Notes on MAT137 Video Playlist 3

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1.1   Define Derivate As Slope

Definition   Let $a \in \mathbb{R}$, and f(x) is defined on $(a-\delta, a+\delta)$, then the **derivative** of f(x) at a is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Definition   If function is **differentiable** at point x = a, if and only if, there exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Interpretation $f'(a)$ is the slope of tangent line a x = a.

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1.2   Calculate f'(x) by definition

Example $f(x) = 4x - x^2$, find $f'(1)$:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{4(h+1) - (h+1)^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{4h + 4 - 3 - h^2 - 2h - 1}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h}$$

$$= \lim_{h \to 0} -h + 2 = 2$$

1.3   Rate of Change

Definition   Define derivative as rate of change. Let $x = f(t)$, then $f'(x)$ can be represented as,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = f'(t) = \frac{dx}{dt}$$

1.4   The Product Rule (Formal Version)

Let $a \in \mathbb{R}$, f and g are functions defined at $(a - \delta, a + \delta)$, let $h(x) = f(x)g(x)$. Then, if $f(x), g(x)$ are differentiable at a, we have,

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

1.5   Differentiable $\implies$ Continuous

Recall   f(x) is **differentiable** at a:

$$\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

Recall   f(x) is **continuous** at a:

$$\lim_{x \to a} f(x) = f(a) \tag{2}$$

Proof.

$$\text{Since f(x) is differentiable at a}$$

$$(1) \iff \exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{And } \lim_{x \to a} (x - a) = 0$$

$$\implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} x - a = 0$$

$$\implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} x - a = 0$$

$$\implies \lim_{x \to a} f(x) - f(a) = 0$$

$$\implies \lim_{x \to a} f(x) = f(a)$$

$$\blacksquare$$

1.6 Proof of product rule for derivative.

$(fg)' = f'g + fg'$, see above for a formal definition.

$$\text{Let } h = fg$$

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) + f(a)g(x) - f(a)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))}{x - a}$$

$$= \lim_{x \to a} g(x) \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} f(a) \frac{g(x) - g(a)}{x - a}$$

$$= g(a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x)g(a)}{x - a}$$

$$= g(a)f'(a) + f(a)g'(a)$$

$$\blacksquare$$

1.7 Partial proof of differentiation rule

WTS $\frac{d}{dx} x^c = cx^{c-1}, \forall c \in \mathbb{R}$

Here we only prove statements is true $\forall c \in \mathbb{Z}^+$

Proof.

$$\textbf{Base: c = 1}$$
$$f(x) = x$$
$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$= \lim_{x \to a} 1 = 1$$
$$\textbf{Induction step}$$
$$\text{Assume } \frac{d}{dx}[x^k] = kx^{k-1}|_{x=a}$$
$$\text{For } f(x) = x^{k+1}$$
$$f'(x) = \frac{d}{dx}[x * x^k]$$
$$= x^k + xkx^{k-1}$$
$$= (k+1)x^k$$
$$\blacksquare$$

1.8 Higher Order Derivatives: Notations

Original function: $f(x)$

- **Lagrange** notation: $f^{(n)}$
- **Leibnitz** notation: $\frac{d^n f}{dx^n}$

1.9 Continuous But Not differentiable

Definition Function $f(x)$ is **non-differentiable** at a.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \textbf{ DNE}$$

Example 1 **Corner/Kink** $f(x) = |x|$ at 0.

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{|x|}{x} = -1$$
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = 1$$
$$\lim_{x \to 0^-} \neq \lim_{x \to 0^+}$$
$$\implies \lim_{x \to 0} \frac{f(x) - f(0)}{x} \textbf{ DNE}$$

Example 2   Vertical Tangent Line $g(x) = x^{\frac{1}{3}}$ at 0,

$$g'(0) = \lim_{x \to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty \textbf{(DNE)}$$

Caution   Difference between **vertical asymptote** and **vertical tangent line**

- Vertical asymptote: $f(a) = \infty$ (f(a) is not defined)

- Vertical tangent line: $f(a)$ is defined, $f'(a)$ is undefined.

1.10   Chain Rule

Derivation

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$$
$$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$$

Attention: we could only apply the operation above if $f(x) \neq f(a)$ during the process of $x \to a$. This holds for majority of functions we operate in calculus.

$$= \lim_{f(x) \to f(a)} \frac{g(f(x)) - g(f(a))}{x - a} f'(a)$$
$$= g'(f(a)) \cdot f'(a)$$
∎

Formal Theorem of Chain Rule   Let $a \in \mathbb{R}$, let f and g be functions. If f is differentiable at a and g is differentiable at $f(a)$, then, $(g \circ f)$ is differentiable at a,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

1.11   Derivatives of Trig Functions

Basic 6 results

1. $\frac{d}{dx} sin(x) = cos(x)$

2. $\frac{d}{dx} cos(x) = -sin(x)$

3. $\frac{d}{dx} tan(x) = sec^2(x)$

4. $\frac{d}{dx} cot(x) = -csc^2(x)$

5. $\frac{d}{dx} sec(x) = sec(x) tan(x)$

6. $\frac{d}{dx} csc(x) = -csc(x) cot(x)$

Proof.   Prove (i) and (ii) and use (i), (ii) and quotient rule to derive (iii), (iv), (v) and (vi).

Proof. (i)   **WTS** $f(x) = sin(x)$, then $f'(x) = cos(x)$

$$f'(x) = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h}$$
$$= \lim_{h \to 0} \frac{sin(x)cos(h) + cos(x)sin(h) - sin(x)}{h}$$
$$= \lim_{h \to 0} \frac{sin(x)(cos(h) - 1) + cos(x)sin(x)}{h}$$
$$= \lim_{h \to 0} cos(x)\frac{sin(h)}{h}$$
$$= cos(x)$$

■   (3)

Proof. (ii)   **WTS** $f(x) = cos(x)$, then $f'(x) = -sin(x)$

$$f'(x) = \lim_{h \to 0} \frac{cos(x+h) - cos(x)}{h}$$
$$= \lim_{h \to 0} \frac{cos(x)cos(h) - sin(h)sin(x) - cos(x)}{h}$$
$$= \lim_{h \to 0} \frac{(cos(h) - 1)cos(x) - sin(h)sin(x)}{h}$$
$$= \lim_{h \to 0} -\frac{sin(h)}{h}sin(x)$$
$$= -sin(x)$$

■   (4)

Recall   Compound angle formula:

1. $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + sin(\beta)cos(\alpha)$

2. $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - sin(\beta)cos(\alpha)$

3. $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$

4. $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$

1.12   Implicit Differentiation

Key   Use chain rule.

1.13   Derivative of Exponential Functions

Let $f(x) = a^x$ $(a > 0)$, find $f'(x)$, by definition,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h \to 0} \frac{(a^n - 1)a^x}{h}
\end{aligned}
$$

By property of limit, h is the only variable, so that a^x is a constant

$$
= a^x \lim_{h \to 0} \frac{a^h - 1}{h}
$$

$$(5)$$

Equivalently, $\frac{d}{dx} a^x = L_a a^x$

Definition e is the only positive number, such that,

$$
\lim_{h \to 0} \frac{e^h - 1}{h} = 1
$$

So that, $\frac{d}{dx} e^x = e^x$

1.14   Properties of logarithms

Definition   Let $a > 0, a \neq 1, x > 0, y \in \mathbb{R}$,

$$
\log_a x = y \iff a^y = x
$$

Properties

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a x = \frac{\log_b x}{\log_b a}$

4. $\log_a xy = \log_a x + \log_a y$

5. $\log_a \frac{x}{y} = \log_a x - \log_a y$

6. $\log_a x^r = r \log_a x$

Proof. (i)   let $a > 0, a \neq 1, let x, y > 0,$ **WTS** $\log_a xy = \log_a x + \log_a y$

$$\text{Let } p = \log_a x \iff a^p = x$$
$$\text{Let } q = \log_a y \iff a^q = y$$
$$\text{We have } a^p a^q = xy$$
$$\iff a^{p+q} = xy$$
$$\iff \log_a xy = p + q = \log_a x + \log_a y$$
$$\blacksquare$$

1.15   The derivatives of logarithm functions

For $\ln x$ $\frac{d}{dx} \ln x = \frac{1}{x}$

$$e^{\ln x} = x$$
$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$
$$\frac{d}{d \ln x} e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$$
$$x \frac{d \ln x}{dx} = 1$$
$$\frac{d}{dx} \ln x = \frac{1}{x}$$
$$\blacksquare$$

1.16   Derivative of other exponentials

WTS $\frac{d}{dx} a^x = \ln a \cdot a^x,$

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$
$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$
$$= \frac{d}{dx} e^{x \ln a} \cdot \frac{d}{dx} \ln a$$
$$= e^{x \ln a} \ln a$$
$$= \ln a \cdot a^x$$
$$\blacksquare$$

1.17   The power rule, complete proof

WTS $x^c = cx^{c-1}$

$$x^c = (e^{\ln x})^c = e^{c\ln x}$$
$$\text{So that } \frac{d}{dx}x^c = \frac{d}{dx}e^{c\ln x}$$
$$= \frac{de^{c\ln x}}{d\ln xc} \cdot \frac{\ln xc}{d\ln x} \cdot \frac{d\ln x}{dx}$$
$$= e^{c\ln x} \cdot c \cdot \frac{1}{x}$$
$$= c \cdot x^c \cdot \frac{1}{x}$$
$$= cx^{c-1}$$

$$\blacksquare$$

1.18   Logarithmic Differentiation

Example $f(x) = cos(x)^{sin(x)}(\star)$, find $f'(x)$
   Step1. Take ln on both sides of $(\star)$

$$\ln f(x) = \ln cos(x)^{sin(x)} = sin(x)\ln cos(x)$$

Step2. Take derivative.

$$\frac{f'(x)}{f(x)} = cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)}$$

   Step3. Solve for $f'(x)$

$$f'(x) = cos(x)^{sin(x)}(cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)})$$

2   Video Playlist 4

2.1   Functions

In calculus   We assume the domain is the largest subset of $\mathbb{R}$ that makes sense. And assume the codomain is always $\mathbb{R}$.

	Math	Computer Science
Notations	Domain	Domain
	Codomain	Range
	Range	Image

2.2   Inverse Functions

Definition   Let $f : A \to B$ be a function. Function $f^{-1} : B \to A$ is the **inverse function** is and only if

$$\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$$

Properties

- $\forall x \in A, f^{-1}(f(x)) = x$

- $\forall y \in B, f(f^{-1}(y)) = y$

Pre-condition   Function f has inverse function f^{-1} if and only if f is **injective/one-to-one** function.

2.3   Surjective Functions

Why function don't have an inverse: Part 1.

Definition   Function $f(x)$ is **surjective/onto** if $codomain(f(x)) = range(f(x))$.

Problem   If $f(x)$ is not surjective, then some points in codomain has no corresponding point in domain, then f^{-1} is not a function.

Solution   **Shrink** the codomain to range.

Example   Let $f(x) = e^x$, $g(x) = \ln x$, then we have,

-
  - $Domain(f(x)) = \mathbb{R}$
  - $Codomain(f(x)) = \mathbb{R}$
  - $Range(f(x)) = (0, \infty)$
-
  - $Domaing(x) = (0, \infty)$
  - $Codomaing(x) = \mathbb{R}$
  - $Rangeg(x) = \mathbb{R}$

Definition   Definition of inverse in calculus (*simplified, we don't consider codomain here.*)
   Let $f(x)$ be a function, and $f^{-1}(x)$ be the **inverse** of it. Then,

- $Domain(f^{-1}(x)) = Range(f(x))$

- $Range(f^{-1}(x)) = Domain(f(x))$

also,

$$\forall x \in Domain(f(x)), \forall y \in Range(f(x)), x = f^{-1}(y) \iff y = f(x)$$

and,

$$\forall x \in Domain(f(x)), f^{-1}(f(x)) = x$$
$$\forall y \in Range(f(x)), f(f^{-1}(y)) = y$$

2.4   Injective function

Definition   Let $f(x)$ be a function, with $Domain(f(x)) = A$, we say $f(x)$ is **injective/one-to-one** when,

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

equivalently (contrapositive)

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Theorem   Function f has an inverse if and only if f is **injective**.

Example $f(x) = x^2$ has no inverse, but we could take it's inverse by <u>shrinking the domain</u>.

- Take domain $= [0, \infty)$, $f^{-1}(x) = \sqrt{x}$
- Take domain $= (-\infty, 0]$, $f^{-1}(x) = -\sqrt{x}$

2.5   Some theorems

Let $f(x)$ be a function with domain I.

Theorem 1   Function f has an inverse function f^{-1} if and only if f is injective.

Theorem 2   For function f, if

1. f is **continuous** (*This means, f is continuous on its domain.*).

2. I is an **interval**.

then, $f^{-1}(x)$ is continuous.

Theorem 3   If

1. f is **differentiable**.

2. $\forall x \in I, f'(x) \neq 0$ (*This ensures the inverse function does not have a vertical tangent line, which causes non-differentiability*).

then, $f^{-1}(x)$ is differentiable.

Theorem 4 $\forall x \in I$ with $y = f(x)$, we have

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Proof.

$$f(f^{-1}(y)) = y$$
$$\frac{d}{dy}f(f^{-1}(y)) = \frac{d}{dy}y$$
$$\frac{d}{dy}f(f^{-1}(y)) = 1$$
$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$
$$f'(x) \cdot (f^{-1})'(y) = 1$$
$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$$\blacksquare$$

2.6   ArcSin

Note $ArcSin$ is **NOT** the inverse of Sin. $y = sin(x)$ has $domain = \mathbb{R}$ and $range = [-1, 1]$, so that, it is **not injective**.

Definition $ArcSin$ is the inverse function to the **restriction** of sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So that, $Domain(ArcSin) = Range(Sin) = [-1, 1]$, and, $Range(ArcSin) = Domain(Sin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Meaning $ArcSin(\frac{1}{2}) = t$ means:

$$\begin{cases} sin(t) = \frac{1}{2} \\ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{cases}$$

Composite

$$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], ArcSin(Sin(x)) = x$$
$$\forall y \in [-1, 1], Sin(ArcSin(y)) = y$$

2.7   Derivative of ArcSin

Result

$$\frac{dArcSin(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Derive.

$$\forall x \in [-1, 1]$$

$$Sin(ArcSin(x)) = x$$

$$\frac{d}{dx}Sin(ArcSin(x)) = \frac{d}{dx}x$$

$$Cos(ArcSin(x)) \cdot \frac{d}{dx}ArcSin(x) = 1$$

$$\frac{d}{dx}ArcSin(x) = \frac{1}{Cos(ArcSin(x))}$$

$$\text{Let } \theta = ArcSin(x)$$

$$Cos^2(\theta) = 1 - Sin^2(\theta)$$

$$Cos(\theta) = \pm\sqrt{(1 - x^2)}$$

$$\text{Since } \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], Sin(\theta) \geq 0$$

$$\implies Cos(\theta) = +\sqrt{1 - x^2}$$

$$\implies \frac{d}{dx}ArcSin(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\blacksquare$$

2.8   Other inverse trig functions

2.8.1 $y = Cos(x)$

Definition $ArcCos$ is the inverse function to the restriction of $Cos(x)$ to $[0, \pi]$, and,

$$\forall x \in [-1, 1], \forall y \in [0, \pi], x = ArcCos(y) \iff Cos(y) = x$$

Result

$$\frac{d}{dx}ArcCos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

2.8.2 $y = Tan(x)$

Definition $ArcTan(x)$ is the inverse function to the restriction of $Tan(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and,

$$\forall y \in \mathbb{R}, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], x = ArcTan(y) \iff Tan(x) = y$$

3   Video Playlist 5

3.1   Usage of MVT

Theorem   Let I be an open interval. Let f be a function defined on I. If $\forall x \in I, f'(x) = 0$ then f is a constant function.

If we want to prove this theorem, we need mean value theorem

3.2   Local Extreme Theorem

Definition   Let f be a function with domain I, let $c \in I$.

- f takes **maximum** at c if $\forall x \in I, f(x) \leq f(c)$.

- f takes **local maximum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \leq f(c)$.

Definition   Let f be a function with domain I, let $c \in I$.

- f takes **minimum** at c if $\forall x \in I, f(x) \geq f(c)$.

- f takes **local minimum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \geq f(c)$.

End-point cannot be a local extremum since the definition of local extremum requires a open interval at both left and right sides around point c.

Theorem (Local EVT)   Let f be a function with domain I as an interval. Let $c \in I$, then if,

1. $f(c)$ is an extremum.

2. c is an interior point.

then, $f'(c) = 0$ or DNE.

Definition   Point $c \in I$ for function f is a **critical point** if $f'(c) = 0$ or it does not exist.

Proof. (Local EVT)   Proof is in two parts: (1) f has maximum at c, (2) f has minimum at c.

$$\text{Part1: } f(c) \text{ is a maximum}$$
$$\text{Take left and right side limits}$$
$$\text{As } x \to c^+, x - c > 0$$
$$\text{As } x \to c^-, x - c < 0$$
$$\text{By definition of maximum} f(x) - f(c) \le 0$$
$$\text{Left limit}$$
$$x - c < 0 \wedge f(x) - f(c) \le 0$$
$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \ge 0$$
$$\text{Right limit}$$
$$x - c > 0 \wedge f(x) - f(c) \le 0$$
$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \le 0$$
$$\text{For limit to exist}$$
$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \le 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \ge 0$$
$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$
$$\iff f'(c) = 0$$
$$\text{Part2: } f(c) \text{ is a minimum}$$
$$\text{Take left and right side limits}$$
$$\text{As } x \to c^+, x - c > 0$$
$$\text{As } x \to c^-, x - c < 0$$
$$\text{By definition of maximum} f(x) - f(c) \ge 0$$
$$\text{Left limit}$$
$$x - c < 0 \wedge f(x) - f(c) \ge 0$$
$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \le 0$$
$$\text{Right limit}$$
$$x - c > 0 \wedge f(x) - f(c) \ge 0$$
$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \ge 0$$
$$\text{For limit to exist}$$
$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \ge 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \le 0$$
$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$
$$\iff f'(c) = 0$$
$$\blacksquare$$

3.3   Find Extremum

Example   find extremum of function $f(x) = x^3 - 3x^2 - 9x + 3$ for $I = [-4, 4]$
   Steps

1. Ensure existence of extremum. f is polynomial and therefore continuous, and $[-4, 4]$ is a compact set. By EVT, extremum exist.

2. Find all *critical points* and *end-points*.

3. Compare values at candidate points.

3.4   Rolle's Theorem

Theorem   let $a < b$, let f be a function defined on a closed interval $[a, b]$ (Compact set). Then, if,

1. $f(x)$ is continuous on $[a, b]$.

2. $(\wedge)$ $f(x)$ is differentiable on (a, b).

3. $(\wedge)$ $f(a) = f(b)$.

then,
$$\exists c \in (a, b) \, s.t. \, f'(c) = 0$$

   Proof.

$$\text{By EVT, } f(x) \text{ has extremum in } [a, b].$$
$$\textbf{Case1} \text{Interior Extremum Point.} (c \in (a, b))$$
$$\text{By Local EVT, } f'(c) = 0 \vee f'(c) DNE$$
$$\text{By (ii)} f'(c) = 0$$
$$\textbf{Case2} \text{End-point Extremum}$$
$$\text{Since (iii)} f(a) = f(b)$$
$$\forall x \in (a, b)$$
$$f(x) \leq max(f(a), f(b))$$
$$f(x) \geq min(f(a), f(b))$$
$$\implies f(x) \text{ is constant.}$$
$$\implies \forall c \in (a, b), f(c) = 0$$
$$\blacksquare$$

3.5   Application of Rolle's Theorem

Application   How many zeros does a function have.

Step 1   Use IVT to prove it has *at least* n zeros.

Step 2 Use Rolle's theorem to prove it has *at most* n zeros.

Example
$$g(x) = x^6 + x^2 + x - 2$$

IVT Applied

$$g(-2) = 64$$
$$g(0) = -2$$
$$g(1) = 1$$

So that, $g(x)$ has at least 2 zeros.

Rolle's theorem applied Assume $f(x_1) = f(x_2) = 0$, by Rolle's theorem, there must exits a $a \in (x_1, x_2)$ such that $f'(a) = 0$

Conclusion 1 Between any two zeros of f there must be *at least* one zero of f'.

Conclusion 2 $\sharp$ of zeros of $f' \geq \sharp$ of zeros of f - 1

Conclusion 2' $\sharp$ of zeros of f $\leq \sharp$ of zeros of $f' + 1$

$$g'(x) = 6x^5 + 2x + 1$$
$$g''(x) = 30x^4 + 2$$
$$g''(x) \text{ has no zeros}$$

3.6 (Lagrange)Mean Value Theorem

Theorem Let $a < b$, let f be a function defined on $[a, b]$, if,

1. f is continuous on $[a, b]$.

2. f is differentiable on (a, b).

then,

$$\exists c \in (a, b) s.t. f'(c) = \frac{f(b) - f(a)}{b - a}$$

3.7   Proof. of MVT

$$\text{Let } m = \frac{f(b) - f(a)}{b - a}$$
$$\text{Let } g(x) = f(x) - f(a) - m(x - a)$$
$$\text{Satisfies } g(a) = f(a) - f(a) - m(a - a) = 0$$
$$\wedge g(b) = f(b) - f(a) - m(b - a) = 0$$
$$\text{By Rolle's Theorem}$$
$$g(a) = g(b) = 0$$
$$\exists c \in (a, b) s.t. g'(c) = 0$$
$$\implies \frac{d}{x}[f(x) - f(a) - m(x - a)] = 0$$
$$\implies f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\blacksquare$$

3.8   Zero-derivative implies constant

Theorem   Let $a < b$. Let f be a function defined on $[a, b]$, then,

$$\forall x \in (a, b), f'(x) = 0 \wedge \ f \text{ is continuous on } [a, b] \implies \ f \text{ is constant on [a,b]}.$$

    proof.

$$\text{Let } x_1, x_2 \in [a, b] \wedge x_1 < x_2$$
$$\text{By MVT, } \exists c \in (x_1, x_2), s.t.$$
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$\because f'(c) = 0$$
$$\therefore f(x_1) = f(x_2)$$

3.9   Monotonicity of functions

Definition   Let f be a function defined on an interval I.

- f is **increasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2)$$

- f is **non-decreasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

Theorem   Let $a < b$. Let f be a function defined on (a, b). Then,

$$\forall x \in (a, b), f'(x) > 0 \implies f \text{ is increasing on (a,b)}$$

Theorem   Let $a < b$. Let f be a function defined on $[a, b]$. Then,

$$\forall x \in (a, b), f'(x) > 0 \wedge f \text{ is continuous on } [a, b] \implies f \text{ is increasing on } [a, b]$$

Short summary   On an <u>open interval</u>

- $f' = 0 \implies f$ constant.

- $f' > 0 \implies f$ increasing.

- $f' < 0 \implies f$ decreasing.

4   Video Playlist 7

4.1   Integral

Integral   Let $a < b$, let f be a <u>positive</u> function, then *integral of f from a to b* is denoted as:

$$\int_a^b f(x) \ dx$$

this is represented as the area of region under function f from $x = a$ to $x = b$.

4.2   Sigma Notation

Sigma Notation   The sigma notation, with **index** i, could be represented in the following form:

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_N$$

4.3   Supremum and Infimum

Definitions   Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Upper bound**: a is a <u>upper bound</u> of A means $\forall x \in A, \ x \leq a$.

- **Least upper bound(l.u.b) / Supremum**: a is the <u>least upper bound</u> or <u>supremum(sup)</u> of A iff a is an upper bound of A and $\forall b \in \{$upper bound of A$\}, \ a \leq b$.

- **Maximum**: if supremum of $A \in A$, it's <u>maximum</u> of A.

- **Bounded above**: A is <u>bounded above</u> if A has (at least) one upper bound.

Definitions (counter-part)   Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Lower bound**: a is a <u>lower bound</u> of A means $\forall x \in A, \; x \geq a$.

- **Greatest lower bound(g.l.b) / Infimum**: a is the <u>greatest lower bound (g.l.b)</u> or <u>infimum(inf)</u> of A iff a is a lower bound of A and $\forall b \in \{$Lower bound of A$\}$, $a \geq b$.

- **Minimum**: if infimum of $A \in A$, it's the <u>minimum</u> of A.

- **Bounded below**: A is <u>bounded below</u> if A has (at least) one lower bound.

Theorem: The l.u.b. principle   Let $A \subseteq \mathbb{R}$, if A is <u>bounded above</u> and $A \neq \emptyset$, then, A has a least upper bound(supremum).

Theorem: The g.l.b principle   Let $A \subseteq \mathbb{R}$, if A is <u>bounded below</u> and $A \neq \emptyset$, then, A has a greatest lower bound(infimum).

4.4   Supremum and Infimum of a function

Definition   Supremum of a function f on a domain I is defined as:

$$\sup_{x \in I} f(x) = \sup\{f(x) \mid x \in I\}$$

Theorem   Let f be a function defined on domain $I \neq \emptyset$, if f is <u>bounded above</u>, then $\exists \sup_{x \in I} f(x)$. Similarly, if f is <u>bounded below</u>, then $\exists \inf_{x \in I} f(x)$.

Theorem(EVT)   Let $a < b$, let f defined on $[a, b]$, if f is <u>continuous</u> on $[a, b]$, then f has a maximum and a minimum on $[a, b]$.

4.5   Definition of Integral (i)

Definition   A **partition** of the interval $[a, b]$ is a finite set P, s.t. $\{a, b\} \subseteq P$.

Notation $P = \{x_0, x_1, \ldots x_N\}$ on $[a, b]$. Implicitly, x_i are <u>ordered</u>, such that, $a = x_0 < x_1 < \cdots < x_N = b$.

Let f be bounded on $[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$, let $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, and $\Delta x_i = x_i - x_{i-1}$.

Definition   P-Lower sum of f is defined as:

$$L_P(f) = \sum_{i=1}^{N} (m_i \Delta x_i)$$

Definition   P-Upper sum of f is defined as:

$$U_P(f) = \sum_{i=1}^{N} (M_i \Delta x_i)$$

Property   For all partition P on interval $[a, b]$, the lower sum and upper sum satisfy the following inequality,

$$L_P(f) \leq \int_a^b f(x) \ dx \leq U_P(f)$$

4.6   Definition of Integral (ii): Properties of $U_P(f)$ and $L_P(f)$

Let f be a <u>bounded</u> function on $[a, b]$, let P and Q be partitions of $[a, b]$, the lower sums and upper sums have the following properties.

1. (Always) $L_P(f) \leq U_P(f)$.

2. If $P \subseteq Q$ (<u>Q is a *finer* partition</u>), then $L_P(f) \leq L_Q(f) \wedge U_P(f) \geq U_Q(f)$.

3. (Always) $L_P(f) \leq U_Q(f)$

   Proof

$$\text{Let } R = P \cup Q,$$
$$\text{so that, } P \subseteq R \wedge Q \subseteq R. \ \textit{(R is finer than both P and Q)}$$
$$L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f)$$
$$\implies L_P(f) \leq U_Q(f)$$

$\blacksquare$

4.7   Definition of Integral (iii): Upper Integral and Lower Integral

Definition   Let f be a <u>bounded</u> function on $[a, b]$, then, <u>lower integral of f from a to b</u> is defined as,

$$\underline{I_a^b(f)} = \sup\{\text{lower sums of } f\}$$

and the <u>upper integral of f from a to b</u> is defined as,

$$\overline{I_a^b(f)} = \inf\{\text{upper sums of } f\}$$

Then if $\underline{I_a^b(f)} < \overline{I_a^b(f)}$, then f is **non-integrable** on $[a, b]$.

4.8   An example of integrable function

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \quad \text{on } [-1, 1]$$

4.9   An example of non-integrable function

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases} \quad \text{on } [-1, 1]$$

4.10   Integrals as limits

Definition   Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$, the **norm** of P is defined as:

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_N\}$$

Theorem - Lower Integrals   For lower integrals, we have,

$$\underline{I_a^b(f)} = \lim_{\|P\| \to 0} L_P(f) = \sup\{\text{lower sums of } f\}$$

alternatively, using $\delta - \epsilon$ expression,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall P \text{ over } [a, b], \|P\| < \delta \implies |L_P(f) - \underline{I_a^b(f)}| < \epsilon$$

theorem - Upper Integrals   For upper integrals, we have,

$$\overline{I_a^b(f)} = \lim_{\|P\| \to 0} U_P(f)$$

4.11   Riemann Sums

Definition   Fix a partition P on $[a, b]$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, pick $x_i^\star \in [x_{i-1}, x_i]$, so that,

$$m_i \leq f(x_i^\star) \leq M_i$$
$$\implies m_i \Delta x_i \leq f(x_i^\star) \Delta x_i \leq M_i \Delta x_i$$
$$\implies L_P(f) = \sum_{i=1}^{N} (m_i \Delta x_i) \leq \sum_{i=1}^{N} (f(x_i^\star) * \Delta x_i) \leq \sum_{i=1}^{N} (M_i \Delta x_i) = U_P(f)$$

where the term $\sum_{i=1}^{N} (f(x_i^\star) \Delta x_i)$ is called a **Riemann sum**.

Definition   Let f be a <u>bounded</u> function on $[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition on $[a, b]$, for each i, pick **any** point $x_i^\star \in [x_{i-1}, x_i]$. then,

$$S_P^\star(f) = \sum_{i=1}^{N} (f(x_i^\star) * \Delta x_i)$$

is <u>a</u> **Riemann sum** for f and P. *(There are infinitely many Riemann sum).*

In general, we have,
$$L_P(f) \le S_P^\star(f) \le U_P(f)$$

and also,
$$\lim_{\|P\| \to 0} L_P(f) = \underline{I_a^b(f)}$$

$$\lim_{\|P\| \to 0} U_P(f) = \overline{I_a^b(f)}$$

and if f is **integrable**, then

$$\lim_{\|P\| \to 0} L_P(f) = \lim_{\|P\| \to 0} U_P(f) = \int_a^b f(x) \ dx$$

By <u>Squeeze Theorem</u>,

$$\lim_{\|P\| \to 0} S_P^\star(f) = \int_a^b f(x) \ dx$$

4.12   Properties of the integral

Property 1

$$\int_a^b [f(x) + g(x)] \ dx = \int_a^b f(x) \ dx + \int_a^b g(x) \ dx$$

Property 2
$$\int_a^b [cf(x)] \ dx = c \int_a^b f(x) \ dx$$

Property 3   If f is bounded on $[a, c]$, and f is integrable on $[a, b]$ and integrable on $[b, c]$, then,
$$\int_a^c f(x) \ dx = \int_a^b f(x) \ dx + \int_b^c f(x) \ dx$$

Property 4: Backward Integrals

$$\int_b^a f(x) \ dx = - \int_a^b f(x) \ dx$$

Negative function f   Integral for negative function is the negative area.

$$\int_a^b f(x) \ dx$$

5   Video Playlist 8

5.1   Anti-dervatives

Notations

- **Definite integral** $\int_a^b f(x)\ dx$

- **Indefinite integral** $\int f(x)\ dx$

Definition   Let f be a function defined on an interval, *an **anti-dervative*** of f is any function F that
$$F' = f$$

Note   As a consequence of MVT, if two functions have same dervative on an interval, then they differ by a constant.

5.2   Functions Defined as Integrals

Consider integrable function f, define function F as the definite integral from a, a fixed point in domain of f, to another point x in domain of f, that's,

$$F(x) = \int_a^x f(t)\ dt$$

Methodology   Let I be an interval, let $a \in I$ and let f be a function integrable on I, then for each $x \in I$, compute $F(x) = \int_a^x f(t)\ dt$ as a number.

5.3   The Fundamental Theorem of Calculus: Part 1

This provides connections between definite integrals and anti-dervatives

Theorem: FTC(part 1)

- Let I be an interval,

- Let $a \in I$,

- Let f be a function on I.

Define $F(x)$ as
$$F(x) = \int_a^x f(t)\ dt$$

If f is continuous, then F is differentiable and $F' = f$, that's,

$$F'(x) = f(x) \quad \forall x \in I$$

5.4   A Proof of Part 1 of the FTC

Proof.

$$\text{Let(fix) } x \in I$$

$$\text{WTS. } F'(x) = f(x)$$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} [\frac{1}{h}(F(x+h) - F(x))]$$

$$= \lim_{h \to 0} [\frac{1}{h}(\int_a^{x+h} f(t) \ dt - \int_a^x f(t) \ dt)]$$

$$= \lim_{h \to 0} [\frac{1}{h} \int_x^{x+h} f(t) \ dt]$$

Consider $h > 0$ (for negative h, the proof would be similar)

$$\text{Let } M_h = \sup_{[x,x+h]} (f)$$

$$\text{Let } m_h = \inf_{[x,x+h]} (f)$$

Then we have, by definition of infimum and supremum,

$$m_h \leq \frac{1}{h} \int_x^{x+h} f(t) \ dt \leq M_h$$

Since f is continuous on $[x, x+h]$, by EVT, it has maximum and minimum on this interval.

$$\exists c_h \in [x, x+h] \ s.t. \ M_h = f(c_h)$$

$$\exists d_h \in [x, x+h] \ s.t. \ m_h = f(d_h)$$

$$\because \lim_{h \to 0} c_h = x \wedge \lim_{h \to 0} d_h = x$$

$$\therefore \lim_{h \to 0} M_h = \lim_{h \to 0, c_h \to x} f(c_h) = f(x) \text{ (since } f \text{ is continuous.)}$$

$$\text{Similarly, } \lim_{h \to 0} m_h = \lim_{h \to 0, d_h \to x} f(d_h) = f(x)$$

$$\text{By Squeeze Theorem, } \lim_{h \to 0} [\frac{1}{h} \int_x^{x+h} f(t) \ dt] = f(x)$$

$$\therefore F'(x) = f(x) \ \forall x \in I$$

∎

5.5   The Fundamental Theorem of Calculus: Part 2

This provides a quick way to compute definite integrals.

Theorem: FTC(part 2)

- Let $a < b \in \mathbb{R}$,

- let f be continuous on $[a,b]$,

then,

$$\int_a^b f(x) \ dx = G(b) - G(a)$$

where G is any anti-dervative of f.

Notation

$$G(b) - G(a) = G(x)|_{x=a}^{x=b} = G(x)|_a^b$$

5.6   A Proof of Part 2 of the FTC

Proof.

We know that, from the first part of FTC, $G' = f$,

$$\text{WTS. } \int_a^b f(x) = G(b) - G(a)$$

$$\text{Define } F(x) = \int_a^x f(t) \ dt$$

$$\text{WTS. } F(b) = G(b) - G(a)$$

$$\text{Since } f \text{ is continuous,} F' = f$$

$$\text{By the consequence of MVT,}$$

$$F' = G' \implies \exists C \in \mathbb{R} s.t. F - G = C \forall x \in [a,b]$$

$$\text{at } x = a, F(a) = 0 \implies C = -G(a)$$

$$\implies \forall x \in [a,b] F(x) = G(x) - G(a)$$

$$\text{at } x = b, F(b) = G(b) - G(a)$$

$\blacksquare$

5.7   Summary: Definite and indefinite integrals, notation, definitions and theorems.

5.7.1   Definite Integral.

$$\int_a^b f(x) \ dx$$

Theorem (Formal definite)   if $\overline{I}_a^b(f) = \underline{I}_a^b(f)$ then $\int_a^b f(x) \ dx = \overline{I}_a^b(f) = \underline{I}_a^b(f)$.

Theorem (FTC: part 2)   Choose <u>one</u> anti-dervative $G(x)$ of $f(x)$, then compute the definite integral as $\int_a^b f(x)\ dx = G(b) - G(a)$.

5.7.2   Indefinite Integral

$$\int f(x)\ dx \text{ A \underline{collection} of functions.}$$

Find indefinite integral   Find $G(x)$ as <u>one</u> anti-dervative, by the consequence of MVT, then the indefinite integral of f could be constructed as,

$$F(x) = \{G(x) + C \mid C \in \mathbb{R}\}$$

5.7.3   Function Defined by an Integral.

$$F(x) = \int_a^x f(t)\ dt \text{ This is \underline{one} function with fixed value of } a.$$

Theorem (FTC: part 1)   if f is continuous, then $F'(x) = f(x)$

6   Video Playlist 9

6.1   Integration By Substitution: derivation of the formula

Backwards usage of chain rule.

If $\int f(x)\ dx = F(x)$ is the anti-derivative of $f(x)$, then

$$F(g(x)) = \int f(g(x))g'(x)\ dx = F(g(x))$$

6.2   Example 2

6.3   Example 3

6.4   Example 4

Theorem   Let $a < b$, let f be a continuous function, let g be a function with <u>continuous derivative</u> in $[a, b]$, assume the range of g on $[a, b]$ is contained in the domain of f. Then,

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

6.5   Integration by parts

Backwards product rule

Let f and g be two differentiable function, by product rule of differentiation, we have,

$$f'(x)g(x) + f(x)g'(x) = \frac{d}{dx}f(x)g(x)$$

$$\implies \int f'(x)g(x) + f(x)g'(x) \ dx = f(x)g(x) + C$$

$$\implies \int f'(x)g(x) \ dx + \int f(x)g'(x) \ dx = f(x) + g(x) + C$$

$$\implies \int f'(x)g(x) \ dx = f(x) + g(x) - \int f(x)g'(x) \ dx$$

The integral constant is implicitly contained in the integral term.

6.6   Examples

Example 1

$$\int x^2 e^2 \ dx$$

Example 2

$$\int e^2 \sin x \ dx$$

Use integration by parts twice.

Example 3

$$\int \arctan x \ dx$$

Consider the form $1 \times f(x)$ as partition method.

6.7   Integration of products of trigonometric functions

Types

$$\int sin^n x \ cos^m x \ dx$$

$$\int sec^n x \ tan^m x \ dx$$

Keys

$$sin^2(x) + cos^2(x) = 1$$
$$sec^2(x) = 1 + tan^2(x)$$

Summary I   Consider the integral in the following form

$$\int sin^n x \ cos^m x \ dx$$

- If **m is odd** then try $u = sin(x)$, then $du = cos(x)dx$

- If **n is odd** then try $u = cos(x)$, then $du = -sin(x)dx$

7   Video Playlist 11

7.1   What Is a Sequence

Definition   A **sequence** is a <u>function</u> with domain $\mathbb{N}$.

7.1.1   Conventions

Functions   function with domain <u>interval</u>.

- x as variable.

- $f(x)$ as value at x.

Sequence   function with domain $\mathbb{N}$.

- n as variable.

- a_n as value at n.

   A sequence is not a set.

7.1.2   Describe sequences

Equation $a_n = \frac{2^n n!}{n+1}$

First few values $\{1, 2, 4, 8, 16, \dots\}$

Words $p_n = $ n-th prime.

Recurrence relation   e.g. Fibonacci Sequence.

$$\{F_n\}_{n=0}^{\infty} : F_0 = F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \ \forall n \geq 2$$

A general definition   A sequence is a function with domain $\{n \in \mathbb{Z} \mid n \geq n_0\}$ for some fixed $n_0 \in \mathbb{Z}$.

7.2   The Limit of a Sequence

Example

$$\{\frac{n}{n+1}\}_{n=0}^{\infty} \quad \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Definition(Limit)   We say that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to the number $L \in \mathbb{R}$ when

$$\forall \epsilon > 0, \ \exists n_0 \in \mathbb{N} s.t. \ \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \epsilon$$

denoted as

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L$$

Tail: all terms of the sequence after the first few terms.
Every interval centred at L contains a tail of the sequence.

Definition   A sequence is **convergent** if it has a limit. This sequence is **divergent** if it does not have a limit.

7.3   Properties of Limits of Sequences

Properties from the limit of functions

- Limit laws: Yes

- Squeeze theorem: Yes

- *L'Hôpital's Rule*: No

7.3.1   Sequence from a function

Let $c \in \mathbb{Z}$ and function f defined on $[c, \infty)$, and define the seuqnce $\{a_n\}_{n=c}^{\infty}$ as

$$a_n = f(n)$$

We have if $\lim_{n\to\infty} f(n) = L$ then $\lim_{n\to\infty} a_n = L$. If $\lim_{n\to\infty} f(n)$ DNE, then $\lim_{n\to\infty} a_n$ <u>may or may not exist</u>.

7.3.2   Composite of sequence and function

Theorem   If $a_n \to L$ and f is continuous at L then

$$f(a_n) \to f(L)$$

7.4   Monotonic and Bounded Sequences

7.4.1   Monotonic Sequences

Definition   We say $\{a_n\}_{n=0}^{\infty}$ is **increasing** if

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n < a_m$$

Also, we say this sequence is **non-decreasing** if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n \leq a_m$$

Definition   We say $\{a_n\}_{n=0}^{\infty}$ is **decreasing** if

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n > a_m$$

Also, if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n \geq a_m$$

we say this sequence is **non-increasing**.

Definition   We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **monotonic** is if is has any of the four properties above.

Definition $\{a_n\}_{n=0}^{\infty}$ is **eventually decreasing** if

$$\exists n_0 \in \mathbb{N}, \ s.t. \forall n \in \mathbb{N}, n \geq n_0 \implies a_n > a_{n+1}$$

7.4.2   Bounded Sequences

Definition   We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **bounded below** if

$$\exists A \in \mathbb{R} s.t. \forall n \in \mathbb{N}, \ A \leq a_n$$

Similarly, the sequence is **bounded above** if

$$\exists B \in \mathbb{R}. s.t. \forall n \in \mathbb{N}, \ B \geq a_n$$

Definition   We say a sequence is **bounded** if and only if it is both bounded above and below.

Theorem   If a sequence is convergent then it is bounded.

Theorem 2A(The monotone convergence theorem for sequence)   If a sequence is eventually increasing and bounded above, then it is convergent

Theorem   If a sequence is eventually increasing and not bounded above then it divergent to ∞.

Remark for a sequence:

$$\text{Sequence} \begin{cases} \text{Convergent} \\ \text{Divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \textit{Oscillating} \end{cases} \end{cases}$$