Introduction to Real Analysis

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1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R}\ s.t.\ \forall a \in A,\ u \geq a \tag{1.1}$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R}\ s.t.\ \forall a \in A,\ l \leq a \tag{1.2}$$

Example 1.1. The set of integers, $\mathbb{Z}$, is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subset \mathbb{R}$, we use $A^{\uparrow}$ and $A^{\downarrow}$ to denote collections of upper bounds of A and lower bounds of A. Note that $A^{\uparrow}$ and $A^{\downarrow}$ are potentially empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound(supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^{\uparrow}$ and $\forall u \in A^{\uparrow},\ s \leq u$. Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if $f \in A^{\downarrow}$ and $\forall l \in A^{\downarrow},\ l \leq f$. Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property)**.** $\forall \varnothing \neq A \subset \mathbb{R}$ such that $A^{\uparrow} \neq \varnothing$, $\exists \sup A$.

Definition 1.4. Let $\varnothing \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

Proposition 1.1. Let $\varnothing \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \tag{1.3}$$

Proof. Step 1: Show $c + \sup A \in (c + A)^{\uparrow}$*:*
Let $x \in c+A$, $\exists\ a \in A\ s.t.\ x = c+a$. Then, $x = c+a \leq c+\sup A$. Therefore, $x \leq c+\sup A\ \forall x \in A$, which implies what desired.
Step 2: Show $\forall u \in (c + A)^{\uparrow},\ c + \sup A \leq u$*:*
Let $u \in (c + A)^{\uparrow}$, then $u \geq c + a\ \forall a \in A \implies u - c \geq a\ \forall a \in A \implies u - c \in A \uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.
Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum)**.** Let $s \in A^{\uparrow}$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

(i) $s = \sup A$;

(ii) $\forall \varepsilon, \exists a \in A,\ s.t.\ a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^{\uparrow}$).

Proof. Immediately. ■

Theorem 1.1 (Nested Interval Property)**.** Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \ \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing \tag{1.5}$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k, by the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$. Since $a^* \in (a_n)^{\uparrow}$, $a^* \geq a_n \ \forall n \in \mathbb{N}$. Further, because a^* is the *least* upper bound, then for every upper bound b_n, it must be $a^* \leq b_n \ \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n] \ \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Theorem 1.2 (Archimedean Property)**.**

(i) $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}\ s.t.\ n > x$;

(ii) $\forall y \in \mathbb{R}_{++},\ \exists n \in \mathbb{N}\ s.t. \frac{1}{n} < y$.

Archimedean property *of natural numbers* can be interpreted as *there is no real number that bounds* $\mathbb{N}$. This interpretation can be seen by considering the negations of above statements:

(i) $\exists x \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N},\ n \leq x$;

(ii) $\exists y \in \mathbb{R}_{++}\ s.t.\ \forall n \in \mathbb{N},\ y \leq \frac{1}{n}$.

Proof of (i) by Contradiction. Suppose the negated statement (i) is true, $\mathbb{N}$ is bounded above. By the completeness axiom, there exists $a^* := \sup \mathbb{N}$. $\exists n \in \mathbb{N}\ s.t.\ a^* - 1 < n$. In this case, $a^* < n + 1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^{\uparrow}$ and leads to a contradiction. ■

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ■

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The above theorem says $\mathbb{Q}$ is in fact **dense** in $\mathbb{R}$. More generally, one says a set $A \subset X$ is dense whenever the closure of A, $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.
Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ■

Theorem 1.4. $\exists \alpha \in \mathbb{R}\ s.t. \alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in $\mathbb{R}$ bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^{\uparrow}$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ■