbf{x} \in K$ such that $\mathbf{p} \cdot \mathbf{x} = \mu_k(p)$ if and only if μ_K is differentiable at $\mathbf{p}$. Moreover, in this case

$$\nabla \mu_K(\mathbf{p}) = \mathbf{x} \tag{3.29}$$

3.6 Relationships between Demand, Indirect Utility, and Expenditure Functions

Proposition 3.21 (3.G.1, Shephard's lemma)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p,u)$ satisfies

$$h(p,u) = \nabla_p e(p,u) \tag{3.30}$$

Proof using Envelope Theorem. The Lagrangian function for EMP is $\mathcal{L} := p \cdot x - \lambda(u - \overline{u})$. By the envelope theorem, at the optimum,

$$\partial_p e(p,u) = \partial_p \mathcal{L}(p, h(p,u), \overline{u}, \lambda) \tag{3.31}$$
$$\implies \nabla_p e(p,u) = h(p,u) \tag{3.32}$$

■

Remark 3.9 (Interpretation)**.** The above proposition says if we are at an optimum in the EMP, the changes in demand caused by price changes have no first-order effect on the consumer's expenditure.

Proposition 3.22 (3.G.2)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Suppose also that h is continuously differentiable at (p,u), then $D_p h(p,u) \in M_{L\times L}$ satisfies

(i) $D_p h(p,u) = D_p^2 e(p,u)$;

(ii) $D_p h(p,u)$ is a negative semidefinite matrix;

(iii) $D_p h(p,u)$ is symmetric;

(iv) $D_p h(p,u) \cdot p = 0$.

Proposition 3.23 (3.G.3, the Slutsky Equation)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T \tag{3.33}$$

Proof. Let $p \gg 0, w > 0, u > u(0)$, $h(p,u) = x(p, e(p,u))$, and $w = e(p,u)$. Differentiate both sides of this identity gives

$$D_p h(p,u) = D_p x(p, e(p,u)) + D_w x(p, e(p,u))\ D_p e(p,u) \tag{3.34}$$
$$= D_p x(p, e(p,u)) + D_w x(p, e(p,u))\ h(p,u) \text{ (Shephard's lemma)} \tag{3.35}$$
$$= D_p x(p, e(p,u)) + D_w x(p, e(p,u))\ x(p,w) \tag{3.36}$$

■

Definition 3.21. The **Slutsky substitution matrix** is defined as

$$S(p,w) := D_p h(p,u) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix} \tag{3.37}$$

where

$$\underbrace{s_{\ell k}(p,w)}_{\text{SE}} = \underbrace{\partial x_\ell(p,w)/\partial p_k}_{\text{TE}} + \underbrace{[\partial x_\ell(p,w)/\partial w]\, x_k(p,w)}_{\text{IE}} \tag{3.38}$$

3.6.1 Walrasian Demand and the Indirect Utility Function

Proposition 3.24 (3.G.4 Roy's Identity)**.** Suppose that u is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(p, w) \gg 0$. Then

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w) \tag{3.39}$$

Proof. Apply the envelope theorem to UMP,

$$\nabla_p v(p,w) = \partial_p[u(x^*) - \lambda^*(w - p\cdot x^*)] = \lambda^* x^* \tag{3.40}$$
$$\nabla_w v(p,w) = \partial_w[u(x^*) - \lambda^*(w - p\cdot x^*)] = -\lambda^* \tag{3.41}$$
$$\implies x^* = -\frac{1}{\nabla_w v(p,w)} Inabla_p v(p,w) \tag{3.42}$$

■

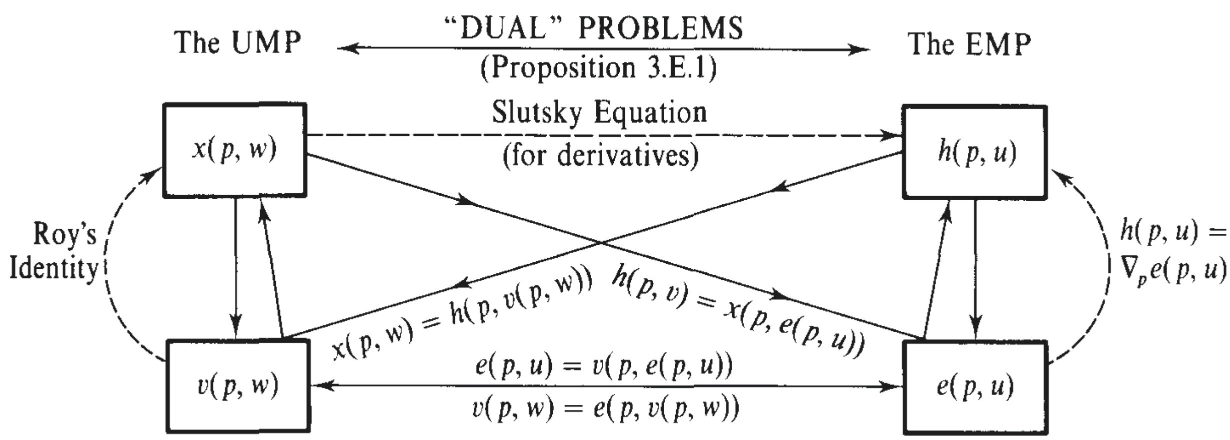

Figure 3.1: Fig. 3.G.3 in Mas-Colell: A Summary

3.7 Integrability

TODO: this subsection needs revision.

Remark 3.10 (the Central Question)**.** Given a continuously differentiable demand function $x(p, w)$ satisfying homogenous of degree zero, Walras' law, and possessing negative semidefinite substitution matrix $S(p, w)$, can we find preferences rationalizing $x(p, w)$.

Proposition 3.25. Suppose the demand $x(p, w)$ satisfying the weak axiom, homogeneity of degree zero and Walras' law, then it can be rationalized by preferences if and only if it has a symmetric substitution matrix $S(p, w)$.

3.7.1 Recovering Preferences from the Expenditure Function

Construction For each utility level u, we can construct a **at-least-as-good-as set** $V_u \subset \mathbb{R}^L$ such that $e(p, u)$ is the minimal expenditure required for the consumer to purchase a bundle in V_u at all price $p \gg 0$. That's V_u satisfies

$$\forall p \gg 0,\ e(p,u) = \min_{x\geq 0} p\cdot x \ s.t.\ x \in V_u \tag{3.43}$$

Proposition 3.26 (3.H.1)**.** Suppose that $e(p,u)$ is strictly increasing in u and is continuous, increasing, homogeneous of degree one, concave, and differentiable in p. Then, for every utility level u, $e(p,u)$ is the expenditure associated with at-least-as-good-as set

$$V_u = \left\{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p,u) \text{ for all } p \gg 0\right\} \tag{3.44}$$

That is, $e(p,u) = \min\{p \cdot x : x \in V_u\}\ \forall p \gg 0$.

Then, for each $u > u(0)$, we can construct such V_u and define $\succsim$ on $X = \mathbb{R}^L_+$ with V_u. That's, for a given $x \in \mathbb{R}^L_+$ such that $u(x) \leftarrow \overline{u}$, then for each $y \in \mathbb{R}^L_+$,

$$y \succsim x \iff y \in V_{\overline{u}} \tag{3.45}$$

3.7.2 Recovering the Expenditure Function from Demand

Construction Suppose $x(p,w)$ is known, and by Shephard's lemma, expenditure function $e(p,w)$ can be solved from the system of partial differential equations

$$\begin{aligned} &\tfrac{\partial e(p)}{\partial p_1} = x_1(p, e(p)) \\ &\vdots \\ &\tfrac{\partial e(p)}{\partial p_L} = x_L(p, e(p)) \end{aligned} \tag{3.46}$$

and initial conditions.

Proposition 3.27. The necessary and sufficient condition for the recovery of an underlying expenditure function is the symmetry and negative semi-definiteness of the Slutsky matrix.

3.8 Welfare Evaluation of Economic Changes

Context We assume that the consumer has a fixed wealth level $w > 0$ and that the price vector is initially p^0. We wish to evaluate the impact on the consumer's welfare of a change from p^0 to a new price vector p^1.

Definition 3.22. Money metric indirect utility functions measure welfare change expressed in dollar units. A money metric can be formed at an arbitrary price vector $\overline{p} \gg 0$, and the welfare change expressed in dollar units is

$$e\left(\overline{p}, v\left(p^1, w\right)\right) - e\left(\overline{p}, v\left(p^0, w\right)\right) \tag{3.47}$$

Definition 3.23. Let $u^0 = v(p^0, w)$, $u^1 = v(p^1, w)$, and noting that $e(p^0, u^0) = e(p^1, u^1) = w$. When $\overline{p} = p^0$, the money metric is called **equivalence variation**

$$EV\left(p^0, p^1, w\right) = e\left(p^0, u^1\right) - e\left(p^0, u^0\right) = e\left(p^0, u^1\right) - w \tag{3.48}$$

when $\overline{p} = p^1$, then it is referred to as

$$CV\left(p^0, p^1, w\right) = e\left(p^1, u^1\right) - e\left(p^1, u^0\right) = w - e\left(p^1, u^0\right) \tag{3.49}$$

Remark 3.11 (Interpretation of EV)**.** The equivalent variation can be thought of as the dollar amount that the consumer would be indifferent about accepting in lieu of the price change; that is, it is the change in her wealth that would be equivalent to the price change in terms of its welfare impact (so it is negative if the price change would make the consumer worse off). Therefore,

$$v(p^0, w + EV) = u^1 \tag{3.50}$$

Remark 3.12 (interpretation of CV)**.** The compensating variation, on the other hand, measures the net revenue of a planner who must compensate the consumer for the price change after it occurs, bringing her back to her original utility level u^0 (Hence, the compensating variation is negative if the planner would have to pay the consumer a positive level of compensation because the price change makes her worse off.)

$$v\left(p^1, w - CV\right) = u^0 \tag{3.51}$$

Proposition 3.28. Suppose p_1 changes from p_1^0 to p_1^1, and $p_\ell^0 = p_\ell^1 = \overline{p}_\ell \ \forall \ell \neq 1$,

$$
\begin{aligned}
EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - w && (3.52)\\
&= e\left(p^0, u^1\right) - e\left(p^1, u^1\right) && (3.53)\\
&= \int_{p_1^1}^{p_1^0} h_1\left(p_1, \overline{p}_{-1}, u^1\right) dp_1 && (3.54)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
CV\left(p^0, p^1, w\right) &= w - e(p^1, u^0) && (3.55)\\
&= e(p^0, u^0) - e(p^1, u^0) && (3.56)\\
&= \int_{p_1}^{p_1^\circ} h_1\left(p_1, \overline{p}_{-1}, u^0\right) dp_1 && (3.57)
\end{aligned}
$$

Remark 3.13. However, if there is no wealth effect for good 1 (e.g., if the underlying preferences are quasilinear with respect to some good $\ell \neq 1$), the CV and EV measures are the same.

Definition 3.24. The value of area lying between p_0^1 and p_1^1 to the left of the market (Walrasian) demand curve for good 1, $\int_{p_1^1}^{p_1^0} x_1(p_1, \overline{p}_{-1}, w)\ dp_1$, measures the change in **Marshallian consumer surplus**. Such welfare measure is a special case of money metric when the wealth effects are absent.

Definition 3.25. Suppose the government imposes commodity tax of t for each unit of good 1, let T denote the amount of tax collected, then the **deadweight loss of commodity taxation** is

$$
\begin{aligned}
(-T) - EV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^0, u^1\right) - T && (3.58)\\
&= \int_{p_1^0}^{p_1^0+t} h_1\left(p_1, \overline{p}_{-1}, u^1\right) dp_1 - th_1\left(p_1^0 + t, \overline{p}_{-1}, u^1\right) && (3.59)\\
&= \int_{p_1^0}^{p_1^0+t} \left[h_1\left(p_1, \overline{p}_{-1}, u^1\right) - h_1\left(p_1^0 + t, \overline{p}_{-1}, u^1\right)\right] dp_1 && (3.60)
\end{aligned}
$$

It measures the extra amount by which the consumer is made worse off by commodity taxation above what is necessary to raise the same revenue through a lump-sum tax.

3.8.1 Welfare Analysis with Partial Information

Proposition 3.29 (3.I.1)**.** Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better off under price-wealth situation (p^1, w) than (p^0, w).

Proof. By Walras' law, $p^0 \cdot x^0 = w$. Therefore, $x^0 \in B_{p_1,w}^{int}$. Because x^0 is in the interior of budget set, so $x^0 \neq x^1$ by local nonsatiation. As a result, $x^1 \succ^* x^0$ is revealed. ∎

Proposition 3.30 (3.I.2)**.** Suppose that the consumer has a differentiable expenditure function. then if $\left(p^1 - p^0\right) \cdot x^0 > 0$, there is a sufficiently small $\overline{\alpha} \in (0, 1)$ such that for all $\alpha < \overline{\alpha}$, we have $\left(p^1 - p^0\right) \cdot x^0 > 0$, and so the consumer is strictly better off under price-wealth situation (p^0, w) than under $\left((1-\alpha)p^0 + \alpha p^1, w\right)$

Definition 3.26. The **area variation** measure (AV) provides an estimate of welfare change when only Walrasian demand is observable

$$
AV\left(p^0, p^1, w\right) = \int_{p_1^1}^{p_1^0} x_1\left(p_1, \overline{p}_{-1}, w\right) dp_1 \qquad (3.61)
$$

Remark 3.14. When there are no wealth effects, $AV = EV = CV$.

Remark 3.15. Marshall argued that if a good is just one commodity among many, then because one extra unit of wealth will spread itself around, the wealth effect for the commodity is bound to be small; therefore, no significant errors will be made by evaluating the welfare effects of price changes for that good using the area measure.

Remark 3.16. So when Δp is small, the error involved using the area variation measure becomes small as a fraction fo the true welfare change, and therefore, can be used as an accurate approximation of the actual welfare change.

3.9 The Strong Axiom of Revealed Preference

Definition 3.27 (3.J.1)**.** The market demand function $x(p, w)$ satisfies the **strong axiom of revealed preference** (the SA) if for any sequence

$$\left(p^1, w^1\right), \ldots, \left(p^N, w^N\right) \tag{3.62}$$

with $x\left(p^{n+1}, w^{n+1}\right) \neq x\left(p^n, w^n\right)$ for all $n \leq N-1$, we have

$$\underbrace{p^n \cdot x\left(p^{n+1}, w^{n+1}\right) \leq w^n}_{x^n \succ^* x^{n+1} \text{ revealed}} \ \forall n \leq N-1 \implies p^N \cdot x\left(p^1, w^1\right) > w^N \tag{3.63}$$

Proposition 3.31 (3.J.1)**.** If the Walrasian demand function $x(p, w)$ satisfies the strong axiom of revealed preference then there is a rational preference relation $\succsim$ that rationalizes $x(p, w)$, that is, such that for all (p, w),

$$x(p, w) \succ y \ \forall y \in B_{p,w} \ s.t. \ y \neq x(p, w) \tag{3.64}$$

Proposition 3.32 (1.D.2 (WARP $\rightarrow$ Rational))**.** If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) The weak axiom is satisfied;

(ii) $\mathscr{B}$ includes all subsets of X up to three elements.

Then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$.

Remark 3.17. Proposition 3.J.1 tells us that a choice-based theory of demand founded on the strong axiom is essentially equivalent to the preference-based theory of demand presented in this chapter.

4 Aggregate Demand

4.1 Aggregate Demand and Aggregate Wealth

Context Suppose there are I consumers with rational preference relations $\succsim_i$ and corresponding Walrasian demand function $x_i(p, w_i)$. We want to write the **aggregate demand**

$$x\left(p, w_1, \ldots, w_I\right) = \sum_{i=1}^{I} x_i\left(p, w_i\right) \tag{4.1}$$

as a function of price level p and aggregate wealth $\sum_i w_i$, but independent to the wealth distribution. Specifically, for any two wealth distribution (w_i) and (w_i'),

$$\sum_i w_i = \sum_i w_i' \implies \sum_i x_i(p, w_i) = \sum_i x_i(p, w_i') \tag{4.2}$$

Proposition 4.1. The above property can be written using differential change in wealth (dw_i) such that $\sum_i dw_i = 0$. Assuming the differentiability of Walrasian demand functions, the invariant condition is equivalent to

$$\sum_i \frac{\partial x_{\ell i}\left(p, w_i\right)}{\partial w_i} dw_i = 0 \quad \text{ for every } \ell \tag{4.3}$$

which holds if and only if

$$\frac{\partial x_{\ell i}\left(p, w_i\right)}{\partial w_i}=\frac{\partial x_{\ell j}\left(p, w_j\right)}{\partial w_j} \tag{4.4}$$

for every commodity ℓ, pair of individuals i, j, and all wealth distribution (w_i).

Proof. ($\Longrightarrow$) for given ℓ, i, j, we can define $d\mathbf{w} := \mathbf{e}_i - \mathbf{e}_j$ and substitute it into the identity.
($\Longleftarrow$) if all partial derivatives with respect to w are equal, we can partial them out from the summation, $\sum_i \frac{\partial x_{\ell i}(p,w_i)}{\partial w_i} dw_i = \eta^* \sum_i dw_i$. And because $\sum_i dw_i = 0$, $\sum_i \frac{\partial x_{\ell i}(p,w_i)}{\partial w_i} dw_i$. ∎

Remark 4.1. The above proposition says aggregate demand is independent to the particular wealth distribution if the individual demand changes arising from any wealth redistribution across consumers will cancel out. Geometrically, the condition is equivalent to *all consumers' wealth expansion paths are parallel string lines.*

Proposition 4.2 (4.B.1)**.** A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that their preferences admit indirect utility functions of the **Gorman form** with the coefficients on w_i identical for every consumer, that's,

$$v_i\left(p, w_i\right)=a_i(p)+b(p) w_i \tag{4.5}$$

Definition 4.1. A family of functions $(w_1(p,w), \ldots, w_I(p,w))$ with $\sum_i w_i(p,w) = w$ for all (p,w) is called a **wealth distribution rule**.

Proposition 4.3. If the individual wealth levels are generated by a wealth distribution rule, we can always write the aggregate demand $x(p,w)$ as a function depends only on price and aggregate wealth.

5 Production

5.1 Production Sets

Definition 5.1. A **production vector/production plan** is a vector $y \in \mathbb{R}^L$ that describes the net outputs of L commodities from a production process. Where positive numbers denote outputs and negative numbers denote inputs. The set of all *technologically possible* production plans is defined as the **production set** $Y \subset \mathbb{R}^L$.

Remark 5.1. The set of feasible production plans is limited first and foremost by technological constraints. However, in any particular model, legal restrictions or prior contractual commitments may also contribute to the determination of the production set.

Definition 5.2. The **transformation function** $F : \mathbb{R}^L \to \mathbb{R}$ associated with production plan Y is defined such that

$$Y=\left\{y \in \mathbb{R}^L: F(y) \leq 0\right\} \tag{5.1}$$

and $F(y)$ if and only if $y \in \partial\, Y$. The set $\left\{y \in \mathbb{R}^L : F(y) = 0\right\}$ is called the **transformation frontier**.

Definition 5.3. The **marginal rate of transformation (MRT) of good ℓ for good k at $\overline{y}$** is defined as

$$MRT_{\ell k}(\overline{y}):=\frac{\partial F(\overline{y}) / \partial y_{\ell}}{\partial F(\overline{y}) / \partial y_k}\left(=\frac{d y_k}{d y_{\ell}}\right) \tag{5.2}$$

it is a measure of how much the (net) output of good k can increase if the firm decreases the (net) output of good ℓ by one marginal unit.

Definition 5.4. For single-output technologies with distinct inputs and output, the **production function** $f(z)$ is defined as the maximum amount q of output that can be produced using input amounts $(z_1, \cdots, z_{L-1}) \geq 0$. So the production set generated by f is

$$Y=\left\{\left(-z_1, \ldots,-z_{L-1}, q\right): q-f\left(z_1, \ldots, z_{L-1}\right) \leq 0 \wedge\left(z_1, \ldots, z_{L-1}\right) \geq 0\right\} \tag{5.3}$$

Definition 5.5. The **marginal rate of technical substitution (MRTS) of input ℓ for input k at** $\overline{z} \geq 0$ is defined to be

$$MRTS_{\ell k}(\overline{z}) := \frac{\partial f(\overline{z})/\partial z_\ell}{\partial f(\overline{z})/\partial z_k} \tag{5.4}$$

Assumption 5.1. Common properties of production sets assumed are

(i) $Y \neq \varnothing$;

(ii) Y is closed (*primarily for technical issues*);

(iii) *No free lunch*: $y \in Y \wedge y \geq 0 \implies y = 0$, more succinctly, $Y \cap \mathbb{R}_+^L \subset \{0\}$;

(iv) *Possibility of inaction*: $0 \in Y$;

(v) *Free disposal*: $Y - \mathbb{R}_+^L \subset Y$;

(vi) *Irreversibility*: $\forall y \in Y \setminus \neq 0, -y \notin Y$;

(vii) *Non-increasing returns to scale*: $y \in Y \wedge \alpha \in [0,1] \implies \alpha y \in Y$;

(viii) *Non-decreasing returns to scale*: $y \in Y \wedge \alpha \geq 1 \implies \alpha y \in Y$;

(ix) *Constant returns to scale*: $y \in Y \wedge \alpha \geq 0 \implies \alpha y \in Y$, therefore, Y is a cone;

(x) *Additivity/Free entry*: $Y + Y \subset Y$;

(xi) *Convexity*: Y is convex;

(xii) Y is a convex cone: $\forall y_1, y_2 \in Y, \alpha, \beta \geq 0,\ \alpha y_1 + \beta y_2 \in Y$.

Definition 5.6. A subset C of a vector space V is a **cone** (or sometimes called a linear cone) if $\forall x \in C, \alpha \geq 0,\ \alpha x \in C$.

Proposition 5.1 (Exercise 5.B.2)**.** For single-output technologies with distinct inputs and output, Y satisfies constant return to scale if and only if $f(\cdot)$ is homogenous of degree 1.

Proposition 5.2 (5.B.1)**.** The production set Y is additive and satisfies the non-increasing returns condition if and only if it is a convex cone.

Proposition 5.3 (5.B.2, adding the missing entrepreneurship input factor)**.** For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that

$$Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\} \tag{5.5}$$

Remark 5.2. In a competitive environment, the return to the entrepreneurial factor is the firm's profit.

5.2 Profit Maximization and Cost Minimization

Assumption 5.2 (price-taking assumption)**.** Assuming the prices quoted for the L goods, $p \gg 0$, are independent of the production plans of the firm.

Assumption 5.3. We assume that the firm's production set Y satisfies the *non-emptiness*, *closedness*, and *free disposal*.

5.2.1 The Profit Maximization Problem

Definition 5.7 (PMP)**.**

$$\max_{y\in\mathbb{R}^L} \ p\cdot y \ s.t. \ y\in Y \tag{5.6}$$

equivalently

$$\max_{y\in\mathbb{R}^L} \ p\cdot y \ s.t. \ F(y)\leq 0 \tag{5.7}$$

define the **profit function** and **supply correspondence** as

$$\pi(p) := \max\{p\cdot : y\in Y\} \tag{5.8}$$
$$y(p) := \text{argmax}\{p\cdot : y\in Y\} \tag{5.9}$$

If the transformation function $F(\cdot)$ is differentiable, then the necessary condition for $y^*\in y(p)$ is

$$p = \lambda\nabla F(y^*) \tag{5.10}$$

Definition 5.8 (PMP with single-output technology)**.** Suppose the input and output sets are disjoint. Let $f:\mathbb{R}_+^{L-1}\to\mathbb{R}_+$ denote the production function, let w and p denote the prices of inputs and the output commodity respectively. The PMP can be written as

$$\max_{z\geq 0} pf(z) - w\cdot z \tag{5.11}$$

the necessary conditions for $z^*\in$ argmax are

$$\begin{cases} p\ \nabla f(z^*)\leq w \\ [p\ \nabla f(z^*) - w]\odot z^* = 0 \end{cases} \tag{5.12}$$

Interpretation: the marginal product of every input ℓ actually used must equal its price in terms of output, w_ℓ/p.

Proposition 5.4 (5.C.1)**.** Suppose $\pi(p)$ and $y(p)$ are the profit function and supply correspondence with production set Y. Further, suppose Y is closed and satisfies the free disposal property. Then

(i) π is homogeneous of degree one;

(ii) π is convex;

(iii) If Y is convex, then $Y = \{y\in\mathbb{R}^L : p\cdot y\leq\pi(p)\ \forall p\gg 0$;

(iv) y is homogeneous of degree zero;

(v) If Y is convex, then $y(p)$ is a convex set for all p. If Y is strictly convex, then $y(p)$ is a singleton (if nonempty);

(vi) (**Hotelling's Lemma**) If $y(\overline{p})$ is a singleton, then pi is differentiable at p and $\nabla\pi(\overline{p}) = y(\overline{p})$;

(vii) (**Lay of Supply**) If y is a function differentiable at $\overline{p}$, then $Dy(\overline{p}) = D^2\pi(\overline{p})$ is a symmetric and positive semidefinite matrix with $Dy(\overline{p})\overline{p}) = 0$. Where Dy is called the **supply substitution matrix**.

Proposition 5.5 (Law of Supply)**.** Discrete case:

$$(p-p')\cdot(y-y')\geq 0 \tag{5.13}$$

Differential case, when $p=\overline{p}$ and $y=y(\overline{p})$:

$$dp\,\underbrace{Dy(p)dp}_{dy} = dp\cdot dy\geq 0 \tag{5.14}$$

5.2.2 Cost Minimization

Definition 5.9 (CMP)**.** Suppose the firm has a single-output technology, where input and output sets are disjoint. Let $(w, p) \gg 0$ denote the prices of inputs and the output. For each output level $q \geq 0$, the cost minimization problem is

$$\min_{z \geq 0} w \cdot z \ s.t. \ f(z) \geq q \tag{5.15}$$

The value function $c(w, q)$ is called the **cost function** and the solution set $z(w, q)$ is known as the **conditional factor demand correspondence**. The necessary conditions for $z^* \in z(w, q)$ are

$$\begin{cases} w \geq \lambda \nabla f(z^*) \\ [w - \lambda \nabla f(z^*)] \odot z^* = 0 \end{cases} \tag{5.16}$$

By the envelope theorem, $\lambda^* = \frac{\partial c(z^*, q)}{\partial q}$, which can be interpreted as the marginal value of relaxing the constraint $f(z^*) \geq q$. Therefore, λ is the *marginal cost of production*.

Proposition 5.6 (5.C.2)**.** Suppose that $c(w, q)$ and $z(w, q)$ are the cost function and conditional factor demand correspondence corresponding to a single-output technology Y with production function f. Suppose Y is closed and satisfies the free disposal property. Then

(i) c is homogeneous of degree one in w and nondecreasing in q;

(ii) c is a concave function of w;

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q) \ \forall w \gg 0\}$;

(iv) z is homogenous of degree zero in w;

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. If $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is a singleton;

(vi) (**Shepard's Lemma**) If $z(\overline{w}, q)$ is a singleton, then c is differentiable with respect to w at $\overline{w}$, and $\nabla_w c(\overline{w}, q) = z(\overline{w}, q)$;

(vii) If z is differentiable at $\overline{w}$ then $D_w z(\overline{w}, q) = D_w^2(\overline{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\overline{w}, q)\overline{w} = 0$;

(viii) If f is homogeneous of degree one (i.e. constant return to scale), then c and z are homogenous of degree one in q;

(ix) If f is concave, then c is a convex function of q.

Corollary 5.1. Alternative form of CMP, determining the profit-maximizing production level

$$\max_{q \geq 0} pq - c(w, q) \tag{5.17}$$

where the necessary conditions for $q^* \in$ argmax are

$$p \leq \frac{\partial c(w, q^*)}{\partial q} \text{ with equality if } q^* > 0 \tag{5.18}$$

At an interior optimum, price equals marginal cost. Note that if c is convex in q (i.e. $pq - c(w, q)$ is concave in q), the the necessary condition is also sufficient.

5.3 The Geometry of Cost and Supply in the Single-Output Case

Definition 5.10. The cost function excluding any prior input commitments is called the **long-run cost function**.

Remark 5.3. Suppose input z_k is fixed at $\overline{z}_k$, in the short run, then for each $q \geq 0$, the **short-run cost function** is defined as

$$C(q|\overline{z}_k) := \min_{z \geq 0}\{w \cdot z : f(z) \geq q, z_2 = \overline{z}_2\} \tag{5.19}$$

the corresponding long run cost function at q is defined as

$$C(q) := \min_{z \geq 0}\{w \cdot z : f(z) \geq q\} \tag{5.20}$$

given the relaxation of constraint, for all $q \geq 0$, $C(q) \leq C(q|\overline{z}_k)$. Geometrically, $C(\cdot)$ is the *lower envelope* of the family of short-run cost functions $\{C(q|\overline{z}_k) : \overline{z}_k \geq 0\}$.

5.4 Aggregation

Remark 5.4. The absence of budget constraint in the profit maximization implies that individual supply is not subject to wealth effects. As prices change, there are only substitution effects along the production frontier.

Definition 5.11. Suppose there are J production units in the economy with production set (Y_j). Assuming each Y_j is nonempty, closed, and satisfies the free disposal property. Then the **aggregate supply correspondence** is defined as

$$y(p) = \sum_{j=1}^{J} y_j(p) = \left\{ y \in \mathbb{R}^L : y = \sum_j y_j \text{ where } y_j \in y_j(p), \ \forall j = 1, \cdots, J \right\} \tag{5.21}$$

Proposition 5.7. If every y_j is single valued, since $Dy_j(p)$ is symmetric and positive semidefinite, then $D_y(p)$ is also symmetric and positive semidefinite.

Proposition 5.8 (Law of supply in the aggregate)**.** By adding over the *lay of individual supply*, we have

$$(p - p') \cdot [y(p) - y(p')] \geq 0 \tag{5.22}$$

Definition 5.12. The **aggregate production set** is defined as

$$Y := \sum_j Y_j = \left\{ y \in \mathbb{R}^L : y = \sum_j y_j \text{ where } y_j \in Y_j, \ \forall j = 1, \cdots, J \right\} \tag{5.23}$$

Proposition 5.9 (5.E.1)**.** For all $p \gg 0$, we have

(i) $\pi^*(p) = \sum_j \pi_j(p)$;

(ii) $y^*(p) = \sum_j y_j(p) = \left\{ \sum_j y_j : y_j \in y_j(p) \ \forall j \right\}$.

Interpretation: the aggregate profit obtained by each production unit maximizing profit separately taking prices as given is the same as that which would be obtained if they were to coordinate their actions in a joint profit maximizing decision. Therefore, to find the solution of the aggregate profit maximization problem for given prices p, it is enough to add the solutions of the corresponding individual problems. Moreover, given the minimizing aggregate output $q = \sum_j q_j$, the total cost $c(w, q)$ is exactly the value of **aggregate cost function** (the cost function corresponding to the aggregate production set Y). Thus, *the allocation of production of output level q among the firms is cost minimizing.* As a result, $q(p)$ satisfies necessary conditions $p \leq c'(q)$ and $(p - c'(q))q = 0$.

5.5 Efficient Production

Definition 5.13 (5.F.1)**.** A production vector $y \in Y$ is **efficient** if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. That is, $Y \cap (\{y\} + \mathbb{R}^L_+) = \{y\}$.

Proposition 5.10 (5.F.1, A simplified version of the first fundamental theorem of welfare economics)**.** If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient. *Interpretation: if a collection of firms each independently maximizes profits with respect to the same fixed price vector $p \gg 0$, then the aggregate production is socially efficient.*

$$\text{Competitive Equilibrium} \implies \text{Pareto Optimal} \tag{5.24}$$

Remark 5.5. The above proposition is valid even if the production set is non-convex.

Proposition 5.11 (5.F.2, A simplified version of the second fundamental theorem of welfare economics)**.** Suppose Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.

$$\text{Pareto Optimal} \overset{\text{reallocation}}{\Longrightarrow} \text{Competitive Equilibirum} \tag{5.25}$$