Forecasting and Time Series Econometrics
ECO374 Winter 2019

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Contents

1 Introduction and Statistics Review

Definition 1.1. Given random variable X, the k^{th} **non-central moment** is defined as

$$\mathbb{E}[X^k] \tag{1.1}$$

Definition 1.2. Given random variable X, the k^{th} **central moment** is defined as

$$\mathbb{E}[(X - \mathbb{E}[X])^k] \tag{1.2}$$

Remark 1.1. Moments of order higher than a certain k may not exist for certain distribution.

Definition 1.3. Given the **joint density** $f(X, Y)$ of two *continuous* random variables, the **conditional density** of random Y conditioned on X is

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y, x)}{f_X(x)} \tag{1.3}$$

Definition 1.4. Given discrete variables X and Y, the **conditional density** of Y conditioned on X is defined as

$$P(Y = y|X = x) = \frac{P(Y = y \wedge X = x)}{P(X = x)} \tag{1.4}$$

Assumption 1.1. Assumptions on linear regression on time series data:

(i) **Linearity**

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + u \tag{1.5}$$

(ii) **Zero Conditional Mean**

$$\mathbb{E}[u|X_1, X_2, \ldots, X_k] = 0 \tag{1.6}$$

(iii) **Homoscedasitcity**

$$\mathbb{V}[u|X_1, X_2, \dots, X_k] = \sigma_u^2 \tag{1.7}$$

(iv) **No Serial Correlation**

$$Cov(u_t, u_s) = 0 \ \forall t \neq s \in \mathbb{Z} \tag{1.8}$$

(v) **No Perfect Collinearity**

(vi) **Sample Variation in Regressors**

$$\mathbb{V}[X_j] > 0 \ \forall j \tag{1.9}$$

Theorem 1.1 (Gauss-Markov Theorem)**.** Under assumptions 1.1, the OLS estimators $\hat{\beta}_j$ are *best linear unbiased estimators* of the unknown population regression coefficients β_j.

Remark 1.2. The *no serial correlation* assumption is typically not satisfied for time series data. And the *linearity* assumption is also too restrictive for time series featuring complex dynamics. Hence, for time series data we typically use other models than linear regression with OLS.

2 Statistics and Time Series

2.1 Stochastic Processes

Definition 2.1 (1.1)**.** A **stochastic process** (or **time series process**) is a family (collection) random variables indexed by $t \in \mathcal{T}$ and defined on some given probability space $(\Omega, \mathcal{F}, P)$.

$$\{Y_t\} = Y_1, \dots, Y_T \tag{2.1}$$

Definition 2.2 (1.2)**.** The function $t \to y_t$ which assigns to each point in time $t \in \mathcal{T}$ the realization of the random variable Y_t, y_t is called a **realization** or a **trajectory** or an **outcome** of the stochastic process.

Definition 2.3. An *outcome* of a stochastic process

$$\{y_t\} = y_1, \dots, y_T \tag{2.2}$$

is a **time series**.

Definition 2.4 (1.3)**.** A **time series model** or a **model** for the observations (data), $\{y_t\}$, is a specification of the *joint distribution* of $\{Y_t\}$ for which $\{y_t\}$ is a realization.

Assumption 2.1. The **ergodicity** assumption requires the observations cover in principle all possible events.

Definition 2.5. A stochastic process $\{Y_t\}$ is **first order strongly stationary** if all random variables $Y_t \in \{Y_t\}$ has the *same probability density function.*

Definition 2.6 (1.7)**.** A stochastic process $\{Y_t\}$ is **strictly stationary** if for all $h, n \geq 1$, $(X_1, \dots, X_n)$ and $(X_{1+h}, \dots, X_{n+h}$ have the same distribution.

Definition 2.7. A stochastic process $\{Y_t\}$ is **first order weakly stationary** if

$$\forall t \in \mathcal{T}, \ \mu_{Y_t} \equiv \mathbb{E}[Y_t] = \bar{\mu} \tag{2.3}$$

Definition 2.8. A stochastic process $\{Y_t\}$ is **second order weakly stationary**, or **covariance stationary** if all random variables $\{Y_t\}$ have the same mean and variance. And the covariances do not depend on t. That's, for all $t \in \mathcal{T}$,

(i) $\mathbb{E}[Y_t] = \mu \ \forall t$

(ii) $\mathbb{V}[Y_t] = \sigma^2 < \infty \ \forall t$

(iii) $Cov(Y_t, Y_s) = Cov(Y_{t+r}, Y_{s+r}) \ \forall t, s, r \in \mathbb{Z}$

2.2 Auto-correlations

Definition 2.9. Let $\{Y_t\}$ be a stochastic process with $\mathbb{V}[Y_t] < \infty \ \forall t \in \mathcal{T}$, the **auto-covariance function** is defined as

$$\gamma_Y(t, s) \equiv Cov(Y_t, Y_s) \tag{2.4}$$
$$= \mathbb{E}[(Y_t - \mathbb{E}[Y_t])(Y_s - \mathbb{E}[Y_s])] \tag{2.5}$$
$$= \mathbb{E}[Y_t Y_s] - \mathbb{E}[Y_t]\mathbb{E}[Y_s] \tag{2.6}$$

Lemma 2.1. If $\{Y_t\}$ is stationary, then the auto-covariance function does not depend on specific time point t. We can write the $h \in \mathbb{Z}$ degree auto-covariance as

$$\gamma_Y(h) \equiv \gamma_X(t, t+h) \ \forall t \in \mathcal{T} \tag{2.7}$$

Proposition 2.1. By the symmetry of covariance,

$$\gamma_Y(h) = \gamma_Y(-h) \tag{2.8}$$

Definition 2.10. The **auto-correlation coefficient** of order k is given by

$$\rho_{Y_t, Y_{t-k}} = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{\mathbb{V}[Y_t]}\sqrt{\mathbb{V}[Y_{t-k}]}} \tag{2.9}$$

Definition 2.11. Let $\{Y_t\}$ be a *stationary process* and the **auto-correlation function** (ACF) is a mapping from *order* of auto-correlation coefficient to the coefficient $\rho_Y : k \to \rho_{Y_t, Y_{t-k}}$, defined as

$$\rho_Y(k) \equiv \frac{\gamma(k)}{\gamma(0)} = corr(Y_{t+k}, Y_t) \tag{2.10}$$

Proposition 2.2. Note that

$$\rho_k = \rho_{-k} = \rho_{|k|} \tag{2.11}$$

so the ACF for stationary process can be simplified to a mapping

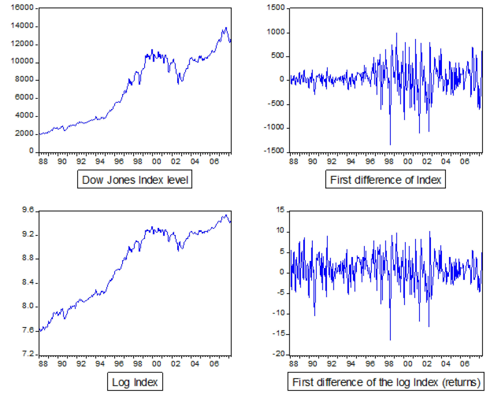
$$\rho : k \to \rho_{|k|} \tag{2.12}$$

Remark 2.1. Strong stationarity is difficult to test so we will focus on weak stationarity only.

Proposition 2.3. For a non-stationary stochastic process $\{Y_t\}$, $\{\Delta Y_t\}$ becomes *first order weakly stationary* and $\{\Delta \log(Y_t)\}$ becomes *second order weakly stationary.*

Definition 2.12 (1.8)**.** A stochastic process $\{Y_t\}$ is called a **Gaussian process** if all *finite* dimensional distribution from the process are multivariate normally distributed. That's

$$\forall n \in \mathbb{Z}_{>0}, \ \forall (t_1, \dots, t_n) \in \mathcal{T}^n, (Y_{t_1}, \dots, Y_{t_n}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \tag{2.13}$$



Notation 2.1. Consider the problem of forecasting Y_{T+1} from observations $\{Y_t\}_{t=1}^T$, the *best linear predictor* is denoted as

$$\mathbb{P}_T Y_{T+1} = \sum_{i=1}^{T} a_i L^i Y_{T+1} \tag{2.14}$$

And Y_{T+1} can be expressed as

$$Y_{T+1} = \mathbb{P}_T Y_{T+1} + Z_{T+1} \tag{2.15}$$

where Z_{T+1} denotes the forecast error which is *uncorrelated* with $X_T, \ldots, X_1$.

Definition 2.13 (3.3)**.** The **partial auto-correlation function** (PACT) $\alpha(h)$ with $h \in \mathbb{Z}_{\geq 0}$ of a *stationary* process is defined as

$$\alpha(0) = 1 \tag{2.16}$$

$$\alpha(1) = corr(Y_2, Y_1) = \rho(1) \tag{2.17}$$

$$\alpha(h) = corr\Big(Y_{h+1} - \mathbb{P}(Y_{h+1}|1, Y_2, \ldots, Y_h), X_1 - \mathbb{P}(Y_1|1, Y_2, \ldots, Y_h)\Big) \tag{2.18}$$

Remark 2.2 (Interpretation of PACF)**.** partial auto-correlation r_k only measures correlation between two variables Y_t and Y_{t+k} while controlling $(Y_{t+1}, \ldots, Y_{t+k-1})$.

Remark 2.3. Properties of ACF and PACF

processes	ACF	PACF
$AR(p)$	Declines exponentially (monotonic or oscillating) to zero	$\alpha(h) = 0 \ \forall h > p$
$MA(q)$	$\rho(h) = 0 \ \forall h > q$	Declines exponentially (monotonic or oscillating) to zero

Test for Auto-correlation To test single auto-correlation with

$$H_0 : \rho_k = 0 \tag{2.19}$$

we can use usual t-statistic. While testing the joint hypothesis

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_k = 0 \tag{2.20}$$

we are using the **Ljung-Box Q-statistic**:

$$Q_k = T(T+1)\sum_{j=1}^{k}\frac{\hat{\rho}_j^2}{T-j} \sim \chi_k^2 \tag{2.21}$$