

# MAT237: Multivariable Calculus

Tianyu Du

Saturday 27<sup>th</sup> April, 2019

This work is licensed under a Creative Commons “Attribution-NonCommercial 4.0 International” license.



- GitHub: [https://github.com/TianyuDu/Spikey\\_UofT\\_Notes](https://github.com/TianyuDu/Spikey_UofT_Notes)
- Website: [TianyuDu.com/notes](https://TianyuDu.com/notes)

## Contents

<b>1</b>	<b>Limits, continuity, and related topics</b>	<b>2</b>
1.1	Open and Closed Sets, Boundary, Interior and Closure . . . . .	2
1.2	Limits and Continuity . . . . .	3
1.2.1	Limits of Multivariable Functions . . . . .	3
1.2.2	Continuity . . . . .	4
1.2.3	Continuous Functions and Open Sets . . . . .	4
1.3	Sequences and Completeness . . . . .	5
1.4	Compactness . . . . .	6
1.4.1	Compactness . . . . .	6
1.4.2	the Extreme Value Theorem . . . . .	7
1.4.3	Uniform Continuity . . . . .	8
1.5	the Intermediate Value Theorem . . . . .	8
<b>2</b>	<b>Differentiation and related topics</b>	<b>9</b>
2.1	Differentiation of Real-Valued Functions . . . . .	9
2.1.1	Single Variable Case . . . . .	9
2.1.2	Differentiability of Real-valued Functions Defined on $\mathbb{R}^n$ . . . . .	9
2.1.3	Partial Differentiability . . . . .	10
2.1.4	Directional Derivatives . . . . .	10
2.2	Differentiation . . . . .	11
2.3	the Chain Rule . . . . .	11
2.4	the Mean Value Theorem . . . . .	12
2.5	Higher Order Derivatives . . . . .	13
2.6	Taylor’s Theorem . . . . .	14
2.7	Critical Points . . . . .	15
2.8	Optimization . . . . .	16

<b>3</b>	<b>The Implicit and Inverse Function Theorems</b>	<b>17</b>
3.1	The Implicit Function Theorem I . . . . .	17
3.2	Geometric Content of the Implicit Function Theorem . . . . .	17
3.3	Transformations, and the Inverse Function Theorem . . . . .	18
<b>4</b>	<b>Integration</b>	<b>19</b>
4.1	Basics . . . . .	19
4.2	Integration on Higher Dimensions . . . . .	19
4.3	Iterated Integrals . . . . .	21
4.4	Change of Variables . . . . .	21
4.5	Further Aspects . . . . .	22
4.5.1	Exchanging Differentiation and Integration . . . . .	22
4.5.2	Improper Integrals . . . . .	22
<b>5</b>	<b>Vector Calculus</b>	<b>24</b>
5.1	Line Integrals . . . . .	24
5.1.1	Arc Length . . . . .	24
5.1.2	Line Integrals of Scalar Functions . . . . .	24
5.1.3	Line Integrals of Vector Fields . . . . .	25
5.1.4	Rectifiable Curves . . . . .	25
5.2	Green's Theorem . . . . .	25
5.2.1	Preliminary Definitions . . . . .	25
5.3	Surface Integrals . . . . .	27
5.3.1	Surface Areas and Surface Integrals . . . . .	27
5.3.2	An invariance property . . . . .	28
5.3.3	Volume and Area (Optional) . . . . .	28
5.4	Divergence, Gradient and Curl . . . . .	29
5.5	Divergence Theorem . . . . .	29
5.6	Stokes Theorem . . . . .	31
5.6.1	The Stoke Boundary . . . . .	31
5.6.2	The Orientation . . . . .	31
5.6.3	Stokes Theorem in $\mathbb{R}^3$ . . . . .	32
5.6.4	The Generalization(Optional) . . . . .	33
5.7	Vector Fields that are Gradients or Curls . . . . .	35
5.7.1	Conservative Vector Field . . . . .	35
5.7.2	Vector Fields that are Curls . . . . .	36
5.8	Summary . . . . .	36
5.8.1	Line Integrals . . . . .	36
5.9	Surface Integrals . . . . .	36

# 1 Limits, continuity, and related topics

## 1.1 Open and Closed Sets, Boundary, Interior and Closure

**Definition 1.1.1.** Let  $\mathbf{a} \in \mathbb{R}^n$ , and  $r > 0$ . The **open ball with centre  $\mathbf{a}$  and radius  $r$**  is defined as

$$\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\}. \quad (1.1.1)$$

**Definition 1.1.2.** The **sphere with centre  $\mathbf{a}$  and radius  $r$**  is defined as

$$\partial\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| = r\} \quad (1.1.2)$$

**Definition 1.1.3.** Let  $S \subset \mathbb{R}^n$ ,  $S$  is **bounded** if

$$\exists r > 0 \text{ s.t. } S \subset \mathcal{B}(r, \mathbf{0}) \quad (1.1.3)$$

**Definition 1.1.4.** Let  $S \subset \mathbb{R}^n$ , then the **complement** of  $S$  is defined as

$$S^c := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin S\} \quad (1.1.4)$$

**Definition 1.1.5.** Let  $S \subset \mathbb{R}^n$ , the **interior** of  $S$  is defined as

$$S^{int} := \{\mathbf{x} \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ s.t. } \mathcal{B}(\varepsilon, \mathbf{x}) \subset S\} \quad (1.1.5)$$

**Definition 1.1.6.** The **boundary** of  $S$  is defined as

$$\partial S := \{\mathbf{x} \in \mathbb{R}^n : \forall \varepsilon > 0 \mathcal{B}(\varepsilon, \mathbf{x}) \cap S \neq \emptyset \wedge \mathcal{B}(\varepsilon, \mathbf{x}) \cap S^c \neq \emptyset\} \quad (1.1.6)$$

**Theorem 1.1.1.** A point  $\mathbf{x} \in S$  is either a *boundary point* or a *interior point*.

**Definition 1.1.7.** The **closure** of  $S$  is defined as

$$\overline{S} := S^{int} \cup \partial S \quad (1.1.7)$$

**Theorem 1.1.2.** For any  $S \subset \mathbb{R}^n$

$$S^{int} \subset S \subset \overline{S} \quad (1.1.8)$$

**Theorem 1.1.3.** For any  $S \subset \mathbb{R}^n$

$$\partial S = \partial(S^c) \quad (1.1.9)$$

**Definition 1.1.8.** A set  $S \subset \mathbb{R}^n$  is **open** if  $S = S^{int}$ .  $S$  is **closed** if  $S = \overline{S}$ .

**Theorem 1.1.4.**

$$S \text{ is closed} \iff S^c \text{ is open} \quad (1.1.10)$$

*Proof.*

$$S \text{ is closed} \iff \partial S \subset S \iff \partial(S^c) \subset S \quad (1.1.11)$$

$$\iff \text{no point of } S^c \text{ is a boundary point} \iff S^c \text{ is open} \quad (1.1.12)$$

■

**Proposition 1.1.1.** A set  $S$  is *closed* if it contains all limit points. That's, every convergent sequence in  $S$  converges to a limit point in  $S$ .

**Theorem 1.1.5** (Homework 1.1). The limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|x|^{r_1} |y|^{r_2}}{|x|^{s_1} + |y|^{s_2}} \quad (1.1.13)$$

exists if and only if

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} > 1 \quad (1.1.14)$$

## 1.2 Limits and Continuity

### 1.2.1 Limits of Multivariable Functions

**Definition 1.2.1.** Let  $S \subset \mathbb{R}^n$ ,  $\mathbf{f}: S \rightarrow \mathbb{R}^k$ , and  $\mathbf{a} \in S$ , then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^k \quad (1.2.1)$$

is defined as

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \mathbf{x} \in S, \underbrace{0 < \|\mathbf{x} - \mathbf{a}\| < \delta}_{\text{punctured neighbourhood}} \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon \quad (1.2.2)$$

For this definition to be non-trivial, we need  $\mathbf{a}$  not be an isolated point,

$$\forall \delta > 0, \exists \mathbf{x} \in S \text{ s.t. } \|\mathbf{x} - \mathbf{a}\| \in (0, \delta) \quad (1.2.3)$$

**Theorem 1.2.1** (Limit Laws of Real Valued Functions). Let  $S \subset \mathbb{R}^n$  and  $\mathbf{a} \in \mathbb{R}^n$  satisfying (1.3.3) And  $f, g: S \rightarrow \mathbb{R}$ ,  $L, M \in \mathbb{R}$  such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L \quad (1.2.4)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = M \quad (1.2.5)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) + g(\mathbf{x})] = L + M \quad (1.2.6)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) \cdot g(\mathbf{x})] = LM \quad (1.2.7)$$

**Theorem 1.2.2** (Squeeze Theorem on Real Valued Functions). Let  $S \subset \mathbb{R}^n$ ,  $\mathbf{a} \in \mathbb{R}^n$  satisfies (1.3.3). Suppose that  $f, g, h: S \rightarrow \mathbb{R}$  and there exists  $p > 0$  and  $L \in \mathbb{R}$  such that

$$\forall \mathbf{x} \in S \cap \mathcal{B}(p, \mathbf{a}) \quad f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x}) \quad (1.2.8)$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = L \quad (1.2.9)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = L \quad (1.2.10)$$

**Corollary 1.2.1.** Let  $g, h : S \rightarrow \mathbb{R}$  and

$$|g(\mathbf{x})| \leq h(\mathbf{x}) \quad \forall \mathbf{x} \in S \quad (1.2.11)$$

$$\text{and } \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = 0 \quad (1.2.12)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = 0 \quad (1.2.13)$$

**Theorem 1.2.3.** Assume that  $S \subset \mathbb{R}^n$  and let  $\mathbf{a} \in \mathbb{R}^n$  satisfying (1.3.3). Let  $\mathbf{f} : S \rightarrow \mathbb{R}^k$ , then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \iff \lim_{\mathbf{x} \rightarrow \mathbf{a}} f_j(\mathbf{x}) = L_j \quad \forall j \quad (1.2.14)$$

### 1.2.2 Continuity

**Definition 1.2.2.** Let  $S \subset \mathbb{R}^n$  and  $\mathbf{f} : S \rightarrow \mathbb{R}^k$ .  $\mathbf{f}$  is **continuous at**  $\mathbf{a} \in S$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) \quad (1.2.15)$$

and  $\mathbf{f}$  is **continuous** if  $\mathbf{f}$  is continuous at every point in  $S$ .

**Theorem 1.2.4** (Basic Properties of Continuity). Assume that  $S \subset \mathbb{R}^n$  and  $\mathbf{a} \in S$ ,

- (i) If  $\mathbf{f} : S \rightarrow \mathbb{R}^k$  is continuous at  $\mathbf{a}$ , then every component of  $\mathbf{f}$ ,  $f_j : S \rightarrow \mathbb{R}$ , is continuous at  $\mathbf{a}$ .
- (ii) If  $\mathbf{f}, \mathbf{g} : S \rightarrow \mathbb{R}^k$  are continuous at  $\mathbf{a}$ , then  $\mathbf{f} + \mathbf{g}$  is continuous at  $\mathbf{a}$ .
- (iii) If  $f, g : S \rightarrow \mathbb{R}$  continuous, then  $fg$  is continuous and  $\frac{f}{g}$  is continuous given  $g(\mathbf{a}) \neq 0$ .
- (iv) A composition of continuous functions is continuous.
- (v) The elementary functions of a single variable (trigonometric functions and their inverses, polynomials, exponential and log) are continuous on their domains.

### 1.2.3 Continuous Functions and Open Sets

**Theorem 1.2.5.** Assume that  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , then the following are equivalent

- (i)  $\mathbf{f}$  is continuous;
- (ii) For every open set  $\mathcal{O} \subset \mathbb{R}^k$ ,  $\mathbf{f}^{-1}(\mathcal{O})$  is also open;
- (iii) For every closed set  $\mathcal{C} \subset \mathbb{R}^k$ ,  $\mathbf{f}^{-1}(\mathcal{C})$  is also closed.

*Openness and closeness are preserved under pre-image operations of continuous functions.*

**Theorem 1.2.6.** An arbitrary union of open sets is open.

**Theorem 1.2.7.** A finite intersection of open sets is open.

**Theorem 1.2.8.** A finite union of closed sets is closed.

**Theorem 1.2.9.** An arbitrary intersection of closed sets is closed.

### 1.3 Sequences and Completeness

**Definition 1.3.1.** A sequence  $\{\mathbf{a}_j\}_{j \in \mathbb{Z}_{++}}$  in  $\mathbb{R}^n$  **converges to the limit**  $\mathbf{L} \in \mathbb{R}^n$  if

$$\forall \varepsilon > 0 \exists J \in \mathbb{N}, \text{ s.t. } \forall j \geq J \implies \|\mathbf{a}_j - \mathbf{L}\| < \varepsilon \quad (1.3.1)$$

**Theorem 1.3.1.**

$$\lim_{j \rightarrow \infty} \mathbf{a}_j = \mathbf{L} \iff \lim_{j \rightarrow \infty} \|\mathbf{a}_j - \mathbf{L}\| = 0 \quad (1.3.2)$$

**Theorem 1.3.2.** Let  $\{a_{jk}\}_j$  be a sequence in  $\mathbb{R}^n$  where  $k \in [n]$ , and let  $\mathbf{L} = (L_1, \dots, L_n) \in \mathbb{R}^n$ , then

$$\lim_{j \rightarrow \infty} \mathbf{a}_j = \mathbf{L} \iff \lim_{j \rightarrow \infty} a_{jk} = L_k \quad \forall k \in [n] \quad (1.3.3)$$

*Proof Idea.*

$$\forall j \in [n], |a_j - L_j| \leq \|\mathbf{a} - \mathbf{L}\| \leq n \cdot \max_{k \in [n]} |a_k - L_k| \quad (1.3.4)$$

■

**Axiom 1.1** (the Completeness Axiom). Every bounded and nonempty set of real numbers has a least upper bound (**supremum**) and a greatest lower bound (**infimum**).

**Theorem 1.3.3** (Monotone Sequence Theorem). Every bounded nondecreasing sequence of real numbers converges to a limit.

*Proof Idea.* Note that such sequence converges to its supremum  $S$ .

Let  $\varepsilon > 0$ , there exists  $j^*$  such that

$$S - \varepsilon < a_{j^*} \leq S \quad (1.3.5)$$

take such  $j^*$  and by the nondecreasing property,

$$\forall j \geq j^* \quad a_j > S - \varepsilon \quad (1.3.6)$$

which implies  $|S - a_j| < \varepsilon$ .

■

**Theorem 1.3.4** (Monotone Sequence Theorem). Every bounded monotone sequence in  $\mathbb{R}$  is convergent.

**Definition 1.3.2.** A **subsequence** of a sequence  $\{\mathbf{a}_j\}_{j \geq j_0}$  in  $\mathbb{R}^n$  is a sequence constructed as  $\{a_{k_j}\}_j$ , such that  $\{k_j\}_j$  is a *strictly increasing* sequence bounded below by  $j_0$ .

**Proposition 1.3.1.** If  $\{\mathbf{a}_j\}_j$  is a sequence in  $\mathbb{R}^n$  converges to  $\mathbf{L}$ , then (i) any subsequence of it converges to the (ii) same limit.

*Proof Idea.* Suppose not and reach a contradiction.

■

**Theorem 1.3.5** (Bounded Sequence Theorem in  $\mathbb{R}$ ). Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

*Proof.* Let  $\{a_j\}_j$  be a bounded sequence.

For each  $j \in \mathbb{N}$ , define  $b_{k_j} := \inf_{k > k_j} a_k$ .

Note that  $\{b_j\}$  is non-decreasing and bounded, so it converges to some limit  $\ell$ .

Let  $\{a_{k_j}\}_j$  denote a subsequence of the original sequence, define  $k_0 = j_0$ , and indices are constructed in a recurrent way.

Suppose every index before  $k_j$  has been chosen, we choose  $k_{j+1}$  to be the index such that

$$b_{k_j} \leq a_{k_{j+1}} < b_{k_j} + \frac{1}{j} \quad (1.3.7)$$

by construction,  $\{a_{k_j}\}_j$  is bounded by both  $\{b_{k_j}\}_j$  and  $\{b_{k_j} + \frac{1}{j}\}_j$ , and both bounding sequences converge to  $\ell$ . So  $\{a_{k_j}\}_j$  converges to  $\ell$  by *squeeze theorem*. ■

**Theorem 1.3.6** (Bounded Sequence Theorem). Every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

*Proof.* Let  $\{\mathbf{a}_j\}_j$  be a bounded sequence.

Applying the previous theorem iteratively, we can construct a subsequence of  $\{\mathbf{a}_{k_j}\}_j$  such that  $\{\mathbf{a}_{k_j} \cdot \mathbf{e}_1\}_j$  is bounded and convergent.

Then we apply the previous theorem iteratively on the constructed convergent subsequences to construct new subsequences with more convergent components. ■

**Theorem 1.3.7** (Nested Interval Theorem). Given a sequence of *closed intervals* in  $\mathbb{R}$ ,

$$\{I_k\}_{k \in \mathcal{A}} \text{ s.t. } I_k = [a_k, b_k] \subset \mathbb{R} \quad (1.3.8)$$

and

$$\cdots I_{k+1} \subseteq I_k \subseteq \cdots I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \quad (1.3.9)$$

Then

$$\bigcap_{k \in \mathcal{A}} I_k \neq \emptyset \quad (1.3.10)$$

## 1.4 Compactness

### 1.4.1 Compactness

**Definition 1.4.1** (Heine-Borel Property). A set  $S$  is **compact** if every *open* covering of  $S$  has a *finite* sub-covering.

**Definition 1.4.2** (Sequentially Compact). A set  $S \subset \mathbb{R}^n$  is **compact** if every sequence in  $S$  has a subsequence that converges to a limit in  $S$ .

**Proposition 1.4.1.** If  $\{\mathbf{x}_j\}_j$  is a *convergent* sequence in a *closed* set  $S \subset \mathbb{R}^n$ , then the limit of this sequence is in  $S$ .

*Proof Idea.* Let  $\mathbf{x} := \lim_{j \rightarrow \infty} \mathbf{x}_j$ , and we wish to show  $\mathbf{x} \in S$ .

Equivalently, we can show  $\mathbf{x} \in \overline{S}$ , and that's

$$\forall \varepsilon > 0 \quad \mathcal{B}(\varepsilon, \mathbf{x}) \cap S \neq \emptyset \quad (1.4.1)$$

this is immediately true by the definition of sequence convergence. There must be some points in the sequence, thus in  $S$ , belongs to  $\mathcal{B}(\varepsilon, \mathbf{x})$ . ■

**Theorem 1.4.1** (Bolzano-Weierstrass). Let  $S \subset \mathbb{R}^n$ ,

$$S \text{ is compact} \iff S \text{ is closed and bounded} \quad (1.4.2)$$

*Proof Idea.*

( $\Leftarrow$ ) Suppose  $S$  is closed and bounded, boundedness ensures such sequence converges, and closeness ensures the limit point of sequence is in  $S$ .

( $\Rightarrow$ ) Prove by *modus tollens*.

**Case (i):**  $S$  is not bounded, then

$$\forall R > 0 \exists \mathbf{x} \in S \setminus \mathcal{B}(R, \mathbf{0}) \quad (1.4.3)$$

and above  $\mathbf{x}(R)$  depends on  $R$ , we can construct a sequence using  $\mathbf{x}(j)$  such that the  $\|\mathbf{x}\|$  is ever increasing and it does not have a limit.

**Case (ii):**  $S$  is not closed, we can construct a sequence with subsequence converges to  $\mathbf{x} \in \partial S \setminus S$ , which is nonempty because  $S$  is not closed. ■

**Theorem 1.4.2.** Let  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a continuous function. If  $K \subset \mathbb{R}^n$  is compact, then the image  $\mathbf{f}(K)$  is compact.

## 1.4.2 the Extreme Value Theorem

**Theorem 1.4.3** (the Extreme Value Theorem). Assume  $K$  is a compact subset of  $\mathbb{R}^n$  and  $f : K \rightarrow \mathbb{R}$  is continuous.

Then (i)

$$f(K) \text{ is compact} \quad (1.4.4)$$

and (ii) the infimum and supremum of  $f(\mathbf{x})$  on  $K$  are attainable.

$$\exists \bar{\mathbf{x}}, \underline{\mathbf{x}} \in K \text{ s.t. } \begin{cases} f(\bar{\mathbf{x}}) = \sup_{\mathbf{x} \in K} f(\mathbf{x}) \\ f(\underline{\mathbf{x}}) = \inf_{\mathbf{x} \in K} f(\mathbf{x}) \end{cases} \quad (1.4.5)$$

*Proof.* Let  $\{y_j\}_j$  be a sequence in  $f(K)$ , and we can find a sequence  $\{\mathbf{z}_j\}_j$  in  $K$  such that  $y_j = f(\mathbf{z}_j)$  (by definition of image). Because  $K$  is compact, there exists a subsequence of  $\{\mathbf{z}_j\}_j$  converges to  $\mathbf{z}^* \in K$ . Since  $f$  is continuous, we can conclude there a subsequence, sharing the same indices, such that  $f(\mathbf{z}_j) \rightarrow f(\mathbf{z}^*)$ . Obviously  $f(\mathbf{z}^*) \in f(K)$ , so  $f(K)$  is compact.

Since  $f(K)$  is compact,  $\sup_{\mathbf{x} \in K} f(\mathbf{x}) \in f(K)$ . By definition of image,  $\exists \mathbf{x} \in K$  such that  $f(\mathbf{x}) = \sup_{\mathbf{x} \in K} f(\mathbf{x})$ , supremum attainability shown.

Proof for infimum attainability is the same. ■

**Proposition 1.4.2.** Assume that  $\{\mathbf{z}_j\}_j$  is a sequence in a set  $S \subset \mathbb{R}^k$ , and  $f$  is a continuous real-valued function defined on  $S$ , then

$$\mathbf{z}_j \rightarrow \mathbf{z} \implies f(\mathbf{z}_j) \rightarrow f(\mathbf{z}) \quad (1.4.6)$$

**Proposition 1.4.3.** If  $S$  is a compact set in  $\mathbb{R}$ , then  $\sup S$  and  $\inf S$  both in  $S$ .



*Proof Idea.* Suppose  $\sup S \notin S$ , by definition of supremum,

$$\forall \varepsilon \exists x \in S \text{ s.t. } \sup S - \varepsilon < x \leq \sup S \quad (1.4.7)$$

note that such  $x \in \mathcal{B}(\varepsilon, \sup S)$ . Also, similarly,

$$\forall \varepsilon > 0 \exists x \notin S \text{ s.t. } \sup S < x < \sup S + \varepsilon \quad (1.4.8)$$

so such  $x \in \mathcal{B}(\varepsilon, \sup S)$ . We conclude

$$\forall \varepsilon > 0 \mathcal{B}(\varepsilon, \sup S) \cap S \neq \emptyset \wedge \mathcal{B}(\varepsilon, \sup S) \cap S^c \neq \emptyset \quad (1.4.9)$$

which means  $\sup S \in \partial S$ . Thus if  $\sup S \notin S$ ,  $S$  cannot be closed and this contradicts our assumption that  $S$  is compact.

The proof for  $\inf S \in S$  is similar. ■

### 1.4.3 Uniform Continuity

**Definition 1.4.3.** Let  $S \subset \mathbb{R}^n$ , a function  $\mathbf{f} : S \rightarrow \mathbb{R}^k$  is **uniformly continuous** if

$$\underbrace{\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \mathbf{x}, \mathbf{y} \in S, \|\mathbf{x} - \mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon}_{\text{cts: } \forall \mathbf{x} \in S, \varepsilon > 0, \exists \delta > 0} \quad (1.4.10)$$

**Remark 1.4.1.** In the definition of *continuity*, value of  $\delta$  can depend on  $\mathbf{x}$ . But in the definition of *uniform continuity*, one  $\delta$  has to work for every  $\mathbf{x}$ .

**Theorem 1.4.4.** If  $K$  is a compact subset of  $\mathbb{R}^n$ , and  $\mathbf{f} : K \rightarrow \mathbb{R}^k$  is continuous, then  $\mathbf{f}$  is uniformly continuous.

## 1.5 the Intermediate Value Theorem

**Definition 1.5.1.** A set  $S \subset \mathbb{R}^n$  is **path-connected** (arcwise connected/ pathwise connected) if for every  $\mathbf{x}, \mathbf{y} \in S$ , there exists a **continuous** function  $\gamma : [0, 1] \rightarrow S$  such that

$$\gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y} \quad (1.5.1)$$

**Example 1.5.1.** Convex sets are path-connected, a path can be constructed using the convex combination,

$$\gamma(t) := (1 - t)\mathbf{x} + t\mathbf{y} \quad (1.5.2)$$

**Proposition 1.5.1.** Let  $S_1, S_2 \subset \mathbb{R}^n$  be two path-connected sets, and  $S_1 \cap S_2 \neq \emptyset$ . Then  $S_1 \cup S_2$  is path-connected.

*Proof.* Take  $\mathbf{z} \in S_1 \cap S_2$ , and let  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  be two connecting paths between  $\mathbf{x}, \mathbf{z}$  and  $\mathbf{z}, \mathbf{y}$  respectively. Then define  $\gamma : [0, 1] \rightarrow S_1 \cup S_2$  as

$$\gamma(t) := \mathbb{1}\{t \in [0, \frac{1}{2}]\} \times \tilde{\gamma}_1(2t) + \mathbb{1}\{t \in [\frac{1}{2}, 1]\} \times \tilde{\gamma}_2(2(t - \frac{1}{2})) \quad (1.5.3)$$

■

**Theorem 1.5.1.** Let  $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^k$  be a **continuous** function, and  $S \subset \mathbb{R}^m$  be a **path-connected** set. Then,  $\mathbf{f}(S)$  is path-connected.

*Proof Idea.* Take the composite  $\mathbf{f} \circ \gamma$ . ■

**Theorem 1.5.2** (the Intermediate Value Theorem). Assume that  $S$  is a **path-connected** subset of  $\mathbb{R}^n$  and that  $f : S \rightarrow \mathbb{R}$  is **continuous**. Let  $\mathbf{a}, \mathbf{b} \in S$ . Then for every  $t \in (\min\{f(\mathbf{a}), f(\mathbf{b})\}, \max\{f(\mathbf{a}), f(\mathbf{b})\})$ , there exists  $\mathbf{c} \in S$  such that  $f(\mathbf{c}) = t$ .

*Proof.* Let  $\mathbf{a}, \mathbf{b} \in S$ . WLOG, assume  $f(\mathbf{a}) < f(\mathbf{b})$ . Let  $t$  be an arbitrary value in  $(f(\mathbf{a}), f(\mathbf{b}))$ . Since  $S$  is path-connected, let  $\vec{\varphi} : [0, 1] \rightarrow S$  be a continuous function such that  $\vec{\varphi}(0) = \mathbf{a}$  and  $\vec{\varphi}(1) = \mathbf{b}$ .

Then we can construct composite  $f \circ \vec{\varphi} : [0, 1] \rightarrow \mathbb{R}$ , then apply the Intermediate Value Theorem in  $\mathbb{R}$ . We can conclude that  $\exists \eta \in (0, 1)$  s.t.  $f \circ \vec{\varphi}(\eta) = t$ . And  $\vec{\varphi}(\eta) \in S$  is the point desired. ■

## 2 Differentiation and related topics

### 2.1 Differentiation of Real-Valued Functions

#### 2.1.1 Single Variable Case

**Definition 2.1.1** (Equivalent Definitions of Differentiability). Let  $S \subset \mathbb{R}$  open, and  $f : S \rightarrow \mathbb{R}$  is said to be **differentiable at**  $x \in S$  if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ exists} \quad (2.1.1)$$

or there exists  $m \in \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - mh}{h} = 0 \quad (2.1.2)$$

or there exists  $m \in \mathbb{R}$  and  $E(h) : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x+h) = f(x) + mh + E(h), \quad \lim_{h \rightarrow 0} \frac{E(h)}{h} = 0 \quad (2.1.3)$$

If  $f$  is differentiable at  $x$ , we define the **derivative**  $f'(x) := m$ .

#### 2.1.2 Differentiability of Real-valued Functions Defined on $\mathbb{R}^n$

**Definition 2.1.2.** Let  $S$  be an open subset of  $\mathbb{R}^n$ , and  $f : S \rightarrow \mathbb{R}$  is **differentiable at**  $\mathbf{x} \in S$  if

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\|\mathbf{h}\|} \text{ exists} \quad (2.1.4)$$

or there exists  $\mathbf{m} \in M_{1 \times n}$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \mathbf{m} \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0 \quad (2.1.5)$$

or there exists  $\mathbf{m} \in M_{1 \times n}$  and  $E(\mathbf{h})$  such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{m} \cdot \mathbf{h} + E(\mathbf{h}), \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{E(\mathbf{h})}{\|\mathbf{h}\|} = 0 \quad (2.1.6)$$

If  $f$  is differentiable at  $\mathbf{x}$ , we define its gradient as  $\nabla f(\mathbf{x}) := \mathbf{m}$ .

**Theorem 2.1.1.** Assume that  $f : S \rightarrow \mathbb{R}$ , where  $S$  is an open subset of  $\mathbb{R}^n$ , and that  $\mathbf{x} \in S$ . If  $f$  is *differentiable* at  $\mathbf{x}$ , then  $f$  is continuous at  $\mathbf{x}$ .

*Proof.*

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{m} \cdot \mathbf{h} + E(\mathbf{h}) \quad (2.1.7)$$

Note that when  $\|\mathbf{h}\| \leq 1$ ,

$$E(\mathbf{h}) \leq \frac{|E(\mathbf{h})|}{\|\mathbf{h}\|} \quad (2.1.8)$$

By the *Squeeze Theorem*,  $\lim_{\mathbf{h} \rightarrow \mathbf{0}} E(\mathbf{h}) = 0$ . Also,  $\lim_{\mathbf{h} \rightarrow \mathbf{0}} \mathbf{m} \cdot \mathbf{h} = 0$ . Thus

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = 0 \quad (2.1.9)$$

so  $f$  is continuous at  $\mathbf{x}$ . ■

### 2.1.3 Partial Differentiability

**Definition 2.1.3.** Let  $S$  be an open subset of  $\mathbb{R}^n$ , and  $f : S \rightarrow \mathbb{R}$ . The  *$j$ -th partial derivative of  $f$  at  $\mathbf{x}$*  is defined as

$$\frac{\partial f(\mathbf{x})}{\partial x_j} := \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \quad (2.1.10)$$

**Theorem 2.1.2.** Let  $f$  be a function  $S \rightarrow \mathbb{R}$ , where  $S$  is an open subset of  $\mathbb{R}^n$ . If  $f$  is differentiable at a point  $\mathbf{x} \in S$ , then (i)  $\frac{\partial f}{\partial x_j}$  exists at  $\mathbf{x}$  for every  $j \in [n]$  and (ii)

$$\nabla f(\mathbf{x}) := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)(\mathbf{x}) \quad (2.1.11)$$

**Theorem 2.1.3.** Assume  $f$  is a function  $S \rightarrow \mathbb{R}$  for some open  $S \subset \mathbb{R}^n$ . If all partial derivatives of  $f$  **exist and are continuous** at every point of  $S$ , then  $f$  is differentiable in  $S$ .

**Definition 2.1.4.** A function  $f : S \rightarrow \mathbb{R}$  is said to be **of class  $C^1$**  if all partial derivatives of  $f$  exist and continuous at every point of  $S$ .

### 2.1.4 Directional Derivatives

**Definition 2.1.5.** A **direction** in  $\mathbb{R}^n$  is represented by a unit vector  $\mathbf{u}$ . And given such a unit vector, the **directional derivative of  $f$  at  $\mathbf{x}$  in the direction of  $\mathbf{u}$**  is defined as

$$\partial_{\mathbf{u}} f(\mathbf{x}) := \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \quad (2.1.12)$$

**Theorem 2.1.4.** If  $f$  is differentiable at a point  $\mathbf{x}$ , then  $\partial_{\mathbf{u}} f(\mathbf{x})$  exists for every unit vector  $\mathbf{u}$ , and moreover

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \mathbf{u} \cdot \nabla f(\mathbf{x}) \quad (2.1.13)$$

## 2.2 Differentiation

**Definition 2.2.1.** Assume  $S$  is an open subset of  $\mathbb{R}^n$ . Given function  $\mathbf{f} : S \rightarrow \mathbb{R}^m$ , we say that  $\mathbf{f}$  is differentiable at a point  $\mathbf{a} \in S$  if there exists  $M \in M_{m \times n}$  such that

$$\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) = M\mathbf{h} + \mathbf{E}(\mathbf{h}), \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\mathbf{E}(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0} \in \mathbb{R}^m \quad (2.2.1)$$

If such  $M$  exists, we define the **Jacobian matrix** of  $\mathbf{f}$  at  $\mathbf{a}$  as

$$D\mathbf{f}(\mathbf{a}) := M \quad (2.2.2)$$

**Definition 2.2.2.** Given a differentiable function  $f : S \rightarrow \mathbb{R}$ , where  $S$  is an open subset of  $\mathbb{R}^n$ , at a point  $\mathbf{a}$  we define the **differential of  $f$  at  $\mathbf{a}$**  as

$$df|_{\mathbf{a}}(\mathbf{h}) := \nabla f(\mathbf{a}) \cdot \mathbf{h} \quad (2.2.3)$$

**Remark 2.2.1.** The notion of differential applies to real-valued functions only (in this textbook).

**Remark 2.2.2** (First Order Taylor Polynomial). The differential can be used for linear approximations for small  $\mathbf{h}$ .

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + df|_{\mathbf{a}}(\mathbf{h}) \quad (2.2.4)$$

## 2.3 the Chain Rule

**Theorem 2.3.1** (the Chain Rule). Let  $S_n \subset \mathbb{R}^n$  and  $T_m \subset \mathbb{R}^m$ , given functions  $\mathbf{g} : S_n \rightarrow \mathbb{R}^m$  and  $\mathbf{f} : T_m \rightarrow \mathbb{R}^\ell$ . Also let  $\mathbf{a} \in S_n$  such that  $\mathbf{g}$  is differentiable at  $\mathbf{a}$  and  $\mathbf{f}$  is differentiable at  $\mathbf{g}(\mathbf{a})^1$ . Then

$$\underbrace{D(\mathbf{f} \circ \mathbf{g})(\mathbf{a})}_{\ell \times n} = \underbrace{D(\mathbf{f})(\mathbf{g}(\mathbf{a}))}_{\ell \times m} \underbrace{D\mathbf{g}(\mathbf{a})}_{m \times n} \quad (2.3.1)$$

**Example 2.3.1.**

$$\frac{d}{d\mathbf{x}} \|\mathbf{x}\| = \frac{\mathbf{x}}{\|\mathbf{x}\|} \quad (2.3.2)$$

**Definition 2.3.1.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called **homogeneous of degree  $\alpha$**  if

$$f(\lambda \mathbf{x}) = \lambda^\alpha f(\mathbf{x}) \quad \forall \mathbf{x} \neq \mathbf{0}, \lambda \in \mathbb{R}_{++} \quad (2.3.3)$$

**Theorem 2.3.2** (the Euler's Theorem of Homogeneous Functions). If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a homogeneous equation of degree  $\alpha$ , then

$$\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x}) \quad (2.3.4)$$

*Proof.*

$$\begin{cases} \frac{\partial f(\lambda \mathbf{x})}{\partial \lambda} = \nabla f(\lambda \mathbf{x}) \cdot \mathbf{x} \\ \frac{\partial f(\lambda \mathbf{x})}{\partial \lambda} = \frac{\partial \lambda^\alpha f(\mathbf{x})}{\partial \lambda} = \alpha \lambda^{\alpha-1} f(\mathbf{x}) \end{cases} \quad (2.3.5)$$

$$\implies \nabla f(\lambda \mathbf{x}) \cdot \mathbf{x} = \alpha \lambda^{\alpha-1} f(\mathbf{x}) \quad (2.3.6)$$

$$\implies \nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x}) \text{ evaluated at } \lambda = 1 \quad (2.3.7)$$

■

---

<sup>1</sup>Also all functions  $\mathbf{f}$  and  $\mathbf{g}$  and  $\mathbf{f} \circ \mathbf{g}$  are well-defined near  $\mathbf{a}$  and  $\mathbf{g}(\mathbf{a})$ .

**Definition 2.3.2.** Let  $C$  be the level set of  $f : S \rightarrow \mathbb{R}$  at  $\mathbf{a} \in S$  defined as

$$C := \{\mathbf{x} \in S : f(\mathbf{x}) = f(\mathbf{a})\} \quad (2.3.8)$$

and a vector  $\mathbf{v}$  is **tangent to  $C$  at  $\mathbf{a}$**  if there exists a function  $\gamma : I \rightarrow C$  defined on interval  $I$  containing 0, such that

$$\gamma(0) = \mathbf{a} \quad (2.3.9)$$

and

$$\mathbf{v} = \gamma'(0) \quad (2.3.10)$$

**Theorem 2.3.3.** Let  $S \subset \mathbb{R}^n$  be an open set, and  $f : S \rightarrow \mathbb{R}$  is differentiable at  $\mathbf{a}$ . Then  $\nabla f(\mathbf{a})$  is orthogonal to the level set of  $f$  passes through  $\mathbf{a}$ .

*Proof Idea.* Let  $\mathbf{v}$  be an arbitrary tangent vector to  $C$  at  $\mathbf{a}$ , there must exists a function  $\gamma : I \rightarrow C$ . And define

$$h(t) := f \circ \gamma(t) \quad (2.3.11)$$

by definition of  $\gamma$ ,  $h(I) = \{f(\mathbf{a})\}$ . Thus

$$\frac{d}{dt}h(t) = \frac{d}{dt}f \circ \gamma(t) \text{ evaluated at } t = 0 \quad (2.3.12)$$

$$= \nabla f(\gamma(0)) \cdot \gamma'(0) \quad (2.3.13)$$

$$= \nabla f(\mathbf{a}) \cdot \gamma'(0) \quad (2.3.14)$$

$$= \nabla f(\mathbf{a}) \cdot \mathbf{v} = 0 \quad (2.3.15)$$

So  $\nabla f(\mathbf{a})$  is orthogonal to any tangent vector of  $C$  at  $\mathbf{a}$ , which means  $\nabla f(\mathbf{a})$  is orthogonal to  $C$ . ■

## 2.4 the Mean Value Theorem

**Theorem 2.4.1** (the Mean Value Theorem). Assume  $f : S \rightarrow \mathbb{R}$ , where  $S$  is a **convex** and **open** subset of  $\mathbb{R}^n$ , of class  $C^1$ , then

$$\forall \mathbf{a}, \mathbf{b} \in S, \exists \lambda \in [0, 1] \text{ s.t. } \mathbf{c} = \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \quad (2.4.1)$$

$$\nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = f(\mathbf{b}) - f(\mathbf{a}) \quad (2.4.2)$$

*Proof Idea.* Define  $\gamma(t) := t\mathbf{a} + (1 - t)\mathbf{b}$ . Construct  $h : [0, 1] \rightarrow \mathbb{R}$  defined as  $h := f(\gamma(t))$  then apply one dimensional mean value theorem on  $h$ . ■

**Definition 2.4.1.** A set  $S \subset \mathbb{R}^n$  is **convex** if

$$\forall \mathbf{a}, \mathbf{b} \in S, \lambda \in [0, 1], \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \in S \quad (2.4.3)$$

**Theorem 2.4.2.** Assume that  $S$  is an open and convex subset of  $\mathbb{R}^n$  and that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable in  $S$  such that

$$\|\nabla f(\mathbf{x})\| \leq M \quad \forall \mathbf{x} \in S \quad (2.4.4)$$

then for every  $\mathbf{a}, \mathbf{b} \in S$ ,

$$|f(\mathbf{b}) - f(\mathbf{a})| \leq M \|\mathbf{b} - \mathbf{a}\| \quad (2.4.5)$$

*Proof Idea.* Use *Cauchy's Inequality*. ■

**Theorem 2.4.3.** Assume that  $S$  is an open and convex subset of  $\mathbb{R}^n$ , and  $f : S \rightarrow \mathbb{R}$  is a function differentiable on  $S$ . If  $\nabla f(\mathbf{x}) = \mathbf{0} \ \forall \mathbf{x} \in S$ , then  $f$  is constant on  $S$ .

*Proof Idea.* Take two arbitrary  $\mathbf{a}, \mathbf{b} \in S$ , then use mean value theorem to show  $f(\mathbf{a}) = f(\mathbf{b})$ . ■

**Theorem 2.4.4** (Generalization of the Previous Theorem). Assume that  $S$  is an open and path-connected subset of  $\mathbb{R}^n$ , and  $f : S \rightarrow \mathbb{R}$  is a function differentiable on  $S$ . If  $\nabla f(\mathbf{x}) = \mathbf{0} \ \forall \mathbf{x} \in S$ , then  $f$  is constant on  $S$ .

*Proof.* Any path-connected set can be written as a countable union of convex sets  $S = \cup_{i \in \mathcal{A}} C_i$  such that

$$\forall \alpha \subset \mathcal{A} \text{ s.t. } \alpha \neq \emptyset, \cup_{i \in \alpha} C_i \cap \cup_{i \in \alpha^c} C_i \neq \emptyset \quad (2.4.6)$$

then apply the previous theorem. ■

## 2.5 Higher Order Derivatives

**Definition 2.5.1.** A function  $f$  defined on  $S$  is **of class  $C^k$**  if all of its  $k^{th}$  order partial derivatives exists and continuous everywhere in  $S$ .

**Theorem 2.5.1.** Assume that  $S$  is an open subset of  $\mathbb{R}^n$  and that  $f : S \rightarrow \mathbb{R}$  is  $C^k$ . Let  $\alpha \in [n]^k$ , and let  $\beta$  be any permutation of  $\alpha$ ,

$$\partial^\alpha f = \partial^\beta f \quad (2.5.1)$$

**Definition 2.5.2.** A **multi-index** is an  $n$ -tuple of nonnegative integers. And we define

$$\partial^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f, \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \quad (2.5.2)$$

where the **order** of multi-index is defined as

$$|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n \quad (2.5.3)$$

**Theorem 2.5.2** (the Multinomial Theorem).

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{x}^\alpha \quad (2.5.4)$$

*Proof Idea.* Prove by induction on  $n$ , with Binomial Theorem. ■

**Definition 2.5.3.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^2$  function, then its **Hessian matrix** is defined as

$$H_f := \begin{pmatrix} \partial_1 \partial_1 f & \cdots & \partial_n \partial_1 f \\ \vdots & \ddots & \vdots \\ \partial_1 \partial_n f & \cdots & \partial_n \partial_n f \end{pmatrix} \quad (2.5.5)$$

## 2.6 Taylor's Theorem

**Definition 2.6.1.** Let  $f : I \rightarrow \mathbb{R}$ , where  $I$  is an open subset of  $\mathbb{R}$ , be  $C^k$ . Let  $a \in I$ . then the  $k^{th}$  **order Taylor polynomial of  $f$  at  $a$**  is the unique polynomial of order at most  $k$ , denoted  $P_{a,k}(h)$  such that

$$f^{(j)}(a) = P_{a,k}^{(j)}(0) \quad \forall j \in \{0, 1, \dots, k\} \quad (2.6.1)$$

Note

$$P_{a,k}^{(j)}(h) = \sum_{j=0}^k \frac{h^j}{j!} f^{(j)}(a) \quad (2.6.2)$$

**Theorem 2.6.1** (Taylor's Theorem in 1 Dimension). Assume that  $I \subset \mathbb{R}$  is an open interval and that  $f : I \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $I$ . For  $a \in I$  and  $h \in \mathbb{R}$  such that  $a + h \in I$ . Define the **remainder**

$$R_{a,k}(h) := f(a + h) - P_{a,k}(h) \quad (2.6.3)$$

Then

$$\lim_{h \rightarrow 0} \frac{R_{a,k}(h)}{h^{\underline{k}}} = 0 \quad (2.6.4)$$

**Proposition 2.6.1.** Assume that  $I \subset \mathbb{R}$  is an open interval and that  $f : I \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $I$ . For  $a \in I$  and  $h \in \mathbb{R}$  such that  $a + h \in I$ , there exists some  $\theta \in (0, 1)$  such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \dots + \frac{h^{k-1}}{(k-1)!}f^{(k-1)}(a) + \frac{h^k}{k!}f^{(k)}(a + \theta h) \quad (2.6.5)$$

**Definition 2.6.2.** Assume that  $S \subset \mathbb{R}^n$  is an open interval and that  $f : S \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $S$ . For a point  $\mathbf{a} \in S$ , the  $k^{th}$  **order Taylor polynomial of  $f : S \rightarrow \mathbb{R}$  is a polynomial of order at most  $k$** , denoted  $P_{\mathbf{a},k}(\mathbf{h})$  satisfying

$$f(\mathbf{a}) = P_{\mathbf{a},k}(\mathbf{0}) \quad (2.6.6)$$

$$\partial^\alpha f(\mathbf{a}) = \partial^\alpha P_{\mathbf{a},k}(\mathbf{0}) \quad \forall \alpha \text{ s.t. } |\alpha| \leq k \quad (2.6.7)$$

**Theorem 2.6.2** (Taylor's Theorem in  $n$  Dimensions). Assume that  $S \subset \mathbb{R}^n$  is an open set and that  $f : S \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $S$ . For  $\mathbf{a} \in S$  and  $\mathbf{h} \in \mathbb{R}^n$  such that  $\mathbf{a} + \mathbf{h} \in S$ . Define the **remainder**

$$R_{\mathbf{a},k}(\mathbf{h}) := f(\mathbf{a} + \mathbf{h}) - P_{\mathbf{a},k}(\mathbf{h}) \quad (2.6.8)$$

Then

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{R_{\mathbf{a},k}(\mathbf{h})}{\|\mathbf{h}\|^{\underline{k}}} = 0 \quad (2.6.9)$$

**Theorem 2.6.3** (the Quadratic Case).

$$P_{\mathbf{a},2}(\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} \mathbf{h}^T H_f(\mathbf{a}) \mathbf{h} \quad (2.6.10)$$

$$\exists \theta \in (0, 1) \text{ s.t. } f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} \mathbf{h}^T H_f(\mathbf{a} + \theta \mathbf{h}) \mathbf{h} \quad (2.6.11)$$

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{R_{\mathbf{a},2}(\mathbf{h})}{\|\mathbf{h}\|^2} = 0 \quad (2.6.12)$$

**Definition 2.6.3** (the General Taylor's Polynomial).

$$P_{\mathbf{a},k}(\mathbf{h}) = \sum_{\{\alpha: |\alpha| \leq k\}} \frac{\mathbf{h}^\alpha}{\alpha!} \partial^\alpha f(\mathbf{a}) \quad (2.6.13)$$

## 2.7 Critical Points

**Definition 2.7.1.** A **symmetric**  $n \times n$  matrix  $A$  is said to be

- **Positive definite** if  $\mathbf{x}^T A \mathbf{x} > 0$  for all  $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ .
- **Non-negative definite/Positive semi-definite** if  $\mathbf{x}^T A \mathbf{x} \geq 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ .
- **Negative definite** if  $\mathbf{x}^T A \mathbf{x} < 0$  for all  $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ .
- **Non-positive definite/Negative semi-definite** if  $\mathbf{x}^T A \mathbf{x} \leq 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

and **indefinite** otherwise.

**Theorem 2.7.1.** Assume  $A$  is a symmetric matrix. Then

$$\begin{aligned} A \text{ is positive definite} &\iff \text{all its eigenvalues are positive} \\ &\iff \exists \lambda_i > 0 \text{ such that } \mathbf{x}^T A \mathbf{x} \geq \lambda_i \|\mathbf{x}\|^2 \text{ for all } \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

and

$$A \text{ is nonnegative definite} \iff \text{all its eigenvalues are nonnegative.} \quad (2.7.1)$$

and

$$A \text{ is indefinite} \iff A \text{ has both positive and negative eigenvalues.} \quad (2.7.2)$$

**Lemma 2.7.1.** Let  $A$  be a symmetric matrix, then

$$\text{the smallest eigenvalue of } A = \min_{\{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\|=1\}} \mathbf{u}^T A \mathbf{u} \quad (2.7.3)$$

**Definition 2.7.2.** A point  $\mathbf{a} \in S$  is a **local minimum point** for  $f : S \rightarrow \mathbb{R}$  if

$$\exists \varepsilon > 0 \text{ s.t. } \forall \mathbf{x} \in \mathcal{B}(\varepsilon, \mathbf{a}) \quad f(\mathbf{a}) \leq f(\mathbf{x}) \quad (2.7.4)$$

**Definition 2.7.3.** A point  $\mathbf{a} \in S$  is a **local maximum point** for  $f : S \rightarrow \mathbb{R}$  if

$$\exists \varepsilon > 0 \text{ s.t. } \forall \mathbf{x} \in \mathcal{B}(\varepsilon, \mathbf{a}) \quad f(\mathbf{a}) \geq f(\mathbf{x}) \quad (2.7.5)$$

**Definition 2.7.4.** Let  $f : S \rightarrow \mathbb{R}$  be differentiable on the open subset  $S \subset \mathbb{R}^n$ , then a point  $\mathbf{a} \in S$  is a **critical point** if

$$\nabla f(\mathbf{a}) = \mathbf{0} \quad (2.7.6)$$

**Definition 2.7.5.** Let  $\mathbf{a} \in S$  be a critical point of  $f$ , then  $\mathbf{a}$  is a **saddle point** if  $H_f(\mathbf{a})$  is indefinite.

**Theorem 2.7.2** (First Derivative Test). If  $f : S \rightarrow \mathbb{R}$  is differentiable, then

$$\text{local extremum} \implies \text{critical point} \quad (2.7.7)$$

**Theorem 2.7.3** (**Necessary Condition for a Local Minimum**). If  $f : S \rightarrow \mathbb{R}$  is  $C^2$  and  $\mathbf{a}$  is a local minimum point for  $f$ , then

- $\mathbf{a}$  is critical point of  $f$ ;
- $H_f(\mathbf{a})$  is positive semi-definite.



**Theorem 2.7.4** (Sufficient Condition for a Local Minimum). If

- (i)  $\mathbf{a}$  is a critical point of  $f$ ;
- (ii)  $H_f(\mathbf{a})$  is positive definite.

Then  $\mathbf{a}$  is a local minimum for  $f$ .

**Corollary 2.7.1.** Assume  $f$  is  $C^2$  and  $\nabla f(\mathbf{a}) = \mathbf{0}$ , then

- (i) If  $H_f(\mathbf{a})$  is positive definite, then  $\mathbf{a}$  is a local minimum;
- (ii) If  $H_f(\mathbf{a})$  is negative definite, then  $\mathbf{a}$  is a local maximum;
- (iii) If  $H_f(\mathbf{a})$  is indefinite, then  $\mathbf{a}$  is a saddle point.

*If none of the above holds (semi-definiteness), then we cannot determine the character of the critical point without further thought.*

**Definition 2.7.6.** A critical point  $\mathbf{a}$  of  $f$  is **degenerate** if  $\det H_f(\mathbf{a}) = 0$ , and **non-degenerate** if  $\det H_f(\mathbf{a}) \neq 0$ .

## 2.8 Optimization

**Theorem 2.8.1.** Let  $S \subset \mathbb{R}^n$  be an open set and  $f, g : S \rightarrow \mathbb{R}$  be  $C^1$  functions. If  $\mathbf{x}$  is a *local extremal* satisfying  $g(\mathbf{x}) = 0$ , and  $\nabla g(\mathbf{x}) \neq \mathbf{0}$ , then

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \quad (2.8.1)$$

**Proposition 2.8.1.** Equations (2.8.1)  $\implies \nabla f(\mathbf{x}) \perp \{\mathbf{x} \in S : g(\mathbf{x}) = 0\}$  at  $\mathbf{x}$ .

**Theorem 2.8.2.** Let  $S \subseteq \mathbb{R}^n$  be an open set, and  $f, \{g_i\}_{i=1}^k : S \rightarrow \mathbb{R}$  be  $C^1$  functions. Define  $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k := (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_k(\mathbf{x}))$ .

If  $\mathbf{x} \in S$  is a *local extremal* of  $f$  such that  $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ , and  $\{\nabla g_i(\mathbf{x})\}$  are linearly independent (i.e.  $\text{rank}(D\mathbf{g}(\mathbf{x})) = k$ ), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \quad (2.8.2)$$

**Remark 2.8.1.** Procedure of optimization on *open sets*:

- (i) Find all critical points;
- (ii) Find optimizers (classify them) among critical points.

**Remark 2.8.2.** Procedure of optimization with *inequality constraints*:

- (i) Find critical points *in* the constraint set from global critical points;
- (ii) Find critical points *on* the boundary of constraint set;
- (iii) Find optimizers among candidates.

### 3 The Implicit and Inverse Function Theorems

#### 3.1 The Implicit Function Theorem I

**Theorem 3.1.1** (Implicit Function Theorem). Let  $S \subseteq \mathbb{R}^{n+k}$  be an open set, and function  $\mathbf{F} : S \rightarrow \mathbb{R}^k$  be a  $C^1$  function. Suppose there exists point  $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$  such that

$$\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad (3.1.1)$$

If

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.2)$$

then there exists  $r_0, r_1 > 0$  and a  $C^1$  function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$  such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}), \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge \mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \quad (3.1.3)$$

and define  $\mathbf{y} := \mathbf{f}(\mathbf{x})$ , the derivative of  $\mathbf{f}$  can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y}) \quad (3.1.4)$$

**Remark 3.1.1.** Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (3.1.5)$$

near  $(\mathbf{a}, \mathbf{b})$ .

(i) Verify  $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ .

(ii) Assert

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.6)$$

(iii) Approximate solution  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  using

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \quad (3.1.7)$$

$$= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b})\mathbf{h} \quad (3.1.8)$$

#### 3.2 Geometric Content of the Implicit Function Theorem

**Definition 3.2.1.** Let  $S \subseteq \mathbb{R}^n$  and  $\mathbf{a} \in S$ .  $S$  is **singular** at  $\mathbf{a}$  if

$$\forall r > 0 \ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \quad (3.2.1)$$

$S$  is **regular** at  $\mathbf{a}$  if it is not singular there.

**Theorem 3.2.1** ( $k$  dimensional manifold as level set). Let  $U \subseteq \mathbb{R}^n$  and let  $\mathbf{F} : U \rightarrow \mathbb{R}^{n-k}$  be a  $C^1$  function.

$$S := \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\} \quad (3.2.2)$$

Let  $\mathbf{a} \in U$ , if

$$\text{rank}(D\mathbf{F}(\mathbf{a})) = n - k \quad (3.2.3)$$

then  $\exists r > 0$  such that the *level set of  $\mathbf{F}$  near  $\mathbf{a}$*

$$\mathcal{B}(r, \mathbf{a}) \cap S \quad (3.2.4)$$

can be represented as a  $C^1$  graph.

**Remark 3.2.1** (Interpretation). In  $\mathbb{R}^n$ , there are  $n$  degree of freedom before imposing any constraint, after adding constraints  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$  with  $n - k$  effective constraints, there are  $k$  degree of freedom left. What's left is a  $k$  dimensional manifold.

**Theorem 3.2.2** ( $k$  dimensional manifold as parameterization). Let  $T \subseteq \mathbb{R}^k$  and let  $\mathbf{f} : U \rightarrow \mathbb{R}^n$  be a  $C^1$  function.

$$S := \mathbf{f}(T) \quad (3.2.5)$$

Let  $\mathbf{t} \in T$ , if

$$\text{rank}(D\mathbf{f}(\mathbf{t})) = k \quad (3.2.6)$$

then  $\exists r > 0$  such that the parameterization of  $\mathbf{f}$  near  $\mathbf{t}$

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \quad (3.2.7)$$

can be represented as a  $C^1$  graph.

### 3.3 Transformations, and the Inverse Function Theorem

**Example 3.3.1** (Polar coordinate in  $\mathbb{R}^2$ ). Let

$$U := \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \quad (3.3.1)$$

$$V := \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \quad (3.3.2)$$

Define  $\mathbf{f} : U \rightarrow V$  as

$$\mathbf{f}(r, \theta) := \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} \quad (3.3.3)$$

**Example 3.3.2** (Spherical coordinate in  $\mathbb{R}^3$ ). Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix} \quad (3.3.4)$$

**Example 3.3.3** (Cylindrical coordinate in  $\mathbb{R}^3$ ). Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{pmatrix} \quad (3.3.5)$$

**Theorem 3.3.1** (Inverse Function Theorem). Let  $U$  and  $V$  be open subsets in  $\mathbb{R}^n$ , and  $\mathbf{f} : U \rightarrow V$ . Let  $\mathbf{a} \in U$  and define  $\mathbf{b} := \mathbf{f}(\mathbf{a}) \in V$ . If

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0 \quad (3.3.6)$$

then there exists  $M \subseteq U$  and  $N \subseteq V$  such that

- (i)  $\mathbf{a} \in M$  and  $\mathbf{b} \in N$ ,
- (ii)  $\mathbf{f}$  is bijective between  $M$  and  $N$ ,
- (iii)  $\mathbf{f}^{-1} : N \rightarrow M$  is  $C^1$ ,

and for all  $\mathbf{x} \in M$  such  $\mathbf{y} := \mathbf{f}(\mathbf{x}) \in N$ ,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \quad (3.3.7)$$

## 4 Integration

### 4.1 Basics

**Theorem 4.1.1 (Properties of infimum and supremum).** Let  $A \subseteq \mathbb{R}^n$  and  $A \neq \emptyset$ , and  $f, g : A \rightarrow \mathbb{R}$  are bounded functions. Let  $m$  and  $M$  denote the infimum and supremum respectively, then

- (i)  $m_A f + m_A g \leq m_A(f + g) \leq M_A(f + g) \leq M_A f + M_A g$
- (ii) If  $A' \subseteq A$ , then  $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$
- (iii) If  $f(\mathbf{x}) \leq g(\mathbf{x}) \forall \mathbf{x} \in A$ , then  $m_A f \leq m_A g$  and  $M_A f \leq M_A g$
- (iv)  $|M_A f| \leq M_A |f|$
- (v)  $M_A |f| - m_A |f| \leq M_A f - m_A f$
- (vi)  $\forall c \in \mathbb{R}, M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$
- (vii)  $M_A f - m_A f = \sup\{f(x) - f(y) : x, y \in A\}$

### 4.2 Integration on Higher Dimensions

**Definition 4.2.1.** A rectangle  $\mathcal{R} \subseteq \mathbb{R}^n$  ( $n$ -orthotope) is defined as

$$\mathcal{R} := \prod_{i=1}^n [a_i, b_i] \quad (4.2.1)$$

where  $a_i, b_i \in \mathbb{R}$  and  $a_i < b_i$ .

**Definition 4.2.2.** A **partition**  $P$  of rectangle  $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$  is a list of  $n$  **finite** and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \quad (4.2.2)$$

where  $L_i = \{e_j\}_{j=0}^{T_i}$  such that

$$a_i = e_0 < e_1 < \dots < e_{T_i} = b_i \quad (4.2.3)$$

and such partition generates a set of rectangles(boxes)  $\mathcal{B}(P) := \{B_j\}_{j=1}^J \subseteq \mathcal{R}$ .

**Definition 4.2.3.** Let  $P$  and  $P'$  be two partitions of  $\mathcal{R}$ . Then  $P'$  is a **refinement** of  $P$  if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee (B_j'^{\text{int}} \cap B_j^{\text{int}} = \emptyset) \quad (4.2.4)$$

**Definition 4.2.4.** Define the **volume** of rectangle  $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$  as

$$V^n(\mathcal{R}) := \prod_{i=1}^n (b_i - a_i) \quad (4.2.5)$$

**Definition 4.2.5.** The **lower Riemann sum** of  $f$  with partition  $P$  on  $\mathcal{R}$  is defined as

$$L_P f := \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.6)$$

and the **upper Riemann sum** is defined as

$$U_P f := \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.7)$$

**Definition 4.2.6.** The **upper integral** and **lower integral** of  $f$  on  $\mathcal{R}$  are defined as

$$\bar{I}_{\mathcal{R}}f := \inf_P U_P f \quad (4.2.8)$$

$$\underline{I}_{\mathcal{R}}f := \sup_P L_P f \quad (4.2.9)$$

**Definition 4.2.7.** A bounded real-valued function  $f$  defined on  $\mathcal{R}$  is **integrable** if

$$\underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.10)$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \, dV^n := \underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.11)$$

**Lemma 4.2.1.** Let  $f$  be a bounded real-valued function defined on  $\mathcal{R}$ ,  $f$  is integrable if and only if  $\forall \epsilon > 0$ , there exists a partition  $P$  of  $\mathcal{R}$  such that

$$U_P f - L_P f < \epsilon \quad (4.2.12)$$

**Theorem 4.2.1.** Let  $f$  and  $g$  be two integrable functions on  $\mathcal{R} \subseteq \mathbb{R}^n$ , let  $c \in \mathbb{R}$ ,

- (i)  $f + g : \mathcal{R} \rightarrow \mathbb{R}$  is integrable and  $\int_{\mathcal{R}}(f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$
- (ii)  $c \cdot f$  is integrable and  $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$
- (iii)  $f(\mathbf{x}) \geq g(\mathbf{x}) \, \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$
- (iv)  $|f|$  is integrable and  $|\int_{\mathcal{R}} f| \leq \int_{\mathcal{R}} |f|$

**Definition 4.2.8.** Let  $S \subseteq \mathbb{R}^n$  be a bounded set, and there exists rectangle  $\mathcal{R}$  covers  $S$ , the **indicator function** of  $S$  is  $\chi_S : \mathcal{R} \rightarrow \{0, 1\}$ , defined as

$$\chi_S(\mathbf{x}) := \mathbf{1}\{\mathbf{x} \in S\} \quad (4.2.13)$$

**Definition 4.2.9.** Let  $S \subseteq \mathbb{R}^n$  be a bounded set, and there exists rectangle  $\mathcal{R}$  covers  $S$ . Let  $f : \mathcal{R} \rightarrow \mathbb{R}$  be a bounded function, then  $f$  is **integrable on  $S$**  if  $\chi_S f$  is integrable on  $\mathcal{R}$ . And

$$\int \cdots \int_S f \, dV^n := \int \cdots \int_{\mathcal{R}} \chi_S f \, dV^n \quad (4.2.14)$$

**Definition 4.2.10.** Let  $Z \subseteq \mathbb{R}^n$ ,  $Z$  has **zero content** if for all  $\epsilon > 0$ , there exists a finite set of rectangles  $\{R_\ell\}_{\ell=1}^L$  covers  $Z$  and

$$\sum_{\ell=1}^L V^n(R_\ell) < \epsilon \quad (4.2.15)$$

**Definition 4.2.11.** A set  $S \subset \mathbb{R}^n$  is **Jordan measurable** if it is bounded and its boundary has zero content.

**Proposition 4.2.1.** Let  $Z \subseteq \mathbb{R}^n$  has zero content, then

- (i) For any  $Z' \subseteq Z$ ,  $Z'$  has zero content.
- (ii) Finite union of content zero sets has zero content.
- (iii) Let  $f : [a, b] \rightarrow \mathbb{R}$  be an integrable function, its graph  $\{(x, f(x)) : x \in [a, b]\}$  has zero content.

(iv) Let  $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^2$  be a  $C^1$  function, the parameterization  $\mathbf{f}([a, b])$  has zero content.

**Theorem 4.2.2.** Let  $\mathcal{R}$  be a rectangle in  $\mathbb{R}^n$  and  $f$  is integrable on  $\mathcal{R}$  if the set

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \quad (4.2.16)$$

has zero content.

**Proposition 4.2.2** (Folland 4.22). Suppose  $Z \subseteq \mathbb{R}^n$  has zero content. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded, then  $f$  is integrable on  $Z$  and  $\int_Z f \, dV^n = 0$ .

**Remark 4.2.1** (Measures and Contents). A measure  $\mu$  is required to be *countably* additivity on disjoint sets, but a content only requires *finite* additivity on disjoint sets. Therefore, if a set has content zero based on some content function, such a content function is also a measure function, thus such set has measure zero as well.

$$\text{content zero} \implies \text{measure zero} \quad (4.2.17)$$

### 4.3 Iterated Integrals

**Theorem 4.3.1** (Fubini's Theorem). Let  $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$  and  $f : \mathcal{R} \rightarrow \mathbb{R}$  is bounded. Assuming that

- (i)  $f$  is integrable on  $\mathcal{R}$ .
- (ii) for each  $y \in [c, d]$ , the function  $f_y(x) := f(x, y)$  is integrable on  $[a, b]$ .
- (iii) Define  $g(y) := \int_a^b f(x, y) dy$  is integrable on  $[c, d]$ .

Then

$$\iint_{\mathcal{R}} f \, dA = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy \quad (4.3.1)$$

**Proposition 4.3.1.** Let  $S \subseteq \mathbb{R}^n$  be an unbounded set, and  $f : S \rightarrow \mathbb{R}$ . Then improper integral  $\int \cdots \int_S f \, d^n \mathbf{x}$  is absolutely convergent on  $\mathbb{R}^n$  if and only if  $\int \cdots \int_{\mathbb{R}^n} \chi_S f \, d^n \mathbf{x}$  is absolutely convergent.

### 4.4 Change of Variables

**Theorem 4.4.1** (Change of Variable). Let  $U$  and  $V$  be two open subsets of  $\mathbb{R}^n$ , and let  $\mathbf{G} : U \rightarrow V$  be a  $C^1$  bijection. Let  $T \subset U$  and  $S \subset V$ . Suppose  $\mathbf{G}(T) = S$ , then

$$\underbrace{\int \cdots \int_S f \, d\Omega}_{\text{Target Integral}} = \int \cdots \int_{T \equiv \mathbf{G}^{-1}(S)} f \circ \mathbf{G} \, |\det D\mathbf{G}| \, d\Theta \quad (4.4.1)$$

**Corollary 4.4.1.** Let  $S$  be a region in  $\mathbb{R}^n$ , suppose  $S$  can be parameterized by  $\mathbf{G} : T \rightarrow S$ . By the change of variable formula, consider the special case  $f(\mathbf{x}) = 1$ ,

$$|S| = \int \cdots \int_S 1 \, d\Omega = \int \cdots \int_T 1 \, |\det D\mathbf{G}(\mathbf{u})| \, d\Theta \quad (4.4.2)$$

**Example 4.4.1** (Polar Coordinate). Define the coordinate transformation mapping from polar to Cartesian,

$$\mathbf{P}(r, \theta) := (x, y) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad (4.4.3)$$

and  $|\det D\mathbf{P}(r, \theta)| = r$ .

**Example 4.4.2** (Cylindrical Coordinate). Define the coordinate transformation mapping from cylindrical to Cartesian as

$$\mathbf{C}(r, \theta, z) := (x, y, z) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad z \in \mathbb{R} \quad (4.4.4)$$

and  $|\det DC(r, \theta, z)| = r$ .

**Example 4.4.3** (Spherical Coordinate). Define the coordinate transformation mapping from spherical to Cartesian as

$$\mathbf{S}(r, \theta, \varphi) = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} \quad (4.4.5)$$

and  $|\det DS(r, \theta, \varphi)| = r^2 \sin \varphi$

## 4.5 Further Aspects

### 4.5.1 Exchanging Differentiation and Integration

**Theorem 4.5.1** (**Exchanging Differentiation and Integration**). Let  $f(\mathbf{x}, \mathbf{t}) : S \times T \rightarrow \mathbb{R}$  and define  $F(\mathbf{x}) : S \rightarrow \mathbb{R}$  as

$$F(\mathbf{x}) := \int \cdots \int_T f(\mathbf{x}, \mathbf{t}) \, d\mathbf{t} \quad (4.5.1)$$

If

- (i)  $S$  is open and  $T$  is compact;
- (ii)  $f$  and  $F$  are continuous;
- (iii) and  $\forall x_j \in \mathbf{x}, \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j}$  is continuous (differentiable w.r.t  $\mathbf{x}$ ),

then  $F$  is  $C^1$  in  $S$  and for every  $j$ ,

$$\frac{\partial F(\mathbf{x})}{\partial x_j} = \int \cdots \int_T \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j} \, d\mathbf{t} \quad (4.5.2)$$

**Corollary 4.5.1** (**Exchanging Limits and Integration**). By the definition of partial derivative, above theorem is equivalent to

$$\lim_{h \rightarrow 0} \int \cdots \int_T \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} \, d\mathbf{t} = \int \cdots \int_T \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} \, d\mathbf{t} \quad (4.5.3)$$

### 4.5.2 Improper Integrals

**Definition 4.5.1** (Unbounded Domains). An **improper integral** with unbounded domain  $\int \cdots \int_{\mathbb{R}^n} f \, d\Omega$  is **absolutely convergent** if there exists  $L \in \mathbb{R}$  such that

$$\forall \varepsilon > 0 \quad \exists R > 0 \text{ s.t. } \forall S \subseteq \mathbb{R}^n \quad \mathcal{B}(R, \mathbf{0}) \subset S \implies \left| \int \cdots \int_S f \, d\Omega - L \right| < \varepsilon \quad (4.5.4)$$

**Theorem 4.5.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function, and that

$$\lim_{R \rightarrow \infty} \int \cdots \int_{\mathcal{B}(R, \mathbf{0})} |f| d\Omega \text{ exists} \quad (4.5.5)$$

then  $\int \cdots \int_{\mathbb{R}^n} f d\Omega$  is absolutely convergent.

**Corollary 4.5.2** (Equivalence). Above improper integral  $\int \cdots \int_{\mathbb{R}^n} f d\Omega$  is absolutely convergent if set

$$\left\{ \int \cdots \int_{\mathcal{B}(R, \mathbf{0})} |f| d\Omega : R \in \mathbb{R}_{++} \right\} \quad (4.5.6)$$

is bounded.

**Corollary 4.5.3.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be an continuous function, if

$$\exists p > n, C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{1}{\|\mathbf{x}\|^p} \forall \mathbf{x} \in \mathbb{R}^n \quad (4.5.7)$$

then  $\int \cdots \int_{\mathbb{R}^n} f d\Omega$  is absolutely convergent.

**Definition 4.5.2** (Unbounded Function). Let  $S \subset \mathbb{R}^n$ ,  $\mathbf{a} \in \mathbb{R}^n$ . Consider a function  $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$ . Then the improper integral  $\int \cdots \int_S f d\Omega$  is absolutely convergent if

$$\exists L \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0 \exists r > 0 \text{ s.t. } \forall U \subset S \text{ s.t. } \mathbf{a} \in U^{\text{int}} \wedge U \subset \mathcal{B}(r, \mathbf{a}), \left| \int \cdots \int_{S \setminus U} f d\Omega - L \right| < \varepsilon \quad (4.5.8)$$

**Theorem 4.5.3.** Let  $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$ , if

$$\lim_{r \rightarrow 0} \int \cdots \int_{S \setminus \mathcal{B}(r, \mathbf{a})} |f| d\Omega \text{ exists} \quad (4.5.9)$$

then  $\int \cdots \int_S f d\Omega$  is absolutely convergent.

**Corollary 4.5.4** (Equivalence). If the set

$$\left\{ \int \cdots \int_{S \setminus \mathcal{B}(r, \mathbf{a})} |f| d\Omega : r \in \mathbb{R}_{++} \right\} \quad (4.5.10)$$

is bounded, then  $\int \cdots \int_S f d\Omega$  is absolutely convergent.

**Corollary 4.5.5.** Let  $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$ , if

$$\exists p < n, C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x} - \mathbf{a}\|^p} \forall \mathbf{x} \in S \setminus \{\mathbf{a}\} \quad (4.5.11)$$

then the improper integral  $\int \cdots \int_S f d\Omega$  is absolutely convergent.



## 5 Vector Calculus

### 5.1 Line Integrals

#### 5.1.1 Arc Length

**Definition 5.1.1.** Let  $C$  be a smooth curve in  $\mathbb{R}^n$  parameterized by  $C^1$  function  $\mathbf{g}$  such that  $\mathbf{g}'(t) \neq \mathbf{0}$  for every appropriate  $t$ .

$$C := \{\mathbf{g}(t) : t \in [a, b]\} \quad (5.1.1)$$

and the **arc length** of  $C$  is defined as

$$\int_C d^n \mathbf{x} := \int_C ds := \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.2)$$

**Proposition 5.1.1.** The arc length of a curve  $C$  is an intrinsic property of the geometric object  $C$  and should not depend on the particular parameterization we use.

*Proof.* Let  $\varphi : [c, d] \rightarrow [a, b]$  be a bijection, so that  $\mathbf{h} := \mathbf{g} \circ \varphi$  is also a valid parameterization of  $C$  such that

$$C := \{\mathbf{h}(u) : u \in [c, d]\} \quad (5.1.3)$$

The arc length of  $C$  can be computed using

$$\int_C ds = \int_c^d \|\mathbf{h}'(u)\| du \quad (5.1.4)$$

$$= \int_c^d \|\mathbf{g}'(\varphi(u))\| \cdot |\varphi'(u)| du \quad (5.1.5)$$

$$= \int_a^b \|\mathbf{g}'(t)\| dt \text{ by change of variable formula.} \quad (5.1.6)$$

■

**Remark 5.1.1** (Interpretations). Suppose  $\mathbf{g}$  is a parameterization of  $C$ .

- (i)  $\int_a^b \mathbf{g}'(t) dt = \mathbf{g}(b) - \mathbf{g}(a)$  measures the distance between two endpoints of  $C$ .
- (ii) Choosing a parameterization is effectively choosing an **orientation** for the curve  $C$ .

**Definition 5.1.2.** A function  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  is called **piecewise smooth** if

- (i) it's *continuous*, and
- (ii) it's derivate exists and is continuous except at finitely many points  $t_j$ , at which the one-sided limits exist.

#### 5.1.2 Line Integrals of Scalar Functions

**Definition 5.1.3.** Let smooth curve  $C \subseteq \mathbb{R}^n$ ,  $f : C \rightarrow \mathbb{R}$  and  $\mathbf{g}$  be a parameterization of  $C$ , then

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \quad (5.1.7)$$

**Remark 5.1.2.** The line integrals of scalar functions are also independent from the choices of parameterizations.

**Definition 5.1.4.**

$$\text{Average of } f \text{ over } C := \frac{\int_C f ds}{\int_C ds} \quad (5.1.8)$$

### 5.1.3 Line Integrals of Vector Fields

**Definition 5.1.5.** Let smooth  $C \in \mathbb{R}^n$  with parameterization  $\mathbf{g}$  and  $\mathbf{F} : C \rightarrow \mathbb{R}^n$  defined on it, the **line integral** of  $\mathbf{F}$  over  $C$  is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt \quad (5.1.9)$$

**Proposition 5.1.2.** The line integral  $\int_C \mathbf{F} \cdot d\mathbf{x}$  is independent of the parameterization *as long as the orientation is unchanged*.

**Theorem 5.1.1** (The Fundamental Theorem of Line Integral). Let  $f : C \rightarrow \mathbb{R}$  defined on smooth curve  $C$  parameterized by  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ , then

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.10)$$

*Proof.*

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial \mathbf{g}(t)} \cdot \mathbf{g}'(t) dt \quad (5.1.11)$$

$$= \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial t} dt = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.12)$$

■

### 5.1.4 Rectifiable Curves

**Remark 5.1.3.** Let  $C$  be a curve in  $\mathbb{R}^n$  parameterized by injection  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  such that  $\mathbf{g}'(t) \neq \mathbf{0}$ . Let  $P$  be a partition of  $[a, b]$ . Denote

$$L_P(C) := \sum_j \|\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})\| \quad (5.1.13)$$

**Definition 5.1.6.** A curve  $C$  is **rectifiable** if the set  $\{L_P(C) : P\}$  is bounded. And the arc length of  $C$  is defined as

$$L(C) := \sup\{L_P(C) : P\} \quad (5.1.14)$$

**Theorem 5.1.2.** The supremum found above,  $L(C)$  is precisely the arc length of  $C$ :

$$L(C) = \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.15)$$

## 5.2 Green's Theorem

### 5.2.1 Preliminary Definitions

**Definition 5.2.1.** A **simple closed curve** is a curve with parameterization  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  where

- (i)  $\mathbf{g}$  is continuous;
- (ii)  $\mathbf{g}(a) = \mathbf{g}(b)$ ;
- (iii)  $\mathbf{g}$  is injective with its domain restricted to  $(a, b)$  (so the curve does not cross).

**Definition 5.2.2.** A *simple closed curve* is **piecewise smooth** if it has a parameterization  $\mathbf{g}$  such that

- (i)  $\mathbf{g}$  is continuously differentiable with  $\mathbf{g}'(t) \neq \mathbf{0}$  except finitely many breakpoints;
- (ii)  $\mathbf{g}'(t)$  is *one side continuous* at breakpoints of the curve.

**Definition 5.2.3.** A **regular region**  $S \subseteq \mathbb{R}^n$  is a set satisfying both

- (i)  $S$  is compact;
- (ii)  $\overline{S^{int}} = S$ .

**Definition 5.2.4.** Let  $S \subseteq \mathbb{R}^2$ ,  $S$  has **piecewise smooth boundary** if  $\partial S$  consists of one or more *disjoint, piecewise smooth, simple closed curve*.

**Definition 5.2.5.** Let  $S \subseteq \mathbb{R}^2$ , then **positive orientation** on  $\partial S$  is the orientation on each of the closed curves that make up the boundary such that the region is on the *left* with respect to the positive direction on the curve.

**Theorem 5.2.1** (Green's Theorem). Suppose  $S \subseteq \mathbb{R}^2$  is a regular region with piecewise smooth boundary  $\partial S$ . Suppose  $\mathbf{F}$  is a  $C^1$  vector field defined on  $\overline{S}$ , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left( \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA \quad (5.2.1)$$

**Corollary 5.2.1.** Suppose  $S$  is a regular region in  $\mathbb{R}^2$  with piecewise smooth boundary  $\partial S$ , and let  $\mathbf{n}(\mathbf{x})$  be the *unit outward normal* vector to  $\partial S$  at  $\mathbf{x} \in \partial S$ . Suppose also that  $\mathbf{F}$  is a vector field defined on  $\overline{S}$ , then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \left( \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA \quad (5.2.2)$$

*Proof.* Let  $\mathbf{g}(t)$  be a parameterization of boundary  $\partial S$ . Then the tangent vector would be  $\mathbf{g}'(t)$  and we can conclude the *outer normal vector*  $\mathbf{n}$  is  $\frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|}$ . Then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \int_T \mathbf{F} \circ \mathbf{g} \cdot \frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|} \|\mathbf{g}'(t)\| \, dt \quad (5.2.3)$$

$$= \int_T F_1 g'_2(t) - F_2 g'_1(t) \, dt \quad (5.2.4)$$

$$= \int_T \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot \begin{pmatrix} g'_1(t) \\ g'_2(t) \end{pmatrix} \, dt \quad (5.2.5)$$

$$= \int_{\partial S} \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot d^2\mathbf{x} \quad (5.2.6)$$

$$= \iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \, dA \text{ By Green's Theorem} \quad (5.2.7)$$

■

**Theorem 5.2.2** (Application: Moving the Curve). Suppose we wish to compute

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} \quad (5.2.8)$$

for some piecewise smooth curve  $C_1$ . For any piecewise smooth curve  $C_2$  such that  $C_1 \cup C_2$  and  $C_1 \cap C_2$  has zero content, such that  $C_1 \cup C_2 = \partial S$  for some regular region  $S \subset \mathbb{R}^2$ . We can compute the desired line integral using

$$\iint_S \left( \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x} \quad (5.2.9)$$

$$= \int_{C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{x} \quad (5.2.10)$$

$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{x} + \int_{C_2} \mathbf{F} \cdot d\mathbf{x} \quad (5.2.11)$$

### 5.3 Surface Integrals

#### 5.3.1 Surface Areas and Surface Integrals

**Definition 5.3.1.** Suppose  $S$  is a surface in  $\mathbb{R}^3$  and parameterized by

$$\mathbf{G}(\mathbf{u}) : R \rightarrow S \quad (5.3.1)$$

where  $\text{rank}(D\mathbf{G}(\mathbf{u})) = 2$  for every  $\mathbf{u} \in R \setminus Z$  where  $Z$  is a probably empty set with zero content. If  $\|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\|$  is integrable, then

$$\text{Area}(S) := \iint_R \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.2)$$

**Definition 5.3.2.** Let  $f : S \rightarrow \mathbb{R}$  be a real-valued continuous function defined on a super set of  $S$ , the **integral of a real-valued function on a surface** is defined as

$$\iint_S f(\mathbf{x}) dA := \iint_R f(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.3)$$

**Definition 5.3.3.** Let  $\mathbf{F} : S \rightarrow \mathbb{R}^3$  be a continuous vector field defined on a super set of  $S$ , the **integral of vector field on a surface** is defined as

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} dA := \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (5.3.4)$$

**Remark 5.3.1.** Surface integrals of **real-valued functions** are independent of the choice of parametrization.

**Remark 5.3.2.** But the choice of parameterization can change the sign of surface integrals of **vector fields**. We need to **choose the direction of the normal,  $\mathbf{n}$** .

**Definition 5.3.4.** Let  $S \subseteq \mathbb{R}^3$  be a two dimensional sub-manifold, and  $f$  is a real-valued function defined on a super set of  $S$ . Define the **average of  $f$  over  $S$**  as

$$\text{aver of } f \text{ over } S := \frac{\iint_S f dA}{\iint_S 1 dA} \quad (5.3.5)$$

**Remark 5.3.3.** A note on the relation between integrals of a vector field and a real-valued function. The surface of vector field  $\mathbf{F}$  on  $S$  is defined by *reducing  $\mathbf{F}$  to a real-valued function  $\mathbf{F} \cdot \mathbf{n}$*  and then

follow the definition of conventional real-valued function on  $S$ . Define  $h := \mathbf{F} \cdot \mathbf{n}$ ,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_S h \, dA \quad (5.3.6)$$

$$:= \iint_R h(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.7)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \mathbf{n}(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.8)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \frac{\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}}{\left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\|} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.9)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (5.3.10)$$

where  $\mathbf{G} : R \rightarrow S$  is a parameterization of  $S$ .

### 5.3.2 An invariance property

**Remark 5.3.4.** As mentioned above, given  $\mathbf{n}(\mathbf{x})$  fixed, we can define the surface integral of vector field as the surface integral of a real-valued function defined as  $h(\mathbf{x}) := \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ . And as argued before, one  $\mathbf{n}$  is fixed (i.e. orientation is fixed), the value of integral is deterministic. Therefore we can conclude **the integral of a vector field  $\mathbf{F}$  over a surface  $S$  depends on the orientation of  $S$  but otherwise independent of the parameterization.**

**Remark 5.3.5.** Let  $S \subseteq \mathbb{R}^2$  be a two dimensional sub-manifold parameterized by  $\mathbf{G} : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that  $\text{rank}(\mathbf{G}(\mathbf{u})) = 2$  for all but zero-content sets on its domain.

Let  $\varphi : W \subseteq \mathbb{R}^2 \rightarrow R$  be a bijection such that  $\mathbf{H} := \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$  is another parameterization of  $S$ .

Now consider the integral of vector field  $\mathbf{F}$  under parameterization  $\mathbf{H}$ ,

$$\iint_S \mathbf{F} \cdot \mathbf{u} \, dA = \iint_W \mathbf{F}(\mathbf{H}) \cdot \left( \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right) d\Theta \quad (5.3.11)$$

$$= \iint_W \mathbf{F} \circ \mathbf{G} \circ \varphi \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) \text{det } D\varphi \, d\Theta \quad (5.3.12)$$

$$= \pm \iint_R \mathbf{F} \circ \mathbf{G} \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (\text{change of variable}) \quad (5.3.13)$$

**Theorem 5.3.1** (Invariance). Let  $\mathbf{G} : R \rightarrow \mathbb{R}^3$  and  $\mathbf{H} := \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$  be two parameterizations of  $S$ , then

$$\iint_R f \circ \mathbf{G} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta = \iint_W f \circ \mathbf{H} \left\| \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right\| d\Theta \quad (5.3.14)$$

and

$$\iint_R \mathbf{F} \circ \mathbf{G} \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta = \pm \iint_W \mathbf{F} \circ \mathbf{H} \cdot \left( \frac{\partial \mathbf{H}}{\partial u} \times \frac{\partial \mathbf{H}}{\partial v} \right) d\Theta \quad (5.3.15)$$

### 5.3.3 Volume and Area (Optional)

**Theorem 5.3.2.** Let  $R$  be an arbitrary regular region in  $\mathbb{R}^3$ , and let  $S$  be the boundary surface of  $R$ , define

$$S_h := \{\mathbf{x} + \delta \mathbf{n} : \mathbf{x} \in S \wedge \delta \in [0, h]\} \quad (5.3.16)$$

where  $S_h$  can be interpreted as a shell of region  $R$  with thickness  $h$ . Then the surface area of  $S$  is

$$\text{area}(S) = \lim_{h \rightarrow 0} \frac{|S_h|}{h} \quad (5.3.17)$$

## 5.4 Divergence, Gradient and Curl

**Definition 5.4.1.** Let  $U \subseteq \mathbb{R}^n$  be an open set, and define real-valued function  $f : U \rightarrow \mathbb{R}$  (0-form) and vector field  $\mathbf{F} : U \rightarrow \mathbb{R}^n$ . Then we define

1. (1-form) The **gradient** of  $f$  as  $\nabla f$ ;
2. (2-form) The **curl** of  $\mathbf{F}$  as  $\nabla \times \mathbf{F}$ ;
3. (3-form) The **divergence** of  $\mathbf{F}$  as  $\nabla \cdot \mathbf{F}$ .

**Definition 5.4.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^1$  real-valued function, define the **Laplacian** of  $f$  as a mapping from *real-valued functional space* to *real-valued functional space* defined as

$$\text{div}(\text{grad})f := \sum_j \partial_j^2 f = \Delta f = \nabla^2 f \quad (5.4.1)$$

**Theorem 5.4.1.** For every  $C^2$  real valued function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,

$$\text{curl}(\text{grad}f) = \mathbf{0} \quad (5.4.2)$$

For every  $C^2$  vector field defined in  $\mathbb{R}^3$  or a subset of it,

$$\text{div}(\text{curl}\mathbf{F}) = 0 \quad (5.4.3)$$

Note that the domain of  $f$  and  $\mathbf{F}$  must be  $\mathbb{R}^3$  or a subset of it, otherwise the curl operation is not well-defined.

**Theorem 5.4.2** (Product rules).

$$\text{grad}(fg) = f \text{ grad}g + \text{grad}f \cdot g \quad (5.4.4)$$

$$\text{curl}(f\mathbf{G}) = f \text{ curl}\mathbf{G} + \text{grad}f \times \mathbf{G} \quad (5.4.5)$$

$$\text{div}(f\mathbf{G}) = f \text{ div}\mathbf{G} + \text{grad}f \cdot \mathbf{G} \quad (5.4.6)$$

## 5.5 Divergence Theorem

**Remark 5.5.1.** vector field integral on boundary (2-dimensional sub-manifold) of region in  $\mathbb{R}^3$  ( $\mathbf{F} \cdot \mathbf{n} \, dA$  as 2-form) and scalar valued function ( $\text{div}(\mathbf{F}) \, dV$  as 3-form) in a region (3-dimensional sub-manifold).

**Theorem 5.5.1** (Divergence Theorem). Let  $R \subseteq \mathbb{R}^3$  be a regular region with piecewise smooth boundary  $\partial S$ . And  $\mathbf{n}$  is the outer normal vector on  $\partial S$ , then,

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_S \text{div}(\mathbf{F}) \, dV \quad (5.5.1)$$

*Proof.*

**Definition 5.5.1.** A region  $R \subseteq \mathbb{R}^3$  is said to be  **$xy$ -simple** if and only if it can be expressed as the following form

$$R = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\} \quad (5.5.2)$$

Suppose  $S$  is simple in terms of all combinations of  $x, y, z$ .

Then

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_1 n_1 + F_2 n_2 + F_3 n_3 \, dA \quad (5.5.3)$$

Consider  $\iint_{\partial S} F_3 n_3 \, dA$ , since  $R$  is  $xy$ -simple,

$$\iint_{\partial S} F_3 n_3 \, dA = \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA \quad (5.5.4)$$

Note that except the bottom and top sides, which are parameterized by  $\mathbf{G}_1(x, y) = (x, y, \varphi_1(x, y))$  and  $\mathbf{G}_2(x, y) = (x, y, \varphi_2(x, y))$ , the outer normal vector of those region has form  $(\cdot, \cdot, 0)$ , and therefore  $\mathbf{n} \cdot \mathbf{k} = 0$  for every  $\mathbf{x}$  on those regions, and contribute nothing to the integral.

Therefore, to evaluate  $\iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA$ , we only need to consider the upper and bottom surfaces. Also note that  $\mathbf{n}$  has opposite  $z$  component on those two surfaces.

Moreover, the undirected  $\mathbf{n}$  on those two surfaces is

$$\tilde{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ \partial_x \varphi_i \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y \varphi_i \end{pmatrix} = \begin{pmatrix} -\partial_x \varphi_i \\ \partial_x \varphi_i - \partial_y \varphi_i \\ 1 \end{pmatrix} \quad (5.5.5)$$

$$\implies \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_3 \, dA \quad (5.5.6)$$

$$= \iint_{\text{upper } \partial S} F_3 \, dA - \iint_{\text{lower } \partial S} F_3 \, dA \quad (5.5.7)$$

$$= \iint_W F_3(x, y, \varphi_2(x, y)) \, dx dy - \iint_W F_3(x, y, \varphi_1(x, y)) \, dx dy \quad (5.5.8)$$

$$= \iint_W \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} \partial_3 F_3 \, dz dx dy \quad (5.5.9)$$

$$= \iiint_S \partial_3 F_3 \, dV \quad (5.5.10)$$

We can prove the equalities involving the other two components, and the proof is completed by the fact that any open set in  $\mathbb{R}^n$  can be written as a countable union of *almost disjoint* cubes, which are simple and the boundary of  $S$  has zero content. ■

**Proposition 5.5.1** (Geometric Interpretation of Divergence). Let  $S \subset \mathbb{R}^3$ ,  $\mathbf{F} : S \rightarrow \mathbb{R}^3$ ,  $\mathbf{a} \in S$ ,

$$\operatorname{div}(\mathbf{F})(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \iiint_{\mathcal{B}(\mathbf{a}, r)} \operatorname{div}(\mathbf{F})(\mathbf{x}) \, d\mathbf{x} \quad (5.5.11)$$

$$= \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \underbrace{\iint_{\partial \mathcal{B}(\mathbf{a}, r)} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{x}}_{\text{flux through boundary}} \quad (5.5.12)$$

thus  $\operatorname{div}(\mathbf{F})(\mathbf{a}) > 0$  if and only if at point  $\mathbf{a}$ , matters are flowing *away* from this point.

**Theorem 5.5.2** (Application: Moving the Surface). Given a regular region  $R \subset \mathbb{R}^3$  with piecewise smooth boundary  $\partial R$  consisting of  $S$  and  $R'$ . And let  $\mathbf{F}$  be a vector field defined on an open set containing  $R$ . We can find

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA \quad (5.5.13)$$

by

$$\iiint_R \nabla \cdot \mathbf{F} dV = \iint_{\partial R} \mathbf{F} \cdot \mathbf{n} dA = \iint_S \mathbf{F} \cdot \mathbf{n} dA + \iint_{S'} \mathbf{F} \cdot \mathbf{n} dA \quad (5.5.14)$$

so

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_R \nabla \cdot \mathbf{F} dV - \iint_{S'} \mathbf{F} \cdot \mathbf{n} dA \quad (5.5.15)$$

**Corollary 5.5.1** (Green's Formula). Suppose  $R \subset \mathbb{R}^3$  and  $f, g : R \rightarrow \mathbb{R}$  are  $C^2$  functions, then

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} dA = \iiint_S \nabla f \cdot \nabla g + f \nabla^2 g dV \quad (5.5.16)$$

$$\iint_{\partial S} (f \nabla g - g \nabla f) dA = \iiint_S (f \nabla^2 g - g \nabla^2 f) dV \quad (5.5.17)$$

*Proof.*

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} dA = \iiint_S \operatorname{div}(f \nabla g) dA \quad (5.5.18)$$

$$= \iiint_S f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g dV = \iiint_S f \nabla f \cdot \nabla g + \nabla^2 g dV \quad (5.5.19)$$

The second formula can be proved directly using divergence theorem the first formula. ■

## 5.6 Stokes Theorem

### 5.6.1 The Stoke Boundary

**Definition 5.6.1.** Let  $S$  be a surface that is a subset of a smooth (or piecewise smooth) surface  $S_0$ , and the *boundary of  $S$*  is interpreted as the *boundary of  $S$  within  $S_0$* . Then the  $\mathbf{x} \in S_0$  belongs to the **Stoke boundary** of  $S$  within  $S_0$  if

$$\forall r > 0, \mathcal{B}(r, \mathbf{x}) \cap S \neq \emptyset \wedge \mathcal{B}(r, \mathbf{x}) \cap (S_0 \setminus S) \neq \emptyset \quad (5.6.1)$$

### 5.6.2 The Orientation

**Definition 5.6.2.** Given a smooth surface  $S \subset S_0$ , at a point  $\mathbf{x}$  in the Stoke's boundary of  $S$ ,  $\partial S$ , if  $\mathbf{n}$  denotes the **positive orientation normal** to  $S_0$  and  $\mathbf{t}$  denotes the tangent to  $\partial S$  at  $\mathbf{x}$  with the **Stokes orientation**, then

$$\mathbf{n} \times \mathbf{t} \text{ points into } S \quad (5.6.2)$$

**Remark 5.6.1** (Informal Interpretation of Stoke's Orientation). The orientation for  $\partial S$  may be described by saying if someone circles  $\partial S$  with their head pointing in the direction of the unit normal  $\mathbf{n}$ , and with  $S$  on the left, then the Stokes orientation corresponds to the direction in which they travel around  $\partial S$ .



### 5.6.3 Stokes Theorem in $\mathbb{R}^3$

**Theorem 5.6.1** (Stokes Theorem, Special Case). Assume that  $S$  is a piecewise smooth surface in  $\mathbb{R}^3$  with boundary  $\partial S$ , that  $S$  is oriented the unit normal  $\mathbf{n}$  and that  $\partial S$  has the compatible (Stoke) orientation ( $\mathbf{n} \times \mathbf{t}$  points into  $S$ ). Assume also that  $\mathbf{F}$  is any vector field that is  $C^1$  in an open set containing  $S$ . Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x} \quad (5.6.3)$$

**Remark 5.6.2.** In above theorem,  $\omega := \mathbf{F} \cdot d\mathbf{x}$  is a 1-form in  $\mathbb{R}^3$  and  $d\omega := \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA$  is a 2-form in  $\mathbb{R}^3$ .

**Corollary 5.6.1.** Let  $S$  be a closed surface in  $\mathbb{R}^3$ , that's,  $\partial S = \emptyset$ , and let  $\mathbf{n}$  denote the outer normal vector, and  $\mathbf{F}$  is a  $C^1$  vector field. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = 0 \quad (5.6.4)$$

*Proof.* We can construct a *small* simple closed curve  $C$  on  $S$  and divide  $S$  into two regions sharing the same boundary. And note that given orientation fixed on  $S$ , the orientation on  $\partial S_1$  and  $\partial S_2$  are opposite. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_1} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA + \iint_{S_2} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA \quad (5.6.5)$$

$$= \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{x} - \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot d\mathbf{x} - \int_C \mathbf{F} \cdot d\mathbf{x} = 0 \quad (5.6.6)$$

■

**Definition 5.6.3.** A surface  $S \subset \mathbb{R}^3$  is **closed** if it has no Stokes boundary.

**Definition 5.6.4.** A piecewise smooth surface  $S \subset \mathbb{R}^3$  is **closed** if there exists subsets  $S_1, S_2$  such that  $\partial S_1 = \partial S_2 = S_1 \cap S_2$ , also,  $\partial S_1$  and  $\partial S_2$  are piecewise smooth and have opposite orientations, where  $S_1, S_2$  have the same orientation as  $S$ .

**Theorem 5.6.2.** If  $S$  is a piecewise smooth closed surface in  $\mathbb{R}^3$  and  $\mathbf{F}$  is a vector field that is  $C^1$  near  $S$ , then

$$\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dA = 0 \quad (5.6.7)$$

*Proof.*

$$\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dA = \iint_{S_1} \text{curl} \mathbf{F} \cdot \mathbf{n} dA + \iint_{S_2} \text{curl} \mathbf{F} \cdot \mathbf{n} dA = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{x} + \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{x} = 0 \quad (5.6.8)$$

■

**Proposition 5.6.1** (Geometric Interpretation of Curl). Let  $R \subset \mathbb{R}^3$  be a 2 dimensional sub-manifold with  $\mathbf{n}$  as outer normal vector on it, and  $\mathbf{a} \in R$ ,

$$\text{curl}(F)(\mathbf{a}) \cdot \mathbf{n}(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \iint_{\mathcal{D}(\mathbf{a}, r)} \text{curl}(F)(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, dA \quad (5.6.9)$$

$$= \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \int_{\partial \mathcal{D}(\mathbf{a}, r)} \mathbf{F} \cdot d\mathbf{x} \quad (5.6.10)$$

$$(5.6.11)$$

If we think of  $\mathbf{F}$  as a force field, then  $\int_{\partial\mathcal{D}(\mathbf{a},r)} \mathbf{F} \cdot d\mathbf{x}$  represents the work done by  $\mathbf{F}$  on a particle moves around  $\partial\mathcal{D}(\mathbf{a},r)$ . Thus  $\text{curl}(\mathbf{F}) \cdot \mathbf{u}$  represents the **tendency of the force  $\mathbf{F}$  to push the particle around  $\partial\mathcal{D}(\mathbf{a},r)$  in a direction compatible with  $\mathbf{n}$ .**

**Theorem 5.6.3** (Application: Moving the Surface). Let  $S$  and  $S'$  be two oriented surfaces such that  $\partial S = \partial S'$  (have the same orientation), then

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial S'} \mathbf{F} \cdot d\mathbf{x} = \iint_{S'} \text{curl } \mathbf{F} \cdot \mathbf{n} dA \quad (5.6.12)$$

#### 5.6.4 The Generalization(Optional)

**Proposition 5.6.2** (Properties of Exterior Products). Let  $\alpha_1, \alpha_2$  and  $\beta$  be 1-forms on  $\mathbb{R}^n$  and  $f_1, f_2$  are continuous functions defined on  $\mathbb{R}^n$ ,

##### 1. Distributive

$$(f_1\alpha_1 + f_2\alpha_2) \wedge \beta = f_1(\alpha_1 \wedge \beta) + f_2(\alpha_2 \wedge \beta) \quad (5.6.13)$$

$$\beta \wedge (f_1\alpha_1 + f_2\alpha_2) = f_1(\beta \wedge \alpha_1) + f_2(\beta \wedge \alpha_2) \quad (5.6.14)$$

##### 2. Anti-commutative

$$\beta \wedge \alpha = -\alpha \wedge \beta \quad (5.6.15)$$

**Corollary 5.6.2.** By the *anti-commutative* property,

$$\alpha \wedge \alpha = 0 \quad (5.6.16)$$

**Theorem 5.6.4** (Divergence Theorem in  $\mathbb{R}^n$ ). Let  $R$  be a regular region in  $\mathbb{R}^n$  bounded by a piecewise smooth hyper-surface  $\partial R$ . Note here  $R$  is a  $n$  dimensional sub-manifold and  $\partial R$  is a  $n-1$  dimension sub-manifold. Then

$$\int \cdots \int_{\partial R} \mathbf{F} \cdot \mathbf{n} dV^{n-1} = \iint \cdots \int_R \text{div}(\mathbf{F}) dV^n \quad (5.6.17)$$

where if  $\partial R$  is parameterized by  $\mathbf{G}(u_1, \dots, u_{n-1})$ , then

$$\mathbf{n} dV^{n-1} = \det \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \partial_1 G_1 & \cdots & \partial_1 G_n \\ \vdots & & \vdots \\ \partial_{n-1} G_1 & \cdots & \partial_{n-1} G_n \end{pmatrix} \quad (5.6.18)$$

**Definition 5.6.5.** A 0-form on  $\mathbb{R}^n$  is a real valued function  $f$ .

**Remark 5.6.3.** While writing the basis elements  $dx_i \wedge dx_j$  with the variables in *cyclic order*. That's  $dx$  before  $dy$  before  $dz$  before  $dx$  in  $\mathbb{R}^3$  case.

**Definition 5.6.6.** A  $k$ -form in  $\mathbb{R}^n$  takes the expression of linear combination of  $C(n, k)$  basis elements  $\{\beta_i\}_i$ .

**Example 5.6.1.** A 2-form  $\omega$  in  $\mathbb{R}^3$  can be expressed using a 3-element basis

$$\omega = \sum_{1 \leq i < j \leq 3} C_{ij}(\mathbf{x}) \beta_{ij} \quad (5.6.19)$$

$$\beta_{ij} \in \{dx \wedge dy, dy \wedge dz, dx \wedge dz\} \quad (5.6.20)$$

**Definition 5.6.7.** Let  $\omega = \sum_{j=1}^{C(n,k)} f_j \beta_j$  be a  $k$ -form in  $\mathbb{R}^n$ , then its **exterior derivative** is defined to be the  $(k+1)$ -form in  $\mathbb{R}^n$  defined as

$$d\omega := \sum_j df_j \wedge \beta_j \quad (5.6.21)$$

where  $df_j$  can be computed using *total derivative*.

**Example 5.6.2.** In  $\mathbb{R}^3$ , the *exterior derivative* for a 0-form  $f$  is its **gradient**, which is a 1-form. And the exterior derivative of a 1-form in  $\mathbb{R}^3$  is its curl

$$\omega := F_1 dx + F_2 dy + F_3 dz \quad (5.6.22)$$

$$\implies d\omega = dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \quad (5.6.23)$$

$$= (\partial_1 F_1 dx + \partial_2 F_1 dy + \partial_3 F_1 dz) \wedge dx \quad (5.6.24)$$

$$+ (\partial_1 F_2 dx + \partial_2 F_2 dy + \partial_3 F_2 dz) \wedge dy \quad (5.6.25)$$

$$+ (\partial_1 F_3 dx + \partial_2 F_3 dy + \partial_3 F_3 dz) \wedge dz \quad (5.6.26)$$

$$= (\partial_1 F_2 - \partial_2 F_1) dx \wedge dy + (\partial_2 F_3 - \partial_3 F_2) dy \wedge dz + (\partial_3 F_1 - \partial_1 F_3) dz \wedge dx \quad (5.6.27)$$

$$= \text{curl}(\mathbf{F}) \quad (5.6.28)$$

The exterior derivative of a 2-form in  $\mathbb{R}^3$  is its divergence

$$\omega := A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \quad (5.6.29)$$

$$\implies d\omega = (\partial_1 A dx + \partial_2 A dy + \partial_3 A dz) \wedge dy \wedge dz \quad (5.6.30)$$

$$+ (\partial_1 B dx + \partial_2 B dy + \partial_3 B dz) \wedge dz \wedge dx \quad (5.6.31)$$

$$+ (\partial_1 C dx + \partial_2 C dy + \partial_3 C dz) \wedge dx \wedge dy \quad (5.6.32)$$

$$= (\partial_1 A + \partial_2 B + \partial_3 C) dx \wedge dy \wedge dz \quad (5.6.33)$$

$$= \text{div}(\mathbf{F}) \quad (5.6.34)$$

**Theorem 5.6.5** (Stokes Theorem, 5.77). Let  $M$  be a smooth, oriented  $k$  dimensional sub-manifold of  $\mathbb{R}^n$  with a piecewise smooth boundary  $\partial M$ , and let  $\partial M$  carry the orientation that is (in a suitable sense) compatible with the one on  $M$ . If  $\omega$  is a  $(k-1)$ -form of class  $C^1$  on an open set containing  $M$ , then

$$\int \cdots \int_{\partial M} \omega = \iint \cdots \int_M d\omega \quad (5.6.35)$$

**Theorem 5.6.6.** The *boundary* of a (smoothly bounded) region  $M$  in a  $k$  dimensional manifold is a  $(k-1)$  dimensional manifold with no boundary.

That's let  $M$  be a  $k$  dimensional manifold with piecewise smooth boundary  $\partial M$ , then

$$\partial(\partial M) = \emptyset \quad (5.6.36)$$

**Theorem 5.6.7.** For any  $k$ -form  $\omega$  on  $\mathbb{R}^n$ ,

$$d(d\omega) = 0 \quad (5.6.37)$$

*Proof.* Let  $M$  be a  $k$  dimensional sub-manifold in  $\mathbb{R}^n$  with piecewise smooth boundary, and  $\omega$  is a  $(k-2)$ -form on  $\mathbb{R}^n$ , so  $d(d\omega)$  is a  $k$ -form on  $\mathbb{R}^n$ . And

$$\iiint \cdots \int_M d(d\omega) = \iint \cdots \int_{\partial M} d\omega \quad (5.6.38)$$

$$= \int \cdots \int_{\partial(\partial M)} \omega = \int \cdots \int_{\emptyset} \omega = 0 \quad (5.6.39)$$

■

## 5.7 Vector Fields that are Gradients or Curls

### 5.7.1 Conservative Vector Field

**Definition 5.7.1.** A continuous vector field  $\mathbf{G}$  defined on open  $U \subset \mathbb{R}^n$  is **conservative** if there exists a function  $f : U \rightarrow \mathbb{R}$  of class  $C^1$  such that  $\mathbf{G} = \nabla f$ .

**Theorem 5.7.1.** Let  $U \subset \mathbb{R}^n$  open, and let  $\mathbf{G} : U \rightarrow \mathbb{R}^n$  be continuous. Then the following are equivalent:

- (i) There exists a function  $f : U \rightarrow \mathbb{R}$  of class  $C^1$  such that  $\mathbf{G} = \nabla f$ ;
- (ii)  $\int_C \mathbf{G} \cdot d\mathbf{x} = 0$  for any closed piecewise smooth oriented curve  $C \subset U$ ;
- (iii)  $\int_{C_1} \mathbf{G} \cdot d\mathbf{x} = \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$  for any two piecewise smooth oriented curves  $C_1, C_2$  in  $U$  that shares starting and ending points.

*Proof.*

(i)  $\implies$  (ii) by the *Fundamental Theorem of Line Integrals*.

(ii)  $\implies$  (iii) Given  $C_1$  and  $C_2$  both from  $\mathbf{p}$  to  $\mathbf{q}$ . Constructing  $C_2'$  from  $\mathbf{q}$  to  $\mathbf{p}$  by reversing the orientation of  $C_2$ . And  $C_1 \cup C_2'$  form a closed curve.

$$\int_C \mathbf{G} \cdot d\mathbf{x} = \int_{C_1} \mathbf{G} \cdot d\mathbf{x} - \int_{C_2} \mathbf{G} \cdot d\mathbf{x} \quad (5.7.1)$$

(iii)  $\implies$  (i), pick an arbitrary point  $\mathbf{p} \in U$ , and for every  $\mathbf{q} \in U$ , define  $f$  as

$$f(\mathbf{q}) = \int_{C_{\mathbf{p},\mathbf{q}}} \mathbf{G} \cdot d\mathbf{x} \quad (5.7.2)$$

where  $C_{\mathbf{p},\mathbf{q}}$  is an arbitrary curve starting at  $\mathbf{p}$  and ending at  $\mathbf{q}$ . ■

**Theorem 5.7.2.** If  $\mathbf{G}$  is a conservative vector field of class  $C^1$  on an open set  $U \subset \mathbb{R}^3$ , then  $\text{curl} \mathbf{G} = \mathbf{0}$ .

**Theorem 5.7.3** (the Converse). If  $U \subset \mathbb{R}^3$  is convex and if  $\mathbf{G}$  is a  $C^1$  vector field defined on  $U$  and  $\text{curl} \mathbf{G} = \mathbf{0}$  on  $U$ , then  $\mathbf{G}$  is conservative.

**Remark 5.7.1.** That's possible for a non-conservative vector field  $\mathbf{G}$  defined on some non-convex set to have  $\text{curl} \mathbf{G} = \mathbf{0}$

**Proposition 5.7.1** (Finding  $f$  for Conservative Vector Fields). If the domain of a conservative vector field  $\mathbf{G}$  is convex, then we can construct  $f$  following

- (i) Choose  $\mathbf{p} = (a, b, c) \in U$ ;
- (ii) Define  $f(x, y, z)$  as

$$f(x, y, z) := \int_a^x G_1(t, b, c) dt + \int_b^y G_2(x, t, c) dt + \int_c^z G_3(x, y, t) dt \quad (5.7.3)$$

**Corollary 5.7.1.** If the domain of  $\mathbf{G}$  is a convex set containing the origin, then we can define  $\mathbf{p} = (a, b, c) = (0, 0, 0)$ , and use a straight line  $\mathbf{g}(t) : [0, 1] \rightarrow \text{Domain}(\mathbf{G}) := (xt, yt, zt)$ :

$$f(x, y, z) = \int_0^1 \mathbf{G}(tx, ty, tz) \cdot (x, y, z) dt \quad (5.7.4)$$

### 5.7.2 Vector Fields that are Curls

**Theorem 5.7.4.** If  $\mathbf{G} : U \rightarrow \mathbb{R}^3$  is a vector field of class  $C^1$  and  $\mathbf{G} = \text{curl}\mathbf{F}$  for some vector field  $\mathbf{F} : U \rightarrow \mathbb{R}^3$  of class  $C^2$ , then  $\text{div}\mathbf{G} = 0$ .

**Theorem 5.7.5.** Suppose  $\mathbf{G}$  is a  $C^1$  vector field in an open set  $U \subset \mathbb{R}^3$  such that  $\text{div}\mathbf{G} = 0$ . If  $U$  is convex, then there exists a vector field  $\mathbf{F}$  such that  $\text{curl}\mathbf{F} = \mathbf{G}$ .

**Definition 5.7.2.** Given a closed curve  $C \subset \mathbb{R}^3$ , the **minimal surface** is the surface  $S$  with the smallest possible area, given that  $\partial S = C$ .

## 5.8 Summary

### 5.8.1 Line Integrals

$$\text{Arc length} \quad \int_C 1 \, d\mathbf{x} = \int_a^b \|\mathbf{g}'(t)\| \, dt \quad (5.8.1)$$

$$\text{Integral of scalar valued function on curve} \quad \int_C f(\mathbf{x}) \, d\mathbf{x} = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt \quad (5.8.2)$$

$$\text{Integral of vector field on curve} \quad \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt \quad (5.8.3)$$

## 5.9 Surface Integrals

$$\text{Surface area} \quad \iint_S 1 \, dA = \iint_W \|\partial_u \mathbf{G} \times \partial_v \mathbf{G}\| \, dudv \quad (5.9.1)$$

$$\text{Scalar function on surface} \quad \iint_S f(\mathbf{x}) \, dA = \iint_W f(\mathbf{G}(\mathbf{u})) \|\partial_u \mathbf{G} \times \partial_v \mathbf{G}\| \, dudv \quad (5.9.2)$$

$$\text{Vector field on surface} \quad \iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} \, dA = \iint_W \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\partial_u \mathbf{G} \times \partial_v \mathbf{G}) \, dudv \quad (5.9.3)$$