ECO326 Advanced Microeconomic Theory

A Course in Game Theory

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`Github Page` `https://github.com/TianyuDu/Spikey_UofT_Notes`
`Note Page` `TianyuDu.com/notes`

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory*.

Contents

1 Lecture 1. Jan. 7 2019
Games and Dominant Strategies

Game Theory Choice environment where individual choices impact others.

	W	S
W	$(1-c, 1-c)$	$(1-c, 1)$
S	$(1, 1-c)$	$(0, 0)$

Figure 1.1: Payoff Matrix for Example 1

Example 1.1.

Suppose $c \in (0, 1)$. In this game,

i $N = \{i, j\}$,

ii $A_i = A_j = \{W, S\}$,

Definition 1.1 (pg.7)**.** A **preference relation** is a complete reflexive and transitive binary relation.

Definition 1.2 (11.1, lec.1)**.** A **(strategic) game** consists of

i a finite set of **players** N, with $|N| \geq 2$.

ii for each player $i \in N$, an **actions** $A_i \neq \emptyset$.

iii for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i)\rangle$.

Definition 1.3 (Equivalent definition of game)**.** For each player $i \in N$ we can define a utility function, $u_i : A \to \mathbb{R}$ such that

$$\forall (a_i), (a_i)' \in A,\ u_i((a_i)) \geq u_i((a_i)') \iff (a_i) \succsim_i (a_i)' \tag{1.1}$$

So the game can be defined as a triple $\langle N, (A_i), (u_i)\rangle$.

Definition 1.4 (lec.1)**.** An **action profile** is a n-tuple of actions $a_i \in A_i$ for each player $i \in N$ and denoted as

$$(a_i)_{i \in N} \text{ or } (a_i)$$

The **action profile space** is defined as

$$A \equiv \times_{i \in N} A_i$$

Definition 1.5 (lec.1)**.** Action $a_i \in A_i$ is **strictly dominated** by action $\tilde{a}_i \in A_i$ if

$$\forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) < u_i(\tilde{a}_i, a_{-i})$$

And a_i is **weakly dominated** by $\tilde{a}_i$ if

$$\forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) \leq u_i(\tilde{a}_i, a_{-i})$$

and

$$\exists a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) < u_i(\tilde{a}_i, a_{-i})$$

Corollary 1.1 (Consequence of RCT)**.** It is irrational to play strictly dominated actions. So rational choice theory suggests a player would never play strictly dominated strategies.

Definition 1.6. Action $a_i \in A_i$ is **strictly dominant** if it strictly dominates all other actions.

Definition 1.7. Action $a_i \in A_i$ is **weakly dominant** if it weakly dominates all other actions.

Definition 1.8. Action $a_i \in A_i$ is **weakly/strictly dominated** if there exists another strategy weakly/strictly dominates a_i.

	S	C
S	(-1, -1)	(-10, 0)
C	(0, -10)	(-5, -5)

Figure 1.2: Payoff matrix for example 2

Example 1.2 (Prisoner Dilemma)**.** Note that S is strictly dominated by C. Therefore C is strictly dominant for both players.

	L	C	R
U	(2, 2)	(5, 0)	(3, 0)
M	(2, 7)	(2, 5)	(2, 6)
D	(5, 3)	(4, 2)	(3, 1)

Figure 1.3: Payoff matrix for example 2

Example 1.3. So in this game, for player 2, L is strictly dominant.
For player 1, M is strictly dominated by D. And M is weakly dominated by U.

Example 1.4. There are three candidates, $\{A, B, C\}$. And there are 50 players (voters, note that $\emptyset \notin A_i$ since they must vote). And

$$\forall i \in N,\ A_i = \{A, B, C\}$$

Each individual has strictly preference over A, B, C. If tie is encountered, randomization would be taken.

i $A \succ B \succ C$,

ii $A \succ AC_{tie} \succ C$

Claim 1: There are no weakly or strictly dominant actions.

Proof. Let $a_i \in \{V_A, V_B, V_C\}$ denote the action taken by player $i \in N$,
Note that weak dominance is a necessary condition for strict dominance,
So above claim is reduced to *there are no weakly dominant actions.*
The reduced claim is equivalent to the following statement,

$$\forall a_i \in A_i,\ \exists \tilde{a}_i \in A_i\ s.t.\ a_i \neq \tilde{a}_i$$
$$s.t.\ \exists a_{-i} \in A_{-i}\ s.t.\ u_i(a_i, a_{-i}) > u_i(\tilde{a}_i, a_{-i}) \vee \forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) = u_i(\tilde{a}_i, a_{-i})$$

Let n^j_{-i} denote the number of voters other than i voting for candidate j. Clearly each $a_{-i} \in A_{-i}$ would induce an outcome as a triple $(n^A_{-i}, n^B_{-i}, n^C_{-i})$.
Consider action V_A, and a_{-i} induces

$$(n^A_{-i}, n^B_{-i}, n^C_{-i}) = (1, 24, 24)$$

then

$$(V_B, a_{-i}) \succ_i (V_A, a_{-i})$$

So V_A failed to be a dominant strategy of any kind.
Similarly, consider action V_B, if a_{-i} induces

$$(n^A_{-i}, n^B_{-i}, n^C_{-i}) = (24, 1, 24)$$

then

$$(V_A, a_{-i}) \succsim_i (V_B, a_{-i})$$

So V_B failed to be a dominant strategy.
Similarly, consider action V_C, if a_{-i} induces

$$(n^A_{-i}, n^B_{-i}, n^C_{-i}) = (24, 24, 1)$$

then

$$(V_A, a_{-i}) \succsim_i (V_C, a_{-i})$$

So V_B failed to be a dominant strategy. ■

Claim 2: Only voting for your least preferred candidate is weakly dominated.

Proof. We are going to show there exists a strategy (voting for B) weakly dominates voting for C.

Vote A	Cases	Vote C
A	$n^A_{-i} > n^B_{-i}, n^C_{-i}$	A, AC
B	$n^B_{-i} > n^A_{-i}, n^C_{-i}$	B, BC
C, BC	$n^C_{-i} > n^A_{-i}, n^B_{-i}$	C
B	$n^A_{-i} = n^B_{-i} > n^C_{-i}$	AB
A	$n^A_{-i} = n^C_{-i} > n^B_{-i}$	C
BC	$n^C_{-i} = n^B_{-i} > n^A_{-i}$	C

Figure 1.4: Voting for A versus Voting for C

■

Definition 1.9 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2 Lecture 2. Jan. 14 2019
Iterated Elimination and Rationalizability

Example 2.1 (Bubble Game)**.** Consider a player game

$$\langle N, (A_i), (u_i) \rangle \tag{2.1}$$

where

$$A_i = \{0, \dots, 100\},\ \forall i \tag{2.2}$$

and

$$u_i(a_i, a_{-i}) = a_i - penalty_i(a_i, a_{-i}) \tag{2.3}$$

$$penalty_i = \begin{cases} 0 \text{ if } a_i < \max_{j \neq i} a_j - 1 \\ 10(a_i - \max_{j \neq i} a_j + 1) \text{ if } a_i \geq \max_{j \neq i} -1 \end{cases} \tag{2.4}$$

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (IESD)**.** Given the initial game,

$$G_0 = \langle N, (A_i^0), (u_i) \rangle$$

At stage $k \in \mathbb{N}$,

$$G_k = \langle N, (A_i^k), (u_i) \rangle$$

In stage k, for all $i \in N$, find the set of strictly dominated actions, $D_i^k \subsetneq A_i^k$.

i) If $\forall i \in N\ s.t.\ D_i^k = \emptyset$, conclude the profile

$$(A_i^k)$$

to be the set of action profiles survive from IESD.

ii) If $\exists i \in N\ s.t.\ D_i^k \neq \emptyset$, define

$$\forall i \in N,\ A_i^{k+1} := A_i^k \backslash D_i^k$$

Example 2.2. Action profile (M, R) survives the IESD.

Proof.

$$\begin{aligned} k = 0,\ A_1^0 = \{U, M, D\},\ A_2^0 = \{L, R\} \\ k = 1,\ A_1^1 = \{U, M\},\ A_2^1 = \{L, R\} \\ k = 2,\ A_1^2 = \{U, M\},\ A_2^2 = \{R\} \\ k = 3,\ A_1^3 = \{M\},\ A_2^3 = \{R\} \end{aligned}$$

∎

Example 2.3 (Hotelling Model of Politics)**.** Players maximize their votes by choosing where to stand along a natural number line.

	L	R
U	4,0	2,2
M	1, 2	5,3
D	0,5	1,4

Figure 2.1: Game for Example 2.1

- Player $N = \{1, 2\}$
- Action set $A_i = \{1, \dots, M\}$, with $2 \nmid M$ and $M > 3$.
- Payoff

$$u_i(a_i, a_{-i}) = \begin{cases} a_i + \frac{1}{2}(a_{-i} - a_i - 1) \text{ if } a_i < a_{-i} \\ \frac{M}{2} \text{ if } a_i = a_{-i} \\ M - [a_{-i} + \frac{1}{2}(a_i - a_{-i} - 1)] \text{ if } a_i > a_{-i} \end{cases} \tag{2.5}$$

Claim i. $a_i = 1$ is strictly dominated by $a_i = 2$.

Proof.

$$u_i(a_i = 1, a_{-i}) = \begin{cases} \frac{M}{2} \text{ if } a_{-i} = 1 \\ \frac{a_{-i}}{2} \text{ if } a_{-i} > 1 \end{cases} \tag{2.6}$$

$$u_i(a_i = 2, a_{-i}) = \begin{cases} M - 1 \text{ if } a_{-i} = 1 \\ \frac{M}{2} \text{ if } a_{-i} = 2 \\ \frac{a_{-i}}{2} + \frac{1}{2} \text{ if } a_{-i} > 2 \end{cases} \tag{2.7}$$

■

Claim ii. $\lfloor \frac{n}{2} \rfloor + 1$ is the only action survives.

Proof. Similarly, we can eliminate all edge-values iteratively. ■

Definition 2.2. For each $i \in N$, the **best-response function** of this player is a correspondence $B_i : A_{-i} \rightrightarrows A_i$ defined as

$$B_i(a_{-i}) := \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \ \forall a_i' \in A_i\} \tag{2.8}$$

Definition 2.3. A **belief** of player i (about the actions of the other players) is a probability measure, α_i, on $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$.
α_i is a mapping such that

- $\alpha_i : A_{-i} \to [0, 1]$.
- $\alpha_i(A_{-i}) = 1$.
- For all countable piece-wise disjoint collection

  $$\{E_j\}_{j \in \mathcal{J}} \in \mathcal{P}(A_{-i})$$

  α_i satisfies the *countable additivity property*:

  $$\alpha_i(\bigcup_{i \in I} E_i) = \sum_{i \in I} \alpha_i(E_i)$$

Definition 2.4. a_i is a **best response** to the belief α_i if

$$\forall a_i' \in A_i,\ \sum_{a_{-i}} u_i(a_i, a_{-i})\alpha_i(a_{-i}) \geq \sum_{a_{-i}} u_i(a_i', a_{-i})\alpha_i(a_{-i}) \tag{2.9}$$

or, more generally,

$$\forall a_i' \in A_i, \mathbb{E}[u_i(a_i, a_{-i})|\alpha_i] \geq \mathbb{E}[u_i(a_i', a_{-i})|\alpha_i] \tag{2.10}$$

Definition 2.5. $a_i \in A_i$ is a **never best response** if it is not a best response given any belief α_i.

Corollary 2.1. *Iterative Elimination of Never Best Response*: same procedures but D_i^k is the set of never best responses for player i at game G^k.

Example 2.4. For player 1, D is not strictly dominated, but it is a never best response.

Proof. Let α be a probability measure on $\{L, R\}$ such that $\alpha(L) = p \in [0, 1]$.

$$\mathbb{E}[u_1|U, \alpha] = 10p \tag{2.11}$$

$$\mathbb{E}[u_1|M, \alpha] = 10 - 10p \tag{2.12}$$

$$\mathbb{E}[u_1|D, \alpha] = 1 \tag{2.13}$$

Case i

$$p \geq 0.5 \implies \mathbb{E}[u_1|U, \alpha] \geq 5 \tag{2.14}$$

Case ii

$$p < 0.5 \implies \mathbb{E}[u_1|M, \alpha] > 5 \tag{2.15}$$

Therefore, for any belief α, D cannot be a best response. So D is a never best response. ■

	L	R
U	10,0	0,0
M	0,0	10,0
D	1,0	1,0

Definition 2.6. An action $a_i \in A_i$ is **rationalizable** if it survives *iterative elimination of never best responses*.

Lemma 2.1 (i385.3)**.** In a two player game, a_i is strictly dominated if and only if it is a never best response.

Assumption 2.1 (Common knowledge rationality)**.** We assume our players of game all acknowledge that other players are playing the game in rational ways.

3 Lecture 3. Jan. 21 2019

Definition 3.1. A **pure strategy Nash equilibrium** is a strategy profile (a_i) such that

$$\forall i \in N,\ a_i \in Br_i(a_{-i}) \tag{3.1}$$

Remark 3.1. That's, a NE is a situation that if player i knows what other players do, the action given by the NE profile is still a best response.

Remark 3.2 (Interpretations)**.** A Nash equilibrium is

i) An action profile induces a stable outcome,

ii) A creditable agreement, such that no player has incentive to break the agreement.

Example 3.1 (Cournot Duopoly)**.** Consider a game with

i) Player $N = \{1, 2\}$

ii) Action set $A_i = [0, \infty)\ \forall i \in N$

And revenue R_i defined by

$$R_i = p_i q_i \tag{3.2}$$

where price is linear in quantity supplied,

$$p_i = \alpha - (q_i + q_{-i}),\ \ \alpha \in \mathbb{R} \tag{3.3}$$

and firms face fixed cost $c \in \mathbb{R}$, with the assumption that $\alpha > c$. So the cost function is given by

$$C_i(q_i) = cq_i \tag{3.4}$$

The profit function is given by

$$\begin{aligned} \Pi_i(q_i, q_{-i}) &= (\alpha - (q_i + q_{-i}) - c)q_i &(3.5)\\ &= (\alpha - c - q_{-i})q_i - q_i^2 &(3.6)\end{aligned}$$

Given $q_{-i} \in A_{-i}$, the best response is given by

$$\begin{aligned} Br_i(q_{-i}) &= \text{argmax}_{q_i \in [0,\infty)} \Pi_i(q_i, q_{-i}) &(3.7)\\ &= \max\{0, \frac{\alpha - c - q_{-i}}{2}\}\ \forall i \in N &(3.8)\end{aligned}$$

Considering the case that both players are producing positive quantities, we can solve q_i^* by

$$\begin{aligned} Br_i \circ Br_{-i}(q_i) &= \frac{\alpha - c - \frac{\alpha - c - q_i}{2}}{2} = q_i &(3.9)\\ \implies 2q_i - \frac{q_i}{2} &= \frac{\alpha - c}{2} &(3.10)\\ \implies q_i^* &= \frac{\alpha - c}{3} &(3.11)\end{aligned}$$

Remark 3.3. If fixed cost presents, even if the game is symmetric, the Nash equilibrium could be asymmetric. (e.g. one firm is out of market and the other firm produces the monopoly amount)

Remark 3.4. Nash equilibrium induces an *individual level optimality* instead of the common wealth optimality.

Example 3.2 (Continue Cournot Duopoly)**.** Note that in the Cournot Duopoly game, for each $i \in N$, the Nash equilibrium profit is

$$\Pi^*_{NE} = (\alpha - c - \frac{2(\alpha - c)}{3})\frac{\alpha - c}{3} \tag{3.12}$$
$$= \frac{(\alpha - c)^2}{9} \tag{3.13}$$

So the total profit for the two firms is $\frac{2(\alpha-c)^2}{9}$.
Now considering if the two firms form a Cartel, the *aggregate* quantity produced is

$$Q^* = \text{argmax}_{Q \in \mathbb{R}_{\geq 0}}(\alpha - c - Q)Q \tag{3.14}$$
$$= \frac{\alpha - c}{2} \tag{3.15}$$
$$\implies \Pi^*_{Cartel} = (\alpha - c - \frac{\alpha - c}{2})\frac{\alpha - c}{2} \tag{3.16}$$
$$= \frac{(\alpha - c)^2}{4} > \frac{2(\alpha - c)^2}{9} \tag{3.17}$$

The fact $\Pi^*_{Cartel} > 2 \times \Pi^*_{NE}$ suggests the Nash equilibrium action profile did not induce the optimal common wealth outcome. However, the Cartel action profile is not a stable outcome since every player has incentive to increase their production level.

Example 3.3 (Prisoner's Dilemma)**.** The Nash equilibrium(Confess, Confess) is not the best outcome for the two players as a group. The optimal action profile for a group is (Silent, Silent).

Proposition 3.1. No strategy that is eliminated during iterated elimination of *never best response* can be played in a Nash equilibrium.

4 Lecture 4. Jan. 28. 2019

Example 4.1 (From last lecture)**.** Consider the payoff matrix Both (A, A) and (B, B) are Nash

	A	B
A	(1,1)	(0,0)
B	(0,0)	(0,0)

equilibria. But in the former NE, for each $i \in N$,

$$Br_i(a_{-i} = A) = \{A\} \tag{4.1}$$

which is a singleton.
In the second NE, for each $i \in N$,

$$Br_i(a_{-i} = B) = \{A, B\} \tag{4.2}$$

We call Nash equilibria of the first type *strict Nash equilibria* and the later one *weak Nash equilibria.* More formal definitions of these two types of Nash equilibria are given below.

Definition 4.1. A **strict Nash equilibrium** is an action profile (a_i) such that

$$\forall i \in N, \ |Br_i(a_{-i})| = 1$$

Definition 4.2. A **weak Nash equilibrium** is a Nash equilibrium that is not strict. That's, a weak Nash equilibrium is an action profile (a_i) such that

$$\exists i \in N,\ |Br_i(a_{-i})| > 1$$

Example 4.2 (Cournot with n firms)**.** Consider the game

$$\langle N, (\mathbb{R}_{\geq 0}), (\pi_i) \rangle \tag{4.3}$$

Where $|N| = n$ and each firm picks a quantity $q_i \in A_i \equiv \mathbb{R}_{\geq 0}$.
Define

$$Q \equiv \sum_j q_j \tag{4.4}$$

For each $i \in N$, define

$$Q_{-i} \equiv \sum_{j \neq i} q_j \tag{4.5}$$

And the market has *linear demand curve*

$$P(\{q_i\}_{i \in N}) = \alpha - \sum_j q_j = \alpha - Q \tag{4.6}$$

And firms face fixed cost

$$\forall i \in N,\ C(q_i) = cq_i \text{ where } 0 < c < \alpha \tag{4.7}$$

The profit function is

$$\forall i \in N,\ \pi_i(q_i, Q_{-i}) = (\alpha - c - Q_{-i})q_i - q_i^2 \tag{4.8}$$

Question What are the Nash equilibria in this environment?
For each firm $i \in N$, the best response correspondence is

$$Br_i(Q_{-i}) = \max\{0, \operatorname*{argmax}_{q_i \in \mathbb{R}_{\geq 0}} \pi_i(q_i, Q_{-i})\} \tag{4.9}$$

$$= \max\{0, \frac{\alpha - c - Q_{-i}}{2}\} \tag{4.10}$$

Assume We have a symmetric Nash equilibrium,

$$\forall i \in N,\ q_i^* = q^* = \frac{Q}{n} \tag{4.11}$$

$$\implies q^* = Br_i(\frac{n-1}{n}Q) \tag{4.12}$$

$$\implies 2q^* = \alpha - c - (n-1)q^* \tag{4.13}$$

$$\implies q^* = \frac{\alpha - c}{n+1} \tag{4.14}$$

Check Then check the validity of symmetric Nash equilibrium by asserting for every player, if all other players are playing the action suggested by the symmetric Nash equilibrium, then this player should also play it. That's

$$\begin{aligned}
q^* &= Br_i(\{q_j\}_{j\neq i}) && (4.15)\\
&= \frac{\alpha - c}{2} - \frac{1}{2}\frac{n-1}{n+1}(\alpha - c) && (4.16)\\
&= \frac{1}{2}((\alpha - c) - \frac{n-1}{n+1}(\alpha - c)) && (4.17)\\
&= \frac{1}{2}\frac{2}{n+1}(\alpha - c) && (4.18)\\
&= \frac{\alpha - c}{n+1} = q^* && (4.19)
\end{aligned}$$

Uniqueness We are going to show the symmetric Nash equilibrium is the only possible equilibrium action profile.

Proof. Suppose there exists some non-symmetric Nash equilibrium.
For concreteness, assuming there exists $\epsilon > 0$ such that

$$\exists i, j \in N,\ q_i = q_j + \epsilon \tag{4.20}$$

Define $Q_{-ij} \equiv Q - q_i - q_j$.
For firm i,

$$\begin{aligned}
q_i = Br_i(Q_{-i}) &= \frac{1}{2}(\alpha - c - Q_{-i}) && (4.21)\\
&= \frac{1}{2}(\alpha - c - Q_{-ij} - q_j) && (4.22)\\
&= \frac{1}{2}(\alpha - c - Q_{-ij} - q_i + \epsilon) && (4.23)\\
\implies 3q_i &= \alpha - c - Q_{-ij} + \epsilon && (4.24)\\
\implies 3q_i &= \alpha - c - Q_{-j} + q_i + \epsilon && (4.25)\\
\implies 2q_i - \epsilon &= \alpha - c - Q_{-j} && (4.26)\\
\implies q_i - \frac{\epsilon}{2} &= Br(Q_{-j}) = q_j && (4.27)\\
&= q_i - \epsilon && (4.28)\\
\implies \epsilon &= 2\epsilon && (4.29)
\end{aligned}$$

which contradicts the assumption that $\epsilon > 0$.
Therefore we conclude that the symmetric Nash equilibrium is the only Nash equilibrium in this environment. ■

Example 4.3 (Cournot duopoly with fixed cost)**.** Consider the game

$$\mathcal{G} = \langle N = (1, 2), (A_i = \mathbb{R}_{\geq 0}), (\pi_i)\rangle \tag{4.30}$$

Where the cost function is defined as

$$C_i(q_i) = \begin{cases} cq_i + f & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases} \tag{4.31}$$

where $\alpha > c > 0$.
For each firm $i \in N$, the best response function, conditioned on $q_i > 0$, is

$$q_i = Br_i(q_{-i}) = \frac{\alpha - c - q_{-i}}{2} \tag{4.32}$$

We have to **check the profit** to assert the profit is non-negative while firm i is producing above quantity. Because, otherwise, this firm could always derivate to $q_i = 0$ to avoid loss (earning zero profit).

$$\pi_i(Br_{q_{-i}}, q_{-i}) = (\alpha - q_{-i} - \frac{\alpha - c - q_{-i}}{2} - c)\frac{\alpha - c - q_{-i}}{2} - f \tag{4.33}$$

$$= \frac{(\alpha - c - q_{-i})^2}{4} - f \tag{4.34}$$

$$\pi_i(q_{-i}) \geq 0 \iff \alpha - c - q_{-i} \geq 2\sqrt{f} \tag{4.35}$$

$$\iff q_{-i} \leq \alpha - c - 2\sqrt{f} \tag{4.36}$$

Therefore we can modify our **best response correspondence** to

$$Br_i(q_{-i}) = \begin{cases} \frac{\alpha - c - q_{-i}}{2} & \text{if } q_{-i} \leq \alpha - c - 2\sqrt{f} \\ 0 & \text{if } q_{-i} \geq \alpha - c - 2\sqrt{f} \end{cases} \tag{4.37}$$

Monopoly: the monopoly amount, conditioned on the firm decides to produce, is $\frac{\alpha - c}{2}$. In the monopoly case, suppose the NE action profile is (the opposite case can be shown by symmetry)

$$(\frac{\alpha - c}{2}, 0) \tag{4.38}$$

We have to assert both

$$\begin{cases} \frac{\alpha - c}{2} \in Br_1(0) \\ 0 \in Br_2(\frac{\alpha - c}{2}) \end{cases} \tag{4.39}$$

$$\implies \begin{cases} 0 \leq \alpha - c - 2\sqrt{f} & \text{for firm 1 to produce positive amount.} \\ \frac{\alpha - c}{2} \geq \alpha - c - 2\sqrt{f} & \text{for firm 2 to produce zero.} \end{cases} \tag{4.40}$$

$$\implies 0 \leq \alpha - c - 2\sqrt{f} \leq \frac{\alpha - c}{2} \tag{4.41}$$

$$\implies -(\alpha - c) \leq -2\sqrt{f} \leq -\frac{\alpha - c}{2} \tag{4.42}$$

$$\implies \sqrt{f} \in [\frac{\alpha - c}{4}, \frac{\alpha - c}{2}] \tag{4.43}$$

Positive symmetric equilibrium: we've proven, in the general case, it's impossible for both firms to produce positive but different amounts. Therefore we have to assert

$$\frac{\alpha - c}{3} \in Br_i(\frac{\alpha - c}{3}) \; \forall i \in N \tag{4.44}$$

$$\implies \frac{\alpha - c}{3} \leq \alpha - c - 2\sqrt{f} \tag{4.45}$$

$$\implies \sqrt{f} \leq \frac{1}{3}(\alpha - c) \tag{4.46}$$

Zero symmetric equilibrium: we have to assert

$$
\begin{aligned}
& 0 \in Br_i(0)\ \forall i \in N && (4.47)\\
\implies\ & 0 \geq \alpha - c - 2\sqrt{f} && (4.48)\\
\implies\ & \sqrt{f} \geq \frac{\alpha - c}{2} && (4.49)
\end{aligned}
$$

Example 4.4 (Discrete price Bertrand duopoly)**.** Consider the following game

$$
\mathcal{G} = \langle N = \{1,2\}, (A_i = \{k\epsilon : k \in \mathbb{Z}_{\geq 0}\}), (\pi_i)\rangle \tag{4.50}
$$

The profit function can be derived as

$$
\pi_i(p_i, p_{-i}) = \begin{cases} (\alpha - p_i)(p_i - c) & \text{if } p_i < p_{-i} \\ \frac{\alpha - p_i}{2}(p_i - c) & \text{if } p_i = p_{-i} \\ 0 & \text{if } p_i > p_{-i} \end{cases} \tag{4.51}
$$

Claim the only Nash equilibria are (c,c) and $(c+\epsilon, c+\epsilon)$.
Justify (c,c), consider any firm $i \in N$, currently $\pi_i = 0$

$$
\begin{aligned}
\uparrow p_i \implies \pi_i \leftarrow \pi_i' = 0\ ✗ && (4.52)\\
\downarrow p_i \implies \pi_i \leftarrow \pi_i' < 0\ ✗ && (4.53)
\end{aligned}
$$

So no firm has incentive to derivate from this action profile, so (c,c) is a Nash equilibrium by definition.
Consider the action profile $(c+\epsilon, c+\epsilon)$, both firms are earning a positive profit $\pi_i > 0$. For any firm $i \in N$,

$$
\begin{aligned}
\uparrow p_i \implies \pi_i \leftarrow \pi_i' = 0\ ✗ && (4.54)\\
\downarrow p_i \implies \pi_i \leftarrow \pi_i' = 0\ ✗ && (4.55)
\end{aligned}
$$

So $(c+\epsilon, c+\epsilon)$ is a Nash equilibrium by definition.
No other Nash equilibrium
Claim: (p_1, p_2) cannot be a Nash equilibrium if any $p_i < c$.
Obviously, set $p_i < c$ would induce negative profit and firm i do better off by setting $p_i \leftarrow c$.
Claim: the symmetric profit (p,p) with $p > c + \epsilon$ cannot be a Nash equilibrium. For both firms, the current profit is

$$
\pi_1 = \pi_2 = \frac{1}{2}(\alpha - c - k\epsilon)k\epsilon \tag{4.56}
$$

And reducing price by ϵ leads to a profit of

$$
\pi_i' = (\alpha - c - k\epsilon + \epsilon)(k-1)\epsilon > \pi_i \tag{4.57}
$$

so such action profile cannot be Nash equilibrium.
Claim (p_1, p_2) with $p_1 \neq p_2$ and $p_1, p_2 > c$ cannot be a Nash equilibrium. The firm charges higher price can always reduce it's price to the price charged by the other firm and gain a positive profit.
Claim (c, p_2) with $p_2 \geq c + \epsilon$ cannot be a Nash equilibrium. The firm charging $p = c$ can always increase its price to $c + \epsilon$ to earn positive profit.
Therefore there's no Nash equilibrium other than (c,c) and $(c+\epsilon, c+\epsilon)$.

Example 4.5 (Bertrand duopoly with differentiated products)**.** Consumers are uniformly distributed on a preference line $[0, 1]$.
For a firm $i \in N$, let $x_i \in [0, 1]$ measure consumer's preference towards firm i's products.
Define

$$x_{-i} \equiv 1 - x_i \sim Unif(0, 1) \tag{4.58}$$

Consumer buy product i if

$$x_i - p_i \geq x_{-i} - p_{-i} \tag{4.59}$$

and purchases product 1 if

$$x_i - p_i \leq x_{-i} - p_{-i} \tag{4.60}$$

Then solve the boundary $x^* \in [0, 1]$ such that consumers at x^* are indifferent between two products.

$$x + p_i = (1 - x) + p_{-i} \tag{4.61}$$

$$\implies x^* = \frac{1 - p_i + p_{-i}}{2} \tag{4.62}$$

So any consumer with $x_i > x^*$ would choose firm i's product, also because consumers are uniformly distributed on $x_i \in [0, 1]$. So the portion of consumers buying firm i's product is

$$1 - x_i = \frac{1 + p_i - p_{-i}}{2} \tag{4.63}$$

And, clearly, The demand function for firm i is

$$D_i(q_i, q_{-i}) = \begin{cases} 0 & \text{if } p_i \geq p_{-i} + 1 \\ \frac{1+p_i-p_{-i}}{2} & \text{if } p_i \in [p_{-i} - 1, p_{-i} + 1] \\ 1 & \text{if } p_i \leq p_{-i} - 1 \end{cases} \tag{4.64}$$

Consider the case where $|p_i - p_{-i}| \leq 1$,

$$\pi_i = D_i(p_i)(p_i - c) = \frac{1 + p_i - p_{-i}}{2}(p_i - c) \tag{4.65}$$

Take the first order condition

$$\frac{\partial \pi_i}{\partial p_i} = 0 \tag{4.66}$$

$$\implies p_i = \frac{p_{-i} + c - 1}{2} \tag{4.67}$$

5 Lecture 5 Feb. 7 2019

Example 5.1. Matching pennies: an example of game without pure strategy Nash equilibrium.

	H	T
H	(1,-1)	(-1,1)
L	(-1,1)	(1,-1)

Definition 5.1. Suppose player i has a *finite* set of pure strategies A_i, then a **mixed strategy** $\sigma_i \in \Delta(A_i)$ is a probability distribution over A_i.

Definition 5.2. The **support** of σ_i is defined as

$$\mathcal{S}(\sigma_i) \equiv \{a \in A_i : \sigma_i(a) > 0\} \tag{5.1}$$

Notation 5.1. Given action set $A_i \equiv \{a_i\}$, a mixed strategy can be denoted as A_i^α where α is a multi-index.

Remark 5.1. A pure strategy $a_i \in A_i$ is a mixed strategy with

$$\sigma_i(a_i) = 1 \tag{5.2}$$

So mixed strategy is a generalization of pure strategy.

Proposition 5.1. In a finite game, given the independence of randomization, the probability of the action profile $a = (a_i)$ to be realized given mixed strategy profile (σ_i) is

$$\sigma(a) \equiv \mathbb{P}[(a_j)_{j\in N}] = \prod_{j\in N} \sigma_j(a_j) \tag{5.3}$$

and for player i, the **expected payoff** on profile $(\sigma_j)_{j\in N}$ is

$$U_i((\sigma_j)_{j\in N}) = \sum_{a\in A}(\prod_{j\in N}\sigma_j(a_j))u_i(a) = \mathbb{E}[u_i(a)|(\sigma_j)_{j\in N}] \tag{5.4}$$

Proposition 5.2. The **expected payoff** from mixed strategy profile $(\sigma_i) \equiv (\sigma_i, \sigma_{-i})$ is

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}[u_i(a)|(\sigma_i)] = \sum_{a_i\in A_i}\sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i})\sigma_{-i}(a_{-i})\sigma_i(a_i)$$

CONTENTS NOT COVERED IN LECTURE

Definition 5.3 (60.2)**.** The set $X \subseteq A$ of outcomes of a finite strategic game $\langle N, (A_i), (u_i)\rangle$ **survives iterated elimination of strictly dominated actions** if $X = \times_{j\in N} X_j$ and there is a collection $((X_j^t)_{j\in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \dots, T-1$.
- For each $t = 0, \dots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\langle N, (X_i^t), (u_i^t)\rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j\in N} X_j^t$.
- No action in X_t^T is strictly dominated in game $\langle N, (X_i^T), (u_i^T)\rangle$.

Proposition 5.3 (61.2)**.** If $X = \times_{j\in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\langle N, (A_i), (u_i)\rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

5.1 Rationalizability

Definition 5.4 (pg.15)**.** The **best-response function** for a player i is defined as

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i})\ \forall a_i' \in A_i\}$$

Remark 5.2. The best-response of a_{-i} can be written as

$$B_i(a_{-i}) = \bigcap_{a_i' \in A_i} \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i})\}$$

where each of them is the upper contour set of a_i'.
Thus, if $\succsim_i$ is quasi-concave, then $B_i(a_{-i})$ is an intersection of convex sets and therefore itself convex.

Definition 5.5 (pg.54)**.** A **belief** of player i (about the actions of the other players) is a probability measure, μ_i, on $A_{-i} = \times_{j\in N\backslash\{i\}} A_j$.
μ_i is a mapping such that

- $\mu_i : A_{-i} \to [0,1]$.
- $\mu_i(A_{-i}) = 1$.
- For all countable piece-wise disjoint collection $\{E_i\}_{i\in I}$, it satisfies the *countable additivity property*:

$$\mu_i(\bigcup_{i\in I} E_i) = \sum_{i\in I} \mu_i(E_i)$$

Definition 5.6 (lec.2)**.** For a player $i \in N$, $a_i^* \in A_i$ is the **best response to belief** μ_i in a strategic game $\langle N, (A_i), (u_i)\rangle$ if and only if

$$\forall a_i \in A_i,\ \sum_{a_{-i}\in A_{-i}} u_i(a_i^*, a_{-i})\mu_i(a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i})\mu_i(a_{-i})$$

Equivalently,

$$\forall a_i \in A_i,\ \mathbb{E}[u_i(a_i^*, a_{-i})|\mu_i] \geq \mathbb{E}[u_i(a_i, a_{-i})|\mu_i]$$

Definition 5.7 (59.1)**.** An action of player i in a strategic game is a **never best response** if it is not a best response to any belief of player i.

Definition 5.8 (lec.2)**.** For player $i \in N$, action $a_i \in A_i$ is **rationalizable** if it survives from the iterated elimination of never best responses.

Definition 5.9 (59.2)**.** The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **strictly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

6 Lecture 3. Nash Equilibrium

Definition 6.1 (14.1)**.** A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $a^* \in A$ of actions with property that for every player $i \in N$

$$(a_i^*, a_{-i}^*) \succsim_i (a_i, a_{-i}^*) \ \forall a_i \in A_i$$

Proposition 6.1 (pg.15, equivalent definition of Nash equilibrium)**.** So a Nash equilibrium is a profile $a^* \in A$ such that

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

Proposition 6.2 (lec.3)**.** No strategy that is eliminated during iterated deletion of never best response can be played in Nash equilibrium.

Lemma 6.1 (pg.19)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if equivalent to the following statement:
Define set-valued function $B : A \to A$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and there exists $a^* \in A$ such that $a^* \in B(a^*)$.

Lemma 6.2 (20.1 Kakutani's fixed point theorem)**.** Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- for all $x \in X$ the set $f(x)$ is non-empty and convex.
- the graph of f is closed. *(i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$ then $y \in f(x)$)*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Definition 6.2 (pg.20)**.** A preference relation $\succsim_i$ over A is quasi-concave on A_i if for every $a^* \in A$ the upper contour set over a_i^*, given other players' strategies

$$\{a_i \in A_i : (a_{-i}^*, a_i) \succsim_i a^*\}$$

is convex.

Proposition 6.3 (20.3)**.** The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if for all $i \in N$,

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous
- quasi-concave on A_i.

Proof. Let $B : A \to A$ be a correspondence defined as

$$B(a) := \times_{i \in N} B_i(a_{-i})$$

Note that for each $a \in A$ and for each $i \in N$,
$B_i(a_{-i}) \neq \emptyset$ since preference $\succsim_i$ is continuous and A_i is compact (EVT).
Also $B_i(a_{-i})$ is convex since it's basically an intersection of upper contour sets and each of those upper contour is convex since $\succsim_i$ is quasi-concave.
So the Cartesian product of the finite collection of B_i is non-empty and convex.
Also the graph B is closed since $\succsim_i$ is continuous.
So there exists $a^* \in A$ such that $a^* \in B(a^*)$.
So Nash equilibrium presents. ∎

Definition 6.3 (lec.3)**.** A **strict Nash equilibrium** is an action profile $a^* \in A$ where all players are playing their unique best response. That is, for every player $i \in N$, the image of their best response $B_i(a^*_{-i})$ is singleton,

$$\forall i \in N \;\; B_i(a^*_{-i}) = \{a_i^*\}$$

Definition 6.4 (lec.3)**.** Otherwise, a Nash equilibrium is a **weak Nash equilibrium**.

7 Lecture 4. Nash Equilibrium: Examples

8 Lecture 5. Mixed Strategies

Notation 8.1 (pg.32)**.** Let $\Delta(A_i)$ denote the set of probability measures/distributions on set A_i.

Definition 8.1 (lec.5)**.** For player $i \in N$, a **mixed strategy** σ_i is a member in $\Delta(A_i)$ and it is a probability distribution over A_i.

Remark 8.1 (lec.5)**.** A pure strategy $a_i \in A_i$ is a mixed strategy with

$$\sigma_i(a_i) = 1$$

So mixed strategy is a generalization of pure strategy.

Definition 8.2 (pg.32)**.** A profile $(\sigma_j)_{j \in N}$ of mixed strategies induces a probability distribution over the set A.

Proposition 8.1 (pg.32)**.** In a finite game,(i.e., each A_i is finite), then given the independence of randomization, the probability of the action profile $a = (a_j)_{j \in N}$ to be realized given mixed strategy profile $(\sigma_j)_{j \in N}$ is

$$Pr((a_j)_{j \in N}) = \prod_{j \in N} \sigma_j(a_j)$$

and for player i, the **expected payoff** on profile $(\sigma_j)_{j\in N}$ is

$$U_i((\sigma_j)_{j\in N}) = \sum_{a\in A}(\prod_{j\in N}\sigma_j(a_j))u_i(a) = \mathbb{E}[u_i(a)|(\sigma_j)_{j\in N}]$$

Proposition 8.2 (lec.5, equivalent)**.** The **expected payoff** from mixed strategy profile $(\sigma_i) \equiv (\sigma_i, \sigma_{-i})$ is

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}[u_i(a)|(\sigma_i)] = \sum_{a_i\in A_i}\sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i})\sigma_{-i}(a_{-i})\sigma_i(a_i)$$

Definition 8.3 (32.1)**.** The **mixed extension** of the strategic game $\langle N, (A_i), (u_i)\rangle$ is the strategic game $\langle N, (\Delta(A_i)), (U_i)\rangle$ in which $\Delta(A_i)$ is the set of probability distributions over A_i and $U_i : \times_{j\in N}\Delta(A_i) \to \mathbb{R}$ assigns to each $(\sigma_i)_{i\in N} \in \times_{j\in N}\Delta(A_i)$ the expected value under u_i of the lottery over A that is induced by $(\sigma_i)_{i\in N}$.

Remark 8.2 (pg.32, notes on above definition)**.** If the game is finite, that is, for each $i \in N$, the set A_i is finite, then

$$U_i(\sigma) = \sum_{a\in A}(\prod_{j\in N}\sigma_j(a_j))u_i(a)$$

Definition 8.4 (32.3)**.** A **mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension.

Proposition 8.3 (33.1)**.** Every finite strategic game has a mixed strategy Nash equilibrium.

Lemma 8.1 (33.2)**.** Let $G = \langle N, (A_i), (u_i)\rangle$ be a finite strategic game. Then $\sigma^* \in \times_{i\in N}\Delta(A_i)$ is a mixed strategy Nash equilibrium of G is and only if for every player $i \in N$ every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*

Assumption 8.1 (lec.5)**.** Assuming all agents follows Von-Neumann Morgenstern theorem.

Definition 8.5 (lec.5)**.** An action a_i is **strictly dominated** by mixed strategy σ_i if and only if

$$\forall a_{-i} \in A_{-i}\ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i})$$

where σ_i could be a pure strategy.

Definition 8.6 (lec.5)**.** A mixed strategy σ_i is a **best response** to σ_{-i} if and only if

$$\forall \sigma_i' \in \Delta(A_i)\ U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

Definition 8.7 (lec.5)**.** The **support** of a mixed strategy $\sigma_i \in \Delta(A_i)$ is the set

$$supp(\sigma_i) = \{a_i \in A_i : \sigma_i(a_i) \neq 0\}$$

Proposition 8.4 (lec.5)**.** A mixed strategy σ_i is a **best response** to an strategy profile σ_{-i} if and only if

(a) Player i is indifferent between all a_i in the support of σ_i,

$$\forall a_j, a_j' \in supp(\sigma_i) \quad a_j \sim_i a_j'$$

(b) and player i weakly prefers all actions in the support of σ_i to those not in the support of σ_i. That's

$$\forall a_j \in supp(\sigma_i),\ \forall a_j' \notin supp(\sigma_i) \quad a_j \succsim_i a_j'$$

Proof. ($\Longrightarrow$) show the if parts by proving it's contraposition.
Suppose (a) is not true, then

$$\exists a_i, a_j \in supp(\sigma_i) \ s.t. \ a_i \not\sim_i a_j$$

WLOG, suppose

$$u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

then σ_i would not be the best response since we can refine it by assigning

$$\begin{cases} \sigma_i'(a_i) = \sigma_i(a_i) + \sigma_i(a_j) \\ \sigma_i'(a_j) = 0 \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

and σ_i' would provides higher expected payoff.
Suppose (b) does not hold,

$$\exists a_i \notin supp(\sigma_i) \ s.t. \ \exists a_j \in supp(\sigma_i) \ s.t \ u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

Then σ_i could not be a best response since we can construct another mixed strategy σ_i' strictly dominating σ_i by setting

$$\begin{cases} \sigma_i'(a_j) = 0 \\ \sigma_i'(a_i) = \sigma_i(a_j) \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

($\Longleftarrow$) Assuming σ_i is not a best response towards σ_{-i},
then there exists $\sigma_i' \in \Delta(A_i)$ such that

$$\begin{aligned} & U_i(\sigma_i', \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i}) \\ \iff & \mathbb{E}[u_i(a)|(\sigma_i', \sigma_{-i})] > \mathbb{E}[u_i(a)|(\sigma_i, \sigma_{-i})] \\ \iff & \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i'(a_i)\sigma_{-i}(a_{-i}) > \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) \end{aligned}$$

Probability measures σ_i and σ_i' could only be different in two aspects, their supports and the values assigned on elements in their supports, this fails assumption (a).
The following argument needs to be revised.
Case 1 suppose $supp(\sigma_i) = supp(\sigma_i')$, then the strictly inequality in expected payoffs implies redistributing probabilities does affect the expected payoffs.
So player i cannot be indifferent between any two actions in the support.
Case 2 suppose $supp(\sigma_i) \neq supp(\sigma_i')$ and $supp(\sigma_i') \not\subseteq supp(\sigma_i)$. That's

$$\exists a_i \in supp(\sigma_i') \wedge \notin supp(\sigma_i)$$

Then extending the support to a_i of σ_i gives higher expected payoff, this fails the assumption (b).
Case 3 suppose $supp(\sigma_i') \subsetneq supp(\sigma_i)$. Then the expected payoff can be strictly increased by eliminating actions in $supp(\sigma_i) \backslash supp(\sigma_i')$. Then those actions eliminated must be strictly dominated by actions in $supp(\sigma_i')$. This fails assumption (a). ∎

Proposition 8.5 (lec.5 equivalent proposition)**.** All actions in the support are best responses. (i.e.*best response mixed strategy is a mixture of best response pure actions*)

Remark 8.3 (lec.5 Intuition of proposition)**.** If the requirements of above proposition are not satisfied, the player can reduce the probability assigned to the non-best-response pure action and better off.

Theorem 8.1 (lec.5 Nash's Theorem)**.** Any player $i \in N$ in finite game $\langle N, (A_i), (\succsim_i) \rangle$ has a mixed strategy Nash equilibrium.

9 Lecture 6. Extensive Form Games and Subgame Perfection

9.1 Extensive Form Game

Definition 9.1 (89.1)**.** An **extensive game with perfect information** has the following components.

- A set N of **players**.
- A set H of sequences (finite or infinite) of **histories** with properties:
  - $\emptyset \in H$.
  - For all $L < K$, $(a^k)_{k=1,2,\ldots,K} \in H \implies (a^k)_{k=1,2,\ldots,L} \in H$.
  - For infinite sequence $(a^k)_{k=1}^{\infty}$,
    $(a^k)_{k=1,2,\ldots,L} \in H, \ \forall L \in \mathbb{Z}_{++} \implies (a^k)_{k=1}^{\infty} \in H$.

  And each component of history $h \in H$ is an **action** taken by a player.
- A function $P : H \backslash Z \to N$, where for $h \in H$, $P(h) \in N$ is defined by the player who takes an action after the history h.
- For each player $i \in N$ a **preference relation** $\succsim_i$ defined on Z.

Notation 9.1 (pg.90)**.** An extensive game with perfect information can be represented by a 4-tuple, $\langle N, H, P, (\succsim_i) \rangle$. *Sometimes it is convenient to specify the structure of an extensive game without specifying the players' preference, as* $\langle N, H, P \rangle$.

Definition 9.2 (pg.90)**.** A history $(a^k)_{k=1,2,\ldots,K} \in H$ is **terminal** if

1. it is infinite,
2. or (i.e. it cannot be extended to another valid history sequence)

$$\forall a^{K+1}, \ (a^k)_{k=1,2,\ldots,K+1} \notin H$$

The set of terminal histories is denoted by Z.

Notation 9.2 (pg.90, the action set)**.** After any nonterminal history, $h \in H \backslash Z$, the player $P(h)$ chooses an action from set

$$A(h) = \{a : (h, a) \in H\}$$

Remark 9.1. Note that all player function, action set and player preference relation are defined on H. Thus, unlike a normal form game, which was *player oriented*, we'd better consider an extensive form game as *history oriented*.

Definition 9.3 (pg.90)**.** We refer to the empty set, which is required to be an element of H, as the **initial history**.

Definition 9.4 (92.1)**.** A **strategy of player** $i \in N$, s_i, in an extensive game with perfect information $\langle N, H, P, (\succsim_i)\rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Remark 9.2 (pg.92)**.** A strategy specifies the action chosen by a player for *every* history after which it is his turn to move, *even for histories that is, if the strategy is followed, are never reached.*

Definition 9.5 (pg.93)**.** For each strategy profile $s = (s_i)_{i \in N}$ in the extensive game $\langle N, H, P, (\succsim_i)\rangle$, the **outcome** of s, $O(s)$, is defined as the terminal history that results when each player $i \in N$ follows the precepts of s_i. That is, $O(s)$ is the (possibly infinite) history

$$(a^1, \ldots, a^K) \in Z$$

such that

$$\forall k \in \{0, 1, \ldots K-1\},\ s_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k) = a^{k+1}$$

Definition 9.6 (lec.6)**.** A extensive game $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is finite if and only if

(a) N is finite.

(b) (A_i) are all finite.

(c) All $h \in H$ reach the terminal state with finite length.

Definition 9.7 (93.1)**.** A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$\forall s_i \in S_i,\ O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

Definition 9.8 (94.1)**.** The **strategic form of the extensive game with perfect information**, $\Gamma = \langle N, H, P, (\succsim_i)\rangle$, is the strategic game $\langle N, (S_i), (\succsim'_i)\rangle$ in which for each player $i \in N$

- S_i is the **set of strategies** of player i in Γ.
- $\succsim'_i$ is defined on $\times_{i \in N} S_i$ and defined by

$$\forall s, s' \in \times_{i \in N} S_i,\ s \succsim'_i s' \iff O(s) \succsim_i O(s')$$

Definition 9.9 (pg.94)**.** A **reduced strategy** of player i is defined to be a function f_i whose domain is a *subset* of $\{h \in H : P(h) = i\}$ and has the following properties

1. it associates with every history h in the domain of f_i an action in $A(h)$.
2. a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i. (i.e., for any $h = (a^k)$ and for any $h' = (a^k)_{k=1}^{L}$ as a subsequence of h such that $P(h') = i$, $f_i(h') = a^{L+1}$.)

Remark 9.3 (pg.94)**.** Each **reduced strategy** of player i corresponds to a set of strategies of player i, such that for each vector of strategies of the other players each strategy in this set yields the same outcome. (strategies in the same set are **outcome-equivalent**.)
That's, for each strategy $s_i \in S_i$, its reduced strategy can be defined with an outcome equivalence class, $[s_i]$,

$$[s_i] \equiv \{s'_i \in S_i : \forall s_{-i} \in \times_{j\in N\backslash\{i\}} S_j,\ O(s_{-i}, s_i) = O(s_{-i}, s'_i)\}$$

But in some other game, the definition of outcome-equivalence is more general and defined by generating the same payoff (through possibly difference outcomes), then the reduced strategy is defined as

$$[s_i] \equiv \{s'_i \in S_i : \forall s_{-i} \in \times_{j\in N\backslash\{i\}} S_j,\ \forall j \in N,\ O(s_{-i}, s_i) \sim_j O(s_{-i}, s'_i)\}$$

Definition 9.10 (95.1.1)**.** Let $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim'_i)\rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i, s'_i \in S_i$ to be **equivalent** if

$$\forall s_{-i} \in S_{-i},\ \forall j \in N,\ (s_{-i}, s_i) \sim'_j (s_{-i}, s'_i)$$

Definition 9.11 (95.1.2)**.** The **reduced strategic form of** Γ is the strategic game $\langle N, (S'_i), (\succsim''_i)\rangle$ in which for each $i \in N$ each set S'_i contains one member of each set of equivalent strategies in S_i and $\succsim''_i$ is the preference ordering over $\times_{j\in N} S'_j$ induced by $\succsim'_i$.

9.2 Subgame Perfection

Definition 9.12 (97.1)**.** The **subgame of extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h)\rangle$ where

- $H|_h$ is the set of sequences h' such that $(h, h') \in H$.
- $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$.
- $\succsim_i |_h$ is defined by $h' \succsim_i |_h h'' \iff (h, h') \succsim_i (h, h'') \in Z$.

Notation 9.3 (pg.97)**.** Given strategy $s_i \in S_i$ and $h \in H \in \Gamma$, $s_i|_h$ represents the **strategy that** s_i **induces in the subgame** $\Gamma(h)$. That's, for each $h' \in H_h$

$$s_i|_h(h') \equiv s_i(h, h')$$

Notation 9.4. Let O_h denote the **outcome function of** $\Gamma(h)$, that's, for all $h' \in H|_h$,

$$O_h(h') \equiv O(h, h')$$

Definition 9.13 (97.2)**.** A **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H\backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i |_h O_h(s^*_{-i}|_h, s_i|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.

Definition 9.14 (pg.97)**.** Equivalently, define SPNE to be a strategy profile s^* in Γ for which for any history $h \in H$ the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Remark 9.4 (pg. 97)**.** The notion of SPNE requires the action prescribed by each player's strategy to be optimal, given other players' strategies, after *every* history.

Proposition 9.1 (99.2)**.** Every finite extensive game with perfect information has a subgame perfect equilibrium.

10 Lecture 7. Extensive Form Games: Examples

11 Lecture 8. Repeated Games

12 Lecture 9. Game with Incomplete Information

13 Lecture 10. Game with Incomplete Information II

14 Lecture 11. Auctions