STA447: Stochastic Processes

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Contents

1 Markov Chain Probabilities

Definition 1.1. A **discrete-time, discrete-space, and time-homogenous Markov chain** is a triple of (S, v, p) in which

(i) S represents the *state space*, which is nonempty and countable;

(ii) *initial probability* v, which is a distribution on S;

(iii) and *transition probability* p_{ij}.

Definition 1.2. A Markov chain satisfies the **time-homogenous property** if

$$P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i) = p_{ij} \quad \forall n \in \mathbb{N} \tag{1.1}$$

Definition 1.3. A Markov chain satisfies the **Markov property** if

$$P(X_{n+1} = j | X_n = i_n, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_{n+1} = j | X_n = i_n) \tag{1.2}$$

That is, the chain is *memoryless*.

Proposition 1.1 (Multistep Arrival Probability)**.** Let $m = |S|$ and $\mu_i^{(n)} := P(X_n = i)$ denote the probability that the state ends up at i after n step. By the law of total expectation,

$$\begin{aligned} P(X_n = i) &= \sum_{j \in S} P(X_n = i, X_{n-1} = j) && (1.3)\\ &= \sum_{j \in S} P(X_n = i | X_{n-1} = j) P(X_{n-1} = j) && (1.4)\\ &= \sum_{j \in S} P(X_{n-1} = j) p_{ij} && (1.5)\\ &= \sum_{j \in S} \mu_j^{(n)} p_{ij} && (1.6) \end{aligned}$$

Let $\mu^{(n)} := \left[\mu_1^{(n)}, \mu_2^{(n)}, \cdots, \mu_m^{(n)}\right] \in \mathbb{R}^{1 \times m}$ and $P = [p_{ij}] \in \mathbb{R}^{m \times m}$. In matrix notation:

$$\mu^{(n)} = \mu^{(n-1)} P \tag{1.7}$$

where $\mu^{(0)} = v = [v_1, v_2, \cdots, v_m]$. Define $P^0 = I_m$, then

$$\mu^{(n)} = vP^n \tag{1.8}$$

Proposition 1.2 (Multistep Transition Probability)**.** Define $p_{ij}^{(n)} := P(X_{m+n} = j | X_m = i)$ to be the probability of arriving state j after n steps, starting from state i. By the time-homogenous property,

$$p_{ij}^{(n)} = P(X_{m+n} = j | X_m = i) \quad \forall m \in \mathbb{N} \tag{1.9}$$

Let $P^{(n)} := [p_{ij}^{(n)}] \in \mathbb{R}^{m \times m}$.
Initial Step: for $n = 1$, $P^{(1)} = P$ by definition.

Inductive Step: for $n \in \mathbb{N}$,

$$\begin{aligned} p_{ij}^{(n+1)} &= P(X_{n+1} = j | X_0 = i) && (1.10)\\ &= \sum_{k \in S} P(X_{n+1} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i) && (1.11)\\ &= \sum_{k \in S} P(X_{n+1} = j | X_n = k) p_{ik}^{(n)} && (1.12)\\ &= \sum_{k \in S} p_{ik}^{(n)} p_{kj} && (1.13)\\ &= [P^{(n)} P]_{ij} && (1.14) \end{aligned}$$

Therefore,

$$P^{(n+1)} = P^{(n)} P \tag{1.15}$$

and

$$P^{(n)} = P^n \tag{1.16}$$

Theorem 1.1 (Chapman-Kolmogorov Equations)**.** Let $n = (n_1, n_2, \cdots, n_k)$ be a multi-set of non-negative integers, then

$$P^{(\sum_{i=1}^k n_i)} = \prod_{i=1}^k P^{(n_i)} \quad (\dagger) \tag{1.17}$$

Proof. Prove by induction on the size of multi-set:
Base case is trivial for $k = 1$.
Inductive step for $k > 1$, suppose $(\dagger)$ holds for every set of length k, consider another multi-set with length $k+1$: $n' = (n_1, n_2, \cdots, n_k, n_{k+1})$. Let $\delta := \sum_{i=1}^k n_i$.

$$\begin{aligned} P_{ij}^{(\delta + n_{k+1})} &= P(X_{\delta + n_{k+1}} = j | X_0 = i) && (1.18)\\ &= \sum_{k \in S} P(X_{\delta + n_{k+1}} = j | X_\delta = k, X_0 = i) P(X_\delta | X_0 = i) && (1.19)\\ &= \sum_{k \in S} P(X_{\delta + n_{k+1}} = j | X_\delta = k) P(X_\delta | X_0 = i) && (1.20)\\ &= \sum_{k \in S} P(X_{n_{k+1}} = j | X_0 = k) P(X_\delta = k | X_0 = i) && (1.21)\\ &= \sum_{k \in S} p_{kj}^{n_{k+1}} p_{ik}^{(\delta)} && (1.22)\\ &= [P^{(\delta)} P^{(n_{k+1})}]_{ij} && (1.23)\\ \implies P^{(\delta + n_{k+1})} &= P^{(\delta)} P^{(n_{k+1})} && (1.24) \end{aligned}$$

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Corollary 1.1 (Chapman-Kolmogorov Inequality)**.** For every $k \in S$,

$$p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)} \tag{1.25}$$

For $k, \ell \in S$,

$$p_{ij}^{(m+s+n)} \geq p_{ik}^{(m)} p_{k\ell}^{(s)} p_{\ell j}^{(n)} \tag{1.26}$$

Notation 1.1. Let $N(i) := |\{n \geq 1 : X_n = i\}|$ denote the number of arrivals to state i of the chain.

Definition 1.4. Define the **return probability** from state i to j, f_{ij}, as the probability of arriving state j starting from state i. That is,

$$f_{ij} = P(\exists n \geq 1 \ s.t. \ X_n = j | X_0 = i) \tag{1.27}$$

$$= P(N(j) \geq 1 | X_0 = i) \tag{1.28}$$

Proposition 1.3. The probability of firstly arriving j, then arriving k (denoted as event E) starting from i equals

$$P(E|X_0 = i) = f_{ij} f_{jk} \tag{1.29}$$

Proof. The proof follows the time-homogenous property. ■

Corollary 1.2.

$$P(N(i) \geq k | X_0 = i) = (f_{ii})^k \tag{1.30}$$

$$P(N(j) \geq k | X_0 = i) = f_{ij} (f_{jj})^{k-1} \tag{1.31}$$

Definition 1.5. A state i in a Markov chain is **recurrent** if $f_{ii} = 1$. Otherwise, this state is **transient**.

Theorem 1.2 (Recurrent State Theorem)**.** The following statements are equivalent:

(i) State i is recurrent;

(ii) $P(N(i) = \infty | X_0 = i) = 1$, that is, state i will be visited infinitely often;

(iii) $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.

The following statements are equivalent:

(a) State i is transient;

(b) $P(N(i) = \infty | X_0 = i) = 0$, that is, state i will only be visited finitely many times;

(c) $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

Proof. We only show the equivalence of $(i) \sim (iii)$, $(a) \sim (c)$ are simply the negation of previous statements. $(i) \iff (ii)$:

$$P(N(i) = \infty | X_0 = i) = P(\lim_{k \to \infty} N(i) \geq k | X_0 = i) \tag{1.32}$$

$$= \lim_{k \to \infty} P(N(i) \geq k | X_0 = i) \tag{1.33}$$

$$= \lim_{k \to \infty} (f_{ii})^k = 1 \text{ if and only if } f_{ii} = 1 \tag{1.34}$$

$(i) \iff (iii)$:

$$
\begin{aligned}
\sum_{n=1}^{\infty} p_{ii}^{(n)} &= \sum_{n=1}^{\infty} P(X_n = i | X_0 = i) && (1.35)\\
&= \sum_{n=1}^{\infty} \mathbb{E}(1_{X_n=i} | X_0 = i) && (1.36)\\
&= \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{X_n=i} \middle| X_0 = i \right) && (1.37)\\
&= \mathbb{E}(N(i) | X_0 = i) && (1.38)\\
&= \sum_{n=k}^{\infty} kP(N(i) = k | X_0 = i) && (1.39)\\
&= \sum_{n=k}^{\infty} P(N(i) \geq k | X_0 = i) && (1.40)\\
&= \sum_{n=k}^{\infty} (f_{ii})^k && (1.41)\\
&= \infty \text{ if and only if } f_{ii} = 1 && (1.42)
\end{aligned}
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