MAT 344 Lecture Notes

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Contents

1 Strings, Sets, and Binomial Coefficients

1.1 Strings and Sets

Notation 1.1. Let $n \in \mathbb{Z}_{++}$, and we use $[n]$ to denote the n-element set $\{1, 2, \dots, n\}$.

Definition 1.1. Let X be a set, then an X**-string of length** (or a **word/array**) n is a function $s : [n] \to X$, and X is called the **alphabet** of the string, and each $x \in X$ is called a **character** or letter.

Remark 1.1. An X-string defined by $s : [n] \to X$ with length n can be equivalently defined as a **sequence** consisting elements in X.

$$s(1)s(2)\dots s(n) \tag{1.1}$$

Definition 1.2. In the case $X = \{0, 1\}$, strings generated from X are called **binary strings**. When $X = \{0, 1, 2\}$, strings are called **ternary strings**.

Definition 1.3. Let X be a *finite* set and let $n \in \mathbb{Z}_{++}$. An X-string $s = x_1 x_2 \dots x_n$ is a **permutation** of size m if $x_i \neq x_j \ \forall x_i, x_j \in s$.

Proposition 1.1. If X is an m-element set and $m \geq n \in \mathbb{Z}_{++}$, then the number of X-strings of length n that are permutations is

$$P(m, n) \equiv \frac{m!}{(m-n)!} \tag{1.2}$$

Definition 1.4. Let X be a *finite* set and let $0 \leq k \leq |X|$. Then $S \subseteq X$ with $|S| = k$ is a **combination** of size k.

Proposition 1.2. Let $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$, then the number of combinations is

$$\binom{n}{k} \equiv \frac{P(n,k)}{n!} = \frac{n!}{k!(n-k)!} \tag{1.3}$$

Proposition 1.3. For all integers n and k with $0 \leq k \leq n$

$$\binom{n}{k} = \binom{n}{n-k} \tag{1.4}$$

Example 1.1. Binomial coefficients can be used to find the number of integer solutions of

$$\sum_{i=1}^{k} x_i \leq N \tag{1.5}$$

given appropriate integers $k, N \in \mathbb{Z}$.

(i) $x_i > 0 \ \forall i \in [k]$ and equality holds, then $C(N-1, k-1)$.

(ii) $x_i \geq 0 \ \forall i \in [k]$ and equality holds, then $C(N+k-1, k-1)$.[1]

(iii) $x_i > 0 \ \forall i \neq j, x_j = Z$ and equality holds, then $C(N-Z+k-2, k-2)$.

(iv) $x_i > 0 \ \forall i \in [k]$ and strict inequality holds, then $C(N-1, k)$.[2]

(v) $x_i \geq 0 \ \forall i \in [k]$ and strict inequality holds, then $C(N+k-1, k)$.

(vi) $x_i \geq 0 \ \forall i \in [k]$ and *weak* inequality holds, $C(N+k, k)$ [3].

$$\binom{N+k-1}{k-1} + \binom{N+k-1}{k} = \binom{N+k}{k} \tag{1.6}$$

[1] Simulate choosing $x_i + 1$ instead of x_i.
[2] Image there is a placeholder $x_{k+1} > 0$.
[3] This can be calculated by adding case (ii) and case (v) together, and apply Pascal's identity

Definition 1.5. Define a **plane** as $\mathbb{Z}^2$, then a **lattice path** in the plane is a *sequence* of elements in $\mathbb{Z}^2$

$$((x_i, y_i))_{i=1}^t \tag{1.7}$$

such that for every $i \in \{1, \dots, t-1\}$, either

(i) (*Horizontal move*) $x_{i+1} = x_i + 1 \wedge y_{i+1} = y_i$

(ii) Or (*vertical move*) $x_{i+1} = x_i \wedge y_{i+1} = y_i + 1$

Lemma 1.1. Let $(p, q), (m, n) \in \mathbb{Z}^2$, then the number of lattice paths from (p, q) to (m, n) is

$$\binom{(p-m)+(q-n)}{p-m} \tag{1.8}$$

Proof. The lattice is isomorphic to a H, V-string with length $(p-m)+(q-n)$. There are exactly $p-m$ horizontal moves as well as exactly $q-n$ vertical moves. ∎

Theorem 1.1. Given $n \in \mathbb{Z}_+$, the number of lattice paths from $(0, 0)$ to (n, n) which *never go above the diagonal line* is the **Catalan number**

$$C(n) \equiv \frac{1}{n+1}\binom{2n}{n} \tag{1.9}$$

Proof. Omitted ∎

Theorem 1.2 (Binomial Theorem)**.** Let $x, y \in \mathbb{R}$, then $\forall n \in \mathbb{Z}_+$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \tag{1.10}$$

Theorem 1.3 (Multinomial Theorem)**.** Let $r \in \mathbb{Z}_+$, $\{x_i\}_{i=1}^r \in \mathcal{P}(\mathbb{R})$. Then for every $n \in \mathbb{Z}_+$,

$$(\sum_{i=1}^{r} x_i)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} (x_i)^\alpha \tag{1.11}$$

where $\alpha \equiv (\alpha_i)_{i=1}^r,\ \alpha_i \in \mathbb{Z}_{++}\ \forall i$ is a **multi-index**, and

$$(x_i)^\alpha \equiv \sum_{i=1}^{r} x_i^{\alpha_i} \tag{1.12}$$

$$|\alpha| \equiv \sum_{i=1}^{r} \alpha_i \tag{1.13}$$

$$\binom{n}{\alpha} \equiv \frac{n!}{\alpha_1! \alpha_2! \dots \alpha_r!} \tag{1.14}$$