MAT224 Linear Algebra II
Lecture Notes

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Contents

1   Lecture1 Jan.9 2018

1.1   Vector spaces

Definition   A <u>real</u> [1] **vector space** is a set V together with two vector operations vector addition and scalar multiplication such that

  1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$

  2. **C** Commutative: $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$

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[1]A vector space is real if scalar which defines scalar multiplication is real.

3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

4. **Z** Zero Vector: $\exists \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$

5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists -\vec{x} \in V s.t. \vec{x} + (-\vec{x}) = \vec{0}$

6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$

7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c+d)\vec{x} = c\vec{x} + d\vec{x}$

9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$

10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

Note For V to be a vector space, need to know or be given operations of vector additions multiplication and check <u>all</u> 10 properties hold.

1.2 Examples of vector spaces

Example 1 $\mathbb{R}^n$ w.r.t.[1] usual component-wise addition and scalar multiplication.

Example 2 $\mathbb{M}_{m \times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

Example 3 $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree <u>less or equal</u> to n, w.r.t. usual degree-wise polynomial addition and scalar multiplication.

Note If define $\mathbb{P}_n^\star(\mathbb{R})$ as set of all polynomials of degree <u>exactly equal</u> to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

Explanation: $(1 + x^n), (1 - x^n) \in \mathbb{P}_n^\star(\mathbb{R})$ but $(1 + x^n) + (1 - x^n) = 2 \notin \mathbb{P}_n^\star(\mathbb{R})$

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[1]w.r.t. is the abbreviation of "with respect to".

Example 4   Something unusual, define V as

$$V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

This is a vector space.

1.3   Some properties of vector spaces

Suppose V is a vector space, then it has the following properties.

Property 1   The zero vector is unique.
proof.

$$\text{Assume } \vec{0}, \vec{0^\star} \text{ are two zero vectors in } V$$
$$\text{WTS: } \vec{0} = \vec{0^\star}$$
$$\text{Since } \vec{0} \text{ is the zero vector, by Z } \vec{0^\star} + \vec{0} = \vec{0^\star}$$
$$\text{Similarly, } \vec{0} + \vec{0^\star} = \vec{0}$$
$$\text{Also, } \vec{0} + \vec{0^\star} = \vec{0^\star} + \vec{0} \text{ by commutative vector addition.}$$
$$\text{So, } \vec{0^\star} = \vec{0}$$

∎

Property 2 $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.
proof.
Exercise. (By Cancellation Law)

Property 3 $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.
proof.

$$\text{By property of number 0: } 0\vec{x} = (0+0)\vec{x}$$
$$\text{By DSA: } 0\vec{x} = 0\vec{x} + 0\vec{x}$$
$$\text{By AI, } \exists(-0\vec{x})s.t.$$
$$0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x})$$
$$\text{By AA}$$
$$\implies 0\vec{x} = \vec{0}$$

Property 4 $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$
proof.

$$c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0}$$

2   Lecture2 Jan.10 2018

2.1   Some properties of vector spaces-Cont'd

Property 5   For a vector space V, $\forall \vec{x} \in V$, $(-1)\vec{x} = (-\vec{x})$. (*we could use this property to find the* <u>*additive inverse*</u> *with scalar multiplication with* (-1))[1].
proof.

$$(-\vec{x}) = (-\vec{x}) + \vec{0} \quad \text{By property of zero vector}$$
$$= (-\vec{x}) + 0\vec{x} \quad \text{By property3}$$
$$= (-\vec{x}) + (1 + (-1))\vec{x} \quad \text{By property of zero as real number}$$
$$= (-\vec{x}) + 1\vec{x} + (-1)\vec{x}$$
$$= \vec{0} + (-1)\vec{x}$$
$$= (-1)\vec{x}$$
$$\blacksquare$$

Property 6   For a vector space V, let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \lor \vec{x} = \vec{0}$$

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[1]The scalar multiplication here is the one defined in vector space V.

proof.

$$\text{if } c = 0 \implies True$$
$$\text{else } c^{-1}c\vec{x} = c^{-1} = \vec{0}$$
$$\implies (c^{-1}c)\vec{x} = \vec{0}$$
$$\implies 1\vec{x} = \vec{0}$$
$$\implies \vec{x} = \vec{0}$$
$$\implies True$$

∎

2.2   Subspaces

Loosely   A subspace is a space contained within a vector space.

Definition   Let V be a vector space and $W \subseteq V$, W is a **subspace** of V if W is itself a vector space w.r.t. operations of vector addition and scalar multiplication from V.

Theorem   Let V be a vector space, and $W \subseteq V$, W has the same[1] operations of vector addition and scalar multiplication as in V. Then, W is a subspace of V iff:

1. W is non-empty. $W \neq \emptyset$.

2. W is closed under addition. $\forall \vec{x}, \vec{y} \in W,\ \vec{x} + \vec{y} \in W$.

3. W us closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

Proof.

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[1]Other properties of vector spaces related to vector addition and scalar multiplication are immediately inherited from the parent vector space.

Forward:

If W is a subspace

$\implies \vec{0} \in W$

$\implies W \neq \emptyset$

Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$

WTS. 10 axioms in definition of vector space hold

$(ii) \implies$ Additive Closure

$(iii) \implies$ Scalar Multiplication Clousure

Because $W \subseteq V$, and V is a vector space, so properties hold $\forall \vec{w} \in W$.

Additive inverse: by property 5 and scalar multiplication closure,

$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W$.

Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W$.

2.3   Examples of subspaces

Example 1   Let $V = \mathbb{M}_{n \times n}(\mathbb{R})$, V is a subspace.

Example 2   Define W as

$$W = \{A \in \mathbb{M}_{n \times n}(\mathbb{R}) | A \text{ is } \underline{\text{not}} \text{ symmetric}\}$$

Explanation: Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but

$A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$.

Since there's no additive identity in set W, so W failed to be a vector space, therefore W is not a subspace.

Example 3   Let $V = \mathbb{P}_2(\mathbb{R})$, is W defined as following,

$$W = \{p(x) \in V | p(1) = 0\}$$

a subspace of V ?

proof.

    WTS: (i)

    Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$

    $\implies W \neq \emptyset$

    WTS: (ii)

    Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$

    $(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$

    $\implies p_1 + p_2 \in W$

    $\implies W$ is closed under addition.

    WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$

    $\implies p(1) = 0$

    Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$

    $\implies cp \in W$.

    So W is a subspace of V.

        ■

2.4   Recall from MAT223

Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of $\mathbb{R}^n$ and $Col(A)$ is a subspace of $\mathbb{R}^m$.

3   Lecture3 Jan.16 2018

3.1   Linear Combination

Definition   Let V be a vector space, $\vec{v_1}, \ldots, \vec{v_n} \in V$, $a_1, \ldots, a_n \in \mathbb{R}$ the expression

$$c_1 \vec{v_1} + \cdots + c_n \vec{v_n}$$

is called a **linear combination** of $\vec{v_1}, \ldots, \vec{v_n}$.

Theorem   Let V be a vector space, W is a subspace of V, $\forall \vec{w_1}, \ldots \vec{w_k} \in W, c_1, \ldots, c_k \in \mathbb{R}$, we have

$$c_1 \vec{w_1} + \cdots + c_k \vec{w_k} \in W$$

Subspaces are <u>closed under linear combinations</u>, since subspaces are closed under scalar multiplication and vector addition.

Theorem   Let V be a vector space, let $\vec{v_1}, \ldots, \vec{v_k} \in V$ then the set of all linear combination of $\vec{v_1}, \ldots, \vec{v_k}$

$$W = \{\sum_{i=1}^{k} c_i \vec{v_i} | c_i \in \mathbb{R} \forall \ i\}$$

is a subspace of V.
proof.

$$\text{Consider } \vec{0} \in W$$
$$\text{So, } W \neq \emptyset$$
$$\text{Let } c \in \mathbb{R}, \text{ Let } \vec{x} \in W \wedge \vec{y} \in W$$
$$\text{By definition of span, we have,}$$
$$\vec{x} = \sum_{i=1}^{k} a_i \vec{v_i}, \quad \vec{y} = \sum_{i=1}^{k} b_i \vec{v_i}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\vec{x} + c\vec{y} = \sum_{i=1}^{k} a_i \vec{v_i} + c \sum_{i=1}^{k} b_i \vec{v_i} = \sum_{i=1}^{k} (a_i + cb_i) \vec{v_i} \in W$$

$\blacksquare$

Definition   Let V be a vector space, $\vec{v_1}, \ldots, \vec{v_k} \in V$, **span** of the set of vectors $\{\vec{v_i}\}_{i=1}^{k}$ is defined as the collection of all possible linear combinations of $\{\vec{v_i}\}_{i=1}^{k}$. By pervious theorem, span is a subspace.

3.2   Combination of subspaces

Definition   Let W_1, W_2 be two sets, then the **union** of W_1, W_2 is defined as:
$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

the **intersection** of W_1, W_2 is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

Now consider W_1, W_2 to be two subspaces of vector space V, then we have,

1. $W_1 \cup W_2$ is **not** a subspace.

2. $W_1 \cap W_2$ is a subspace.

proof.

Falsify the statement by providing counter-example:

Consider,

$$W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$
$$W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$$
$$\text{But,} \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$$

∎

proof.

Because W_1 and W_2 are both subspaces, so

$$\vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset$$
$$\text{Let } \vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\text{Sine } W_1, W_2 \text{ are subspaces,}$$
$$\vec{x} + c\vec{y} \in W_1 \wedge \vec{x} + c\vec{y} \in W_2$$
$$\implies \vec{x} + c\vec{y} \in W_1 \cap W_2$$
$$\text{So, } W_1 \cap W_2 \text{ is a subspace.}$$

∎

Definition   Let W_1, W_2 be subspaces of vector space V, define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \wedge \vec{y} \in W_2\}$$

Note   Let $\vec{x} = \vec{0} \in W_1$, $\forall \vec{y} \in W_2$, $\vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = 0 \in W_2$, $\forall \vec{x} \in W_1$, $\vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2$, $\vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

Note $W_1 + W_2$ is a subspace of V.
proof.

$$\text{Let } \vec{x_1}, \vec{x_2} \in W_1, \vec{y_1}, \vec{y_2} \in W_2$$
$$\text{By properties of subspaces,}$$
$$\forall c \in \mathbb{R}, \vec{x_1} + c\vec{x_1} \in W_1 \wedge \vec{y_2} + c\vec{y_2} \in W_2$$
$$\text{Consider, } \vec{x_1} + \vec{y_1} \in W_1 + W_2, \vec{x_2} + \vec{y_2} \in W_1 + W_2$$
$$(\vec{x_1} + \vec{y_1}) + c(\vec{x_2} + \vec{y_2})$$
$$= (\vec{x_1} + c\vec{x_2}) + (\vec{y_1} + c\vec{y_2}) \in W_1 + W_2$$

$\blacksquare$

Definition(Unique Representation)   Let W_1, W_2 be subspaces of vector space V, say V is **direct sum** of W_1 and W_2, written as $V = W_1 \oplus W_2$, if every $\vec{x} \in V$ can be written <u>uniquely</u> as $\vec{x} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in W_1$ and $\vec{w_2} \in W_2$.

Equivalently Let W_1 and W_2 be subspaces of V, $V = W_1 \oplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.

4   Lecture4 Jan.17 2018

4.1   Cont'd

Cont'd Proof of Theorem
proof.

(Forward direction) Suppose $V = W_1 \oplus W_2$

WTS. $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

Let $V = W_1 \oplus W_2$

$\implies \forall \vec{x} \in V,$ can be written uniquely as

$\vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1,\ \vec{w_2} \in W_2$

$\implies V = W_1 + W_2$ by definition of *sum.*

Let $\vec{x} \in W_1 \cap W_2$

Decomposition, let $\vec{z} \in W_1 \cap W_2 \subseteq V$

$\vec{z} = \vec{z} + \vec{0},\ \vec{z} \in W_1, \vec{0} \in W_2$

$\vec{z} = \vec{0} + \vec{z},\ \vec{0} \in W_1, \vec{z} \in W_2$

Since decomposition is unique, $\vec{z} = \vec{0}$

So, $W_1 \cap W_2 = \{\vec{0}\}$

(Backward direction) Suppose $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

WTS. $V = W_1 \oplus W_2$

Assume $\vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1, \vec{w_2} \in W_2$

$\vec{x} = \vec{w_1'} + \vec{w_2'},\ \vec{w_1'} \in W_1, \vec{w_2'} \in W_2$

$\implies \vec{w_1} + \vec{w_2} = \vec{w_1'} + \vec{w_2'}$

$\implies \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2}$

Where, by definition of subspace, $\vec{w_1} - \vec{w_1'} \in W_1 \wedge \vec{w_2'} - \vec{w_2} \in W_2$

So, $\vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2} \in W_1 \cap W_2$

Since $W_1 \cap W_2 = \{\vec{0}\}$

$\implies \vec{w_1} = \vec{w_1'} \wedge \vec{w_2} = \vec{w_2'}$

So the decomposition is unique.

$\blacksquare$

4.2   Linear Independence

Theorem (Redundancy theorem)   Let V be a vector space, $\{\vec{x_1}, \ldots \vec{x_n}\}$, let $\vec{x} \in \{\vec{x_1}, \ldots \vec{x_n}\}$, then

$$span\{\vec{x_1}, \ldots \vec{x_n}, \vec{x}\} = span\{\vec{x_1}, \ldots \vec{x_n}\}$$

we say $\vec{x}$ *is the* **redundant** *vector that contributes nothing to the span.*
proof.

$$\text{let } \vec{x} \in span\{\vec{x}, \ldots, \vec{x_n}\}$$

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{x_i} \text{ for } c_i \in \mathbb{R} \; \forall i$$

$$\text{So, } span\{\vec{x_1}, \ldots, \vec{x_n}, \vec{x}\} = \{\sum_{i=1}^{n} a_i \vec{x_i} + z\vec{x} \mid a_i, z \in \mathbb{R} \forall i\}$$

$$= \{\sum_{i=1}^{n} a_i \vec{x_i} + z \sum_{i=1}^{n} c_i \vec{x_i} \mid a_i, c_i \in \mathbb{R} \forall i\}$$

$$= \{\sum_{i=1}^{n} (a_i + zc_i)\vec{x_i} \mid a_i, c_i \in \mathbb{R} \forall i\}$$

$$\text{Let } d_i = a_i + zc_i \in \mathbb{R}$$

$$= \{\sum_{i=1}^{n} d_i \vec{x_i} \mid d_i \in \mathbb{R} \forall i\}$$

$$= span\{\vec{x_1}, \ldots \vec{x_n}\}$$

$\blacksquare$

Definition   Let V be a vector space, let $\{\vec{x_1}, \ldots, \vec{x_n}\} \in V$, we say $\{v_i\}_{i=1}^{n}$ is **linearly independent** if the only set of scalars $\{c_1, \cdots, c_n\}$ that satisfies,

$$\sum_{i=1}^{n} c_i \vec{x_i} = 0$$

is $\{0, \ldots, 0\}$.

Definition   In contrast, we say a set of vector, with size n, is **linearly dependent** if

$$\exists \vec{c} \neq \vec{0} \in \mathbb{R}^n, \; s.t. \; \sum_{i=1}^{n} c_i \vec{v_i} = 0$$

Theorem   Let V be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly dependent* if and only if,
$$\exists \vec{x} \in \{\vec{v_i}\}_{i=1}^n \ s.t. \ \vec{x_j} \in span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

Theorem   Let V be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly independent* if and only if,
$$\forall \vec{x} \in \{\vec{v_i}\}_{i=1}^n, \ \vec{x_j} \notin span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

5   Lecture5 Jan.23 2018

5.1   Linear independence, recall definitions

Acknowledgement: special thanks to Frank Zhao.

Definition   Let $\{\vec{x_1}, \ldots \vec{x_k}\}$ is **linearly independent** if only scalars $c_1 \ldots c_k$ s.t.
$$\sum_{i=1}^k c_1 \vec{x_k} = 0 (\star)$$

are $c_1 = \cdots = c_k = 0$
linearly dependent means <u>at least one</u> $c_i \neq 0$, $(\star)$ still holds.

5.1.1   Alternative definitions of linear independency

Definition(Alternative.1) $\{\vec{x_1} \ldots \vec{x_k}\}$ is **linearly independent** <u>iff</u> none of them can be written as a linear combination of the remaining $k - 1$ vectors.[1]

Definition(Alternative.2) $\{\vec{x_1} \ldots \vec{x_k}\}$ is **linearly dependent** <u>iff</u> at least one of them can be written as a linear combination of the remaining $k - 1$ vectors. [2]

5.2   Basis

Definition   Let V be a vector space, a non-empty[3] set S of vectors from V is a **basis** for V if

1. $V = span\{S\}$

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[1]See theorem from the pervious lecture.
[2]See theorem from the pervious lecture.
[3]Specially, for an empty set, we define span $\emptyset = \{\vec{0}\}$

2. S is linearly independent.

Theorem (characterization of basis)   A non-empty subset $S = \{\vec{x_i}\}_{i=1}^n$ of vector space V is basis for V iff every $\vec{x} \in V$ can be written <u>uniquely</u> as linear combination for vectors in S.

proof.

Forwards

Suppose S is a basis for V

So every $\vec{x} \in V$ can be written as a linear combination of vectors in S

To prove the uniqueness, assume two expressions of $\vec{x} \in V$

$$\vec{x} = \begin{cases} c_1\vec{x_1} + \cdots + c_k\vec{x_k} \\ b_1\vec{x_1} + \cdots + d_k\vec{x_k} \end{cases}$$

Consider,

$$c_1\vec{x_1} + \cdots + c_k\vec{x_k} - (b_1\vec{x_1} + \cdots + d_k\vec{x_k}) = \vec{0}$$

$$\iff \sum_{i=1}^{k}(c_i - b_i)\vec{x_1} = \vec{0}$$

Since vectors in basis S are linear independent,

$$c_i = b_i \forall i \in \mathbb{Z} \cap [1, k]$$

So the representation is unique.

Backwards

Suppose every $\vec{x} \in V$ can be written uniquely as linear combination of vectors in S.

WTS: $V = span\{S\} \wedge S$ is linearly independent

By the assumption, spanning set is shown.

All we need to show is linear independence.

Consider,

$$\sum_{i=1}^{n} c_i\vec{x_i} = \vec{0}$$

Also, we know

$$\sum_{i=1}^{n} 0\vec{x_i} = \vec{0}$$

By the uniqueness of representation

We have identical expression $\sum_{i=1}^{n} c_i\vec{x_i} = \sum_{i=1}^{n} 0\vec{x_i}$

$$\therefore c_i = 0 \ \forall i \in \mathbb{Z} \cap [1, n]$$

∎

Example

$$V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

Show that $\{(1, 0), (6, 3)\}$ is a basis of V.

By theorem, $\{(1, 0), (6, 3)\}$ is basis if every $(a, b) \in V$ can be written uniquely as linear combination of $\{(1, 0), (6, 3)\}$.

$$\exists \text{ unique scalars } c_1, c_2 \in \mathbb{R} \ s.t. \ c_1(1, 0) + c_2(6, 3) = (a, b)$$

proof.

By definition of scalar multiplication and vector addition in this space,

$$\text{Consider} (a, b) = c_1(1, 0) + c_2(6, 3) = (2c_1 - 1, c_1 - 1) + (7c_2 - 1, 4c_2 - 1)$$

$$= (2c_1 + 7c_2 - 1, c_1 + 4c_2 - 1)$$

Consider the coefficients of variables

$$\begin{cases} 2c_1 + 7c_2 - 1 = a \\ c_1 + 4c_2 - 1 = b \end{cases}$$

WTS, the above system of linear equations has unique solution for all a, b

The system has a unique solution $\forall a, b \in \mathbb{R}$

Since the coefficient matrix has rank 2

$$rank\left(\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}\right) = 2$$

Since obviously the columns are linearly independent.

■

5.3   Dimensions

Definition   For a vector space V, the **dimension** of V is the minimum number of vectors required to span V.

Fundamental Theorem   if V vector space is spanned by m vectors, then any set of more than m vectors from V must be linearly dependent.

Fundamental Theorem (Alternative)   If V is vector space spanned by m vectors, then any linearly independent set in V must contain less or equal to m vectors.

5.3.1   Consequences of fundamental theorem

Theorem   if $S = \{\vec{v}_i\}_{i=1}^{k}$ and $T = \{\vec{w}_i\}_{i=1}^{l}$ are two bases of vector space V then $l = k$. *Bases have the same size.*

proof.

$$\text{Since } S \text{ spans } V \text{ and } T \text{ is linearly independent}$$
$$\therefore l \leq k$$
$$\text{(flip) Since } T \text{ spans } V \text{ and } S \text{ is linearly independent}$$
$$\therefore k \leq l$$
$$\implies l \leq k \wedge k \leq l$$
$$\implies k = l$$

$\blacksquare$

Definition   So we can define the **dimension** of V, as $dim(V)$ as the number vectors in <u>any</u> basis for V. For special case $V = \{\vec{0}\}$, $dim(V) = 0$.

Example

- $dim(\mathbb{R}^n) = n$

- $dim(\mathbb{P}_n(\mathbb{R})) = n + 1$

- $dim(\mathbb{M}_{m \times n}(\mathbb{R})) = m \times n$

5.3.2   Use dimension to prove facts about linearly (in)dependent sets and subspaces

Theorem   If V is a vector space, $dim(V) = n$, $S = \{\vec{x_k}\}_{i=1}^{k}$ is subset of V, if $k > n$ then S is <u>linearly dependent</u>.

Note $k \leq n \nRightarrow S$ is linear dependent.

Theorem   If W is subspace of vector space V, then

1. $dim(W) \leq dim(V)$

2. $dim(W) = dim(V) \iff W = V$

proof.

$$(1) \text{ Suppose } dim(V) = n, dim(W) = k$$

$$\text{WTS, } k \le n$$

Any basis for W is a linearly independent set of k vectors from V.

Since V is spanned by n vectors, since $dim(V) = n$

By fundamental theorem, $k \le n$

$$\Longleftrightarrow dim(W) \le dim(V)$$

$$(2) \text{ By contradiction, assume } dim(V) = dim(W) = n \text{ but } V \neq W$$

$$\text{Then } \exists \vec{x} \in V \wedge \vec{x} \notin W$$

Take S as a basis of W, then $\vec{x} \notin span\{S\}$

Then $S \cup \vec{x}$ is linearly independent

$$\Longrightarrow S \cup \{\vec{x}\} \text{ is linearly independent in } V \text{ containing } n+1 \text{ vectors}$$

This contradicts the assumption by fundamental theorem since

$dim(V) = n$ so it could not contain more than n linearly independent vectors

∎

6   Lecture6 Jan.24 2018

6.1   Basis and Dimension

Theorem   Let V be a vector space , S is a spanning set of V, and I is a linearly independent subset of V, s.t. $I \subseteq S$, then $\exists$ basis B for V s.t. $I \subseteq B \subseteq S$.

Explaining

1. *Any spanning set for V cab be **reduced** to basis for V by removing the linearly dependent(redundant) vector in the spanning set, using* <u>*redundancy theorem*</u> *to get a linearly independent spanning set.*

2. *Linear independent set can be **enlarged** to a basis for V.*

proof.

<u>omitted.</u>

∎

Corollary   Let V be a vector space and $dim(V) = n$, any set of n linearly independent vectors from V is a basis for V.

proof. If n linearly independent vectors did not span V, then could be enlarged to a basis of V by pervious theorem, but then have a basis containing more than n vectors from V, which is impossible by the fundamental theorem since we given the $dim(V) = n$, proven by contradiction.

Example   Let $V = P_2(\mathbb{R})$, $p_1(x) = 2 - 5x$, $p_2(x) = 2 - 5x + 4x^2$, find $p_3 \in P_2(\mathbb{R})$ s.t. $\{p_1(x), p_2(x), p_3(x)\}$ is basis for $P_2(\mathbb{R})$

Note Since $dim(P_2(\mathbb{R})) = 3$ so any 3 linearly independent vectors from $P_2(\mathbb{R})$ will be a basis for $P_2(\mathbb{R})$.

Solutions e.g. constant function $p_3(x) = 1$, since $1 \notin span\{p_1(x), p_2(x)\}$, so $\{p_1(x), p_2(x), p_3(x)\}$ is a basis of $P_2(\mathbb{R})$. e.g. $p_3(x) = x$, since $x \notin span\{p_1(x), p_2(x)\}$

Theorem   Let U and W be subspaces of vector space V, then we have

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

proof.

Let $\{\vec{v_i}\}_1^k$ be basis for $U \cap W$

$\implies dim(U \cap W) = k$

Since $\{\vec{v_i}\}_1^k$ is basis for $U \cap W$ then it's a linearly independent subset of U

So it could be enlarged ti basis for U, $\{\vec{v_1}, \ldots, \vec{v_k}, \vec{y_1}, \ldots, \vec{y_r}\}$

So $dim(U) = k + r$

We also could enlarge a basis for W $\{\vec{v_1}, \ldots, \vec{v_k}, \vec{z_1}, \ldots, \vec{z_s}\}$

$\implies dim(V) = k + s$

WTS. $\{\vec{v_1}, \ldots, \vec{v_k}, \ldots, \vec{y_1}, \ldots, \vec{y_r}, \vec{z_1}, \ldots, \vec{z_s}\}$ is a basis for $U + W$

(If we could show this) $dim(U + W) = k + r + s = (k + r) + (k + s) - k$

$= dim(U) + dim(W) - dim(U \cap W)$

Obviously, the above set spans $U + W$

WTS. $\{\vec{v_1}, \ldots, \vec{v_k}, \ldots, \vec{y_1}, \ldots, \vec{y_r}, \vec{z_1}, \ldots, \vec{z_s}\}$ is linearly independent

Consider $a_1\vec{v_1} + \cdots + a_k\vec{v_k} + b_1\vec{y_1} + \ldots b_r\vec{y_r} + c_1\vec{z_1} + \cdots + c_s\vec{z_s} = \vec{0}$ $(\star)$

From $(\star) \implies \sum(c_i\vec{z_i}) = -\sum(a_i\vec{v_i}) - \sum b_i\vec{y_i}$

$\implies \sum(c_i\vec{z_i}) \in U \wedge \sum(c_i\vec{z_i}) \in W$

$\iff \sum(c_i\vec{z_i}) \in U \cap W$

Since $\{\vec{v_i}\}$ is a basis for $U \cap W$

$\implies \sum(c_i\vec{z_i}) = \sum(d_i\vec{v_i})$

$\iff \sum(c_i\vec{z_i}) - \sum(d_i\vec{v_i}) = \vec{0} \in W$

$\implies c_i = d_i = 0$ since $\{\vec{z_i}, \vec{v_i}\}$ is a basis

Rewrite $(\star)$

$\sum(a_i\vec{v_i}) + \sum b_i\vec{y_i} = 0 \in U$

$\implies a_i = b_i = 0$ since $\{\vec{v_i}, \vec{y_i}\}$ is a basis for U

$\blacksquare$

Corollary For direct sum, since the intersection is $\{\vec{0}\}$

$$dim(U \oplus W) = dim(U) + dim(W)$$

Example Let U, W are subspaces of $\mathbb{R}^3$ such shat $dim(U) = dim(W) = 2$, why is $U \cap W \neq \{\vec{0}\}$

Solutions Geometrically, U and W are planes through origin then the intersection would be a line through origin$(U \neq W)$ or a plane through origin.$(U = W)$, so shown.

Question V is a vector space, $dim(V) = n$, $U \neq W$ are subspaces of V but $dim(U) = dim(V) = (n-1)$, proof:

1. $V = U + W$

2. $dim(U \cap W) = (n - z)$

7   Lecture7 Jan.30. 2018

7.1   Linear Transformations

Definition Let V, W be vector spaces, a function $T : V \to W$ is a **linear transformation**[1] if

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) \; \forall \vec{x}, \vec{y} \in V$ [2]

2. $T(c\vec{x}) = cT(\vec{x}) \; \forall \vec{x} \in V, \; c \in \mathbb{R}$ [3]

Linear transformation preserves <u>vector additions ans saclar multiplications</u> on vector spaces.

Theorem(Alternative definition) Transformation $T : V \to W$ is linear if and only if

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y}), \; \forall \vec{x}, \vec{y} \in V, c, d \in \mathbb{R}$$

Linear transformations preserves <u>linear combinations</u>.

Example (form 223) <u>Rotation</u> through angle θ about the origin in $\mathbb{R}^2$.

---

[1] In some textbooks, this is annotated as **linear mapping**.

[2] Notice that the vector additions on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

[3] Notice that the scalar multiplication on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

Example   (from 223) Matrix transformation, let $A \in M_{m \times n}(\mathbb{R})$, transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$T(\vec{x}) = A\vec{x}$$

is linear.

Example   Derivative $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by

$$T(\vec{p}(x)) = \vec{p}\,'(x)$$

Example   Matrix transpose $T : M_{m \times n}(\mathbb{R}) \to M_{n \times m}(\mathbb{R})$ defined by

$$T(A) = A^T$$

7.2   Properties of linear transformations

Property(i)   Linear transformation $T : V \to W$ are uniquely defined by their values on any basis for V.

  proof.

$$\text{Let}\{\vec{v_1}, \ldots, \vec{v_k}\} \text{ be any basis for } V$$

Every vector $\vec{x} \in V$ can be uniquely written as some linear combination of the $\{\vec{v_i}\}_{i=1}^k$

$$\vec{x} = \sum_{i=1}^k c_i \vec{v_i}, \ c_i \in \mathbb{R}, \ \text{and } c_i \text{ are uniquely determined } \forall \vec{x} \in V$$

$$\implies T(\vec{x}) = T(\sum_{i=1}^k c_i \vec{v_i})$$

$$= \sum_{i=1}^k c_i T(\vec{v_i}) \text{ since the transformation } T \text{ is linear.}$$

Since c_is are uniquely determined by $\{\vec{v_i}\}_{i=1}^k$

so the value of $T(\vec{x})$ is uniquely determined by its value on basis vectors $\{\vec{v_i}\}_{i=1}^k$.

$$\blacksquare$$

Property(ii)   Let $T : V \to W$ be a linear transformation, let A be a subspace of vector space V, then the **image** $T(A)$ defined as

$$T(A) = \{T(\vec{x}) \mid \vec{x} \in A\}$$

called the image of A under linear transformation T is a subspace of W. *Linear transformation maps subspaces of V to subspaces of W.*

proof.

Since A is a subspace so it's non-empty, therefore $\exists T(\vec{x}), \ \vec{x} \in A$

So T(A)$\neq \emptyset$

Let $\vec{w_1}, \vec{w_2} \in T(A)$

$\implies \vec{w_1} = T(\vec{x_1}), \vec{w_2} = T(\vec{x_2}), \ \vec{x_1}, \vec{x_2} \in A$

$\implies \vec{w_1} + \vec{w_2} = T(\vec{x_1}) + T(\vec{x_2}) = T(\vec{x_1} + \vec{x_2})$ since T is linear.

Since $\vec{x_1} + \vec{x_2} \in A$ by the definition of subspaces.

$\implies \vec{w_1} + \vec{w_2} \in T(A)$

So $T(A)$ is closed under vector addition.

Let $\vec{w} \in T(A)$

$\implies \vec{w} = T(\vec{x}), \vec{x} \in A$

Let $c \in \mathbb{R}$

Consider $c\vec{w} = cT(\vec{x}) = T(c\vec{x})$

Since $c\vec{x} \in A$

So $c\vec{w} \in T(A)$

So $T(A)$ is closed under scalar multiplication.

$\blacksquare$

Property(derived from the definition)   For all linear transformation $T : V \to W$, we have [1]
$$T(\vec{0}) = \vec{0}$$

Property(iii)   Let transformation $T : V \to W$ be linear, let B be a subspace of W, then its **pre-image** defined as

$$T^{-1}(B) = \{\vec{x} \in V \mid T(x) \in B\}$$

is a subspace of V. [2]

---

[1]In the equation, clearly, the zero vector on the left side of the equation is in space V and the zero vector on the right side is in space W.

[2]The pre-image and inverse share the same notation, but in this case, transformation T is not necessarily invertible.

proof.

$$\text{Let } \vec{w_1}, \vec{w_2} \in T^{-1}(B)$$
$$\implies T(\vec{w_1}), T(\vec{w_2}) \in B$$
$$\implies aT(\vec{w_1}) + b(\vec{w_2}) \in B, \ \forall a, b \in \mathbb{R} \text{ since } B \text{ is a subspace.}$$
$$\implies T(a\vec{w_1} + b\vec{w_2}) \in B$$
$$\implies a\vec{w_1} + b\vec{w_2} \in T^{-1}(B)$$

So $T^{-1}(B)$ is closed under both vector addition and scalar multiplication,

So $T^{-1}(B)$ is a subspace.

■

7.3   Definitions

Let $T : V \to W$ to be a linear transformation,

Definition   the **Image** of transformation T is defined as

$$Im(T) = T(V) = \{T(\vec{x}) \mid \vec{x} \in V\}$$

Definition   the **Rank** of transformation T is defined as

$$Rank(T) = dim(Im(T))$$

Definition   the **Kernel** of transformation T is defined as

$$Ker(T) = T^{-1}(\{\vec{0}\}) = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

Definition   the **Nullity** of transformation T is defined as

$$Nullity(T) = dim(ker(T))$$

Example $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is <u>linear</u> defined by

$$T(\vec{p}(x)) = \vec{p}(2x + 1) - 8\vec{p}(x)$$

find $Ker(T)$.

Theorem   Let $T : V \to W$ be a linear transformation, let $\{\vec{v_1}, \ldots, \vec{v_k}\}$ be the spanning set of V [1], then $\{T(\vec{v_1}), \ldots, T(\vec{v_k})\}$ spans $Im(T)$

proof.

$$\text{Let } \vec{w} \in Im(T)$$
$$\text{Since } V = span\{\vec{v_1}, \ldots, \vec{v_k}\}$$
$$\text{For any } \vec{x} \in V \text{ can be written as}$$

$$\vec{x} = \sum_{i=1}^{k} c_i \vec{v_i}, \ c_i \in \mathbb{R}$$

$$\implies \vec{w} = T(\vec{x}) = T(\sum_{i=1}^{k} c_i \vec{v_i})$$

$$= \sum_{i=1}^{k} c_i T(\vec{v_i})$$

$$\text{as a linear combination of } \{T(\vec{v_1}), \ldots, T(\vec{v_k})\}$$
$$\text{So } Im(T) = span\{T(\vec{v_1}), \ldots, T(\vec{v_k})\}$$

$\blacksquare$

8   Lecture8 Jan.31 2018

8.1   Linear Transformations

Example $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(p(x)) = p(2x + 1) - 8p(x)$$

Find the image of T.
We know $B = \{1, x, x^2, x^3\}$ is the standard basis for $P_3(\mathbb{R})$, consider the set $P(B)$

$$P(B) = \{-7, 1 - 6x, 1 + 4x - 4x^2, 1 + 6x + 12x^2\}$$

spans $Im(T)$. Notice the first three vectors in the set is linearly independent, the last vector is clearly dependent to the pervious three.[2]. So by the redundancy theorem we could remove the last vector. There we have

$$Im(T) = span\{-7, 1 - 6x, 1 + 4x - 4x^2\}$$

---

[1]The set is only the spanning set of V, it's not necessarily to be a basis of V.
[2]Notice that the first three vectors is a basis of $P_2(\mathbb{R})$.

as basis.

In this example, the dimension of $Ker(T)$ is 1 and the dimension of $Im(T)$ is 3, and dimension of $P_3(\mathbb{R})$ is 4. We have, $dim(P_3(\mathbb{R})) = Nullity(T) + Rank(T)$

Theorem(Dimension Theorem)   Let $T : V \to W$ be a linear transformation,

$$dim(V) = Nullity(T) + Rank(T)$$

Proof.

$$\text{Say } dim(V) = n$$

$$\text{Let } \{\vec{v_1}, \ldots, \vec{v_k}\} \text{ be a basis for } Ker(T)$$

$$\text{Since } Ker(T) \text{ is a subspace of } V, \text{ the set } \{\vec{v_i}\}_1^k \text{ is a subset of } V,$$

$$\text{It can be extended to a basis } \{\vec{v_i}\}_1^k \cup \{\vec{v_i}\}_{k+1}^n \text{ for V.}$$

$$\text{Claim: } \{T(\vec{v_{k+1}}), \ldots, T(\vec{v_n})\} \text{ is basis for } Im(T)$$

$$\text{If the claim is true, this prove the theorem since}$$

$$dim(Ker(T)) + dim(Im(T)) = k + n - k = n = dim(V)$$

$$\because T(\vec{v_i}) = \vec{0}, \ \forall i \in \mathbb{Z}_1^k$$

$$\text{and by the definition of kernel of linear transformation,}$$

$$\therefore \{T(\vec{v_i})\}_{k+1}^n \text{ spans } Im(T)$$

$$\text{Show if } \sum_{i=k+1}^{n} c_i T(\vec{v_i}) = \vec{0} \implies c_i = 0$$

$$\implies T\left(\sum_{i=k+1}^{n} c_i \vec{v_i}\right) = \vec{0}$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} \in Ker(T)$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} = \sum_{i=1}^{k} c_i \vec{v_i}$$

$$\implies c_1 \vec{v_1} + \cdots + c_k \vec{v_k} - c_{k+1} \vec{v_{k+1}} - \cdots - c_n \vec{v_n} = \vec{0}$$

$$\text{Since } \{\vec{v_i}\}_i^n \text{ is a basis for } V.$$

$$\implies c_i = 0 \ \forall i$$

8.2   Applications of dimension theorem

Definition   A linear transformation $T : V \to W$ is called **injective**(one-to-one) if and only if

$$T(\vec{v_1}) = T(\vec{v_2}) \implies \vec{v_1} = \vec{v_2}$$

Definition   A linear transformation $T : V \to W$ is called **surjective**(onto) if and only if

$$Im(T) = W$$

Every vector in W has a pre-image in V.

Definition   A linear transformation $T : V \to W$ is called **bijective** if it's both injective and surjective.

Theorem   Let transformation $T : V \to W$ is linear, T is injective if and only if $dim(Ker(T)) = 0$.

   Proof.
<div align="center">

Exercise

</div>

∎

Theorem T is surjective if and only if $dim(Im(T)) = dim(W)$.

Example $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$$

is T injective? surjective?
Not injective but surjective.
Solution

$$Ker(T) = span\{(x-1)(x-2)\}$$

So T has nullity of 1 and since $dim(P_2(\mathbb{R})) = 3$, by the dimension theorem we have $Rank(T) = 2$ and since $Im(T)$ is a subspace of $\mathbb{R}^2$ which has dimension of 2, we could conclude that $Im(T) = \mathbb{R}^2$.

9   Lecture9 Feb.6 2018

9.1   Applications of dimension theorem

Example $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(p(x)) = (p(1), p(2), p(3))$$

Take $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$, $p(x) \in Ker(T)$ iff $T(p(x)) = \vec{0}$.
Let $p(x) \in Ker(T)$,
Obviously the only solution for the system

$$\begin{cases} a + b + c = 0 \\ a + 2b + 4c = 0 \\ a + 3b + 9c = 0 \end{cases}$$

is $a = b = c = 0$, i.e. $p = \vec{0} \in P_2(\mathbb{R})$. So $dim(Ker(T)) = 0$. Therefore, T is
injective.
By *dimension theorem*,

$$dim(P_2(\mathbb{R})) = 3 = 0 + dim(Im(T)) \implies dim(Im(T)) = 3 = dim(\mathbb{R}^3)$$

therefore T is **surjective**. Therefore, T is **bijective**.

Question $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

$$T(p(x)) = xp'(x)$$

  Solution <u>Not injective</u> because any *constant function* in $P_n(\mathbb{R})$ is mapped
to $\vec{0} \in P_n(\mathbb{R})$, therefore $Ker(T) \neq \{\vec{0}\}$. Also <u>not surjective</u> by the dimension
theorem.

Theorem   Let $T : V \to W$ be an <u>injective</u> linear transformation, if $\{\vec{v_i}\}_{i=1}^k$
is linearly independent in V, then the set $\{T(\vec{v_i})\}_{i=1}^k$ is linearly independent
in W. *Injective transformation maps linearly independent set to linear in-*
dependent set.

Proof. Consider $\sum_i c_i T(\vec{v_i}) = \vec{0}$, then we have $T(\sum_i c_i \vec{v_i}) = \vec{0}$, which implies
$\sum_i c_i v_i \in Ker(T)$. By definition of injective transformation, $\sum_i c_i v_i = \vec{0}$.
Since $\{\vec{v_i}\}_{i=1}^k$ is linearly independent, so $c_i = 0$, $\forall i$ Therefore $\{T(\vec{v_i})\}_{i=1}^k$ is
linearly independent.   ∎

Theorem   Let $T : V \to W$ be a linearly transformation, $\{\vec{v_i}\}_{i=1}^n$ is a basis for V, then if $\{T(\vec{v_i})\}_{i=1}^n$ is linear independent, then T is <u>injective</u>.
A criteria for T to be injective baed on image of a basis.

   Proof.

$$\text{Let } \{\vec{v_i}\}_{i=1}^n \text{ be a basis of } V$$
$$\text{Consider } T(\vec{x}) = \vec{0}$$
$$\text{Since } \{\vec{v_i}\}_{i=1}^n \text{ is a basis}$$
$$\text{Let } \vec{x} = \sum c_i \vec{v_i}$$
$$\text{Assume } \vec{x} \in Ker(T)$$
$$T(\vec{x}) = \vec{0} \iff T(\sum c_i \vec{v_i}) = \vec{0}$$
$$\implies \sum c_i T(\vec{v_i}) = \vec{0}$$
$$\text{Since } \{T(\vec{v_i})\}_{i=1}^n \text{ are linearly independent.}$$
$$\implies c_i = 0$$
$$\text{Therefore } \vec{x} = \sum 0\vec{v_i} = \vec{0}$$
$$\text{Therefore } Ker(T) = \{\vec{0}\}$$
$$\text{Therefore } dim(Ker(T)) = 0$$
$$\implies \text{injective}$$

                                                             ■

Theorem   Let $T : V \to W$ be a linear transformation, [1]

1. If $dim(V) > dim(W)$, then T <u>cannot</u> be injective.

2. If $dim(V) < dim(W)$, then T <u>cannot</u> be surjective.

---

[1]Consider the contrapositive predicates of this theorem.

Lemma   For a linear transformation between spaces with different dimensions, it could not be bijective.

Proof.

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$
$$\because dim(Im(T)) \leq dim(W)$$
$$\therefore dim(V) \leq dim(Ker(T) + dim(W)$$
$$\implies dim(Ker(T)) \geq dim(V) - dim(W)$$
$$\implies dim(Ker(T)) > 0$$

So T could not be injective

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$
$$\because dim(Ker(T)) \geq 0$$
$$\therefore dim(V) \geq dim(Im(T))$$
$$\implies dim(Im(T)) < dim(W)$$

So T could not be surjective

∎

Proof 2.

Consider a transformation $T : V \to W$ is bijective.

By the contrapositive form of above theorem,

$$\text{Injective} \implies dim(V) \leq dim(W)$$
$$\text{Surjective} \implies dim(V) \geq dim(W)$$

Therefore bijective

$$\implies dim(V) \leq dim(W) \wedge dim(V) \geq dim(W) \iff dim(V) = dim(W)$$
$$\text{Therefore bijective} \implies dim(V) = dim(W)$$

So, take contrapositive, $dim(V) \neq dim(W) \implies$ not bijective.

∎

Theorem   *(Half is good enough)* Let $T : V \to W$ is linear, and $dim(V) = dim(W)$. Then T is injective <u>if and only if</u> surjective.

Proof.

$$\text{By dimension theorem}$$
$$dim(V) = dim(Ker(T)) + dim(Im(T)) = dim(W)$$
$$\text{If injective } dim(Ker(T)) = 0$$
$$\implies dim(Im(T)) = dim(W)$$
$$\text{So surjective}$$
$$\text{If surjective } dim(Im(T)) = dim(W) = dim(V)$$
$$\implies dim(Ker(T)) = 0$$
$$\text{So injective}$$

∎

9.2   Isomorphisms

Definition   If $T : V \to W$ is <u>bijective</u>, we call T an **isomorphism**. If there exists an isomorphism $T : V \to W$ say V and W are **isomorphic** vector spaces.

Theorem V, W are isomorphic <u>iff</u> $dim(V) = dim(W)$.

Proof.

$$\to \ V, W \text{ isomorphic} \implies dim(V) = dim(W)$$

Isomorphic means there exists a bijective transformation T

By dimension theorem $dim(V) = dim(Ker(T)) + dim(Im(T))$

$$= 0 + dim(W)$$

$$\leftarrow \ dim(V) = dim(W) \implies V, W \text{ isomorphic}$$

Equivalently, find a isomorphism(bijective) transformation

Let $\{\vec{v_i}\}_{i=1}^n$ be basis for V

Let $\{\vec{w_i}\}_{i=1}^n$ be basis for W

Claim $T : V \to W$ defined by

$T(\vec{v_i}) = \vec{w_i}$ is an isomorphism.

If $\vec{x} \in Ker(T) \subseteq V$

$$\vec{x} = \sum c_i \vec{v_i}$$

$$\vec{0} = T(\vec{x})$$

$$= \sum c_i T(\vec{v_i})$$

$$= \sum (c_i \vec{w_i})$$

$$\implies c_i = 0 \text{ since } \vec{w_i} \text{ are basis.}$$

$$\implies \vec{x} = \vec{0}$$

$$\implies dim(Ker(T)) = 0$$

$$\implies \text{injective} \iff \text{surjective}$$

Therefore V and W are isomorphic vector spaces.

∎

Note   if $T : V \to W$ is an isomorphism, then T maps a basis for V to a basis for W.

Example $T : P_2(\mathbb{R}) \to \mathbb{R}^3$,

$$T(p(x)) = (p(1), p(2), p(3))$$

is an isomorphism. And $P_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic.

Example $T : P_2(\mathbb{R}) \to \mathbb{R}^3$,

$$T(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ T(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ T(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an isomorphism.

Example $M_{2\times 2}(\mathbb{R})$, $P_3(\mathbb{R})$ and $\mathbb{R}^4$ are isomorphic.

Theorem   Any n-dim vector space V is isomorphic to $\mathbb{R}^n$. What is an isomorphism $T : V \to \mathbb{R}^n$

Procedure:

Let $\{\vec{v_i}\}_{i=1}^n$ be <u>any</u> basis for V

We know that $\forall \vec{x} \in V$,

By property of basis,

$$\vec{x} = \sum c_i \vec{v_i}$$

Then transformation T defined by

$$T(\vec{x}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n \text{ is an isomorphism.}$$

9.3   Coordinates

Definition   Let V be a vector space, $\alpha = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be nay basis for V, $\forall \vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

then $c_1, \ldots, c_n$ is called the **<u>coordinates</u>** for $\vec{x}$ relative to basis α, with notation

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \iff \vec{x} = \sum c_i \vec{v_i}$$

Claim $[\vec{x} + c\vec{y}]_\alpha = [\vec{x}]_\alpha + c[\vec{y}]_\alpha \quad \forall \vec{x}, \vec{y} \in V, \; c \in \mathbb{R}$.

Remark   if α, α' are any two bases for V then *generally* $[\vec{x}]_\alpha \neq [\vec{x}]_{\alpha'}$ (except $\vec{0}$)[1].

10   Lecture10 Feb.7 2018

10.1   Matrix of linear transformation

Recall   Let V be a vector space, let α be any basis for V.

$$\forall \vec{x} \in V, x = \sum c_i \vec{v}_i$$

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

So transformation $\vec{x} \to [\vec{x}]_\alpha$ is an isomorphism that $V \to \mathbb{R}^n$.

Definition   Let W be a vector space and let $\beta = \{\vec{w}_i\}_{i=1}^m$ be any basis of W, let $T : V \to W$ be a linear operator.

$$T(\vec{x}) = \sum c_i T(\vec{v}_i)$$

So that

$$[T(\vec{x})]_\beta = \left[\sum c_i T(\vec{v}_i)\right]_\beta = \sum c_i [T(\vec{v}_i)]_\beta$$

$$= \begin{bmatrix} [T(\vec{v_1})]_\beta & \cdots & [T(\vec{v_n})]_\beta \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$\begin{bmatrix} [T(\vec{v_1})]_\beta & \cdots & [T(\vec{v_n})]_\beta \end{bmatrix}$ is called the <u>the **matrix of** T w.r.t. bases α, β.</u> Denoted as $[T]_\alpha^\beta$, and by definition we have

$$[T(\vec{x})]_\beta = [T]_\alpha^\beta [\vec{x}]_\alpha$$

---

[1] $[\vec{0}]_\beta = \vec{0} \in \mathbb{R}^n$, $\forall \beta$ as basis for V.

Example $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(p(x)) = xp(x)$$

$$\alpha = \{1 - x, 1 - x^2, x\}, \ \ \beta = \{1, 1 + x, 1 + x + x^2, 1 - x^3\}$$

Find $[T]_\alpha^\beta$.

$$T(1 - x) = x(1 - x) = x - x^2$$
$$x - x^2 = (-1)(1) + 2(1 + x) + (-1)(1 + x + x^2) + 0(1 - x^3)$$
$$[T(1 - x)]_\beta = (-1, 2, -1, 0)$$
$$T(1 - x^2) = x - x^3$$
$$[T(1 - x^2)]_\beta = (-2, 1, 0, 1)$$
$$[T(x)] = x^2$$
$$[T(x)]_\beta = (0, -1, 1, 0)$$
$$[T]_\alpha^\beta = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Picture   Let V, W be two vectors spaces, $\alpha = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for V and $\beta = \{\vec{w_1}, \ldots, \vec{w_m}\}$ is a basis for W.

$$
\begin{array}{ccc}
V & \xrightarrow{\ T\ } & W \\
\downarrow{\scriptstyle [\ ]_\alpha} & & \downarrow{\scriptstyle [\ ]_\beta} \\
\mathbb{R}^n & \xrightarrow{\ [T]_\alpha^\beta\ } & \mathbb{R}^m
\end{array}
$$

Remark

1. $\vec{x} \in Ker(T) \iff T(\vec{x}) = \vec{0} \iff [T(x)]_\beta = [\vec{0}]_\beta \in \mathbb{R}^m \iff$
   $[T]_\alpha^\beta [\vec{x}]_\alpha = 0 \iff [\vec{x}]_\alpha \in Ker([T]_\alpha^\beta)$

2. $\vec{w} \in Im(T) \iff [\vec{w}]_\beta \in Col([T]_\alpha^\beta)$

Theorem   <u>Rank nullity for transformation matrix</u> Let $T : V \rightarrow W$ be a linear operator and $dim(V) = n$, then

$$dim(Ker([T]_\alpha^\beta)) + dim(Col([T]_\alpha^\beta)) = n$$

Example $T : P_2(\mathbb{R}) \rightarrow M_{2\times 2}(\mathbb{R})$

$$T(a + bx + c^2) = \begin{bmatrix} c & -c \\ a - c & a + c \end{bmatrix}$$

And given bases $\alpha = \{x^2 - x, x - 1, x^2 + 1\}$ and $\beta = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\}$

Solution

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Nul([T]_\alpha^\beta) = span\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$$

$$Nul(T) = span\{2x\}$$

$$Col([T]_\alpha^\beta) = span\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}\}$$

$$Col(T) = span\{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}\}$$

11   lecture11 Feb.13 2018

11.1   Algebra of Transformation

Theorem   Let $T : V \rightarrow W$ be a linear transformation, where $\alpha = \{\vec{v_1}, \ldots, \vec{v_n}\}$ and $\beta = \{\vec{w_1}, \ldots, \vec{w_m}\}$ are bases for V, W respectively.

$$\vec{x} \in Ker(T) \iff [\vec{x}]_\alpha \in Ker([T]_\alpha^\beta)$$

$$\vec{x} \in Im(T) \iff [\vec{x}]_\beta \in Col([T]_\alpha^\beta)$$

Definition $T_1, T_2 : V \to W$ are linear transformations, define **addition** and **scalar multiplication** of transformation as

$$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x}) \ \forall \vec{x} \in V$$

$$(cT_1)(\vec{x}) = c(T_1(\vec{x})) \forall \vec{x} \in V, \ c \in \mathbb{R}$$

Theorem   And, let α and β be bases for V, W respectively, then,

$$[T_1]_\alpha^\beta + [T_2]_\alpha^\beta = [T_1 + T_2]_\alpha^\beta$$

$$c[T_1]_\alpha^\beta = [cT_1]_\alpha^\beta$$

Definition   Let $T : V \to W$ and $S : W \to U$ be two linear transformations, then the **composition** $ST : V \to U$ is defined as

$$(ST)(\vec{x}) = S(T(\vec{x})) \quad \forall \vec{x} \in V$$

Remark   If S, T are *linear* then the composition ST is also *linear*.

Proof.
$$\text{Let } a, b \in \mathbb{R}, \ \vec{x}, \vec{y} \in V$$
$$ST(a\vec{x} + b\vec{y})$$
$$= S(T(a\vec{x} + b\vec{y}))$$
$$= S(aT(\vec{x}) + bT(\vec{y}))$$
$$= a(ST(\vec{x})) + b(ST(\vec{y}))$$

$\blacksquare$

11.2   Matrix of composition of transformations

Consider $T : V \to W$ and $S : W \to U$ as linear transformations, let $\alpha, \ \beta, \ \gamma$ be bases of $V, \ W, \ U$ respectively.

We know how to compute $[T]_\alpha^\beta$ and $[S]_\beta^\gamma$. Now want to find $[ST]_\alpha^\gamma$.

$$\forall \vec{x} \in V, [ST]_\alpha^\gamma [\vec{x}]_\alpha$$
$$= [(ST)(\vec{x})]_\gamma$$
$$= [S(T(\vec{x}))]_\gamma$$
$$= [S]_\beta^\gamma [T(\vec{x})]_\beta$$
$$= [S]_\beta^\gamma [T]_\alpha^\beta [\vec{x}]_\alpha$$

This holds true for all $\vec{x} \in V$

$$\therefore [ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$$

Conclusion the matrix of ST = matrix of S × matrix of T.

11.3   Inverse transformations

Theorem $T : V \to W$ is *isomorphism*[1] <u>if and only if</u> there exists function $S : W \to V$ such that

$$(ST)(\vec{v}) = \vec{v} \ \forall \vec{v} \in V \wedge (TS)(\vec{w}) = \vec{w} \ \forall \vec{w} \in W$$

Definition   And the above-mentioned linear operator S is called the **inverse** of T, written as T^{-1}.

proof.($\to$) T is an isomorphism means every vector in W has an unique pre-image in V the function $S : W \to V$ maps *every* vector in W to its *unique* pre-image in V, so S is the inverse of T.

proof.($\leftarrow$) Assume $S : W \to V$ is the inverse of $T : V \to W$ then $T(S(\vec{y})) = \vec{y}$, $\forall \vec{y} \in V$, this means T is <u>surjective</u> since every $\vec{y} \in W$ has pre-image under T, which is $S(\vec{y}) \in V$. Now suppose $T(\vec{x_1}) = T(\vec{x_2})$, apply transformation S on both sides of the equation, $S(T(\vec{x_1})) = S(T(\vec{x_2}))$ we have $\vec{x_1} = \vec{x_2}$. This implies the transformation is <u>injective</u>. Therefore, transformation T is bijective, that's isomorphism.   ∎

---

[1] Recall that isomorphism is equivalent to bijective.

Note $T^{-1}(\vec{y})$ is the <u>unique</u> vector $\vec{x}$, s.t.$T(\vec{x}) = \vec{y}$. That's

$$T(\vec{x}) = \vec{y} \iff T^{-1}(\vec{y}) = \vec{x}$$

Theorem   If $T : V \to W$ is an isomorphism then the inverse of T, $T^{-1} :$ $W \to V$ is linear.[1]

Proof.
WTS $T^{-1}(a\vec{w_1} + b\vec{w_2}) = aT^{-1}(\vec{w_1}) + bT^{-1}(\vec{w_2}) \forall a, b \in \mathbb{R}, \forall \vec{w_1}, \vec{w_2} \in W$

$$T^{-1}(\vec{w_1}) \text{ is the unique } \vec{x_1} \text{ s.t. } T(\vec{x_1}) = \vec{w_1}$$
$$T^{-1}(\vec{w_2}) \text{ is the unique } \vec{x_2} \text{ s.t. } T(\vec{x_2}) = \vec{w_2}$$
$$T^{-1}(a\vec{w_1} + b\vec{w_2}) \text{ is the unique } \vec{x} \text{ s.t. } T(\vec{x}) = a\vec{w_1} + b\vec{w_2}$$
$$\because T(\vec{x}) = a\vec{w_1} + b\vec{w_2}$$
$$= aT(\vec{x_1}) + bT(\vec{x_2})$$
$$= T(a\vec{x_1} + b\vec{x_2})$$
$$\therefore \vec{x} = a\vec{x_1} + b\vec{x_2}$$
$$\text{Also } T(\vec{x}) = a\vec{w_1} + b\vec{w_2}$$
$$\therefore \vec{x} = T^{-1}(a\vec{w_1} + b\vec{w_2}) = a\vec{x_1} + b\vec{x_2}$$
$$= aT^{-1}(\vec{w_1}) + bT^{-1}(\vec{w_2})$$

∎

Theorem $T : V \to W$ is <u>isomorphism</u>, then let α and β are bases of V and W representing then $[T]_\alpha^\beta$ is <u>invertible</u>, and

$$([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$$

Proof.
<u>omitted</u>

11.4   Change of basis

What's the effect of a change of basis on coordinate of a vector and matrix of transformation.

---

[1]Note: the conclusion could be changed into isomorphism.

Theorem   Let α and α' be two bases of V, and $\vec{x} \in V$, then

$$[I]_\alpha^{\alpha'}[\vec{x}]_\alpha = [\vec{x}]_{\alpha'}$$

Proof.
$$\text{Let } \vec{x} \in V$$
$$I(\vec{x}) = \vec{x}$$
$$[I(\vec{x})]_{\alpha'} = [\vec{x}]_{\alpha'}$$
$$[I]_\alpha^{\alpha'}[\vec{x}]_\alpha = [\vec{x}]_{\alpha'}$$

■

Definition   The above-mentioned $[I]_\alpha^{\alpha'}$ is called the **change of basis matrix** from α to α'.

Computation   Let $\alpha = \{\vec{a_1}, \ldots, \vec{a_n}\}$, then[1]

$$[I]_\alpha^{\alpha'} = [[\vec{a_1}]_{\alpha'} \mid \cdots \mid [\vec{a_n}]_{\alpha'}]$$

12   Lecture12 Feb.14 2018

Recall   Let α and β be bases for V and $I : V \to V$ is the identity transformation, then
$$[I]_\alpha^\beta[\vec{x}]_\alpha = [\vec{x}]_\beta$$

Also,
$$[I]_\beta^\alpha[\vec{x}]_\beta = [\vec{x}]_\alpha$$

Example   Let $\alpha = \{x^2, 1+x, x+x^2\}$ and β be a basis for $P_2(\mathbb{R})$ and

$$[I]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } [p(\vec{x})]_\beta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

find the basis β.
Solution <u>omitted</u>

---

[1]Construct column by column.

Theorem   Suppose $T : V \to W$ is linear, α and α' are any two bases for V and β and β' are any two bases of W, then,

$$[T]_{\alpha'}^{\beta'} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha}$$

Proof.

<div align="center">

Recall $T = ITI$

Consider let $\vec{x} \in V$

$[I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha}[\vec{x}]_{\alpha'}$

$= [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[\vec{x}]_{\alpha}$

$= [I]_{\beta}^{\beta'}[T(\vec{x})]_{\beta}$

$= [T(\vec{x})]_{\beta'}$

$= [T]_{\beta'}^{\alpha'}[\vec{x}]_{\alpha'}$

$\implies [T]_{\beta'}^{\alpha'} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha}$

</div>

∎

Also,

$$[T]_{\alpha}^{\beta} = [I]_{\beta'}^{\beta}[T]_{\alpha'}^{\beta'}[I]_{\alpha}^{\alpha'}$$

Special Case   Consider when $V = W$, $\alpha = \beta$ and $\alpha' = \beta'$. we have

$$[T]_{\alpha'}^{\alpha'} = [I]_{\alpha}^{\alpha'}[T]_{\alpha}^{\alpha}[I]_{\alpha'}^{\alpha}$$

where

$$([I]_{\alpha}^{\alpha'})^{-1} = [I]_{\alpha'}^{\alpha}$$

the equation becomes

$$[T]_{\alpha'}^{\alpha'} = ([I]_{\alpha}^{\alpha'})^{-1}[T]_{\alpha}^{\alpha}[I]_{\alpha'}^{\alpha}$$

and can be written in the form of

$$B = P^{-1}AP$$

Definition   Two matrices A and B are **similar** if there exists an <u>invertible</u> matrix P s.t.

$$B = P^{-1}AP$$

<div align="center">43</div>

Interpretation [1] Linear operators A and B are **similar** if and only if A and B representing the same transformation relative to different bases and P is the change of basis matrix.

13   Lecture13 Feb.27 2018

13.1   Diagonalization

Definition   Consider a linear operator $T : V \to V$ is **diagonalizable** if and only $\exists$ a basis β for V s.t.
$$[T]_\beta^\beta$$

is diagonal.

Note   Let $\beta = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis, $T : V \to V$ is diagonalizable if and only if $[T]_\beta^\beta$ is in form $\begin{bmatrix} \lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_n \end{bmatrix}$

Definition $T : V \to V$ is a linear operator on V, a *non-zero* vector $\vec{x} \in V$ is an **eigenvector** of T if and only if $T(\vec{x}) = \lambda \vec{x}$ for some $\lambda \in \mathbb{R}$. λ is called the **eigenvalue** of T corresponding to vector $\vec{x}$.

Theorem   Linear operator $T : V \to V$ is *diagonalizable* if and only exist a basis of V consisting of *eigenvectors* of T. If T is diagonalizable, the diagonal entries of $[T]_\beta^\beta$ are corresponding eigenvalues of T, <u>in the same order</u>.

13.2   How to find eigenvalues and eigenvectors of T

Definition   The **determinant** of T is defined as $det([T]_\alpha^\alpha)$ for <u>any</u> basis α for V.

Remark   The determinant of linear operator T does <u>not</u> depends on the choice of basis of α for V, *since similar matrices have the same determinant.*

---

[1]Could be used as alternative definition for similarity between matrices.

Theorem $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if

$$det(T - \lambda I) = 0$$

Proof.

Let λ be an eigenvalue of T,

Let α be any basis for V,

$\iff \exists \vec{x} \in V, \ \vec{x} \neq \vec{0}, \ s.t. \ T(\vec{x}) = \lambda \vec{x}$

$\iff T(\vec{x}) - \lambda \vec{x} = \vec{0}$

$\iff (T - \lambda I)(\vec{x}) = \vec{0}$

$\iff \vec{x} \in Ker(T - \lambda I)$

$\therefore Ker(T - \lambda I) \neq \{\vec{0}\}$

$\iff (T - \lambda I)$ is not injective

$\iff [T - \lambda I]_\alpha^\alpha$ is not injective and not invertible

$\iff det([T - \lambda I]_\alpha^\alpha) = det(T - \lambda I) = 0$

∎

Definition $det(T - \lambda I) = 0$ is called the **characteristic polynomial** of T, written as $P_T(\lambda) := det(T - \lambda I)$, the degree of $P_T(\lambda)$ is the dimension of V.

Note λ is an eigenvalue $\iff \lambda$ is a root of $P_T(\lambda)$.

Theorem $T : V \to V$ is a linear operator and λ is an eigenvalue of T, $\vec{x}$ is an eigenvector of T corresponding to eigenvalue λ, if and only if

$$\vec{x} \neq \vec{0} \wedge \vec{x} \in Ker(T - \lambda I)$$

Proof.

<u>By definition</u>

∎

Definition $Ker(T - \lambda I)$ is called the **eigenspace** of T corresponding to eigenvalue λ, noted as $E_\lambda(T)$, and it is a subspace of V.

Note   To find eigenvalues and eigenvectors of $T : V \to V$, choose any basis β for V, $\vec{x}$ is an eigenvector with corresponding eigenvalue λ if and only if $[\vec{x}]_\beta$ is an eigenvector of $[T]_\beta^\beta$ with corresponding eigenvalue λ.
That's

$$T(\vec{x}) = \lambda\vec{x}$$
$$\implies [T(\vec{x})]_\beta = [\lambda\vec{x}]_\beta$$
$$\iff [T]_\beta^\beta[\vec{x}]_\beta = \lambda[\vec{x}]_\beta$$

Note   Consider diagonalization in MAT223,

$$D = P^{-1}AP$$

Let D and A representing the same linear operator $[T]_V^V$ and let β be a basis of V consisting of eigenvectors of T and α is another basis of V. Then, the above equation is

$$[T]_\beta^\beta = ([I]_\beta^\alpha)^{-1}[T]_\alpha^\alpha[I]_\beta^\alpha$$

14   Lecture14 Feb.28 2018

Theorem   Suppose λ_0 is an eigenvalue of linear operator $T : V \to V$, let $dim(E_{\lambda 0}) = k$, then $(\lambda - \lambda_0)^k$ divides $P_T(\lambda)$

Proof.

$$\text{Let}\{\vec{v_1}, \ldots, \vec{v_k}\} \text{ be basis for } E_{\lambda_0}$$
$$\text{Since } E_{\lambda_0} \subset V$$
$$\text{Let } dim(V) = n$$
$$\text{Extend basis of } E_{\lambda_0} \text{ to basis of } V.$$
$$\alpha = \{\vec{v_1}, \ldots, \vec{v_k}\} \cup \{\vec{v_{k+1}}, \ldots, \vec{v_n}\}$$
$$\text{Since } \vec{v_i} \in E_{\lambda 0},$$
$$\text{Therefore } T(\vec{v_i}) = \lambda_0 \vec{v_i}$$
$$[T]_\alpha^\alpha = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$
$$\text{Where } A = diag(\lambda_0, \ldots, \lambda_0) \in \mathbb{M}_{k \times k}(\mathbb{R})$$
$$\text{And } B \in \mathbb{M}_{k \times n-k}(\mathbb{R}), D \in \mathbb{M}_{n-k \times n-k}(\mathbb{R})$$
$$P_T(\lambda) = det(A - \lambda I) * det(D - \lambda I)$$
$$= (\lambda_0 - \lambda)^k * det(D - \lambda I)$$
$$\text{Therefore} (\lambda - \lambda_0)^k \mid P_T(\lambda)$$

∎

Definition   The **multiplicity** of eigenvalue λ_0 is the number of times $(\lambda - \lambda_0)$ appears as a factor in $P_T(\lambda)$.

Note   If eigenvalue λ has multiplicity m, the above theorem says

$$1 \leq dim(E_\lambda) \leq m$$

if $m = 1$, then $dim(E_\lambda) = 1$.

Theorem   If $\lambda_1, \ldots, \lambda_k$ are *distinct* eigenvalues of $T : V \to V$ and $\alpha = \{\vec{x_1}, \ldots, \vec{x_k}\}$ are corresponding eigenvectors, then the set α is *linearly inde-*

pendent.

   Proof.

<u>Exercise</u>

∎

(⋆)**Theorem**   <u>Sufficient condition for diagonalizability</u> Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T, suppose the characteristic polynomial is in form

$$P_T(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i}$$

and T is diagonalizable if and only if

$$dim(E_{\lambda_i}) = m_i, \ \forall i$$

Note   Also, $\sum_{i=1}^{k} m_i = dim(V) = n$

Proof.

$$\leftarrow$$

Assume $dim(E_{\lambda_i}) = m_i \ \forall i$

Consider E_{λ_i}

Take basis for E_{λ_i}, note as $\{\vec{v}^i_1, \ldots, \vec{v}^i_{m_i}\}$

Claim: the union of bases of $E_{\lambda_i} \ \forall i$ gives a basis consisting of eigenvectors of T.

$$\text{Note}|\cup_{i=1}^{k} \{\vec{v^i}_1, \ldots, \vec{v^i}_{m_i}\}| = \sum_{i=1}^{k} m_i = dim(V)$$

All we need to show is linear independence.

$$\text{Consider} \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \vec{v^i_j} = \vec{0}(\star)$$

$$\text{Consider} \sum_{j=1}^{m_i} c_{ij} \vec{v^i_j} \in E_{\lambda_i} = \vec{x_i}$$

$$\text{So } (\star) \text{ becomes} \sum_{i=1}^{k} \vec{x_i} = \vec{0} \text{where } \vec{x_i} \in E_{\lambda_i}, \ \forall i$$

Since $\vec{x_i}$ is eigenvectors for T, corresponding to different eigenvalues,

Therefore, $\{\vec{x_{i1}}, \ldots, \vec{x_{ik}}\}$ is linearly independent

So $\vec{x_i} = \vec{0} \ \forall i$

$$\text{That's} \sum_{j=1}^{m_i} c_{ij} \vec{v^i_j} = \vec{x} = \vec{0} \ \forall i$$

$$\implies c_{ij} = 0 \ \forall i, j$$

Therefore linearly independent, so exists basis for V consisting of eigenvectors,

Therefore T is diagonalizable.

$$\rightarrow$$

Suppose T is diagonalizable,

Since T is diagonalizable, then exists basis for V consisting of eigenvectors, say α

$$\text{Consider } [T]_\alpha^\alpha = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_1 & \dots & \dots & 0 \\ 0 & \dots & \lambda_2 & \ddots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Where λ_1 takes first m_1 rows, λ_2 takes the next m_2 rows, etc.

$$P_T(\lambda) = det([T]_\alpha^\alpha - \lambda I)$$

$$= \prod_{i=1}^{k} (\lambda_i - \lambda)^{m_i}$$

Since $1 \le dim(E_{\lambda_i}) \le m_i \ \forall i$

$$\implies dim(E_{\lambda_i}) = m_i \ \forall i$$

∎

15   Lecture15 Mar.6 2018

15.1   Fields

Definition   A **field** is a set F together with two operations, *addition* and *multiplication* that satisfies the following properties.

1. $\forall x, y \in F, x + y = y + x$

2. $\forall x, y, z \in F, (x + y) + z = x + (y + z)$

3. *Additive identity* $\exists 0 \in F, \ s.t. \ \forall x \in F, 0 + x = x$

4. *Additive inverse* $\forall x \in F, \exists (-x) \in F \ s.t. \ x + (-x) = 0$

5. $\forall x, y \in F, xy = yx$

6. $\forall x, y, z \in F, (xy)z = x(yz)$

7. *Multiplicative identity* $\exists 1 \in F, \ s.t. \ \forall x \in F, 1 \times x = x$

8. *Multiplicative inverse* $\forall x \in F, \ x \neq 0, \exists x^{-1} \ s.t. \ x \times x^{-1} = 1$

Note   Every field has at least 2 elements: 0, the *additive identity* and 1, the *multiplicative identity*.

Examples

1. $\mathbb{R}$ is a field.

2. $\mathbb{Z}$ is not a field.

3. $\mathbb{N}$ is not a field.

4. $\mathbb{Q}$ is a field.

5. Irrational numbers is not a field.

15.2   Complex Numbers

Definition   The set of **complex number** $\mathbb{C}$ is the set of <u>ordered pair</u> of real numbers together with the following rules on basic operations.

1. Addition: $(a, b) + (c, d) = (a + c, b + d)$

2. Multiplication: $(a, b) + (c, d) = (ac - bd, ad + bc)$

With set notation we define complex numbers as

$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$

* altogether with operations of addition and multiplication defined above.

Note   (*Connection to* $\mathbb{R}$) Any complex number with second component as 0, $(a, 0)$ is identified as $a \in \mathbb{R}$, i.e. $\mathbb{R} \subsetneq \mathbb{C}$

Alt. notation $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \wedge i^2 = -1\}$

Definition   Let $w, z \in \mathbb{C}$, we define w **equals** z as,

$$w = z \iff (\Re(z) = \Re(w)) \wedge (\Im(z) = \Im(w))$$

Definition   Let $z = a + ib \in \mathbb{C}$ then the **conjugate** of z is $\overline{z} = a + i(-b)$, and if $z \neq 0$, then the **inverse** of z could be computed as

$$z^{-1} = \frac{\overline{z}}{z\overline{z}}$$

Definition   A field F is **algebraically closed** is every polynomial of degree n in F has n roots in F. (Counting multiplicities)

Examples $\mathbb{C}$ is algebraically closed and $\mathbb{R}$ is not.

16   Lecture16 Mar.7 2018

16.1   Vector space over a field

Definition   A vector space over field F is a set V together with two operations, addition and scalar multiplication s.t. [Very similar to those those defining properties for real vector space.]

16.2   Complex vector space

Complex vector space $\mathbb{C}^n = \{(z_1, \ldots, z_n) | z_1, \ldots, z_n \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$, with dimension n and standard basis $\{\vec{e_1}, \ldots, \vec{e_n}\}$

Definition   Let F be a field, then

$$F^n = \{(x_1, \ldots, x_n) | x_1, \ldots, x_n \in F\}$$

and

$$dim(F^n) = n$$

F^n is a *vector space over field F w.r.t. usual coordinate wise addition and scalar multiplication.*

Definition   Let V vector space over field F, then $\{\vec{x_1}, \ldots, \vec{x_n}\}$ is **linearly independent** if and only if

$$\sum_{i=1}^{i} c_i \vec{x_i} = \vec{0}, \ c_1, \ldots, c_n \in F \implies c_1 = \cdots = c_2 = 0 \in F$$

Definition   **span**$\{\vec{x_1}, \ldots, \vec{x_n}\}$ is defined as

$$\{\sum_{i=1}^{n} c_i \vec{x_i} | c_1, \ldots, c_n \in F\}$$

Definition   Consider V, W as two vector spaces over fields F then transformation $T : V(F) \to W(F)$ is **linear** if and only if

$$\forall \vec{v_1}, \vec{v_2} \in V, c, d \in F, \ T(c\vec{v_1} + d\vec{v_2}) = cT(\vec{v_1}) + dT(\vec{v_2})$$

17   Lecture 17 Mar.13 2018

Theorem   Let $T : V \to V$ be a linear operator, and β is a basis for vector space V. Let W_i be the span of first i vectors in β, then $[T]_\beta^\beta$ is *upper-triangular* if and only if

$$T(W_i) \subset W_i, \ \forall i$$

Definition   Let $T : V \to V$ be a linear operator, a subspace W of V is called **invariant** under T (*T-invariant*) if and only if

$$T(W) \subset W$$

Examples   For linear operator $T : V \to V$,[1]

1. V

2. $\{\vec{0}\}$

3. $Ker(T)$

4. $Im(T)$

5. $E_\lambda(T)$ for any eigenvalue λ of T [2]

6. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$T((x, y, z)) = (3x + 2y, y - z, 4x + 2y - z)$$

Then subspace of $\mathbb{R}^3$: $W = \{(x, y, x) \mid x, y \in \mathbb{R}\}$ is T-invariant.

Theorem   Let $T : V \to V$ be a linear operator, $\beta = \{\vec{x_1}, \ldots, \vec{x_k}\}$ is a basis for V, then $[T]_\beta^\beta$ is upper-triangular if and only if W_i, defined as the span of first i vectors in β, is T-invariant for all $i \leq k$.

Note $\{\vec{0}\} \subset W_1 \subset W_2 \subset W_3 \cdots \subset W_k = V$

Definition   Linear operator $T : V \to V$ is said to be **triangularizable** if there exists a basis β for V such that $[T]_\beta^\beta$ is upper-triangular.

---

[1]Proofs are omitted.

[2]As eigenspace is defined as kernel.

Remark   (Consider property of determinant of triangular form matrix) If $[T]_\beta^\beta$ is upper-triangular, the characteristic polynomial $P_T(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$ where λ_i are entries on the main diagonal.

Remark   Entries above main diagonal are **not** uniquely determined by T, it's also depends on the choice of basis β.

18   Lecture 18 Mar.14 2018

18.1   Triangular form

Theorem   Let V be a vector space over field F, let $T : V \to V$ be a linear operator, suppose the characteristic polynomial has $dim(V)$ roots in F,[1] then there exists β as a basis of V such that $[T]_\beta^\beta$ is upper-triangular.

Fact   Any transformation $T : V \to V$ whose eigenvalues all have multiplicity of 1, then T is diagonalizable. (Since there would be $dim(V)$ unique eigenvalues.)

Contra-positive of above fact   non-diagonalizable $\implies \exists\, \lambda_i$ with multiplicity greater than 1.

Consider   *Break down the problem* Linear operator $T : V \to V$

1. **Case 1** T has only eigenvalue 0 with multiplicity of $dim(V)$.

2. **Case 2** T has only eigenvalue λ with multiplicity of $dim(V)$. If T has only eigenvalue λ then $S = (T - \lambda I)$ has eigenvalue 0 only, as in case 1.

3. **Case 3** T has multiple eigenvalues. *the direct sum of single eigenvalue case.*

---

[1]i.e. field F is algebraically closed, e.g. $F = \mathbb{C}$

18.2   Nilpotent transformation

Theorem   Let V be a vector space over $\mathbb{C}$ and linear operator $T : V \to V$ has only eigenvalue 0 if and only if $T^k = \mathbf{0}$ [1] for some $k \in \mathbb{Z}^+$.

Proof.

$$\leftarrow \text{Suppose } T^k = 0 \text{ for some } k \in \mathbb{Z}^+$$

$$\text{Let } \vec{x} \neq \vec{0} \text{ be an eigenvector for } T,$$

$$\text{And } \lambda \text{ is the corresponding eigenvalue,}$$

$$\text{Then } T(\vec{x}) = \lambda \vec{x}$$

$$(\text{Inductively}) \; T^k(\vec{x}) = \lambda^k \vec{x}$$

$$\text{Since } \vec{x} \neq \vec{0} \wedge T^k(\vec{x}) = \vec{0}$$

$$\implies \lambda^k = 0$$

$$\implies \lambda = 0$$

$$\to \text{Suppose only eigenvalue of } T \text{ is 0.}$$

$$\text{We know there exists basis for } V ...$$

$$\text{so the matrix of } T \text{ relative to this basis is upper-triangular...}$$

$$\text{with 0 along diagonal.}$$

$$\text{And matrix of } T^2 \text{ relative to this basis has 0 on the super diagonal}$$

$$\text{And with every composition of additional } T, ...$$

$$\text{the zero diagonal is pushed up for at least one step higher.}$$

$$\text{Eventually, for the worst case we could guarantee } T^{dim(V)} = 0$$

$$\textbf{Note}: \text{ the actual value of } k \text{ might be smaller than } dim(V), ...$$

$$\text{and } k \text{ is bounded above by } dim(V).$$

$$\text{As composition of zero transformations is zero,}$$

$$\text{There must exist } k \leq dim(V) \; s.t. \; T^k = 0$$

$$\blacksquare$$

Definition   A linear operator $T : V \to V$ is called **nilpotent** if

$$\exists k \in \mathbb{Z}^+ \; s.t. \; T^k = 0$$

the *smallest* possible k that $T^k = 0$ is called the **order/index** of T.

---

[1] The 0 here stands for zero transformation.

Theorem (Same as above theorem) A linear operator $T : V \to V$ is nilpotent if and only if T has only eigenvalue 0.

Example 1 Let $T : P_n(\mathbb{C}) \to P_n(\mathbb{C})$ and $T(p(x)) = p'(x)$, T is nilpotent with order $n + 1$.

Example 2 Let $T : P_4(\mathbb{C}) \to P_4(\mathbb{C})$ and $T(p(x)) = p''(x) + p'''(x)$, T is nilpotent with order 3.

Example 3/Theorem If $T^{k-1}(\vec{x}) \neq \vec{0}$ for non-zero $\vec{x}$, and $T^k(\vec{x}) = \vec{0}$,i.e. T is a nilpotent transformation with degree k. Then $\beta = \{T^{k-1}(\vec{x}), \ldots, T(\vec{x}), \vec{x}\}$ is linearly independent. And β is called a **cycle** of T generated by initial vector $\vec{x}$.

Proof.

$$\text{If } (\star) = c_{k-1}T^{k-1}(\vec{x}) + \cdots + c_1 T(\vec{x}) + c_0 \vec{x} = \vec{0}$$

$$\text{Apply } T^{k-1} \text{ on both sides of above equation,}$$

$$\text{That's } T^{k-1}(\star) = T^{k-1}(\vec{0}) = \vec{0}$$

$$\implies c_0 T^{k-1}(\vec{0}) = \vec{0}$$

$$\implies c_0 = 0$$

$$\text{Recursively, } c_i = 0 \; \forall i \in \mathbb{Z}_0^{k-1}$$

$$\text{Therefore } \beta \text{ is linearly independent.}$$

■

Theorem Let $T : V \to V$ be a nilpotent with degree $n = dim(V)$, then there exists $\vec{x} \in V$ *(not necessarily unique)* such that

$$\beta = \{T^{n-1}(\vec{x}), \ldots, T(\vec{x}), \vec{x}\}$$

is a basis for V. And $[T]_\beta^\beta$ is upper-triangular with zero on main diagonal and one on super-diagonal, and zero elsewhere, like,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Proof.

$$\text{Since } T^n = 0 \wedge T^{n-1} \neq 0$$

Therefore β is linearly independent by result from example 3

And β contains n vectors, so β is a basis for V.

$\blacksquare$

19   Lecture 19 Mar.20 2018

Next Goal   If $T : V \to V$ is nilpotent in order between 1 and $dim(V)$, then the matrix of T relative to some basis is in the form of

$$\begin{pmatrix} J_{m_1} & 0 & \ldots & 0 \\ 0 & J_{m_2} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{m_k} \end{pmatrix}$$

where $J_{m_i} \in \mathbb{M}_{m_i \times m_i}(F)$ in the form with ones on super-diagonal and zeros elsewhere.

Essential procedures   Identify vectors

1. in $Ker(T)$

2. in $Ker(T^2) \backslash Ker(T)$

3. in $Ker(T^3) \backslash Ker(T^2)$

4. ...

Claim
$$\{\vec{0}\} \subseteq Ker(T) \subseteq (T^2) \subseteq \cdots \subseteq Ker(T^k) = V$$

Theorem   Let $T : V \to V$ is nilpotent of order k, let W be a subspace of $Ker(T^k)$ *s.t.* $W \cap Ker(T^{k-1}) = \{\vec{0}\}$, then

$$dim(T^i(W)) = dim(W), \; \forall i < k$$

Proof.

Let $\{\vec{w_1}, \ldots, \vec{w_s}\}$ be a basis for subspace W

So $dim(W) = s$

Let $i < k$, know $\{T^i(\vec{w_1}), \ldots, T^i(\vec{w_s})\}$ spans $T^i(W)$

WTS linear independency, so that $\{T^i(\vec{w_j})\}$ is a basis for $T^i(W)$

So that we could show they have same dimension by checking the sizes of their bases.

$$\text{Consider } \sum_{j=1}^{s} c_j T^i(\vec{w_j}) = \vec{0}$$

$$\text{That's } T^i(\sum_{j=1}^{s} c_j \vec{w_j}) = \vec{0}$$

Applying T^{k-i-1} on both side of above equation

$$T^{k-1}(\sum_{j=1}^{s} c_j \vec{w_j}) = \vec{0}$$

$$\text{So } \sum_{j=1}^{s} c_j \vec{w_j} \in W \cap Ker(T^{k-1})$$

$$\text{Therefore } \sum_{j=1}^{s} c_j \vec{w_j} = \vec{0} \in W$$

Since $\{\vec{w_1}, \ldots, \vec{w_s}\}$ is a basis for W

So $c_1 = c_2 = \cdots = c_s = 0$

So $\{\vec{w_1}, \ldots, \vec{w_s}\}$ is a basis for $T^i(W)$

So $dim(T^i(W)) = dim(W) = s$

■

20   Lecture 20 Mar.21 2018

20.1   Nilpotent Transformations

Goal   Show that every nilpotent $T : V \to V$ can be brought into canonical form (in some basis).

Theorem   Two nilpotent transformations are <u>similar</u> (i.e. they represents the same transformations relative to different bases) <u>if and only if</u> they have the <u>same canonical form</u>.

20.2   Canonical Forms for Transformations $T : V \to V$ with Single Eigenvalue λ

If linear operator T has only eigenvalue λ the linear operator $(T-\lambda I)(nilpotent)$ has eigenvalue 0 only. Therefore, linear operator $T : V \to V$ has only eigenvalue λ means operator $(T - \lambda I)$ is nilpotent, so for some bases β of V, $[T - \lambda I]_\beta^\beta$ could be in canonical form.

$$[T - \lambda I]_\beta^\beta = J = \begin{pmatrix} J_{m_1} & 0 & 0 & 0 \\ 0 & J_{m_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & Jm_k \end{pmatrix}$$

and for the matrix of original transformation,

$$[T]_\beta^\beta = J + \lambda I$$

20.3   Graph(Computational aspect)

omitted

21   Lecture21 Mar.27 2018

21.1   Goal

Goal   Prove for all $T : V \to V$ can decompose V into direct sum of two invariant subspaces s.t. on one subspace,[1] T has only single eigenvalue λ and on other no eigenvalue of T is λ.

Definition   Let λ be an eigenvalue of $T : V \to V$, the **generalized eigenspace** corresponding to eigenvalue λ is

$$K_\lambda = \{\vec{x} \in V \mid (T - \lambda I)^i(\vec{x}) = \vec{0} \text{ for some } i \in \mathbb{Z}^+\}$$

In the definition, i might be different for different $\vec{x}$.

---

[1]Transformation T restricted to this particular subspace.

Note 1 $K_\lambda = Ker(T - \lambda I)^k$ for some k. Since

$$\{\vec{0}\} \subset Ker(T - \lambda I) \subset Ker(T - \lambda I)^2 \ldots$$

the chain cannot grow forever must eventually stabilize.[1]   That's, there exists a (smallest) k s.t. $Ker(T - \lambda I)^k = Ker(T - \lambda I)^{k+1}$, more generally the $Ker(T - \lambda I)^l = Ker(T - \lambda I)^k$, $\forall l > k$. *k is the degree where kernel gets stabilized.*

Note 2 K_λ is T invariant. $\iff (\vec{v} \in K_\lambda \implies T(\vec{v}) \in K_\lambda)$

   Proof.

$$\text{Let } \vec{v} \in K_\lambda$$
$$\text{i.e. } (T - \lambda I)^i(\vec{v}) = \vec{0}, \ \forall i \geq k$$
$$\text{Consider } (T - \lambda I)^{k+1}(\vec{v}) = \vec{0}$$
$$\implies (T - \lambda I)^k(T - \lambda I)(\vec{0}) = \vec{0}$$
$$\implies (T - \lambda I)^k T(\vec{v}) - \lambda(T - \lambda I)^k(\vec{v}) = \vec{0}$$
$$\implies (T - \lambda I)^k T(\vec{v}) - \vec{0} = \vec{0}$$
$$\implies (T - \lambda I)^k T(\vec{v}) = \vec{0}$$
$$\text{We have shown that operator } (T - \lambda I)^k \text{ maps } T(\vec{v}) \text{ to } \vec{0}$$
$$\implies T(\vec{v}) \in K_\lambda$$

■

Note 3   The only eigenvalue of T on K_λ is λ. Equivalently,

$$T(\vec{v}) = \mu\vec{v} \implies \mu = \lambda$$

   Proof.
Consider $(T - \lambda I)^i(\vec{v}) = (\mu - \lambda)^i(\vec{v}) = \vec{0}$ by definition of generalized eigenspace.
$$\text{Since } \vec{v} \neq \vec{0} \text{ by definition of eigenvector.}$$
$$\text{So } \mu = \lambda$$

■

---

[1] As kernel is a subspace of V, its dimension could not exceed $dim(V)$.

Note 4 $V = Ker(T - \lambda I)^k \oplus Im(T - \lambda I)^k$
Check: $Im(T - \lambda I)^k$ is T-invariant

Proof.

By dimension theorem,

$$dim(V) = dim(Ker(T - \lambda I)^k) + dim(Im(T - \lambda I)^k)$$

So to prove direct sum only need to show

$$Ker(T - \lambda I)^k \cap Im(T - \lambda I)^k = \{\vec{0}\}$$

Let $\vec{v} \in Ker(T - \lambda I)^k \cap Im(T - \lambda I)^k$

Since $\vec{v}$ is in the image, there exits $\vec{w} \in V$

s.t. $\vec{v} = (T - \lambda I)^k(\vec{w}) \in Ker(T - \lambda I)^k$

Therefore $(T - \lambda I)^k(\vec{v}) = (T - \lambda I)^k((T - \lambda I)^k(\vec{w}))$

$$= (T - \lambda I)^{2k}(\vec{w}) = \vec{0} \text{ since } 2k > k$$

$$\implies \vec{w} \in Ker(T - \lambda I)^{2k} = Ker(T - \lambda I)^k$$

$$\implies \vec{v} = (T - \lambda I)^k(\vec{w}) = \vec{0}$$

$$\implies Ker(T - \lambda I)^k \cap Im(T - \lambda I)^k = \{\vec{0}\}$$

Therefore $V = Ker(T - \lambda I)^k \oplus Im(T - \lambda I)^k$

∎

Note 5 $T : V \to V$ is a linear operator and λ is an eigenvalue of T with multiplicity m, then

$$dim(K_\lambda) = m$$

In generally, the dimension of generalized eigenspace is equal to the multiplicity of λ

   Proof.

$$\text{By Note 4, } V = Ker(T - \lambda I)^k \oplus Im(T - \lambda I)^k$$

$$\text{Let } \alpha, \beta \text{ be respective bases for } Ker(T - \lambda I)^k, \ Im(T - \lambda I)^k$$

$$\implies \gamma = \alpha \cup \beta \text{ is a basis for } V$$

$$\text{Let } Ker \text{ denote } Ker(T - \lambda I)^k$$

$$\text{Let } Im \text{ denote } Im(T - \lambda I)^k$$

$$[T]_\gamma^\gamma = \begin{bmatrix} [T|_{Ker}]_\alpha^\alpha & 0 \\ 0 & [T|_{Im}]_\beta^\beta \end{bmatrix}$$

$$\implies P_T(x) = P_{T|_{Ker}}(x) \times P_{T|_{Im}}(x)$$

$$\text{Since multiplicity of eigenvalue } \lambda \text{ is } m, \text{ factoring out,}$$

$$\implies P_T(x) = (x - \lambda)^m q(x), \ q(x) \neq 0$$

$$\text{Since } \lambda \text{ is the only eigenvalue for } T|_{Ker}$$

$$P_{T|_{Ker}}(x) = (x - \lambda)^l$$

$$\text{Now WTS } m = l$$

$$\text{For } T|_{Im}, \text{ it has no eigenvalue equals } \lambda$$

$$\text{Let } \vec{v} \in Im(T - \lambda I)^k \text{ and } T(\vec{v}) = \lambda \vec{v}$$

$$\vec{v} = (T - \lambda I)^k(\vec{w}) \text{ for some } \vec{w}$$

$$\implies T(\vec{v}) = T(T - \lambda I)^k(\vec{w}) = \lambda(T - \lambda I)^k(\vec{w})$$

$$\implies (T - \lambda I)^k(\vec{w}) \in E_\lambda \subset Ker(T - \lambda I)^k$$

$$\implies (T - \lambda I)^k(\vec{w}) \in Ker(T - \lambda I)^k \cap Im(T - \lambda I)^k = \{\vec{0}\}$$

$$\text{Contradict the fact that eigenvector cannot be zero vector,}$$

$$\text{Therefore } \lambda \text{ cannot be an eigenvalue of } T|_{Im}$$

$$\implies P_{T|_{Im}}(\lambda) \neq 0$$

$$\text{So } (x - \lambda)^m q(x) = (x - \lambda)^l P_{T|_{Im}}$$

$$\text{Where } q(x) \neq 0 \wedge P_{T|_{Im}}(x) \neq 0$$

$$\implies l = m$$

∎

Goal / crucial idea $T : V \to V$ is a linear operator with λ as an eigen-value with multiplicity m, then

$$V = Ker(T - \lambda I)^k \oplus Im(T - \lambda I)^k = K_\lambda \oplus Im(T - \lambda I)^k$$

and both $Ker(T - \lambda I)^k$ and $Im(T - \lambda I)^k$ are invariant under T, the only eigenvalue of $T|_{Ker(T-\lambda I)^k}$ is λ and no eigenvalue of $T|_{Im(T-\lambda I)^k}$ is equal to λ. Also $dim(K_\lambda) = m$.

Implication   Let V be a vector space over $\mathbb{C}$, and $T : V \to V$ be a linear operator with <u>distinct</u> eigenvalues $\{\lambda_1, \ldots, \lambda_l\}$ then

$$V = \bigoplus_{i=1,\ldots,l} K_{\lambda_i}$$

Proof(Sketch).

$$V = K_{\lambda_1} \oplus Im(T - \lambda_1 I)^k$$

Apply induction on $dim(V)$

Keep splitting, one-by-one, until there are no more eigenvalues left.

■

So V is a vector space over $\mathbb{C}$, $T : V \to V$ has matrix (in some basis)

$$\begin{pmatrix} B_{\lambda_1} & 0 & 0 & 0 \\ 0 & B_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_{\lambda_i} \end{pmatrix}$$

where B_{λ_i} is a Jordan block. And the matrix is called **Jordan canonical form** of T, and is unique *up to ordering of Jordan blocks*.

Theorem   Two matrices are similar (i.e. representing same transformation relative to different bases) if and only if they have same JCF.

Note   If T is diagonalizable, then

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_l}$$

Diagonal form is one of Jordan canonical form.

22   Lecture22 Mar.28 2018

22.1   Examples on finding JCF.

Example 1   Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation and T has matrix A relative to standard basis of $\mathbb{R}^4$,

$$A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Find Jordan canonical form for T and a canonical basis.

Solution:

<u>Omitted</u>

■

Example 2   Let $T : \mathbb{R}^6 \to \mathbb{R}^6$ has matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}$$

Find the Jordan Canonical Form of T.

Solution:

<u>Omitted</u>

■