Notes on MAT137 Video Playlist 3

Tianyu Du

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3.1 Define Derivate As Slope

Definition Let $a \in \mathbb{R}$, and f(x) is defined on $(a-\delta, a+\delta)$, then the **derivative** of f(x) at a is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition If function is **differentiable** at point x = a, if and only if, there exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Interpretation $f'(a)$ is the slope of tangent line a x = a.

3.2 Calculate f'(x) by definition

Example $f(x) = 4x - x^2$, find $f'(1)$:

$$\begin{aligned} f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{4(h+1) - (h+1)^2 - 3}{h} \\ = \lim_{h \to 0} \frac{4h + 4 - 3 - h^2 - 2h - 1}{h} &= \lim_{h \to 0} \frac{-h^2 + 2h}{h} \\ &= \lim_{h \to 0} -h + 2 = 2 \end{aligned}$$

3.3 Rate of Change

Definition Define derivative as rate of change. Let $x = f(t)$, then $f'(x)$ can be represented as,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = f'(t) = \frac{dx}{dt}$$

3.4 The Product Rule (Formal Version)

Let $a \in \mathbb{R}$, f and g are functions defined at $(a - \delta, a + \delta)$, let $h(x) = f(x)g(x)$. Then, if $f(x), g(x)$ are differentiable at a, we have,

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

3.5 Differentiable $\implies$ Continuous

Recall f(x) is **differentiable** at a:

$$\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

Recall f(x) is **continuous** at a:

$$\lim_{x \to a} f(x) = f(a) \tag{2}$$

Proof.

$$\begin{aligned} &\text{Since f(x) is differentiable at a} \\ (1) \iff &\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &\text{And } \lim_{x \to a} (x - a) = 0 \\ \implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &\lim_{x \to a} x - a = 0 \\ \implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &x - a = 0 \\ \implies &\lim_{x \to a} f(x) - f(a) = 0 \\ \implies &\lim_{x \to a} f(x) = f(a) \end{aligned}$$

$\blacksquare$

3.6 Proof of product rule for derivative.

$(fg)' = f'g + fg'$, see above for a formal definition.

$$\begin{aligned}
\text{Let } h &= fg \\
h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)+f(a)g(x)-f(a)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{g(x)(f(x)-f(a))+f(a)(g(x)-g(a))}{x-a} \\
&= \lim_{x\to a} g(x)\frac{f(x)-f(a)}{x-a} + \lim_{x\to a} f(a)\frac{g(x)-g(a)}{x-a} \\
&= g(a)\lim_{x\to a} \frac{f(x)-f(a)}{x-a} + f(a)\lim_{x\to a} \frac{g(x)g(a)}{x-a} \\
&= g(a)f'(a)+f(a)g'(a)
\end{aligned}$$

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3.7 Partial proof of differentiation rule

WTS $\frac{d}{dx}x^c = cx^{c-1}, \forall c \in \mathbb{R}$

Here we only prove statements is true $\forall c \in \mathbb{Z}^+$

Proof.

$$\begin{aligned}
&\textbf{Base: c = 1} \\
f(x) &= x \\
f'(x) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} 1 = 1 \\
&\textbf{Induction step} \\
\text{Assume } \frac{d}{dx}[x^k] &= kx^{k-1}|_{x=a} \\
\text{For } f(x) &= x^{k+1} \\
f'(x) &= \frac{d}{dx}[x * x^k] \\
&= x^k + xkx^{k-1} \\
&= (k+1)x^k
\end{aligned}$$

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3.8 Higher Order Derivatives: Notations

Original function: $f(x)$

- **Lagrange** notation: $f^{(n)}$
- **Leibnitz** notation: $\frac{d^n f}{dx^n}$

3.9 Continuous But Not differentiable

Definition Function $f(x)$ is **non-differentiable** at a.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \textbf{ DNE}$$

Example 1 Corner/Kink $f(x) = |x|$ at 0.

$$\begin{aligned}
\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} &= \lim_{x\to 0^-} \frac{|x|}{x} = -1 \\
\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} &= \lim_{x\to 0^+} \frac{|x|}{x} = 1 \\
\lim_{x\to 0^-} &\neq \lim_{x\to 0^+} \\
\implies \lim_{x\to 0} \frac{f(x)-f(0)}{x} &\textbf{ DNE}
\end{aligned}$$

Example 2 Vertical Tangent Line $g(x) = x^{\frac{1}{3}}$ at 0,

$$g'(0) = \lim_{x\to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}} = \infty \textbf{(DNE)}$$

Caution Difference between **vertical asymptote** and **vertical tangent line**

- Vertical asymptote: $f(a) = \infty$ (f(a) is not defined)
- Vertical tangent line: $f(a)$ is defined, $f'(a)$ is undefined.

3.10 Chain Rule

Derivation

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$$
$$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$$

Attention: we could only apply the operation above if $f(x) \neq f(a)$ during the process of $x \to a$. This holds for majority of functions we operate in calculus.

$$= \lim_{f(x) \to f(a)} \frac{g(f(x)) - g(f(a))}{x - a} f'(a)$$
$$= g'(f(a)) \cdot f'(a)$$

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Formal Theorem of Chain Rule Let $a \in \mathbb{R}$, let f and g be functions. If f is differentiable at a and g is differentiable at $f(a)$, then, $(g \circ f)$ is differentiable at a,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

3.11 Derivatives of Trig Functions

Basic 6 results

1. $\frac{d}{dx} sin(x) = cos(x)$
2. $\frac{d}{dx} cos(x) = -sin(x)$
3. $\frac{d}{dx} tan(x) = sec^2(x)$
4. $\frac{d}{dx} cot(x) = -csc^2(x)$
5. $\frac{d}{dx} sec(x) = sec(x) tan(x)$
6. $\frac{d}{dx} csc(x) = -csc(x) cot(x)$

Proof. Prove (i) and (ii) and use (i), (ii) and quotient rule to derive (iii), (iv), (v) and (vi).

Proof. (i) **WTS** $f(x) = sin(x)$, then $f'(x) = cos(x)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{sin(x+h) - sin(x)}{h} \\ &= \lim_{h\to 0} \frac{sin(x)cos(h) + cos(x)sin(h) - sin(x)}{h} \\ &= \lim_{h\to 0} \frac{sin(x)(cos(h) - 1) + cos(x)sin(x)}{h} \\ &= \lim_{h\to 0} cos(x)\frac{sin(h)}{h} \\ &= cos(x) \end{aligned}$$

$$\blacksquare \quad (3)$$

Proof. (ii) **WTS** $f(x) = cos(x)$, then $f'(x) = -sin(x)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{cos(x+h) - cos(x)}{h} \\ &= \lim_{h\to 0} \frac{cos(x)cos(h) - sin(h)sin(x) - cos(x)}{h} \\ &= \lim_{h\to 0} \frac{(cos(h) - 1)cos(x) - sin(h)sin(x)}{h} \\ &= \lim_{h\to 0} -\frac{sin(h)}{h}sin(x) \\ &= -sin(x) \end{aligned}$$

$$\blacksquare \quad (4)$$

Recall Compound angle formula:

1. $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + sin(\beta)cos(\alpha)$
2. $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - sin(\beta)cos(\alpha)$
3. $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$
4. $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$

3.12 Implicit Differentiation

Key Use chain rule.

3.13 Derivative of Exponential Functions

Let $f(x) = a^x$ $(a > 0)$, find $f'(x)$, by definition,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{(a^n - 1)a^x}{h} \\ &\text{By property of limit, h is the only variable, so that } a^x \text{ is a constant} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned} \tag{5}$$

Equivalently, $\frac{d}{dx} a^x = L_a a^x$

Definition e is the only positive number, such that,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

So that, $\frac{d}{dx} e^x = e^x$

3.14 Properties of logarithms

Definition Let $a > 0, a \neq 1, x > 0, y \in \mathbb{R}$,

$$\log_a x = y \iff a^y = x$$

Properties

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x = \frac{\log_b x}{\log ba}$
4. $\log_a xy = \log_a x + \log_a y$
5. $\log_a \frac{x}{y} = \log_a x - \log_a y$
6. $\log_a x^r = r \log_a x$

Proof. (i) let $a > 0, a \neq 1, let x, y > 0$, **WTS** $\log_a xy = \log_a x + \log_a y$

$$\begin{aligned}
\text{Let } p = \log_a x &\iff a^p = x \\
\text{Let } q = \log_a y &\iff a^q = y \\
\text{We have } a^p a^q &= xy \\
\iff a^{p+q} &= xy \\
\iff \log_a xy = p + q &= \log_a x + \log_a y
\end{aligned}$$

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3.15 The derivatives of logarithm functions

For $\ln x$ $\quad \frac{d}{dx} \ln x = \frac{1}{x}$

$$\begin{aligned}
e^{\ln x} &= x \\
\frac{d}{dx} e^{\ln x} &= \frac{d}{dx} x \\
\frac{d}{d \ln x} e^{lnx} \cdot \frac{d}{dx} \ln x &= 1 \\
x \frac{d \ln x}{dx} &= 1 \\
\frac{d}{dx} \ln x &= \frac{1}{x}
\end{aligned}$$

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3.16 Derivative of other exponentials

WTS $\quad \frac{d}{dx} a^x = \ln a \cdot a^x$,

$$\begin{aligned}
a^x = (e^{\ln a})^x &= e^{x \ln a} \\
\frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} \\
= \frac{d}{dx} e^{x \ln a} &\cdot \frac{d}{dx} \ln a \\
&= e^{x \ln a} \ln a \\
&= \ln a \cdot a^x
\end{aligned}$$

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3.17 The power rule, complete proof

WTS $x^c = cx^{c-1}$

$$
\begin{aligned}
x^c &= (e^{\ln x})^c = e^{c\ln x} \\
\text{So that } \frac{d}{dx}x^c &= \frac{d}{dx}e^{c\ln x} \\
&= \frac{de^{c\ln x}}{d\ln xc}\cdot\frac{\ln xc}{d\ln x}\cdot\frac{d\ln x}{dx} \\
&= e^{c\ln x}\cdot c\cdot\frac{1}{x} \\
&= c\cdot x^c\cdot\frac{1}{x} \\
&= cx^{c-1}
\end{aligned}
$$

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3.18 Logarithmic Differentiation

Example $f(x) = cos(x)^{sin(x)}(\star)$, find $f'(x)$

Step1. Take ln on both sides of $(\star)$

$$\ln f(x) = \ln cos(x)^{sin(x)} = sin(x)\ln cos(x)$$

Step2. Take derivative.

$$\frac{f'(x)}{f(x)} = cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)}$$

Step3. Solve for $f'(x)$

$$f'(x) = cos(x)^{sin(x)}(cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)})$$

4 Video Playlist 4

4.1 Functions

In calculus We assume the domain is the largest subset of $\mathbb{R}$ that makes sense. And assume the codomain is always $\mathbb{R}$.

Notations

Math	Computer Science
Domain	Domain
Codomain	Range
Range	Image

4.2 Inverse Functions

Definition Let $f : A \to B$ be a function. Function $f^{-1} : B \to A$ is the **inverse function** is and only if

$$\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$$

Properties

- $\forall x \in A, f^{-1}(f(x)) = x$
- $\forall y \in B, f(f^{-1}(y)) = y$

Pre-condition Function f has inverse function f^{-1} if and only if f is **injective/one-to-one** function.

4.3 Surjective Functions

Why function don't have an inverse: Part 1.

Definition Function $f(x)$ is **surjective/onto** if $codomain(f(x)) = range(f(x))$.

Problem If $f(x)$ is not surjective, then some points in codomain has no corresponding point in domain, then f^{-1} is not a function.

Solution **Shrink** the codomain to range.

Example Let $f(x) = e^x$, $g(x) = \ln x$, then we have,

- - $Domain(f(x)) = \mathbb{R}$
  - $Codomain(f(x)) = \mathbb{R}$
  - $Range(f(x)) = (0, \infty)$
- - $Domaing(x) = (0, \infty)$
  - $Codomaing(x) = \mathbb{R}$
  - $Rangeg(x) = \mathbb{R}$

Definition Definition of inverse in calculus (*simplified, we don't consider codomain here.*)

Let $f(x)$ be a function, and $f^{-1}(x)$ be the **inverse** of it. Then,

- $Domain(f^{-1}(x)) = Range(f(x))$
- $Range(f^{-1}(x)) = Domain(f(x))$

also,

$$\forall x \in Domain(f(x)), \forall y \in Range(f(x)), x = f^{-1}(y) \iff y = f(x)$$

and,

$$\forall x \in Domain(f(x)), f^{-1}(f(x)) = x$$
$$\forall y \in Range(f(x)), f(f^{-1}(y)) = y$$

4.4 Injective function

Definition Let $f(x)$ be a function, with $Domain(f(x)) = A$, we say $f(x)$ is **injective/one-to-one** when,

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

equivalently (contrapositive)

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Theorem Function f has an inverse if and only if f is **injective**.

Example $f(x) = x^2$ has no inverse, but we could take it's inverse by shrinking the domain.

- Take domain $= [0, \infty)$, $f^{-1}(x) = \sqrt{x}$
- Take domain $= (-\infty, 0]$, $f^{-1}(x) = -\sqrt{x}$

4.5 Some theorems

Let $f(x)$ be a function with domain I.

Theorem 1 Function f has an inverse function f^{-1} if and only if f is injective.

Theorem 2 For function f, if

1. f is **continuous** (*This means, f is continuous on its domain.*).
2. I is an **interval**.

then, $f^{-1}(x)$ is continuous.

Theorem 3 If

1. f is **differentiable**.
2. $\forall x \in I, f'(x) \neq 0$ (*This ensures the inverse function does not have a vertical tangent line, which causes non-differentiability*).

then, $f^{-1}(x)$ is differentiable.

Theorem 4 $\forall x \in I$ with $y = f(x)$, we have

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Proof.

$$\begin{aligned} f(f^{-1}(y)) &= y \\ \frac{d}{dy} f(f^{-1}(y)) &= \frac{d}{dy} y \\ \frac{d}{dy} f(f^{-1}(y)) &= 1 \\ f'(f^{-1}(y)) \cdot (f^{-1})'(y) &= 1 \\ f'(x) \cdot (f^{-1})'(y) &= 1 \\ (f^{-1})'(y) &= \frac{1}{f'(x)} \end{aligned}$$

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4.6 ArcSin

Note $ArcSin$ is **NOT** the inverse of Sin. $y = sin(x)$ has $domain = \mathbb{R}$ and $range = [-1, 1]$, so that, it is **not injective**.

Definition $ArcSin$ is the inverse function to the **restriction** of sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So that, $Domain(ArcSin) = Range(Sin) = [-1, 1]$, and, $Range(ArcSin) = Domain(Sin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Meaning $ArcSin(\frac{1}{2}) = t$ means:

$$\begin{cases} sin(t) = \frac{1}{2} \\ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{cases}$$

Composite

$$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], ArcSin(Sin(x)) = x$$

$$\forall y \in [-1, 1], Sin(ArcSin(y)) = y$$

4.7 Derivative of ArcSin

Result

$$\frac{dArcSin(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Derive.

$$\begin{aligned}
\forall x &\in [-1,1] \\
Sin(ArcSin(x)) &= x \\
\frac{d}{dx}Sin(ArcSin(x)) &= \frac{d}{dx}x \\
Cos(ArcSin(x)) \cdot \frac{d}{dx}ArcSin(x) &= 1 \\
\frac{d}{dx}ArcSin(x) &= \frac{1}{Cos(ArcSin(x))} \\
\text{Let } \theta &= ArcSin(x) \\
Cos^2(\theta) &= 1 - Sin^2(\theta) \\
Cos(\theta) &= \pm\sqrt(1-x^2) \\
\text{Since } \forall\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], Sin(\theta) &\geq 0 \\
\implies Cos(\theta) &= +\sqrt{1-x^2} \\
\implies \frac{d}{dx}ArcSin(x) &= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

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4.8 Other inverse trig functions

4.8.1 $y = Cos(x)$

Definition $ArcCos$ is the inverse function to the restriction of $Cos(x)$ to $[0, \pi]$, and,

$$\forall x \in [-1,1], \forall y \in [0,\pi], x = ArcCos(y) \iff Cos(y) = x$$

Result

$$\frac{d}{dx}ArcCos(x) = -\frac{1}{\sqrt{1-x^2}}$$

4.8.2 $y = Tan(x)$

Definition $ArcTan(x)$ is the inverse function to the restriction of $Tan(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and,

$$\forall y \in \mathbb{R}, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], x = ArcTan(y) \iff Tan(x) = y$$

5 Video Playlist 5

5.1 Usage of MVT

Theorem Let I be an open interval. Let f be a function defined on I. If $\forall x \in I, f'(x) = 0$ then f is a constant function.

If we want to prove this theorem, we need mean value theorem

5.2 Local Extreme Theorem

Definition Let f be a function with domain I, let $c \in I$.

- f takes **maximum** at c if $\forall x \in I, f(x) \leq f(c)$.
- f takes **local maximum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \leq f(c)$.

Definition Let f be a function with domain I, let $c \in I$.

- f takes **minimum** at c if $\forall x \in I, f(x) \geq f(c)$.
- f takes **local minimum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \geq f(c)$.

End-point cannot be a local extremum since the definition of local extremum requires a open interval at both left and right sides around point c.

Theorem (Local EVT) Let f be a function with domain I as an interval. Let $c \in I$, then if,

1. $f(c)$ is an extremum.
2. c is an interior point.

then, $f'(c) = 0$ or DNE.

Definition Point $c \in I$ for function f is a **critical point** if $f'(c) = 0$ or it does not exist.

Proof. (Local EVT) Proof is in two parts: (1) f has maximum at c, (2) f has minimum at c.

$$\text{Part1: } f(c) \text{ is a maximum}$$
$$\text{Take left and right side limits}$$
$$\text{As} x \to c^+, x - c > 0$$
$$\text{As} x \to c^-, x - c < 0$$
$$\text{By definition of maximum} f(x) - f(c) \leq 0$$
$$\text{Left limit}$$
$$x - c < 0 \wedge f(x) - f(c) \leq 0$$
$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \geq 0$$
$$\text{Right limit}$$
$$x - c > 0 \wedge f(x) - f(c) \leq 0$$
$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \leq 0$$
$$\text{For limit to exist}$$
$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \leq 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \geq 0$$
$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$
$$\iff f'(c) = 0$$
$$\text{Part2: } f(c) \text{ is a minimum}$$
$$\text{Take left and right side limits}$$
$$\text{As} x \to c^+, x - c > 0$$
$$\text{As} x \to c^-, x - c < 0$$
$$\text{By definition of maximum} f(x) - f(c) \geq 0$$
$$\text{Left limit}$$
$$x - c < 0 \wedge f(x) - f(c) \geq 0$$
$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \leq 0$$
$$\text{Right limit}$$
$$x - c > 0 \wedge f(x) - f(c) \geq 0$$
$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \geq 0$$
$$\text{For limit to exist}$$
$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \geq 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \leq 0$$
$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$
$$\iff f'(c) = 0$$

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5.3 Find Extremum

Example find extremum of function $f(x) = x^3 - 3x^2 - 9x + 3$ for $I = [-4, 4]$
Steps

1. Ensure existence of extremum. f is polynomial and therefore continuous, and $[-4, 4]$ is a compact set. By EVT, extremum exist.
2. Find all *critical points* and *end-points.*
3. Compare values at candidate points.

5.4 Rolle's Theorem

Theorem let $a < b$, let f be a function defined on a closed interval $[a, b]$ (Compact set). Then, if,

1. $f(x)$ is continuous on $[a, b]$.
2. $(\wedge)$ $f(x)$ is differentiable on (a, b).
3. $(\wedge)$ $f(a) = f(b)$.

then,

$$\exists c \in (a, b) s.t. f'(c) = 0$$

Proof.

$$\begin{aligned}
\text{By EVT, } f(x) \text{ has extremum in } [a, b].\\
\textbf{Case1}\text{Interior Extremum Point.}(c \in (a, b))\\
\text{By Local EVT, } f'(c) = 0 \vee f'(c) DNE\\
\text{By (ii)} f'(c) = 0\\
\textbf{Case2}\text{End-point Extremum}\\
\text{Since (iii)} f(a) = f(b)\\
\forall x \in (a, b)\\
f(x) \leq max(f(a), f(b))\\
f(x) \geq min(f(a), f(b))\\
\implies f(x) \text{ is constant.}\\
\implies \forall c \in (a, b), f(c) = 0
\end{aligned}$$

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5.5 Application of Rolle's Theorem

Application How many zeros does a function have.

Step 1 Use IVT to prove it has *at least* n zeros.

Step 2 Use Rolle's theorem to prove it has *at most* n zeros.

Example

$$g(x) = x^6 + x^2 + x - 2$$

IVT Applied

$$g(-2) = 64$$
$$g(0) = -2$$
$$g(1) = 1$$

So that, $g(x)$ has at least 2 zeros.

Rolle's theorem applied Assume $f(x_1) = f(x_2) = 0$, by Rolle's theorem, there must exits a $a \in (x_1, x_2)$ such that $f'(a) = 0$

Conclusion 1 Between any two zeros of f there must be *at least* one zero of f'.

Conclusion 2 $\sharp$ of zeros of $f' \geq \sharp$ of zeros of f - 1

Conclusion 2' $\sharp$ of zeros of f $\leq \sharp$ of zeros of f' + 1

$$g'(x) = 6x^5 + 2x + 1$$
$$g''(x) = 30x^4 + 2$$
$$g''(x) \text{ has no zeros}$$

5.6 (Lagrange)Mean Value Theorem

Theorem Let $a < b$, let f be a function defined on $[a, b]$, if,

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

then,

$$\exists c \in (a, b) s.t. f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.7 Proof. of MVT

$$\begin{aligned}
\text{Let } m &= \frac{f(b)-f(a)}{b-a} \\
\text{Let } g(x) &= f(x) - f(a) - m(x-a) \\
\text{Satisfies } g(a) &= f(a) - f(a) - m(a-a) = 0 \\
\wedge g(b) &= f(b) - f(a) - m(b-a) = 0 \\
&\text{By Rolle's Theorem} \\
g(a) &= g(b) = 0 \\
\exists c &\in (a,b) s.t. g'(c) = 0 \\
\implies \frac{d}{x}[f(x) - f(a) - m(x-a)] &= 0 \\
\implies f'(c) &= \frac{f(b)-f(a)}{b-a} \\
&\blacksquare
\end{aligned}$$

5.8 Zero-derivative implies constant

Theorem Let $a < b$. Let f be a function defined on $[a,b]$, then,

$$\forall x \in (a,b), f'(x) = 0 \wedge \ f \text{ is continuous on } [a,b] \implies f \text{ is constant on [a,b].}$$

proof.

$$\begin{aligned}
\text{Let } x_1, x_2 &\in [a,b] \wedge x_1 < x_2 \\
\text{By MVT, } \exists c &\in (x_1, x_2), s.t. \\
f'(c) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\
&\because f'(c) = 0 \\
&\therefore f(x_1) = f(x_2)
\end{aligned}$$

5.9 Monotonicity of functions

Definition Let f be a function defined on an interval I.

- f is **increasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2)$$

- f is **non-decreasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

Theorem Let $a < b$. Let f be a function defined on (a, b). Then,

$$\forall x \in (a, b), f'(x) > 0 \implies f \text{ is increasing on (a,b)}$$

Theorem Let $a < b$. Let f be a function defined on $[a, b]$. Then,

$$\forall x \in (a, b), f'(x) > 0 \wedge f \text{ is continuous on } [a, b] \implies f \text{ is increasing on } [a, b]$$

Short summary On an <u>open interval</u>

- $f' = 0 \implies f$ constant.
- $f' > 0 \implies f$ increasing.
- $f' < 0 \implies f$ decreasing.

6 Video Playlist 6

Note This chapter focus on *optimization applications*, and there's no video for this topic.

7 Video Playlist 7

7.1 Integral

Integral Let $a < b$, let f be a <u>positive</u> function, then *integral of f from a to b* is denoted as:

$$\int_a^b f(x)\ dx$$

this is represented as the area of region under function f from $x = a$ to $x = b$.

7.2 Sigma Notation

Sigma Notation The sigma notation, with **index** i, could be represented in the following form:

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_N$$

7.3 Supremum and Infimum

Definitions Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Upper bound**: a is a <u>upper bound</u> of A means $\forall x \in A,\ x \leq a$.
- **Least upper bound(l.u.b) / Supremum**: a is the <u>least upper bound</u> or <u>supremum(sup)</u> of A iff a is an upper bound of A and $\forall b \in \{\text{upper bound of A}\},\ a \leq b$.

- **Maximum**: if supremum of $A \in A$, it's maximum of A.
- **Bounded above**: A is bounded above if A has (at least) one upper bound.

Definitions (counter-part) Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Lower bound**: a is a lower bound of A means $\forall x \in A,\ x \geq a$.
- **Greatest lower bound(g.l.b) / Infimum**: a is the greatest lower bound (g.l.b) or infimum(inf) of A iff a is a lower bound of A and $\forall b \in \{\text{Lower bound of A}\},\ a \geq b$.
- **Minimum**: if infimum of $A \in A$, it's the minimum of A.
- **Bounded below**: A is bounded below if A has (at least) one lower bound.

Theorem: The l.u.b. principle Let $A \subseteq \mathbb{R}$, if A is bounded above and $A \neq \emptyset$, then, A has a least upper bound(supremum).

Theorem: The g.l.b principle Let $A \subseteq \mathbb{R}$, if A is bounded below and $A \neq \emptyset$, then, A has a greatest lower bound(infimum).

7.4 Supremum and Infimum of a function

Definition Supremum of a function f on a domain I is defined as:

$$\sup_{x \in I} f(x) = \sup\{f(x) \mid x \in I\}$$

Theorem Let f be a function defined on domain $I \neq \emptyset$, if f is bounded above, then $\exists \sup_{x \in I} f(x)$. Similarly, if f is bounded below, then $\exists \inf_{x \in I} f(x)$.

Theorem(EVT) Let $a < b$, let f defined on $[a, b]$, if f is continuous on $[a, b]$, then f has a maximum and a minimum on $[a, b]$.

7.5 Definition of Integral (i)

Definition A **partition** of the interval $[a, b]$ is a finite set P, s.t. $\{a, b\} \subseteq P$.

Notation $P = \{x_0, x_1, \dots x_N\}$ on $[a, b]$. Implicitly, x_i are ordered, such that, $a = x_0 < x_1 < \cdots < x_N = b$.

Let f be bounded on $[a, b]$, let $P = \{x_0, x_1, \dots, x_N\}$, let $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, and $\Delta x_i = x_i - x_{i-1}$.

Definition P-Lower sum of f is defined as:

$$L_P(f) = \sum_{i=1}^{N}(m_i \Delta x_i)$$

Definition P-Upper sum of f is defined as:

$$U_P(f) = \sum_{i=1}^{N}(M_i \Delta x_i)$$

Property For all partition P on interval $[a, b]$, the lower sum and upper sum satisfy the following inequality,

$$L_P(f) \leq \int_a^b f(x)\ dx \leq U_P(f)$$

7.6 Definition of Integral (ii): Properties of $U_P(f)$ and $L_P(f)$

Let f be a <u>bounded</u> function on $[a, b]$, let P and Q be partitions of $[a, b]$, the lower sums and upper sums have the following properties.

1. (Always) $L_P(f) \leq U_P(f)$.
2. If $P \subseteq Q$ (<u>Q is a *finer* partition</u>), then $L_P(f) \leq L_Q(f) \wedge U_P(f) \geq U_Q(f)$.
3. (Always) $L_P(f) \leq U_Q(f)$

Proof

$$\begin{gathered} \text{Let } R = P \cup Q, \\ \text{so that, } P \subseteq R \wedge Q \subseteq R. \textit{ (R is finer than both P and Q)} \\ L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f) \\ \implies L_P(f) \leq U_Q(f) \end{gathered}$$

■

7.7 Definition of Integral (iii): Upper Integral and Lower Integral

Definition Let f be a <u>bounded</u> function on $[a, b]$, then, <u>lower integral of f from a to b</u> is defined as,

$$\underline{I_a^b(f)} = \sup\{\text{lower sums of } f\}$$

and the <u>upper integral of f from a to b</u> is defined as,

$$\overline{I_a^b(f)} = \inf\{\text{upper sums of } f\}$$

Then if $\underline{I_a^b(f)} < \overline{I_a^b(f)}$, then f is **non-integrable** on $[a, b]$.

7.8 An example of integrable function

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \quad \text{on } [-1, 1]$$

7.9 An example of non-integrable function

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases} \quad \text{on } [-1, 1]$$

7.10 Integrals as limits

Definition Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$, the **norm** of P is defined as:

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_N\}$$

Theorem - Lower Integrals For lower integrals, we have,

$$\underline{I_a^b(f)} = \lim_{\|P\| \to 0} L_P(f) = \sup\{\text{lower sums of } f\}$$

alternatively, using $\delta - \epsilon$ expression,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall P \text{ over } [a, b], \|P\| < \delta \implies |L_P(f) - \underline{I_a^b(f)}| < \epsilon$$

theorem - Upper Integrals For upper integrals, we have,

$$\overline{I_a^b(f)} = \lim_{\|P\| \to 0} U_P(f)$$

7.11 Riemann Sums

Definition Fix a partition P on $[a, b]$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, pick $x_i^\star \in [x_{i-1}, x_i]$, so that,

$$\begin{aligned} m_i \le f(x_i^\star) \le M_i \\ \implies m_i \Delta x_i \le f(x_i^\star) \Delta x_i \le M_i \Delta x_i \\ \implies L_P(f) = \sum_{i=1}^{N} (m_i \Delta x_i) \le \sum_{i=1}^{N} (f(x_i^\star) * \Delta x_i) \le \sum_{i=1}^{N} (M_i \Delta x_i) = U_P(f) \end{aligned}$$

where the term $\sum_{i=1}^{N} (f(x_i^\star) \Delta x_i)$ is called a **Riemann sum**.

Definition Let f be a bounded function on $[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition on $[a, b]$, for each i, pick **any** point $x_i^\star \in [x_{i-1}, x_i]$. then,

$$S_P^\star(f) = \sum_{i=1}^{N} (f(x_i^\star) * \Delta x_i)$$

is a **Riemann sum** for f and P. *(There are infinitely many Riemann sum).*

In general, we have,

$$L_P(f) \leq S_P^\star(f) \leq U_P(f)$$

and also,

$$\lim_{\|P\| \to 0} L_P(f) = \underline{I_a^b(f)}$$

$$\lim_{\|P\| \to 0} U_P(f) = \overline{I_a^b(f)}$$

and if f is **integrable**, then

$$\lim_{\|P\| \to 0} L_P(f) = \lim_{\|P\| \to 0} U_P(f) = \int_a^b f(x)\ dx$$

By Squeeze Theorem,

$$\lim_{\|P\| \to 0} S_P^\star(f) = \int_a^b f(x)\ dx$$

7.12 Properties of the integral

Property 1

$$\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

Property 2

$$\int_a^b [cf(x)]\ dx = c \int_a^b f(x)\ dx$$

Property 3 If f is bounded on $[a, c]$, and f is integrable on $[a, b]$ and integrable on $[b, c]$, then,

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx$$

Property 4: Backward Integrals

$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$$

Negative function f Integral for negative function is the negative area.

$$\int_a^b f(x)\ dx$$

8 Video Playlist 8

8.1 Anti-dervatives

Notations

- **Definite integral** $\int_a^b f(x)\ dx$
- **Indefinite integral** $\int f(x)\ dx$

Definition Let f be a function defined on an interval, *an* **anti-dervative** of f is any function F that

$$F' = f$$

Note As a consequence of MVT, if two functions have same dervative on an interval, then they differ by a constant.

8.2 Functions Defined as Integrals

Consider integrable function f, define function F as the definite integral from a, a fixed point in domain of f, to another point x in domain of f, that's,

$$F(x) = \int_a^x f(t)\ dt$$

Methodology Let I be an interval, let $a \in I$ and let f be a function integrable on I, then for each $x \in I$, compute $F(x) = \int_a^x f(t)\ dt$ as a number.

8.3 The Fundamental Theorem of Calculus: Part 1

This provides connections between definite integrals and anti-dervatives

Theorem: FTC(part 1)

- Let I be an interval,
- Let $a \in I$,
- Let f be a function on I.

Define $F(x)$ as

$$F(x) = \int_a^x f(t)\ dt$$

If f is continuous, then F is differentiable and $F' = f$, that's,

$$F'(x) = f(x) \quad \forall x \in I$$

8.4 A Proof of Part 1 of the FTC

Proof.

$$\text{Let(fix) } x \in I$$
$$\text{WTS. } F'(x) = f(x)$$
$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0} [\frac{1}{h}(F(x+h) - F(x))]$$
$$= \lim_{h\to 0} [\frac{1}{h}(\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt)]$$
$$= \lim_{h\to 0} [\frac{1}{h} \int_x^{x+h} f(t)\ dt]$$

Consider $h > 0$ (for negative h, the proof would be similar)

$$\text{Let } M_h = \sup_{[x,x+h]} (f)$$
$$\text{Let } m_h = \inf_{[x,x+h]} (f)$$

Then we have, by definition of infimum and supremum,

$$m_h \le \frac{1}{h} \int_x^{x+h} f(t)\ dt \le M_h$$

Since f is continuous on $[x, x+h]$, by EVT, it has maximum and minimum on this interval.

$$\exists c_h \in [x, x+h]\ s.t.\ M_h = f(c_h)$$
$$\exists d_h \in [x, x+h]\ s.t.\ m_h = f(d_h)$$
$$\because \lim_{h\to 0} c_h = x \wedge \lim_{h\to 0} d_h = x$$
$$\therefore \lim_{h\to 0} M_h = \lim_{h\to 0, c_h\to x} f(c_h) = f(x) \text{ (since } f \text{ is continuous.)}$$
$$\text{Similarly, } \lim_{h\to 0} m_h = \lim_{h\to 0, d_h\to x} f(d_h) = f(x)$$
$$\text{By Squeeze Theorem, } \lim_{h\to 0} [\frac{1}{h} \int_x^{x+h} f(t)\ dt] = f(x)$$
$$\therefore F'(x) = f(x)\ \forall x \in I$$

■

8.5 The Fundamental Theorem of Calculus: Part 2

This provides a quick way to compute definite integrals.

Theorem: FTC(part 2)

- Let $a < b \in \mathbb{R}$,
- let f be continuous on $[a, b]$,

then,

$$\int_a^b f(x)\ dx = G(b) - G(a)$$

where G is any anti-dervative of f.

Notation

$$G(b) - G(a) = G(x)|_{x=a}^{x=b} = G(x)|_a^b$$

8.6 A Proof of Part 2 of the FTC

Proof.

$$\begin{gathered}
\text{We know that, from the first part of FTC, } G' = f, \\
\text{WTS. } \int_a^b f(x) = G(b) - G(a) \\
\text{Define } F(x) = \int_a^x f(t)\ dt \\
\text{WTS. } F(b) = G(b) - G(a) \\
\text{Since } f \text{ is continuous,} F' = f \\
\text{By the consequence of MVT,} \\
F' = G' \implies \exists C \in \mathbb{R} s.t. F - G = C \forall x \in [a, b] \\
\text{at } x = a, F(a) = 0 \implies C = -G(a) \\
\implies \forall x \in [a, b] F(x) = G(x) - G(a) \\
\text{at } x = b, F(b) = G(b) - G(a)
\end{gathered}$$

■

8.7 Summary: Definite and indefinite integrals, notation, definitions and theorems.

8.7.1 Definite Integral.

$$\int_a^b f(x)\ dx$$

Theorem (Formal definite) if $\overline{I_a^b}(f) = \underline{I_a^b}(f)$ then $\int_a^b f(x)\ dx = \overline{I_a^b}(f) = \underline{I_a^b}(f)$.

Theorem (FTC: part 2) Choose <u>one</u> anti-dervative $G(x)$ of $f(x)$, then compute the definite integral as $\int_a^b f(x)\ dx = G(b) - G(a)$.

8.7.2 Indefinite Integral

$$\int f(x)\ dx \text{ A \underline{collection} of functions.}$$

Find indefinite integral Find $G(x)$ as <u>one</u> anti-dervative, by the consequence of MVT, then the indefinite integral of f could be constructed as,

$$F(x) = \{G(x) + C \mid C \in \mathbb{R}\}$$

8.7.3 Function Defined by an Integral.

$$F(x) = \int_a^x f(t)\ dt \text{ This is \underline{one} function with fixed value of } a.$$

Theorem (FTC: part 1) if f is continuous, then $F'(x) = f(x)$

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9.1 Integration By Substitution: derivation of the formula

Backwards usage of chain rule.

If $\int f(x)\ dx = F(x)$ is the anti-derivative of $f(x)$, then

$$F(g(x)) = \int f(g(x))g'(x)\ dx = F(g(x))$$

9.2 Example 2

9.3 Example 3

9.4 Example 4

Theorem Let $a < b$, let f be a continuous function, let g be a function with <u>continuous derivative</u> in $[a, b]$, assume the range of g on $[a, b]$ is contained in the domain of f. Then,

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

9.5 Integration by parts

Backwards product rule

Let f and g be two differentiable function, by product rule of differentiation, we have,

$$
\begin{aligned}
f'(x)g(x) + f(x)g'(x) &= \frac{d}{dx} f(x)g(x) \\
\implies \int f'(x)g(x) + f(x)g'(x)\ dx &= f(x)g(x) + C \\
\implies \int f'(x)g(x)\ dx + \int f(x)g'(x)\ dx &= f(x) + g(x) + C \\
\implies \int f'(x)g(x)\ dx &= f(x) + g(x) - \int f(x)g'(x)\ dx
\end{aligned}
$$

The integral constant is implicitly contained in the integral term.

9.6 Examples

Example 1

$$\int x^2 e^2\ dx$$

Example 2

$$\int e^2 \sin x\ dx$$

<u>Use integration by parts twice.</u>

Example 3

$$\int \arctan x\ dx$$

<u>Consider the form $1 \times f(x)$ as partition method.</u>

9.7 Integration of products of trigonometric functions

Types

$$\int sin^n x\ cos^m x\ dx$$

$$\int sec^n x\ tan^m x\ dx$$

Keys

$$sin^2(x) + cos^2(x) = 1$$

$$sec^2(x) = 1 + tan^2(x)$$

Summary I Consider the integral in the following form

$$\int sin^n x\ cos^m x\ dx$$

- If **m is odd** then try $u = sin(x)$, then $du = cos(x)dx$
- If **n is odd** then try $u = cos(x)$, then $du = -sin(x)dx$

10 Video Playlist 10

Note This chapter focus on *volumes*, and there's no video for this topic.

11 Video Playlist 11

11.1 What Is a Sequence

Definition A **sequence** is a function with domain $\mathbb{N}$.

11.1.1 Conventions

Functions function with domain interval.

- x as variable.
- $f(x)$ as value at x.

Sequence function with domain $\mathbb{N}$.

- n as variable.
- a_n as value at n.

A sequence is not a set.

11.1.2 Describe sequences

Equation $a_n = \frac{2^n n!}{n+1}$

First few values $\{1, 2, 4, 8, 16, \dots\}$

Words $p_n =$ n-th prime.

Recurrence relation e.g. Fibonacci Sequence.

$$\{F_n\}_{n=0}^{\infty} : F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ \forall n \geq 2$$

A general definition A sequence is a function with domain $\{n \in \mathbb{Z} \mid n \geq n_0\}$ for some fixed $n_0 \in \mathbb{Z}$.

11.2 The Limit of a Sequence

Example

$$\{\frac{n}{n+1}\}_{n=0}^{\infty} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Definition(Limit) We say that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to the number $L \in \mathbb{R}$ when

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N} s.t.\ \forall n \in \mathbb{N},\ n \geq n_0 \implies |L - a_n| < \epsilon$$

denoted as

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L$$

Tail: all terms of the sequence after the first few terms.
Every interval centred at L contains a tail of the sequence.

Definition A sequence is **convergent** if it has a limit. This sequence is **divergent** if it does not have a limit.

11.3 Properties of Limits of Sequences

Properties from the limit of functions

- Limit laws: Yes
- Squeeze theorem: Yes
- $L'H\hat{o}pital's\ Rule$: No

11.3.1 Sequence from a function

Let $c \in \mathbb{Z}$ and function f defined on $[c, \infty)$, and define the seuqnce $\{a_n\}_{n=c}^{\infty}$ as

$$a_n = f(n)$$

We have if $\lim_{n\to\infty} f(n) = L$ then $\lim_{n\to\infty} a_n = L$. If $\lim_{n\to\infty} f(n)$ DNE, then $\lim_{n\to\infty} a_n$ <u>may or may not exist</u>.

11.3.2 Composite of sequence and function

Theorem If $a_n \to L$ and f is continuous at L then

$$f(a_n) \to f(L)$$

11.4 Monotonic and Bounded Sequences

11.4.1 Monotonic Sequences

Definition We say $\{a_n\}_{n=0}^{\infty}$ is **increasing** if

$$\forall n, m \in \mathbb{N},\ n < m \implies a_n < a_m$$

Also, we say this sequence is **non-decreasing** if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N},\ n < m \implies a_n \leq a_m$$

Definition We say $\{a_n\}_{n=0}^{\infty}$ is **decreasing** if

$$\forall n, m \in \mathbb{N},\ n < m \implies a_n > a_m$$

Also, if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N},\ n < m \implies a_n \geq a_m$$

we say this sequence is **non-increasing**.

Definition We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **monotonic** is if is has any of the four properties above.

Definition $\{a_n\}_{n=0}^{\infty}$ is **eventually decreasing** if

$$\exists n_0 \in \mathbb{N},\ s.t. \forall n \in \mathbb{N}, n \geq n_0 \implies a_n > a_{n+1}$$

11.4.2 Bounded Sequences

Definition We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **bounded below** if

$$\exists A \in \mathbb{R} s.t. \forall n \in \mathbb{N},\ A \leq a_n$$

Similarly, the sequence is **bounded above** if

$$\exists B \in \mathbb{R}. s.t. \forall n \in \mathbb{N},\ B \geq a_n$$

Definition We say a sequence is **bounded** if and only if it is <u>both</u> bounded above and below.

Theorem If a sequence is <u>convergent</u> then it is <u>bounded</u>.

Theorem 2A(The monotone convergence theorem for sequence) If a sequence is <u>eventually increasing</u> and <u>bounded above</u>, then it is <u>convergent</u>

Theorem If a sequence is <u>eventually increasing</u> and <u>not bounded above</u> then it <u>divergent to ∞</u>.

Remark for a sequence:

$$\text{Sequence}\begin{cases}\text{Convergent}\\ \text{Divergent}\begin{cases}\text{to } \infty\\ \text{to } -\infty\\ \textit{Oscillating}\end{cases}\end{cases}$$

11.5 Proof: Every convergent sequence is bounded

Theorem Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, if $\{a_n\}_{n=0}^{\infty}$ is convergent then the sequence is bounded. Equivalently,

Proof.

$$\text{Assume sequence } \{a_n\}_{n=0}^{\infty} \text{ is convergent.}$$
$$\text{Let } L \text{ be the limit.}$$
$$\text{By the definition of limit, choose } \epsilon = 10$$
$$\text{So that, } \exists n_0 \in \mathbb{N} s.t. \forall n \in \mathbb{N}, n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$$
$$\text{Take } A = min\{a_0, \ldots, a_{n_0-1}, L - 10\}$$
$$\text{Take } B = max\{a_0, \ldots, a_{n_0-1}, L + 10\}$$
$$\text{By definition of max and min, let } n \in \mathbb{N}$$
$$\text{case } 1 n > n_0 \implies A \leq a_n \leq B$$
$$\text{case } 2 n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$$
$$\text{Since } A \leq L - 10 \wedge B \geq L + 10$$
$$\implies A \leq a_n \leq B \forall n \in \mathbb{N}$$
$$\therefore \{a_n\}_{n=0}^{\infty} \text{ is bounded.}$$

∎

11.6 The monotone convergence theorem of sequences

(General) Theorem If a sequence is (eventually) monotonic and bounded then it is convergent.

(Particular Case) Theorem 1 If a sequence is increasing and bounded above the it's convergent.

Proof.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence that's increasing and bounded above.

Consider $A = \{a_n \mid n \in \mathbb{N}\} \neq \emptyset$

By least upper bound principle, there exists a supremum of set A

Take $L = sup\{A\}$

Let $\epsilon > 0$

By definition of supremum,

$\exists a_{n0} \in A \ s.t. \ a_{n0} > L - \epsilon$

Take this value n_0

Since sequence is increasing,

$\forall n \geq n_0 \ a_n > L - \epsilon$

Also, by definition of supremum, $a_n \leq L$

$\implies a_n \leq L + \epsilon$

Therefore, $\forall n \in \mathbb{N}, n \geq n_0 \implies L - \epsilon < a_n < L + \epsilon$

Therefore, $\lim_{n\to\infty} \{a_n\}_{n=0}^{\infty} = L$

Therefore, $\{a_n\}_{n=0}^{\infty}$ is convergent.

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11.7 the Big theorem of sequences

Definition (for positive sequences only) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be positive sequences.

$$a_n << b_n \iff \lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

say $\{a_n\}$ is **much smaller than** $\{b_n\}$.

Theorem for every $a > 0$ and $c > 1$

$$\ln n << n^a << c^n << n! << n^n$$

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12.1 Improper Integral

12.1.1 Improper integral "type 1" (Unbounded domain)

Definition Let $a \in \mathbb{R}$ and f continuous on $[a, \ \infty]$ the integral of f from a to ∞, denoted as

$$\int_a^{\infty} f(x) \ dx = \lim_{b\to\infty} \int_a^b f(x) \ dx$$

assuming the limit exists. If the limit exists, the integral is called **convergent**, otherwise, it's called **divergent**.

12.2 The most important family if the improper integrals

Let $p \in \mathbb{R}$ consider

$$\int_1^{\infty} \frac{1}{x^p}\ dx$$

Summary

$$\int_1^{\infty} \frac{1}{x^p}\ dx \text{ is } \begin{cases} \textbf{convergent} \text{ if } p > 1 \\ \textbf{divergent to } \infty \text{ if } p \leq 1 \end{cases}$$

12.3 Example

$$\int_0^{\infty} sin(x)\ dx$$

12.4 The most import family of improper integral

Consider

$$I = \int_1^{\infty} \frac{1}{x^p}\ dx$$

Summary

1. $p > 1 \iff I$ converges.
2. $p \leq 1 \iff I$ diverges to ∞.

12.5 Example

Vertical asymptote improper.

$$\int_0^{1} ln(x)\ dx$$

12.6 Doubly improper integrals

General Strategy Assume A has **multiple** improper.

1. Break A into pieces with **single** improper at their endpoints.
2. If each piece convergent **seperately**, then A converges.
3. Else, A diverges, it's *not a number*.

12.7 Basic Comparison Test

Theorem Let $a \in \mathbb{R}$,
Let f and g be *continuous* functions one $[a, \infty)$, and

$$\forall x \geq a, 0 \leq f(x) \leq g(x)$$

we have,

1. $\int_a^\infty g(x)\ dx < \infty \implies \int_a^\infty f(x)\ dx < 0$
2. $\int_a^\infty f(x)\ dx = \infty \implies \int_a^\infty g(x)\ dx = \infty$

12.8 Examples

12.9 Limit Comparison Test

Theorem Let $a \in \mathbb{R}$, f and g are *positive* and *continuous* functions on $[a, \infty)$. And the following limit exists,

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)} \in \mathbb{R}$$

Then, $\int_a^\infty f(x)\ dx$ and $\int_a^\infty g(x)\ dx$ are **both** convergent or **both** divergent.

12.10 Proof of LCT

Omitted

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13.1 Infinite Sums

Nothing.

13.2 Definition of series

Definition Series $\sum_{n=1}^\infty a_n$ is defined as

$$\lim_{k \to \infty} S_k$$

where $S_k = \sum_{n=1}^k a_n$ as finite sum. If the above limit exist, we say series $\sum_{n=1}^\infty a_n$ is convergent (it's a *number*), else series is divergent and it's not a number.

13.3 Example

$$\sum_{n=1}^\infty \frac{1}{n^2+n} = \sum_{n=1}^\infty \frac{1}{n} - \frac{1}{n+1} = \lim_{k \to \infty} 1 - \frac{1}{k+1} = 1$$

13.4 Divergent Series Examples

$$S = \sum_{n=1}^{\infty} 1 = \lim_{k\to\infty} k = \infty$$

$$S = \sum_{n=1}^{\infty} (-1)^n = \begin{cases} 0 \text{ if k is odd} \\ 1 \text{ if k is even} \end{cases} \quad \text{divergent due to osciallation.}$$

13.5 Geometric Series

Let $x \in \mathbb{R}$

$$S = \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=0}^{\infty} x^n$$

Consider

$$S_k = 1 + x + x^2 + \cdots + x^k$$
$$xS_k = x + x^2 + x^3 + \cdots + x^{k+1}$$
$$S_k - xS_k = 1 - x^{k+1}$$

If x = 1, the series is simply divergent to ∞.

$$S_k = \frac{1 - x^{k+1}}{1 - x},\ x \neq 1$$

$$S = \lim_{k\to\infty} 1 - x^{k+1} = \frac{1 - \lim_{k\to\infty}(x^(k+1))}{1 - x} = \begin{cases} \frac{1}{1-x} \iff x \in (-1, 1) \\ \text{Divergent} \begin{cases} \infty \impliedby x > 1 \\ \text{Oscillating} \impliedby x \leq -1 \end{cases} \end{cases}$$

13.6 Linearity of series

Simple form of fact

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} c\ b_n = \sum_{n=0}^{\infty} a_n + c\ b_n,\ \forall c \in \mathbb{R}$$

Theorem If series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent, then $\sum_{n=0}^{\infty} a_n + b_n$ is also convergent and

$$\sum_{n=0}^{\infty} a_n + b_n = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

Proof.

$$\text{Let } \sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$$
$$\text{Let } \sum_{n=0}^{\infty} b_n = \lim_{k\to\infty} T_k$$
$$\text{Let } \sum_{n=0}^{\infty} a_n + b_n = \lim_{k\to\infty} R_k$$
$$\text{Where } S_k = \sum_{n=0}^{k} a_n$$
$$T_k = \sum_{n=0}^{k} b_n$$
$$R_k = \sum_{n=0}^{k} a_n + b_n$$
$$\text{Since } R_k = S_k + T_k,\ \forall k \in \mathbb{N}$$
$$\text{By limit laws, } \lim_{k\to\infty} S_k + \lim_{k\to\infty} T_k = \lim_{k\to\infty} R_k$$
$$\implies \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n$$

■

Theorem If series $\sum_{n=0}^{\infty} a_n$ is convergent, then for any $c \in \mathbb{R}$, series $\sum_{n=0}^{\infty} c\ a_n$ is also convergent and

$$\sum_{n=0}^{\infty} c\ a_n = c \sum_{n=0}^{\infty} a_n$$

General proof procedure

1. Write series as limit of partial sums.
2. Manipulate partial sums (finite).
3. Manipulate limits.

13.7 the Tail of a series

Fact Consider two series

$$\sum_{n=0}^{\infty} a_n \text{ convergent } \iff \sum_{n=1}^{\infty} a_n \text{ convergent}$$

And $\sum_{n=1}^{\infty} a_n$ is a tail of series $\sum_{n=0}^{\infty} a_n$.

Notation We say $\sum_n^\infty a_n$ is convergent or divergent without specifying the starting index of the series.

Specific form of theorem If $\forall n \in \mathbb{N}$ [Condition(s)] then $\sum_n^\infty a_n$ is convergent or divergent.

General form of theorem If $\exists\, n_0 \in \mathbb{N}\, s.t.\, \forall n \in N,\ n \geq n_0 \implies$ [Condition(s)] then $\sum_n^\infty a_n$ is convergent.

13.8 A necessary condition for convergence of series

Fact Series $\sum_{n=0}^\infty a_n$ is convergent if and only if the sequence of its partial sums $\{S_n\}_{n=0}^k$ is convergent.

Theorem If $\sum_{n=0}^\infty a_n$ is convergent then

$$\lim_{n\to\infty} a_n = 0$$

Note The above theorem is often used as it's contrapositive form

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=0}^\infty a_n \text{ is divergent}$$

Proof.

$$\text{Let } S = \sum_{n=0}^\infty a_n \text{ be convergent}$$

$$S = \lim_{k\to\infty} S_k,\ S_k = \sum_{n=0}^k a_n$$

$$S = \lim_{k-1\to\infty} S_{k-1}$$

$$\implies \lim_{k\to\infty} S_k - \lim_{k-1\to\infty} S_{k-1} = 0$$

By the convergence assumption, those two limits above exist.

$$\implies \lim_{k\to\infty} S_k - S_{k-1} = 0$$

$$\implies \lim_{k\to\infty} a_k = 0$$

$$\implies \{a_n\}_{n=1}^\infty \to 0$$

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13.9 Positive series

Definition A series $\sum_{n=0}^\infty a_n$ is positive when $\forall n \in \mathbb{N},\ a_n > 0$. And a series is positive means it could never diverge to $-\infty$ or *oscillating*.

Notation (For positive series only)

1. $\sum_n^\infty a_n = \infty \iff$ divergent.
2. $\sum_n^\infty a_n < \infty \iff$ convergent.

13.10 The Integral Test

Theorem Let $a \in \mathbb{R}$, let f be a continuous, positive and decreasing function on $[a, \infty)$, then

$$\int_a^\infty f(x)\ dx < \infty \iff \sum_n^\infty f(n) < \infty$$

That's the improper integral and series have the same convergence/divergence feature. Note as

$$\int_a^\infty f(x)\ dx \sim \sum_n^\infty f(n)$$