ECO326 Advanced Microeconomic Theory
A Course in Game Theory

Tianyu Du

December 26, 2018
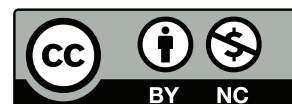
Github Page https://github.com/TianyuDu/Spikey_UofT_Notes
Note Page TianyuDu.com/notes

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory*.

Contents

1 Lecture 1. Games and Dominant Strategies

Assumption 1.1 (pg.4)**.** Assume that each decision-maker is *rational* in the sense that he is aware of his alternatives, forms expectation about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization.

Definition 1.1 (pg.4)**.** A model of **rational choice** consists

- A set A of *actions.*

- A set C of *consequences.*

- A *consequence function* $g : A \to C$.

- A *preference relation* $\succsim$ on C.

Definition 1.2 (pg.7)**.** A **preference relation** is a <u>complete reflexive and transitive</u> binary relation.

Definition 1.3 (11.1)**.** A **strategic game** consists of

- a <u>finite</u> set of **players** N.

- for each player $i \in N$, an **actions** $A_i \neq \emptyset$.

- for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i) \rangle$.

Definition 1.4 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2

2 Lecture 2. Iterated Elimination and Rationalizability

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (60.2)**.** The set $X \subseteq A$ of outcomes of a $\underline{\text{finite}}$ strategic game $\langle N, (A_i), (u_i) \rangle$ **survives iterated elimination of** $\underline{\text{strictly}}$ **dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $\overline{((X_j^t)_{j \in N})_{t=0}^T}$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.

- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \dots, T - 1$.

- For each $t = 0, \dots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is $\underline{\text{strictly}}$ $\underline{\text{dominated}}$ in the game $\langle N, (X_i^t), (u_i^t) \rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.

- No action in X_t^T is strictly dominated in game $\langle N, (X_i^T), (u_i^T) \rangle$.

Proposition 2.1 (61.2)**.** If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a $\underline{\text{finite}}$ strategic game $\langle N, (A_i), (u_i) \rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

2.2 Rationalizability

Definition 2.2 (pg.15)**.** The **best-response function** for a player i is defined as

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i}) \; \forall a_i' \in A_i\}$$

Remark 2.1. The best-response of a_{-i} can be written as

$$B_i(a_{-i}) = \bigcap_{a_i' \in A_i} \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i})\}$$

where each of them is the upper contour set of a_i'.
Thus, if $\succsim_i$ is quasi-concave, then $B_i(a_{-i})$ is an intersection of convex sets and therefore itself convex.

Definition 2.3 (pg.54)**.** A **belief** of player i (about the actions of the other players) is a $\underline{\text{probability measure}}$, μ_i, on $A_{-i} = \times_{j \in N \backslash \{i\}} A_j$.
μ_i is a mapping such that

- $\mu_i : A_{-i} \to [0, 1]$.

- $\mu_i(A_{-i}) = 1$.

- For all countable piece-wise <u>disjoint</u> collection $\{E_i\}_{i \in I}$, it satisfies the *countable additivity property*:

$$\mu_i(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu_i(E_i)$$

Definition 2.4 (lec.2)**.** For a player $i \in N$, $a_i^* \in A_i$ is the **best response to belief** μ_i in a strategic game $\langle N, (A_i), (u_i) \rangle$ if and only if

$$\forall a_i \in A_i, \quad \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i})$$

Equivalently,

$$\forall a_i \in A_i, \ \mathbb{E}[u_i(a_i^*, a_{-i}) | \mu_i] \geq \mathbb{E}[u_i(a_i, a_{-i}) | \mu_i]$$

Definition 2.5 (59.1)**.** An action of player i in a strategic game is a **never best response** if it is not a best response to any <u>belief</u> of player i.

Definition 2.6 (lec.2)**.** For player $i \in N$, action $a_i \in A_i$ is **rationalizable** if it survives from the iterated elimination of never best responses.

Definition 2.7 (59.2)**.** The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **strictly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

3   Lecture 3. Nash Equilibrium

Definition 3.1 (14.1)**.** A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $a^* \in A$ of actions with property that for every player $i \in N$

$$(a_i^*, a_{-i}^*) \succsim_i (a_i, a_{-i}^*) \ \forall a_i \in A_i$$

Proposition 3.1 (pg.15, equivalent definition of Nash equilibrium)**.** So a Nash equilibrium is a profile $a^* \in A$ such that

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

Proposition 3.2 (lec.3)**.** No strategy that is eliminated during iterated deletion of never best response can be played in Nash equilibrium.

Lemma 3.1 (pg.19)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if equivalent to the following statement:
Define set-valued function $B : A \to A$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and there exists $a^* \in A$ such that $a^* \in B(a^*)$.

Lemma 3.2 (20.1 Kakutani's fixed point theorem)**.** Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- for all $x \in X$ the set $f(x)$ is non-empty and convex.

- the graph of f is closed. *(i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$ then $y \in f(x))$*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Definition 3.2 (pg.20)**.** A preference relation $\succsim_i$ over A is quasi-concave on A_i if for every $a^* \in A$ the upper contour set over a_i^*, given other players' strategies

$$\{a_i \in A_i : (a_{-i}^*, a_i) \succsim_i a^*\}$$

is convex.

Proposition 3.3 (20.3)**.** The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if for all $i \in N$,

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous

- quasi-concave on A_i.

Proof. Let $B : A \to A$ be a correspondence defined as

$$B(a) := \times_{i \in N} B_i(a_{-i})$$

Note that for each $a \in A$ and for each $i \in N$,
$B_i(a_{-i}) \neq \emptyset$ since preference $\succsim_i$ is continuous and A_i is compact (EVT).

Also $B_i(a_{-i})$ is convex since it's basically an intersection of upper contour sets and each of those upper contour is convex since $\succsim_i$ is quasi-concave.

So the Cartesian product of the finite collection of B_i is non-empty and convex.

Also the graph B is closed since $\succsim_i$ is continuous.

So there exists $a^* \in A$ such that $a^* \in B(a^*)$.

So Nash equilibrium presents. ∎

Definition 3.3 (lec.3). A **strict Nash equilibrium** is an action profile $a^* \in A$ where all players are playing their <u>unique</u> best response. That is, for every player $i \in N$, the image of their best response $B_i(a^*_{-i})$ is singleton,

$$\forall i \in N \ B_i(a^*_{-i}) = \{a^*_i\}$$

Definition 3.4 (lec.3). Otherwise, a Nash equilibrium is a **weak Nash equilibrium**.

4 Lecture 4. Nash Equilibrium: Examples

5 Lecture 5. Mixed Strategies

Notation 5.1 (pg.32). Let $\Delta(A_i)$ denote the <u>set of probability measures/distributions</u> on set A_i.

Definition 5.1 (lec.5). For player $i \in N$, a **mixed strategy** σ_i is a member in $\Delta(A_i)$ and it is a <u>probability distribution</u> over A_i.

Remark 5.1 (lec.5). A pure strategy $a_i \in A_i$ is a mixed strategy with

$$\sigma_i(a_i) = 1$$

So mixed strategy is a generalization of pure strategy.

Definition 5.2 (pg.32). A profile $(\sigma_j)_{j \in N}$ of mixed strategies induces a probability distribution over the set A.

Proposition 5.1 (pg.32). In a finite game,(i.e., each A_i is finite), then given the independence of randomization, the probability of the action profile $a = (a_j)_{j \in N}$ to be realized given mixed strategy profile $(\sigma_j)_{j \in N}$ is

$$Pr((a_j)_{j \in N}) = \prod_{j \in N} \sigma_j(a_j)$$

6

and for player i, the **expected payoff** on profile $(\sigma_j)_{j \in N}$ is

$$U_i((\sigma_j)_{j \in N}) = \sum_{a \in A} (\prod_{j \in N} \sigma_j(a_j)) u_i(a) = \mathbb{E}[u_i(a) | (\sigma_j)_{j \in N}]$$

Proposition 5.2 (lec.5, equivalent)**.** The **expected payoff** from mixed strategy profile $(\sigma_i) \equiv (\sigma_i, \sigma_{-i})$ is

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}[u_i(a) | (\sigma_i)] = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \sigma_{-i}(a_{-i}) \sigma_i(a_i)$$

Definition 5.3 (32.1)**.** The **mixed extension** of the strategic game $\langle N, (A_i), (u_i) \rangle$ is the strategic game $\langle N, (\Delta(A_i)), (U_i) \rangle$ in which $\Delta(A_i)$ is the set of probability distributions over A_i and $U_i : \times_{j \in N} \Delta(A_i) \to \mathbb{R}$ assigns to each $(\sigma_i)_{i \in N} \in \times_{j \in N} \Delta(A_i)$ the expected value under u_i of the lottery over A that is induced by $(\sigma_i)_{i \in N}$.

Remark 5.2 (pg.32, notes on above definition)**.** If the game is finite, that is, for each $i \in N$, the set A_i is finite, then

$$U_i(\sigma) = \sum_{a \in A} (\prod_{j \in N} \sigma_j(a_j)) u_i(a)$$

Definition 5.4 (32.3)**. A mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension.

Proposition 5.3 (33.1)**.** Every finite strategic game has a mixed strategy Nash equilibrium.

Lemma 5.1 (33.2)**.** Let $G = \langle N, (A_i), (u_i) \rangle$ be a finite strategic game. Then $\sigma^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G is and only if for every player $i \in N$ every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*

Assumption 5.1 (lec.5)**.** Assuming all agents follows Von-Neumann Morgenstern theorem.

Definition 5.5 (lec.5)**.** An action a_i is **strictly dominated** by mixed strategy σ_i if and only if

$$\forall a_{-i} \in A_{-i} \ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i})$$

where σ_i could be a pure strategy.

Definition 5.6 (lec.5). A mixed strategy σ_i is a **best response** to σ_{-i} if and only if

$$\forall \sigma_i' \in \Delta(A_i) \ U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

Definition 5.7 (lec.5). The **support** of a mixed strategy $\sigma_i \in \Delta(A_i)$ is the set

$$supp(\sigma_i) = \{a_i \in A_i : \sigma_i(a_i) \neq 0\}$$

Proposition 5.4 (lec.5). A mixed strategy σ_i is a **best response** to an strategy profile σ_{-i} if and only if

(a) Player i is indifferent between all a_i in the support of σ_i,

$$\forall a_j, a_j' \in supp(\sigma_i) \quad a_j \sim_i a_j'$$

(b) <u>and</u> player i weakly prefers all actions in the support of σ_i to those not in the support of σ_i. That's

$$\forall a_j \in supp(\sigma_i), \ \forall a_j' \notin supp(\sigma_i) \quad a_j \succsim_i a_j'$$

Proof. ($\implies$) show the if parts by proving it's contraposition.
Suppose (a) is not true, then

$$\exists a_i, a_j \in supp(\sigma_i) \ s.t. \ a_i \nsim_i a_j$$

WLOG, suppose

$$u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

then σ_i would not be the best response since we can refine it by assigning

$$\begin{cases} \sigma_i'(a_i) = \sigma_i(a_i) + \sigma_i(a_j) \\ \sigma_i'(a_j) = 0 \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

and σ_i' would provides higher expected payoff.
Suppose (b) does not hold,

$$\exists a_i \notin supp(\sigma_i) \ s.t. \ \exists a_j \in supp(\sigma_i) \ s.t \ u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

Then σ_i could not be a best response since we can construct another mixed strategy σ_i' strictly dominating σ_i by setting

$$\begin{cases} \sigma_i'(a_j) = 0 \\ \sigma_i'(a_i) = \sigma_i(a_j) \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

8

$(\Longleftarrow)$ Assuming σ_i is not a best response towards σ_{-i},
then there exists $\sigma_i' \in \Delta(A_i)$ such that

$$U_i(\sigma_i', \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i})$$
$$\Longleftrightarrow \mathbb{E}[u_i(a)|(\sigma_i', \sigma_{-i})] > \mathbb{E}[u_i(a)|(\sigma_i, \sigma_{-i})]$$
$$\Longleftrightarrow \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i'(a_i)\sigma_{-i}(a_{-i}) > \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i})$$

Probability measures σ_i and σ_i' could only be different in two aspects, their supports and the values assigned on elements in their supports, this fails assumption (a).
The following argument needs to be revised.
Case 1 suppose $supp(\sigma_i) = supp(\sigma_i')$, then the strictly inequality in expected payoffs implies redistributing probabilities does affect the expected payoffs. So player i cannot be indifferent between any two actions in the support.
Case 2 suppose $supp(\sigma_i) \neq supp(\sigma_i')$ and $supp(\sigma_i') \not\subseteq supp(\sigma_i)$. That's

$$\exists a_i \in supp(\sigma_i') \wedge \notin supp(\sigma_i)$$

Then extending the support to a_i of σ_i gives higher expected payoff, this fails the assumption (b).
Case 3 suppose $supp(\sigma_i') \subsetneq supp(\sigma_i)$. Then the expected payoff can be strictly increased by eliminating actions in $supp(\sigma_i)\backslash supp(\sigma_i')$. Then those actions eliminated must be strictly dominated by actions in $supp(\sigma_i')$. This fails assumption (a). ∎

Proposition 5.5 (lec.5 equivalent proposition)**.** All actions in the support are best responses. (i.e.*best response mixed strategy is a mixture of best response pure actions*)

Remark 5.3 (lec.5 Intuition of proposition)**.** If the requirements of above proposition are not satisfied, the player can reduce the probability assigned to the non-best-response pure action and better off.

Theorem 5.1 (lec.5 Nash's Theorem)**.** Any player $i \in N$ in finite game $\langle N, (A_i), (\succsim_i) \rangle$ has a mixed strategy Nash equilibrium.

6 Lecture 6. Extensive Form Games and Subgame Perfection

6.1 Extensive Form Game

Definition 6.1 (89.1)**.** An **extensive game with perfect information** has the following components.

- A set N of **players**.

- A set H of sequences (finite or infinite) of **histories** with properties:

  - $\emptyset \in H$.
  - For all $L < K$, $(a^k)_{k=1,2,...,K} \in H \implies (a^k)_{k=1,2,...,L} \in H$.
  - For infinite sequence $(a^k)_{k=1}^{\infty}$,
    $(a^k)_{k=1,2,...,L} \in H, \ \forall L \in \mathbb{Z}_{++} \implies (a^k)_{k=1}^{\infty} \in H$.

  And each component of history $h \in H$ is an **action** taken by a player.

- A function $P : H\backslash Z \to N$, where for $h \in H$, $P(h) \in N$ is defined by the player who takes an action <u>after</u> the history h.

- For each player $i \in N$ a **preference relation** $\succsim_i$ <u>defined on Z</u>.

Notation 6.1 (pg.90)**.** *An extensive game with perfect information can be represented by a 4-tuple, $\langle N, H, P, (\succsim_i) \rangle$. Sometimes it is convenient to specify the structure of an extensive game without specifying the players' preference, as $\langle N, H, P \rangle$.*

Definition 6.2 (pg.90)**.** A history $(a^k)_{k=1,2,...,K} \in H$ is **terminal** if

1. it is infinite,

2. <u>or</u> (i.e. it cannot be extended to another valid history sequence)

$$\forall a^{K+1}, \ (a^k)_{k=1,2,...,K+1} \notin H$$

The set of terminal histories is denoted by Z.

Notation 6.2 (pg.90, the action set)**.** After any nonterminal history, $h \in H\backslash Z$, the player $P(h)$ chooses an action from set

$$A(h) = \{a : (h, a) \in H\}$$

Remark 6.1. Note that all player function, action set and player preference relation are defined on H. Thus, unlike a normal form game, which was *player oriented*, we'd better consider an extensive form game as *history oriented*.

Definition 6.3 (pg.90)**.** We refer to the empty set, which is required to be an element of H, as the **initial history**.

Definition 6.4 (92.1)**.** A **strategy of player** $i \in N$, s_i, in an extensive game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Remark 6.2 (pg.92)**.** A strategy specifies the action chosen by a player for *every* history after which it is his turn to move, *even for histories that is, if the strategy is followed, are never reached.*

Definition 6.5 (pg.93)**.** For each strategy profile $s = (s_i)_{i \in N}$ in the extensive game $\langle N, H, P, (\succsim_i) \rangle$, the **outcome** of s, $O(s)$, is defined as the terminal history that results when each player $i \in N$ follows the precepts of s_i. That is, $O(s)$ is the (possibly infinite) history

$$(a^1, \ldots, a^K) \in Z$$

such that

$$\forall k \in \{0, 1, \ldots K - 1\}, \ s_{P(a^1, \ldots, a^k)}(a^1, \ldots, a^k) = a^{k+1}$$

Definition 6.6 (lec.6)**.** A extensive game $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is finite if and only if

(a) N is finite.

(b) (A_i) are all finite.

(c) All $h \in H$ reach the terminal state with finite length.

Definition 6.7 (93.1)**.** A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$\forall s_i \in S_i, \ O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

Definition 6.8 (94.1)**.** The **strategic form of the extensive game with perfect information**, $\Gamma = \langle N, H, P, (\succsim_i) \rangle$, is the strategic game $\langle N, (S_i), (\succsim'_i) \rangle$ in which for each player $i \in N$

- S_i is the **set of strategies** of player i in Γ.

- $\succsim_i'$ is defined on $\times_{i \in N} S_i$ and defined by

$$\forall s, s' \in \times_{i \in N} S_i, \ s \succsim_i' s' \iff O(s) \succsim_i O(s')$$

Definition 6.9 (pg.94). A **reduced strategy** of player i is defined to be a function f_i whose domain is a *subset* of $\{h \in H : P(h) = i\}$ and has the following properties

1. it associates with every history h in the domain of f_i an action in $A(h)$.

2. a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i. (i.e., for any $h = (a^k)$ and for any $h' = (a^k)_{k=1}^L$ as a subsequence of h such that $P(h') = i$, $f_i(h') = a^{L+1}$.)

Remark 6.3 (pg.94). Each **reduced strategy** of player i corresponds to a <u>set of strategies of player i</u>, such that for each vector of strategies of the <u>other players</u> each strategy in this set yields the <u>same outcome</u>. (strategies in the same set are **outcome-equivalent**.)
That's, for each strategy $s_i \in S_i$, its reduced strategy can be defined with an outcome equivalence class, $[s_i]$,

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j, \ O(s_{-i}, s_i) \textcolor{red}{=} O(s_{-i}, s_i')\}$$

But in some other game, the definition of outcome-equivalence is more general and defined by <u>generating the same payoff</u> (through possibly difference outcomes), then the reduced strategy is defined as

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j, \ \forall j \in N, \ O(s_{-i}, s_i) \textcolor{red}{\sim_j} O(s_{-i}, s_i')\}$$

Definition 6.10 (95.1.1). Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim_i') \rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i, s_i' \in S_i$ to be **equivalent** if

$$\forall s_{-i} \in S_{-i}, \ \forall j \in N, \ (s_{-i}, s_i) \sim_j' (s_{-i}, s_i')$$

Definition 6.11 (95.1.2). The **reduced strategic form of** Γ is the strategic game $\langle N, (S_i'), (\succsim_i'') \rangle$ in which for each $i \in N$ each set S_i' contains one member of each set of equivalent strategies in S_i and $\succsim_i''$ is the preference ordering over $\times_{j \in N} S_j'$ induced by $\succsim_i'$.

6.2 Subgame Perfection

Definition 6.12 (97.1). The **subgame of extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h) \rangle$ where

- $H|_h$ is the set of sequences h' such that $(h, h') \in H$.

- $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$.

- $\succsim_i |_h$ is defined by $h' \succsim_i |_h h'' \iff (h, h') \succsim_i (h, h'') \in Z$.

Notation 6.3 (pg.97). Given strategy $s_i \in S_i$ and $h \in H \in \Gamma$, $s_i|_h$ represents the **strategy that s_i induces in the subgame** $\Gamma(h)$. That's, for each $h' \in H_h$

$$s_i|_h(h') \equiv s_i(h, h')$$

Notation 6.4. Let O_h denote the **outcome function of** $\Gamma(h)$, that's, for all $h' \in H|_h$,

$$O_h(h') \equiv O(h, h')$$

Definition 6.13 (97.2). A **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i |_h O_h(s^*_{-i}|_h, s_i|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.

Definition 6.14 (pg.97). Equivalently, define SPNE to be a strategy profile s^* in Γ for which for any history $h \in H$ the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Remark 6.4 (pg. 97). The notion of SPNE requires the action prescribed by each player's strategy to be optimal, given other players' strategies, after *every* history.

Proposition 6.1 (99.2). Every finite extensive game with perfect information has a subgame perfect equilibrium.