Notes on MAT223

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1 Sep. 18. Lecture notes @SS2102

Span of $v = \{\vec{v_1}, ..., \vec{v_m}\}$ is the set of all *linear combinations* of vectors in v.

$$span\{\vec{v_1}, ..., \vec{v_m}\} = \{\Sigma_{i=1}^{m} c_i * v_i | \forall i, c_i \in \Re\}$$

$\vec{b} \in span\{\vec{v_1}, ..., \vec{v_m}\} \iff [\vec{v_1} \cdots \vec{v_m} \vec{b}]$ is consistent, that's the right most col. of mat. is not *pivot column.*

Null Vector $\vec{0}$

$$\vec{0} \in span\{\vec{v_1}, ..., \vec{v_m}\}$$

- $\forall i, c_i = 0$
- $\{\vec{v_i}\}$ is *linearly dependent.*

Examples

1. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $span\{\vec{u}\} = \begin{bmatrix} t \\ 2t \end{bmatrix}, t \in \Re$.
2. Let vector set $v = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$, then $span\{v\} = \{ \begin{bmatrix} s \\ t \\ t \end{bmatrix}, s, t \in \Re\}$

Law of Cosine Given a triangle with sides a, b, c. θ is the angle opposite to side c.

$$c^2 = a^2 + b^2 - 2 * a * b * \cos\theta$$

Theorem($\Re^2$ Case) Let $\vec{u}, \vec{v} \in \Re^2$,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| * \|\vec{v}\| * \cos\theta$$

Proof.
Let $AB = \vec{u}$, $AC = \vec{v}$, $CB = \vec{u} - \vec{v}$.
Ref. law of cosine.

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 * \|\vec{u}\| * \|\vec{v}\| * \cos\theta \quad (1)$$
$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \quad (2)$$
$$\cdots = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2 * \vec{u} \cdot \vec{v} \quad (3)$$
$$\text{so that,} - 2\vec{u} \cdot \vec{v} = -2 * \|\vec{u}\| * \cos\theta \quad (4)$$
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| * \|\vec{v}\| * \cos\theta \quad (5)$$

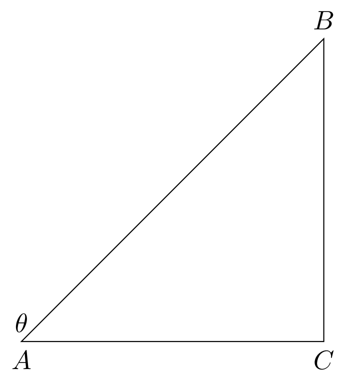

Figure 1: Proof of Theorem.

Corollary For $\|\vec{u}\|, \|\vec{v}\| \neq 0$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| * \|\vec{v}\|}$$

- $\vec{u} \cdot \vec{v} = 0 \rightarrow \theta = \frac{\pi}{2}$.
- $\vec{u} \cdot \vec{v} < 0 \rightarrow \theta \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$
- $\vec{u} \cdot \vec{v} > 0 \rightarrow \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$

Theorem $\Re^n$ Case Let $\vec{u}, \vec{v} \in \Re^n$,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| * \|\vec{v}\| * \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. When $\vec{u}$ and $\vec{v}$ meets at right angle, that is, if $\vec{u} \cdot \vec{v} = 0$, we say $\vec{u}$ and $\vec{v}$ are **orthogonal**.

Cauchy-Schwarz Inequality(CSI) Let $\vec{u}, \vec{v} \in \Re^n$, then

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| * \|\vec{v}\|$$

where equality holds when and only when $\vec{u}$ and $\vec{v}$ are *multiples (*linearly dependent)* of each other.

Proof.

$$
\begin{aligned}
0 &\leq \|(\vec{u} * \|\vec{v}\| \pm \vec{v} * \|\vec{u}\|)\|^2 \\
&= (\vec{u} * \|\vec{v}\| \pm \vec{v} * \|\vec{u}\|) \cdot (\vec{u} * \|\vec{v}\| \pm \vec{v} * \|\vec{u}\|) \\
&= \vec{u} \cdot \vec{u} * \|\vec{v}\|^2 \pm \vec{u} \cdot \vec{v} * \|\vec{v}\| * \|\vec{u}\| \pm \vec{u} \cdot \vec{v} * \|\vec{v}\| * \|\vec{u}\| + \vec{u} \cdot \vec{v} * \|\vec{u}\|^2 \\
&= \|\vec{u}\|^2 * \|\vec{v}\|^2 + \|\vec{u}\|^2 * \|\vec{v}\|^2 \pm 2 * (\vec{u} \cdot \vec{v} * \|\vec{u}\| * \|\vec{v}\|) \\
&\implies 2 * \|\vec{u}\|^2 * \|\vec{v}\|^2 \pm 2 * (\vec{u} \cdot \vec{v} * \|\vec{u}\| * \|\vec{v}\|) \\
&\implies \mp(\vec{u} \cdot \vec{v} * \|\vec{u}\| * \|\vec{v}\|) \leq \|\vec{u}\|^2 * \|\vec{v}\|^2 \\
&\implies \mp(\vec{u} \cdot \vec{v}) \leq \|\vec{u}\| * \|\vec{v}\| \\
&\text{Notice CSI holds if} \vec{v} = \vec{u} = \vec{0}, \text{ so assume} \vec{u}, \vec{v} \neq \vec{0}: \\
&\vec{u} \cdot \vec{v} \leq \|\vec{u}\| * \|\vec{v}\| \text{ and} - \vec{u} \cdot \vec{v} \leq \|\vec{u}\| * \|\vec{v}\| \\
&\implies |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| * \|\vec{v}\| \quad \blacksquare
\end{aligned} \tag{6}
$$

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Continuous Proof. for CSI
proof 1. multiple of each other $\implies$ equality.
Let $\vec{v} = c\vec{u}, \vec{v}, \vec{u} \in \Re^n, c \in \Re$

$$
\begin{aligned}
|\vec{v} \cdot \vec{u}| &= |c\vec{u} \cdot \vec{u}| \\
&= |c| * \|\vec{u}\|^2 \\
&= |c| * \|\vec{u}\| \cdot \|\vec{u}\| \\
&= \|\vec{v}\| \cdot \|\vec{u}\|
\end{aligned} \tag{7}
$$

proof 2. equality $\implies$ multiple of each other.

Distance between two vectors $\vec{u}$ and $\vec{v} \in \Re^n$is defined as

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

- $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$
- $d(\vec{v}, \vec{u}) = 0$

Triangle Inequality(1) Let $\vec{u}, \vec{v} \in \Re^n$

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

proof.

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \\
&= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
\text{(Ref. CSI)}|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\| \text{So that, } \|\vec{u}+\vec{v}\|^2 &\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\
\|\vec{u}+\vec{v}\|^2 &\leq (\|\vec{u}\| + \|\vec{v}\|)^2 \\
\text{Since, } \|\vec{a}\| &\geq 0 \forall \vec{a} \in \Re^n \\
\|\vec{u}+\vec{v}\| &\leq \|\vec{u}\| + \|\vec{v}\| \blacksquare
\end{aligned} \tag{8}$$

Triangle Inequality(2) for $\vec{u}, \vec{v}, \vec{w} \in \Re^n$, we have

$$d(\vec{v}, \vec{u}) \leq d(\vec{v}, \vec{u}) + d(\vec{u}, \vec{w})$$

proof.

$$\begin{aligned}
d(\vec{v}, \vec{u}) &= \|\vec{u} - \vec{v}\| \\
&= \|\vec{u} - \vec{w} + \vec{w} - \vec{v}\| \\
\text{Ref. Triangle Inequality(1)} &\leq \|\vec{u} - \vec{w}\| + \|\vec{w} - \vec{v}\| \\
&= d(\vec{u}, \vec{v}) + d(\vec{w}, \vec{v}) \blacksquare
\end{aligned} \tag{9}$$

Orthogonal sets Let set S = $\{\vec{v_1}, ..., \vec{v_m}\} \in \Re^n$, set s is **orthogonal** if and only if

$$\vec{v_i} \cdot \vec{v_j} = 0, \forall i \neq j \in \{1, 2, ..., m\}$$

Orthonormal sets For an **orthogonal set** s, we say s is **orthonormal** if and only if

$$\|\vec{v_i}\| = 1 \forall \vec{v_i} \in s$$

Orthonormal $\implies$ ***Orthogonal***.

Normalize Given $\vec{v} \in \Re^n$,

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

is a **unit vector** in the same direction.

Projection Let $\vec{v}, \vec{d} \neq \vec{0} \in \Re^n$, there is a new vector called $proj_{\vec{d}}\vec{v}$ such that,

- $proj_{\vec{d}}\vec{v}$ is *parallel* to $\vec{d}$.
- $proj_{\vec{d}}\vec{v}$ has tip *closest* point to $\vec{v}$ along the line in $\vec{d}$ direction.

$proj_{\vec{d}}\vec{v}$ is called the **projection** of $\vec{v}$ onto $\vec{d}$.

$$proj_{\vec{d}}\vec{v} = \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d}$$

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Projection Given $\vec{d} \neq \vec{0} \in \Re^n, \vec{v} \in \Re^n$, the projection of $\vec{v}$ on $\vec{d}$ is

$$proj_{\vec{d}}\vec{v} = \frac{\vec{d} \cdot \vec{v}}{\|\vec{d}\|^2} * \vec{d}$$

Component of $\vec{v}$ along $\vec{d}$ is

$$c = \frac{\vec{d} \cdot \vec{v}}{\|\vec{d}\|^2}$$

Consider system of equations:

$$(\star) \begin{cases} a * x_1 + b * x_2 + c * x_3 = g \\ d * x_1 + e * x_2 + f * x_3 = h \end{cases}$$

is equivalent to system:

$$x_1 \begin{pmatrix} a \\ d \end{pmatrix} + x_2 \begin{pmatrix} b \\ e \end{pmatrix} + x_3 \begin{pmatrix} c \\ f \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

is equivalent to **Matrix-vector multiplication equation**:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

in $\mathbf{A}\vec{x} = \vec{b}$ form.

Solvability The system $(\star)$ is solvable if and only if $\vec{b}$ is in the *span* of *columns* of $\mathbf{A}$, that is:

$$\vec{b} \in span\{columns of \mathbf{A}\}$$

or, $\vec{b}$ is a **linear combination** of *columns* of $\mathbf{A}$.

Matrix-vector multiplication In general, for $\vec{a_1}, \vec{a_2}, ..., \vec{a_n} \in \Re^m$

$$\mathbf{A} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & ... & \vec{a_n} \end{bmatrix}_{m \times n}$$

$\mathbf{A}\vec{x}$ is a *linear combination* of columns of $\mathbf{A}$ with **weights** the entries at $\vec{x}$.
$\mathbf{A}\vec{x}$ could be defined if and only if $\vec{x} \in \Re^{\sharp col.of \mathbf{A}}$.
Generally,

$$\mathbf{A}\colon \Re^{\sharp col.of \mathbf{A}} \rightarrowtail \Re^{\sharp row of \mathbf{A}}$$

Every linear system can be written as matrix equation:

$$\mathbf{A}\vec{x} = \vec{b}$$

where size of $\mathbf{A}$ is $[\sharp equations \times \sharp unknowns]$.
$\mathbf{A}\vec{x} = \vec{b}$ is solvable if and only if $\begin{bmatrix} \vec{a_1} & \vec{a_2} & ... & \vec{a_n} & \mid & \vec{b} \end{bmatrix}$ is the **augmented matrix** for a *consistent* system.

Theorem Let $\mathbf{A}$ is a $[m \times n]$ matrix, the following are equivalent.

1. $\forall \vec{b} \in \Re^m$,$\mathbf{A}\vec{x} = \vec{b}$ is solvable.
2. $\forall \vec{b} \in \Re^m$,$\vec{b}$ is aa *linear combination* of columns of $\mathbf{A}$.
3. Columns of $\mathbf{A}$ *spans/generates* $\Re^m$.
4. Every row of $\mathbf{A}$ has a *pivot position*.

1. **proof.** of (4) $\implies$ (1).
2. Suppose (4) holds, let $\vec{b} \in \Re^m$
3. Aug mat $\begin{bmatrix} \mathbf{A} & \mid & \vec{b} \end{bmatrix}$ has size $[m \times (n+1)]$.
4. Since every row of $\mathbf{A}$ has pivot position.
5. So that, the last column of $\begin{bmatrix} \mathbf{A} & \mid & \vec{b} \end{bmatrix}$ could not be a pivot column cause there is no spot.
6. So that, the system $\begin{bmatrix} \mathbf{A} & \mid & \vec{b} \end{bmatrix}$ is solvable.
7. So, (4) $\implies$ (1). ■

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Identity matrix For each $nin \in \mathbb{Z}^+$ there is a matrix $\mathbf{I}_n$ (*often n is omitted*). So that,

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\text{and, } \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is called } \vec{e_1} \text{ and } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \text{ is called } \vec{e_n}$$

$$\text{Set of vectors}\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}\text{is called } \textbf{Standard basis} \text{ of } \Re^n$$

For an identity matrix, we have:

$$I_n \cdot \vec{x} = I_n \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

Dot product rule of matrix Use dot product to calculate $\mathbf{A}\cdot\vec{x}$ We have:

$$\mathbf{A} \cdot \vec{x} = \begin{bmatrix} row_1(A) \cdot \vec{x} \\ row_2(A) \cdot \vec{x} \\ \vdots \\ row_n(A) \cdot \vec{x} \end{bmatrix}$$

Rules Let $A, B \in M_{m \times n}(\Re)$, we have:

$$\mathbf{A}(\vec{x} + \vec{y}) = \mathbf{A}\vec{x} + \mathbf{A}\vec{y}, \forall \vec{x}, \vec{y} \in \Re^n$$

and,

$$\mathbf{A}c\vec{x} = c\mathbf{A}\vec{x}, \forall c \in \Re$$

Solutions of linear system A linear system is called **homogeneous** if it can be write in the form $\mathbf{A}\vec{x} = \vec{0}$.

Fact Homogeneous $\implies$ consistency.

Explanations

- $\vec{x} = \vec{0}$ solves the system.(Called the **trivial** solution).
- Last column ($\vec{0}$) could be a pivot column.

Non-trivial Non-zero solutions are **non-trivial**, it is not necessary for a linear system to have non-trivial solution.

Theorem A *homogeneous* system $\mathbf{A}\vec{x} = \vec{0}$ has **non-trivial** solution if and only if there's a free variable.
Proof:

1. Homogeneous system is consistent.
2. So there is one unique solution or infinitely many solutions.
3. $\vec{0}$ is always a solution, and a trivial solution.
4. If there exist other solution, there are infinitely many solutions.
5. There should be at least one free variable to create infinitely many solutions.

Example:

$$
\begin{gathered}
\text{Let augmented matrix be:} \\
AugMat = \begin{bmatrix} 1 & -2 & 3 & -2 & 0 \\ 3 & 6 & 4 & 0 & 0 \\ 2 & 4 & 4 & -2 & 0 \end{bmatrix} \\
\text{In the form of } \mathbf{A}\vec{x} = \vec{0} \\
\text{Use reduction algorithm: } \mathbf{A} \sim \begin{bmatrix} 1 & -2 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{So that:} \\
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} \\
= span\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} \}
\end{gathered} \tag{10}
$$

Theorem If a matrix $\mathbf{A}_{m\times n}$ is a *homogeneous* system with more variables than equations, there are infinitely many solutions.

Proof. If $n > m$, then not every variable can be basic, since a pivot would have to go in a row and column, but too many columns. So there's at least one free variables.

5 Sep. 27. Lecture notes @SS2102

Example A **conic** is graph of an equation in form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0\text{a,b,c not all zero}$$

Show that a conic goes through any 5 *non-colinear* points in plane.
Proof.
Consider p_i, q_i where i = 1,2,3,4,5 on the conic curve.

$$ap_i^2 + bp_iq_i + cq_i^2 + dp_i + eq_i + f = 0 \text{ for i = 1,2,3,4,5}$$

so there are **5** equations and **6** variables (a,b,c,d,e,f) which means there are *more variables than equations.* Refer to theorem above (pervious lecture), three are **infinitely many** solutions.
If a,b,c are all zeros, equations are reduced to:

$$dp_i + eq_i + f = 0 \text{ for i = 1,2,3,4,5}$$

which contributes a linear, so the solutions for the system when a,b,c are all zeros, are **co-linear**.*Shown by contradiction.*

5.1 Non-Homogeneous Systems

Example A non-homogeneous system with like:

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

By reduction algorithm, the reduced echelon form of associated augmented matrix is:

$$[\mathbf{A}\vec{b}] \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the solution would be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \text{ denote: } \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \vec{p} \text{ and, } \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = \vec{v_h}$$

and we find:

$$\mathbf{A}\vec{p} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$\vec{p}$ solves the non-homogeneous system. and:

$$\mathbf{A}\vec{v_h} = \vec{0}$$

$\vec{v_h}$ solves the corresponding homogeneous system. so that, $\vec{x} = \vec{p} + \vec{v_h}(t), t \in \mathbb{R}$ solves the non-homogeneous system. And the solution is in a **linear** form. thus, the solution of linear equation above is

$$\vec{x} = \{\vec{p} + \vec{v_h}(t), t \in \mathbb{R}\}$$

this is a line pass through $\vec{p}$ and in direction of $\vec{v_h}$, and is the *shifted version* of $\{t\vec{v_h} | t \in \mathbb{R}\} = span\{\vec{v_h}\}$.

$\vec{p}$ is called the **particular solution** to the system and $\vec{v_h}$ is the solution to the corresponding homogeneous form of the solution. $\vec{x}$ is therefore the **general solution** to the non-homogeneous system of equations.

Theorem If $\mathbf{A}\vec{x} = \vec{b}$ is consistent for a given $\vec{b}$, the solution to this system is $\vec{x} = \vec{p} + \vec{v_h}(t)$, where

$$\mathbf{A}\vec{p} = \vec{b} \text{ and } \mathbf{A}\vec{v_h} = \vec{0}$$

Proof.
Let $\vec{x}$ is a solution to the system and $\vec{p}$ is a *particular* solution solving $\mathbf{A}\vec{x} = \vec{b}$, so that we have:

$$\begin{gathered} \begin{cases} \mathbf{A}\vec{p} = \vec{b} \\ \mathbf{A}\vec{x} = \vec{b} \end{cases} \\ \mathbf{A}\vec{x} - \mathbf{A}\vec{p} = \vec{b} - \vec{b} = \vec{0} \\ \text{So that, } \vec{x} - \vec{p} \text{ solves } \mathbf{A}\vec{v} = \vec{0} \\ \text{Let } \vec{v_h} = \vec{x} - \vec{p} \text{ for a solution set of } \mathbf{A}\vec{v} = \vec{0} \\ \text{So that, } \vec{x} = \vec{v_h} + \vec{p} \quad \blacksquare \end{gathered} \tag{11}$$

6 Sep. 29. Lecture nots @SS2102

Recall $\{\vec{v_1}, ..., .\vec{v_k}\} in \mathbb{R}$, if $\vec{v} \in span(\{\vec{v_1}, ..., \vec{v_k}\})$, then we write:

$$\vec{v} = \sum_{j=1}^{k} c_j * \vec{v_j}$$

WTS if $\{c_i\}$ is *unique*. Equivalently, is there:

$$\{d_i\} \neq \{c_i\} s.t. \vec{v} = \sum_{j=1}^{k} d_j * \vec{v_j}$$

Let $\hat{c_i} = c_i - d_i$, want to show:

$$\sum_{j=1}^{k} \hat{c_j} * \vec{v_j} = \vec{0}$$

Definition Let $\{\vec{v_j}\}_{j=1}^{k} \subseteq \mathbb{R}^n$, if

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}$$

has only *trivial solution*, that's, $\vec{c_i} = \vec{0}$, the set $\{\vec{v_j}\}_{j=1}^{k}$ is *linearly independent*, else, it's called **linearly dependent**.

Proposition Let $\vec{v_1} \& \vec{v_2} \in \mathbb{R}^3$, $\vec{v_1} \& \vec{v_2}$ are *linearly independent* if and only if they are **not** parallel.
proof. Part1(Contrapositive): If they were parallel, then

$$\begin{gathered} \vec{v_1} = c\vec{v_2}, c \in \mathbb{R} \\ \text{If } c = 0, \text{ then, } \vec{v_1} = \vec{0} \\ \text{So that, } k\vec{v_1} + 0\vec{v_2} \neq \vec{0} \implies k \neq 0. \text{ so they are linearly dependent.} \\ \text{If } c \neq 0, \text{then} \vec{v_1} - c\vec{v_2} = \vec{0}, \text{there is non-trivial solution. So linearly dependent.} \end{gathered} \tag{12}$$

proof. Part2(Contrapositive): If $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent.

$$\begin{gathered} \text{When } c_1 \text{ and } c_2 \text{ are not both zero, satisfy that:} \\ c_1\vec{v_1} + c_2\vec{v_2} = \vec{0} \\ \textbf{WLOG,} c_1 \neq 0 \implies \vec{v_1} = -\frac{c_2}{c_1} * \vec{v_2} \implies \vec{v_1} \parallel \vec{v_2} \\ \text{So whenever two vectors in } \mathbb{R}^2 \text{ are linearly dependent, they are parallel.} \end{gathered} \tag{13}$$

Note is $\vec{0}$ is in a set, vectors in the set are **linearly dependent**. Since $\vec{0} * c = \vec{0}, \forall c \in \mathbb{R}$

Theorem Take non-zero vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and given that $\{\vec{v}, \vec{w}\}$ is linearly independent, then,

$$\vec{u} \notin span\{\vec{v}, \vec{w}\} \iff \{\vec{u}, \vec{v}, \vec{w}\}\text{is linearly independent.}$$

proof.(contrapositive, to prove: linear independency $\implies$ In span)

$$\begin{gathered}\text{Suppose } \vec{u} \in span\{\vec{v}, \vec{w}\} \\ \text{If } \vec{u} = c_1\vec{v} + c_2\vec{w} \\ \implies 1\vec{u} - c_1\vec{v} - c_2\vec{w} = \vec{0} \\ \text{There are non-trivial solution, the set is linearly dependent.}\end{gathered} \tag{14}$$

next.(contradiction)

$$\begin{gathered}\text{Suppose: } \vec{u} \notin span\{\vec{v}, \vec{w}\} \\ \text{Consider equation:} \\ c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0} \\ \textbf{Case 1: } c_1 = 0 \implies c_2\vec{v} + c_3\vec{w} = \vec{0} \\ \implies c_2 = c_3 = 0 \\ c_1 = c_2 = c_3 = 0, \text{so that set is linearly independent.} \\ \textbf{Case 2: } c_1 \neq 0 \implies \vec{u} = -\frac{c_2}{c_1}\vec{v} - \frac{c_3}{c_1}\vec{w} \\ \text{So, } \vec{u} \in span\{\vec{v}, \vec{w}\} \\ \text{By contradiction, so}\{\vec{u}, \vec{v}, \vec{w}\} \text{ are linearly independent.}\blacksquare\end{gathered} \tag{15}$$

Theorem For $\{\vec{v_1}, ..., \vec{v_k}\} \in \mathbb{R}^n$, if k ¿ n, then $\{\vec{v_1}, ..., \vec{v_k}\}$ are linearly dependent.
proof. let $\mathbf{A}\vec{x} = \vec{0}$, where $\mathbf{A} = [\vec{v_1}, \ldots, \vec{v_k}]$. Size of $\mathbf{A}$ and the system is a homogeneous system with more variables than equations. As long as it's consistent, where are free variables, which means the existence of infinitely many solutions and non-trivial solutions.

Linear Transformation Consider "Multiplication of $\vec{x}$ by $\mathbf{A}$ and returns $\vec{b}$", and the size of $\mathbf{A}$ is $m \times n$.

$$\mathbf{A}\vec{x} = \vec{b}$$

and represent it by:

$$T_{\mathbf{A}}(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^m = \mathbf{A}\vec{x}$$

where $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^{\ltimes}$ is the **codomain** of linear transformation T_A. **Range** of this linear transformation is defined as:

$$range(T_{\mathbf{A}}) = \{T(\vec{x}) | \vec{x} \in \mathbb{R}^n\} = span\{\text{columns of A}\}$$

and *range is always a subset of codomain.*

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Consider **transformation**

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

where $\mathbb{R}^n$ is the **domain** and $\mathbb{R}^m$ is the **codomain**. The could also be demonstrated as a *matrix multiplication*, where $\mathbf{A}$ is a $m \times n$ matrix.

$$\mathbf{A}\vec{x} = \vec{b}$$

We define **range** of transformation T as

$$Range(T) = \{T(\vec{x}) | \vec{x} \in \mathbb{R}^n\}$$

T could also be written as $T_A(\vec{x})$. Also, range of transformation is the same as the column space of the standards matrix.

$$Range(T_A) = span\{cols.of\,\mathbf{A}\} = Col\{\mathbf{A}\}$$

Definition When we say a transformation is **linear** if and only if for $\vec{x}, \vec{y} \in \mathbb{R}^n$ the following holds:

i. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

ii. $T(c\vec{x}) = cT(\vec{x})\ , \quad \forall c \in \mathbb{R}$

If a linear transformation could be represented by a matrix, then it's linear.

Properties If a transformation is linear.

1. $T(\vec{0}) = T(\vec{x} - \vec{x}) = T(\vec{x}) - T(\vec{x}) = 0$
2. $T(-\vec{x}) = -T(\vec{x})$, so the transformation is **odd**.

Superposition Principle For all $c_i \in \mathbb{R}$ and $\vec{x_i} \in \mathbb{R}^n$ for $i = 1, ..., k$:

$$T(\sum_{i=1}^{k} c_i\vec{x_i}) = \sum_{i=1}^{k} c_i T(\vec{x_i})$$

equivalently, *T(Linear Combination of* $\vec{x_i}$*) = linear combination of* $T(\vec{x_i})$

Theorem A transformation is linear if and only if it's induced by a matrix, in which:

$$\mathbf{A} = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_n}) \end{bmatrix}$$

Induction Suppose transformation T is linear, and $T : \mathbb{R}^n \to \mathbb{R}^m$.

$$\begin{aligned}
T(\vec{x}) &= T(I * \vec{x}) \\
&= T([\vec{e_1}, \dots, \vec{e_n}] * \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) \\
&= T(\sum_{i=1}^{k} x_i \vec{e_i}) \\
&= \sum_{i=1}^{k} x_i T(\vec{e_i}) \\
&= \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}
\end{aligned} \tag{16}$$

So that, we could conclude if T is linear, its **standard matrix** is matrix $\mathbf{A} = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_n}) \end{bmatrix}$ with size $m \times n$.

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Example Use matrix to represent reflect about $y = mx$.

1. R_θ^{CR}: Let $\theta = \arctan m$, rotate for θ clockwise.
2. P: Reflect image about x-axis.
3. R_θ^{CCR}: Rotate for θ counter-clockwise.

$$
\begin{aligned}
Q_m(\vec{x}) &= R_\theta^{CCR}(P(R_\theta^{CR}(\vec{x}))) \\
&= R_\theta^{CCR}(P(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix})) \\
&= R_\theta^{CCR}(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}) \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ x_1\sin\theta - x_2\cos\theta \end{bmatrix} \\
&= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned} \tag{17}
$$

Definition Let $T : \mathbb{R}^n \to \mathbb{R}^m$. T is called **onto(surjective)** if and only if

$$\forall \vec{b} \in \mathbb{R}^m, \exists \vec{x} \in \mathbb{R}^n s.t. T(\vec{x}) = \vec{b}$$

that is, the range of transformation T is

Definition Let $T : \mathbb{R}^n \to \mathbb{R}^m$. T is called **one-to-one(injective)** if and only if $\forall \vec{v} \in \mathbb{R}^m$ is the image of **at most one** $\vec{x} \in \mathbb{R}^n$. That is,

$$\forall \vec{x_1} \neq \vec{x_2} \in \mathbb{R}^n \iff T(\vec{x_1}) \neq T(\vec{x_2})$$

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be **linear**, the T is **one-to-one** if and only if $T(\vec{x}) = \vec{0} \implies \vec{x} = \vec{0}$.

proof.
Suppose T is one to one, then $T(\vec{x}) = \vec{0}$ has at most one **1** solution. For a linear transformation, $T(\vec{0}) = \vec{0}$. So that the only possible value for $\vec{x}$ is $\vec{0}$.

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear and its standard matrix is **A**.

1. $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if and only if columns of A spans $\mathbb{R}^m$.
2. $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one to one** if and only if columns of A is *linearly independent*.

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9.1 Matrices

Let **A** be a matrix with entires a_{ij} has **size** $m \times n$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and **diagonal** of **A** is defined as

$$diag(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{22} & \dots & a_{nn} \end{bmatrix}$$

0 Matrix is defined as

$$a_{ij} = 0, \forall i \in \{1, ..., m\}, \forall j \in \{1, ..., n\}$$

Examples of diagonal matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{are diagonal matrices.}$$

9.2 Matrix Properties

Matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are *equal* if and only if $a_{ij} = b_{ij}$ holds for all i,j.

If sizes of matrices **A** and **B** are equal, then we have

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

$$\forall c \in \mathbb{R}, c\mathbf{A} = [c \times a_{ij}]$$

Properties

1. **A** + **B** = **B** + **A**
2. (**A** + **B**) + **C** = **A** + (**B** + **C**)
3. **A** + **0** = **A**
4. **A** + (-1) **A** = **0**
5. c(**A** + **B**) = c**A** + c**B**
6. cd**A** = c(d**A**)
7. (r + s)**A** = r**A** + s**A**

Suppose, $T : \mathbb{R}^n \to \mathbb{R}^k$ and $S : \mathbb{R}^k \to \mathbb{R}^m$, then we have,

$$T(\vec{x}) = \mathbf{B}\vec{x} \text{ ,and size of } \mathbf{B} = \text{k * n}$$

$$S(\vec{x}) = \mathbf{A}\vec{x} \text{ ,and size of } \mathbf{A} = \text{m * k}$$

so that, $(S \cdot T) : \mathbb{R}^n \to \mathbb{R}^m$, is the **composite** of linear transformations S and T.

$$(S \cdot T)\vec{x} = \mathbf{A}(\mathbf{B}\vec{x}) = \mathbf{AB}\vec{x}$$

Let $B = [\vec{b_1}, \dots, \vec{b_n}]$,

$$\begin{aligned}\mathbf{A}(\mathbf{B}\vec{x}) &= \mathbf{A}(x_1\vec{b_1} + \cdots + x_n\vec{b_n}) \\ &= x_1\mathbf{A}\vec{b_1} + \cdots + x_n\mathbf{A}\vec{b_n} \\ &= (A\vec{b_1}, \dots, A\vec{b_n}) \cdot (x_1, \dots, x_n)^T\end{aligned} \tag{18}$$

9.3 Matrix Multiplication

Definition let **A** has size $m \times k$ and **B** has size $k \times n$, then **A*****B** is a matrix with size $m \times n$, is given by

$$\mathbf{AB} = [\mathbf{A}\vec{b_1}, \dots, \mathbf{A}\vec{b_n}]$$

Computation the (i, j) entry of multiplied matrix **AB** is given by the **dot product** of i^{th} row of **A** and j^{th} column of **B**.

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Inverse $(\star)A^{-1}A = AA^{-1} = I$ and **A** has to be *square*, any A^{-1} satisfying $(\star)$ is the **inverse** of **A**.

Determinant(For 2*2 Matrix) A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, det(A) or $|A|$ or **determinant** of A is defined as:

$$|A| = ad - bc$$

If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $|A| \neq 0$, then A is **invertible**, and the inverse of A is given by:

$$A^{-1} = \frac{1}{|A|} * adj(A)$$

where $adj(A)$ stands for **adjugate** of matrix A. And for matrix A above, its adjugate is given by"

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

proof. Since $|A| \neq 0$, so that, $\frac{1}{|A|} * adj(A)$ is defined.

$$\begin{aligned} \frac{1}{|A|} * adj(A) * A &= \frac{1}{ad-bc} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc} * \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{19}$$

$$\begin{aligned} A * \frac{1}{|A|} * adj(A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \frac{1}{ad-bc} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} * \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{20}$$

Converse is true also

Proof. WTS: For non-zero matrix A, invertible $\implies det(A) \neq 0$

$$\begin{gathered}
\text{Proof by contrapositive:} \\
\text{WTS: det(A) = 0} \implies \text{A not invertible} \\
A * adj(A) = adj(A) * A = |A| * I \\
\text{So if det(A) = 0} \\
\implies A * adj(A) = \mathbf{0} \\
\text{Suppose A is invertible} \\
A^{-1} * A * adj(A) = A^{-1} * \mathbf{0} = \mathbf{0} \\
\implies adj(A) = \mathbf{0} \\
\implies A = \mathbf{0} \\
\text{Contrapositive statement is proven by contradiction} \quad \blacksquare
\end{gathered} \tag{21}$$

Theorem Suppose A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then,

$$|A| \neq 0 \iff \text{Invertibility}$$

and we have:

$$A^{-1} = \frac{1}{|A|} adj(A)$$

Generally If A with size $n \times n$ is **invertible**(or, equivalently, $|A| \neq 0$), then linear system $A\vec{x} = \vec{b}$ could be **uniquely** solved with solution $\vec{x} = A^{-1}\vec{b}$.

Properties

1. If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
2. If A,B are both invertible, and AB is invertible, we have $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Proof.

$$\begin{gathered}
\textbf{(1).} \\
\text{A is invertible, then } A * A^{-1} = A^{-1} * A = I \\
\text{Let C be the inverse of } A^{-1} \\
C * A^{-1} = A^{-1} * C = I \text{ holds only when C = A} \\
\textbf{(2).} \\
B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \\
\text{So that, } (B^{-1}A^{-1}) = (AB)^{-1} \\
\textbf{(3).} \\
A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I
\end{gathered} \tag{22}$$

Symmetric Matrix A matrix A is said to be **symmetric** if $A^T = A$.

Lemma Property (b) extends to arbitrary products of matrices. If matrices $A_1, A_2, ..., A_l$ are invertible, then we have:

$$(\prod_{i=1}^{l} A_i)^{-1} = \prod_{i=l}^{l} (A_i)^{-1}$$

Elementary Matrices An elementary matrix (EM) is a matrix obtained as result of **AN** elementary row operation (ERO) on an identity matrix.

Property Performing an ERO on a matrix is the same as multiplying the matrix by an EM obtained by performing the same ERO on an identity matrix.

Since ERO(s) are reversible, the corresponding EM(s) are invertible. The inverse matrix would be obtained by performing the reverse ERO(s) on an identity matrix.

Examples

$$\begin{bmatrix} 1 & 0 \\ -17 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 17 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem An matrix with size $n \times n$ is **invertible** if and only if A is **row-equivalent** to I, in which case sequence of EROs, taking $A \to I$, takes $I \to A^{-1}$.

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Recall LU Factorization Let $\mathbf{A} \in \mathbb{M}_{m\times n}$ be a matrix can be reduced with or without interchange. A could be written as

$$\mathbf{A} = \mathbf{LU}$$

where $\mathbf{L}$ is a lower triangular matrix with size $m \times m$ and diagonal entries as 1. And $\mathbf{U}$ is the RREF of matrix $\mathbf{A}$ with size $m \times n$.

Subspace of $\mathbb{R}^n$ Let $S \in \mathbb{R}^n$ is **subspace** if $\forall \vec{u}, \vec{v} \in S$, then $c_1\vec{u} + c_2\vec{v} \in S, \forall c_1, c_2 \in \mathbb{R}$. That's, *subspace is closed under addition and multiplication(vector operation).*

And we have, take $c_1 = c_2 = 0 \in \mathbb{R}$, we see, $\vec{0} \in S$. So that, zero vectors is in any subspace of $\mathbb{R}^n$.

1. **Trivial Subspace** $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
2. **zero Subspace** $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.

Attention: $\vec{0}$ is **not** a subspace of $\mathbb{R}^n$, since it is a single vector instead of a set or space.

Theorem S = $span\{\vec{v_1} \dots \vec{v_k}\} \subset \mathbb{R}^n$, then

1. S is a subspace containing each $\vec{v_j}$.
2. If $W \in \mathbb{R}^n$ containing each $\vec{v_j}$, then $S \subset W$.

And $span\{\vec{v_j}\}$ is the **smallest** subspace containing $\vec{v_j}$.
Proof.

$$\begin{gathered}
\textbf{(1)} \\
\text{Obviously,}\vec{v_j} \in span\{\vec{v_i}\}. \\
\text{Let,}\vec{u}, \vec{v} \in S \\
\text{Then,}\vec{u} = \sum_{i=1}^{k} \hat{c_i}\vec{v_i}, \\
\vec{u} = \sum_{i=1}^{k} d_i\vec{v_i} \\
\text{Then,}c_1\vec{u} + c_2\vec{v} = (c_1\hat{c_1} + c_1 d_1)\vec{v_i} + \ldots + (c2\hat{c_k} + c_2 d_k)\vec{v_k} \\
= \sum_{1}^{k} d_i\vec{v_i} \in span\{\vec{v_i}\} \\
\textbf{(2)} \\
\text{Suppose}w \subset \mathbb{R}^n \text{ containing each of } \vec{v_j}. \\
\vec{v_j} \in w, \text{then} \sum_{i=1}^{k} c_i\vec{v_i} \in W.\forall \vec{c_i} \in \mathbb{R} \\
span\{\vec{v_i}\} \subset span\{\vec{v_i}\}
\end{gathered} \tag{23}$$

11.1 Canonical Subspace attached to A

Let A be a matrix with size $m \times n$.

1. **Column Space** $Col(A) = span\{cols\} \subset \mathbb{R}^m$
2. **Kernal Space / Null Space** $Null(A) = \{\vec{x} | \mathbf{A}\vec{x} = \vec{0}\} \subset \mathbb{R}^n$
3. **Row space** $Row(A) = Col(A^T) \subset \mathbb{R}^n$
4. **Eigen Space with eigen value** λ Applied only if A is square $E_\lambda(A) = Ker(A - \lambda I)$ with $\lambda \in \mathbb{R}.E_\lambda(A) \neq \{\vec{0}\} \subset \mathbb{R}^n$.

11.2 Dual Role of Null(A) and Row(A)

Basis A **Basis** for a subspace S is a set of **linearly independent** vectors that spanning space S.
Like:

$$B = \{\vec{\beta_k}\} \subset \mathbb{R}^n \text{is a basis for} S \subset \mathbb{R}^n.$$

B is a basis for S if and only if $span\{\vec{\beta_k}\} = S$ and all vectors in set B are linearly independent.

Standard Basis If $\vec{x} \in \mathbb{R}^n$, then

$$\vec{x} = I\vec{x} = [\vec{e_1}, \ldots, \vec{e_n}] * [\vec{x_1}, \ldots, \vec{x_n}]^T$$

So that, $\mathbb{R}^n = span\{\vec{e_1}, \ldots, \vec{e_n}\}$, also, all vectors in $\{\vec{e_i}\}$ are linearly independent. So that, $\{\vec{e_i}\}$ is called the **standard basis** for $\mathbb{R}^n$.