ECO421: Topics in Economic of Information

Games with Incomplete Information

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1 Knowledge

1.1 Information Structure

Notation 1.1. Let Ω denote the all possible states of world, and let A denote the set of all agents. Let $\omega^* \in \Omega$ denote the true state of world.

Definition 1.1. Every $E \subseteq \Omega$ is called an **event** or a type/piece of **information**.

Definition 1.2. Let $E \subseteq \Omega$ be a piece of information and $i \in A$, and we say agent i **knows** E in state $\omega \in \Omega$ if this agent knows

(i) The true state of world $\omega^* \in E$;

(ii) but, all $\omega \in E$ are considered to be possible.

Remark: the agent is certain about the true state ω^* if and only if $E = \{\omega^*\}$.

Definition 1.3. For an agent $i \in A$, the **information structure** of this agent, $\mathcal{T}_i$, is a partition of Ω into types of information.

Notation 1.2. Each element of the partition corresponds to one piece of information (an event).
(Information encoding) One may define a mapping $T_i(\cdot) : \Omega \to \mathcal{T}_i$, such that for every $\omega \in \Omega$, let $T_i(\omega) \in \mathcal{T}_i$ denote the piece of information known by this agent in state ω.
Equivalently, $T_i(\omega) \subseteq \Omega$ denotes the set of all states that agent i considers possible given her information in state ω.

Remark 1.1 (Interpretation)**.** At some state of world, ω^*, agent i receives information $T_i(\omega^*)$ but not ω^*. This agent only knows $\omega^* \in T_i(\omega^*)$. The information encoding mechanism is a kind of blurring system preventing the agent from identifying the exact ω^*.

1.2 Knowledge Space

Definition 1.4. A **knowledge-based type space** is defined to be a tuple

$$(\Omega, (\mathcal{T}_i)_{i\in A}) \tag{1.1}$$

where $\mathcal{T}_i$ is an information structure of agent i.

Definition 1.5. An event $E \subseteq \Omega$ is **true** in state $\omega^* \in \Omega$ if $\omega^* \subseteq E$.

Definition 1.6 (Equivalent Definition)**.** An agent i is said to **know** event E in state ω^* if and only if

$$T_i(\omega^*) \subseteq E \tag{1.2}$$

That is, the true state $\omega^* \in T_i(\omega) \subseteq E$, so that agent i is certain that E is true.

Definition 1.7. Define $K_i(E)$ to be the set of states in which agent i knows E to be true, that is,

$$K_i(E) := \{\omega : T_i(\omega) \subseteq E\} \subseteq \Omega \tag{1.3}$$

1.3 Learning

Definition 1.8. Given a piece of information $F \subseteq \Omega$ characterizing that $\omega^* \in F$, suppose F is known by all agents, the new **information structure updated by** F is defined to be

$$(\Omega^F, (T_i^F(\cdot))) \tag{1.4}$$

where

$$\Omega^F = \Omega \cap F \tag{1.5}$$
$$T_i^F(\omega) = T_i(\omega) \cap F \quad \forall \omega \in \Omega^F \tag{1.6}$$

Definition 1.9. The set of states where $E \subseteq \Omega^F$ is known given F is

$$K_i(E|F) := \{\omega \in \Omega^F : T_i^F(\omega) \subseteq E\} \tag{1.7}$$
$$= \{\omega \in \Omega^F : T_i(\omega) \cap F \subseteq E\} \tag{1.8}$$

Notation 1.3. For $i, j \in A$, the states in which i knows that player j knows E can be denoted as

$$K_i(K_j(E)) \tag{1.9}$$

1.4 Knowledge Hierarchies and Common knowledges

Definition 1.10. An event $E \subseteq \Omega$ is a **common knowledge** in state ω if everyone knows E and knows everyone else possesses the same information. That is:

$$\omega \in \bigcap_{i \in A} K_i(E) \tag{1.10}$$
$$\omega \in \bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \tag{1.11}$$
$$\omega \in \bigcap_{k \in A} K_k \left[\bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \right] \tag{1.12}$$
$$\vdots \tag{1.13}$$

2 Beliefs

2.1 Type Space

Definition 2.1. A **type space** is a triple $(\Omega, (T_i), (\mu_i(\omega, t_{-i}|t_i)))$.

(i) Ω is the state space.

(ii) T_i is the collection of types for player i.

(iii) The **belief** of type t_i player i is a probability distribution:

$$\mu_i(\omega, t_{-i}|t_i) \in \Delta(\Omega \times T_{-i}) \text{ where } T_{-i} = \prod_{j \neq i} T_j \tag{2.1}$$

Remark 2.1. In the definition, we were overloading the notion of T_i, here T_i is a fixed set representing the collection of all possible types of player i. In the previous section, $T_i(\cdot)$ was a function maps a state $\omega \in \Omega$ to the information possessed by agent i in state ω.

Proposition 2.1 (Relation between type space and knowledge space)**.** For a type space $(\Omega, (T_i), (\mu_i(\omega, t_{-i}|t_i)))$, define

$$\Omega^* := \Omega \times T_1 \times \ldots \times T_n \tag{2.2}$$

and for every $\omega^* \in \Omega^*$, define

$$T_i^*(\omega^*) \equiv T_i^*(\omega, t_{1:n}) = \{(\omega', t'_{1:n}) : t'_i = t_i\} \subseteq \Omega^* \tag{2.3}$$

That is, player i only knows his own type and cannot distinguish types of any other players. And $T_i^*(\cdot)$ defines a partition over Ω^*. Therefore, $(\Omega^*, (T_i^*(\cdot)))$ is a knowledge space induced by the type space.

2.2 Prior and interim beliefs

Definition 2.2. The **interim belief** of player i is defined to be the distribution $\mu_i(\omega, t_{-i}|t_i)$ in the type space.

Definition 2.3. The **posterior belief** of player i is defined as the belief after the player learns the types of all other players:

$$\mu_i(\omega|t_{1:n}) \tag{2.4}$$

Definition 2.4. The **prior belief** μ, is a distribution over (ω, t_i, t_{-i}). Then player i can compute the prior belief on t_i by marginalizing

$$\mu\left(t_i\right) = \sum_{\omega, t_{-i}} \mu_i\left(\omega, t_{-i}, t_i\right) \tag{2.5}$$

Note that the prior belief μ does not have a subscript i because the prior belief is the same for all players.

Remark 2.2. Ex-ante approach:

(i) Player i learn (overall) prior $\mu\left(\omega, t_i, t_{-i}\right)$;

(ii) Player i deduces his type prior following

$$\mu\left(t_i\right) = \sum_{\omega, t_{-i}} \mu_i\left(\omega, t_{-i}, t_i\right) \tag{2.6}$$

(iii) Player i deduces interim belief following

$$\mu_i\left(\omega, t_{-i}|t_i\right) = \frac{\mu\left(\omega, t_i, t_{-i}\right)}{\mu\left(t_i\right)} \tag{2.7}$$

2.3 Bayesian Games

Definition 2.5. A **Bayesian game** consists of

(i) A set of players $i \in \{1, \cdots, N\}$;

(ii) Action space A_i for each player i;

(iii) Type space $(\Omega, (T_i), \mu_i)$;

(iv) Payoff function for each player i:

$$u_i\left(a_i, a_{-i}, t_i, t_{-i}, \omega\right) \tag{2.8}$$

(v) Strategies for each player i, $\sigma_i : T_i \to \Delta(A_i)$.

Assumption 2.1. Players are assumed to maximize their expected payoffs. Moreover, players calculate their expected payoffs with respect to their interim beliefs.

Definition 2.6. The **expected payoff** of player i with type t_i from a pure strategy $a_i \in A_i$ given opponents' strategies σ_{-i} is

$$U_i\left(a_i, \sigma_{-i}, t_i\right) = \sum_{t_{-i}, \omega} u_i\left(a_i, \sigma_{-i}\left(t_{-i}\right), t_i, t_{-i}, \omega\right) \underbrace{\mu_i\left(t_{-i}, \omega | t_i\right)}_{\text{interim belief}} \tag{2.9}$$

Definition 2.7 (Generalization)**.** The **expected payoff** of player i with type t_i from a mixed strategy $\sigma_i \in \Delta(A_i)$ given opponents' strategies σ_{-i} is

$$U_i\left(\sigma_i, \sigma_{-i}, t_i\right) = \sum_{a_i \in \sigma_i(t_i)} \sigma_i(t_i)(a_i) \sum_{t_{-i}, \omega} u_i\left(a_i, \sigma_{-i}\left(t_{-i}\right), t_i, t_{-i}, \omega\right) \mu_i\left(t_{-i}, \omega | t_i\right) \tag{2.10}$$

Definition 2.8. An action a_i is a **best response** for type t_i against σ_{-i} if

$$\forall a_i' \in A_i \backslash \{a_i\},\ U_i\left(a_i, \sigma_{-i}; t_i\right) \geq U_i\left(a_i', \sigma_{-i}; t_i\right) \tag{2.11}$$

Definition 2.9. An action a_i is **strictly dominated** for type t_i if

$$\exists a_i' \in A_i \backslash \{a_i\}\ s.t.\ \forall \sigma_{-i}\ U_i\left(a_i, \sigma_{-i}; t_i\right) < U_i\left(a_i', \sigma_{-i}; t_i\right) \tag{2.12}$$

Definition 2.10. A **Bayesian Nash equilibrium** is a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_l)$ such that for each player i and each type t_i of player i, the action $\sigma_i\left(t_i\right)$ is a best response for type t_i against σ_{-i}.

3 Adverse Selection

3.1 Market for Lemons

Two types of cars

1. high quality $P(h) = \lambda$

2. low quality $P(l) = 1 - \lambda$

Assuming sellers know the quality of car but buyers don't.

1. b_θ: value of car with type $\theta \in \{h, l\}$ for the buyer.
2. s_θ value for the seller.

Assume

1. $b_h > b_l$ and $s_h > s_l$ (defining what high quality is);
2. $b_h > s_h$ and $b_l > s_l$ (benefit to trade).

Therefore, if the car quality is observed, then it is beneficial to trade, and two prices exist such that

$$b_\theta > p_\theta > s_\theta \ \forall \theta \in \{l, h\} \tag{3.1}$$

If the quality is not observed, only one price p exists.
Seller type h sells if and only if $p > s_h$ and type l if and only if $p > s_l$.
Seller behaviours:

(a) Both types sell: $p > s_h$;

(b) Only type h sells: $s_h > p > s_l$;

(c) Nobody wants to sell: $s_l > p$.

Buyer behaviours:

(a) Both types sell: buyers buy if average quality $\lambda b_+ (1 - \lambda) b_l > p$;

(b) Only type l sells, buyers buy if

$$\exists \ p \in \mathbb{R}_+ \ s.t. \ b_l > p > s_l \tag{3.2}$$

Aggregate conditions

(a) Trade with both types if and only if

$$\exists \ p \in \mathbb{R}_+ \ s.t. \lambda b_h + (1 - \lambda) b_l > p > s_h \tag{3.3}$$

That is,

$$\lambda b_h + (1 - \lambda) b_l > s_h \quad (\dagger) \tag{3.4}$$

(b) The second case (low quality trading) is always satisfied given assumption $b_l > s_l$.

The **market failure** occurs because high quality cars are withdrawn from the market.

3.2 Market for Insurances

- p denote the price;
- C denote the compensation (paid out if claim);

- D denote the damage;
- Δ denote value of peace of mind;
- $\pi_h > \pi_l$ probability of damage;
- $\lambda = P(h)$.

Buyer behaviours for type $\theta \in \{l, h\}$:

1. Buy: $\Delta - p + \pi_\theta(C - D)$;
2. Don't buy: $\pi_\theta(-D)$.

Buy if and only if

$$\Delta - p + \pi_\theta(C - D) \geq \pi_\theta(-D) \tag{3.5}$$

$$\Longrightarrow p \leq \Delta + \underbrace{\pi_\theta C}_{\text{expected compensation}} \qquad (\dagger\dagger) \tag{3.6}$$

Not that $(\dagger\dagger)$ is independent from D.

Seller behaviour

(a) If both types buy insurance:

$$\pi = \lambda\pi_h + (1 - \lambda)\pi_l \tag{3.7}$$

Sellers sell if

$$p - \pi C \geq 0 \tag{3.8}$$

Aggregated conditions:

$$\exists\, p \in \mathbb{R}_+ \ s.t.\ \Delta + \pi_l C \geq p \geq \pi C \tag{3.9}$$

$$\Longleftrightarrow \Delta + \pi_l C > \lambda\pi_h + (1 - \lambda)\pi_l C \tag{3.10}$$

$$\Longleftrightarrow \Delta > \lambda(\pi_h - \pi_l)C \tag{3.11}$$

$$\Longleftrightarrow \pi_h - \pi_l < \frac{\Delta}{\lambda C} \quad (\dagger) \tag{3.12}$$

$(\dagger)$ says both types of buyers participate in the market and sellers sell insurances if <u>the difference between two types is insignificant</u>.
If $(\dagger)$ is not satisfied, then there is <u>no trade with both type (only one type trades)</u>.

(b) (<u>Price is high enough so that low type does not want to buy</u>) Suppose only type h trades, insurer sells if and only if

$$\pi_h C < p \tag{3.13}$$

And buyer type h wants

$$\Delta + \pi_h C > p \tag{3.14}$$

Given $\Delta > 0$, there would always be trading between sellers and type h buyers.

3.3 Monopoly Under Adverse Selection

Sellers

1. Decision (p, q) where p denotes price and q denotes quality;
2. Profit $\pi(p, q) = p - 1/2q^2$.

Buyers

1. $u_\theta(p, q) = \theta q - p$;
2. θ denotes the taste for quality: $\theta_h > \theta_l$;
3. $P(\theta_h) = \lambda$.

Case 1: 1th degree price discrimination Monopolist knows type θ for each consumer, and design good such that

$$\max_{(p,q)} p - 1/2q^2 \tag{3.15}$$

$$s.t.\ \theta q - p \geq 0 \text{ (Individual Rationality)} \tag{3.16}$$

Solution:

$$q^* = \theta^*,\ p^* = \theta^{*2} \tag{3.17}$$

Case 2: 2nd degree price discrimination Monopolist does not the θ.
Monopolist design a *menu* of products

$$(p_1, q_1), (p_2, q_2), \cdots, (p_j, q_j) \tag{3.18}$$

Provided there are only two types, it is sufficient to construct a menu with two types:

$$(p_l, q_l), (p_h, q_h) \tag{3.19}$$

Note that it is possible that $(p_l, q_l) = (p_h, q_h)$. Further, it is allowed to exclude certain type of consumers by setting $(p_i, q_i) = (0, 0)$.
Monopolist's problem designing an optimal menu such that both types are willing to purchase and each type only buy the bundle designed for them.

$$\max_{(p_l,q_l)\in\mathbb{R}^2_+,(p_h,q_h)\in\mathbb{R}^2_+} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.20}$$

$$\theta_l q_l - p_l \geq 0 \text{ (Individual rationality for low type)} \tag{3.21}$$

$$\theta_h q_h - p_h \geq 0 \text{ (Individual rationality for high type)} \tag{3.22}$$

$$\theta_h q_h - p_h \geq \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \tag{3.23}$$

$$\theta_l q_l - p_l \geq \theta_l q_h - p_h \text{ (Incentive compatibility for low type)} \tag{3.24}$$

Proposition 3.1 (Step 0)**.**

$$\mathrm{IC}_h \wedge \mathrm{IC}_l \implies q_h \geq q_l \tag{3.25}$$

Proof.

$$\text{IC}_h \iff \theta_h(q_h - q_l) \geq p_h - p_l \tag{3.26}$$
$$\text{IC}_l \iff \theta_l(q_l - q_h) \geq p_l - p_h \tag{3.27}$$

Summing two conditions:

$$(q_h - q_l)(\theta_h - \theta_l) \geq 0 \tag{3.28}$$

Provided that $\theta_h > \theta_l$,

$$q_h \geq q_l \tag{3.29}$$

■

Proposition 3.2 (Step 1)**.**

$$\text{IC}_h \wedge \text{IR}_l \implies \text{IR}_h \tag{3.30}$$

Proof.

$$\text{IC}_h \iff \theta_h q_h - p_h \geq \theta_h q_l - p_l \tag{3.31}$$
$$\implies \theta_h q_h - p_h \geq \theta_h q_l - p_l \geq \theta_l q_l - p_l \geq 0 \text{ (By IR for low type)} \tag{3.32}$$
$$\implies \theta_h q_h - p_h \geq 0 \iff \text{IR}_h \tag{3.33}$$

■

Proposition 3.3 (Step 2)**.** Given step 1, IC_h constrain is binding.

Proposition 3.4 (Step 3)**.** IC_h is binding and $q_h \geq q_l$ imply IC_l.

Proof.

$$\text{binding IC}_h \iff \theta_h(q_h - q_l) = p_h - p_l \tag{3.34}$$
$$\implies \theta_l(q_h - q_l) \leq \theta_h(q_h - q_l) = p_h - p_l \tag{3.35}$$
$$\implies \theta_l(q_h - q_l) \leq p_h - p_l \tag{3.36}$$
$$\iff \theta_l q_l - p_l \geq \theta_l q_h - p_h \tag{3.37}$$
$$\iff \text{IC}_l \tag{3.38}$$

■

Proposition 3.5 (Step 4)**.** IR_l is binding.

Solving the reduced problem

$$\max_{(p_l,q_l)\in\mathbb{R}_+^2,(p_h,q_h)\in\mathbb{R}_+^2} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.39}$$

$$\theta_l q_l - p_l = 0 \text{ (Individual rationality for low type)} \tag{3.40}$$

$$\theta_h q_h - p_h = \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \tag{3.41}$$

The problem reduced to

$$\max_{q_l,q_h\in\mathbb{R}_+} ... \tag{3.42}$$

Solving the problem gives

$$q_h = \theta_h \tag{3.43}$$

$$q_l = \theta_l - \frac{\lambda}{1-\lambda}(\theta_h - \theta_l) \tag{3.44}$$