CSC165 Lecture notes

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1.1 Predicate Logic

Allow you to have a domain of objects that you want to talk about, we want to be express reason about this domain.

Predicate Simplest kind of Predicate Logical Formula is called a **predicate**. A predicate is a function with range $\{0, 1\}$.
Examples

1. *Less-than-or-equal-to:* $\leq: \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$
2. *Equality:* $=: \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$
3. *Prime:* $Prime : \mathbb{N} \to \{0, 1\}$
4. Define: $R : \{a, b\} \times \{1, 2, 3\} \to \{0, 1\}$ as $R(a, 1) = R(b, 1) = R(c, 1) = 1, R = 0$ elsewise.

When you specify/define a predicate you have to specify the domain.

Quantifiers Introduce two quantifiers, **exists**: $\exists$ and **for all**: $\forall$, that let us express,

$\exists x \in \mathbb{DOMAIN}$: There is at least one element in domain of predicate that is true.

equivalently, represent as $\vee$,

$$"\exists" \equiv p(x_0) \vee p(x_1) \vee p(x_2) \ldots$$

$\forall x \in \mathbb{DOMAIN}$: All element in domain of predicate satisfy the predicate.

equivalently, represented as $\wedge$,

$$"\forall" \equiv p(x_0) \wedge p(x_1) \wedge p(x_2) \ldots$$

Negation of quantifier statements

$$\neg(\exists x \in \mathbb{D},\ s.t.\ p(x)) \equiv \forall x \in \mathbb{D}, \neg p(x)$$

$$\neg(\forall x \in \mathbb{D},\ s.t.\ p(x)) \equiv \exists x \in \mathbb{D}, \neg p(x)$$

Nested quantifier more than one variable quantified.
Example

For every natural number x, if x is a power of 2 then $2x$ is a power of 2.

$$\forall x \in \mathbb{N}, (\exists k \in \mathbb{N}\ s.t.\ x = 2^k) \implies (\exists k' \in \mathbb{N}\ s.t.\ 2^{k'} = 2x)$$

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Another example There are infinitely many natural number that are even.

$$\text{Even(...)}:\ \exists y \in \mathbb{N},\ x = 2y$$

$$\forall x \in \mathbb{N},\ \exists y\ s.t.\ y > x \wedge Even(y)$$

$$Q(x):\ \text{some predicate}$$
$$Q(x): \mathbb{N} \to \{0, 1\}$$
$$\forall x \in \mathbb{N}[Q(x) \implies \exists y \in \mathbb{N}[y > x \wedge Q(y)]]$$

Does this express *there are infinitely many numbers that satisfy Q*?
Not, consider a Q that is false for all x, the statement is vacuous truth, but it does not express what we want.
Fix it.

$$\exists z \in \mathbb{N}[Q(z)] \wedge \forall x \in \mathbb{N} \exists y \in \mathbb{N}[y > x \wedge Q(y)]$$

To ensure that there are some elements satisfy Q

Tips

1. Make sure you write down a predicate logic formula with **correct syntax**.
2. Use different variable names for different quantities.
3. When you quantify ($\exists x$ *or* $\forall y$) you must specify the **set**. e.g. We want to quantify over all $x \in \mathbb{N}$ such that $x \geq 5$.
   (a) Method 1: Predefine your set.
   $$\text{Let } S = \{x \mid x \in \mathbb{N},\ x \geq 5\} \quad \forall x \in S, x \geq 3$$
   (b) Method 2: By implication.
   $$\forall x \in \mathbb{N}, \quad x \geq 5 \implies x \geq 3$$

Note We will spend a lot of time defining new predicates using predicate logic and then reasoning about them.

Example: Divisibility Let $n, d \in \mathbb{Z}$, we say that d **divides** n or n is **divisible** by d iff

$$\exists k \in \mathbb{Z}\ s.t.\ n = d * k$$

$$Divides(d, n) : \exists k \in \mathbb{Z}\ [n = d * k]$$

this formula is a predicate logic, some of variables (k) are quantified and others (n, d) are not quantified, called **free variable**. This formula since it has free variables, it represents a predicate.
A formula with no free variable is called a **sentence**, sentences are true or false.

Let's express *For every integer x, if x divides 10 then it also divide 100.*

$$\forall x \in \mathbb{Z}\ [Divides(x, 10) \implies Divides(x, 100)]$$

equivalently,

$$\forall x \in \mathbb{Z}\ [\exists k \in \mathbb{Z}[10 = k * x] \implies \exists k' \in \mathbb{Z}[100 = k' * x]]$$

Note Proposition is a special type of predicate but it's **not** a sentence.

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3.1 Introduction to proofs

Proof A **proof** is a logical argument that convinces another person that a statement is <u>true</u>. Can also have a **disproof** showing that a statement if <u>false</u>.

1. Write down what we want to prove using language of first order logic.
2. Introducing variable(s).
3. Write body of proof.

Example Prove that every natural number n satisfies the inequality $n^2+3n+7 \geq 4$,

Statement in first order logic:

$$\forall n \in \mathbb{N}, n^2 + 3n + 7 \geq 4$$

proof:

Introducing variable(s):

Let $n \in \mathbb{N}$

Body of proof:

Since $n \in \mathbb{N},\ n \geq 0$

Therefore $n^2 \geq 0$

Similarly, $since n \in \mathbb{N}, n \geq 0, 3n \geq 0$

$\therefore n^2 + 3n + 7 \geq 0 + 0 + 4$

$\blacksquare$

Example Prove that for every natural number n greater than 20, n satisfies $1.5n - 4 \geq 3$.

$$\forall n \in \mathbb{N}, [\ n > 20 \implies 1.5n - 4 > 3\ ]$$

proof.

Let $n \in \mathbb{N},$ assume that $n > 20$

Since $n > 20,\ 1.5n > 1.5(20) = 30$

So $1.5n > 30$

$\therefore 1.5n - 4 > 26$

$\therefore 1.5n - 4 > 3$

$\blacksquare$

(More complex)Example Define a natural number to be a **Prime** numbers:

$$Divides(x, n) : \exists xk = n$$

$$Prime(n) : (n > 1) \land (\forall x \in \mathbb{N}, ((x \neq 1 \land x \neq n) \implies \neg Divides(x, n)))$$

Equivalently, take <u>contrapositive</u>

$$Prime(n) : (n > 1) \land (\forall x \in \mathbb{N}, Divides(x, n) \implies (x = 1 \lor x = n))$$

Example For every integer x, $x|x + 1$ then $x|5$

$$\forall x \in \mathbb{Z}, [\ Divides(x, x + 5) \implies Divides(x, 5)\ ]$$

$$\forall x \in \mathbb{Z}, [\ (\exists k \in \mathbb{Z}\ s.t.\ xk = x + 5) \implies (\exists k' \in \mathbb{Z}\ s.t.\ xk' = x)\ ]$$

proof.

$$\begin{gathered}
\text{Let } x \in \mathbb{Z} \\
\text{Assume } k \in \mathbb{Z} \text{ is such that } xk = x + 5 \\
\text{Let } k' = k - 1 \\
\text{Then, } k'x = (k-1)x \\
= kx - x \\
\text{By assumption,} \\
= 5 \\
\text{Therefore, } \exists k' \in \mathbb{Z},\ s.t.\ k'x = 5
\end{gathered}$$

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