Introduction to Real Analysis

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Contents

1 The Axiom of Completeness

Definition 1.1. A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \; s.t. \; \forall a \in A, \; u \geq a \tag{1.1}$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \; s.t. \; \forall a \in A, \; l \leq a \tag{1.2}$$

Example 1.1. The set of integers, $\mathbb{Z}$, is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subset \mathbb{R}$, we use $A^{\uparrow}$ and $A^{\downarrow}$ to denote collections of upper bounds of A and lower bounds of A. Note that $A^{\uparrow}$ and $A^{\downarrow}$ are potentially empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound(supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^{\uparrow}$ and $\forall u \in A^{\uparrow}, \; s \leq u$. Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if $f \in A^{\downarrow}$ and $\forall l \in A^{\downarrow}, \; l \leq f$. Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property)**.** $\forall \varnothing \neq A \subset \mathbb{R}$ such that $A^{\uparrow} \neq \varnothing$, $\exists \sup A$.

Definition 1.4. Let $\varnothing \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

Proposition 1.1. Let $\varnothing \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \tag{1.3}$$

Proof. Step 1: Show $c + \sup A \in (c + A)^{\uparrow}$*:*
Let $x \in c + A$, $\exists \; a \in A \; s.t. \; x = c + a$. Then, $x = c + a \leq c + \sup A$. Therefore, $x \leq c + \sup A \; \forall x \in A$, which implies what desired.
Step 2: Show $\forall u \in (c + A)^{\uparrow}, \; c + \sup A \leq u$*:*
Let $u \in (c + A)^{\uparrow}$, then $u \geq c + a \; \forall a \in A \implies u - c \geq a \; \forall a \in A \implies u - c \in A \uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.
Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum)**.** Let $s \in A^{\uparrow}$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

(i) $s = \sup A$;

(ii) $\forall \varepsilon, \exists a \in A,\ s.t.\ a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^{\uparrow}$).

Proof. Immediately. ■

Theorem 1.1 (Nested Interval Property)**.** Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \ \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing \tag{1.5}$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k, by the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$. Since $a^* \in (a_n)^{\uparrow}$, $a^* \geq a_n \ \forall n \in \mathbb{N}$. Further, because a^* is the *least* upper bound, then for every upper bound b_n, it must be $a^* \leq b_n \ \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n] \ \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Theorem 1.2 (Archimedean Property)**.**

(i) $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}\ s.t.\ n > x$;

(ii) $\forall y \in \mathbb{R}_{++},\ \exists \mathbb{N}\ s.t. \frac{1}{n} < y$.