

MAT237: Multivariable Calculus

Tianyu Du

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- GitHub: https://github.com/TianyuDu/Spikey_UofT_Notes
- Website: TianyuDu.com/notes

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1 Limits, continuity, and related topics

1.1 Open and Closed Sets, Boundary, Interior and Closure

Definition 1.1.1. Let $\mathbf{a} \in \mathbb{R}^n$, and $r > 0$. The **open ball with centre \mathbf{a} and radius r** is defined as

$$\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\}. \quad (1.1.1)$$

Definition 1.1.2. The **sphere with centre \mathbf{a} and radius r** is defined as

$$\partial\mathcal{B}(r, \mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| = r\} \quad (1.1.2)$$

Definition 1.1.3. Let $S \subset \mathbb{R}^n$, S is **bounded** if

$$\exists r > 0 \text{ s.t. } S \subset \mathcal{B}(r, \mathbf{0}) \quad (1.1.3)$$

Definition 1.1.4. Let $S \subset \mathbb{R}^n$, then the **complement** of S is defined as

$$S^c := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin S\} \quad (1.1.4)$$

Definition 1.1.5. Let $S \subset \mathbb{R}^n$, the **interior** of S is defined as

$$S^{int} := \{\mathbf{x} \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ s.t. } \mathcal{B}(\varepsilon, \mathbf{x}) \subset S\} \quad (1.1.5)$$

Definition 1.1.6. The **boundary** of S is defined as

$$\partial S := \{\mathbf{x} \in \mathbb{R}^n : \forall \varepsilon > 0 \mathcal{B}(\varepsilon, \mathbf{x}) \cap S \neq \emptyset \wedge \mathcal{B}(\varepsilon, \mathbf{x}) \cap S^c \neq \emptyset\} \quad (1.1.6)$$

Theorem 1.1.1. A point $\mathbf{x} \in S$ is either a *boundary point* or a *interior point*.

Definition 1.1.7. The **closure** of S is defined as

$$\overline{S} := S^{int} \cup \partial S \quad (1.1.7)$$

Theorem 1.1.2. For any $S \subset \mathbb{R}^n$

$$S^{int} \subset S \subset \overline{S} \quad (1.1.8)$$

Theorem 1.1.3. For any $S \subset \mathbb{R}^n$

$$\partial S = \partial(S^c) \quad (1.1.9)$$

Definition 1.1.8. A set $S \subset \mathbb{R}^n$ is **open** if $S = S^{int}$. S is **closed** if $S = \overline{S}$.

Theorem 1.1.4.

$$S \text{ is closed} \iff S^c \text{ is open} \quad (1.1.10)$$

Proof.

$$S \text{ is closed} \iff \partial S \subset S \iff \partial(S^c) \subset S \quad (1.1.11)$$

$$\iff \text{no point of } S^c \text{ is a boundary point} \iff S^c \text{ is open} \quad (1.1.12)$$

■

Proposition 1.1.1. A set S is *closed* if it contains all limit points. That's, every convergent sequence in S converges to a limit point in S .

Theorem 1.1.5 (Homework 1.1). The limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|x|^{r_1} |y|^{r_2}}{|x|^{s_1} + |y|^{s_2}} \quad (1.1.13)$$

exists if and only if

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} > 1 \quad (1.1.14)$$

1.2 Limits and Continuity

1.2.1 Limits of Multivariable Functions

Definition 1.2.1. Let $S \subset \mathbb{R}^n$, $\mathbf{f} : S \rightarrow \mathbb{R}^k$, and $\mathbf{a} \in S$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in \mathbb{R}^k \quad (1.2.1)$$

is defined as

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \mathbf{x} \in S, \underbrace{0 < \|\mathbf{x} - \mathbf{a}\| < \delta}_{\text{punctured neighbourhood}} \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon \quad (1.2.2)$$

For this definition to be non-trivial, we need \mathbf{a} not be an isolated point,

$$\forall \delta > 0, \exists \mathbf{x} \in S \text{ s.t. } \|\mathbf{x} - \mathbf{a}\| \in (0, \delta) \quad (1.2.3)$$

Theorem 1.2.1 (Limit Laws of Real Valued Functions). Let $S \subset \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$ satisfying (1.3.3) And $f, g : S \rightarrow \mathbb{R}$, $L, M \in \mathbb{R}$ such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L \quad (1.2.4)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = M \quad (1.2.5)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) + g(\mathbf{x})] = L + M \quad (1.2.6)$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} [f(\mathbf{x}) \cdot g(\mathbf{x})] = LM \quad (1.2.7)$$

Theorem 1.2.2 (Squeeze Theorem on Real Valued Functions). Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$ satisfies (1.3.3). Suppose that $f, g, h : S \rightarrow \mathbb{R}$ and there exists $p > 0$ and $L \in \mathbb{R}$ such that

$$\forall \mathbf{x} \in S \cap \mathcal{B}(p, \mathbf{a}) \quad f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x}) \quad (1.2.8)$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = L \quad (1.2.9)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = L \quad (1.2.10)$$

Corollary 1.2.1. Let $g, h : S \rightarrow \mathbb{R}$ and

$$|g(\mathbf{x})| \leq h(\mathbf{x}) \quad \forall \mathbf{x} \in S \quad (1.2.11)$$

$$\text{and } \lim_{\mathbf{x} \rightarrow \mathbf{a}} h(\mathbf{x}) = 0 \quad (1.2.12)$$

then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} g(\mathbf{x}) = 0 \quad (1.2.13)$$

Theorem 1.2.3. Assume that $S \subset \mathbb{R}^n$ and let $\mathbf{a} \in \mathbb{R}^n$ satisfying (1.3.3). Let $\mathbf{f} : S \rightarrow \mathbb{R}^k$, then

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \iff \lim_{\mathbf{x} \rightarrow \mathbf{a}} f_j(\mathbf{x}) = L_j \quad \forall j \quad (1.2.14)$$

1.2.2 Continuity

Definition 1.2.2. Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \rightarrow \mathbb{R}^k$. \mathbf{f} is **continuous at** $\mathbf{a} \in S$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) \quad (1.2.15)$$

and \mathbf{f} is **continuous** if \mathbf{f} is continuous at every point in S .

Theorem 1.2.4 (Basic Properties of Continuity). Assume that $S \subset \mathbb{R}^n$ and $\mathbf{a} \in S$,

- (i) If $\mathbf{f} : S \rightarrow \mathbb{R}^k$ is continuous at \mathbf{a} , then every component of \mathbf{f} , $f_j : S \rightarrow \mathbb{R}$, is continuous at \mathbf{a} .
- (ii) If $\mathbf{f}, \mathbf{g} : S \rightarrow \mathbb{R}^k$ are continuous at \mathbf{a} , then $\mathbf{f} + \mathbf{g}$ is continuous at \mathbf{a} .
- (iii) If $f, g : S \rightarrow \mathbb{R}$ continuous, then fg is continuous and $\frac{f}{g}$ is continuous given $g(\mathbf{a}) \neq 0$.
- (iv) A composition of continuous functions is continuous.
- (v) The elementary functions of a single variable (trigonometric functions and their inverses, polynomials, exponential and log) are continuous on their domains.

1.2.3 Continuous Functions and Open Sets

Theorem 1.2.5. Assume that $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$, then the following are equivalent

- (i) \mathbf{f} is continuous;
- (ii) For every open set $\mathcal{O} \subset \mathbb{R}^k$, $\mathbf{f}^{-1}(\mathcal{O})$ is also open;
- (iii) For every closed set $\mathcal{C} \subset \mathbb{R}^k$, $\mathbf{f}^{-1}(\mathcal{C})$ is also closed.

Openness and closeness are preserved under pre-image operations of continuous functions.

Theorem 1.2.6. An arbitrary union of open sets is open.

Theorem 1.2.7. A finite intersection of open sets is open.

Theorem 1.2.8. A finite union of closed sets is closed.

Theorem 1.2.9. An arbitrary intersection of closed sets is closed.

1.3 Sequences and Completeness

Definition 1.3.1. A sequence $\{\mathbf{a}_j\}_{j \in \mathbb{Z}_{++}}$ in \mathbb{R}^n **converges to the limit** $\mathbf{L} \in \mathbb{R}^n$ if

$$\forall \varepsilon > 0 \exists J \in \mathbb{N}, \text{ s.t. } \forall j \geq J \implies \|\mathbf{a}_j - \mathbf{L}\| < \varepsilon \quad (1.3.1)$$

Theorem 1.3.1.

$$\lim_{j \rightarrow \infty} \mathbf{a}_j = \mathbf{L} \iff \lim_{j \rightarrow \infty} \|\mathbf{a}_j - \mathbf{L}\| = 0 \quad (1.3.2)$$

Theorem 1.3.2. Let $\{a_{jk}\}_j$ be a sequence in \mathbb{R}^n where $k \in [n]$, and let $\mathbf{L} = (L_1, \dots, L_n) \in \mathbb{R}^n$, then

$$\lim_{j \rightarrow \infty} \mathbf{a}_j = \mathbf{L} \iff \lim_{j \rightarrow \infty} a_{jk} = L_k \quad \forall k \in [n] \quad (1.3.3)$$

Proof Idea.

$$\forall j \in [n], |a_j - L_j| \leq \|\mathbf{a} - \mathbf{L}\| \leq n \cdot \max_{k \in [n]} |a_k - L_k| \quad (1.3.4)$$

■

Axiom 1.1 (the Completeness Axiom). Every bounded and nonempty set of real numbers has a least upper bound (**supremum**) and a greatest lower bound (**infimum**).

Theorem 1.3.3 (Monotone Sequence Theorem). Every bounded nondecreasing sequence of real numbers converges to a limit.

Proof Idea. Note that such sequence converges to its supremum S .

Let $\varepsilon > 0$, there exists j^* such that

$$S - \varepsilon < a_{j^*} \leq S \quad (1.3.5)$$

take such j^* and by the nondecreasing property,

$$\forall j \geq j^* \quad a_j > S - \varepsilon \quad (1.3.6)$$

which implies $|S - a_j| < \varepsilon$.

■

Theorem 1.3.4 (Monotone Sequence Theorem). Every bounded monotone sequence in \mathbb{R} is convergent.

Definition 1.3.2. A **subsequence** of a sequence $\{\mathbf{a}_j\}_{j \geq j_0}$ in \mathbb{R}^n is a sequence constructed as $\{a_{k_j}\}_j$, such that $\{k_j\}_j$ is a *strictly increasing* sequence bounded below by j_0 .

Proposition 1.3.1. If $\{\mathbf{a}_j\}_j$ is a sequence in \mathbb{R}^n converges to \mathbf{L} , then (i) any subsequence of it converges to the (ii) same limit.

Proof Idea. Suppose not and reach a contradiction.

■

Theorem 1.3.5 (Bounded Sequence Theorem in \mathbb{R}). Every bounded sequence in \mathbb{R} has a convergent subsequence.

Proof. Let $\{a_j\}_j$ be a bounded sequence.

For each $j \in \mathbb{N}$, define $b_{k_j} := \inf_{k > k_j} a_k$.

Note that $\{b_j\}$ is non-decreasing and bounded, so it converges to some limit ℓ .

Let $\{a_{k_j}\}_j$ denote a subsequence of the original sequence, define $k_0 = j_0$, and indices are constructed in a recurrent way.

Suppose every index before k_j has been chosen, we choose k_{j+1} to be the index such that

$$b_{k_j} \leq a_{k_{j+1}} < b_{k_j} + \frac{1}{j} \quad (1.3.7)$$

by construction, $\{a_{k_j}\}_j$ is bounded by both $\{b_{k_j}\}_j$ and $\{b_{k_j} + \frac{1}{j}\}_j$, and both bounding sequences converge to ℓ . So $\{a_{k_j}\}_j$ converges to ℓ by *squeeze theorem*. ■

Theorem 1.3.6 (Bounded Sequence Theorem). Every bounded sequence in \mathbb{R}^n has a convergent subsequence.

Proof. Let $\{\mathbf{a}_j\}_j$ be a bounded sequence.

Applying the previous theorem iteratively, we can construct a subsequence of $\{\mathbf{a}_{k_j}\}_j$ such that $\{\mathbf{a}_{k_j} \cdot \mathbf{e}_1\}_j$ is bounded and convergent.

Then we apply the previous theorem iteratively on the constructed convergent subsequences to construct new subsequences with more convergent components. ■

Theorem 1.3.7 (Nested Interval Theorem). Given a sequence of *closed intervals* in \mathbb{R} ,

$$\{I_k\}_{k \in \mathcal{A}} \text{ s.t. } I_k = [a_k, b_k] \subset \mathbb{R} \quad (1.3.8)$$

and

$$\cdots I_{k+1} \subseteq I_k \subseteq \cdots I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \quad (1.3.9)$$

Then

$$\bigcap_{k \in \mathcal{A}} I_k \neq \emptyset \quad (1.3.10)$$

1.4 Compactness

1.4.1 Compactness

Definition 1.4.1 (Heine-Borel Property). A set S is **compact** if every *open* covering of S has a *finite* sub-covering.

Definition 1.4.2 (Sequentially Compact). A set $S \subset \mathbb{R}^n$ is **compact** if every sequence in S has a subsequence that converges to a limit in S .

Proposition 1.4.1. If $\{\mathbf{x}_j\}_j$ is a *convergent* sequence in a *closed* set $S \subset \mathbb{R}^n$, then the limit of this sequence is in S .

Proof Idea. Let $\mathbf{x} := \lim_{j \rightarrow \infty} \mathbf{x}_j$, and we wish to show $\mathbf{x} \in S$.

Equivalently, we can show $\mathbf{x} \in \overline{S}$, and that's

$$\forall \varepsilon > 0 \quad \mathcal{B}(\varepsilon, \mathbf{x}) \cap S \neq \emptyset \quad (1.4.1)$$

this is immediately true by the definition of sequence convergence. There must be some points in the sequence, thus in S , belongs to $\mathcal{B}(\varepsilon, \mathbf{x})$. ■

Theorem 1.4.1 (Bolzano-Weierstrass). Let $S \subset \mathbb{R}^n$,

$$S \text{ is compact} \iff S \text{ is closed and bounded} \quad (1.4.2)$$

Proof Idea.

(\Leftarrow) Suppose S is closed and bounded, boundedness ensures such sequence converges, and closeness ensures the limit point of sequence is in S .

(\Rightarrow) Prove by *modus tollens*.

Case (i): S is not bounded, then

$$\forall R > 0 \exists \mathbf{x} \in S \setminus \mathcal{B}(R, \mathbf{0}) \quad (1.4.3)$$

and above $\mathbf{x}(R)$ depends on R , we can construct a sequence using $\mathbf{x}(j)$ such that the $\|\mathbf{x}\|$ is ever increasing and it does not have a limit.

Case (ii): S is not closed, we can construct a sequence with subsequence converges to $\mathbf{x} \in \partial S \setminus S$, which is nonempty because S is not closed. ■

Theorem 1.4.2. Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous function. If $K \subset \mathbb{R}^n$ is compact, then the image $\mathbf{f}(K)$ is compact.

1.4.2 the Extreme Value Theorem

Theorem 1.4.3 (the Extreme Value Theorem). Assume K is a compact subset of \mathbb{R}^n and $f : K \rightarrow \mathbb{R}$ is continuous.

Then (i)

$$f(K) \text{ is compact} \quad (1.4.4)$$

and (ii) the infimum and supremum of $f(\mathbf{x})$ on K are attainable.

$$\exists \bar{\mathbf{x}}, \underline{\mathbf{x}} \in K \text{ s.t. } \begin{cases} f(\bar{\mathbf{x}}) = \sup_{\mathbf{x} \in K} f(\mathbf{x}) \\ f(\underline{\mathbf{x}}) = \inf_{\mathbf{x} \in K} f(\mathbf{x}) \end{cases} \quad (1.4.5)$$

Proof. Let $\{y_j\}_j$ be a sequence in $f(K)$, and we can find a sequence $\{\mathbf{z}_j\}_j$ in K such that $y_j = f(\mathbf{z}_j)$ (by definition of image). Because K is compact, there exists a subsequence of $\{\mathbf{z}_j\}_j$ converges to $\mathbf{z}^* \in K$. Since f is continuous, we can conclude there a subsequence, sharing the same indices, such that $f(\mathbf{z}_j) \rightarrow f(\mathbf{z}^*)$. Obviously $f(\mathbf{z}^*) \in f(K)$, so $f(K)$ is compact.

Since $f(K)$ is compact, $\sup_{\mathbf{x} \in K} f(\mathbf{x}) \in f(K)$. By definition of image, $\exists \mathbf{x} \in K$ such that $f(\mathbf{x}) = \sup_{\mathbf{x} \in K} f(\mathbf{x})$, supremum attainability shown.

Proof for infimum attainability is the same. ■

Proposition 1.4.2. Assume that $\{\mathbf{z}_j\}_j$ is a sequence in a set $S \subset \mathbb{R}^k$, and f is a continuous real-valued function defined on S , then

$$\mathbf{z}_j \rightarrow \mathbf{z} \implies f(\mathbf{z}_j) \rightarrow f(\mathbf{z}) \quad (1.4.6)$$

Proposition 1.4.3. If S is a compact set in \mathbb{R} , then $\sup S$ and $\inf S$ both in S .

Proof Idea. Suppose $\sup S \notin S$, by definition of supremum,

$$\forall \varepsilon \exists x \in S \text{ s.t. } \sup S - \varepsilon < x \leq \sup S \quad (1.4.7)$$

note that such $x \in \mathcal{B}(\varepsilon, \sup S)$. Also, similarly,

$$\forall \varepsilon > 0 \exists x \notin S \text{ s.t. } \sup S < x < \sup S + \varepsilon \quad (1.4.8)$$

so such $x \in \mathcal{B}(\varepsilon, \sup S)$. We conclude

$$\forall \varepsilon > 0 \mathcal{B}(\varepsilon, \sup S) \cap S \neq \emptyset \wedge \mathcal{B}(\varepsilon, \sup S) \cap S^c \neq \emptyset \quad (1.4.9)$$

which means $\sup S \in \partial S$. Thus if $\sup S \notin S$, S cannot be closed and this contradicts our assumption that S is compact.

The proof for $\inf S \in S$ is similar. ■

1.4.3 Uniform Continuity

Definition 1.4.3. Let $S \subset \mathbb{R}^n$, a function $\mathbf{f} : S \rightarrow \mathbb{R}^k$ is **uniformly continuous** if

$$\underbrace{\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall \mathbf{x}, \mathbf{y} \in S, \|\mathbf{x} - \mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon}_{\text{cts: } \forall \mathbf{x} \in S, \varepsilon > 0, \exists \delta > 0} \quad (1.4.10)$$

Remark 1.4.1. In the definition of *continuity*, value of δ can depend on \mathbf{x} . But in the definition of *uniform continuity*, one δ has to work for every \mathbf{x} .

Theorem 1.4.4. If K is a compact subset of \mathbb{R}^n , and $\mathbf{f} : K \rightarrow \mathbb{R}^k$ is continuous, then \mathbf{f} is uniformly continuous.

1.5 the Intermediate Value Theorem

Definition 1.5.1. A set $S \subset \mathbb{R}^n$ is **path-connected** (arcwise connected/ pathwise connected) if for every $\mathbf{x}, \mathbf{y} \in S$, there exists a **continuous** function $\gamma : [0, 1] \rightarrow S$ such that

$$\gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y} \quad (1.5.1)$$

Example 1.5.1. Convex sets are path-connected, a path can be constructed using the convex combination,

$$\gamma(t) := (1 - t)\mathbf{x} + t\mathbf{y} \quad (1.5.2)$$

Proposition 1.5.1. Let $S_1, S_2 \subset \mathbb{R}^n$ be two path-connected sets, and $S_1 \cap S_2 \neq \emptyset$. Then $S_1 \cup S_2$ is path-connected.

Proof. Take $\mathbf{z} \in S_1 \cap S_2$, and let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two connecting paths between \mathbf{x}, \mathbf{z} and \mathbf{z}, \mathbf{y} respectively. Then define $\gamma : [0, 1] \rightarrow S_1 \cup S_2$ as

$$\gamma(t) := \mathbb{1}\{t \in [0, \frac{1}{2}]\} \times \tilde{\gamma}_1(2t) + \mathbb{1}\{t \in [\frac{1}{2}, 1]\} \times \tilde{\gamma}_2(2(t - \frac{1}{2})) \quad (1.5.3)$$

■

Theorem 1.5.1. Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^k$ be a **continuous** function, and $S \subset \mathbb{R}^m$ be a **path-connected** set. Then, $\mathbf{f}(S)$ is path-connected.

Proof Idea. Take the composite $\mathbf{f} \circ \gamma$. ■

Theorem 1.5.2 (the Intermediate Value Theorem). Assume that S is a **path-connected** subset of \mathbb{R}^n and that $f : S \rightarrow \mathbb{R}$ is **continuous**. Let $\mathbf{a}, \mathbf{b} \in S$. Then for every $t \in (\min\{f(\mathbf{a}), f(\mathbf{b})\}, \max\{f(\mathbf{a}), f(\mathbf{b})\})$, there exists $\mathbf{c} \in S$ such that $f(\mathbf{c}) = t$.

Proof. Let $\mathbf{a}, \mathbf{b} \in S$. WLOG, assume $f(\mathbf{a}) < f(\mathbf{b})$. Let t be an arbitrary value in $(f(\mathbf{a}), f(\mathbf{b}))$. Since S is path-connected, let $\vec{\varphi} : [0, 1] \rightarrow S$ be a continuous function such that $\vec{\varphi}(0) = \mathbf{a}$ and $\vec{\varphi}(1) = \mathbf{b}$.

Then we can construct composite $f \circ \vec{\varphi} : [0, 1] \rightarrow \mathbb{R}$, then apply the Intermediate Value Theorem in \mathbb{R} . We can conclude that $\exists \eta \in (0, 1)$ s.t. $f \circ \vec{\varphi}(\eta) = t$. And $\vec{\varphi}(\eta) \in S$ is the point desired. ■

2 Differentiation and related topics

2.1 Differentiation of Real-Valued Functions

2.1.1 Single Variable Case

Definition 2.1.1 (Equivalent Definitions of Differentiability). Let $S \subset \mathbb{R}$ open, and $f : S \rightarrow \mathbb{R}$ is said to be **differentiable at** $x \in S$ if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ exists} \quad (2.1.1)$$

or there exists $m \in \mathbb{R}$ such that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - mh}{h} = 0 \quad (2.1.2)$$

or there exists $m \in \mathbb{R}$ and $E(h) : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+h) = f(x) + mh + E(h), \quad \lim_{h \rightarrow 0} \frac{E(h)}{h} = 0 \quad (2.1.3)$$

If f is differentiable at x , we define the **derivative** $f'(x) := m$.

2.1.2 Differentiability of Real-valued Functions Defined on \mathbb{R}^n

Definition 2.1.2. Let S be an open subset of \mathbb{R}^n , and $f : S \rightarrow \mathbb{R}$ is **differentiable at** $\mathbf{x} \in S$ if

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\|\mathbf{h}\|} \text{ exists} \quad (2.1.4)$$

or there exists $\mathbf{m} \in M_{1 \times n}$ such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \mathbf{m} \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0 \quad (2.1.5)$$

or there exists $\mathbf{m} \in M_{1 \times n}$ and $E(\mathbf{h})$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{m} \cdot \mathbf{h} + E(\mathbf{h}), \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{E(\mathbf{h})}{\|\mathbf{h}\|} = 0 \quad (2.1.6)$$

If f is differentiable at \mathbf{x} , we define its gradient as $\nabla f(\mathbf{x}) := \mathbf{m}$.

Theorem 2.1.1. Assume that $f : S \rightarrow \mathbb{R}$, where S is an open subset of \mathbb{R}^n , and that $\mathbf{x} \in S$. If f is differentiable at \mathbf{x} , then f is continuous at \mathbf{x} .

Proof.

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{m} \cdot \mathbf{h} + E(\mathbf{h}) \quad (2.1.7)$$

Note that when $\|\mathbf{h}\| \leq 1$,

$$E(\mathbf{h}) \leq \frac{|E(\mathbf{h})|}{\|\mathbf{h}\|} \quad (2.1.8)$$

By the *Squeeze Theorem*, $\lim_{\mathbf{h} \rightarrow \mathbf{0}} E(\mathbf{h}) = 0$. Also, $\lim_{\mathbf{h} \rightarrow \mathbf{0}} \mathbf{m} \cdot \mathbf{h} = 0$. Thus

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = 0 \quad (2.1.9)$$

so f is continuous at \mathbf{x} . ■

2.1.3 Partial Differentiability

Definition 2.1.3. Let S be an open subset of \mathbb{R}^n , and $f : S \rightarrow \mathbb{R}$. The j -th partial derivative of f at \mathbf{x} is defined as

$$\frac{\partial f(\mathbf{x})}{\partial x_j} := \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \quad (2.1.10)$$

Theorem 2.1.2. Let f be a function $S \rightarrow \mathbb{R}$, where S is an open subset of \mathbb{R}^n . If f is differentiable at a point $\mathbf{x} \in S$, then (i) $\frac{\partial f}{\partial x_j}$ exists at \mathbf{x} for every $j \in [n]$ and (ii)

$$\nabla f(\mathbf{x}) := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)(\mathbf{x}) \quad (2.1.11)$$

Theorem 2.1.3. Assume f is a function $S \rightarrow \mathbb{R}$ for some open $S \subset \mathbb{R}^n$. If all partial derivatives of f exist and are continuous at every point of S , then f is differentiable in S .

Definition 2.1.4. A function $f : S \rightarrow \mathbb{R}$ is said to be of class C^1 if all partial derivatives of f exist and are continuous at every point of S .

2.1.4 Directional Derivatives

Definition 2.1.5. A direction in \mathbb{R}^n is represented by a unit vector \mathbf{u} . And given such a unit vector, the directional derivative of f at \mathbf{x} in the direction of \mathbf{u} is defined as

$$\partial_{\mathbf{u}} f(\mathbf{x}) := \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \quad (2.1.12)$$

Theorem 2.1.4. If f is differentiable at a point \mathbf{x} , then $\partial_{\mathbf{u}} f(\mathbf{x})$ exists for every unit vector \mathbf{u} , and moreover

$$\partial_{\mathbf{u}} f(\mathbf{x}) = \mathbf{u} \cdot \nabla f(\mathbf{x}) \quad (2.1.13)$$

2.2 Differentiation

Definition 2.2.1. Assume S is an open subset of \mathbb{R}^n . Given function $\mathbf{f} : S \rightarrow \mathbb{R}^m$, we say that \mathbf{f} is differentiable at a point $\mathbf{a} \in S$ if there exists $M \in M_{m \times n}$ such that

$$\mathbf{f}(\mathbf{a} + \mathbf{h}) - \mathbf{f}(\mathbf{a}) = M\mathbf{h} + \mathbf{E}(\mathbf{h}), \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\mathbf{E}(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0} \in \mathbb{R}^m \quad (2.2.1)$$

If such M exists, we define the **Jacobian matrix** of \mathbf{f} at \mathbf{a} as

$$D\mathbf{f}(\mathbf{a}) := M \quad (2.2.2)$$

Definition 2.2.2. Given a differentiable function $f : S \rightarrow \mathbb{R}$, where S is an open subset of \mathbb{R}^n , at a point \mathbf{a} we define the **differential of f at \mathbf{a}** as

$$df|_{\mathbf{a}}(\mathbf{h}) := \nabla f(\mathbf{a}) \cdot \mathbf{h} \quad (2.2.3)$$

Remark 2.2.1. The notion of differential applies to real-valued functions only (in this textbook).

Remark 2.2.2 (First Order Taylor Polynomial). The differential can be used for linear approximations for small \mathbf{h} .

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + df|_{\mathbf{a}}(\mathbf{h}) \quad (2.2.4)$$

2.3 the Chain Rule

Theorem 2.3.1 (the Chain Rule). Let $S_n \subset \mathbb{R}^n$ and $T_m \subset \mathbb{R}^m$, given functions $\mathbf{g} : S_n \rightarrow \mathbb{R}^m$ and $\mathbf{f} : T_m \rightarrow \mathbb{R}^\ell$. Also let $\mathbf{a} \in S_n$ such that \mathbf{g} is differentiable at \mathbf{a} and \mathbf{f} is differentiable at $\mathbf{g}(\mathbf{a})^1$. Then

$$\underbrace{D(\mathbf{f} \circ \mathbf{g})(\mathbf{a})}_{\ell \times n} = \underbrace{D(\mathbf{f})(\mathbf{g}(\mathbf{a}))}_{\ell \times m} \underbrace{D\mathbf{g}(\mathbf{a})}_{m \times n} \quad (2.3.1)$$

Example 2.3.1.

$$\frac{d}{d\mathbf{x}} \|\mathbf{x}\| = \frac{\mathbf{x}}{\|\mathbf{x}\|} \quad (2.3.2)$$

Definition 2.3.1. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **homogeneous of degree α** if

$$f(\lambda \mathbf{x}) = \lambda^\alpha f(\mathbf{x}) \quad \forall \mathbf{x} \neq \mathbf{0}, \lambda \in \mathbb{R}_{++} \quad (2.3.3)$$

Theorem 2.3.2 (the Euler's Theorem of Homogeneous Functions). If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a homogeneous equation of degree α , then

$$\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x}) \quad (2.3.4)$$

Proof.

$$\begin{cases} \frac{\partial f(\lambda \mathbf{x})}{\partial \lambda} = \nabla f(\lambda \mathbf{x}) \cdot \mathbf{x} \\ \frac{\partial f(\lambda \mathbf{x})}{\partial \lambda} = \frac{\partial \lambda^\alpha f(\mathbf{x})}{\partial \lambda} = \alpha \lambda^{\alpha-1} f(\mathbf{x}) \end{cases} \quad (2.3.5)$$

$$\implies \nabla f(\lambda \mathbf{x}) \cdot \mathbf{x} = \alpha \lambda^{\alpha-1} f(\mathbf{x}) \quad (2.3.6)$$

$$\implies \nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x}) \text{ evaluated at } \lambda = 1 \quad (2.3.7)$$

■

¹Also all functions \mathbf{f} and \mathbf{g} and $\mathbf{f} \circ \mathbf{g}$ are well-defined near \mathbf{a} and $\mathbf{g}(\mathbf{a})$.

Definition 2.3.2. Let C be the level set of $f : S \rightarrow \mathbb{R}$ at $\mathbf{a} \in S$ defined as

$$C := \{\mathbf{x} \in S : f(\mathbf{x}) = f(\mathbf{a})\} \quad (2.3.8)$$

and a vector \mathbf{v} is **tangent to C at \mathbf{a}** if there exists a function $\gamma : I \rightarrow C$ defined on interval I containing 0, such that

$$\gamma(0) = \mathbf{a} \quad (2.3.9)$$

and

$$\mathbf{v} = \gamma'(0) \quad (2.3.10)$$

Theorem 2.3.3. Let $S \subset \mathbb{R}^n$ be an open set, and $f : S \rightarrow \mathbb{R}$ is differentiable at \mathbf{a} . Then $\nabla f(\mathbf{a})$ is orthogonal to the level set of f passes through \mathbf{a} .

Proof Idea. Let \mathbf{v} be an arbitrary tangent vector to C at \mathbf{a} , there must exists a function $\gamma : I \rightarrow C$. And define

$$h(t) := f \circ \gamma(t) \quad (2.3.11)$$

by definition of γ , $h(I) = \{f(\mathbf{a})\}$. Thus

$$\frac{d}{dt}h(t) = \frac{d}{dt}f \circ \gamma(t) \text{ evaluated at } t = 0 \quad (2.3.12)$$

$$= \nabla f(\gamma(0)) \cdot \gamma'(0) \quad (2.3.13)$$

$$= \nabla f(\mathbf{a}) \cdot \gamma'(0) \quad (2.3.14)$$

$$= \nabla f(\mathbf{a}) \cdot \mathbf{v} = 0 \quad (2.3.15)$$

So $\nabla f(\mathbf{a})$ is orthogonal to any tangent vector of C at \mathbf{a} , which means $\nabla f(\mathbf{a})$ is orthogonal to C . ■

2.4 the Mean Value Theorem

Theorem 2.4.1 (the Mean Value Theorem). Assume $f : S \rightarrow \mathbb{R}$, where S is a **convex** and **open** subset of \mathbb{R}^n , of class C^1 , then

$$\forall \mathbf{a}, \mathbf{b} \in S, \exists \lambda \in [0, 1] \text{ s.t. } \mathbf{c} = \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \quad (2.4.1)$$

$$\nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = f(\mathbf{b}) - f(\mathbf{a}) \quad (2.4.2)$$

Proof Idea. Define $\gamma(t) := t\mathbf{a} + (1 - t)\mathbf{b}$. Construct $h : [0, 1] \rightarrow \mathbb{R}$ defined as $h := f(\gamma(t))$ then apply one dimensional mean value theorem on h . ■

Definition 2.4.1. A set $S \subset \mathbb{R}^n$ is **convex** if

$$\forall \mathbf{a}, \mathbf{b} \in S, \lambda \in [0, 1], \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \in S \quad (2.4.3)$$

Theorem 2.4.2. Assume that S is an open and convex subset of \mathbb{R}^n and that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable in S such that

$$\|\nabla f(\mathbf{x})\| \leq M \quad \forall \mathbf{x} \in S \quad (2.4.4)$$

then for every $\mathbf{a}, \mathbf{b} \in S$,

$$|f(\mathbf{b}) - f(\mathbf{a})| \leq M \|\mathbf{b} - \mathbf{a}\| \quad (2.4.5)$$

Proof Idea. Use *Cauchy's Inequality*. ■

Theorem 2.4.3. Assume that S is an open and convex subset of \mathbb{R}^n , and $f : S \rightarrow \mathbb{R}$ is a function differentiable on S . If $\nabla f(\mathbf{x}) = \mathbf{0} \ \forall \mathbf{x} \in S$, then f is constant on S .

Proof Idea. Take two arbitrary $\mathbf{a}, \mathbf{b} \in S$, then use mean value theorem to show $f(\mathbf{a}) = f(\mathbf{b})$. ■

Theorem 2.4.4 (Generalization of the Previous Theorem). Assume that S is an open and path-connected subset of \mathbb{R}^n , and $f : S \rightarrow \mathbb{R}$ is a function differentiable on S . If $\nabla f(\mathbf{x}) = \mathbf{0} \ \forall \mathbf{x} \in S$, then f is constant on S .

Proof. Any path-connected set can be written as a countable union of convex sets $S = \cup_{i \in \mathcal{A}} C_i$ such that

$$\forall \alpha \subset \mathcal{A} \text{ s.t. } \alpha \neq \emptyset, \cup_{i \in \alpha} C_i \cap \cup_{i \in \alpha^c} C_i \neq \emptyset \quad (2.4.6)$$

then apply the previous theorem. ■

2.5 Higher Order Derivatives

Definition 2.5.1. A function f defined on S is **of class C^k** if all of its k^{th} order partial derivatives exists and continuous everywhere in S .

Theorem 2.5.1. Assume that S is an open subset of \mathbb{R}^n and that $f : S \rightarrow \mathbb{R}$ is C^k . Let $\alpha \in [n]^k$, and let β be any permutation of α ,

$$\partial^\alpha f = \partial^\beta f \quad (2.5.1)$$

Definition 2.5.2. A **multi-index** is an n -tuple of nonnegative integers. And we define

$$\partial^\alpha f := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f, \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n) \quad (2.5.2)$$

where the **order** of multi-index is defined as

$$|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n \quad (2.5.3)$$

Theorem 2.5.2 (the Multinomial Theorem).

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{x}^\alpha \quad (2.5.4)$$

Proof Idea. Prove by induction on n , with Binomial Theorem. ■

Definition 2.5.3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 function, then its **Hessian matrix** is defined as

$$H_f := \begin{pmatrix} \partial_1 \partial_1 f & \cdots & \partial_n \partial_1 f \\ \vdots & \ddots & \vdots \\ \partial_1 \partial_n f & \cdots & \partial_n \partial_n f \end{pmatrix} \quad (2.5.5)$$

2.6 Taylor's Theorem

Definition 2.6.1. Let $f : I \rightarrow \mathbb{R}$, where I is an open subset of \mathbb{R} , be C^k . Let $a \in I$. then the k^{th} **order Taylor polynomial of f at a** is the unique polynomial of order at most k , denoted $P_{a,k}(h)$ such that

$$f^{(j)}(a) = P_{a,k}^{(j)}(0) \quad \forall j \in \{0, 1, \dots, k\} \quad (2.6.1)$$

Note

$$P_{a,k}^{(j)}(h) = \sum_{j=0}^k \frac{h^j}{j!} f^{(j)}(a) \quad (2.6.2)$$

Theorem 2.6.1 (Taylor's Theorem in 1 Dimension). Assume that $I \subset \mathbb{R}$ is an open interval and that $f : I \rightarrow \mathbb{R}$ is a function of class C^k on I . For $a \in I$ and $h \in \mathbb{R}$ such that $a + h \in I$. Define the **remainder**

$$R_{a,k}(h) := f(a + h) - P_{a,k}(h) \quad (2.6.3)$$

Then

$$\lim_{h \rightarrow 0} \frac{R_{a,k}(h)}{h^{\underline{k}}} = 0 \quad (2.6.4)$$

Proposition 2.6.1. Assume that $I \subset \mathbb{R}$ is an open interval and that $f : I \rightarrow \mathbb{R}$ is a function of class C^k on I . For $a \in I$ and $h \in \mathbb{R}$ such that $a + h \in I$, there exists some $\theta \in (0, 1)$ such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \dots + \frac{h^{k-1}}{(k-1)!}f^{(k-1)}(a) + \frac{h^k}{k!}f^{(k)}(a + \theta h) \quad (2.6.5)$$

Definition 2.6.2. Assume that $S \subset \mathbb{R}^n$ is an open interval and that $f : S \rightarrow \mathbb{R}$ is a function of class C^k on S . For a point $\mathbf{a} \in S$, the k^{th} **order Taylor polynomial of $f : S \rightarrow \mathbb{R}$ is a polynomial of order at most k** , denoted $P_{\mathbf{a},k}(\mathbf{h})$ satisfying

$$f(\mathbf{a}) = P_{\mathbf{a},k}(\mathbf{0}) \quad (2.6.6)$$

$$\partial^\alpha f(\mathbf{a}) = \partial^\alpha P_{\mathbf{a},k}(\mathbf{0}) \quad \forall \alpha \text{ s.t. } |\alpha| \leq k \quad (2.6.7)$$

Theorem 2.6.2 (Taylor's Theorem in n Dimensions). Assume that $S \subset \mathbb{R}^n$ is an open set and that $f : S \rightarrow \mathbb{R}$ is a function of class C^k on S . For $\mathbf{a} \in S$ and $\mathbf{h} \in \mathbb{R}^n$ such that $\mathbf{a} + \mathbf{h} \in S$. Define the **remainder**

$$R_{\mathbf{a},k}(\mathbf{h}) := f(\mathbf{a} + \mathbf{h}) - P_{\mathbf{a},k}(\mathbf{h}) \quad (2.6.8)$$

Then

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{R_{\mathbf{a},k}(\mathbf{h})}{\|\mathbf{h}\|^{\underline{k}}} = 0 \quad (2.6.9)$$

Theorem 2.6.3 (the Quadratic Case).

$$P_{\mathbf{a},2}(\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} \mathbf{h}^T H_f(\mathbf{a}) \mathbf{h} \quad (2.6.10)$$

$$\exists \theta \in (0, 1) \text{ s.t. } f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} \mathbf{h}^T H_f(\mathbf{a} + \theta \mathbf{h}) \mathbf{h} \quad (2.6.11)$$

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{R_{\mathbf{a},2}(\mathbf{h})}{\|\mathbf{h}\|^2} = 0 \quad (2.6.12)$$

Definition 2.6.3 (the General Taylor's Polynomial).

$$P_{\mathbf{a},k}(\mathbf{h}) = \sum_{\{\alpha: |\alpha| \leq k\}} \frac{\mathbf{h}^\alpha}{\alpha!} \partial^\alpha f(\mathbf{a}) \quad (2.6.13)$$

2.7 Critical Points

Definition 2.7.1. A **symmetric** $n \times n$ matrix A is said to be

- **Positive definite** if $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.
- **Non-negative definite/Positive semi-definite** if $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
- **Negative definite** if $\mathbf{x}^T A \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.
- **Non-positive definite/Negative semi-definite** if $\mathbf{x}^T A \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

and **indefinite** otherwise.

Theorem 2.7.1. Assume A is a symmetric matrix. Then

$$\begin{aligned} A \text{ is positive definite} &\iff \text{all its eigenvalues are positive} \\ &\iff \exists \lambda_i > 0 \text{ such that } \mathbf{x}^T A \mathbf{x} \geq \lambda_i \|\mathbf{x}\|^2 \text{ for all } \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

and

$$A \text{ is nonnegative definite} \iff \text{all its eigenvalues are nonnegative.} \quad (2.7.1)$$

and

$$A \text{ is indefinite} \iff A \text{ has both positive and negative eigenvalues.} \quad (2.7.2)$$

Lemma 2.7.1. Let A be a symmetric matrix, then

$$\text{the smallest eigenvalue of } A = \min_{\{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\|=1\}} \mathbf{u}^T A \mathbf{u} \quad (2.7.3)$$

Definition 2.7.2. A point $\mathbf{a} \in S$ is a **local minimum point** for $f : S \rightarrow \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ s.t. } \forall \mathbf{x} \in \mathcal{B}(\varepsilon, \mathbf{a}) \quad f(\mathbf{a}) \leq f(\mathbf{x}) \quad (2.7.4)$$

Definition 2.7.3. A point $\mathbf{a} \in S$ is a **local maximum point** for $f : S \rightarrow \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ s.t. } \forall \mathbf{x} \in \mathcal{B}(\varepsilon, \mathbf{a}) \quad f(\mathbf{a}) \geq f(\mathbf{x}) \quad (2.7.5)$$

Definition 2.7.4. Let $f : S \rightarrow \mathbb{R}$ is differentiable on the open subset $S \subset \mathbb{R}^n$, then a point $\mathbf{a} \in S$ is a **critical point** if

$$\nabla f(\mathbf{a}) = \mathbf{0} \quad (2.7.6)$$

Definition 2.7.5. Let $\mathbf{a} \in S$ be a critical point of f , then \mathbf{a} is a **saddle point** if $H_f(\mathbf{a})$ is indefinite.

Theorem 2.7.2 (First Derivative Test). If $f : S \rightarrow \mathbb{R}$ is differentiable, then

$$\text{local extremum} \implies \text{critical point} \quad (2.7.7)$$

Theorem 2.7.3 (**Necessary Condition for a Local Minimum**). If $f : S \rightarrow \mathbb{R}$ is C^2 and \mathbf{a} is a local minimum point for f , then

- \mathbf{a} is critical point of f ;
- $H_f(\mathbf{a})$ is positive semi-definite.

Theorem 2.7.4 (Sufficient Condition for a Local Minimum). If

- (i) \mathbf{a} is a critical point of f ;
- (ii) $H_f(\mathbf{a})$ is positive definite.

Then \mathbf{a} is a local minimum for f .

Corollary 2.7.1. Assume f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$, then

- (i) If $H_f(\mathbf{a})$ is positive definite, then \mathbf{a} is a local minimum;
- (ii) If $H_f(\mathbf{a})$ is negative definite, then \mathbf{a} is a local maximum;
- (iii) If $H_f(\mathbf{a})$ is indefinite, then \mathbf{a} is a saddle point.

If none of the above holds (semi-definiteness), then we cannot determine the character of the critical point without further thought.

Definition 2.7.6. A critical point \mathbf{a} of f is **degenerate** if $\det H_f(\mathbf{a}) = 0$, and **non-degenerate** if $\det H_f(\mathbf{a}) \neq 0$.

2.8 Optimization

Theorem 2.8.1. Let $S \subset \mathbb{R}^n$ be an open set and $f, g : S \rightarrow \mathbb{R}$ be C^1 functions. If \mathbf{x} is a *local extremal* satisfying $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{x}) \neq \mathbf{0}$, then

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \quad (2.8.1)$$

Proposition 2.8.1. Equations (2.8.1) $\implies \nabla f(\mathbf{x}) \perp \{\mathbf{x} \in S : g(\mathbf{x}) = 0\}$ at \mathbf{x} .

Theorem 2.8.2. Let $S \subseteq \mathbb{R}^n$ be an open set, and $f, \{g_i\}_{i=1}^k : S \rightarrow \mathbb{R}$ be C^1 functions. Define $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k := (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_k(\mathbf{x}))$.

If $\mathbf{x} \in S$ is a *local extremal* of f such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, and $\{\nabla g_i(\mathbf{x})\}$ are linearly independent (i.e. $\text{rank}(D\mathbf{g}(\mathbf{x})) = k$), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \quad (2.8.2)$$

Remark 2.8.1. Procedure of optimization on *open sets*:

- (i) Find all critical points;
- (ii) Find optimizers (classify them) among critical points.

Remark 2.8.2. Procedure of optimization with *inequality constraints*:

- (i) Find critical points *in* the constraint set from global critical points;
- (ii) Find critical points *on* the boundary of constraint set;
- (iii) Find optimizers among candidates.

3 The Implicit and Inverse Function Theorems

3.1 The Implicit Function Theorem I

Theorem 3.1.1 (Implicit Function Theorem). Let $S \subseteq \mathbb{R}^{n+k}$ be an open set, and function $\mathbf{F} : S \rightarrow \mathbb{R}^k$ be a C^1 function. Suppose there exists point $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$ such that

$$\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad (3.1.1)$$

If

$$\det(D_{\mathbf{y}}(\mathbf{F}(\mathbf{a}, \mathbf{b}))) \neq 0 \quad (3.1.2)$$

then there exists $r_0, r_1 > 0$ and a C^1 function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}), \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge \mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \quad (3.1.3)$$

and define $\mathbf{y} := \mathbf{f}(\mathbf{x})$, the derivative of \mathbf{f} can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{x}, \mathbf{y})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{y}) \quad (3.1.4)$$

Remark 3.1.1. Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (3.1.5)$$

near (\mathbf{a}, \mathbf{b}) .

(i) Verify $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

(ii) Assert

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.6)$$

(iii) Approximate solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ using

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \quad (3.1.7)$$

$$= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b})\mathbf{h} \quad (3.1.8)$$

3.2 Geometric Content of the Implicit Function Theorem

Definition 3.2.1. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in S$. S is **singular** at \mathbf{a} if

$$\forall r > 0 \ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \quad (3.2.1)$$

S is **regular** at \mathbf{a} if it is not singular there.

Theorem 3.2.1 (k dimensional manifold as level set). Let $U \subseteq \mathbb{R}^n$ and let $\mathbf{F} : U \rightarrow \mathbb{R}^{n-k}$ be a C^1 function.

$$S := \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{0}\} \quad (3.2.2)$$

Let $\mathbf{a} \in U$, if

$$\text{rank}(D\mathbf{F}(\mathbf{a})) = n - k \quad (3.2.3)$$

then $\exists r > 0$ such that the *level set of \mathbf{F} near \mathbf{a}*

$$\mathcal{B}(r, \mathbf{a}) \cap S \quad (3.2.4)$$

can be represented as a C^1 graph.

Remark 3.2.1 (Interpretation). In \mathbb{R}^n , there are n degree of freedom before imposing any constraint, after adding constraints $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ with $n - k$ effective constraints, there are k degree of freedom left. What's left is a k dimensional manifold.

Theorem 3.2.2 (k dimensional manifold as parameterization). Let $T \subseteq \mathbb{R}^k$ and let $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be a C^1 function.

$$S := \mathbf{f}(T) \quad (3.2.5)$$

Let $\mathbf{t} \in T$, if

$$\text{rank}(D\mathbf{f}(\mathbf{t})) = k \quad (3.2.6)$$

then $\exists r > 0$ such that the parameterization of \mathbf{f} near \mathbf{t}

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \quad (3.2.7)$$

can be represented as a C^1 graph.

3.3 Transformations, and the Inverse Function Theorem

Example 3.3.1 (Polar coordinate in \mathbb{R}^2). Let

$$U := \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \quad (3.3.1)$$

$$V := \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \quad (3.3.2)$$

Define $\mathbf{f} : U \rightarrow V$ as

$$\mathbf{f}(r, \theta) := \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} \quad (3.3.3)$$

Example 3.3.2 (Spherical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix} \quad (3.3.4)$$

Example 3.3.3 (Cylindrical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{pmatrix} \quad (3.3.5)$$

Theorem 3.3.1 (Inverse Function Theorem). Let U and V be open subsets in \mathbb{R}^n , and $\mathbf{f} : U \rightarrow V$. Let $\mathbf{a} \in U$ and define $\mathbf{b} := \mathbf{f}(\mathbf{a}) \in V$. If

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0 \quad (3.3.6)$$

then there exists $M \subseteq U$ and $N \subseteq V$ such that

- (i) $\mathbf{a} \in M$ and $\mathbf{b} \in N$,
- (ii) \mathbf{f} is bijective between M and N ,
- (iii) $\mathbf{f}^{-1} : N \rightarrow M$ is C^1 ,

and for all $\mathbf{x} \in M$ such $\mathbf{y} := \mathbf{f}(\mathbf{x}) \in N$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \quad (3.3.7)$$

4 Integration

4.1 Basics

Theorem 4.1.1 (Properties of infimum and supremum). Let $A \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, and $f, g : A \rightarrow \mathbb{R}$ are bounded functions. Let m and M denote the infimum and supremum respectively, then

- (i) $m_A f + m_A g \leq m_A(f + g) \leq M_A(f + g) \leq M_A f + M_A g$
- (ii) If $A' \subseteq A$, then $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$
- (iii) If $f(\mathbf{x}) \leq g(\mathbf{x}) \forall \mathbf{x} \in A$, then $m_A f \leq m_A g$ and $M_A f \leq M_A g$
- (iv) $|M_A f| \leq M_A |f|$
- (v) $M_A |f| - m_A |f| \leq M_A f - m_A f$
- (vi) $\forall c \in \mathbb{R}, M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$
- (vii) $M_A f - m_A f = \sup\{f(x) - f(y) : x, y \in A\}$

4.2 Integration on Higher Dimensions

Definition 4.2.1. A rectangle $\mathcal{R} \subseteq \mathbb{R}^n$ (n -orthotope) is defined as

$$\mathcal{R} := \prod_{i=1}^n [a_i, b_i] \quad (4.2.1)$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.

Definition 4.2.2. A **partition** P of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ is a list of n **finite** and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \quad (4.2.2)$$

where $L_i = \{e_j\}_{j=0}^{T_i}$ such that

$$a_i = e_0 < e_1 < \dots < e_{T_i} = b_i \quad (4.2.3)$$

and such partition generates a set of rectangles(boxes) $\mathcal{B}(P) := \{B_j\}_{j=1}^J \subseteq \mathcal{R}$.

Definition 4.2.3. Let P and P' be two partitions of \mathcal{R} . Then P' is a **refinement** of P if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee (B_j'^{\text{int}} \cap B_j^{\text{int}} = \emptyset) \quad (4.2.4)$$

Definition 4.2.4. Define the **volume** of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ as

$$V^n(\mathcal{R}) := \prod_{i=1}^n (b_i - a_i) \quad (4.2.5)$$

Definition 4.2.5. The **lower Riemann sum** of f with partition P on \mathcal{R} is defined as

$$L_P f := \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.6)$$

and the **upper Riemann sum** is defined as

$$U_P f := \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.7)$$

Definition 4.2.6. The **upper integral** and **lower integral** of f on \mathcal{R} are defined as

$$\bar{I}_{\mathcal{R}}f := \inf_P U_P f \quad (4.2.8)$$

$$\underline{I}_{\mathcal{R}}f := \sup_P L_P f \quad (4.2.9)$$

Definition 4.2.7. A bounded real-valued function f defined on \mathcal{R} is **integrable** if

$$\underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.10)$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \, dV^n := \underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.11)$$

Lemma 4.2.1. Let f be a bounded real-valued function defined on \mathcal{R} , f is integrable if and only if $\forall \epsilon > 0$, there exists a partition P of \mathcal{R} such that

$$U_P f - L_P f < \epsilon \quad (4.2.12)$$

Theorem 4.2.1. Let f and g be two integrable functions on $\mathcal{R} \subseteq \mathbb{R}^n$, let $c \in \mathbb{R}$,

- (i) $f + g : \mathcal{R} \rightarrow \mathbb{R}$ is integrable and $\int_{\mathcal{R}}(f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$
- (ii) $c \cdot f$ is integrable and $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$
- (iii) $f(\mathbf{x}) \geq g(\mathbf{x}) \, \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$
- (iv) $|f|$ is integrable and $|\int_{\mathcal{R}} f| \leq \int_{\mathcal{R}} |f|$

Definition 4.2.8. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S , the **indicator function** of S is $\chi_S : \mathcal{R} \rightarrow \{0, 1\}$, defined as

$$\chi_S(\mathbf{x}) := \mathbf{1}\{\mathbf{x} \in S\} \quad (4.2.13)$$

Definition 4.2.9. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S . Let $f : \mathcal{R} \rightarrow \mathbb{R}$ be a bounded function, then f is **integrable on S** if $\chi_S f$ is integrable on \mathcal{R} . And

$$\int \cdots \int_S f \, dV^n := \int \cdots \int_{\mathcal{R}} \chi_S f \, dV^n \quad (4.2.14)$$

Definition 4.2.10. Let $Z \subseteq \mathbb{R}^n$, Z has **zero content** if for all $\epsilon > 0$, there exists a finite set of rectangles $\{R_\ell\}_{\ell=1}^L$ covers Z and

$$\sum_{\ell=1}^L V^n(R_\ell) < \epsilon \quad (4.2.15)$$

Definition 4.2.11. A set $S \subset \mathbb{R}^n$ is **Jordan measurable** if it is bounded and its boundary has zero content.

Proposition 4.2.1. Let $Z \subseteq \mathbb{R}^n$ has zero content, then

- (i) For any $Z' \subseteq Z$, Z' has zero content.
- (ii) Finite union of content zero sets has zero content.
- (iii) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function, its graph $\{(x, f(x)) : x \in [a, b]\}$ has zero content.

(iv) Let $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^2$ be a C^1 function, the parameterization $\mathbf{f}([a, b])$ has zero content.

Theorem 4.2.2. Let \mathcal{R} be a rectangle in \mathbb{R}^n and f is integrable on \mathcal{R} if the set

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \quad (4.2.16)$$

has zero content.

Proposition 4.2.2 (Folland 4.22). Suppose $Z \subseteq \mathbb{R}^n$ has zero content. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f \, dV^n = 0$.

Remark 4.2.1 (Measures and Contents). A measure μ is required to be *countably* additivity on disjoint sets, but a content only requires *finite* additivity on disjoint sets. Therefore, if a set has content zero based on some content function, such a content function is also a measure function, thus such set has measure zero as well.

$$\text{content zero} \implies \text{measure zero} \quad (4.2.17)$$

4.3 Iterated Integrals

Theorem 4.3.1 (Fubini's Theorem). Let $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \mathcal{R} \rightarrow \mathbb{R}$ is bounded. Assuming that

- (i) f is integrable on \mathcal{R} .
- (ii) for each $y \in [c, d]$, the function $f_y(x) := f(x, y)$ is integrable on $[a, b]$.
- (iii) Define $g(y) := \int_a^b f(x, y) dy$ is integrable on $[c, d]$.

Then

$$\iint_{\mathcal{R}} f \, dA = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy \quad (4.3.1)$$

Proposition 4.3.1. Let $S \subseteq \mathbb{R}^n$ be an unbounded set, and $f : S \rightarrow \mathbb{R}$. Then improper integral $\int \cdots \int_S f \, d^n \mathbf{x}$ is absolutely convergent on \mathbb{R}^n if and only if $\int \cdots \int_{\mathbb{R}^n} \chi_S f \, d^n \mathbf{x}$ is absolutely convergent.

4.4 Change of Variables

Theorem 4.4.1 (Change of Variable). Let U and V be two open subsets of \mathbb{R}^n , and let $\mathbf{G} : U \rightarrow V$ be a C^1 bijection. Let $T \subset U$ and $S \subset V$. Suppose $\mathbf{G}(T) = S$, then

$$\underbrace{\int \cdots \int_S f \, d\Omega}_{\text{Target Integral}} = \int \cdots \int_{T \equiv \mathbf{G}^{-1}(S)} f \circ \mathbf{G} \, |\det D\mathbf{G}| \, d\Theta \quad (4.4.1)$$

Corollary 4.4.1. Let S be a region in \mathbb{R}^n , suppose S can be parameterized by $\mathbf{G} : T \rightarrow S$. By the change of variable formula, consider the special case $f(\mathbf{x}) = 1$,

$$|S| = \int \cdots \int_S 1 \, d\Omega = \int \cdots \int_T 1 \, |\det D\mathbf{G}(\mathbf{u})| \, d\Theta \quad (4.4.2)$$

Example 4.4.1 (Polar Coordinate). Define the coordinate transformation mapping from polar to Cartesian,

$$\mathbf{P}(r, \theta) := (x, y) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad (4.4.3)$$

and $|\det D\mathbf{P}(r, \theta)| = r$.

Example 4.4.2 (Cylindrical Coordinate). Define the coordinate transformation mapping from cylindrical to Cartesian as

$$\mathbf{C}(r, \theta, z) := (x, y, z) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad z \in \mathbb{R} \quad (4.4.4)$$

and $|\det DC(r, \theta, z)| = r$.

Example 4.4.3 (Spherical Coordinate). Define the coordinate transformation mapping from spherical to Cartesian as

$$\mathbf{S}(r, \theta, \varphi) = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} \quad (4.4.5)$$

and $|\det DS(r, \theta, \varphi)| = r^2 \sin \varphi$

4.5 Further Aspects

4.5.1 Exchanging Differentiation and Integration

Theorem 4.5.1 (**Exchanging Differentiation and Integration**). Let $f(\mathbf{x}, \mathbf{t}) : S \times T \rightarrow \mathbb{R}$ and define $F(\mathbf{x}) : S \rightarrow \mathbb{R}$ as

$$F(\mathbf{x}) := \int \cdots \int_T f(\mathbf{x}, \mathbf{t}) \, d\mathbf{t} \quad (4.5.1)$$

If

- (i) S is open and T is compact;
- (ii) f and F are continuous;
- (iii) and $\forall x_j \in \mathbf{x}, \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j}$ is continuous (differentiable w.r.t \mathbf{x}),

then F is C^1 in S and for every j ,

$$\frac{\partial F(\mathbf{x})}{\partial x_j} = \int \cdots \int_T \frac{\partial f(\mathbf{x}, \mathbf{t})}{\partial x_j} \, d\mathbf{t} \quad (4.5.2)$$

Corollary 4.5.1 (**Exchanging Limits and Integration**). By the definition of partial derivative, above theorem is equivalent to

$$\lim_{h \rightarrow 0} \int \cdots \int_T \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} \, d\mathbf{t} = \int \cdots \int_T \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j, \mathbf{t})}{h} \, d\mathbf{t} \quad (4.5.3)$$

4.5.2 Improper Integrals

Definition 4.5.1 (Unbounded Domains). An **improper integral** with unbounded domain $\int \cdots \int_{\mathbb{R}^n} f \, d\Omega$ is **absolutely convergent** if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \quad \exists R > 0 \text{ s.t. } \forall S \subseteq \mathbb{R}^n \quad \mathcal{B}(R, \mathbf{0}) \subset S \implies \left| \int \cdots \int_S f \, d\Omega - L \right| < \varepsilon \quad (4.5.4)$$

Theorem 4.5.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, and that

$$\lim_{R \rightarrow \infty} \int \cdots \int_{\mathcal{B}(R, \mathbf{0})} |f| d\Omega \text{ exists} \quad (4.5.5)$$

then $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent.

Corollary 4.5.2 (Equivalence). Above improper integral $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent if set

$$\left\{ \int \cdots \int_{\mathcal{B}(R, \mathbf{0})} |f| d\Omega : R \in \mathbb{R}_{++} \right\} \quad (4.5.6)$$

is bounded.

Corollary 4.5.3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an continuous function, if

$$\exists p > n, C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{1}{\|\mathbf{x}\|^p} \forall \mathbf{x} \in \mathbb{R}^n \quad (4.5.7)$$

then $\int \cdots \int_{\mathbb{R}^n} f d\Omega$ is absolutely convergent.

Definition 4.5.2 (Unbounded Function). Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$. Consider a function $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$. Then the improper integral $\int \cdots \int_S f d\Omega$ is absolutely convergent if

$$\exists L \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0 \exists r > 0 \text{ s.t. } \forall U \subset S \text{ s.t. } \mathbf{a} \in U^{\text{int}} \wedge U \subset \mathcal{B}(r, \mathbf{a}), \left| \int \cdots \int_{S \setminus U} f d\Omega - L \right| < \varepsilon \quad (4.5.8)$$

Theorem 4.5.3. Let $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$, if

$$\lim_{r \rightarrow 0} \int \cdots \int_{S \setminus \mathcal{B}(r, \mathbf{a})} |f| d\Omega \text{ exists} \quad (4.5.9)$$

then $\int \cdots \int_S f d\Omega$ is absolutely convergent.

Corollary 4.5.4 (Equivalence). If the set

$$\left\{ \int \cdots \int_{S \setminus \mathcal{B}(r, \mathbf{a})} |f| d\Omega : r \in \mathbb{R}_{++} \right\} \quad (4.5.10)$$

is bounded, then $\int \cdots \int_S f d\Omega$ is absolutely convergent.

Corollary 4.5.5. Let $f : S \setminus \{\mathbf{a}\} \rightarrow \mathbb{R}$, if

$$\exists p < n, C > 0 \text{ s.t. } |f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x} - \mathbf{a}\|^p} \forall \mathbf{x} \in S \setminus \{\mathbf{a}\} \quad (4.5.11)$$

then the improper integral $\int \cdots \int_S f d\Omega$ is absolutely convergent.

5 Vector Calculus

5.1 Line Integrals

5.1.1 Arc Length

Definition 5.1.1. Let C be a smooth curve in \mathbb{R}^n parameterized by C^1 function \mathbf{g} such that $\mathbf{g}'(t) \neq \mathbf{0}$ for every appropriate t .

$$C := \{\mathbf{g}(t) : t \in [a, b]\} \quad (5.1.1)$$

and the **arc length** of C is defined as

$$\int_C d^n \mathbf{x} := \int_C ds := \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.2)$$

Proposition 5.1.1. The arc length of a curve C is an intrinsic property of the geometric object C and should not depend on the particular parameterization we use.

Proof. Let $\varphi : [c, d] \rightarrow [a, b]$ be a bijection, so that $\mathbf{h} := \mathbf{g} \circ \varphi$ is also a valid parameterization of C such that

$$C := \{\mathbf{h}(u) : u \in [c, d]\} \quad (5.1.3)$$

The arc length of C can be computed using

$$\int_C ds = \int_c^d \|\mathbf{h}'(u)\| du \quad (5.1.4)$$

$$= \int_c^d \|\mathbf{g}'(\varphi(u))\| \cdot |\varphi'(u)| du \quad (5.1.5)$$

$$= \int_a^b \|\mathbf{g}'(t)\| dt \text{ by change of variable formula.} \quad (5.1.6)$$

■

Remark 5.1.1 (Interpretations). Suppose \mathbf{g} is a parameterization of C .

- (i) $\int_a^b \mathbf{g}'(t) dt = \mathbf{g}(b) - \mathbf{g}(a)$ measures the distance between two endpoints of C .
- (ii) Choosing a parameterization is effectively choosing an **orientation** for the curve C .

Definition 5.1.2. A function $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ is called **piecewise smooth** if

- (i) it's *continuous*, and
- (ii) it's derivate exists and is continuous except at finitely many points t_j , at which the one-sided limits exist.

5.1.2 Line Integrals of Scalar Functions

Definition 5.1.3. Let smooth curve $C \subseteq \mathbb{R}^n$, $f : C \rightarrow \mathbb{R}$ and \mathbf{g} be a parameterization of C , then

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \quad (5.1.7)$$

Remark 5.1.2. The line integrals of scalar functions are also independent from the choices of parameterizations.

Definition 5.1.4.

$$\text{Average of } f \text{ over } C := \frac{\int_C f ds}{\int_C ds} \quad (5.1.8)$$

5.1.3 Line Integrals of Vector Fields

Definition 5.1.5. Let smooth $C \in \mathbb{R}^n$ with parameterization \mathbf{g} and $\mathbf{F} : C \rightarrow \mathbb{R}^n$ defined on it, the **line integral** of \mathbf{F} over C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt \quad (5.1.9)$$

Proposition 5.1.2. The line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ is independent of the parameterization *as long as the orientation is unchanged*.

Theorem 5.1.1 (The Fundamental Theorem of Line Integral). Let $f : C \rightarrow \mathbb{R}$ defined on smooth curve C parameterized by $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$, then

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.10)$$

Proof.

$$\int_C \nabla f(\mathbf{x}) \cdot d^n \mathbf{x} = \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial \mathbf{g}(t)} \cdot \mathbf{g}'(t) dt \quad (5.1.11)$$

$$= \int_a^b \frac{\partial f(\mathbf{g}(t))}{\partial t} dt = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) \quad (5.1.12)$$

■

5.1.4 Rectifiable Curves

Remark 5.1.3. Let C be a curve in \mathbb{R}^n parameterized by injection $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ such that $\mathbf{g}'(t) \neq \mathbf{0}$. Let P be a partition of $[a, b]$. Denote

$$L_P(C) := \sum_j \|\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})\| \quad (5.1.13)$$

Definition 5.1.6. A curve C is **rectifiable** if the set $\{L_P(C) : P\}$ is bounded. And the arc length of C is defined as

$$L(C) := \sup\{L_P(C) : P\} \quad (5.1.14)$$

Theorem 5.1.2. The supremum found above, $L(C)$ is precisely the arc length of C :

$$L(C) = \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.15)$$

5.2 Green's Theorem

5.2.1 Preliminary Definitions

Definition 5.2.1. A **simple closed curve** is a curve with parameterization $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$ where

- (i) \mathbf{g} is continuous;
- (ii) $\mathbf{g}(a) = \mathbf{g}(b)$;
- (iii) \mathbf{g} is injective with its domain restricted to (a, b) (so the curve does not cross).

Definition 5.2.2. A *simple closed curve* is **piecewise smooth** if it has a parameterization \mathbf{g} such that

- (i) \mathbf{g} is continuously differentiable with $\mathbf{g}'(t) \neq \mathbf{0}$ except finitely many breakpoints;
- (ii) $\mathbf{g}'(t)$ is *one side continuous* at breakpoints of the curve.

Definition 5.2.3. A **regular region** $S \subseteq \mathbb{R}^n$ is a set satisfying both

- (i) S is compact;
- (ii) $\overline{S^{int}} = S$.

Definition 5.2.4. Let $S \subseteq \mathbb{R}^2$, S has **piecewise smooth boundary** if ∂S consists of one or more *disjoint, piecewise smooth, simple closed curve*.

Definition 5.2.5. Let $S \subseteq \mathbb{R}^2$, then **positive orientation** on ∂S is the orientation on each of the closed curves that make up the boundary such that the region is on the *left* with respect to the positive direction on the curve.

Theorem 5.2.1 (Green's Theorem). Suppose $S \subseteq \mathbb{R}^2$ is a regular region with piecewise smooth boundary ∂S . Suppose \mathbf{F} is a C^1 vector field defined on \overline{S} , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA \quad (5.2.1)$$

Corollary 5.2.1. Suppose S is a regular region in \mathbb{R}^2 with piecewise smooth boundary ∂S , and let $\mathbf{n}(\mathbf{x})$ be the *unit outward normal* vector to ∂S at $\mathbf{x} \in \partial S$. Suppose also that \mathbf{F} is a vector field defined on \overline{S} , then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA \quad (5.2.2)$$

Proof. Let $\mathbf{g}(t)$ be a parameterization of boundary ∂S . Then the tangent vector would be $\mathbf{g}'(t)$ and we can conclude the *outer normal vector* \mathbf{n} is $\frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|}$. Then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \int_T \mathbf{F} \circ \mathbf{g} \cdot \frac{(g'_2(t), -g'_1(t))}{\|(g'_2(t), -g'_1(t))\|} \|\mathbf{g}'(t)\| \, dt \quad (5.2.3)$$

$$= \int_T F_1 g'_2(t) - F_2 g'_1(t) \, dt \quad (5.2.4)$$

$$= \int_T \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot \begin{pmatrix} g'_1(t) \\ g'_2(t) \end{pmatrix} \, dt \quad (5.2.5)$$

$$= \int_{\partial S} \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot d^2\mathbf{x} \quad (5.2.6)$$

$$= \iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \, dA \text{ By Green's Theorem} \quad (5.2.7)$$

■

5.3 Surface Integrals

5.3.1 Surface Areas and Surface Integrals

Definition 5.3.1. Suppose S is a surface in \mathbb{R}^3 and parameterized by

$$\mathbf{G}(\mathbf{u}) : R \rightarrow S \quad (5.3.1)$$

where $\text{rank}(D\mathbf{G}(\mathbf{u})) = 2$ for every $\mathbf{u} \in R \setminus Z$ where Z is a probably empty set with zero content. If $\|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\|$ is integrable, then

$$\text{Area}(S) := \iint_R \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.2)$$

Definition 5.3.2. Let $f : S \rightarrow \mathbb{R}$ be a real-valued continuous function defined on a super set of S , the **integral of a real-valued function on a surface** is defined as

$$\iint_S f(\mathbf{x}) dA := \iint_R f(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.3)$$

Definition 5.3.3. Let $\mathbf{F} : S \rightarrow \mathbb{R}^3$ be a continuous vector field defined on a super set of S , the **integral of vector field on a surface** is defined as

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} dA := \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) d\Theta \quad (5.3.4)$$

Remark 5.3.1. Surface integrals of **real-valued functions** are independent of the choice of parametrization.

Remark 5.3.2. But the choice of parameterization can change the sign of surface integrals of **vector fields**. We need to **choose the direction of the normal, \mathbf{n}** .

Definition 5.3.4. Let $S \subseteq \mathbb{R}^3$ be a two dimensional sub-manifold, and f is a real-valued function defined on a super set of S . Define the **average of f over S** as

$$\text{aver of } f \text{ over } S := \frac{\iint_S f dA}{\iint_S 1 dA} \quad (5.3.5)$$

Remark 5.3.3. A note on the relation between integrals of a vector field and a real-valued function. The surface of vector field \mathbf{F} on S is defined by *reducing \mathbf{F} to a real-valued function $\mathbf{F} \cdot \mathbf{n}$* and then follow the definition of conventional real-valued function on S . Define $h := \mathbf{F} \cdot \mathbf{n}$,

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_S h dA \quad (5.3.6)$$

$$:= \iint_R h(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.7)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \mathbf{n}(\mathbf{G}(\mathbf{u})) \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.8)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \frac{\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}}{\|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\|} \|\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}\| d\Theta \quad (5.3.9)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}) d\Theta \quad (5.3.10)$$

where $\mathbf{G} : R \rightarrow S$ is a parameterization of S .

5.3.2 An invariance property

Remark 5.3.4. As mentioned above, given $\mathbf{n}(\mathbf{x})$ fixed, we can define the surface integral of vector field as the surface integral of a real-valued function defined as $h(\mathbf{x}) := \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$. And as argued before, one \mathbf{n} is fixed (i.e. orientation is fixed), the value of integral is deterministic. Therefore we can conclude **the integral of a vector field \mathbf{F} over a surface S depends on the orientation of S but otherwise independent of the parameterization.**

Remark 5.3.5. Let $S \subseteq \mathbb{R}^2$ be a two dimensional sub-manifold parameterized by $\mathbf{G} : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $\text{rank}(\mathbf{G}(\mathbf{u})) = 2$ for all but zero-content sets on its domain.

Let $\varphi : W \subseteq \mathbb{R}^2 \rightarrow R$ be a bijection such that $\mathbf{H} := \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$ is another parameterization of S .

Now consider the integral of vector field \mathbf{F} under parameterization \mathbf{H} ,

$$\iint_S \mathbf{F} \cdot \mathbf{u} \, dA = \iint_W \mathbf{F}(\mathbf{H}) \cdot \left(\frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right) d\Theta \quad (5.3.11)$$

$$= \iint_W \mathbf{F} \circ \mathbf{G} \circ \varphi \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) \text{det } D\varphi \, d\Theta \quad (5.3.12)$$

$$= \pm \iint_R \mathbf{F} \circ \mathbf{G} \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (\text{change of variable}) \quad (5.3.13)$$

Theorem 5.3.1 (Invariance). Let $\mathbf{G} : R \rightarrow \mathbb{R}^3$ and $\mathbf{H} := \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$ be two parameterizations of S , then

$$\iint_R f \circ \mathbf{G} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta = \iint_W f \circ \mathbf{H} \left\| \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right\| d\Theta \quad (5.3.14)$$

and

$$\iint_R \mathbf{F} \circ \mathbf{G} \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta = \pm \iint_W \mathbf{F} \circ \mathbf{H} \cdot \left(\frac{\partial \mathbf{H}}{\partial u} \times \frac{\partial \mathbf{H}}{\partial v} \right) d\Theta \quad (5.3.15)$$

5.3.3 Volume and Area (Optional)

Theorem 5.3.2. Let R be an arbitrary regular region in \mathbb{R}^3 , and let S be the boundary surface of R , define

$$S_h := \{\mathbf{x} + \delta \mathbf{n} : \mathbf{x} \in S \wedge \delta \in [0, h]\} \quad (5.3.16)$$

where S_h can be interpreted as *a shell of region R with thickness h* . Then the surface area of S is

$$\text{area}(S) = \lim_{h \rightarrow 0} \frac{|S_h|}{h} \quad (5.3.17)$$

5.4 Divergence, Gradient and Curl

Definition 5.4.1. Let $U \subseteq \mathbb{R}^n$ be an open set, and define real-valued function $f : U \rightarrow \mathbb{R}$ (0-form) and vector field $\mathbf{F} : U \rightarrow \mathbb{R}^n$. Then we define

1. (1-form) The **gradient** of f as ∇f ;
2. (2-form) The **curl** of \mathbf{F} as $\nabla \times \mathbf{F}$;
3. (3-form) The **divergence** of \mathbf{F} as $\nabla \cdot \mathbf{F}$.

Definition 5.4.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 real-valued function, define the **Laplacian** of f as a mapping from *real-valued functional space* to *real-valued functional space* defined as

$$\operatorname{div}(\operatorname{grad})f := \sum_j \partial_j^2 f = \Delta f = \nabla^2 f \quad (5.4.1)$$

Theorem 5.4.1. For every C^2 real valued function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$\operatorname{curl}(\operatorname{grad}f) = \mathbf{0} \quad (5.4.2)$$

For every C^2 vector field defined in \mathbb{R}^3 or a subset of it,

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0 \quad (5.4.3)$$

Note that the domain of f and \mathbf{F} must be \mathbb{R}^3 or a subset of it, otherwise the curl operation is not well-defined.

Theorem 5.4.2 (Product rules).

$$\operatorname{grad}(fg) = f \operatorname{grad}g + \operatorname{grad}f \cdot g \quad (5.4.4)$$

$$\operatorname{curl}(f\mathbf{G}) = f \operatorname{curl}\mathbf{G} + \operatorname{grad}f \times \mathbf{G} \quad (5.4.5)$$

$$\operatorname{div}(f\mathbf{G}) = f \operatorname{div}\mathbf{G} + \operatorname{grad}f \cdot \mathbf{G} \quad (5.4.6)$$

5.5 Divergence Theorem

Remark 5.5.1. vector field integral on boundary (2-dimensional sub-manifold) of region in \mathbb{R}^3 ($\mathbf{F} \cdot \mathbf{n} \, dA$ as 2-form) and scalar valued function ($\operatorname{div}(\mathbf{F}) \, dV$ as 3-form) in a region (3-dimensional sub-manifold).

Theorem 5.5.1 (Divergence Theorem). Let $R \subseteq \mathbb{R}^3$ be a regular region with piece-wise smooth boundary ∂S . And \mathbf{n} is the outer normal vector on ∂S , then,

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_S \operatorname{div}(\mathbf{F}) \, dV \quad (5.5.1)$$

Proof.

Definition 5.5.1. A region $R \subseteq \mathbb{R}^3$ is said to be ***xy-simple*** if and only if it can be expressed as the following form

$$R = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, \varphi_1(x, y) \leq z \leq \varphi_2(x, y)\} \quad (5.5.2)$$

Suppose S is simple in terms of all combinations of x, y, z .

Then

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_1 n_1 + F_2 n_2 + F_3 n_3 \, dA \quad (5.5.3)$$

Consider $\iint_{\partial S} F_3 n_3 \, dA$, since R is *xy-simple*,

$$\iint_{\partial S} F_3 n_3 \, dA = \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA \quad (5.5.4)$$

Note that except the bottom and top sides, which are parameterized by $\mathbf{G}_1(x, y) = (x, y, \varphi_1(x, y))$ and $\mathbf{G}_2(x, y) = (x, y, \varphi_2(x, y))$, the outer normal vector of those region has form $(\cdot, \cdot, 0)$, and therefore $\mathbf{n} \cdot \mathbf{k} = 0$ for every \mathbf{x} on those regions, and contribute nothing to the integral.

Therefore, to evaluate $\iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA$, we only need to consider the upper and bottom surfaces. Also note that \mathbf{n} has opposite z component on those two surfaces.

Moreover, the undirected \mathbf{n} on those two surfaces is

$$\tilde{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ \partial_x \varphi_i \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y \varphi_i \end{pmatrix} = \begin{pmatrix} -\partial_x \varphi_i \\ \partial_x \varphi_i - \partial_y \varphi_i \\ 1 \end{pmatrix} \quad (5.5.5)$$

$$\implies \iint_{\partial S} F_3 \mathbf{k} \cdot \mathbf{n} \, dA = \iint_{\partial S} F_3 \, dA \quad (5.5.6)$$

$$= \iint_{\text{upper } \partial S} F_3 \, dA - \iint_{\text{lower } \partial S} F_3 \, dA \quad (5.5.7)$$

$$= \iint_W F_3(x, y, \varphi_2(x, y)) \, dx dy - \iint_W F_3(x, y, \varphi_1(x, y)) \, dx dy \quad (5.5.8)$$

$$= \iint_W \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} \partial_3 F_3 \, dz dx dy \quad (5.5.9)$$

$$= \iiint_S \partial_3 F_3 \, dV \quad (5.5.10)$$

We can prove the equalities involving the other two components, and the proof is completed by the fact that any open set in \mathbb{R}^n can be written as a countable union of *almost disjoint* cubes, which are simple and the boundary of S has zero content. \blacksquare

Proposition 5.5.1 (Geometric Interpretation of Divergence). Let $S \subset \mathbb{R}^3$, $\mathbf{F} : S \rightarrow \mathbb{R}^3$, $\mathbf{a} \in S$,

$$\operatorname{div}(\mathbf{F})(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \iiint_{\mathcal{B}(\mathbf{a}, r)} \operatorname{div}(\mathbf{F})(\mathbf{x}) \, d\mathbf{x} \quad (5.5.11)$$

$$= \lim_{r \rightarrow 0} \frac{3}{4\pi r^2} \underbrace{\iint_{\partial \mathcal{B}(\mathbf{a}, r)} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{x}}_{\text{flux through boundary}} \quad (5.5.12)$$

thus $\operatorname{div}(\mathbf{F})(\mathbf{a}) > 0$ if and only if at point \mathbf{a} , matters are flowing *away* from this point.

Corollary 5.5.1 (Green's Formula). Suppose $R \subset \mathbb{R}^3$ and $f, g : R \rightarrow \mathbb{R}$ are C^2 functions, then

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} \, dA = \iiint_S \nabla f \cdot \nabla g + f \nabla^2 g \, dV \quad (5.5.13)$$

$$\iint_{\partial S} (f \nabla g - g \nabla f) \, dA = \iiint_S (f \nabla^2 g - g \nabla^2 f) \, dV \quad (5.5.14)$$

Proof.

$$\iint_{\partial S} f \nabla g \cdot \mathbf{n} \, dA = \iiint_S \operatorname{div}(f \nabla g) \, dV \quad (5.5.15)$$

$$= \iiint_S f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g \, dV = \iiint_S f \nabla^2 g + \nabla f \cdot \nabla g \, dV \quad (5.5.16)$$

The second formula can be proved directly using divergence theorem the first formula. \blacksquare

5.6 Stokes Theorem

5.6.1 The Stoke Boundary

Definition 5.6.1. Let S be a surface that is a subset of a smooth (or piecewise smooth) surface S_0 , and the *boundary of S* is interpreted as the *boundary of S within S_0* . Then the $\mathbf{x} \in S_0$ belongs to the **Stoke boundary** of S within S_0 if

$$\forall r > 0, \mathcal{B}(r, \mathbf{x}) \cap S \neq \emptyset \wedge \mathcal{B}(r, \mathbf{x}) \cap (S_0 \setminus S) \neq \emptyset \quad (5.6.1)$$

5.6.2 The Orientation

Definition 5.6.2. Given a smooth surface $S \subset S_0$, at a point \mathbf{x} in the Stoke's boundary of S , ∂S , if \mathbf{n} denotes the **positive orientation normal** to S_0 and \mathbf{t} denotes the tangent to ∂S at \mathbf{x} with the **Stokes orientation**, then

$$\mathbf{n} \times \mathbf{t} \text{ points into } S \quad (5.6.2)$$

5.6.3 Stokes Theorem in \mathbb{R}^3

Theorem 5.6.1 (Stokes Theorem, Special Case). Let S be a 2 dimensional sub-manifold in \mathbb{R}^3 , and let \mathbf{F} be a vector field defined on some neighbour of S , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA \quad (5.6.3)$$

Remark 5.6.1. In above theorem, $\omega := \mathbf{F} \cdot d\mathbf{x}$ is a 1-form in \mathbb{R}^3 and $d\omega := \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA$ is a 2-form in \mathbb{R}^3 .

Corollary 5.6.1. Let S be a closed surface in \mathbb{R}^3 , that's, $\partial S = \emptyset$, and let \mathbf{n} denote the outer normal vector, and \mathbf{F} is a C^1 vector field. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = 0 \quad (5.6.4)$$

Proof. We can construct a *small* simple closed curve C on S and divide S into two regions sharing the same boundary. And note that given orientation fixed on S , the orientation on ∂S_1 and ∂S_2 are opposite. Then

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_1} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA + \iint_{S_2} \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dA \quad (5.6.5)$$

$$= \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{x} - \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot d\mathbf{x} - \int_C \mathbf{F} \cdot d\mathbf{x} = 0 \quad (5.6.6)$$

■

Definition 5.6.3. A surface $S \subset \mathbb{R}^3$ is **closed** if it has no Stokes boundary.

Definition 5.6.4. A piecewise smooth surface $S \subset \mathbb{R}^3$ is **closed** if there exists subsets S_1, S_2 such that $\partial S_1 = \partial S_2 = S_1 \cap S_2$, and ∂S_1 and ∂S_2 have *opposite orientations*, where S_1, S_2 have the same orientation as S .

Theorem 5.6.2. If S is a piecewise smooth closed surface in \mathbb{R}^3 and \mathbf{F} is a vector field that is C^1 near S , then

$$\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dA = 0 \quad (5.6.7)$$

Proof.

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dA = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{n} dA + \iint_{S_2} \text{curl } \mathbf{F} \cdot \mathbf{n} dA = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{x} + \int_{\partial S_2} \mathbf{F} \cdot d\mathbf{x} = 0 \quad (5.6.8)$$

■

Proposition 5.6.1 (Geometric Interpretation of Curl). Let $R \subset \mathbb{R}^3$ be a 2 dimensional sub-manifold with \mathbf{n} as outer normal vector on it, and $\mathbf{a} \in R$,

$$\text{curl}(F)(\mathbf{a}) \cdot \mathbf{n}(\mathbf{a}) = \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \iint_{\mathcal{D}(\mathbf{a}, r)} \text{curl}(F)(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) dA \quad (5.6.9)$$

$$= \lim_{r \rightarrow 0} \frac{1}{2\pi r^2} \int_{\partial \mathcal{D}(\mathbf{a}, r)} \mathbf{F} \cdot d\mathbf{x} \quad (5.6.10)$$

$$(5.6.11)$$

If we think of \mathbf{F} as a force field, then $\int_{\partial \mathcal{D}(\mathbf{a}, r)} \mathbf{F} \cdot d\mathbf{x}$ represents the work done by \mathbf{F} on a particle moves around $\partial \mathcal{D}(\mathbf{a}, r)$. Thus $\text{curl}(\mathbf{F}) \cdot \mathbf{u}$ represents the **tendency of the force \mathbf{F} to push the particle around $\partial \mathcal{D}(\mathbf{a}, r)$ in a direction compatible with \mathbf{n} .**

Theorem 5.6.3 (Application: Moving the Surface). Let S and S' be two oriented surfaces such that $\partial S = \partial S'$ (share the same orientation), then

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial S'} \mathbf{F} \cdot d\mathbf{x} = \iint_{S'} \text{curl } \mathbf{F} \cdot \mathbf{n} dA \quad (5.6.12)$$

5.6.4 The Generalization(Optional)

Proposition 5.6.2 (Properties of Exterior Products). Let α_1, α_2 and β be 1-forms on \mathbb{R}^n and f_1, f_2 are continuous functions defined on \mathbb{R}^n ,

1. Distributive

$$(f_1\alpha_1 + f_2\alpha_2) \wedge \beta = f_1(\alpha_1 \wedge \beta) + f_2(\alpha_2 \wedge \beta) \quad (5.6.13)$$

$$\beta \wedge (f_1\alpha_1 + f_2\alpha_2) = f_1(\beta \wedge \alpha_1) + f_2(\beta \wedge \alpha_2) \quad (5.6.14)$$

2. Anti-commutative

$$\beta \wedge \alpha = -\alpha \wedge \beta \quad (5.6.15)$$

Corollary 5.6.2. By the *anti-commutative* property,

$$\alpha \wedge \alpha = 0 \quad (5.6.16)$$

Theorem 5.6.4 (Divergence Theorem in \mathbb{R}^n). Let R be a regular region in \mathbb{R}^n bounded by a piecewise smooth hyper-surface ∂R . Note here R is a n dimensional sub-manifold and ∂R is a $n-1$ dimension sub-manifold. Then

$$\int \cdots \int_{\partial R} \mathbf{F} \cdot \mathbf{n} dV^{n-1} = \iint \cdots \int_R \text{div}(\mathbf{F}) dV^n \quad (5.6.17)$$

where if ∂R is parameterized by $\mathbf{G}(u_1, \dots, u_{n-1})$, then

$$\mathbf{n} dV^{n-1} = \det \begin{pmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ \partial_1 G_1 & \cdots & \partial_1 G_n \\ \vdots & & \vdots \\ \partial_{n-1} G_1 & \cdots & \partial_{n-1} G_n \end{pmatrix} \quad (5.6.18)$$

Definition 5.6.5. A 0-form on \mathbb{R}^n is a real valued function f .

Remark 5.6.2. While writing the basis elements $dx_i \wedge dx_j$ with the variables in *cyclic order*. That's dx before dy before dz before dx in \mathbb{R}^3 case.

Definition 5.6.6. A k -form in \mathbb{R}^n takes the expression of linear combination of $C(n, k)$ basis elements $\{\beta_i\}_i$.

Example 5.6.1. A 2-form ω in \mathbb{R}^3 can be expressed using a 3-element basis

$$\omega = \sum_{1 \leq i < j \leq 3} C_{ij}(\mathbf{x}) \beta_{ij} \quad (5.6.19)$$

$$\beta_{ij} \in \{dx \wedge dy, dy \wedge dz, dx \wedge dz\} \quad (5.6.20)$$

Definition 5.6.7. Let $\omega = \sum_{j=1}^{C(n,k)} f_j \beta_j$ be a k -form in \mathbb{R}^n , then it's **exterior derivative** is defined to be the $(k+1)$ -form in \mathbb{R}^n defined as

$$d\omega := \sum_j df_j \wedge \beta_j \quad (5.6.21)$$

where df_j can be computed using *total derivative*.

Example 5.6.2. In \mathbb{R}^3 , the *exterior derivative* for a 0-form f is its **gradient**, which is a 1-form. And the exterior derivate of a 1-form in \mathbb{R}^3 is its curl

$$\omega := F_1 dx + F_2 dy + F_3 dz \quad (5.6.22)$$

$$\implies d\omega = dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \quad (5.6.23)$$

$$= (\partial_1 F_1 dx + \partial_2 F_1 dy + \partial_3 F_1 dz) \wedge dx \quad (5.6.24)$$

$$+ (\partial_1 F_2 dx + \partial_2 F_2 dy + \partial_3 F_2 dz) \wedge dy \quad (5.6.25)$$

$$+ (\partial_1 F_3 dx + \partial_2 F_3 dy + \partial_3 F_3 dz) \wedge dz \quad (5.6.26)$$

$$= (\partial_1 F_2 - \partial_2 F_1) dx \wedge dy + (\partial_2 F_3 - \partial_3 F_2) dy \wedge dz + (\partial_3 F_1 - \partial_1 F_3) dz \wedge dx \quad (5.6.27)$$

$$= \text{curl}(\mathbf{F}) \quad (5.6.28)$$

The exterior derivate of a 2-form in \mathbb{R}^3 is its divergence

$$\omega := A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \quad (5.6.29)$$

$$\implies d\omega = (\partial_1 A dx + \partial_2 A dy + \partial_3 A dz) \wedge dy \wedge dz \quad (5.6.30)$$

$$+ (\partial_1 B dx + \partial_2 B dy + \partial_3 B dz) \wedge dz \wedge dx \quad (5.6.31)$$

$$+ (\partial_1 C dx + \partial_2 C dy + \partial_3 C dz) \wedge dx \wedge dy \quad (5.6.32)$$

$$= (\partial_1 A + \partial_2 B + \partial_3 C) dx \wedge dy \wedge dz \quad (5.6.33)$$

$$= \text{div}(\mathbf{F}) \quad (5.6.34)$$

Theorem 5.6.5 (Stokes Theorem, 5.77). Let M be a smooth, oriented k dimensional sub-manifold of \mathbb{R}^n with a piecewise smooth boundary ∂M , and let ∂M carry the orientation that is (in a suitable sense) compatible with the one on M . If ω is a $(k-1)$ -form of class C^1 on an open set containing M , then

$$\int \cdots \int_{\partial M} \omega = \iint \cdots \int_M d\omega \quad (5.6.35)$$

Theorem 5.6.6. The *boundary* of a (smoothly bounded) region M in a k dimensional manifold is a $(k - 1)$ dimensional manifold with no boundary.

That's let M be a k dimensional manifold with piecewise smooth boundary ∂M , then

$$\partial(\partial M) = \emptyset \quad (5.6.36)$$

Theorem 5.6.7. For any k -form ω on \mathbb{R}^n ,

$$d(d\omega) = 0 \quad (5.6.37)$$

Proof. Let M be a k dimensional sub-manifold in \mathbb{R}^n with piecewise smooth boundary, and ω is a $(k - 2)$ -form on \mathbb{R}^n , so $d(d\omega)$ is a k -form on \mathbb{R}^n . And

$$\iiint \cdots \int_M d(d\omega) = \iint \cdots \int_{\partial M} d\omega \quad (5.6.38)$$

$$= \int \cdots \int_{\partial(\partial M)} \omega = \int \cdots \int_{\emptyset} \omega = 0 \quad (5.6.39)$$

■

5.7 Vector Fields that are Gradients or Curls

5.7.1 Conservative Vector Field

Definition 5.7.1. A continuous vector field \mathbf{G} defined on open $U \subset \mathbb{R}^n$ is **conservative** if there exists a function $f : U \rightarrow \mathbb{R}$ of class C^1 such that $\mathbf{G} = \nabla f$.

Theorem 5.7.1. Let $U \subset \mathbb{R}^n$ open, and let $\mathbf{G} : U \rightarrow \mathbb{R}^n$ be continuous. Then the following are equivalent:

- (i) There exists a function $f : U \rightarrow \mathbb{R}$ of class C^1 such that $\mathbf{G} = \nabla f$;
- (ii) $\int_C \mathbf{G} \cdot d\mathbf{x} = 0$ for any closed piecewise smooth oriented curve $C \subset U$;
- (iii) $\int_{C_1} \mathbf{G} \cdot d\mathbf{x} = \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$ for any two piecewise smooth oriented curves C_1, C_2 in U that shares starting and ending points.

Proof.

(i) \implies (ii) by the *Fundamental Theorem of Line Integrals*.

(ii) \implies (iii) Given C_1 and C_2 both from \mathbf{p} to \mathbf{q} . Constructing C_2' from \mathbf{q} to \mathbf{p} by reversing the orientation of C_2 . And $C_1 \cup C_2'$ form a closed curve.

$$\int_C \mathbf{G} \cdot d\mathbf{x} = \int_{C_1} \mathbf{G} \cdot d\mathbf{x} - \int_{C_2} \mathbf{G} \cdot d\mathbf{x} \quad (5.7.1)$$

(iii) \implies (i), pick an arbitrary point $\mathbf{p} \in U$, and for every $\mathbf{q} \in U$, define f as

$$f(\mathbf{q}) = \int_{C_{\mathbf{p},\mathbf{q}}} \mathbf{G} \cdot d\mathbf{x} \quad (5.7.2)$$

where $C_{\mathbf{p},\mathbf{q}}$ is an arbitrary curve starting at \mathbf{p} and ending at \mathbf{q} . ■

Theorem 5.7.2. If \mathbf{G} is a conservative vector field of class C^1 on an open set $U \subset \mathbb{R}^3$, then $\text{curl} \mathbf{G} = \mathbf{0}$.

Theorem 5.7.3. If $U \subset \mathbb{R}^3$ is convex and if \mathbf{G} is a C^1 vector field defined on U and $\text{curl}\mathbf{G} = \mathbf{0}$ on U , then \mathbf{G} is conservative.

Remark 5.7.1. That's possible for a non-conservative vector field \mathbf{G} defined on some non-convex set to have $\text{curl}\mathbf{G} = \mathbf{0}$

Proposition 5.7.1 (Finding f for Conservative Vector Fields). If the domain of a conservative vector field \mathbf{G} is convex, then we can construct f following

- (i) Choose $\mathbf{p} = (a, b, c) \in U$;
- (ii) Define $f(x, y, z)$ as

$$f(x, y, z) := \int_a^x G_1(t, b, c) dt + \int_b^y G_2(x, t, c) dt + \int_c^z G_3(x, y, t) dt \quad (5.7.3)$$

Corollary 5.7.1. If the domain of \mathbf{G} is a convex set containing the origin, then we can define $\mathbf{p} = (a, b, c) = (0, 0, 0)$,

$$f(x, y, z) = \int_0^1 \mathbf{G}(tx, ty, tz) \cdot (x, y, z) dt \quad (5.7.4)$$

5.7.2 Vector Fields that are Curls

Theorem 5.7.4. If $\mathbf{G} : U \rightarrow \mathbb{R}^3$ is a vector field of class C^1 and $\mathbf{G} = \text{curl}\mathbf{F}$ for some vector field $\mathbf{F} : U \rightarrow \mathbb{R}^3$ of class C^2 , then $\text{div}\mathbf{G} = 0$.

Theorem 5.7.5. Suppose \mathbf{G} is a C^1 vector field in an open set $U \subset \mathbb{R}^3$ such that $\text{div}\mathbf{G} = 0$. If U is convex, then there exists a vector field \mathbf{F} such that $\text{curl}\mathbf{F} = \mathbf{G}$.

Definition 5.7.2. Given a closed curve $C \subset \mathbb{R}^3$, the **minimal surface** is the surface S with the smallest possible area, given that $\partial S = C$.

5.8 Summary

5.8.1 Line Integrals

$$\text{Arc length} \quad \int_C 1 \, d\mathbf{x} = \int_a^b \|\mathbf{g}'(t)\| \, dt \quad (5.8.1)$$

$$\text{Integral of scalar valued function on curve} \quad \int_C f(\mathbf{x}) \, d\mathbf{x} = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt \quad (5.8.2)$$

$$\text{Integral of vector field on curve} \quad \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt \quad (5.8.3)$$

5.9 Surface Integrals

$$\text{Surface area} \quad \iint_S 1 \, dA = \iint_W \|\partial_u \mathbf{G} \times \partial_v \mathbf{G}\| \, dudv \quad (5.9.1)$$

$$\text{Scalar function on surface} \quad \iint_S f(\mathbf{x}) \, dA = \iint_W f(\mathbf{G}(\mathbf{u})) \|\partial_u \mathbf{G} \times \partial_v \mathbf{G}\| \, dudv \quad (5.9.2)$$

$$\text{Vector field on surface} \quad \iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} \, dA = \iint_W \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot (\partial_u \mathbf{G} \times \partial_v \mathbf{G}) \, dudv \quad (5.9.3)$$