ECO325: Lecture Notes

Advanced Economic Theory: Macro

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Notes

Github `https://github.com/TianyuDu/Spikey_UofT_Notes`

Color notations

- Important equations for model setup.
- Important equations as results from model.
- Implications of model result.

Contents

1 Lecture 1. September 6. 2018

Definition 1.1. A **growth miracle** are episodes where thee growth in a country far exceeds the world average over a extended period of time. Result the country experiencing the miracle moves up the wold income distribution.

Definition 1.2. A **growth disaster** is an episodes where the growth in a country falls short at the world average for an extended period of time. Result the country moves down in the world income distribution.

Facts Data from the 20^{th} century suggest that

1. Real output grows at a (more or less) constant rate.
2. Stock of real capital grows at a (more or less) constant rate (but it grows faster than labor input).
3. Growth rates of real output and the stock of capital are about the same.
4. The rate of growth of output per capita varies greatly across countries.

1.1 Solow Growth Model (continuous time version)

Solow growth model decomposes the growth in output per capita into portions accounted for by increase in inputs and the portion contributed to increases in productivity.

In the baseline model we denote K as capital, L as labor and A as technology.

1.1.1 Production Function

Remark 1.1. Harrod-neutral technology here, refer to Uzawa's theorem.

Definition 1.3. The **effective labor input** is defined as $A(t)L(t)$

Definition 1.4. The production function is defined as

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

Typically Cobb-Douglas form is taken

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha},\ \alpha \in (0,1)$$

Property 1.1. Properties on production function.

1. CRS in K and AL: $Y(cK, cAL) = cY(K, AL),\ \forall c \geq 0$ implies
   - All gains from specialization have been exhausted.
   - Inputs other than K and AL are unimportant.

Definition 1.5. Define $c := \frac{1}{AL}$, the **intensive form** of production function is

$$y(t) = \frac{Y(t)}{A(t)L(t)} = f(k(t))$$

where $y := \frac{Y}{AL}$ denotes the **output per unit of effective labor** and $k := \frac{K}{AL}$ denote the capital stock per unit of effective labor.

2 Lecture 2 September 13. 2018

2.1 Solow Growth Model: Setup

Definition 2.1. Production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ maps input factors:

- $K(t) :=$ aggregate capital stock at time t.
- $L(t) :=$ aggregate labor supple at time t.
- $A(t) :=$ labor argument technology (effectiveness of labor) at time t.

to output values ($Y(t) :=$ aggregate output at time t.) The production function takes the form of

$$Y(t) = F(K(t), A(t)L(t))$$

Assumption 2.1 (Assumptions on Production Function)**.** The production function are assumed to satisfy the following assumptions:

- Constant Return to Scale:

$$cF(K(t), A(t)L(t)) = F(cK(t), cA(t)L(t)), \ \forall c > 0$$

Definition 2.2. The **intensive form of production function** is defined as the output per effective unit of labor. Let $f(t) := \frac{Y(t)}{A(t)L(t)}$ and $k(t) := \frac{K(t)}{A(t)L(t)}$ be the output and capital per unit of effective labor respectively. By the assumption of *CRS* on *aggregate* production function, take $c = \frac{1}{A(t)L(t)}$. The *intensive form* production function can be expressed as

$$y(t) = f(k(t)) \tag{1}$$

Assumption 2.2 (Assumptions on Intensive Form Production Function)**.** the function $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy *Inada Conditions.*

1. $f(0) = 0$: capital is necessary for production.
2. $f'(k) > 0, \ \forall k \in \mathbb{R}_+$: the marginal return of capital per effective unit of labor is positive.
3. $f''(k) < 0, \ \forall k \in \mathbb{R}_+$: capital per effective unit of labor is experiencing diminishing marginal return.

4. $\lim_{k\to 0} f'(k) = \infty$

5. $\lim_{k\to\infty} f''(k) = 0$

The role of above conditions is to ensure that the path of the economy does not diverge.

Example 2.1 (Cobb-Douglas Production Function)**.** Consider the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha},\ \alpha \in (0,1)$$

Check. Let $c \in \mathbb{R}_+$,

$$\begin{aligned} F(cK, cAL) &= (cK)^{\alpha}(cAL)^{1-\alpha} \\ &= c^{\alpha}c^{1-\alpha}K^{\alpha}AL^{1-\alpha} \\ &= cK^{\alpha}AL^{1-\alpha} = cF(K, AL) \end{aligned}$$

CRS on aggregate form is shown.
Notice that $f(k) = k^{\alpha}$
And

1. $f(0) = 0^{\alpha} = 0$

2. $f'(k) = \alpha k^{\alpha-1} > 0, \forall k \in \mathbb{R}_+$

3. $f''(k) = (\alpha - 1)\alpha k^{\alpha-2} < 0,\ \forall k \in \mathbb{R}_+$

4. $\lim_{k\to 0} \alpha \frac{1}{k^{1-\alpha}} = \infty$

5. $\lim_{k\to\infty} \alpha \frac{1}{k^{1-\alpha}} = 0$

Inada conditions on intensive form are shown.

∎

Assumption 2.3 (Assumptions on the Economy)**.** Assume the initial values of K, A, L are given and strictly positive. Labor and Knowledge are assumed to grow at an exogenously given constant rate, denoted as n, g respective.

$$\dot{L(t)} = nL(t),\ n > 0 \tag{2}$$

$$\dot{A(t)} = gA(t),\ g > 0 \tag{3}$$

Proposition 2.1. Notice the growth rate of variable X is given by

$$g_X := \frac{\dot{X(t)}}{X(t)} = \frac{\partial \ln X(t)}{\partial t}$$

Proof.

$$\frac{\partial \ln X(t)}{\partial t} = \frac{\partial \ln X(t)}{\partial X(t)} \frac{\partial X(t)}{\partial t}$$
$$= \frac{1}{X(t)} \dot{X(t)} = \frac{\dot{X(t)}}{X(t)} = g_X$$

■

Proposition 2.2. The functional form of technology and labor at time t can be found by solving ODEs

$$L(t) = e^{nt} L(0) \tag{4}$$
$$A(t) = e^{gt} A(0) \tag{5}$$

Assume there is *no government* and the Solow economy is a *closed economy*. The output is divided between *consumption* and *investment* as

$$Y(t) = C(t) + I(t)$$

And given δ as depreciation rate of capital, in discrete time (let $\Delta t = 1$)we have

$$K(t+1) = (1-\delta)K(t) + I(t)$$
$$\iff I(t) = K(t+1) - K(t) + \delta K(t)$$
$$\text{As } \Delta \to 0 \text{ (convert to continuous time)}$$
$$I(t) = \dot{K}(t) + \delta K(t)$$

Assumption 2.4. Assume investment equals saving and a constant friction $s \in [0,1]$ of output is saved at each epoch. The marginal propensity to save, s is given exogenously.

Therefore,

$$I(t) = sY(t) \implies \dot{K}(t) + \delta K(t) = sY(t)$$
$$\implies \dot{K}(t) = sY(t) - \delta(K(t))$$

2.2 Dynamics of $k(t)$

For simplicity, assuming $n, g, \delta > 0$ and the dynamics of capital per effective unit of labor follows:

$$\begin{aligned}\dot{k}(t) := \frac{\partial k(t)}{\partial t} &= \frac{\partial}{\partial t}\frac{K(t)}{A(t)L(t)} \\ &= \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2} \\ &= \frac{\dot{K}}{AL} - \frac{K\dot{A}L}{(AL)^2} - \frac{KA\dot{L}}{(AL)^2} \\ &= \frac{sY - \delta K}{AL} - \frac{\dot{A}}{A}\frac{K}{AL} - \frac{\dot{L}}{L}\frac{K}{AL} \\ &= sy(t) - (n + g + \delta)k(t)\end{aligned}$$

Where $sy(t)$ is the **actual investment** per unit effective unit of labor and $(n + g + \delta)k(t)$ is the **break-even investment** per effective unit of labor.

Remark 2.1. The convergence speed is inversely correlated with the value of $||k(t) - k^*||$, where k^* denotes the steady state level of capital stock per effective unit of labor.

Remark 2.2. With convex production function ($f''(k) > 0$), then $k(t) < k^* \implies \dot{k} < 0$ and $k(t) > k^* \implies \dot{k} > 0$. The steady state value k^* is steady but not stable (with $k(t) \neq k^*$, k does not automatically converge to k^*).

3 Lecture 3 September 20. 2018

3.1 Dynamic Transitions

Remark 3.1. For the dynamic transition function of capital per unit of effective labor:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \tag{1}$$

And dynamic transition and phase diagram can be expressed as

Definition 3.1. Steady level of capital per unit of effective labor(k^*) is defined as the level of capital per unit of effective labor that equates break-even investment per unit of effective labor and actual investment per unit of effective labor. [1]

$$k^* := \{k \in \mathbb{R}_+ : sf(k) = (n + g + \delta)k\}$$

Remark 3.2. The values of other endogenous variables at steady state are also derived from k^*.

[1]The definition can also be expressed as $k^* := \{k \in \mathbb{R}_+ : \hat{k} = 0\}$

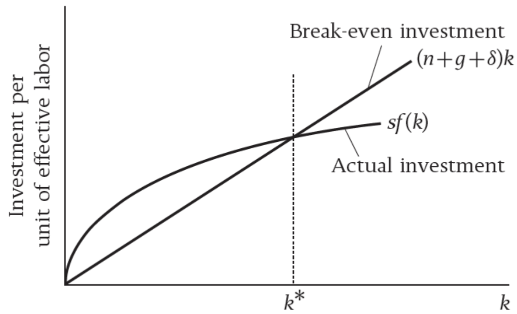

Figure 1: Dynamic Transition of Capital Per Unit of Effective Labor

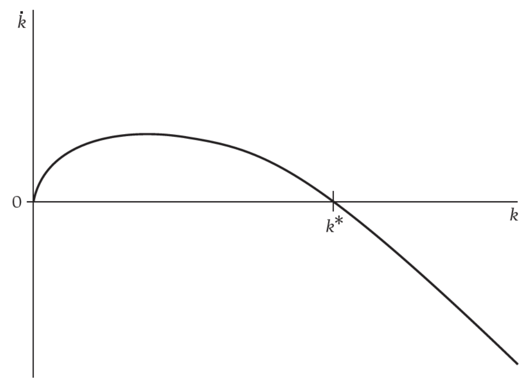

Figure 2: Phase Diagram of Capital Per Unit of Effective Labor

Example 3.1.

$$y^* = f(k^*) \tag{2}$$

$$i^* = sf(k^*) = (n + g + \delta)k^* \tag{3}$$

$$c^* = y^* - i^* = f(k^*) - (n + g + \delta)k^* = (1 - s)f(k^*) \tag{4}$$

For the growth rate of each endogenous variable (per unit of effective labor).

$$\frac{\partial k(t)}{\partial t}|_{k=k^*} = 0 \tag{5}$$

$$\frac{\partial c(t)}{\partial t}|_{k=k^*} = 0 \tag{6}$$

$$\frac{\partial y(t)}{\partial t}|_{k=k^*} = 0 \tag{7}$$

above relations are equivalent to

$$\text{On steady state} \begin{cases} \dot{c}(t) = 0 \\ \dot{k}(t) = 0 \\ \dot{y}(t) = 0 \end{cases} \tag{8}$$

Proof. By definition of consumption per unit of effective labor,
$c(\cdot) = (1-s)f(k(t))$
$\implies \dot{c}(t) := \frac{\partial c(\cdot)}{\partial t} = (1-s)f'(k(t))\dot{k(t)}$ by chain rule
Since $\dot{k}|_{k=k^*} = 0$ and $(1-s)f'(k(t)) < \infty$
Thus $\dot{c}(t)|_{k=k^*} = 0$ ■

3.2 Balanced Growth Path

Definition 3.2. A **balanced growth path** is a situation where each variable in the model are all growing at a constant rate. [2] [3]

3.2.1 Growth Rates on Balanced Growth Path

Population and Technology By definition of population and technological progress,

$$g_A := \frac{\dot{A}}{A} = g \tag{9}$$

$$g_L := \frac{\dot{L}}{L} = n \tag{10}$$

Capital per person Since $\frac{K(t)}{L(t)} = \frac{k(t)A(t)L(t)}{L(t)} = k(t)A(t)$, and the growth rate of $x(t)$ can be found as $\frac{\partial \ln x(t)}{\partial t}$. Then

Solution.

$$\begin{aligned} \frac{\partial \ln \frac{K(t)}{L(t)}}{\partial t} &= \frac{\partial k(t)A(t)}{\partial t} \\ &= \frac{\partial \ln k(t)}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\ &= \frac{\dot{k}(t)}{k(t)} + g \end{aligned}$$

And at the steady state, by definition, $\dot{k}(t)|_{k-k^*} = 0$, therefore

$$g^*_{\frac{K}{L}} = g \tag{11}$$

■

[2]Variables are not required to grow at the same rate by this definition.
[3]Variables remaining fixed are also considered as growing at a constant rate ($g = 0$).

Output and Consumption per person Similarly,

Solution.

$$\begin{aligned}
\frac{Y(t)}{L(t)} &= y(t)A(t) \\
g_{\frac{Y}{L}} &= \frac{\partial \ln y(t) + \ln A(t)}{\partial t} \\
&= \frac{\partial \ln y}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\
&= g + \frac{\dot{y}}{y}
\end{aligned}$$

and for consumption per person,

$$\begin{aligned}
\frac{C(t)}{L(t)} &= c(t)A(t) \\
g_{\frac{C}{L}} &= g + \frac{\dot{c}}{c}
\end{aligned}$$

Thus, on the balanced growth path, [4]

$$g^*_{\frac{Y}{L}} = g \tag{12}$$

$$g^*_{\frac{C}{L}} = g \tag{13}$$

■

Proposition 3.1. Along the balanced growth path, consumption and output per person also growth at rate g.

Proposition 3.2. Along the balanced growth path, aggregate variables, $Y(t), I(t), C(t)$ are all growing at a rate $n + g$.

$$g^*_Y = g^*_C = g^*_I = n + g \tag{14}$$

Proof.

$$\begin{aligned}
g_K &= \frac{\partial \ln K(t)}{\partial t} \\
&= \frac{\partial \ln A(t)L(t)k(t)}{\partial t} \\
&= \frac{\partial \ln A(t)}{\partial t} + \frac{\partial \ln L(t)}{\partial t} + \frac{\partial \ln k(t)}{\partial t} \\
&= g + n + \frac{\dot{k}}{k}
\end{aligned}$$

[4] g^*_X denotes the growth rate of variable X on the balanced growth path.

and at balanced growth path, $\frac{\dot{k}}{k}|_{k=k^*} = 0$, therefore

$$g_K^* = n + g \tag{15}$$

and proof for $C(t)$ and $I(t)$ follows the same path. ■

Definition 3.3. The **golden rule level of capital per unit of effective labor** (k_G) is the steady state level of capital per unit of effective labor that maximizes steady state consumption per unit of effective labor.

$$k_G = argmax_{k^* \in \mathbf{k}^*(\Theta)} \{c^* = f(k^*) - (n + g + \delta)k^*\}$$

First Order Necessary Condition. Solve

$$\begin{aligned} &\frac{\partial c^*(k^*)}{\partial k^*} = 0 \\ \implies &\frac{\partial f(k^*) - (n + g + \delta)k^*}{\partial k^*} = 0 \\ \implies &f'(k^*) = (n + g + \delta) \end{aligned}$$

Thus, golden rule level of capital stock per unit of effective labor k_G can be expressed as [5]

$$k_G = \{k \in \mathbb{R}_+ : f'(k) = (n + g + \delta)\} \tag{16}$$

■

3.3 Experiment

3.3.1 Impact of Change in the Saving Rate ($s_1 > s_0$)

Suppose at time t_0, the saving rate parameter increases discretely: $s_0 \to s_1$.

Remark 3.3. The relation of c_0^* and c_1^* depends on the relative position of s_1 and the golden rule level of saving rate s_G.

3.3.2 Derive the Effect of Change in s Mathematically

Goal Find $\frac{\partial k^*}{\partial s}$. And notice that $k^*(n + g + \delta) = sf(k^*)$ for any steady state capital level k^*. And the steady state level of capital per unit of effective labor can be written as a function of parameters, as $k^*(n, g, \delta, s)$.

[5] Notice that the zero solution, $k^* = 0$ is a trivial steady state and we ignore this case during this course.

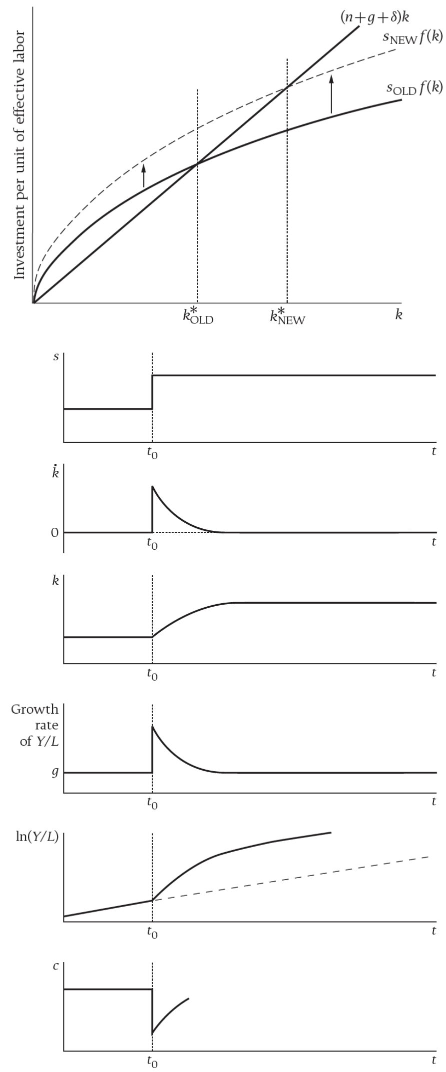

Figure 3: Effect of an Increase in Saving Rate.

Impact on k^*

Solution. At any steady state level, k^* satisfies

$$sf(k^*(n,g,\delta,s)) = (n+g+\delta)k^*(n,g,\delta,s)$$

Differentiate both sides with respect to s,
We have

$$sf'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n+g+\delta)\frac{\partial k^*}{\partial s}$$

Rearrange and get

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)}$$

Notice that the slope of break-even investment is greater than the slope of the actual investment at the steady state, therefore

$$\frac{\partial k^*}{\partial s} > 0$$

■

Impact on y^*

Solution. Using chain rule we have

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s}$$
$$= \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial s} > 0, \ \forall k^* \in \mathbf{k}^*(\Theta)$$

■

To get a sense on how much y^* changes wit respect to change in s, we could look at the elasticity.

$$\eta = \frac{\partial y^*}{\partial s}\frac{s}{y^*} = f'(k^*)\frac{\partial k^*}{\partial s}\frac{s}{f(k^*)} = \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)}$$

Recall that $(n+g+\delta) = \frac{sf(k^*)}{k^*}$ and rearrange the elasticity

$$\begin{aligned}
\eta &= \frac{\partial y^*}{\partial s}\frac{s}{y^*} \\
&= \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)} \\
&= \frac{sf'(k^*)}{\frac{sf(k^*)}{k^*} - sf'(k^*)} \\
&= \frac{f'(k^*)}{\frac{f(k^*)}{k^*} - f'(k^*)} \\
&= \frac{f'(k^*)\frac{k^*}{f(k^*)}}{1 - f'(k^*)\frac{k^*}{f(k^*)}} \\
&= \frac{\alpha_K}{1-\alpha_K}
\end{aligned}$$

Remark 3.4. α_K denotes the elasticity of output per unit of effective unit labor with respect to capital stock per unit of effective labor, along the balanced growth path. And

$$\alpha_K \approx \frac{1}{3}$$

Remark 3.5. If the production function is in the Cobb-Douglas form, then $\alpha_K = \alpha$.

Example 3.2. If $\alpha_K \approx \frac{1}{3}$ then $\frac{\partial y^*}{\partial s} sy^* \approx \frac{1}{2}$.

Impact on c^* Notice that on the balanced growth path $c^* = y^* - i^*$.

$$c^* = f(k^*) - (n+g+\delta)k^* \tag{17}$$

and differentiate with respect to s

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}$$

And notice that the sign of $\frac{\partial c^*}{\partial s}$ depends on the relative slope of production function and break-even investment. By the first order condition of golden rule level of capital per unit of effective labor, $(n+g+\delta) = f'(k_G)$

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - f'(k_G)]\frac{\partial k^*}{\partial s}$$

And

$$\begin{cases}
k^* = k_G \implies f'(k^*) = f'(k_G) \implies \frac{\partial c^*}{\partial s} = 0 \\
k^* < k_G \implies f'(k^*) > f'(k_G) \implies \frac{\partial c^*}{\partial s} > 0 \\
k^* > k_G \implies f'(k^*) < f'(k_G) \implies \frac{\partial c^*}{\partial s} < 0
\end{cases}$$

4 Lecture 4 September 27. 2018

4.1 Speed of Convergence

Linearize the model