

# APM462: Nonlinear Optimization

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# 1 Preliminaries

## 1.1 Mean Value Theorems and Taylor Approximations.

**Definition 1.1.** Let  $f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , the **gradient** of  $f$  at  $x \in S$ , if exists, is a vector  $\nabla f(x) \in \mathbb{R}^n$  characterized by the property

$$\lim_{v \rightarrow 0} \frac{f(x+v) - f(x) - \nabla f(x) \cdot v}{\|v\|} = 0 \quad (1.1)$$

**Theorem 1.1** (The First Order Mean Value Theorem). Let  $f \in C^1(\mathbb{R}^n, \mathbb{R})$ , then for any  $x, v \in \mathbb{R}^n$ , there exists some  $\theta \in (0, 1)$  such that

$$f(x+v) = f(x) + \nabla f(x+\theta v) \cdot v \quad (1.2)$$

*Proof.* Let  $x, v \in \mathbb{R}^n$ , define  $g(t) := f(x+tv) \in C^1(\mathbb{R}, \mathbb{R})$ .

By the mean value theorem on  $\mathbb{R}$ , there exists  $\theta \in (0, 1)$  such that  $g(0+1) = g(0) + g'(\theta)(1-0)$ , that is,  $f(x+v) = f(x) + g'(\theta)$ . Note that  $g'(\theta) = \nabla f(x+\theta v) \cdot v$ . ■

**Proposition 1.1** (The First Order Taylor Approximation). Let  $f \in C^1(\mathbb{R}^n, \mathbb{R})$ , then

$$f(x+v) = f(x) + \nabla f(x) \cdot v + o(\|v\|) \quad (1.3)$$

that is

$$\lim_{\|v\| \rightarrow 0} \frac{f(x+v) - f(x) - \nabla f(x) \cdot v}{\|v\|} = 0 \quad (1.4)$$

*Proof.* By the mean value theorem,  $\exists \theta \in (0, 1)$  such that  $f(x+v) - f(x) = \nabla f(x+\theta v) \cdot v$ .

The limit becomes  $\lim_{\|v\| \rightarrow 0} \frac{[\nabla f(x+\theta v) - \nabla f(x)] \cdot v}{\|v\|} = \lim_{\|v\| \rightarrow 0; x+\theta v \rightarrow x} \frac{[\nabla f(x+\theta v) - \nabla f(x)] \cdot v}{\|v\|}$ .

Since  $f \in C^1$ ,  $\lim_{x+\theta v \rightarrow x} \nabla f(x+\theta v) = \nabla f(x)$ .

And  $\frac{v}{\|v\|}$  is a unit vector, and every component of it is bounded, as the result, the limit of inner product vanishes instead of explodes. ■

**Theorem 1.2** (The Second Order Mean Value Theorem). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^2$  function, then for any  $x, v \in \mathbb{R}^n$ , there exists  $\theta \in (0, 1)$  satisfying

$$f(x+v) = f(x) + \nabla f(x) \cdot v + \frac{1}{2} v^T \nabla^2 f(x+\theta v) v \quad (1.5)$$

**Proposition 1.2** (The Second Order Taylor Approximation). Let  $f : C^2(\mathbb{R}^n, \mathbb{R})$  function, and  $x, v \in \mathbb{R}^n$ , then

$$f(x+v) = f(x) + \nabla f(x) \cdot v + \frac{1}{2} v^T \nabla^2 f(x+\theta v) v + o(\|v\|^2) \quad (1.6)$$

that is

$$\lim_{\|v\| \rightarrow 0} \frac{f(x+v) - f(x) - \nabla f(x) \cdot v - \frac{1}{2} v^T \nabla^2 f(x) v}{\|v\|^2} = 0 \quad (1.7)$$

*Proof.* By the second mean value theorem, there exists  $\theta \in (0, 1)$  such that the limit is equivalent to

$$\lim_{\|v\| \rightarrow 0} \frac{1}{2} \left( \frac{v}{\|v\|} \right)^T [\nabla^2 f(x+\theta v) - \nabla^2 f(x)] \frac{v}{\|v\|} \quad (1.8)$$

Since  $f \in C^2$ , the limit of  $[H_f(x+\theta v) - H_f(x)]$  is in fact  $\mathbf{0}_{n \times n}$ . And every component of unit vector  $\frac{v}{\|v\|}$  is bounded, the quadratic form converges to zero as an immediate result. ■

It is often noted that the gradient at a particular  $x_0 \in \text{dom}(f) \subseteq \mathbb{R}^n$  gives the direction  $f$  increases most rapidly. Let  $x_0 \in \text{dom}(f)$ , and  $v$  be a unit vector representing a *feasible direction* of change. That is, there exists  $\delta > 0$  such that  $x_0 + tv \in \text{dom}(f) \forall t \in [0, \delta]$ . Then the rate of change of  $f$  along feasible direction  $v$  can be written as

$$\left. \frac{d}{dt} \right|_{t=0} f(x_0 + tv) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \|v\| \cos(\theta) \quad (1.9)$$

where  $\theta = \angle(v, \nabla f(x_0))$ . And the derivative is maximized when  $\theta = 0$ , that is, when  $v$  and  $\nabla f$  point the same direction.

## 1.2 Implicit Function Theorem

**Theorem 1.3** (Implicit Function Theorem). Let  $f : C^1(\mathbb{R}^{n+1}, \mathbb{R})$ , let  $(a, b) \in \mathbb{R}^n \times \mathbb{R}$  such that  $f(a, b) = 0$ . If  $\nabla f(a, b) \neq 0$ , then  $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f(x, y) = 0\}$  is locally a graph of a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ .

**Remark 1.1.**  $\nabla f(x_0) \perp$  level set of  $f$  near  $x_0$ .

## 2 Convexity

### 2.1 Terminologies

**Definition 2.1.** Set  $\Omega \subseteq \mathbb{R}^n$  is **convex** if and only if

$$\forall x_1, x_2 \in \Omega, \lambda \in [0, 1], \lambda x_1 + (1 - \lambda)x_2 \in \Omega \quad (2.1)$$

**Definition 2.2.** A function  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **convex** if and only if  $\Omega$  is convex, and

$$\forall x_1, x_2 \in \Omega, \lambda \in [0, 1], f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (2.2)$$

**Definition 2.3.** A function  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is **strictly convex** if and only if  $\Omega$  is convex and

$$\forall x_1, x_2 \in \Omega, \lambda \in (0, 1), f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (2.3)$$

### 2.2 Basic Properties of Convex Functions

**Definition 2.4.** A function  $f : \Omega \rightarrow \mathbb{R}$  is **concave** if and only if  $-f$  is **convex**.

**Proposition 2.1.** Properties of convex functions:

- (i) If  $f_1, f_2$  are convex on  $\Omega$ , so is  $f_1 + f_2$ ;
- (ii) If  $f$  is convex on  $\Omega$ , then for any  $a > 0$ ,  $af$  is also convex on  $\Omega$ ;
- (iii) Any **sub-level/lower contour set** of a convex function  $f$

$$\mathcal{L}(c) := \{x \in \mathbb{R}^n : f(x) \leq c\} \quad (2.4)$$

is convex.

*Proof of (iii).* Let  $c \in \mathbb{R}$ , and  $x_1, x_2 \in SL(c)$ . Let  $s \in [0, 1]$ . Since  $x_1, x_2 \in \mathcal{L}(c)$ , and  $f(\cdot)$  is convex,  $f(sx_1 + (1-s)x_2) \leq sf(x_1) + (1-s)f(x_2) \leq sc + (1-s)c = c$ . Which implies  $sx_1 + (1-s)x_2 \in \mathcal{L}(c)$ . ■

**Example 2.1.**  $\ell_2$  norm  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R} := \|x\|_2$  is convex.

*Proof.* Note that for any  $u, v \in \mathbb{R}^n$ , by triangle inequality,  $\|u - (-v)\| \leq \|u - 0\| + \|0 - (-v)\| = \|u\| + \|v\|$ . Consequently, let  $u, v \in \mathbb{R}^n$  and  $s \in [0, 1]$ , then  $\|su + (1-s)v\| \leq \|su\| + \|(1-s)v\| = s\|u\| + (1-s)\|v\|$ . Therefore,  $\|\cdot\|$  is convex. ■

**Proposition 2.2.** Any norm function  $\|\cdot\|$  defined on a vector space  $\mathcal{X}(\mathbb{R})$  is convex.

*Proof.* The proof follows the defining properties of norm,

$$\|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\| \quad (2.5)$$

$$= \lambda\|x\| + (1-\lambda)\|y\| \quad (2.6)$$

■

## 2.3 Characteristics of $C^1$ Convex Functions

**Theorem 2.1** ( $C^1$  criteria for convexity). Let  $f \in C^1$ , then  $f$  is convex on a convex set  $\Omega$  if and only if

$$\forall x, y \in \Omega, f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \quad (2.7)$$

that is, *the linear approximation is an underestimate of  $f$ .*

*Proof.* ( $\implies$ ) Suppose  $f$  is convex on a convex set  $\Omega$ . Then  $f(sy + (1-s)x) \leq sf(y) + (1-s)f(x)$  for every  $x, y \in \Omega$  and  $s \in [0, 1]$ , which implies, for every  $s \in (0, 1]$ :

$$\frac{f(sy + (1-s)x) - f(x)}{s} \leq f(y) - f(x) \quad (2.8)$$

By taking the limit of  $s \rightarrow 0$ ,

$$\lim_{s \rightarrow 0} \frac{f(x + s(y-x)) - f(x)}{s} \leq f(y) - f(x) \quad (2.9)$$

$$\implies \left. \frac{d}{ds} \right|_{s=0} f(x + s(y-x)) \leq f(y) - f(x) \quad (2.10)$$

$$\implies \nabla f(x) \cdot (y-x) \leq f(y) - f(x) \quad (2.11)$$

( $\impliedby$ ) Let  $x_0, x_1 \in \Omega$ , let  $s \in [0, 1]$ . Define  $x^* := sx_0 + (1-s)x_1$ , then

$$f(x_0) \geq f(x^*) + \nabla f(x^*) \cdot (x_0 - x^*) \quad (2.12)$$

$$\implies f(x_0) \geq f(x^*) + \nabla f(x^*) \cdot [(1-s)(x_0 - x_1)] \quad (2.13)$$

Similarly,

$$f(x_1) \geq f(x^*) + \nabla f(x^*) \cdot (x_1 - x^*) \quad (2.14)$$

$$\implies f(x_1) \geq f(x^*) + \nabla f(x^*) \cdot [s(x_1 - x_0)] \quad (2.15)$$

Therefore,  $sf(x_0) + (1-s)f(x_1) \geq f(x^*)$ . ■

**Theorem 2.2** ( $C^2$  criterion for convexity).  $f \in C^2$  is a convex function on a convex set  $\Omega \subseteq \mathbb{R}^n$  if and only if  $\nabla^2 f(x) \succcurlyeq 0$  (i.e. positive semidefinite) for all  $x \in \Omega$ .

**Corollary 2.1.** When  $f$  is defined on  $\mathbb{R}$ , the  $C^2$  criterion becomes  $f''(x) \geq 0$ .

*Proof.* ( $\Leftarrow$ ) Suppose  $\nabla^2 f(x) \succcurlyeq 0$  for every  $x \in \Omega$ , let  $x, y \in \Omega$ . By the second order MVT,

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x + s(y - x))(y - x) \text{ for some } s \in [0, 1] \quad (2.16)$$

$$\implies f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \quad (2.17)$$

So  $f$  is convex by the  $C^1$  criterion of convexity.

( $\Rightarrow$ ) Let  $v \in \mathbb{R}^n$ . Suppose, for contradiction, that for some  $x \in \Omega$ ,  $\nabla^2 f(x) \not\succcurlyeq 0$ .

If such  $x \in \partial\Omega$ , note that  $v^T \nabla^2 f(\cdot)v$  is continuous because  $f \in C^2$ , then there exists  $\varepsilon > 0$  such that  $\forall x' \in V_\varepsilon(x) \cap \Omega^{int}$ ,  $v^T \nabla^2 f(x')v < 0$ .

Hence, one may assume with loss of generality that such  $x \in \Omega^{int}$ .

Because  $x \in \Omega^{int}$ , exists  $\varepsilon' > 0$ , such that  $V_{\varepsilon'}(x) \subseteq \Omega^{int}$ .

Define  $\hat{v} := \frac{v}{\sqrt{\varepsilon'}}$ , then for every  $s \in [0, 1]$ ,  $\hat{v}^T \nabla^2 f(x + s\hat{v})\hat{v} < 0$ .

Let  $y = x + \hat{v}$ , by the mean value theorem,

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f[x + s(y - x)](y - x) \quad (2.18)$$

for some  $s \in [0, 1]$ .

This implies  $f(y) < f(x) + \nabla f(x) \cdot (y - x)$ , which contradicts the  $C^1$  criterion for convexity. ■

## 2.4 Minimum and Maximum of Convex Functions

**Theorem 2.3.** Let  $\Omega \subseteq \mathbb{R}^n$  be a convex set, and  $f : \Omega \rightarrow \mathbb{R}$  is a convex function. Let

$$\Gamma := \left\{ x \in \Omega : f(x) = \min_{x \in \Omega} f(x) \right\} \equiv \underset{x \in \Omega}{\operatorname{argmin}} f(x) \quad (2.19)$$

If  $\Gamma \neq \emptyset$ , then

- (i)  $\Gamma$  is convex;
- (ii) any local minimum of  $f$  is the global minimum.

*Proof (i).* Let  $x, y \in \Gamma$ ,  $s \in [0, 1]$ , then  $sx + (1-s)y \in \Omega$  because  $\Omega$  is convex. Since  $f$  is convex,  $f(sx + (1-s)y) \leq sf(x) + (1-s)f(y) = \min_{x \in \Omega} f(x)$ . The inequality must be equality since it would contradict the fact that  $x, y \in \Gamma$ . Therefore,  $sx + (1-s)y \in \Gamma$ . ■

*Proof (ii).* Let  $x \in \Omega$  be a local minimizer for  $f$ , but assume, for contradiction, it is not a global minimizer. That is, there exists some other  $y$  such that  $f(y) < f(x)$ . Since  $f$  is convex,

$$f(x + t(y - x)) = f((1-t)x + ty) \leq (1-t)f(x) + tf(y) < f(x) \quad (2.20)$$

for every  $t \in (0, 1]$ . Therefore, for every  $\varepsilon > 0$ , there exists  $t^* \in (0, 1]$  such that  $x + t^*(y - x) \in V_\varepsilon(x)$  and  $f(x + t^*(y - x)) < f(x)$ , this contradicts the fact that  $x$  is a local minimum. ■

**Theorem 2.4.** Let  $\Omega \subseteq \mathbb{R}^n$  be a convex and compact set, and  $f : \Omega \rightarrow \mathbb{R}$  is a convex function. Then

$$\max_{x \in \Omega} f(x) = \max_{x \in \partial\Omega} f(x) \quad (2.21)$$

*Proof.* As we assumed,  $\Omega$  is closed, therefore  $\partial\Omega \subseteq \Omega$ . Hence,  $\max_{x \in \Omega} f \geq \max_{x \in \partial\Omega} f$ .

Suppose, for contradiction,  $\max_{x \in \Omega} f > \max_{x \in \partial\Omega} f$ , then let  $x^* \in \operatorname{argmax}_{x \in \Omega} f$ .

Then we can construct a straight line through  $x^*$  and intersects  $\partial\Omega$  at two points,  $y_1, y_2 \in \partial\Omega$ , such that  $x^* = sy_1 + (1 - s)y_2$  for some  $s \in (0, 1)$ . Further, since  $f$  is convex,  $\max_{x \in \Omega} f(x) = f(x^*) \leq sf(y_1) + (1 - s)f(y_2) \leq s \max_{\partial\Omega} f + (1 - s) \max_{\partial\Omega} f = \max_{\partial\Omega} f$ , which leads to a contradiction.

Therefore,  $\max_{x \in \Omega} f = \max_{x \in \partial\Omega} f$ . ■

**Proposition 2.3.** For  $p, g > 1$  satisfying  $\frac{1}{p} + \frac{1}{g} = 1$ ,

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{g}|b|^g \quad (2.22)$$

*Proof.*

$$(-\log)|ab| = (-\log)|a| + (-\log)|b| \quad (2.23)$$

$$= \frac{1}{p}(-\log)|a|^p + \frac{1}{g}(-\log)|b|^g \quad (2.24)$$

$$(\because (-\log) \text{ is convex}) \geq (-\log) \left( \frac{1}{p}|a|^p + \frac{1}{g}|b|^g \right) \quad (2.25)$$

And since  $(-\log)$  is monotonically decreasing,

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{g}|b|^g \quad (2.26)$$

■

**Corollary 2.2.**

$$|ab| \leq \frac{|a|^2 + |b|^2}{2} \quad (2.27)$$

## 3 Finite Dimensional Optimization

### 3.1 Unconstraint Optimization

**Theorem 3.1** (Extreme Value Theorem). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and  $K \subseteq \mathbb{R}^n$  be a compact set, then the minimization problem  $\min_{x \in K} f(x)$  has a solution.

*Proof.* Suppose  $f$  is continuous, let  $K \subseteq \mathbb{R}^n$  be a compact set, then  $f(K)$  is compact as well. Therefore, the  $\sup f(K)$  exists (b/c bounded) and is attainable (b/c closed). ■

**Remark 3.1.**  $f : \Omega \rightarrow \mathbb{R}$  is convex does not imply  $f$  is continuous.

**Proposition 3.1.** A convex function  $f$  defined on a convex open set is continuous.

*Proof.* Let  $f : \Omega \rightarrow \mathbb{R}$  be a convex function, where  $\Omega \subseteq \mathbb{R}^n$  is open. **TODO:** *Is this true?* ■

**Proposition 3.2.** A convex function  $f$  defined on an open interval in  $\mathbb{R}$  is continuous.

*Proof.* See homework 1. The proof involves squeeze theorem. ■

*Detailed Proof of EVT..* Let  $f : K \rightarrow \mathbb{R}$  be a continuous function defined on a compact set  $K$ .

WLOG, we only prove the existence of  $\min f$ , since the existence of  $\max$  can be easily proven by applying the exact same argument on  $-f$ .

That is, we claim the infimum of  $f(K)$  is attained within  $K$ .

Because  $K$  is compact, the continuity of  $f$  implies  $f(K)$  is compact.

By the completeness axiom of  $\mathbb{R}$ ,  $m := \inf_{x \in K} f(x)$  is well-defined. There exists a sequence  $(x_i) \subseteq K$ , such that  $(f(x_i)) \rightarrow m$ . Because  $K$  is compact, there exists a subsequence  $(x_{i_k})$  of  $(x_i)$  converges to some limit  $x^* \in K$ .

Since  $f$  is continuous,  $(f(x_{i_k})) \rightarrow f(x^*)$ , which is a subsequence of the convergent sequence  $(f(x_i))$ , and they must converge to the same limit. Hence,  $f(x^*) = m$ , and the infimum is attained at  $x^* \in K$ . ■

**Theorem 3.2** (Heine–Borel). Let  $K \subseteq \mathbb{R}^n$ , then the following are equivalent:

- (i)  $K$  is compact (every open cover of  $K$  has a finite sub-cover);
- (ii)  $K$  is closed and bounded;
- (iii) Every sequence in  $K$  has a convergent subsequence converges to a point in  $K$ .

**Proposition 3.3.** Let  $\{h_i\}$  and  $\{g_j\}$  be sets of continuous functions on  $\mathbb{R}^n$ , the the set of all points in  $\mathbb{R}^n$  that satisfy

$$\begin{cases} h_i(x) = 0 \ \forall i \\ g_j(x) \leq 0 \ \forall j \end{cases} \quad (3.1)$$

is a closed set.

*Proof.* For every equality constraint  $h_i$ , it can be represented as the conjunction of two inequality constraint, namely  $h_i^\alpha(x) := -h_i(x) \leq 0 \wedge h_i^\beta(x) := h_i(x) \leq 0$ . Then the constraint collection is equivalent to

$$\begin{cases} h_i^\alpha(x) \leq 0 \ \forall i \\ h_i^\beta(x) \leq 0 \ \forall i \\ g_j(x) \leq 0 \ \forall j \end{cases} \quad (3.2)$$

The subset of  $\mathbb{R}^n$  qualified by each individual constraint is closed by the property of continuous functions (i.e. a continuous function's pre-image of closed set is closed). And the intersection of arbitrarily many closed sets is closed. ■



The optimization problems investigated in this section can be formulated as

$$y \min_{x \in \Omega} f(x) \quad (3.3)$$

where  $\Omega \subseteq \mathbb{R}^n$ . Typically, for simplicity,  $\Omega$  are often  $\mathbb{R}^n$ , an open subset of  $\mathbb{R}^n$ , or the closure of some open subset of  $\mathbb{R}^n$ .

Everything above minimization discussed in this section is applicable to maximization as well using the proposition below.

**Proposition 3.4.** When  $\Omega = \mathbb{R}^n$ , the unconstrained minimization has the following properties

- (i)  $\operatorname{argmax} f = \operatorname{argmin}(-f)$ ;
- (ii)  $\max f = -\min(-f)$

*Proof.* Immediate by applying definitions of maximum and minimum. ■

**Definition 3.1.** A function  $f : \Omega \rightarrow \mathbb{R}$  has **local minimum** at  $x_0 \in \Omega$  if

$$\exists \varepsilon > 0 \text{ s.t. } \forall x \in V_\varepsilon(x_0) \cap \Omega, f(x_0) \leq f(x) \quad (3.4)$$

$f$  attains **strictly local minimum** at  $x_0$  if

$$\exists \varepsilon > 0 \text{ s.t. } \forall x \in V_\varepsilon(x_0) \cap \Omega \setminus \{x_0\} f(x_0) < f(x) \quad (3.5)$$

$f$  attains **global minimum** at  $x_0$  if

$$\forall x \in \Omega f(x_0) \leq f(x) \quad (3.6)$$

$f$  attains **strict global minimum** at  $x_0$  if

$$\forall x \in \Omega \setminus \{x_0\} f(x_0) < f(x) \quad (3.7)$$

Note that strict global minimum is always unique by its definition.

**Theorem 3.3** (Necessary Condition for Local Minimum). Let  $f \in C^1(\Omega, \mathbb{R})$ , let  $x_0 \in \Omega$  be a local minimum of  $f$ , then for every *feasible direction*  $v$  at  $x_0$ ,

$$\nabla f(x_0) \cdot v \geq 0 \quad (3.8)$$

*This theorem serves as the primary defining property of local minimum.*

**Definition 3.2.** For  $x_0 \in \Omega \subseteq \mathbb{R}^n$ ,  $v \in \mathbb{R}^n$  is a **feasible direction** at  $x_0$  if

$$\exists \bar{s} > 0 \text{ s.t. } \forall s \in [0, \bar{s}], x_0 + sv \in \Omega \quad (3.9)$$

*Proof of Necessary Condition.* Let  $x_0 \in \Omega$  be a local minimum, and let  $v$  be a feasible direction. Let  $g(s) := f(x_0 + sv)$ . And since  $g$  attains minimum at  $s = 0$ , there exists some  $\bar{s} > 0$  such that

$$g(s) - g(0) \geq 0 \quad \forall s \in V_{\bar{s}}(0) \quad (3.10)$$

Therefore

$$g'(0) := \lim_{s \rightarrow 0^+} \frac{g(s) - g(0)}{s - 0} \geq 0 \quad (3.11)$$

The alternative form of derivative can be derived using chain rule as

$$g'(0) = \nabla f(x + sv) \cdot v \big|_{s=0} = \nabla f(x) \cdot v \quad (3.12)$$

By combining the two identities above,  $\nabla f(x) \cdot v \geq 0$ . ■

*Alternative Proof of Necessary Condition (not that rigorous).* The prove is almost immediate, if there exists a feasible direction  $v^*$  such that  $\nabla f(x_0) \cdot v^* < 0$ , for every  $\varepsilon > 0$ , one can construct  $x' := x^* + sv^*$  with sufficiently small  $s$  so that  $x' \in V_\varepsilon(x^*) \cap \Omega$  and  $f(x') < f(x^*)$ . ■

**Corollary 3.1.** When  $\Omega$  is open, then  $x_0$  is a local minimum  $\implies \nabla f(x_0) = 0$ .

*Proof.* Since  $\Omega$  is open, any sufficiently small  $v \neq 0$  such that both  $v$  and  $-v$  are feasible directions at  $x_0$ , applying the necessary condition on both  $v$  and  $-v$  provides the equality. ■

**Theorem 3.4** (Second Order Necessary Condition for Local Minimum). Let  $f : \in C^2(\Omega, \mathbb{R})$ , let  $x_0 \in \Omega$  be a local minimum of  $f$ , then for every non-zero feasible direction  $v$  at  $x_0$ ,

$$(i) \quad \nabla f(x_0) \cdot v \geq 0;$$

$$(ii) \quad \nabla f(x_0) \cdot v = 0 \implies v^T \nabla^2 f(x_0) v \geq 0.$$

*Proof.* Let  $x_0$  be a local minimum and  $v$  be a feasible direction at  $\Omega$ , and  $s \in (0, \bar{s}]$ . The first statement is the immediate result of the first order necessary condition. Now suppose  $\nabla f(x_0) = 0$ , by the Taylor's theorem,

$$0 \leq f(x_0 + sv) - f(x_0) = s \nabla f(x_0) \cdot v + \frac{s^2}{2} v^T \nabla^2 f(x_0) v + o(s^2) \quad (3.13)$$

$$= \frac{s^2}{2} v^T \nabla^2 f(x_0) v + o(s^2) \quad (3.14)$$

Since  $s^2 > 0$ , divide both sides by  $s^2$  and take limit,

$$\lim_{s \rightarrow 0} \frac{f(x_0 + sv) - f(x_0)}{s^2} = \lim_{s \rightarrow 0} \left\{ \frac{1}{2} v^T \nabla^2 f(x_0) v + \frac{o(s^2)}{s^2} \right\} \quad (3.15)$$

$$= \frac{1}{2} v^T \nabla^2 f(x_0) v + \lim_{s \rightarrow 0} \frac{o(s^2)}{s^2} \quad (3.16)$$

$$= \frac{1}{2} v^T \nabla^2 f(x_0) v \geq 0 \quad (3.17)$$

■

**Example 3.1.**  $f(x, y) = x^2 - xy + y^2 - 3y : \Omega = \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then at  $(x_0, y_0) = (1, 2)$ ,

$$\nabla f(x_0, y_0) = (2x_0 - y, -x_0 + 2y_0 - 3) = (0, 0) \quad (3.18)$$

$$\nabla^2 f(x_0, y_0) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \succcurlyeq 0 \quad (3.19)$$

**Definition 3.3.** Let  $A \in \mathbb{R}^{n \times n}$ ,  $A$  is

- (i) **Positive definite** (denoted as  $A \succ 0$ ) if  $x^T A x > 0 \forall x \neq 0$ , if and only if all eigenvalues  $\lambda_i > 0$ ;
- (ii) **Positive Semi-definite** (denoted as  $A \succeq 0$ ) if  $x^T A x \geq 0 \forall x \in \mathbb{R}^n$ , if and only if all eigenvalues  $\lambda_i \geq 0$ .

**Theorem 3.5** (Sylvester's Criterion). Let  $A \in \mathbb{R}^{n \times n}$  be a Hermitian matrix (i.e.  $A = \overline{A^T}^1$ ), then

1.  $A \succ 0 \iff$  all leading principal minors have positive determinants;
2.  $A \succeq 0 \iff$  all leading principal minors have non-negative determinants.

**Theorem 3.6** (Second Order Sufficient Condition for Interior Local Minima). Let  $f : C^2(\Omega, \mathbb{R})$ , for some  $x_0 \in \Omega$ , if

- (i)  $\nabla f(x_0) = 0$ ,
- (ii) (and)  $\nabla^2 f(x_0) \succ 0$ .

then  $x_0$  is a strictly local minimizer.

**Lemma 3.1.** Suppose  $\nabla^2 f(x_0)$  is positive definite, then

$$\exists a > 0 \text{ s.t. } v^T \nabla^2 f(x_0) v \geq a \|v\|^2 \quad \forall v \quad (3.20)$$

That is, the quadratic form of a positive definite matrix is bounded away from zero.

*Proof of the Lemma.* Recall that a squared matrix  $Q$  is called **orthogonal** when every column and row of it is an orthogonal unit vector. So that for every orthogonal matrix  $Q$ ,  $Q^T Q = I$ , which implies  $Q^T = Q^{-1}$ . Further, note that

$$\|Qv\|^2 = (Qv)^T (Qv) = v^T Q^T Q v = \|v\|^2 \quad (3.21)$$

$$\implies \|Qv\| = \|v\| \quad \forall v \in \mathbb{R}^n \quad (3.22)$$

Let  $v \in \mathbb{R}^n$ , consider the eigenvector decomposition of  $\nabla^2 f(x_0)$ , let  $w$  satisfy  $v = Qw$ :

$$Q^T \nabla^2 f(x_0) Q = \text{diag}(\lambda_1, \dots, \lambda_n) \quad (3.23)$$

$$\implies v^T \nabla^2 f(x_0) v = (Qw)^T \nabla^2 f(x_0) (Qw) \quad (3.24)$$

$$= w^T Q^T \nabla^2 f(x_0) Q w \quad (3.25)$$

$$= w^T \text{diag}(\lambda_1, \dots, \lambda_n) w \quad (3.26)$$

$$= \lambda_1 w_1^2 + \dots + \lambda_n w_n^2 \quad (3.27)$$

Let  $a := \min\{\lambda_1, \dots, \lambda_n\}$ ,

$$\dots \geq a \|w\|^2 = a \|Q^T v\|^2 = a \|v\|^2 \quad (3.28)$$

■

---

<sup>1</sup> $\overline{A^T}$  denotes the complex conjugate of the transpose, a matrix with *real entries* is Hermitian if and only if it is symmetric.

*Proof of the Theorem.* Let  $x \in \Omega$ , suppose  $\nabla f(x_0) = 0$  and  $\nabla^2 f(x_0) \succ 0$ . By the second order Taylor approximation,

$$f(x_0 + v) - f(x_0) = \nabla f(x_0)^T v + \frac{1}{2} v^T \nabla^2 f(x_0) v + o(\|v\|^2) \quad (3.29)$$

$$= \frac{1}{2} v^T \nabla^2 f(x_0) v + o(\|v\|^2) \quad (3.30)$$

$$\geq \frac{a}{2} \|v\|^2 + o(\|v\|^2) \text{ for some } a > 0 \quad (3.31)$$

$$= \|v\|^2 \left( \frac{a}{2} + \frac{o(\|v\|^2)}{\|v\|} \right) \quad (3.32)$$

$$> 0 \text{ for sufficiently small } v \quad (3.33)$$

Therefore,  $f(x_0) < f(x) \forall x \in V_\varepsilon(x_0)$ . ■

## 3.2 Equality Constraints: Lagrangian Multiplier

### 3.2.1 Tangent Space to a (Hyper) Surface at a Point

**Definition 3.4.** A surface  $\mathcal{M} \subseteq \mathbb{R}^n$  is defined as

$$\mathcal{M} := \{x \in \mathbb{R}^n : h_i(x) = 0 \forall i\} \quad (3.34)$$

where  $h_i$  are all  $C^1$  functions.

**Definition 3.5.** A differentiable curve on a surface  $\mathcal{M}$  is a  $C^1$  function mapping from  $(-\varepsilon, \varepsilon)$  to  $\mathcal{M}$ .

*Remark: in previous calculus courses, differentiable curves are often referred to as parameterizations.*

Let  $x(s)$  be a differentiable curve on  $\mathcal{M}$  passes through  $x_0 \in \mathcal{M}$ , re-parameterize it so that  $x(0) = x_0$ . Then vector

$$v := \left. \frac{d}{ds} \right|_{s=0} x(s) \quad (3.35)$$

touches  $\mathcal{M}$  tangentially.

**Definition 3.6.** Any vector  $v$  generated by some differentiable curve on  $\mathcal{M}$  and takes above form is a **tangent vector** on  $\mathcal{M}$  through  $x_0$ .

**Definition 3.7.** The **tangent space** to  $\mathcal{M}$  at  $x_0$  is defined to be the set of all tangent vectors:

$$T_{x_0} \mathcal{M} := \left\{ v \in \mathbb{R}^n : v := \left. \frac{d}{ds} \right|_{s=0} x(s) \text{ for some } x \in C^1(V_\varepsilon(0), \mathcal{M}) \text{ s.t. } x(0) = x_0 \right\} \quad (3.36)$$

**Example 3.2.** Define

$$\mathcal{M} := \{x \in \mathbb{R}^2 : \|x\|_2 = 1\} \quad (3.37)$$

By defining  $C^1$  functions  $g(x) := \|x\|_2^2 - 1$ ,  $\mathcal{M}$  is a surface. The tangent space of  $\mathcal{M}$  at  $x_0$  is

$$T_{x_0} \mathcal{M} = \{v \in \mathbb{R}^n : \langle v, x_0 \rangle = 0\} \quad (3.38)$$

**Definition 3.8.** Let  $\mathcal{M}$  be a surface defined using  $C^1$  functions, a point  $x_0 \in \mathcal{M}$  is a regular point of the constraints if

$$\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\} \quad (3.39)$$

are linearly independent.

*Remark: if there is only one constraint  $h$ , then  $x_0$  is regular if and only if  $\nabla h(x_0) \neq 0$ .*

**Notation 3.1.** Define the  $T$  space on equality constraint as

$$T_{x_0} := \{x \in \mathbb{R}^n : \langle x_0, \nabla h_i(x_0) \rangle = 0 \ \forall i \in [k]\} = \{\nabla_i(x_0)\}^\perp \quad (3.40)$$

**Example 3.3** (Counter example). Define

$$\mathcal{M} := \{(x, y) \in \mathbb{R}^2 : h(x, y) = xy = 0\} \quad (3.41)$$

Then it is easy to verify that  $(0, 0)$  is not a regular point. And

$$T_{0,0} = \{(x, y) \in \mathbb{R}^2 : (x, y) \cdot (0, 0) = 0\} = \mathbb{R}^2 \quad (3.42)$$

$$\neq T_{0,0}\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\} \quad (3.43)$$

**Theorem 3.7.** Suppose  $x_0$  is a regular point of  $\mathcal{M} := \{h_i(x) = 0, i = 1, \dots, k\}$ , then  $T_{x_0} = T_{x_0}\mathcal{M}$ .

*Proof.* Show  $T_{x_0}\mathcal{M} \subseteq T_{x_0}$ .

Suppose  $x_0$  is a regular point of  $\mathcal{M}$ . Let  $v \in T_{x_0}\mathcal{M}$ , then there exists some differentiable curve  $x(\cdot) : V_\varepsilon(0) \rightarrow \mathcal{M}$  such that  $x(0) = x_0$ , such that

$$v = \left. \frac{d}{ds} \right|_{s=0} x(s) \quad (3.44)$$

Note that  $h_i(x(s)) = 0$  is constant for every  $i \in [k]$ , therefore

$$\left. \frac{d}{ds} \right|_{s=0} h_i(x(s)) \quad (3.45)$$

By the chain rule,

$$\nabla h_i(x_0) \cdot v = 0 \ \forall i \quad (3.46)$$

Therefore  $v \in T_{x_0}$ .

Show  $T_{x_0} \subseteq T_{x_0}\mathcal{M}$ .

(i)  $x_0$  is regular  $\implies T_{x_0}\mathcal{M}$  is a vector space;

(ii)  $T_{x_0} = \text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}^\perp$ .

Show  $T_{x_0} \subseteq \text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}^\perp$ :

Let  $v \in T_{x_0}$ , then  $v \perp \nabla h_i(x_0)$  for every  $i$ . Therefore  $v$  is orthogonal to every linear combination of  $\nabla h_i(x_0)$ , and therefore orthogonal to the span.

Show  $\text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}^\perp \subseteq T_{x_0}$ :

Let  $v$  in the perp of the span, then  $v$  is orthogonal to every basis of the span, so  $v \in T_{x_0}$ . ■

**Lemma 3.2.** Let  $f, h_1, \dots, h_k \in C^1$  defined on open subset  $\Omega \subseteq \mathbb{R}^n$ . Define  $\mathcal{M} := \{x \in \mathbb{R}^n : h_i(x) = 0 \ \forall i\}$ . Suppose  $x_0 \in \mathcal{M}$  is a local minimum of  $f$  on  $\mathcal{M}$ , then

$$\nabla f(x_0) \perp T_{x_0}\mathcal{M} \quad (3.47)$$

*Proof.* WLOG  $\Omega = \mathbb{R}^n$ , take  $v \in T_{x_0}\mathcal{M}$ . Then there exists some differentiable curve  $x$  on  $\mathcal{M}$  satisfying  $v = x'(0)$ . Because  $x_0$  is a local minimum of  $f$  on  $\Omega$ ,  $s = 0$  is a local minimum of  $f(x(s))$ , moreover, it is an interior minimum. By chain rule and the necessary condition of local minimum,

$$Df(x(0)) = \nabla f(x(0)) \cdot x'(0) = 0 \quad (3.48)$$

$$\implies \nabla f(x_0) \cdot v = 0 \quad (3.49)$$

Therefore  $\nabla f(x_0) \perp T_{x_0}\mathcal{M}$ . ■

**Theorem 3.8** (Lagrange Multipliers: First Order Necessary Condition). Let  $f, h_1, \dots, h_k \in C^1$  defined on open subset  $\Omega \subseteq \mathbb{R}^n$ . Let  $x_0$  be a regular point of the constraint set  $\mathcal{M} := \bigcap_{i=1}^k h_i^{-1}(0)$ . Suppose  $x_0$  is a local minimum of  $\mathcal{M}$ , then there exists  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  such that

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) = 0 \quad (3.50)$$

*Remark:* if we define Lagrangian  $\mathcal{L}(x, \lambda_i) := f(x) + \sum_{i=1}^k \lambda_i h_i(x)$ , then the theorem says the local minimum is a critical point of  $\mathcal{L}$ .

*Proof.* Because  $x_0$  is a regular point, then by previous lemma,  $\nabla f(x_0) \perp T_{x_0}\mathcal{M}$ . Moreover,

$$T_{x_0}\mathcal{M} = T_{x_0} = (\text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\})^\perp \quad (3.51)$$

Also, because  $x_0$  is a local minimum,

$$\nabla f(x_0) \perp T_{x_0}\mathcal{M} \quad (3.52)$$

Therefore,  $\nabla f(x_0) \in (T_{x_0}\mathcal{M})^\perp = (\text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\})^{\perp\perp} = \text{span}\{\nabla h_1(x_0), \dots, \nabla h_k(x_0)\}$ , where the last equality holds in finite dimensional cases. Hence, it is obvious that we can write  $\nabla f(x_0)$  as a linear combination of  $\{\nabla h_i(x_0)\}$ . ■

**Theorem 3.9** (Second Order Necessary Condition). Let  $f, h_i \in C^2$ , if  $x_0$  is a local minimum on previously defined surface  $\mathcal{M}$ , then there exists Lagrangian multipliers  $\{\lambda_i\}$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) = 0$  ( $\nabla_x \mathcal{L} = 0$ );
- (ii) And  $\nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \succcurlyeq 0$  on  $T_{x_0}\mathcal{M}$  ( $\nabla_x^2 \mathcal{L} \succcurlyeq 0$ ).

*Remark:* whenever  $x_0$  is a local minimum, it must be a critical point of  $\mathcal{L}$ , and  $\mathcal{L}$  is positive semidefinite on the tangent space at  $x_0$ .

*Proof.* The first result is exactly the same as the first order condition proven above.

To show the second result, let  $x(s) \in \mathcal{M}$  be an arbitrary differentiable curve on  $\mathcal{M}$  such that  $x(0) = x_0$ . Then,

$$\frac{d}{ds}f(x(s)) = \nabla f(x(s)) \cdot x'(s) \quad (3.53)$$

$$\frac{d^2}{ds^2}f(x(s)) = x'(s)^T \nabla^2 f(x(s)) x'(s) + \nabla f(x(s)) x''(s) \quad (3.54)$$

By the second order Taylor theorem, for every  $s$  such that  $x(s) \in \mathcal{M}$ ,

$$f(x(s)) - f(x_0) = s \nabla f(x_0) \cdot x'(0) + \frac{s^2}{2} [x'(0)^T \nabla^2 f(x_0) x'(0) + \nabla f(x_0) x''(0)] + o(s^2) \quad (3.55)$$

Note that by definition,  $x'(0)$  is in the tangent space at  $x_0$ . Also, we've shown previously that  $\nabla f(x_0)$  is orthogonal to the tangent space at  $x_0$ , therefore,

$$f(x(s)) - f(x_0) = \frac{s^2}{2} [x'(0)^T \nabla^2 f(x_0) x'(0) + \nabla f(x_0) x''(0)] + o(s^2) \quad (3.56)$$

Also, by the definition of  $\mathcal{M}$ , all constraints hold with equality:

$$f(x_0) = f(x_0) + \sum_{i=1}^k \lambda_i h_i(x_0) \quad (3.57)$$

where  $\lambda_i$ 's are from the first result. Hence,

$$f(x(s)) - f(x_0) = \frac{s^2}{2} \left[ x'(0)^T \left( \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \right) x'(0) + \left( \nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) \right) x''(0) \right] + o(s^2) \quad (3.58)$$

$$= \frac{s^2}{2} x'(0)^T \left( \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \right) x'(0) + o(s^2) \quad (3.59)$$

And above expression is greater or equal to zero because  $x_0$  is a local minimum,

$$\frac{s^2}{2} x'(0)^T \left( \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \right) x'(0) + o(s^2) \geq 0 \quad (3.60)$$

$$\implies x'(0)^T \left( \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \right) x'(0) + \frac{o(s^2)}{s^2} \geq 0 \quad (3.61)$$

$$\xrightarrow{s \rightarrow 0} x'(0)^T \left( \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \right) x'(0) \geq 0 \quad (3.62)$$

Where  $x'(0)$  is a vector in the tangent space at  $x_0$  by definition. Moreover, the curve  $x(s)$  was chosen arbitrarily, so the argument works for every curve and therefore every tangent vector, and what's desired is shown. ■

**Example 3.4.**

$$\min f(x, y) = x^2 - y^2 \quad (3.63)$$

$$s.t. \ h(x, y) = y = 0 \quad (3.64)$$

First order condition suggests  $(x_0, y_0) = (0, 0)$  Note that the tangent space at  $(x_0, y_0)$  is  $\text{span}\{\nabla h_i\}^\perp$ :

$$T_{x_0}\mathcal{M} = \{(u, 0) : u \in \mathbb{R}\} \quad (3.65)$$

and

$$\nabla_x^2 \mathcal{L} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad (3.66)$$

is obviously positive semidefinite (actually positive definition) on the tangent space.

**Theorem 3.10** (Second Order Sufficient Conditions). Let  $f, h_i \in C^2$  on open  $\Omega \subseteq \mathbb{R}^n$ , and  $x_0 \in \mathcal{M}$  is a regular point, if there exists  $\lambda_i \in \mathbb{R}$  such that

$$(i) \ \nabla_x \mathcal{L}(x_0, \lambda_i) = 0;$$

$$(ii) \ \nabla_x^2 \mathcal{L}(x_0, \lambda_i) \succ 0 \text{ on } T_{x_0}\mathcal{M},$$

then  $x_0$  is a *strict* local minimum.

*Proof.* Recall that  $\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)$  positive definite on  $T_{x_0}\mathcal{M}$  implies there exists  $a > 0$  ( $a$  is taken to be equal to the least eigenvalue of  $\nabla_x^2 \mathcal{L}$ ) such that

$$v^T [\nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)] v \geq a \|v\|^2 \quad \forall v \in T_{x_0}\mathcal{M} \quad (3.67)$$

Let  $x(s) \in \mathcal{M}$  be a curve such that  $x(0) = x_0$  and  $v = x'(0)$ . WLOG,  $\|x'(0)\| = 1$ . By the second order Taylor expansion,

$$f(x(s)) - f(x(0)) = s \left. \frac{d}{ds} \right|_{s=0} f(x(s)) + \frac{s^2}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} f(x(s)) + o(s^2) \quad (3.68)$$

$$= s \left. \frac{d}{ds} \right|_{s=0} \left[ f(x(s)) + \sum \lambda_i h_i(x(s)) \right] + \frac{s^2}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} \left[ f(x(s)) + \sum \lambda_i h_i(x(s)) \right] + o(s^2) \quad (3.69)$$

$$= s \nabla_x \mathcal{L}(x_0, \lambda_i) \cdot x'(0) + \frac{s^2}{2} [x'(0)^T \nabla_x^2 \mathcal{L}(x_0, \lambda_i) x'(0) + \nabla_x \mathcal{L}(x_0, \lambda_i) x''(0)] + o(s^2) \quad (3.70)$$

$$= \frac{s^2}{2} x'(0)^T \nabla_x^2 \mathcal{L}(x_0, \lambda_i) x'(0) + o(s^2) \quad (3.71)$$

$$\geq \frac{s^2}{2} a \|x'(0)\|^2 + o(s^2) \quad \text{where } a > 0 \quad (3.72)$$

$$= s^2 \left( \frac{a}{2} + \frac{o(s^2)}{s^2} \right) \quad (3.73)$$

$$\stackrel{s \rightarrow 0}{>} 0 \quad (3.74)$$

Therefore, for sufficiently small  $s$ ,  $f(x(s)) - f(x(0)) > 0$ . And this is true for every curve  $x$  on  $\mathcal{M}$ . So  $x(0)$  is a strict local minimum. ■



### 3.3 Remark on the Connection Between Constrained and Unconstrained Optimizations

**Example 3.5.** Consider

$$\min f(x, y, z) \quad (3.75)$$

$$s.t. g(x, y, z) = z - h(x, y) = 0 \quad (3.76)$$

where  $\mathcal{M}$  is the graph of  $h$ . Using Lagrangian multiplier provides necessary condition:  $\nabla f + \lambda \nabla g = 0$ ,

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} + \lambda \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix} = 0 \quad (3.77)$$

Convert the constrained optimization into an unconstrained optimization as

$$\min_{(x,y) \in \mathbb{R}^2} F(x, y) = f(x, y, h(x, y)) \quad (3.78)$$

The necessary condition for unconstrained optimization is

$$\nabla F(x, y) = \begin{pmatrix} f_x + f_z h_x \\ f_y + f_z h_y \end{pmatrix} \quad (3.79)$$

$$= \begin{pmatrix} f_x \\ f_y \end{pmatrix} - f_z \begin{pmatrix} -h_x \\ -h_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.80)$$

Define  $\lambda := -f_z$ .

$$\nabla F(x, y) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} + \lambda \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.81)$$

### 3.4 Inequality Constraints

**Definition 3.9.** Let  $x_0$  satisfy the set of constraints

$$(\dagger) \begin{cases} h_i(x) = 0 & i \in \{1, \dots, k\} \\ g_j(x) \leq 0 & j \in \{1, \dots, \ell\} \end{cases} \quad (3.82)$$

we say that the constraint  $g_i$  is **active** at  $x_0$  if  $g_i(x_0) = 0$ , and is **inactive** at  $x_0$  if  $g_i(x_0) < 0$ .

**Definition 3.10.** Split the collection of inequality constraints into active and inactive constraints, let  $\Theta(x_0)$  denote the collection of active indices, that's:

$$g_j(x_0) = 0 \quad \forall j \in \Theta(x_0) \quad (3.83)$$

$$g_j(x_0) < 0 \quad \forall j \notin \Theta(x_0) \quad (3.84)$$

Then  $x_0$  is said to be a **regular point** of the constraint if

$$\{\nabla h_i(x_0) \forall i \in \{1, \dots, k\}; \underbrace{\nabla g_j(x_0) \forall j \in \Theta(x_0)}_{\text{Active Constraints}}\} \quad (3.85)$$

is linearly independent.

**Theorem 3.11** (The First Order Necessary Condition for Local Minimum: Kuhn-Tucker Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  with constraints  $h_i$  and  $g_i$  to be  $C^1$  on  $\Omega$ . Suppose  $x_0 \in \Omega$  is a regular point with respect to constraints, further suppose  $x_0$  is a local minimum, then there exists some  $\lambda_i \in \mathbb{R}$  and  $\mu_j \in \mathbb{R}_+$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0$  (i.e.  $\nabla_x \mathcal{L}(x, \lambda, \mu) = 0$ );
- (ii)  $\mu_j g_j(x_0) = 0$  (*Complementary slackness*).

*Remark 1: by complementary slackness, all  $\mu_j$  corresponding to inactive inequality constraints are zero.*

*Remark 2: it is possible for an active constraint to have zero multiplier.*

*Proof.* Let  $x_0$  be a local minimum for  $f$  satisfying constraints, equivalently, it is a local minimum for equality constraints and active inequality constraints.

By the first order necessary condition for local minimum with equality constraints, there exists  $\lambda_i, \mu_j \in \mathbb{R}$  such that

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j \in \Theta(x_0)} \mu_j \nabla g_j(x_0) = 0 \quad (3.86)$$

Then by setting  $\mu_j = 0$  for all  $j \notin \Theta(x_0)$  one have

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0 \quad (3.87)$$

By construction, the complementary slackness is satisfied. At this stage, we have construct  $\lambda_i \in \mathbb{R}$  and  $\mu_j \in \mathbb{R}$  satisfying both conditions, we still need to argue that  $\mu_j \geq 0$  for every  $j$ . ■

**Theorem 3.12** (The Second Order Necessary Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , and  $f, h_1, \dots, h_k, g_1, \dots, g_\ell \in C^2(\mathbb{R}^n, \mathbb{R})$ . Let  $x_0$  be a regular point of the constraints ( $\dagger$ ). Suppose  $x_0$  is a local minimum of  $f$  subject to constraint ( $\dagger$ ), then there exists  $\lambda_i \in \mathbb{R}$  and  $\mu_j \geq 0$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0$ ;
- (ii)  $\mu_j g_j(x_0) = 0$ ;
- (iii)  $\nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla^2 g_j(x_0)$  is positive semidefinite on the tangent space to activate constraints at  $x_0$ .

*Proof.* (i) and (ii) are immediate result from the first order necessary condition.

Suppose  $x_0$  is a local minimum for ( $\dagger$ ), then  $x_0$  is a local minimum for active constraints at  $x_0$ .

Therefore,  $\nabla^2 \hat{\mathcal{L}} = \nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) + \sum_{j \in I(x_0)} \mu_j \nabla^2 g_j(x_0)$  is positive semidefinite on the tangent space to active constraints. Note that because  $\mu_j = 0$  for inactive constraints, therefore  $\nabla^2 \hat{\mathcal{L}} = \nabla^2 \mathcal{L}$  at  $x_0$ , and both of them are positive semidefinite on the tangent space corresponding to active constraints. ■

**Theorem 3.13** (The Second Order Sufficient Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , let  $f, h_i, q_j \in C^2(\Omega)$ . Consider minimizing  $f(x)$  with the constraint

$$(\dagger) \begin{cases} h_i(x) = 0 & \forall i \\ g_j(x) \leq 0 & \forall j \\ x \in \Omega \end{cases} \quad (3.88)$$

Suppose there exists a feasible  $x_0$  satisfying  $(\dagger)$  and  $\lambda_i \in \mathbb{R}$  and  $\mu_j \in \mathbb{R}_+$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^{\ell} \mu_j \nabla g_j(x_0) = 0$ ;
- (ii)  $\mu_j g_j(x_0) = 0$  (*Complementary slackness*).

If the Hessian matrix for Lagrangian  $\nabla_x^2 \mathcal{L}(x_0)$  is positive definite on  $\tilde{T}_{x_0}$ , the space of **strongly active** constraints at  $x_0$ , then  $x_0$  is a strict local minimum.

**Definition 3.11.** A constraint  $g_j$  is **strongly active** at  $x_0$  if  $g_j(x_0) = 0$  (so it is active) and  $\mu_j > 0$ .

**Notation 3.2.** For convenience, we can rearrange the collection of constraints such that, among the  $\ell$  constraints in total, the first  $\ell'$  constraints are *active* at  $x_0$  and the first  $\ell''$  constraints are *strongly active*. Note that  $\ell'' \leq \ell' \leq \ell$ .

Define

$$\tilde{T}_{x_0} := \{v \cdot \nabla h_i(x_0) = 0 \ \forall i \wedge v \cdot \nabla g_j(x_0) = 0 \text{ for all } g_j \text{ active.}\} \quad (3.89)$$

$$\tilde{\tilde{T}}_{x_0} := \{v \cdot \nabla h_i(x_0) = 0 \ \forall i \wedge v \cdot \nabla g_j(x_0) = 0 \text{ for all } g_j \text{ strongly active.}\} \quad (3.90)$$

Clearly,  $\tilde{T}_{x_0} \subseteq \tilde{\tilde{T}}_{x_0}$  because there are (weakly) more active constraints than strongly active constraints.

*Proof of the Sufficient Condition.* Suppose, for contradiction,  $x_0$  is not a strict local minimum.

**Claim 1:** There exists unit vector  $v \in \mathbb{R}^n$  such that

- (a)  $\nabla f(x_0) \cdot v \leq 0$ ;
- (b)  $\nabla h_i(x_0) \cdot v = 0$  for every  $i$ ;
- (c)  $\nabla g_j(x_0) \cdot v \leq 0$  for all  $j \leq \ell'$  (active constraints).

*Proof of Claim 1.* Because  $x_0$  is not a strictly local minimum, one can construct a sequence of feasible points  $(x_k) \rightarrow x_0$  by setting  $\varepsilon = \frac{1}{k}$  for every  $k \in \mathbb{N}$  such that  $f(x_k) \leq f(x_0)$ .

Let  $v_k := \frac{x_k - x_0}{\|x_k - x_0\|}$ ,  $s_k := \|x_k - x_0\|$ . Note that every  $v_k$  is in unit sphere, which is compact. Therefore, there exists a subsequence of  $(v_k)$  converges to some unit vector  $v$ .

$$0 \geq f(x_k) - f(x_0) = f(x_0 + s_k v_k) - f(x_0) \ \forall k \in \mathbb{N} \quad (3.91)$$

The first order Taylor series suggests the following holds for every  $k \in \mathbb{N}$ :

$$0 \geq f(x_0 + s_k v_k) - f(x_0) \quad (3.92)$$

$$= s_k \nabla f(x_0) \cdot v_k + o(s_k) \quad (3.93)$$

$$0 = h_i(x_0 + s_k v_k) - h_i(x_0) = s_k \nabla h_i(x_0) \cdot v_k + o(s_k) \quad (3.94)$$

$$0 \geq g_j(x_0 + s_k v_k) - g_j(x_0) = s_k \nabla g_j(x_0) \cdot v_k + o(s_k) \quad \forall j \leq \ell' \quad (3.95)$$

Above inequalities are preserved by limit operation, therefore,

$$\nabla f(x_0) \cdot v_k + \frac{o(s_k)}{s_k} \rightarrow \nabla f(x_0) \cdot v \leq 0 \quad (3.96)$$

$$\nabla h_i(x_0) \cdot v_k + \frac{o(s_k)}{s_k} \rightarrow \nabla h_i(x_0) \cdot v = 0 \quad (3.97)$$

$$\nabla g_j(x_0) \cdot v_k + \frac{o(s_k)}{s_k} \rightarrow \nabla g_j(x_0) \cdot v \leq 0 \quad \forall j \leq \ell' \quad (3.98)$$

■

**Claim 2:**  $\nabla g_j(x_0) \cdot v = 0$  for  $j = 1, \dots, \ell''$ .

*Proof of Claim 2.* Suppose not, there exists  $j \in \{1, \dots, \ell''\}$  such that  $\nabla g_j(x_0) \cdot v < 0$ . Then by (i),

$$0 \geq \nabla f(x_0) \cdot v = - \sum_{i=1}^k \lambda_i \nabla h_i(x_0) \cdot v - \sum_{j=1}^{\ell} \mu_j \nabla g_j(x_0) \cdot v \quad (3.99)$$

$$= - \sum_{j=1}^{\ell} \mu_j \nabla g_j(x_0) \cdot v > 0 \quad (3.100)$$

the last inequality is from the fact that  $\mu_j \nabla g_j(x_0) \cdot v \leq 0$  for all active constraints and  $\mu_j = 0$  for all inactive constraints.

■

(b) and claim 2 suggests  $v \in \tilde{T}_{x_0}$ .

By the second order Taylor approximation,

$$0 \geq f(x_k) - f(x_0) = s_k \nabla f(x_0) \cdot v_k + \frac{s_k^2}{2} v_k \cdot \nabla^2 f(x_0) \cdot v_k + o(s_k^2) \quad (3.101)$$

$$0 = h_i(x_k) - h_i(x_0) = s_k \nabla h_i(x_0) \cdot v_k + \frac{s_k^2}{2} v_k \cdot \nabla^2 h_i(x_0) \cdot v_k + o(s_k^2) \quad \forall i \quad (3.102)$$

$$0 \geq g(x_k) - g(x_0) = s_k \nabla g_j(x_0) \cdot v_k + \frac{s_k^2}{2} v_k \cdot \nabla^2 g_j(x_0) \cdot v_k + o(s_k^2) \quad \forall j \leq \ell' \quad (3.103)$$

Multiply the second equation by  $\lambda_i$  and third equation by  $\mu_j$ , and use the fact that  $\mu_j = 0$  for every  $j > \ell'$ . Also, given  $\nabla \mathcal{L} = 0$  in (i):

$$0 \geq \frac{s_k^2}{2} v_k \cdot \nabla^2 \mathcal{L} \cdot v_k + o(s_k^2) \quad (3.104)$$

Divide by  $s_k^2$  and take the limit  $(v_k) \rightarrow v$ :

$$v \cdot \nabla^2 \mathcal{L} \cdot v \leq 0 \quad (3.105)$$

which contradicts the assumption that  $\nabla^2 \mathcal{L}$  is positive definite in  $\tilde{T}_{x_0}$  because we've shown that  $v \in \tilde{T}_{x_0}$ . ■

## 4 Iterative Algorithms for Optimization

### 4.1 Newton's Method

**Example 4.1** (Motivation: a second order iterative algorithm). Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  where  $I$  is an open interval. Let  $x_i \in I$  be a starting point, consider the second order linear approximation of  $f$  at  $x_0$ :

$$g(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 \quad (4.1)$$

By construction, the second order Taylor polynomial,  $g(x)$ , is the best second order approximation to  $f$  at  $x_0$  in the following sense:

$$g(x_0) = f(x_0) \quad (4.2)$$

$$g'(x_0) = f'(x_0) \quad (4.3)$$

$$g''(x_0) = f''(x_0) \quad (4.4)$$

The Newton's method aims to solve the critical point of  $g(x)$  and define  $x_1$  to be the critical point found:

$$g'(x_1) = f'(x_0) + f''(x_0)(x_1 - x_0) = 0 \quad (4.5)$$

$$\implies x_1 \leftarrow x_0 - \frac{f'(x_0)}{f''(x_0)} \quad (4.6)$$

**Algorithm 4.1** (Newton's Method in  $\mathbb{R}$ ). Given initial point  $x_0 \in I$ , while not terminated:

$$x_{n+1} \leftarrow x_n - \frac{f'(x_n)}{f''(x_n)} \quad (4.7)$$

**Theorem 4.1.** Let  $f \in C^3$  on open interval  $I \subseteq \mathbb{R}$ . Suppose  $x_* \in I$  satisfies  $f'(x_*) = 0$  and  $f''(x_*) \neq 0$ , then the sequence of points  $(x_n)$  generated by Newton's method converges to  $x_*$  if  $x_0$  is sufficiently close to  $x_*$ .

**Example 4.2.** Let  $f(x) = x^2$ , then  $\frac{f'(x)}{f''(x)} = \frac{2x}{2} = x$ . For any starting point  $x_0$ ,  $x_1 = x_0 - \frac{2x_0}{2} = 0$ . That is, the algorithm converges to the global minimum in one iteration.

*Proof.* Let  $g(x) = f'(x)$  so that  $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ .

Because  $f \in C^3$ , then  $g \in C^2$ .

Note that by  $g \in C^2$ ,  $g' = f''$  is bounded away from zero near  $x_*$ .

And by continuity again,  $g'' = f^{(3)}$  is bounded near the bounded region  $V_\varepsilon(x_*)$ .

That is, within small region near  $x_*$ ,  $V_\delta(x_*)$ , there exists a sufficiently small  $\alpha > 0$  such that

$$\begin{cases} |g'(x_1)| > \alpha \quad \forall x_1 \in V_\delta(x_*) \\ |g''(x_2)| < \frac{1}{\alpha} \quad \forall x_2 \in V_\delta(x_*) \end{cases} \quad (4.8)$$

Further, note that  $g(x_*) = f'(x_*) = 0$ .

WLOG, let  $n \in \mathbb{N}$ , suppose  $x_n > x_*$ :

$$x_{n+1} - x_* = x_n - \frac{g(x_n)}{g'(x_n)} - x_* \quad (4.9)$$

$$= x_n - x_* - \frac{g(x_n) - g(x_*)}{g'(x_n)} \quad (4.10)$$

$$= -\frac{g(x_n) - g(x_*) - g'(x_n)(x_n - x_*)}{g'(x_n)} \quad (4.11)$$

$$= -\frac{1}{2} \frac{g''(\xi)}{g'(x_n)} (x_n - x_*)^2 \quad \text{for some } \xi \in (x_*, x_n) \quad (4.12)$$

By taking the absolute values on both sides:

$$|x_{n+1} - x_*| = \frac{1}{2} \frac{|g''(\xi)|}{|g'(x_n)|} |x_n - x_*|^2 \quad (4.13)$$

$$< \frac{1}{2\alpha^2} |x_n - x_*|^2 \quad (4.14)$$

Let  $\rho := \frac{1}{\alpha^2} |x_0 - x_*|^2$ , choose  $x_0$  sufficiently close to  $x_*$  such that  $\rho < 1$ .

*Remark: we are showing the iterative algorithm induces a contraction map.*

Then,

$$|x_1 - x_*| < \frac{1}{2\alpha^2} |x_0 - x_*|^2 \quad (4.15)$$

$$= \frac{1}{2\alpha^2} |x_0 - x_*| |x_0 - x_*| \quad (4.16)$$

$$= \rho |x_0 - x_*| \quad (4.17)$$

Inductively,

$$|x_2 - x_*| < \frac{1}{2\alpha^2} |x_1 - x_*|^2 \quad (4.18)$$

$$< \frac{1}{2\alpha^2} \rho^2 |x_0 - x_*|^2 \quad (4.19)$$

$$= \rho^3 |x_0 - x_*| \quad (4.20)$$

$$< \rho^2 |x_0 - x_*| \quad (4.21)$$

By induction,

$$|x_n - x_*| < \rho^2 |x_0 - x_*| \quad (4.22)$$

Therefore, as  $n \rightarrow \infty$ ,  $(x_n) \rightarrow x_*$ . ■

**Theorem 4.2** (2nd Order MVT).

$$g(x) = g(y) + g'(y)(x - y) + \frac{1}{2} g''(\xi)(x - y)^2 \quad \xi \in (x, y) \quad (4.23)$$

**Algorithm 4.2** (Newton's Method in  $\mathbb{R}^n$ ). Let  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  where  $\Omega$  is open, let initial point  $x_0 \in \Omega$ . Suppose  $\nabla^2 f(x_n)$  is invertible for every generated  $n$ , and  $\nabla f(x_*) = 0$  so that algorithm stops at minimum.

The iterative algorithm is defined as following:

$$x_{n+1} \leftarrow x_n - [\nabla^2 f(x_n)]^{-1} \nabla f(x_n) \quad (4.24)$$

**Theorem 4.3** (Generalization). Suppose  $x_* \in \Omega$  and  $f \in C^3(\Omega, \mathbb{R})$  such that  $\nabla f(x_*) = 0$  and  $\nabla^2 f(x_*)$  is invertible. **TODO: check this** Then if initial point  $x_0$  is sufficiently closed to  $x_*$ , then Newton's method converges to  $x_*$ .

*Proof.* The basic idea is the same as the  $\mathbb{R}$  case: prove the iterative algorithm induces a contraction mapping. ■

**Example 4.3** (Newton's Method Fails to Converge). Even if  $f$  has an unique global minimum  $x_*$ , and  $x_0$  is arbitrarily close to the  $x_*$ , Newton's method could fail to converge. Consider

$$f(x) = \frac{2}{3} |x|^{\frac{3}{2}} \quad (4.25)$$

Note that

$$f(x) = \begin{cases} \frac{2}{3} x^{\frac{3}{2}} & x \geq 0 \\ -\frac{2}{3} x^{\frac{3}{2}} & x < 0 \end{cases} \quad (4.26)$$

$$f'(x) = \begin{cases} x^{\frac{1}{2}} & x \geq 0 \\ -x^{\frac{1}{2}} & x < 0 \end{cases} \quad (4.27)$$

$$f''(x) = \begin{cases} \frac{1}{2} x^{-\frac{1}{2}} & x > 0 \\ -\frac{1}{2} x^{-\frac{1}{2}} & x < 0 \\ \text{DNE} & x = 0 \end{cases} \quad (4.28)$$

Therefore  $f \notin C^2$ .

Let  $\delta > 0$  arbitrarily small, take initial point  $x_0 \in V_\delta(0)$ . WLOG,  $x_0 = \varepsilon \in V_\delta(0)$  with  $\varepsilon > 0$ . The algorithm will oscillate between  $\pm\varepsilon$  and never converge.

**Remark 4.1.** Newton's method does not necessarily converge to a global minimum, it may converge to local minimum or local maximum or even saddle point.

**Example 4.4** (Newton's Method Converges to a Saddle Point). Consider  $f(x) = x^3$ ,  $x_{n+1} \rightarrow \frac{x_n}{2}$ , which converges to 0 (a saddle point).

**Example 4.5** (Newton's Method on Quadratic Function). Let  $Q$  be a symmetric  $n \times n$  invertible matrix. Define quadratic form  $f(x) := \frac{1}{2} x^T Q x : \mathbb{R}^n \rightarrow \mathbb{R}$ . The optimal is  $x = 0$ . Let  $x_0 \in \mathbb{R}^n$ , then  $x_1 := x_0 - H_f(x_0)^{-1} \nabla f(x_0) = x_0 - Q^{-1} Q x_0 = 0$ . *Therefore, Newton's method converges in one iteration.*

## 4.2 Steepest/Gradient Descent

**Algorithm 4.3** (Steepest Descent). Let  $f : \Omega \rightarrow \mathbb{R}$  where  $\Omega$  is an open subset of  $\mathbb{R}^n$ . Let initial point  $x_0 \in \Omega$ .

To minimize  $f$  on  $\Omega$ , iteratively update  $x$  follows at each step  $k$ :

$$x_{k+1} \leftarrow x_k - \alpha_k \nabla f(x_k) \quad (4.29)$$

where  $\alpha_k = \operatorname{argmin}_{\alpha \geq 0} f(x_k - \alpha \nabla f(x_k))$ .

*Remark:* There might be multiple minimizing  $\alpha$ , in real world implementations, we take the least minimizer found.

**Theorem 4.4** (Gradient Descending is Descending). At every step  $k$ , if  $\nabla f(x_k) = 0$ , the algorithm terminates. Otherwise,

$$f(x_{k+1}) < f(x_k) \quad (4.30)$$

*Proof.* Suppose  $\nabla f(x_k) \neq 0$ .

Note that for the first minimizing  $\alpha_k$  found:

$$f(x_{k+1}) = f(x_k - \alpha_k \nabla f(x_k)) \quad (4.31)$$

$$\leq f(x_k - \alpha \nabla f(x_k)) \quad \forall 0 \leq \alpha \leq \alpha_k \quad (4.32)$$

Recall that

$$\left. \frac{d}{ds} \right|_{s=0} f(x_k - s \nabla f(x_k)) = -\nabla f(x_k) \cdot \nabla f(x_k) = -\|\nabla f(x_k)\|_2^2 < 0 \quad (4.33)$$

Therefore,

$$f(x_{k+1}) \leq f(x_k - \alpha \nabla f(x_k)) < f(x_k) \text{ for small } \alpha \quad (4.34)$$

■

**Theorem 4.5** (Gradient Descending Induces Perpendicular Steps). The consecutive steps induced by gradient descending are perpendicular. That is

$$(x_{k+2} - x_{k+1}) \cdot (x_{k+1} - x_k) = 0 \quad (4.35)$$

*Proof.* Note that

$$(x_{k+2} - x_{k+1}) \cdot (x_{k+1} - x_k) = (-\alpha_{k+1} \nabla f(x_{k+1})) \cdot (-\alpha_k \nabla f(x_k)) \quad (4.36)$$

$$= \alpha_k \alpha_{k+1} \nabla f(x_k) \cdot \nabla f(x_{k+1}) \quad (4.37)$$

If  $\alpha_k = 0$ , done.



If  $\alpha_k > 0$ ,

$$f(x_{k+1}) = f(x_k) - \alpha_k \nabla f(x_k) \quad (4.38)$$

$$= \min_{\alpha \geq 0} \{f(x_k - \alpha \nabla f(x_k))\} \quad (4.39)$$

$$= \min_{\alpha > 0} \{f(x_k - \alpha \nabla f(x_k))\} \quad (4.40)$$

$$\implies \left. \frac{\partial}{\partial \alpha} \right|_{\alpha=\alpha_k} f(x_k - \alpha \nabla f(x_k)) = 0 \quad (4.41)$$

$$\implies -\nabla f(x_k - \alpha_k \nabla f(x_k)) \cdot \nabla f(x_k) = 0 \quad (4.42)$$

$$\implies -\nabla f(x_{k+1}) \cdot \nabla f(x_k) = 0 \quad (4.43)$$

■

**Theorem 4.6** (Sufficient Condition for Gradient Descent to Converge). Let  $f \in C^1$  on open  $\Omega \subseteq \mathbb{R}^n$ .

Let  $\{x_k\}$  be the sequence generated by gradient descent:  $x_{k+1} \leftarrow x_k - \alpha_k \nabla f(x_k)$ .

If  $(x_k)$  is bounded in  $\Omega$ , that is, there exists a compact set  $K \subseteq \Omega$  such that  $(x_k) \subseteq K$ , then every convergent subsequence of  $(x_k)$  converges to a critical point  $x_* \in \Omega$  of  $f$ .

*Proof.* **TODO:** Need to fix this proof. Let  $x_k \in K$  compact.

Then there exists subsequence  $x_{k_i} \rightarrow x_* \in K$ .

Show:  $\nabla f(x_*) = 0$ .

Note that  $f(x_k) \geq f(x_{k+1})$  for every  $k \in \mathbb{N}$ , therefore  $f(x_{k_i}) \searrow f(x_*)$ . Therefore,  $f(x_k) \searrow f(x_*)$ . **TODO:**

Show this. Suppose, for contradiction,  $\nabla f(x_*) \neq 0$ .

By continuity of  $\nabla f$ ,  $(\nabla f(x_{k_i})) \rightarrow \nabla f(x_*)$ .

Let  $y_{k_i} := x_{k_i} - \alpha_{k_i} \nabla f(x_{k_i}) = x_{k_{i+1}}$ .

Note that  $y_{k_i}$  has a convergent subsequence converging to  $y_*$ .

WLOG,  $(y_{k_i}) \rightarrow y_*$ .

Observe

$$\alpha_{k_i} = \frac{|y_{k_i} - x_{k_i}|}{\|\nabla f(x_{k_i})\|} \quad (4.44)$$

$$\implies \lim_{k_i \rightarrow \infty} \alpha_{k_i} = \frac{|y_* - x_*|}{\|\nabla f(x_*)\|} =: \alpha_* \quad (4.45)$$

Put back:  $y_* = x_* - \alpha_* \nabla f(x_*)$ .

Now  $f(y_{k_i}) = f(x_{k_{i+1}}) = \min_{\alpha \geq 0} f(x_{k_i} - \alpha \nabla f(x_{k_i}))$ , which implies

$$f(y_{k_i}) \leq f(x_{k_i} - \alpha \nabla f(x_{k_i})) \quad \forall \alpha \geq 0 \quad (4.46)$$

$$\forall \alpha \geq 0 \quad \lim_{i \rightarrow \infty} f(y_{k_i}) = f(y_*) \leq \lim_{i \rightarrow \infty} f(x_{k_i} - \alpha \nabla f(x_{k_i})) = f(x_* - \alpha \nabla f(x_*)) \quad (4.47)$$

$$\implies f(y_*) \leq \min_{\alpha \geq 0} f(x_* - \alpha \nabla f(x_*)) < f(x_*) \quad (4.48)$$

Further note that

$$f(y_*) = \lim_{i \rightarrow \infty} f(y_{k_i}) = \lim_{i \rightarrow \infty} f(x_{k_{i+1}}) = f(x_*) \quad (4.49)$$

Contradiction. ■

#### 4.2.1 Steepest Descent: the Quadratic Case

**Example 4.6.** Let  $f$  follow the general quadratic form

$$f(x) = \frac{1}{2}x^T Qx - b^T x \quad (4.50)$$

with  $b, x \in \mathbb{R}^n$  and  $Q$  is positive definite.

Let  $0 < \lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \Lambda$  be eigenvalues of  $Q$ .

**Proposition 4.1.** Gradient descent on strictly convex(concave) quadratic functions is guaranteed to converge to the unique global minimum(maximum).

Essentially, we are going to define an auxiliary function  $g$ , preserving the optimizing behaviour in a sense that

$$\operatorname{argmin} g(x) = \operatorname{argmin} f(x) \quad (4.51)$$

and then show the convergence property on  $f(x)$  indirectly by showing converging property on  $g(x)$ .

**Lemma 4.1.** Recall that positive definite  $Q$  implies the existence of unique minimizer  $x_*$ . The minimizer satisfies the first order necessary condition.

$$Qx_* - b = 0 \quad (4.52)$$

$$\iff x_* = Q^{-1}b \quad (4.53)$$

Where the second equation came from the invertibility of positive definite matrices. One can rewrite auxiliary function capturing the optimization behaviours  $f$  as following

$$g(x) := \frac{1}{2}(x - x_*)^T Q(x - x_*) \quad (4.54)$$

$$= \underbrace{\frac{1}{2}x^T Qx - \overbrace{x^T Qx_*}^{x^T b}}_{f(x)} + \frac{1}{2}x_*^T Qx_* \quad (4.55)$$

Because  $Q$  is positive definite:

$$g(x) \geq 0 \quad \forall x \in \mathbb{R}^n \quad (4.56)$$

$$g(x) = 0 \iff x = x_* \quad (4.57)$$

$$\nabla f(x) = \nabla g(x) = Qx - b \quad (4.58)$$

Where the last equation came from the fact that  $f(x)$  and  $g(x)$  differ by a constant.

Therefore, using the method of steepest descent,

$$x_{k+1} = x_k - \alpha_k \nabla g(x_k) \quad (4.59)$$

where  $\alpha_k \in \operatorname{argmin}_{\alpha \geq 0} f(x_k - \alpha \nabla g(x_k))$ .

The necessary condition for minimizations suggests  $\alpha_k$  must satisfy

$$0 = \left. \frac{d}{d\alpha} \right|_{\alpha=\alpha_k} f(x_k - \alpha \nabla g(x_k)) = \nabla f(x_k - \alpha_k \nabla g(x_k)) \cdot (-\nabla g(x_k)) \quad (4.60)$$

$$= -[Q(x_k - \alpha_k \nabla g(x_k)) - b] \cdot \nabla g(x_k) \quad (4.61)$$

$$= -[Qx_k - \alpha_k Q \nabla g(x_k) - b] \cdot \nabla g(x_k) \quad (4.62)$$

$$= -[\nabla g(x_k) - \alpha_k Q \nabla g(x_k)] \cdot \nabla g(x_k) \quad (4.63)$$

$$= -\|\nabla g(x_k)\|_2^2 + \alpha_k \nabla g(x_k)^T Q \nabla g(x_k) \quad (4.64)$$

$$\implies \alpha_k = \frac{\|\nabla g(x_k)\|_2^2}{\nabla g(x_k)^T Q \nabla g(x_k)} \quad (4.65)$$

**Assumption 4.1.** TODO: Need to check if this assumption is required.

Assume  $Q$  is symmetric.

**Lemma 4.2.** The iterative updating from gradient descent on  $g(x)$  is a contraction mapping. That is,

$$g(x_{k+1}) = \underbrace{\left(1 - \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-1} \nabla g(x_k)]}\right)}_{\in [-1, 1]} g(x_k) \quad (4.66)$$

*Proof.*

$$g(x_{k+1}) \equiv g(x_k - \alpha_k \nabla g(x_k)) \quad (4.67)$$

$$\equiv \frac{1}{2} [x_k - \alpha_k \nabla g(x_k) - x_*]^T Q [x_k - \alpha_k \nabla g(x_k) - x_*] \quad (4.68)$$

$$= \frac{1}{2} [x_k - x_* - \alpha_k \nabla g(x_k)]^T Q [x_k - x_* - \alpha_k \nabla g(x_k)] \quad (4.69)$$

$$= \underbrace{\frac{1}{2} (x_k - x_*)^T Q (x_k - x_*)}_{g(x_k)} - \alpha_k \nabla g(x_k)^T Q (x_k - x_*) + \frac{1}{2} \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k) \quad (4.70)$$

$$\implies g(x_k) - g(x_{k+1}) = -\frac{1}{2} \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k) + \alpha_k \nabla g(x_k)^T Q \underbrace{(x_k - x_*)}_{=: y_k} \quad (4.71)$$

$$\implies \frac{g(x_k) - g(x_{k+1})}{g(x_k)} = \frac{-\frac{1}{2} \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k) + \alpha_k \nabla g(x_k)^T Q y_k}{\frac{1}{2} y_k^T Q y_k} \quad (4.72)$$

$$= \frac{2\alpha_k \nabla g(x_k)^T Q y_k - \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k)}{y_k^T Q y_k} \quad (4.73)$$

Note that the first order condition implies  $Qx_* = b$ .

Therefore,  $\nabla g(x_k) = Qx_k - b = Qx_k - Qx_* = Qy_k$ , which implies  $y_k = Q^{-1} \nabla g(x_k)$ .

$$\frac{2\alpha_k \nabla g(x_k)^T Q y_k - \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k)}{y_k^T Q y_k} = \frac{2\alpha_k \nabla g(x_k)^T Q Q^{-1} \nabla g(x_k) - \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k)}{\nabla g(x_k)^T Q^{-T} Q Q^{-1} \nabla g(x_k)} \quad (4.74)$$

$$= \frac{2\alpha_k \|\nabla g(x_k)\|_2^2 - \alpha_k^2 \nabla g(x_k)^T Q \nabla g(x_k)}{\nabla g(x_k)^T Q^{-T} \nabla g(x_k)} \quad (4.75)$$

Plug in the  $\alpha_k$  computed before:

$$\dots = \frac{2 \frac{\|\nabla g(x_k)\|_2^2}{\nabla g(x_k)^T Q \nabla g(x_k)} \|\nabla g(x_k)\|_2^2 - \frac{\|\nabla g(x_k)\|_2^4}{(\nabla g(x_k)^T Q \nabla g(x_k))^2} \nabla g(x_k)^T Q \nabla g(x_k)}{\nabla g(x_k)^T Q^{-T} \nabla g(x_k)} \quad (4.76)$$

$$= \frac{2 \frac{\|\nabla g(x_k)\|_2^4}{\nabla g(x_k)^T Q \nabla g(x_k)} - \frac{\|\nabla g(x_k)\|_2^4}{\nabla g(x_k)^T Q \nabla g(x_k)}}{\nabla g(x_k)^T Q^{-T} \nabla g(x_k)} \quad (4.77)$$

$$= \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-T} \nabla g(x_k)]} \quad (4.78)$$

$$= \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-1} \nabla g(x_k)]} \quad \because Q \in \mathbb{S}^n \quad (4.79)$$

$$\implies g(x_k) - g(x_{k+1}) = \left\{ \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-1} \nabla g(x_k)]} \right\} g(x_k) \quad (4.80)$$

$$\implies g(x_{k+1}) = \left\{ 1 - \left[ \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-1} \nabla g(x_k)]} \right] \right\} g(x_k) \quad (4.81)$$

**Lemma 4.3** (Kantorovich Inequality). Let  $Q$  be a  $n \times n$  positive definite symmetric matrix with eigenvalues  $0 < \lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \Lambda$ . Then, for any  $v \in \mathbb{R}^n$ :

$$\frac{\|v\|_2^4}{(v^T Q v)(v^T Q^{-1} v)} \geq \frac{4\lambda\Lambda}{(\lambda + \Lambda)^2} \quad (4.82)$$

*Proof of Kantorovich Inequality.* **TODO:** Prove this lemma ■

Therefore,

$$g(x_{k+1}) = \left\{ 1 - \left[ \frac{\|\nabla g(x_k)\|_2^4}{[\nabla g(x_k)^T Q \nabla g(x_k)][\nabla g(x_k)^T Q^{-1} \nabla g(x_k)]} \right] \right\} g(x_k) \quad (4.83)$$

$$\leq \left\{ 1 - \frac{4\lambda\Lambda}{(\lambda + \Lambda)^2} \right\} g(x_k) \quad (4.84)$$

$$= \frac{(\lambda - \Lambda)^2}{(\lambda + \Lambda)^2} g(x_k) \quad (4.85)$$

■

**Theorem 4.7.** For any initial point  $x_0 \in \mathbb{R}^n$ , gradient descent converges to the unique minimum point  $x_*$  of the quadratic  $f(x) = x^T Q x - b^T x$ .

*Proof.* Define  $q(x) := \frac{1}{2}(x - x_*)^T Q (x - x_*)$ .

Note that  $q(x)$  and  $f(x)$  differ by a constant, therefore  $\operatorname{argmin} q(x) = \operatorname{argmin} f(x)$ .

Moreover, we've shown:

$$q(x_{k+1}) \leq \underbrace{\left( \frac{\Lambda - \lambda}{\Lambda + \lambda} \right)^2}_{\in [0,1)} q(x_k) \quad (4.86)$$

It is easy to notice that

$$q(x_k) \leq r q(x_{k-1}) \quad (4.87)$$

$$\implies q(x_k) \leq r^k q(x_0) \quad (4.88)$$

$$\implies q(x_k) \in \{x \in \mathbb{R}^k : q(x) \leq r^k q(x_0)\} =: \mathcal{L}_k \quad (4.89)$$

Note that the sub-level set  $\mathcal{L}_k$  is strictly decreasing (i.e.  $\mathcal{L}_{k+1} \subsetneq \mathcal{L}_k$ ).

Further, note that  $x_*$  is the only point satisfying the inequality at the limit:

$$q(x_*) = 0 = \lim_{k \rightarrow \infty} q(x_0) \quad (4.90)$$

Therefore,  $\lim_{k \rightarrow \infty} \mathcal{L} = \{0\}$ , and  $(x_k) \rightarrow x_*$ . ■

**Remark 4.2.** Note that

$$r = \left( \frac{\frac{\Lambda}{\lambda} - 1}{\frac{\Lambda}{\lambda} + 1} \right)^2 \in [0, 1) \quad (4.91)$$

$$= \left( \frac{C - 1}{C + 1} \right)^2 \quad (4.92)$$

where  $C = \frac{\Lambda}{\lambda}$  is the **condition number** of  $Q$ .

Clearly, when  $\lambda = \Lambda$ ,  $r = 0$  and gradient descent converges to the unique global minimum after only one epoch.

While  $C \gg 1$ ,  $r \approx 1$  and the worst case of gradient descent converges slowly.

### 4.3 Method of Conjugate Directions

**Motivation** Method of conjugate directions is designed for quadratic functions with form  $\frac{1}{2}x^T Q x - b^T x$ . For other functional forms, one can approximate the function using quadratic form firstly and then apply method of conjugate directions.

**Definition 4.1.** Let  $Q \in \mathbb{S}^n$ ,  $d, d' \in \mathbb{R}^n$  are  **$Q$ -orthogonal** or  **$Q$ -conjugate** if

$$d^T Q d' = 0 \quad (4.93)$$

*Remark: note that  $d^T Q d' = d'^T Q d$  because  $Q$  is symmetric.*

**Definition 4.2.** Finite set  $D = (d_0, \dots, d_k) \subseteq \mathbb{R}^n$  is a  **$Q$ -orthogonal** set if

$$\forall i \neq j, d_i^T Q d_j = 0 \quad (4.94)$$

That is,  $D$  is *orthogonal in pairs*.

**Example 4.7.** When  $Q = I_n$ , the notional of  $Q$ -orthogonal becomes the conventional notion of orthogonality.

**Proposition 4.2.** Let  $d, d'$  be two eigenvectors of  $Q$  with different eigenvalues  $\lambda, \lambda'$ , then  $d, d'$  are  $Q$ -orthogonal.

*Proof.* Let  $Qv = \lambda v$  and  $Qw = \lambda' w$  where  $\lambda \neq \lambda'$ .

Note that

$$\langle v, Qw \rangle = \langle v, \lambda' w \rangle \quad (4.95)$$

$$= \lambda' \langle v, w \rangle \quad (4.96)$$

because inner product is bilinear.

Similarly,

$$\langle v, Qw \rangle = \langle Q^T v, w \rangle \quad (4.97)$$

$$= \langle Qv, w \rangle \quad (4.98)$$

$$= \langle \lambda v, w \rangle \quad (4.99)$$

$$= \lambda \langle v, w \rangle \quad (4.100)$$

$$\implies (\lambda' - \lambda) \langle v, w \rangle = 0 \quad (4.101)$$

where  $\lambda' - \lambda \neq 0$ .

Therefore,  $\langle v, w \rangle = 0$ .

Further,

$$v^T Qw = \langle v, Qw \rangle \quad (4.102)$$

$$= \langle v, \lambda w \rangle \quad (4.103)$$

$$= \lambda \langle v, w \rangle = 0 \quad (4.104)$$

So  $v, w$  are  $Q$ -orthogonal. ■

**Proposition 4.3.** Let  $Q \in \mathbb{S}^n$ , then there exists a set of orthogonal eigen-basis of  $Q$ .

**Corollary 4.1.** The orthogonal eigen-basis of  $Q$  is also  $Q$ -orthogonal.

*Proof.*

$$\forall i \neq j, \quad d_i^T Q d_j = d_i^T \lambda_j d_j = \lambda_j \langle d_i, d_j \rangle = 0 \quad (4.105)$$

■

**Proposition 4.4.** Let  $Q \in \mathbb{S}_+^n$ , let  $d_0, \dots, d_k \neq 0$  be a set of  $Q$ -orthogonal vectors with  $k \leq n - 1$ , then  $d_0, \dots, d_k$  are linearly independent.

*Proof.* Suppose

$$\alpha_0 d_0 + \alpha_1 d_1 + \dots + \alpha_k d_k = 0 \quad (4.106)$$

For every  $i \in \{0, 1, \dots, k\}$ , multiply  $d_i^T Q$  on both sides of the equation:

$$\underbrace{\alpha_0 d_i^T Q d_0 \alpha_1 + d_i^T Q d_1 + \dots + \alpha_i d_i^T Q d_i}_{=0} + \underbrace{\dots + \alpha_k d_i^T Q d_k}_{=0} = 0 \quad (4.107)$$

Further,  $d_i^T Q d_i > 0$  because  $Q$  is positive definite. Hence,  $\alpha_i = 0$  for every  $i$ , and  $d_0, \dots, d_k$  are linearly independent. ■

**Lemma 4.4** (Theorems Covered so Far). Recall that

- (i)  $d_i, d_j$  are  $Q$ -orthogonal if  $d_i^T Q d_j = 0$ ;
- (ii) Eigen-vectors with different eigenvalues are  $Q$ -orthogonal;
- (iii)  $Q$  symmetric  $\implies$  there exists an orthogonal basis  $\implies$  the set of basis is  $Q$ -orthogonal as well;
- (iv)  $Q$ -orthogonal vectors are linearly independent.

**Example 4.8** (Special Case: Method of Conjugate Direction on Quadratic Functions). Let  $Q \in \mathbb{S}_{++}^n$  and minimizing the quadratic function

$$\min f(x) = \frac{1}{2} x^T Q x - b^T \quad (4.108)$$

Recall that the unique global minimum is  $x^* = Q^{-1}b$ .

Let  $d_0, d_1, \dots, d_{n-1}$  be non-zero  $Q$ -orthogonal vectors.

Note that they are linearly independent by the previous theorem.

Therefore, they form a basis of  $\mathbb{R}^n$ .

The global minimum can be represented as

$$x^* = \sum_{j=1}^{n-1} \alpha_j d_j \quad \alpha_j \in \mathbb{R} \quad (4.109)$$

For every  $j$ , the following holds

$$d_j^T Q x^* = \alpha_j d_j^T Q d_j \quad (4.110)$$

$$\implies \alpha_j = \frac{d_j^T Q x^*}{d_j^T Q d_j} \quad (4.111)$$

**Algorithm 4.4** (Method of Conjugate Directions).

- (i) Let  $Q \in \mathbb{S}_{++}^n$  and  $\{d_j\}_{j=0}^{n-1}$  be a set of non-zero  $Q$ -orthogonal vectors, note that they form a basis of  $\mathbb{R}^n$ .
- (ii) Given initial point  $x_0 \in \mathbb{R}^n$ , the method of conjugate direction generates a sequence of points  $\{x_k\}_{k=0}^n$  as the following:

$$x_{k+1} \leftarrow x_k + \alpha_k d_k \quad (4.112)$$

$$\alpha_k := -\frac{\langle g_k, d_k \rangle}{d_k^T Q d_k} \quad g_k := \nabla f(x_k) \quad (4.113)$$

**Theorem 4.8.** Given the method of conjugate, the sequence of points generated eventually reaches the global minimum. That is,  $x_n = x^*$ .

*Proof.* Let  $x^*, x_0 \in \mathbb{R}^n$ , consider

$$x^* - x_0 = \sum_{j=0}^{n-1} \beta_j d_j \quad (4.114)$$

$$\iff x^* = x_0 + \sum_{j=0}^{n-1} \beta_j d_j \quad (4.115)$$

$$d_j^T Q(x^* - x_0) = \beta_j d_j^T Q d_j \quad (4.116)$$

$$\implies \beta_j = \frac{d_j^T Q(x^* - x_0)}{d_j^T Q d_j} \quad (4.117)$$

Note that the algorithm generates the sequence as following:

$$x_k = x_0 + \sum_{j=0}^{k-1} \alpha_j d_j \quad (4.118)$$

$$\implies (x_k - x_0) = \sum_{j=0}^{k-1} \alpha_j d_j \quad (4.119)$$

$$\implies d_k^T Q(x_k - x_0) = \sum_{j=0}^{k-1} \alpha_j d_k^T Q d_j = 0 \quad (4.120)$$

Therefore,

$$\beta_k = \frac{d_k^T Q(x^* - x_0)}{d_k^T Q d_k} \quad (4.121)$$

$$= \frac{d_k^T Q(x^* - x_0) - d_k^T Q(x_k - x_0)}{d_k^T Q d_k} \quad (4.122)$$

$$= \frac{d_k^T Q(x^* - x_k)}{d_k^T Q d_k} \quad (4.123)$$

$$= \frac{d_k^T (Qx^* - Qx_k)}{d_k^T Q d_k} \quad (4.124)$$

The first order necessary condition suggests  $Qx_* = b$ ,

$$\beta_k = \frac{d_k^T (Qx^* - Qx_k)}{d_k^T Q d_k} \quad (4.125)$$

$$= \frac{d_k^T (b - Qx_k)}{d_k^T Q d_k} \quad (4.126)$$

$$= - \frac{d_k^T (Qx_k - b)}{d_k^T Q d_k} \quad (4.127)$$

$$= - \frac{d_k^T \nabla f(x_k)}{d_k^T Q d_k} = \alpha_k \quad (4.128)$$



Consequently,

$$x^* = x_0 + \sum_{j=0}^{n-1} \beta_j d_j \quad (4.129)$$

$$= x_0 + \sum_{j=0}^{n-1} \alpha_j d_j \quad (4.130)$$

$$= x_n \quad (4.131)$$

■

#### 4.4 Geometric Interpretations of Method of Conjugate Directions

**Theorem 4.9.** Let  $f \in C^1(\Omega, \mathbb{R})$ , where  $\Omega$  is a convex subset of  $\mathbb{R}^n$ , then  $x_0$  is a local minimum of  $f$  on  $\Omega$  if and only if

$$\nabla f(x_0) \cdot (y - x_0) \geq 0 \quad \forall y \in \Omega \quad (4.132)$$

**Corollary 4.2.** Now consider the special case in which  $\Omega$  is an affine hyperplane, that is,

$$\Omega = \{x \in \mathbb{R}^n : cx + b = 0\} \quad (4.133)$$

where  $\dim(\Omega)$  is  $n - 1$ .

Note that for every  $y \in \Omega$ ,  $\nabla f(x_0) \cdot (y - x_0) \geq 0$ . For any feasible direction  $a$  at point  $x_0$ , by the definition of hyperplane,  $-a$  is a feasible direction as well.

Consequently,  $a \cdot \nabla f(x_0) = 0$  for every feasible direction. That is,  $\nabla f(x_0) \perp \Omega$ .

**Notation 4.1.** Let  $d_0, d_1, \dots, d_{n-1}$  be a set of non-zero  $Q$ -orthogonal vectors in  $\mathbb{R}^n$ . For every  $k \in \{0, 1, \dots, n\}$ , let

$$\mathcal{B}_k = \text{span}\{d_0, \dots, d_{k-1}\} \quad (4.134)$$

In particular,  $\mathcal{B}_0 = \{0\}$  and  $\mathcal{B}_n = \mathbb{R}^n$ .

**Theorem 4.10.** The sequence  $\{x_k\}$  generated from  $x_0 \in \mathbb{R}^n$  by conjugate direction method has the property that  $x_k$  minimizes  $f(x) = \frac{1}{2}x^T Qx - b^T x$  on the affine hyperplane  $x_0 + \mathcal{B}_k$ .

That is,

$$x_k \in \underset{x \in x_0 + \mathcal{B}_k}{\operatorname{argmin}} f(x) \quad (4.135)$$

*Proof.* Recall that  $x_k$  is the the minimizer of  $f$  on the affine hyperplane if and only if  $\nabla f(x_k) \perp x_0 + \mathcal{B}_k$ . It is enough to show that  $\nabla f(x_k) \perp \mathcal{B}_k$ . *Base Case:*  $g_0 := \nabla f(x_0) \perp \{0\}$  trivially.

*Inductive Step:* Assume  $g_k \perp \mathcal{B}_k$ , show  $g_{k+1} \perp \mathcal{B}_{k+1}$ :

$$g_k \perp \mathcal{B}_k \quad (4.136)$$

$$\implies g_k \perp \text{span}\{d_0, \dots, d_{k-1}\} \quad (4.137)$$

$$\langle g_{k+1}, d_k \rangle = \langle g_k + \alpha_k Q d_k, d_k \rangle \quad (4.138)$$

$$= \langle g_k, d_k \rangle + \alpha_k \langle Q d_k, d_k \rangle \quad (4.139)$$

$$= \langle g_k, d_k \rangle - \frac{\langle g_k, d_k \rangle}{d_k^T Q d_k} d_k^T Q d_k = 0 \quad (4.140)$$

$$\implies g_{k+1} \perp d_k \quad (4.141)$$

$$\langle g_{k+1}, d_i \rangle = \langle g_k + \alpha_k Q d_k, d_i \rangle \quad 0 \leq i < k \quad (4.142)$$

$$= \langle g_k, d_i \rangle + \langle \alpha_k Q d_k, d_i \rangle \quad (4.143)$$

$$= 0 + 0 = 0 \quad (4.144)$$

$$\implies g_{k+1} \perp \mathcal{B}_k \quad (4.145)$$

■

**Corollary 4.3.**  $x_n$  (the final output of the method of conjugate gradient) minimizes  $f(x)$  on  $x_0 + \mathcal{B}_n = \mathbb{R}^n$ .

**Corollary 4.4.** Let  $q(\cdot)$  be a quadratic function, then

$$0 \leq q(x_k) = \min_{x \in x_0 + \mathcal{B}_k} q(x) \leq q(x_k) = \min_{x \in x_0 + \mathcal{B}_{k-1}} q(x) \quad (4.146)$$

*Proof.* The result is immediate by noting that  $x_0 + \mathcal{B}_{k-1} \subseteq x_0 + \mathcal{B}_k$ . ■

## 5 Infinite Dimensional Optimization

### 5.1 Calculus of Variation

#### 5.1.1 The Canonical Example

**Problem Setup** Let

$$\mathcal{A} := \{u : [0, 1] \rightarrow \mathbb{R} : u \in C^1 \wedge u(0) = u(1) = 1\} \quad (5.1)$$

And let  $F[\cdot] : \mathcal{A} \rightarrow \mathbb{R}$  denote the **functional** defined as

$$F[u(\cdot)] := \frac{1}{2} \int_0^1 u^2(x) + u'^2(x) \, dx \quad (5.2)$$

Consider the minimization problem over the *space of functions*:

$$\min F[u] \text{ s.t. } u \in \mathcal{A} \quad (5.3)$$

**Optimizing via Disturbance (Test Functions)** The general idea of solving such infinite dimensional problem would be reducing it to a collection of one dimensional problems.

Let  $v(\cdot) \in C^1([0, 1], \mathbb{R})$  such that  $v(0) = v(1) = 0$ .

Suppose  $u^*(\cdot)$  is the minimizer of the problem stated above.

Consider

$$u^*(\cdot) + sv(\cdot) \quad s \in \mathbb{R} \quad (5.4)$$

Notice that by construction,  $u^*(\cdot) + sv(\cdot) \in \mathcal{A}$  for every  $s \in \mathbb{R}$ .

Define the auxiliary function

$$f(s) := F[u^* + sv] \in \mathbb{R} \quad (5.5)$$

The attained when  $s = 0$ , the first order necessary condition of minimization suggests:

$$f'(0) = \left. \frac{d}{ds} \right|_{s=0} F[u^*(\cdot) + sv(\cdot)] \quad (5.6)$$

$$= \left. \frac{d}{ds} \right|_{s=0} \frac{1}{2} \int_0^1 [u_*(x) + sv(x)]^2 + [u'_*(x) + sv'(x)]^2 dx \quad (5.7)$$

$$= \left. \frac{d}{ds} \right|_{s=0} \left\{ \frac{1}{2} \int_0^1 [u_*(x)^2 + u'_*(x)^2] dx + \int_0^1 s[u_*(x)v(x) + u'_*(x)v'(x)] dx + \frac{s^2}{2} \int_0^1 v^2(x) + v'^2(x) dx \right\} \quad (5.8)$$

$$= \left. \frac{d}{ds} \right|_{s=0} \int_0^1 s[u_*(x)v(x) + u'_*(x)v'(x)] dx \quad (5.9)$$

$$= \int_0^1 \left. \frac{d}{ds} \right|_{s=0} s[u_*(x)v(x) + u'_*(x)v'(x)] dx \quad (5.10)$$

$$= \int_0^1 u_*(x)v(x) + u'_*(x)v'(x) dx = 0 \quad (\dagger') \quad (5.11)$$

Where  $(\dagger')$  is the primitive form of the first order necessary condition.

Furthermore,

$$\int_0^1 u_*(x)v(x) + u'_*(x)v'(x) dx = \int_0^1 u_*(x)v(x) dx + \int_0^1 u'_*(x)v'(x) dx \quad (5.12)$$

$$= \int_0^1 u_*(x)v(x) dx + u_*(x)v(x)|_0^1 + \int_0^1 u''_*(x)v(x) dx \quad (5.13)$$

$$= \int_0^1 u_*(x)v(x) dx + \int_0^1 u''_*(x)v(x) dx \quad (5.14)$$

$$= \int_0^1 [u_*(x) + u''_*(x)]v(x) dx \quad (5.15)$$

The first order necessary condition is amount to

$$\int_0^1 [u_*(x) + u''_*(x)]v(x) dx = 0 \quad (5.16)$$

for every function  $v \in C^1([0, 1], \mathbb{R})$  such that  $v(0) = v(1) = 0$ .

**Lemma 5.1** (Du Bois-Reymond Lemma). Provided that  $u_*(\cdot) \in \mathcal{A}$  is  $C^1$ , altogether with the fact that it is

the minimizer,  $u_*(\cdot)$  is automatically  $C^2$ .

**Notation 5.1.** We call all  $v \in C^1([0, 1], \mathbb{R})$  such that  $v(0) = v(1) = 0$  **test functions**.

**Lemma 5.2** (The Fundamental Lemma of Calculus of Variation). Suppose  $g$  is continuous on interval  $[a, b]$  such that

$$\int_a^b g(x)v(x) dx = 0 \quad \forall \text{ test function } v(\cdot) \quad (5.17)$$

Then  $g(x) \equiv 0$  on  $[a, b]$ .

*Proof.* Suppose, for contradiction,  $g(\eta) \neq 0$  for some  $\eta \in [0, 1]$ .

WLOG, assume  $g(\eta) > 0$ .

Note that if  $\eta$  is at either endpoint, by continuity of  $g$ , one may choose another  $\eta' \in V_\varepsilon(\eta)$  such that  $\eta' \in (a, b)$ .

Therefore, we may assume  $\eta \in (a, b)$  without loss of generality.

Then there exists an open interval  $(c, d) \subsetneq (a, b)$  such that  $g(x) > 0 \quad \forall x \in (c, d)$ .

Define

$$v(x) := \begin{cases} (x-c)^2(x-d)^2 & \text{if } x \in (c, d) \\ 0 & \text{otherwise} \end{cases} \quad (5.18)$$

Clearly,  $v(\cdot) \in C^1([0, 1], \mathbb{R})$  by noticing

$$v'(x) = \begin{cases} 2(x-c)(x-d)^2 + 2(x-c)^2(x-d) & \text{if } x \in (c, d) \\ 0 & \text{otherwise} \end{cases} \quad (5.19)$$

Obviously,

$$\int_a^b g(x)v(x) dx = \int_c^d g(x)v(x) dx > 0 \quad (5.20)$$

$\Rightarrow \times$

**Proposition 5.1.**

$$\int_0^1 (u_*(x) + u_*''(x))v(x) dx = 0 \quad \forall v \implies u_*(\cdot) + u_*''(\cdot) \equiv 0 \quad (5.21)$$

*Proof.* By Du Bois-Reymond Lemma,  $u_*(\cdot) + u_*''(\cdot)$  is continuous. Because  $v(\cdot)$  is a test function and by the fundamental lemma of calculus of variation,  $u_*(\cdot) + u_*''(\cdot) \equiv 0$ . ■

## 5.2 A Classical Example from Physics: Brachistochrone

**Brachistochrone** (Galileo, 1638) Find the curve connecting two points  $A$  and  $B$  on which a point mass moves without friction under the influence of gravity moves from  $A$  to  $B$  in the least time possible.

### 5.3 General Class of Problems in Calculus of Variations

#### Constraint

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B\} \quad (5.22)$$

#### Objective function

$$F[u(\cdot)] := \int_a^b L(x, u(x), u'(x)) \, dx \quad (5.23)$$

where  $L(x, z, p) : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ .

#### Example 5.1.

$$L(x, z, p) = \frac{1}{2}(z^2 + p^2) \quad (5.24)$$

$$F[u(\cdot)] = \int_0^1 \frac{u(x)^2 + u'(x)^2}{2} \, dx \quad (5.25)$$

**Definition 5.1.** Given  $u(\cdot) \in \mathcal{A}$ , suppose  $\exists$  function  $g(\cdot)$  on  $[a, b]$  such that

$$\left. \frac{d}{ds} \right|_{s=0} F[u(\cdot) + sv(\cdot)] = \int_a^b g(x)v(x) \, dx \quad \forall \text{ test functions } v(\cdot) \quad (5.26)$$

Then  $g(\cdot)$  is called the **variational derivative** of  $F$  at  $u(\cdot)$ , often denoted as  $\frac{\delta F}{\delta u}(u)(\cdot)$ .

**Remark 5.1** (Analogy to the finite dimensional case). In finite dimensional case:

$$\left. \frac{d}{ds} \right|_{s=0} f(u + sv) = \nabla f(u) \cdot v = \sum_{i=1}^n \nabla_i f(u) v_i \quad \forall v \in \mathbb{R}^n \quad (5.27)$$

And in the infinite dimensional case:

$$\left. \frac{d}{ds} \right|_{s=0} F[u(\cdot) + sv(\cdot)] = \int_a^b \frac{\delta F}{\delta u}(u)(x)v(x) \, dx \quad \forall v \in \mathcal{T} \quad (5.28)$$

**Lemma 5.3** (Necessary condition on the variational derivative). Let

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B\} \quad (5.29)$$

If  $u_*(\cdot) \in \mathcal{A}$  minimizes  $F$  over  $\mathcal{A}$ , and if  $\frac{\delta F}{\delta u}(u_*)(\cdot)$  exists and is continuous, then it must satisfy

$$\frac{\delta F}{\delta u}(u_*)(\cdot) \equiv 0 \quad (5.30)$$

*Proof.* Note that for every test function  $v$ ,  $u_*(\cdot) + sv(\cdot) \in \mathcal{A}$ .

Suppose  $u_*$  minimizes  $F$  over  $\mathcal{A}$ , equivalently,

$$f(0) = F[u_*] \leq F[u_* + sv] = f(s) \text{ for every test function } v \quad (5.31)$$

$$\iff f(0) \leq f(s) \quad \forall s \in \mathbb{R} \quad (5.32)$$

$$\implies 0 = f'(0) = \left. \frac{d}{ds} \right|_{s=0} f(s) = \left. \frac{d}{ds} \right|_{s=0} F[u_* + sv] \quad (5.33)$$

$$= \int_a^b \frac{\delta F}{\delta u}(u)(x) v(x) \, dx \quad (5.34)$$

$$\implies \frac{\delta F}{\delta u}(u)(\cdot) \equiv 0 \quad (5.35)$$

■

**Theorem 5.1** (Euler-Lagrange). Let

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B\} \quad (5.36)$$

Let  $L \in C^2$  such that

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) \, dx \quad (5.37)$$

Then, if  $u(\cdot) \in C^1$ , then  $\frac{\delta F}{\delta u}(u)(\cdot)$  exists and is continuous.

Moreover,

$$\frac{\delta F}{\delta u}(u)(\cdot)(x) = -\frac{d}{dx}[L_p(x, u(x), u'(x))] + L_z(x, u(x), u'(x)) \quad (5.38)$$

This equation is often referred to as the **Euler-Lagrange equation**.

*Proof.* Let  $v$  be a test function.

$$\left. \frac{d}{ds} \right|_{s=0} F[u(\cdot) + sv(\cdot)] = \left. \frac{d}{ds} \right|_{s=0} \int_a^b L(x, u(x) + sv(x), u'(x) + sv'(x)) \, dx \quad (5.39)$$

$$= \int_a^b \left. \frac{d}{ds} \right|_{s=0} L(x, z, p) \, dx \quad (5.40)$$

$$= \int_a^b L_z(\cdot)v(x) + L_p(\cdot)v'(x) \, dx \quad (5.41)$$

$$= \int_a^b L_z(\cdot)v(x) \, dx + \int_a^b L_p(\cdot)v'(x) \, dx \quad (5.42)$$

$$= \int_a^b L_z(\cdot)v(x) \, dx + L_p(\cdot)v(x) \Big|_a^b - \int_a^b \frac{d}{dx} L_p(\cdot)v(x) \, dx \quad (5.43)$$

$$= \int_a^b \left[ -\frac{d}{dx} L_p(\cdot) + L_p(\cdot) \right] v(x) \, dx \quad \forall \text{ test functions } v(\cdot) \quad (5.44)$$

The result follows definition of variational derivative.

Further, because  $L(\cdot) \in C^2$ ,  $-\frac{d}{dx} L_p(\cdot)$  is continuous. Moreover,  $u(\cdot)$  and  $u'(\cdot)$  are continuous, so is the composite function. Hence, the variational derivative is continuous. ■

**Example 5.2.** In the model example,

$$L(x, z, p) = \frac{1}{2}(z^2 + p^2) \quad (5.45)$$

$$L_p(x, z, p) = p \implies L_p(x, u(x), u'(x)) = u'(x) \quad (5.46)$$

$$L_z(x, z, p) = z \implies L_z(x, u(x), u'(x)) = u(x) \quad (5.47)$$

$$\frac{\delta F}{\delta u}(u)(\cdot) = -\frac{d}{dx}[u'(x)] + u(x) = -u''(x) + u(x) \quad (5.48)$$

By above theorem, if  $u_*$  is the minimizer, it must be

$$-u''(x) + u(x) \equiv 0 \quad (5.49)$$

**Example 5.3** (Min Arclength). Suppose we are trying to minimize

$$F[u] = \int_a^b \sqrt{1 + u'(x)^2} dx = \text{arclength of } u(\cdot) \quad (5.50)$$

where

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B\} \quad (5.51)$$

and

$$L(x, z, p) = \sqrt{1 + p^2} \quad (5.52)$$

$$L_z = 0 \quad (5.53)$$

$$L_p = \frac{p}{\sqrt{1 + p^2}} \quad (5.54)$$

$$\implies \frac{\delta F}{\delta u}(u)(\cdot) = -\frac{d}{dx} \frac{u'(x)}{\sqrt{1 + u'(x)^2}} \equiv 0 \quad (5.55)$$

$$\implies \frac{u'(x)}{\sqrt{1 + u'(x)^2}} = C \quad (5.56)$$

$$\implies u'(x)^2 = C(1 + u'(x)^2) \quad (5.57)$$

$$\implies u'(x)^2 = C \quad (5.58)$$

$$\implies u'(x) = \alpha \quad (5.59)$$

$$\implies u(x) = \alpha x + \beta \quad (5.60)$$

**Example 5.4** (Surface Area of Revolution). Suppose  $u(\cdot) \in C^1$  on  $[a, b]$ , the surface area of rotating the curve  $u$  connecting  $a$  and  $b$  can be computed as

$$F[u(\cdot)] = \int_a^b 2\pi u(x) \sqrt{1 + u'(x)^2} dx \quad (5.61)$$

For simplicity, assume  $u > 0$ .

In this example, the space of feasible functions is

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B, u > 0\} \quad (5.62)$$

If  $u(\cdot)$  solves the minimization problem, it must be the case that

$$\frac{\delta F}{\delta u}(u)(\cdot) \equiv 0 \quad (\dagger) \quad (5.63)$$

Notice

$$L(x, z, p) = 2\pi z \sqrt{1 + p^2} \quad (5.64)$$

$$L_z(x, z, p) = 2\pi \sqrt{1 + p^2} \quad (5.65)$$

$$L_p(x, z, p) = 2\pi z \frac{p}{\sqrt{1 + p^2}} \quad (5.66)$$

**Claim:** the family of  $u(\cdot) = \beta \cosh\left(\frac{x-\alpha}{\beta}\right)$  solves the necessary condition  $(\dagger)$ .

**Instance 1** When  $a = 0, b = 1, A = B = 1$ , plugging in the initial condition gives

$$\begin{cases} \beta \cosh\left(\frac{0-\alpha}{\beta}\right) &= 1 \\ \beta \cosh\left(\frac{1-\alpha}{\beta}\right) &= 1 \end{cases} \quad (5.67)$$

solving above system of equations provides the solution.

**Instance 2** When  $a = 0, b = 1, A = 1, B = 0$ , plugging in these initial conditions gives

$$\begin{cases} \beta \cosh\left(\frac{0-\alpha}{\beta}\right) &= 1 \\ \beta \cosh\left(\frac{1-\alpha}{\beta}\right) &= 0 \end{cases} \quad (5.68)$$

because  $\cosh > 0$ , the second equation suggests  $\beta = 0$ , but in this case the first equation would never hold. Therefore, there is no solution to this calculus of variation.

In face, the surface area is minimized by

$$u(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.69)$$

## 5.4 Euler-Lagrange Equations in $\mathbb{R}^n$

**Setup**

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) \, dx \quad (5.70)$$

$$u : [a, b] \rightarrow \mathbb{R}^n \quad (5.71)$$

$$L(x, z, p) : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (5.72)$$

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R}^n : u \in C^1, u(a) = \mathbf{A}, u(b) = \mathbf{B}\} \quad (5.73)$$

**Theorem 5.2** (Euler-Lagrange Equations in Vector Forms).

$$-\frac{d}{dx} \nabla_p L(x, z, p) + \nabla_z L(x, z, p) = \mathbf{0} \in \mathbb{R}^n \quad (\dagger) \quad (5.74)$$



**Example 5.5** (Classical Lagrangian Mechanics).

$$V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \text{ potential energy} \quad (5.75)$$

$$\frac{1}{2}m\|v\|_2^2 \text{ kinetic energy} \quad (5.76)$$

$$L(t, x, v) := \frac{1}{2}m\|v\|_2^2 - V(x) \text{ difference between KE and PE} \quad (5.77)$$

Consider a path  $x(t)$  in  $\mathbb{R}^n$ , define objective function as

$$F[x(\cdot)] = \int_a^b L(t, x(t), x'(t)) dt \quad (5.78)$$

$$= \int_a^b \frac{1}{2}m\|\dot{x}(t)\|_2^2 - V(x(t)) dt \quad (5.79)$$

The Euler-Lagrange equation in vector form implies

$$-\frac{d}{dt}\nabla_{(3)}L(t, x(t), \dot{x}(t)) + \nabla_{(2)}L(t, x(t), \dot{x}(t)) = 0 \quad (5.80)$$

$$\implies -\frac{d}{dt}m\dot{x}(t) - \nabla V(x(t)) = 0 \quad (5.81)$$

$$\implies m\ddot{x}(t) = \nabla V(x(t)) \quad (\dagger\dagger) \quad (5.82)$$

**Remark 5.2.**  $(\dagger\dagger)$  is often referred to as *Newton's second law*: object moves along the path on which the total conversion between kinetic and potential energies is minimized.

**Example 5.6** (3-Dimensional Pendulum). Suppose the pendulum is moving on a path such that the total conversion between kinetic and potential energies is minimized, that is

$$\min \int_a^b L(t) dt = \int_a^b \frac{1}{2}m(\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2) - mgz(t) dt \quad (5.83)$$

with the restriction that  $\|\mathbf{x}(t)\| = \ell$ , where  $\ell$  is the radius of the sphere.

The restriction can be embodied by framing the problem using spherical coordinates:

$$x := \ell \cos \varphi \sin \theta \quad (5.84)$$

$$y := \ell \sin \varphi \sin \theta \quad (5.85)$$

$$z := -\ell \cos \theta \quad (5.86)$$

where the path of motion can be characterized using  $(\theta(t), \varphi(t))$ .

The objective function is therefore

$$L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2(\theta)) + mg\ell \cos \theta \quad (5.87)$$

So the Euler-Lagrange equation can be written as

$$-\frac{d}{dt}\nabla_{(3)}L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) + \nabla_{(2)}L\left(t, \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix}, \begin{pmatrix} \dot{\theta}(t) \\ \dot{\varphi}(t) \end{pmatrix}\right) = \mathbf{0} \quad (5.88)$$

### 5.5 Equality Constraints: Isoperimetric Case

**Recall: Finite Dimensional Case** Given  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , consider

$$\begin{cases} \min_{x \in \mathbb{R}^n} f(x) \\ g(x) = \text{constant} \end{cases} \quad (5.89)$$

Suppose  $x_*$  is a regular point ( $\nabla g(x_*) \neq 0$ ) is a minimizer, then there exists  $\lambda \in \mathbb{R}$  such that

$$\nabla f(x_*) + \lambda \nabla g(x_*) = 0 \quad (\text{Lagrange Multiplier}) \quad (5.90)$$

**Remark 5.3.**  $x_*$  is a critical point minimizing  $f + \lambda g$  as well. Effectively, the method of Lagrange multiplier converts the constrained optimization to an unconstrained optimization problem.

**Infinite Dimensional Case** Consider the objective and constraint functions

$$F[u(\cdot)] = \int_a^b L^F(x, u(x), u'(x)) \, dx \quad (5.91)$$

$$G[u(\cdot)] = \int_a^b L^G(x, u(x), u'(x)) \, dx \quad (5.92)$$

and the optimization problem

$$\begin{cases} \min_{u(\cdot) \in \mathcal{A}} F[u(\cdot)] \\ G[u(\cdot)] = \text{constant} \end{cases} \quad (5.93)$$

**Theorem 5.3.** Suppose  $u_*(\cdot)$  is a regular point, that is, the variational derivative  $\frac{\delta G}{\delta u}(u_*) \neq 0$ . Further, if  $u_*(\cdot)$  is a minimizer of above constrained optimization problem, then  $\exists \lambda \in \mathbb{R}$  such that:

$$\frac{\delta F}{\delta u}[u_*] + \lambda \frac{\delta G}{\delta u}[u_*] \equiv 0 \quad (5.94)$$

**Example 5.7.**

$$\mathcal{A} := \{u : [-a, a] \rightarrow \mathbb{R}, u \in C^1, u(-a) = u(a) = 0\} \quad (5.95)$$

$$L^G(x, z, p) = \sqrt{1 + p^2} \quad (5.96)$$

$$G[u(\cdot)] = \int_a^b \sqrt{1 + u'(x)^2} \, dx = \ell > 0 \quad (5.97)$$

$$L^F(x, z, p) = z \quad (5.98)$$

$$L^{-F}(x, z, p) = -z \quad (5.99)$$

$$F[u(\cdot)] = \int_a^b u(x) \, dx \quad (5.100)$$

$$\begin{cases} \min_{u(\cdot) \in \mathcal{A}} (-F)[u(\cdot)] \\ G[u(\cdot)] = \ell \end{cases} \quad (5.101)$$

Let  $u_*(\cdot)$  be a minimizer, then Euler-Lagrange equation suggests

$$\frac{\delta(-F)}{\delta u} = -\frac{d}{dx}L_p^{-F} + L_z^{-F} \equiv 0 \quad (5.102)$$

$$\frac{\delta G}{\delta u} = -\frac{d}{dx}L_p^G + L_z^G \equiv 0 \quad (5.103)$$

By the necessary condition of minimization,

$$\frac{\delta(-F)}{\delta u} + \lambda \frac{\delta G}{\delta u} \equiv 0 \quad (5.104)$$

$$\implies -1 + \lambda \left( -\frac{d}{dx} \frac{p}{\sqrt{1+p^2}} + 0 \right) \equiv 0 \quad (5.105)$$

$$\implies \lambda \frac{d}{dx} \frac{p}{\sqrt{1+p^2}} \equiv -1 \quad (5.106)$$

$$\implies \lambda \frac{p}{\sqrt{1+p^2}} \equiv -x + C_1 \quad (5.107)$$

$$\implies \lambda^2 \frac{u'_*(x)^2}{1 + u'_*(x)^2} \equiv (C_1 - x)^2 \quad (\dagger) \quad (5.108)$$

**Claim:** Note that any solution  $u_*(\cdot)$  to  $(\dagger)$  satisfies

$$(x - C_1)^2 + (u_*(x) - C_2)^2 = \lambda^2 \quad (5.109)$$

*Check:*

$$\frac{d}{dx} [2(x - C_1) + 2(u_*(x) - C_2)u'_*(x)] = 0 \quad (5.110)$$

which implies

$$u'_*(x) = -\frac{x - C_1}{u_*(x) - C_2} \quad (5.111)$$

$$\implies u'_*(x)^2 = \frac{(x - C_1)^2}{(u_*(x) - C_2)^2} \quad (\S) \quad (5.112)$$

Also,

$$(u'_*(x)^2)(u_*(x) - C_2)^2 = (x - C_1)^2 + (u_*(x) - C_2)^2 = \lambda^2 \quad (5.113)$$

$$\implies (u_*(x) - C_2)^2 = \frac{\lambda^2}{1 + u'_*(x)^2} \quad (\S\S) \quad (5.114)$$

Combine  $(\S)$  and  $(\S\S)$ ,

$$\frac{\lambda^2}{1 + u'_*(x)^2} u'_*(x)^2 = (x - C_1)^2 \quad (5.115)$$

## 5.6 Equality Constraints: Holonomic Case

**Setup (3-Dim Special Case)** Minimize

$$F[x(\cdot), y(\cdot), z(\cdot)] = \int_a^b L(t, x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)) dt \quad (5.116)$$

with constraint

$$H(x(t), y(t), z(t)) \equiv 0 \quad (5.117)$$

**Theorem 5.4** (Euler-Lagrange Equations). Let  $\mathbf{x}_*(t) := \begin{pmatrix} x_*(t) \\ y_*(t) \\ z_*(t) \end{pmatrix}$  be the minimizer subject to constraint, then

$$\begin{pmatrix} \frac{\delta F}{\delta x} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta y} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta z} [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} + \lambda(t) \begin{pmatrix} H_x [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_y [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_z [x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} = 0 \quad \forall t \in \mathbb{R} \quad (5.118)$$

where  $\lambda(t)$  is a function here.

**Example 5.8** (3-Dim Pendulum).

$$\min_{\mathbf{x}(\cdot) \in \mathcal{A}} F[x, y, z] = \int_a^b \frac{1}{2} m (\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2) - mgz(t) dt \quad (5.119)$$

$$s.t. H = \frac{1}{2} (x^2 + y^2 + z^2 - \ell^2) = 0 \quad (5.120)$$

In this case,

$$L(t, z_1, z_2, z_3, p_1, p_2, p_3) = \frac{1}{2} m (p_1^2 + p_2^2 + p_3^2) - mgz_3 \quad (5.121)$$

$$\begin{cases} L_{p_1} = mp_1 & L_{p_2} = mp_2 & L_{p_3} = mp_3 \\ L_{z_1} = 0 & L_{z_2} = 0 & L_{z_3} = -mg \end{cases} \quad (5.122)$$

Because

$$\frac{\delta F}{\delta x_i(t)} = -\frac{d}{dt} L_{p_i} + L_{z_i} \quad (5.123)$$

The Euler-Lagrange equation in this case is

$$\begin{pmatrix} -m\ddot{x}(t) \\ -m\ddot{y}(t) \\ -m\ddot{z}(t) - mg \end{pmatrix} + \lambda(t) \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \equiv 0 \quad (\dagger\dagger) \quad (5.124)$$

Combing (††) and the constraint

$$\begin{cases} m\ddot{x}(t) = x(t) \\ m\ddot{y}(t) = y(t) \\ m\ddot{z}(t) = z(t) \\ x^2 + y^2 + z^2 = \ell^2 \end{cases} \quad (5.125)$$

Solving above system of equations provides the minimizer function to the proposed problem.

## 5.7 Geodesics on the Cylinder

**Problem** Find the string connecting two points on a cylinder of radius  $r$  such that the arc-length is minimized:

$$\min F[x(t), y(t), z(t)] := \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} dt \quad (5.126)$$

$$s.t. \ H := x^2 + y^2 - r^2 = 0 \quad (5.127)$$

The Euler-Lagrange equations in this case are

$$\begin{cases} -\frac{d}{dt}L_{\dot{x}} + L_x = -2\lambda(t)x(t) \\ -\frac{d}{dt}L_{\dot{y}} + L_y = -2\lambda(t)y(t) \\ -\frac{d}{dt}L_{\dot{z}} + L_z = -2\lambda(t)z(t) \end{cases} \quad (5.128)$$

$$\text{where } L := \sqrt{p_1^2 + p_2^2 + p_3^2} \quad (5.129)$$

The Euler-Lagrange equations can be reduced to

$$\frac{d}{dt} \begin{pmatrix} x' / \sqrt{x'^2 + y'^2 + z'^2} \\ y' / \sqrt{x'^2 + y'^2 + z'^2} \\ z' / \sqrt{x'^2 + y'^2 + z'^2} \end{pmatrix} = \lambda(t) \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \quad (\dagger) \quad (5.130)$$

**Definition 5.2.** A curve is called **geodesic** if it satisfies the Euler-Lagrange necessary condition (†).

**Definition 5.3.** Given two points on the cylinder, being geodesic is not a sufficient condition to show one curve is actually the distance minimizing curve connecting these two points.

**Example 5.9.** Spirals are geodesics.

*Proof.* A spiral can be parametrized using time  $t$  as

$$\begin{cases} x(t) = r \cos(t) \\ y(t) = r \sin(t) \\ z(t) = r\alpha \end{cases} \implies \begin{cases} x'(t) = -r \sin(t) \\ y'(t) = r \cos(t) \\ z'(t) = \alpha \end{cases} \quad (5.131)$$

Therefore,

$$\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{r^2 + \alpha^2} = \text{constant} \quad (5.132)$$

The Euler-Lagrange equations become

$$\frac{1}{\text{constant}} \frac{d}{dt} \begin{pmatrix} -r \sin(t) \\ r \cos(t) \\ \alpha \end{pmatrix} = 2\lambda(t) \begin{pmatrix} r \cos(t) \\ r \sin(t) \\ 0 \end{pmatrix} \quad (5.133)$$

$$\Rightarrow \frac{1}{\text{constant}} \begin{pmatrix} -r \cos(t) \\ -r \sin(t) \\ \alpha \end{pmatrix} = 2\lambda(t) \begin{pmatrix} r \cos(t) \\ r \sin(t) \\ 0 \end{pmatrix} \quad (5.134)$$

$$\Rightarrow \lambda(t) = -\frac{1}{2 \text{ constant}} \quad (5.135)$$

$$= -\frac{1}{2\sqrt{r^2 + \alpha^2}} \quad (5.136)$$

■

**Example 5.10.** Straight line is also geodesic.

*Proof.* Consider the straight line parameterized as

$$\begin{cases} x(t) = \alpha \\ y(t) = \beta \\ z(t) = \gamma t \end{cases} \Rightarrow \begin{cases} x'(t) = 0 \\ y'(t) = 0 \\ z'(t) = \gamma \end{cases} \quad (5.137)$$

In this case,  $\sqrt{x'^2 + y'^2 + z'^2} = \gamma$ .

The Euler-Lagrange equations are

$$\frac{d}{dt} \frac{1}{\gamma} \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} = 2\lambda(t) \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \quad (5.138)$$

$$\Rightarrow \lambda(t) = 0 \quad (5.139)$$

■

**Example 5.11** (Another Example of Isoperimetric Constraints). Consider

$$F[u(\cdot)] = \int_0^a u(x) \sqrt{1 + u'(x)^2} dx \quad (5.140)$$

$$L^F = z \sqrt{1 + p^2} \quad (5.141)$$

$$u \in \mathcal{A} := \{u : [0, a] \rightarrow \mathbb{R} : u \in C^1, u(x) \geq 0, u(0) = b, u(a) = 0\} \quad (5.142)$$

$$G[u(\cdot)] = \int_0^a u(X) dx = \frac{ab}{2} \quad (5.143)$$

$$L^G = z - \frac{b}{2} \quad (5.144)$$

**Part 1:** Derive the Euler-Lagrange equations.

$$\frac{\delta F}{\delta u} = -\frac{d}{dx} \frac{zp}{\sqrt{1+p^2}} + \sqrt{1+p^2} \quad (5.145)$$

$$\frac{\delta G}{\delta u} = 1 \quad (5.146)$$

The Euler-Lagrange equations are

$$\frac{d}{dx} \frac{zp}{\sqrt{1+p^2}} - \sqrt{1+p^2} = \lambda \quad (5.147)$$

**Part 2:** Show that the following function satisfies the necessary condition.

$$u(x) = b \left(1 - \frac{x}{a}\right) \quad (5.148)$$

$$u'(x) = -\frac{b}{a} \quad (5.149)$$

Therefore,

$$\frac{d}{dx} \frac{zp}{\sqrt{1+p^2}} - \sqrt{1+p^2} = \frac{p^2}{\sqrt{1+p^2}} - \sqrt{1+p^2} \quad (5.150)$$

$$= \frac{\frac{b^2}{a^2}}{\sqrt{1+\frac{b^2}{a^2}}} - \sqrt{1+\frac{b^2}{a^2}} \quad (5.151)$$

$$= \frac{\frac{b^2}{a}}{\sqrt{a^2+b^2}} - \frac{1}{a} \sqrt{a^2+b^2} \quad (5.152)$$

$$= \frac{b^2}{a\sqrt{a^2+b^2}} - \frac{a^2+b^2}{a\sqrt{a^2+b^2}} \quad (5.153)$$

$$= \frac{-a}{\sqrt{a^2+b^2}} = \lambda \quad (5.154)$$

Obviously,  $u(\cdot) \in C^1$  and  $u(0) = b, u(a) = 0$ .

Also,

$$G[u(\cdot)] = \int_0^a b \left(1 - \frac{x}{a}\right) dx \quad (5.155)$$

$$= \int_0^a b - \frac{bx}{a} dx \quad (5.156)$$

$$= bx - \frac{bx^2}{2a} \Big|_0^a \quad (5.157)$$

$$= ab - \frac{ba^2}{2a} \quad (5.158)$$

$$= \frac{ab}{2} \quad (5.159)$$

**Example 5.12** (The Brachistochrone Problem). Parametrize the problem using  $c : [0, T] \rightarrow \mathbb{R}^2$  to describe the curve such that  $c(0) = A = (0, a)$  and  $c(T) = B = (b, 0)$ .

In particular, for each path of motion  $x(t)$ , the parametric representation can be constructed as

$$c(t) = (x(t), y(t)) = (x(t), u(x(t))) \quad (5.160)$$

$$\implies v(t) = \frac{d}{dt}c(t) = \dot{x}(t) \begin{pmatrix} 1 \\ \dot{u}(x(t)) \end{pmatrix} \quad (5.161)$$

The kinetic energy at time  $t$  is

$$T(t) = \frac{1}{2}m\dot{x}(t)^2[1 + \dot{u}(x(t))^2] \quad (5.162)$$

The potential energy at time  $t$  is

$$V(t) = mgu(x(t)) \quad (5.163)$$

By the conservation of energy, the total energy  $E(t) := T(t) + V(t)$  is a constant function.

Re-write the total energy function

$$E(t) = \frac{1}{2}m\dot{x}(t)^2[1 + \dot{u}(x(t))^2] + mgu(x(t)) = mga \quad \forall t \in [0, T] \quad (5.164)$$

$$\implies \dot{x}(t) = \sqrt{\frac{2g(a - u(x(t)))}{1 + \dot{u}(x(t))^2}} \quad (5.165)$$

The total time for the point mass to travel from  $A$  to  $B$  is  $T$ :

$$T = \int_0^T 1 \, dt = \int_0^T \frac{1}{\dot{x}(t)} \dot{x}(t) \, dt \quad (5.166)$$

$$= \int_0^T \sqrt{\frac{1 + \dot{u}(x(t))^2}{2g(a - u(x(t)))}} \dot{x}(t) \, dt \quad (5.167)$$

$$= \int_{x(t_0)}^{x(t_1)} \sqrt{\frac{1 + \dot{u}(x(t))^2}{2g(a - u(x(t)))}} \, dx \quad (5.168)$$

$$= \int_0^b \sqrt{\frac{1 + \dot{u}(x(t))^2}{2g(a - u(x(t)))}} \, dx \quad (5.169)$$

The minimization problem becomes

$$\min F[u(\cdot)] = \int_0^b \sqrt{\frac{1 + \dot{u}(x(t))^2}{2g(a - u(x(t)))}} \, dx \quad (5.170)$$

$$s.t. \, u(\cdot) \in \mathcal{A} = \{u : [0, b] \rightarrow \mathbb{R} : u \in C^1, u(0) = a, u(b) = 0\} \quad (5.171)$$

The Euler-Lagrange equation is **TODO: verify this point**

$$(1 + u'(x)^2)(a - u(x))c^2 = 1 \text{ where } c = \text{constant} \quad (5.172)$$

**Claim:** the solution to above differentiable equation takes the form

$$u(x(t)) = a - k(1 - \cos t) \quad (5.173)$$



To verify the validation of the proposed solution

$$\frac{d}{dt} u'(x(t)) x'(t) = -k \sin t \quad (5.174)$$

$$\implies u'(x(t)) = -\frac{k \sin t}{x'(t)} \quad (5.175)$$

$$\implies c^2 \left( 1 + \frac{k^2 \sin^2 t}{x'(t)^2} \right) k(1 - \cos t) = 1 \quad (5.176)$$

In particular, choose  $c^2 k = \frac{1}{2}$ :

$$\left( 1 + \frac{k^2 \sin^2 t}{x'(t)^2} \right) (1 - \cos t) = 2 \quad (5.177)$$

$$\implies k^2 \sin^2 t (1 - \cos t) = x'(t)^2 (1 + \cos t) \quad (5.178)$$

$$\implies k^2 \sin^2 t (1 - \cos t)^2 = x'(t)^2 (1 + \cos t)(1 - \cos t) \quad (5.179)$$

$$= x'(t)^2 (1 - \cos^2 t) = x'(t)^2 \sin^2 t \quad (5.180)$$

$$\implies k^2 (1 - \cos t)^2 = x'(t)^2 \quad (5.181)$$

$$\implies k(1 - \cos t) = x'(t) \quad (5.182)$$

$$\implies kt - k \sin t = x(t) + C \quad (5.183)$$

$$\implies c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ u(x(t)) \end{pmatrix} \quad (5.184)$$

$$= \begin{pmatrix} kt - k \sin t \\ a - k(1 - \cos t) \end{pmatrix} \quad (5.185)$$

With initial conditions:

$$c(0) = (0, a) = A \quad (5.186)$$

$$c(T) = \begin{pmatrix} k(T - \sin T) \\ a - k(1 - \cos T) \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \quad (5.187)$$

## 6 Appendix I: Some Basic ODEs

$$u'(x) = 0 \implies u(x) = c \quad (6.1)$$

$$u'(x) = u(x) \implies u(x) = Ae^x \quad (6.2)$$

$$u''(x) = 0 \implies u(x) = Ax + B \quad (6.3)$$

$$u''(x) = u(x) \implies u(x) = Ae^x + Be^{-x} \quad (6.4)$$

$$u''(x) = -u(x) \implies u(x) = A \sin x + B \cos x \quad (6.5)$$

## 7 Appendix II: List of Theorems

### 7.1 Finite Dimensional: Unconstrained

**Theorem 7.1** (Necessary Condition for Local Minimum). Let  $f \in C^1(\Omega, \mathbb{R})$ , let  $x_0 \in \Omega$  be a local minimum of  $f$ , then for every *feasible direction*  $v$  at  $x_0$ ,

$$\nabla f(x_0) \cdot v \geq 0 \quad (7.1)$$

*This theorem serves as the primary defining property of local minimum.*

**Theorem 7.2** (Second Order Necessary Condition for Local Minimum). Let  $f : \Omega \rightarrow \mathbb{R}$ , let  $x_0 \in \Omega$  be a local minimum of  $f$ , then for every non-zero feasible direction  $v$  at  $x_0$ ,

- (i)  $\nabla f(x_0) \cdot v \geq 0$ ;
- (ii)  $\nabla f(x_0) \cdot v = 0 \implies v^T \nabla^2 f(x_0) v \geq 0$ .

**Theorem 7.3** (Second Order Sufficient Condition for Interior Local Minima). Let  $f : C^2(\Omega, \mathbb{R})$ , for some  $x_0 \in \Omega$ , if

- (i)  $\nabla f(x_0) = 0$ ,
- (ii) (and)  $\nabla^2 f(x_0) \succ 0$ .

then  $x_0$  is a strictly local minimizer.

### 7.2 Finite Dimensional: Equality Constraints

**Definition 7.1.** The **tangent space** to  $\mathcal{M}$  at  $x_0$  is defined to be the set of all tangent vectors:

$$T_{x_0} \mathcal{M} := \left\{ v \in \mathbb{R}^n : v := \frac{d}{ds} \Big|_{s=0} x(s) \text{ for some } x \in C^1(V_\varepsilon(0), \mathcal{M}) \text{ s.t. } x(0) = x_0 \right\} \quad (7.2)$$

**Notation 7.1.** Define the  $T$  space on equality constraint as

$$T_{x_0} := \{x \in \mathbb{R}^n : \langle x_0, \nabla h_i(x_0) \rangle = 0 \forall i \in [k]\} = \{\nabla_i(x_0)\}^\perp \quad (7.3)$$

**Theorem 7.4.** Suppose  $x_0$  is a *regular point* of  $\mathcal{M} := \{h_i(x) = 0, i = 1, \dots, k\}$ , then  $T_{x_0} = T_{x_0} \mathcal{M}$ .

**Theorem 7.5** (Lagrange Multipliers: First Order Necessary Condition). Let  $f, h_1, \dots, h_k \in C^1$  defined on open subset  $\Omega \subseteq \mathbb{R}^n$ . Let  $x_0$  be a regular point of the constraint set  $\mathcal{M} := \bigcap_{i=1}^k h_i^{-1}(0)$ . Suppose  $x_0$  is a local minimum of  $\mathcal{M}$ , then there exists  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  such that

$$\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) = 0 \quad (7.4)$$

*Remark: if we define Lagrangian  $\mathcal{L}(x, \lambda_i) := f(x) + \sum_{i=1}^k \lambda_i h_i(x)$ , then the theorem says the local minimum is a critical point of  $\mathcal{L}$ .*

**Theorem 7.6** (Second Order Necessary Condition). Let  $f, h_i \in C^2$ , if  $x_0$  is a local minimum on previously defined surface  $\mathcal{M}$ , then there exists Lagrangian multipliers  $\{\lambda_i\}$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) = 0$  ( $\nabla_x \mathcal{L} = 0$ );
- (ii) And  $\nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) \succcurlyeq 0$  **on**  $T_{x_0} \mathcal{M}$  ( $\nabla_x^2 \mathcal{L} \succcurlyeq 0$ ).

*Remark: whenever  $x_0$  is a local minimum, it must be a critical point of  $\mathcal{L}$ , and  $\mathcal{L}$  is positive semidefinite on the tangent space at  $x_0$ .*

**Theorem 7.7** (Second Order Sufficient Conditions). Let  $f, h_i \in C^2$  on open  $\Omega \subseteq \mathbb{R}^n$ , and  $x_0 \in \mathcal{M}$  is a regular point, if there exists  $\lambda_i \in \mathbb{R}$  such that

- (i)  $\nabla_x \mathcal{L}(x_0, \lambda_i) = 0$ ;
- (ii)  $\nabla_x^2 \mathcal{L}(x_0, \lambda_i) \succ 0$  **on**  $T_{x_0} \mathcal{M}$ ,

then  $x_0$  is a *strict* local minimum.

### 7.3 Finite Dimensional: Inequality Constraints

**Theorem 7.8** (The First Order Necessary Condition for Local Minimum: Kuhn-Tucker Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  with constraints  $h_i$  and  $g_j$  to be  $C^1$  on  $\Omega$ . Suppose  $x_0 \in \Omega$  is a regular point with respect to constraints, further suppose  $x_0$  is a local minimum, then there exists some  $\lambda_i \in \mathbb{R}$  and  $\mu_j \in \mathbb{R}_+$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0$  (i.e.  $\nabla_x \mathcal{L}(x, \lambda, \mu) = 0$ );
- (ii)  $\mu_j g_j(x_0) = 0$  (*Complementary slackness*).

*Remark 1: by complementary slackness, all  $\mu_j$  corresponding to inactive inequality constraints are zero.*

*Remark 2: it is possible for an active constraint to have zero multiplier.*

**Theorem 7.9** (The Second Order Necessary Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , and  $f, h_1, \dots, h_k, g_1, \dots, g_\ell \in C^2(\mathbb{R}^n, \mathbb{R})$ . Let  $x_0$  be a regular point of the constraints  $(\dagger)$ . Suppose  $x_0$  is a local minimum of  $f$  subject to constraint  $(\dagger)$ , then there exists  $\lambda_i \in \mathbb{R}$  and  $\mu_j \geq 0$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0$ ;
- (ii)  $\mu_j g_j(x_0) = 0$ ;
- (iii)  $\nabla^2 f(x_0) + \sum_{i=1}^k \lambda_i \nabla^2 h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla^2 g_j(x_0)$  is positive semidefinite on the tangent space to activate constraints at  $x_0$ .

**Theorem 7.10** (The Second Order Sufficient Conditions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , let  $f, h_i, q_j \in C^2(\Omega)$ . Consider minimizing  $f(x)$  with the constraint

$$(\dagger) \begin{cases} h_i(x) = 0 & \forall i \\ g_j(x) \leq 0 & \forall j \\ x \in \Omega \end{cases} \quad (7.5)$$

Suppose there exists a feasible  $x_0$  satisfying  $(\dagger)$  and  $\lambda_i \in \mathbb{R}$  and  $\mu_j \in \mathbb{R}_+$  such that

- (i)  $\nabla f(x_0) + \sum_{i=1}^k \lambda_i \nabla h_i(x_0) + \sum_{j=1}^\ell \mu_j \nabla g_j(x_0) = 0;$
- (ii)  $\mu_j g_j(x_0) = 0$  (*Complementary slackness*).

If the Hessian matrix for Lagrangian  $\nabla_x^2 \mathcal{L}(x_0)$  is positive definite on  $\tilde{T}_{x_0}$ , the space of **strongly active** constraints at  $x_0$ , then  $x_0$  is a strict local minimum.

## 7.4 Iterative Algorithms

**Algorithm 7.1** (Newton's Method in  $\mathbb{R}$ ). Given initial point  $x_0 \in I$ , while not terminated:

$$x_{n+1} \leftarrow x_n - \frac{f'(x_n)}{f''(x_n)} \quad (7.6)$$

**Theorem 7.11.** Let  $f \in C^3$  on open interval  $I \subseteq \mathbb{R}$ . Suppose  $x_* \in I$  satisfies  $f'(x_*) = 0$  and  $f''(x_*) \neq 0$ , then the sequence of points  $(x_n)$  generated by Newton's method converges to  $x_*$  if  $x_0$  is *sufficiently close* to  $x_*$ .

**Algorithm 7.2** (Newton's Method in  $\mathbb{R}^n$ ). Let  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  where  $\Omega$  is open, let initial point  $x_0 \in \Omega$ . Suppose  $\nabla^2 f(x_n)$  is invertible for every generated  $n$ , and  $\nabla f(x_*) = 0$  so that algorithm stops at minimum. The iterative algorithm is defined as following:

$$x_{n+1} \leftarrow x_n - [\nabla^2 f(x_n)]^{-1} \nabla f(x_n) \quad (7.7)$$

**Algorithm 7.3** (Steepest Descent). Let  $f : \Omega \rightarrow \mathbb{R}$  where  $\Omega$  is an open subset of  $\mathbb{R}^n$ . Let initial point  $x_0 \in \Omega$ .

To minimize  $f$  on  $\Omega$ , iteratively update  $x$  follows at each step  $k$ :

$$x_{k+1} \leftarrow x_k - \alpha_k \nabla f(x_k) \quad (7.8)$$

where  $\alpha_k = \operatorname{argmin}_{\alpha \geq 0} f(x_k - \alpha \nabla f(x_k))$ .

*Remark: There might be multiple minimizing  $\alpha$ , in real world implementations, we take the least minimizer found.*

**Theorem 7.12.** For any initial point  $x_0 \in \mathbb{R}^n$ , gradient descent converges to the unique minimum point  $x_*$  of the quadratic  $f(x) = x^T Q x - b^T x$ .

**Theorem 7.13.** Given the method of conjugate, the sequence of points generated eventually reaches the global minimum. That is,  $x_n = x^*$ .

**Algorithm 7.4** (Method of Conjugate Directions).

- (i) Let  $Q \in \mathbb{S}_{++}^n$  and  $\{d_j\}_{j=0}^{n-1}$  be a set of non-zero  $Q$ -orthogonal vectors, note that they form a basis of  $\mathbb{R}^n$ .
- (ii) Given initial point  $x_0 \in \mathbb{R}^n$ , the method of conjugate direction generates a sequence of points  $\{x_k\}_{k=0}^n$  as the following:

$$x_{k+1} \leftarrow x_k + \alpha_k d_k \quad (7.9)$$

$$\alpha_k := -\frac{\langle g_k, d_k \rangle}{d_k^T Q d_k} \quad g_k := \nabla f(x_k) \quad (7.10)$$

## 7.5 Infinite Dimensional Analysis: Calculus of Variation

**Lemma 7.1** (The Fundamental Lemma of Calculus of Variation). Suppose  $g$  is continuous on interval  $[a, b]$  such that

$$\int_a^b g(x)v(x) dx = 0 \quad \forall \text{ test function } v(\cdot) \quad (7.11)$$

Then  $g(x) \equiv 0$  on  $[a, b]$ .

**Definition 7.2.** Given  $u(\cdot) \in \mathcal{A}$ , suppose  $\exists$  function  $g(\cdot)$  on  $[a, b]$  such that

$$\left. \frac{d}{ds} \right|_{s=0} F[u(\cdot) + sv(\cdot)] = \int_a^b g(x)v(x) dx \quad \forall \text{ test functions } v(\cdot) \quad (7.12)$$

Then  $g(\cdot)$  is called the **variational derivative** of  $F$  at  $u(\cdot)$ , often denoted as  $\frac{\delta F}{\delta u}(u)(\cdot)$ .

**Theorem 7.14** (Euler-Lagrange). Let

$$\mathcal{A} := \{u : [a, b] \rightarrow \mathbb{R} : u \in C^1, u(a) = A, u(b) = B\} \quad (7.13)$$

Let  $L \in C^2$  such that

$$F[u(\cdot)] = \int_a^b L(x, u(x), u'(x)) dx \quad (7.14)$$

Then, if  $u(\cdot) \in C^1$ , then  $\frac{\delta F}{\delta u}(u)(\cdot)$  exists and is continuous.

Moreover,

$$\frac{\delta F}{\delta u}(u)(\cdot)(x) = -\frac{d}{dx}[L_p(x, u(x), u'(x))] + L_z(x, u(x), u'(x)) \quad (7.15)$$

This equation is often referred to as the **Euler-Lagrange equation**.

**Theorem 7.15** (Euler-Lagrange Equations in Vector Forms).

$$-\frac{d}{dx} \nabla_p L(x, z, p) + \nabla_z L(x, z, p) = \mathbf{0} \in \mathbb{R}^n \quad (\dagger) \quad (7.16)$$

**Theorem 7.16.** Suppose  $u_*(\cdot)$  is a regular point, that is, the variational derivative  $\frac{\delta G}{\delta u}(u_*) \neq 0$ .

Further, if  $u_*(\cdot)$  is a minimizer of above constrained optimization problem, then  $\exists \lambda \in \mathbb{R}$  such that:

$$\frac{\delta F}{\delta u}[u_*] + \lambda \frac{\delta G}{\delta u}[u_*] \equiv 0 \quad (7.17)$$

**Theorem 7.17** (Euler-Lagrange Equations). Let  $\mathbf{x}_*(t) := \begin{pmatrix} x_*(t) \\ y_*(t) \\ z_*(t) \end{pmatrix}$  be the minimizer subject to constraint,

then

$$\begin{pmatrix} \frac{\delta F}{\delta x}[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta y}[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ \frac{\delta F}{\delta z}[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} + \lambda(t) \begin{pmatrix} H_x[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_y[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \\ H_z[x_*(\cdot), y_*(\cdot), z_*(\cdot)](t) \end{pmatrix} = 0 \quad \forall t \in \mathbb{R} \quad (7.18)$$

where  $\lambda(t)$  is a function here.