

# Introduction to Real Analysis

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# 1 The Axiom of Completeness

## 1.1 Preliminaries

**Definition 1.1.** A set  $A \subseteq \mathbb{R}$  is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

**Example 1.1.** The set of integers,  $\mathbb{Z}$ , is neither bounded from above nor below. Sets  $\{1, 2, 3\}$  and  $\{\frac{1}{n} : n \in \mathbb{N}\}$  are bounded from both above and below.

**Notation 1.1.** Let  $A \subseteq \mathbb{R}$ , we use  $A^\uparrow$  and  $A^\downarrow$  to denote collections of upper bounds of  $A$  and lower bounds of  $A$ . When  $A$  is bounded, either  $A^\uparrow$  or  $A^\downarrow$  is empty.

**Definition 1.2.** A real number  $s \in \mathbb{R}$  is the **least upper bound (supremum)** for a set  $A \subseteq \mathbb{R}$  if

- (i)  $s \in A^\uparrow$ ;
- (ii) and  $\forall u \in A^\uparrow, s \leq u$ .

Such  $s$  is denoted as  $s := \sup A$ .

**Definition 1.3.** A real number  $f \in \mathbb{R}$  is the **greatest lower bound (infimum)** for  $A$  if

- (i)  $f \in A^\downarrow$ ;
- (ii) and  $\forall l \in A^\downarrow, l \leq f$ .

Such  $f$  is often written as  $f := \inf A$ .

**Axiom 1.1** (The Axiom of Completeness/Least Upper Bounded Property).  $\forall \emptyset \neq A \subseteq \mathbb{R}$  such that  $A^\uparrow \neq \emptyset$ ,  $\exists \mathbb{R} \ni u = \sup A$ .

**Definition 1.4.** Let  $\emptyset \neq A \subseteq \mathbb{R}$ ,  $a_0 \in A$  is the **maximum** of  $A$  if  $\forall a \in A, a_0 \geq a$ ;  $a_1 \in A$  is the **minimum** of  $A$  if  $\forall a \in A, a_1 \leq a$ .

**Example 1.2.**  $\mathbb{Q} \subseteq \mathbb{R}$  does not satisfy the axiom of completeness. Let  $A = \{r \in \mathbb{Q} : r < \sqrt{2}\}$ , clearly  $A$  is bounded above, but for every  $r' \in \mathbb{Q} \cap A^\uparrow$ , there exists  $r'' \in (\sqrt{2}, r') \cap A^\uparrow$ .

**Proposition 1.1.** Let  $\emptyset \neq A \subseteq \mathbb{R}$  bounded above, and  $c \in \mathbb{R}$ . Define  $c + A := \{a + c : a \in A\}$ . Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

*Proof. Step 1: Show  $c + \sup A \in (c + A)^\uparrow$ :*

Let  $x \in c + A$ ,  $\exists a \in A$  s.t.  $x = c + a$ . Then,  $x = c + a \leq c + \sup A$ . Therefore,  $x \leq c + \sup A \forall x \in A$ , which implies what desired.

*Step 2: Show  $\forall u \in (c + A)^\uparrow$ ,  $c + \sup A \leq u$ :*

Let  $u \in (c + A)^\uparrow$ , then  $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$ .

Hence,  $\sup(c + A) = c + \sup A$ . ■

**Lemma 1.1** (Alternative Definition of Supremum). Let  $s \in A^\uparrow$  for some nonempty  $A \subseteq \mathbb{R}$ . The following statements are equivalent:

- (i)  $s = \sup A$ ;
- (ii)  $\forall \varepsilon, \exists a \in A$ , s.t.  $a > s - \varepsilon$  (i.e.  $s - \varepsilon \notin A^\uparrow$ ).

*Proof.* The proof is immediate by the definition of supremum as the least upper bound. ■

**Theorem 1.1** (Nested Interval Property). Let  $(I_n)_{n \in \mathbb{N}}$  be a sequence of closed intervals  $I_n := [a_n, b_n]$  such that these intervals are *nested* in a sense that

$$I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \tag{1.5}$$

*Proof.* Note that the sequence  $(a_n)_{n \in \mathbb{N}}$  is bounded above by any  $b_k$ .

By the completeness axiom, there exists  $a^* := \sup_{n \in \mathbb{N}} a_n$ .

Since  $a^* \in (a_n)^\uparrow$ ,  $a^* \geq a_n \forall n \in \mathbb{N}$ .

Further, because  $a^*$  is the *least* upper bound, then for every upper bound  $b_n$ , it must be  $a^* \leq b_n \forall n \in \mathbb{N}$ . Therefore,  $a^* \in [a_n, b_n] \forall n \in \mathbb{N}$ . That is,  $a^* \in \bigcap_{n \in \mathbb{N}} I_n$ . ■

**Remark 1.1.** Note that NIP requires all intervals to be closed. One instance when this fails to hold:  $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$ .

**Theorem 1.2** (Archimedean Property).

- (i)  $\forall x \in \mathbb{R}$ ,  $\exists n \in \mathbb{N}$  s.t.  $n > x$ ;
- (ii)  $\forall y \in \mathbb{R}_{++}$ ,  $\exists n \in \mathbb{N}$  s.t.  $\frac{1}{n} < y$ .

Archimedean property of natural numbers can be interpreted as *there is no real number that bounds  $\mathbb{N}$* . This interpretation can be seen by considering the negations of above statements:

- (i)  $\exists x \in \mathbb{R}$  s.t.  $\forall n \in \mathbb{N}$ ,  $n \leq x$ ;
- (ii)  $\exists y \in \mathbb{R}_{++}$  s.t.  $\forall n \in \mathbb{N}$ ,  $y \leq \frac{1}{n}$  (i.e.  $n \leq \frac{1}{y}$ ).

*Proof of (i).* Suppose, for contradiction, (i) is not true, then  $\mathbb{N}$  is bounded above in  $\mathbb{R}$ .

By the completeness axiom, there exists  $a^* := \sup \mathbb{N}$ .

Therefore,  $\exists n \in \mathbb{N}$  s.t.  $a^* - 1 < n$ .

In this case,  $a^* < n + 1 \in \mathbb{N}$ , which means  $a^* \notin \mathbb{N}^\uparrow$  and leads to a contradiction. ■

*Proof of (ii).* Let  $y^* \in \mathbb{R}_{++}$ , take  $x = \frac{1}{y}$ . By statement (i), there exists  $n^* \in \mathbb{N}$  such that  $n > \frac{1}{y}$ . Because  $y > 0$ ,  $\frac{1}{n} < y$ . ■

**Remark 1.2.** The two statements of Archimedean property are equivalent.

## 1.2 Density of Rational Numbers

**Theorem 1.3.** For every  $a, b \in \mathbb{R}$  such that  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .

**Remark 1.3.** The above theorem says  $\mathbb{Q}$  is in fact **dense** in  $\mathbb{R}$ . More generally, one says a set  $A \subseteq X$  is dense whenever the closure of  $A$ ,  $\overline{A} = X$ .

*Proof. Step 1:* Since  $b - a > 0$ , by the first Archimedean property, there exists  $n \in \mathbb{N}$  such that  $n > \frac{1}{b-a}$ . Such natural number satisfies  $\frac{1}{n} < b - a$ .

*Step 2:* Let  $m$  be smallest integer such that  $m > an$ . That is,  $m - 1 \leq an < m$ . Obviously,  $a < \frac{m}{n}$  since  $n > 0$ . Further, since  $m \leq an + 1$ , with results from step (i),  $m < bn - 1 + 1 = bn$ , and  $\frac{m}{n} < b$ . Therefore  $\frac{m}{n} \in (a, b)$ . ■

**Theorem 1.4.**  $\exists \alpha \in \mathbb{R}$  s.t.  $\alpha^2 = 2$ .

*Proof.* Let  $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$ , which is obviously a set in  $\mathbb{R}$  bounded from above. By the completeness axiom,  $\Omega$  possesses a supremum, and we claim  $\alpha := \sup \Omega$  satisfies  $\alpha^2 = 2$ . Suppose  $\alpha^2 > 2$ , then there exists  $\varepsilon > 0$  such that  $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$ . Therefore,  $\alpha > \alpha - \varepsilon \in \Omega^\uparrow$ , which contradicts the fact that  $\alpha$  is the least upper bound. Suppose  $\alpha^2 < 2$ , then there exists some  $\varepsilon > 0$  such that  $\alpha + \varepsilon \in \Omega$ , which contradicts the assumption that  $\alpha$  is an upper bound. Hence, it must be the case that  $\alpha^2 = 2$ . ■

## 2 Sequences

### 2.1 Definitions

**Theorem 2.1** (Triangle Inequality). Let  $a, b \in \mathbb{R}$ , then  $|a + b| \leq |a| + |b|$ .

**Corollary 2.1.** Let  $a, b \in \mathbb{R}$ , then

$$||a| - |b|| \leq |a - b| \quad (2.1)$$

*Proof.* Note that  $|a| = |a - b + b| \leq |a - b| + |b|$ , which implies  $|a| - |b| \leq |a - b|$ .

Similarly,  $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$ , which implies  $|b| - |a| \leq |a - b|$ .

Therefore, by taking the absolute value,  $||a| - |b|| \leq |a - b|$ . ■

**Definition 2.1.** A sequence  $(a_n) \subseteq \mathbb{R}$  **converges** to  $a \in \mathbb{R}$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies a_n \in V_\varepsilon(a) \quad (2.2)$$

**Definition 2.2.** Let  $a \in \mathbb{R}$  and  $\varepsilon > 0$ , the open ball centred at  $a$  with radius  $\varepsilon$  is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \quad (2.3)$$

**Theorem 2.2.** The limit of any convergent sequence is unique.

*Proof.* Let  $(a_n)$  be a convergent sequence, assume, for contradiction, that  $(a_n) \rightarrow L_1$  and  $(a_n) \rightarrow L_2$  such that  $L_1 \neq L_2$ .

Let  $\varepsilon = \frac{|L_1 - L_2|}{3}$ , because  $(a_n) \rightarrow L_1$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$ . Therefore, for every  $n \geq N$ ,

$$|a_n - L_2| = |a_n - L_1 - (L_2 - L_1)| \quad (2.4)$$

$$\geq ||a_n - L_1| - |L_2 - L_1|| \quad (2.5)$$

$$= ||L_1 - L_2| - |a_n - L_1|| \quad (2.6)$$

$$= 3\varepsilon - |a_n - L_1| \quad (2.7)$$

$$> 2\varepsilon \quad (2.8)$$

Therefore, there does not exist any  $N' \in \mathbb{N}$  such that  $|a_n - L_2| < \varepsilon$  for every  $n \geq N'$ . ■

**Definition 2.3.** A sequence  $(a_n)$  is **divergent** if it does not converge.

**Example 2.1.** The sequence  $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \dots)$  is divergent.

*Proof.* Let  $\varepsilon := \frac{2}{5 \times 3}$ , assume, for contradiction, that  $(a_n) \rightarrow L$  for some  $L \in \mathbb{R}$ . Then there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $|a_n - L| < \frac{2}{15}$ . Since the sequence is alternating, it must be the case that  $|L - \frac{1}{5}| < \frac{2}{15}$ . Similarly,

$$\left| -\frac{1}{5} - L \right| = \left| \frac{1}{5} + L \right| \quad (2.9)$$

$$= \left| \frac{1}{5} + L - \frac{1}{5} + \frac{1}{5} \right| \quad (2.10)$$

$$= \left| (L - \frac{1}{5}) - (-\frac{2}{5}) \right| \quad (2.11)$$

$$\geq \left| \left| L - \frac{1}{5} \right| - \frac{6}{15} \right| \quad (2.12)$$

$$= \frac{6}{15} - \left| L - \frac{1}{5} \right| \quad (2.13)$$

$$> \frac{4}{15} \quad (2.14)$$

$$> \varepsilon \quad (2.15)$$

the strict inequality suggests there cannot be a  $M \in \mathbb{N}$  such that  $|a_n - L| < \varepsilon$  for every  $n \geq M$ . ■

*Alternative Proof.* If  $(a_n)$  is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of  $(a_n)$  converging to  $\frac{1}{5}$  and  $-\frac{1}{5}$  respectively, this leads to a contradiction. ■

**Definition 2.4.** A sequence is **bounded** if  $\exists M \in \mathbb{R}$  such that  $\forall n \in \mathbb{N}, |a_n| < M$ .

**Theorem 2.3.** Every convergent sequence is bounded.

*Proof.* Let  $(a_n) \rightarrow L$ , take  $\varepsilon = 1$ , then there exists  $N \in \mathbb{N}$  such that  $|a_n - L| < 1$  for every  $n > N$ . Note that  $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$ , which implies  $|a_n| < |L| + 1$ . Let  $Q := \max_{n < N} a_n$ , take  $M := \max\{Q, |L| + 1\}$ , then  $M$  bounds  $(a_n)$ . ■

## 2.2 Limit Theorems

**Theorem 2.4** (Algebraic Limit Theorem). Let  $(a_n) \rightarrow a, (b_n) \rightarrow b$  be convergent sequences, and  $c \in \mathbb{R}$ , then

- (i)  $(ca_n) \rightarrow ca$ ;
- (ii)  $(a_n + b_n) \rightarrow a + b$ ;
- (iii)  $(a_nb_n) \rightarrow ab$ ;
- (iv)  $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$ , provided  $(b_n), b \neq 0$ .

*Proof (i).* Let  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $\forall n \geq N, |a_n - a| < \frac{\varepsilon}{|c|}$ . Then, for every  $n \geq N$ ,  $|ca_n - ca| = |c||a_n - a| < \varepsilon$ . ■

*Proof (ii).* Let  $\varepsilon > 0$ , there exists  $N_1, N_2 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\varepsilon}{3} \forall n \geq N_1$  and  $|b_n - b| < \frac{\varepsilon}{3} \forall n \geq N_2$ . Take  $N := \max\{N_1, N_2\}$ , let  $n \geq N$ ,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \quad (2.16)$$

■

*Proof (iii).* Note that

$$|a_nb_n - ab| = |a_nb_n + a_nb - a_nb - ab| \quad (2.17)$$

$$\leq |a_nb_n - a_nb| + |a_nb - ab| \quad (2.18)$$

$$\leq |a_n||b_n - b| + |b||a_n - a| \quad (2.19)$$

Let  $N_1 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\varepsilon}{2|b|}$  for every  $n \geq N_1$ . Because  $(a_n)$  is convergent, let  $M$  denote its bound such that  $|a_n| < M \forall n \in \mathbb{N}$ . Let  $N_2 \in \mathbb{N}$  such that  $|b_n - b| < \frac{\varepsilon}{2M}$ . Then for every  $n \geq N_3 := \max\{N_1, N_2\}$ ,  $|a_nb_n - ab| < \varepsilon$ . ■

*Proof (iv).* *Claim i:* when  $n$  is sufficiently larger,  $|b_n| > 0$  is bounded away from zero by  $M$ .

Let  $\varepsilon = \frac{|b|}{10}$ , then there exists  $N_1 \in \mathbb{N}$  such that for every  $n \geq N_1$ ,  $|b_n - b| < \frac{|b|}{10}$ . Note that for every such  $n$ ,

$$|b_n| = |b_n - b - (-b)| \quad (2.20)$$

$$\geq ||b_n - b| - |b|| \quad (2.21)$$

$$\geq |b| - |b_n - b| \quad (2.22)$$

$$> \frac{9|b|}{10} \quad (2.23)$$

*Claim ii:*  $\left(\frac{1}{b_n}\right) \rightarrow \frac{1}{b}$ . Let  $\varepsilon > 0$ , note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_nb} - \frac{b_n}{b_nb}\right| \quad (2.24)$$

$$= \frac{1}{|b_n||b|}|b_n - b| \quad (2.25)$$

from the first claim,  $\frac{1}{|b_n|} < \frac{10}{9|b|}$  for every  $n \geq N_1$ . Since  $(b_n) \rightarrow b$ , there exists  $N_2 \in \mathbb{N}$  such that for every  $n \geq N_2$ ,  $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$ . Consequently, for every  $n \geq N_3 := \max\{N_1, N_2\}$ ,  $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$ . Then the result is immediate from property (iii) in the algebraic limit theorem. ■

**Theorem 2.5** (Order Limit Theorem). Let  $(a_n) \rightarrow a$  and  $(b_n) \rightarrow b$ , then

- (i)  $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$ ;
- (ii)  $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$ ;
- (iii)  $\exists c \in \mathbb{R} \text{ s.t. } c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$ ;
- (iv)  $\exists c \in \mathbb{R} \text{ s.t. } a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$ .

*Proof.* (i) Assume, for contradiction,  $a < 0$ . Take  $\varepsilon = \frac{|a|}{2}$ , then for some  $N \in \mathbb{N}$ , for every  $n \geq N$   $a_n \in V_\varepsilon(a)$ . However, this contradicts the fact that  $a_n \geq 0$ .

(ii) Consider sequence  $(b_n - a_n)$  in which  $b_n - a_n \geq 0$  for every  $n \in \mathbb{N}$ .  $(b_n - a_n) \rightarrow (b - a)$  by the algebraic limit theorem. By property (i),  $b - a \geq 0$ .

(iii) and (iv) Consider constant sequence defined as  $(c_n)$  such that  $c_n = c$  for every  $n \in \mathbb{N}$ , the results are immediate by applying (ii). ■

**Theorem 2.6** (Squeeze Theorem). Let  $(x_n) \rightarrow L$  and  $(z_n) \rightarrow \ell$ . If for every  $n \in \mathbb{N}$ ,  $x_n \leq y_n \leq z_n$ , then  $(y_n) \rightarrow \ell$ .

*Remark:* squeeze theorem does not impose the prior that  $(y_n)$  is convergent.

*Proof.* Let  $\varepsilon > 0$ , because both  $(x_n) \rightarrow \ell$  and  $(y_n) \rightarrow \ell$ ,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \quad (2.26)$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \quad (2.27)$$



Take  $N_3 := \max\{N_1, N_2\}$ , then for every  $n \geq N_3$ ,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \quad (2.28)$$

$$\implies y_n \in V_\varepsilon(\ell) \quad (2.29)$$

therefore  $(y_n) \rightarrow \ell$  by definition. ■

### 2.3 Monotone Convergence Theorem

**Definition 2.5.** A sequence  $(a_n)$  is said to be **monotone** if it is either increasing ( $a_{n+1} \geq a_n \forall n \in \mathbb{N}$ ) or decreasing ( $a_{n+1} \leq a_n \forall n \in \mathbb{N}$ ).

**Theorem 2.7** (Monotone Convergence Theorem). If a monotone sequence  $(a_n)$  is bounded, then it converges.

*Proof.* WLOG, assume  $(a_n)$  is increasing, let  $\Gamma := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ , because  $\Gamma$  is bounded,  $s := \sup_n \Gamma$  is well-defined by the completeness of real numbers.

*Claim:*  $(a_n) \rightarrow s$ . Let  $\varepsilon > 0$ , by the definition of supremum,  $\exists N \in \mathbb{N}$  such that  $a_N > s - \varepsilon$ . Because the sequence is increasing and  $s + \varepsilon \in \Gamma^\uparrow$ ,  $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$ .  $(a_n) \rightarrow s$  by definition. ■

### 2.4 Series

**Definition 2.6.** Let  $(a_i)$  be a sequence, then the  $n$ -th **partial sum** is defined as  $s_n := \sum_{i=1}^n a_i$ . And the **infinite sum/series** of  $(a_n)$  is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \rightarrow s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \quad (2.30)$$

**Example 2.2.**  $\sum_{i=1}^{\infty} \frac{1}{i^2}$  converges.

*Proof.* Obviously the corresponding partial sums are increasing because the sequence  $(\frac{1}{i^2})$  is positive.

**Claim:**  $(s_n)$  is bounded from above. Let  $n \in \mathbb{N}$ , observe

$$\sum_{i=1}^n \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \quad (2.31)$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \quad (2.32)$$

$$= 2 - \frac{1}{n} \leq 2 \quad (2.33)$$

The result is immediate by the monotone convergence theorem. ■

**Example 2.3** (Harmonic Series).  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

*Proof. Claim:* there exists a subsequence of  $(s_n)$  diverges, so  $(s_n)$  cannot be convergent. Consider the subsequence  $(s_k)$  constructed by defining  $s_k := s_{2^k}$ . Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \quad (2.34)$$

$$> 1 + \frac{1}{2}k \quad (2.35)$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. Therefore, the original sequence of partial sums cannot be convergent. ■

**Definition 2.7.** Let  $(a_n)$  be a sequence, then for every strictly increasing sequence  $(n_i)_i$  in  $\mathbb{N}$ ,  $(a_{n_i})$  is a **subsequence** of  $(a_n)$ .

**Theorem 2.8.** All subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Let  $(a_n) \rightarrow \ell$ , let  $(a_{n_k})$  be a subsequence of  $(a_n)$ . Let  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies a_n \in V_\varepsilon(\ell)$ . By the definition of subsequences, there exists some  $K \in \mathbb{N}$  such that  $n_K \geq N$ . Take such  $K$ , then for every  $k \geq K$ , it must be  $n_k \geq N$ . Therefore  $a_{n_k} \in V_\varepsilon(\ell)$  for every  $k \geq K$ , and  $(a_{n_k}) \rightarrow \ell$  by definition. ■

**Remark 2.1.** Note the implication of above theorem is two-fold:

- (i) Every subsequence of a convergent sequence is convergent;
- (ii) All subsequences converge to the same limit.

**Corollary 2.2.** A sequence  $(a_n)$  must be divergent if there exists two subsequences of it converge to two different limits.

*Proof.* Immediate by taking the contrapositive form of above theorem. ■

**Theorem 2.9** (Bolzano–Weierstrass). Every bounded sequence contains a convergent subsequence.

*Proof.* Suppose  $(a_n)$  is bounded by certain  $M > 0$ , that's, for every  $n \in \mathbb{N}$ ,  $-M < a_n < M$ . Consider the split  $I_1^\ell := [-M, 0]$  and  $I_1^u := [0, M]$ . At least one of above closed intervals contain an infinitely many elements of  $(a_n)$ .

Define the interval as  $I_2$ . At each  $I_n$ , one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and  $I_{n+1}$  is defined to be such closed interval containing infinitely many elements.

Note that the sequence  $(I_n)$  is nested by construction. By the nested interval property, one can show that  $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$ .

Also,  $\lim_{n \rightarrow \infty} |I_n| = 0$ . Then  $\cap_{n \in \mathbb{N}} I_n$  must be a singleton with  $a$  in it. One can construct such that  $a_{n_k} \in I_k$ . Note that  $|I_n| = \frac{1}{2^{n-1}}$ , therefore, for every  $\varepsilon > 0$ , one can take  $N \geq \log_2 \left(\frac{1}{\varepsilon}\right) + 1$ , so that for every  $k \geq N$ , by definition of subsequences,  $n_k \geq n$ , so that  $a_{n_k}, a \in I_N$ . This implies  $a_{n_k} \in V_\varepsilon(a)$  and  $(a_{n_k}) \rightarrow a$ . ■

## 2.5 Cauchy Criterion

**Definition 2.8.** A sequence  $(a_n)$  is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n \geq N \implies |a_n - a_m| < \varepsilon \quad (2.36)$$

**Proposition 2.1.** Every convergent sequence is Cauchy.

*Proof.* Let  $(a_n) \rightarrow \ell$ , let  $\varepsilon > 0$ . By the convergence of sequence,  $\exists N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $|a_n - \ell| < \frac{\varepsilon}{2}$ , which turns out to imply  $a_n, a_m \in V_\varepsilon(\ell)$ . ■

**Lemma 2.1.** Every Cauchy sequence is bounded.

*Proof.* Let  $(a_n)$  be a Cauchy sequence, take  $\varepsilon = 1$ , then there exists  $N \in \mathbb{N}$  such that for every  $m, n \geq N$ ,  $|a_n - a_m| < 1$ . In particular, take  $m = N$ , for every  $n \geq N$ ,  $|a_n - a_N| < 1$ , and  $|a_n| \leq |a_N| + 1$ . Then  $(a_n)$  is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \quad (2.37)$$

■

**Theorem 2.10** (Cauchy Criterion). A sequence in  $\mathbb{R}$  is convergent if and only if it's Cauchy.

*Proof.* ( $\Leftarrow$ ) Suppose  $(a_n)$  is Cauchy, by the lemma established above,  $(a_n)$  is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence  $(a_{n_k}) \rightarrow \ell$ .

*Claim:*  $(a_n) \rightarrow \ell$ . Let  $\varepsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that for every  $n_k, n \geq N_1$ ,  $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$ . And there exists another  $N_2 \in \mathbb{N}$  such that for every  $n_k \geq N_2$ ,  $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$ .

Take  $N_3 := \max\{N_1, N_2\}$ .

Note that for every  $n \geq N_3$ , one can choose some  $n_k \geq n$  as leverage and derive

$$|a_n - \ell| = |a_n - a_{n_k} + a_{n_k} - \ell| \quad (2.38)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| \quad (2.39)$$

$$< \varepsilon \quad (2.40)$$

( $\Rightarrow$ ) Already shown in previous proposition. ■

## 2.6 Convergence Test for Series

**Theorem 2.11** ( $n$ -th term test).

$$\sum_{i=1}^{\infty} a_n \text{ converges} \implies \lim_{n \rightarrow \infty} a_n = 0 \quad (2.41)$$

*Remark:* this theorem is only a necessary condition for convergence of series.

*Proof.* Suppose the partial sums converges to  $\ell$ , by the definition of partial sums,  $a_n = s_{n+1} - s_n$ . Further, the convergence of partial sums guarantees the convergence of  $(a_n)$ . By taking limit on both sides of above identity, it can be shown  $\lim_{n \rightarrow \infty} a_n = 0$ . ■

**Theorem 2.12** (Cauchy Criterion for Series). For series  $\sum_{n=1}^{\infty} a_n$ , the following are equivalent:

- (i) Series converges;
- (ii)  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N, \left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$  (i.e. *tail* sum sequence converges);
- (iii)  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\forall m > n \geq N, \left| \sum_{k=n+1}^m a_k \right| < \varepsilon$ . (i.e. partial sum is Cauchy)

*Proof.* (i)  $\implies$  (ii): Suppose  $(S_n)$  converges, let  $\varepsilon > 0, \exists N$  s.t.  $\forall n \geq N, |S_n - L| < \varepsilon$ . Note that

$$L - S_n = \lim_{m \rightarrow \infty} \sum_{k=1}^m a_k - S_n \quad (2.42)$$

$$= \lim_{m \rightarrow \infty} \left[ \sum_{k=1}^m a_k - S_n \right] \quad (2.43)$$

$$= \lim_{m \rightarrow \infty} \sum_{k=n+1}^m a_k \quad (2.44)$$

which implies the convergence of tail sums.

(ii)  $\implies$  (iii): Suppose the tail sum converges, let  $\varepsilon > 0$ , note that

$$\left| \sum_{k=n+1}^m a_k \right| = \left| \sum_{k=m+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k \right| \quad (2.45)$$

$$\leq \left| \sum_{k=m+1}^{\infty} a_k \right| + \left| \sum_{k=n+1}^{\infty} a_k \right| \quad (2.46)$$

Both terms can be made arbitrarily small by (ii), specifically, one can choose  $N_1$  and  $N_2$  such that both terms are strictly bounded by  $\frac{\varepsilon}{2}$ , and  $N_3 := \max\{N_1, N_2\}$  is the desired value.

(iii)  $\implies$  (i): Since the partial sum is a Cauchy sequence in a complete space, it must converges, so the series is well-defined. ■

### 2.6.1 The Comparison Test

**Definition 2.9.** A sequence  $(a_n)$  is a **geometric sequence** with coefficient  $r$  if  $a_{n+1} = ra_n$ .

**Proposition 2.2.** Geometric sequences whenever  $r \in (-1, 1)$ . Note that when  $r = -1$ , the sequence becomes an alternating sequence, and the convergence property is indefinite.

**Proposition 2.3.** Let  $(a_n)$  be a geometric sequence with coefficient  $r$ , then for every  $m \in \mathbb{N}$ ,

$$rS_m^a = ra_0 + r^2a_0 + \cdots + r^{n+1}a_0 \quad (2.47)$$

$$\implies (r-1)S_m^a = r^{n+1}a_0 - a_0 \quad (2.48)$$

$$\implies S_m^a = a_0 \frac{1 - r^{m+1}}{1 - r} \quad (2.49)$$

**Theorem 2.13** (The Comparison Test). Let  $(a_n)$  and  $(b_n)$  be two sequences satisfy  $|a_n| \leq b_n$  for every  $n \in \mathbb{N}$ . Then

- (i)  $\sum_{i=1}^{\infty} b_n$  converges  $\implies \sum_{i=1}^{\infty} a_n$  converges;
- (ii)  $\sum_{i=1}^{\infty} a_n$  diverges  $\implies \sum_{i=1}^{\infty} b_n$ .

*Proof. Part 1:* Suppose  $(b_n)$  converges, it is therefore Cauchy. Let  $\varepsilon > 0$ . Note that for every  $m > n$ :

$$|S_m^a - S_n^a| = \left| \sum_{k=n+1}^m a_k \right| \quad (2.50)$$

$$\leq \sum_{k=n+1}^m |a_k| \quad (2.51)$$

$$\leq \sum_{k=n+1}^m b_k \quad (2.52)$$

Therefore exists  $N \in \mathbb{N}$  such that  $\sum_{k=n+1}^m b_k \leq \left| \sum_{k=n+1}^m b_k \right| < \varepsilon$  for every  $m, n \geq N$ . Taking such  $N$  provides the cutoff needed for  $(S_n^a)$  to be Cauchy. Because  $(S_n^a) \subseteq \mathbb{R}$ , it converges.

*Part 2:* The result is immediate by taking the contrapositive form of the previous statement. ■

## 2.6.2 The Root Test

**Definition 2.10.** Let  $(a_n)$  be a bounded sequence, then

$$\limsup(a_n) := \sup_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.53)$$

$$\liminf(a_n) := \inf_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.54)$$

$$(2.55)$$

**Theorem 2.14** (The Root Test). Let  $(a_n)$  be a sequence in which  $a_n \geq 0$  for every  $n \in \mathbb{N}$ , let  $\ell = \limsup a_n^{1/n}$ , then

(i) If  $\ell < 1$ , then  $(S_n^a)$  converges;

(ii) If  $\ell > 1$ , then  $(S_n^a)$  diverges;

(iii) If  $\ell = 0$ , inconclusive.

*Proof. Part 1:*(Idea: compare with geometric series with  $r < 1$ ) Suppose  $\ell < 1$ , pick  $r \in (\ell, 1)$ , and let  $\varepsilon = r - \ell$ . By the convergence of supremum, there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,

$$\left| \sup_{k \geq n} a_k^{1/k} - \ell \right| < \varepsilon \quad (2.56)$$

$$\implies a_n^{1/n} \leq \sup_{k \geq n} a_k^{1/k} < \ell + \varepsilon =: r \quad (2.57)$$

Therefore, for every  $n \geq N$ ,  $a_n < r^n$ . Because  $(a_n)$  is assumed to be a non-negative sequence, then  $|a_n| < r^n$ . Construct new sequences:

$$b_k = \begin{cases} a_k & \forall k < N \\ r^k & \forall k \geq N \end{cases} \quad (2.58)$$

Then, clearly  $|a_n| \leq b_k$  for every  $k \in \mathbb{N}$ . And  $(b_n)$  is a sequence with geometric tails (which has coefficient less than one). So  $\sum_{k=1}^{\infty} b_k$  converges, which implies  $\sum_{k=1}^{\infty} a_k$  converges by the comparison test.

*Part 2:* Suppose  $\ell > 1$ .

Note that the necessary condition for  $\sum a_n^{1/n}$  to converge is  $\lim_{n \rightarrow \infty} a_n^{1/n} = 0$ , which implies every subsequence of  $(a_n^{1/n})$  converges to zero. We are going to prove the divergence of series by constructing a subsequence of  $(a_n^{1/n})$  does not converge to zero.

Take  $\varepsilon = \ell - 1 > 0$ , there exists  $N$  such that for every  $n \geq N$ :

$$\ell - \varepsilon < \sup_{k \geq n} a_k^{1/k} \quad (2.59)$$

$$\implies 1 < \sup_{k \geq n} a_k^{1/k} \quad (2.60)$$

By definition of supremum, there exists  $n_1 \geq n$  such that

$$a_{n_1}^{1/n_1} > 1 \quad (2.61)$$

For every  $n \geq N$ , we can construct a subsequence of  $(a_n^{1/n})$  such that every term in it is strictly greater than 1, which means it cannot converge to 0. Therefore, series diverges. ■

### 2.6.3 Other Tests

**Theorem 2.15** (Limit Comparison Test). Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  satisfy:

- (i)  $b_n \geq 0$ ;
- (ii)  $\limsup \frac{|a_n|}{b_n} < \infty$ ;
- (iii)  $\sum_{n=1}^{\infty} b_n$  converges.

Then  $\sum_{n=1}^{\infty} a_n$  converges as well.

**Theorem 2.16** (Ratio Test). Given sequence  $(a_n)_{n=1}^{\infty}$  such that  $a_n \geq 0$ , then

- 1. If  $\limsup \frac{a_{n+1}}{a_n} < 1$ ,  $\sum_{n=1}^{\infty} a_n$  converges;
- 2. If  $\limsup \frac{a_{n+1}}{a_n} > 1$ ,  $\sum_{n=1}^{\infty} a_n$  diverges.

**Theorem 2.17** (Integral Test). Let  $f(x)$  be a *positive* and *monotone decreasing* function on  $[1, \infty)$ . Consider  $(f(x_n))$ , then

$$\sum_{n=1}^{\infty} f(n) \text{ convergent} \iff \int_1^{\infty} f(x) dx < \infty \quad (2.62)$$

**Theorem 2.18** (Alternating Series Test). For an alternating sequence  $\sum_{n=1}^{\infty} (-1)^n a_n$ , if  $(a_n) \searrow 0$ , then the series converges.

*Proof.* **TODO** ■

## 2.7 Absolute and Conditional Convergence

**Corollary 2.3** (Corollary of Comparison Test). If  $\sum_{i=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

**Definition 2.11.** For any series  $\sum_{n=1}^{\infty} a_n$ , if

1.  $\sum_{i=1}^{\infty} |a_n|$  converges,  $\sum_{n=1}^{\infty} a_n$  **converges absolutely**;
2.  $\sum_{i=1}^{\infty} |a_n|$  does not converge, then  $\sum_{n=1}^{\infty} a_n$  **converges conditionally**.

**Example 2.4.** Alternating harmonic series converges conditionally.

However,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  converges absolutely.

**Definition 2.12.**  $\sum_{n=1}^{\infty} b_n$  is called a **rearrangement** of series  $\sum_{n=1}^{\infty} a_n$  if there exists  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f$  is a bijection and  $b_{f(k)} = a_k$  for every  $k \in \mathbb{N}$ .

**Theorem 2.19** (Riemann Series Theorem). If series  $\sum_{n=1}^{\infty} a_n$  converges conditionally, for every  $\alpha \in \mathbb{R}$ , there exists a rearrangement  $\sum_{n=1}^{\infty} b_n$  converges to  $\alpha$ .

*Proof.* The proof is non-trivial and omitted. ■

**Theorem 2.20.** If series  $\sum_{n=1}^{\infty} a_n$  converges absolutely to some value  $A \in \mathbb{R}$ , then every rearrangement  $\sum_{n=1}^{\infty} b_n$  converges to  $A$ .

*Proof.* Define partial sum sequences

$$S_n := \sum_{k=1}^n a_k \quad T_m := \sum_{k=1}^m b_k \quad (2.63)$$

Suppose  $(S_n) \rightarrow A$ , want to show:  $(T_n) \rightarrow A$ .

Let  $\varepsilon > 0$  fixed.

By convergence of  $(S_n)$ , there exists  $N_1 \in \mathbb{N}$  such that

$$n \geq N_1 \implies |S_n - A| < \frac{\varepsilon}{2} \quad (2.64)$$

Because  $\sum_{n=1}^{\infty} a_n$  converges absolutely, by the Cauchy criterion for convergent series (i.e. partial sum sequence is Cauchy), there exists  $N_2 \in \mathbb{N}$  such that

$$n > m \geq N_2 \implies \sum_{k=n+1}^m |a_k| < \frac{\varepsilon}{2} \quad (2.65)$$

Define  $N := \max\{N_1, N_2\}$ ,  $M := \max\{f(k) : 1 \leq k \leq N\}$ .

$$|T_m - S_N| = |b_1 + \cdots + b_m - a_1 - \cdots - a_N| \quad (2.66)$$

$$= |b_1 + \cdots + b_m - b_{f(1)} - \cdots - b_{f(N)}| \quad (2.67)$$

Note that for every  $m \geq M$ , by construction,  $\{b_{f(1)}, \dots, b_{f(N)}\} \subseteq \{b_1, \dots, b_m\}$ .

Note that for each  $b_{f(k)} \in \{b_1, \dots, b_m\}$ , either  $k > N$  or  $k \leq N$ . But all  $b_{f(k)}$  with  $k \leq N$  were subtracted, so  $b_{f(k)}$  elements left are all from  $\{a_k : k \geq N+1\}$ .

$$\dots = \left| \sum_{k \in \mathcal{I} \geq N+1} a_k \right| \quad (2.68)$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad (2.69)$$

Therefore, for all  $m \geq M$ ,

$$|T_m - A| = |T_M - S_n + S_n - A| \quad (2.70)$$

$$\leq |T_M - S_n| + |S_n - A| \quad (2.71)$$

$$< \varepsilon \quad (2.72)$$

The desired result is immediate. ■

## 3 Topology in $\mathbb{R}$

### 3.1 Definitions

**Definition 3.1.** A set  $\mathcal{O} \subseteq \mathbb{R}$  is **open** if

$$\forall x \in \mathcal{O} \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(x) \subseteq \mathcal{O} \quad (3.1)$$

**Theorem 3.1.** Arbitrary union of open sets is open; Any finite intersection of open sets is open.

*Proof.* Let  $\mathcal{O}_\alpha$  open for all  $\alpha \in \mathcal{A}$ . Let  $\mathcal{O} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha$ . If  $x \in \mathcal{O}$ , there exists some  $\alpha \in \mathcal{A}$  such that  $x \in \mathcal{O}_\alpha$ . There exists  $V_\varepsilon(x) \subseteq \mathcal{O}_\alpha \subseteq \mathcal{O}$ . Hence  $\mathcal{O}$  is open.

Let  $\{\mathcal{O}_i : 1 \leq i \leq n\}$  be a collection of open sets, let  $\mathcal{O} := \bigcap_{i=1}^{\infty} \mathcal{O}_i$ . If  $x \in \mathcal{O}$ , there exists  $\varepsilon_i > 0$  such that  $V_{\varepsilon_i}(x) \subseteq \mathcal{O}_i$  for every  $i$ . Take  $\varepsilon := \min\{\varepsilon_i\}$ , which exists and is strictly positive by finiteness of index set. Therefore  $V_\varepsilon(x) \subseteq \mathcal{O}_i$  for every  $i$ , and therefore in  $\mathcal{O}$ . ■



**Definition 3.2.**  $x$  is a **limit point** of  $A$  if  $\forall \varepsilon > 0$ ,

$$V_\varepsilon(x) \cap A - \{x\} \neq \emptyset \quad (3.2)$$

*Remark:* this definition does not require  $x$  to be an element of  $A$ .

**Theorem 3.2.**  $x$  is a limit point of  $A$  if and only if there exists a sequence  $(a_n)_{n=1}^\infty \subseteq A$  such that  $a_n \neq x \forall n \in \mathbb{N}$  and  $(a_n)_{n=1}^\infty \rightarrow x$ .

*Proof.* ( $\implies$ ) Let  $x$  be a limit point, take  $\varepsilon = \frac{1}{n}$ , immediate by the definition of limit point.

( $\impliedby$ ) Trivially by definition of sequential convergence. ■

**Definition 3.3.**  $X \subseteq \mathbb{R}$  is **closed** if it contains all its limit points.

**Definition 3.4.**  $x \in A$  is an **isolated point** if it is not a limit point of  $A$ .

**Definition 3.5.** The **closure** of  $A$ , denoted as  $\overline{A}$ , is defined to be the union of  $A$  and all limit points of  $A$ .

**Definition 3.6.**  $A \subseteq X$  is **dense** in  $X$  if  $\overline{A} = X$ .

**Theorem 3.3.** Let  $x \in \mathbb{R}$ , there exists a sequence  $(q_n)_{n=1}^\infty \subseteq \mathbb{Q}$  such that  $(q_n)_{n=1}^\infty \rightarrow x$ .

*Proof.* Let  $x \in \mathbb{R}$ . Note that  $\forall u < v \in \mathbb{R}$ , there exists  $q \in (u, v) \cap \mathbb{Q}$ . Hence, for every  $n \in \mathbb{N}$ ,  $\exists q_n \in \mathbb{Q}$  such that  $x - \frac{1}{n} < q_n < x + \frac{1}{n}$ . It is evident that  $(q_n)_{n=1}^\infty \rightarrow x$ . ■

**Lemma 3.1.**  $\overline{A}$  is the smallest closed set containing  $A$ .

*Proof.* It is evident that  $\overline{A}$  is a closed set containing  $A$ .

Now show the closure is in fact the smallest closed set. Let  $B \subsetneq \overline{A}$  be a proper subset of the closure, we are going to show that  $B$  is not closed. Let  $x \in \overline{A} - B \neq \emptyset$ .

Note that  $\overline{A} \equiv A \cup A'$ , then either  $x \in A$  or  $x \in A'$ . If  $x \in A$ , then  $B$  does not contain  $A$ . If  $x \in A'$ , then  $B$  does not contain all limit points of  $A$ , so it is not closed. ■

**Theorem 3.4.** Equivalent definitions of openness and closedness:

(i)  $\mathcal{O}$  is open if and only if  $\mathcal{O}^c$  is closed;

(ii)  $\mathcal{F}$  is closed if and only if  $\mathcal{F}^c$  is open.

*Proof.* ( $\implies$ ) Let  $\mathcal{O}$  be an open set, let  $(x_n) \rightarrow x$  be a convergent sequence in  $\mathcal{O}^c$ . It is evident that if  $x \in \mathcal{O}$ , infinitely many elements in the tail of  $(x_n)$  would be in  $V_\varepsilon(x) \subseteq \mathcal{O}$ , which leads to a contradiction. Therefore  $\mathcal{O}^c$  contains all of its limit points, and  $\mathcal{O}^c$  is therefore closed.

( $\impliedby$ ) Let  $\mathcal{F}^c$  be a closed set, suppose  $\mathcal{F}$  is not open, there exists  $x \in \mathcal{F}$  such that for all  $\varepsilon > 0$ ,  $V_\varepsilon(x) \cap \mathcal{F}^c \neq \emptyset$ . Then we can construct a sequence in  $\mathcal{F}^c$  converge to  $x$ , which leads to a contradiction that there is a limit point of a sequence in  $\mathcal{F}^c$  not contained by  $\mathcal{F}^c$ .

The second part is immediate. ■

**Theorem 3.5.** Any intersection of closed sets is closed; any finite union of closed sets is closed.

*Proof.* Direct result from De Morgan's law and the previous theorem. ■

*Remark:* Limit points and boundary points are completely different. Example: let  $\Omega = [1, 2] \cup 3$ , then 3 is a boundary point but not a limit point (i.e. it is isolated). And 0.5 is a limit point but not a boundary point.

### 3.2 Compactness

**Definition 3.7.** A set  $K \subseteq \mathbb{R}$  is **compact** if every sequence in  $K$  has a convergent subsequence converges to some limit  $x \in K$ .

**Theorem 3.6.** A set  $K \subseteq \mathbb{R}$  is compact if and only if it is closed and bounded.

*Proof.* ( $\implies$ ) Suppose  $K \subseteq \mathbb{R}$  is compact.

*Show  $K$  is bounded:* suppose, for contradiction,  $K$  is unbounded, then for every  $N \in \mathbb{N}$ , one can construct a sequence as following:  $a_1 \in K$  and  $a_{n+1} > \max\{a_n, n\}$ . Such sequence diverges to positive infinity, and every subsequence of it converges to infinity as well (easy to verify). This leads to a contradiction to the compactness of  $K$ .

*Show  $K$  is closed:* Suppose, for contradiction,  $K$  is not closed, then there exists some limit point of  $K$  say  $x \notin K$ . Consider the sequence  $(x_n) \rightarrow x$  in  $K$ , because every subsequence of such convergent sequence converges to the same limit  $x \notin K$ , which leads to a contradiction of compactness.

( $\impliedby$ ) Let  $(x_n) \subseteq K$ , then  $(x_n)$  is bounded and therefore possesses a convergent subsequence by Bolzano-Weierstrass Theorem. Further, because  $K$  is closed, then the limit point must be in  $K$ . ■

**Theorem 3.7** (Nested Compact Set Property). Let  $\mathbb{R}^n \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ , where  $K_n \neq \emptyset$  are all compact sets, then

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset \quad (3.3)$$

*Proof.* Construct a sequence such that  $x_n \in K_n$  for every  $n \in \mathbb{N}$ . In particular,  $(x_n) \subseteq K_1$ . Because  $K_1$  is compact, it has a convergent subsequence  $(x_{n_k}) \rightarrow x \in K_1$ . Then every subsequence of  $(x_{n_k})$  converges to the same limit  $x$ .

Note that by dropping out the first element of the subsequence, the resulted sequence starts with  $x_{n_2}$ . By the definition of subsequences,  $n_2 \geq 2$ , therefore, the truncated subsequence is contained in  $K_2$  because of the compactness of  $K_2$ . As a result,  $x \in K_2$ . Applying the same argument on all natural numbers, it is immediate that  $x \in K_n \forall n \in \mathbb{N}$ . So  $x \in \bigcap_{n \in \mathbb{N}} K_n$ . ■

*Proof. (Cantor's Argument).* Suppose, for contradiction, the intersection is empty. Define  $U_n := K_1 \setminus K_n$ . Note that  $U_n = K_1 \cap K_n^c = K_n^c$ , which is open. Further,  $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} K_1 \cap K_n^c = K_1 \cap (\bigcup_{n \in \mathbb{N}} K_n^c) = K_1 \cap (\bigcap_{n \in \mathbb{N}} K_n)^c = K_1 \setminus \bigcap_{n \in \mathbb{N}} K_n = K_1$ . Therefore,  $\mathcal{C} = \{U_n : n \in \mathbb{N}\}$  is an open cover of  $K_1$ . Because  $K_1$  is compact, there exists a finite subcover of  $\mathcal{C}$ . Take  $n^*$  to be the greatest index in this finite subcover, then for every  $x' \in K_{n^*+1} \subseteq K_1$ ,  $x'$  is not in the union of the constructed subcover, which leads to a contradiction. ■

**Example 3.1.** Note that the closedness itself is not sufficient for the nest compact set property to hold. For instance, the following sequence of closed sets are nested:  $F_n := [n, \infty)$ , but indeed, for every  $x \in \mathbb{R}$ , there exists a natural number  $n > x$ , so that  $x \notin \bigcap_{n \in \mathbb{N}} F_n$ . Therefore,  $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$ .

**Definition 3.8.** Let  $A \subseteq \mathbb{R}$ , an **open cover** for  $A$  is a collection of open sets  $\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$  such that  $A \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ .

**Theorem 3.8** (Heine-Borel). Let  $K \subseteq \mathbb{R}$ , then the following are equivalent:

- (i)  $K$  is (sequentially) compact;
- (ii)  $K$  is closed and bounded;
- (iii) Every open cover of  $K$  has a finite subcover.

*Proof.* The equivalence of (i) and (ii) has been proven previously.

*Show (iii)  $\implies$  (ii):* suppose every open cover of  $K$  has a finite subcover, consider the following cover of  $K$ :  $\mathcal{C} = \{[-n, n] : n \in \mathbb{N}\}$ . Let  $M$  be the greatest index in the finite subcover  $\mathcal{C}$ , and obviously  $K$  is bounded by  $M$ .

Suppose, for contradiction, that  $K$  is not closed. Let  $y$  be a limit point of  $K$  but  $y \notin K$ . Then, for every  $\varepsilon > 0$ ,  $V_\varepsilon^o(y) \cap K \neq \emptyset$ . We've shown that  $K$  is bounded, take  $M \in \mathbb{R}$  such that  $(-M, M) \supset K$ . Define the following cover:

$$\mathcal{C} := \left\{ (-M, M) \setminus \overline{V_\varepsilon(y)} : \varepsilon \in \mathbb{R}_{++} \right\} \quad (3.4)$$

Because  $K$  is compact, there exists a finite subcover of  $\mathcal{C}$ , which is clearly a contradiction.

*Show (ii)  $\implies$  (iii):* Suppose  $K$  is closed and bounded, because of the transitivity of covering, it is sufficient to show that for every  $M \in \mathbb{R}_+$ , every open cover of  $[-M, M]$  has a finite subcover.

Let  $M \in \mathbb{R}_+$ , and  $\mathcal{C} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$  is an open cover of  $[-M, M]$ . Suppose, for contradiction, there is no finite subcover. Then either  $[-M, 0]$  or  $[0, M]$  does not have a finite subcover from  $\mathcal{C}$ . Define such interval as  $I_1$ . Interval  $I_n$  is defined inductively from  $I_{n-1}$  by firstly bisecting  $I_{n-1}$  into two closed intervals and then taking the partition that cannot be covered by any finite subcover of  $\mathcal{C}$ . Note that  $(I_n)$  is a sequence of nested compact sets, by Cantor's intersection theorem, there intersection is nonempty. Further, because the length of interval shrinks to zero as  $n \rightarrow \infty$ , the intersection must be a singleton. Let  $\{x\} = \bigcap_{n \in \mathbb{N}} I_n$ , there exists some  $\lambda \in \Lambda$ , such that  $x \in \mathcal{O}_\lambda$ . Because  $\mathcal{O}_\lambda$  is open, there exists  $\varepsilon > 0$  such that  $V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$ . Take  $k \in \mathbb{N}$  such that  $|I_k| < 2\varepsilon$ , clearly  $I_k \subseteq V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$ . Then  $\mathcal{O}_\lambda$  is a finite subcover of  $I_k$ , which leads to a contradiction. ■

### 3.3 Connected Sets

**Definition 3.9.**  $\emptyset \neq A, B \subseteq \mathbb{R}$  are **separated** if and only if  $\overline{A} \cap B = \emptyset$  and  $A \cap \overline{B} = \emptyset$ .

**Definition 3.10.**  $E \subseteq \mathbb{R}$  is **disconnected** if  $E = A \cup B$  where  $A, B$  are nonempty separated sets.

**Proposition 3.1** (Equivalent Definiton).  $E \subseteq \mathbb{R}$  is disconnected if and only if it can be expressed as the union of two *nonempty disjoint open* (open in  $E$ ) sets.

**Theorem 3.9.** A set  $E \subseteq \mathbb{R}$  is connected if for every nonempty disjoint sets  $A, B$  such that  $E = A \cup B$ , then there exists a sequence  $(a_n) \subseteq A$  converges to some point  $a \in B$ , or a sequence  $(b_n) \subseteq B$  converges to some point  $b \in A$ .

*Remark:* Essentially, a set  $E$  is connected if, no matter how it is partitioned into two nonempty disjoint sets, it is always possible to show that at least one of the sets contains a limit point of the other.

*Proof.* Suppose  $E$  is connected, then for any partition  $E = A \cup B$ , either  $\overline{A} \cap B \neq \emptyset$  or  $A \cap \overline{B} \neq \emptyset$ . WLOG,  $\overline{A} \cap B \neq \emptyset$ , take  $a \in \overline{A} \cap B \subseteq B$ , and there exists a convergent sequence in  $A$  converging to  $a \in B$ . ■

**Theorem 3.10.** Let  $E \subseteq \mathbb{R}$ , the following are equivalent:

- (i)  $E$  is connected;
- (ii) For every  $a < c < b$ ,  $a, b \in E \implies c \in E$  (intervals).

*Remark:* the collection of connected subsets of  $\mathbb{R}$  is exactly the collection of all intervals.

*Proof.* ( $\implies$ ) Suppose  $E$  is connected, considering the following sets

$$A := (-\infty, c) \cap E \tag{3.5}$$

$$B := (c, \infty) \cap E \tag{3.6}$$

Note that  $a \in A$  and  $b \in B$ , so both of them are nonempty. And  $A$  and  $B$  are separated. Suppose, for contradiction,  $c \notin E$ ,  $E = A \cup B$ , which leads to a contradiction to the assumption that  $E$  is connected.

( $\impliedby$ ) Suppose (ii), show  $E$  is connected. Let  $A$  and  $B$  be two nonempty set such that  $A \cup B = E$  and  $A \cap B = \emptyset$ . We are going to show that  $A$  and  $B$  must be separated in this case. Let  $a_0 \in A$  and  $b_0 \in B$ , WLOG, suppose  $a_0 < b_0$ . By (ii), the entire interval  $[a_0, b_0] \subseteq E$ . Split  $[a_0, b_0]$  into two half intervals  $[\alpha, \beta]$  and  $[\beta, \gamma]$ . Note that it is impossible for  $\{\beta\}$  to be the only point intersect both  $A$  and  $B$ , because in this case  $A$  and  $B$  cannot be disjoint. Take the one intersects both  $A$  and  $B$ , denoted as  $[a_1, b_1]$ .

One can construct a sequence of closed intervals inductively, such that every  $I_n$  intersects both  $A$  and  $B$ . Also, previous result shows that  $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$ , and is in fact a singleton. Let  $x \in \cap_{n \in \mathbb{N}} I_n$ , if  $x \in A$ , then there exists  $(b_n) \subseteq B$  such that  $(b_n) \rightarrow x$ . Similarly, if  $x \in B$ , there exists  $(a_n) \subseteq A$  such that  $(a_n) \rightarrow x$ . As a result, either  $\overline{A} \cap B \neq \emptyset$  or  $A \cap \overline{B} \neq \emptyset$ . Therefore,  $E$  is connected. ■

### 3.4 Cantor Set

**Definition 3.11.** Define sequence of sets

$$S_0 = [0, 1] \tag{3.7}$$

$$S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{3.8}$$

inductively, where  $S_n$  is defined by removing the mid-one-third of elements from each component of  $S_{n-1}$ . The **Cantor set** is defined as

$$\mathcal{C} := \bigcap_{n \in \mathbb{N}} S_n \neq \emptyset \quad (3.9)$$

$\mathcal{C}$  is nonempty because each  $S_n$  is a finite union of closed set. Altogether with the fact that each of  $S_n$  is bounded, so  $\mathcal{C}$  is an intersection of nested compact sets. Therefore,  $\mathcal{C}$  is nonempty by Cantor's intersection theorem.

**Definition 3.12.** A set is called **perfect** if it is closed and has no isolated point.

**Proposition 3.2.**  $\mathcal{C}$  has measure zero.

*Proof.* Note that on while constructing  $S_n$ , intervals with total length of  $\frac{2^n}{3^{n+1}}$  are removed from  $S_{n-1}$ . To construct a Cantor set, the total length of intervals from  $[0, 1]$  equals

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 \quad (3.10)$$

Therefore the length left for Cantor set is zero. ■

**Proposition 3.3.**  $\mathcal{C}^{int} = \emptyset$ .

*Proof.* Note that for any set to have nonempty interior, it must contains some open intervals. Claim: for every open interval  $(a, b)$ , it cannot be contained in  $\mathcal{C}$ . Let  $a < b$ , note that for every partition of  $S_n$  has length  $\frac{1}{3^n}$ . Then there exists  $n \in \mathbb{N}$  such that  $\frac{1}{3^n} < b - a$ . Therefore,  $(b - a) \not\subseteq S_n$  for such  $n$ . So that  $\mathcal{C}$  cannot contain any open interval. ■

**Proposition 3.4.**  $\mathcal{C}$  is closed.

*Proof.*  $\mathcal{C}$  is the intersection of infinitely many closed sets, so it is closed. ■

**Proposition 3.5.**  $\mathcal{C}$  is compact.

*Proof.*  $\mathcal{C}$  is bounded by  $[0, 1]$  and closed by previous proposition. Therefore,  $\mathcal{C} \subseteq \mathbb{R}$  is compact. ■

**Proposition 3.6.**  $\mathcal{C}$  is perfect.

*Proof.* We are going to show that every point  $x \in \mathcal{C}$  is the limit of some sequence in  $\mathcal{C}$ .

*Case 1:*  $x$  is not the right endpoint of any intervals in  $S_n$  for any  $n \in \mathbb{N}$ . Then for every  $n \in \mathbb{N}$ , let  $x_n$  be the right endpoint of the interval in  $S_n$  containing  $x$ . Obviously,  $(x_n) \rightarrow x$ .

*Case 2:*  $x$  is the right endpoint of some closed interval in some  $S_n$  (the implication of this observation is that we cannot simply take the right endpoint, so take the left endpoint). For every  $n \in \mathbb{N}$ , take  $x_n$  to be the left end of  $S_n$  containing  $x$ . Clearly,  $(x_n) \rightarrow x$ . ■

**Theorem 3.11.** Any nonempty perfect set  $P$  is uncountable.

*Proof.* Note that  $P$  is obviously not finite. Suppose, for contradiction,  $P$ , then there exists an enumeration of  $P = \{x_1, x_2, \dots, x_n, \dots\}$ . Construct a sequence of compact sets as following: take  $\varepsilon > 0$ , there exists  $y_1 \neq x_1$  such that  $y_1 \in P \cap [x_1 - \varepsilon, x_1 + \varepsilon]$ . Let  $\delta_1 := \frac{|y_1 - x_1|}{2}$ , and take  $K_1 := [y_1 - \delta_1, y_1 + \delta_1] \cap P$ . **TODO: Show  $K_1$  is compact.** Note that  $x_1 \notin K_1$ .

Apply the same argument on  $K_1$  to construct  $K_2$  such that  $x_2 \notin K_2$ , so that  $P \supset K_1 \supset K_2 \supset \dots$ . By construction, no points in  $P$  is in the intersection  $\bigcap_{n \in \mathbb{N}} K_n$ . However, the intersection is nonempty and the element belongs to the intersection is clearly in  $P$ , which is a contradiction. ■

**Proposition 3.7.**  $\mathcal{C}$  is uncountable.

*Proof.*  $\mathcal{C}$  is a nonempty perfect set, so it is uncountable. ■

*Another Direct Proof.* Consider the ternary expansions of all numbers in  $\mathcal{C}$ , it is easy to observe that for every  $x \in \mathcal{C}$ , if the ternary expansion of  $x$  contains 1 somewhere,  $x$  cannot be in  $\mathcal{C}$ .

Therefore,  $\mathcal{C}$  has the same cardinality as the collection of all binary expansions, which has the same cardinality of  $\mathbb{R}$  (refer to the construction of real numbers). ■

## 4 Functional Limits and Continuity

**Definition 4.1.** Let  $f : A \rightarrow \mathbb{R}$  be a function, let  $c$  be a limit point of domain  $A$ , then  $\lim_{x \rightarrow c} f(x) = L$  if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } x \in V_\delta^o(c) \implies f(x) \in V_\varepsilon(L) \quad (4.1)$$

*Remark:* The definition of continuity is stated in terms of punctuated ball, however, it is often easier to argue  $x \in V_\delta(c) \implies f(x) \in V_\varepsilon(L)$ .

**Example 4.1.** Let  $g(x) = x^2$ , show that  $\lim_{x \rightarrow 2} g(x) = 4$ .

*Proof.* Let  $\varepsilon > 0$ , note that for all  $\delta < 1$ , for all  $x \in V_\delta^o(2)$ ,

$$|x^2 - 4| = |x - 2| |x + 2| \quad (4.2)$$

$$|x| = |x - 2 + 2| \leq |x - 2| + 2 < 3 \quad (4.3)$$

$$|x + 2| \leq |x| + 2 < 5 \quad (4.4)$$

$$\implies |x^2 - 4| < 5\delta \quad (4.5)$$

Take  $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{5}\}$ , both inequality reasoning (because  $\delta < 1$ ) and  $\varepsilon$  requirement are valid. ■

**Theorem 4.1** (Sequential Criterion for Functional Limits). Given a function  $f : A \rightarrow \mathbb{R}$  and  $c \in A'$ , then the following are equivalent:

- (i)  $\lim_{x \rightarrow c} f(x) = L$ ;
- (ii)  $\forall (x_n) \subseteq A \setminus \{c\}$  such that  $(x_n) \rightarrow c$ ,  $(f(x_n)) \rightarrow L$ .

*Proof.* (i)  $\implies$  (ii): assume  $f(x) \rightarrow L$ , let  $(x_n) \subseteq A \setminus \{c\}$  be an arbitrary convergent sequence with limit  $c$ .

Let  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every  $x \in V_\delta^0(c)$ ,  $f(x) \in V_\varepsilon(L)$ .

Consider such  $\delta$ , by the convergence of sequence, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n \in V_\delta(c)$ .

Moreover, note that  $x_n \neq c \ \forall n \in \mathbb{N}$ , therefore  $n \geq N \implies x_n \in V_\delta^o(c)$ , which further implies  $f(x_n) \in V_\varepsilon(L)$  by the limit property of  $f$ .

(ii)  $\implies$  (i): assume, for contradiction,  $\lim_{x \rightarrow c} f(x) \neq L$ .

Negating the definition of functional limit gives

$$\exists \varepsilon^* > 0 \text{ s.t. } \forall \delta > 0 \exists x_\delta \in V_\delta^o(c) \text{ s.t. } f(x_\delta) \notin V_{\varepsilon^*}(L) \quad (4.6)$$

For every  $n \in \mathbb{N}$ , take  $\delta = \frac{1}{n}$ , and define  $x_n := x_\delta$  from above statement.

Clearly,  $(x_n) \rightarrow c$  by construction, but  $(f(x_n))$  is bounded away from  $L$  by  $\varepsilon^* > 0$ . This leads to a contradiction of (ii).  $\blacksquare$

**Theorem 4.2** (Convergence Criterion for Functional Limits). Let  $f : A \rightarrow \mathbb{R}$  and  $c \in A'$ . If there exists two sequences  $(x_n), (y_n) \subseteq A \setminus \{c\}$  converging to  $c$ , but  $\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n)$ , then  $\lim_{x \rightarrow c} f(x)$  does not exist.

*Proof.* In the previous theorem, the negation of (ii) proposes exactly the existence of two convergent sequences in  $A \setminus \{c\}$  converging to the same limit  $c$  but their image sequences does not converge to the same limit. The result is immediate by taking the contraposition of (i)  $\implies$  (ii) part.  $\blacksquare$

**Example 4.2.** Limit of  $f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$  at 0 does not exist.

**Example 4.3** (Dirichlet Function). Limit of  $f(x) := \mathbb{1}\{x \in \mathbb{Q}\}$  does not exist anywhere in  $\mathbb{R}$ .

**Example 4.4.** Limit of  $f(x) := x\mathbb{1}\{x \in \mathbb{Q}\}$  only exists at  $x = 0$ .

**Theorem 4.3** (Characterizations of Continuity: Alternative Notations). Let  $f : A \rightarrow \mathbb{R}$ ,  $c \in A$ , then  $f$  is continuous at  $c$  if and only if one of the following holds:

- (i)  $\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon;$
- (ii)  $\forall V_\varepsilon(f(c)) \exists V_\delta(c) \text{ s.t. } x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(c));$
- (iii)  $\forall A \supseteq (x_n) \rightarrow c \in A' (f(x_n)) \rightarrow f(c).$

**Proposition 4.1** (Criterion of Discontinuity). Let  $f : A \rightarrow \mathbb{R}$ ,  $c \in A'$ , if there exists sequence  $(x_n) \subseteq A$  converges to  $c$  but  $(f(x_n)) \not\rightarrow f(c)$ , then  $f$  is not continuous at  $c$ .

**Example 4.5** (Thomae's Function). Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \wedge \gcd(p, q) = 1 \\ 1 & \text{if } x = 0 \end{cases} \quad (4.7)$$

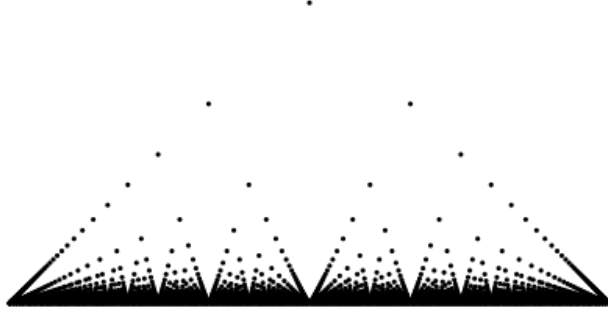


Figure 1: Thomae's Function in the Unit Interval

**Proposition 4.2.** For every  $a \in \mathbb{R}$ ,  $\lim_{x \rightarrow a} f(x) = 0$ . As a result,  $D_f = \mathbb{Q}$ .

*Proof.* WLOG, consider the domain  $a \in (0, 1)$  only, show:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in V_\delta^o(a), f(x) \in V_\varepsilon(0) \quad (4.8)$$

Fix  $\varepsilon > 0$ .

Note that there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies \left| \frac{1}{n} \right| < \varepsilon$ .

Because  $\mathbb{Q}$  is countable, define finite set  $L$  following Cantor's diagonal order

$$L := \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{N-2}{N-1} \right\} \setminus \{a\} \quad (4.9)$$

That is,  $L$  contains every rational number such that its denominator in the lowest form is less than  $N$  excluding  $a$  (if  $a$  is rational).

Define  $m := \min_{q_i \in L} |a - q_i|$ , which is well-defined because  $L$  is finite.

Take  $\delta := \frac{m}{2}$ , note that  $V_\delta(a) \cap \mathbb{Q} \cap L = \emptyset$  by construction.

Let  $x \in V_\delta^o(a)$ , either

- (i)  $x \in \mathbb{Q} \implies x \notin L \implies x = \frac{p}{q}$  where  $q \geq N$ , which implies  $f(x) = \frac{1}{q} < \varepsilon$ ;
- (ii) or  $x \notin \mathbb{Q} \implies f(x) = 0 < \varepsilon$ .

Either case implies the limit to be zero.

Therefore  $f$  is discontinuous on  $\mathbb{Q}$ . ■

**Theorem 4.4.** Composition of continuous functions is continuous.

Given  $f : A \rightarrow \mathbb{R}$ ,  $g : B \rightarrow \mathbb{R}$  such that the range  $f(A) \subseteq B$ . If  $f$  is continuous at  $c \in A$ , and if  $g$  is continuous at  $f(c) \in B$ . Then  $g \circ f$  is continuous at  $c$ .

*Proof.* Let  $\varepsilon > 0$ , and  $g(x)$  is continuous at  $f(c)$ .

$$\exists \tilde{\delta} \text{ s.t. } f(x) \in V_{\tilde{\delta}}(f(c)) \implies g \circ f(x) \in V_\varepsilon(g \circ f(c)) \quad (4.10)$$

$$\exists \delta > 0 \text{ s.t. } x \in V_\delta(c) \implies f(x) \in V_{\tilde{\delta}}(f(c)) \quad (4.11)$$

■



## 4.1 Continuous Functions on Compact Sets

**Theorem 4.5.** Let  $K$  be a compact set in  $\mathbb{R}$ , and  $f : K \rightarrow \mathbb{R}$  is a continuous function, then  $f(K)$  is compact.

*Proof.* Let  $(y_n) \subseteq K$ . Consider  $f^{-1}(y_n) \neq \emptyset$ , take  $x_n \in f^{-1}(y_n)$  to construct a sequence. Because  $K$  is compact, there exists a subsequence of  $(x_n)$  converges to  $x \in K$ . Since  $f$  is continuous,  $f(x_n)$  converges to  $f(x) \in f(K)$ . ■

**Theorem 4.6** (Extreme Value Theorem). If  $f : K \rightarrow \mathbb{R}$  is continuous on a compact set  $K \subset \mathbb{R}$ . Then  $f$  attains a maximum and minimum value. That is,

$$\exists x_0, x_1 \in K, \text{ s.t. } f(x_0) \leq f(x) \leq f(x_1) \forall x \in K \quad (4.12)$$

*Proof.*  $f(K)$  is compact, in particular it is bounded in  $\mathbb{R}$ . So it possesses a supremum  $\sup f(K)$ . Then there exists a sequence converging to the supremum. Because  $f(K)$  is bounded as well, its limit can be attained in  $K$ . Therefore the maximum is attainable (minimum case is similar). ■

## 4.2 Uniform Continuity

**Example 4.6** (Uniformly Continuous).  $f(x) = 3x + 1$  is continuous for every  $c \in \mathbb{R}$ .

*Proof.* Let  $\varepsilon > 0$ , take  $\delta = \frac{\varepsilon}{3}$  gives the definition of continuity. Note that  $\delta$  does not depend on particular  $c$ . ■

**Example 4.7** (Non-uniformly Continuous).  $y = x^2$  is continuous for every  $c \in \mathbb{R}$ .

*Proof.* Let  $\varepsilon > 0$ ,

Note that  $|x + c| = |x - c + 2c| \leq |x - c| + 2|c|$ .

Suppose  $\delta \leq 1$  (the following argument works for  $x \in V_1(c)$  only):

$$|x^2 - c^2| = |x - c| |x + c| \quad (4.13)$$

$$\leq \delta |x + c| \quad (4.14)$$

$$\leq \delta(|x - c| + 2|c|) \quad (4.15)$$

$$\leq \delta(1 + 2|c|) < \varepsilon \quad (4.16)$$

Take  $\delta := \min \left\{ 1, \frac{1}{1+2|c|} \right\}$ . Note that  $\delta$  depends on particular realization  $c$ . ■

**Definition 4.2.** A function  $f$  is **uniformly continuous** on  $A \subset \mathbb{R}$  if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x, y \in \mathbb{R} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad (4.17)$$

**Theorem 4.7** (Sequential Criterion for Absence of Uniform Continuity). Let  $f : A \rightarrow \mathbb{R}$ , the following are equivalent:

- (i)  $f$  fails to be uniformly continuous on  $A$ ;

- (ii)  $\exists \varepsilon_0 > 0$  and two sequences  $(x_n), (y_n) \subseteq A$  such that  $|x_n - y_n| \rightarrow 0$  but  $|f(x_n) - f(y_n)| \geq \varepsilon_0$  for every  $n \in \mathbb{N}$ .

*Proof.*  $f : A \rightarrow \mathbb{R}$  is not uniformly continuous,

iff  $\exists \varepsilon > 0$  s.t.  $\forall \delta > 0 \exists x, y \in \mathbb{R}$  s.t.  $|x_\delta - y_\delta| < \delta \wedge |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$ .

For each  $n \in \mathbb{N}$ , take  $\delta = \frac{1}{n}$  and construct two sequences. ■

**Example 4.8.** Let  $f(x) = \sin(x^{-1})$  defined on  $A = (0, 1)$ .

Take  $x_n := \frac{1}{2n + \frac{\pi}{2}}$  and  $y_n = \frac{1}{2n + \frac{3\pi}{2}}$ . It is evident that  $f(x_n) = 1$  and  $f(y_n) = -1$  for every  $n \in \mathbb{N}$ . By taking  $\varepsilon_0 = 2$ , the proof is complete.

**Example 4.9.**  $f(x) = x^2$  is not uniformly continuous.

Consider two sequences  $(x_n) = (n)$  and  $(y_n) = (n + \frac{1}{n})$ .

Note that  $|x_n - y_n| = |\frac{1}{n}| \rightarrow 0$ .

Moreover,  $|f(x_n) - f(y_n)| = |2 + \frac{1}{n^2}| > 2$ .

Therefore  $f$  is not uniformly continuous.

**Theorem 4.8** (Uniform Continuity on Compact Set). A continuous function  $f$  on a compact  $K$  is uniformly continuous.

*Proof.* Suppose, for contradiction,  $f$  is continuous but not uniformly continuous.

Then there exists  $\varepsilon_0 > 0$  and  $(x_n), (y_n) \subseteq K$  such that  $|x_n - y_n| \rightarrow 0$  and  $|f(x_n) - f(y_n)| \geq \varepsilon_0$ .

Because  $K$  is compact, there exists subsequences of  $(x_n)$  and  $(y_n)$  converging to  $x, y \in K$ .

Suppose  $x \neq y$ , it contradicts  $|x_n - y_n| \rightarrow 0$  because this property holds for subsequences as well.

Therefore,  $(x_{n_k})$  and  $(y_{n_k})$  converge to the same limit.

Because  $f$  is continuous,  $\lim_{k \rightarrow \infty} [f(x_{n_k}) - f(y_{n_k})] = 0 < \varepsilon_0$ , contradiction. ■

### 4.3 Intermediate Value Theorem

**Theorem 4.9.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, if  $L \in \mathbb{R}$  such that either

(i)  $L \in (f(a), f(b))$ ,

(ii) or  $L \in (f(b), f(a))$ .

Then  $\exists c \in (a, b)$  such that  $f(c) = L$ .

**Lemma 4.1.** All the connected subsets of  $\mathbb{R}$  are intervals.

**Theorem 4.10.** Continuous map preserves connectedness. That is, for some  $f : G \rightarrow \mathbb{R}$  continuous with  $G \subseteq \mathbb{R}$ , then  $E \subseteq G$  connected  $\implies f(E)$  connected.

*Proof.* Suppose  $f(E) = A \cup B$ , where  $A, B \neq \emptyset$  and disjoint.

Note that  $E = f^{-1}(A) \cup f^{-1}(B)$  with  $f^{-1}(A) \cap f^{-1}(B) = \emptyset$  (evident using contradiction).

Because  $E$  is connected, either there exists a convergent sequence in  $f^{-1}(A)$  converges to somewhere in  $f^{-1}(B)$  or a convergent sequence in  $f^{-1}(B)$  converges to somewhere in  $f^{-1}(A)$ .

WLOG, suppose there exists a sequence  $(x_n) \in f^{-1}(A)$  converges to some point  $x \in f^{-1}(B)$ .

Because  $f$  is continuous,  $\lim_{n \rightarrow \infty} f(x_n) = f(x) \in B$ .

That is, there exists sequence  $(f(x_n)) \subseteq A$  converges to  $f(x) \in B$ .

$E$  is connected. ■

*Proof of IVT.* Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function.

WLOG, suppose  $f(a) < f(b)$ .

Let  $L \in (f(a), f(b))$ .

Note that  $f([a, b])$  is connected by previous theorem, therefore  $f([a, b])$  is an interval by previous lemma.

hence,  $f([a, b])$  contains every point between  $f(a)$  and  $f(b)$ , in particular  $L$ .

Therefore,  $\exists c \in [a, b]$  with  $f(c) = L$ .

Note that  $f(a), f(b) \neq L$ , so  $c \neq a, b$ .

Therefore, the chosen  $c \in (a, b)$ . ■

**Remark 4.1.** EVT says the continuous functions' image of a compact set is compact.

IVT says the continuous functions' image of a connected set is connected.

**Definition 4.3.** A function  $f$  possesses the **intermediate value property** on closed interval  $[a, b]$  if for every  $x < y$  in  $[a, b]$  and  $f(x) < L < f(y)$ , it is always possible to find  $c \in (a, b)$  such that  $f(c) = L$ .

**Theorem 4.11** (IVT restated).  $f$  is continuous  $\implies f$  satisfies IVP.

**Remark 4.2.** The converse of IVT is not true.

Consider

$$g(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (4.18)$$

$g(x) : [-1, 1] \rightarrow [-1, 1]$  satisfies IVP, but  $g(x)$  is not continuous on  $[-1, 1]$ , in particular,  $g(x)$  is discontinuous at 0.

**Proposition 4.3.** If a sequence of functions  $(f_n)$  converges uniformly to  $f$ , then  $(f_n)$  converges point-wise to  $f$  as well.

**Example 4.10.**  $f_n(x) - \frac{(x^2 + nx)}{n}$  has point-wise limit to  $f(x) = x$  but it does not converge uniformly.

## 5 Sequences and Series of Functions

### 5.1 Sequences of Functions

**Definition 5.1.** For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on  $A \subseteq \mathbb{R}$ . The sequence  $(f_n)_{n \in \mathbb{N}}$  **converges point-wise** on  $A$  to a function  $f : A \rightarrow \mathbb{R}$  if

$$\forall x \in A, (f_n(x)) \rightarrow f(x) \quad (5.1)$$

That is,

$$\forall x \in A, \forall \varepsilon > 0 \exists N_x \in \mathbb{N} \text{ s.t. } \forall n \geq N_x, f_n(x) \in V_\varepsilon(f(x)) \quad (5.2)$$

That is, the sequence induced at each  $x \in A$  is convergent, and the value of  $N_x$  can depend on specific  $x$ .

**Definition 5.2.** Let  $(f_n)$  be a sequence of functions defined on  $A \subseteq \mathbb{R}$ . Then  $(f_n)$  **converges uniformly** on  $A$  to  $f : A \rightarrow \mathbb{R}$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall x \in A \forall n \geq N, f_n(x) \in V_\varepsilon(f(x)) \quad (5.3)$$

The uniform convergence requires a single  $N$  to work for every  $x \in A$ .

**Theorem 5.1** (Cauchy Criterion). Given a sequence of functions  $(f_n)$  defined on  $A \subseteq \mathbb{R}$ , the following are equivalent:

- (i)  $(f_n)$  converges uniformly on  $A$ ;
- (ii) (**Uniformly Cauchy**)  $\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon \forall m, n \geq N, \forall x \in A$ .

*Proof.* (  $\implies$  )

The sufficient condition is immediate by triangle inequality.

Suppose  $(f_n) \xrightarrow{\text{unif}} f$ , let  $\varepsilon > 0$ , take  $N \in \mathbb{N}$  such that  $\forall n \geq N, |f_n(x) - f(x)| < \frac{\varepsilon}{2}$  for every  $x \in A$ .

Take the same  $N$  and let  $m, n \geq N$ , let  $x \in A$ ,

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \quad (5.4)$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \quad (5.5)$$

$$< \varepsilon \quad (5.6)$$

(  $\impliedby$  )

Assume  $(f_n)$  is uniformly Cauchy.

Let  $\varepsilon > 0$ .

$\forall x \in A, (f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ .

By the completeness of  $\mathbb{R}$ ,  $(f_n(x)) \rightarrow f(x)$  for some point-wise limit  $f(x)$ .

We are going to show  $(f_n)$  converges uniformly to the point-wise limit.

By uniform Cauchy,  $\exists N \in \mathbb{N}$  such that  $\forall m \geq N, n = m + k \geq N$ , and  $\forall x \in A$ ,

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2} \quad (5.7)$$

Fix  $x \in A$ ,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (5.8)$$

$$\implies |f_m(x) - f(x)| = \lim_{n \rightarrow \infty} |f_m(x) - f_n(x)| \quad (5.9)$$

$$= \lim_{k \rightarrow \infty} |f_m(x) - f_{m+k}(x)| \quad (5.10)$$

$$\leq \frac{\varepsilon}{2} < \varepsilon \text{ (order limit theorem)} \quad (5.11)$$

■

**Theorem 5.2** (Continuous Limit Theorem). Let  $(f_n)$  be a sequence of functions defined on  $A \subseteq \mathbb{R}$  that converges uniformly on  $A$  to a function  $f$ . If each  $f_n$  is continuous at  $c \in A$ , then  $f$  is continuous at  $c$ .

*Proof.* Let  $(f_n)$  be a sequence of continuous functions, let  $c \in A$ , note that for arbitrary  $N \in \mathbb{N}$ ,

$$|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \quad (5.12)$$

$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \quad (5.13)$$

Where the last term can be made arbitrarily small given the uniform convergence property of  $(f_n)$ . And the first two terms can be made arbitrarily small as well given the continuity properties.

This concludes the continuity of  $f$ . ■

**Theorem 5.3** (Differentiable Limit Theorem). Let  $(f_n) \rightarrow f$  pointwise on closed interval  $[a, b]$ , and assume that each  $f_n$  is differentiable. If  $(f'_n) \rightarrow g$  uniformly on  $[a, b]$ , then the limit  $f$  is differentiable and  $f' = g$ .

*Proof.* Let  $c \in [a, b]$ , note that

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad (5.14)$$

The following holds for every  $n \in \mathbb{N}$ :

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \quad (5.15)$$

$$= \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) + f'_n(c) - g(c) \right| \quad (5.16)$$

$$\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| + |f'_n(c) - g(c)| \quad (5.17)$$

In particular, one may choose  $N_1 \in \mathbb{N}$  such that

$$|f'_n(c) - g(c)| < \frac{\varepsilon}{3} \quad (5.18)$$

for every  $n \geq N_1$ .

Moreover, given the Cauchy property of uniform convergence, there exists  $N_2 \in \mathbb{N}$  such that  $\forall m, n \geq N_2, |f'_m(x) - f'_n(x)| < \frac{\varepsilon}{3}$  for every  $x \in A$ .

Define  $N = \max\{N_1, N_2\}$ .

Note that  $\exists \delta > 0$  such that

$$x \in V_\delta^o(c) \implies \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\varepsilon}{3} \quad (5.19)$$

Let  $m \geq N$ , for an arbitrary  $x \in V_\delta^o(c)$ , WLOG, suppose  $x < c$ . Applying the mean value theorem on  $f_m - f_N$  gives

$$f'_m(\eta) - f'_N(\eta) = \frac{f_m(c) - f_N(c) - [f_m(x) - f_N(x)]}{c - x} \quad (5.20)$$

$$\implies \frac{\varepsilon}{3} > |f'_m(\eta) - f'_N(\eta)| = \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| \quad (5.21)$$

$$\implies \lim_{m \rightarrow \infty} \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| = \left| \frac{f(c) - f(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| \quad (5.22)$$

$$\leq \frac{\varepsilon}{3} \text{ (order limit theorem)} \quad (5.23)$$

Therefore, for every  $x \in V_\delta^0(c)$ ,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon \quad (5.24)$$

■

**Theorem 5.4.** Let  $(f_n)$  be a sequence of differentiable functions defined on the closed interval  $[a, b]$ , and assume  $(f'_n)$  converges uniformly on  $[a, b]$ . If there exists a point  $x_0 \in [a, b]$  where  $f_n(x_0)$  converges, then  $(f_n)$  converges uniformly on  $[a, b]$ .

**Theorem 5.5** (Stronger Version). Let  $(f_n)$  be a sequence of differentiable functions defined on the closed interval  $[a, b]$ , and assume  $(f'_n)$  converges uniformly to a function  $g$  on  $[a, b]$ .

If there exists a point  $x_0 \in [a, b]$  for which  $f_n(x_0)$  converges, then  $(f_n)$  converges uniformly,

Furthermore, the limit function  $f$  is differentiable and satisfies  $f' = g$ .

## 5.2 Series of Functions

**Definition 5.3.** A series of function  $\sum_{i=1}^{\infty} f_n(x)$  converges point-wise/uniformly if  $\sum_{i=1}^k f_n(x)$  converges point-wise/uniformly.

**Example 5.1.** Note that  $\sum_{i=1}^{\infty} \frac{\sin(nx)}{n^2}$  converges uniformly.

$$\left| \sum_{i=1}^k \frac{\sin(nx)}{n^2} - \sum_{i=1}^{\ell} \frac{\sin(nx)}{n^2} \right| \leq \sum_{n=\ell+1}^k \frac{1}{n^2} < \varepsilon \quad (5.25)$$

Hence, uniformly Cauchy.

**Theorem 5.6.** If  $f_n : S \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^m$  continuous, then if  $\sum_{n=1}^{\infty} f_n \rightarrow f$  uniformly, then  $f$  is also continuous.

**Definition 5.4.** Let  $f_k : S \rightarrow \mathbb{R}^m$ , then  $(f_k)$  is **uniformly Cauchy** on  $S$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \sup_{x \in S} \left\| \sum_{i=k+1}^{\ell} f_i(x) \right\| \leq \varepsilon \quad \forall \ell > k \geq N \quad (5.26)$$

**Theorem 5.7.**  $(f_k)$  is uniformly Cauchy  $\iff \sum f_k \rightarrow f$  uniformly.

*This is the same as the theorem mentioned in previous section. Simply define  $g_n := \sum_{i=1}^n s f_i(x)$ , and  $(g_n)$  is uniformly Cauchy if and only if it converges to  $g_{\infty} \equiv \sum_{n=1}^{\infty} f_n(x)$  uniformly.*

*Proof.* ( $\Leftarrow$ )

Let  $g_m := \sum_{k=1}^m f_k(x)$ , suppose  $g_k \rightarrow f$  uniformly. Then,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |g_k - f| < \frac{\varepsilon}{2} \quad \forall k \geq N, \forall x \in S \quad (5.27)$$

For every  $k > \ell \geq N$ ,

$$|g_k - g_{\ell}| \leq |g_k - f| + |f - g_{\ell}| < \varepsilon \quad \forall x \in S \quad (5.28)$$

Therefore,  $(f_k)$  is uniformly Cauchy.

( $\Rightarrow$ )

Suppose  $(f_k)$  is uniformly Cauchy. Then for every fixed  $x \in S$ ,  $(g_k(x))$  is Cauchy.

So  $f(x) = \lim_{k \rightarrow \infty} g_k(x)$  exists.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |g_k(x) - g_{\ell}(x)| < \varepsilon \quad \forall k > \ell \geq N, \forall x \in S \quad (5.29)$$

$$\implies |f(x) - g_k(x)| = \lim_{\ell \rightarrow \infty} |g_{\ell}(x) - g_k(x)| \leq \varepsilon \quad \forall x \in S \quad (5.30)$$

$$\implies g_k \rightarrow f \text{ uniformly} \quad (5.31)$$

■

**Theorem 5.8** (Term-by-term Continuity Theorem). Let  $f_n$  be continuous functions defined on a set  $A \subseteq \mathbb{R}$ , suppose  $\sum_{n=1}^{\infty} f_n \rightarrow f$  uniformly, then  $f$  is continuous on  $A$ .

**Theorem 5.9** (Term-by-term Differentiability Theorem). Let  $f_n$  be continuous functions defined on  $A \subseteq \mathbb{R}$ , and suppose

(i)  $\sum_{n=1}^{\infty} f'_n \rightarrow g$  uniformly;

(ii) There exists a point  $x_0$  such that the series converges at  $x_0$ .

Then,  $\sum_{n=1}^{\infty} f_n \rightarrow f$  converges uniformly to a differentiable function  $f$  with  $f'(x) = g(x)$ .

That is

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (5.32)$$

$$f'(x) = \sum_{n=1}^{\infty} f'_n(x) \quad (5.33)$$

**Theorem 5.10** (Weierstrass M-test). Let  $a_n : S \rightarrow \mathbb{R}^m$ , and sequence  $M_n \in \mathbb{R}$ . If

$$\exists N \in \mathbb{N} \forall k \geq N \sup_{x \in S} |a_n(x)| \leq M_n \quad (5.34)$$

then  $\sum_{n=1}^{\infty} M_n$  converges implies  $\sum_{n=1}^{\infty} a_n$  converges uniformly.

*Proof.* The desired result can be shown by showing the sequence of partial sums is Cauchy.

Let  $\varepsilon > 0$ , show that there exists  $k \in \mathbb{N}$  such that

$$\forall m, n \geq k, \quad |S_m(x) - S_n(x)| < \varepsilon \quad (5.35)$$

WLOG, suppose  $m > n$ , then

$$|S_m(x) - S_n(x)| = |a_{n+1}(x) + \cdots a_m(x)| \quad (5.36)$$

$$\leq |M_{n+1} + \cdots + M_m| \quad (5.37)$$

■

*Proof.*

$$\forall x \in S \sum_{n=1}^{\infty} |a_n(x)| \leq \sum_{n=1}^{\infty} \sup_{x \in S} |a_n(x)| \quad (5.38)$$

$$\leq \sum_{n=1}^N \sup_{x \in S} |a_n(x)| + \sum_{n=N+1}^{\infty} M_n < \infty \quad (5.39)$$

Therefore,  $f(x) = \sum_{n=1}^{\infty} a_n(x)$  exists.



Note that  $\forall x \in S, \forall \ell \geq N$ :

$$\left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| = \left| \sum_{k=\ell+1}^{\infty} a_k(x) \right| \quad (5.40)$$

$$\leq \sum_{k=\ell+1}^{\infty} |a_k(x)| \quad (5.41)$$

$$\leq \sum_{k=\ell+1}^{\infty} M_k < \infty \quad (5.42)$$

$$\Rightarrow \lim_{\ell \rightarrow \infty} \sup_{x \in S} \left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| \leq \lim_{\ell \rightarrow \infty} \sum_{k=\ell+1}^{\infty} M_n = 0 \quad (5.43)$$

Therefore  $\sum a_k \rightarrow f$  uniformly. ■

**Example 5.2.** Consider  $\sum \frac{x^n}{n!}$  defined on  $x \in [-A, A]$ . Note that

$$\sup_{x \in S} \left| \frac{x^n}{n!} \right| \leq \frac{A^n}{n!} =: M_n \quad (5.44)$$

By ratio test,  $\sum M_n$  converges.

By M-test  $\sum \frac{x^n}{n!}$  converges uniformly on  $[-A, A]$ .

Therefore,  $\sum \frac{x^n}{n!}$  is continuous on  $[-A, A]$ .

**Example 5.3** (Geometric Series).

$$\sum_{n=1}^{\infty} (-x^2)^n \quad (5.45)$$

By ratio test, the series converges only on  $(-1, 1)$ . In particular, it converges to  $\frac{1}{1+x^2}$ .

For every  $0 < r < 1$ , for  $x \in [-r, r]$ , then

$$\sup_{x \in [-r, r]} |(-x^2)^n| = r^{2n} \quad (5.46)$$

and  $\sum_{n=1}^{\infty} r^{2n} = \frac{1}{1+r^2}$  implies  $\sum_{n=1}^{\infty} (-x^2)^n$  converges uniformly on  $[-r, r]$ .

**Example 5.4** (Weierstrass Function). Define

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2^n} \cos(10^n \pi x)}_{f_n(x)} \quad (5.47)$$

Note that  $\sup_{x \in S} |f_n(x)| \leq \frac{1}{2^n}$ , and  $\sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow 1$ . Therefore,  $\sum f_n \rightarrow f$  uniformly on  $\mathbb{R}$ , so that  $f$  is continuous.

**Proposition 5.1.**  $f$  is nowhere differentiable.

*Proof.* Let  $x \in \mathbb{R}$  with decimal expansion  $x_0 \cdot x_1 x_2 \dots$ .

Construct sequence  $(z_n) \rightarrow x$  as following:

Fix  $n$ , let  $y_0 = x_0 \cdot x_1 \cdot x_n$  and  $y_1 = y_0 + \frac{1}{10^n}$ .

Note that  $10^n \pi y_i$  is an integer multiplied by  $\pi$ , which implies  $f(y_0) = \frac{(-1)^{x_n}}{2^n}$ ,  $f(y_1) = \frac{(-1)^{x_{n+1}}}{2^n}$  such that

$$\left| \frac{f(z_n) - f(x)}{z_n - x} \right| \rightarrow \infty \quad (5.48)$$

■

**Example 5.5.** Let

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \equiv \sum_{n=1}^{\infty} h_n(x) \quad (5.49)$$

Claim:  $h(x)$  is continuous and differentiable on  $\mathbb{R}$ .

*Continuity.* Note that each  $h_n(x)$  is continuous, so is each partial sum  $S_k := \sum_{n=1}^k h_n(x)$ .

By Weierstrass M-test, each  $|h_n(x)| \leq \frac{1}{n^2} \equiv M_n$ , the series converges to  $h(x)$  uniformly.

Therefore, by the definition of series convergence, the sequence of partial sums, in which all elements are continuous, converges uniformly to  $h(x)$ , hence  $h(x)$  is continuous as well. ■

*Differentiability.* We've already shown the sequence of partial sum converges uniformly to  $h(x)$ .

All we need to show is the derivative of partial sums converges uniformly to some function.

Equivalently, we are showing the uniform convergence of series of derivatives.

In particular,

$$h'_n(x) = -\frac{2x}{(x^2 + n^2)^2} \quad (5.50)$$

$$S'_k(x) = \sum_{n=1}^k h'_n(x) = -\sum_{n=1}^k \frac{2x}{(x^2 + n^2)^2} \quad (5.51)$$

Apply M-Test again,

$$|h'_n(x)| = \left| -\frac{2x}{(x^2 + n^2)^2} \right| \quad (5.52)$$

$$= \frac{2}{\frac{x^2 + n^2}{x}(x^2 + n^2)} \quad (5.53)$$

$$= \frac{2}{(x + \frac{n^2}{x})(x^2 + n^2)} \quad (5.54)$$

Note that

$$\left(\sqrt{x} - \frac{n}{\sqrt{x}}\right)^2 = x - 2n + \frac{n^2}{x} \geq 0 \quad (5.55)$$

$$\implies \left(x + \frac{n^2}{x}\right) \geq 2n \quad (5.56)$$

Therefore,

$$|h'_n(x)| \leq \frac{2}{2n(x^2 + n^2)} \quad (5.57)$$

$$\leq \frac{1}{n^3} \quad (5.58)$$

Therefore,

$$\sum_{n=1}^{\infty} h'_n(x) \rightarrow g(x) \text{ uniformly} \quad (5.59)$$

Hence,  $h(x)$  is differentiable, in particular,  $h'(x) = g(x)$ . ■

### 5.3 Power Series

**Definition 5.5.** A **power series** takes the form of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5.60)$$

in which  $f(0) < \infty$  trivially.

**Theorem 5.11.** If a power series converges at some point  $x_0 \in \mathbb{R}$ , then it it converges absolutely for any  $x$  such that  $|x| \leq |x_0|$ . TODO: verify the inequality, strictly?

*Proof.* The result can be easily established via comparison test.

Suppose  $f(x_0)$  converges, let  $x \in \overline{V}_{|x_0|}(0)$ .

By the convergence of  $f(x_0)$ ,  $(a_n x_0^n) \rightarrow 0$ , so that  $|a_n x_0^n| \leq M \forall n \in \mathbb{N}$ .

$$|a_n x^n| = |a_n| \left| \frac{x^n}{x_0^n} x_0^n \right| \quad (5.61)$$

$$= |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \quad (5.62)$$

$$\leq M \left| \frac{x}{x_0} \right|^n \quad (5.63)$$

In which  $\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$  converges as a power series with ratio less than 1. Therefore,  $\sum_{n=0}^{\infty} a_n x^n$  is convergent by comparison test. ■

**Definition 5.6. Radius of convergence and interval of convergence** **TODO:** *Do we need to define this?*

**Theorem 5.12** (Abel's Theorem). Suppose  $g(x) = \sum_{n=0}^{\infty} a_n x^n$  converge somewhere  $x = R > 0$ , then  $g(x)$  converges uniformly (as a series of functions) on  $[0, R]$ .

The same holds for  $-R$ : if  $g(-R)$  converges, then  $g(x)$  converges uniformly on  $[-R, 0]$ .

**Theorem 5.13.** If a power series  $g(x) = \sum_{n=0}^{\infty} a_n x^n$  converges point-wisely on  $A \subseteq \mathbb{R}$ , then it converges uniformly on any compact subset  $K \subseteq A$ .

*Proof.* **TODO:** *Complete this proof.* ■

## 6 Appendix I: Midterms

**Proposition 6.1** (Midterm 1, Q7). Let  $(a_n)$  be a bounded sequence, let  $S := \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}$ . Then  $\exists(a_{n_k}) \rightarrow s := \sup S$ .

**Lemma 6.1.**  $S$  is bounded above, therefore  $s := \sup S$  is well-defined.

*Proof.* Every element in  $S$  is bounded from above by infinitely many elements from  $(a_n)$ , which are themselves bounded. Therefore  $S$  is bounded from above.

So  $s$  is well-defined by the completeness axiom. ■

*Proof. Case 1.* Suppose there are infinitely many  $a_n > s$ .

Then either  $\forall k \in \mathbb{N} \exists a_{n_k} \in [s, s + \frac{1}{k})$ , such subsequence  $(a_{n_k}) \rightarrow s$ .

Or  $\exists k \in \mathbb{N}$  such that no  $a_n$  in  $[s, s + \frac{1}{k})$ , this leads to a contradiction to the assumption that  $s$  is the least upper bound. ■

*Proof. Case 2.* Suppose there are only finitely many  $a_n > s$ .

Consider  $(s - \frac{1}{k}, s]$ , by the definition of supremum,  $\forall k \in \mathbb{N}, \exists s_k \in (s - \frac{1}{k}, s] \cap S$ .

But  $s_k < a_n$  for infinitely many  $a_n$  by definition of  $S$ .

In particular, because there are only finitely many  $a_n > s$ , there must be infinitely many  $a_n$  clustered in  $(s - \frac{1}{k}, s]$  for every  $k \in \mathbb{N}$ .

For each  $k \in \mathbb{N}$ , one can pick  $a_{n_k} \in (s - \frac{1}{k}, s]$  and  $(a_{n_k}) \rightarrow s$ . ■

**Proposition 6.2** (Another Midterm Question). The open clopen set in  $\mathbb{R}$  is  $\emptyset$  and  $\mathbb{R}$ .

*Proof.* Suppose, for contradiction, there exists  $U, V \neq \emptyset$  and  $V = U^c$ , further  $U$  is clopen.

By definition of closedness and openness,  $V$  is clopen as well.

Pick  $a \in U$  and  $b \in V$ . WLOG, assume  $a < b$ .

Construct  $X \subseteq U$  as

$$X := \{x \in U : x < b\} \tag{6.1}$$

$X$  is nonempty because  $a \in X$ .

$X$  is certainly bounded above by  $b$ , hence, by the completeness axiom,  $\exists \alpha = \sup X$ .

**Calim:**  $\alpha \in \partial U$ .

By the definition of supremum, for every  $\varepsilon > 0$ ,

$$\forall \varepsilon > 0, V_\varepsilon(\alpha) \cap X \neq \emptyset \tag{6.2}$$

$$\implies V_\varepsilon(\alpha) \cap U \neq \emptyset \tag{6.3}$$

*Case 1:*

If  $V_\varepsilon(\alpha) \cap V = \emptyset$  for some  $\varepsilon > 0$ , then  $V_\varepsilon(\alpha) \subseteq U$ .

*Subcase 1:*

If  $\alpha + \frac{\varepsilon}{2} \leq b$ , then  $\alpha \neq \sup X$  (contradiction);

*Subcase 2:*

If  $\alpha + \frac{\varepsilon}{2} > b$ , then  $b \in V_{\varepsilon/2}(\alpha) \cap V \subseteq V_\varepsilon(\alpha) \cap V = \emptyset$  (contradiction).

*Case 2:*  $\forall \varepsilon > 0$ ,  $V_\varepsilon(\alpha) \cap V \neq \emptyset$ , then by definition,  $\alpha \in \partial U \subseteq U$  and  $\alpha \in \partial U \subseteq U$  because  $U$  and  $V$  are closed.

This leads to a contradiction that  $U \cap V = \emptyset$ . ■

**Proposition 6.3** (Sample Midterm 2). Let  $\mathcal{K} = \{K_\lambda : \lambda \in \Lambda\}$  be a collection of compact subsets of  $\mathbb{R}$  such that the intersection of every finite sub-collection is nonempty. Then  $\bigcap_{\lambda \in \Lambda} K_\lambda \neq \emptyset$ .

*Proof.* Suppose, for contradiction,  $\bigcap_{\lambda \in \Lambda} K_\lambda = \emptyset$ .

Take an arbitrary  $K_0 \in \mathcal{K}$ , which is obviously non-empty.

Then,

$$K_0 = K_0 \cap \emptyset^c \tag{6.4}$$

$$= K_0 \cap \left( \bigcap_{\lambda \in \Lambda} K_\lambda \right)^c \tag{6.5}$$

$$= K_0 \cap \bigcup_{\lambda \in \Lambda} K_\lambda^c \tag{6.6}$$

$$= \bigcup_{\lambda \in \Lambda} K_0 \cap K_\lambda^c \tag{6.7}$$

For an arbitrary  $\lambda \in \Lambda$ , let  $x \in K_0 \cap K_\lambda^c \neq \emptyset$ .

Because  $K_\lambda^c$  is open in  $\mathbb{R}$ ,  $\exists V_\varepsilon(x) \subseteq K_\lambda^c$ .

Note that because  $K_\lambda^c$  is open in  $\mathbb{R}$ , therefore  $K_0 \cap K_\lambda^c$  is open in  $K_0$ .

Hence,  $\bigcup_{\lambda \in \Lambda} K_0 \cap K_\lambda^c$  is an open cover of  $K_0$ . The compactness of  $K_0$  suggests it has a finite subcover, say  $\bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c$ . That is,

$$K_0 = \bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c \tag{6.8}$$

$$= K_0 \cap \bigcup_{\lambda=1}^n K_\lambda^c \tag{6.9}$$

$$= K_0 \setminus \bigcap_{\lambda=1}^n K_\lambda \tag{6.10}$$

$$\implies \bigcap_{\lambda=1}^n K_\lambda = \emptyset \tag{6.11}$$

$\Rightarrow \Leftarrow$  ■

**Theorem 6.1.** If  $\sum_{n=0}^{\infty} a_n x^n$  converges for all  $x \in (-R, R)$ , then the differentiated series  $\sum_{n=0}^{\infty} n a_n x^{n-1}$  converges for all  $x \in (-R, R)$  as well.

Further, the differentiated series converges uniformly for every compact  $K \subseteq (-R, R)$ .

Moreover, provided the uniform convergence of differentiated series, the limit function  $\sum_{n=0}^{\infty} a_n x^n$

is differentiable. Therefore,

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad (6.12)$$

*Remark:* Differentiating a power series does not change its radius of convergence.

*Proof.* Prove using comparison test. ■

## 7 Taylor's Series

**Idea** Given  $f(x) \in C^\infty$ , we are trying express  $f(x)$  as a series  $\sum_{n=0}^{\infty} a_n x^n$  such that the series converges somewhere else other than  $x = 0$ . If such series exists, it must satisfy

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (7.1)$$

**Example 7.1.**

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (7.2)$$

This series has infinity radius of convergence, specifically,

$$\left| \frac{x^{n+1}}{(n+1)!} / \frac{x^n}{n!} \right| = \left| \frac{x}{n+1} \right| \quad (7.3)$$

$$\implies \limsup \left| \frac{x}{n+1} \right| = 0 < 1 \quad (7.4)$$

**Theorem 7.1** (Lagrangian Remainder Theorem). Let  $f$  be differentiable for  $N + 1$  times on  $(-R, R)$ , let

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \in \{0, \dots, N\} \quad (7.5)$$

And the **Taylor's polynomial** of order  $N$  is defined as

$$S_N(x) := \sum_{n=0}^N a_n x^n \quad (7.6)$$

For every  $x \neq 0$  in  $(-R, R)$ , define the **error term** associated with the Taylor's polynomial of order  $N$ :

$$E_N(x) := f(x) - S_N(x) \quad (7.7)$$

Then there exists  $c \in V_x(0)$  such that

$$E_N(x) := \frac{f^{N+1}(c)}{(N+1)!} x^{N+1} \quad (7.8)$$

**Example 7.2** (Bound Estimation Error). Consider  $f(x) = \sin(x)$  and its Taylor's polynomial of order 5 in  $V_2(0)$ :

$$S_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \quad (7.9)$$

$$|E_5(x)| = \left| \sin(x) - x + \frac{x^3}{3} - \frac{x^5}{5} \right| \quad (7.10)$$

$$= \left| \frac{\frac{d^6 \sin(x)}{dx^6}}{6!} x^6 \right| \text{ for some } c \in V_2(0) \quad (7.11)$$

$$\leq \frac{|x^6|}{6!} \approx 0.089 \quad (7.12)$$

In general, for every fixed  $N \in \mathbb{N}$ , the tail of Taylor's series

$$|\sin(x) - S_N(x)| = \frac{\left| \frac{d^N \sin(x)}{dx^N} \right|_{x=c} |x|^N}{(N+1)!} \quad (7.13)$$

$$\leq \frac{R^N}{(N+1)!} \rightarrow 0 \text{ uniformly by Cauchy's criterion} \quad (7.14)$$

**Theorem 7.2** (Cauchy Mean Value Theorem). Let  $f$  and  $g$  be two continuous functions defined on  $[a, b]$ , and differentiable on  $(a, b)$ . Then, there exists  $c \in (a, b)$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (7.15)$$

*Proof.* Define

$$h(x) := [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x) \quad (7.16)$$

Apply the Lagrange's mean value theorem on  $h(x)$ , there exists  $c \in (a, b)$  such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} \quad (7.17)$$

$$= \frac{f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) - f(b)g(a) + f(a)g(a) + g(b)f(a) - g(a)f(a)}{b - a} = 0 \quad (7.18)$$



Therefore,

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0 \quad (7.19)$$

$$\implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad (7.20)$$

■

*Proof of Lagrange's Remainder Theorem.* Apply Cauchy MVT on  $E_N(x)$  and  $x^{N+1}$  on  $[0, x]$ , there exists  $x_1 \in (0, x)$  such that

$$\frac{E_N(x) - E_N(0)}{x^{N+1} - 0} = \frac{E_N(x)}{x^{N+1}} \quad (7.21)$$

$$= \frac{E'_N(x_1)}{(N+1)x_1^N} \quad (7.22)$$

Apply the GMT again on  $[0, x_1]$ , there exists  $x_2 \in (0, x_1)$  such that

$$\frac{E'_N(x_1) - E'_N(0)}{(N+1)x_1^N - 0} = \frac{E'_N(x_1)}{(N+1)x_1^N} \quad (7.23)$$

$$= \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \quad (7.24)$$

where the first equality holds because

$$E_N^{(k)}(0) = f^{(k)} - k!a_k = 0 \quad (7.25)$$

Applying the GMT iteratively suggests that there exists  $c \in (0, x)$  such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \quad (7.26)$$

$$= \frac{E_N^{(3)}(x_3)}{(N-1)N(N+1)x_3^{N-2}} \quad (7.27)$$

$$= \dots \quad (7.28)$$

$$= \frac{E_N^{(N+1)}(c)}{(N+1)!} \quad (7.29)$$

$$\implies E_N(x) = \frac{E_N^{(N+1)}(c)}{(N+1)!} x^{N+1} \quad (7.30)$$

Further, observe that  $E_N(x) = f(x) - S_N(x)$  and  $E_N^{(N+1)}(x) = f^{(N+1)}(x)$ .  
Therefore,

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \quad (7.31)$$

■

**Remark 7.1** (Motivation for Taylor's Series). For function  $f(\cdot)$ , we want a power series converges to  $f(\cdot)$  on some interval.

As mentioned before, it must be the case

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (7.32)$$

Let  $f(\cdot) \in C^\infty$ , can we always find a suitable power series?

Potential problems are

- (i) The found series converges at  $\{0\}$ ;
- (ii) Converges to some function  $g(\cdot) \neq f(\cdot)$ .

**Example 7.3.**

$$g(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (7.33)$$

observe that  $\eta_0 = g(0) = 0$ , and

$$\eta_1 = g'(0) = \lim_{h \rightarrow 0} \frac{\exp(-1/h^2)}{h} = 0 \quad (7.34)$$

The same argument is applicable for higher order terms  $\eta_k$ , therefore,

$$S_\infty(x) = \sum_{k=1}^{\infty} \frac{\eta_k}{k!} f^{(k)}(0) x^k = 0 \quad (7.35)$$

The Taylor's series converges to zero function, which is clearly not  $g$ .

## 8 Integration

**Remark 8.1.** In this section, we are assuming  $f$  to be bounded unless otherwise mentioned.

### 8.1 Riemann Integrable Functions

**Definition 8.1.** A **partition**  $P$  of interval  $[a, b]$  is a finite set of points that has  $a, b$  as its elements, and

$$a = x_0 < x_1 < \cdots < x_n = b \quad (8.1)$$

**Notation 8.1.** Given a function partition  $P$ , for every  $x_k \in P$ , define

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad (8.2)$$

$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad (8.3)$$

**Definition 8.2.** The **lower sum** and **upper sum** of a function  $f$  and a partition  $P$  on  $[a, b]$  are defined

$$L(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) m_k \quad (8.4)$$

$$U(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) M_k \quad (8.5)$$

**Example 8.1.** For every partition  $P$  of every interval  $I \subseteq \mathbb{R}$ ,

$$U(\mathbf{1}\{x \in \mathbb{Q}\}, P) = 1 \quad (8.6)$$

$$L(\mathbf{1}\{x \in \mathbb{Q}\}, P) = 0 \quad (8.7)$$

**Definition 8.3.** A partition  $Q$  is a **refinement** of another partition  $P$  if

$$P \subseteq Q \quad (8.8)$$

**Lemma 8.1.** If  $P \subseteq Q$ , then for every function  $f$ ,

$$L(f, P) \leq L(f, Q) \quad (8.9)$$

$$U(f, P) \geq U(f, Q) \quad (8.10)$$

*Proof.* Suppose  $Q$  contains one extra split  $z \in [x_{k-1}, x_k]$ , then

$$m_k(x_k - x_{k-1}) = m_k(x_k - z + z - x_{k-1}) \quad (8.11)$$

$$= m_k(x_k - z) + m_k(z - x_{k-1}) \quad (8.12)$$

$$\geq (z - x_{k-1}) \inf_{x \in [x_{k-1}, z]} f(x) + (x_k - z) \inf_{x \in [z, x_k]} f(x) \quad (8.13)$$

■

**Lemma 8.2.** Let  $P_1, P_2$  be any two partitions,

$$L(f, P_1) \leq U(f, P_2) \quad (8.14)$$

*Therefore, lower sums are bounded above and upper sums are bounded from below.*

*Proof.*

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \quad (8.15)$$

$$\leq U(f, P_1 \cup P_2) \quad (8.16)$$

$$\leq U(f, P_2) \quad (8.17)$$

■

**Definition 8.4.** Define the **upper** and **lower integrals** as

$$U(f) := \inf_P U(f, P) \quad (8.18)$$

$$L(f) := \sup_P L(f, P) \quad (8.19)$$

note that the infimum and supremum over partitions are well-defined by the previous lemma.  
 $f$  is **(Riemann) integrable** if  $U(f) = L(f)$

**Lemma 8.3.** For any bounded function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$U(f) \geq L(f) \quad (8.20)$$

*Proof.* Fix an arbitrary partition  $P$ .

For every partition  $P'$

$$U(f, P') \geq L(f, P) \quad (8.21)$$

$$\implies \sup_{P'} L(f, P') \leq U(f, P) \quad (8.22)$$

$$\iff L(f) \leq U(f, P) \quad (8.23)$$

The last inequality holds for every partition  $P$ , hence  $L(f) \in \{U(f, P) : P\}^\downarrow$  and

$$L(f) \leq \inf_P U(f, P) \quad (8.24)$$

$$\iff L(f) = U(f) \quad (8.25)$$

■

**Theorem 8.1** (Criterion of Integrability). A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if

$$\forall \varepsilon > 0 \exists P_\varepsilon \text{ s.t. } U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \quad (8.26)$$

*Proof.* (  $\implies$  ) Suppose  $f$  is integrable, let  $\varepsilon > 0$ .

Because  $U(f)$  is the infimum of upper sums,

$$\exists P_1 \text{ s.t. } U(f) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \quad (8.27)$$

And  $L(f)$  is the the supremum of lower sums,

$$\exists P_2 \text{ s.t. } L(f) - \frac{\varepsilon}{2} < L(f, P_2) \leq L(f) \quad (8.28)$$

Take  $P = P_1 \cup P_2$ ,

$$L(f) - \frac{\varepsilon}{2} < L(f, P_2) \leq L(f, P) \leq U(f, P) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \quad (8.29)$$

$$\implies U(f, P) - L(f, P) < \varepsilon \quad (8.30)$$

( $\Leftarrow$ ) Suppose, for contradiction,  $f$  is not integrable:

$$L(f) < U(f) \quad (8.31)$$

Take

$$\varepsilon = \frac{U(f) - L(f)}{2} \quad (8.32)$$

In this case, for every partition  $P$ ,

$$L(f, P) \leq L(f) < U(f) \leq U(f, P) \quad (8.33)$$

$$\implies U(f, P) - L(f, P) \geq U(f) - L(f) > \varepsilon \quad (8.34)$$

Contradiction. ■

**Theorem 8.2.** If  $f$  is continuous on  $[a, b]$ , then it is integrable.

*Proof.* Because  $[a, b]$  is compact,  $f([a, b])$  is bounded and  $f$  is uniformly continuous on  $[a, b]$ .

Let  $\varepsilon > 0$ , by the definition of continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, b], |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a} \quad (8.35)$$

Construct partition  $P$  using by splitting  $[a, b]$  into intervals of length  $\delta/2$ :  $I_k := [a + k\frac{\delta}{2}, a + (k+1)\frac{\delta}{2}]$  so that there are  $\frac{2(b-a)}{\delta}$  intervals in total.

$$U(f, P) - L(f, P) = \sum_k [\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)] |I_k| \quad (8.36)$$

$$= \sum_k [\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)] \frac{\delta}{2} \quad (8.37)$$

$$< \frac{\varepsilon}{b - a} \frac{2(b - a)}{\delta} \frac{\delta}{2} = \varepsilon \quad (8.38)$$

By the criterion of integrability,  $f$  is integrable on  $[a, b]$ . ■

**Example 8.2** (Functions with Point Discontinuity). Define  $f$  on  $[0, 2]$  as

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \quad (8.39)$$

Let  $\varepsilon > 0$ , define partition

$$P_\varepsilon = \left\{ 0, 1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}, 1 \right\} \quad (8.40)$$

Note that  $U(f, P_\varepsilon) = 2$  and

$$L(f, P_\varepsilon) = \left(1 - \frac{\varepsilon}{4}\right) * 1 + \frac{\varepsilon}{2} * 0 + \left(1 - \frac{\varepsilon}{4}\right) = 2 - \frac{\varepsilon}{2} \quad (8.41)$$

and

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \frac{\varepsilon}{2} < \varepsilon \quad (8.42)$$

**Theorem 8.3.** If  $f$  is bounded on  $[a, b]$  and integrable on  $[c, d]$  for every  $c \in (a, b)$ .

*Proof.* Suppose  $|f(x)| \leq M \forall x \in [a, b]$ , let  $\varepsilon > 0$ .

Take

$$c = a + \frac{\varepsilon}{4M} \quad (8.43)$$

Note that because  $f$  is integrable on  $[c, b]$ , there exists another partition  $P'$  of  $[c, b]$  such that

$$U(f, P') - L(f, P') < \frac{\varepsilon}{2} \quad (8.44)$$

Construct  $P_\varepsilon$  as

$$P_\varepsilon = \{a\} \cup P' \quad (8.45)$$

so that  $P_\varepsilon$  is a partition of  $[a, b]$ , and

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \left[ \sup_{x \in [a, c]} f(x) - \inf_{x \in [a, c]} f(x) \right] \frac{\varepsilon}{4M} + U(f, P') - L(f, P') \quad (8.46)$$

$$< 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} \quad (8.47)$$

$$< \varepsilon \quad (8.48)$$

Therefore, by the criterion of integrability,  $f$  is integrable on  $[a, b]$ . ■

## 8.2 Properties of the Integral

**Theorem 8.4.** Assume  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and let  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, b]$  iff  $f$  is integrable on  $[a, c]$  and  $[c, b]$ .

*Proof.* ( $\implies$ ) Suppose  $f$  is integrable on  $[a, b]$ .

Let  $\varepsilon > 0$ , there exists  $P'_\varepsilon$  of  $[a, b]$  such that

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon \quad (8.49)$$

Define  $P_\varepsilon := P'_\varepsilon \cup \{a, c, b\}$  so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon \quad (8.50)$$

one can obtain partitions of  $[a, c]$  and  $[c, b]$ ,  $P_\varepsilon^1$  and  $P_\varepsilon^2$ , following

$$P_\varepsilon^1 := \{x \in P_\varepsilon : x \leq c\} \quad (8.51)$$

$$P_\varepsilon^2 := \{x \in P_\varepsilon : x \geq c\} \quad (8.52)$$

In the case,

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = U(f, P_\varepsilon^1) - L(f, P_\varepsilon^1) + U(f, P_\varepsilon^2) - L(f, P_\varepsilon^2) < \varepsilon \quad (8.53)$$

Because  $U(f, P_\varepsilon^i) - L(f, P_\varepsilon^i) \geq 0$  for  $i \in \{1, 2\}$ ,

$$U(f, P_\varepsilon^i) - L(f, P_\varepsilon^i) \leq U(f, P_\varepsilon^1) - L(f, P_\varepsilon^1) + U(f, P_\varepsilon^2) - L(f, P_\varepsilon^2) < \varepsilon \quad \forall i \in \{1, 2\} \quad (8.54)$$

( $\Leftarrow$ ) Assume  $f$  is integrable on both  $[a, c]$  and  $[c, b]$ .

Let  $\varepsilon > 0$  be given,  $\exists P'_\varepsilon$  of  $[a, c]$  and  $P''_\varepsilon$  of  $[c, b]$  such that

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \frac{\varepsilon}{2} \quad (8.55)$$

$$U(f, P''_\varepsilon) - L(f, P''_\varepsilon) < \frac{\varepsilon}{2} \quad (8.56)$$

Take  $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$  so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \quad (8.57)$$

**Lemma 8.4.**

$$\int_a^b f = \int_a^c f + \int_c^b f \quad (8.58)$$

*Proof.* Recall

$$\int_a^b f = U(f) = L(f) \quad (8.59)$$

Given any  $\varepsilon > 0$ , there exists partition  $P$  such that

$$\int_a^b f < L(f, P) + \varepsilon \quad (8.60)$$

$$= L(f, P_1) + L(f, P_2) + \varepsilon \quad (8.61)$$

$$\leq \int_a^c f + \int_a^b f + \varepsilon \quad (8.62)$$

$$\Rightarrow \int_a^b f \leq \int_a^c f + \int_c^b f \quad (8.63)$$

Further, note that

$$\int_a^c f + \int_c^b f < L(f, P_1) + L(f, P_2) + \varepsilon \quad (8.64)$$

$$= L(f, P) + \varepsilon \quad (8.65)$$

$$\leq \int_a^b f + \varepsilon \quad (8.66)$$

$$\Rightarrow \int_a^c f + \int_c^b f \leq \int_a^b f \quad (8.67)$$

■

Hence, for any  $\varepsilon > 0$ ,  $\exists P_1, P_2$  of  $[a, c]$  and  $[c, b]$  such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad (8.68)$$

$$U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2} \quad (8.69)$$

■

**Theorem 8.5.** Assume  $f$  and  $g$  are integrable on  $[a, b]$ ,

(i)  $f + g$  is integrable on  $[a, b]$  and

$$\int_a^b f + g = \int_a^b f + \int_a^b g \quad (8.70)$$

(ii) For any  $k \in \mathbb{R}$ ,  $kf$  is integrable and

$$\int_a^b kf = k \int_a^b f \quad (8.71)$$

(iii) If  $m \leq f \leq M$  on  $[a, b]$ , then

$$m(b - a) \leq \int_a^b f \leq M(b - a) \quad (8.72)$$