ECO426H1 Market Design: Auctions and Matching Markets

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February 25, 2020

Contents

1 Preliminary: Auctions

Definition 1.1. An **auction** is an informational environment consisting of

(i) **Bidding format rules**: the form of the bids, which can be price only, multi-attribute, price and quantity, or quantity only;

(ii) **Bidding process rules**: Closing/timing rules, available information, rules for bid improvements/counter-bids, closing conditions;

(iii) **Price and allocation rules**: final prices, quantities, winners.

Auctions are commonly referred to as a market mechanism as well as a price discovery mechanism

Definition 1.2. A **market mechanism** uses prices to determine allocations.

Definition 1.3. An auction is a **private value** auction if agents' valuations do not dependent on other buyers' valuations. Otherwise, the auction is called a **interdependent / common value** auction.

Assumption 1.1. In a private value auction, we shall impose the following assumption on bidders' valuations:

(i) Each bidder's valuation is independently and identically distributed on some interval $[0, \omega]$ according to a distribution function F:

$$V_i \overset{i.i.d.}{\sim} F \ s.t. \ \text{supp}(F) = \mathbb{R}_+ \tag{1.1}$$

(ii) F belongs to the common knowledge in this system;

(iii) Bidders' valuations have finite expectations:

$$\mathbb{E}\left[V_i\right] < \infty \tag{1.2}$$

Assumption 1.2. Moreover, we assume bidders' behaviours to satisfy the following properties:

(i) Bidders are risk neutral, they are maximizing expected profits;

(ii) Each bidder it both willing and able to pay up to his or her value.

Definition 1.4. A **strategy** of a bidder is a mapping from the space of his/her valuation to a bid:

$$s : [0, \omega] \to \mathbb{R}_+ \tag{1.3}$$

Definition 1.5. An equilibrium of auction is **symmetric** if all bidders are following the same bidding strategy s.

Definition 1.6. A bidder is **bidding sincerely / truthfully** if he bids his true value.

Definition 1.7. An asymmetric game where played have private information is said to be **strategy proof** if it is a weakly-dominant strategy for every player to reveal his/her private information.

Definition 1.8. An auction selling one item is a **standard auction** if the bidder with highest value is always the winner. That is, a standard auction maximizes social value.

2 Ascending Auctions: Extensive Form Games

Definition 2.1. In an **English outcry auction**, bidders announce the prices,

(i) Bidders announce prices,

(ii) with minimum increment between two bids (i.e., the ticking price).

(iii) The auction ends when there's no further bid or when a time limit is reached.

(iv) The winner is the one bidding the highest price.

(v) The winner pays his bid.

Remark 2.1. Bidding speed matters in English outcry auctions: two bidders cannot announce the same bid at the same time.

Definition 2.2. In an **English auction / Japanese button auction**,

(i) The auctioneer announces prices, the price goes up by the ticking price each round;

(ii) in each round, bidders who feel this price is acceptable remain active, other bidders become inactive;

(iii) bidders cannot be reactivated.

(iv) the auction ends when there's no active bidder.

(v) the winner is the last bidder becomes inactive, if there's a tie, winner is randomly chosen.

(vi) the price paid is the last announce price (the price corresponds to no active bidder).

Remark 2.2. In an English auction, the winner is the one with <u>the highest valuation</u>, but the price is that of <u>the second highest valuation plus the ticking price</u>.

Remark 2.3. In the English auction, the auctioneer learns (at the end) the valuations of all bidders except the valuation of the highest bidder.

3 Second Price Sealed-Bid Auction with Private Values

Definition 3.1. In the **Vickrey auction / second price sealed-bid auction**,

(i) Buyers submit a sealed-bid;

(ii) The winner is the one with the highest bid,

(iii) the winner pays the 2nd highest bid.

Remark 3.1. The second price sealed-bid auction and an English auction with negligible ticking price generate the same outcome.
However, second price auction is a strategic for game, but English auction is an extensive form game. They are not exactly identical.

Proposition 3.1. In a symmetric equilibrium of the <u>second-price</u> auction, $s(v) = v$ is a weakly dominant strategy.

Proof. For a fixed valuation $v_i \in [0, \omega]$ of bidder i.
Let $p := \max_{j \neq i} b_j$ be highest bidding price by other bidders.
Let $\pi_i(b, p)$ denote bidder i's profit when bidding b given the highest price from other bidders to be p.
Part 1: consider another bidding $z_i < v_i$, the following cases are possible:

(i) $v_i < p \implies z_i < v_i < p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i losses anyway).

(ii) $v_i = p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i is indifferent).

(iii) $v_i > p$:

  (a) $v_i > z_i > p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = v_i - p$;

  (b) $v_i > z_i = p \implies \pi_i(v_i, p) \geq \pi_i(z_i, p)$;

  (c) $v_i > p > z_i \implies \pi_i(v_i, p) > \pi_i(z_i, p)$.

Hence, bidding v_i weakly dominates bidding any value below it.
Part 2: for $z_i > v_i$, the argument is similar.
Therefore, bidding v_i weakly dominates bidding any other values. ■

Remark 3.2. Refer to the general k^{th} price sealed-bid auction with private values for an alternative proof to this proposition.

4 First Price Sealed-Bid Auction with Private Values

Notation 4.1. Let $\beta^K(v)$ denote the symmetric equilibrium strategy in a k-th price auction.

Remark 4.1. For every continuous distribution F, the probability for a tie to happen is zero. Therefore, we ignore the tie for now.

Definition 4.1 (First Price Auction)**.** Let N denote the set of bidders such that $|N| = n$. For each bidder $i \in N$, his valuation of the auctioned item V_i follows some distribution F. Further assume that $V_i \perp V_j$ for every $i \neq j$.
Let $W(b, v_i)$ denote the event that player i, who has valuation v_i, wins by bidding $b \in \mathbb{R}_+$, define

$$W(b, v_i) \iff b > \max_{j \neq i} b_j \tag{4.1}$$

The payoff (utility) of bidder i, who has valuation v_i, is

$$U(b, v_i) = \begin{cases} v_i - b & \text{if } W(b, v_i) \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

4.1 Symmetric Equilibrium Behaviour

Consider a symmetric environment such that all bidders are using the same <u>strictly increasing</u> strategy $s(\cdot)$ such that $s(\cdot)$ is invertible.

Equilibrium Strategy

Proposition 4.1. In a symmetric equilibrium of the first-price auction, equilibrium bidding strategies are given by

$$s(v_i) = \mathbb{E}\left[\max_{j\neq i} v_j | v_j \leq v_i\right] \tag{4.3}$$

which is the expected second highest valuation conditional on v_i being the highest valuation.

Proof. Let $s(v)$ denote an equilibrium strategy.

Lemma 4.1. For any agent, bidding more than $s(\omega)$ can never be optimal. Bidding $b > s(\omega)$ makes this agent win for sure. In such case, bidding $b' \in (s(\omega), b)$ strictly dominates bidding b.

Lemma 4.2. For any agent, $s(0) = 0$. Bidding any positive number would cause negative payoff with positive probability, and therefore, leads to a negative expected profit.

Lemma 4.3. Because s is monotonically increasing, therefore,

$$\max_{j\neq i} s(v_j) = s(\max_{j\neq i} v_j) \tag{4.4}$$

Let p denote the highest price among all other $N-1$ bidders and let $F^{(N-1)}(x)$ denote the distribution of p.
The expected profit of bidder i by bidding an arbitrary $b \in \mathbb{R}_+$ is

$$\pi_i(b, v_i) = P(b > p)(v_i - s(v_i)) + P(b = p)(v_i - s(v_i)) + P(b < p)0 \tag{4.5}$$

Note that $b > p = s(\max_{j\neq i} v_j)$ if and only if $s^{-1}(b) > \max_{j\neq i} v_j$. It follows

$$P(b > p) = P(\max_{j\neq i} v_j < s^{-1}(b)) = F^{(N-1)}(s^{-1}(b)) \tag{4.6}$$

Therefore,

$$\pi_i(b, v_i) = F^{(N-1)}(s^{-1}(b))(v_i - b) \tag{4.7}$$

The first order condition implies

$$\begin{aligned} \frac{\partial \pi_i}{\partial b}\pi_i(b, v_i) &= \frac{\partial \pi_i}{\partial b} F^{N-1}(s^{-1}(b))v_i - F^{N-1}(s^{-1}(b))b && (4.8) \\ &= f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 && (4.9) \end{aligned}$$

For a symmetric equilibrium, all other bidders are following the same strategy s so that $s(v_i) = b$, therefore,

$$\begin{aligned}
f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 && (4.10)\\
\implies f^{(N-1)}(s^{-1}(b))(v_i - b) - F^{(N-1)}(s^{-1}(b))s'(v_i) = 0 && (4.11)\\
\implies f^{(N-1)}(s^{-1}(b))v_i = F^{(N-1)}(s^{-1}(b))s'(v_i) + f^{(N-1)}(s^{-1}(b))s(v_i) && (4.12)\\
\implies f^{(N-1)}(v_i)v_i = \frac{d}{dv_i}\left[F^{(N-1)}(v_i)s(v_i)\right] && (4.13)\\
\implies \int_0^{v_i} f^{(N-1)}(y)y\ dy = F^{(N-1)}(v_i)s(v_i) - F^{(N-1)}(0)s(0) && (4.14)\\
\implies F^{(N-1)}(v_i)s(v_i) = \int_0^{v_i} f^{(N-1)}(y)y\ dy && (4.15)\\
\implies s(v_i) = \frac{1}{F^{(N-1)}(v_i)}\int_0^{v_i} f^{(N-1)}(y)y\ dy && (4.16)\\
\implies s(v_i) = \mathbb{E}\left[\max_{j\neq i} v_j \,\middle|\, \max_{j\neq i} v_j < v_i\right] && (4.17)
\end{aligned}$$

■

When $F = Unif(0,1)$.

$$\beta^I(v) = \frac{n-1}{n}v \tag{4.18}$$

Probability of Winning

$$\begin{aligned}
P(W(b,v_i)) &= P(b > \max_{j\neq i} s(v_j)) && (4.19)\\
&= P(b > s(\max_{j\neq i} v_j)) && (4.20)\\
&= P(\max_{j\neq i} v_j \leq s^{-1}(b))) && (4.21)\\
&= F(s^{-1}(b))^{n-1} && (4.22)\\
&= F(v_i)^{n-1} \text{ because } b = s(v_i) && (4.23)
\end{aligned}$$

When $F = Unif(0,1)$,

$$P(W(b,v_i)) = v_i^{n-1} \tag{4.24}$$

Expected Payment from Bidder i with v_i Conditioned on Winning Suppose bidder i is following strategy $s(\cdot)$. Then,

$$\mathbb{E}\left[Payment_i | v_i, W(b,v_i)\right] = b = s(v_i) \tag{4.25}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Payment_i | v_i, W(b,v_i)\right] = \frac{n-1}{n}v_i \tag{4.26}$$

Unconditional Payment from Bidder i with v_i

$$\begin{aligned}\mathbb{E}\left[Payment_i|v_i\right] &= P(W(b,v_i))\mathbb{E}\left[Payment_i|v_i, W(b,v_i)\right] + P(Loss)\times 0 && (4.27)\\ &= P(W(b,v_i))\mathbb{E}\left[Payment_i|v_i, W(b,v_i)\right] && (4.28)\\ &= F(v_i)^{n-1}s(v_i) && (4.29)\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Payment_i|v_i\right] = \frac{n-1}{n}v_i^n \tag{4.30}$$

Expected Payoff of Bidder i with v_i

$$\begin{aligned}\mathbb{E}\left[U|v_i\right] &= P(W(s(v_i),v_i))v_i - \mathbb{E}\left[Payment_i|v_i\right] && (4.31)\\ &= F(v_i)^{n-1}v_i - F(v_i)^{n-1}s(v_i) && (4.32)\\ &= F(v_i)^{n-1}[v_i - s(v_i)] && (4.33)\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[U|v_i\right] = \frac{v_i^n}{n} \tag{4.34}$$

Unconditional Payment from Bidder i This is the same as the expected revenue from bidder i:

$$\begin{aligned}\mathbb{E}[Payment_i] &= \int_{\mathbb{R}_+} \mathbb{E}\left[Payment_i|v_i\right] dF && (4.35)\\ &= \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f(v_i)\ dv_i && (4.36)\end{aligned}$$

When $F = Unif(0,1)$,

$$\begin{aligned}\mathbb{E}[Payment_i] &= \int_0^1 \frac{n-1}{n}v_i^n\ dv_i && (4.37)\\ &= \frac{n-1}{n(n+1)} && (4.38)\end{aligned}$$

Auctioneer's Expected Revenue Since all bidders are the same,

$$\begin{aligned}\mathbb{E}\left[Revenue\right] &= n\ \mathbb{E}[Payment_i] && (4.39)\\ &= n\ \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f_i\ dv_i && (4.40)\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Revenue\right] = \frac{n-1}{n+1} \tag{4.41}$$

5 Generalization: k^{th} Price Sealed Bid Private Value Auction

5.1 Uniform Values

In a k^{th} price auction, the bidder with the highest bidding wins, and pays the k^{th} highest bid. Let n denote the number of bidders.

Proposition 5.1. Assume $v_i \overset{i.i.d.}{\sim} Unif(0,1)$, the following strategy forms a symmetric equilibrium in k^{th} price auction:

$$\beta^k(v) = \frac{n-1}{n-k+1}v \tag{5.1}$$

Proof. We are going to verify the proposed strategy indeed forms an equilibrium.
Assume the optimal strategy is linear in v, say αv with $\alpha \in [0,1]$, and all bidders other than i are following this strategy.
The expected payoff of bidder i with value v_i from bidding b is

$$\begin{aligned}
U(b, v_i) &= \mathbb{E}P(W(b, v_i))(v_i - b_{n:k}) && (5.2)\\
&= \mathbb{E}P(b \geq \alpha v_j\ \forall j \neq i)(v_i - b_{n:k}) && (5.3)\\
&= \mathbb{E}P\left(v_j \leq \frac{b}{\alpha}\right)^{n-1}(v_i - b_{n:k}) && (5.4)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} \mathbb{E}\left[b_{n:k} | v_j \leq \frac{b}{\alpha}\ \forall j \neq i\right] && (5.5)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} \alpha\mathbb{E}\left[v_{n-1:k-1} | v_j \leq \frac{b}{\alpha}\ \forall j \neq i\right] && (5.6)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} b\mathbb{E}\left[v_{n-1:k-1} | v_j \leq 1\ \forall j \neq i\right] && (5.7)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} b\mathbb{E}\left[v_{n-1:k-1}\right] && (5.8)\\
& && (5.9)
\end{aligned}$$

Note that each individual $v_j \sim Unif(0,1)$ for every j.
The density function of $v_{n-1:k-1}$ is

$$\begin{aligned}
f_{v_{n-1:k-1}}(x) &= (n-1)\binom{n-2}{k-2}F(x)^{n-k}(1-F(x))^{k-2}f(x) && (5.10)\\
&= (n-1)\binom{n-2}{k-2}x^{n-k}(1-x)^{k-2} && (5.11)
\end{aligned}$$

Taking the expectation

$$\mathbb{E}\left[v_{n-1:k-1}\right] = \int_0^1 x f_{v_{n-1:k-1}}(x)\ dx \tag{5.12}$$

$$= (n-1)\binom{n-2}{k-2}\int_0^1 x^{n-k+1}(1-x)^{k-2}\ dx \tag{5.13}$$

$$= (n-1)\binom{n-2}{k-2}\frac{\Gamma(k-1)\Gamma(-k+n+2)}{\Gamma(n+1)} \tag{5.14}$$

$$= \frac{-k+n+1}{n} \tag{5.15}$$

Therefore,

$$U(b, v_i) = \left(\frac{b}{\alpha}\right)^{n-1}\left(v_i - b\frac{-k+n+1}{n}\right) \tag{5.16}$$

Proposition 5.2. Let $X_i \overset{i.i.d.}{\sim} Unif(0,1)$ for $i = 1, \cdots n$, then

$$\mathbb{E}\left[X_{n:k}\right] = \frac{n-k+1}{n} \tag{5.17}$$

Taking the first order condition,

$$\frac{\partial}{\partial b}U(b, v_i) = \frac{1}{\alpha^{n-1}}\frac{\partial}{\partial b}\left(b^{n-1}\left(v_i - b\frac{-k+n+1}{n}\right)\right) = 0 \tag{5.18}$$

$$\implies (n-1)b^{n-2}\left(v_i - b\frac{-k+n+1}{n}\right) - \frac{-k+n+1}{n}b^{n-1} = 0 \tag{5.19}$$

$$\implies (n-1)\left(v_i - b\frac{-k+n+1}{n}\right) - \frac{-k+n+1}{n}b = 0 \tag{5.20}$$

$$\implies (n-1)v_i - (n-1)b\frac{-k+n+1}{n} - \frac{-k+n+1}{n}b = 0 \tag{5.21}$$

$$\implies (n-1)v_i = (n-k+1)b \tag{5.22}$$

$$\implies \beta^K(v_i) = \frac{n-1}{n-k+1} \tag{5.23}$$

∎

Probability of Winning Given v_i

$$P(W(s(v_i), v_i)) = v_i^{n-1} \tag{5.24}$$

Probability of Payment Given v_i Conditioned on Winning Given that v_i is the highest value, the payment conditioned on winning is

$$\mathbb{E}\left[s(v_{n-1:k-1}) | v_j \leq v_i\ \forall j \neq i\right] \tag{5.25}$$

6 Dutch/Descending Price Auction: An Extensive Form Game

7 Revenue Equivalence Theorem

8 Reserve Price

9 Common Value Auction

10 Combinatorial Auction: The VCG Mechanism

Definition 10.1. A Vickrey-Clarke-Groves (VCG) auction consists of a set of items to be sold X. Each bidder $i \in N$ has a private **value** for each possible bundle of items:

$$v_i : \mathcal{P}(X) \to \mathbb{R} \tag{10.1}$$

Each bidder submits a (sealed) **bidding** for every possible bundle of items:

$$b_i : \mathcal{P}(X) \to \mathbb{R} \tag{10.2}$$

An **assignment** characterize the allocation of items to bidders:

$$\mu : N \to \mathcal{P}(X) \tag{10.3}$$

such that no item is shared between two bidders:

$$\mu(i) \cap \mu(j) = \varnothing \ \forall i \neq j \tag{10.4}$$

The **outcome** assignment seeks to maximize the social value.
Note that the auctioneer does not know v_j's, this social value is computed based on biddings b_j instead of bidders' actual values.

$$\mu^* = \underset{\mu}{\operatorname{argmax}} \sum_{i \in N} b_i(\mu(i)) \tag{10.5}$$

The **price** paid by bidder i is the externality this bidder imposes on other bidders.:

$$p_i = \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) - \sum_{j \neq i} b_j(\mu^*(j)) \tag{10.6}$$

Remark 10.1. The auctioneer does not have to allocate all items in X, that is, μ is not necessary a partition of X. $\bigcup_{i \in N} \mu(i)$ is not necessary X.

Remark 10.2. When $|X| = 1$, VCG mechanism is the second price auction.

Proposition 10.1. Submitting one's true valuation function (i.e., $b_i = v_i$) is a dominate strategy in the VCG auction, that is, VCG auction is strategy proof.

Proof. Suppose all other bidders are bidding b_j.

Let μ^* be the allocation when bidder i bid $b_i = v_i$ while all other bidders bid b_j:

$$\mu^* = \underset{\mu}{\operatorname{argmax}}\, v_i(\mu(i)) + \sum_{j \neq i} b_j(\mu(j)) \quad (\dagger) \tag{10.7}$$

Then, for bidder i, the payoff by bidding v_i is

$$v_i(\mu^*(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\mu^*(j)) \tag{10.8}$$

Alternatively, bidder i could bid $b_i \neq v_i$, let

$$\hat{\mu} = \underset{\mu}{\operatorname{argmax}} \sum_{i \in N} b_i(\mu_i(i)) \tag{10.9}$$

The payoff from bidding b_i instead is

$$v_i(\hat{\mu}(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \tag{10.10}$$

Take the difference between two payoffs:

$$\begin{aligned}
& v_i(\mu^*(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\mu^*(j)) - \left(v_i(\hat{\mu}(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \right) && (10.11) \\
& = v_i(\mu^*(i)) + \sum_{j \neq i} b_j(\mu^*(j)) - \left(v_i(\hat{\mu}(i)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \right) && (10.12) \\
& \geq 0 \text{ by } (\dagger) && (10.13)
\end{aligned}$$

Therefore, bidding one's own value function is dominant. ■

Proposition 10.2. The price paid by any bidder in VCG auctions is non-negative.

Proof.

$$p_i = \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) - \sum_{j \neq i} b_j(\mu^*(j)) \geq 0 \tag{10.14}$$

■

11 Keyword Auctions

12 Matching Market

The second half.

13 Appendix A: Order Statistics

Definition 13.1. Let $(X_1, \cdots, X_n)$ be n random variables on the probability space $(\Omega, \mathcal{F}, P)$, further assume they are iid following distribution function $F(\cdot)$. For each $\omega \in \Omega$, realizations of above random variables can be sorted as

$$X_{(n)}(\omega) \leq X_{(n-1)}(\omega) \leq \cdots \leq X_{(1)}(\omega) \tag{13.1}$$

For each ω, the random variable $X_{n:k}$ is defined such that $X_{n:k}(\omega)$ equals the k-th largest value, $X_{(k)}(\omega)$.

Distribution Function Let $x \in X(\Omega)$, then

$$\begin{aligned}
X_{n:k} \leq x &\iff (\text{no } X_i > x) \bigcup (\text{exactly } 1 \ X_i > x) \bigcup \cdots \bigcup (\text{exactly } k-1 \ X_i > x) && (13.2)\\
&\iff (X_i \leq x \ \forall i) \bigcup (\text{exactly } n-1 \ X_i \leq x) \bigcup \cdots \bigcup (\text{exactly } n-k+1 \ X_i \leq x) && (13.3)\\
&\iff \bigcup_{j=n-k+1}^{n} (\text{exactly } j \ X_i \leq x) && (13.4)
\end{aligned}$$

Note that events in the union are mutually exclusive, therefore,

$$\begin{aligned}
F_{n:k}(x) = P(X_{n:k} \leq x) &= \sum_{j=n-k+1}^{n} P\left(\text{exactly } j \ X_i \leq x\right) && (13.5)\\
&= \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} && (13.6)
\end{aligned}$$

Density Function

$$
\begin{aligned}
f_{n:k}(x) &= \frac{d}{dx} F_{n:k}(x) && (13.7)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} && (13.8)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} F(x)^j (1-F(x))^{n-j} && (13.9)\\
&= \sum_{j=n-k+1}^{n} \left[\frac{n!}{j!(n-j)!} jF(x)^{j-1}(1-F(x))^{n-j} - \frac{n!}{j!(n-j)!}(n-j)F(x)^j(1-F(x))^{n-j-1} \right] f(x) && (13.10)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} jF(x)^{j-1}(1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j)!}(n-j)F(x)^j(1-F(x))^{n-j-1} f(x) && (13.11)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j(1-F(x))^{n-j-1} f(x) && (13.12)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (13.13)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x)\\
&\quad - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j(1-F(x))^{n-j-1} f(x)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (13.14)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1}(1-F(x))^{n-j} f(x)\\
&\quad - \sum_{i=n-k+2}^{n} \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1}(1-F(x))^{n-i} f(x) \ \text{(substitute } j = i-1)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (13.15)\\
&= n\frac{(n-1)!}{(n-k)!(k-1)!} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (13.16)\\
&= n\binom{n-1}{k-1} F(x)^{n-k}(1-F(x))^{k-1} f(x) && (13.17)
\end{aligned}
$$