MAT246: Concepts in Abstract Mathematics:
Lecture 0101 Notes

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Contents

1 Lecture 1 Sep. 7 2018

Definition 1.1. Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of **natural numbers**.

Theorem 1.1 (Principle of Mathematical Induction)**.** Suppose S is a set of natural numbers, $S \subseteq \mathbb{N}$. If

1. $1 \in S$
2. $k \in S \implies k + 1 \in S, \ \forall k \in \mathbb{N}$

then, $S = \mathbb{N}$

Example 1.1. Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \ \forall n \in \mathbb{N}$$

Proof. ■

2 Lecture 2 Sep. 10 2018

Theorem 2.1 (Extended Principle of Mathematical Induction)**.** Suppose set $S \subseteq \mathbb{N}$ and let $n_0 \in \mathbb{N}$ fixed, if

1. $n_0 \in S$
2. $\forall k \geq n_0, k \in S \implies k + 1 \in S$

then $\{n_0, n_0 + 1, n_0 + 2, \dots\} \subseteq S$

Example 2.1. Show that

$$n! \geq 3^n \ \forall n \geq 7$$

Proof. ■

Theorem 2.2 (Well-Ordering Principle)**.** Every non-empty subset of natural number has a smallest element.

Proof. (Principle of Mathematical Induction)
Let $S \subseteq \mathbb{N}$
Suppose $1 \in S \wedge (k \in S \implies k + 1 \in S, \forall k \in \mathbb{N})$
Show: $S = \mathbb{N}$
Let $T = \mathbb{N} \backslash S$

Suppose $T \neq \emptyset$
By Well-Ordering Principle, there exists a smallest element of T, denoted as $t_0 \in \mathbb{N}$.
Since $1 \in S$, therefore $t_0 \neq 1$.
Therefore $t_0 > 2$.
Thus $t_0 - 1 \in \mathbb{N}$ and since $t_0 = \min T$, $t_0 - 1 \notin T$
Therefore $t_0 - 1 \in S$, then, $t_0 - 1 + 1 = t_0 \in S$,
Contradict the assumption that $t_0 \in T$.
Thus $T = \emptyset$ and $S = \mathbb{N}$.

■

Remark 2.1. We can use principle of Mathematical Induction to prove Well-Ordering Principle as well.

3 Lecture 3 Sep. 12 2018

Definition 3.1. Let $a, b \in \mathbb{N}$ and a **divides** b, written as $a|b$ if

$$\exists c \in \mathbb{N} \ s.t. \ b = ac$$

And a is a **divisor** of b.

Definition 3.2. A natural number p (except 1) is called **prime** if the only divisors of p are 1 and p.

Lemma 3.1 (Prime numbers are building blocks of natural numbers)**.** Every natural number other than 1 is a *product*[1] of prime numbers.

Theorem 3.1 (Principle of Complete Induction)**.** Suppose $S \subseteq \mathbb{N}$ and if

1. $n_0 \in S$
2. $n_0, n_0 + 1, \ldots, k \in S \implies k + 1 \in S, \ \forall k \geq n_0$

then

$$\{n_0, n_0 + 1, \ldots\} \subseteq S$$

Proof of Lemma. Let $S \subseteq \mathbb{N}$ for which the lemma is true,
Want to show: $S = \mathbb{N} \backslash \{1\}$
(Base Case) For 2 it's a product of prime. Thus $2 \in S$
(Inductive Step) Suppose $\{2, 3, \ldots k\} \subseteq S$

[1] Product could mean the product of a single number.

Consider $k+1$, if $k+1$ is a prime then $k+1$ can be written as a product of itself, as a product of one single prime.
Else, if $k+1$ is not a prime, then $\exists 1 < m, n < k+1$ $s.t.$ $k+1 = mn$.
By induction hypothesis of strong induction, m, n can both be written as product of primes.
$m = \prod_{i=1}^{\ell} p_i,\ n = \prod_{i=1}^{t} q_i$ where p_i, q_i are all primes.
and $k+1 = \prod_{i=1}^{t} q_i \prod_{i=1}^{\ell} p_i$
thus $k+1 \in S$
by principle of strong induction, $\{2, 3, \ldots, \} \subseteq S$. ■

Theorem 3.2. There is no largest prime number.

Proof. (By contradiction)
Assume there is a largest prime p,
then $\{2, 3, 5, \cdots, p\}$ is the set of all primes
Let $M := (2 * 3 * 5 * \cdots * p) + 1 \in \mathbb{N}$
M is either prime or not.
Suppose M is not a prime, then by Lemma 3.1, $\exists p'$ dividing M.
Obviously $\forall i \in \{2 * 3 * 5 * \cdots * p\},\ i \not| M$.
There is no prime dividing M, which contradict Lemma 3.1
Thus M is a prime, and $M > p$, which contradicts assumption
Therefore there is no largest prime. ■