MAT395 Independent Reading in Mathematical Economics

Individual Decision Making, Market Equilibrium, Market Failure, and Other Topics.

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- GitHub: `https://github.com/TianyuDu/Spikey_UofT_Notes`
- Website: `TianyuDu.com/notes`

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1 Chapter 1. Preference and Choice

1.1 Preference Relations

Definition 1.1.

(i) The **strict preference** relation, $\succ$, is defined by

$$x \succ y \iff x \succsim y \wedge \neg(y \succsim x) \tag{1.1}$$

(ii) The **indifference** relation, $\sim$, is defined by

$$x \sim y \iff x \succsim y \wedge y \succsim x \tag{1.2}$$

Definition 1.2 (1.B.1)**.** The preference relation $\succsim$ is **rational** if it possesses the following two properties

(i) *Completeness*
$$\forall x, y \in X,\ x \succsim y \vee y \succsim x \tag{1.3}$$

(ii) *Transitivity*
$$\forall x, y, z \in X,\ x \succsim y \wedge y \succsim z \implies x \succsim z \tag{1.4}$$

Proposition 1.1 (1.B.1)**.** If $\succsim$ is rational, then

(i) $\succ$ is both **reflexive** ($\neg\ x \succ x$) and **transitive** ($x \succ y \wedge y \succ z \implies x \succ z$);

(ii) $\sim$ is both **reflexive** and **transitive**;

(iii) $x \succ y \succsim z \implies x \succ z$.

Example 1.1. Typical scenarios when transitivity of preference is violated:

(i) *Just perceptible differences*;

(ii) *Framing problem*;

(iii) *Observed preference might from the result of the interaction of several more primitive rational preferences (Condorcet paradox)*;

(iv) *Change of tastes.*

Definition 1.3 (1.B.2)**.** A function $u : X \to \mathbb{R}$ is a **utility function representing preference relation** $\succsim$ if
$$\forall x, y \in X,\ x \succsim y \iff u(x) \geq u(y) \tag{1.5}$$

Proposition 1.2 (1.B.2)**.** If a preference relation $\succsim$ can be represented by a utility function, then $\succsim$ is rational.

1.2 Choice Rules

Definition 1.4. A **choice structure**, $(\mathscr{B}, C(\cdot))$, is a tuple consists of

(i) The collection of **budget sets** $\mathscr{B}$, which is a set of nonempty subsets of X.

(ii) The **choice rule**, $C(B) \subset B$, is a *correspondence* for every $B \subset \mathscr{B}$ denotes the individual's choice from among the alternatives in B. If $C(B)$ is not a singleton, it can be interpreted as the *acceptable alternatives* in B, which the individual would actually chosen if the decision-making process is run repeatedly.

Definition 1.5 (1.C.1)**.** The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the **weak axiom of revealed preference** if

$$\Big(\underbrace{\exists B \in \mathscr{B} \ s.t. \ x, y \in B \land x \in C(B)}_{x \succsim *y \text{ revealed.}}\Big) \implies \Big(\forall B' \in \mathscr{B} \ s.t. \ x, y \in B', \ y \in C(B') \implies x \in C(B')\Big) \tag{1.6}$$

Definition 1.6. Given a choice structure $(\mathscr{B}, C(\cdot))$, the **revealed preference relation** $\succsim^*$ is defined as

$$x \succsim^* y \iff \exists B \in \mathscr{B} \ s.t. \ x, y \in B \land x \in C(B) \tag{1.7}$$

Remark 1.1 (Interpretation on the definition of WARP)**.** If x is *revealed* at least as good as y, then y cannot be revealed preferred to x.

1.3 The Relationship between Preference Relations and Choice Rules

Definition 1.7. Given rational preference relation $\succsim$ on X, the **preference-maximizing choice rule** is defined as

$$C^*(B, \succsim) := \{x \in B : x \succsim y \ \forall y \in B\} \ \forall B \in \mathscr{B} \tag{1.8}$$

We say the rational preference relation **generates** the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

Assumption 1.1. Assume $C^*(B, \succsim) \neq \varnothing$ for all $B \in \mathscr{B}$.

Proposition 1.3 (1.D.1 (Rational $\to$ WARP))**.** Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom.

Definition 1.8 (1.D.1)**.** Given choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ **rationalizes** $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim) \ \forall B \in \mathscr{B} \tag{1.9}$$

That is, $\succsim$ *generates the choice structure* $(\mathscr{B}, C(\cdot))$.

Remark 1.2. In general, for a given choice structure $(\mathscr{B}, C(\cdot))$, there may be more than one rational preference relation $\succsim$ rationalizing it.

Proposition 1.4 (1.D.2 (WARP $\to$ Rational))**.** If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) The weak axiom is satisfied;

(ii) $\mathscr{B}$ includes all subsets of X up to three elements.

Then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$.

2 Chapter 2. Consumer Choice

2.1 Commodities

Definition 2.1. Assume the number of **commodities** is finite and equal to L. In general, a **commodity vector** or **commodity bundle** is an element in a **commodity space**, typically $\mathbb{R}^L$.

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L \tag{2.1}$$

Remark 2.1 (Time Aggregation)**.** The time/location of commodity matters in some scenarios, and can be built into the definition of a commodity.

Remark 2.2. We should also note that in some contexts it becomes convenient, and even necessary, to expand the set of commodities to include goods and services that may potentially be available for purchase but are not actually so and even some that may be available by means other than market exchange.

2.2 The Consumption Set

Definition 2.2. The **consumption set** is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by his environment.

Assumption 2.1. For simplicity, we assume the consumption set to be $\mathbb{R}^L_+$, which is *convex*.

$$X := \mathbb{R}^L_+ = \{\mathbf{x} \in \mathbb{R}^L : x_\ell \geq 0,\ \forall \ell \in [L]\} \tag{2.2}$$

2.3 Competitive Budgets

Definition 2.3. A **price vector** is defined as

$$\mathbf{p} := \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L \tag{2.3}$$

For simplicity, here we always assume

(i) *Positive price*: $\mathbf{p} \gg \mathbf{0}$;

(ii) *Price-taking assumption*: $\mathbf{p}$ is beyond the influence of the consumer.

Definition 2.4 (2.D.1)**.** The **Walrasian**, or **competitive budget set** is defined as

$$B_{\mathbf{p},w} := \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} \leq w\} \tag{2.4}$$

where w is the *wealth* of consumer, and assumed to be positive.

Definition 2.5. The **consumer's problem** is choosing a consumption bundle $\mathbf{x} \in B_{\mathbf{p},w}$, for each given $(\mathbf{p}, w) \in \mathbb{R}^L_{++}$.

Definition 2.6. The set $\{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} = w\}$ is called the **budget hyperplane**.

Proposition 2.1. The price vector $\mathbf{p}$ is orthogonal to the budget hyperplane.

Proposition 2.2. The Walrasian budget set $B_{\mathbf{p},w}$ is a *convex* set.

2.4 Demand Functions and Comparative Statics

Definition 2.7. The consumer's **Walrasian demand correspondence** $x(\mathbf{p}, w) : \mathbb{R}^{L+1}_{++} \rightrightarrows \mathbb{R}^L_+$ assigns a *set* of chosen consumption bundles for each price-wealth pair $(\mathbf{p}, w)$. When $x(\mathbf{p}, w)$ is single-valued, we refer to it as a **demand function**

$$\mathbf{x}(\mathbf{p}, w) = \begin{bmatrix} x_1(\mathbf{p}, w) \\ x_2(\mathbf{p}, w) \\ \vdots \\ x_L(\mathbf{p}, w) \end{bmatrix} \tag{2.5}$$

Definition 2.8 (2.E.1)**.** The Walrasian demand correspondence $x(\mathbf{p}, w) : \mathbb{R}^{L+1}_{++} \rightrightarrows \mathbb{R}^L_+$ is **homogenous of degree zero** if

$$x(\alpha\mathbf{p}, \alpha w) = x(\mathbf{p}, w)\ \forall (\mathbf{p}, w, \alpha) \in \mathbb{R}^{L+2}_{++} \tag{2.6}$$

Also note that

$$B_{\mathbf{p},w} = B_{\alpha\mathbf{p},\alpha w}\ \forall (\mathbf{p}, w, \alpha) \in \mathbb{R}^{L+2}_{++} \tag{2.7}$$

Definition 2.9 (2.E.2)**.** The Walrasian demand correspondence $x(\mathbf{p}, w)$ satisfies **Walras' law** if

$$\forall (\mathbf{p}, w) \gg \mathbf{0},\ \forall \mathbf{x} \in x(\mathbf{p}, w),\ \mathbf{p} \cdot \mathbf{x} = w \tag{2.8}$$

Assumption 2.2. For simplicity, we assume $x(\mathbf{p}, w)$ is always *single-valued, continuous and differentiable.*

Proposition 2.3. The **family of Walrasian budget sets** defined as

$$\mathscr{B}^{\mathscr{W}} := \{B_{\mathbf{p},w} : \mathbf{p}, w \gg \mathbf{0}\} \tag{2.9}$$

altogether with Walrasian demand homogeneous to degree zero forms a *choice structure*

$$(\mathscr{B}^{\mathscr{W}}, x(\cdot)) \tag{2.10}$$

Definition 2.10. For fixed prices $\overline{\mathbf{p}} \in \mathbb{R}^L_{++}$, the function of wealth $\mathbf{x}(\overline{\mathbf{p}}, w)$ is called consumer's **Engel function**. Its image in $\mathbb{R}^L_+$,

$$E_{\overline{\mathbf{p}}} := \{\mathbf{x}(\overline{\mathbf{p}}, w) : w \in \mathbb{R}_{++}\} \subset \mathbb{R}^L_+ \tag{2.11}$$

is defined as the **wealth expansion path**.

Definition 2.11. Given $(\mathbf{p}, w)$, the **wealth effect** is defined as

$$D_w\mathbf{x}(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial w} \\ \frac{\partial x_2(\mathbf{p},w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(\mathbf{p},w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L \tag{2.12}$$

For the ℓ-th commodity, it's called **normal** at $(\mathbf{p}, w)$ if $\frac{\partial x_\ell(\mathbf{p},w)}{\partial w} \geq 0$, and **inferior** otherwise. And the ℓ-th commodity is normal/inferior if its normal/inferior every where in $\mathbb{R}^{L+1}_{++}$.

Definition 2.12. The **offer curve** is defined as the locus

$$\{\mathbf{x}(\mathbf{p}, w) : p_j > 0\} \tag{2.13}$$

for any chosen j.

Definition 2.13. Good ℓ is said to be a **Giffen good** at $(\mathbf{p}, w)$ if

$$\frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_\ell} > 0 \tag{2.14}$$

Definition 2.14. The **price effects** at $(\mathbf{p}, w)$ is defined as

$$D_{\mathbf{p}}(\mathbf{p}, w) = \begin{bmatrix} \frac{\partial x_1(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_1(\mathbf{p},w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(\mathbf{p},w)}{\partial p_1} & \cdots & \frac{\partial x_L(\mathbf{p},w)}{\partial p_L} \end{bmatrix} \tag{2.15}$$

Proposition 2.4 (2.E.1)**.** If the Walrasian demand function $x(\mathbf{p}, w)$ is homogenous of degree zero, then for all $\mathbf{p}$ and w, then

$$\sum_{k=1}^{L} \frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_k} p_k + \frac{\partial x_\ell(\mathbf{p}, w)}{\partial w} w = 0 \text{ for } \ell = 1, \ldots, L \tag{2.16}$$

Equivalently,

$$D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ \mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ w = \mathbf{0} \tag{2.17}$$

Proof. Apply *Euler's theorem* on homogenous functions to each component x_ℓ.

$$\underbrace{D_{(\mathbf{p},w)}\mathbf{x}(\mathbf{p}, w)}_{L\times(L+1)} \cdot \underbrace{(\mathbf{p}, w)}_{(L+1)\times 1} = 0\ \mathbf{x}(\mathbf{p}, w) = \mathbf{0} \tag{2.18}$$

$$\implies [\underbrace{D_{\mathbf{p}}(\mathbf{p}, w)}_{L\times L} \,|\, \underbrace{D_w\mathbf{x}(\mathbf{p}, w)}_{L\times 1}] \cdot (\mathbf{p}, w) = D_{\mathbf{p}}(\mathbf{p}, w)\ \mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ w = \mathbf{0} \tag{2.19}$$

■

Definition 2.15. The **elasticities of demand ℓ with respect to price k and wealth** is defined as

$$\varepsilon_{\ell,k}(\mathbf{p},w) := \frac{\partial x_\ell(\mathbf{p},w)}{\partial p_k}\frac{p_l}{x_\ell(\mathbf{p},w)} \tag{2.20}$$

$$\varepsilon_{\ell,w}(\mathbf{p},w) := \frac{\partial x_\ell(\mathbf{p},w)}{\partial w}\frac{w}{x_\ell(\mathbf{p},w)} \tag{2.21}$$

Corollary 2.1. Dividing both sides of the equality in proposition (2.E.1) by x_ℓ gives

$$\sum_{k=1}^{L} \varepsilon_{\ell,k}(\mathbf{p},w) + \varepsilon_{\ell,w}(\mathbf{p},w) = 0 \ \forall \ell \in \{1,\dots,L\} \tag{2.22}$$

Proposition 2.5 (2.E.2 Cournot Aggregation)**.** If the Walrasian demand function $x(\mathbf{p},w)$ satisfies *Walras' law*, then for every $(\mathbf{p},w)$,

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(\mathbf{p},w)}{\partial p_k} + x_k(\mathbf{p},w) = 0 \quad \text{for } k = 1,\dots,L \tag{2.23}$$

Equivalently,

$$\mathbf{p}^T D_{\mathbf{p}}\mathbf{x}(\mathbf{p},w) + \mathbf{x}(\mathbf{p},w)^T = \mathbf{0}^T \tag{2.24}$$

Proof. Differentiate both sides of Walras' law identity $\mathbf{p}^T\mathbf{x} = w$ with respect to $\mathbf{p}$. ■

Proposition 2.6 (2.E.3. Engel Aggregation)**.** If the Walrasian demand function $x(\mathbf{p},w)$ satisfies *Walras' law*, then for every $(\mathbf{p},w)$,

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(\mathbf{p},w)}{\partial w} = 1 \tag{2.25}$$

or equivalently

$$\mathbf{p}\cdot D_w x(\mathbf{p},w) = 1 \tag{2.26}$$

Proof. Differentiate both sides of Walras' law identity $\mathbf{p}^T\mathbf{x} = w$ with respect to w. ■

Proposition 2.7 (Exer. 2.E.2)**.**

$$\sum_{\ell=1}^{L} b_\ell(\mathbf{p},w)\varepsilon_{\ell k}(\mathbf{p},w) + b_k(\mathbf{p},w) = 0 \tag{2.27}$$

and

$$\sum_{\ell=1}^{L} b_\ell(\mathbf{p},w)\varepsilon_{\ell w}(\mathbf{p},w) = 1 \tag{2.28}$$

where $b_\ell := \frac{x_\ell p_\ell}{w}$ is defined to be the portion of wealth spent on commodity ℓ.

2.5 The Weak Axiom of Revealed Preference and the Law of Demand

Assumption 2.3. In the section, we assume $\mathbf{x}(\mathbf{p},w)$ is

(i) Single-valued;

(ii) homogeneous to degree zero;

(iii) satisfies Walras' law.

Definition 2.16 (2.F.1)**.** The Walrasian demand function $\mathbf{x}(\mathbf{p}, w)$ satisfies the **weak axiom of revealed preference** if for every two $(\mathbf{p}, w), (\mathbf{p}', w') \in \mathbb{R}^{L+1}_{++}$,

$$\underbrace{\mathbf{p} \cdot \mathbf{x}(\mathbf{p}', w') \leq w \wedge \mathbf{x}(\mathbf{p}, w) \neq \mathbf{x}(\mathbf{p}', w')}_{\text{revealed: } \mathbf{x}(\mathbf{p},w) \succ^* \mathbf{x}(\mathbf{p}',w')} \implies \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w) > w' \tag{2.29}$$

Equivalently,

$$\mathbf{x}(\mathbf{p}', w') \in B_{\mathbf{p},w} \wedge \mathbf{x}(\mathbf{p}', w') \notin C(B_{\mathbf{p},w}) \implies \mathbf{x}(\mathbf{p}, w) \notin C(B_{\mathbf{p}',w'}) \tag{2.30}$$

Corollary 2.2 (Equivalent Definition)**.** The weak axiom says, given our assumptions and $\mathbf{x}(\mathbf{p}_1, w_1) \neq \mathbf{x}(\mathbf{p}_2, w_2)$, we <u>cannot</u> have both

$$\mathbf{x}(\mathbf{p}_1, w_1) \in B_{\mathbf{p}_2,w_2} \wedge \mathbf{x}(\mathbf{p}_2, w_2) \in B_{\mathbf{p}_1,w_1} \tag{2.31}$$

Definition 2.17. A price change $\Delta\mathbf{p}$ is a **Slutsky compensated price change** if the consumer is given a **Slutsky wealth compensation** with amount

$$\Delta w = \Delta\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, w) \tag{2.32}$$

such that the consumer's initial consumption is just affordable at the new price.

Proposition 2.8 (2.F.1)**.** Suppose that the Walrasian demand function $\mathbf{x}(\mathbf{p}', w')$ is homogenous of degree zero and satisfies Walras' law. Then $\mathbf{x}(\mathbf{p}', w')$ satisfies the weak axiom <u>if and only if</u> the following property holds:
For any *compensated price change* from $(\mathbf{p}, w)$ to $(\mathbf{p}', w' := \mathbf{p}' \cdot \mathbf{x}(\mathbf{p}, w))$,

$$\Delta\mathbf{p} \cdot \Delta\mathbf{x} \leq 0 \tag{2.33}$$

with strict inequality whenever $\mathbf{x}(\mathbf{p}, w) \neq \mathbf{x}(\mathbf{p}', w')$.

Corollary 2.3 (Compensated Law of Demand)**.** $\Delta\mathbf{p} \cdot \Delta\mathbf{x} \leq 0$ says demand and price move in opposite directions, *under Slutsky compensation.*

Definition 2.18. At infinitesimal price change, the Slutsky compensation can be written as

$$dw = \mathbf{x}(\mathbf{p}, w) \cdot d\mathbf{p} \tag{2.34}$$

and the compensated law of demand becomes

$$d\mathbf{p} \cdot d\mathbf{x} \leq 0 \tag{2.35}$$

Then the total derivative of $\mathbf{x}$ is

$$\begin{aligned}
d\mathbf{x} &= D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ d\mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ dw && (2.36)\\
&= D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)\ d\mathbf{p} + D_w\mathbf{x}(\mathbf{p}, w)\ [\mathbf{x}(\mathbf{p}, w) \cdot d\mathbf{p}] && (2.37)\\
&= [\underbrace{D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w)}_{L\times L} + \underbrace{D_w\mathbf{x}(\mathbf{p}, w)}_{L\times 1}\underbrace{\mathbf{x}(\mathbf{p}, w)^T}_{1\times L}]d\mathbf{p} && (2.38)\\
\implies d\mathbf{p}^T &\underbrace{[D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w) + D_w\mathbf{x}(\mathbf{p}, w)\mathbf{x}(\mathbf{p}, w)^T]}_{L\times L} d\mathbf{p} \leq 0 && (2.39)
\end{aligned}$$

and the **Slutsky/substitution matrix** is defined as

$$\begin{aligned}
S(\mathbf{p}, w) &:= [D_{\mathbf{p}}\mathbf{x}(\mathbf{p}, w) + D_w\mathbf{x}(\mathbf{p}, w)\mathbf{x}(\mathbf{p}, w)^T] && (2.40)\\
s_{\ell k} &= \underbrace{\frac{\partial x_\ell(\mathbf{p}, w)}{\partial p_k}}_{\text{total effect}} + \underbrace{\frac{\partial x_\ell(\mathbf{p}, w)}{\partial w} x_k(p, w)}_{\text{wealth effect}} && (2.41)
\end{aligned}$$

where $s_{\ell k}$ is the **substitution effect**.

Remark 2.3. The above identity (Slutsky equation) suggests the total impact of price change in p_k on demand for x_ℓ can be decomposed into two portions, substitution effect and income effect.

Corollary 2.4 (Slutsky Equation)**.**

$$\frac{\partial x_i(\mathbf{p},w)}{\partial p_j} = \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} - \frac{\partial x_i(\mathbf{p},w)}{\partial w} x_j(\mathbf{p},w) \tag{2.42}$$

Remark 2.4. Consider the scenario when only p_k changes, with Slutsky compensation, consumer's wealth changes by $dw = x_k(\mathbf{p},w)dp_k$. So the wealth effect on x_ℓ is $\frac{\partial x_\ell}{\partial w}dw = \frac{\partial x_\ell}{\partial w}x_k(\mathbf{p},w)dp_k$.

Proposition 2.9 (2.F.2)**.** If a differentiable Walrasian demand function $\mathbf{x}(\mathbf{p},w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any $(\mathbf{p},w)$, the Slutsky matrix $S(\mathbf{p},w)$ is negative semi-definite.

Corollary 2.5. Given $S(\mathbf{p},w)$ is negative semi-definite, we have

$$\mathbf{e}_\ell^T S(\mathbf{p},w)\mathbf{e}_\ell \le 0 \ \forall \ell \in \{1,\dots,L\} \tag{2.43}$$

$$\implies s_{\ell\ell} \le 0 \ \forall \ell \in \{1,\dots,L\} \tag{2.44}$$

which suggests the *substitution effect of good ℓ with respect to its own price is always negative.*

Remark 2.5. Proposition 2.F.2 does *not* imply, in general, that the matrix $S(\mathbf{p},w)$ is symmetric.

Proposition 2.10 (2.F.3)**.** Suppose that the Walrasian demand function $\mathbf{x}(\mathbf{p},w)$ is differentiable, homogenous of degree zero, and satisfies Walras' law. Then for every $(\mathbf{p},w)$

$$\mathbf{p}^T S(\mathbf{p},w) = \mathbf{0} \wedge S(\mathbf{p},w)\mathbf{p} = \mathbf{0} \tag{2.45}$$

Proof. By propositions 2.E.1 to 2.E.3. ∎

3 Chapter 3. Classical Demand Theory

3.1 Preference Relations: Basic Properties

Definition 3.1 (3.B.1)**.** The preference relation $\succsim$ on X is **rational** if it possesses the following two properties

(i) *Completeness.* $\forall \mathbf{x},\mathbf{y} \in X,\ \mathbf{x} \succsim \mathbf{y} \vee \mathbf{y} \succsim \mathbf{x}$;

(ii) *Transitivity.* $\forall \mathbf{x},\mathbf{y},\mathbf{z} \in X,\ \mathbf{x} \succsim \mathbf{y} \wedge \mathbf{y} \succsim \mathbf{z} \implies \mathbf{x} \succsim \mathbf{z}$.

3.1.1 Desirability Assumptions

Definition 3.2 (3.B.2)**.** The preference relation $\succsim$ on X is **monotone** if $\mathbf{x} \in X$ and $\mathbf{y} \gg \mathbf{x} \implies \mathbf{y} \succ \mathbf{x}$; It is **strongly monotone** if $\mathbf{y} \ge \mathbf{x} \wedge \mathbf{x} \ne \mathbf{y}$

Remark 3.1. If $\succsim$ is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities.

Definition 3.3 (3.B.3)**.** A preference relation $\succsim$ on X is **locally nonsatiated** if

$$\forall \mathbf{x} \in X, \varepsilon > 0,\ \exists \mathbf{y} \in \overline{\mathcal{B}}(\mathbf{x},\varepsilon) \cap X \ s.t.\ \mathbf{y} \succ \mathbf{x} \tag{3.1}$$

Remark 3.2. Local nonsatiation rules out the extreme situation in which all commodities are bads, since in that case no consumption at all (the point $\mathbf{x} = \mathbf{0}$) would be a satiation point.

Proposition 3.1 (Exercise 3.B.1)**.**

$$\text{Strongly Monotone} \implies \text{Monotone} \implies \text{Locally Non-satiation} \tag{3.2}$$

Definition 3.4. The **indifference set** containing point $\mathbf{x}$ is defined as $\{\mathbf{y} \in X : \mathbf{x} \sim \mathbf{y}\}$. The **upper contour set** of bundle $\mathbf{x}$ is $\{\mathbf{y} \in X : \mathbf{y} \succsim \mathbf{x}\}$. The **lower contour set** of $\mathbf{x}$ is defined as $\{\mathbf{y} \in X : \mathbf{x} \succsim \mathbf{y}\}$.

Remark 3.3 (Implication of Local Nonsatiation)**.** One implication of local nonsatiation (and, hence, of monotonicity) is that it rules out "thick" indifference sets.

3.1.2 Convexity Assumptions

Definition 3.5 (3.B.4)**.** The preference relation $\succsim$ on X is **convex** if for every $\mathbf{x} \in X$, the upper contour set if $\mathbf{x}$ is convex.

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \mathbf{y} \succsim \mathbf{x} \wedge \mathbf{z} \succsim \mathbf{x} \implies \alpha\mathbf{y} + (1-\alpha)\mathbf{z} \succsim \mathbf{x}\ \forall \alpha \in [0,1] \tag{3.3}$$

Remark 3.4 (Implication of Convexity)**.** Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification.

Remark 3.5. The convex assumption can hold only if X is convex.

Definition 3.6 (3.B.5)**.** The preference relation $\succsim$ on X is **strictly convex** if

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X,\ \mathbf{y} \succ \mathbf{x} \wedge \mathbf{z} \succ \mathbf{x} \wedge \mathbf{y} \neq \mathbf{z} \implies \alpha\mathbf{y} + (1-\alpha)\mathbf{z} \succ \mathbf{x}\ \forall \alpha \in (0,1) \tag{3.4}$$

Definition 3.7 (3.B.6)**.** A monotone preference relation $\succsim$ on $X = \mathbb{R}_+^L$ is **homothetic** if

$$\forall \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \sim \mathbf{y} \implies \alpha\mathbf{x} \sim \alpha\mathbf{y},\ \forall \alpha \in \mathbb{R}_+ \tag{3.5}$$

Definition 3.8 (3.B.7)**.** The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is **quasilinear** with respect to commodity 1 (the **numeraire** commodity) if $\forall \mathbf{x}, \mathbf{y} \in X$

(i) $\mathbf{x} \sim \mathbf{y} \implies \mathbf{x} + \alpha\mathbf{e}_1 \sim \mathbf{y} + \alpha\mathbf{e}_1\ \forall \alpha \in \mathbb{R}$;

(ii) *Good 1 is desirable*: $\forall \mathbf{x} \in X, \alpha \in \mathbb{R}_{++},\ \mathbf{x} + \alpha\mathbf{e}_1 \succ \mathbf{x}$.

3.2 Preference and Utility

Definition 3.9 (Example 3.C.1)**.** The **lexicographic preference relation** on $X = \mathbb{R}_+^2$ defines $x \succsim y$ if either $x_1 > y_1$ or $x_1 = y_1 \wedge x_2 \geq y_2$.

Definition 3.10 (3.C.1)**.** The preference relation $\succsim$ on X is **continuous** if it is *preserved under limits*. That's

$$\forall ((x^n, y^n)_n)_{n=1}^{\infty}\ s.t.\ x := \lim_{n\to\infty} x^n,\ y := \lim_{n\to\infty} y^n, \quad x^n \succsim y^n\ \forall n \implies x \succsim y \tag{3.6}$$

Proposition 3.2 (Equivalent Definition)**.** A preference relation $\succsim$ is continuous if and only if for all $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ and lower contour set $\{y \in X : x \succsim y\}$ are closed.

Proof. Suppose $\succsim$ is continuous, fix $x \in X$. Then for any sequence in the upper contour set of x, the limit point is also in the upper contour set of x. As a result, for every $x \in X$, U_x contains all limit points, so it is closed. ■

Proposition 3.3. Lexicographic preference relation is *not* continuous.

Proof.

$$x^n := (1/n, 0) \text{ and } y^n := (0, 1) \tag{3.7}$$

■

Proposition 3.4.][3.C.1] Let $\succsim$ be a continuous preference relation on X, there is a continuous utility function $u : X \to \mathbb{R}$ representing $\succsim$.

Proof. Construction of utility function:

(i) For each $x \in X$, by monotonicity and continuity of $\succsim$, there exists an unique $\alpha(x)$ such that

$$\alpha(x)e \sim x \tag{3.8}$$

(ii) Take $\alpha(x)$ as the utility function.

■

Remark 3.6. Above proposition guarantees the existence of continuous utility function for any continuous $\succsim$. But, not all utility functions representing $\succsim$ are continuous. We can construct discontinuous utility function by compositing a continuous utility function with a discontinuous but strictly increasing transformation.

Remark 3.7. It is possible for continuous preferences *not* to be representable by a differentiable (but still continuous) utility function (*Leontief*).

Lemma 3.1. The upper contour set of a quasi-concave function is convex.

Proposition 3.5. [?] is this bi-conditional? If $\succsim$ is (strictly) convex, then $u(\cdot)$ representing $\succsim$ is (strictly) quasi-concave.

Proposition 3.6. A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if and only if it admits a utility function u homogeneous of degree one.

Proposition 3.7. A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is *quasilinear* with respect to the first commodity (numeraire) if and only if it admits a utility function u in the form $u(x) = x_1 + \phi(x_2, \dots, x_L)$.

Remark 3.8. Increasingness and quasi-concavity are ordinal properties of u; they are preserved for any arbitrary increasing transformation of the utility index. In contrast, the special forms of the utility representations in above propositions are not preserved; they are cardinal properties that are simply convenient choices for a utility representation.

3.3 The Utility Maximization Problem

Definition 3.11. Suppose a consumer chooses her most *preferred consumption bundle* given prices $p \gg 0$ and wealth level $w > 0$, then the **utility maximization problem**(UMP) of this consumer is

$$\max_{x \geq 0} u(x) \ s.t. \ p \cdot x \leq w \tag{3.9}$$

Proposition 3.8 (3.D.1)**.** If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proof. Note $B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$ is compact. The proposition is an immediate consequence of the extreme value theorem. ■

3.3.1 The Walrasian Demand Correspondence/Function

Definition 3.12. The **Walrasian demand correspondence**, $x(p, w)$, is the set of solutions to consumer's UMP. When the solution is unique, it is referred to as the walrasian demand function.

Proposition 3.9 (3.D.2)**.** Suppose u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on $X := \mathbb{R}^L_+$. The the Walrasian demand correspondence, $x(p, w)$, satisfies

(i) *Homogeneous of degree zero* in (p, w);

(ii) *Walras' law*: $p \cdot x = w$;

(iii) *Convexity* if $\succsim$ is convex (i.e. u is quasi-concave), then $x(p, w)$ is convex;

(iv) *Uniqueness* if $\succsim$ is strictly convex (i.e. u is strictly quasi-concave), then $x(p, w)$ is a singleton.[1]

[1] A singleton set is trivially convex.

Proposition 3.10 (Kuhn-Tucker Necessary Condition)**.** Let $x^* \in x(p,w)$, then there exists a *Lagrangian multiplier* $\lambda \geq 0$ such that

$$\nabla u\left(x^*\right) \leq \lambda p \tag{3.10}$$

$$x^* \cdot \left[\nabla u\left(x^*\right) - \lambda p\right] = 0 \text{ (complementary slackness)} \tag{3.11}$$

As a result, for any interior optimum ($x^* \gg 0$),

$$\nabla u\left(x^*\right) = \lambda p \tag{3.12}$$

Corollary 3.1. If $\nabla u\left(x^*\right) \gg 0$, then the first order necessary condition for an interior optimum to UMP is equivalent to

$$\frac{\partial u\left(x^*\right)/\partial x_\ell}{\partial u\left(x^*\right)/\partial x_k} = \frac{p_\ell}{p_k} \tag{3.13}$$

for every ℓ, k.

Definition 3.13. The left hand side of above equality is the **marginal rate of substitution** of good ℓ for good k at x^*, $MRS_{\ell k}$ at (x^*). It tells us the amount of good k that the consumer must be given to compensate her for a one-unit marginal reduction in her consumption of good ℓ ($\frac{dx_k}{dx_\ell}$).

Proposition 3.11 (Interpretation of λ)**.** The Lagrangian multiplier λ gives the **shadow price** of relaxing the wealth constraint in UMP. Therefore it equals the *marginal utility value of wealth* at the optimum.

Proof. This is an immediate consequence of the envelope theorem. ∎

Proposition 3.12. If u is quasi-concave and monotone, and has $\nabla u(x) \neq 0 \; \forall x \in \mathbb{R}^L_+$, then the Kuhn-Tucker conditions are indeed sufficient.

Proposition 3.13. Indeed, if preferences are continuous, strictly convex, and locally nonsatiated on the consumption set $\mathbb{R}^L_+$,then $x(p,w)$ (which is then a function) is always continuous at all $(p,w) \gg 0$.

3.3.2 The Indirect Utility Function

Definition 3.14. The value function of consumer's UMP, $v(p,w) := u(x^*(p,w))$, is called the **indirect utility function**.

Proposition 3.14 (3.D.3)**.** Suppose u is a continuous utility function representing a locally nonsatiated $\succsim$ on $\mathbb{R}^L_+$, then $v(p,w)$ satisfies

(i) Homogeneous of degree zero;

(ii) Strictly increasing in w and non-increasing in p_ℓ for every ℓ;

(iii) Quasi-convex (i.e. its lower contour set is convex);

(iv) Continuous in (p,w).

Proof. Show quasi-convexity of $v(p,w)$. Let $\overline{v} \in \mathbb{R}$ be an attainable utility level, the corresponding lower contour is $L := \{(p,w) : v(p,w) \leq \overline{v}\}$. Let $(p,w),(p',w') \in L$, $\alpha \in [0,1]$. Show $(p'',w'') := \alpha(p,w) + (1-\alpha)(p',w') \in L$ by showing $u(x) \leq \overline{v}$ for every $p'' \cdot x \leq w''$. Suppose $p'' \cdot x \leq w''$, then

$$\alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w' \tag{3.14}$$

$$\implies p \cdot x \leq w \vee p' \cdot x \leq w' \tag{3.15}$$

which implies either $u(x) \leq v(p,w)$ or $u(x) \leq v(p',w')$, by the definition of value function of maximization problems. Since both $v(p,w), v(p',w') \leq \overline{v}$, then $u(x) \leq \overline{v}$. Therefore $v(p'',w'') \leq \overline{v}$. So $(p'',w'') \in L$, and L is convex. ∎

Proposition 3.15 (Transformation on v)**.** [?] Does this require f to be strictly increasing? Note that the indirect utility function depends on the utility representation chosen. In particular, if $v(p,w)$ is the indirect utility function when the consumer's utility function is u, then the indirect utility function corresponding to utility representation $\tilde{u}(x) = f \circ u(x)$ is $\tilde{v}(p,w) = f \circ v(p,w)$.

Proof. the maximizer of such an optimization problem is invariant under such a monotonically increasing transformation f. ∎

3.4 The Expenditure Minimization Problem

Definition 3.15. Suppose a consumer chooses her most *preferred consumption bundle* given prices $p \gg 0$ and wealth level $u > u(0)$, then the **expenditure minimization problem**(EMP) of this consumer is

$$\min_{x \geq 0} p \cdot x \ s.t. \ u(x) \geq u \tag{3.16}$$

Definition 3.16. The value function of above optimization problem is called the **expenditure function**, denoted as $e(p, u)$.

Assumption 3.1. We assume that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$.

Proposition 3.16 (3.E.1, the Duality)**.** Suppose u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$, and $p \gg 0$. Then,

(i) If x^* is optimal in the UMP when wealth $w > 0$, then x^* is the in the EMP with utility level $u(x^*)$, and the minimal expenditure is w;

(ii) If x^* is optimal in the EMP with utility level $u > u(0)$, then x^* is optimal in the UMP with wealth level $p \cdot x^*$, and the attained maximal utility is u.

Proof. Contradiction. ■

Corollary 3.2. For any $p \gg 0$, $w > 0$, and $u > u(0)$,

$$e(p, v(p, w)) = w \tag{3.17}$$

$$v(p, e(p, u)) = u \tag{3.18}$$

Corollary 3.3.

$$h(p, u) = x(p, e(p, u)) \tag{3.19}$$

$$x(p, w) = h(p, v(p, w)) \tag{3.20}$$

Proposition 3.17 (3.E.2)**.** Suppose that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$. Then the expenditure function $e(p, u)$ possesses the following properties

(i) Homogeneous of degree one in p;

(ii) Strictly increasing in u and nondecreasing in p_ℓ for every ℓ;

(iii) Concave in p;

(iv) Continuous in (p, u).

Proof. Show the concavity of e, let $p, p' \gg 0$, $\alpha \in [0, 1]$, and $\overline{u} > u(0)$. Define $p'' := \alpha p + (1 - \alpha)p'$, then

$$e(p'', \overline{u}) = p'' \cdot x'' \tag{3.21}$$

$$= \alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \tag{3.22}$$

$$\geq \alpha e(p, \overline{u}) + (1 - \alpha)e(p', \overline{u}) \tag{3.23}$$

■

3.4.1 The Hicksian (or Compensated) Demand Function

Definition 3.17. The set of solutions to EMP, $h(p,u) \subseteq \mathbb{R}_+^L$, is known as the **Hicksian, or compensated, demand correspondence, or function** if single-valued.

Proposition 3.18 (3.E.3)**.** Suppose that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$. Then for any $p \gg 0$, the Hicksian demand $h(p,u)$ possess the following properties

(i) *Homogeneous of degree zero* in p;

(ii) *No excess utility*: $\forall x \in h(p,u),\ u(x) = u$;

(iii) *Convexity*: if $\succsim$ is convex, then $h(p,u)$ is convex;

(iv) *Uniqueness*: if $\succsim$ is strictly convex, then $h(p,u)$ is a singleton.

Definition 3.18. As prices vary, $h(p,u)$ gives precisely the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u. The amount of wealth compensated to ensure the original utility level attainable is referred to as the **Hicksian wealth compensation**.

$$\Delta w_{\text{ Hicks }} = e\left(p^{\prime}, u\right) - w \tag{3.24}$$

3.4.2 Hicksian Demand and the Compensated Law of Demand

Proposition 3.19 (3.E.4, the Compensated Law of Demand)**.** Suppose that u is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}_+^L$. And suppose $h(p,u)$ is single-valued everywhere, then for all p' and p'',

$$\left(p^{\prime\prime} - p^{\prime}\right) \cdot \left[h\left(p^{\prime\prime}, u\right) - h\left(p^{\prime}, u\right)\right] \leq 0 \tag{3.25}$$

That's, *Demand and price move in opposite directions for price changes that are accompanied by Hicksian wealth compensation.*

Proof. Suppose $u > u(0)$, and the price changes from p' tp p'', then

$$p^{\prime\prime} \cdot h(p^{\prime\prime}, u) \leq p^{\prime\prime} \cdot h(p^{\prime}, u) \tag{3.26}$$
$$p^{\prime} \cdot h(p^{\prime\prime}, u) \geq p^{\prime} \cdot h(p^{\prime}, u) \tag{3.27}$$

subtracting above inequalities gives the desired result. ∎

3.5 Duality: A Mathematical Introduction

Definition 3.19. A **half-space** is a set of the form $\left\{x \in \mathbb{R}^L : p \cdot x \geq c\right\}$ for some $p \in \mathbb{R}^L$, $p \neq 0$ is called the **normal vector** to the half-space. Its boundary $\left\{x \in \mathbb{R}^L : p \cdot x = c\right\}$ is called a **hyperplane**.

Definition 3.20 (3.F.1)**.** For any nonempty closed set $K \subset \mathbb{R}^L$, the **support function** of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\} \tag{3.28}$$

Proposition 3.20 (3.F.1, The Duality Theorem)**.** Let K be a nonempty closed set, and let μ_K be its support function. Then there is a unique $\mathbf{x} \in K$ such that $\mathbf{p} \cdot \mathbf{x} = \mu_k(p)$ if and only if μ_K is differentiable at $\mathbf{p}$. Moreover, in this case

$$\nabla \mu_K(\mathbf{p}) = \mathbf{x} \tag{3.29}$$

3.6 Relationships between Demand, Indirect Utility, and Expenditure Functions

Proposition 3.21 (3.G.1, Shephard's lemma)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p,u)$ satisfies

$$h(p,u) = \nabla_p e(p,u) \tag{3.30}$$

Proof using Envelope Theorem. The Lagrangian function for EMP is $\mathcal{L} := p \cdot x - \lambda(u - \overline{u})$. By the envelope theorem, at the optimum,

$$\partial_p e(p,u) = \partial_p \mathcal{L}(p, h(p,u), \overline{u}, \lambda) \tag{3.31}$$

$$\implies \nabla_p e(p,u) = h(p,u) \tag{3.32}$$

■

Remark 3.9 (Interpretation)**.** The above proposition says if we are at an optimum in the EMP, the changes in demand caused by price changes have no first-order effect on the consumer's expenditure.

Proposition 3.22 (3.G.2)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Suppose also that h is continuously differentiable at (p,u), then $D_p h(p,u) \in M_{L\times L}$ satisfies

(i) $D_p h(p,u) = D^2_p e(p,u)$;

(ii) $D_p h(p,u)$ is a negative semidefinite matrix;

(iii) $D_p h(p,u)$ is symmetric;

(iv) $D_p h(p,u) \cdot p = 0$.

Proposition 3.23 (3.G.3, the Slutsky Equation)**.** Suppose that u is a <u>continuous</u> utility function representing a <u>locally nonsatiated</u> and <u>strictly convex</u> preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T \tag{3.33}$$

Proof. Let $p \gg 0, w > 0, u > u(0)$, $h(p,u) = x(p, e(p,u))$, and $w = e(p,u)$. Differentiate both sides of this identity gives

$$D_p h(p,u) = D_p x(p,e(p,u)) + D_w x(p,e(p,u)) \; D_p e(p,u) \tag{3.34}$$

$$= D_p x(p,e(p,u)) + D_w x(p,e(p,u)) \; h(p,u) \text{ (Shephard's lemma)} \tag{3.35}$$

$$= D_p x(p,e(p,u)) + D_w x(p,e(p,u)) \; x(p,w) \tag{3.36}$$

■

Definition 3.21. The **Slutsky substitution matrix** is defined as

$$S(p,w) := D_p h(p,u) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix} \tag{3.37}$$

where

$$\underbrace{s_{\ell k}(p,w)}_{\text{SE}} = \underbrace{\partial x_\ell(p,w)/\partial p_k}_{\text{TE}} + \underbrace{[\partial x_\ell(p,w)/\partial w]\, x_k(p,w)}_{\text{IE}} \tag{3.38}$$

3.6.1 Walrasian Demand and the Indirect Utility Function

Proposition 3.24 (3.G.4 Roy's Identity)**.** Suppose that u is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X := \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(p, w) \gg 0$. Then

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w) \tag{3.39}$$

Proof. Apply the envelope theorem to UMP,

$$\nabla_p v(p,w) = \partial_p[u(x^*) - \lambda^*(w - p \cdot x^*)] = \lambda^* x^* \tag{3.40}$$
$$\nabla_w v(p,w) = \partial_w[u(x^*) - \lambda^*(w - p \cdot x^*)] = -\lambda^* \tag{3.41}$$
$$\implies x^* = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w) \tag{3.42}$$

■

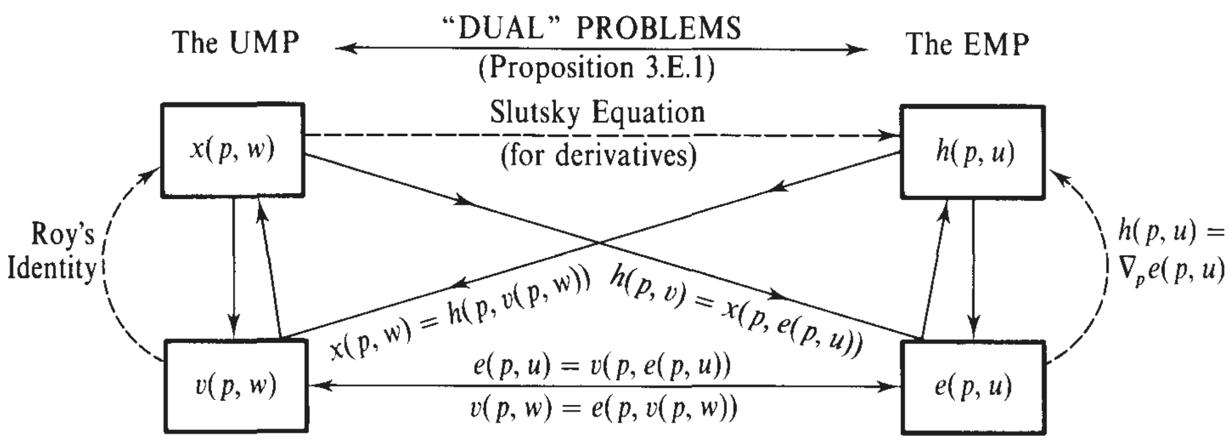

Figure 3.1: Summary