# Introduction to Real Analysis

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## Contents

# 1 The Axiom of Completeness

## 1.1 Preliminaries

**Definition 1.1.** A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \; s.t. \; \forall a \in A, \; u \geq a \tag{1.1}$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \; s.t. \; \forall a \in A, \; l \leq a \tag{1.2}$$

**Example 1.1.** The set of integers, $\mathbb{Z}$, is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

**Notation 1.1.** Let $A \subset \mathbb{R}$, we use $A^{\uparrow}$ and $A^{\downarrow}$ to denote collections of upper bounds of $A$ and lower bounds of $A$. Note that $A^{\uparrow}$ and $A^{\downarrow}$ are potentially empty.

**Definition 1.2.** A real number $s \in \mathbb{R}$ is the **least upper bound(supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^{\uparrow}$ and $\forall u \in A^{\uparrow}, \; s \leq u$. Such $s$ is denoted as $s := \sup A$.

**Definition 1.3.** A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for $A$ if $f \in A^{\downarrow}$ and $\forall l \in A^{\downarrow}, \; l \leq f$. Such $f$ is often written as $f := \inf A$.

**Axiom 1.1** (The Axiom of Completeness/Least Upper Bounded Property)**.** $\forall \varnothing \neq A \subset \mathbb{R}$ such that $A^{\uparrow} \neq \varnothing$, $\exists \sup A$.

**Definition 1.4.** Let $\varnothing \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of $A$ if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of $A$ if $\forall a \in A, a_1 \leq a$.

**Example 1.2.** $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

**Proposition 1.1.** Let $\varnothing \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \tag{1.3}$$

*Proof. Step 1: Show* $c + \sup A \in (c + A)^{\uparrow}$*:*
Let $x \in c+A$, $\exists \; a \in A \; s.t. \; x = c+a$. Then, $x = c+a \leq c+\sup A$. Therefore, $x \leq c+\sup A \; \forall x \in A$, which implies what desired.
*Step 2: Show* $\forall u \in (c + A)^{\uparrow}, \; c + \sup A \leq u$*:*
Let $u \in (c + A)^{\uparrow}$, then $u \geq c + a \; \forall a \in A \implies u - c \geq a \; \forall a \in A \implies u - c \in A \uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.
Hence, $\sup(c + A) = c + \sup A$. ■

**Lemma 1.1** (Alternative Definition of Supremum)**.** Let $s \in A^{\uparrow}$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

(i) $s = \sup A$;

(ii) $\forall \varepsilon, \exists a \in A,\ s.t.\ a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^{\uparrow}$).

*Proof.* Immediately. ∎

**Theorem 1.1** (Nested Interval Property)**.** Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \ \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing \tag{1.5}$$

*Proof.* Note that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded above by any $b_k$, by the completeness axiom, there exists $a^* := \sup_{n\in\mathbb{N}} a_n$. Since $a^* \in (a_n)^{\uparrow}$, $a^* \geq a_n \ \forall n \in \mathbb{N}$. Further, because $a^*$ is the *least* upper bound, then for every upper bound $b_n$, it must be $a^* \leq b_n \ \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n] \ \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n\in\mathbb{N}} I_n$. ∎

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

**Theorem 1.2** (Archimedean Property)**.**

(i) $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}\ s.t.\ n > x$;

(ii) $\forall y \in \mathbb{R}_{++},\ \exists n \in \mathbb{N}\ s.t. \frac{1}{n} < y$.

Archimedean property *of natural numbers* can be interpreted as *there is no real number that bounds* $\mathbb{N}$. This interpretation can be seen by considering the negations of above statements:

(i) $\exists x \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N},\ n \leq x$;

(ii) $\exists y \in \mathbb{R}_{++}\ s.t.\ \forall n \in \mathbb{N},\ y \leq \frac{1}{n}$.

*Proof of (i) by Contradiction.* Suppose the negated statement (i) is true, $\mathbb{N}$ is bounded above. By the completeness axiom, there exists $a^* := \sup \mathbb{N}$. $\exists n \in \mathbb{N}\ s.t.\ a^* - 1 < n$. In this case, $a^* < n+1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^{\uparrow}$ and leads to a contradiction. ∎

*Proof of (ii).* Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ∎

### 1.2 Density of Rational Numbers

**Theorem 1.3.** For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The above theorem says $\mathbb{Q}$ is in fact **dense** in $\mathbb{R}$. More generally, one says a set $A \subset X$ is dense whenever the closure of $A$, $\overline{A} = X$.

*Proof.* *Step 1:* Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.
*Step 2:* Let $m$ be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ∎

**Theorem 1.4.** $\exists \alpha \in \mathbb{R}$ *s.t.* $\alpha^2 = 2$.

*Proof.* Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in $\mathbb{R}$ bounded from above. By the completeness axiom, $\Omega$ possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^{\uparrow}$, which contradicts the fact that $\alpha$ is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that $\alpha$ is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ∎

## 2 Sequences

**Theorem 2.1** (Triangle Inequality)**.** Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

**Corollary 2.1.** Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \tag{2.1}$$

*Proof.* Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$. And $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$. Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ∎

**Definition 2.1.** A sequence $(a_n) \subset \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geq N \implies |a_n - a| < \varepsilon \tag{2.2}$$

Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at $a$ with radius $\varepsilon$ is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{2.3}$$

**Theorem 2.2.** The limit of any convergent sequence is unique.

*Proof.* Let $(a_n)$ be a convergent sequence, assume, for contradiction, that $(a_n) \to L_1$ and $(a_n) \to L_2$ such that $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \to L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$\begin{aligned}
|a_n - L_2| &= |a_n - L_1 - (L_2 - L_1)| && (2.4)\\
&\geq ||a_n - L_1| - |L_2 - L_1|| && (2.5)\\
&= ||L_1 - L_2| - |a_n - L_1|| && (2.6)\\
&= 3\varepsilon - |a_n - L_1| && (2.7)\\
&> 2\varepsilon && (2.8)
\end{aligned}$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq \mathbb{N}$. ■

**Definition 2.2.** A sequence $(a_n)$ is **divergent** if it does not converge.

**Example 2.1.** The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \cdots)$ is divergent.

*Proof.* Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \to L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $\left|L - \frac{1}{5}\right| < \frac{2}{15}$. Similarly,

$$\begin{aligned}
\left|-\frac{1}{5} - L\right| &= \left|\frac{1}{5} + L\right| && (2.9)\\
&= \left|\frac{1}{5} + L - \frac{1}{5} + \frac{1}{5}\right| && (2.10)\\
&= \left|(L - \frac{1}{5}) - (-\frac{2}{5})\right| && (2.11)\\
&\geq \left|\left|L - \frac{1}{5}\right| - \frac{6}{15}\right| && (2.12)\\
&= \frac{6}{15} - \left|L - \frac{1}{5}\right| && (2.13)\\
&> \frac{4}{15} && (2.14)\\
&> \varepsilon && (2.15)
\end{aligned}$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ■

*Alternative Proof.* If $(a_n)$ is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of $(a_n)$ converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ■

**Definition 2.3.** A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N},\ |a_n| < M$.

**Theorem 2.3.** Every convergent sequence is bounded.

*Proof.* Let $(a_n) \to L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n<N} a_n$, take $M := \max\{Q, |L| + 1\}$, then $M$ bounds $(a_n)$. ■

**Theorem 2.4** (Algebraic Limit Theorem)**.** Let $(a_n) \to a, (b_n) \to b$ be convergent sequences, and $c \in \mathbb{R}$, then

(i) $(ca_n) \to ca$;

(ii) $(a_n + b_n) \to a + b$;

(iii) $(a_n b_n) \to ab$;

(iv) $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$, provided $b_n, b \neq 0$.

*Proof (i).* Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

*Proof (ii).* Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3}$ $\forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3}$ $\forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \tag{2.16}$$

■

*Proof (iii).* Note that

$$|a_n b_n - ab| = |a_n b_n + a_n b - a_n b - ab| \tag{2.17}$$
$$\leq |a_n b_n - a_n b| + |a_n b - ab| \tag{2.18}$$
$$\leq |a_n||b_n - b| + |b||a_n - a| \tag{2.19}$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because $(a_n)$ is convergent, let $M$ denote its bound such that $|a_n| < M$ $\forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_n b_n - ab| < \varepsilon$. ■

*Proof (iv).* **Claim i:** when $n$ is sufficiently larger, $|b_n| > 0$ is bounded away from zero by $M$. Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such $n$,

$$|b_n| = |b_n - b - (-b)| \tag{2.20}$$
$$\geq ||b_n - b| - |b|| \tag{2.21}$$
$$\geq |b| - |b_n - b| \tag{2.22}$$
$$> \frac{9|b|}{10} \tag{2.23}$$

**Claim ii:** $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_n b} - \frac{b_n}{b_n b}\right| \tag{2.24}$$

$$= \frac{1}{|b_n||b|}|b_n - b| \tag{2.25}$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

**Theorem 2.5** (Order Limit Theorem)**.** Let $(a_n) \to a$ and $(b_n) \to b$, then

(i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;

(ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;

(iii) $\exists c \in \mathbb{R} \ s.t. \ c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;

(iv) $\exists c \in \mathbb{R} \ s.t. \ a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

*Proof.* (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.
(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \to (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.
(iii) and (iv) Consider constant sequence defined as $(c_n)$ such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

**Theorem 2.6** (Squeeze Theorem)**.** Let $(x_n) \to L$ and $(z_n) \to \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \to \ell$.

*Proof.* Let $\varepsilon > 0$, because both $(x_n) \to \ell$ and $(y_n) \to \ell$,

$$\exists N_1 \ s.t. \ n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \tag{2.26}$$

$$\exists N_2 \ s.t. \ n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \tag{2.27}$$

Take $N_3 := \max\{N_1, N_2\}$, then for every $n \geq N_3$,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \tag{2.28}$$

$$\implies y_n \in V_\varepsilon(\ell) \tag{2.29}$$

therefore $(y_n) \to \ell$ by definition. ■

## 2.1 Monotone Convergence Theorem

**Definition 2.4.** A sequence $(a_n)$ is said to be **monotone** if it is either increasing ($a_{n+1} \geq a_n \ \forall n \in \mathbb{N}$) or decreasing ($a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$).

**Theorem 2.7** (Monotone Convergence Theorem)**.** If a sequence $(a_n)$ is bounded, then it converges.

*Proof.* WLOG, assume $(a_n)$ is increasing, let $\Gamma := \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$, because $\Gamma$ is bounded, $s := \sup_n \Gamma$ is well-defined by the completeness of real numbers.
**Claim:** $(a_n) \to s$. Let $\varepsilon > 0$, by the definition of supremum, $\exists N \in \mathbb{N}$ such that $a_N > s - \varepsilon$. Because the sequence is increasing and $s + \varepsilon \in \Gamma^{\uparrow}$, $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$. $(a_n) \to s$ by definition. ■

## 2.2 Series

**Definition 2.5.** Let $(a_i)$ be a sequence, then the $n$-th **partial sum** is defined as $s_n := \sum_{i=1}^n a_i$. And the **infinite sum/series** of $(a_n)$ is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \to s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \tag{2.30}$$

**Example 2.2.** $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

*Proof.* Obviously the corresponding partial sums are increasing because the sequence $(\frac{1}{i^2})$ is positive.
**Claim:** $(s_n)$ is bounded from above. Let $n \in \mathbb{N}$, observe

$$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \tag{2.31}$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \tag{2.32}$$

$$= 2 - \frac{1}{n} \leq 2 \tag{2.33}$$

The result is immediate by the monotone convergence theorem. ■

**Example 2.3** (Harmonic Series)**.** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

*Proof.* **Claim:** there exists a subsequence of $(s_n)$ diverges, so $(s_n)$ cannot be convergent. Consider the subsequence $(s_k)$ constructed by defining $s_k := s_{2^k}$. Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \tag{2.34}$$

$$> 1 + \frac{1}{2}k \tag{2.35}$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. ■

**Definition 2.6.** Let $(a_n)$ be a sequence, then for every strictly increasing sequence $(n_i)_i$ in $\mathbb{N}$, $(a_{n_i})$ is a **subsequence** of $(a_n)$.

**Theorem 2.8.** All subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Let $(a_n) \to \ell$, let $(a_{n_k})$ be a subsequence of $(a_n)$. Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\varepsilon(\ell)$. By the definition of subsequences, there exists some $K \in \mathbb{N}$ such that $n_K = N$. Take such $K$, then for every $k \geq K$, it must be $n_k \geq N$. Therefore $a_{n_k} \in V_\varepsilon(\ell)$ for every $k \geq K$, and $(a_{n_k}) \to \ell$ by definition. ■

**Corollary 2.2.** A sequence $(a_n)$ must be divergent if there exists two subsequences of it converge to two different limits.

*Proof.* Immediate by taking the contrapositive form of above theorem. ■

**Theorem 2.9** (Bolzano–Weierstrass)**.** Every bounded sequence contains a convergent subsequence.

*Proof.* Suppose $(a_n)$ is bounded by certain $M > 0$, that's, for every $n \in \mathbb{N}$, $-M < a_n < M$. Consider the split $I_1^\ell := [-M, 0]$ and $I_1^u := [0, M]$. At least one of above closed intervals contain an infinitely many elements of $(a_n)$. Define the interval as $I_2$. At each $I_n$, one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and $I_{n+1}$ is defined to be such sequence. Note that the sequence of closed intervals constructed from above recursive procedure is in fact nested. Obviously $\lim_{n\to\infty} |I_n| = 0$. Further, by the nested interval property, one can show that $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$. Then $\cap_{n\in\mathbb{N}} I_n$ must be a singleton with $a$ in it. One can construct such that $a_{n_k} \in I_k$, and clearly $(a_{n_k}) \to a$. ■