

MAT237: Multivariable Calculus

Tianyu Du

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Contents

1 Limits, continuity, and related topics

1.1 Open and Closed Sets, Boundary, Interior and Closure

Definition 1.1.1. Let $\mathbf{a} \in \mathbb{R}^n$, and $r > 0$. The **open ball with centre \mathbf{a} and radius r** is defined as

$$\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\}. \quad (1.1.1)$$

Definition 1.1.2. The **sphere with centre \mathbf{a} and radius r** is defined as

$$\partial\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| = r\} \quad (1.1.2)$$

Definition 1.1.3. Let $S \subset \mathbb{R}^n$, S is **bounded** if

$$\exists r > 0 \text{ s.t. } S \subset \mathcal{B}(r, \mathbf{0}) \quad (1.1.3)$$

Definition 1.1.4. Let $S \subset \mathbb{R}^n$, then the **complement** of S is defined as

$$S^c := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin S\} \quad (1.1.4)$$

Definition 1.1.5. Let $S \subset \mathbb{R}^n$, the **interior** of S is defined as

$$S^{int} := \{\mathbf{x} \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ s.t. } \mathcal{B}(\varepsilon, \mathbf{x}) \subset S\} \quad (1.1.5)$$

Definition 1.1.6. The **boundary** of S is defined as

$$\partial S := \{\mathbf{x} \in \mathbb{R}^n : \forall \varepsilon > 0 \mathcal{B}(\varepsilon, \mathbf{x}) \cap S \neq \emptyset \wedge \mathcal{B}(\varepsilon, \mathbf{x}) \cap S^c \neq \emptyset\} \quad (1.1.6)$$

Theorem 1.1.1. A point $\mathbf{x} \in S$ is either a *boundary point* or a *interior point*.

Definition 1.1.7. The **closure** of S is defined as

$$\overline{S} := S^{int} \cup \partial S \quad (1.1.7)$$

Theorem 1.1.2. For any $S \subset \mathbb{R}^n$

$$S^{int} \subset S \subset \overline{S} \quad (1.1.8)$$

Theorem 1.1.3. For any $S \subset \mathbb{R}^n$

$$\partial S = \partial(S^c) \quad (1.1.9)$$

Definition 1.1.8. A set $S \subset \mathbb{R}^n$ is **open** if $S = S^{int}$. S is **closed** if $S = \overline{S}$.

Theorem 1.1.4.

$$S \text{ is closed} \iff S^c \text{ is open} \quad (1.1.10)$$

Proof.

$$S \text{ is closed} \iff \partial S \subset S \iff \partial(S^c) \subset S \quad (1.1.11)$$

$$\iff \text{no point of } S^c \text{ is a boundary point} \iff S^c \text{ is open} \quad (1.1.12)$$

■

1.2 Limits and Continuity

1.3 Limits of Multivariable Functions

Definition 1.3.1. Let $S \subset \mathbb{R}^n$, $\mathbf{f}: S \rightarrow \mathbb{R}^k$, and $\mathbf{a} \in S$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^k \quad (1.3.1)$$

is defined as

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \mathbf{x} \in S, 0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon \quad (1.3.2)$$

For this definition to be non-trivial, we need \mathbf{a} not be an isolated point,

$$\forall \delta > 0, \exists \mathbf{x} \in S \text{ s.t. } \|\mathbf{x} - \mathbf{a}\| \in (0, \delta) \quad (1.3.3)$$

Theorem 1.3.1 (Limit Laws). Let $S \subset \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$ satisfying (1.3.3). And $f, g: S \rightarrow \mathbb{R}$, $L, M \in \mathbb{R}$ such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L \quad (1.3.4)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = M \quad (1.3.5)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) + g(\mathbf{x})] = L + M \quad (1.3.6)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) \cdot g(\mathbf{x})] = LM \quad (1.3.7)$$

Theorem 1.3.2 (Squeeze Theorem). Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$ satisfies (1.3.3). Suppose that $f, g, h: S \rightarrow \mathbb{R}$ and there exists $p > 0$ and $L \in \mathbb{R}$ such that

$$\forall \mathbf{x} \in S \cap \mathcal{B}(p, \mathbf{a}) \quad f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x}) \quad (1.3.8)$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = L \quad (1.3.9)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = L \quad (1.3.10)$$

Corollary 1.3.1. Let $g, h: S \rightarrow \mathbb{R}$ and

$$|g(\mathbf{x})| \leq h(\mathbf{x}) \quad \forall \mathbf{x} \in S \quad (1.3.11)$$

$$\text{and } \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = 0 \quad (1.3.12)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = 0 \quad (1.3.13)$$

Theorem 1.3.3. Assume that $S \subset \mathbb{R}^n$ and let $\mathbf{a} \in \mathbb{R}^n$ satisfying (1.3.3). Let $\mathbf{f}: S \rightarrow \mathbb{R}^k$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \iff \lim_{\mathbf{x} \rightarrow \mathbf{a}} f_j(\mathbf{x}) = L_j \quad \forall j \quad (1.3.14)$$

1.3.1 Continuity

Definition 1.3.2. Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \rightarrow \mathbb{R}^k$. \mathbf{f} is **continuous at** $\mathbf{a} \in S$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) \quad (1.3.15)$$

and \mathbf{f} is **continuous** if \mathbf{f} is continuous at every point in S .

Theorem 1.3.4 (Basic Properties of Continuity). Assume that $S \subset \mathbb{R}^n$ and $\mathbf{a} \in S$,

- (i) If $\mathbf{f} : S \rightarrow \mathbb{R}^k$ is continuous at \mathbf{a} , then every component of \mathbf{f} , $f_j : S \rightarrow \mathbb{R}$, is continuous at \mathbf{a} .
- (ii) If $\mathbf{f}, \mathbf{g} : S \rightarrow \mathbb{R}^k$ are continuous at \mathbf{a} , then $\mathbf{f} + \mathbf{g}$ is continuous at \mathbf{a} .
- (iii) If $f, g : S \rightarrow \mathbb{R}$ continuous, then fg is continuous and $\frac{f}{g}$ is continuous given $g(\mathbf{a}) \neq 0$.
- (iv) A composition of continuous functions is continuous.
- (v) The elementary functions of a single variable (trigonometric functions and their inverses, polynomials, exponential and log) are continuous on their domains.

1.3.2 Continuous Functions and Open Sets

Theorem 1.3.5. Assume that $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$, then the following are equivalent

- (i) \mathbf{f} is continuous;
- (ii) For every open set $\mathcal{O} \subset \mathbb{R}^k$, $\mathbf{f}^{-1}(\mathcal{O})$ is also open;
- (iii) For every closed set $\mathcal{C} \subset \mathbb{R}^k$, $\mathbf{f}^{-1}(\mathcal{C})$ is also closed.

1.4 Sequences and Completeness

1.5 Compactness

1.6 the Intermediate Value Theorem

2 Differentiation and related topics

2.1

2.2

2.3

2.4

2.5

2.6

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2.8 Optimization

Theorem 2.8.1. Let $S \subset \mathbb{R}^n$ be an open set and $f, g : S \rightarrow \mathbb{R}$ be C^1 functions. If \mathbf{x} is a *local extremal* satisfying $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{x}) \neq \mathbf{0}$, then

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \quad (2.8.1)$$

Lemma 2.8.1. $\nabla g(\mathbf{x})$ is orthogonal to the constraint set $g^{-1}(0)$.

Proposition 2.8.1. Equations (2.8.1) $\implies \nabla f(\mathbf{x}) \perp g^{-1}(0)$ at \mathbf{x} .

Theorem 2.8.2. Let $S \subseteq \mathbb{R}^n$ be an open set, and $f, \{g_i\}_{i=1}^k : S \rightarrow \mathbb{R}$ be C^1 functions. Define $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_k(\mathbf{x}))$.

If $\mathbf{x} \in S$ is a *local extremal* of f such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, and $\{\nabla g_i(\mathbf{x})\}$ are linearly independent (i.e. $\text{rank}(D\mathbf{g}(\mathbf{x})) = k$), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \quad (2.8.2)$$

Remark 2.8.1. Procedure of optimization on *open sets*:

- (i) Find all critical points.
- (ii) Find optimizers among critical points.

Remark 2.8.2. Procedure of optimization with *inequality constraints*:

- (i) Find critical points without the constraints.
- (ii) Find critical points on the constraints.
- (iii) Find optimizers among candidates.

3 The Implicit and Inverse Function Theorems

3.1 The Implicit Function Theorem I

Theorem 3.1.1 (Implicit Function Theorem). Let $S \subseteq \mathbb{R}^{n+k}$ be an open set, and function $F : S \rightarrow \mathbb{R}^k$ be a C^1 function. Suppose there exists point $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$ such that

$$F(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad (3.1.1)$$

If

$$\det(D_{\mathbf{y}}F(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.2)$$

then there exists $r_0, r_1 > 0$ and a C^1 function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}), \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \quad (3.1.3)$$

and define $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$, the derivative of \mathbf{f} can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}}F(\mathbf{x}, \mathbf{y})]^{-1} D_{\mathbf{x}}F(\mathbf{x}, \mathbf{y}) \quad (3.1.4)$$

Remark 3.1.1. Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (3.1.5)$$

near (\mathbf{a}, \mathbf{b}) .

- (i) Verify $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.
- (ii) Assert

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.6)$$

- (iii) Approximate solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ using

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \quad (3.1.7)$$

$$= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \quad (3.1.8)$$

3.2 Geometric content of the Implicit Function Theorem

Definition 3.2.1. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in S$. S is **singular** at \mathbf{a} if

$$\forall r > 0 \ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \quad (3.2.1)$$

S is **regular** at \mathbf{a} if it is not singular there.

Theorem 3.2.1 (k dimensional manifold as level set). Let $U \subseteq \mathbb{R}^n$ and let $\mathbf{F} : U \rightarrow \mathbb{R}^{n-k}$ be a C^1 function.

$$S \equiv \mathbf{F}^{-1}(\mathbf{0}) \quad (3.2.2)$$

Let $\mathbf{a} \in U$, if

$$\text{rank}(D\mathbf{F}(\mathbf{a})) = n - k \quad (3.2.3)$$

then $\exists r > 0$ such that the *level set of \mathbf{F} near \mathbf{a}*

$$\mathcal{B}(r, \mathbf{a}) \cap S \quad (3.2.4)$$

can be represented as a C^1 graph.

Theorem 3.2.2 (k dimensional manifold as parameterization). Let $T \subseteq \mathbb{R}^k$ and let $\mathbf{f} : T \rightarrow \mathbb{R}^n$ be a C^1 function.

$$S \equiv \mathbf{f}(T) \quad (3.2.5)$$

Let $\mathbf{t} \in T$, if

$$\text{rank}(\mathbf{f}'(\mathbf{t})) = k \quad (3.2.6)$$

then $\exists r > 0$ such that the *parameterization of \mathbf{f} near \mathbf{t}*

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \quad (3.2.7)$$

can be represented as a C^1 graph.

3.3 Transformations, and the Inverse Function Theorem

Example 3.3.1 (Polar coordinate in \mathbb{R}^2). Let

$$U \equiv \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \quad (3.3.1)$$

$$V \equiv \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \quad (3.3.2)$$

Define $\mathbf{f} : U \rightarrow V$ as

$$\mathbf{f}(r, \theta) \equiv \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} \quad (3.3.3)$$

Example 3.3.2 (Spherical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix} \quad (3.3.4)$$

Example 3.3.3 (Cylindrical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{pmatrix} \quad (3.3.5)$$

Theorem 3.3.1 (Inverse Function Theorem). Let U and V be open subsets in \mathbb{R}^n , and $\mathbf{f} : U \rightarrow V$. Let $\mathbf{a} \in U$ and define $\mathbf{b} \equiv \mathbf{f}(\mathbf{a}) \in V$. If

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0 \quad (3.3.6)$$

then there exists $M \subseteq U$ and $N \subseteq V$ such that

- (i) $\mathbf{a} \in M$ and $\mathbf{b} \in N$,
- (ii) \mathbf{f} is bijective between M and N ,
- (iii) $\mathbf{f}^{-1} : N \rightarrow M$ is C^1 ,

and for all $\mathbf{x} \in M$ such $\mathbf{y} \equiv \mathbf{f}(\mathbf{x}) \in N$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \quad (3.3.7)$$

4 Integration

4.1 Basics

Theorem 4.1.1 (Properties of infimum and supremum). Let $A \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, and $f, g : A \rightarrow \mathbb{R}$ are bounded functions. Let m and M denote the infimum and supremum respectively, then

- (i) $m_A f + m_A g \leq m_A(f + g) \leq M_A(f + g) \leq M_A f + M_A g$
- (ii) If $A' \subseteq A$, then $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$
- (iii) If $f(\mathbf{x}) \leq g(\mathbf{x}) \forall \mathbf{x} \in A$, then $m_A f \leq m_A g$ and $M_A f \leq M_A g$
- (iv) $|M_A f| \leq M_A |f|$
- (v) $M_A |f| - m_A |f| \leq M_A f - m_A f$
- (vi) $\forall c \in \mathbb{R}, M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$
- (vii) $M_A f - m_A f = \sup\{f(x) - f(y) : x, y \in A\}$

4.2 Integration on Higher Dimensions

Definition 4.2.1. A rectangle $\mathcal{R} \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{R} \equiv \prod_{i=1}^n [a_i, b_i] \quad (4.2.1)$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.

Definition 4.2.2. A partition P of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ is a list of n finite and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \quad (4.2.2)$$

where $L_i = \{e_j\}_{j=0}^{T_i}$ such that

$$a_i = e_0 < e_1 < \dots < e_{T_i} = b_i \quad (4.2.3)$$

and such partition induces a set of rectangles(boxes) $\mathcal{B}(P) \equiv \{B_j\}_{j=1}^J \subseteq \mathcal{R}$.

Definition 4.2.3. Let P and P' be two partitions of \mathcal{R} . Then P' is a **refinement** of P if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee B_j'^{int} \cap B_j^{int} = \emptyset \quad (4.2.4)$$

Definition 4.2.4. Define the **volume** of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ as

$$V^n(\mathcal{R}) \equiv \prod_{i=1}^n (b_i - a_i) \quad (4.2.5)$$

Definition 4.2.5. The **lower Riemann sum** of f with partition P on \mathcal{R} is defined as

$$L_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.6)$$

and the **upper Riemann sum** is defined as

$$U_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.7)$$

Definition 4.2.6. The **upper integral** and **lower integral** of f on \mathcal{R} are defined as

$$\bar{I}_{\mathcal{R}} f \equiv \inf_P U_P f \quad (4.2.8)$$

$$\underline{I}_{\mathcal{R}} f \equiv \sup_P L_P f \quad (4.2.9)$$

Definition 4.2.7. A bounded real-valued function f defined on \mathcal{R} is **integrable** if

$$\underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \quad (4.2.10)$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \, dV^n \equiv \underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \quad (4.2.11)$$

Lemma 4.2.1. Let f be a bounded real-valued function defined on \mathcal{R} , f is integrable if and only if $\forall \epsilon > 0$, there exists a partition P of \mathcal{R} such that

$$U_P f - L_P f < \epsilon \quad (4.2.12)$$

Theorem 4.2.1. Let f and g be two integrable functions on $\mathcal{R} \subseteq \mathbb{R}^n$, let $c \in \mathbb{R}$,

- (i) $f + g : \mathcal{R} \rightarrow \mathbb{R}$ is integrable and $\int_{\mathcal{R}} (f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$
- (ii) $c \cdot f$ is integrable and $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$
- (iii) $f(\mathbf{x}) \geq g(\mathbf{x}) \, \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$
- (iv) $|f|$ is integrable and $|\int_{\mathcal{R}} f| \leq \int_{\mathcal{R}} |f|$

Definition 4.2.8. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S , the **indicator function** of S is $\chi_S : \mathcal{R} \rightarrow \{0, 1\}$, defined as

$$\chi_S(\mathbf{x}) \equiv \mathbb{I}(\mathbf{x} \in S) \quad (4.2.13)$$

Definition 4.2.9. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S . Let $f : \mathcal{R} \rightarrow \mathbb{R}$ be a bounded function, then f is **integrable on S** if $\chi_S f$ is integrable on \mathcal{R} . And

$$\int \cdots \int_S f \, dV^n \equiv \int \cdots \int_{\mathcal{R}} \chi_S f \, dV^n \quad (4.2.14)$$

Definition 4.2.10. Let $Z \subseteq \mathbb{R}^n$, Z has **zero content** if for all $\epsilon > 0$, there exists a finite set of rectangles $\{R_\ell\}_{\ell=1}^L$ covers Z and

$$\sum_{\ell=1}^L V^n(R_\ell) < \epsilon \quad (4.2.15)$$

Proposition 4.2.1. Let $Z \subseteq \mathbb{R}^n$ has zero content, then

- (i) For any $Z' \subseteq Z$, Z' has zero content.
- (ii) Finite union of content zero sets has zero content.
- (iii) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function, it's graph $\{(x, f(x)) : x \in [a, b]\}$ has zero content.
- (iv) Let $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^2$ be a C^1 function, the parameterization $\mathbf{f}([a, b])$ has zero content.

Theorem 4.2.2. Let \mathcal{R} be a rectangle in \mathbb{R}^n and f is integrable on \mathcal{R} if

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \quad (4.2.16)$$

has zero content.

Proposition 4.2.2 (Folland 4.22). Suppose $Z \subseteq \mathbb{R}^n$ has zero content. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f \, dV^n = 0$.

4.3 Iterated Integrals

Theorem 4.3.1 (Fubini's Theorem). Let $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \mathcal{R} \rightarrow \mathbb{R}$ is bounded. Assuming that

- (i) f is integrable on \mathcal{R} .
- (ii) for each $y \in [c, d]$, the function $f_y(x) \equiv f(x, y)$ is integrable on $[a, b]$.
- (iii) Define $g(y) \equiv \int_a^b f(x, y) dx$ is integrable on $[c, d]$.

Then

$$\iint_{\mathcal{R}} f \, dA = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy \quad (4.3.1)$$

Proposition 4.3.1. Let $S \subseteq \mathbb{R}^n$ be an unbounded set, and $f : S \rightarrow \mathbb{R}$. Then improper integral $\int \cdots \int_S f \, d^n \mathbf{x}$ is absolutely convergent on \mathbb{R}^n if and only if $\int \cdots \int_{\mathbb{R}^n} \chi_S f \, d^n \mathbf{x}$ is absolutely convergent.

4.4 Change of Variables

Theorem 4.4.1 (Change of Variable). Let U and V be two open subsets of \mathbb{R}^n , and let $\mathbf{G} : U \rightarrow V$ be a C^1 bijection. Let $T \subset U$ and $S \subset V$. Suppose $\mathbf{G}(T) = S$, then

$$\int \cdots \int_S f \, d\Omega = \int \cdots \int_T f \circ \mathbf{G} \, |\det D\mathbf{G}| \, d\Theta \quad (4.4.1)$$

Corollary 4.4.1. Let S be a region in \mathbb{R}^n , suppose S can be parameterized by $\mathbf{G} : T \rightarrow S$. By the change of variable formula, consider the special case $f(\mathbf{x}) = 1$,

$$|S| = \int \cdots \int_S 1 \, d\Omega = \int \cdots \int_T 1 \, |\det D\mathbf{G}(\mathbf{u})| \, d\Theta \quad (4.4.2)$$

Example 4.4.1 (Polar Coordinate). Define the coordinate transformation mapping from polar to Cartesian,

$$\mathbf{P}(r, \theta) \equiv (x, y) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad (4.4.3)$$

and $|\det D\mathbf{P}(r, \theta)| = r$.

Example 4.4.2 (Cylindrical Coordinate). Define the coordinate transformation mapping from cylindrical to Cartesian as

$$\mathbf{C}(r, \theta, z) \equiv (x, y, z) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad z \in \mathbb{R} \quad (4.4.4)$$

and $|\det D\mathbf{C}(r, \theta, z)| = r$.

Example 4.4.3 (Spherical Coordinate). Define the coordinate transformation mapping from spherical to Cartesian as

$$\mathbf{S}(r, \theta, \varphi) = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} \quad (4.4.5)$$

and $|\det D\mathbf{S}(r, \theta, \varphi)| = r^2 \sin \varphi$

4.5 Further Aspects

4.5.1 Exchanging Differentiation and Integration

Theorem 4.5.1 (Exchanging Differentiation and Integration). Let $f(\mathbf{x}, \mathbf{t}) : S \times T \rightarrow \mathbb{R}$ and define $F(\mathbf{x}) : S \rightarrow \mathbb{R}$ as

$$F(\mathbf{x}) \equiv \int \cdots \int_T f(\mathbf{x}, \mathbf{t}) \, d\Omega \quad (4.5.1)$$

If

- (i) S is open and T is compact and bounded;
- (ii) f and F are continuous on their domains;
- (iii) and $\forall x_j \in \mathbf{x}$, $\frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j}$ is continuous,

then F is C^1 in S and for every j ,

$$\frac{\partial F(\mathbf{x})}{\partial x_j} = \int \cdots \int_T \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j} d\Omega \quad (4.5.2)$$

Corollary 4.5.1. By the definition of partial derivative, above theorem is equivalent to

$$\lim_{h \rightarrow 0} \int \cdots \int_T \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} d\Omega = \int \cdots \int_T \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} d\Omega \quad (4.5.3)$$

4.5.2 Improper Integrals

Definition 4.5.1 (Unbounded Domains). An **improper integral** with unbounded domain $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is **absolutely convergent** if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \exists R > 0 \text{ s.t. } \forall S \subseteq \mathbb{R}^n \ B(R, \mathbf{0}) \subset S \implies \left| \int \cdots \int_S f d\Omega - L \right| < \varepsilon \quad (4.5.4)$$

Theorem 4.5.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, and that

$$\lim_{R \rightarrow \infty} \int \cdots \int_{B(R, \mathbf{0})} |f| d\Omega \text{ exists} \quad (4.5.5)$$

then $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent.

Corollary 4.5.2 (Equivalence). Above improper integral $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent if set

$$\left\{ \int \cdots \int_{B(R, \mathbf{0})} |f| d\Omega : R \in \mathbb{R}_{++} \right\} \quad (4.5.6)$$

is bounded.

Corollary 4.5.3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, if

$$\exists p > n, \ C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{1}{\|\mathbf{x}\|^p} \ \forall \mathbf{x} \in \mathbb{R}^n \quad (4.5.7)$$

then $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent.

Definition 4.5.2 (Unbounded Function). Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$. Consider a function $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$. Then the improper integral $\int \cdots \int_S f d\Omega$ is absolutely convergent if

$$\exists L \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0 \exists r > 0 \text{ s.t. } \forall U \subset S \text{ s.t. } \mathbf{a} \in U^{\text{int}} \wedge U \subset B(r, \mathbf{a}), \left| \int \cdots \int_{S \setminus U} f d\Omega - L \right| < \varepsilon \quad (4.5.8)$$

Theorem 4.5.3. Let $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$, if

$$\lim_{r \rightarrow 0} \int \cdots \int_{S \setminus B(r, \mathbf{a})} |f| d\Omega \text{ exists} \quad (4.5.9)$$

then $\int \cdots \int_S f d\Omega$ is absolutely convergent.

Corollary 4.5.4 (Equivalence). If the set

$$\left\{ \int \cdots \int_{S \setminus B(r, \mathbf{a})} |f| d\Omega : r \in \mathbb{R}_{++} \right\} \quad (4.5.10)$$

is bounded, then $\int \cdots \int_S f d\Omega$ is absolutely convergent.

Corollary 4.5.5. Let $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$, if

$$\exists p < n, \ C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x} - \mathbf{a}\|^p} \ \forall \mathbf{x} \in S \setminus \{\mathbf{a}\} \quad (4.5.11)$$

then the improper integral $\int \cdots \int_S f d\Omega$ is absolutely convergent.

5 Vector Calculus

5.1 Line Integrals

5.1.1 Arc Length

Definition 5.1.1. Let C be a smooth curve in \mathbb{R}^n parameterized by C^1 function \mathbf{g} such that $\mathbf{g}'(t) \neq \mathbf{0}$ for every appropriate t .

$$C \equiv \{\mathbf{g}(t) : t \in [a, b]\} \quad (5.1.1)$$

and the **arc length** of C is defined as

$$\int_C d^n \mathbf{x} \equiv \int_C ds \equiv \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.2)$$

Proposition 5.1.1. The arc length of a curve C is an intrinsic property of the geometric object C and should not depend on the particular parameterization we use.

Proof. Let $\varphi : [c, d] \rightarrow [a, b]$ be a bijection, so that $\mathbf{h} \equiv \mathbf{g} \circ \varphi$ is also a valid parameterization of C such that

$$C \equiv \{\mathbf{h}(u) : u \in [c, d]\} \quad (5.1.3)$$

The arc length of C can be computed using

$$\int_C ds = \int_c^d \|\mathbf{h}'(u)\| du \quad (5.1.4)$$

$$= \int_c^d \|\mathbf{g}'(\varphi(u))\| \times \|\varphi'(u)\| du \quad (5.1.5)$$

$$= \int_a^b \|\mathbf{g}'(t)\| dt \text{ by change of variable formula.} \quad (5.1.6)$$

■

Remark 5.1.1 (Interpretations). Suppose \mathbf{g} is a parameterization of C .

- (i) $\int_a^b \mathbf{g}'(t) dt = \mathbf{g}(b) - \mathbf{g}(a)$ measures the distance between two endpoints of C .
- (ii) Choosing a parameterization is effectively choosing an **orientation** for the curve C .

Definition 5.1.2. A function $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ is called **piecewise smooth** if

- (i) it's *continuous*, and
- (ii) it's derivate exists and is continuous except at finitely many points t_j , at which the one-sided limits exists.

5.1.2 Line Integrals of Scalar Functions

Definition 5.1.3. Let smooth curve $C \subseteq \mathbb{R}^n$, $f : C \rightarrow \mathbb{R}$ and \mathbf{g} be a parameterization of C , then

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \quad (5.1.7)$$

Remark 5.1.2. The line integrals of scalar functions are also independent from the choices of parameterizations.

Definition 5.1.4.

$$\text{Average of } f \text{ over } C \equiv \frac{\int_C f ds}{\int_C ds} \quad (5.1.8)$$

5.1.3 Line Integrals of Vector Fields

Definition 5.1.5. Let smooth $C \in \mathbb{R}^n$ with parameterization \mathbf{g} and $\mathbf{F} : C \rightarrow \mathbb{R}^n$ defined on it, the **line integral** of \mathbf{F} over C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt \quad (5.1.9)$$

Proposition 5.1.2. The line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ is independent of the parameterization *as long as the orientation is unchanged*.

Theorem 5.1.1 (The Fundamental Theorem of Line Integral). Let $f : C \rightarrow \mathbb{R}$ defined on smooth curve C parameterized by $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$, then

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.10)$$

Proof.

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial \mathbf{g}(t)} \cdot \mathbf{g}'(t) dt \quad (5.1.11)$$

$$= \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial t} dt = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.12)$$

■

5.1.4 Rectifiable Curves

Remark 5.1.3. Let C be a curve in \mathbb{R}^n parameterized by injection $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ such that $\mathbf{g}'(t) \neq \mathbf{0}$. Let P be a partition of $[a, b]$. Denote

$$L_P(C) \equiv \sum_j \|\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})\| \quad (5.1.13)$$

Definition 5.1.6. A curve C is **rectifiable** if the set $\{L_P(C) : P\}$ is bounded. And the arc length of C is defined as

$$L(C) \equiv \sup\{L_P(C) : P\} \quad (5.1.14)$$

Theorem 5.1.2. The supremum found above, $L(C)$ is precisely the arc length of C :

$$L(C) = \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.15)$$

5.2 Green's Theorem

5.2.1 Preliminary Definitions

Definition 5.2.1. A **simple closed curve** is a curve with parameterization $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ where

- (i) \mathbf{g} is continuous;
- (ii) $\mathbf{g}(a) = \mathbf{g}(b)$;
- (iii) \mathbf{g} is injective with its domain restricted to (a, b) .

Definition 5.2.2. A *simple closed curve* is **piecewise smooth** if it has a parameterization \mathbf{g} such that

- (i) \mathbf{g} is continuously differentiable with $\mathbf{g}'(t) \neq \mathbf{0}$ except finitely many breakpoints;
- (ii) $\mathbf{g}'(t)$ is *one side continuous* at breakpoints of the curve.

Definition 5.2.3. A **regular region** $S \subseteq \mathbb{R}^n$ is a set satisfying both

- (i) S is compact;
- (ii) $\overline{S^{int}} = S$.

Definition 5.2.4. Let $S \subseteq \mathbb{R}^2$, S has **piecewise smooth boundary** if ∂S consists of one or more *disjoint, piecewise smooth, simple closed curve*.

Definition 5.2.5. Let $S \subseteq \mathbb{R}^2$, then **positive orientation** on ∂S is the orientation on each of the closed curves that make up the boundary such that the region is on the *left* with respect to the positive direction on the curve.

Theorem 5.2.1 (Green's Theorem). Suppose $S \subseteq \mathbb{R}^2$ is a regular region with piecewise smooth region ∂S . Suppose \mathbf{F} is a C^1 vector field defined on \overline{S} , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA \quad (5.2.1)$$

Corollary 5.2.1. Suppose S is a regular region in \mathbb{R}^2 with piecewise smooth boundary ∂S , and let $\mathbf{n}(\mathbf{x})$ be the *unit outward normal* vector to ∂S at $\mathbf{x} \in \partial S$. Suppose also that \mathbf{F} is a vector field defined on \overline{S} , then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA \quad (5.2.2)$$

Proof. Let $\mathbf{g}(t)$ be a parameterization of boundary ∂S . Then the tangent vector would be $\mathbf{g}'(t)$ and we can conclude the *outer normal vector* \mathbf{n} is $\frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|}$. Then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \int_T \mathbf{F} \circ \mathbf{g} \cdot \frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|} \|\mathbf{g}'(t)\| \, dt \quad (5.2.3)$$

$$= \int_T F_1 g'_2(t) - F_2 g'_1(t) \, dt \quad (5.2.4)$$

$$= \int_T \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot \begin{pmatrix} g'_1(t) \\ g'_2(t) \end{pmatrix} \, dt \quad (5.2.5)$$

$$= \int_{\partial S} \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot d^2\mathbf{x} \quad (5.2.6)$$

$$= \iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \, dA \text{ By Green's Theorem} \quad (5.2.7)$$

■

5.3 Surface Integrals

5.3.1 Surface Areas and Surface Integrals

Definition 5.3.1. Suppose S is a surface in \mathbb{R}^3 and parameterized by

$$\mathbf{G}(\mathbf{u}) : R \rightarrow S \quad (5.3.1)$$

where $\text{rank}(D\mathbf{G}(\mathbf{u})) = 2$ for every $\mathbf{u} \in R \setminus Z$ where Z is a probably empty set with zero content. If $\|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\|$ is integrable, then

$$\text{Area}(S) \equiv \iint_R \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.2)$$

Definition 5.3.2. Let $f : S \rightarrow \mathbb{R}$ be a real-valued continuous function defined on a super set of S , the **integral of a real-valued function on a surface** is defined as

$$\iint_S f(\mathbf{x}) dA \equiv \iint_R f(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.3)$$

Definition 5.3.3. Let $\mathbf{F} : S \rightarrow \mathbb{R}^3$ be a continuous vector field defined on a super set of S , the **integral of vector field on a surface** is defined as

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} dA \equiv \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) d\Theta \quad (5.3.4)$$

Remark 5.3.1. Surface integrals of **real-valued functions** are independent of the choice of parametrization.

Remark 5.3.2. But the choice of parameterization can change the sign of surface integrals of **vector fields**. We need to **choose the direction of the normal, \mathbf{n}** .

Definition 5.3.4. Let $S \subseteq \mathbb{R}^3$ be a two dimensional sub-manifold, and f is a real-valued function defined on a super set of S . Define the **average of f over S** as

$$\text{aver of } f \text{ over } S \equiv \frac{\iint_S f dA}{\iint_S 1 dA} \quad (5.3.5)$$

Remark 5.3.3. A note on the relation between integrals of a vector field and a real-valued function. The surface of vector field \mathbf{F} on S is defined by *reducing \mathbf{F} to a real-valued function $\mathbf{F} \cdot \mathbf{n}$* and then follow the definition of conventional real-valued function on S . Define $h \equiv \mathbf{F} \cdot \mathbf{n}$,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_S h dA \quad (5.3.6)$$

$$\equiv \iint_R h(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.7)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \mathbf{n}(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.8)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \frac{\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}}{\|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\|} \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.9)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) d\Theta \quad (5.3.10)$$

5.3.2 An invariance property

Remark 5.3.4. As mentioned above, given $\mathbf{n}(\mathbf{x})$ fixed, we can define the surface integral of vector field as the surface integral of a real-valued function defined as $h(\mathbf{x}) \equiv \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$. And as argued before, one \mathbf{n} is fixed (i.e. orientation is fixed), the value of integral is deterministic. Therefore we can conclude the integral of a vector field \mathbf{F} over a surface S depends on the orientation of S but otherwise independent of the parameterization.

Remark 5.3.5. Let $S \subseteq \mathbb{R}^3$ be a two dimensional sub-manifold parameterized by $\mathbf{G} : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $\text{rank}(\mathbf{G}(\mathbf{u})) = 2$ for all but zero-content sets on its domain.

Let $\varphi : W \subseteq \mathbb{R}^2 \rightarrow R$ be a bijection such that $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$ is another parameterization of S .

Now consider the integral of vector field \mathbf{F} under parameterization \mathbf{H} ,

$$\iint_S \mathbf{F} \cdot \mathbf{u} \, dA = \iint_W \mathbf{F}(\mathbf{H}) \cdot \left(\frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right) d\Theta \quad (5.3.11)$$

$$= \iint_W \mathbf{F} \circ \mathbf{G} \circ \varphi \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) \text{det } D\varphi \, d\Theta \quad (5.3.12)$$

$$= \pm \iint_R \mathbf{F} \circ \mathbf{G} \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (\text{change of variable}) \quad (5.3.13)$$

Theorem 5.3.1 (Invariance). Let $\mathbf{G} : R \rightarrow \mathbb{R}^3$ and $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$ be two parameterizations of S , then

$$\iint_R f \circ \mathbf{G} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta = \iint_W f \circ \mathbf{H} \left\| \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right\| d\Theta \quad (5.3.14)$$

and

$$\iint_R \mathbf{F} \circ \mathbf{G} \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta = \pm \iint_W \mathbf{F} \circ \mathbf{H} \cdot \left(\frac{\partial \mathbf{H}}{\partial u} \times \frac{\partial \mathbf{H}}{\partial v} \right) d\Theta \quad (5.3.15)$$

5.3.3 Volume and Area

Theorem 5.3.2. Let R be an arbitrary regular region in \mathbb{R}^3 , and let S be the boundary surface of R , define

$$S_h \equiv \{\mathbf{x} + \delta \mathbf{n} : \mathbf{x} \in S \wedge \delta \in [0, h]\} \quad (5.3.16)$$

where S_h can be interpreted as *a shell of region R with thickness h* . Then the surface area of S is

$$\text{area}(S) = \lim_{h \rightarrow 0} \frac{|S_h|}{h} \quad (5.3.17)$$

5.4 Divergence, Gradient and Curl

Definition 5.4.1. Let $U \subseteq \mathbb{R}^n$ be an open set, and define real-valued function $f : U \rightarrow \mathbb{R}$ and vector field $\mathbf{F} : U \rightarrow \mathbb{R}^n$. Then we define

1. The **gradient** of f as ∇f ;
2. The **divergence** of \mathbf{F} as $\nabla \cdot \mathbf{F}$;
3. The **curl** of \mathbf{F} as $\nabla \times \mathbf{F}$.

Definition 5.4.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 real-valued function, define the **Laplacian** of f as a mapping from *real-valued functional space* to *real-valued functional space* defined as

$$\operatorname{div}(\operatorname{grad})f \equiv \sum_j \partial_j^2 f = \Delta f = \nabla^2 f \quad (5.4.1)$$

Theorem 5.4.1. For every C^2 real valued function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$\operatorname{curl}(\operatorname{grad}f) = \mathbf{0} \quad (5.4.2)$$

For every C^2 vector field defined in \mathbb{R}^3 or a subset of it,

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0 \quad (5.4.3)$$

Note that the domain of f and \mathbf{F} must be \mathbb{R}^3 or a subset of it, otherwise the curl operation is not well-defined.

Theorem 5.4.2 (Product rules).

$$\operatorname{grad}(fg) = f \operatorname{grad}g + \operatorname{grad}f \cdot g \quad (5.4.4)$$

$$\operatorname{div}(f\mathbf{G}) = f \operatorname{div}\mathbf{G} + \operatorname{grad}f \cdot \mathbf{G} \quad (5.4.5)$$

$$\operatorname{curl}(f\mathbf{G}) = f \operatorname{curl}\mathbf{G} + \operatorname{grad}f \times \mathbf{G} \quad (5.4.6)$$

5.5 Divergence Theorem

Remark 5.5.1. vector field integral on boundary (2-dimensional sub-manifold) of region in \mathbb{R}^3 ($\mathbf{F} \cdot \mathbf{n} \, dA$ 2-form) and scalar valued function ($\operatorname{div}(\mathbf{F}) \, dV$ 3-form) in a region (3-dimensional sub-manifold).

Theorem 5.5.1 (Divergence Theorem). Let $R \subseteq \mathbb{R}^3$ be a *regular region* with *piece-wise smooth boundary* ∂S . And \mathbf{n} is the *outer normal vector* on ∂S , then,

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_S \operatorname{div}(\mathbf{F}) \, dV \quad (5.5.1)$$

Proof.

Definition 5.5.1. A region $R \subseteq \mathbb{R}^3$ is said to be ***xy-simple*** if and only if it can be expressed as the following form

$$R = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\} \quad (5.5.2)$$

Suppose S is simple in terms of all combinations of x, y, z .

Then

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_1 n_1 + F_2 n_2 + F_3 n_3 \, dA \quad (5.5.3)$$

Consider $\iint_{\partial S} F_3 n_3 \, dA$, since R is *xy-simple*,

$$\iint_{\partial S} F_3 n_3 \, dA = \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA \quad (5.5.4)$$

Note that except the bottom and top sides, which are parameterized by $\mathbf{G}_1(x, y) = (x, y, \varphi_1(x, y))$ and $\mathbf{G}_2(x, y) = (x, y, \varphi_2(x, y))$, the outer normal vector of those region has form $(\cdot, \cdot, 0)$, and therefore $\mathbf{n} \cdot \mathbf{k} = 0$ for every \mathbf{x} on those regions, and contribute nothing to the integral.

Therefore, to evaluate $\iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA$, we only need to consider the upper and bottom surfaces. Also note that \mathbf{n} has opposite z component on those two surfaces.

Moreover, the undirected \mathbf{n} on those two surfaces is

$$\tilde{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ \partial_x \varphi_i \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y \varphi_i \end{pmatrix} = \begin{pmatrix} -\partial_x \varphi_i \\ \partial_x \varphi_i - \partial_y \varphi_i \\ 1 \end{pmatrix} \quad (5.5.5)$$

$$\implies \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_3 \, dA \quad (5.5.6)$$

$$= \iint_{\text{upper } \partial S} F_3 \, dA - \iint_{\text{lower } \partial S} F_3 \, dA \quad (5.5.7)$$

$$= \iint_W F_3(x, y, \varphi_2(x, y)) \, dx dy - \iint_W F_3(x, y, \varphi_1(x, y)) \, dx dy \quad (5.5.8)$$

$$= \iint_W \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} \partial_3 F_3 \, dz dx dy = \iiint_S \partial_3 F_3 \, dV \quad (5.5.9)$$

We can prove the equalities involving the other two components, and the proof is completed by the fact that any open set in \mathbb{R}^n can be written as a countable union of *almost disjoint* cubes, which are simple and the boundary of S has zero content. \blacksquare

Proposition 5.5.1 (Geometric Interpretation of Divergence). Let $S \subset \mathbb{R}^3$, $\mathbf{F} : S \rightarrow \mathbb{R}^3$, $\mathbf{a} \in S$,

$$\operatorname{div}(\mathbf{F})(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \iiint_{\mathcal{B}(\mathbf{a}, r)} \operatorname{div}(\mathbf{F})(\mathbf{x}) \, d\mathbf{x} \quad (5.5.10)$$

$$= \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \underbrace{\iint_{\partial \mathcal{B}(\mathbf{a}, r)} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{x}}_{\text{flux through boundary}} \quad (5.5.11)$$

thus $\operatorname{div}(\mathbf{F})(\mathbf{a}) > 0$ if and only if at point \mathbf{a} , matters are flowing away from this point.

Corollary 5.5.1 (Green's Formula). Suppose $R \subset \mathbb{R}^3$ and $f, g : R \rightarrow \mathbb{R}$ are C^2 functions, then

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} \, dA = \iiint_S \nabla f \cdot \nabla g + f \nabla^2 g \, dV \quad (5.5.12)$$

$$\iint_{\partial S} (f \nabla g - g \nabla f) \, dA = \iiint_S (f \nabla^2 g - g \nabla^2 f) \, dV \quad (5.5.13)$$

Proof.

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} \, dA = \iiint_S \operatorname{div}(f \nabla g) \, dV \quad (5.5.14)$$

$$= \iiint_S f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g \, dV = \iiint_S f \nabla^2 g + \nabla f \cdot \nabla g \, dV \quad (5.5.15)$$

The second formula can be proved directly using divergence theorem the first formula. \blacksquare

5.6 Stokes Theorem

5.6.1 Stokes Theorem in \mathbb{R}^3

Theorem 5.6.1 (Stokes Theorem, Special Case). Let S be a 2 dimensional sub-manifold in \mathbb{R}^3 , and let \mathbf{F} be a vector field defined on some neighbour of S , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA \quad (5.6.1)$$

Remark 5.6.1. In above theorem, $\omega \equiv \mathbf{F} \cdot d\mathbf{x}$ is a 1-form in \mathbb{R}^3 and $d\omega \equiv \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA$ is a 2-form in \mathbb{R}^3 .

Corollary 5.6.1. Let S be a closed surface in \mathbb{R}^3 , that's, $\partial S = \emptyset$, and let \mathbf{n} denote the outer normal vector, and \mathbf{F} is a C^1 vector field. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = 0 \quad (5.6.2)$$

Proof. We can construct a *small* simple closed curve C on S and divide S into two regions sharing the same boundary. And note that given orientation fixed on S , the orientation on ∂S_1 and ∂S_2 are opposite. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_1} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA + \iint_{S_2} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA \quad (5.6.3)$$

$$= \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{x} - \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot d\mathbf{x} - \int_C \mathbf{F} \cdot d\mathbf{x} = 0 \quad (5.6.4)$$

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Proposition 5.6.1 (Geometric Interpretation of Curl). Let $R \subset \mathbb{R}^3$ be a 2 dimensional sub-manifold with \mathbf{n} as outer normal vector on it, and $\mathbf{a} \in R$,

$$\text{curl}(F)(\mathbf{a}) \cdot \mathbf{n}(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \iint_{\mathcal{D}(\mathbf{a}, r)} \text{curl}(F)(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, dA \quad (5.6.5)$$

$$= \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \int_{\partial \mathcal{D}(\mathbf{a}, r)} \mathbf{F} \cdot d\mathbf{x} \quad (5.6.6)$$

$$(5.6.7)$$

If we think of \mathbf{F} as a force field, then $\int_{\partial \mathcal{D}(\mathbf{a}, r)} \mathbf{F} \cdot d\mathbf{x}$ represents the work done by \mathbf{F} on a particle moves around $\partial \mathcal{D}(\mathbf{a}, r)$. Thus $\text{curl}(\mathbf{F}) \cdot \mathbf{n}$ represents the tendency of the force \mathbf{F} to push the particle around $\partial \mathcal{D}(\mathbf{a}, r)$ in a direction compatible with \mathbf{n} .

5.6.2 The Generalization

Proposition 5.6.2 (Properties of Exterior Products). Let α_1, α_2 and β be 1-forms on \mathbb{R}^n and f_1, f_2 are continuous functions defined on \mathbb{R}^n ,

1. Distributive

$$(f_1\alpha_1 + f_2\alpha_2) \wedge \beta = f_1(\alpha_1 \wedge \beta) + f_2(\alpha_2 \wedge \beta) \quad (5.6.8)$$

$$\beta \wedge (f_1\alpha_1 + f_2\alpha_2) = f_1(\beta \wedge \alpha_1) + f_2(\beta \wedge \alpha_2) \quad (5.6.9)$$

2. Anti-commutative

$$\beta \wedge \alpha = -\alpha \wedge \beta \quad (5.6.10)$$

Theorem 5.6.2 (Divergence Theorem in \mathbb{R}^n). Let R be a regular region in \mathbb{R}^n bounded by a piecewise smooth hyper-surface ∂R . Note here R is a n dimensional sub-manifold and ∂R is a $n-1$ dimension sub-manifold. Then

$$\int \cdots \int_{\partial R} \mathbf{F} \cdot \mathbf{n} dV^{n-1} = \iiint \cdots \int_R \operatorname{div}(\mathbf{F}) dV^n \quad (5.6.11)$$

where if ∂R is parameterized by $\mathbf{G}(u_1, \dots, u_{n-1})$, then

$$\mathbf{n} dV^{n-1} = \det \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \partial_1 G_1 & \cdots & \partial_1 G_n \\ \vdots & & \vdots \\ \partial_{n-1} G_1 & \cdots & \partial_{n-1} G_n \end{pmatrix} \quad (5.6.12)$$

Definition 5.6.1. A 0-form on \mathbb{R}^n is a real valued function f .

Remark 5.6.2. While writing the basis elements $dx_i \wedge dx_j$ with the variables in *cyclic order*. That's dx before dy before dz before dx in \mathbb{R}^3 case.

Definition 5.6.2. A k -form in \mathbb{R}^n takes the expression of linear combination of $C(n, k)$ basis elements $\{\beta_i\}_i$.

Example 5.6.1. A 2-form ω in \mathbb{R}^3 can be expressed using a 3-element basis

$$\omega = \sum_{1 \leq i < j \leq 3} C_{ij}(\mathbf{x}) \beta_{ij} \quad (5.6.13)$$

$$\beta_{ij} \in \{dx \wedge dy, dy \wedge dz, dx \wedge dz\} \quad (5.6.14)$$

Definition 5.6.3. Let $\omega = \sum_{j=1}^{C(n,k)} f_j \beta_j$ be a k -form in \mathbb{R}^n , then it's **exterior derivative** is defined to be the $(k+1)$ -form in \mathbb{R}^n defined as

$$d\omega \equiv \sum_j df_j \wedge \beta_j \quad (5.6.15)$$

where df_j can be computed using *total derivative*.

Example 5.6.2. In \mathbb{R}^3 , the *exterior derivative* for a 0-form f is its **gradient**, which is a 1-form. And the exterior derivate of a 1-form in \mathbb{R}^3 is its curl

$$\omega := F_1 dx + F_2 dy + F_3 dz \quad (5.6.16)$$

$$\implies d\omega = dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \quad (5.6.17)$$

$$= (\partial_1 F_1 dx + \partial_2 F_1 dy + \partial_3 F_1 dz) \wedge dx \quad (5.6.18)$$

$$+ (\partial_1 F_2 dx + \partial_2 F_2 dy + \partial_3 F_2 dz) \wedge dy \quad (5.6.19)$$

$$+ (\partial_1 F_3 dx + \partial_2 F_3 dy + \partial_3 F_3 dz) \wedge dz \quad (5.6.20)$$

$$= (\partial_1 F_2 - \partial_2 F_1) dx \wedge dy + (\partial_2 F_3 - \partial_3 F_2) dy \wedge dz + (\partial_3 F_1 - \partial_1 F_3) dz \wedge dx \quad (5.6.21)$$

$$= \operatorname{curl}(\mathbf{F}) \quad (5.6.22)$$

The exterior derivative of a 2-form in \mathbb{R}^3 is its divergence

$$\omega := A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \quad (5.6.23)$$

$$\implies d\omega = (\partial_1 A dx + \partial_2 A dy + \partial_3 A dz) \wedge dy \wedge dz \quad (5.6.24)$$

$$+ (\partial_1 B dx + \partial_2 B dy + \partial_3 B dz) \wedge dz \wedge dx \quad (5.6.25)$$

$$+ (\partial_1 C dx + \partial_2 C dy + \partial_3 C dz) \wedge dx \wedge dy \quad (5.6.26)$$

$$= (\partial_1 A + \partial_2 B + \partial_3 C) dx \wedge dy \wedge dz \quad (5.6.27)$$

$$= \operatorname{div}(\mathbf{F}) \quad (5.6.28)$$

Theorem 5.6.3 (Stokes Theorem, 5.77). Let M be a smooth, oriented k dimensional sub-manifold of \mathbb{R}^n with a piecewise smooth boundary ∂M , and let ∂M carry the orientation that is (in a suitable sense) compatible with the one on M . If ω is a $(k-1)$ -form of class C^1 on an open set containing M , then

$$\int \cdots \int_{\partial M} \omega = \iint \cdots \int_M d\omega \quad (5.6.29)$$

Theorem 5.6.4. The *boundary* of a (smoothly bounded) region M in a k dimensional manifold is a $(k-1)$ dimensional manifold with no boundary.

That's let M be a k dimensional manifold with piecewise smooth boundary ∂M , then

$$\partial(\partial M) = \emptyset \quad (5.6.30)$$

Theorem 5.6.5. For any k -form ω on \mathbb{R}^n ,

$$d(d\omega) = 0 \quad (5.6.31)$$

Proof. Let M be a k dimensional sub-manifold in \mathbb{R}^n with piecewise smooth boundary, and ω is a $(k-2)$ -form on \mathbb{R}^n , so $d(d\omega)$ is a k -form on \mathbb{R}^n . And

$$\iiint \cdots \int_M d(d\omega) = \iint \cdots \int_{\partial M} d\omega \quad (5.6.32)$$

$$= \int \cdots \int_{\partial(\partial M)} \omega = \int \cdots \int_{\emptyset} \omega = 0 \quad (5.6.33)$$

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