

Introduction to Real Analysis

Tianyu Du

Monday 30th September, 2019

This work is licensed under a Creative Commons “Attribution-NonCommercial 4.0 International” license.



Contents

1	The Axiom of Completeness	2
1.1	Preliminaries	2
1.2	Density of Rational Numbers	4
2	Sequences	4
2.1	Monotone Convergence Theorem	7
2.2	Series	8
2.3	Cauchy Criterion	9
2.4	Convergence Test for Series	10
2.4.1	The Comparison Test	11
2.4.2	The Root Test	12
2.4.3	Other Tests	13
2.5	Absolute and Conditional Convergence	13
3	Topology in \mathbb{R}	15

1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

Example 1.1. The set of integers, \mathbb{Z} , is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subset \mathbb{R}$, we use A^\uparrow and A^\downarrow to denote collections of upper bounds of A and lower bounds of A . Note that A^\uparrow and A^\downarrow are potentially empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound (supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^\uparrow$ and $\forall u \in A^\uparrow, s \leq u$. Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if $f \in A^\downarrow$ and $\forall l \in A^\downarrow, l \leq f$. Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property). $\forall \emptyset \neq A \subset \mathbb{R}$ such that $A^\uparrow \neq \emptyset, \exists \sup A$.

Definition 1.4. Let $\emptyset \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

Proposition 1.1. Let $\emptyset \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

Proof. Step 1: Show $c + \sup A \in (c + A)^\uparrow$:

Let $x \in c + A, \exists a \in A \text{ s.t. } x = c + a$. Then, $x = c + a \leq c + \sup A$. Therefore, $x \leq c + \sup A \forall x \in c + A$, which implies what desired.

Step 2: Show $\forall u \in (c + A)^\uparrow, c + \sup A \leq u$:

Let $u \in (c + A)^\uparrow$, then $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.

Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum). Let $s \in A^\uparrow$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

- (i) $s = \sup A$;
- (ii) $\forall \varepsilon, \exists a \in A, \text{ s.t. } a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^\uparrow$).

Proof. Immediately. ■

Theorem 1.1 (Nested Interval Property). Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \quad \forall n \in \mathbb{N} \quad (1.4)$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.5)$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k , by the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$. Since $a^* \in (a_n)^\uparrow$, $a^* \geq a_n \quad \forall n \in \mathbb{N}$. Further, because a^* is the *least* upper bound, then for every upper bound b_n , it must be $a^* \leq b_n \quad \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n] \quad \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$.

Theorem 1.2 (Archimedean Property).

- (i) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$;
- (ii) $\forall y \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < y$.

Archimedean property of *natural numbers* can be interpreted as *there is no real number that bounds \mathbb{N}* . This interpretation can be seen by considering the negations of above statements:

- (i) $\exists x \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, n \leq x$;
- (ii) $\exists y \in \mathbb{R}_{++} \text{ s.t. } \forall n \in \mathbb{N}, y \leq \frac{1}{n}$.

Proof of (i) by Contradiction. Suppose the negated statement (i) is true, \mathbb{N} is bounded above. By the completeness axiom, there exists $a^* := \sup \mathbb{N}$. $\exists n \in \mathbb{N} \text{ s.t. } a^* - 1 < n$. In this case, $a^* < n+1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^\uparrow$ and leads to a contradiction. ■

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ■

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The above theorem says \mathbb{Q} is in fact **dense** in \mathbb{R} . More generally, one says a set $A \subset X$ is dense whenever the closure of A , $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.

Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ■

Theorem 1.4. $\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in \mathbb{R} bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^\uparrow$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ■

2 Sequences

Theorem 2.1 (Triangle Inequality). Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.1. Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \quad (2.1)$$

Proof. Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$. And $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$. Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ■

Definition 2.1. A sequence $(a_n) \subset \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |a_n - a| < \varepsilon \quad (2.2)$$

Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at a with radius ε is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \quad (2.3)$$

Theorem 2.2. The limit of any convergent sequence is unique.

Proof. Let (a_n) be a convergent sequence, assume, for contradiction, that $(a_n) \rightarrow L_1$ and $(a_n) \rightarrow L_2$ such that $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \rightarrow L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$|a_n - L_2| = |a_n - L_1 - (L_2 - L_1)| \quad (2.4)$$

$$\geq ||a_n - L_1| - |L_2 - L_1|| \quad (2.5)$$

$$= ||L_1 - L_2| - |a_n - L_1|| \quad (2.6)$$

$$= 3\varepsilon - |a_n - L_1| \quad (2.7)$$

$$> 2\varepsilon \quad (2.8)$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq N$. ■

Definition 2.2. A sequence (a_n) is **divergent** if it does not converge.

Example 2.1. The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \dots)$ is divergent.

Proof. Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \rightarrow L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $|L - \frac{1}{5}| < \frac{2}{15}$. Similarly,

$$\left| -\frac{1}{5} - L \right| = \left| \frac{1}{5} + L \right| \quad (2.9)$$

$$= \left| \frac{1}{5} + L - \frac{1}{5} + \frac{1}{5} \right| \quad (2.10)$$

$$= \left| \left(L - \frac{1}{5} \right) - \left(-\frac{2}{5} \right) \right| \quad (2.11)$$

$$\geq \left| \left| L - \frac{1}{5} \right| - \frac{6}{15} \right| \quad (2.12)$$

$$= \frac{6}{15} - \left| L - \frac{1}{5} \right| \quad (2.13)$$

$$> \frac{4}{15} \quad (2.14)$$

$$> \varepsilon \quad (2.15)$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ■

Alternative Proof. If (a_n) is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of (a_n) converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ■

Definition 2.3. A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $|a_n| < M$.

Theorem 2.3. Every convergent sequence is bounded.

Proof. Let $(a_n) \rightarrow L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n < N} a_n$, take $M := \max\{Q, |L| + 1\}$, then M bounds (a_n) . ■

Theorem 2.4 (Algebraic Limit Theorem). Let $(a_n) \rightarrow a, (b_n) \rightarrow b$ be convergent sequences, and $c \in \mathbb{R}$, then

- (i) $(ca_n) \rightarrow ca$;
- (ii) $(a_n + b_n) \rightarrow a + b$;
- (iii) $(a_nb_n) \rightarrow ab$;
- (iv) $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$, provided $b_n, b \neq 0$.

Proof (i). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

Proof (ii). Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3} \forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3} \forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \quad (2.16)$$

■

Proof (iii). Note that

$$|a_nb_n - ab| = |a_nb_n + a_nb - a_nb - ab| \quad (2.17)$$

$$\leq |a_nb_n - a_nb| + |a_nb - ab| \quad (2.18)$$

$$\leq |a_n||b_n - b| + |b||a_n - a| \quad (2.19)$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because (a_n) is convergent, let M denote its bound such that $|a_n| < M \forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_nb_n - ab| < \varepsilon$. ■

Proof (iv). **Claim i:** when n is sufficiently larger, $|b_n| > 0$ is bounded away from zero by M . Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such n ,

$$|b_n| = |b_n - b - (-b)| \quad (2.20)$$

$$\geq ||b_n - b| - |b|| \quad (2.21)$$

$$\geq |b| - |b_n - b| \quad (2.22)$$

$$> \frac{9|b|}{10} \quad (2.23)$$

Claim ii: $\left(\frac{1}{b_n}\right) \rightarrow \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_nb} - \frac{b_n}{b_nb}\right| \quad (2.24)$$

$$= \frac{1}{|b_n||b|} |b_n - b| \quad (2.25)$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \rightarrow b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

Theorem 2.5 (Order Limit Theorem). Let $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$, then

- (i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;
- (ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;
- (iii) $\exists c \in \mathbb{R} \text{ s.t. } c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;
- (iv) $\exists c \in \mathbb{R} \text{ s.t. } a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

Proof. (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.

(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \rightarrow (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.

(iii) and (iv) Consider constant sequence defined as (c_n) such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

Theorem 2.6 (Squeeze Theorem). Let $(x_n) \rightarrow L$ and $(z_n) \rightarrow \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \rightarrow \ell$.

Proof. Let $\varepsilon > 0$, because both $(x_n) \rightarrow \ell$ and $(y_n) \rightarrow \ell$,

$$\exists N_1 \text{ s.t. } n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \quad (2.26)$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \quad (2.27)$$

Take $N_3 := \max\{N_1, N_2\}$, then for every $n \geq N_3$,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \quad (2.28)$$

$$\implies y_n \in V_\varepsilon(\ell) \quad (2.29)$$

therefore $(y_n) \rightarrow \ell$ by definition. ■

2.1 Monotone Convergence Theorem

Definition 2.4. A sequence (a_n) is said to be **monotone** if it is either increasing ($a_{n+1} \geq a_n \ \forall n \in \mathbb{N}$) or decreasing ($a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$).

Theorem 2.7 (Monotone Convergence Theorem). If a sequence (a_n) is bounded, then it converges.

Proof. WLOG, assume (a_n) is increasing, let $\Gamma := \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$, because Γ is bounded, $s := \sup_n \Gamma$ is well-defined by the completeness of real numbers.

Claim: $(a_n) \rightarrow s$. Let $\varepsilon > 0$, by the definition of supremum, $\exists N \in \mathbb{N}$ such that $a_N > s - \varepsilon$. Because the sequence is increasing and $s + \varepsilon \in \Gamma^\uparrow$, $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$. $(a_n) \rightarrow s$ by definition. ■

2.2 Series

Definition 2.5. Let (a_i) be a sequence, then the n -th **partial sum** is defined as $s_n := \sum_{i=1}^n a_i$. And the **infinite sum/series** of (a_n) is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \rightarrow s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \quad (2.30)$$

Example 2.2. $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

Proof. Obviously the corresponding partial sums are increasing because the sequence $(\frac{1}{i^2})$ is positive.

Claim: (s_n) is bounded from above. Let $n \in \mathbb{N}$, observe

$$\sum_{i=1}^n \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \quad (2.31)$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \quad (2.32)$$

$$= 2 - \frac{1}{n} \leq 2 \quad (2.33)$$

The result is immediate by the monotone convergence theorem. ■

Example 2.3 (Harmonic Series). $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. **Claim:** there exists a subsequence of (s_n) diverges, so (s_n) cannot be convergent. Consider the subsequence (s_k) constructed by defining $s_k := s_{2^k}$. Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \quad (2.34)$$

$$> 1 + \frac{1}{2}k \quad (2.35)$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. ■

Definition 2.6. Let (a_n) be a sequence, then for every strictly increasing sequence $(n_i)_i$ in \mathbb{N} , (a_{n_i}) is a **subsequence** of (a_n) .

Theorem 2.8. All subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \rightarrow \ell$, let (a_{n_k}) be a subsequence of (a_n) . Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\varepsilon(\ell)$. By the definition of subsequences, there exists some $K \in \mathbb{N}$ such that $n_K = N$. Take such K , then for every $k \geq K$, it must be $n_k \geq N$. Therefore $a_{n_k} \in V_\varepsilon(\ell)$ for every $k \geq K$, and $(a_{n_k}) \rightarrow \ell$ by definition. ■

Corollary 2.2. A sequence (a_n) must be divergent if there exists two subsequences of it converge to two different limits.

Proof. Immediate by taking the contrapositive form of above theorem. ■

Theorem 2.9 (Bolzano–Weierstrass). Every bounded sequence contains a convergent subsequence.

Proof. Suppose (a_n) is bounded by certain $M > 0$, that's, for every $n \in \mathbb{N}$, $-M < a_n < M$. Consider the split $I_1^\ell := [-M, 0]$ and $I_1^u := [0, M]$. At least one of above closed intervals contain an infinitely many elements of (a_n) . Define the interval as I_2 . At each I_n , one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and I_{n+1} is defined to be such sequence. Note that the sequence of closed intervals constructed from above recursive procedure is in fact nested. Obviously $\lim_{n \rightarrow \infty} |I_n| = 0$. Further, by the nested interval property, one can show that $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$. Then $\cap_{n \in \mathbb{N}} I_n$ must be a singleton with a in it. One can construct such that $a_{n_k} \in I_k$. Note that $|I_n| = \frac{1}{2^{n-1}}$, therefore, for every $\varepsilon > 0$, one can take $N \geq \log_2 \left(\frac{1}{\varepsilon} \right) + 1$, so that for every $k \geq N$, by definition of subsequences, $n_k \geq n$, so that $a_{n_k}, a \in I_N$. This implies $a_{n_k} \in V_\varepsilon(a)$ and $(a_{n_k}) \rightarrow a$. ■

2.3 Cauchy Criterion

Definition 2.7. A sequence (a_n) is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n \geq N \implies |a_n - a_m| < \varepsilon \quad (2.36)$$

Proposition 2.1. Every convergent sequence is Cauchy.

Proof. Let $(a_n) \rightarrow \ell$, let $\varepsilon > 0$. By the convergence of sequence, $\exists N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - \ell| < \frac{\varepsilon}{2}$, which turns out to imply $a_n, a_m \in V_\varepsilon(\ell)$. ■

Lemma 2.1. Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, $|a_n - a_m| < 1$. In particular, take $m = N$, for every $n \geq N$, $|a_n - a_N| < 1$, and $|a_n| \leq |a_N| + 1$. Then (a_n) is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \quad (2.37)$$

Theorem 2.10 (Cauchy Criterion). A sequence of real numbers is convergent if and only if it's Cauchy.

Proof. (\Leftarrow) Suppose (a_n) is Cauchy, by the lemma established above, (a_n) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence $(a_{n_k}) \rightarrow \ell$.

Claim: $(a_n) \rightarrow \ell$. Let $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $n_k, n \geq N_1$, $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$. And there exists another $N_2 \in \mathbb{N}$ such that for every $n_k \geq N_2$, $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$. Take $N_3 := \max\{N_1, N_2\}$. Note that for every $n \geq N_3$, one can choose some $n_k \geq n$ and derive

$$|a_n - \ell| = |a_n - a_{n_k} + a_{n_k} - \ell| \quad (2.38)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| \quad (2.39)$$

$$< \varepsilon \quad (2.40)$$

(\Rightarrow) Already shown in previous proposition. ■

2.4 Convergence Test for Series

Theorem 2.11 (n -th term test; necessary condition for convergent series). Series $\sum_{i=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$.

Proof. Suppose the partial sums converges to ℓ , by the definition of partial sums, $a_n = s_{n+1} - s_n$. Further, the convergence of partial sums guarantees the convergence of (a_n) . By taking limit on both sides of above identity, it can be shown $\lim_{n \rightarrow \infty} a_n = 0$. ■

Theorem 2.12 (Cauchy Criterion for Series). For series $\sum_{n=1}^{\infty} a_n$, TFAE:

- (i) Series converges;
- (ii) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, \left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$ (i.e. tail sum sequence converges);
- (iii) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m > n \geq N, \left| \sum_{k=n+1}^m a_k \right| < \varepsilon$. (i.e. partial sum is Cauchy)

Proof. (i) \Rightarrow (ii): Suppose (S_n) converges, let $\varepsilon > 0, \exists N$ s.t. $\forall n \geq N, |S_n - L| < \varepsilon$. Note that

$$L - S_n = \lim_{m \rightarrow \infty} \sum_{k=1}^m a_k - S_n \quad (2.41)$$

$$= \lim_{m \rightarrow \infty} \left[\sum_{k=1}^m a_k - S_n \right] \quad (2.42)$$

$$= \lim_{m \rightarrow \infty} \sum_{k=n+1}^m a_k \quad (2.43)$$

which implies the convergence of tail sums.

(ii) \Rightarrow (iii): Suppose the tail sum converges, let $\varepsilon > 0$, note that

$$\left| \sum_{k=n+1}^m a_k \right| = \left| \sum_{k=m+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k \right| \quad (2.44)$$

$$\leq \left| \sum_{k=m+1}^{\infty} a_k \right| + \left| \sum_{k=n+1}^{\infty} a_k \right| \quad (2.45)$$

Both terms can be made arbitrarily small by (ii), specifically, one can choose N_1 and N_2 such that both terms are strictly bounded by $\frac{\varepsilon}{2}$, and $N_3 := \max\{N_1, N_2\}$ is the desired value.

(iii) \implies (i): Since the partial sum is a Cauchy sequence in a complete space, it must converge, so the series is well-defined. ■

2.4.1 The Comparison Test

Definition 2.8. A sequence (a_n) is a **geometric sequence** with coefficient r if $a_{n+1} = ra_n$.

Proposition 2.2. Geometric sequences whenever $r \in (-1, 1)$. Note that when $r = -1$, the sequence becomes an alternating sequence, and the convergence property is indefinite.

Proposition 2.3. Let (a_n) be a geometric sequence with coefficient r , then for every $m \in \mathbb{N}$,

$$rS_m^a = ra_0 + r^2a_0 + \cdots + r^{m+1}a_0 \quad (2.46)$$

$$\implies (r-1)S_m^a = r^{m+1}a_0 - a_0 \quad (2.47)$$

$$\implies S_m^a = a_0 \frac{1 - r^{m+1}}{1 - r} \quad (2.48)$$

Theorem 2.13 (The Comparison Test). Let (a_n) and (b_n) be two sequences satisfy $|a_n| \leq b_n$ for every $n \in \mathbb{N}$. Then

$$(i) \sum_{i=1}^{\infty} b_n \text{ converges} \implies \sum_{i=1}^{\infty} a_n \text{ converges};$$

$$(ii) \sum_{i=1}^{\infty} a_n \text{ diverges} \implies \sum_{i=1}^{\infty} b_n.$$

Proof. Part 1: Suppose (b_n) converges, it is therefore Cauchy. Let $\varepsilon > 0$. Note that for every $m > n$:

$$|S_m^a - S_n^a| = \left| \sum_{k=n+1}^m a_k \right| \quad (2.49)$$

$$\leq \sum_{k=n+1}^m |a_k| \quad (2.50)$$

$$\leq \sum_{k=n+1}^m b_k \quad (2.51)$$

Therefore exists $N \in \mathbb{N}$ such that $\sum_{k=n+1}^m b_k \leq \left| \sum_{k=n+1}^m b_k \right| < \varepsilon$ for every $m, n \geq N$. Taking such N provides the cutoff needed for (S_n^a) to be Cauchy. Because $(S_n^a) \subset \mathbb{R}$, it converges.

Part 2: The result is immediate by taking the contrapositive form of the previous statement. ■

2.4.2 The Root Test

Definition 2.9. Let (a_n) be a bounded sequence, then

$$\limsup(a_n) := \sup_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.52)$$

$$\liminf(a_n) := \inf_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.53)$$

$$(2.54)$$

Theorem 2.14 (The Root Test). Let (a_n) be a sequence in which $a_n \geq 0$ for every $n \in \mathbb{N}$, let $\ell = \limsup a_n^{1/n}$, then

- (i) If $\ell < 1$, then (S_n^a) converges;
- (ii) If $\ell > 1$, then (S_n^a) diverges;
- (iii) If $\ell = 0$, inconclusive.

Proof. Part 1: (Idea: compare with geometric series with $r < 1$) Suppose $\ell < 1$, pick $r \in (\ell, 1)$, and let $\varepsilon = r - \ell$. By the convergence of supremum, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\left| \sup_{k \geq n} a_k^{1/k} - \ell \right| < \varepsilon \quad (2.55)$$

$$\implies a_n^{1/n} \leq \sup_{k \geq n} a_k^{1/k} < \ell + \varepsilon =: r \quad (2.56)$$

Therefore, for every $n \geq N$, $a_n < r^n$. Because (a_n) is assumed to be a non-negative sequence, then $|a_n| < r^n$. Construct new sequences:

$$b_k = \begin{cases} a_k & \forall k < N \\ r^k & \forall k \geq N \end{cases} \quad (2.57)$$

Then, clearly $|a_n| \leq b_k$ for every $k \in \mathbb{N}$. And (b_n) is a sequence with geometric tails (which has coefficient less than one). So $\sum^\infty b_k$ converges, which implies $\sum^\infty a_k$ converges by the comparison test.

Part 2: Suppose $\ell > 1$.

Note that the necessary condition for $\sum a_n^{1/n}$ to converge is $\lim_{n \rightarrow \infty} a_n^{1/n} = 0$, which implies every subsequence of $(a_n^{1/n})$ converges to zero. We are going to prove the divergence of series by constructing a subsequence of $(a_n^{1/n})$ does not converge to zero.

Take $\varepsilon = \ell - 1 > 0$, there exists N such that for every $n \geq N$:

$$\ell - \varepsilon < \sup_{k \geq n} a_k^{1/k} \quad (2.58)$$

$$\implies 1 < \sup_{k \geq n} a_k^{1/k} \quad (2.59)$$

By definition of supremum, there exists $n_1 \geq n$ such that

$$a_{n_1}^{1/n_1} > 1 \quad (2.60)$$

For every $n \geq N$, we can construct a subsequence of $(a_n^{1/n})$ such that every term in it is strictly greater than 1, which means it cannot converge to 0. Therefore, series diverges. ■

2.4.3 Other Tests

Theorem 2.15 (Limit Comparison Test). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ satisfy:

- (i) $b_n \geq 0$;
- (ii) $\limsup_n \frac{|a_n|}{b_n} < \infty$;
- (iii) $\sum_{n=1}^{\infty} b_n$ converges.

Then $\sum_{n=1}^{\infty} a_n$ converges as well.

Theorem 2.16 (Ratio Test). Given sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \geq 0$, then

- 1. If $\limsup \frac{a_{n+1}}{a_n} < 1$, $\sum_{n=1}^{\infty} a_n$ converges;
- 2. If $\limsup \frac{a_{n+1}}{a_n} > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 2.17 (Integral Test). Let $f(x)$ be a *positive* and *monotone decreasing* function on $[1, \infty)$. Consider $(f(x_n))$, then

$$\sum_{n=1}^{\infty} f(n) \text{ conv} \iff \int_1^{\infty} f(x) dx < \infty \quad (2.61)$$

Theorem 2.18 (Alternating Series Test). For an alternating sequence $\sum_{n=1}^{\infty} (-1)^n a_n$, if $(a_n) \searrow 0$, then the series converges.

Proof. **TODO** ■

2.5 Absolute and Conditional Convergence

Corollary 2.3 (Corollary of Comparison Test). If $\sum_{i=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition 2.10. For any series $\sum_{n=1}^{\infty} a_n$, if

- 1. $\sum_{i=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ **converges absolutely**;
- 2. $\sum_{i=1}^{\infty} |a_n|$ does not converge, then $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 2.4. Alternating harmonic series converges conditionally.

However, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

Definition 2.11. $\sum_{n=1}^{\infty} b_n$ is called a **rearrangement** of series $\sum_{n=1}^{\infty} a_n$ if there exists $f : \mathbb{N} \rightarrow \mathbb{N}$ such that f is a bijection and $b_{f(k)} = a_k$ for every $k \in \mathbb{N}$.

Theorem 2.19 (Riemann Series Theorem). If series $\sum_{n=1}^{\infty} a_n$ converges conditionally, for every $\alpha \in \mathbb{R}$, there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ converges to α .

Proof. The proof is non-trivial and omitted. ■

Theorem 2.20. If series $\sum_{n=1}^{\infty} a_n$ converges absolutely to some value $A \in \mathbb{R}$, then every rearrangement $\sum_{n=1}^{\infty} b_n$ converges to A .

Proof. Define partial sum sequences

$$S_n := \sum_{k=1}^n a_k \quad T_m := \sum_{k=1}^m b_k \quad (2.62)$$

Suppose $(S_n) \rightarrow A$, want to show: $(T_n) \rightarrow A$.

Let $\varepsilon > 0$ fixed.

By convergence of (S_n) , there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |S_n - A| < \frac{\varepsilon}{2} \quad (2.63)$$

Because $\sum_{n=1}^{\infty} a_n$ converges absolutely, by the Cauchy criterion for convergent series (i.e. partial sum sequence is Cauchy), there exists $N_2 \in \mathbb{N}$ such that

$$n > m \geq N_2 \implies \sum_{k=m+1}^n |a_k| < \frac{\varepsilon}{2} \quad (2.64)$$

Define $N := \max\{N_1, N_2\}$, $M := \max\{f(k) : 1 \leq k \leq N\}$.

$$|T_m - S_N| = |b_1 + \cdots + b_m - a_1 - \cdots - a_N| \quad (2.65)$$

$$= |b_1 + \cdots + b_m - b_{f(1)} - \cdots - b_{f(N)}| \quad (2.66)$$

Note that for every $m \geq M$, by construction, $\{b_{f(1)}, \dots, b_{f(N)}\} \subset \{b_1, \dots, b_m\}$.

Note that for each $b_{f(k)} \in \{b_1, \dots, b_m\}$, either $k > N$ or $k \leq N$. But all $b_{f(k)}$ with $k \leq N$ were subtracted, so $b_{f(k)}$ elements left are all from $\{a_k : k \geq N+1\}$.

$$\dots = \left| \sum_{k \in \mathbb{I} \geq N+1} a_k \right| \quad (2.67)$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad (2.68)$$

Therefore, for all $m \geq M$,

$$|T_m - A| = |T_M - S_n + S_n - A| \quad (2.69)$$

$$\leq |T_M - S_n| + |S_n - A| \quad (2.70)$$

$$< \varepsilon \quad (2.71)$$

The desired result is immediate. ■

3 Topology in \mathbb{R}

Definition 3.1.

$$V_\varepsilon(x_0) := \{x \in \mathbb{R} : ||x, x_0|| < \varepsilon\} \quad (3.1)$$

Definition 3.2. A subset of $\mathcal{O} \subset \mathbb{R}$ is **open** if

$$\forall x \in \mathcal{O} \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(x) \subset \mathcal{O} \quad (3.2)$$

Theorem 3.1. Arbitrary union of open sets is open; Any finite intersection of open sets is open.

Proof. Let \mathcal{O}_α open for all $\alpha \in \mathcal{A}$. Let $\mathcal{O} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha$. If $x \in \mathcal{O}$, there exists some $\alpha \in \mathcal{A}$ such that $x \in \mathcal{O}_\alpha$. There exists $V_\varepsilon(x) \subset \mathcal{O}_\alpha \subset \mathcal{O}$. Hence \mathcal{O} is open.

Let $\{\mathcal{O}_i : 1 \leq i \leq n\}$ be a collection of open sets, let $\mathcal{O} := \bigcap_{i=1}^n \mathcal{O}_i$. If $x \in \mathcal{O}$, there exists $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(x) \subset \mathcal{O}_i$ for every i . Take $\varepsilon := \max\{\varepsilon_i\}$, which exists and is strictly positive by finiteness of index set. Therefore $V_\varepsilon(x) \subset \mathcal{O}_i$ for every i , and therefore in \mathcal{O} . ■

Definition 3.3. x is a **limit point** of A if $\forall \varepsilon > 0$,

$$V_\varepsilon(x) \cap A - \{x\} \neq \emptyset \quad (3.3)$$

Remark: this definition does not require x to be an element of A .

Theorem 3.2. x is a limit point A if and only if there exists a sequence $(a_n)_{n=1}^\infty \subset A$ such that $a_n \neq x \forall n \in \mathbb{N}$ and $(a_n)_{n=1}^\infty \rightarrow x$.

Proof. (\implies) Let x be a limit point, take $\varepsilon = \frac{1}{n}$, immediate by the definition of limit point.

(\impliedby) Trivially by definition of sequential convergence. ■

Definition 3.4. $X \subset \mathbb{R}$ is **closed** if it contains all its limit points.

Definition 3.5. $x \in A$ is an **isolated point** if it is not a limit point of A .

Example 3.1. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, every point in A is isolated, and 0 is the limit point.

Definition 3.6. $A \subset X$ is **dense** in X if $\overline{A} = X$.

Theorem 3.3. Let $x \in \mathbb{R}$, there exists a sequence $(q_n)_{n=1}^\infty \subset \mathbb{Q}$ such that $(q_n)_{n=1}^\infty \rightarrow x$.

Proof. Let $x \in \mathbb{R}$. Note that $\forall u < v \in \mathbb{R}$, there exists $q \in (u, v) \cap \mathbb{Q}$. Hence, for every $n \in \mathbb{N}$, $\exists q_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < q_n < x + \frac{1}{n}$. It is evident that $(q_n)_{n=1}^{\infty} \rightarrow x$. ■

Definition 3.7. The **closure** of A , denoted as \overline{A} , is defined to be the union of A and all limit points of A .

Lemma 3.1. \overline{A} is the smallest closed set containing A .

Proof. It is evident that \overline{A} is a closed set containing A .

Now show the closure is in fact the smallest closed set. Let $B \subsetneq \overline{A}$ be a proper subset of the closure, we are going to show that B is not closed. Let $x \in \overline{A} - B \neq \emptyset$.

Note that $\overline{A} \equiv A \cup A'$, then either $x \in A$ or $x \in A'$. If $x \in A$, then B does not contain A . If $x \in A'$, then B does not contain all limit points of A , so it is not closed. ■

Theorem 3.4. Equivalent definitions of openness and closedness:

- (i) \mathcal{O} is open if and only if \mathcal{O}^c is closed;
- (ii) \mathcal{O} is closed if and only if \mathcal{O}^c is open.

Proof. (\implies) Let \mathcal{O} be an open set, let $(x_n) \rightarrow x$ be a convergent sequence in \mathcal{O}^c . It is evident that if $x \in \mathcal{O}$, infinitely many elements in the tail of (x_n) would be in $V_\varepsilon(x) \subset \mathcal{O}$, which leads to a contradiction. Therefore \mathcal{O}^c contains all of its limit points, and \mathcal{O}^c is therefore closed.

(\impliedby) Let \mathcal{O}^c be a closed set, suppose \mathcal{O} is not open, there exists $x \in \mathcal{O}$ such that for all $\varepsilon > 0$, $V_\varepsilon(x) \cap \mathcal{O}^c \neq \emptyset$. Then we can construct a sequence in \mathcal{O}^c converge to x , which leads to a contradiction that there is a limit point of a sequence in \mathcal{O}^c not contained by \mathcal{O}^c .

The second part is immediate. ■

Theorem 3.5. Any intersection of closed sets is closed; any finite union of closed sets is closed.

Proof. Direct result from De Morgan's law and the previous theorem. ■