MAT224 Notes

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1 Lecture1 Jan.9 2018

1.1 Vector spaces

Definition A real [1] **vector space** is a set V together with two vector operations vector addition and scalar multiplication such that

1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x}+\vec{y} \in V$
2. **C** Commutative: $\forall \vec{v}, \vec{y} \in V, \vec{x}+\vec{y} = \vec{y}+\vec{x}$
3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
4. **Z** Zero Vector: $\exists\ \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x}+\vec{0} = \vec{x}$
5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists -\vec{x} \in V s.t. \vec{x}+(-\vec{x}) = \vec{0}$
6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$
7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$

[1] A vector space is real if scalar which defines scalar multiplication is real.

8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c+d)\vec{x} = c\vec{x}+d\vec{x}$
9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$
10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

Note For V to be a vector space, need to know or be given operations of vector additions multiplication and check all 10 properties hold.

1.2 Examples of vector spaces

Example 1 $\mathbb{R}^n$ w.r.t.[2] usual component-wise addition and scalar multiplication.

Example 2 $\mathbb{M}_{m\times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

Example 3 $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree less or equal to n, w.r.t. usual degree-wise polynomial addition and scalar multiplication.

Note If define $\mathbb{P}^{\star}_n(\mathbb{R})$ as set of all polynomials of degree exactly equal to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

Explanation: $(1 + x^n), (1 - x^n) \in \mathbb{P}^{\star}_n(\mathbb{R})$ but $(1 + x^n) + (1 - x^n) = 2 \notin \mathbb{P}^{\star}_n(\mathbb{R})$

Example 4 Something unusual, define V as

$$V = \{(x_1, x_2)|x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

This is a vector space.

[2] w.r.t. is the abbreviation of "with respect to".

1.3 Some properties of vector spaces

Suppose V is a vector space, then it has the following properties.

Property 1 The zero vector is unique.
proof.

$$\text{Assume } \vec{0}, \vec{0^\star} \text{ are two zero vectors in } V$$
$$\text{WTS: } \vec{0} = \vec{0^\star}$$
$$\text{Since } \vec{0} \text{ is the zero vector, by Z } \vec{0^\star} + \vec{0} = \vec{0^\star}$$
$$\text{Similarly, } \vec{0} + \vec{0^\star} = \vec{0}$$
$$\text{Also, } \vec{0} + \vec{0^\star} = \vec{0^\star} + \vec{0} \text{ by commutative vector addition.}$$
$$\text{So, } \vec{0^\star} = \vec{0}$$

■

Property 2 $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.
proof.
Exercise. (By Cancellation Law)

Property 3 $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.
proof.

$$\text{By property of number 0: } 0\vec{x} = (0+0)\vec{x}$$
$$\text{By DSA: } 0\vec{x} = 0\vec{x} + 0\vec{x}$$
$$\text{By AI, } \exists(-0\vec{x}) s.t.$$
$$0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x})$$
$$\text{By AA}$$
$$\implies 0\vec{x} = \vec{0}$$

Property 4 $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$
proof.

$$c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0}$$

2 Lecture2 Jan.10 2018

2.1 Some properties of vector spaces-Cont'd

Property 5 For a vector space V, $\forall \vec{x} \in V$, $(-1)\vec{x} = (-\vec{x})$. (*we could use this property to find the additive inverse with scalar multiplication with (-1)*)[3].

proof.

$$\begin{aligned}(-\vec{x}) &= (-\vec{x}) + \vec{0} \quad \text{By property of zero vector} \\ &= (-\vec{x}) + 0\vec{x} \quad \text{By property3} \\ &= (-\vec{x}) + (1 + (-1))\vec{x} \quad \text{By property of zero as real number} \\ &= (-\vec{x}) + 1\vec{x} + (-1)\vec{x} \\ &= \vec{0} + (-1)\vec{x} \\ &= (-1)\vec{x}\end{aligned}$$

■

Property 6 For a vector space V, let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \vee \vec{x} = \vec{0}$$

proof.

$$\begin{gathered}\text{if } c = 0 \implies True \\ \text{else } c^{-1}c\vec{x} = c^{-1} = \vec{0} \\ \implies (c^{-1}c)\vec{x} = \vec{0} \\ \implies 1\vec{x} = \vec{0} \\ \implies \vec{x} = \vec{0} \\ \implies True\end{gathered}$$

■

2.2 Subspaces

Loosely A subspace is a space contained within a vector space.

[3] The scalar multiplication here is the one defined in vector space V.

Definition Let V be a vector space and $W \subseteq V$, W is a **subspace** of V if W is itself a vector space w.r.t. operations of vector addition and scalar multiplication from V.

Theorem Let V be a vector space, and $W \subseteq V$, W has the same[4] operations of vector addition and scalar multiplication as in V. Then, W is a subspace of V iff:

1. W is non-empty. $W \neq \emptyset$.
2. W is closed under addition. $\forall \vec{x}, \vec{y} \in W,\ \vec{x} + \vec{y} \in W$.
3. W us closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

Proof.
Forward:

If W is a subspace
$\implies \vec{0} \in W$
$\implies W \neq \emptyset$
Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$
WTS. 10 axioms in definition of vector space hold
$(ii) \implies$ Additive Closure
$(iii) \implies$ Scalar Multiplication Clousure
Because $W \subseteq V$, and V is a vector space, so properties hold $\forall \vec{w} \in W$.
Additive inverse: by property 5 and scalar multiplication closure,
$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W.$
Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W.$

2.3 Examples of subspaces

Example 1 Let $V = \mathbb{M}_{n\times n}(\mathbb{R})$, V is a subspace.

[4] Other properties of vector spaces related to vector addition and scalar multiplication are immediately inherited from the parent vector space.

Example 2 Define W as

$$W = \{A \in \mathbb{M}_{n\times n}(\mathbb{R}) | A \text{ is } \underline{\text{not}} \text{ symmetric}\}$$

Explanation: Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but $A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$.
Since there's no additive identity in set W, so W failed to be a vector space, therefore W is not a subspace.

Example 3 Let $V = \mathbb{P}_2(\mathbb{R})$, is W defined as following,

$$W = \{p(x) \in V | p(1) = 0\}$$

a subspace of V ?
proof.
WTS: (i)
Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$
$\implies W \neq \emptyset$
WTS: (ii)
Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$
$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$
$\implies p_1 + p_2 \in W$
$\implies W$ is closed under addition.
WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$
$\implies p(1) = 0$
Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$
$\implies cp \in W$.
So W is a subspace of V.
■

2.4 Recall from MAT223

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of $\mathbb{R}^n$ and $Col(A)$ is a subspace of $\mathbb{R}^m$.

3 Lecture3 Jan.16 2018

3.1 Linear Combination

Definition Let V be a vector space, $\vec{v_1}, \ldots, \vec{v_n} \in V$, $a_1, \ldots, a_n \in \mathbb{R}$ the expression

$$c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

is called a **linear combination** of $\vec{v_1}, \ldots, \vec{v_n}$.

Theorem Let V be a vector space, W is a subspace of V, $\forall \vec{w_1}, \ldots \vec{w_k} \in W, c_1, \ldots, c_k \in \mathbb{R}$, we have

$$c_1\vec{w_1} + \cdots + c_k\vec{w_k} \in W$$

Subspaces are closed under linear combinations, since subspaces are closed under scalar multiplication and vector addition.

Theorem Let V be a vector space, let $\vec{v_1}, \ldots, \vec{v_k} \in V$ then the set of all linear combination of $\vec{v_1}, \ldots, \vec{v_k}$

$$W = \{\sum_{i=1}^{k} c_i\vec{v_i} | c_i \in \mathbb{R} \forall\ i\}$$

is a subspace of V.
proof.

$$\text{Consider } \vec{0} \in W$$
$$\text{So, } W \neq \emptyset$$
$$\text{Let } c \in \mathbb{R}, \text{ Let } \vec{x} \in W \wedge \vec{y} \in W$$
$$\text{By definition of span, we have,}$$
$$\vec{x} = \sum_{i=1}^{k} a_i\vec{v_i}, \quad \vec{y} = \sum_{i=1}^{k} b_i\vec{v_i}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\vec{x} + c\vec{y} = \sum_{i=1}^{k} a_i\vec{v_i} + c\sum_{i=1}^{k} b_i\vec{v_i} = \sum_{i=1}^{k} (a_i + cb_i)\vec{v_i} \in W$$

■

Definition Let V be a vector space, $\vec{v_1}, \dots, \vec{v_k} \in V$, **span** of the set of vectors $\{\vec{v}_i\}_{i=1}^k$ is defined as the collection of all possible linear combinations of $\{\vec{v}_i\}_{i=1}^k$. By pervious theorem, span is a subspace.

3.2 Combination of subspaces

Definition Let W_1, W_2 be two sets, then the **union** of W_1, W_2 is defined as:

$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

the **intersection** of W_1, W_2 is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

Now consider W_1, W_2 to be two subspaces of vector space V, then we have,

1. $W_1 \cup W_2$ is **not** a subspace.
2. $W_1 \cap W_2$ is a subspace.

proof.

Falsify the statement by providing counter-example:

Consider,

$$W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$
$$W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$$
$$\text{But, } \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$$

■

proof.

$$\text{Because } W_1 \text{ and } W_2 \text{ are both subspaces, so}$$
$$\vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset$$
$$\text{Let } \vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\text{Sine } W_1, W_2 \text{ are subspaces,}$$
$$\vec{x} + c\vec{y} \in W_1 \wedge \vec{x} + c\vec{y} \in W_2$$
$$\implies \vec{x} + c\vec{y} \in W_1 \cap W_2$$
$$\text{So, } W_1 \cap W_2 \text{ is a subspace.}$$

■

Definition Let W_1, W_2 be subspaces of vector space V, define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \wedge \vec{y} \in W_2\}$$

Note Let $\vec{x} = \vec{0} \in W_1$, $\forall \vec{y} \in W_2,\ \vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = 0 \in W_2$, $\forall \vec{x} \in W_1,\ \vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2,\ \vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

Note $W_1 + W_2$ is a subspace of V.
proof.

$$\text{Let } \vec{x_1}, \vec{x_2} \in W_1, \vec{y_1}, \vec{y_2} \in W_2$$
$$\text{By properties of subspaces,}$$
$$\forall c \in \mathbb{R}, \vec{x_1} + c\vec{x_1} \in W_1 \wedge \vec{y_2} + c\vec{y_2} \in W_2$$
$$\text{Consider, } \vec{x_1} + \vec{y_1} \in W_1 + W_2, \vec{x_2} + \vec{y_2} \in W_1 + W_2$$
$$(\vec{x_1} + \vec{y_1}) + c(\vec{x_2} + \vec{y_2})$$
$$= (\vec{x_1} + c\vec{x_2}) + (\vec{y_1} + c\vec{y_2}) \in W_1 + W_2$$

■

Definition(Unique Representation) Let W_1, W_2 be subspaces of vector space V, say V is **direct sum** of W_1 and W_2, written as $V = W_1 \bigoplus W_2$, if every $\vec{x} \in V$ can be written uniquely as $\vec{x} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in W_1$ and $\vec{w_2} \in W_2$.
Equivalently Let W_1 and W_2 be subspaces of V, $V = W_1 \bigoplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.

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4.1 Cont'd

Cont'd Proof of Theorem
proof.

$$\text{(Forward direction) Suppose } V = W_1 \bigoplus W_2$$
$$\text{WTS. } V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{Let } V = W_1 \bigoplus W_2$$
$$\implies \forall \vec{x} \in V, \text{ can be written uniquely as}$$
$$\vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1,\ \vec{w_2} \in W_2$$
$$\implies V = W_1 + W_2 \text{ by definition of } \textit{sum}.$$
$$\text{Let } \vec{x} \in W_1 \cap W_2$$
$$\text{Decomposition, let } \vec{z} \in W_1 \cap W_2 \subseteq V$$
$$\vec{z} = \vec{z} + \vec{0},\ \vec{z} \in W_1, \vec{0} \in W_2$$
$$\vec{z} = \vec{0} + \vec{z},\ \vec{0} \in W_1, \vec{z} \in W_2$$
$$\text{Since decomposition is unique, } \vec{z} = \vec{0}$$
$$\text{So, } W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{(Backward direction) Suppose } V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{WTS. } V = W_1 \bigoplus W_2$$
$$\text{Assume } \vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1, \vec{w_2} \in W_2$$
$$\vec{x} = \vec{w_1'} + \vec{w_2'},\ \vec{w_1'} \in W_1, \vec{w_2'} \in W_2$$
$$\implies \vec{w_1} + \vec{w_2} = \vec{w_1'} + \vec{w_2'}$$
$$\implies \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2}$$
$$\text{Where, by definition of subspace, } \vec{w_1} - \vec{w_1'} \in W_1 \wedge \vec{w_2'} - \vec{w_2} \in W_2$$
$$\text{So, } \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2} \in W_1 \cap W_2$$
$$\text{Since } W_1 \cap W_2 = \{\vec{0}\}$$
$$\implies \vec{w_1} = \vec{w_1'} \wedge \vec{w_2} = \vec{w_2'}$$
$$\text{So the decomposition is unique.}$$

■

4.2 Linear Independence

Theorem (Redundancy theorem) Let V be a vector space, $\{\vec{x_1}, \dots \vec{x_n}\}$, let $\vec{x} \in \{\vec{x_1}, \dots \vec{x_n}\}$, then

$$span\{\vec{x_1}, \dots \vec{x_n}, \vec{x}\} = span\{\vec{x_1}, \dots \vec{x_n}\}$$

we say $\vec{x}$ *is the* ***redundant*** *vector that contributes nothing to the span.*
proof.

$$\begin{aligned}
&\text{let } \vec{x} \in span\{\vec{x}, \dots, \vec{x_n}\} \\
&\vec{x} = \sum_{i=1}^{n} c_i \vec{x_i} \text{ for } c_i \in \mathbb{R}\ \forall i \\
\text{So, } span\{\vec{x_1}, \dots, \vec{x_n}, \vec{x}\} &= \{\sum_{i=1}^{n} a_i \vec{x_i} + z\vec{x} \mid a_i, z \in \mathbb{R}\forall i\} \\
&= \{\sum_{i=1}^{n} a_i \vec{x_i} + z \sum_{i=1}^{n} c_i \vec{x_i} \mid a_i, c_i \in \mathbb{R}\forall i\} \\
&= \{\sum_{i=1}^{n} (a_i + zc_i) \vec{x_i} \mid a_i, c_i \in \mathbb{R}\forall i\} \\
&\text{Let } d_i = a_i + zc_i \in \mathbb{R} \\
&= \{\sum_{i=1}^{n} d_i \vec{x_i} \mid d_i \in \mathbb{R}\forall i\} \\
&= span\{\vec{x_1}, \dots \vec{x_n}\}
\end{aligned}$$

■

Definition Let V be a vector space, let $\{\vec{x_1}, \dots, \vec{x_n}\} \in V$, we say $\{v_i\}_{i=1}^n$ is **linearly independent** if the only set of scalars $\{c_1, \cdots, c_n\}$ that satisfies,

$$\sum_{i=1}^{n} c_i \vec{x_i} = 0$$

is $\{0, \dots, 0\}$.

Definition In contrast, we say a set of vector, with size n, is **linearly dependent** if

$$\exists \vec{c} \neq \vec{0} \in \mathbb{R}^n,\ s.t.\ \sum_{i=1}^{n} c_i \vec{v_i} = 0$$

Theorem Let V be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly dependent* if and only if,

$$\exists \vec{x} \in \{\vec{v_i}\}_{i=1}^n \ s.t. \ \vec{x_j} \in span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

Theorem Let V be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly independent* if and only if,

$$\forall \vec{x} \in \{\vec{v_i}\}_{i=1}^n, \ \vec{x_j} \notin span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

5 Lecture5 Jan.23 2018

5.1 Linear independence, recall definitions

Acknowledgement: special thanks to Frank Zhao.

Definition Let $\{\vec{x_1}, \dots \vec{x_k}\}$ is **linearly independent** if only scalars $c_1 \dots c_k$ s.t.

$$\sum_{i=1}^{k} c_1 \vec{x_k} = 0(\star)$$

are $c_1 = \cdots = c_k = 0$
linearly dependent means <u>at least one</u> $c_i \neq 0$, $(\star)$ still holds.

5.1.1 Alternative definitions of linear independency

Definition(Alternative.1) $\{\vec{x_1} \dots \vec{x_k}\}$ is **linearly independent** <u>iff</u> none of them can be written as a linear combination of the remaining $k-1$ vectors.[5]

Definition(Alternative.2) $\{\vec{x_1} \dots \vec{x_k}\}$ is **linearly dependent** <u>iff</u> at least one of them can be written as a linear combination of the remaining $k-1$ vectors. [6]

5.2 Basis

Definition Let V be a vector space, a non-empty[7] set S of vectors from V is a **basis** for V if

1. $V = span\{S\}$

[5] See theorem from the pervious lecture.
[6] See theorem from the pervious lecture.
[7] Specially, for an empty set, we define $span\{\emptyset\} = \{\vec{0}\}$

2. S is linearly independent.

Theorem (characterization of basis) A non-empty subset $S = \{\vec{x_i}\}_{i=1}^{n}$ of vector space V is basis for V iff every $\vec{x} \in V$ can be written uniquely as linear combination for vectors in S.

proof.

Forwards

Suppose S is a basis for V

So every $\vec{x} \in V$ can be written as a linear combination of vectors in S

To prove the uniqueness, assume two expressions of $\vec{x} \in V$

$$\vec{x} = \begin{cases} c_1\vec{x_1} + \cdots + c_k\vec{x_k} \\ b_1\vec{x_1} + \cdots + d_k\vec{x_k} \end{cases}$$

Consider,

$$c_1\vec{x_1} + \cdots + c_k\vec{x_k} - (b_1\vec{x_1} + \cdots + d_k\vec{x_k}) = \vec{0}$$

$$\iff \sum_{i=1}^{k} (c_i - b_i)\vec{x_1} = \vec{0}$$

Since vectors in basis S are linear independent,

$$c_i = b_i \forall i \in \mathbb{Z} \cap [1, k]$$

So the representation is unique.

Backwards

Suppose every $\vec{x} \in V$ can be written uniquely as linear combination of vectors in S.

WTS: $V = span\{S\} \wedge S$ is linearly independent

By the assumption, spanning set is shown.

All we need to show is linear independence.

Consider,

$$\sum_{i=1}^{n} c_i\vec{x}_i = \vec{0}$$

Also, we know

$$\sum_{i=1}^{n} 0\vec{x_i} = \vec{0}$$

By the uniqueness of representation

We have identical expression $\displaystyle\sum_{i=1}^{n} c_i\vec{x}_i = \sum_{i=1}^{n} 0\vec{x}_i$

$$\therefore c_i = 0 \ \forall i \in \mathbb{Z} \cap [1, n]$$

■

Example

$$V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$
$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$
$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

Show that $\{(1,0),(6,3)\}$ is a basis of V.

By theorem, $\{(1,0),(6,3)\}$ is basis if every $(a,b) \in V$ can be written uniquely as linear combination of $\{(1,0),(6,3)\}$.

$$\exists \text{ unique scalars } c_1, c_2 \in \mathbb{R} \ s.t. \ c_1(1,0) + c_2(6,3) = (a,b)$$

proof.

By definition of scalar multiplication and vector addition in this space,

$$\text{Consider}(a,b) = c_1(1,0) + c_2(6,3) = (2c_1 - 1, c_1 - 1) + (7c_2 - 1, 4c_2 - 1)$$
$$= (2c_1 + 7c_2 - 1, c_1 + 4c_2 - 1)$$

Consider the coefficients of variables

$$\begin{cases} 2c_1 + 7c_2 - 1 = a \\ c_1 + 4c_2 - 1 = b \end{cases}$$

WTS, the above system of linear equations has unique solution for all a, b

The system has a unique solution $\forall a, b \in \mathbb{R}$

Since the coefficient matrix has rank 2

$$rank(\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}) = 2$$

Since obviously the columns are linearly independent.

■

5.3 Dimensions

Definition For a vector space V, the **dimension** of V is the minimum number of vectors required to span V.

Fundamental Theorem if V vector space is spanned by m vectors, then any set of more than m vectors from V must be <u>linearly dependent</u>.

Fundamental Theorem (Alternative) If V is vector space spanned by m vectors, then any <u>linearly independent</u> set in V must contain less or equal to m vectors.

5.3.1 Consequences of fundamental theorem

Theorem if $S = \{\vec{v}_i\}_{i=1}^{k}$ and $T = \{\vec{w}_i\}_{i=1}^{l}$ are two bases of vector space V then $l = k$. *Bases have the same size.*

proof.

$$\begin{gathered}\text{Since } S \text{ spans } V \text{ and } T \text{ is linearly independent}\\ \therefore l \leq k\\ \text{(flip) Since } T \text{ spans } V \text{ and } S \text{ is linearly independent}\\ \therefore k \leq l\\ \implies l \leq k \wedge k \leq l\\ \implies k = l\end{gathered}$$

∎

Definition So we can define the **dimension** of V, as $dim(V)$ as the number vectors in <u>any</u> basis for V. For special case $V = \{\vec{0}\}$, $dim(V) = 0$.

Example

- $dim(\mathbb{R}^n) = n$
- $dim(\mathbb{P}_n(\mathbb{R})) = n + 1$
- $dim(\mathbb{M}_{m\times n}(\mathbb{R})) = m \times n$

5.3.2 Use dimension to prove facts about linearly (in)dependent sets and subspaces

Theorem If V is a vector space, $dim(V) = n$, $S = \{\vec{x_k}\}_{i=1}^{k}$ is subset of V, if $k > n$ then S is <u>linearly dependent</u>.

Note $k \leq n \nRightarrow S$ is linear dependent.

Theorem If W is subspace of vector space V, then

1. $dim(W) \leq dim(V)$
2. $dim(W) = dim(V) \iff W = V$

proof.

(1) Suppose $dim(V) = n, dim(W) = k$

WTS, $k \leq n$

Any basis for W is a linearly independent set of k vectors from V.

Since V is spanned by n vectors, since $dim(V) = n$

By fundamental theorem, $k \leq n$

$\iff dim(W) \leq dim(V)$

(2) By contradiction, assume $dim(V) = dim(W) = n$ but $V \neq W$

Then $\exists \vec{x} \in V \wedge \vec{x} \notin W$

Take S as a basis of W, then $\vec{x} \notin span\{S\}$

Then $S \cup \vec{x}$ is linearly independent

$\implies S \cup \{\vec{x}\}$ is linearly independent in V containing $n+1$ vectors

This contradicts the assumption by fundamental theorem since

$dim(V) = n$ so it could not contain more than n linearly independent vectors

■

6 Lecture6 Jan.24 2018

6.1 Basis and Dimension

Theorem Let V be a vector space , S is a spanning set of V, and I is a linearly independent subset of V, s.t. $I \subseteq S$, then $\exists$ basis B for V s.t. $I \subseteq B \subseteq S$.

Explaining

1. *Any spanning set for V cab be* ***reduced*** *to basis for V by removing the linearly dependent(redundant) vector in the spanning set, using* <u>*redundancy theorem*</u> *to get a linearly independent spanning set.*

2. *Linear independent set can be* ***enlarged*** *to a basis for V.*

proof.

<u>omitted.</u>

■

Corollary Let V be a vector space and $dim(V) = n$, any set of n linearly independent vectors from V is a basis for V.

proof. If n linearly independent vectors did not span V, then could be enlarged to a basis of V by pervious theorem, but then have a basis containing more than n vectors from V, which is impossible by the fundamental theorem since we given the $dim(V) = n$, proven by contradiction.

Example Let $V = P_2(\mathbb{R})$, $p_1(x) = 2 - 5x$, $p_2(x) = 2 - 5x + 4x^2$, find $p_3 \in P_2(\mathbb{R})$ s.t. $\{p_1(x), p_2(x), p_3(x)\}$ is basis for $P_2(\mathbb{R})$

Note Since $dim(P_2(\mathbb{R})) = 3$ so any 3 linearly independent vectors from $P_2(\mathbb{R})$ will be a basis for $P_2(\mathbb{R})$.

Solutions e.g. constant function $p_3(x) = 1$, since $1 \notin span\{p_1(x), p_2(x)\}$, so $\{p_1(x), p_2(x), p_3(x)\}$ is a basis of $P_2(\mathbb{R})$. e.g. $p_3(x) = x$, since $x \notin span\{p_1(x), p_2(x)\}$

Theorem Let U and W be subspaces of vector space V, then we have

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

proof.

$$\text{Let } \{\vec{v_i}\}_1^k \text{ be basis for } U \cap W$$
$$\implies dim(U \cap W) = k$$
$$\text{Since } \{\vec{v_i}\}_1^k \text{ is basis for } U \cap W \text{ then it's a linearly independent subset of U}$$
$$\text{So it could be enlarged ti basis for } U,\ \{\vec{v_1}, \ldots, \vec{v_k}, \vec{y_1}, \ldots, \vec{y_r}\}$$
$$\text{So } dim(U) = k + r$$
$$\text{We also could enlarge a basis for } W\ \{\vec{v_1}, \ldots, \vec{v_k}, \vec{z_1}, \ldots, \vec{z_s}\}$$
$$\implies dim(V) = k + s$$
$$\text{WTS. } \{\vec{v_1}, \ldots, \vec{v_k}, \ldots, \vec{y_1}, \ldots, \vec{y_r}, \vec{z_1}, \ldots, \vec{z_s}\} \text{ is a basis for } U + W$$
$$\text{(If we could show this) } dim(U + W) = k + r + s = (k + r) + (k + s) - k$$
$$= dim(U) + dim(W) - dim(U \cap W)$$
$$\text{Obviously, the above set spans } U + W$$
$$\text{WTS. } \{\vec{v_1}, \ldots, \vec{v_k}, \ldots, \vec{y_1}, \ldots, \vec{y_r}, \vec{z_1}, \ldots, \vec{z_s}\} \text{ is linearly independent}$$
$$\text{Consider } a_1\vec{v_1} + \cdots + a_k\vec{v_k} + b_1\vec{y_1} + \ldots b_r\vec{y_r} + c_1\vec{z_1} + \cdots + c_s\vec{z_s} = \vec{0}\ (\star)$$
$$\text{From } (\star) \implies \sum(c_i\vec{z_i}) = -\sum(a_i\vec{v_i}) - \sum b_i\vec{y_i}$$
$$\implies \sum(c_i\vec{z_i}) \in U \wedge \sum(c_i\vec{z_i}) \in W$$
$$\iff \sum(c_i\vec{z_i}) \in U \cap W$$
$$\text{Since } \{\vec{v_i}\} \text{ is a basis for } U \cap W$$
$$\implies \sum(c_i\vec{z_i}) = \sum(d_i\vec{v_i})$$
$$\iff \sum(c_i\vec{z_i}) - \sum(d_i\vec{v_i}) = \vec{0} \in W$$
$$\implies c_i = d_i = 0 \text{ since } \{\vec{z_i}, \vec{v_i}\} \text{ is a basis}$$
$$\text{Rewrite } (\star)$$
$$\sum(a_i\vec{v_i}) + \sum b_i\vec{y_i} = 0 \in U$$
$$\implies a_i = b_i = 0 \text{ since } \{\vec{v_i}, \vec{y_i}\} \text{ is a basis for } U$$

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Corollary For direct sum, since the intersection is $\{\vec{0}\}$

$$dim(U \bigoplus W) = dim(U) + dim(W)$$

Example Let U, W are subspaces of $\mathbb{R}^3$ such shat $dim(U) = dim(W) = 2$, why is $U \cap W \neq \{\vec{0}\}$

Solutions Geometrically, U and W are planes through origin then the intersection would be a line through origin($U \neq W$) or a plane through origin.($U = W$), so shown.

Question V is a vector space, $dim(V) = n$, $U \neq W$ are subspaces of V but $dim(U) = dim(V) = (n-1)$, proof:

1. $V = U + W$
2. $dim(U \cap W) = (n - z)$

7 Lecture7 Jan.30. 2018

7.1 Linear Transformations

Definition Let V, W be vector spaces, a function $T : V \to W$ is a **linear transformation**[8] if

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})\ \forall \vec{x}, \vec{y} \in V$[9]
2. $T(c\vec{x}) = cT(\vec{x})\ \forall \vec{x} \in V,\ c \in \mathbb{R}$[10]

Linear transformation preserves vector additions ans saclar multiplications on vector spaces.

Theorem(Alternative definition) Transformation $T : V \to W$ is linear if and only if

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y}),\ \forall \vec{x}, \vec{y} \in V, c, d \in \mathbb{R}$$

Linear transformations preserves linear combinations.

Example (form 223) Rotation through angle θ about the origin in $\mathbb{R}^2$.

[8] In some textbooks, this is annotated as **linear mapping**.

[9] Notice that the vector additions on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

[10] Notice that the scalar multiplication on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

Example (from 223) Matrix transformation, let $A \in M_{m\times n}(\mathbb{R})$, transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$T(\vec{x}) = A\vec{x}$$

is linear.

Example Derivative $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by

$$T(\vec{p}(x)) = \vec{p}\,'(x)$$

Example Matrix transpose $T : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ defined by

$$T(A) = A^T$$

7.2 Properties of linear transformations

Property(i) Linear transformation $T : V \to W$ are uniquely defined by their values on any basis for V.

proof.

$$\text{Let}\{\vec{v_1}, \dots, \vec{v_k}\} \text{ be any basis for } V$$

Every vector $\vec{x} \in V$ can be uniquely written as some linear combination of the $\{\vec{v_i}\}_{i=1}^k$

$$\vec{x} = \sum_{i=1}^k c_i\vec{v_i},\ c_i \in \mathbb{R},\ \text{and } c_i \text{ are uniquely determined } \forall \vec{x} \in V$$

$$\implies T(\vec{x}) = T(\sum_{i=1}^k c_i\vec{v_i})$$

$$= \sum_{i=1}^k c_iT(\vec{v_i}) \text{ since the transformation } T \text{ is linear.}$$

$$\text{Since } c_i\text{s are uniquely determined by } \{\vec{v_i}\}_{i=1}^k$$

so the value of $T(\vec{x})$ is uniquely determined by its value on basis vectors $\{\vec{v_i}\}_{i=1}^k$.

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Property(ii) Let $T : V \to W$ be a linear transformation, let A be a subspace of vector space V, then the **image** $T(A)$ defined as

$$T(A) = \{T(\vec{x}) \mid \vec{x} \in A\}$$

called the image of A under linear transformation T is a subspace of W. *Linear transformation maps subspaces of V to subspaces of W.*

proof.

Since A is a subspace so it's non-empty, therefore $\exists T(\vec{x}),\ \vec{x} \in A$

So T(A)$\neq \emptyset$

Let $\vec{w_1}, \vec{w_2} \in T(A)$

$\implies \vec{w_1} = T(\vec{x_1}), \vec{w_2} = T(\vec{x_2}),\ \vec{x_1}, \vec{x_2} \in A$

$\implies \vec{w_1} + \vec{w_2} = T(\vec{x_1}) + T(\vec{x_2}) = T(\vec{x_1} + \vec{x_2})$ since T is linear.

Since $\vec{x_1} + \vec{x_2} \in A$ by the definition of subspaces.

$\implies \vec{w_1} + \vec{w_2} \in T(A)$

So $T(A)$ is closed under vector addition.

Let $\vec{w} \in T(A)$

$\implies \vec{w} = T(\vec{x}), \vec{x} \in A$

Let $c \in \mathbb{R}$

Consider $c\vec{w} = cT(\vec{x}) = T(c\vec{x})$

Since $c\vec{x} \in A$

So $c\vec{w} \in T(A)$

So $T(A)$ is closed under scalar multiplication.

■

Property(derived from the definition) For all linear transformation $T : V \to W$, we have [11]

$$T(\vec{0}) = \vec{0}$$

Property(iii) Let transformation $T : V \to W$ be linear, let B be a subspace of W, then its **pre-image** defined as

$$T^{-1}(B) = \{\vec{x} \in V \mid T(x) \in B\}$$

is a subspace of V. [12]

[11] In the equation, clearly, the zero vector on the left side of the equation is in space V and the zero vector on the right side is in space W.

[12] The pre-image and inverse share the same notation, but in this case, transformation T is not necessarily invertible.

proof.

$$\text{Let } \vec{w_1}, \vec{w_2} \in T^{-1}(B)$$
$$\implies T(\vec{w_1}), T(\vec{w_2}) \in B$$
$$\implies aT(\vec{w_1}) + b(\vec{w_2}) \in B,\ \forall a, b \in \mathbb{R} \text{ since } B \text{ is a subspace.}$$
$$\implies T(a\vec{w_1} + b\vec{w_2}) \in B$$
$$\implies a\vec{w_1} + b\vec{w_2} \in T^{-1}(B)$$

So $T^{-1}(B)$ is closed under both vector addition and scalar multiplication,

So $T^{-1}(B)$ is a subspace.

■

7.3 Definitions

Let $T : V \to W$ to be a linear transformation,

Definition the **Image** of transformation T is defined as

$$Im(T) = T(V) = \{T(\vec{x}) \mid \vec{x} \in V\}$$

Definition the **Rank** of transformation T is defined as

$$Rank(T) = dim(Im(T))$$

Definition the **Kernel** of transformation T is defined as

$$Ker(T) = T^{-1}(\{\vec{0}\}) = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

Definition the **Nullity** of transformation T is defined as

$$Nullity(T) = dim(ker(T))$$

Example $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is <u>linear</u> defined by

$$T(\vec{p}(x)) = \vec{p}(2x+1) - 8\vec{p}(x)$$

find $Ker(T)$.

Theorem Let $T : V \to W$ be a linear transformation, let $\{\vec{v_1}, \dots, \vec{v_k}\}$ be the spanning set of V [13], then $\{T(\vec{v_1}), \dots, T(\vec{v_k})\}$ spans $Im(T)$

proof.

$$\begin{gathered} \text{Let } \vec{w} \in Im(T) \\ \text{Since } V = span\{\vec{v_1}, \dots, \vec{v_k}\} \\ \text{For any } \vec{x} \in V \text{ can be written as} \\ \vec{x} = \sum_{i=1}^{k} c_i \vec{v_i},\ c_i \in \mathbb{R} \\ \implies \vec{w} = T(\vec{x}) = T(\sum_{i=1}^{k} c_i \vec{v_i}) \\ = \sum_{i=1}^{k} c_i T(\vec{v_i}) \\ \text{as a linear combination of } \{T(\vec{v_1}), \dots, T(\vec{v_k})\} \\ \text{So } Im(T) = span\{T(\vec{v_1}), \dots, T(\vec{v_k})\} \end{gathered}$$

■

8 Lecture8 Jan.31 2018

8.1 Linear Transformations

Example $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(p(x)) = p(2x + 1) - 8p(x)$$

Find the image of T.
We know $B = \{1, x, x^2, x^3\}$ is the standard basis for $P_3(\mathbb{R})$, consider the set $P(B)$

$$P(B) = \{-7, 1 - 6x, 1 + 4x - 4x^2, 1 + 6x + 12x^2\}$$

spans $Im(T)$. Notice the first three vectors in the set is linearly independent, the last vector is clearly dependent to the pervious three.[14]. So by the redundancy theorem we could remove the last vector. There we have

$$Im(T) = span\{-7, 1 - 6x, 1 + 4x - 4x^2\}$$

[13] The set is only the spanning set of V, it's not necessarily to be a basis of V.
[14] Notice that the first three vectors is a basis of $P_2(\mathbb{R})$.

as basis.
In this example, the dimension of $Ker(T)$ is 1 and the dimension of $Im(T)$ is 3, and dimension of $P_3(\mathbb{R})$ is 4. We have, $dim(P_3(\mathbb{R})) = Nullity(T) + Rank(T)$

Theorem(Dimension Theorem) Let $T : V \to W$ be a linear transformation,

$$dim(V) = Nullity(T) + Rank(T)$$

Proof.

$$\text{Say } dim(V) = n$$

$$\text{Let } \{\vec{v_1}, \dots, \vec{v_k}\} \text{ be a basis for } Ker(T)$$

$$\text{Since } Ker(T) \text{ is a subspace of } V\text{, the set } \{\vec{v_i}\}_1^k \text{ is a subset of } V,$$

$$\text{It can be extended to a basis } \{\vec{v_i}\}_1^k \cup \{\vec{v_i}\}_{k+1}^n \text{ for V.}$$

$$\text{Claim: } \{T(\vec{v_{k+1}}), \dots, T(\vec{v_n})\} \text{ is basis for } Im(T)$$

$$\text{If the claim is true, this prove the theorem since}$$

$$dim(Ker(T)) + dim(Im(T)) = k + n - k = n = dim(V)$$

$$\because T(\vec{v_i}) = \vec{0},\ \forall i \in \mathbb{Z}_1^k$$

$$\text{and by the definition of kernel of linear transformation,}$$

$$\therefore \{T(\vec{v_i})\}_{k+1}^n \text{ spans } Im(T)$$

$$\text{Show if } \sum_{i=k+1}^{n} c_i T(\vec{v_i}) = \vec{0} \implies c_i = 0$$

$$\implies T(\sum_{i=k+1}^{n} c_i \vec{v_i}) = \vec{0}$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} \in Ker(T)$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} = \sum_{i=1}^{k} c_i \vec{v_i}$$

$$\implies c_1\vec{v_1} + \cdots + c_k\vec{v_k} - c_{k+1}\vec{v_{k+1}} - \cdots - c_n\vec{v_n} = \vec{0}$$

$$\text{Since } \{\vec{v_i}\}_i^n \text{ is a basis for } V.$$

$$\implies c_i = 0\ \forall i$$

8.2 Applications of dimension theorem

Definition A linear transformation $T : V \to W$ is called **injective**(one-to-one) if and only if

$$T(\vec{v_1}) = T(\vec{v_2}) \implies \vec{v_1} = \vec{v_2}$$

Definition A linear transformation $T : V \to W$ is called **surjective**(onto) if and only if

$$Im(T) = W$$

Every vector in W has a pre-image in V.

Definition A linear transformation $T : V \to W$ is called **bijective** if it's both injective and surjective.

Theorem Let transformation $T : V \to W$ is linear, T is injective if and only if $dim(Ker(T)) = 0$.

Proof.

Exercise

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Theorem T is surjective if and only if $dim(Im(T)) = dim(W)$.

Example $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$$

is T injective? surjective?
Not injective but surjective.
Solution

$$Ker(T) = span\{(x-1)(x-2)\}$$

So T has nullity of 1 and since $dim(P_2(\mathbb{R})) = 3$, by the dimension theorem we have $Rank(T) = 2$ and since $Im(T)$ is a subspace of $\mathbb{R}^2$ which has dimension of 2, we could conclude that $Im(T) = \mathbb{R}^2$.