# MAT224 Notes

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## Contents

# 1 Lecture1 Jan.9 2018

## 1.1 Vector spaces

**Definition** A real [1] **vector space** is a set $V$ together with two vector operations vector addition and scalar multiplication such that

1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$
2. **C** Commutative: $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. **Z** Zero Vector: $\exists\ \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$
5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists - \vec{x} \in V s.t. \vec{x} + (-\vec{x}) = \vec{0}$
6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$
7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c+d)\vec{x} = c\vec{x} + d\vec{x}$
9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$
10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

**Note** For $V$ to be a vector space, need to know or be given operations of vector additions multiplication and check all 10 properties hold.

## 1.2 Examples of vector spaces

**Example 1** $\mathbb{R}^n$ w.r.t.[2] usual component-wise addition and scalar multiplication.

**Example 2** $\mathbb{M}_{m \times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

[1] A vector space is real if scalar which defines scalar multiplication is real.
[2] w.r.t. is the abbreviation of "with respect to".

**Example 3** $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree less or equal to n, w.r.t. usual degree-wise polynomial addition and scalar multiplication.

**Note** If define $\mathbb{P}_n^\star(\mathbb{R})$ as set of all polynomials of degree exactly equal to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

**Explanation**: $(1+x^n),(1-x^n) \in \mathbb{P}_n^\star(\mathbb{R})$ but $(1+x^n)+(1-x^n) = 2 \notin \mathbb{P}_n^\star(\mathbb{R})$

**Example 4** Something unusual, define $V$ as

$$V = \{(x_1, x_2)|x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

This is a vector space.

### 1.3 Some properties of vector spaces

Suppose $V$ is a vector space, then it has the following properties.

**Property 1** The zero vector is unique.
*proof.*

Assume $\vec{0}, \vec{0^\star}$ are two zero vectors in $V$

WTS: $\vec{0} = \vec{0^\star}$

Since $\vec{0}$ is the zero vector, by Z $\vec{0^\star} + \vec{0} = \vec{0^\star}$

Similarly, $\vec{0} + \vec{0^\star} = \vec{0}$

Also, $\vec{0} + \vec{0^\star} = \vec{0^\star} + \vec{0}$ by commutative vector addition.

So, $\vec{0^\star} = \vec{0}$

∎

**Property 2** $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.
*proof.*
Exercise. (By Cancellation Law)

**Property 3** $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.
*proof.*

$$\begin{gathered}\text{By property of number 0: } 0\vec{x} = (0+0)\vec{x} \\ \text{By DSA: } 0\vec{x} = 0\vec{x} + 0\vec{x} \\ \text{By AI, } \exists(-0\vec{x}) s.t. \\ 0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x}) \\ \text{By AA} \\ \implies 0\vec{x} = \vec{0}\end{gathered}$$

**Property 4** $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$
*proof.*

$$c\vec{0} = c(\vec{0} + \vec{0}) = c\vec{0} + c\vec{0}$$

# 2 Lecture2 Jan.10 2018

## 2.1 Some properties of vector spaces-Cont'd

**Property 5** For a vector space $V$, $\forall \vec{x} \in V$, $(-1)\vec{x} = (-\vec{x})$. (*we could use this property to find the additive inverse with scalar multiplication with (-1)*)[3].
*proof.*

$$\begin{aligned}(-\vec{x}) &= (-\vec{x}) + \vec{0} \quad \text{By property of zero vector} \\ &= (-\vec{x}) + 0\vec{x} \quad \text{By property3} \\ &= (-\vec{x}) + (1 + (-1))\vec{x} \quad \text{By property of zero as real number} \\ &= (-\vec{x}) + 1\vec{x} + (-1)\vec{x} \\ &= \vec{0} + (-1)\vec{x} \\ &= (-1)\vec{x}\end{aligned}$$

■

[3] The scalar multiplication here is the one defined in vector space $V$.

**Property 6** For a vector space $V$, let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \vee \vec{x} = \vec{0}$$

*proof.*

$$\begin{gathered}
\text{if } c = 0 \implies True \\
\text{else } c^{-1}c\vec{x} = c^{-1} = \vec{0} \\
\implies (c^{-1}c)\vec{x} = \vec{0} \\
\implies 1\vec{x} = \vec{0} \\
\implies \vec{x} = \vec{0} \\
\implies True
\end{gathered}$$

■

### 2.2 Subspaces

**Loosely** A subspace is a space contained within a vector space.

**Definition** Let $V$ be a vector space and $W \subseteq V$, $W$ is a **subspace** of $V$ if $W$ is itself a vector space w.r.t. operations of vector addition and scalar multiplication from $V$.

**Theorem** Let $V$ be a vector space, and $W \subseteq V$, $W$ has the same[4] operations of vector addition and scalar multiplication as in $V$. Then, $W$ is a subspace of $V$ iff:

1. $W$ is non-empty. $W \neq \emptyset$.
2. $W$ is closed under addition. $\forall \vec{x}, \vec{y} \in W,\ \vec{x} + \vec{y} \in W$.
3. $W$ us closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

*Proof.*

[4] Other properties of vector spaces related to vector addition and scalar multiplication are immediately inherited from the parent vector space.

Forward:

If $W$ is a subspace
$\implies \vec{0} \in W$
$\implies W \neq \emptyset$
Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$
WTS. 10 axioms in definition of vector space hold
$(ii) \implies$ Additive Closure
$(iii) \implies$ Scalar Multiplication Clousure
Because $W \subseteq V$, and $V$ is a vector space, so properties hold $\forall \vec{w} \in W$.
Additive inverse: by property 5 and scalar multiplication closure,
$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W.$
Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W.$

## 2.3 Examples of subspaces

**Example 1** Let $V = \mathbb{M}_{n\times n}(\mathbb{R})$, $V$ is a subspace.

**Example 2** Define $W$ as

$$W = \{A \in \mathbb{M}_{n\times n}(\mathbb{R}) | A \text{ is } \underline{\text{not}} \text{ symmetric}\}$$

*Explanation:* Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but $A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W.$
Since there's no additive identity in set $W$, so $W$ failed to be a vector space, therefore $W$ is not a subspace.

**Example 3** Let $V = \mathbb{P}_2(\mathbb{R})$, is $W$ defined as following,

$$W = \{p(x) \in V | p(1) = 0\}$$

a subspace of $V$ ?

*proof.*

WTS: (i)

Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$

$\implies W \neq \emptyset$

WTS: (ii)

Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$

$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$

$\implies p_1 + p_2 \in W$

$\implies W$ is closed under addition.

WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$

$\implies p(1) = 0$

Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$

$\implies cp \in W$.

So $W$ is a subspace of $V$.

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### 2.4 Recall from MAT223

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of $\mathbb{R}^n$ and $Col(A)$ is a subspace of $\mathbb{R}^m$.

## 3 Lecture3 Jan.16 2018

### 3.1 Linear Combination

**Definition** Let $V$ be a vector space, $\vec{v_1}, \dots, \vec{v_n} \in V$, $a_1, \dots, a_n \in \mathbb{R}$ the expression

$$c_1\vec{v_1} + \dots + c_n\vec{v_n}$$

is called a **linear combination** of $\vec{v_1}, \dots, \vec{v_n}$.

**Theorem** Let $V$ be a vector space, $W$ is a subspace of $V$, $\forall \vec{w_1}, \dots \vec{w_k} \in W, c_1, \dots, c_k \in \mathbb{R}$, we have

$$c_1\vec{w_1} + \dots + c_k\vec{w_k} \in W$$

*Subspaces are <u>closed under linear combinations</u>, since subspaces are closed under scalar multiplication and vector addition.*

**Theorem** Let $V$ be a vector space, let $\vec{v_1}, \dots, \vec{v_k} \in V$ then the set of all linear combination of $\vec{v_1}, \dots, \vec{v_k}$

$$W = \{\sum_{i=1}^{k} c_i \vec{v_i} | c_i \in \mathbb{R} \forall\ i\}$$

is a subspace of $V$.
*proof.*

$$\text{Consider } \vec{0} \in W$$
$$\text{So, } W \neq \emptyset$$
$$\text{Let } c \in \mathbb{R}, \text{ Let } \vec{x} \in W \wedge \vec{y} \in W$$
$$\text{By definition of span, we have,}$$
$$\vec{x} = \sum_{i=1}^{k} a_i \vec{v_i}, \quad \vec{y} = \sum_{i=1}^{k} b_i \vec{v_i}$$
$$\text{Consider, } \vec{x} + c\vec{y}$$
$$\vec{x} + c\vec{y} = \sum_{i=1}^{k} a_i \vec{v_i} + c \sum_{i=1}^{k} b_i \vec{v_i} = \sum_{i=1}^{k} (a_i + cb_i)\vec{v_i} \in W$$

■

**Definition** Let $V$ be a vector space, $\vec{v_1}, \dots, \vec{v_k} \in V$, **span** of the set of vectors $\{\vec{v_i}\}_{i=1}^{k}$ is defined as the collection of all possible linear combinations of $\{\vec{v_i}\}_{i=1}^{k}$. By pervious theorem, span is a subspace.

### 3.2 Combination of subspaces

**Definition** Let $W_1, W_2$ be two sets, then the **union** of $W_1, W_2$ is defined as:

$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

the **intersection** of $W_1, W_2$ is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

Now consider $W_1, W_2$ to be two subspaces of vector space $V$, then we have,

1. $W_1 \cup W_2$ is **not** a subspace.

2. $W_1 \cap W_2$ is a subspace.

*proof.*

$$\begin{gathered}\text{Falsify the statement by providing counter-example:}\\ \text{Consider,}\\ W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}\\ W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}\\ \begin{pmatrix}0\\1\end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix}1\\0\end{pmatrix} \in W_1 \cup W_2\\ \text{But, } \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \notin W_1 \cup W_2\end{gathered}$$

■

*proof.*

$$\begin{gathered}\text{Because } W_1 \text{ and } W_2 \text{ are both subspaces, so}\\ \vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset\\ \text{Let } \vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}\\ \text{Consider, } \vec{x} + c\vec{y}\\ \text{Sine } W_1, W_2 \text{ are subspaces,}\\ \vec{x} + c\vec{y} \in W_1 \wedge \vec{x} + c\vec{y} \in W_2\\ \implies \vec{x} + c\vec{y} \in W_1 \cap W_2\\ \text{So, } W_1 \cap W_2 \text{ is a subspace.}\end{gathered}$$

■

**Definition** Let $W_1, W_2$ be subspaces of vector space $V$, define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \wedge \vec{y} \in W_2\}$$

**Note** Let $\vec{x} = \vec{0} \in W_1$, $\forall \vec{y} \in W_2,\ \vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = 0 \in W_2$, $\forall \vec{x} \in W_1,\ \vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2,\ \vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

**Note** $W_1 + W_2$ is a subspace of $V$.
*proof.*

$$\begin{gathered}
\text{Let } \vec{x_1}, \vec{x_2} \in W_1, \vec{y_1}, \vec{y_2} \in W_2 \\
\text{By properties of subspaces,} \\
\forall c \in \mathbb{R}, \vec{x_1} + c\vec{x_1} \in W_1 \wedge \vec{y_2} + c\vec{y_2} \in W_2 \\
\text{Consider, } \vec{x_1} + \vec{y_1} \in W_1 + W_2, \vec{x_2} + \vec{y_2} \in W_1 + W_2 \\
(\vec{x_1} + \vec{y_1}) + c(\vec{x_2} + \vec{y_2}) \\
= (\vec{x_1} + c\vec{x_2}) + (\vec{y_1} + c\vec{y_2}) \in W_1 + W_2
\end{gathered}$$

■

**Definition(Unique Representation)** Let $W_1, W_2$ be subspaces of vector space $V$, say $V$ is **direct sum** of $W_1$ and $W_2$, written as $V = W_1 \bigoplus W_2$, if every $\vec{x} \in V$ can be written uniquely as $\vec{x} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in W_1$ and $\vec{w_2} \in W_2$.
**Equivalently** Let $W_1$ and $W_2$ be subspaces of $V$, $V = W_1 \bigoplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.

# 4 Lecture4 Jan.17 2018

## 4.1 Cont'd

**Cont'd Proof of Theorem**

*proof.*

$$\text{(Forward direction) Suppose } V = W_1 \bigoplus W_2$$
$$\text{WTS. } V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{Let } V = W_1 \bigoplus W_2$$
$$\implies \forall \vec{x} \in V, \text{ can be written uniquely as}$$
$$\vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1,\ \vec{w_2} \in W_2$$
$$\implies V = W_1 + W_2 \text{ by definition of } \textit{sum}.$$
$$\text{Let } \vec{x} \in W_1 \cap W_2$$
$$\text{Decomposition, let } \vec{z} \in W_1 \cap W_2 \subseteq V$$
$$\vec{z} = \vec{z} + \vec{0},\ \vec{z} \in W_1, \vec{0} \in W_2$$
$$\vec{z} = \vec{0} + \vec{z},\ \vec{0} \in W_1, \vec{z} \in W_2$$
$$\text{Since decomposition is unique, } \vec{z} = \vec{0}$$
$$\text{So, } W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{(Backward direction) Suppose } V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$$
$$\text{WTS. } V = W_1 \bigoplus W_2$$
$$\text{Assume } \vec{x} = \vec{w_1} + \vec{w_2},\ \vec{w_1} \in W_1, \vec{w_2} \in W_2$$
$$\vec{x} = \vec{w_1'} + \vec{w_2'},\ \vec{w_1'} \in W_1, \vec{w_2'} \in W_2$$
$$\implies \vec{w_1} + \vec{w_2} = \vec{w_1'} + \vec{w_2'}$$
$$\implies \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2}$$
$$\text{Where, by definition of subspace, } \vec{w_1} - \vec{w_1'} \in W_1 \wedge \vec{w_2'} - \vec{w_2} \in W_2$$
$$\text{So, } \vec{w_1} - \vec{w_1'} = \vec{w_2'} - \vec{w_2} \in W_1 \cap W_2$$
$$\text{Since } W_1 \cap W_2 = \{\vec{0}\}$$
$$\implies \vec{w_1} = \vec{w_1'} \wedge \vec{w_2} = \vec{w_2'}$$
$$\text{So the decomposition is unique.}$$

■

## 4.2 Linear Independence

**Theorem (Redundancy theorem)** Let $V$ be a vector space, $\{\vec{x_1}, \dots \vec{x_n}\}$, let $\vec{x} \in \{\vec{x_1}, \dots \vec{x_n}\}$, then

$$span\{\vec{x_1}, \dots \vec{x_n}, \vec{x}\} = span\{\vec{x_1}, \dots \vec{x_n}\}$$

we say $\vec{x}$ *is the* ***redundant*** *vector that contributes nothing to the span.*
*proof.*

$$\begin{gathered}
\text{let } \vec{x} \in span\{\vec{x}, \dots, \vec{x_n}\} \\
\vec{x} = \sum_{i=1}^{n} c_i \vec{x_i} \text{ for } c_i \in \mathbb{R}\ \forall i \\
\text{So, } span\{\vec{x_1}, \dots, \vec{x_n}, \vec{x}\} = \{\sum_{i=1}^{n} a_i \vec{x_i} + z\vec{x} \mid a_i, z \in \mathbb{R} \forall i\} \\
= \{\sum_{i=1}^{n} a_i \vec{x_i} + z \sum_{i=1}^{n} c_i \vec{x_i} \mid a_i, c_i \in \mathbb{R} \forall i\} \\
= \{\sum_{i=1}^{n} (a_i + zc_i) \vec{x_i} \mid a_i, c_i \in \mathbb{R} \forall i\} \\
\text{Let } d_i = a_i + zc_i \in \mathbb{R} \\
= \{\sum_{i=1}^{n} d_i \vec{x_i} \mid d_i \in \mathbb{R} \forall i\} \\
= span\{\vec{x_1}, \dots \vec{x_n}\}
\end{gathered}$$

■

**Definition** Let $V$ be a vector space, let $\{\vec{x_1}, \dots, \vec{x_n}\} \in V$, we say $\{v_i\}_{i=1}^{n}$ is **linearly independent** if the only set of scalars $\{c_1, \cdots, c_n\}$ that satisfies,

$$\sum_{i=1}^{n} c_i \vec{x_i} = 0$$

is $\{0, \dots, 0\}$.

**Definition** In contrast, we say a set of vector, with size $n$, is **linearly dependent** if

$$\exists \vec{c} \neq \vec{0} \in \mathbb{R}^n, \ s.t. \ \sum_{i=1}^{n} c_i \vec{v_i} = 0$$

**Theorem** Let $V$ be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly dependent* if and only if,

$$\exists \vec{x} \in \{\vec{v_i}\}_{i=1}^n \ s.t. \ \vec{x_j} \in span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

**Theorem** Let $V$ be a vector space, $\{\vec{v_i}\}_{i=1}^n \in V$ is *linearly independent* if and only if,

$$\forall \vec{x} \in \{\vec{v_i}\}_{i=1}^n, \ \vec{x_j} \notin span\{\{\vec{v_i}\}_{i=1}^n \backslash \{\vec{x}\}\}$$

# 5 Lecture5 Jan.23 2018

## 5.1 Linear independence, recall definitions

*Acknowledgement: special thanks to Frank Zhao.*

**Definition** Let $\{\vec{x_1}, \ldots \vec{x_k}\}$ is **linearly independent** if only scalars $c_1 \ldots c_k$ s.t.

$$\sum_{i=1}^{k} c_1 \vec{x_k} = 0 (\star)$$

are $c_1 = \cdots = c_k = 0$

**linearly dependent** means <u>at least one</u> $c_i \neq 0$, $(\star)$ still holds.

### 5.1.1 Alternative definitions of linear independency

**Definition(Alternative.1)** $\{\vec{x_1} \ldots \vec{x_k}\}$ is **linearly independent** <u>iff</u> none of them can be written as a linear combination of the remaining $k-1$ vectors.[5]

**Definition(Alternative.2)** $\{\vec{x_1} \ldots \vec{x_k}\}$ is **linearly dependent** <u>iff</u> at least one of them can be written as a linear combination of the remaining $k-1$ vectors. [6]

## 5.2 Basis

**Definition** Let $V$ be a vector space, a non-empty[7] set $S$ of vectors from $V$ is a **basis** for $V$ if

1. $V = span\{S\}$

[5] See theorem from the pervious lecture.
[6] See theorem from the pervious lecture.
[7] Specially, for an empty set, we define $span\{\emptyset\} = \{\vec{0}\}$

2. $S$ is linearly independent.

**Theorem (characterization of basis)** A non-empty subset $S = \{\vec{x_i}\}_{i=1}^n$ of vector space $V$ is basis for $V$ iff every $\vec{x} \in V$ can be written uniquely as linear combination for vectors in $S$.

*proof.*

**Forwards**

Suppose $S$ is a basis for $V$

So every $\vec{x} \in V$ can be written as a linear combination of vectors in $S$

To prove the uniqueness, assume two expressions of $\vec{x} \in V$

$$\vec{x} = \begin{cases} c_1\vec{x_1} + \cdots + c_k\vec{x_k} \\ b_1\vec{x_1} + \cdots + d_k\vec{x_k} \end{cases}$$

Consider,

$$c_1\vec{x_1} + \cdots + c_k\vec{x_k} - (b_1\vec{x_1} + \cdots + d_k\vec{x_k}) = \vec{0}$$

$$\iff \sum_{i=1}^{k} (c_i - b_i)\vec{x_1} = \vec{0}$$

Since vectors in basis $S$ are linear independent,

$$c_i = b_i \forall i \in \mathbb{Z} \cap [1, k]$$

So the representation is unique.

**Backwards**

Suppose every $\vec{x} \in V$ can be written uniquely as linear combination of vectors in $S$.

WTS: $V = span\{S\} \wedge S$ is linearly independent

By the assumption, spanning set is shown.

All we need to show is linear independence.

Consider,

$$\sum_{i=1}^{n} c_i\vec{x}_i = \vec{0}$$

Also, we know

$$\sum_{i=1}^{n} 0\vec{x_i} = \vec{0}$$

By the uniqueness of representation

We have identical expression $\sum_{i=1}^{n} c_i\vec{x}_i = \sum_{i=1}^{n} 0\vec{x}_i$

$$\therefore c_i = 0 \ \forall i \in \mathbb{Z} \cap [1, n]$$

■

**Example**

$$V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$
$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$
$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

Show that $\{(1,0),(6,3)\}$ is a basis of $V$.

By theorem, $\{(1,0),(6,3)\}$ is basis if every $(a,b) \in V$ can be written uniquely as linear combination of $\{(1,0),(6,3)\}$.

$$\exists \text{ unique scalars } c_1, c_2 \in \mathbb{R} \; s.t. \; c_1(1,0) + c_2(6,3) = (a,b)$$

*proof.*

By definition of scalar multiplication and vector addition in this space,

$$\text{Consider}(a,b) = c_1(1,0) + c_2(6,3) = (2c_1 - 1, c_1 - 1) + (7c_2 - 1, 4c_2 - 1)$$
$$= (2c_1 + 7c_2 - 1, c_1 + 4c_2 - 1)$$

Consider the coefficients of variables

$$\begin{cases} 2c_1 + 7c_2 - 1 = a \\ c_1 + 4c_2 - 1 = b \end{cases}$$

WTS, the above system of linear equations has unique solution for all $a, b$

The system has a unique solution $\forall a, b \in \mathbb{R}$

Since the coefficient matrix has rank 2

$$rank(\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}) = 2$$

Since obviously the columns are linearly independent.

∎

## 5.3 Dimensions

**Definition** For a vector space $V$, the **dimension** of $V$ is the minimum number of vectors required to span $V$.

**Fundamental Theorem** if $V$ vector space is spanned by $m$ vectors, then any set of more than $m$ vectors from $V$ must be linearly dependent.

**Fundamental Theorem (Alternative)** If $V$ is vector space spanned by $m$ vectors, then any linearly independent set in $V$ must contain less or equal to $m$ vectors.

### 5.3.1 Consequences of fundamental theorem

**Theorem** if $S = \{\vec{v}_i\}_{i=1}^{k}$ and $T = \{\vec{w}_i\}_{i=1}^{l}$ are two bases of vector space $V$ then $l = k$. *Bases have the same size.*

*proof.*

$$\text{Since } S \text{ spans } V \text{ and } T \text{ is linearly independent}$$
$$\therefore l \leq k$$
$$\text{(flip) Since } T \text{ spans } V \text{ and } S \text{ is linearly independent}$$
$$\therefore k \leq l$$
$$\implies l \leq k \wedge k \leq l$$
$$\implies k = l$$

■

**Definition** So we can define the **dimension** of $V$, as $dim(V)$ as the number vectors in <u>any</u> basis for $V$. For special case $V = \{\vec{0}\}$, $dim(V) = 0$.

**Example**

- $dim(\mathbb{R}^n) = n$
- $dim(\mathbb{P}_n(\mathbb{R})) = n + 1$
- $dim(\mathbb{M}_{m\times n}(\mathbb{R})) = m \times n$

### 5.3.2 Use dimension to prove facts about linearly (in)dependent sets and subspaces

**Theorem** If $V$ is a vector space, $dim(V) = n$, $S = \{\vec{x_k}\}_{i=1}^{k}$ is subset of $V$, if $k > n$ then $S$ is <u>linearly dependent</u>.

**Note** $k \leq n \nRightarrow S$ is linear dependent.

**Theorem** If $W$ is subspace of vector space $V$, then

1. $dim(W) \leq dim(V)$
2. $dim(W) = dim(V) \iff W = V$

*proof.*

(1) Suppose $dim(V) = n, dim(W) = k$

WTS, $k \leq n$

Any basis for $W$ is a linearly independent set of $k$ vectors from $V$.

Since $V$ is spanned by $n$ vectors, since $dim(V) = n$

By fundamental theorem, $k \leq n$

$\iff dim(W) \leq dim(V)$

(2) By contradiction, assume $dim(V) = dim(W) = n$ but $V \neq W$

Then $\exists \vec{x} \in V \wedge \vec{x} \notin W$

Take $S$ as a basis of $W$, then $\vec{x} \notin span\{S\}$

Then $S \cup \vec{x}$ is linearly independent

$\implies S \cup \{\vec{x}\}$ is linearly independent in $V$ containing $n+1$ vectors

This contradicts the assumption by fundamental theorem since $dim(V) = n$ so it could not contain more than $n$ linearly independent vectors

■

# 6 Lecture6 Jan.24 2018

## 6.1 Basis and Dimension

**Theorem** Let $V$ be a vector space , $S$ is a spanning set of $V$, and $I$ is a linearly independent subset of $V$, s.t. $I \subseteq S$, then $\exists$ basis $B$ for $V$ s.t. $I \subseteq B \subseteq S$.

**Explaining**

1. *Any spanning set for $V$ cab be **reduced** to basis for $V$ by removing the linearly dependent(redundant) vector in the spanning set, using redundancy theorem to get a linearly independent spanning set.*

2. *Linear independent set can be **enlarged** to a basis for $V$.*

*proof.*

omitted.

■

**Corollary** Let $V$ be a vector space and $dim(V) = n$, any set of $n$ linearly independent vectors from $V$ is a basis for $V$.

*proof.* If $n$ linearly independent vectors did not span V, then could be enlarged to a basis of $V$ by pervious theorem, but then have a basis containing more than $n$ vectors from $V$, which is impossible by the fundamental theorem since we given the $dim(V) = n$, proven by contradiction.

**Example** Let $V = P_2(\mathbb{R})$, $p_1(x) = 2 - 5x$, $p_2(x) = 2 - 5x + 4x^2$, find $p_3 \in P_2(\mathbb{R})$ s.t. $\{p_1(x), p_2(x), p_3(x)\}$ is basis for $P_2(\mathbb{R})$

**Note** Since $dim(P_2(\mathbb{R})) = 3$ so any 3 linearly independent vectors from $P_2(\mathbb{R})$ will be a basis for $P_2(\mathbb{R})$.

**Solutions** e.g. constant function $p_3(x) = 1$, since $1 \notin span\{p_1(x), p_2(x)\}$, so $\{p_1(x), p_2(x), p_3(x)\}$ is a basis of $P_2(\mathbb{R})$. e.g. $p_3(x) = x$, since $x \notin span\{p_1(x), p_2(x)\}$

**Theorem** Let $U$ and $W$ be subspaces of vector space $V$, then we have

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

*proof.*

$$\text{Let } \{\vec{v_i}\}_1^k \text{ be basis for } U \cap W$$
$$\implies dim(U \cap W) = k$$
$$\text{Since } \{\vec{v_i}\}_1^k \text{ is basis for } U \cap W \text{ then it's a linearly independent subset of U}$$
$$\text{So it could be enlarged ti basis for } U,\ \{\vec{v_1}, \dots, \vec{v_k}, \vec{y_1}, \dots, \vec{y_r}\}$$
$$\text{So } dim(U) = k + r$$
$$\text{We also could enlarge a basis for } W\ \{\vec{v_1}, \dots, \vec{v_k}, \vec{z_1}, \dots, \vec{z_s}\}$$
$$\implies dim(V) = k + s$$
$$\text{WTS. } \{\vec{v_1}, \dots, \vec{v_k}, \dots, \vec{y_1}, \dots, \vec{y_r}, \vec{z_1}, \dots, \vec{z_s}\} \text{ is a basis for } U + W$$
$$\text{(If we could show this) } dim(U + W) = k + r + s = (k + r) + (k + s) - k$$
$$= dim(U) + dim(W) - dim(U \cap W)$$
$$\text{Obviously, the above set spans } U + W$$
$$\text{WTS. } \{\vec{v_1}, \dots, \vec{v_k}, \dots, \vec{y_1}, \dots, \vec{y_r}, \vec{z_1}, \dots, \vec{z_s}\} \text{ is linearly independent}$$
$$\text{Consider } a_1\vec{v_1} + \dots + a_k\vec{v_k} + b_1\vec{y_1} + \dots b_r\vec{y_r} + c_1\vec{z_1} + \dots + c_s\vec{z_s} = \vec{0}\ (\star)$$
$$\text{From } (\star) \implies \sum(c_i\vec{z_i}) = -\sum(a_i\vec{v_i}) - \sum b_i\vec{y_i}$$
$$\implies \sum(c_i\vec{z_i}) \in U \wedge \sum(c_i\vec{z_i}) \in W$$
$$\iff \sum(c_i\vec{z_i}) \in U \cap W$$
$$\text{Since } \{\vec{v_i}\} \text{ is a basis for } U \cap W$$
$$\implies \sum(c_i\vec{z_i}) = \sum(d_i\vec{v_i})$$
$$\iff \sum(c_i\vec{z_i}) - \sum(d_i\vec{v_i}) = \vec{0} \in W$$
$$\implies c_i = d_i = 0 \text{ since } \{\vec{z_i}, \vec{v_i}\} \text{ is a basis}$$
$$\text{Rewrite } (\star)$$
$$\sum(a_i\vec{v_i}) + \sum b_i\vec{y_i} = 0 \in U$$
$$\implies a_i = b_i = 0 \text{ since } \{\vec{v_i}, \vec{y_i}\} \text{ is a basis for } U$$

■

**Corollary** For direct sum, since the intersection is $\{\vec{0}\}$

$$dim(U \bigoplus W) = dim(U) + dim(W)$$

**Example** Let $U, W$ are subspaces of $\mathbb{R}^3$ such shat $dim(U) = dim(W) = 2$, why is $U \cap W \neq \{\vec{0}\}$

**Solutions** Geometrically, $U$ and $W$ are planes through origin then the intersection would be a line through origin($U \neq W$) or a plane through origin.($U = W$), so shown.

**Question** $V$ is a vector space, $dim(V) = n$, $U \neq W$ are subspaces of $V$ but $dim(U) = dim(V) = (n-1)$, proof:

1. $V = U + W$
2. $dim(U \cap W) = (n - z)$

# 7 Lecture7 Jan.30. 2018

## 7.1 Linear Transformations

**Definition** Let $V, W$ be vector spaces, a function $T : V \to W$ is a **linear transformation**[8] if

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})\ \forall \vec{x}, \vec{y} \in V$[9]
2. $T(c\vec{x}) = cT(\vec{x})\ \forall \vec{x} \in V,\ c \in \mathbb{R}$[10]

*Linear transformation preserves <u>vector additions ans saclar multiplications</u> on vector spaces.*

**Theorem(Alternative definition)** Transformation $T : V \to W$ is linear if and only if

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y}),\ \forall \vec{x}, \vec{y} \in V, c, d \in \mathbb{R}$$

*Linear transformations preserves <u>linear combinations</u>.*

**Example** (form 223) <u>Rotation</u> through angle $\theta$ about the origin in $\mathbb{R}^2$.

[8] In some textbooks, this is annotated as **linear mapping**.

[9] Notice that the vector additions on the left and right sides of the equation are defined in different vector spaces, in $V$ and $W$ respectively.

[10] Notice that the scalar multiplication on the left and right sides of the equation are defined in different vector spaces, in $V$ and $W$ respectively.

**Example** (from 223) Matrix transformation, let $A \in M_{m\times n}(\mathbb{R})$, transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$T(\vec{x}) = A\vec{x}$$

is linear.

**Example** Derivative $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by

$$T(\vec{p}(x)) = \vec{p}\,'(x)$$

**Example** Matrix transpose $T : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ defined by

$$T(A) = A^T$$

## 7.2 Properties of linear transformations

**Property(i)** Linear transformation $T : V \to W$ are uniquely defined by their values on any basis for $V$.

*proof.*

$$\text{Let}\{\vec{v_1}, \dots, \vec{v_k}\} \text{ be any basis for } V$$

Every vector $\vec{x} \in V$ can be uniquely written as some linear combination of the $\{\vec{v_i}\}_{i=1}^k$

$$\vec{x} = \sum_{i=1}^{k} c_i\vec{v_i},\ c_i \in \mathbb{R},\ \text{and } c_i \text{ are uniquely determined } \forall \vec{x} \in V$$

$$\implies T(\vec{x}) = T(\sum_{i=1}^{k} c_i\vec{v_i})$$

$$= \sum_{i=1}^{k} c_i T(\vec{v_i}) \text{ since the transformation } T \text{ is linear.}$$

$$\text{Since } c_i\text{s are uniquely determined by } \{\vec{v_i}\}_{i=1}^k$$

so the value of $T(\vec{x})$ is uniquely determined by its value on basis vectors $\{\vec{v_i}\}_{i=1}^k$.

■

**Property(ii)** Let $T : V \to W$ be a linear transformation, let $A$ be a subspace of vector space $V$, then the **image** $T(A)$ defined as

$$T(A) = \{T(\vec{x}) \mid \vec{x} \in A\}$$

called the image of $A$ under linear transformation $T$ is a subspace of $W$. *Linear transformation maps subspaces of $V$ to subspaces of $W$.*

*proof.*

$$\text{Since } A \text{ is a subspace so it's non-empty, therefore } \exists T(\vec{x}),\ \vec{x} \in A$$
$$\text{So T(A)} \neq \emptyset$$
$$\text{Let } \vec{w_1}, \vec{w_2} \in T(A)$$
$$\implies \vec{w_1} = T(\vec{x_1}), \vec{w_2} = T(\vec{x_2}),\ \vec{x_1}, \vec{x_2} \in A$$
$$\implies \vec{w_1} + \vec{w_2} = T(\vec{x_1}) + T(\vec{x_2}) = T(\vec{x_1} + \vec{x_2}) \text{ since } T \text{ is linear.}$$
$$\text{Since } \vec{x_1} + \vec{x_2} \in A \text{ by the definition of subspaces.}$$
$$\implies \vec{w_1} + \vec{w_2} \in T(A)$$
$$\text{So } T(A) \text{ is closed under vector addition.}$$
$$\text{Let } \vec{w} \in T(A)$$
$$\implies \vec{w} = T(\vec{x}), \vec{x} \in A$$
$$\text{Let } c \in \mathbb{R}$$
$$\text{Consider } c\vec{w} = cT(\vec{x}) = T(c\vec{x})$$
$$\text{Since } c\vec{x} \in A$$
$$\text{So } c\vec{w} \in T(A)$$
$$\text{So } T(A) \text{ is closed under scalar multiplication.}$$

■

**Property(derived from the definition)** For all linear transformation $T : V \to W$, we have [11]

$$T(\vec{0}) = \vec{0}$$

**Property(iii)** Let transformation $T : V \to W$ be linear, let $B$ be a subspace of $W$, then its **pre-image** defined as

$$T^{-1}(B) = \{\vec{x} \in V \mid T(x) \in B\}$$

is a subspace of $V$. [12]

[11] In the equation, clearly, the zero vector on the left side of the equation is in space $V$ and the zero vector on the right side is in space $W$.

[12] The pre-image and inverse share the same notation, but in this case, transformation $T$ is not necessarily invertible.

*proof.*

$$\text{Let } \vec{w_1}, \vec{w_2} \in T^{-1}(B)$$
$$\implies T(\vec{w_1}), T(\vec{w_2}) \in B$$
$$\implies aT(\vec{w_1}) + b(\vec{w_2}) \in B,\ \forall a, b \in \mathbb{R} \text{ since } B \text{ is a subspace.}$$
$$\implies T(a\vec{w_1} + b\vec{w_2}) \in B$$
$$\implies a\vec{w_1} + b\vec{w_2} \in T^{-1}(B)$$

So $T^{-1}(B)$ is closed under both vector addition and scalar multiplication,

So $T^{-1}(B)$ is a subspace.

■

### 7.3 Definitions

Let $T : V \to W$ to be a linear transformation,

**Definition** the **Image** of transformation $T$ is defined as

$$Im(T) = T(V) = \{T(\vec{x}) \mid \vec{x} \in V\}$$

**Definition** the **Rank** of transformation $T$ is defined as

$$Rank(T) = dim(Im(T))$$

**Definition** the **Kernel** of transformation $T$ is defined as

$$Ker(T) = T^{-1}(\{\vec{0}\}) = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

**Definition** the **Nullity** of transformation $T$ is defined as

$$Nullity(T) = dim(ker(T))$$

**Example** $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ is <u>linear</u> defined by

$$T(\vec{p}(x)) = \vec{p}(2x + 1) - 8\vec{p}(x)$$

find $Ker(T)$.

**Theorem** Let $T : V \to W$ be a linear transformation, let $\{\vec{v_1}, \dots, \vec{v_k}\}$ be the <u>spanning set</u> of $V$ [13], then $\{T(\vec{v_1}), \dots, T(\vec{v_k})\}$ spans $Im(T)$

*proof.*

$$\text{Let } \vec{w} \in Im(T)$$
$$\text{Since } V = span\{\vec{v_1}, \dots, \vec{v_k}\}$$
$$\text{For any } \vec{x} \in V \text{ can be written as}$$
$$\vec{x} = \sum_{i=1}^{k} c_i \vec{v_i},\ \ c_i \in \mathbb{R}$$
$$\implies \vec{w} = T(\vec{x}) = T(\sum_{i=1}^{k} c_i \vec{v_i})$$
$$= \sum_{i=1}^{k} c_i T(\vec{v_i})$$
$$\text{as a linear combination of } \{T(\vec{v_1}), \dots, T(\vec{v_k})\}$$
$$\text{So } Im(T) = span\{T(\vec{v_1}), \dots, T(\vec{v_k})\}$$

■

# 8 Lecture8 Jan.31 2018

## 8.1 Linear Transformations

**Example** $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(p(x)) = p(2x + 1) - 8p(x)$$

Find the image of $T$.
We know $B = \{1, x, x^2, x^3\}$ is the standard basis for $P_3(\mathbb{R})$, consider the set $P(B)$

$$P(B) = \{-7, 1 - 6x, 1 + 4x - 4x^2, 1 + 6x + 12x^2\}$$

spans $Im(T)$. Notice the first three vectors in the set is linearly independent, the last vector is clearly dependent to the pervious three.[14]. So by the <u>redundancy theorem</u> we could remove the last vector. There we have

$$Im(T) = span\{-7, 1 - 6x, 1 + 4x - 4x^2\}$$

[13] The set is only the spanning set of $V$, it's not necessarily to be a basis of $V$.
[14] Notice that the first three vectors is a basis of $P_2(\mathbb{R})$.

as basis.
In this example, the dimension of $Ker(T)$ is 1 and the dimension of $Im(T)$ is 3, and dimension of $P_3(\mathbb{R})$ is 4. We have, $dim(P_3(\mathbb{R})) = Nullity(T) + Rank(T)$

**Theorem(Dimension Theorem)** Let $T : V \to W$ be a linear transformation,

$$dim(V) = Nullity(T) + Rank(T)$$

*Proof.*

$$\text{Say } dim(V) = n$$

$$\text{Let } \{\vec{v_1}, \ldots, \vec{v_k}\} \text{ be a basis for } Ker(T)$$

$$\text{Since } Ker(T) \text{ is a subspace of } V, \text{ the set } \{\vec{v_i}\}_1^k \text{ is a subset of } V,$$

$$\text{It can be extended to a basis } \{\vec{v_i}\}_1^k \cup \{\vec{v_i}\}_{k+1}^n \text{ for V.}$$

$$\text{Claim: } \{T(\vec{v_{k+1}}), \ldots, T(\vec{v_n})\} \text{ is basis for } Im(T)$$

$$\text{If the claim is true, this prove the theorem since}$$

$$dim(Ker(T)) + dim(Im(T)) = k + n - k = n = dim(V)$$

$$\because T(\vec{v_i}) = \vec{0}, \ \forall i \in \mathbb{Z}_1^k$$

$$\text{and by the definition of kernel of linear transformation,}$$

$$\therefore \{T(\vec{v_i})\}_{k+1}^n \text{ spans } Im(T)$$

$$\text{Show if } \sum_{i=k+1}^{n} c_i T(\vec{v_i}) = \vec{0} \implies c_i = 0$$

$$\implies T(\sum_{i=k+1}^{n} c_i \vec{v_i}) = \vec{0}$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} \in Ker(T)$$

$$\implies \sum_{i=k+1}^{n} c_i \vec{v_i} = \sum_{i=1}^{k} c_i \vec{v_i}$$

$$\implies c_1\vec{v_1} + \cdots + c_k\vec{v_k} - c_{k+1}\vec{v_{k+1}} - \cdots - c_n\vec{v_n} = \vec{0}$$

$$\text{Since } \{\vec{v_i}\}_i^n \text{ is a basis for } V.$$

$$\implies c_i = 0 \ \forall i$$

### 8.2 Applications of dimension theorem

**Definition** A linear transformation $T : V \to W$ is called **injective**(one-to-one) if and only if

$$T(\vec{v_1}) = T(\vec{v_2}) \implies \vec{v_1} = \vec{v_2}$$

**Definition** A linear transformation $T : V \to W$ is called **surjective**(onto) if and only if

$$Im(T) = W$$

*Every vector in $W$ has a pre-image in $V$.*

**Definition** A linear transformation $T : V \to W$ is called **bijective** if it's both injective and surjective.

**Theorem** Let transformation $T : V \to W$ is linear, $T$ is injective if and only if $dim(Ker(T)) = 0$.

*Proof.*

Exercise

■

**Theorem** $T$ is surjective if and only if $dim(Im(T)) = dim(W)$.

**Example** $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$$

is $T$ injective? surjective?
Not injective but surjective.
**Solution**

$$Ker(T) = span\{(x-1)(x-2)\}$$

So T has nullity of 1 and since $dim(P_2(\mathbb{R})) = 3$, by the dimension theorem we have $Rank(T) = 2$ and since $Im(T)$ is a subspace of $\mathbb{R}^2$ which has dimension of 2, we could conclude that $Im(T) = \mathbb{R}^2$.

# 9 Lecture9 Feb.6 2018

## 9.1 Applications of dimension theorem

**Recall Dimension Theorem** $T : V \to W$ is linear transformation,

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$

**Recall** $T$ is **injective** if and only if $dim(Ker(T)) = 0$.

**Recall** $T$ is **surjective** if and only if $dim(Im(T)) = dim(W)$.

**Example** $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(p(x)) = (p(1), p(2), p(3))$$

Take $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$, $p(x) \in Ker(T)$ iff $T(p(x)) \in \vec{0}$.
Let $p(x) \in Ker(T)$,
Obviously the only solution for the system

$$\begin{cases} a + b + c = 0 \\ a + 2b + 4c = 0 \\ a + 3b + 9c = 0 \end{cases}$$

is $a = b = c = 0$, So $dim(Ker(T)) = 0$. Therefore, $T$ is **injective**.
By *dimension theorem*,

$$dim(V) = 3 = 0 + dim(Im(T)) \implies dim(Im(T)) = 3 = dim(\mathbb{R}^3)$$

therefore $T$ is **surjective**. Therefore, $T$ is called **bijective**.

**Question** $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

$$T(p(x)) = xp'(x)$$

**Solution Not injective** because any *constant function* in $P_n(\mathbb{R})$ is mapped onto $\vec{0} \in P_n(\mathbb{R})$. Also **not surjective** by the dimension theorem.

**Theorem** Let $T : V \to W$ be an injective linear transformation, if $\{\vec{v_i}\}_{i=1}^{k}$ is linearly independent in $V$, then the set $\{T(\vec{v_i})\}_{i=1}^{k}$ is linearly independent in $W$.
*Injective transformation maps linearly independent set to linear independent set.*
*Proof.*
If $\sum c_i T(\vec{v_i}) = \vec{0}$, then we have $T(\sum c_i \vec{v_i}) = \vec{0}$, which means $\sum c_i v_i \in Ker(T)$. By definition of injective transformation, $\sum c_i v_i = \vec{0}$. Since $\{\vec{v_i}\}_{i=1}^{k}$ is linearly independent, so $c_i = 0,\ \forall i$. ∎

**Theorem** $T : V \to W$ is a linearly transformation, $\{\vec{v_i}\}_{i=1}^{n}$ is a basis for $V$ then, if $\{T(\vec{v_i})\}_{i=1}^{n}$ is linear independent, then $T$ is injective.
*A criteria for T to be injective baed on image of a basis.*

*Proof.*

$$\begin{gathered}
\text{Let } \{\vec{v_i}\}_{i=1}^{n} \text{ be a basis of } V \\
\text{Consider } T(\vec{x}) = \vec{0} \\
\text{Since } \{\vec{v_i}\}_{i=1}^{n} \text{ is a basis} \\
\text{Let } x = \sum c_i \vec{v_i} \\
T(\vec{x}) = \vec{0} \iff T(\sum c_i \vec{v_i}) = \vec{0} \\
\implies \sum c_i T(\vec{v_i}) = \vec{0} \implies c_i = 0 \\
\therefore \vec{x} = \sum 0 \vec{v_i} = \vec{0} \\
\text{Therefore } Ker(T) = \{\vec{0}\} \\
\text{Therefore } dim(Ker(T)) = 0 \\
\implies \text{injective}
\end{gathered}$$

∎

**Theorem** Let $T : V \to W$ be a linear transformation,

1. If $dim(V) > dim(W)$, then $T$ cannot be injective.
2. If $dim(V) < dim(W)$, then $T$ cannot be surjective.

*For a linear transformation between spaces with different dimension, it could not be bijective.*

*Proof.*

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$
$$\because dim(Im(T)) \leq dim(W)$$
$$\therefore dim(V) \leq dim(Ker(T) + dim(W)$$
$$\implies dim(Ker(T)) \geq dim(V) - dim(W)$$
$$\implies dim(Ker(T)) > 0$$
$$\text{So } T \text{ could not be injective}$$
$$dim(V) = dim(Ker(T)) + dim(Im(T))$$
$$\because dim(Ker(T)) \geq 0$$
$$\therefore dim(V) \geq dim(Im(T))$$
$$\implies dim(Im(T)) < dim(W)$$
$$\text{So } T \text{ could not be surjective}$$

■

**Theorem *Half is good enough*** Let $T : V \to W$ is linear, and $dim(V) = dim(W)$. $T$ is injective if and only if surjective.

*Proof.*

$$\text{By dimension theorem}$$
$$dim(V) = dim(Ker(T)) + dim(Im(T)) = dim(W)$$
$$\text{If injective} dim(Ker(T)) = 0$$
$$\implies dim(Im(T)) = dim(W)$$
$$\text{So surjective}$$
$$\text{If surjective} dim(Im(T)) = dim(W) = dim(V)$$
$$\implies dim(Ker(T)) = 0$$
$$\text{So injective}$$

■

## 9.2 Isomorphisms

**Recall** If $T : V \to W$ is both injective and surjective, say $T$ is bijective.

**Definition** If $T : V \to W$ is bijective, we call $T$ an **isomorphism**. If there exists an isomorphism $T : V \to W$ say $V$ and $W$ are **isomorphic** vector spaces.

**Theorem** $V, W$ are isomorphic iff $dim(V) = dim(W)$.

*Proof.*

$$\to \; V, W \text{ isomorphic } \implies dim(V) = dim(W)$$

Isomorphic means there exists a bijective transformation $T$

By dimension theorem $dim(V) = dim(Ker(T)) + dim(Im(T))$

$$= 0 + dim(W)$$

$$\leftarrow \; dim(V) = dim(W) \implies V, W \text{ isomorphic}$$

Equivalently, find a bijective transformation

Let $\{\vec{v_i}\}_{i=1}^{n}$ be basis for $V$

Let $\{\vec{w_i}\}_{i=1}^{n}$ be basis for $W$

Claim $T : V \to W$ is linear and $s.t.$

$T(\vec{v_i}) = \vec{w_i}$ is an isomorphism.

If $\vec{x} \in Ker(T) \subseteq V$

$$x = \sum c_i \vec{v_i}$$

$$\vec{0} = T(\vec{x})$$

$$= \sum c_i T(\vec{v_i})$$

$$= \sum (c_i \vec{w_i})$$

$\implies c_i = 0$ since $\vec{w_i}$ are basis.

$$\implies \vec{x} = \vec{0}$$

$$\implies dim(Ker(T)) = 0$$

$\implies$ injective $\iff$ surjective

■

**Note** if $T : V \to W$ is an isomorphism, then $T$ maps a basis for $V$ to a basis for $W$.

**Example** $T : P_2(\mathbb{R}) \to \mathbb{R}^3$,

$$T(p(x)) = (p(1), p(2), p(3))$$

is an isomorphism. And $P_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic.

**Example** $T : P_2(\mathbb{R}) \to \mathbb{R}^3$,

$$T(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ T(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ T(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an isomorphism.

**Example** $M_{2\times 2}(\mathbb{R})$, $P_3(\mathbb{R})$ and $\mathbb{R}^4$ are isomorphic.

**Theorem** Any n-dim vector space $V$ is isomorphic to $\mathbb{R}^n$. What is an isomorphism $T : V \to \mathbb{R}^n$

*Procedure:*

$$\begin{gathered} \text{Let } \{\vec{v_i}\}_{i=1}^{n} \text{ be } \underline{\text{any}} \text{ basis for V} \\ \text{We know that } \forall \vec{x} \in V, \\ \text{By property of basis,} \\ \vec{x} = \sum c_i \vec{v_i} \\ \text{Then } T(\vec{x}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n \text{ is an isomorphism.} \end{gathered}$$

## 9.3 Coordinates

**Definition** Let $V$ be a vector space, $\alpha = \{\vec{v_1}, \dots, \vec{v_n}\}$ be nay basis for $V$, $\forall \vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

then $c1, \dots, c_n$ is called the <u>**coordinates** for $\vec{x}$ relative to $\alpha$</u>, with notation

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \iff \vec{x} = \sum c_i \vec{v_i}$$

**Claim** $[\vec{x}+c\vec{y}]_\alpha = [\vec{x}]_\alpha + c[\vec{y}]_\alpha \quad \forall \vec{x},\vec{y} \in V,\ c \in \mathbb{R}$.

**Note** if $\alpha, \alpha'$ are any two bases for $V$ then generally $[\vec{x}]_\alpha \neq [\vec{x}]_{\alpha'}$ (except $\vec{0}$).

# 10 Lecture10 Feb.7 2018

## 10.1 Matrix of linear transformation

**Recall** Let $V$ be a vector space, let $\alpha$ be any basis for $V$.

$$\forall \vec{x} \in V, x = \sum c_i \vec{v_i}$$

$$[\vec{x}]_\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

So transformation $\vec{x} \to [\vec{x}]_\alpha$ is an isomorphism that $V \to \mathbb{R}^n$.

Say $W$ is a vector space and let $\beta = \{\vec{w_i}\}_1^m$ be any basis of $W$, say $T : V \to W$ is linear.

$$T(\vec{x}) = \sum c_i T(\vec{v_i})$$

So that

$$[T(\vec{x})]_\beta = [\sum c_i T(\vec{v_i})]_\beta = \sum c_i [T(\vec{v_i})]_\beta$$

$$= \begin{bmatrix} [T(\vec{v_1})]_\beta & \dots & [T(\vec{v_n})]_\beta \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$\begin{bmatrix} [T(\vec{v_1})]_\beta & \dots & [T(\vec{v_n})]_\beta \end{bmatrix}$ is called the <u>the matrix of T w.r.t. $\alpha, \beta$.</u> Denoted as $[T]_\alpha^\beta$

$$[T(\vec{x})]_\beta = [T]_\alpha^\beta [\vec{x}]_\alpha$$

**Example** $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$

$$T(p(x)) = xp(x)$$

$$\alpha = \{1-x, 1-x^2, x\},\ \beta = \{1, 1+x, 1+x+x^2, 1-x^3\}$$

Find $[T]_\alpha^\beta$.

$$\begin{gathered}
T(1-x) = x(1-x) = x - x^2 \\
x - x^2 = (-1)(1) + 2(1+x) + (-1)(1+x+x^2) + 0(1-x^3) \\
[T(1-x)]_\beta = (-1, 2, -1, 0) \\
T(1-x^2) = x - x^3 \\
[T(1-x^2)]_\beta = (-2, 1, 0, 1) \\
[T(x)] = x^2 \\
[T(x)]_\beta = (0, -1, 1, 0) \\
[T]_\alpha^\beta = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\end{gathered}$$

**Picture** $V, W$ are vectors spaces, $\alpha = \{\vec{v_1}, \ldots, vecv_n\}$ is a basis for $V$ and $\beta = \{\vec{w_1}, \ldots, vecw_m\}$ is a basis for $W$.

$$\begin{gathered}
V \longrightarrow^T W \\
\downarrow^{[\ ]_\alpha} \qquad \downarrow^{[\ ]_\beta} \\
\mathbb{R}^n \rightarrowtail^{[T]_\alpha^\beta} \mathbb{R}^m
\end{gathered}$$

**Note**

1. $\vec{x} \in Ker(T) \iff T(\vec{x}) = \vec{0} \iff [T(x)]_\beta = [\vec{0}]_\beta \in \mathbb{R}^m \iff [T]_\alpha^\beta [\vec{x}]_\alpha = 0 \iff [\vec{x}]_\alpha \in Ker([T]_\alpha^\beta)$
2. $\vec{w} \in Im(T) \iff [\vec{w}]_\beta \in Col([T]_\alpha^\beta)$

**Theorem(Rank nullity for transformation matrix)**

$$dim(Ker([T]_\alpha^\beta)) + dim(Col([T]_\alpha^\beta)) = n$$

**Example** $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$$T(a + bx + c^2) = \begin{bmatrix} c & -c \\ a-c & a+c \end{bmatrix}$$

And given bases $\alpha = \{x^2-x, x-1, x^2+1\}$ and $\beta = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\}$

**Answer**

$$[T]^\beta_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Nul([T]^\beta_\alpha) = span\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$$

$$Nul(T) = span\{2x\}$$

$$Col([T]^\beta_\alpha) = span\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}\}$$

$$Col(T) = span\{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}\}$$

# 11 lecture11 Feb. 13 2018

## 11.1 Algebra of Transformation

**Recall** Let $T : V \to W$ be a linear transformation, where $\alpha = \{\vec{v_1}, \dots, \vec{v_n}\}$ and $\beta = \{\vec{w_1}, \dots, \vec{w_m}\}$ are bases for $V, W$ respectively.

$$\vec{x} \in Ker(T) \iff [\vec{x}]_\alpha \in Ker([T]^\beta_\alpha)$$

$$\vec{x} \in Im(T) \iff [\vec{x}]_\beta \in Col([T]^\beta_\alpha)$$

**Definition** $T_1, T_2 : V \to W$ are linear transformations, define

$$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x}) \forall \vec{x} \in V$$

$$(cT_1)(\vec{x}) = c(T_1(\vec{x})) \forall \vec{x} \in V,\ c \in \mathbb{R}$$

And, let $\alpha$ and $\beta$ be bases for $V, W$ respectively, then,

$$[T_1]^\beta_\alpha + [T_2]^\beta_\alpha = [T_1 + T_2]^\beta_\alpha$$

$$c[T_1]^\beta_\alpha = [cT_1]^\beta_\alpha$$

**Definition** $T : V \to W$ and $S : W \to U$ are linear transformations, then the **composition** $ST : V \to U$ is defined as

$$(ST)(\vec{x}) = S(T(\vec{x})) \quad \forall \vec{x} \in V$$

**Note** If $S, T$ are *linear* then the composition $ST$ is also *linear*.
*Check*

$$\begin{gathered} \text{Let } a, b \in \mathbb{R},\ \vec{x}, \vec{y} \in V \\ ST(a\vec{x} + b\vec{y}) \\ = S(T(a\vec{x} + b\vec{y})) \\ = S(aT(\vec{x}) + bT(\vec{y})) \\ = a(ST(\vec{x})) + b(ST(\vec{y})) \end{gathered}$$

**Example**
omitted

### 11.2 Matrix of composition

Consider $T : V \to W$ and $S : W \to U$ as linear transformations, let $\alpha,\ \beta,\ \gamma$ be bases of $V,\ W,\ U$ respectively.
We know how to compute $[T]_\alpha^\beta$ and $[S]_\beta^\gamma$. Now want to find $[ST]_\alpha^\gamma$.

$$\begin{gathered} \forall \vec{x} \in V, [ST]_\alpha^\gamma [\vec{x}]_\alpha \\ = [(ST)(\vec{x})]_\gamma \\ = [S(T(\vec{x}))]_\gamma \\ = [S]_\beta^\gamma [T(\vec{x})]_\beta \\ = [S]_\beta^\gamma [T]_\alpha^\beta [\vec{x}]_\alpha \\ \text{This holds true for all } \vec{x} \in V \\ \therefore [ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta \end{gathered}$$

**Conclusion** the matrix of $ST$ = matrix of $S \times$ matrix of $T$.

## 11.3 Inverse transformations

**Definition** $T : V \to W$ is *isomorphism*[15] if and only if there exists function $S : W \to V$ such that

$$(ST)(\vec{v}) = \vec{v}\ \forall \vec{v} \in V \wedge (TS)(\vec{w}) = \vec{w}\ \forall \vec{w} \in W$$

And $S$ is called the **inverse** of $T$, written as $T^{-1}$.

*proof.* $\to$ $T$ is an isomorphism means every vector in $W$ has an unique pre-image in $V$ the function $S : W \to V$ maps *every* vector in $W$ to its *unique* pre-image in $V$, so $S$ is the inverse of $T$.

*proof.* $\leftarrow$ Assume $S : W \to V$ is the inverse of $T : V \to W$ then $T(S(\vec{y})) = \vec{y}\ \forall \vec{y} \in V$, this means $T$ is surjective since every $\vec{y} \in W$ has pre-image under $T$, that's $S(\vec{y}) \in V$. Now suppose $T(\vec{x_1}) = T(\vec{x_2})$, apply transformation $S$ on both sides of the equation, $S(T(\vec{x_1})) = S(T(\vec{x_2}))$ we have $\vec{x_1} = \vec{x_2}$. This implies the transformation is injective. Therefore, transformation $T$ is bijective, that's isomorphism. ■

**Note** $T^{-1}(\vec{y})$ is the unique vector $\vec{x}$, s.t.$T(\vec{x}) = \vec{y}$. That's

$$T(\vec{x}) = \vec{y} \iff T^{-1}(\vec{y}) = \vec{x}$$

[15] Recall that isomorphism is equivalent to bijective.

**Theorem** If $T : V \to W$ is an isomorphism then the inverse of $T$, $T^{-1}$, then $T - 1 : W \to V$ is linear.[16]

*Proof.*

$$\begin{gathered}
\text{WTS } T^{-1}(a\vec{w_1} + b\vec{w_2}) = aT^{-1}(\vec{w_1}) + bT^{-1}(\vec{w_2}) \forall a, b \in \mathbb{R}, \forall \vec{w_1}, \vec{w_2} \in W \\
T^{-1}(\vec{w_1}) \text{ is the unique } \vec{x_1} \ s.t. \ T(\vec{x_1}) = \vec{w_1} \\
T^{-1}(\vec{w_2}) \text{ is the unique } \vec{x_2} \ s.t. \ T(\vec{x_2}) = \vec{w_2} \\
T^{-1}(a\vec{w_1} + b\vec{w_2}) \text{ is the unique } \vec{x} \ s.t. \ T(\vec{x}) = a\vec{w_1} + b\vec{w_2} \\
\because T(\vec{x}) = a\vec{w_1} + b\vec{w_2} \\
= aT(\vec{x_1}) + bT(\vec{x_2}) \\
= T(a\vec{x_1} + b\vec{x_2}) \\
\therefore \vec{x} = a\vec{x_1} + b\vec{x_2} \\
\text{Also } T(\vec{x}) = a\vec{w_1} + b\vec{w_2} \\
\therefore \vec{x} = T^{-1}(a\vec{w_1} + b\vec{w_2}) = a\vec{x_1} + b\vec{x_2} \\
= aT^{-1}(\vec{w_1}) + bT^{-1}(\vec{w_2})
\end{gathered}$$

■

**Theorem** $T : V \to W$ is <u>isomorphism</u>, then let $\alpha$ and $\beta$ are bases of $V$ and $W$ representing then $[T]_\alpha^\beta$ is <u>invertible</u>, and

$$([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$$

*Proof.*
<u>omitted</u>

### 11.4 Change of basis

*What's the effect of a change of basis on coordinate of a vector and matrix of transformation.*

**Theorem** Let $\alpha$ and $\alpha'$ be two bases of $V$, then

$$[I]_\alpha^{\alpha'}[\vec{x}]_\alpha = [\vec{x}]_{\alpha'}$$

[16]Note: the conclusion could be changed into isomorphism.

*Proof.*

$$\text{Let } \vec{x} \in V$$
$$I(\vec{x}) = \vec{x}$$
$$[I(\vec{x})]_{\alpha'} = [\vec{x}]_{\alpha'}$$
$$[I]_{\alpha}^{\alpha'}[\vec{x}]_{\alpha} = [\vec{x}]_{\alpha'}$$

■

$[I]_{\alpha}^{\alpha'}$ is called the change of basis matrix from $\alpha$ to $\alpha'$.

**Computation** Let $\alpha = \{\vec{a_1}, \ldots, \vec{a_n}\}$, then

$$[I]_{\alpha}^{\alpha'} = [[\vec{a_1}]_{\alpha'} \mid \cdots \mid [\vec{a_n}]_{\alpha'}]$$

## 12 Lecture12 Feb. 14 2018

**Recall** Let $\alpha$ and $\beta$ be bases for $V$ and $I : \to V$ is the identity transformation, then

$$[I]_{\alpha}^{\beta}[\vec{x}]_{\alpha} = [\vec{x}]_{\beta}$$

Also,

$$[I]_{\beta}^{\alpha}[\vec{x}]_{\beta} = [\vec{x}]_{\alpha}$$

**Example** Let $\alpha = \{x^2, 1+x, x+x^2\}$ and $\beta$ be bases for $P_2(\mathbb{R})$ and

$$[I]_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } [p(\vec{x})]_{\beta} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

**Solution** <u>omitted</u>

**Theorem** Suppose $[T]_{V}^{W}$ is linear, $\alpha$ and $\alpha'$ are any two bases for $V$ and $\beta$ and $\beta'$ are any two bases of $W$, then,

$$[T]_{\alpha'}^{\beta'} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha}$$

*Proof.*

$$\textbf{Recall } T = ITI$$

$$\text{Consider let } \vec{x} \in V$$

$$\begin{aligned} &[I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha}[\vec{x}]_{\alpha'} \\ &= [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[\vec{x}]_{\alpha} \\ &= [I]_{\beta}^{\beta'}[T(\vec{x})]_{\beta} \\ &= [T(\vec{x})]_{\beta'} \\ &= [T]_{\beta'}^{\alpha'}[\vec{x}]_{\alpha'} \\ \implies &[T]_{\beta'}^{\alpha'} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha'}^{\alpha} \end{aligned}$$

■

Also,

$$[T]_{\alpha}^{\beta} = [I]_{\beta'}^{\beta}[T]_{\alpha'}^{\beta'}[I]_{\alpha}^{\alpha'}$$

**Special Case** Consider when $V = W$, $\alpha = \beta$ and $\alpha' = \beta'$. we have

$$[T]_{\alpha'}^{\alpha'} = [I]_{\alpha}^{\alpha'}[T]_{\alpha}^{\alpha}[I]_{\alpha'}^{\alpha}$$

where

$$([I]_{\alpha}^{\alpha'})^{-1} = [I]_{\alpha'}^{\alpha}$$

the equation becomes

$$[T]_{\alpha'}^{\alpha'} = ([I]_{\alpha}^{\alpha'})^{-1}[T]_{\alpha}^{\alpha}[I]_{\alpha'}^{\alpha}$$

and can be written in the form of

$$B = P^{-1}AP$$

**Definition** Two matrices $A$ and $B$ are **similar** if there exists an invertible matrix $P$ s.t.

$$B = P^{-1}AP$$

$A$ and $B$ representing the same transformation relative to different bases and $P$ is the change of basis matrix if and only if $A$ and $B$ are similar.

**Example** Omitted