

MAT224 Notes

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Info.

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1 Lecture1 Jan.9 2018

1.1 Vector spaces

Definition A real¹ **vector space** is a set V together with two vector operations vector addition and scalar multiplication such that

1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$
2. **C** Commutative: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. **Z** Zero Vector: $\exists \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$
5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists -\vec{x} \in V s.t. \vec{x} + (-\vec{x}) = \vec{0}$
6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$
7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c+d)\vec{x} = c\vec{x} + d\vec{x}$
9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$
10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

Note For V to be a vector space, need to know or be given operations of vector additions multiplication and check all 10 properties hold.

¹A vector space is real if scalar which defines scalar multiplication is real.

1.2 Examples of vector spaces

Example 1 \mathbb{R}^n w.r.t.² usual component-wise addition and scalar multiplication.

Example 2 $M_{m \times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

Example 3 $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree less or equal to n , w.r.t. usual degree-wise polynomial addition and scalar multiplication.

Note If define $\mathbb{P}_n^*(\mathbb{R})$ as set of all polynomials of degree exactly equal to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

Explanation: $(1 + x^n), (1 - x^n) \in \mathbb{P}_n^*(\mathbb{R})$ but $(1 + x^n) + (1 - x^n) = 2 \notin \mathbb{P}_n^*(\mathbb{R})$

Example 4 Something unusual, define V as

$$V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

This is a vector space.

1.3 Some properties of vector spaces

Suppose V is a vector space, then it has the following properties.

²w.r.t. is the abbreviation of "with respect to".

Property 1 The zero vector is unique.

proof.

Assume $\vec{0}, \vec{0}^*$ are two zero vectors in V

WTS: $\vec{0} = \vec{0}^*$

Since $\vec{0}$ is the zero vector, by Z $\vec{0}^* + \vec{0} = \vec{0}^*$

Similarly, $\vec{0} + \vec{0}^* = \vec{0}$

Also, $\vec{0} + \vec{0}^* = \vec{0}^* + \vec{0}$ by commutative vector addition.

So, $\vec{0}^* = \vec{0}$

■

Property 2 $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.

proof.

Exercise. (By Cancellation Law)

Property 3 $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.

proof.

By property of number 0: $0\vec{x} = (0 + 0)\vec{x}$

By DSA: $0\vec{x} = 0\vec{x} + 0\vec{x}$

By AI, $\exists(-0\vec{x})$ s.t.

$0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x})$

By AA

$\implies 0\vec{x} = \vec{0}$

Property 4 $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$

proof.

$c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0}$

2 Lecture2 Jan.10 2018

2.1 Some properties of vector spaces-Cont'd

Property 5 For a vector space V , $\forall \vec{x} \in V$, $(-1)\vec{x} = (-\vec{x})$. (we could use this property to find the additive inverse with scalar multiplication with (-1))³.

proof.

$$\begin{aligned}
 (-\vec{x}) &= (-\vec{x}) + \vec{0} && \text{By property of zero vector} \\
 &= (-\vec{x}) + 0\vec{x} && \text{By property 3} \\
 &= (-\vec{x}) + (1 + (-1))\vec{x} && \text{By property of zero as real number} \\
 &= (-\vec{x}) + 1\vec{x} + (-1)\vec{x} \\
 &= \vec{0} + (-1)\vec{x} \\
 &= (-1)\vec{x}
 \end{aligned}$$

■

Property 6 For a vector space V , let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \vee \vec{x} = \vec{0}$$

proof.

$$\begin{aligned}
 &\text{if } c = 0 \implies \text{True} \\
 &\text{else } c^{-1}c\vec{x} = c^{-1}\vec{0} \\
 &\implies (c^{-1}c)\vec{x} = \vec{0} \\
 &\implies 1\vec{x} = \vec{0} \\
 &\implies \vec{x} = \vec{0} \\
 &\implies \text{True}
 \end{aligned}$$

■

2.2 Subspaces

Loosely A subspace is a space contained within a vector space.

³The scalar multiplication here is the one defined in vector space V .

Definition Let V be a vector space and $W \subseteq V$, W is a **subspace** of V if W is itself a vector space w.r.t. operations of vector addition and scalar multiplication from V .

Theorem Let V be a vector space, and $W \subseteq V$, W has the same⁴ operations of vector addition and scalar multiplication as in V . Then, W is a subspace of V iff:

1. W is non-empty. $W \neq \emptyset$.
2. W is closed under addition. $\forall \vec{x}, \vec{y} \in W, \vec{x} + \vec{y} \in W$.
3. W is closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

Proof.

Forward:

If W is a subspace

$$\implies \vec{0} \in W$$

$$\implies W \neq \emptyset$$

Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$

WTS. 10 axioms in definition of vector space hold

$(ii) \implies$ Additive Closure

$(iii) \implies$ Scalar Multiplication Closure

Because $W \subseteq V$, and V is a vector space, so properties hold $\forall \vec{w} \in W$.

Additive inverse: by property 5 and scalar multiplication closure,

$$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W.$$

Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W$.

2.3 Examples of subspaces

Example 1 Let $V = \mathbb{M}_{n \times n}(\mathbb{R})$, V is a subspace.

⁴Other properties of vector spaces related to vector addition and scalar multiplication are immediately inherited from the parent vector space.

Example 2 Define W as

$$W = \{A \in \mathbb{M}_{n \times n}(\mathbb{R}) \mid A \text{ is not symmetric}\}$$

Explanation: Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but

$$A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W.$$

Since there's no additive identity in set W , so W failed to be a vector space, therefore W is not a subspace.

Example 3 Let $V = \mathbb{P}_2(\mathbb{R})$, is W defined as following,

$$W = \{p(x) \in V \mid p(1) = 0\}$$

a subspace of V ?

proof.

WTS: (i)

Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$

$$\implies W \neq \emptyset$$

WTS: (ii)

Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$

$$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$$

$$\implies p_1 + p_2 \in W$$

$$\implies W \text{ is closed under addition.}$$

WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$

$$\implies p(1) = 0$$

Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$

$$\implies cp \in W.$$

So W is a subspace of V .



2.4 Recall from MAT223

Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of \mathbb{R}^n and $Col(A)$ is a subspace of \mathbb{R}^m .

3 Lecture3 Jan.16 2018

3.1 Linear Combination

Definition Let V be a vector space, $\vec{v}_1, \dots, \vec{v}_n \in V$, $a_1, \dots, a_n \in \mathbb{R}$ the expression

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n$$

is called a **linear combination** of $\vec{v}_1, \dots, \vec{v}_n$.

Theorem Let V be a vector space, W is a subspace of V , $\forall \vec{w}_1, \dots, \vec{w}_k \in W$, $c_1, \dots, c_k \in \mathbb{R}$, we have

$$c_1\vec{w}_1 + \dots + c_k\vec{w}_k \in W$$

Subspaces are closed under linear combinations, since subspaces are closed under scalar multiplication and vector addition.

Theorem Let V be a vector space, let $\vec{v}_1, \dots, \vec{v}_k \in V$ then the set of all linear combination of $\vec{v}_1, \dots, \vec{v}_k$

$$W = \left\{ \sum_{i=1}^k c_i \vec{v}_i \mid c_i \in \mathbb{R} \forall i \right\}$$

is a subspace of V .

proof.

Consider $\vec{0} \in W$

So, $W \neq \emptyset$

Let $c \in \mathbb{R}$, Let $\vec{x} \in W \wedge \vec{y} \in W$

By definition of span, we have,

$$\vec{x} = \sum_{i=1}^k a_i \vec{v}_i, \quad \vec{y} = \sum_{i=1}^k b_i \vec{v}_i$$

Consider, $\vec{x} + c\vec{y}$

$$\vec{x} + c\vec{y} = \sum_{i=1}^k a_i \vec{v}_i + c \sum_{i=1}^k b_i \vec{v}_i = \sum_{i=1}^k (a_i + cb_i) \vec{v}_i \in W$$

■

Definition Let V be a vector space, $\vec{v}_1, \dots, \vec{v}_k \in V$, **span** of the set of vectors $\{\vec{v}_i\}_{i=1}^k$ is defined as the collection of all possible linear combinations of $\{\vec{v}_i\}_{i=1}^k$. By pervious theorem, span is a subspace.

3.2 Combination of subspaces

Definition Let W_1, W_2 be two sets, then the **union** of W_1, W_2 is defined as:

$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

the **intersection** of W_1, W_2 is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

Now consider W_1, W_2 to be two subspaces of vector space V , then we have,

1. $W_1 \cup W_2$ is **not** a subspace.
2. $W_1 \cap W_2$ is a subspace.

proof.

Falsify the statement by providing counter-example:

Consider,

$$W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$

$$W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$$

$$\text{But, } \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$$

■

proof.

Because W_1 and W_2 are both subspaces, so

$$\vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset$$

Let $\vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}$

Consider, $\vec{x} + c\vec{y}$

Sine W_1, W_2 are subspaces,

$$\vec{x} + c\vec{y} \in W_1 \wedge \vec{x} + c\vec{y} \in W_2$$

$$\implies \vec{x} + c\vec{y} \in W_1 \cap W_2$$

So, $W_1 \cap W_2$ is a subspace.

■

Definition Let W_1, W_2 be subspaces of vector space V , define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \wedge \vec{y} \in W_2\}$$

Note Let $\vec{x} = \vec{0} \in W_1, \forall \vec{y} \in W_2, \vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = \vec{0} \in W_2, \forall \vec{x} \in W_1, \vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2, \vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

Note $W_1 + W_2$ is a subspace of V .

proof.

Let $\vec{x}_1, \vec{x}_2 \in W_1, \vec{y}_1, \vec{y}_2 \in W_2$

By properties of subspaces,

$$\forall c \in \mathbb{R}, \vec{x}_1 + c\vec{x}_1 \in W_1 \wedge \vec{y}_2 + c\vec{y}_2 \in W_2$$

Consider, $\vec{x}_1 + \vec{y}_1 \in W_1 + W_2, \vec{x}_2 + \vec{y}_2 \in W_1 + W_2$

$$\begin{aligned} & (\vec{x}_1 + \vec{y}_1) + c(\vec{x}_2 + \vec{y}_2) \\ &= (\vec{x}_1 + c\vec{x}_2) + (\vec{y}_1 + c\vec{y}_2) \in W_1 + W_2 \end{aligned}$$

■

Definition(Unique Representation) Let W_1, W_2 be subspaces of vector space V , say V is **direct sum** of W_1 and W_2 , written as $V = W_1 \oplus W_2$, if every $\vec{x} \in V$ can be written uniquely as $\vec{x} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$.

Equivalently Let W_1 and W_2 be subspaces of V , $V = W_1 \oplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.

4 Lecture4 Jan.17 2018

4.1 Cont'd

Cont'd Proof of Theorem

proof.

(Forward direction) Suppose $V = W_1 \oplus W_2$

WTS. $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

Let $V = W_1 \oplus W_2$

$\implies \forall \vec{x} \in V$, can be written uniquely as

$$\vec{x} = \vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$$

$\implies V = W_1 + W_2$ by definition of *sum*.

Let $\vec{x} \in W_1 \cap W_2$

Decomposition, let $\vec{z} \in W_1 \cap W_2 \subseteq V$

$$\vec{z} = \vec{z} + \vec{0}, \vec{z} \in W_1, \vec{0} \in W_2$$

$$\vec{z} = \vec{0} + \vec{z}, \vec{0} \in W_1, \vec{z} \in W_2$$

Since decomposition is unique, $\vec{z} = \vec{0}$

So, $W_1 \cap W_2 = \{\vec{0}\}$

(Backward direction) Suppose $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

WTS. $V = W_1 \oplus W_2$

Assume $\vec{x} = \vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$

$$\vec{x} = \vec{w}'_1 + \vec{w}'_2, \vec{w}'_1 \in W_1, \vec{w}'_2 \in W_2$$

$$\implies \vec{w}_1 + \vec{w}_2 = \vec{w}'_1 + \vec{w}'_2$$

$$\implies \vec{w}_1 - \vec{w}'_1 = \vec{w}'_2 - \vec{w}_2$$

Where, by definition of subspace, $\vec{w}_1 - \vec{w}'_1 \in W_1 \wedge \vec{w}'_2 - \vec{w}_2 \in W_2$

$$\text{So, } \vec{w}_1 - \vec{w}'_1 = \vec{w}'_2 - \vec{w}_2 \in W_1 \cap W_2$$

Since $W_1 \cap W_2 = \{\vec{0}\}$

$$\implies \vec{w}_1 = \vec{w}'_1 \wedge \vec{w}_2 = \vec{w}'_2$$

So the decomposition is unique.

■

4.2 Linear Independence

Theorem (Redundancy theorem) Let V be a vector space, $\{\vec{x}_1, \dots, \vec{x}_n\}$, let $\vec{x} \in \{\vec{x}_1, \dots, \vec{x}_n\}$, then

$$\text{span}\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}\} = \text{span}\{\vec{x}_1, \dots, \vec{x}_n\}$$

we say \vec{x} is the **redundant** vector that contributes nothing to the span.
proof.

$$\begin{aligned} \text{let } \vec{x} &\in \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} \\ \vec{x} &= \sum_{i=1}^n c_i \vec{x}_i \text{ for } c_i \in \mathbb{R} \forall i \\ \text{So, } \text{span}\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}\} &= \left\{ \sum_{i=1}^n a_i \vec{x}_i + z \vec{x} \mid a_i, z \in \mathbb{R} \forall i \right\} \\ &= \left\{ \sum_{i=1}^n a_i \vec{x}_i + z \sum_{i=1}^n c_i \vec{x}_i \mid a_i, c_i \in \mathbb{R} \forall i \right\} \\ &= \left\{ \sum_{i=1}^n (a_i + z c_i) \vec{x}_i \mid a_i, c_i \in \mathbb{R} \forall i \right\} \\ \text{Let } d_i &= a_i + z c_i \in \mathbb{R} \\ &= \left\{ \sum_{i=1}^n d_i \vec{x}_i \mid d_i \in \mathbb{R} \forall i \right\} \\ &= \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} \end{aligned}$$

■

Definition Let V be a vector space, let $\{\vec{x}_1, \dots, \vec{x}_n\} \in V$, we say $\{v_i\}_{i=1}^n$ is **linearly independent** if the only set of scalars $\{c_1, \dots, c_n\}$ that satisfies,

$$\sum_{i=1}^n c_i \vec{x}_i = 0$$

is $\{0, \dots, 0\}$.

Definition In contrast, we say a set of vector, with size n , is **linearly dependent** if

$$\exists \vec{c} \neq \vec{0} \in \mathbb{R}^n, \text{ s.t. } \sum_{i=1}^n c_i \vec{v}_i = 0$$

Theorem Let V be a vector space, $\{\vec{v}_i\}_{i=1}^n \in V$ is *linearly dependent* if and only if,

$$\exists \vec{x} \in \{\vec{v}_i\}_{i=1}^n \text{ s.t. } \vec{x} \in \text{span}\{\{\vec{v}_i\}_{i=1}^n \setminus \{\vec{x}\}\}$$

Theorem Let V be a vector space, $\{\vec{v}_i\}_{i=1}^n \in V$ is *linearly independent* if and only if,

$$\forall \vec{x} \in \{\vec{v}_i\}_{i=1}^n, \vec{x} \notin \text{span}\{\{\vec{v}_i\}_{i=1}^n \setminus \{\vec{x}\}\}$$

5 Lecture5 Jan.23 2018

5.1 Linear independence, recall definitions

Acknowledgement: special thanks to Frank Zhao.

Definition Let $\{\vec{x}_1, \dots, \vec{x}_k\}$ is **linearly independent** if only scalars $c_1 \dots c_k$ s.t.

$$\sum_{i=1}^k c_i \vec{x}_i = 0(\star)$$

are $c_1 = \dots = c_k = 0$

linearly dependent means at least one $c_i \neq 0$, (\star) still holds.

5.1.1 Alternative definitions of linear independency

Definition(Alternative.1) $\{\vec{x}_1 \dots \vec{x}_k\}$ is **linearly independent** iff none of them can be written as a linear combination of the remaining $k - 1$ vectors.⁵

Definition(Alternative.2) $\{\vec{x}_1 \dots \vec{x}_k\}$ is **linearly dependent** iff at least one of them can be written as a linear combination of the remaining $k - 1$ vectors.⁶

5.2 Basis

Definition Let V be a vector space, a non-empty⁷ set S of vectors from V is a **basis** for V if

1. $V = \text{span}\{S\}$

⁵See theorem from the pervious lecture.

⁶See theorem from the pervious lecture.

⁷Specially, for an empty set, we define $\text{span}\{\emptyset\} = \{\vec{0}\}$

2. S is linearly independent.

Theorem (characterization of basis) A non-empty subset $S = \{\vec{x}_i\}_{i=1}^n$ of vector space V is basis for V iff every $\vec{x} \in V$ can be written uniquely as linear combination for vectors in S .

proof.

Forwards

Suppose S is a basis for V

So every $\vec{x} \in V$ can be written as a linear combination of vectors in S

To prove the uniqueness, assume two expressions of $\vec{x} \in V$

$$\vec{x} = \begin{cases} c_1\vec{x}_1 + \cdots + c_k\vec{x}_k \\ b_1\vec{x}_1 + \cdots + d_k\vec{x}_k \end{cases}$$

Consider,

$$c_1\vec{x}_1 + \cdots + c_k\vec{x}_k - (b_1\vec{x}_1 + \cdots + d_k\vec{x}_k) = \vec{0}$$

$$\iff \sum_{i=1}^k (c_i - b_i)\vec{x}_i = \vec{0}$$

Since vectors in basis S are linear independent,

$$c_i = b_i \forall i \in \mathbb{Z} \cap [1, k]$$

So the representation is unique.

Backwards

Suppose every $\vec{x} \in V$ can be written uniquely as linear combination of vectors in S .

WTS: $V = \text{span}\{S\} \wedge S$ is linearly independent

By the assumption, spanning set is shown.

All we need to show is linear independence.

Consider,

$$\sum_{i=1}^n c_i \vec{x}_i = \vec{0}$$

Also, we know

$$\sum_{i=1}^n 0\vec{x}_i = \vec{0}$$

By the uniqueness of representation

$$\text{We have identical expression } \sum_{i=1}^n c_i \vec{x}_i = \sum_{i=1}^n 0\vec{x}_i$$

$$\therefore c_i = 0 \forall i \in \mathbb{Z} \cap [1, n]$$

■

Example

$$\begin{aligned}
 V &= \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \\
 (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1 + 1, x_2 + y_2 + 1) \\
 c(x_1, x_2) &= (cx_1 + c - 1, cx_2 + c - 1)
 \end{aligned}$$

Show that $\{(1, 0), (6, 3)\}$ is a basis of V .

By theorem, $\{(1, 0), (6, 3)\}$ is basis if every $(a, b) \in V$ can be written uniquely as linear combination of $\{(1, 0), (6, 3)\}$.

$$\exists \text{ unique scalars } c_1, c_2 \in \mathbb{R} \text{ s.t. } c_1(1, 0) + c_2(6, 3) = (a, b)$$

proof.

By definition of scalar multiplication and vector addition in this space,

$$\begin{aligned}
 \text{Consider } (a, b) &= c_1(1, 0) + c_2(6, 3) = (2c_1 - 1, c_1 - 1) + (7c_2 - 1, 4c_2 - 1) \\
 &= (2c_1 + 7c_2 - 1, c_1 + 4c_2 - 1)
 \end{aligned}$$

Consider the coefficients of variables

$$\begin{cases} 2c_1 + 7c_2 - 1 = a \\ c_1 + 4c_2 - 1 = b \end{cases}$$

WTS, the above system of linear equations has unique solution for all a, b

The system has a unique solution $\forall a, b \in \mathbb{R}$

Since the coefficient matrix has rank 2

$$\text{rank}\left(\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}\right) = 2$$

Since obviously the columns are linearly independent.

■

5.3 Dimensions

Definition For a vector space V , the **dimension** of V is the minimum number of vectors required to span V .

Fundamental Theorem if V vector space is spanned by m vectors, then any set of more than m vectors from V must be linearly dependent.

Fundamental Theorem (Alternative) If V is vector space spanned by m vectors, then any linearly independent set in V must contain less or equal to m vectors.

5.3.1 Consequences of fundamental theorem

Theorem if $S = \{\vec{v}_i\}_{i=1}^k$ and $T = \{\vec{w}_i\}_{i=1}^l$ are two bases of vector space V then $l = k$. *Bases have the same size.*

proof.

Since S spans V and T is linearly independent

$$\therefore l \leq k$$

(flip) Since T spans V and S is linearly independent

$$\therefore k \leq l$$

$$\implies l \leq k \wedge k \leq l$$

$$\implies k = l$$

■

Definition So we can define the **dimension** of V , as $\dim(V)$ as the number vectors in any basis for V . For special case $V = \{\vec{0}\}$, $\dim(V) = 0$.

Example

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{P}_n(\mathbb{R})) = n + 1$
- $\dim(\mathbb{M}_{m \times n}(\mathbb{R})) = m \times n$

5.3.2 Use dimension to prove facts about linearly (in)dependent sets and subspaces

Theorem If V is a vector space, $\dim(V) = n$, $S = \{\vec{x}_k\}_{i=1}^k$ is subset of V , if $k > n$ then S is linearly dependent.

Note $k \leq n \nRightarrow S$ is linear dependent.

Theorem If W is subspace of vector space V , then

1. $\dim(W) \leq \dim(V)$
2. $\dim(W) = \dim(V) \iff W = V$

proof.

(1) Suppose $\dim(V) = n, \dim(W) = k$

WTS, $k \leq n$

Any basis for W is a linearly independent set of k vectors from V .

Since V is spanned by n vectors, since $\dim(V) = n$

By fundamental theorem, $k \leq n$

$$\iff \dim(W) \leq \dim(V)$$

(2) By contradiction, assume $\dim(V) = \dim(W) = n$ but $V \neq W$

Then $\exists \vec{x} \in V \wedge \vec{x} \notin W$

Take S as a basis of W , then $\vec{x} \notin \text{span}\{S\}$

Then $S \cup \vec{x}$ is linearly independent

$\implies S \cup \{\vec{x}\}$ is linearly independent in V containing $n + 1$ vectors

This contradicts the assumption by fundamental theorem since $\dim(V) = n$ so it could not contain more than n linearly independent vectors

■

6 Lecture6 Jan.24 2018

6.1 Basis and Dimension

Theorem Let V be a vector space, S is a spanning set of V , and I is a linearly independent subset of V , s.t. $I \subseteq S$, then \exists basis B for V s.t. $I \subseteq B \subseteq S$.

Explaining

1. Any spanning set for V can be **reduced** to basis for V by removing the linearly dependent(redundant) vector in the spanning set, using redundancy theorem to get a linearly independent spanning set.
2. Linear independent set can be **enlarged** to a basis for V .

proof.

omitted.

■

Corollary Let V be a vector space and $\dim(V) = n$, any set of n linearly independent vectors from V is a basis for V .

proof. If n linearly independent vectors did not span V , then could be enlarged to a basis of V by previous theorem, but then have a basis containing more than n vectors from V , which is impossible by the fundamental theorem since we given the $\dim(V) = n$, proven by contradiction.

Example Let $V = P_2(\mathbb{R})$, $p_1(x) = 2 - 5x$, $p_2(x) = 2 - 5x + 4x^2$, find $p_3 \in P_2(\mathbb{R})$ s.t. $\{p_1(x), p_2(x), p_3(x)\}$ is basis for $P_2(\mathbb{R})$

Note Since $\dim(P_2(\mathbb{R})) = 3$ so any 3 linearly independent vectors from $P_2(\mathbb{R})$ will be a basis for $P_2(\mathbb{R})$.

Solutions e.g. constant function $p_3(x) = 1$, since $1 \notin \text{span}\{p_1(x), p_2(x)\}$, so $\{p_1(x), p_2(x), p_3(x)\}$ is a basis of $P_2(\mathbb{R})$. e.g. $p_3(x) = x$, since $x \notin \text{span}\{p_1(x), p_2(x)\}$

Theorem Let U and W be subspaces of vector space V , then we have

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

proof.

Let $\{\vec{v}_i\}_1^k$ be basis for $U \cap W$

$$\implies \dim(U \cap W) = k$$

Since $\{\vec{v}_i\}_1^k$ is basis for $U \cap W$ then it's a linearly independent subset of U

So it could be enlarged to basis for U , $\{\vec{v}_1, \dots, \vec{v}_k, \vec{y}_1, \dots, \vec{y}_r\}$

$$\text{So } \dim(U) = k + r$$

We also could enlarge a basis for W $\{\vec{v}_1, \dots, \vec{v}_k, \vec{z}_1, \dots, \vec{z}_s\}$

$$\implies \dim(W) = k + s$$

WTS. $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{y}_1, \dots, \vec{y}_r, \vec{z}_1, \dots, \vec{z}_s\}$ is a basis for $U + W$

$$\begin{aligned} \text{(If we could show this)} \quad \dim(U + W) &= k + r + s = (k + r) + (k + s) - k \\ &= \dim(U) + \dim(W) - \dim(U \cap W) \end{aligned}$$

Obviously, the above set spans $U + W$

WTS. $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{y}_1, \dots, \vec{y}_r, \vec{z}_1, \dots, \vec{z}_s\}$ is linearly independent

$$\text{Consider } a_1 \vec{v}_1 + \dots + a_k \vec{v}_k + b_1 \vec{y}_1 + \dots + b_r \vec{y}_r + c_1 \vec{z}_1 + \dots + c_s \vec{z}_s = \vec{0} \quad (\star)$$

$$\text{From } (\star) \implies \sum (c_i \vec{z}_i) = -\sum (a_i \vec{v}_i) - \sum b_i \vec{y}_i$$

$$\implies \sum (c_i \vec{z}_i) \in U \wedge \sum (c_i \vec{z}_i) \in W$$

$$\iff \sum (c_i \vec{z}_i) \in U \cap W$$

Since $\{\vec{v}_i\}$ is a basis for $U \cap W$

$$\implies \sum (c_i \vec{z}_i) = \sum (d_i \vec{v}_i)$$

$$\iff \sum (c_i \vec{z}_i) - \sum (d_i \vec{v}_i) = \vec{0} \in W$$

$$\implies c_i = d_i = 0 \text{ since } \{\vec{z}_i, \vec{v}_i\} \text{ is a basis}$$

Rewrite (\star)

$$\sum (a_i \vec{v}_i) + \sum b_i \vec{y}_i = \vec{0} \in U$$

$$\implies a_i = b_i = 0 \text{ since } \{\vec{v}_i, \vec{y}_i\} \text{ is a basis for } U$$

■

Corollary For direct sum, since the intersection is $\{\vec{0}\}$

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

Example Let U, W are subspaces of \mathbb{R}^3 such that $\dim(U) = \dim(W) = 2$, why is $U \cap W \neq \{\vec{0}\}$

Solutions Geometrically, U and W are planes through origin then the intersection would be a line through origin ($U \neq W$) or a plane through origin ($U = W$), so shown.

Question V is a vector space, $\dim(V) = n$, $U \neq W$ are subspaces of V but $\dim(U) = \dim(W) = (n - 1)$, proof:

1. $V = U + W$
2. $\dim(U \cap W) = (n - 2)$

7 Lecture7 Jan.30. 2018

7.1 Linear Transformations

Definition Let V, W be vector spaces, a function $T : V \rightarrow W$ is a **linear transformation**⁸ if

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) \quad \forall \vec{x}, \vec{y} \in V$ ⁹
2. $T(c\vec{x}) = cT(\vec{x}) \quad \forall \vec{x} \in V, c \in \mathbb{R}$ ¹⁰

Linear transformation preserves vector additions and scalar multiplications on vector spaces.

Theorem(Alternative definition) Transformation $T : V \rightarrow W$ is linear if and only if

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y}), \quad \forall \vec{x}, \vec{y} \in V, c, d \in \mathbb{R}$$

Linear transformations preserves linear combinations.

Example (form 223) Rotation through angle θ about the origin in \mathbb{R}^2 .

⁸In some textbooks, this is annotated as **linear mapping**.

⁹Notice that the vector additions on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

¹⁰Notice that the scalar multiplication on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

Example (from 223) Matrix transformation, let $A \in M_{m \times n}(\mathbb{R})$, transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as

$$T(\vec{x}) = A\vec{x}$$

is linear.

Example Derivative $T : P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$ defined by

$$T(\vec{p}(x)) = \vec{p}'(x)$$

Example Matrix transpose $T : M_{m \times n}(\mathbb{R}) \rightarrow M_{n \times m}(\mathbb{R})$ defined by

$$T(A) = A^T$$

7.2 Properties of linear transformations

Property(i) Linear transformation $T : V \rightarrow W$ are uniquely defined by their values on any basis for V .

proof.

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be any basis for V

Every vector $\vec{x} \in V$ can be uniquely written as some linear combination of the $\{\vec{v}_i\}_{i=1}^k$

$$\vec{x} = \sum_{i=1}^k c_i \vec{v}_i, \quad c_i \in \mathbb{R}, \quad \text{and } c_i \text{ are uniquely determined } \forall \vec{x} \in V$$

$$\implies T(\vec{x}) = T\left(\sum_{i=1}^k c_i \vec{v}_i\right)$$

$$= \sum_{i=1}^k c_i T(\vec{v}_i) \text{ since the transformation } T \text{ is linear.}$$

Since c_i s are uniquely determined by $\{\vec{v}_i\}_{i=1}^k$

so the value of $T(\vec{x})$ is uniquely determined by its value on basis vectors $\{\vec{v}_i\}_{i=1}^k$. ■

Property(ii) Let $T : V \rightarrow W$ be a linear transformation, let A be a subspace of vector space V , then the **image** $T(A)$ defined as

$$T(A) = \{T(\vec{x}) \mid \vec{x} \in A\}$$

called the image of A under linear transformation T is a subspace of W .
Linear transformation maps subspaces of V to subspaces of W .

proof.

Since A is a subspace so it's non-empty, therefore $\exists T(\vec{x}), \vec{x} \in A$

So $T(A) \neq \emptyset$

Let $\vec{w}_1, \vec{w}_2 \in T(A)$

$\implies \vec{w}_1 = T(\vec{x}_1), \vec{w}_2 = T(\vec{x}_2), \vec{x}_1, \vec{x}_2 \in A$

$\implies \vec{w}_1 + \vec{w}_2 = T(\vec{x}_1) + T(\vec{x}_2) = T(\vec{x}_1 + \vec{x}_2)$ since T is linear.

Since $\vec{x}_1 + \vec{x}_2 \in A$ by the definition of subspaces.

$\implies \vec{w}_1 + \vec{w}_2 \in T(A)$

So $T(A)$ is closed under vector addition.

Let $\vec{w} \in T(A)$

$\implies \vec{w} = T(\vec{x}), \vec{x} \in A$

Let $c \in \mathbb{R}$

Consider $c\vec{w} = cT(\vec{x}) = T(c\vec{x})$

Since $c\vec{x} \in A$

So $c\vec{w} \in T(A)$

So $T(A)$ is closed under scalar multiplication. ■

Property(derived from the definition) For all linear transformation $T : V \rightarrow W$, we have ¹¹

$$T(\vec{0}) = \vec{0}$$

Property(iii) Let transformation $T : V \rightarrow W$ be linear, let B be a subspace of W , then its **pre-image** defined as

$$T^{-1}(B) = \{\vec{x} \in V \mid T(\vec{x}) \in B\}$$

is a subspace of V . ¹²

¹¹In the equation, clearly, the zero vector on the left side of the equation is in space V and the zero vector on the right side is in space W .

¹²The pre-image and inverse share the same notation, but in this case, transformation T is not necessarily invertible.

proof.

$$\begin{aligned}
 & \text{Let } \vec{w}_1, \vec{w}_2 \in T^{-1}(B) \\
 & \implies T(\vec{w}_1), T(\vec{w}_2) \in B \\
 & \implies aT(\vec{w}_1) + bT(\vec{w}_2) \in B, \forall a, b \in \mathbb{R} \text{ since } B \text{ is a subspace.} \\
 & \implies T(a\vec{w}_1 + b\vec{w}_2) \in B \\
 & \implies a\vec{w}_1 + b\vec{w}_2 \in T^{-1}(B)
 \end{aligned}$$

So $T^{-1}(B)$ is closed under both vector addition and scalar multiplication,

So $T^{-1}(B)$ is a subspace. ■

7.3 Definitions

Let $T : V \rightarrow W$ to be a linear transformation,

Definition the **Image** of transformation T is defined as

$$Im(T) = T(V) = \{T(\vec{x}) \mid \vec{x} \in V\}$$

Definition the **Rank** of transformation T is defined as

$$Rank(T) = dim(Im(T))$$

Definition the **Kernel** of transformation T is defined as

$$Ker(T) = T^{-1}(\{\vec{0}\}) = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

Definition the **Nullity** of transformation T is defined as

$$Nullity(T) = dim(ker(T))$$

Example $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ is linear defined by

$$T(\vec{p}(x)) = \vec{p}(2x + 1) - 8\vec{p}(x)$$

find $Ker(T)$.

Theorem Let $T : V \rightarrow W$ be a linear transformation, let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be the spanning set of V ¹³, then $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ spans $Im(T)$

proof.

Let $\vec{w} \in Im(T)$

Since $V = span\{\vec{v}_1, \dots, \vec{v}_k\}$

For any $\vec{x} \in V$ can be written as

$$\vec{x} = \sum_{i=1}^k c_i \vec{v}_i, \quad c_i \in \mathbb{R}$$

$$\implies \vec{w} = T(\vec{x}) = T\left(\sum_{i=1}^k c_i \vec{v}_i\right)$$

$$= \sum_{i=1}^k c_i T(\vec{v}_i)$$

as a linear combination of $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$

So $Im(T) = span\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$

■

8 Lecture8 Jan.31 2018

8.1 Linear Transformations

Example $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$

$$T(p(x)) = p(2x + 1) - 8p(x)$$

Find the image of T .

We know $B = \{1, x, x^2, x^3\}$ is the standard basis for $P_3(\mathbb{R})$, consider the set $P(B)$

$$P(B) = \{-7, 1 - 6x, 1 + 4x - 4x^2, 1 + 6x + 12x^2\}$$

spans $Im(T)$. Notice the first three vectors in the set is linearly independent, the last vector is clearly dependent to the pervious three.¹⁴ So by the redundancy theorem we could remove the last vector. There we have

$$Im(T) = span\{-7, 1 - 6x, 1 + 4x - 4x^2\}$$

¹³The set is only the spanning set of V , it's not necessarily to be a basis of V .

¹⁴Notice that the first three vectors is a basis of $P_2(\mathbb{R})$.

as basis.

In this example, the dimension of $\text{Ker}(T)$ is 1 and the dimension of $\text{Im}(T)$ is 3, and dimension of $P_3(\mathbb{R})$ is 4. We have, $\dim(P_3(\mathbb{R})) = \text{Nullity}(T) + \text{Rank}(T)$

Theorem(Dimension Theorem) Let $T : V \rightarrow W$ be a linear transformation,

$$\dim(V) = \text{Nullity}(T) + \text{Rank}(T)$$

Proof.

Say $\dim(V) = n$

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\text{Ker}(T)$

Since $\text{Ker}(T)$ is a subspace of V , the set $\{\vec{v}_i\}_1^k$ is a subset of V ,

It can be extended to a basis $\{\vec{v}_i\}_1^k \cup \{\vec{v}_i\}_{k+1}^n$ for V .

Claim: $\{T(\vec{v}_{k+1}), \dots, T(\vec{v}_n)\}$ is basis for $\text{Im}(T)$

If the claim is true, this prove the theorem since

$$\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = k + n - k = n = \dim(V)$$

$$\because T(\vec{v}_i) = \vec{0}, \forall i \in \mathbb{Z}_1^k$$

and by the definition of kernel of linear transformation,

$$\therefore \{T(\vec{v}_i)\}_{k+1}^n \text{ spans } \text{Im}(T)$$

$$\text{Show if } \sum_{i=k+1}^n c_i T(\vec{v}_i) = \vec{0} \implies c_i = 0$$

$$\implies T\left(\sum_{i=k+1}^n c_i \vec{v}_i\right) = \vec{0}$$

$$\implies \sum_{i=k+1}^n c_i \vec{v}_i \in \text{Ker}(T)$$

$$\implies \sum_{i=k+1}^n c_i \vec{v}_i = \sum_{i=1}^k c_i \vec{v}_i$$

$$\implies c_1 \vec{v}_1 + \dots + c_k \vec{v}_k - c_{k+1} \vec{v}_{k+1} - \dots - c_n \vec{v}_n = \vec{0}$$

Since $\{\vec{v}_i\}_i^n$ is a basis for V .

$$\implies c_i = 0 \forall i$$

8.2 Applications of dimension theorem

Definition A linear transformation $T : V \rightarrow W$ is called **injective**(one-to-one) if and only if

$$T(\vec{v}_1) = T(\vec{v}_2) \implies \vec{v}_1 = \vec{v}_2$$

Definition A linear transformation $T : V \rightarrow W$ is called **surjective**(onto) if and only if

$$Im(T) = W$$

Every vector in W has a pre-image in V .

Definition A linear transformation $T : V \rightarrow W$ is called **bijective** if it's both injective and surjective.

Theorem Let transformation $T : V \rightarrow W$ is linear, T is injective if and only if $dim(Ker(T)) = 0$.

Proof.

Exercise

■

Theorem T is surjective if and only if $dim(Im(T)) = dim(W)$.

Example $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$$

is T injective? surjective?

Not injective but surjective.

Solution

$$Ker(T) = span\{(x-1)(x-2)\}$$

So T has nullity of 1 and since $dim(P_2(\mathbb{R})) = 3$, by the dimension theorem we have $Rank(T) = 2$ and since $Im(T)$ is a subspace of \mathbb{R}^2 which has dimension of 2, we could conclude that $Im(T) = \mathbb{R}^2$.

9 Lecture9 Feb.6 2018

9.1 Applications of dimension theorem

Recall Dimension Theorem $T : V \rightarrow W$ is linear transformation,

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$$

Recall T is **injective** if and only if $\dim(\text{Ker}(T)) = 0$.

Recall T is **surjective** if and only if $\dim(\text{Im}(T)) = \dim(W)$.

Example $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$ defined by

$$T(p(x)) = (p(1), p(2), p(3))$$

Take $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$, $p(x) \in \text{Ker}(T)$ iff $T(p(x)) \in \vec{0}$.

Let $p(x) \in \text{Ker}(T)$,

Obviously the only solution for the system

$$\begin{cases} a + b + c = 0 \\ a + 2b + 4c = 0 \\ a + 3b + 9c = 0 \end{cases}$$

is $a = b = c = 0$, So $\dim(\text{Ker}(T)) = 0$. Therefore, T is **injective**.

By *dimension theorem*,

$$\dim(V) = 3 = 0 + \dim(\text{Im}(T)) \implies \dim(\text{Im}(T)) = 3 = \dim(\mathbb{R}^3)$$

therefore T is **surjective**. Therefore, T is called **bijective**.

Question $T : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$

$$T(p(x)) = xp'(x)$$

Solution Not injective because any *constant function* in $P_n(\mathbb{R})$ is mapped onto $\vec{0} \in P_n(\mathbb{R})$. Also **not surjective** by the dimension theorem.

Theorem Let $T : V \rightarrow W$ be an injective linear transformation, if $\{\vec{v}_i\}_{i=1}^k$ is linearly independent in V , then the set $\{T(\vec{v}_i)\}_{i=1}^k$ is linearly independent in W .

Injective transformation maps linearly independent set to linear independent set.

Proof.

If $\sum c_i T(\vec{v}_i) = \vec{0}$, then we have $T(\sum c_i \vec{v}_i) = \vec{0}$, which means $\sum c_i \vec{v}_i \in \text{Ker}(T)$. By definition of injective transformation, $\sum c_i \vec{v}_i = \vec{0}$. Since $\{\vec{v}_i\}_{i=1}^k$ is linearly independent, so $c_i = 0, \forall i$. ■

Theorem $T : V \rightarrow W$ is a linearly transformation, $\{\vec{v}_i\}_{i=1}^n$ is a basis for V then, if $\{T(\vec{v}_i)\}_{i=1}^n$ is linear independent, then T is injective.
A criteria for T to be injective based on image of a basis.

Proof.

Let $\{\vec{v}_i\}_{i=1}^n$ be a basis of V

Consider $T(\vec{x}) = \vec{0}$

Since $\{\vec{v}_i\}_{i=1}^n$ is a basis

Let $x = \sum c_i \vec{v}_i$

$$T(\vec{x}) = \vec{0} \iff T(\sum c_i \vec{v}_i) = \vec{0}$$

$$\implies \sum c_i T(\vec{v}_i) = \vec{0} \implies c_i = 0$$

$$\therefore \vec{x} = \sum 0 \vec{v}_i = \vec{0}$$

Therefore $\text{Ker}(T) = \{\vec{0}\}$

Therefore $\dim(\text{Ker}(T)) = 0$

\implies injective

■

Theorem Let $T : V \rightarrow W$ be a linear transformation,

1. If $\dim(V) > \dim(W)$, then T cannot be injective.
2. If $\dim(V) < \dim(W)$, then T cannot be surjective.

For a linear transformation between spaces with different dimension, it could not be bijective.

Proof.

$$\begin{aligned} \dim(V) &= \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) \\ &\because \dim(\text{Im}(T)) \leq \dim(W) \\ \therefore \dim(V) &\leq \dim(\text{Ker}(T)) + \dim(W) \\ \implies \dim(\text{Ker}(T)) &\geq \dim(V) - \dim(W) \\ &\implies \dim(\text{Ker}(T)) > 0 \end{aligned}$$

So T could not be injective

$$\begin{aligned} \dim(V) &= \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) \\ &\because \dim(\text{Ker}(T)) \geq 0 \\ \therefore \dim(V) &\geq \dim(\text{Im}(T)) \\ \implies \dim(\text{Im}(T)) &< \dim(W) \end{aligned}$$

So T could not be surjective

■

Theorem *Half is good enough* Let $T : V \rightarrow W$ is linear, and $\dim(V) = \dim(W)$. T is injective if and only if surjective.

Proof.

By dimension theorem

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(W)$$

If injective $\dim(\text{Ker}(T)) = 0$

$$\implies \dim(\text{Im}(T)) = \dim(W)$$

So surjective

If surjective $\dim(\text{Im}(T)) = \dim(W) = \dim(V)$

$$\implies \dim(\text{Ker}(T)) = 0$$

So injective

■

9.2 Isomorphisms

Recall If $T : V \rightarrow W$ is both injective and surjective, say T is bijective.

Definition If $T : V \rightarrow W$ is bijective, we call T an **isomorphism**. If there exists an isomorphism $T : V \rightarrow W$ say V and W are **isomorphic** vector spaces.

Theorem V, W are isomorphic iff $\dim(V) = \dim(W)$.

Proof.

$$\rightarrow V, W \text{ isomorphic} \implies \dim(V) = \dim(W)$$

Isomorphic means there exists a bijective transformation T

$$\begin{aligned} \text{By dimension theorem } \dim(V) &= \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) \\ &= 0 + \dim(W) \end{aligned}$$

$$\leftarrow \dim(V) = \dim(W) \implies V, W \text{ isomorphic}$$

Equivalently, find a bijective transformation

Let $\{\vec{v}_i\}_{i=1}^n$ be basis for V

Let $\{\vec{w}_i\}_{i=1}^n$ be basis for W

Claim $T : V \rightarrow W$ is linear and s.t.

$T(\vec{v}_i) = \vec{w}_i$ is an isomorphism.

If $\vec{x} \in \text{Ker}(T) \subseteq V$

$$x = \sum c_i \vec{v}_i$$

$$\vec{0} = T(\vec{x})$$

$$= \sum c_i T(\vec{v}_i)$$

$$= \sum (c_i \vec{w}_i)$$

$$\implies c_i = 0 \text{ since } \vec{w}_i \text{ are basis.}$$

$$\implies \vec{x} = \vec{0}$$

$$\implies \dim(\text{Ker}(T)) = 0$$

$$\implies \text{injective} \iff \text{surjective}$$

■

Note if $T : V \rightarrow W$ is an isomorphism, then T maps a basis for V to a basis for W .

Example $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$,

$$T(p(x)) = (p(1), p(2), p(3))$$

is an isomorphism. And $P_2(\mathbb{R})$ and \mathbb{R}^3 are isomorphic.

Example $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$,

$$T(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an isomorphism.

Example $M_{2 \times 2}(\mathbb{R})$, $P_3(\mathbb{R})$ and \mathbb{R}^4 are isomorphic.

Theorem Any n-dim vector space V is isomorphic to \mathbb{R}^n . What is an isomorphism $T : V \rightarrow \mathbb{R}^n$

Procedure:

Let $\{\vec{v}_i\}_{i=1}^n$ be any basis for V

We know that $\forall \vec{x} \in V$,

By property of basis,

$$\vec{x} = \sum c_i \vec{v}_i$$

Then $T(\vec{x}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ is an isomorphism.

9.3 Coordinates

Definition Let V be a vector space, $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ be any basis for V , $\forall \vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

then c_1, \dots, c_n is called the coordinates for \vec{x} relative to α , with notation

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \iff \vec{x} = \sum c_i \vec{v}_i$$

Claim $[\vec{x} + c\vec{y}]_\alpha = [\vec{x}]_\alpha + c[\vec{y}]_\alpha \quad \forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}.$

Note if α, α' are any two bases for V then generally $[\vec{x}]_\alpha \neq [\vec{x}]_{\alpha'}$ (except $\vec{0}$).

10 Lecture10 Feb.7 2018

10.1 Matrix of linear transformation

Recall Let V be a vector space, let α be any basis for V .

$$\forall \vec{x} \in V, x = \sum c_i \vec{v}_i$$

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

So transformation $\vec{x} \rightarrow [\vec{x}]_\alpha$ is an isomorphism that $V \rightarrow \mathbb{R}^n$.

Say W is a vector space and let $\beta = \{\vec{w}_i\}_1^m$ be any basis of W , say $T : V \rightarrow W$ is linear.

$$T(\vec{x}) = \sum c_i T(\vec{v}_i)$$

So that

$$\begin{aligned} [T(\vec{x})]_\beta &= [\sum c_i T(\vec{v}_i)]_\beta = \sum c_i [T(\vec{v}_i)]_\beta \\ &= \begin{bmatrix} [T(\vec{v}_1)]_\beta & \dots & [T(\vec{v}_n)]_\beta \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \end{aligned}$$

$\begin{bmatrix} [T(\vec{v}_1)]_\beta & \dots & [T(\vec{v}_n)]_\beta \end{bmatrix}$ is called the the matrix of T w.r.t. α, β . Denoted as $[T]_\alpha^\beta$

$$[T(\vec{x})]_\beta = [T]_\alpha^\beta [\vec{x}]_\alpha$$

Example $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$

$$T(p(x)) = xp(x)$$

$$\alpha = \{1 - x, 1 - x^2, x\}, \quad \beta = \{1, 1 + x, 1 + x + x^2, 1 - x^3\}$$

Find $[T]_{\alpha}^{\beta}$.

$$T(1-x) = x(1-x) = x - x^2$$

$$x - x^2 = (-1)(1) + 2(1+x) + (-1)(1+x+x^2) + 0(1-x^3)$$

$$[T(1-x)]_{\beta} = (-1, 2, -1, 0)$$

$$T(1-x^2) = x - x^3$$

$$[T(1-x^2)]_{\beta} = (-2, 1, 0, 1)$$

$$[T(x)] = x^2$$

$$[T(x)]_{\beta} = (0, -1, 1, 0)$$

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Picture V, W are vectors spaces, $\alpha = \{\vec{v}_1, \dots, vecv_n\}$ is a basis for V and $\beta = \{\vec{w}_1, \dots, vecw_m\}$ is a basis for W .

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow [\]_{\alpha} & & \downarrow [\]_{\beta} \\ \mathbb{R}^n & \xrightarrow{[T]_{\alpha}^{\beta}} & \mathbb{R}^m \end{array}$$

Note

1. $\vec{x} \in Ker(T) \iff T(\vec{x}) = \vec{0} \iff [T(x)]_{\beta} = [\vec{0}]_{\beta} \in \mathbb{R}^m \iff [T]_{\alpha}^{\beta}[\vec{x}]_{\alpha} = 0 \iff [\vec{x}]_{\alpha} \in Ker([T]_{\alpha}^{\beta})$
2. $\vec{w} \in Im(T) \iff [\vec{w}]_{\beta} \in Col([T]_{\alpha}^{\beta})$

Theorem(Rank nullity for transformation matrix)

$$dim(Ker([T]_{\alpha}^{\beta})) + dim(Col([T]_{\alpha}^{\beta})) = n$$

Example $T : P_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$

$$T(a + bx + c^2) = \begin{bmatrix} c & -c \\ a - c & a + c \end{bmatrix}$$

And given bases $\alpha = \{x^2 - x, x - 1, x^2 + 1\}$ and $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Answer

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}([T]_{\alpha}^{\beta}) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Nul}(T) = \text{span}\{2x\}$$

$$\text{Col}([T]_{\alpha}^{\beta}) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\text{Col}(T) = \text{span}\left\{ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \right\}$$

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11.1 Algebra of Transformation

Recall Let $T : V \rightarrow W$ be a linear transformation, where $\alpha = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \dots, \vec{w}_m\}$ are bases for V, W respectively.

$$\vec{x} \in \text{Ker}(T) \iff [\vec{x}]_{\alpha} \in \text{Ker}([T]_{\alpha}^{\beta})$$

$$\vec{x} \in \text{Im}(T) \iff [\vec{x}]_{\alpha} \in \text{Col}([T]_{\alpha}^{\beta})$$

Definition $T_1, T_2 : V \rightarrow W$ are linear transformations, define

$$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x}) \forall \vec{x} \in V$$

$$(cT_1)(\vec{x}) = c(T_1(\vec{x})) \forall \vec{x} \in V, c \in \mathbb{R}$$

And, let α and β be bases for V, W respectively, then,

$$[T_1]_{\alpha}^{\beta} + [T_2]_{\alpha}^{\beta} = [T_1 + T_2]_{\alpha}^{\beta}$$

$$c[T_1]_{\alpha}^{\beta} = [cT_1]_{\alpha}^{\beta}$$

Definition $T : V \rightarrow W$ and $S : W \rightarrow U$ are linear transformations, then the **composition** $ST : V \rightarrow U$ is defined as

$$(ST)(\vec{x}) = S(T(\vec{x})) \quad \forall \vec{x} \in V$$

Note If S, T are *linear* then the composition ST is also *linear*.
Check

$$\begin{aligned} \text{Let } a, b \in \mathbb{R}, \vec{x}, \vec{y} \in V \\ ST(a\vec{x} + b\vec{y}) \\ &= S(T(a\vec{x} + b\vec{y})) \\ &= S(aT(\vec{x}) + bT(\vec{y})) \\ &= a(ST(\vec{x})) + b(ST(\vec{y})) \end{aligned}$$

Example
omitted

11.2 Matrix of composition

Consider $T : V \rightarrow W$ and $S : W \rightarrow U$ as linear transformations, let α, β, γ be bases of V, W, U respectively.
 We know how to compute $[T]_{\alpha}^{\beta}$ and $[S]_{\beta}^{\gamma}$. Now want to find $[ST]_{\alpha}^{\gamma}$.

$$\begin{aligned} \forall \vec{x} \in V, [ST]_{\alpha}^{\gamma}[\vec{x}]_{\alpha} \\ &= [(ST)(\vec{x})]_{\gamma} \\ &= [S(T(\vec{x}))]_{\gamma} \\ &= [S]_{\beta}^{\gamma}[T(\vec{x})]_{\beta} \\ &= [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}[\vec{x}]_{\alpha} \end{aligned}$$

This holds true for all $\vec{x} \in V$

$$\therefore [ST]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$$

Conclusion the matrix of ST = matrix of $S \times$ matrix of T .

11.3 Inverse transformations

Definition $T : V \rightarrow W$ is *isomorphism*¹⁵ if and only if there exists function $S : W \rightarrow V$ such that

$$(ST)(\vec{v}) = \vec{v} \quad \forall \vec{v} \in V \quad \wedge \quad (TS)(\vec{w}) = \vec{w} \quad \forall \vec{w} \in W$$

And S is called the **inverse** of T , written as T^{-1} .

proof. \rightarrow T is an isomorphism means every vector in W has an unique pre-image in V the function $S : W \rightarrow V$ maps *every* vector in W to its *unique* pre-image in V , so S is the inverse of T .

proof. \leftarrow Assume $S : W \rightarrow V$ is the inverse of $T : V \rightarrow W$ then $T(S(\vec{y})) = \vec{y} \quad \forall \vec{y} \in W$, this means T is surjective since every $\vec{y} \in W$ has pre-image under T , that's $S(\vec{y}) \in V$. Now suppose $T(\vec{x}_1) = T(\vec{x}_2)$, apply transformation S on both sides of the equation, $S(T(\vec{x}_1)) = S(T(\vec{x}_2))$ we have $\vec{x}_1 = \vec{x}_2$. This implies the transformation is injective. Therefore, transformation T is bijective, that's isomorphism. ■

Note $T^{-1}(\vec{y})$ is the unique vector \vec{x} , s.t. $T(\vec{x}) = \vec{y}$. That's

$$T(\vec{x}) = \vec{y} \iff T^{-1}(\vec{y}) = \vec{x}$$

¹⁵Recall that isomorphism is equivalent to bijective.

Theorem If $T : V \rightarrow W$ is an isomorphism then the inverse of T , T^{-1} , then $T^{-1} : W \rightarrow V$ is linear.¹⁶

Proof.

$$\text{WTS } T^{-1}(a\vec{w}_1 + b\vec{w}_2) = aT^{-1}(\vec{w}_1) + bT^{-1}(\vec{w}_2) \forall a, b \in \mathbb{R}, \forall \vec{w}_1, \vec{w}_2 \in W$$

$$T^{-1}(\vec{w}_1) \text{ is the unique } \vec{x}_1 \text{ s.t. } T(\vec{x}_1) = \vec{w}_1$$

$$T^{-1}(\vec{w}_2) \text{ is the unique } \vec{x}_2 \text{ s.t. } T(\vec{x}_2) = \vec{w}_2$$

$$T^{-1}(a\vec{w}_1 + b\vec{w}_2) \text{ is the unique } \vec{x} \text{ s.t. } T(\vec{x}) = a\vec{w}_1 + b\vec{w}_2$$

$$\because T(\vec{x}) = a\vec{w}_1 + b\vec{w}_2$$

$$= aT(\vec{x}_1) + bT(\vec{x}_2)$$

$$= T(a\vec{x}_1 + b\vec{x}_2)$$

$$\therefore \vec{x} = a\vec{x}_1 + b\vec{x}_2$$

$$\text{Also } T(\vec{x}) = a\vec{w}_1 + b\vec{w}_2$$

$$\therefore \vec{x} = T^{-1}(a\vec{w}_1 + b\vec{w}_2) = a\vec{x}_1 + b\vec{x}_2$$

$$= aT^{-1}(\vec{w}_1) + bT^{-1}(\vec{w}_2)$$

■

Theorem $T : V \rightarrow W$ is isomorphism, then let α and β are bases of V and W representing then $[T]_{\alpha}^{\beta}$ is invertible, and

$$([T]_{\alpha}^{\beta})^{-1} = [T^{-1}]_{\alpha}^{\beta}$$

Proof.

omitted

11.4 Change of basis

What's the effect of a change of basis on coordinate of a vector and matrix of transformation.

Theorem Let α and α' be two bases of V , then

$$[I]_{\alpha}^{\alpha'} [\vec{x}]_{\alpha} = [\vec{x}]_{\alpha'}$$

¹⁶Note: the conclusion could be changed into isomorphism.

Proof.

$$\begin{aligned} \text{Let } \vec{x} &\in V \\ I(\vec{x}) &= \vec{x} \\ [I(\vec{x})]_{\alpha'} &= [\vec{x}]_{\alpha'} \\ [I]_{\alpha}^{\alpha'} [\vec{x}]_{\alpha} &= [\vec{x}]_{\alpha'} \end{aligned}$$

■

$[I]_{\alpha}^{\alpha'}$ is called the change of basis matrix from α to α' .

Computation Let $\alpha = \{\vec{a}_1, \dots, \vec{a}_n\}$, then

$$[I]_{\alpha}^{\alpha'} = [[\vec{a}_1]_{\alpha'} \mid \dots \mid [\vec{a}_n]_{\alpha'}]$$

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Recall Let α and β be bases for V and $I : V \rightarrow V$ is the identity transformation, then

$$[I]_{\alpha}^{\beta} [\vec{x}]_{\alpha} = [\vec{x}]_{\beta}$$

Also,

$$[I]_{\beta}^{\alpha} [\vec{x}]_{\beta} = [\vec{x}]_{\alpha}$$

Example Let $\alpha = \{x^2, 1+x, x+x^2\}$ and β be bases for $P_2(\mathbb{R})$ and

$$[I]_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } [p(\vec{x})]_{\beta} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution omitted

Theorem Suppose $[T]_V^W$ is linear, α and α' are any two bases for V and β and β' are any two bases of W , then,

$$[T]_{\alpha'}^{\beta'} = [I]_{\beta}^{\beta'} [T]_{\alpha}^{\beta} [I]_{\alpha'}^{\alpha}$$

Proof.

Recall $T = ITI$

Consider let $\vec{x} \in V$

$$\begin{aligned}
 [I]_{\beta}^{\beta'} [T]_{\alpha}^{\beta} [I]_{\alpha'}^{\alpha} [\vec{x}]_{\alpha'} & \\
 &= [I]_{\beta}^{\beta'} [T]_{\alpha}^{\beta} [\vec{x}]_{\alpha} \\
 &= [I]_{\beta}^{\beta'} [T(\vec{x})]_{\beta} \\
 &= [T(\vec{x})]_{\beta'} \\
 &= [T]_{\beta'}^{\alpha'} [\vec{x}]_{\alpha'} \\
 \implies [T]_{\beta'}^{\alpha'} &= [I]_{\beta}^{\beta'} [T]_{\alpha}^{\beta} [I]_{\alpha'}^{\alpha}
 \end{aligned}$$

■

Also,

$$[T]_{\alpha}^{\beta} = [I]_{\beta'}^{\beta} [T]_{\alpha'}^{\beta'} [I]_{\alpha}^{\alpha'}$$

Special Case Consider when $V = W$, $\alpha = \beta$ and $\alpha' = \beta'$. we have

$$[T]_{\alpha'}^{\alpha'} = [I]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I]_{\alpha'}^{\alpha}$$

where

$$([I]_{\alpha}^{\alpha'})^{-1} = [I]_{\alpha'}^{\alpha}$$

the equation becomes

$$[T]_{\alpha'}^{\alpha'} = ([I]_{\alpha}^{\alpha'})^{-1} [T]_{\alpha}^{\alpha} [I]_{\alpha'}^{\alpha}$$

and can be written in the form of

$$B = P^{-1}AP$$

Definition Two matrices A and B are **similar** if there exists an invertible matrix P s.t.

$$B = P^{-1}AP$$

A and B representing the same transformation relative to different bases and P is the change of basis matrix if and only if A and B are similar.

Example Omitted

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13.1 Recall

Omitted.

13.2 Diagonalization

Definition Consider a *linear operator* $T : V \rightarrow V$ is **diagonalizable** if and only \exists a basis β for V s.t.

$$[T]_{\beta}^{\beta}$$

is diagonal.

Note Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis, $T : V \rightarrow V$ is diagonalizable if and only it's in form
$$\begin{bmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

Definition $T : V \rightarrow V$ is a linear operator on V , a *non-zero* vector $\vec{x} \in V$ is an **eigenvector** of $T(\vec{x}) = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$. λ is called the **eigenvalue** of T corresponding to vector \vec{x} .

Theorem Linear operator $T : V \rightarrow V$ is *diagonalizable* if and only exist a basis of V consisting of *eigenvectors* of T . If T is diagonalizable, the diagonal entries of $[T]_{\beta}^{\beta}$ are corresponding eigenvalues of T , in the same order.

13.3 How to find eigenvalues and eigenvectors of T

Definition The **determinant** of T is defined as $\det([T]_{\alpha}^{\alpha})$ for any basis α for V .

Note The determinant of linear operator T does not depends on the choice of basis of α for V , *since similar matrices have the same determinant*.

Theorem $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if

$$\det(T - \lambda I) = 0$$

Proof. λ is an eigenvalue of T

$$\begin{aligned}
 &\iff \exists \vec{x} \in V, \vec{x} \neq \vec{0}, \text{ s.t. } T(\vec{x}) = \lambda \vec{x} \\
 &\iff T(\vec{x}) - \lambda \vec{x} = \vec{0} \\
 &\iff (T - \lambda I)(\vec{x}) = \vec{0} \\
 &\iff \vec{x} \in \text{Ker}(T - \lambda I) \\
 &\therefore \text{Ker}(T - \lambda I) \neq \{\vec{0}\} \\
 &\iff (T - \lambda I) \text{ is not injective} \\
 &\iff [T - \lambda I]_{\alpha}^{\alpha} \text{ is not injective} \\
 &\iff \det([T - \lambda I]_{\alpha}^{\alpha}) = \det(T - \lambda I) = 0
 \end{aligned}$$

■

Note $\det(T - \lambda I) = 0$ is called the **characteristic polynomial** of T , written as $P_T(\lambda) := \det(T - \lambda I)$, the degree of $P_T(\lambda)$ is the dimension of V .

Note λ is an eigenvalue $\iff \lambda$ is a root of $P_T(\lambda)$.

Theorem $T : V \rightarrow V$ is a linear operator and λ is an eigenvalue of T , \vec{x} is an eigenvector of T corresponding to eigenvalue λ , if and only if

$$\vec{x} \neq \vec{0} \wedge \vec{x} \in \text{Ker}(T - \lambda I)$$

Proof.

Exercise, using definition

■

Definition $\text{Ker}(T - \lambda I)$ is called the **eigenspace** of T corresponding to eigenvalue λ , noted as $E_{\lambda}(T)$, which is a subspace of V .

Note To find eigenvalues and eigenvectors of $T : V \rightarrow V$, choose any basis β for V , \vec{x} is an eigenvector with corresponding eigenvalue λ if and only if $[\vec{x}]_{\beta}$ is an eigenvector of $[T]_{\beta}^{\beta}$ with corresponding eigenvalue λ .

That's

$$\begin{aligned} T(\vec{x}) &= \lambda \vec{x} \\ \implies [T(\vec{x})]_\beta &= [\lambda \vec{x}]_\beta \\ \iff [T]_\beta^\beta [\vec{x}]_\beta &= \lambda [\vec{x}]_\beta \end{aligned}$$

Note Consider diagonalization in MAT223,

$$D = P^{-1}AP$$

Let D and A representing the same linear operator $[T]_V^V$ and let β be a basis of V consisting of eigenvectors of T and α is another basis of V . Then, the above equation is

$$[T]_\beta^\beta = ([I]_\beta^\alpha)^{-1} [T]_\alpha^\alpha [I]_\beta^\alpha$$

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Theorem Suppose λ_0 is an eigenvalue of linear operator $T : V \rightarrow V$, let $\dim(E_{\lambda_0}) = k$, then $(\lambda - \lambda_0)^k$ divides $P_T(\lambda)$

Proof.

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be basis for E_{λ_0}

Let $\dim(V) = n$

Extend basis of E_{λ_0} to basis of V .

$$\alpha = \{\vec{v}_1, \dots, \vec{v}_k\} \cup \{\vec{v}_{k+1}, \dots, \vec{v}_n\}$$

Since $\vec{v}_i \in E_{\lambda_0}$,

$$\text{Therefore } T(\vec{v}_i) = \lambda_0 \vec{v}_i$$

$$[T]_\alpha^\alpha = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

Where $A = \text{diag}(\lambda_0, \dots, \lambda_0) \in \mathbb{M}_{k \times k}(\mathbb{R})$

And $B \in \mathbb{M}_{k \times n-k}(\mathbb{R}), D \in \mathbb{M}_{n-k \times n-k}(\mathbb{R})$

$$P_T(\lambda) = \det(A - \lambda I) * \det(D - \lambda I)$$

$$= (\lambda_0 - \lambda)^k * \det(D - \lambda I)$$

$$\text{Therefore } (\lambda - \lambda_0)^k \mid P_T(\lambda)$$

■

Definition The **multiplicity** of eigenvalue λ_0 is the number of times $(\lambda - \lambda_0)$ appears as a factor in $P_T(\lambda)$.

Note If eigenvalue λ has multiplicity m , the above theorem says

$$1 \leq \dim(E_\lambda) \leq m$$

if $m = 1$, then $\dim(E_\lambda) = 1$.

Theorem If $\lambda_1, \dots, \lambda_k$ are *distinct* eigenvalues of $T : V \rightarrow V$ and $\alpha = \{\vec{x}_1, \dots, \vec{x}_k\}$ are corresponding eigenvectors, then the set α is *linearly independent*.

Proof.

Exercise

■

(★)**Theorem** Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T , suppose the characteristic polynomial is in form

$$P_T(\lambda) = \prod_{i=1}^k (\lambda - \lambda_i)^{m_i}$$

the T is diagonalizable if and only if

$$\dim(E_{\lambda_i}) = m_i, \quad \forall i$$

Note Also, $\sum_{i=1}^k m_i = \dim(V) = n$

Proof.

←

Assume $\dim(E_{\lambda_i}) = m_i \forall i$

Consider E_{λ_i}

Take basis for E_{λ_i} , as $\{\vec{v}_1^i, \dots, \vec{v}_{m_i}^i\}$

Claim: the union of bases of $E_{\lambda_i} \forall i$ gives a basis consisting of eigenvectors of T .

$$\text{Note } |\cup_{i=1}^k \{\vec{v}_1^i, \dots, \vec{v}_{m_i}^i\}| = \sum_{i=1}^k m_i = \dim(V)$$

All we need to show is linear independence.

$$\text{Consider } \sum_{i=1}^k \sum_{j=1}^{m_i} c_{ij} \vec{v}_j^i = \vec{0}(\star)$$

$$\text{Consider } \sum_{j=1}^{m_i} c_{ij} \vec{v}_j^i \in E_{\lambda_i} = \vec{x}_i$$

$$\text{So } (\star) \text{ becomes } \sum_{i=1}^k \vec{x}_i = \vec{0} \text{ where } \vec{x}_i \in E_{\lambda_i}, \forall i$$

Since \vec{x}_i is eigenvectors for T , corresponding to different eigenvalues,

Therefore, $\{\vec{x}_1, \dots, \vec{x}_k\}$ is linearly independent

$$\text{So } \vec{x}_i = \vec{0} \forall i$$

$$\text{That's } \sum_{j=1}^{m_i} c_{ij} \vec{v}_j^i = \vec{x} = \vec{0} \forall i$$

$$\implies c_{ij} = 0 \forall i, j$$

Therefore linearly independent, so exists basis for V consisting of eigenvectors,

Therefore T is diagonalizable.

→

Suppose T is diagonalizable,

Since T is diagonalizable, then exists basis for V consisting of eigenvectors, say α

$$\text{Consider } [T]_{\alpha}^{\alpha} = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_1 & \dots & \dots & 0 \\ 0 & \dots & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Where λ_1 takes first m_1 rows, λ_2 takes the next m_2 rows, etc.

$$\begin{aligned}
P_T(\lambda) &= \det([T]_\alpha^\alpha - \lambda I) \\
&= \prod_{i=1}^k (\lambda_i - \lambda)^{m_i} \\
\text{Since } 1 &\leq \dim(E_{\lambda_i}) \leq m_i \quad \forall i \\
&\implies \dim(E_{\lambda_i}) = m_i \quad \forall i
\end{aligned}$$

■

15 Lecture16 Mar. 7 2018

15.1 Vector space over a field

Definition A vector space over field F is a set V together with two operations, addition and scalar multiplication s.t. [Very similar to those those defining properties for real vector space.]

15.2 Complex vector space

Complex vector space $\mathbb{C}^n = \{(z_1, \dots, z_n) | z_1, \dots, z_n \in \mathbb{C}\}$ is a vector space over \mathbb{C} , with dimension n and standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$

Let F be a field, then

$$F^n = \{(x_1, \dots, x_n) | x_1, \dots, x_n \in F\}$$

and

$$\dim(F^n) = n$$

F^n is a *vector space over field F w.r.t. usual coordinate wise addition and scalar multiplication.*

Definition Let V vector space over field F , then $\{\vec{x}_1, \dots, \vec{x}_n\}$ is **linearly independent** if and only if

$$\sum_{i=1}^i c_i \vec{x}_i = \vec{0}, \quad c_1, \dots, c_n \in F \implies c_1 = \dots = c_n = 0 \in F$$

Definition $\text{span}\{\vec{x}_1, \dots, \vec{x}_n\}$ is defined as

$$\left\{ \sum_{i=1}^n c_i \vec{x}_i \mid c_1, \dots, c_n \in F \right\}$$

Definition Consider V, W as two vector spaces over fields F then transformation $T : V \rightarrow F$ is **linear** if and only if

$$\forall \vec{v}_1, \vec{v}_2 \in V, c, d \in F, T(c\vec{v}_1 + d\vec{v}_2) = cT(\vec{v}_1) + dT(\vec{v}_2)$$

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Theorem Let $T : V \rightarrow V$ be a linear operator, and β is a basis for vector space V . Let W_i be the span of first i vectors in β , then $[T]_\beta^\beta$ is *upper-triangular* if and only if

$$T(W_i) \subset W_i, \forall i$$

Definition Let $T : V \rightarrow V$ be a linear operator, a subspace W of V is called **invariant** under T (T -invariant) if $T(W) \subset W$.

Examples For linear operator T ,¹⁷

1. (Trivial invariant subspaces) $V, \{\vec{0}\}$
2. $\text{Ker}(T)$
3. $\text{Im}(T)$
4. $E_\lambda(T)$ for any eigenvalue λ of T
5. $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$T((x, y, z)) = (3x + 2y, y - z, 4x + 2y - z)$$

Then subspace of \mathbb{R}^3 : $W = \{(x, y, x) \mid x, y \in \mathbb{R}\}$ is T -invariant.

Theorem Let $T : V \rightarrow V$ be a linear operator, $\beta = \{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis for V , then $[T]_\beta^\beta$ is upper-triangular if and only if W_i is T -invariant for all $i \leq k$.

Note $\{\vec{0}\} \subset W_1 \subset W_2 \subset W_3 \cdots \subset W_k = V$

Definition Linear operator $T : V \rightarrow V$ is said to be **triangularizable** if there exists a basis β for V such that $[T]_\beta^\beta$ is upper-triangular.

¹⁷Proofs are omitted.

Note If $[T]_{\beta}^{\beta}$ is upper-triangular, the characteristic polynomial $P_T(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$ where λ_i are entries on the main diagonal.

Problem Entries above main diagonal are **not** uniquely determined by T , it's also depends on the choice of basis β .