MAT237: Multivariable Calculus

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2.8 Optimization

Theorem 2.8.1. Let $S \subset \mathbb{R}^n$ be an open set and $f, g : S \to \mathbb{R}$ be C^1 functions. If $\mathbf{x}$ is a *local extremal* satisfying $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{x}) \neq 0$, then

$$\exists \lambda \in \mathbb{R} \ s.t. \ \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \tag{2.8.1}$$

Lemma 2.8.1. $\nabla g(\mathbf{x})$ is orthogonal to the constraint set $g^{-1}(0)$.

Proposition 2.8.1. Equations (2.8.1) $\implies$ $\nabla f(\mathbf{x}) \perp g^{-1}(0)$ at $\mathbf{x}$.

Theorem 2.8.2. Let $S \subseteq \mathbb{R}^n$ be an open set, and $f, \{g_i\}_{i=1}^k : S \to \mathbb{R}$ be C^1 functions. Define $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^k \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_k(\mathbf{x}))$.
If $\mathbf{x} \in S$ is a *local extremal* of f such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, and $\{\nabla g_i(\mathbf{x})\}$ are linearly independent (i.e. $rank(D\mathbf{g}(\mathbf{x})) = k$), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \ s.t. \ \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \tag{2.8.2}$$

Remark 2.8.1. Procedure of optimization on *open sets*:

(i) Find all critical points.

(ii) Find optimizers among critical points.

Remark 2.8.2. Procedure of optimization with *inequality constraints*:

(i) Find critical points without the constraints.

(ii) Find critical points on the constraints.

(iii) Find optimizers among candidates.

3 The Implicit and Inverse Function Theorems

3.1 The Implicit Function Theorem I

Theorem 3.1.1 (Implicit Function Theorem)**.** Let $S \subseteq \mathbb{R}^{n+k}$ be an open set, and function $F : S \to \mathbb{R}^k$ be a C^1 function. Suppose there exists point $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$ such that

$$F(\mathbf{a}, \mathbf{b}) = \mathbf{0} \tag{3.1.1}$$

If

$$det(D_{\mathbf{y}}(F(\mathbf{a}, \mathbf{b}))) \neq 0 \tag{3.1.2}$$

then there exists $r_0, r_1 > 0$ and a C^1 function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}),\ \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \tag{3.1.3}$$

and define $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$, the derivative of $\mathbf{f}$ can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}} F(\mathbf{x}, \mathbf{y})]^{-1} D_{\mathbf{x}} F(\mathbf{x}, \mathbf{y}) \tag{3.1.4}$$

Remark 3.1.1. Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \tag{3.1.5}$$

near $(\mathbf{a}, \mathbf{b})$.

(i) Verify $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

(ii) Assert

$$det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \tag{3.1.6}$$

(iii) Approximate solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ using

$$\begin{align} \mathbf{f}(\mathbf{x} + \mathbf{h}) &\approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \tag{3.1.7} \\ &= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \tag{3.1.8} \end{align}$$

3.2 Geometric content of the Implicit Function Theorem

Definition 3.2.1. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in S$. S is **singular** at $\mathbf{a}$ if

$$\forall r > 0\ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \tag{3.2.1}$$

S is **regular** at $\mathbf{a}$ is its not singular there.

Theorem 3.2.1 (k dimensional manifold as level set)**.** Let $U \subseteq \mathbb{R}^n$ and let $\mathbf{F} : U \to \mathbb{R}^{n-k}$ be a C^1 function.

$$S \equiv \mathbf{F}^{-1}(\mathbf{0}) \tag{3.2.2}$$

Let $\mathbf{a} \in U$, if

$$rank(D\mathbf{F}(\mathbf{a})) = n - k \tag{3.2.3}$$

then $\exists r > 0$ such that the *level set of **F** near **a***

$$\mathcal{B}(r, \mathbf{a}) \cap S \tag{3.2.4}$$

can be represented as a C^1 graph.

Theorem 3.2.2 (k dimensional manifold as parameterization)**.** Let $T \subseteq \mathbb{R}^k$ and let $\mathbf{f} : U \to \mathbb{R}^n$ be a C^1 function.

$$S \equiv \mathbf{f}(T) \tag{3.2.5}$$

Let $\mathbf{t} \in T$, if

$$rank(\mathbf{f}(\mathbf{t})) = k \tag{3.2.6}$$

then $\exists r > 0$ such that the *parameterization of* $\boldsymbol{f}$ *near* $\boldsymbol{t}$

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \tag{3.2.7}$$

can be represented as a C^1 graph.

3.3 Transformations, and the Inverse Function Theorem

Example 3.3.1 (Polar coordinate in $\mathbb{R}^2$)**.** Let

$$U \equiv \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \tag{3.3.1}$$

$$V \equiv \mathbb{R}^2 \backslash \{(x, 0) : x \leq 0\} \tag{3.3.2}$$

Define $\mathbf{f} : U \to V$ as

$$\mathbf{f}(r, \theta) \equiv \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} \tag{3.3.3}$$

Example 3.3.2 (Spherical coordinate in $\mathbb{R}^3$)**.** Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix} \tag{3.3.4}$$

Example 3.3.3 (Cylindrical coordinate in $\mathbb{R}^3$)**.** Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{pmatrix} \tag{3.3.5}$$

Theorem 3.3.1 (Inverse Function Theorem)**.** Let U and V be open subsets in $\mathbb{R}^n$, and $\mathbf{f} : U \to V$. Let $\mathbf{a} \in U$ and define $\mathbf{b} \equiv \mathbf{f}(\mathbf{a}) \in V$. If

$$det(D\mathbf{f}(\mathbf{a})) \neq 0 \tag{3.3.6}$$

then there exists $M \subseteq U$ and $N \subseteq V$ such that

(i) $\mathbf{a} \in M$ and $\mathbf{b} \in N$,

(ii) $\mathbf{f}$ is bijective between M and N,

(iii) $\mathbf{f}^{-1} : N \to M$ is C^1,

and for all $\mathbf{x} \in M$ such $\mathbf{y} \equiv \mathbf{f}(\mathbf{x}) \in N$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \tag{3.3.7}$$

4 Integration

4.1 Basics

Theorem 4.1.1 (Properties of infimum and supremum)**.** Let $A \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, and $f, g : A \to \mathbb{R}$ are bounded functions. Let m and M denote the infimum and supremum respectively, then

(i) $m_A f + m_A g \leq m_A(f+g) \leq M_A(f+g) \leq M_A f + M_A g$

(ii) If $A' \subseteq A$, then $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$

(iii) If $f(\mathbf{x}) \leq g(\mathbf{x})$ $\forall \mathbf{x} \in A$, then $m_A f \leq m_A g$ and $M_A f \leq M_A g$

(iv) $|M_A f| \leq M_A |f|$

(v) $M_A|f| - m_A|f| \leq M_A f - m_A f$

(vi) $\forall c \in \mathbb{R}$, $M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$

(vii) $M_A f - m_A f = sup\{f(x) - f(y) : x, y \in A\}$

4.2 Integration on Higher Dimensions

Definition 4.2.1. A **rectangle** $\mathcal{R} \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{R} \equiv \prod_{i=1}^{n} [a_i, b_i] \tag{4.2.1}$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.

Definition 4.2.2. A **partition** P of rectangle $\mathcal{R} = \prod_{i=1}^{n}[a_i, b_i]$ is a list of n finite and increasing list of real numbers

$$P = \{L_1, L_2, \ldots, L_n\} \tag{4.2.2}$$

where $L_i = \{e_j\}_{j=0}^{T_i}$ such that

$$a_i = e_0 < e_1 < \cdots < e_{T_i} = b_i \tag{4.2.3}$$

and such partition induces a set of rectangles(boxes) $\mathcal{B}(P) \equiv \{B_j\}_{j=1}^{J} \subseteq \mathcal{R}$.

Definition 4.2.3. Let P and P' be two partitions of $\mathcal{R}$. Then P' is a **refinement** of P if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee B_j'^{int} \cap B_j^{int} = \emptyset \tag{4.2.4}$$

Definition 4.2.4. Define the **volume** of rectangle $\mathcal{R} = \prod_{i=1}^{n}[a_i, b_i]$ as

$$V^n(\mathcal{R}) \equiv \prod_{i=1}^{n}(b_i - a_i) \tag{4.2.5}$$

Definition 4.2.5. The **lower Riemann sum** of f with partition P on $\mathcal{R}$ is defined as

$$L_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \tag{4.2.6}$$

and the **upper Riemann sum** is defined as

$$U_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \tag{4.2.7}$$

Definition 4.2.6. The **upper integral** and **lower integral** of f on $\mathcal{R}$ are defined as

$$\bar{I}_{\mathcal{R}} f \equiv \inf_P U_P f \tag{4.2.8}$$

$$\underline{I}_{\mathcal{R}} f \equiv \sup_P L_P f \tag{4.2.9}$$

Definition 4.2.7. A bounded real-valued function f defined on $\mathcal{R}$ is **integrable** if

$$\underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \tag{4.2.10}$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \ dV^n \equiv \underline{I}_{\mathcal{R}} f = \bar{I}_{\mathcal{R}} f \tag{4.2.11}$$

Lemma 4.2.1. Let f be a bounded real-valued function defined on $\mathcal{R}$, f is integrable if and only if $\forall \epsilon > 0$, there exists a partition P of $\mathcal{R}$ such that

$$U_P f - L_P f < \epsilon \tag{4.2.12}$$

Theorem 4.2.1. Let f and g be two integrable functions on $\mathcal{R} \subseteq \mathbb{R}^n$, let $c \in \mathbb{R}$,

(i) $f + g : \mathcal{R} \to \mathbb{R}$ is integrable and $\int_{\mathcal{R}} (f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$

(ii) $c \cdot f$ is integrable and $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$

(iii) $f(\mathbf{x}) \geq g(\mathbf{x}) \ \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$

(iv) $|f|$ is integrable and $| \int_R f| \leq \int_R |f|$

Definition 4.2.8. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle $\mathcal{R}$ covers S, the **indicator function** of S is $\chi_S : \mathcal{R} \to \{0, 1\}$, defined as

$$\chi_S(\mathbf{x}) \equiv \mathbb{I}(\mathbf{x} \in S) \tag{4.2.13}$$

Definition 4.2.9. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle $\mathcal{R}$ covers S. Let $f : \mathcal{R} \to \mathbb{R}$ be a bounded function, then f is **integrable on** S if $\chi_S f$ is integrable on $\mathcal{R}$. And

$$\int \cdots \int_S f \ dV^n \equiv \int \cdots \int_{\mathcal{R}} \chi_S f \ dV^n \tag{4.2.14}$$

Definition 4.2.10. Let $Z \subseteq \mathbb{R}^n$, Z has **zero content** if for all $\epsilon > 0$, there exists a finite set of rectangles $\{R_\ell\}_{\ell=1}^L$ covers Z and

$$\sum_{\ell=1}^{L} V^n(R_\ell) < \epsilon \tag{4.2.15}$$

Proposition 4.2.1. Let $Z \subseteq \mathbb{R}^n$ has zero content, then

(i) For any $Z' \subseteq Z$, Z' has zero content.

(ii) Finite union of content zero sets has zero content.

(iii) Let $f : [a, b] \to \mathbb{R}$ be an integrable function, it's graph $\{(x, f(x)) : x \in [a, b]\}$ has zero content.

(iv) Let $\mathbf{f} : [a, b] \to \mathbb{R}^2$ be a C^1 function, the parameterization $\mathbf{f}([a, b])$ has zero content.

Theorem 4.2.2. Let $\mathcal{R}$ be a rectangle in $\mathbb{R}^n$ and f is integrable on $\mathcal{R}$ if

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \tag{4.2.16}$$

has zero content.

Proposition 4.2.2 (Folland 4.22)**.** Suppose $Z \subseteq \mathbb{R}^n$ has zero content. If $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f \ dV^n = 0$.

4.3 Iterated Integrals

Theorem 4.3.1 (Fubini's Theorem)**.** Let $\mathcal{R} = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and $f : \mathcal{R} \to \mathbb{R}$ is bounded. Assuming that

(i) f is integrable on $\mathcal{R}$.

(ii) for each $y \in [c,d]$, the function $f_y(x) \equiv f(x,y)$ is integrable on $[a,b]$.

(iii) Define $g(y) \equiv \int_a^b f(x,y)dy$ is integrable on $[c,d]$.

Then

$$\iint_{\mathcal{R}} f \; dA = \int_c^d \Big(\int_a^b f(x,y) \; dx \Big) dy \tag{4.3.1}$$

Proposition 4.3.1. Let $S \subseteq \mathbb{R}^n$ be an unbounded set, and $f : S \to \mathbb{R}$. Then improper integral $\int \cdots \int_S f \; d^n\mathbf{x}$ is absolutely convergent on $\mathbb{R}^n$ <u>if and only if</u> $\int \cdots \int_{\mathbb{R}^n} \chi_S f \; d^n\mathbf{x}$ is absolutely convergent.

4.4 Change of Variables

Theorem 4.4.1 (Change of Variable)**.** Let U and V be two open subsets of $\mathbb{R}^n$, and let $\mathbf{G} : U \to V$ be a C^1 bijection. Let $T \subset U$ and $S \subset V$. Suppose $\mathbf{G}(T) = S$, then

$$\int \cdots \int_S f \; d\Omega = \int \cdots \int_T f \circ \mathbf{G} \; |\text{det} D\mathbf{G}| \; d\Theta \tag{4.4.1}$$

Corollary 4.4.1. Let S be a region in $\mathbb{R}^n$, suppose S can be parameterized by $\mathbf{G} : T \to S$. By the change of variable formula, consider the special case $f(\mathbf{x}) = 1$,

$$|S| = \int \cdots \int_S 1 \; d\Omega = \int \cdots \int_T 1 \; |\det \; D\mathbf{G}(\mathbf{u})| \; d\Theta \tag{4.4.2}$$

Example 4.4.1 (Polar Coordinate)**.** Define the coordinate transformation mapping from polar to Cartesian,

$$\mathbf{P}(r,\theta) \equiv (x,y) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \;\; \theta \in [0, 2\pi] \; r \in \mathbb{R}_+ \tag{4.4.3}$$

and $|\det \; D\mathbf{P}(r,\theta)| = r$.

Example 4.4.2 (Cylindrical Coordinate)**.** Define the coordinate transformation mapping from cylindrical to Cartesian as

$$\mathbf{C}(r,\theta,z) \equiv (x,y,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}, \;\; \theta \in [0, 2\pi] \; r \in \mathbb{R}_+ \; z \in \mathbb{R} \tag{4.4.4}$$

and $|\det \; D\mathbf{C}(r,\theta,z)| = r$.

Example 4.4.3 (Spherical Coordinate)**.** Define the coordinate transformation mapping from spherical to Cartesian as

$$\mathbf{S}(r,\theta,\varphi) = \begin{pmatrix} r\cos\theta\sin\varphi \\ r\sin\theta\sin\varphi \\ r\cos\varphi \end{pmatrix} \tag{4.4.5}$$

and $|\det \; D\mathbf{S}(r,\theta,\varphi)| = r^2 \sin\varphi$

4.5 Further Aspects

4.5.1 Exchanging Differentiation and Integration

Theorem 4.5.1 (Exchanging Differentiation and Integration)**.** Let $f(\mathbf{x}, \mathbf{t}) : S \times T \to \mathbb{R}$ and define $F(\mathbf{x}) : S \to \mathbb{R}$ as

$$F(\mathbf{x}) \equiv \int \cdots \int_T f(\mathbf{x}, \mathbf{t})\ d\Omega \tag{4.5.1}$$

If both

(i) f and F are continuous on their domains;

(ii) and $\forall\ x_j \in \mathbf{x}$, $\frac{\partial f(\mathbf{x},\mathbf{t})}{\partial x_j}$ is continuous,

then F is C^1 in S and for every j,

$$\frac{\partial F(\mathbf{x})}{\partial x_j} = \int \cdots \int_T \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j}\ d\Omega \tag{4.5.2}$$

Corollary 4.5.1. By the definition of partial derivative, above theorem is equivalent to

$$\lim_{h \to 0} \int \cdots \int_T \frac{f(\mathbf{x}, \mathbf{t})}{h}\ d\Omega = \int \cdots \int_T \lim_{h \to 0} \frac{f(\mathbf{x}, \mathbf{t})}{h}\ d\Omega \tag{4.5.3}$$

4.5.2 Improper Integrals

Definition 4.5.1 (Unbounded Domains)**.** An **improper integral** with unbounded domain $\int \cdots \int_{\mathbb{R}^n} f\ d\Omega$ is **absolutely convergent** if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0\ \exists R > 0\ s.t.\ \forall S \subseteq \mathbb{R}^n\ B(R, \mathbf{0}) \subset S \implies \left| \int \cdots \int_S f\ d\Omega - L \right| < \varepsilon \tag{4.5.4}$$

Theorem 4.5.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and that

$$\lim_{R \to \infty} \int \cdots \int_{B(R, \mathbf{0})} |f|\ d\Omega \text{ exists} \tag{4.5.5}$$

then $\int \cdots \int_{\mathbb{R}^n} f\ d\Omega$ is absolutely convergent.

Corollary 4.5.2 (Equivalence)**.** Above improper integral $\int \cdots \int_{\mathbb{R}^n} f\ d\Omega$ is absolutely convergent if set

$$\{ \int \cdots \int_{B(R, \mathbf{0})} |f|\ d\Omega : R \in \mathbb{R}_{++} \} \tag{4.5.6}$$

is bounded.

Corollary 4.5.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an continuous function, if

$$\exists p > n,\ C > 0\ s.t.\ |f(\mathbf{x})| \leq \frac{1}{||\mathbf{x}||^p}\ \forall \mathbf{x} \in \mathbb{R}^n \tag{4.5.7}$$

then $\int \cdots \int_{\mathbb{R}^n} f\ d\Omega$ is absolutely convergent.

Definition 4.5.2 (Unbounded Function)**.** Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$. Consider a function $f : S\backslash\{\mathbf{a}\} \to \mathbb{R}$. Then the improper integral $\int \cdots \int_S f d\Omega$ is absolutely convergent if

$$\exists L \in \mathbb{R}\ s.t\ \forall \varepsilon > 0\ \exists r > 0\ s.t.\ \forall U \subset S\ s.t.\ \mathbf{a} \in U^{int} \wedge U \subset B(r, \mathbf{a}),\ \left| \int \cdots \int_{S\backslash U} f\ d\Omega - L \right| < \varepsilon \quad (4.5.8)$$

Theorem 4.5.3. Let $f : S\backslash\{\mathbf{a}\} \to \mathbb{R}$, if

$$\lim_{r\to 0} \int \cdots \int_{S\backslash B(r,\mathbf{a})} |f|\ d\Omega \text{ exists} \quad (4.5.9)$$

then $\int \cdots \int_S f\ d\Omega$ is absolutely convergent.

Corollary 4.5.4 (Equivalence)**.** If the set

$$\{\iint_{S\backslash B(r,\mathbf{a})} |f|\ d\Omega : r \in \mathbb{R}_{++}\} \quad (4.5.10)$$

is bounded, then $\int \cdots \int_S f\ d\Omega$ is absolutely convergent.

Corollary 4.5.5. Let $f : S\backslash\{\mathbf{a}\} \to \mathbb{R}$, if

$$\exists p < n,\ C > 0\ s.t. |f(\mathbf{x})| \leq \frac{C}{||\mathbf{x} - \mathbf{a}||^p}\ \forall \mathbf{x} \in S\backslash\{\mathbf{a}\} \quad (4.5.11)$$

then the improper integral $\int \cdots \int_S f\ d\Omega$ is absolutely convergent.

5 Vector Calculus

5.1 Line Integrals

5.1.1 Arc Length

Definition 5.1.1. Let C be a smooth curve in $\mathbb{R}^n$ parameterized by C^1 function $\mathbf{g}$ such that $\mathbf{g}'(t) \neq \mathbf{0}$ for every appropriate t.

$$C \equiv \{\mathbf{g}(t) : t \in [a, b]\} \quad (5.1.1)$$

and the **arc length** of C is defined as

$$\int_C d^n\mathbf{x} \equiv \int_C ds \equiv \int_a^b ||\mathbf{g}'(t)||\ dt \quad (5.1.2)$$

Proposition 5.1.1. The arc length of a curve C is an intrinsic property of the geometric object C and should not depend on the particular parameterization we use.

Proof. Let $\varphi : [c, d] \to [a, b]$ be a bijection, so that $\mathbf{h} \equiv \mathbf{g} \circ \varphi$ is also a valid parameterization of C such that

$$C \equiv \{\mathbf{h}(u) : u \in [c, d]\} \quad (5.1.3)$$

The arc length of C can be computed using

$$\int_C ds = \int_c^d ||\mathbf{h}'(u)||\ du \quad (5.1.4)$$

$$= \int_c^d ||\mathbf{g}'(\varphi(u))|| \times ||\varphi'(u)||\ du \quad (5.1.5)$$

$$= \int_a^b ||\mathbf{g}'(t)||\ dt \text{ by change of variable formula.} \quad (5.1.6)$$

■

Remark 5.1.1 (Interpretations)**.** Suppose $\mathbf{g}$ is a parameterization of C.

(i) $\int_a^b \mathbf{g}'(t)\ dt = \mathbf{g}(b) - \mathbf{g}(a)$ measures the distance between two endpoints of C.

(ii) Choosing a parameterization is effectively choosing an **orientation** for the curve C.

Definition 5.1.2. A function $\mathbf{g} : [a, b] \to \mathbb{R}^n$ is called **piecewise smooth** if

(i) it's *continuous*, and

(ii) it's derivate exists and is continuous except at finitely many points t_j, at which the one-sided limits exists.

5.1.2 Line Integrals of Scalar Functions

Definition 5.1.3. Let smooth curve $C \subseteq \mathbb{R}^n$, $f : C \to \mathbb{R}$ and $\mathbf{g}$ be a parameterization of C, then

$$\int_C f\ ds = \int_a^b f(\mathbf{g}(t))\ ||\mathbf{g}'(t)||\ dt \tag{5.1.7}$$

Remark 5.1.2. The line integrals of scalar functions are also independent from the choices of parameterizations.

Definition 5.1.4.

$$\text{Average of } f \text{ over } C \equiv \frac{\int_C f\ ds}{\int_C\ ds} \tag{5.1.8}$$

5.1.3 Line Integrals of Vector Fields

Definition 5.1.5. Let smooth $C \in \mathbb{R}^n$ with parameterization $\mathbf{g}$ and $\mathbf{F} : C \to \mathbb{R}^n$ defined on it, the **line integral** of $\mathbf{F}$ over C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\ dt \tag{5.1.9}$$

Proposition 5.1.2. The line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ is independent of the parameterization *as long as the orientation is unchanged.*

5.1.4 Rectifiable Curves

Remark 5.1.3. Let C be a curve in $\mathbb{R}^n$ parameterized by injection $\mathbf{g} : [a, b] \to \mathbb{R}^n$ such that $\mathbf{g}'(t) \neq \mathbf{0}$. Let P be a partition of $[a, b]$. Denote

$$L_P(C) \equiv \sum_j ||\mathbf{g}(t_j) - \mathbf{g}(t_{j-})|| \tag{5.1.10}$$

Definition 5.1.6. A curve C is **rectifiable** if the set $\{L_P(C) : P\}$ is bounded. And the arc length of C s defined as

$$L(C) \equiv \sup\{L_P(C) : P\} \tag{5.1.11}$$

Theorem 5.1.1. The supremum found above, $L(C)$ is the precisely the arc length of C:

$$L(C) = \int_a^b ||\mathbf{g}'(t)||\ dt \tag{5.1.12}$$

5.2 Green's Theorem

5.2.1 Preliminary Definitions

Definition 5.2.1. A **simple closed curve** is a curve with parameterization $\mathbf{g} : [a, b] \to \mathbb{R}^n$ where

(i) $\mathbf{g}$ is continuous;

(ii) $\mathbf{g}(a) = \mathbf{g}(b)$;

(iii) $\mathbf{g}$ is injective with its domain restricted to (a, b).

Definition 5.2.2. A *simple closed curve* is **piecewise smooth** if it has a parameterization $\mathbf{g}$ such that

(i) $\mathbf{g}$ is continuously differentiable with $\mathbf{g}'(t) \neq \mathbf{0}$ except finitely many breakpoints;

(ii) $\mathbf{g}'(t)$ is *one side continuous* at breakpoints of the curve.

Definition 5.2.3. A **regular region** $S \subseteq \mathbb{R}^n$ is a set satisfying both

(i) S is compact;

(ii) $\overline{S^{int}} = S$.

Definition 5.2.4. Let $S \subseteq \mathbb{R}^2$, S has **piecewise smooth boundary** if ∂S consists of one or more *disjoint, piecewise smooth, simple closed curve.*

Definition 5.2.5. Let $S \subseteq \mathbb{R}^2$, then **positive orientation** on ∂S is the orientation on each of the closed curves that make up the boundary such that the region is on the *left* with respect to the positive direction on the curve.

Theorem 5.2.1 (Green's Theorem)**.** Suppose $S \subseteq \mathbb{R}^2$ is a regular region with piecewise smooth region ∂S. Suppose $\mathbf{F}$ is a C^1 vector field defined on $\overline{S}$, then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA \tag{5.2.1}$$

Corollary 5.2.1. Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S, and let $\mathbf{n}(\mathbf{x})$ be the *unit outward normal* vector to ∂S at $\mathbf{x} \in \partial S$. Suppose also that $\mathbf{F}$ is a vector field defined on $\overline{S}$, then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\ ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) dA \tag{5.2.2}$$

5.3 Surface Integrals

5.3.1 Surface Areas and Surface Integrals

Definition 5.3.1. Suppose S is a surface in $\mathbb{R}^3$ and parameterized by

$$\mathbf{G}(\mathbf{u}) : R \to S \tag{5.3.1}$$

where $rank(D\mathbf{G}(\mathbf{u})) = 2$ for every $\mathbf{u} \in R \backslash Z$ where Z is a probably empty set with zero content. If $||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||$ is integrable, then

$$Area(S) \equiv \iint_R ||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||\ d\Theta \tag{5.3.2}$$

Definition 5.3.2. Let $f : S \to \mathbb{R}$ be a real-valued continuous function defined on a super set of S, the **integral of a real-valued function on a surface** is defined as

$$\iint_S f(\mathbf{x})\ dA \equiv \iint_R f(\mathbf{G}(\mathbf{u}))||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||\ d\Theta \tag{5.3.3}$$

Definition 5.3.3. Let $\mathbf{F} : S \to \mathbb{R}^3$ be a continuous vector field defined on a super set of S, the **integral of vector field on a surface** is defined as

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}\ dA \equiv \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v})\ d\Theta \tag{5.3.4}$$

Remark 5.3.1. Surface integrals of real-valued functions are independent of the choice of parametrization.

Remark 5.3.2. But the choice of parameterization can change the sign of surface integrals of vector fields. We need to choose the direction of the normal, $\mathbf{n}$.

Definition 5.3.4. Let $S \subseteq \mathbb{R}^3$ be a two dimensional sub-manifold, and f is a real-valued function defined on a super set of S. Define the **average of f over S** as

$$\text{aver of } f \text{ over } S \equiv \frac{\iint_S f\ dA}{\iint_S 1\ dA} \tag{5.3.5}$$

Remark 5.3.3. A note on the relation between integrals of a vector field and a real-valued function. The surface of vector field $\mathbf{F}$ on S is defined by *reducing $\boldsymbol{F}$ to a real-valued function $\boldsymbol{F} \cdot \boldsymbol{n}$* and then follow the definition of ordinary real-valued function on S. Define $h \equiv \mathbf{F} \cdot \mathbf{n}$,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\ dA &= \iint_S h\ dA && (5.3.6)\\
&\equiv \iint_R h(\mathbf{G}(\mathbf{u}))||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||\ d\Theta && (5.3.7)\\
&= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \mathbf{n}(\mathbf{G}(\mathbf{u}))||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||\ d\Theta && (5.3.8)\\
&= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \frac{\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}}{||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||}||\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}||\ d\Theta && (5.3.9)\\
&= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v})\ d\Theta && (5.3.10)
\end{aligned}$$

5.3.2 An invariance property

Remark 5.3.4. As mentioned above, given $\mathbf{n}(\mathbf{x})$ fixed, we can define the surface integral of vector field as the surface integral of a real-valued function defined as $h(\mathbf{x}) \equiv \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$. And as argued before, one $\mathbf{n}$ is fixed (i.e. orientation is fixed), the value of integral is deterministic. Therefore we can conclude the integral of a vector field $\mathbf{F}$ over a surface S depends on the **orientation** of S but otherwise independent of the parameterization.

Remark 5.3.5. Let $S \subseteq \mathbb{R}^2$ be a two dimensional sub-manifold parameterized by $\mathbf{G} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ such that $rank(\mathbf{G}(\mathbf{u})) = 2$ for all but zero-content sets on its domain.
Let $\varphi : W \subseteq \mathbb{R}^2 \to R$ be a bijection such that $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \to \mathbb{R}^3$ is another parameterization of

S.
Now consider the integral of vector field $\mathbf{F}$ under parameterization $\mathbf{H}$,

$$\iint_S \mathbf{F} \cdot \mathbf{u} \, dA = \iint_W \mathbf{F}(\mathbf{H}) \cdot (\frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t}) \, d\Theta \tag{5.3.11}$$

$$= \iint_W \mathbf{F} \circ \mathbf{G} \circ \varphi \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) det \; D\varphi \, d\Theta \tag{5.3.12}$$

$$= \pm \iint_R \mathbf{F} \circ \mathbf{G} \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) \, d\Theta \text{ (change of variable)} \tag{5.3.13}$$

Theorem 5.3.1 (Invariance)**.** Let $\mathbf{G} : R \to \mathbb{R}^3$ and $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \to \mathbb{R}^3$ be two parameterizations of S, then

$$\iint_R f \circ \mathbf{G} || \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} || \, d\Theta = \iint_W f \circ \mathbf{H} || \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} || \, d\Theta \tag{5.3.14}$$

and

$$\iint_R \mathbf{F} \circ \mathbf{G} \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) \, d\Theta = \pm \iint_W \mathbf{F} \circ \mathbf{H} \cdot (\frac{\partial \mathbf{H}}{\partial u} \times \frac{\partial \mathbf{H}}{\partial v}) \, d\Theta \tag{5.3.15}$$

5.3.3 Volume and Area

Theorem 5.3.2. Let R be an arbitrary regular region in $\mathbb{R}^3$, and let S be the boundary surface of R, define

$$S_h \equiv \{\mathbf{x} + \delta \mathbf{n} : \mathbf{x} \in S \wedge \delta \in [0, h]\} \tag{5.3.16}$$

where S_h can be interpreted as *a shell of region R with thickness h*. Then the surface area of S is

$$\text{area}(S) = \lim_{h \to 0} \frac{|S_h|}{h} \tag{5.3.17}$$

5.4 Divergence, Gradient and Curl

Definition 5.4.1. Let $U \subseteq \mathbb{R}^n$ be an open set, and define real-valued function $f : U \to \mathbb{R}$ and vector field $\mathbf{F} : U \to \mathbb{R}^n$. Then we define

1. The **gradient** of f as ∇f;
2. The **divergence** of $\mathbf{F}$ as $\nabla \cdot \mathbf{F}$;
3. The **curl** of $\mathbf{F}$ as $\nabla \times \mathbf{F}$.

Definition 5.4.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 real-valued function, define the **Laplacian** of f as a mapping from *real-valued functional space* to *real-valued functional space* defined as

$$\text{div}(\text{grad}) f \equiv \sum_j \partial_j^2 f = \Delta f = \nabla^2 f \tag{5.4.1}$$

Theorem 5.4.1. For every C^2 real valued function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\text{curl}(\text{grad} f) = \mathbf{0} \tag{5.4.2}$$

For every C^2 vector field defined in $\mathbb{R}^3$ or a subset of it,

$$\text{div}(\text{curl}\mathbf{F}) = 0 \tag{5.4.3}$$

Note that the domain of f and $\mathbf{F}$ must be $\mathbb{R}^3$ or a subset of it, otherwise the curl operation is not well-defined.

Theorem 5.4.2 (Product rules)**.**

$$\operatorname{grad}(fg) = f\operatorname{grad}g + g\operatorname{grad}f \tag{5.4.4}$$

$$\operatorname{div}(f\mathbf{G}) = f\operatorname{div}G + \operatorname{grad}f \cdot \mathbf{G} \tag{5.4.5}$$

$$\operatorname{curl}(f\mathbf{G}) = f\operatorname{curl}G + \operatorname{grad}f \times \mathbf{G} \tag{5.4.6}$$