MAT224 Notes

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1.1 Vector spaces

Definition A <u>real</u> [1] **vector space** is a set V together with two vector operations vector addition and scalar multiplication such that

1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$

[1] A vector space is real if scalar which defines scalar multiplication is real.

2. **C** Commutative: $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. **Z** Zero Vector: $\exists\ \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$
5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists - \vec{x} \in V s.t. \vec{x} + (-\vec{x}) = \vec{0}$
6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$
7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c + d)\vec{x} = c\vec{x} + d\vec{x}$
9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$
10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

Note For V to be a vector space, need to know or be given operations of vector additions multiplication and check <u>all</u> 10 properties hold.

1.2 Examples of vector spaces

Example 1 $\mathbb{R}^n$ w.r.t. usual component-wise addition and scalar multiplication.

Example 2 $\mathbb{M}_{m\times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

Example 3 $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree <u>less or equal</u> to n, w.r.t. usual degree-wise polynomial addition and scalar multiplication.

Note If define $\mathbb{P}^\star_n(\mathbb{R})$ as set of all polynomials of degree <u>exactly equal</u> to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

Explanation: $(1+x^n), (1-x^n) \in \mathbb{P}^\star_n(\mathbb{R})$ but $(1+x^n)+(1-x^n) = 2 \notin \mathbb{P}^\star_n(\mathbb{R})$

Example 4 Something unusual, define V as

$$V = \{(x_1, x_2)|x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

<u>This is a vector space.</u>

1.3 Some properties of vector spaces

Suppose V is a vector space, then it has the following properties.

Property 1 The zero vector is unique.
proof.

$$\text{Assume } \vec{0}, \vec{0^\star} \text{ are two zero vectors in } V$$
$$\text{WTS: } \vec{0} = \vec{0^\star}$$
$$\text{Since } \vec{0} \text{ is the zero vector, by Z } \vec{0^\star} + \vec{0} = \vec{0^\star}$$
$$\text{Similarly, } \vec{0} + \vec{0^\star} = \vec{0}$$
$$\text{Also, } \vec{0} + \vec{0^\star} = \vec{0^\star} + \vec{0} \text{ by commutative vector addition.}$$
$$\text{So, } \vec{0^\star} = \vec{0}$$

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Property 2 $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.
proof.
Exercise.

Property 3 $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.
proof.

$$\text{By property of number 0: } 0\vec{x} = (0+0)\vec{x}$$
$$\text{By DSA: } 0\vec{x} = 0\vec{x} + 0\vec{x}$$
$$\text{By AI, } \exists(-0\vec{x}) s.t.$$
$$0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x})$$
$$\text{By AA}$$
$$\implies 0\vec{x} = \vec{0}$$

Property 4 $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$
proof.
Exercise.

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2.1 Some properties of vector spaces-Cont'd

Property 5 For a vector space V, $\forall \vec{x} \in V,\ (-1)\vec{x} = (-\vec{x})$. (*we could use this property to find the additive inverse with scalar multiplication with (-1)*).
proof.

$$\begin{aligned}(-\vec{x}) &= (-\vec{x}) + \vec{0} \quad \text{By property of zero vector}\\ &= (-\vec{x}) + 0\vec{x} \quad \text{By property3}\\ &= (-\vec{x}) + (1 + (-1))\vec{x} \quad \text{By property of zero as real number}\\ &= (-\vec{x}) + 1\vec{x} + (-1)\vec{x}\\ &= \vec{0} + (-1)\vec{x}\\ &= (-1)\vec{x}\end{aligned}$$

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Property 6 For a vector space V, let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \vee \vec{x} = \vec{0}$$

proof.
Exercise.

2.2 Subspaces

Loosely A subspace is a space contained within a vector space.

Definition Let V be a vector space and $W \subseteq V$, W is a **subspace** of V if W is itself a vector space w.r.t. operations of vector addition and scalar multiplication from V.

Theorem Let V be a vector space, and $W \subseteq V$, W has the same operations of vector addition and scalar multiplication as in V. Then, W is a subspace of V iff:

1. W is non-empty. $W \neq \emptyset$.
2. W is closed under addition. $\forall \vec{x}, \vec{y} \in W,\ \vec{x} + \vec{y} \in W$.
3. W us closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

Proof.

Forward:

If W is a subspace
$\implies \vec{0} \in W$
$\implies W \neq \emptyset$
Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$
WTS. 10 axioms in definition of vector space hold
$(ii) \implies$ Additive Closure
$(iii) \implies$ Scalar Multiplication Clousure
Because $W \subseteq V$, and V is a vector space, so properties hold $\forall \vec{w} \in W$.
Additive inverse: by property 5 and scalar multiplication closure,
$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W$.
Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W$.

2.3 Examples of subspaces

Example 1 Let $V = \mathbb{M}_{n\times n}(\mathbb{R})$, V is a subspace.

Example 2 Define W as

$$W = \{A \in \mathbb{M}_{n\times n}(\mathbb{R}) | A \text{ is } \underline{\text{not}} \text{ symmetric}\}$$

Explanation: Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but $A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$.

Example 3 Let $V = \mathbb{P}_2(\mathbb{R})$, is W defined as following,

$$W = \{p(x) \in V | p(1) = 0\}$$

a subspace of V ?
proof.
WTS: (i)
Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$
$\implies W \neq \emptyset$
WTS: (ii)
Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$

$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$
$\implies p_1 + p_2 \in W$
$\implies W$ is closed under addition.
WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$
$\implies p(1) = 0$
Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$
$\implies cp \in W$.
So W is a subspace of V.
$\blacksquare$

2.4 Recall from MAT223

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of $\mathbb{R}^n$ and $Col(A)$ is a subspace of $\mathbb{R}^m$.

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3.1 Linear Combination

Definition Let V be a vector space, $\vec{v_1}, \dots, \vec{v_n} \in V$, $a_1, \dots, a_n \in \mathbb{R}$ the expression

$$c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

is called a **linear combination** of $\vec{v_1}, \dots, \vec{v_n}$.

Theorem Let V be a vector space, W is a subspace of V, $\forall \vec{w_1}, \dots \vec{w_k} \in W, c_1, \dots, c_k \in \mathbb{R}$, we have

$$c_1\vec{w_1} + \cdots + c_k\vec{w_k} \in W$$

Subspaces are <u>*closed*</u> *under linear combinations, since subspaces are closed under scalar multiplication and vector addition.*

Theorem Let V be a vector space, let $\vec{v_1}, \dots, \vec{v_k} \in V$ then the set of all linear combination of $\vec{v_1}, \dots, \vec{v_k}$

$$W = \{\sum_{i=i}^{k} c_i\vec{v_i} | c_i \in \mathbb{R} \forall\ i\}$$

is a subspace of V.
proof.

$$\text{Consider } \vec{0} \in W$$
$$\text{So, } W \neq \emptyset$$
$$\text{Let } c \in \mathbb{R}, \text{ Let } \vec{x} \in W \wedge \vec{y} \in W$$
$$\text{By definition of span, we have,}$$
$$\vec{x} = \sum_{i=1}^{k} a_i \vec{v_i}, \quad \vec{y} = \sum_{i=1}^{k} b_i \vec{v_i}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\vec{x} + c\vec{y} = \sum_{i=1}^{k} a_i \vec{v_i} + c \sum_{i=1}^{k} b_i \vec{v_i} = \sum_{i=1}^{k} (a_i + cb_i)\vec{v_i} \in W$$

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Definition Let V be a vector space, $\vec{v_1}, \ldots, \vec{v_k} \in V$, **span** of the set of vectors $\{\vec{v}_i\}_{i=1}^{k}$ is defined as the collection of all possible linear combinations of $\{\vec{v}_i\}_{i=1}^{k}$. By pervious theorem, span is a subspace.

3.2 Combination of subspaces

Definition Let W_1, W_2 be two sets, then the **union** of W_1, W_2 is defined as:

$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

the **intersection** of W_1, W_2 is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

Now consider W_1, W_2 to be two subspaces of vector space V, then we have,

1. $W_1 \cup W_2$ is **not** a subspace.
2. $W_1 \cap W_2$ is a subspace.

proof.

$$\text{Falsify the statement by providing counter-example:}$$
$$\text{Consider,}$$
$$W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$
$$W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$$
$$\text{But, } \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$$

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proof.

$$\text{Because } W_1 \text{ and } W_2 \text{ are both subspaces, so}$$
$$\vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset$$
$$\text{Let } \vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\text{Sine } W_1, W_2 \text{ are subspaces,}$$
$$\vec{x} + c\vec{y} \in W_1 \land \vec{x} + c\vec{y} \in W_2$$
$$\implies \vec{x} + c\vec{y} \in W_1 \cap W_2$$
$$\text{So, } W_1 \cap W_2 \text{ is a subspace.}$$

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Definition Let W_1, W_2 be subspaces of vector space V, define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \land \vec{y} \in W_2\}$$

Note Let $\vec{x} = \vec{0} \in W_1$, $\forall \vec{y} \in W_2,\ \vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = 0 \in W_2$, $\forall \vec{x} \in W_1,\ \vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2,\ \vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

Note $W_1 + W_2$ is a subspace of V.
proof.

$$\begin{gathered}\text{Let } \vec{x_1}, \vec{x_2} \in W_1, \vec{y_1}, \vec{y_2} \in W_2 \\ \text{By properties of subspaces,} \\ \forall c \in \mathbb{R}, \vec{x_1} + c\vec{x_1} \in W_1 \wedge \vec{y_2} + c\vec{y_2} \in W_2 \\ \text{Consider, } \vec{x_1} + \vec{y_1} \in W_1 + W_2, \vec{x_2} + \vec{y_2} \in W_1 + W_2 \\ (\vec{x_1} + \vec{y_1}) + c(\vec{x_2} + \vec{y_2}) \\ = (\vec{x_1} + c\vec{x_2}) + (\vec{y_1} + c\vec{y_2}) \in W_1 + W_2\end{gathered}$$

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Definition Let W_1, W_2 be subspaces of vector space V, say V is **direct sum** of W_1 and W_2, written as $V = W_1 \bigoplus W_2$, if every $\vec{x} \in V$ can be written <u>uniquely</u> as $\vec{x} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in W_1$ and $\vec{w_2} \in W_2$.
Equivalently Let W_1 and W_2 be subspaces of V, $V = W_1 \bigoplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.