MAT237: Multivariable Calculus

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2.8 Optimization

Theorem 2.8.1. Let $S \subset \mathbb{R}^n$ be an open set and $f, g : S \to \mathbb{R}$ be C^1 functions. If $\mathbf{x}$ is a *local extremal* satisfying $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{x}) \neq 0$, then

$$\exists \lambda \in \mathbb{R} \ s.t. \ \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \tag{2.8.1}$$

Lemma 2.8.1. $\nabla g(\mathbf{x})$ is orthogonal to the constraint set $g^{-1}(0)$.

Proposition 2.8.1. Equations (2.8.1) $\implies$ $\nabla f(\mathbf{x}) \perp g^{-1}(0)$ at $\mathbf{x}$.

Theorem 2.8.2. Let $S \subseteq \mathbb{R}^n$ be an open set, and $f, \{g_i\}_{i=1}^k : S \to \mathbb{R}$ be C^1 functions. Define $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^k \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_k(\mathbf{x}))$.
If $\mathbf{x} \in S$ is a *local extremal* of f such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, and $\{\nabla g_i(\mathbf{x})\}$ are linearly independent (i.e. $rank(D\mathbf{g}(\mathbf{x})) = k$), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \ s.t. \ \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \tag{2.8.2}$$

Remark 2.8.1. Procedure of optimization on *open sets*:

(i) Find all critical points.

(ii) Find optimizers among critical points.

Remark 2.8.2. Procedure of optimization with *inequality constraints*:

(i) Find critical points without the constraints.

(ii) Find critical points on the constraints.

(iii) Find optimizers among candidates.

3 The Implicit and Inverse Function Theorems

3.1 The Implicit Function Theorem I

Theorem 3.1.1 (Implicit Function Theorem)**.** Let $S \subseteq \mathbb{R}^{n+k}$ be an open set, and function $F : S \to \mathbb{R}^k$ be a C^1 function. Suppose there exists point $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$ such that

$$F(\mathbf{a}, \mathbf{b}) = \mathbf{0} \tag{3.1.1}$$

If

$$det(D_{\mathbf{y}}(F(\mathbf{a}, \mathbf{b}))) \neq 0 \tag{3.1.2}$$

then there exists $r_0, r_1 > 0$ and a C^1 function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}),\ \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \tag{3.1.3}$$

and define $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$, the derivative of $\mathbf{f}$ can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}} F(\mathbf{x}, \mathbf{y})]^{-1} D_{\mathbf{x}} F(\mathbf{x}, \mathbf{y}) \tag{3.1.4}$$

Remark 3.1.1. Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \tag{3.1.5}$$

near $(\mathbf{a}, \mathbf{b})$.

(i) Verify $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

(ii) Assert
$$det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \tag{3.1.6}$$

(iii) Approximate solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ using
$$\begin{aligned} \mathbf{f}(\mathbf{x} + \mathbf{h}) &\approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} & (3.1.7)\\ &= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b}) & (3.1.8) \end{aligned}$$

3.2 Geometric content of the Implicit Function Theorem

Definition 3.2.1. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in S$. S is **singular** at $\mathbf{a}$ if

$$\forall r > 0\ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \tag{3.2.1}$$

S is **regular** at $\mathbf{a}$ is its not singular there.

Theorem 3.2.1 (k dimensional manifold as level set)**.** Let $U \subseteq \mathbb{R}^n$ and let $\mathbf{F} : U \to \mathbb{R}^{n-k}$ be a C^1 function.

$$S \equiv \mathbf{F}^{-1}(\mathbf{0}) \tag{3.2.2}$$

Let $\mathbf{a} \in U$, if

$$rank(D\mathbf{F}(\mathbf{a})) = n - k \tag{3.2.3}$$

then $\exists r > 0$ such that the *level set of $\boldsymbol{F}$ near $\boldsymbol{a}$*

$$\mathcal{B}(r, \mathbf{a}) \cap S \tag{3.2.4}$$

can be represented as a C^1 graph.

Theorem 3.2.2 (k dimensional manifold as parameterization)**.** Let $T \subseteq \mathbb{R}^k$ and let $\mathbf{f} : U \to \mathbb{R}^n$ be a C^1 function.

$$S \equiv \mathbf{f}(T) \tag{3.2.5}$$

Let $\mathbf{t} \in T$, if

$$rank(\mathbf{f}(\mathbf{t})) = k \tag{3.2.6}$$

then $\exists r > 0$ such that the *parameterization of* $\boldsymbol{f}$ *near* $\boldsymbol{t}$

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \tag{3.2.7}$$

can be represented as a C^1 graph.

3.3 Transformations, and the Inverse Function Theorem

Example 3.3.1 (Polar coordinate in $\mathbb{R}^2$)**.** Let

$$U \equiv \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \tag{3.3.1}$$

$$V \equiv \mathbb{R}^2 \backslash \{(x, 0) : x \leq 0\} \tag{3.3.2}$$

Define $\mathbf{f} : U \to V$ as

$$\mathbf{f}(r, \theta) \equiv \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} \tag{3.3.3}$$

Example 3.3.2 (Spherical coordinate in $\mathbb{R}^3$)**.** Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix} \tag{3.3.4}$$

Example 3.3.3 (Cylindrical coordinate in $\mathbb{R}^3$)**.** Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{pmatrix} \tag{3.3.5}$$

Theorem 3.3.1 (Inverse Function Theorem)**.** Let U and V be open subsets in $\mathbb{R}^n$, and $\mathbf{f} : U \to V$. Let $\mathbf{a} \in U$ and define $\mathbf{b} \equiv \mathbf{f}(\mathbf{a}) \in V$. If

$$det(D\mathbf{f}(\mathbf{a})) \neq 0 \tag{3.3.6}$$

then there exists $M \subseteq U$ and $N \subseteq V$ such that

(i) $\mathbf{a} \in M$ and $\mathbf{b} \in N$,

(ii) $\mathbf{f}$ is bijective between M and N,

(iii) $\mathbf{f}^{-1} : N \to M$ is C^1,

and for all $\mathbf{x} \in M$ such $\mathbf{y} \equiv \mathbf{f}(\mathbf{x}) \in N$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \tag{3.3.7}$$

4 Integration

4.1 Basics

Theorem 4.1.1 (Properties of infimum and supremum)**.** Let $A \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, and $f, g : A \to \mathbb{R}$ are bounded functions. Let m and M denote the infimum and supremum respectively, then

(i) $m_A f + m_A g \leq m_A(f+g) \leq M_A(f+g) \leq M_A f + M_A g$

(ii) If $A' \subseteq A$, then $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$

(iii) If $f(\mathbf{x}) \leq g(\mathbf{x})\ \forall \mathbf{x} \in A$, then $m_A f \leq m_A g$ and $M_A f \leq M_A g$

(iv) $|M_A f| \leq M_A |f|$

(v) $M_A|f| - m_A|f| \leq M_A f - m_A f$

(vi) $\forall c \in \mathbb{R}$, $M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$

(vii) $M_A f - m_A f = sup\{f(x) - f(y) : x, y \in A\}$

4.2 Integration on Higher Dimensions

Definition 4.2.1. A **rectangle** $\mathcal{R} \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{R} \equiv \prod_{i=1}^{n} [a_i, b_i] \tag{4.2.1}$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.

Definition 4.2.2. A **partition** P of rectangle $\mathcal{R} = \prod_{i=1}^{n} [a_i, b_i]$ is a list of n finite and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \tag{4.2.2}$$

where $L_i = \{e_j\}_{j=0}^{T_i}$ such that

$$a_i = e_0 < e_1 < \cdots < e_{T_i} = b_i \tag{4.2.3}$$

and such partition induces a set of rectangles(boxes) $\mathcal{B}(P) \equiv \{B_j\}_{j=1}^{J} \subseteq \mathcal{R}$.

Definition 4.2.3. Let P and P' be two partitions of $\mathcal{R}$. Then P' is a **refinement** of P if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee B_j'^{int} \cap B_j^{int} = \emptyset \tag{4.2.4}$$

Definition 4.2.4. Define the **volume** of rectangle $\mathcal{R} = \prod_{i=1}^{n} [a_i, b_i]$ as

$$V^n(\mathcal{R}) \equiv \prod_{i=1}^{n} (b_i - a_i) \tag{4.2.5}$$

Definition 4.2.5. The **lower Riemann sum** of f with partition P on $\mathcal{R}$ is defined as

$$L_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \tag{4.2.6}$$

and the **upper Riemann sum** is defined as

$$U_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \tag{4.2.7}$$

Definition 4.2.6. The **upper integral** and **lower integral** of f on $\mathcal{R}$ are defined as

$$\bar{I}_{\mathcal{R}} f \equiv \inf_P U_P f \tag{4.2.8}$$

$$\underline{I}_{\mathcal{R}} f \equiv \sup_P L_P f \tag{4.2.9}$$

Definition 4.2.7. A bounded real-valued function f defined on $\mathcal{R}$ is **integrable** if

$$\underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \tag{4.2.10}$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \ dV^n \equiv \underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \tag{4.2.11}$$

Lemma 4.2.1. Let f be a bounded real-valued function defined on $\mathcal{R}$, f is integrable if and only if $\forall \epsilon > 0$, there exists a partition P of $\mathcal{R}$ such that

$$U_P f - L_P f < \epsilon \tag{4.2.12}$$

Theorem 4.2.1. Let f and g be two integrable functions on $\mathcal{R} \subseteq \mathbb{R}^n$, let $c \in \mathbb{R}$,

(i) $f + g : \mathcal{R} \to \mathbb{R}$ is integrable and $\int_{\mathcal{R}} (f+g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$

(ii) $c \cdot f$ is integrable and $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$

(iii) $f(\mathbf{x}) \geq g(\mathbf{x}) \ \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$

(iv) $|f|$ is integrable and $|\int_R f| \leq \int_R |f|$

Definition 4.2.8. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle $\mathcal{R}$ covers S, the **indicator function** of S is $\chi_S : \mathcal{R} \to \{0, 1\}$, defined as

$$\chi_S(\mathbf{x}) \equiv \mathbb{I}(\mathbf{x} \in S) \tag{4.2.13}$$

Definition 4.2.9. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle $\mathcal{R}$ covers S. Let $f : \mathcal{R} \to \mathbb{R}$ be a bounded function, then f is **integrable on** S if $\chi_S f$ is integrable on $\mathcal{R}$. And

$$\int \cdots \int_S f \ dV^n \equiv \int \cdots \int_{\mathcal{R}} \chi_S f \ dV^n \tag{4.2.14}$$

Definition 4.2.10. Let $Z \subseteq \mathbb{R}^n$, Z has **zero content** if for all $\epsilon > 0$, there exists a finite set of rectangles $\{R_\ell\}_{\ell=1}^L$ covers Z and

$$\sum_{\ell=1}^{L} V^n(R_\ell) < \epsilon \tag{4.2.15}$$

Proposition 4.2.1. Let $Z \subseteq \mathbb{R}^n$ has zero content, then

(i) For any $Z' \subseteq Z$, Z' has zero content.

(ii) Finite union of content zero sets has zero content.

(iii) Let $f : [a, b] \to \mathbb{R}$ be an integrable function, it's graph $\{(x, f(x)) : x \in [a, b]\}$ has zero content.

(iv) Let $\mathbf{f} : [a, b] \to \mathbb{R}^2$ be a C^1 function, the parameterization $\mathbf{f}([a, b])$ has zero content.

Theorem 4.2.2. Let $\mathcal{R}$ be a rectangle in $\mathbb{R}^n$ and f is integrable on $\mathcal{R}$ if

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \tag{4.2.16}$$

has zero content.

Proposition 4.2.2 (Folland 4.22)**.** Suppose $Z \subseteq \mathbb{R}^n$ has zero content. If $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f \ dV^n = 0$.

4.3 Iterated Integrals

Theorem 4.3.1 (Fubini's Theorem)**.** Let $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \mathcal{R} \to \mathbb{R}$ is bounded. Assuming that

(i) f is integrable on $\mathcal{R}$.

(ii) for each $y \in [c, d]$, the function $f_y(x) \equiv f(x, y)$ is integrable on $[a, b]$.

(iii) Define $g(y) \equiv \int_a^b f(x, y)dy$ is integrable on $[c, d]$.

Then

$$\iint_{\mathcal{R}} f \; dA = \int_c^d \Big(\int_a^b f(x, y) \; dx \Big) dy \tag{4.3.1}$$

Proposition 4.3.1. Let $S \subseteq \mathbb{R}^n$ be an unbounded set, and $f : S \to \mathbb{R}$. Then improper integral $\int \cdots \int_S f \; d^n\mathbf{x}$ is absolutely convergent on $\mathbb{R}^n$ <u>if and only if</u> $\int \cdots \int_{\mathbb{R}^n} \chi_S f \; d^n\mathbf{x}$ is absolutely convergent.