

# Elements of Real Analysis

Based on Lecture Notes for MAT337: Introduction to Real Analysis (2019Winter)

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## TO-DO

1. Add Dedekind cut to section 1.
2. Refine subsection titles.

## Contents

<b>1</b>	<b>Real Numbers</b>	<b>2</b>
1.1	Definitions . . . . .	2
1.2	The Axiom of Completeness . . . . .	2
<b>2</b>	<b>Sequences and Series</b>	<b>3</b>
<b>3</b>	<b>Convergence of Series</b>	<b>6</b>
<b>4</b>	<b>Order and Converging Sequences</b>	<b>7</b>

# 1 Real Numbers

## 1.1 Definitions

**Definition 1.1.** Reals are proper initial segments of  $\mathbb{Q}$  with no maximum.

**Definition 1.2.** A subset  $A \subset \mathbb{Q}$  is an **initial segment** if

$$y \in A, x \in \mathbb{Q}, x < y \implies x \in A \quad (1.1)$$

**Definition 1.3.**  $A$  is **proper** if  $A \neq \mathbb{Q}$ .

**Definition 1.4.**  $A$  has no maximal elements if

$$\forall x \in A, \exists y \in A \text{ s.t. } y > x \quad (1.2)$$

**Example 1.1.**

$$\sqrt{2} \approx A_{\sqrt{2}} := \{q \in \mathbb{Q} : q < \sqrt{2}\} \equiv \{q \in \mathbb{Q} : q \leq 0 \vee q^2 < 2\} \quad (1.3)$$

$$x \approx A_x := \{q \in \mathbb{Q} : q < x\} \quad (1.4)$$

## 1.2 The Axiom of Completeness

**Axiom 1.1** (Axiom of Completeness). Every non-empty subset  $B \subset \mathbb{R}$  that is bounded has a supremum (i.e. the least upper bound). That's

$$\forall B \subset \mathbb{R}, \text{ s.t. } B \neq \emptyset \exists b \in \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in B, x \leq b \text{ (upper bound)} \\ \forall c \in \mathbb{R} (\forall x \in B, x \leq c) \implies b \leq c \text{ (least upper bound)} \end{cases} \quad (1.5)$$

**Theorem 1.1.**  $\mathbb{Q}$  is *dense* in  $\mathbb{R}$ , that's

$$\forall x < y \in \mathbb{R}, \exists q \in \mathbb{Q} \text{ s.t. } x < q < y \quad (1.6)$$

**Theorem 1.2** (Cardinality). Let  $A, B$  be non-empty subsets of  $\mathbb{R}$ , then the following statements are equivalent:

- (i)  $\exists h : A \rightarrow B$  such that  $h$  is bijective;
- (ii)  $\exists f : A \rightarrow B$  and  $g : B \rightarrow A$  such that both  $f$  and  $g$  are injective.

*Proof.* (i) is the definition for sets  $A$  and  $B$  to have the same cardinality. And the existence of injection from  $A$  to  $B$  implies the cardinality of  $A$  cannot be greater than the cardinality of  $B$ . Similarly, the existence of injection from  $B$  to  $A$  implies the cardinality of  $B$  cannot be greater than the cardinality of  $A$ . Therefore  $A$  and  $B$  share the same cardinality. ■

**Theorem 1.3** (Nested Intervals). Let  $(I_n)$  be a sequence of closed and non-empty intervals in  $\mathbb{R}$  such that

$$I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \quad (1.7)$$

then

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.8)$$

*Proof.* Claim:

$$x := \sup\{\min(I_n) : n \in \mathbb{N}\} \in \bigcap_{n \in \mathbb{N}} I_n \quad (1.9)$$

Let  $n \in \mathbb{N}$ , then  $x \geq \min(I_n)$ . Now show  $x \leq \max(I_n) \forall n \in \mathbb{N}$ . Suppose not, then  $\exists k \in \mathbb{N}$  such that  $x > \max(I_k)$ . Then by the definition of supremum, there exists  $j \in \mathbb{N}$  such that  $\max(I_k) < \min(I_j)$ . Note that if  $k = j$ , this leads to a contradiction. If  $k < j$ , then because  $I_k \supset I_j$ ,  $\max(I_k) \geq \max(I_j) \geq \min(I_j) \geq \min(I_k)$ , this leads to a contradiction. If  $k > j$ , then  $I_k \subset I_j$ , thus  $\min(I_j) \leq \min(I_k) \leq \max(I_k) \leq \max(I_j)$ , which also leads to a contradiction.

Therefore we conclude

$$\min(I_n) \leq x \leq \max(I_n) \forall n \in \mathbb{N} \quad (1.10)$$

therefore  $x \in I_n \forall n \in \mathbb{N}$ , so  $x \in \bigcap_{n \in \mathbb{N}} I_n$ . ■

**Theorem 1.4.** There exists no injection from  $\mathbb{R}$  to  $\mathbb{N}$ .

*Proof.*  $\mathbb{R}$  has cardinality  $c$  but  $\mathbb{N}$  has cardinality  $\aleph_0$ . ■

## 2 Sequences and Series

**Definition 2.1.** A sequence  $(a_n)_{n=1}^{\infty}$  of real numbers **converges** to a real number  $a$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \ |a_n - a| < \varepsilon \quad (2.1)$$

If there does not exist such  $a$ , we conclude  $(a_n)_{n=1}^{\infty}$  is **divergent**.

**Theorem 2.1.** Every convergent sequence is bounded.

*Proof.* Let  $(a_n)_{n=1}^{\infty}$  be a convergent sequence in  $\mathbb{R}$  with limit point  $a$ . Then take  $\varepsilon = 1$ , there exists  $N \in \mathbb{N}$  such that  $n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1$ . Take

$$M := \max\{\max_{n \leq N}\{|a_n|\}, |a| + 1\} \quad (2.2)$$

and the sequence is bounded by  $M$ . ■

**Definition 2.2.** Let  $(a_n)_{n=1}^{\infty}$  be a sequence, then a sub-sequence of  $(a_n)$  is any sequence in the form  $(a_{n_k})_{k=1}^{\infty}$  such that  $n_1 < n_2 < \dots < n_k < \dots$ .

**Remark 2.1.** A sub-sequence can be generated with a strictly increasing function defined on  $\mathbb{N}$  and a sequence  $(a_n)$ .

**Theorem 2.2** (Bolzano-Weierstrass). Every bounded sequence has a convergent sub-sequence.

*Proof.* Let  $(a_n)_{n=1}^{\infty}$  be a bounded sequence bounded by  $M > 0$ . Define

$$I_0 := [-M, M] \quad (2.3)$$

$$J^0 := [-M, 0] \quad (2.4)$$

$$J^1 := [0, M] \quad (2.5)$$

$$X^0 := \{n \in \mathbb{N} : a_n \in J^0\} \quad (2.6)$$

$$X^1 := \{n \in \mathbb{N} : a_n \in J^1\} \quad (2.7)$$

therefore  $\mathbb{N} = X^0 \cup X^1$ . Thus at least one of  $X^0$  and  $X^1$  is infinite. If  $X^0$  is infinite, define  $I_1 := J^0$ , otherwise, define  $I_1 := J^1$ . Let

$$A := \{x \in \mathbb{R} : \{n \in \mathbb{N} : x < a_n\} \text{ is infinite}\} \quad (2.8)$$

which is the lower bound of selected infinite half intervals. And define  $a := \inf(A)$ . we can construct a sub-sequence, for each  $n \in \mathbb{N}$ , take  $a_n \in I_n$ . And by the nested interval theorem, the intersection of all those selected intervals is non-empty. And  $a$  is the limit point of the constructed sequence. So a convergent sub-sequence exists. ■

**Definition 2.3.** A sequence  $(a_n)_{n=1}^{\infty}$  is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N, |a_n - a_m| < \varepsilon \quad (2.9)$$

**Theorem 2.3** (Convergent  $\implies$  Cauchy). Every convergent sequence is a Cauchy sequence.

*Proof.* Let  $(a_n)$  be a convergent sequence, fix  $\varepsilon > 0$ . Suppose  $(a_n) \rightarrow a$ , take  $\varepsilon^* = \varepsilon/2$ . Thus, there exists  $N \in \mathbb{N}$  such that  $\forall n > N, |a_n - a| < \varepsilon^* = \varepsilon/2$ . By taking such  $N, \forall n, m > N$ , both  $|a_n - a|$  and  $|a_m - a| < \varepsilon/2$ . By triangle inequality,  $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ . Hence, we've shown that for an arbitrary  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $\forall m, n > N, |a_n - a_m| < \varepsilon$ . Therefore  $(a_n)$  is Cauchy. ■

**Theorem 2.4** (Cauchy  $\implies$  Convergent). Every Cauchy sequence is convergent.

*Proof.* Let  $(a_n)$  be a Cauchy sequence.

Claim:  $(a_n)$  is bounded.

*Proof. Bounded.* Take  $\varepsilon = 1$ , then  $\exists N \in \mathbb{N}$  such that  $\forall m, n > N, |a_n - a_m| < 1$ . Take  $m = N + 1$  and define  $a^* := a_m$ . Then we have  $\forall n > N, |a_n - a^*| < 1$ , which implies  $|a_n| < |a^*| + 1$ . Define

$$M := \max\{\max\{a_n : n \leq N\}, |a^*| + 1\} \quad (2.10)$$

So  $(a_n)$  is bounded by  $M$ . ■

Then by Bolzano-Weierstrass Theorem, there exists a sub-sequence  $(a_{n_k})_{k=1}^{\infty}$  converges to some limit point  $a \in \mathbb{R}$ . We are going to show  $(a_n) \rightarrow a$ . Fix  $\varepsilon > 0$ , by the convergence of the

sub-sequence

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |a_{n_k} - a| < \frac{\varepsilon}{2} \quad (2.11)$$

Also since the sequence itself is Cauchy,

$$\exists N_2 \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N_2, |a_n - a_m| < \frac{\varepsilon}{2} \quad (2.12)$$

Take  $N^* := \max\{N_1, N_2\}$ . Show  $|a_n - a| < \varepsilon \forall n \geq N^*$ . Note that

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \quad (2.13)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \quad (2.14)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (2.15)$$

since  $n_k \geq n$  by the definition of sub-sequences. ■

**Corollary 2.1.** A sequence is Cauchy if and only if it is convergent.

*Proof.* Let  $(a_n)$  be a Cauchy sequence.

Claim:  $(a_n)$  is bounded.

*Proof.* Take  $\varepsilon = 1$ , then there exists  $N \in \mathbb{N}$  such that

$$\forall m, n > N, |a_n - a_m| < 1 \quad (2.16)$$

take  $m := N + 1$ , define  $a^* := a_{m+1}$ , then

$$\forall n > N, |a_n - a^*| < 1 \implies |a_n| \leq |a^*| + 1 \quad (2.17)$$

Define  $M := \max\{\max_{n \leq N}\{a_n\}, |a^*| + 1\}$ , and  $(a_n)$  is bounded by  $M$ . ■

By the Bolzano-Weierstrass Theorem, there exists a sub-sequence  $(a_{n_k})_{k=1}^{\infty}$  converges to some limit point  $a \in \mathbb{R}$ . Show  $(a_n) \rightarrow a$  as well.

Let  $\varepsilon > 0$ , by convergence of the sub-sequence,

$$\exists N_1 \in \mathbb{N}, \text{ s.t. } \forall n \geq N_1, |a_{n_k} - a| < \varepsilon/2 \quad (2.18)$$

By the Cauchy property of  $(a_n)$ ,

$$\exists N_2 \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N_2, |a_n - a_m| < \varepsilon/2 \quad (2.19)$$

take  $N^* := \max\{N_1, N_2\}$ . Let  $n \geq N^*$  and note that  $n_k \geq n \geq N^*$

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \quad (2.20)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \quad (2.21)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (2.22)$$

then take such  $N^*$  for the fixed  $\varepsilon > 0$ . Convergence of  $(a_n)$  shown. ■

**Theorem 2.5** (the Uniqueness of the Limit Point). If  $(a_n) \rightarrow a$  and  $(a_n) \rightarrow b$ , then  $a = b$ .

*Proof.* Suppose  $a \neq b$ , define  $s := |a - b| > 0$ . Take  $\varepsilon = \frac{s}{2}$ , there does not exist such  $N \in \mathbb{N}$  satisfying

$$\forall n \geq N, \begin{cases} |a_n - a| < \varepsilon \\ |a_n - b| < \varepsilon \end{cases} \quad (2.23)$$

above notion indicates that the sequence is converging to two separate limit points simultaneously. ■

**Theorem 2.6** (Properties of Limits). If  $(a_n) \rightarrow a$ ,  $(b_n) \rightarrow b$ , and  $c \in \mathbb{R}$ , then

- (i)  $(c \cdot a_n) \rightarrow c \cdot a$ ;
- (ii)  $(a_n + c) \rightarrow a + c$ ;
- (iii)  $(a_n + b_n) \rightarrow a + b$ ;
- (iv)  $(a_n \cdot b_n) \rightarrow a \cdot b$ .

### 3 Convergence of Series

**Definition 3.1.** A series  $\sum_{n=1}^{\infty} a_n$  is **convergent** if

$$\exists a \in \mathbb{R} \text{ s.t. } \sum_{n=1}^{\infty} a_n = a \quad (3.1)$$

**Definition 3.2** (Alternative Definition). Let  $(S_n) := (\sum_{i=1}^n a_i)_{n=1}^{\infty}$  denote the *sequence of partial sums* associated with series  $\sum_{n=1}^{\infty} a_n$ , then the series is convergent if and only if its partial sum converges to a real number.

**Theorem 3.1** (Cauchy Criterion). A series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall m \geq n \geq N, \left| \sum_{i=n}^m a_i \right| < \varepsilon \quad (3.2)$$

That's, the partial sum sequence is Cauchy.

**Corollary 3.1.** If  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$

**Corollary 3.2** (Absolute Convergence Test). If  $\sum_{n=1}^{\infty} |a_n|$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is also convergent.

**Corollary 3.3.** If  $\sum_{n=1}^{\infty} |a_n|$  is convergent, and, let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection, then

$$\sum_{n=1}^{\infty} a_{f(n)} \quad (3.3)$$

is convergent.

*Given the absolute convergence, the rearrangement of sequence does not affect the convergence of series.*

**Theorem 3.2.** Suppose  $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$  and  $a_n \rightarrow 0$ , then  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  is convergent.

**Theorem 3.3.** If  $\sum_{n=1}^{\infty} |a_n|$  is convergent, let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection, then  $\sum_{n=1}^{\infty} |a_{f(n)}|$  is also convergent.

**Theorem 3.4.** Suppose  $\sum_{n=1}^{\infty} |a_n|$  is convergent, let  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  be two bijections, then

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_{g(n)} \quad (3.4)$$

**Theorem 3.5** (Monotone Convergence). Every monotone sequence, which is bounded, is convergent.

**Corollary 3.4.** Given sequence  $(a_n) \subset \mathbb{R}_{++}$  and series  $\sum_{n=1}^{\infty} a_n$ , the sequence of partial sums is therefore a monotonically increasing sequence, so the partial sum  $(S_n)$  is convergent if it is bounded.

**Example 3.1.**  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent.

*Proof.* Let  $m \in \mathbb{N}$ , so

$$S_m = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{m \times m} \quad (3.5)$$

$$< 1 + \frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \cdots + \frac{1}{m \times (m-1)} \quad (3.6)$$

$$= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) \quad (3.7)$$

$$= 2 - \frac{1}{m} < 2 \quad (3.8)$$

therefore  $(S_m)$  is non-decreasing and bounded above by 2. So  $(S_m)$  is convergent, so is  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . ■

## 4 Order and Converging Sequences

**Proposition 4.1.** If  $(a_n) \geq 0$  is convergent to  $a \in \mathbb{R}$ , then  $a \geq 0$ .

*Proof.* By contradiction. ■

**Proposition 4.2.** If  $(a_n) \leq (b_n)$  are convergent to  $a$  and  $b$ , respectively, then  $a \leq b$ .

*Proof.* Construct sequence  $(b_n - a_n) \geq 0$  and apply the previous proposition. ■

**Definition 4.1** (limsup). Let  $(a_n)$  be a bounded sequence, for each  $m \in \mathbb{N}$ , define

$$b_m := \sup_{n \geq m} a_n \quad (4.1)$$

For any  $m_0 \leq m_1 \in \mathbb{N}$ , it by the definition of supremum, it must be the case  $b_{m_0} \geq b_{m_1}$ . Therefore,  $(b_m)$  is a monotonically non-decreasing sequence. Also since  $(a_n)$  is bounded,  $(b_m)$  is bounded as well. Thus, according to the monotone sequence theorem,  $(b_m)$  converges to some limit  $b \in \mathbb{R}$ . Define

$$\limsup_{n \rightarrow \infty} a_n := b \quad (4.2)$$

**Definition 4.2** (liminf). Let  $(a_n)$  be a bounded sequence, for each  $m \in \mathbb{N}$ , define

$$b_m := \inf_{n \geq m} a_n \quad (4.3)$$

For any  $m_0 \leq m_1 \in \mathbb{N}$ , it by the definition of infimum, it must be the case  $b_{m_0} \leq b_{m_1}$ . Therefore,  $(b_m)$  is a monotonically non-increasing sequence. Also since  $(a_n)$  is bounded,  $(b_m)$  is bounded as well. Thus, according to the monotone sequence theorem,  $(b_m)$  converges to some limit  $b \in \mathbb{R}$ . Define

$$\liminf_{n \rightarrow \infty} a_n := b \quad (4.4)$$

**Theorem 4.1.**

$$\lim_{n \rightarrow \infty} a_n = a \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = a \quad (4.5)$$

*Proof.* ( $\implies$ ) Suppose  $(a_n) \rightarrow a$ , for each  $m \in \mathbb{N}$ , define  $b_m := \sup_{n \geq m} a_n$  and  $c_m := \inf_{n \geq m} a_n$ . By the definitions of infimum, supremum, and, the convergence of sequence. For each  $\varepsilon > 0$ , for large enough  $m \in \mathbb{N}$ , for every  $n \geq m$ , we can bound  $a_n$  in the range  $(a - \varepsilon, a + \varepsilon)$ , so are the supremum and infimum.

$$\forall \varepsilon > 0, \exists m \in \mathbb{N}, \text{ s.t. } \begin{cases} b_m < a + \varepsilon \\ c_m > a - \varepsilon \end{cases} \quad (4.6)$$

Also, by the convergence of  $(a_n)$ , there exists  $N^* \in \mathbb{N}$  such that  $\forall n \geq N^*, |a_n - a| < \frac{\varepsilon}{2}$ , which means  $a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$ . Therefore,

$$a - \frac{\varepsilon}{2} \leq \underbrace{\inf_{n \geq N} a_n}_{c_N} \leq \underbrace{\sup_{n \geq N} a_n}_{b_N} \leq a + \frac{\varepsilon}{2} \quad (4.7)$$

so, since  $c_N$  is increasing, and  $b_N$  is decreasing,  $(c_N) \rightarrow a$  and  $(b_N) \rightarrow a$ . ■

**Definition 4.3** (Double Index Sequence). A sequence is said to be in **double index form** (i.e. indexed by  $\mathbb{N}^2$ , which is also countable) if it can be written as

$$(a_{m,n}), \quad m, n \in \mathbb{N} \quad (4.8)$$



and  $\lim_{m \rightarrow \infty, n \rightarrow \infty} a_{m,n} = r$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N, |a_{m,n} - r| < \varepsilon \quad (4.9)$$

**Theorem 4.2.** Suppose, for sequence  $(a_{m,n})$ ,

$$\lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{m,n} \right) = a \quad (4.10)$$

$$\lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{m,n} \right) = b \quad (4.11)$$

$$\lim_{m \rightarrow \infty, n \rightarrow \infty} a_{m,n} = r \quad (4.12)$$

if  $a, b, r$  all exist, then  $a = b = r$ .

**Remark 4.1.** The theorem extends to sequence with countably many indices.