ECO426H1 Market Design: Auctions and Matching Markets

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1 Auctions

Definition 1.1. An **auction** is an informational environment consisting of

(i) **Bidding format rules**: the form of the bids, which can be price only, multi-attribute, price and quantity, or quantity only;

(ii) **Bidding process rules**: Closing/timing rules, available information, rules for bid improvements/counter-bids, closing conditions;

(iii) **Price and allocation rules**: final prices, quantities, winners.

Auctions are commonly referred to as a market mechanism as well as a price discovery mechanism

Definition 1.2. A **market mechanism** uses prices to determine allocations.

Definition 1.3. An auction is a **private value** auction if agents' valuations do not dependent on other buyers' valuations. Otherwise, the auction is called a **interdependent / common value** auction.

1.1 Private Value Auctions

Assumption 1.1. In this chapter, we shall impose the following assumption on bidders' valuations:

(i) Each bidder's valuation is independently and identically distributed on some interval $[0, \omega]$ according to a distribution function F:

$$V_i \overset{i.i.d.}{\sim} F \ s.t. \ \text{supp}(F) = \mathbb{R}_+ \tag{1.1}$$

(ii) F belongs to the common knowledge in this system;

(iii) Bidders' valuations have finite expectations:

$$\mathbb{E}[V_i] < \infty \tag{1.2}$$

Assumption 1.2. Moreover, we assume bidders' behaviours to satisfy the following properties:

(i) Bidders are risk neutral, they are maximizing expected profits;

(ii) Each bidder it both willing and able to pay up to his or her value.

Definition 1.4. A **strategy** of a bidder is a mapping from the space of his/her valuation to a bid:

$$s : [0, \omega] \to \mathbb{R}_+ \tag{1.3}$$

Definition 1.5. An equilibrium of auction is **symmetric** if all bidders are following the same bidding strategy s.

Definition 1.6. A bidder is **bidding sincerely / truthfully** if he bids his true value.

Proposition 1.1. In a symmetric equilibrium of the second-price auction, $s(v) = v$ is a weakly dominant strategy.

Proof. For a fixed valuation $v_i \in [0, \omega]$ of bidder i.
Let $p := \max_{j \neq i} b_j$ be highest bidding price by other bidders.
Let $\pi_i(b, p)$ denote bidder i's profit when bidding b given the highest price from other bidders to be p.
Part 1: consider another bidding $z_i < v_i$, the following cases are possible:

(i) $v_i < p \implies z_i < v_i < p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i losses anyway).

(ii) $v_i = p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i is indifferent).

(iii) $v_i > p$:

 (a) $v_i > z_i > p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = v_i - p$;

 (b) $v_i > z_i = p \implies \pi_i(v_i, p) \geq \pi_i(z_i, p)$;

 (c) $v_i > p > z_i \implies \pi_i(v_i, p) > \pi_i(z_i, p)$.

Hence, bidding v_i weakly dominates bidding any value below it.
Part 2: for $z_i > v_i$, the following cases are possible:

(i)

d

Therefore, bidding v_i weakly dominates bidding any other values. ■

Proposition 1.2. In a symmetric equilibrium of the first-price auction, equilibrium bidding strategies are given by

$$s(v_i) = \mathbb{E}[\max_{j \neq i} v_j | v_j \leq v_i] \tag{1.4}$$

which is the *expected second highest valuation conditional on v_i being the highest valuation.*

Proof. Let $s(v)$ denote an equilibrium strategy.

Lemma 1.1. For any agent, bidding more than $s(\omega)$ can never be optimal. Bidding $b > s(\omega)$ makes this agent win for sure. In such case, bidding $b' \in (s(\omega), b)$ strictly dominates bidding b.

Lemma 1.2. For any agent, $s(0) = 0$. Bidding any positive number would cause negative payoff with positive probability, and therefore, leads to a negative expected profit.

Lemma 1.3. Because s is monotonically increasing, therefore,

$$\max_{j \neq i} s(v_j) = s(\max_{j \neq i} v_j) \tag{1.5}$$

Let p denote the highest price among all other $N-1$ bidders and let $F^{(N-1)}(x)$ denote the distribution of p.
The expected profit of bidder i by bidding an arbitrary $b \in \mathbb{R}_+$ is

$$\pi_i(b, v_i) = P(b > p)(v_i - s(v_i)) + P(b = p)(v_i - s(v_i)) + P(b < p)0 \tag{1.6}$$

Note that $b > p = s(\max_{j \neq i} v_j)$ if and only if $s^{-1}(b) > \max_{j \neq i} v_j$. It follows

$$P(b > p) = P(\max_{j \neq i} v_j < s^{-1}(b)) = F^{(N-1)}(s^{-1}(b)) \tag{1.7}$$

Therefore,

$$\pi_i(b, v_i) = F^{(N-1)}(s^{-1}(b))(v_i - b) \tag{1.8}$$

The first order condition implies

$$\frac{\partial \pi_i}{\partial b}\pi_i(b, v_i) = \frac{\partial \pi_i}{\partial b} F^{N-1}(s^{-1}(b))v_i - F^{N-1}(s^{-1}(b))b \tag{1.9}$$

$$= f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{1.10}$$

For a symmetric equilibrium, all other bidders are following the same strategy s so that $s(v_i) = b$, therefore,

$$f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{1.11}$$

$$\implies f^{(N-1)}(s^{-1}(b))(v_i - b) - F^{(N-1)}(s^{-1}(b))s'(v_i) = 0 \tag{1.12}$$

$$\implies f^{(N-1)}(s^{-1}(b))v_i = F^{(N-1)}(s^{-1}(b))s'(v_i) + f^{(N-1)}(s^{-1}(b))s(v_i) \tag{1.13}$$

$$\implies f^{(N-1)}(v_i)v_i = \frac{d}{dv_i}\left[F^{(N-1)}(v_i)s(v_i)\right] \tag{1.14}$$

$$\implies \int_0^{v_i} f^{(N-1)}(y)y\ dy = F^{(N-1)}(v_i)s(v_i) - F^{(N-1)}(0)s(0) \tag{1.15}$$

$$\implies F^{(N-1)}(v_i)s(v_i) = \int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{1.16}$$

$$\implies s(v_i) = \frac{1}{F^{(N-1)}(v_i)}\int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{1.17}$$

$$\implies s(v_i) = \mathbb{E}\left[\max_{j\neq i} v_j \,\middle|\, \max_{j\neq i} v_j < v_i\right] \tag{1.18}$$

■

2 First Price Private Value Auction

Remark 2.1. For every continuous distribution F, the probability for a tie to happen is zero. Therefore, we ignore the tie for now.

Problem Setup Let N denote the set of bidders such that $|N| = n$. For each bidder $i \in N$, his valuation of the auctioned item V_i follows some distribution F. Further assume that $V_i \perp V_j$ for every $i \neq j$.
Let $W(b, v_i)$ denote the event that player i, who has valuation v_i, wins by bidding $b \in \mathbb{R}_+$, define

$$W(b, v_i) \iff b > \max_{j\neq i} b_j \tag{2.1}$$

The payoff (utility) of bidder i, who has valuation v_i, is

$$U(b, v_i) = \begin{cases} v_i - b & \text{if } W(b, v_i) \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

2.1 Symmetric Equilibrium Behaviour

Consider a symmetric environment such that all bidders are using the same strictly increasing strategy $s(\cdot)$ such that $s(\cdot)$ is invertible.

Equilibrium Strategy when $F = Unif(0,1)$.

$$\beta^I(v) = \frac{n-1}{n} v \tag{2.3}$$

Probability of Winning

$$\begin{aligned}
P(W(b, v_i)) &= P(b > \max_{j \neq i} s(v_j)) && (2.4)\\
&= P(b > s(\max_{j \neq i} v_j)) && (2.5)\\
&= P(\max_{j \neq i} v_j \leq s^{-1}(b))) && (2.6)\\
&= F(s^{-1}(b))^{n-1} && (2.7)\\
&= F(v_i)^{n-1} \text{ because } b = s(v_i) && (2.8)
\end{aligned}$$

When $F = Unif(0,1)$,

$$P(W(b, v_i)) = v_i^{n-1} \tag{2.9}$$

Expected Payment from Bidder i with v_i Conditioned on Winning Suppose bidder i is following strategy $s(\cdot)$. Then,

$$\mathbb{E}[Payment_i | v_i, W(b, v_i)] = b = s(v_i) \tag{2.10}$$

When $F = Unif(0,1)$,

$$\mathbb{E}[Payment_i | v_i, W(b, v_i)] = \frac{n-1}{n} v_i \tag{2.11}$$

Unconditional Payment from Bidder i with v_i

$$\begin{aligned}
\mathbb{E}[Payment_i | v_i] &= P(W(b, v_i)) \mathbb{E}[Payment_i | v_i, W(b, v_i)] + P(Loss) \times 0 && (2.12)\\
&= P(W(b, v_i)) \mathbb{E}[Payment_i | v_i, W(b, v_i)] && (2.13)\\
&= F(v_i)^{n-1} s(v_i) && (2.14)
\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}[Payment_i | v_i] = \frac{n-1}{n} v_i^n \tag{2.15}$$

Expected Payoff of Bidder i with v_i Stopped Here

$$\begin{aligned}
\mathbb{E}[U | v_i] &= P(W(s(v_i), v_i)) v_i - \mathbb{E}[Payment_i | v_i] && (2.16)\\
&= && (2.17)
\end{aligned}$$

Unconditional Payment from Bidder i This is the same as the expected revenue from bidder i:

$$\mathbb{E}[Payment_i] = \int_{\mathbb{R}_+} \mathbb{E}[Payment_i|v_i]dF \tag{2.18}$$
$$= \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f(v_i)\ dv_i \tag{2.19}$$

When $F = Unif(0,1)$,

$$\mathbb{E}[Payment_i] = \int_0^1 \frac{n-1}{n} v_i^n\ dv_i \tag{2.20}$$
$$= \frac{n-1}{n(n+1)} \tag{2.21}$$

Auctioneer's Expected Revenue Since all bidders are the same,

$$\mathbb{E}[Revenue] = n\ \mathbb{E}[Payment_i] \tag{2.22}$$
$$= n \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f_i\ dv_i \tag{2.23}$$

When $F = Unif(0,1)$,

$$\mathbb{E}[Revenue] = \frac{n-1}{n+1} \tag{2.24}$$

3 Second Price Private Value Auction

4 The General Case: k^{th} Price Private Value Auction

5 Common Value Auction

6 Combinatorial Auction: The VCG Mechanism

7 Matching Market

8 Appendix A: Order Statistics

Definition 8.1. Let $(X_1, \cdots, X_n)$ be n random variables on the probability space $(\Omega, \mathcal{F}, P)$, further assume they are iid following distribution function $F(\cdot)$. For each $\omega \in \Omega$, realizations of above random variables can be sorted as

$$X_{(n)}(\omega) \leq X_{(n-1)}(\omega) \leq \cdots \leq X_{(1)}(\omega) \tag{8.1}$$

For each ω, the random variable $X_{n:k}$ is defined such that $X_{n:k}(\omega)$ equals the k-th largest value, $X_{(k)}(\omega)$.

Distribution Function Let $x \in X(\Omega)$, then

$$X_{n:k} \leq x \iff (\text{no } X_i > x) \bigcup (\text{exactly } 1\ X_i > x) \bigcup \cdots \bigcup (\text{exactly } k-1\ X_i > x) \tag{8.2}$$

$$\iff (X_i \leq x\ \forall i) \bigcup (\text{exactly } n-1\ X_i \leq x) \bigcup \cdots \bigcup (\text{exactly } n-k+1\ X_i \leq x) \tag{8.3}$$

$$\iff \bigcup_{j=n-k+1}^{n} (\text{exactly } j\ X_i \leq x) \tag{8.4}$$

Note that events in the union are mutually exclusive, therefore,

$$F_{n:k}(x) = P(X_{n:k} \leq x) = \sum_{j=n-k+1}^{n} P\left(\text{exactly } j\ X_i \leq x\right) \tag{8.5}$$

$$= \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \tag{8.6}$$

Density Function

$$
\begin{aligned}
f_{n:k}(x) &= \frac{d}{dx} F_{n:k}(x) && (8.7)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} && (8.8)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} F(x)^j (1-F(x))^{n-j} && (8.9)\\
&= \sum_{j=n-k+1}^{n} \left[\frac{n!}{j!(n-j)!} jF(x)^{j-1}(1-F(x))^{n-j} - \frac{n!}{j!(n-j)!}(n-j)F(x)^j(1-F(x))^{n-j-1} \right] f(x) && (8.10)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} jF(x)^{j-1}(1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j)!}(n-j)F(x)^j(1-F(x))^{n-j-1} f(x) && (8.11)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j(1-F(x))^{n-j-1} f(x) && (8.12)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (8.13)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x)\\
&\quad - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j(1-F(x))^{n-j-1} f(x)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (8.14)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x)\\
&\quad - \sum_{i=n-k+2}^{n} \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1}(1-F(x))^{n-i} f(x) \ \text{(substitute } j=i-1\text{)}\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (8.15)\\
&= n\frac{(n-1)!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (8.16)\\
&= n\binom{n-1}{k-1} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (8.17)
\end{aligned}
$$