# ECO220: Quantitative Methods in Economics
# Lecture Notes (0201)

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## Contents

# 1 Lecture 1 May. 08 2018

## 1.1 Notations

| Variable | Population | Sample |
|---|---|---|
| Size | $N$ | $n$ |
| Mean | $\mu$ | $\overline{x}$ |
| Std | $\sigma$ | $s$ |

## 1.2 From Sample to Population

Let $p$ denote the percentage of qualified people in <u>population</u> and let $\hat{p}$ denote the percentage of qualified people in <u>sample</u>. Then, $p$ has an <u>unknown</u> value and the value $\hat{p}$ can be calculated from sample data. We say *$\hat{p}$ is an **estimator** for $p$*, and the value of $p$ is still unknown and can only be estimated.

$p$ is a **fixed value** (i.e. $p$ is fixed once population is fixed, we can measure the exact and certain value of $p$ if we traverse the whole population). But $\hat{p}$ will change from sample to sample. We call $\hat{p}$ an **estimator** (or **sample statistic**).

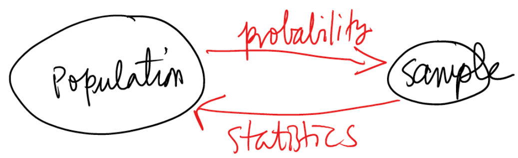
The value of sample statistic will change from sample to sample. And, therefore, we call $\hat{p}$ a **random value**.

# 2 Lecture 2 May. 09 2018

## 2.1 What is Statistics?

$$\text{Statistics} \begin{cases} \text{Descriptive Statistics} \begin{cases} \text{Graphs} \\ \text{Numerical measures} \end{cases} \\ \text{Inferential Statistics} \quad \textit{Draw conclusions in a population based on sample data.} \end{cases}$$

*Inferential Statistics involves uncertainties. To deal with the uncertainties, we need* ***probability***




## 2.2 Data

$$\text{Data} \begin{cases} \text{Quantitative data} \begin{cases} \text{Discrete} \\ \text{Continuous} \end{cases} \\ \text{Qualitative data (Categorical data)} \end{cases}$$

## 2.3 Descriptive Statistics - Graphs

## 2.4 Descriptive Statistics - Numerical measures

### 2.4.1 Measures of centre

**Mean** Let $\{x_1, \ldots, x_N\}$ be measurements for the population with size $N$. The population mean is denoted by $\mu$ and defined as

$$\mu = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Let $\{x_1, \ldots, x_n\}$ be measurements for the <u>sample</u> of size $n$. The sample mean is denoted by $\overline{x}$ and defined as

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

**Note** *The mean is sensitive to extreme values.*

**Median** is the value in the middle when all data are put in order of magnitude. (For data with even size, median is defined as the average of the two values in the middle.)

**Mode** is value(s) with highest frequency.

**Percentiles** the $k^{th}$ percentile is a number such that $k\%$ of data fall below this number.

# 3 Lecture 3 May. 15 2018

## 3.1 Measures of Variation(Spread)

**Variance and Standard Derivation** Let $\{x_1, \dots, x_N\}$ denote the population with size $N$ and let $\{x_1, \dots, x_n\}$ denote the sample with size $n$. Then

| Measures | Population | Sample |
|---|---|---|
| Size | $N$ | $n$ |
| Mean | $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$ | $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ |
| Variance | $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$ | $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ |
| Std | $\sigma = +\sqrt{\sigma^2}$ | $s = +\sqrt{s^2}$ |

**Note** When calculate the sample variance, use $n-1$ as denominator.

**Note** mathematically,

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n-1}[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}(\sum_{i=1}^{n} x_i)^2]$$

**Range** is defined as the difference between the largest value and the smallest value.

# 4 Lecture 4 May. 16 2018

## 4.1 Covariance and Correlation: on Populations

Consider two sets of data (population) with size $N$, denoted as $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_N\}$, where $x$ and $y$ measure the age and income of observation, respectively.

**Denote** $\mu_x :=$ mean of $x$, $\mu_y :=$ mean of $y$
$\sigma_x :=$ std dev of $x$ and $\sigma_y :=$ std dev of $y$. *When $x$ changes, does $y$ change?*

**Covariance** defined covariance between two datasets, $x$ and $y$ as,

$$Cov(x, y) = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu_x)(y_i - \mu_y)$$

**Correlation coefficient** the correlation coefficient $\rho$ between datasets $x$ and $y$ is defined as

$$\rho = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

## 4.2 Covariance and Correlation: on Samples

When $N$ is too large, we select a sample of size $n$.

Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ denote the selected samples with size $n$, $\overline{x}, \overline{y}$ denote the sample means, and $s_x, s_y$ denote the sample std dev.

**Covariance** between two sample is defined as

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

**Correlation coefficient** the sample correlation $r$ is defined as

$$r = \frac{Cov(x, y)}{s_x s_y}$$

## 4.3 Interpretations

### 4.3.1 Interpreting covariance

**Example** Consider samples $x$ and $y$ with

$$Cov(x, y) = -25.31$$

The <u>negative sign</u> means $x$ and $y$ have a <u>negative linear relationship</u>. As $x$ increases, $y$ tends to decrease. The <u>magnitude</u> 25.31 has **no** meaning.

### 4.3.2 Interpreting correlation coefficient

**Example** Consider samples $x$ and $y$ with

$$r = -0.94$$

The <u>negative sign</u> means $x$ and $y$ have <u>negative linear relationship</u>. As $x$ increases, $y$ decreases. The <u>magnitude</u> 0.94 means the <u>linear relationship is strong</u>. When $r$ is close to 1 or -1, the string line relation is strong, when $r$ is close to 0, the relation is weak.

**Note** $\rho \in [-1,1]$ and $r \in [-1,1]$

# 5 Lecture 5 May. 22 2018

## 5.1 Introduction to Simple Regression

Let the linear estimator to be $\hat{y} = b_0 + b_1 x$ and let $y_i$ denote the actual value at $x_i$, $\hat{y}$ is the estimated $y$ value at $x_i$. Then, $e_i := y_i - \hat{y_i}$ is the error of y value at $x_i$ (a.k.a. **residual**).

**Note** notice that $\sum_{i=1}^{n} e_i \equiv 0$.

**SSE** **Sum of Squared Error**(SSE) as

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

**OLS** Minimize $SSE$ with respect to $b_0$ and $b_1$, we have the FOC as

$$\begin{cases} \frac{\partial SSE}{\partial b_0} = 0 \\ \frac{\partial SSE}{\partial b_1} = 0 \end{cases}$$

By solving the first order conditions, we have

$$\begin{cases} b_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2} \\ b_0 = \overline{y} - b_1 \overline{x} \end{cases}$$

The above method to find $b_0$ and $b_1$ is called the <u>method of least square</u>, or <u>method of Ordinary Least Square (OLS)</u>.

## 5.2 Relationship between $b_1$ and $r$

$$b_1 = \frac{Cov(x,y)}{Var(x)} = \frac{Cov(x,y)}{std(x)std(y)} \frac{std(y)}{std(x)} = r \frac{s_y}{s_x}$$

## 5.3 Analysis of Variance (ANOVA)

Let $y_i$ denote the actual $y$ value at $x_i$ and $\hat{y}_i$ denote the estimated $y$ value at $x_i$.

**Definition**

$$SST = \sum_{i=1}^{n}(y_i - \overline{y})^2$$

$$SSR = \sum_{i=1}^{n}(\hat{y}_i - \overline{y})^2$$

$$SSE = \sum_{i=1}^{n}(y_i - \hat{y}_1)^2$$

**Notice** $SST = SSR + SSE$

**Anova Table**

| | SS | df | MS | F |
|---|---|---|---|---|
| Regression | SSR | 1 | MSR | MSR/MSE |
| Error(Residual) | SSE | MSE | $n-2$ | |
| Total | SST | $n-1$ | | |

where MS stands for **mean square** and is defined as

$$MS = \frac{SS}{df}$$

$$MSR = \frac{SSR}{1}$$

$$MSE = \frac{SSE}{n-2}$$

# 6 Lecture 6 May. 23 2018

## 6.1 OLS, continued.

**R-square** **coefficient of determination** is defined as

$$R^2 = \frac{SSR}{SST}$$

and notice that $R^2 \in [0, 1]$ and can be interpreted as <u>% of variation in $y$ explained by $x$ (via the linear model)</u>

**Note** in ECO220, we use $R^2$ or $r^2$ to represent the same thing.

## 6.2 Sample space, Event and Probability

**Experiment** an experiment is a process that creates <u>two or more</u> outcomes.

**Random Experiment** a random experiment is an experiment such that the outputs *cannot* be determined <u>with certainty</u> before the end of the experiment.

**Sample Space** a sample space is the set of all possible outcomes in a random experiment.

**Event** an event is a subset of a sample space.

**Prob** Let $S$ be the sample space, let $E$ be an event, then the **probability of** $E$ $P(E)$ is defined as

$$P(E) = \text{probability of } E = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S}$$

*assuming that each outcome in $S$ has equal likelihood to be chosen into $E$.*

### 6.3 Some Rules of Probability

Let $E$ be an event in sample space $S$, then

- $P(E) \in [0, 1]$.
- $P(S) = 1$.
- Let $E^c$ denote the **complementary** of $E$, then $P(E^c) = 1 - P(E)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Addition Rule)

## 7 Lecture 7 May. 29 2018

**Mutually Exclusive Event** If $A \cap B = \emptyset$, we say events $A$ and $B$ are **mutually exclusive/disjoint**. Then, if $A, B$ are disjoint, we have

$$P(A \cup B) = P(A) + P(B)$$

### 7.1 Conditional Probability

**Conditional Prob** In general, if $A$ and $B$ are events in sample space $S$, the conditional probability of $A$ given $B$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

**Multiplication rule**

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

## 7.2 Independent Event

**Independent Event** We say two events, $A$ and $B$ are **independent** if any of the following is true. (those definitions below are equivalent.)

- $P(A|B) = P(A)$ or
- $P(B|A) = P(B)$ or
- $P(A \cap B) = P(A)P(B)$

# 8 Lecture 8 May. 30 2018

## 8.1 Bayes Theorem

Let $A$ and $B$ be two events. Then,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

*Proven by definition of conditional probability.*

## 8.2 Random Variable and Prob. Distributions

**Prob. distribution**

**Cumulative Prob. distribution**

## 8.3 Expected Values

**Expected Value** Let $X$ be a random variable with probability distribution $P(X)$. Then defined the expected value of $X$, $\mathbb{E}(X)$ as

$$\mu = \mathbb{E}(X) = \sum_x xP(X = x)$$

**Variance of Random Variable** For random variable $X$, we have

$$\sigma^2 = Var(X) = \sum_x (x - \mu)^2 P(X = x) = \mathbb{E}(X - \mu)^2$$

# 9 Lecture 9 June. 5 2018

## 9.1 Expected Value of a Random Variable

**Mean** $\mu = \mathbb{E}(X) = \sum_x xP(X = x)$.

**Variance** $\sigma^2 = \mathbb{E}(x - \mu)^2 = \mathbb{E}(X^2) - \mu^2$.

### 9.2 Laws of Expectation

In general, let $X$ be a random variable, and let $a, c \in \mathbb{R}$, then

$$\mathbb{E}(aX + c) = a\mathbb{E}(X) + c$$

$$Var(aX + c) = Var(aX) = a^2 Var(X)$$

Let $X$ and $Y$ be random variables, and let $a, b, c \in \mathbb{R}$, then

$$\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$$

$$Var(aX + bY + c) = Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\ Cov(X, Y)$$

**Note** if $X$ and $Y$ are independent, then $\rho = Cov(X, Y) = 0$ and

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

### 9.3 Binomial Distribution

In general, let $n$ be the number of independent trails and $p = P(\#success)$. Let $X$ be a random variable which is the number of successes in $n$ trails, we have

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},\ \text{for } x = \{0, 1, 2, \ldots, n\}$$

$$\mu = \mathbb{E}(X) = np$$

$$\sigma^2 = Var(X) = npq,\ q = 1 - p$$

## 10 Lecture 10 June. 6 2018

### 10.1 Uniform Distribution

Let $X$ be uniform from $a$ to $b$. $f(x) = \frac{1}{b-a}, a \leq x \leq b$

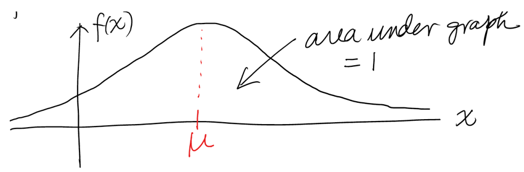
$$\mu = \mathbb{E}(X) = \int_a^b x f(x) dx = \frac{a+b}{2}$$

$$\sigma^2 = Var(X) = \mathbb{E}(X^2) - \mu^2$$

### 10.2 Normal Distribution

Let $X$ be a continuous random variable, satisfying $-\infty < x < \infty$. The mean of $X$ is $\mu$ and the variance of $X$ is $\sigma^2$. The graph of $X$ is The graph is symmetric at $\mu$ and the variance $\sigma^2$ determines the shape(spread) of $X$. We say $X$ <u>follows a normal distribution with mean $\mu$ and variance $\sigma^2$.</u> And denote as

$$X \sim N(\mu, \sigma^2)$$




**Standard Normal Distribution** A standard normal distribution is a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$. Denote the standard normal distribution as

$$Z \sim N(0, 1)$$

# 11 Lecture 11 June. 12 2018

## 11.1 Applying normal distribution

**Theorem** Let $X \sim N(\mu, \sigma^2)$, then

$$z = \frac{x - \mu}{\sigma} \sim N(0, 1)$$

## 11.2 Normal Approximation to Binomial

Consider a random variable $X \sim B(n, p)$, then we can approximate the binomial with a normal distribution $X \approx N(np, npq)$.

# 12 Lecture 12 June. 13 2018

## 12.1 Sampling Distributions

Consider population with size $N$ has $p$ as percentage of *success* (qualified) and sample with size $n$ with $\hat{p} = \frac{x}{n}$ as percentage of *success*.

$p$ is a parameter which has a fixed value. In real life, the value of $p$ is usually unknown. $\hat{p}$ is a sample statistic, which does not have fixed value (random variable, value of $\hat{p}$ vary from sample to sample). Also, $\mu$ and $\sigma$ are parameters, which are fixed but usually unknown. $\overline{x}$ is a sample statistic, and is random.

Suppose we know $p$ for population, then we can conclude about random variables from a random sample,

1. $\mathbb{E}(\hat{p}) = p$.
2. $Var(\hat{p}) = \frac{pq}{n}$, $q = 1 - p$

3. When sample size $n$ is large, the distribution of $\hat{p}$ is approximately normal (**Central Limit Theorem in proportion**) [1]

That's

$$\hat{p} \approx\sim N(p, \frac{pq}{n}), \text{ when } n \text{ is large.}$$

**Example** Given $p_{success} = 0.3$ for the whole population and find the probability that at least 320 *success* found in a sample of size $n = 1000$. i.e. Let $X$ denote the number of success in sample with $n = 1000$, find $P(X \geq 320)$.

**Method 1** Use Central Limit Theorem, check $np = 300 \geq 10 \wedge nq = 700 \geq 10$, thus $n$ is *large*. And approximate $\hat{p}$ of sample as

$$\hat{p} \sim N(p, \frac{pq}{n})$$

*Soln.*

$$P(X \geq 320) = P(\hat{p} \geq 0.32)$$
$$= P(\frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \geq \frac{0.32 - 0.3}{\sqrt{\frac{0.3*0.7}{1000}}}) = P(z \geq \frac{0.02}{\sqrt{\frac{0.21}{1000}}})$$
$$\text{Find } z \text{ in z-table}$$

■

**Method 2** Use Normal Approximation to Binomial. $p = 0.3$ and $n = 1000$.

$$X \approx Y \sim N(300, 210)$$

*Soln.*

$$P(X \geq 320) = P(Y > 319.5)$$
$$= P(\frac{Y - \mu}{\sigma} > \frac{319.5 - 300}{\sqrt{210}})$$
$$= P(z > 1.35) \text{ find in z table}$$

■

**Note** *methods 1 and 2 do* ***not*** *give exactly same answer, but the answers should be close.*

[1] As a rule of thumb, $n$ is considered to be large when $np \geq 10 \wedge nq \geq 10$.

## 12.2 Sampling distribution of $\overline{X}$, the sample mean

1. $\mathbb{E}(\overline{X}) = \mu$.
2. $Var(\overline{X}) = \frac{\sigma^2}{n}$.
3. When $n$ is large, the distribution of $\overline{X}$ is approximately normal. (**Central Limit Theorem in Mean**).
4. When population is normal, the distribution of $\overline{X}$ is exactly normal, regardless of the sample size $n$.

Putting together,

$$\overline{X} \sim N(\mu, \frac{\sigma^2}{n}), \text{ when } n \text{ is large.}$$