

ECO325: Lecture Notes

Advanced Economic Theory: Macro

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cense.



Notes

Github https://github.com/TianyuDu/Spikey_UofT_Notes

Direct Download TianyuDu.com/notes

Color notations

- Important equations for model setup.
- Important equations as results from model.
- Implications of model result.

Revisions

- Revise October 2. 2018. Midterm 1. Lec1-Lec4
- Revise October 21. 2018. Lec5-Lec6
- Revise

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1 Lecture 1 September 6. 2018

Definition 1.1. A **growth miracle** are episodes where the growth in a country far exceeds the world average over an extended period of time. Result the country experiencing the miracle moves up the world income distribution.

Definition 1.2. A **growth disaster** is an episode where the growth in a country falls short of the world average for an extended period of time. Result the country moves down in the world income distribution.

Facts (with corresponding result from Solow growth model.)

1. Real output(Y) grows at a (more or less) constant rate $(n + g)$.
2. Stock of real capital(K) grows at a (more or less) constant rate $(n + g)$ (but it grows faster than labor input(L)).
3. Growth rates of real output and the stock of capital are about the same. (both $n + g$)
4. The rate of growth of output per capita($\frac{Y}{L}$) varies greatly across countries. (g varies across countries)

1.1 Solow Growth Model (continuous time version)

Intro. Solow growth model decomposes the growth in output per capita into portions accounted for by increase in inputs and the portion contributed to increases in productivity.

Notations In the baseline model we denote K as capital, L as labor and A as technology.

1.1.1 Production Function

Remark 1.1. ~~Harrod-neutral technology here, refer to Uzawa's theorem.~~

Definition 1.3. The **effective labor input** (total units of effective labor) is defined as $A(t)L(t)$

Definition 1.4. The production function is defined as a real-valued mapping from input factor space to an output level.

$$Y(t) = F(K(t), A(t)L(t)) \quad (1)$$

Example 1.1. Cobb-Douglas form of production function.

$$Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha}, \quad \alpha \in (0, 1)$$

Assumption 1.1. The production function is assumed to be constant return to scale in K and AL .

$$Y(cK, cAL) = cY(K, AL), \forall c \geq 0$$

This CRS assumption is the result of two separate assumptions.

1. *The economy is big enough that the gains from specialization have been exhausted.* \implies There is **no** increasing return to scale.
2. *Inputs other than capital, labor, and the effectiveness of labor are relatively unimportant.* \implies There is **no** decreasing return to scale.

Definition 1.5. Define $c := \frac{1}{AL}$, the **intensive form** of production function is

$$y(t) = \frac{Y(t)}{A(t)L(t)} = f(k(t))$$

where $y := \frac{Y}{AL}$ denotes the **output per unit of effective labor** and $k := \frac{K}{AL}$ denote the capital stock per unit of effective labor.

2 Lecture 2 September 13. 2018

2.1 Solow Growth Model: Setup

Definition 2.1. Production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ maps input factors:

- $K(t) :=$ aggregate capital stock at time t .
- $L(t) :=$ aggregate labor supply at time t .
- $A(t) :=$ labor argument technology¹ (effectiveness of labor) at time t .

to output values ($Y(t) :=$ aggregate output at time t .) The production function takes the form of

$$Y(t) = F(K(t), A(t)L(t))$$

Assumption 2.1 (Assumptions on Production Function). The production function are assumed to be constant return to scale in $A(t)L(t)$ and $K(t)$.

$$cF(K(t), A(t)L(t)) = F(cK(t), cA(t)L(t)), \forall c > 0$$

Definition 2.2. The **intensive form of production function** is defined as the output per unit of effective labor.

Let

$$f(t) := \frac{Y(t)}{A(t)L(t)}$$

¹Harrod-neutral technology

and

$$k(t) := \frac{K(t)}{A(t)L(t)}$$

denote the output and capital per unit of effective labor respectively. By the assumption of *CRS* on *aggregate* production function, take $c = \frac{1}{A(t)L(t)}$. The *intensive form* production function can be expressed as

$$y(t) = f(k(t)) \quad (1)$$

Assumption 2.2 (Assumptions on Intensive Form Production Function). the function $f(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is assumed to satisfy *Inada Conditions*.

1. $f(0) = 0$: capital is necessary for production.
2. $f'(k) > 0, \forall k \in \mathbb{R}_+$: the marginal return of capital per effective unit of labor is positive.
3. $f''(k) < 0, \forall k \in \mathbb{R}_+$: capital per effective unit of labor is experiencing diminishing marginal return.
4. $\lim_{k \rightarrow 0} f'(k) = \infty$
5. $\lim_{k \rightarrow \infty} f''(k) = 0$

Remark 2.1. The role of assumption 2.2 is to ensure that the path of the economy does not diverge.

Example 2.1 (Cobb-Douglas Production Function). Consider the Cobb-Douglas production function

$$Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha}, \quad \alpha \in (0, 1)$$

Check. Let $c \in \mathbb{R}_+$,

$$\begin{aligned} F(cK, cAL) &= (cK)^\alpha (cAL)^{1-\alpha} \\ &= c^\alpha c^{1-\alpha} K^\alpha AL^{1-\alpha} \\ &= cK^\alpha AL^{1-\alpha} = cF(K, AL) \end{aligned}$$

CRS on aggregate form is shown.

Notice that $f(k) = k^\alpha$

And

1. $f(0) = 0^\alpha = 0$
2. $f'(k) = \alpha k^{\alpha-1} > 0, \forall k \in \mathbb{R}_+$
3. $f''(k) = (\alpha - 1)\alpha k^{\alpha-2} < 0, \forall k \in \mathbb{R}_+$
4. $\lim_{k \rightarrow 0} \alpha \frac{1}{k^{1-\alpha}} = \infty$

5. $\lim_{k \rightarrow \infty} \alpha \frac{1}{k^{1-\alpha}} = 0$

Inada conditions on intensive form are shown. ■

Assumption 2.3 (Assumptions on the Economy). Assume the initial values of K, A, L are given and strictly positive. Labor and Knowledge are assumed to grow at an exogenously given constant rate, denoted as n, g respective.

$$\dot{L}(t) = nL(t), \quad n > 0 \quad (2)$$

$$\dot{A}(t) = gA(t), \quad g > 0 \quad (3)$$

Proposition 2.1. Notice the growth rate of variable $X(t)$ is given by

$$g_X := \frac{\dot{X}(t)}{X(t)} = \frac{\partial \ln X(t)}{\partial t}$$

Proof.

$$\begin{aligned} \frac{\partial \ln X(t)}{\partial t} &= \frac{\partial \ln X(t)}{\partial X(t)} \frac{\partial X(t)}{\partial t} \\ &= \frac{1}{X(t)} \dot{X}(t) = \frac{\dot{X}(t)}{X(t)} = g_X \end{aligned}$$

■

Proposition 2.2. The functional form of technology and labor at time t can be found by solving ODEs

$$L(t) = e^{nt} L(0) \quad (4)$$

$$A(t) = e^{gt} A(0) \quad (5)$$

Assume there is *no government* and the Solow economy is a *closed economy*. The output is divided between *consumption* and *investment* as

$$Y(t) = C(t) + I(t)$$

And given δ as depreciation rate of capital, in discrete time (let $\Delta t = 1$) we have

$$K(t+1) = (1 - \delta)K(t) + I(t)$$

$$\iff I(t) = K(t+1) - K(t) + \delta K(t)$$

As $\Delta \rightarrow 0$ (convert to continuous time)

$$I(t) = \dot{K}(t) + \delta K(t)$$

Assumption 2.4. Assume investment equals saving and a constant friction $s \in [0, 1]$ of output is saved at each epoch. The marginal propensity to save, s is given exogenously.

Therefore,

$$\begin{aligned} I(t) = sY(t) &\implies \dot{K}(t) + \delta K(t) = sY(t) \\ &\implies \dot{K}(t) = sY(t) - \delta(K(t)) \end{aligned}$$

2.2 Dynamics of $k(t)$

For simplicity, assuming $n, g, \delta > 0$ and the dynamics of capital per effective unit of labor follows:

$$\begin{aligned} \dot{k}(t) &:= \frac{\partial k(t)}{\partial t} = \frac{\partial}{\partial t} \frac{K(t)}{A(t)L(t)} \\ &= \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2} \\ &= \frac{\dot{K}}{AL} - \frac{K\dot{A}L}{(AL)^2} - \frac{KA\dot{L}}{(AL)^2} \\ &= \frac{sY - \delta K}{AL} - \frac{\dot{A}}{A} \frac{K}{AL} - \frac{\dot{L}}{L} \frac{K}{AL} \\ &= sy(t) - (n + g + \delta)k(t) \end{aligned}$$

Where $sy(t)$ is the **actual investment** per unit of effective labor and $(n + g + \delta)k(t)$ is the **break-even investment** per unit of effective labor.

Remark 2.2. The **convergence speed** is inversely correlated with the value of $|k(t) - k^*|$, where k^* denotes the steady state level of capital stock per effective unit of labor.

Remark 2.3. With convex production function ($f''(k) > 0$), then $k(t) < k^* \implies \dot{k} < 0$ and $k(t) > k^* \implies \dot{k} > 0$. The steady state value k^* is steady but not stable (with $k(t) \neq k^*$, k does not automatically converge to k^*).

3 Lecture 3 September 20. 2018

3.1 Dynamic Transitions

Remark 3.1. For the dynamic transition function of capital per unit of effective labor:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \quad (1)$$

And dynamic transition and phase diagram can be expressed as

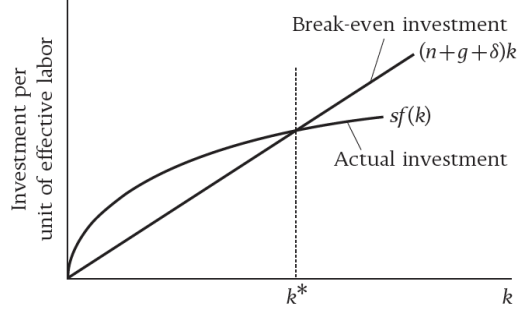


Figure 1: Dynamic Transition of Capital Per Unit of Effective Labor

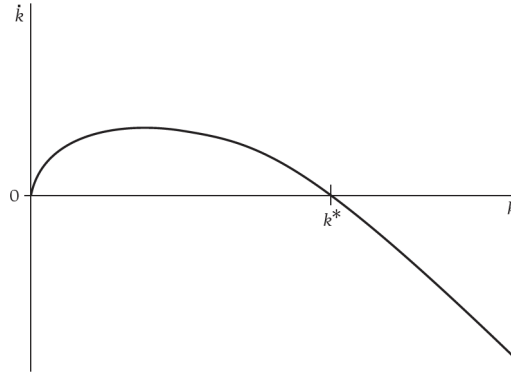


Figure 2: Phase Diagram of Capital Per Unit of Effective Labor

Definition 3.1. Steady level of capital per unit of effective labor(k^*) is defined as the level of capital per unit of effective labor that equates break-even investment per unit of effective labor and actual investment per unit of effective labor. So that k does not deviate from k^* .²

$$k^* := \{k \in \mathbb{R}_+ : sf(k) = (n + g + \delta)k\}$$

Remark 3.2. The values of other endogenous variables at steady state are derived from k^* .

Example 3.1. Find the steady state growth rate of investment, consumption and output per unit of effective labor.

$$y^* = f(k^*) \tag{2}$$

$$i^* = sf(k^*) = (n + g + \delta)k^* \tag{3}$$

$$c^* = y^* - i^* = f(k^*) - (n + g + \delta)k^* = (1 - s)f(k^*) \tag{4}$$

²The definition can also be expressed as $k^* := \{k \in \mathbb{R}_+ : \dot{k}(k) = 0\}$

For the growth rate of each endogenous variable (per unit of effective labor).

$$\frac{\partial i(t)}{\partial t} \Big|_{k=k^*} = 0 \quad (5)$$

$$\frac{\partial c(t)}{\partial t} \Big|_{k=k^*} = 0 \quad (6)$$

$$\frac{\partial y(t)}{\partial t} \Big|_{k=k^*} = 0 \quad (7)$$

above relations are equivalent to

$$\text{On steady state} \begin{cases} \dot{i}(t) = 0 \\ \dot{c}(t) = 0 \\ \dot{y}(t) = 0 \end{cases} \quad (8)$$

Proof. By definition of consumption per unit of effective labor,

$$c(\cdot) = (1 - s)f(k(t))$$

$$\implies \dot{c}(t) := \frac{\partial c(\cdot)}{\partial t} = (1 - s)f'(k(t))k\dot{t} \text{ by chain rule}$$

Since $\dot{k}|_{k=k^*} = 0$ and $(1 - s)f'(k(t)) < \infty$

Thus $\dot{c}(t)|_{k=k^*} = 0$

And $i(t) = sf(k(t))$, which is constant at $sf(k^*)$ at steady state. ■

3.2 Balanced Growth Path

Definition 3.2. A **balanced growth path** is a situation where each variable in the model are all growing at a constant rate.^{3 4}

3.2.1 Growth Rates on Balanced Growth Path

Population and Technology By definition of population and technological progress,

$$g_A := \frac{\dot{A}}{A} = g \quad (9)$$

$$g_L := \frac{\dot{L}}{L} = n \quad (10)$$

Capital per person Since $\frac{K(t)}{L(t)} = \frac{k(t)A(t)L(t)}{L(t)} = k(t)A(t)$, and the growth rate of $x(t)$ can be found as $\frac{\partial \ln x(t)}{\partial t}$. Then

³Variables are not required to grow at the same rate by this definition.

⁴Variables remaining fixed are also considered as growing at a constant rate ($g = 0$).

Solution.

$$\begin{aligned}\frac{\partial \ln \frac{K(t)}{L(t)}}{\partial t} &= \frac{\partial k(t)A(t)}{\partial t} \\ &= \frac{\partial \ln k(t)}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\ &= \frac{\dot{k}(t)}{k(t)} + g\end{aligned}$$

And at the steady state, by definition, $\dot{k}(t)|_{k=k^*} = 0$, therefore

$$g_{\frac{K}{L}}^* = g \quad (11)$$

■

Output and Consumption per person Similarly,

Solution.

$$\begin{aligned}\frac{Y(t)}{L(t)} &= y(t)A(t) \\ g_{\frac{Y}{L}} &= \frac{\partial \ln y(t) + \ln A(t)}{\partial t} \\ &= \frac{\partial \ln y}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\ &= g + \frac{\dot{y}}{y}\end{aligned}$$

and for consumption per person,

$$\begin{aligned}\frac{C(t)}{L(t)} &= c(t)A(t) \\ g_{\frac{C}{L}} &= g + \frac{\dot{c}}{c}\end{aligned}$$

Thus, on the balanced growth path,⁵

$$g_{\frac{Y}{L}}^* = g \quad (12)$$

$$g_{\frac{C}{L}}^* = g \quad (13)$$

■

Proposition 3.1. Along the balanced growth path, consumption and output per person also grow at rate g .

⁵ g_X^* denotes the growth rate of variable X on the balanced growth path.

Proposition 3.2. Along the balanced growth path, aggregate variables, $Y(t), I(t), C(t)$ are all growing at a rate $n + g$.

$$g_Y^* = g_C^* = g_I^* = n + g \quad (14)$$

Proof.

$$\begin{aligned} g_K &= \frac{\partial \ln K(t)}{\partial t} \\ &= \frac{\partial \ln A(t)L(t)k(t)}{\partial t} \\ &= \frac{\partial \ln A(t)}{\partial t} + \frac{\partial \ln L(t)}{\partial t} + \frac{\partial \ln k(t)}{\partial t} \\ &= g + n + \frac{\dot{k}}{k} \end{aligned}$$

and at balanced growth path, $\frac{\dot{k}}{k}|_{k=k^*} = 0$, therefore

$$g_K^* = n + g \quad (15)$$

and proof for $C(t)$ and $I(t)$ follows the same path. ■

Definition 3.3. The **golden rule level of capital per unit of effective labor** (k_G) is the steady state level of capital per unit of effective labor that maximizes steady state consumption per unit of effective labor.

$$k_G = \operatorname{argmax}_{k^* \in \mathbf{k}^*(\Theta)} \{c^* = f(k^*) - (n + g + \delta)k^*\}$$

Definition 3.4. The **golden rule level of saving rate** s_G is the saving rate such that the golden rule level of capital per unit of effective labor is achieved.

Proof. (First Order Necessary Condition for k_G).

$$\begin{aligned} \frac{\partial c^*(k^*)}{\partial k^*} &= 0 \\ \implies \frac{\partial f(k^*) - (n + g + \delta)k^*}{\partial k^*} &= 0 \\ \implies f'(k^*) &= (n + g + \delta) \end{aligned}$$

Thus, golden rule level of capital stock per unit of effective labor k_G can be expressed as ⁶

$$k_G = \{k \in \mathbb{R}_+ : f'(k) = (n + g + \delta)\} \quad (16)$$

⁶Notice that the zero solution, $k^* = 0$ is a trivial steady state and we ignore this case during this course. ■

3.3 Experiment

3.3.1 Impact of Change in the Saving Rate ($s_1 > s_0$)

Suppose at time t_0 , the saving rate parameter increases discretely: $s_0 \rightarrow s_1$.

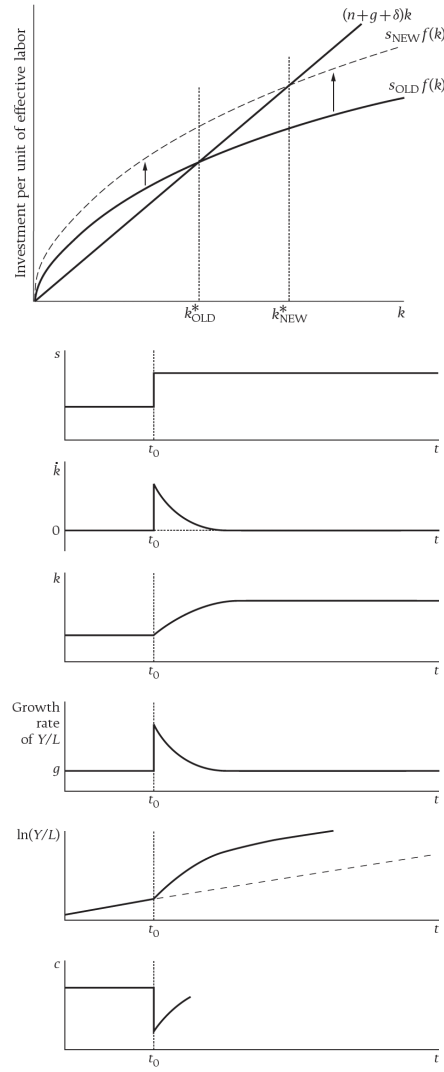


Figure 3: Effect of an Increase in Saving Rate.

Remark 3.3. The relation of c_0^* and c_1^* depends on the relative position of s_1 and the golden rule level of saving rate s_G .

3.3.2 Derive the Effect of Change in s Mathematically

Goal Find $\frac{\partial k^*}{\partial s}$. And notice that $k^*(n + g + \delta) = sf(k^*)$ for any steady state capital level k^* . And the steady state level of capital per unit of effective labor can be written as a function of parameters, as $k^*(n, g, \delta, s)$.

Impact on k^*

Solution. At any steady state level, k^* satisfies

$$sf(k^*(n, g, \delta, s)) = (n + g + \delta)k^*(n, g, \delta, s)$$

Differentiate both sides with respect to s ,

We have

$$sf'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n + g + \delta)\frac{\partial k^*}{\partial s}$$

Rearrange and get

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n + g + \delta) - sf'(k^*)}$$

Notice that the slope of break-even investment is greater than the slope of the actual investment at the steady state, therefore

$$\frac{\partial k^*}{\partial s} > 0$$

■

Impact on y^*

Solution. Using chain rule we have

$$\begin{aligned} \frac{\partial y^*}{\partial s} &= \frac{\partial f(k^*)}{\partial s} \\ &= \frac{\partial f(k^*)}{\partial k^*} \frac{\partial k^*}{\partial s} > 0, \forall k^* \in \mathbf{k}^*(\Theta) \end{aligned}$$

■

To get a sense on how much y^* changes with respect to change in s , we could look at the elasticity.

$$\eta = \frac{\partial y^*}{\partial s} \frac{s}{y^*} = f'(k^*) \frac{\partial k^*}{\partial s} \frac{s}{f(k^*)} = \frac{f'(k^*)s}{(n + g + \delta) - sf'(k^*)}$$

Recall that $(n + g + \delta) = \frac{sf(k^*)}{k^*}$ and rearrange the elasticity

$$\begin{aligned}
\eta &= \frac{\partial y^*}{\partial s} \frac{s}{y^*} \\
&= \frac{f'(k^*)s}{(n + g + \delta) - sf'(k^*)} \\
&= \frac{sf'(k^*)}{\frac{sf(k^*)}{k^*} - sf'(k^*)} \\
&= \frac{f'(k^*)}{\frac{f(k^*)}{k^*} - f'(k^*)} \\
&= \frac{f'(k^*) \frac{k^*}{f(k^*)}}{1 - f'(k^*) \frac{k^*}{f(k^*)}} \\
&= \frac{\alpha_K}{1 - \alpha_K}
\end{aligned}$$

Remark 3.4. α_K denotes the elasticity of output per unit of effective unit labor with respect to capital stock per unit of effective labor, along the balanced growth path. And

$$\alpha_K \approx \frac{1}{3}$$

Remark 3.5. If the production function is in the Cobb-Douglas form, then $\alpha_K = \alpha$.

Example 3.2. If $\alpha_K \approx \frac{1}{3}$ then

$$\eta = \frac{\partial y^*}{\partial s} \frac{s}{y^*} \approx \frac{1}{2}$$

Impact on c^* Notice that on the balanced growth path $c^* = y^* - i^*$.

$$c^* = f(k^*) - (n + g + \delta)k^* \quad (17)$$

and differentiate with respect to s

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n + g + \delta)] \frac{\partial k^*}{\partial s}$$

And notice that the sign of $\frac{\partial c^*}{\partial s}$ depends on the relative slope of production function and break-even investment. By the first order condition of golden rule level of capital per unit of effective labor, $(n + g + \delta) = f'(k_G)$

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - f'(k_G)] \frac{\partial k^*}{\partial s}$$

And

$$\begin{cases} k^* = k_G \implies f'(k^*) = f'(k_G) \implies \frac{\partial c^*}{\partial s} = 0 \\ k^* < k_G \implies f'(k^*) > f'(k_G) \implies \frac{\partial c^*}{\partial s} > 0 \\ k^* > k_G \implies f'(k^*) < f'(k_G) \implies \frac{\partial c^*}{\partial s} < 0 \end{cases}$$

4 Lecture 4 September 27. 2018

4.1 Speed of Convergence

Methodology Look at the change in k and linearize using first order Taylor's expansion.

Recall $\dot{k}(t)$ is a function of $k(t)$ since

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \quad (1)$$

And the first order Taylor series approximation of a function $f(x)$ around the point $x = x_0$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Then

$$\begin{aligned} \dot{k}(k) &\approx \dot{k}(k^*) + \frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} (k - k^*) \\ &= 0 + \frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} (k - k^*) \end{aligned}$$

Differentiating the both sides of equation (1) with respect to k .

$$\begin{aligned} \frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} &= sf'(k^*) - (n + g + \delta) \\ &= \frac{(n + g + \delta)k^*}{f(k^*)} f'(k^*) - (n + g + \delta) \\ &= (n + g + \delta) \left[\frac{f'(k^*)k^*}{f(k^*)} - 1 \right] \\ &= (n + g + \delta)(\alpha(k^*) - 1) \end{aligned}$$

where

$$\alpha_k(k^*) = f'(k^*) \frac{k^*}{f(k^*)} \quad (2)$$

denotes the elasticity of y with respect to k at steady state. So

$$\begin{aligned} \dot{k}(k(t)) &\approx (n + g + \delta)(\alpha(k^*) - 1)(k(t) - k^*) \\ \implies \frac{\partial(k - k^*(k))}{\partial t} &= \dot{k}(k(t)) \approx (n + g + \delta)(\alpha(k^*) - 1)(k(t) - k^*) \end{aligned}$$

Let $\lambda := (n + g + \delta)(1 - \alpha(k^*))$ then

$$k(t) - k^* \approx e^{-\lambda t}(k(0) - k^*) \quad (3)$$

Remark 4.1. *Derive.* (above equation)

Let $X(t) := k(t) - k^*$

And since $\frac{\partial k(t)}{\partial t} = \frac{\partial (k(t) - k^*)}{\partial t}$
Therefore $\dot{X}(t) = \dot{k}(t) \approx -\lambda X(t)$
 $\implies X(t) \approx X(0)e^{-\lambda t}$
 $\iff k(t) - k^* \approx (k(0) - k^*)e^{-\lambda t}$ ■

Then note that 3

$$\begin{aligned}
y(t) &= f(k(t)) \\
\implies \dot{y}(t) &= f'(k(t))\dot{k}(t) \\
(\text{Take the first order Taylor series approximation around } k = k^*) \\
\implies y(t) &\approx f(k^*) + f'(k^*)(k(t) - k^*) \\
\implies y(t) - y^* &\approx f'(k^*)(k(t) - k^*) \\
\implies \frac{\dot{y}(t)}{y(t) - y^*} &= \frac{f'(k^*)\dot{k}(t)}{f'(k^*)(k(t) - k^*)} = \frac{\dot{k}(t)}{k(t) - k^*} \approx -\lambda \\
\implies y(t) - y^* &\approx e^{-\lambda t}(y(0) - y^*)
\end{aligned}$$

Example 4.1. How long does it take to move 1/2 way to the balance growth path. Assuming population growth rate is 2%, growth in output per worker is 2% and depreciation is 2% and $\alpha_K = \frac{1}{3}$.

Solution. $\lambda = (1 - \alpha_K)(n + g + \delta)$ Since we know along the balanced growth path,

$$\begin{aligned}
\frac{Y(t)}{L(t)} &= y^* A(t) \\
\implies \frac{\partial \ln \frac{Y(t)}{L(t)}}{\partial t} &= g
\end{aligned}$$

Therefore $g = 0.02$ and therefore $\lambda = 0.04$.

To find the date where we have moved half way we need to solve

$$\begin{aligned}
\frac{y(\tilde{t}) - y^*}{y(0) - y^*} &= 0.5 \approx e^{-\lambda \tilde{t}} \\
\implies \ln(0.5) &\approx -\lambda \tilde{t} \\
\implies \tilde{t} &= \frac{-\ln(0.5)}{0.04} \approx 17.33
\end{aligned}$$

■

4.2 General Statements

Solow growth model identifies 2 sources of output per worker,

1. Differences in the among of capital per worker.
2. Differences in the effectiveness of productivity of labor A .

Notice that the output per worker

$$\frac{Y(t)}{L(t)} = \frac{F(K(t), A(t)L(t))}{L(t)} = F\left(\frac{K(t)}{L(t)}, A(t)\right)$$

Notice that in the long run balanced growth path

$$\begin{aligned} \frac{K(t)}{L(t)} &= k(t)A(t) \\ \implies \frac{\partial \ln\left(\frac{K(t)}{L(t)}\right)}{\partial t} &= \frac{\dot{k}(t)}{k(t)} + \frac{\dot{A}(t)}{A(t)} \end{aligned}$$

Therefore along the balanced growth path $\dot{k}(t) = 0$ so only the growth in A matters.

4.3 Growth Accounting

Consider the growth rate of aggregate output $Y(t)$, take the total differential and get

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)} \frac{\partial K(t)}{\partial t} + \frac{\partial Y(t)}{\partial L(t)} \frac{\partial L(t)}{\partial t} + \frac{\partial Y(t)}{\partial A(t)} \frac{\partial A(t)}{\partial t} \quad (4)$$

$$\implies \frac{\dot{Y}(t)}{Y(t)} = \frac{\partial Y(t)}{\partial K(t)} \frac{1}{Y(t)} \frac{\partial K(t)}{\partial t} + \frac{\partial Y(t)}{\partial L(t)} \frac{1}{Y(t)} \frac{\partial L(t)}{\partial t} + \frac{\partial Y(t)}{\partial A(t)} \frac{1}{Y(t)} \frac{\partial A(t)}{\partial t} \quad (5)$$

Then express the equation in terms of growth rates in K, L, A variables,

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\partial Y(t)}{\partial K(t)} \frac{K(t)}{Y(t)} \frac{\frac{\partial K(t)}{\partial t}}{K(t)} + \frac{\partial Y(t)}{\partial L(t)} \frac{L(t)}{Y(t)} \frac{\frac{\partial L(t)}{\partial t}}{L(t)} + \frac{\partial Y(t)}{\partial A(t)} \frac{A(t)}{Y(t)} \frac{\frac{\partial A(t)}{\partial t}}{A(t)} \quad (6)$$

$$= \alpha_K \frac{\dot{K}(t)}{K(t)} + \alpha_L \frac{\dot{L}(t)}{L(t)} + R(t) \quad (7)$$

where $R(t)$ is the **Solow Residual** and

$$R(t) = \frac{\partial Y(t)}{\partial A(t)} \frac{A(t)}{Y(t)} \frac{\dot{A}(t)}{A(t)} \quad (8)$$

And $\alpha_K(t)$ and $\alpha_L(t)$ denote the elasticity of output with respect to capital and labor respectively.

Example 4.2. Assume the output growth is 40% and capital growth is 20% and labor growth is 30%. If $\alpha_K = 0.3$ and $\alpha_L = 0.7$. What's the contribution to output growth of capital?

$$\alpha_K \frac{\dot{K}(t)}{K(t)} = 0.3 \times 20\% = 0.06$$

and the contribution from labor is

$$\alpha_L \frac{\dot{L}(t)}{L(t)} = 0.7 \times 30\% = 0.21$$

and the Solow residual is

$$R(t) = g_Y - 6\% - 21\% = 0.4 - 0.06 - 0.21 = 0.13$$

Example 4.3. Let's assume the economy is on its balanced growth path. Assume that a change in medicine increase survival rate during child birth. What would the effect of this be on steady state k^*, y^*, c^*, i^* . First show the growth that despite break-even investment and actual investment. Label the steady state values.

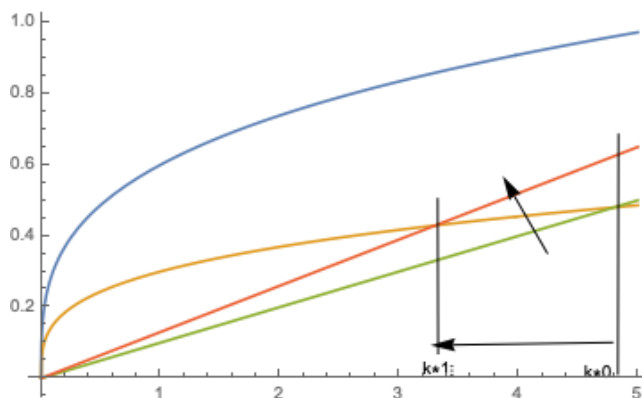
Solution. **The effect would be an increase in n**

Suppose $n_0 \rightarrow n_1$ with $n_0 < n_1$.

Therefore k^* falls, and y^* falls.

Since consumption and actual investment are constant fractions of y^* ,

Therefore both c^*, i^* falls. ■



5 Lecture 5 October 11. 2018

5.1 Economy Setup

- Infinite horizon continuous time model.
- Exogenous growth rates of technology/productivity and population.

$$A(t) = A(0)e^{gt} \quad (1)$$

$$L(t) = L(0)e^{nt} \quad (2)$$

- (Endogenous saving decisions) A key difference of endogenous growth model from the Solow model is that the saving decision and the capital stock are determined by the interaction of utility/profit maximizing households and firms.

5.2 Households

5.2.1 Assumptions and Behaviours

Assumption 5.1. (Setup)

- There are a large numbers of identical households in the economy (H).
To ensure that no household has market power.
- And the size of each household are assumed to grow at rate n .
- Each household has $\frac{L(t)}{H}$ members in it at time t .
- Each household has *initial* capital holding of $\frac{K(0)}{H}$.
- Each member of the household supplies 1 unit of labor at each point in time inelastically. (*No uncertainty*)
- For simplicity, there's no depreciation of capital stock. ($\delta = 0$)
- Capital is rented to firms at rate $r(t)$. (Note that investment is covered by the capital accumulation and interest rate.)
- Labor is hired at wage rate $W(t) = w(t)A(t)$. Where $W(t)$ is the wage per unit of labor and $w(t)$ is the wage per unit of effective labor.

5.2.2 Objective Functions

Household's *objective function* is given by the **lifetime utility function**

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt, \quad \rho > 0 \quad (3)$$

where ρ is the **discount rate** and $C(t)$ is the **consumption per person**.

Remark 5.1. ρ measures consumers' attitude between present and future utilities. When $\rho = 0$, household values utility in the future equally. When $\rho > 0$, household values future utility less than present utility. The greater ρ is, the less value household puts on future consumption/utility (*i.e. consumers are less patient*). Notice that ρ cannot be negative in this model, since with $\rho < 0$ household values utility at $t = \infty$ infinitely more than current utility, the lifetime utility does not converge. For instance, with $\rho > 0$, $e^{-\rho t}$ is increasing in t , then infinite-horizon household could attain infinite utility by allocating consumption at $t = \infty$.⁷

⁷Intuitively, if $\rho < 0$, households could save everything and spend their saving when $t > \infty$. This scenario is certainly unrealistic.

5.2.3 Household Budget Constraints

Household's budget is given in terms of present discounted value. Since the household has $\frac{L(t)}{H}$ members, labor income at time t is

$$A(t)w(t)\frac{L(t)}{H} = W(t)\frac{L(t)}{H}$$

The amount of consumption for the household at the time t is

$$C(t)\frac{L(t)}{H}$$

And let $R(t)$ denote the **continuously compounding interest rate**:

$$R(t) := \int_{\tau=0}^t r(\tau) d\tau \quad (4)$$

Remark 5.2. We know under continuously compounding interest, one unit of output invested at $\tau = 0$ is worth $e^{R(t)}$ units at $\tau = t$. Conversely, 1 unit of output at $\tau = t$ is worth $e^{-R(t)}$ at $\tau = 0$.

Remark 5.3. The budget constraint implies that the present discounted value of lifetime consumption must not exceed the present discounted value of labor income plus initial wealth/capital.

And the household lifetime budget constraint is given by

$$\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt \quad (5)$$

Then normalize budget constraint (5) into *per unit of effective labor variables* to get an equivalent form.

$$\begin{aligned} & \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt \\ \iff & \int_{t=0}^{\infty} e^{-R(t)} c(t) \frac{A(t)L(t)}{H} dt \leq \frac{k(0)A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt \\ \iff & \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} \frac{A(0)L(0)}{H} dt \leq \frac{k(0)A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} \frac{A(0)L(0)}{H} dt \\ \iff & \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t) dt \end{aligned}$$

The **assets** owned by household on date $t = s$, $\frac{K(s)}{H}$, is given by *reverse dis-*

counting saving and initial capital to $\tau = s$.

$$\begin{aligned}
\frac{K(s)}{H} &= e^{R(s)} \frac{K(0)}{H} + \int_{t=0}^s e^{-R(t)+R(s)} \left\{ \frac{w(t)A(t)L(t)}{H} - \frac{c(t)A(t)L(t)}{H} \right\} dt \\
&= e^{R(s)} \frac{K(0)}{H} + \int_{t=0}^s e^{-R(t)+R(s)} \{w(t) - c(t)\} \frac{A(t)L(t)}{H} dt \\
\implies e^{-R(s)} \frac{K(s)}{H} &:= \frac{K(0)}{H} + \int_{t=0}^s e^{-R(t)} \{w(t) - c(t)\} \frac{A(t)L(t)}{H} dt
\end{aligned}$$

Substitute to the budget constraint,

$$\begin{aligned}
\frac{K(0)}{H} + \int_{t=0}^{\infty} \frac{w(t)A(t)L(t)}{H} dt - \int_{t=0}^{\infty} \frac{c(t)A(t)L(t)}{H} dt &\geq 0 \\
\iff \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} \{w(t) - c(t)\} \frac{A(t)L(t)}{H} dt &\geq 0 \\
\iff \lim_{s \rightarrow \infty} \left\{ \frac{K(0)}{H} + \int_{t=0}^s e^{-R(t)} \{w(t) - c(t)\} \frac{A(t)L(t)}{H} dt \right\} &\geq 0 \\
\implies \lim_{s \rightarrow \infty} e^{-R(s)} \frac{K(s)}{H} &\geq 0
\end{aligned} \tag{6}$$

These implies that the household's asset holding cannot be negative at the limit. And this is a **no-Ponzi game** condition.

Definition 5.1. A **Ponzi game** is a scheme in which someone issues debt and rolls it over forever. In other words, the person always pays off debt by issuing more debt.

Remark 5.4. If households could run a Ponzi scheme, then the present value of lifetime consumption can exceed the present value of lifetime resources.

5.3 Instantaneous Utility Function

Assumption 5.2. The household **instantaneous utility function** is assumed to be in the form of constant relative risk aversion (CRRA) as

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \tag{7}$$

And lifetime utility function becomes

$$\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t} \frac{(A(t)c(t))^{1-\theta}}{1-\theta} e^{nt} \frac{L(0)}{H} dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} A(0)^{1-\theta} e^{(1-\theta)gt} e^{nt} \frac{L(0)}{H} dt \\
&= A(0)^{1-\theta} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-\rho t + nt + (1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} dt \\
&= A(0)^{1-\theta} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-(\rho - n - (1-\theta)g)t} \frac{c(t)^{1-\theta}}{1-\theta} dt
\end{aligned}$$

$$B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \quad (8)$$

where

$$B := A(0)^{1-\theta} \frac{L(0)}{H} \text{ and } \beta := \rho - n - (1-\theta)g \quad (9)$$

Assumption 5.3. Assuming $\theta > 0$ and $\beta = \rho - n - (1-\theta)g > 0$.

Remark 5.5. If $\beta \leq 0$ then the integral would diverge and the maximization problem does not have a well-defined solution.

6 Lecture 6 Oct 18. 2018

6.1 Firm Setup

Assumption 6.1. Firms

- are all *identical* and
- employ stocks of capital and labor
- perfectly competitive in factor markets (no control over wage and rent)
- maximize profits
- each firm has access to the production function $Y = F(K, AL)$

Notation 6.1. Let

- $w(t)$ denote the *real* wage per unit of effective labor
- $r(t)$ denote the rental rate of capital
- $A(t)$ denotes the labor augmenting technology/ knowledge

Remark 6.1. The profit is given by

$$\pi(t) = F(K(t), A(t)L(t)) - w(t)A(t)L(t) - r(t)K(t)$$

Assumption 6.2. Technology in this model, $A(t)$, is given exogenously so it isn't a choice for firms.

6.2 Firm's Optimization

Maximization ⁸

$$\max_{K(t), L(t)} \pi(t) = F(K(t), A(t)L(t)) - w(t)A(t)L(t) - r(t)K(t)$$

⁸Note since $A(t)$ is given exogenously, the arguments of maximization could also be written as $\max_{K(t), A(t)L(t)}$

First Order Condition

$$\begin{aligned} K(t) : \frac{\partial F(K(t), A(t)L(t))}{\partial K(t)} - r(t) &= 0 \\ L(t) : \frac{\partial F(K(t), A(t)L(t))}{\partial L(t)} - w(t)A(t) &= 0 \end{aligned}$$

Example 6.1 (Cobb-Douglas Form). Consider

$$\begin{aligned} F(K(t), A(t)L(t)) &= K^\alpha (AL)^{1-\alpha} \\ K(t) : \alpha K^{\alpha-1} (AL)^{1-\alpha} &= r(t) \\ L(t) : (1-\alpha) K^\alpha (AL)^{-\alpha} A(t) &= A(t)w(t) \end{aligned}$$

In per unit of effective form

$$\begin{aligned} \alpha k(t)^{\alpha-1} &= r(t) \\ (1-\alpha)k(t)^\alpha &= w(t) \end{aligned}$$

In general we could have the following first order necessary conditions in terms of the per unit of effective labor variables.

$$\begin{aligned} f'(k(t)) &= r(t) \\ f(k(t)) - k(t)f'(k(t)) &= w(t) \end{aligned}$$

Proof.

$$\begin{aligned} r(t) &= \frac{\partial F(\cdot)}{\partial K} = \frac{\partial ALf(\frac{K}{AL})}{\partial K} \\ &= AL \frac{\partial f(k)}{\partial k} \frac{\partial \frac{K}{AL}}{\partial K} = \frac{\partial f(k)}{\partial k} = f'(k) \end{aligned}$$

■

Proof.

$$\begin{aligned} A(t)w(t) &= \frac{\partial F(K, AL)}{\partial L} = \frac{\partial ALf(\frac{K}{AL})}{\partial L} \\ &= Af(k) + AL \frac{\partial f(\frac{K}{AL})}{\partial AL} \frac{\partial AL}{\partial L} \\ &= Af(k) + AL \frac{\partial f(\frac{K}{AL})}{\partial k} \frac{\partial \frac{K}{AL}}{\partial AL} \frac{\partial AL}{\partial L} \\ &= Af(k) + A^2 L f'(k) \frac{-K}{(AL)^2} \\ &= Af(k) - kf'(k) \frac{A^2 L}{AL} \\ &= Af(k) - kf'(k)A \\ \implies w(t) &= f(k) - f'(k) \end{aligned}$$

■

6.3 Household Behaviour

Household are going to chose a function $C(t) : [0, \infty) \rightarrow \mathbb{R}$ to maximize lifetime utility subjected to their lifetime budget constraint.

$$\begin{aligned} & \max_{\{c(t)\}_{t=0}^{\infty}} B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \\ & s.t. \ 0 \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t)) dt \end{aligned}$$

Setup Lagrangian at each time point,

$$\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t)) dt \right] \quad (1)$$

First Order Necessary Conditions On an arbitrary time period t ,

$$c(t) : B e^{-\beta t} c(t)^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0 \quad (2)$$

$$\lambda : k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t)) dt = 0 \quad (3)$$

6.4 The Behaviour of Consumption

The the log of both size of (2) gives

$$\begin{aligned} \ln(B) + \ln(e^{-\beta t}) + \ln(c(t)^{-\theta}) &= \ln(\lambda) + \ln(e^{-R(t)}) + \ln(e^{(n+g)t}) \\ \iff \ln(B) - \beta t - \theta \ln(c(t)) &= \ln(\lambda) - R(t) + (n+g)t \end{aligned}$$

Since above equation holds for all t , take the derivative with respect to t

$$\begin{aligned} \implies -\beta - \theta \frac{\dot{c}(t)}{c(t)} &= -r(t) + (n+g) \\ \iff \frac{\dot{c}(t)}{c(t)} &= \frac{r(t) - \beta - (n+g)}{\theta} \\ \because \beta &= \rho - n - (1-\theta)g \\ \iff \frac{\dot{c}(t)}{c(t)} &= \frac{r(t) - \rho - \theta g}{\theta} \end{aligned}$$

Remark 6.2. The growth rate of consumption per person is given by

$$\frac{\partial \ln(C(t))}{\partial t} = \frac{r(t) - \rho}{\theta} \quad (4)$$

Remark 6.3. Go backwards by solving the ODE

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &= \frac{r(t) - \rho - \theta g}{\theta} = \frac{\partial \ln(c(t))}{\partial t} \\ \implies c(t) &= c(0) e^{\frac{R(t) - (\rho + \theta g)t}{\theta}} \end{aligned}$$

Using the budget constraint to solve for $c(0)$.

$$\begin{aligned}
k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t) dt &= \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt \\
&= \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(0) e^{\frac{R(t) - (\rho + \theta g)t}{\theta}} dt \\
&= c(0) \int_{t=0}^{\infty} e^{(\frac{1}{\theta} - 1)R(t)} e^{(n - \frac{\rho}{\theta})t} dt \\
\Rightarrow c(0) &= \frac{k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t) dt}{\int_{t=0}^{\infty} e^{(\frac{1}{\theta} - 1)R(t)} e^{(n - \frac{\rho}{\theta})t} dt}
\end{aligned}$$

6.5 Dynamics of the Economy

6.5.1 Dynamic of c

From the firms problem, $r(t) = f'(k(t))$, so at equilibrium,

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

In the long run, $\dot{c}(t) = 0$. This happens if and only if (note that we have $\delta = 0$.)

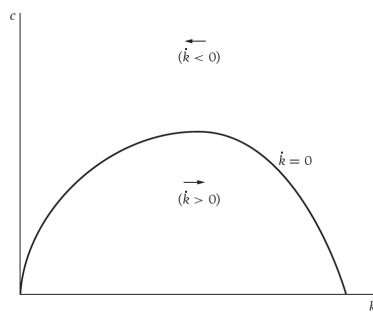
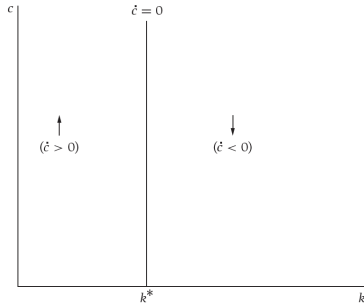
$$f'(k^*) = \rho + \theta g$$

Therefore

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - f'(k^*)}{\theta} \quad (5)$$

That's, $k(t) < k^* \Rightarrow f'(k(t)) > f'(k^*) \Rightarrow \dot{c}(t) > 0$.

Figure 4: Phase diagram of \dot{c} and \dot{k}



6.5.2 Dynamic of k

Assuming there's no depreciation ($\delta = 0$)

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t) \quad (6)$$

Below the $\dot{k} = 0$ curve, $\dot{k} > 0$ since actual investment per unit of effective labor exceeds the break-even level of investment per unit of effective labor.

Putting two phase diagrams together, The **steady state** is given at point E ,

Figure 5: Combined phase diagram

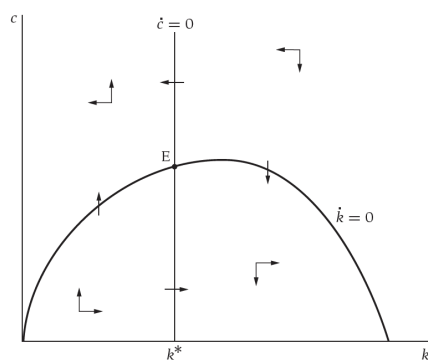


FIGURE 2.3 The dynamics of c and k

where $\dot{c} = \dot{k} = 0$.

6.5.3 Initial Point Problem (Saddle Path)

For any given value of $k(0)$ and the budget constraint satisfied, there exists an unique value of $c(0)$, so that one economy starts with $k(0), c(0)$ would evolve and converge to the steady state E . Such path of convergence is called **saddle path**.

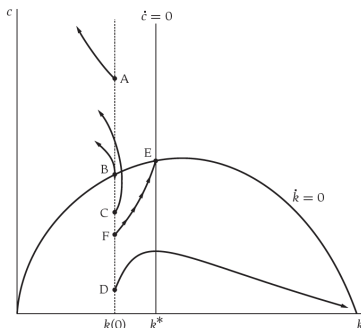


FIGURE 2.4 The behavior of c and k for various initial values of c

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Remark 7.1 (Conditions on Phase Diagram). The phase diagram and saddle path are constructed from the following conditions,

1. Euler equation of c :

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

2. Dynamic transition of k :

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

3. Life time budget constraint.

4. $(c, k) \in \mathbb{R}_{\geq 0}^2$

7.1 Welfare

The *First Welfare Theorem* says if

1. markets are complete,
2. and competitive,
3. No externality

then the decentralized equilibrium is **Pareto efficient** if the number of agents is finite. So it's impossible to make anyone better off without making someone worse off.

7.2 In the Long Run: Balanced Growth Path

At steady state, (i.e. $\dot{c} = \dot{k} = 0$) we have an economy that behaves in an identical way to one in the Solow model.

Per unit of effective labor:

- $\dot{k} = 0 \implies : k$ constant.
- $\dot{c} = 0 \implies : c$ constant.
- $\dot{y}(k) = 0$: y is constant, since k is constant.
- $i^* = y^* - c^*$ investment per unit of effective labor also constant.
- $s = \frac{y-c}{y}$ is constant.

And for **per capita** variable, they are growing at a constant rate g .

Example 7.1.

$$\begin{aligned} \frac{\partial \ln\left(\frac{K(t)}{L(t)}\right)}{\partial t} &= \frac{\partial \ln(k^* A(t))}{\partial t} \\ &= \frac{\partial \ln(k^*) + \ln(A(t))}{\partial t} = \frac{\dot{A}(t)}{A(t)} = g \end{aligned}$$

The **aggregate** variable are growing at a constant rate of $n + g$.

7.3 Modified Golden Rule Level of Capital

The main difference between Ramsey and Solow model is that k^* cannot exceed k_G in Ramsey model.

Proof. Consider the definition of k_G ,

$$f'(k_G) = n + g$$

and for k^* ,

$$f'(k^*) = \rho + \theta g$$

And take the difference

$$f'(k^*) - f'(k_G) = \beta > 0 \implies f'(k^*) > f'(k_G)$$

Thus $k^* < k_G$. ■

And k^* is called the **modified golden rule level of capital**. *It's the optimal but not the best.*

7.4 Experiment: A Fall in the Discount Rate

Assumption 7.1. Suppose the change in unexpected ⁹

Remark 7.2. With $\delta = G = 0$, the locus of $\dot{k}(t) = 0$ is defined by

$$\{(k, c) : c(t) = f(k) - (n + g)k\} \perp \rho$$

thus there's no effect on capital accumulation locus.

But for $\dot{c}(t) = 0$ locus, it's defined by

$$\{(k, c) : f'(k) = \rho + \theta g\}$$

Therefore, consumption locus will shift right.

Impact on k^* and c^*

Proof. Let $\rho_0 > \rho_1$.

$$\begin{aligned} f'(k_0^*) &= \rho_0 + \theta g \wedge f'(k_1^*) = \rho_1 + \theta g \\ \implies f'(k_0^*) &> f'(k_1^*) \\ \because \rho_0 > \rho_1 \wedge f''(\cdot) < 0 \\ \implies k_1^* &> k_0^* \end{aligned}$$

■

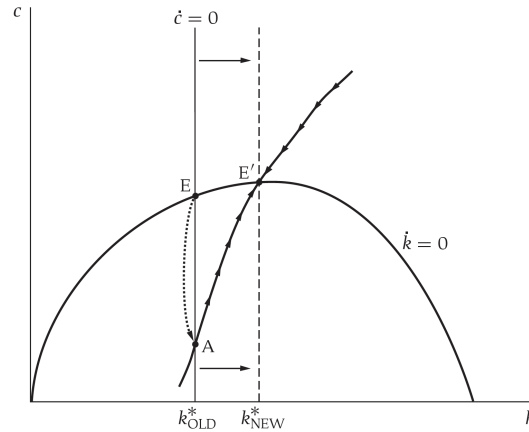


FIGURE 2.6 The effects of a fall in the discount rate

Remark 7.3 (Result). c jump down so that the economy is on the new saddle path (point A), and then the economy moves along the new saddle path till E' .

The effects of a fall in ρ are similar to the effects of a rise in the saving rate in the Solow model with a capital stock below the golden-rule level.

⁹If a change is expected, households may alter their behaviour before the change occurs.

7.5 Experiment: The Effects of Government Purchases

Assumption 7.2. Assume the government buys output at a rate of $G(t)$ per unit of effective labor.

Further, assume

1. Government purchases are assumed **not to affect utility from private consumption**. This can occur if
 - (a) the government devotes the expenditure to an activity consumers don't care about,
 - (b) or utility equals the sum of utility from private consumption and utility from government provided goods.
2. **not to affect future output**, i.e. the government devotes expenditure to public consumption rather than public investment.
3. We will find government purchases with **lump-sum** taxes and the government runs a **balanced budget**.

Impact Thus, there's no impact on the $\dot{c} = 0$ curve since the household problem still gives us $\frac{\dot{c}}{c} = \frac{f'(k) - \rho - \theta g}{\theta}$.

However, there is a change in the $\dot{k} = 0$ curve, because

$$\dot{k}(t) = y(t) - c(t) - (n + g)k(t) - G(t)$$

To see there is not effect on $\dot{c} = 0$ curve, we need to look at the household's problem.

Let's assume that the Household does not care about $G(t)$. So

$$U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c^{1-\theta}}{1-\theta} dt$$

And the change comes in the *budget constraint*. Here when we found government expenditure period by period with lump-sum taxes, the government sets tax equal expenses. (Since no government saving)

$$T(t)A(t)L(t) = G(t)A(t)L(t)$$

The Household's budget constraint then becomes

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) \frac{A(t)L(t)}{H} dt + \int_{t=0}^{\infty} e^{-R(t)} T(t) \frac{A(t)L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)L(t)}{H} dt$$

Effectively two means that in terms of the per unit of effective labor variable we got

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt + \int_{t=0}^{\infty} e^{-R(t)} T(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$$

Given $G(t) = T(t)$, $\forall t$, equivalently

$$\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt + \int_{t=0}^{\infty} e^{-R(t)} G(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$$

This means the household's problem can be written as

$$\begin{aligned} & \max_{\{c(t)\}} B \int_{t=0}^{\infty} e^{-\beta t} \frac{c^{1-\theta}}{1-\theta} dt \\ \text{s.t. } & \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt + \int_{t=0}^{\infty} e^{-R(t)} G(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt \end{aligned}$$

And the Lagrangian becomes

$$\begin{aligned} \mathcal{L} = & B \int_{t=0}^{\infty} e^{-\beta t} \frac{c^{1-\theta}}{1-\theta} dt \\ & + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt + \int_{t=0}^{\infty} e^{-R(t)} G(t) e^{(n+g)t} dt \right] \end{aligned}$$

By solving for the first order condition w.r.t. $c(t)$ we have

$$\frac{\partial \mathcal{L}}{\partial c(t)} = B e^{\beta t} c^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0$$

Since the changing in G does not impact the firms problem we still have $r(t) = f'(k(t))$.

So at equilibrium we still have

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

Consider a shock that *permanently* increases $G(t)$ from $G(t) = G_L \forall t$ to $G(t) = G_H \forall t$, where $G_L < G_H$.

Since $\frac{\dot{c}(t)}{c(t)}$ does not depends on G so there's no change in $\dot{c} = 0$ curve.

But for $\dot{k} = 0$ curve is affected as

$$\begin{aligned} \dot{k}_{old} = 0 & \iff 0 = f(k(t)) - c(t) - (n+g)k(t) - G_L \\ \dot{k}_{new} = 0 & \iff 0 = f(k(t)) - c(t) - (n+g)k(t) - G_H \end{aligned}$$

The $\dot{k} = 0$ curve drops down by $\Delta G = G_H - G_L$.

The c^* falls by ΔG but k^* does not change.

Optimally, at the time of the shock, the economy jumps from point E (before) to point E_1 . We move immediately from one steady state to the other. (No smoothing could be done here)

Since there's no change in k^* , there's no change in the rate of return on capital ($f'(k(t))$).

Remark 7.4. For temporary changes, where would be smoothing.

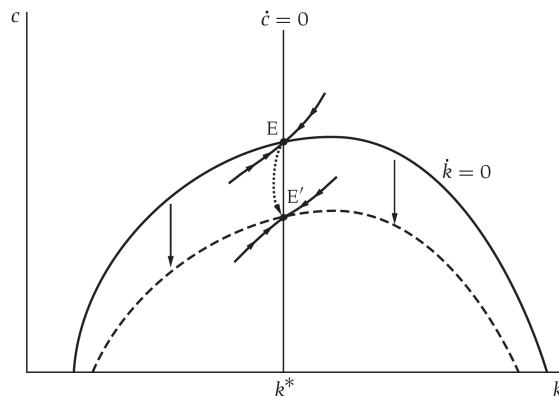


FIGURE 2.8 The effects of a permanent increase in government purchases

8 Lecture 8 Nov. 1 2018

8.1 Government Spending: without bonds

Notation 8.1. Let $G(t)$ and $T(t)$ denote the government expenditure and taxation (in form of lump-sum tax) per **effective unit of labor** at time period t .

Assumption 8.1. Government satisfies its budget constraint in every period,

$$G(t)A(t)L(t) = T(t)A(t)L(t), \quad \forall t \in [0, \infty) \quad (1)$$

Household budget constraint (normalized by $\frac{A(0)L(0)}{H}$) can be expressed as

$$k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [w(t) - c(t) - T(t)] dt \geq 0 \quad (2)$$

$$\iff k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [w(t) - c(t) - G(t)] dt \geq 0 \quad (3)$$

State equation

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t) \quad (4)$$

8.1.1 Temporary shock in G

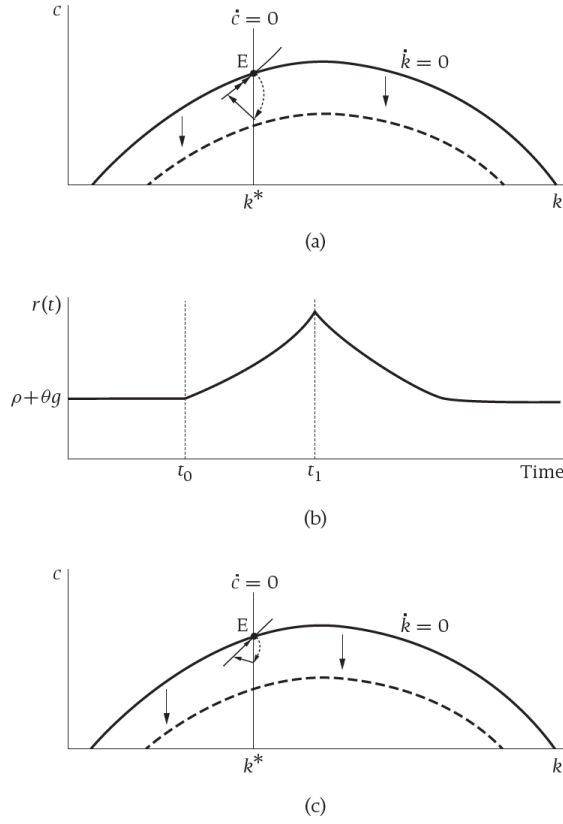


Figure 6: The effects of a temporary increase in government purchases

8.2 Government Spending: with bonds (voluntary taxation)

Household Budget Constraint

$$k(0) + b(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [w(t) - c(t) - T(t)] dt \geq 0 \quad (5)$$

where government bonds $b(0)$ is considered as a part of *initial asset* of households.

Government Budget Constraint

$$-b(0)A(0)L(0) + \int_{t=0}^{\infty} e^{-R(t)} A(t)L(t)[T(t) - G(t)] dt \geq 0 \quad (6)$$

$$\implies -b(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [T(t) - G(t)] dt \geq 0 \quad (7)$$

$$\iff b(s)e^{(n+g)s} = b(0)e^{R(s)} - \int_{t=0}^{\infty} e^{-R(t)+R(s)} e^{(n+g)t} [T(t) - G(t)] dt \leq 0 \quad (8)$$

$$\iff \lim_{s \rightarrow \infty} b(s)e^{-R(s)} \leq 0 \quad (9)$$

Proposition 8.1. The effects of government spending on household optimization are the same with and without bonds.

Proof. Adding household budget (5) and government budget (7)

$$k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [w(t) - c(t) - G(t)] dt \geq 0 \quad (10)$$

Note that (10) is identical to the household budget constraint (3) in last section and taxation and bonds are assumed not affecting household's utility. Therefore household's optimization problem is unchanged under two taxation schemes. ■

Assumption 8.2 (Perfect Financial Institute). Government and household are facing the same interest rate $r(t)$.

Definition 8.1 (Ricardian Equivalence). *Path of taxation* does not affect household behaviour, only *path of government expenditure* does.

8.3 Problems of Ricardian Equivalence

8.3.1 Turn Over of Population

- Old people die before high taxes.
- Remedies: **Intergenerational Altruism** generations behave as if they are one infinitely living individual.

8.3.2 Liquidity Constraint

- People cannot borrow earlier. (Imperfect financial market)
- Response: Government can save for individuals.

8.3.3 No Lump-sum Tax in Real World

One critical assumption for Ricardian Equivalence to hold is the tax is flat. But in real world, taxation (e.g. consumption tax) can introduce significant distortion to household budget constraint and thus household behaviour.

8.3.4 Rule of Thumb: Imperfect Foreseeing

People cannot foresee future events perfectly, instead, they follow rule of thumb (e.g. same 20% of income). Therefore they do not always act optimally.