ECO421: Topics in Economic of Information

Games with Incomplete Information

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Contents

1 Knowledge

1.1 Information Structure

Notation 1.1. Let Ω denote the all possible states of world, and let A denote the set of all agents. Let $\omega^* \in \Omega$ denote the true state of world.

Definition 1.1. Every $E \subseteq \Omega$ is called an **event** or a type/piece of **information**.

Definition 1.2. Let $E \subseteq \Omega$ be a piece of information and $i \in A$, and we say agent i **knows** E in state $\omega \in \Omega$ if this agent knows

(i) The true state of world $\omega^* \in E$;

(ii) but, all $\omega \in E$ are considered to be possible.

Remark: the agent is certain about the true state ω^* if and only if $E = \{\omega^*\}$.

Definition 1.3. For an agent $i \in A$, the **information structure** of this agent, $\mathcal{T}_i$, is a partition of Ω into types of information.

Notation 1.2. Each element of the partition corresponds to one piece of information (an event).
(Information encoding) One may define a mapping $T_i(\cdot) : \Omega \to \mathcal{T}_i$, such that for every $\omega \in \Omega$, let $T_i(\omega) \in \mathcal{T}_i$ denote the piece of information known by this agent in state ω.
Equivalently, $T_i(\omega) \subseteq \Omega$ denotes the set of all states that agent i considers possible given her information in state ω.

Remark 1.1 (Interpretation)**.** At some state of world, ω^*, agent i receives information $T_i(\omega^*)$ but not ω^*. This agent only knows $\omega^* \in T_i(\omega^*)$. The information encoding mechanism is a kind of blurring system preventing the agent from identifying the exact ω^*.

1.2 Knowledge Space

Definition 1.4. A **knowledge-based type space** is defined to be a tuple

$$(\Omega, (\mathcal{T}_i)_{i\in A}) \tag{1.1}$$

where $\mathcal{T}_i$ is an information structure of agent i.

Definition 1.5. An event $E \subseteq \Omega$ is **true** in state $\omega^* \in \Omega$ if $\omega^* \subseteq E$.

Definition 1.6 (Equivalent Definition)**.** An agent i is said to **know** event E in state ω^* if and only if

$$T_i(\omega^*) \subseteq E \tag{1.2}$$

That is, the true state $\omega^* \in T_i(\omega) \subseteq E$, so that agent i is certain that E is true.

Definition 1.7. Define $K_i(E)$ to be the set of states in which agent i knows E to be true, that is,

$$K_i(E) := \{\omega : T_i(\omega) \subseteq E\} \subseteq \Omega \tag{1.3}$$

1.3 Learning

Definition 1.8. Given a piece of information $F \subseteq \Omega$ characterizing that $\omega^* \in F$, suppose F is known by all agents, the new **information structure updated by** F is defined to be

$$(\Omega^F, (T_i^F(\cdot))) \tag{1.4}$$

where

$$\Omega^F = \Omega \cap F \tag{1.5}$$
$$T_i^F(\omega) = T_i(\omega) \cap F \quad \forall \omega \in \Omega^F \tag{1.6}$$

Definition 1.9. The set of states where $E \subseteq \Omega^F$ is known given F is

$$K_i(E|F) := \left\{\omega \in \Omega^F : T_i^F(\omega) \subseteq E\right\} \tag{1.7}$$
$$= \left\{\omega \in \Omega^F : T_i(\omega) \cap F \subseteq E\right\} \tag{1.8}$$

Notation 1.3. For $i, j \in A$, the states in which i knows that player j knows E can be denoted as

$$K_i(K_j(E)) \tag{1.9}$$

1.4 Knowledge Hierarchies and Common knowledges

Definition 1.10. An event $E \subseteq \Omega$ is a **common knowledge** in state ω if everyone knows E and knows everyone else possesses the same information. That is:

$$\omega \in \bigcap_{i \in A} K_i(E) \tag{1.10}$$
$$\omega \in \bigcap_{j \in A} K_j\left(\bigcap_{i \in A} K_i(E)\right) \tag{1.11}$$
$$\omega \in \bigcap_{k \in A} K_k\left[\bigcap_{j \in A} K_j\left(\bigcap_{i \in A} K_i(E)\right)\right] \tag{1.12}$$
$$\vdots \tag{1.13}$$

2 Beliefs

2.1 Type Space

Definition 2.1. A **type space** is a triple $(\Omega, (T_i), (\mu_i(\omega, t_{-i}|t_i)))$.

(i) Ω is the state space.

(ii) T_i is the collection of types for player i.

(iii) The **belief** of type t_i player i is a probability distribution:

$$\mu_i(\omega, t_{-i}|t_i) \in \Delta(\Omega \times T_{-i}) \text{ where } T_{-i} = \prod_{j \neq i} T_j \tag{2.1}$$

Remark 2.1. In the definition, we were overloading the notion of T_i, here T_i is a fixed set representing the collection of all possible types of player i. In the previous section, $T_i(\cdot)$ was a function maps a state $\omega \in \Omega$ to the information possessed by agent i in state ω.

Proposition 2.1 (Relation between type space and knowledge space)**.** For a type space $(\Omega, (T_i), (\mu_i(\omega, t_{-i}|t_i)))$, define

$$\Omega^* := \Omega \times T_1 \times \ldots \times T_n \tag{2.2}$$

and for every $\omega^* \in \Omega^*$, define

$$T_i^*(\omega^*) \equiv T_i^*(\omega, t_{1:n}) = \{(\omega', t'_{1:n}) : t'_i = t_i\} \subseteq \Omega^* \tag{2.3}$$

That is, player i only knows his own type and cannot distinguish types of any other players. And $T_i^*(\cdot)$ defines a partition over Ω^*. Therefore, $(\Omega^*, (T_i^*(\cdot)))$ is a knowledge space induced by the type space.

2.2 Prior and interim beliefs

Definition 2.2. The **interim belief** of player i is defined to be the distribution $\mu_i(\omega, t_{-i}|t_i)$ in the type space.

Definition 2.3. The **posterior belief** of player i is defined as the belief after the player learns the types of all other players:

$$\mu_i(\omega|t_{1:n}) \tag{2.4}$$

Definition 2.4. The **prior belief** μ, is a distribution over (ω, t_i, t_{-i}). Then player i can compute the prior belief on t_i by marginalizing

$$\mu\left(t_i\right) = \sum_{\omega, t_{-i}} \mu_i\left(\omega, t_{-i}, t_i\right) \tag{2.5}$$

Note that the prior belief μ does not have a subscript i because the prior belief is the same for all players.

Remark 2.2. Ex-ante approach:

(i) Player i learn (overall) prior $\mu\left(\omega, t_i, t_{-i}\right)$;

(ii) Player i deduces his type prior following

$$\mu\left(t_i\right) = \sum_{\omega, t_{-i}} \mu_i\left(\omega, t_{-i}, t_i\right) \tag{2.6}$$

(iii) Player i deduces interim belief following

$$\mu_i\left(\omega, t_{-i}|t_i\right) = \frac{\mu\left(\omega, t_i, t_{-i}\right)}{\mu\left(t_i\right)} \tag{2.7}$$

2.3 Bayesian Games

Definition 2.5. A **Bayesian game** consists of

(i) A set of players $i \in \{1, \cdots, N\}$;

(ii) Action space A_i for each player i;

(iii) Type space $(\Omega, (T_i), \mu_i)$;

(iv) Payoff function for each player i:

$$u_i\left(a_i, a_{-i}, t_i, t_{-i}, \omega\right) \tag{2.8}$$

(v) Strategies for each player i, $\sigma_i : T_i \to \Delta(A_i)$.

Assumption 2.1. Players are assumed to maximize their expected payoffs. Moreover, players calculate their expected payoffs with respect to their interim beliefs.

Definition 2.6. The **expected payoff** of player i with type t_i from a pure strategy $a_i \in A_i$ given opponents' strategies σ_{-i} is

$$U_i\left(a_i, \sigma_{-i}, t_i\right) = \sum_{t_{-i}, \omega} u_i\left(a_i, \sigma_{-i}\left(t_{-i}\right), t_i, t_{-i}, \omega\right) \underbrace{\mu_i\left(t_{-i}, \omega | t_i\right)}_{\text{interim belief}} \tag{2.9}$$

Definition 2.7 (Generalization)**.** The **expected payoff** of player i with type t_i from a mixed strategy $\sigma_i \in \Delta(A_i)$ given opponents' strategies σ_{-i} is

$$U_i\left(\sigma_i, \sigma_{-i}, t_i\right) = \sum_{a_i \in \sigma_i(t_i)} \sigma_i(t_i)(a_i) \sum_{t_{-i}, \omega} u_i\left(a_i, \sigma_{-i}\left(t_{-i}\right), t_i, t_{-i}, \omega\right) \mu_i\left(t_{-i}, \omega | t_i\right) \tag{2.10}$$

Definition 2.8. An action a_i is a **best response** for type t_i against σ_{-i} if

$$\forall a_i' \in A_i \backslash \{a_i\},\ U_i\left(a_i, \sigma_{-i}; t_i\right) \geq U_i\left(a_i', \sigma_{-i}; t_i\right) \tag{2.11}$$

Definition 2.9. An action a_i is **strictly dominated** for type t_i if

$$\exists a_i' \in A_i \backslash \{a_i\}\ s.t.\ \forall \sigma_{-i}\ U_i\left(a_i, \sigma_{-i}; t_i\right) < U_i\left(a_i', \sigma_{-i}; t_i\right) \tag{2.12}$$

Definition 2.10. A **Bayesian Nash equilibrium** is a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_l)$ such that for each player i and each type t_i of player i, the action $\sigma_i\left(t_i\right)$ is a best response for type t_i against σ_{-i}.

3 Adverse Selection

3.1 Market for Lemons

Two types of cars

1. high quality $P(h) = \lambda$

2. low quality $P(l) = 1 - \lambda$

Assuming sellers know the quality of car but buyers don't.

1. b_θ: value of car with type $\theta \in \{h, l\}$ for the buyer.
2. s_θ value for the seller.

Assume

1. $b_h > b_l$ and $s_h > s_l$ (defining what high quality is);
2. $b_h > s_h$ and $b_l > s_l$ (benefit to trade).

Therefore, if the car quality is observed, then it is beneficial to trade, and two prices exist such that

$$b_\theta > p_\theta > s_\theta \ \forall \theta \in \{l, h\} \tag{3.1}$$

If the quality is not observed, only one price p exists.
Seller type h sells if and only if $p > s_h$ and type l if and only if $p > s_l$.
Seller behaviours:

(a) Both types sell: $p > s_h$;

(b) Only type h sells: $s_h > p > s_l$;

(c) Nobody wants to sell: $s_l > p$.

Buyer behaviours:

(a) Both types sell: buyers buy if average quality $\lambda b_{+}(1-\lambda)b_l > p$;

(b) Only type l sells, buyers buy if

$$\exists\ p \in \mathbb{R}_+ \ s.t.\ b_l > p > s_l \tag{3.2}$$

Aggregate conditions

(a) Trade with both types if and only if

$$\exists\ p \in \mathbb{R}_+ \ s.t. \lambda b_h + (1-\lambda)b_l > p > s_h \tag{3.3}$$

That is,

$$\lambda b_h + (1-\lambda)b_l > s_h \quad (\dagger) \tag{3.4}$$

(b) The second case (low quality trading) is always satisfied given assumption $b_l > s_l$.

The **market failure** occurs because high quality cars are withdrawn from the market.

3.2 Market for Insurances

- p denote the price;
- C denote the compensation (paid out if claim);

- D denote the damage;
- Δ denote value of peace of mind;
- $\pi_h > \pi_l$ probability of damage;
- $\lambda = P(h)$.

Buyer behaviours for type $\theta \in \{l, h\}$:

1. Buy: $\Delta - p + \pi_\theta(C - D)$;
2. Don't buy: $\pi_\theta(-D)$.

Buy if and only if

$$\Delta - p + \pi_\theta(C - D) \geq \pi_\theta(-D) \tag{3.5}$$

$$\implies p \leq \Delta + \underbrace{\pi_\theta C}_{\text{expected compensation}} \quad (\dagger\dagger) \tag{3.6}$$

Not that $(\dagger\dagger)$ is independent from D.

Seller behaviour

(a) If both types buy insurance:

$$\pi = \lambda\pi_h + (1 - \lambda)\pi_l \tag{3.7}$$

Sellers sell if

$$p - \pi C \geq 0 \tag{3.8}$$

Aggregated conditions:

$$\exists\, p \in \mathbb{R}_+ \; s.t. \; \Delta + \pi_l C \geq p \geq \pi C \tag{3.9}$$

$$\iff \Delta + \pi_l C > \lambda\pi_h + (1 - \lambda)\pi_l C \tag{3.10}$$

$$\iff \Delta > \lambda(\pi_h - \pi_l)C \tag{3.11}$$

$$\iff \pi_h - \pi_l < \frac{\Delta}{\lambda C} \quad (\dagger) \tag{3.12}$$

$(\dagger)$ says both types of buyers participate in the market and sellers sell insurances if <u>the difference between two types is insignificant</u>.
If $(\dagger)$ is not satisfied, then there is <u>no trade with both type (only one type trades)</u>.

(b) (<u>Price is high enough so that low type does not want to buy</u>) Suppose only type h trades, insurer sells if and only if

$$\pi_h C < p \tag{3.13}$$

And buyer type h wants

$$\Delta + \pi_h C > p \tag{3.14}$$

Given $\Delta > 0$, there would always be trading between sellers and type h buyers.

3.3 Monopoly Under Adverse Selection

Sellers

1. Decision (p, q) where p denotes price and q denotes quality;
2. Profit $\pi(p, q) = p - 1/2q^2$.

Buyers

1. $u_\theta(p, q) = \theta q - p$;
2. θ denotes the taste for quality: $\theta_h > \theta_l$;
3. $P(\theta_h) = \lambda$.

Case 1: 1th degree price discrimination Monopolist knows type θ for each consumer, and design good such that

$$\max_{(p,q)} p - 1/2q^2 \tag{3.15}$$

$$s.t.\ \theta q - p \geq 0 \text{ (Individual Rationality)} \tag{3.16}$$

Solution:

$$q^* = \theta^*,\ p^* = \theta^{*2} \tag{3.17}$$

Case 2: 2nd degree price discrimination Monopolist does not the θ.
Monopolist design a *menu* of products

$$(p_1, q_1), (p_2, q_2), \cdots, (p_j, q_j) \tag{3.18}$$

Provided there are only two types, it is sufficient to construct a menu with two types:

$$(p_l, q_l), (p_h, q_h) \tag{3.19}$$

Note that it is possible that $(p_l, q_l) = (p_h, q_h)$. Further, it is allowed to exclude certain type of consumers by setting $(p_i, q_i) = (0, 0)$.
Monopolist's problem designing an optimal menu such that both types are willing to purchase and each type only buy the bundle designed for them.

$$\max_{(p_l,q_l)\in\mathbb{R}^2_+,(p_h,q_h)\in\mathbb{R}^2_+} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.20}$$

$$\theta_l q_l - p_l \geq 0 \text{ (Individual rationality for low type)} \tag{3.21}$$

$$\theta_h q_h - p_h \geq 0 \text{ (Individual rationality for high type)} \tag{3.22}$$

$$\theta_h q_h - p_h \geq \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \tag{3.23}$$

$$\theta_l q_l - p_l \geq \theta_l q_h - p_h \text{ (Incentive compatibility for low type)} \tag{3.24}$$

Proposition 3.1 (Step 0)**.**

$$\mathrm{IC}_h \wedge \mathrm{IC}_l \implies q_h \geq q_l \tag{3.25}$$

Proof.

$$\mathrm{IC}_h \iff \theta_h(q_h - q_l) \geq p_h - p_l \tag{3.26}$$
$$\mathrm{IC}_l \iff \theta_l(q_l - q_h) \geq p_l - p_h \tag{3.27}$$

Summing two conditions:

$$(q_h - q_l)(\theta_h - \theta_l) \geq 0 \tag{3.28}$$

Provided that $\theta_h > \theta_l$,

$$q_h \geq q_l \tag{3.29}$$

■

Proposition 3.2 (Step 1)**.**

$$\mathrm{IC}_h \wedge \mathrm{IR}_l \implies \mathrm{IR}_h \tag{3.30}$$

Proof.

$$\mathrm{IC}_h \iff \theta_h q_h - p_h \geq \theta_h q_l - p_l \tag{3.31}$$
$$\implies \theta_h q_h - p_h \geq \theta_h q_l - p_l \geq \theta_l q_l - p_l \geq 0 \text{ (By IR for low type)} \tag{3.32}$$
$$\implies \theta_h q_h - p_h \geq 0 \iff \mathrm{IR}_h \tag{3.33}$$

■

Proposition 3.3 (Step 2)**.** Given step 1, IC_h constrain is binding.

Proposition 3.4 (Step 3)**.** IC_h is binding and $q_h \geq q_l$ imply IC_l.

Proof.

$$\text{binding } \mathrm{IC}_h \iff \theta_h(q_h - q_l) = p_h - p_l \tag{3.34}$$
$$\implies \theta_l(q_h - q_l) \leq \theta_h(q_h - q_l) = p_h - p_l \tag{3.35}$$
$$\implies \theta_l(q_h - q_l) \leq p_h - p_l \tag{3.36}$$
$$\iff \theta_l q_l - p_l \geq \theta_l q_h - p_h \tag{3.37}$$
$$\iff \mathrm{IC}_l \tag{3.38}$$

■

Proposition 3.5 (Step 4)**.** IR_l is binding.

Solving the reduced problem

$$\max_{(p_l,q_l)\in\mathbb{R}_+^2,(p_h,q_h)\in\mathbb{R}_+^2} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.39}$$

$$\theta_l q_l - p_l = 0 \text{ (Individual rationality for low type)} \tag{3.40}$$

$$\theta_h q_h - p_h = \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \tag{3.41}$$

The problem reduced to

$$\max_{q_l,q_h\in\mathbb{R}_+} ... \tag{3.42}$$

Solving the problem gives

$$q_h = \theta_h \tag{3.43}$$

$$q_l = \theta_l - \frac{\lambda}{1-\lambda}(\theta_h - \theta_l) \tag{3.44}$$

4 Communication

4.1 Cheap Talk

Remark 4.1 (Difference between communication and signalling)**.** The message is costless in communication setup, payoffs of sender and receiver depend only on their actions and the state of the world, but independent from the message. In contrast, signalling is costly and the payoff of the sender is in general lower if he chooses to send signal.

Definition 4.1. A communication is a **cheap talk** if messages have no cost, no payoff consequences at all.

4.2 Verifiable Talk

Definition 4.2. A communication is a **verifiable talk** if the sender has a full-proof evidence and may choose to show it.