ECO326 Advanced Microeconomic Theory
A Course in Game Theory

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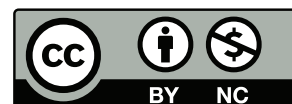
Github Page https://github.com/TianyuDu/Spikey_UofT_Notes
Note Page TianyuDu.com/notes

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory*.

Contents

1 Lecture 1. Games and Dominant Strategies

Assumption 1.1 (pg.4)**.** Assume that each decision-maker is *rational* in the sense that he is aware of his alternatives, forms expectation about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization.

Definition 1.1 (pg.4)**.** A model of **rational choice** consists

- A set A of *actions.*

- A set C of *consequences.*

- A *consequence function* $g : A \to C$.

- A *preference relation* $\succsim$ on C.

Definition 1.2 (pg.7)**.** A **preference relation** is a <u>complete reflexive and transitive</u> binary relation.

Definition 1.3 (11.1)**.** A **strategic game** consists of

- a <u>finite</u> set of **players** N.

- for each player $i \in N$, an **actions** $A_i \neq \emptyset$.

- for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i) \rangle$.

Definition 1.4 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2 Lecture 2. Iterated Elimination and Rationalizability

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (60.2). The set $X \subseteq A$ of outcomes of a <u>finite</u> strategic game $\langle N, (A_i), (u_i) \rangle$ **survives iterated elimination of** <u>strictly</u> **dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $\overline{((X_j^t)_{j \in N})_{t=0}^T}$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.

- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \ldots, T-1$.

- For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is <u>strictly</u> <u>dominated</u> in the game $\langle N, (X_i^t), (u_i^t) \rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.

- No action in X_t^T is strictly dominated in game $\langle N, (X_i^T), (u_i^T) \rangle$.

Proposition 2.1 (61.2). If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a <u>finite</u> strategic game $\langle N, (A_i), (u_i) \rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

2.2 Rationalizability

Definition 2.2 (pg.15). The **best-response function** for a player i is defined as

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i}) \ \forall a_i' \in A_i\}$$

Remark 2.1. The best-response of a_{-i} can be written as

$$B_i(a_{-i}) = \bigcap_{a_i' \in A_i} \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i})\}$$

where each of them is the upper contour set of a_i'.
Thus, if $\succsim_i$ is quasi-concave, then $B_i(a_{-i})$ is an intersection of convex sets and therefore itself convex.

Definition 2.3 (pg.54). A **belief** of player i (about the actions of the other players) is a <u>probability measure</u>, μ_i, on $A_{-i} = \times_{j \in N \backslash \{i\}} A_j$.
μ_i is a mapping such that

- $\mu_i : A_{-i} \to [0, 1]$.

- $\mu_i(A_{-i}) = 1$.

- For all countable piece-wise <u>disjoint</u> collection $\{E_i\}_{i \in I}$, it satisfies the *countable additivity property*:

$$\mu_i(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu_i(E_i)$$

Definition 2.4 (lec.2)**.** For a player $i \in N$, $a_i^* \in A_i$ is the **best response to belief** μ_i in a strategic game $\langle N, (A_i), (u_i) \rangle$ if and only if

$$\forall a_i \in A_i, \quad \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i})$$

Equivalently,

$$\forall a_i \in A_i, \ \mathbb{E}[u_i(a_i^*, a_{-i}) | \mu_i] \geq \mathbb{E}[u_i(a_i, a_{-i}) | \mu_i]$$

Definition 2.5 (59.1)**.** An action of player i in a strategic game is a **never best response** if it is not a best response to any <u>belief</u> of player i.

Definition 2.6 (lec.2)**.** For player $i \in N$, action $a_i \in A_i$ is **rationalizable** if it survives from the iterated elimination of never best responses.

Definition 2.7 (59.2)**.** The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **strictly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

3 Lecture 3. Nash Equilibrium

Definition 3.1 (14.1)**.** A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $a^* \in A$ of actions with property that for every player $i \in N$

$$(a_i^*, a_{-i}^*) \succsim_i (a_i, a_{-i}^*) \ \forall a_i \in A_i$$

Proposition 3.1 (pg.15, equivalent definition of Nash equilibrium)**.** So a Nash equilibrium is a profile $a^* \in A$ such that

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

Proposition 3.2 (lec.3)**.** No strategy that is eliminated during iterated deletion of never best response can be played in Nash equilibrium.

Lemma 3.1 (pg.19)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if equivalent to the following statement:
Define set-valued function $B : A \to A$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and there exists $a^* \in A$ such that $a^* \in B(a^*)$.

Lemma 3.2 (20.1 Kakutani's fixed point theorem)**.** Let X be a <u>compact convex subset</u> of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- for all $x \in X$ the set $f(x)$ is non-empty and convex.

- the graph of f is closed. *(i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$ then $y \in f(x)$)*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Definition 3.2 (pg.20)**.** A preference relation $\succsim_i$ over A is quasi-concave on A_i if for every $a^* \in A$ the upper contour set over a_i^*, given other players' strategies

$$\{a_i \in A_i : (a_{-i}^*, a_i) \succsim_i a^*\}$$

is convex.

Proposition 3.3 (20.3)**.** The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a Nash equilibrium if for all $i \in N$,

- the set A_i of actions of player i is a nonempty <u>compact convex</u> subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous

- quasi-concave on A_i.

Proof. Let $B : A \to A$ be a correspondence defined as

$$B(a) := \times_{i \in N} B_i(a_{-i})$$

Note that for each $a \in A$ and for each $i \in N$,
$B_i(a_{-i}) \neq \emptyset$ since preference $\succsim_i$ is continuous and A_i is compact (EVT).

Also $B_i(a_{-i})$ is convex since it's basically an intersection of upper contour sets and each of those upper contour is convex since $\succsim_i$ is quasi-concave.

So the Cartesian product of the finite collection of B_i is non-empty and convex.

Also the graph B is closed since $\succsim_i$ is continuous.

So there exists $a^* \in A$ such that $a^* \in B(a^*)$.

So Nash equilibrium presents. ∎

Definition 3.3 (lec.3). A **strict Nash equilibrium** is an action profile $a^* \in A$ where all players are playing their <u>unique</u> best response. That is, for every player $i \in N$, the image of their best response $B_i(a^*_{-i})$ is singleton,

$$\forall i \in N \ B_i(a^*_{-i}) = \{a^*_i\}$$

Definition 3.4 (lec.3). Otherwise, a Nash equilibrium is a **weak Nash equilibrium**.

4 Lecture 4. Nash Equilibrium: Examples

5 Lecture 5. Mixed Strategies

Definition 5.1 (lec.5). For player $i \in N$, a **mixed strategy** σ_i is a <u>probability distribution</u> over A_i.

Remark 5.1 (lec.5). A pure strategy $a_i \in A_i$ is a mixed strategy with

$$\sigma_i(a_i) = 1$$

So mixed strategy is a generalization of pure strategy.

Proposition 5.1 (lec.5). The **expected payoff** from mixed strategy profile $(\sigma_i) \equiv (\sigma_i, \sigma_{-i})$ is

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}[u_i(a)|(\sigma_i)] = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i})$$

Assumption 5.1 (lec.5). Assuming all agents follows Von-Neumann Morgenstern theorem.

Definition 5.2 (lec.5). An action a_i is **strictly dominated** by mixed strategy σ_i if and only if

$$\forall a_{-i} \in A_{-i} \ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i})$$

where σ_i could be a pure strategy.

Notation 5.1. Let A be a set and let $\Delta(A)$ denote the <u>set of probability measures</u> on set A.

Definition 5.3 (lec.5)**.** A mixed strategy σ_i is a **best response** to σ_{-i} if and only if

$$\forall \sigma_i' \in \Delta(A_i) \ U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

Definition 5.4 (lec.5)**.** The **support** of a mixed strategy $\sigma_i \in \Delta(A_i)$ is the set

$$supp(\sigma_i) = \{a_i \in A_i : \sigma_i(a_i) \neq 0\}$$

Proposition 5.2 (lec.5)**.** A mixed strategy σ_i is a **best response** to an strategy profile σ_{-i} if and only if

(a) Player i is indifferent between all a_i in the support of σ_i,

$$\forall a_j, a_j' \in supp(\sigma_i) \quad a_j \sim_i a_j'$$

(b) <u>and</u> player i weakly prefers all actions in the support of σ_i to those not in the support of σ_i. That's

$$\forall a_j \in supp(\sigma_i), \ \forall a_j' \notin supp(\sigma_i) \quad a_j \gtrsim_i a_j'$$

Proof. ($\implies$) show the if parts by proving it's contraposition.
Suppose (a) is not true, then

$$\exists a_i, a_j \in supp(\sigma_i) \ s.t. \ a_i \not\sim_i a_j$$

WLOG, suppose

$$u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

then σ_i would not be the best response since we can refine it by assigning

$$\begin{cases} \sigma_i'(a_i) = \sigma_i(a_i) + \sigma_i(a_j) \\ \sigma_i'(a_j) = 0 \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

and σ_i' would provides higher expected payoff.
Suppose (b) does not hold,

$$\exists a_i \notin supp(\sigma_i) \ s.t. \ \exists a_j \in supp(\sigma_i) \ s.t \ u_i(a_i, \sigma_{-i}) > u_i(a_j, \sigma_{-i})$$

Then σ_i could not be a best response since we can construct another mixed strategy σ_i' strictly dominating σ_i by setting

$$\begin{cases} \sigma_i'(a_j) = 0 \\ \sigma_i'(a_i) = \sigma_i(a_j) \\ \sigma_i'(a_k) = \sigma_i(a_k) \text{ otherwise} \end{cases}$$

($\Longleftarrow$) Assuming σ_i is not a best response towards σ_{-i}, then there exists $\sigma_i' \in \Delta(A_i)$ such that

$$U_i(\sigma_i', \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i})$$
$$\Longleftrightarrow \mathbb{E}[u_i(a)|(\sigma_i', \sigma_{-i})] > \mathbb{E}[u_i(a)|(\sigma_i, \sigma_{-i})]$$
$$\Longleftrightarrow \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i'(a_i)\sigma_{-i}(a_{-i}) > \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i})$$

Probability measures σ_i and σ_i' could only be different in two aspects, their supports and the values assigned on elements in their supports, this fails assumption (a).

The following argument needs to be revised.

Case 1 suppose $supp(\sigma_i) = supp(\sigma_i')$, then the strictly inequality in expected payoffs implies redistributing probabilities does affect the expected payoffs. So player i cannot be indifferent between any two actions in the support.

Case 2 suppose $supp(\sigma_i) \neq supp(\sigma_i')$ and $supp(\sigma_i') \nsubseteq supp(\sigma_i)$. That's

$$\exists a_i \in supp(\sigma_i') \wedge \notin supp(\sigma_i)$$

Then extending the support to a_i of σ_i gives higher expected payoff, this fails the assumption (b).

Case 3 suppose $supp(\sigma_i') \subsetneq supp(\sigma_i)$. Then the expected payoff can be strictly increased by eliminating actions in $supp(\sigma_i)\backslash supp(\sigma_i')$. Then those actions eliminated must be strictly dominated by actions in $supp(\sigma_i')$. This fails assumption (a). ∎

Proposition 5.3 (lec.5 equivalent proposition). All actions in the support are best responses. (i.e.*best response mixed strategy is a mixture of best response pure actions*)

Remark 5.2 (lec.5 Intuition of proposition). If the requirements of above proposition are not satisfied, the player can reduce the probability assigned to the non-best-response pure action and better off.

Theorem 5.1 (lec.5 Nash's Theorem). Any player $i \in N$ in finite game $\langle N, (A_i), (\succsim_i) \rangle$ has a mixed strategy Nash equilibrium.

6 Lecture 6. Extensive Form Games and Subgame Perfection

6.1 Extensive Form Game

Definition 6.1 (89.1). An **extensive game with perfect information** has the following components.

- A set N of **players**.

- A set H of sequences (finite or infinite) of **histories** with properties:

    - $\emptyset \in H$.
    - For all $L < K$, $(a^k)_{k=1,2,\ldots,K} \in H \implies (a^k)_{k=1,2,\ldots,L} \in H$.
    - For infinite sequence $(a^k)_{k=1}^\infty$,
      $(a^k)_{k=1,2,\ldots,L} \in H, \ \forall L \in \mathbb{Z}_{++} \implies (a^k)_{k=1}^\infty \in H$.

    And each component of history $h \in H$ is an **action** taken by a player.

- A function $P : H\backslash Z \to N$, where for $h \in H$, $P(h) \in N$ is defined by the player who takes an action <u>after</u> the history h.

- For each player $i \in N$ a **preference relation** $\succsim_i$ <u>defined on Z</u>.

Notation 6.1 (pg.90). An extensive game with perfect information can be represented by a 4-tuple, $\langle N, H, P, (\succsim_i) \rangle$. *Sometimes it is convenient to specify the structure of an extensive game without specifying the players' preference, as $\langle N, H, P \rangle$.*

Definition 6.2 (pg.90). A history $(a^k)_{k=1,2,\ldots,K} \in H$ is **terminal** if

1. it is infinite,

2. <u>or</u> (i.e. it cannot be extended to another valid history sequence)

$$\forall a^{K+1}, \ (a^k)_{k=1,2,\ldots,K+1} \notin H$$

The set of terminal histories is denoted by Z.

Notation 6.2 (pg.90, the action set). After any nonterminal history, $h \in H\backslash Z$, the player $P(h)$ chooses an action from set

$$A(h) = \{a : (h, a) \in H\}$$

Remark 6.1. Note that all player function, action set and player preference relation are defined on H. Thus, unlike a normal form game, which was *player oriented*, we'd better consider an extensive form game as *history oriented*.

Definition 6.3 (pg.90)**.** We refer to the empty set, which is required to be an element of H, as the **initial history**.

Definition 6.4 (92.1)**.** A **strategy of player** $i \in N$, s_i, in an extensive game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Remark 6.2 (pg.92)**.** A strategy specifies the action chosen by a player for *every* history after which it is his turn to move, *even for histories that is, if the strategy is followed, are never reached.*

Definition 6.5 (pg.93)**.** For each strategy profile $s = (s_i)_{i \in N}$ in the extensive game $\langle N, H, P, (\succsim_i) \rangle$, the **outcome** of s, $O(s)$, is defined as the <u>terminal history</u> that results when each player $i \in N$ follows the precepts of s_i. That is, $O(s)$ is the (possibly infinite) history

$$(a^1, \ldots, a^K) \in Z$$

such that

$$\forall k \in \{0, 1, \ldots K-1\}, \ s_{P(a^1, \ldots, a^k)}(a^1, \ldots, a^k) = a^{k+1}$$

Definition 6.6 (lec.6)**.** A extensive game $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is finite if and only if

(a) N is finite.

(b) (A_i) are all finite.

(c) All $h \in H$ reach the terminal state with finite length.

Definition 6.7 (93.1)**.** A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$\forall s_i \in S_i, \ O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

Definition 6.8 (94.1)**.** The **strategic form of the extensive game with perfect information**, $\Gamma = \langle N, H, P, (\succsim_i) \rangle$, is the strategic game $\langle N, (S_i), (\succsim'_i) \rangle$ in which for each player $i \in N$

- S_i is the **set of strategies** of player i in Γ.

- $\succsim_i'$ is defined on $\times_{i \in N} S_i$ and defined by

$$\forall s, s' \in \times_{i \in N} S_i, \ s \succsim_i' s' \iff O(s) \succsim_i O(s')$$

Definition 6.9 (pg.94). A **reduced strategy** of player i is defined to be a function f_i whose domain is a *subset* of $\{h \in H : P(h) = i\}$ and has the following properties

1. it associates with every history h in the domain of f_i an action in $A(h)$.

2. a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i. (i.e., for any $h = (a^k)$ and for any $h' = (a^k)_{k=1}^L$ as a subsequence of h such that $P(h') = i$, $f_i(h') = a^{L+1}$.)

Remark 6.3 (pg.94). Each **reduced strategy** of player i corresponds to a set of strategies of player i, such that for each vector of strategies of the other players each strategy in this set yields the same outcome. (strategies in the same set are **outcome-equivalent**.)
That's, for each strategy $s_i \in S_i$, its reduced strategy can be defined with an outcome equivalence class, $[s_i]$,

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j, \ O(s_{-i}, s_i){=}O(s_{-i}, s_i')\}$$

But in some other game, the definition of outcome-equivalence is more general and defined by generating the same payoff (through possibly difference outcomes), then the reduced strategy is defined as

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j, \ \forall j \in N, \ O(s_{-i}, s_i){\sim_j}O(s_{-i}, s_i')\}$$

Definition 6.10 (95.1.1). Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim_i') \rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i, s_i' \in S_i$ to be **equivalent** if

$$\forall s_{-i} \in S_{-i}, \ \forall j \in N, \ (s_{-i}, s_i) \sim_j' (s_{-i}, s_i')$$

Definition 6.11 (95.1.2). The **reduced strategic form of** Γ is the strategic game $\langle N, (S_i'), (\succsim_i'') \rangle$ in which for each $i \in N$ each set S_i' contains one member of each set of equivalent strategies in S_i and $\succsim_i''$ is the preference ordering over $\times_{j \in N} S_j'$ induced by $\succsim_i'$.

6.2 Subgame Perfection

Definition 6.12 (97.1). The **subgame of extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h) \rangle$ where

- $H|_h$ is the set of sequences h' such that $(h, h') \in H$.

- $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$.

- $\succsim_i |_h$ is defined by $h' \succsim_i |_h h'' \iff (h, h') \succsim_i (h, h'') \in Z$.

Notation 6.3 (pg.97). Given strategy $s_i \in S_i$ and $h \in H \in \Gamma$, $s_i|_h$ represents the **strategy that s_i induces in the subgame** $\Gamma(h)$. That's, for each $h' \in H_h$

$$s_i|_h(h') \equiv s_i(h, h')$$

Notation 6.4. Let O_h denote the **outcome function of $\Gamma(h)$**, that's, for all $h' \in H|_h$,

$$O_h(h') \equiv O(h, h')$$

Definition 6.13 (97.2). A **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i |_h O_h(s^*_{-i}|_h, s_i|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.

Definition 6.14 (pg.97). Equivalently, define SPNE to be a strategy profile s^* in Γ for which for any history $h \in H$ the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Remark 6.4 (pg. 97). The notion of SPNE requires the action prescribed by each player's strategy to be optimal, given other players' strategies, after *every* history.

Proposition 6.1 (99.2). Every finite extensive game with perfect information has a subgame perfect equilibrium.