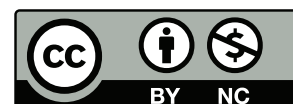


# Introduction to Real Analysis

Tianyu Du

Tuesday 24<sup>th</sup> September, 2019

This work is licensed under a Creative Commons “Attribution-NonCommercial 4.0 International” license.



## Contents

|          |  |          |
|----------|--|----------|
| <b>1</b> | <b>The Axiom of Completeness</b>       | <b>2</b> |
| 1.1      | Preliminaries . . . . .                | 2        |
| 1.2      | Density of Rational Numbers . . . . .  | 4        |
| <b>2</b> | <b>Sequences</b>                       | <b>4</b> |
| 2.1      | Monotone Convergence Theorem . . . . . | 7        |
| 2.2      | Series . . . . .                       | 8        |
| 2.3      | Cauchy Criterion . . . . .             | 9        |
| 2.4      | Convergence Test for Series . . . . .  | 10       |
| 2.4.1    | The Comparison Test . . . . .          | 11       |
| 2.4.2    | The Root Test . . . . .                | 12       |

# 1 The Axiom of Completeness

## 1.1 Preliminaries

**Definition 1.1.** A set  $A \subset \mathbb{R}$  is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

**Example 1.1.** The set of integers,  $\mathbb{Z}$ , is neither bounded from above nor below. Sets  $\{1, 2, 3\}$  and  $\{\frac{1}{n} : n \in \mathbb{N}\}$  are bounded from both above and below.

**Notation 1.1.** Let  $A \subset \mathbb{R}$ , we use  $A^\uparrow$  and  $A^\downarrow$  to denote collections of upper bounds of  $A$  and lower bounds of  $A$ . Note that  $A^\uparrow$  and  $A^\downarrow$  are potentially empty.

**Definition 1.2.** A real number  $s \in \mathbb{R}$  is the **least upper bound (supremum)** for a set  $A \subset \mathbb{R}$  if  $s \in A^\uparrow$  and  $\forall u \in A^\uparrow, s \leq u$ . Such  $s$  is denoted as  $s := \sup A$ .

**Definition 1.3.** A real number  $f \in \mathbb{R}$  is the **greatest lower bound (infimum)** for  $A$  if  $f \in A^\downarrow$  and  $\forall l \in A^\downarrow, l \leq f$ . Such  $f$  is often written as  $f := \inf A$ .

**Axiom 1.1** (The Axiom of Completeness/Least Upper Bounded Property).  $\forall \emptyset \neq A \subset \mathbb{R}$  such that  $A^\uparrow \neq \emptyset$ ,  $\exists \sup A$ .

**Definition 1.4.** Let  $\emptyset \neq A \subset \mathbb{R}$ ,  $a_0 \in A$  is the **maximum** of  $A$  if  $\forall a \in A, a_0 \geq a$ ;  $a_1 \in A$  is the **minimum** of  $A$  if  $\forall a \in A, a_1 \leq a$ .

**Example 1.2.**  $\mathbb{Q} \subset \mathbb{R}$  does not satisfy the axiom of completeness.

**Proposition 1.1.** Let  $\emptyset \neq A \subset \mathbb{R}$  bounded above, and  $c \in \mathbb{R}$ . Define  $c + A := \{a + c : a \in A\}$ . Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

*Proof. Step 1: Show  $c + \sup A \in (c + A)^\uparrow$ :*

Let  $x \in c + A$ ,  $\exists a \in A$  s.t.  $x = c + a$ . Then,  $x = c + a \leq c + \sup A$ . Therefore,  $x \leq c + \sup A \forall x \in c + A$ , which implies what desired.

*Step 2: Show  $\forall u \in (c + A)^\uparrow, c + \sup A \leq u$ :*

Let  $u \in (c + A)^\uparrow$ , then  $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$ .

Hence,  $\sup(c + A) = c + \sup A$ . ■

**Lemma 1.1** (Alternative Definition of Supremum). Let  $s \in A^\uparrow$  for some nonempty  $A \subset \mathbb{R}$ . The following statements are equivalent:

- (i)  $s = \sup A$ ;
- (ii)  $\forall \varepsilon, \exists a \in A, \text{ s.t. } a > s - \varepsilon$  (i.e.  $s - \varepsilon \notin A^\uparrow$ ).

*Proof.* Immediately. ■

**Theorem 1.1** (Nested Interval Property). Let  $(I_n)_n$  be a sequence of closed intervals  $I_n := [a_n, b_n]$  such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \quad \forall n \in \mathbb{N} \quad (1.4)$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.5)$$

*Proof.* Note that the sequence  $(a_n)_{n \in \mathbb{N}}$  is bounded above by any  $b_k$ , by the completeness axiom, there exists  $a^* := \sup_{n \in \mathbb{N}} a_n$ . Since  $a^* \in (a_n)^\uparrow$ ,  $a^* \geq a_n \quad \forall n \in \mathbb{N}$ . Further, because  $a^*$  is the *least* upper bound, then for every upper bound  $b_n$ , it must be  $a^* \leq b_n \quad \forall n \in \mathbb{N}$ . Therefore,  $x^* \in [a_n, b_n] \quad \forall n \in \mathbb{N}$ . That is,  $x^* \in \bigcap_{n \in \mathbb{N}} I_n$ . ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold:  $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$ .

**Theorem 1.2** (Archimedean Property).

- (i)  $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$ ;
- (ii)  $\forall y \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < y$ .

Archimedean property of *natural numbers* can be interpreted as *there is no real number that bounds  $\mathbb{N}$* . This interpretation can be seen by considering the negations of above statements:

- (i)  $\exists x \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, n \leq x$ ;
- (ii)  $\exists y \in \mathbb{R}_{++} \text{ s.t. } \forall n \in \mathbb{N}, y \leq \frac{1}{n}$ .

*Proof of (i) by Contradiction.* Suppose the negated statement (i) is true,  $\mathbb{N}$  is bounded above. By the completeness axiom, there exists  $a^* := \sup \mathbb{N}$ .  $\exists n \in \mathbb{N} \text{ s.t. } a^* - 1 < n$ . In this case,  $a^* < n+1 \in \mathbb{N}$ , which means  $a^* \notin \mathbb{N}^\uparrow$  and leads to a contradiction. ■

*Proof of (ii).* Let  $y^* \in \mathbb{R}_{++}$ , take  $x = \frac{1}{y}$ . By statement (i), there exists  $n^* \in \mathbb{N}$  such that  $n > \frac{1}{y}$ . Because  $y > 0$ ,  $\frac{1}{n} < y$ . ■

## 1.2 Density of Rational Numbers

**Theorem 1.3.** For every  $a, b \in \mathbb{R}$  such that  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .

The above theorem says  $\mathbb{Q}$  is in fact **dense** in  $\mathbb{R}$ . More generally, one says a set  $A \subset X$  is dense whenever the closure of  $A$ ,  $\overline{A} = X$ .

*Proof. Step 1:* Since  $b - a > 0$ , by the first Archimedean property, there exists  $n \in \mathbb{N}$  such that  $n > \frac{1}{b-a}$ . Such natural number satisfies  $\frac{1}{n} < b - a$ .

*Step 2:* Let  $m$  be smallest integer such that  $m > an$ . That is,  $m - 1 \leq an < m$ . Obviously,  $a < \frac{m}{n}$  since  $n > 0$ . Further, since  $m \leq an + 1$ , with results from step (i),  $m < bn - 1 + 1 = bn$ , and  $\frac{m}{n} < b$ . Therefore  $\frac{m}{n} \in (a, b)$ . ■

**Theorem 1.4.**  $\exists \alpha \in \mathbb{R}$  s.t.  $\alpha^2 = 2$ .

*Proof.* Let  $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$ , which is obviously a set in  $\mathbb{R}$  bounded from above. By the completeness axiom,  $\Omega$  possesses a supremum, and we claim  $\alpha := \sup \Omega$  satisfies  $\alpha^2 = 2$ . Suppose  $\alpha^2 > 2$ , then there exists  $\varepsilon > 0$  such that  $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$ . Therefore,  $\alpha > \alpha - \varepsilon \in \Omega^\uparrow$ , which contradicts the fact that  $\alpha$  is the least upper bound. Suppose  $\alpha^2 < 2$ , then there exists some  $\varepsilon > 0$  such that  $\alpha + \varepsilon \in \Omega$ , which contradicts the assumption that  $\alpha$  is an upper bound. Hence, it must be the case that  $\alpha^2 = 2$ . ■

## 2 Sequences

**Theorem 2.1** (Triangle Inequality). Let  $a, b \in \mathbb{R}$ , then  $|a + b| \leq |a| + |b|$ .

**Corollary 2.1.** Let  $a, b \in \mathbb{R}$ , then

$$||a| - |b|| \leq |a - b| \quad (2.1)$$

*Proof.* Note that  $|a| = |a - b + b| \leq |a - b| + |b|$ , which implies  $|a| - |b| \leq |a - b|$ . And  $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$ , which implies  $|b| - |a| \leq |a - b|$ . Therefore, by taking the absolute value,  $||a| - |b|| \leq |a - b|$ . ■

**Definition 2.1.** A sequence  $(a_n) \subset \mathbb{R}$  **converges** to  $a \in \mathbb{R}$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |a_n - a| < \varepsilon \quad (2.2)$$

Let  $a \in \mathbb{R}$  and  $\varepsilon > 0$ , the open ball centred at  $a$  with radius  $\varepsilon$  is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \quad (2.3)$$

**Theorem 2.2.** The limit of any convergent sequence is unique.

*Proof.* Let  $(a_n)$  be a convergent sequence, assume, for contradiction, that  $(a_n) \rightarrow L_1$  and  $(a_n) \rightarrow L_2$  such that  $L_1 \neq L_2$ . Let  $\varepsilon = \frac{|L_1 - L_2|}{3}$ , because  $(a_n) \rightarrow L_1$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$ . Therefore, for every  $n \geq N$ ,

$$|a_n - L_2| = |a_n - L_1 - (L_2 - L_1)| \quad (2.4)$$

$$\geq ||a_n - L_1| - |L_2 - L_1|| \quad (2.5)$$

$$= ||L_1 - L_2| - |a_n - L_1|| \quad (2.6)$$

$$= 3\varepsilon - |a_n - L_1| \quad (2.7)$$

$$> 2\varepsilon \quad (2.8)$$

Therefore, there does not exist any  $N' \in \mathbb{N}$  such that  $|a_n - L_2| < \varepsilon$  for every  $n \geq N$ . ■

**Definition 2.2.** A sequence  $(a_n)$  is **divergent** if it does not converge.

**Example 2.1.** The sequence  $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \dots)$  is divergent.

*Proof.* Let  $\varepsilon := \frac{2}{5 \times 3}$ , assume, for contradiction, that  $(a_n) \rightarrow L$  for some  $L \in \mathbb{R}$ . Then there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $|a_n - L| < \frac{2}{15}$ . Since the sequence is alternating, it must be the case that  $|L - \frac{1}{5}| < \frac{2}{15}$ . Similarly,

$$\left| -\frac{1}{5} - L \right| = \left| \frac{1}{5} + L \right| \quad (2.9)$$

$$= \left| \frac{1}{5} + L - \frac{1}{5} + \frac{1}{5} \right| \quad (2.10)$$

$$= \left| \left( L - \frac{1}{5} \right) - \left( -\frac{2}{5} \right) \right| \quad (2.11)$$

$$\geq \left| \left| L - \frac{1}{5} \right| - \frac{6}{15} \right| \quad (2.12)$$

$$= \frac{6}{15} - \left| L - \frac{1}{5} \right| \quad (2.13)$$

$$> \frac{4}{15} \quad (2.14)$$

$$> \varepsilon \quad (2.15)$$

the strict inequality suggests there cannot be a  $M \in \mathbb{N}$  such that  $|a_n - L| < \varepsilon$  for every  $n \geq M$ . ■

*Alternative Proof.* If  $(a_n)$  is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of  $(a_n)$  converging to  $\frac{1}{5}$  and  $-\frac{1}{5}$  respectively, this leads to a contradiction. ■

**Definition 2.3.** A sequence is **bounded** if  $\exists M \in \mathbb{R}$  such that  $\forall n \in \mathbb{N}$ ,  $|a_n| < M$ .

**Theorem 2.3.** Every convergent sequence is bounded.

*Proof.* Let  $(a_n) \rightarrow L$ , take  $\varepsilon = 1$ , then there exists  $N \in \mathbb{N}$  such that  $|a_n - L| < 1$  for every  $n > N$ . Note that  $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$ , which implies  $|a_n| < |L| + 1$ . Let  $Q := \max_{n < N} a_n$ , take  $M := \max\{Q, |L| + 1\}$ , then  $M$  bounds  $(a_n)$ . ■

**Theorem 2.4** (Algebraic Limit Theorem). Let  $(a_n) \rightarrow a, (b_n) \rightarrow b$  be convergent sequences, and  $c \in \mathbb{R}$ , then

- (i)  $(ca_n) \rightarrow ca$ ;
- (ii)  $(a_n + b_n) \rightarrow a + b$ ;
- (iii)  $(a_nb_n) \rightarrow ab$ ;
- (iv)  $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$ , provided  $b_n, b \neq 0$ .

*Proof (i).* Let  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $\forall n \geq N, |a_n - a| < \frac{\varepsilon}{|c|}$ . Then, for every  $n \geq N$ ,  $|ca_n - ca| = |c||a_n - a| < \varepsilon$ . ■

*Proof (ii).* Let  $\varepsilon > 0$ , there exists  $N_1, N_2 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\varepsilon}{3} \forall n \geq N_1$  and  $|b_n - b| < \frac{\varepsilon}{3} \forall n \geq N_2$ . Take  $N := \max\{N_1, N_2\}$ , let  $n \geq N$ ,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \quad (2.16)$$

■

*Proof (iii).* Note that

$$|a_nb_n - ab| = |a_nb_n + a_nb - a_nb - ab| \quad (2.17)$$

$$\leq |a_nb_n - a_nb| + |a_nb - ab| \quad (2.18)$$

$$\leq |a_n||b_n - b| + |b||a_n - a| \quad (2.19)$$

Let  $N_1 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\varepsilon}{2|b|}$  for every  $n \geq N_1$ . Because  $(a_n)$  is convergent, let  $M$  denote its bound such that  $|a_n| < M \forall n \in \mathbb{N}$ . Let  $N_2 \in \mathbb{N}$  such that  $|b_n - b| < \frac{\varepsilon}{2M}$ . Then for every  $n \geq N_3 := \max\{N_1, N_2\}$ ,  $|a_nb_n - ab| < \varepsilon$ . ■

*Proof (iv).* **Claim i:** when  $n$  is sufficiently larger,  $|b_n| > 0$  is bounded away from zero by  $M$ . Let  $\varepsilon = \frac{|b|}{10}$ , then there exists  $N_1 \in \mathbb{N}$  such that for every  $n \geq N_1$ ,  $|b_n - b| < \frac{|b|}{10}$ . Note that for every such  $n$ ,

$$|b_n| = |b_n - b - (-b)| \quad (2.20)$$

$$\geq ||b_n - b| - |b|| \quad (2.21)$$

$$\geq |b| - |b_n - b| \quad (2.22)$$

$$> \frac{9|b|}{10} \quad (2.23)$$

**Claim ii:**  $\left(\frac{1}{b_n}\right) \rightarrow \frac{1}{b}$ . Let  $\varepsilon > 0$ , note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_nb} - \frac{b_n}{b_nb}\right| \quad (2.24)$$

$$= \frac{1}{|b_n||b|} |b_n - b| \quad (2.25)$$

from the first claim,  $\frac{1}{|b_n|} < \frac{10}{9|b|}$  for every  $n \geq N_1$ . Since  $(b_n) \rightarrow b$ , there exists  $N_2 \in \mathbb{N}$  such that for every  $n \geq N_2$ ,  $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$ . Consequently, for every  $n \geq N_3 := \max\{N_1, N_2\}$ ,  $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$ . Then the result is immediate from property (iii) in the algebraic limit theorem. ■

**Theorem 2.5** (Order Limit Theorem). Let  $(a_n) \rightarrow a$  and  $(b_n) \rightarrow b$ , then

- (i)  $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$ ;
- (ii)  $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$ ;
- (iii)  $\exists c \in \mathbb{R} \text{ s.t. } c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$ ;
- (iv)  $\exists c \in \mathbb{R} \text{ s.t. } a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$ .

*Proof.* (i) Assume, for contradiction,  $a < 0$ . Take  $\varepsilon = \frac{|a|}{2}$ , then for some  $N \in \mathbb{N}$ , for every  $n \geq N$   $a_n \in V_\varepsilon(a)$ . However, this contradicts the fact that  $a_n \geq 0$ .

(ii) Consider sequence  $(b_n - a_n)$  in which  $b_n - a_n \geq 0$  for every  $n \in \mathbb{N}$ .  $(b_n - a_n) \rightarrow (b - a)$  by the algebraic limit theorem. By property (i),  $b - a \geq 0$ .

(iii) and (iv) Consider constant sequence defined as  $(c_n)$  such that  $c_n = c$  for every  $n \in \mathbb{N}$ , the results are immediate by applying (ii). ■

**Theorem 2.6** (Squeeze Theorem). Let  $(x_n) \rightarrow L$  and  $(z_n) \rightarrow \ell$ . If for every  $n \in \mathbb{N}$ ,  $x_n \leq y_n \leq z_n$ , then  $(y_n) \rightarrow \ell$ .

*Proof.* Let  $\varepsilon > 0$ , because both  $(x_n) \rightarrow \ell$  and  $(y_n) \rightarrow \ell$ ,

$$\exists N_1 \text{ s.t. } n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \quad (2.26)$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \quad (2.27)$$

Take  $N_3 := \max\{N_1, N_2\}$ , then for every  $n \geq N_3$ ,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \quad (2.28)$$

$$\implies y_n \in V_\varepsilon(\ell) \quad (2.29)$$

therefore  $(y_n) \rightarrow \ell$  by definition. ■

## 2.1 Monotone Convergence Theorem

**Definition 2.4.** A sequence  $(a_n)$  is said to be **monotone** if it is either increasing ( $a_{n+1} \geq a_n \ \forall n \in \mathbb{N}$ ) or decreasing ( $a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$ ).

**Theorem 2.7** (Monotone Convergence Theorem). If a sequence  $(a_n)$  is bounded, then it converges.

*Proof.* WLOG, assume  $(a_n)$  is increasing, let  $\Gamma := \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$ , because  $\Gamma$  is bounded,  $s := \sup_n \Gamma$  is well-defined by the completeness of real numbers.

**Claim:**  $(a_n) \rightarrow s$ . Let  $\varepsilon > 0$ , by the definition of supremum,  $\exists N \in \mathbb{N}$  such that  $a_N > s - \varepsilon$ . Because the sequence is increasing and  $s + \varepsilon \in \Gamma^\uparrow$ ,  $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$ .  $(a_n) \rightarrow s$  by definition. ■

## 2.2 Series

**Definition 2.5.** Let  $(a_i)$  be a sequence, then the  $n$ -th **partial sum** is defined as  $s_n := \sum_{i=1}^n a_i$ . And the **infinite sum/series** of  $(a_n)$  is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \rightarrow s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \quad (2.30)$$

**Example 2.2.**  $\sum_{i=1}^{\infty} \frac{1}{i^2}$  converges.

*Proof.* Obviously the corresponding partial sums are increasing because the sequence  $(\frac{1}{i^2})$  is positive.

**Claim:**  $(s_n)$  is bounded from above. Let  $n \in \mathbb{N}$ , observe

$$\sum_{i=1}^n \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \quad (2.31)$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \quad (2.32)$$

$$= 2 - \frac{1}{n} \leq 2 \quad (2.33)$$

The result is immediate by the monotone convergence theorem. ■

**Example 2.3** (Harmonic Series).  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

*Proof.* **Claim:** there exists a subsequence of  $(s_n)$  diverges, so  $(s_n)$  cannot be convergent. Consider the subsequence  $(s_k)$  constructed by defining  $s_k := s_{2^k}$ . Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \quad (2.34)$$

$$> 1 + \frac{1}{2}k \quad (2.35)$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. ■

**Definition 2.6.** Let  $(a_n)$  be a sequence, then for every strictly increasing sequence  $(n_i)_i$  in  $\mathbb{N}$ ,  $(a_{n_i})$  is a **subsequence** of  $(a_n)$ .

**Theorem 2.8.** All subsequences of a convergent sequence converge to the same limit as the original sequence.



*Proof.* Let  $(a_n) \rightarrow \ell$ , let  $(a_{n_k})$  be a subsequence of  $(a_n)$ . Let  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies a_n \in V_\varepsilon(\ell)$ . By the definition of subsequences, there exists some  $K \in \mathbb{N}$  such that  $n_K = N$ . Take such  $K$ , then for every  $k \geq K$ , it must be  $n_k \geq N$ . Therefore  $a_{n_k} \in V_\varepsilon(\ell)$  for every  $k \geq K$ , and  $(a_{n_k}) \rightarrow \ell$  by definition. ■

**Corollary 2.2.** A sequence  $(a_n)$  must be divergent if there exists two subsequences of it converge to two different limits.

*Proof.* Immediate by taking the contrapositive form of above theorem. ■

**Theorem 2.9** (Bolzano–Weierstrass). Every bounded sequence contains a convergent subsequence.

*Proof.* Suppose  $(a_n)$  is bounded by certain  $M > 0$ , that's, for every  $n \in \mathbb{N}$ ,  $-M < a_n < M$ . Consider the split  $I_1^\ell := [-M, 0]$  and  $I_1^u := [0, M]$ . At least one of above closed intervals contain an infinitely many elements of  $(a_n)$ . Define the interval as  $I_2$ . At each  $I_n$ , one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and  $I_{n+1}$  is defined to be such sequence. Note that the sequence of closed intervals constructed from above recursive procedure is in fact nested. Obviously  $\lim_{n \rightarrow \infty} |I_n| = 0$ . Further, by the nested interval property, one can show that  $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$ . Then  $\cap_{n \in \mathbb{N}} I_n$  must be a singleton with  $a$  in it. One can construct such that  $a_{n_k} \in I_k$ . Note that  $|I_n| = \frac{1}{2^{n-1}}$ , therefore, for every  $\varepsilon > 0$ , one can take  $N \geq \log_2 \left( \frac{1}{\varepsilon} \right) + 1$ , so that for every  $k \geq N$ , by definition of subsequences,  $n_k \geq n$ , so that  $a_{n_k}, a \in I_N$ . This implies  $a_{n_k} \in V_\varepsilon(a)$  and  $(a_{n_k}) \rightarrow a$ . ■

### 2.3 Cauchy Criterion

**Definition 2.7.** A sequence  $(a_n)$  is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n \geq N \implies |a_n - a_m| < \varepsilon \quad (2.36)$$

**Proposition 2.1.** Every convergent sequence is Cauchy.

*Proof.* Let  $(a_n) \rightarrow \ell$ , let  $\varepsilon > 0$ . By the convergence of sequence,  $\exists N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $|a_n - \ell| < \frac{\varepsilon}{2}$ , which turns out to imply  $a_n, a_m \in V_\varepsilon(\ell)$ . ■

**Lemma 2.1.** Every Cauchy sequence is bounded.

*Proof.* Let  $(a_n)$  be a Cauchy sequence, take  $\varepsilon = 1$ , then there exists  $N \in \mathbb{N}$  such that for every  $m, n \geq N$ ,  $|a_n - a_m| < 1$ . In particular, take  $m = N$ , for every  $n \geq N$ ,  $|a_n - a_N| < 1$ , and  $|a_n| \leq |a_N| + 1$ . Then  $(a_n)$  is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \quad (2.37)$$

■

**Theorem 2.10** (Cauchy Criterion). A sequence of real numbers is convergent if and only if it's Cauchy.

*Proof.* ( $\Leftarrow$ ) Suppose  $(a_n)$  is Cauchy, by the lemma established above,  $(a_n)$  is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence  $(a_{n_k}) \rightarrow \ell$ .

**Claim:**  $(a_n) \rightarrow \ell$ . Let  $\varepsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that for every  $n_k, n \geq N_1$ ,  $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$ . And there exists another  $N_2 \in \mathbb{N}$  such that for every  $n_k \geq N_2$ ,  $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$ . Take  $N_3 := \max\{N_1, N_2\}$ . Note that for every  $n \geq N_3$ , one can choose some  $n_k \geq n$  and derive

$$|a_n - \ell| = |a_n - a_{n_k} + a_{n_k} - \ell| \quad (2.38)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| \quad (2.39)$$

$$< \varepsilon \quad (2.40)$$

( $\Rightarrow$ ) Already shown in previous proposition. ■

## 2.4 Convergence Test for Series

**Theorem 2.11** ( $n$ -th term test; necessary condition for convergent series). Series  $\sum_{i=1}^{\infty} a_n$  converges  $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* Suppose the partial sums converges to  $\ell$ , by the definition of partial sums,  $a_n = s_{n+1} - s_n$ . Further, the convergence of partial sums guarantees the convergence of  $(a_n)$ . By taking limit on both sides of above identity, it can be shown  $\lim_{n \rightarrow \infty} a_n = 0$ . ■

**Theorem 2.12** (Cauchy Criterion for Series). For series  $\sum_{n=1}^{\infty} a_n$ , TFAE:

- (i) Series converges;
- (ii)  $\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$  (i.e. tail sum sequence converges);
- (iii)  $\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m > n \geq N, \left| \sum_{k=n+1}^m a_k \right| < \varepsilon$ . (i.e. partial sum is Cauchy)

*Proof.* (i)  $\Rightarrow$  (ii): Suppose  $(S_n)$  converges, let  $\varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |S_n - L| < \varepsilon$ . Note that

$$L - S_n = \lim_{m \rightarrow \infty} \sum_{k=1}^m a_k - S_n \quad (2.41)$$

$$= \lim_{m \rightarrow \infty} \left[ \sum_{k=1}^m a_k - S_n \right] \quad (2.42)$$

$$= \lim_{m \rightarrow \infty} \sum_{k=n+1}^m a_k \quad (2.43)$$

which implies the convergence of tail sums.

(ii)  $\Rightarrow$  (iii): Suppose the tail sum converges, let  $\varepsilon > 0$ , note that

$$\left| \sum_{k=n+1}^m a_k \right| = \left| \sum_{k=m+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k \right| \quad (2.44)$$

$$\leq \left| \sum_{k=m+1}^{\infty} a_k \right| + \left| \sum_{k=n+1}^{\infty} a_k \right| \quad (2.45)$$

Both terms can be made arbitrarily small by (ii), specifically, one can choose  $N_1$  and  $N_2$  such that both terms are strictly bounded by  $\frac{\varepsilon}{2}$ , and  $N_3 := \max\{N_1, N_2\}$  is the desired value.

(iii)  $\implies$  (i): Since the partial sum is a Cauchy sequence in a complete space, it must converge, so the series is well-defined.  $\blacksquare$

### 2.4.1 The Comparison Test

**Definition 2.8.** A sequence  $(a_n)$  is a **geometric sequence** with coefficient  $r$  if  $a_{n+1} = ra_n$ .

**Proposition 2.2.** Geometric sequences whenever  $r \in (-1, 1)$ . Note that when  $r = -1$ , the sequence becomes an alternating sequence, and the convergence property is indefinite.

**Proposition 2.3.** Let  $(a_n)$  be a geometric sequence with coefficient  $r$ , then for every  $m \in \mathbb{N}$ ,

$$rS_m^a = ra_0 + r^2a_0 + \cdots + r^{n+1}a_0 \quad (2.46)$$

$$\implies (r-1)S_m^a = r^{n+1}a_0 - a_0 \quad (2.47)$$

$$\implies S_m^a = a_0 \frac{1 - r^{m+1}}{1 - r} \quad (2.48)$$

**Theorem 2.13** (The Comparison Test). Let  $(a_n)$  and  $(b_n)$  be two sequences satisfy  $|a_n| \leq b_n$  for every  $n \in \mathbb{N}$ . Then

$$(i) \sum_{i=1}^{\infty} b_n \text{ converges} \implies \sum_{i=1}^{\infty} a_n \text{ converges};$$

$$(ii) \sum_{i=1}^{\infty} a_n \text{ diverges} \implies \sum_{i=1}^{\infty} b_n.$$

*Proof. Part 1:* Suppose  $(b_n)$  converges, it is therefore Cauchy. Let  $\varepsilon > 0$ . Note that for every  $m > n$ :

$$|S_m^a - S_n^a| = \left| \sum_{k=n+1}^m a_k \right| \quad (2.49)$$

$$\leq \sum_{k=n+1}^m |a_k| \quad (2.50)$$

$$\leq \sum_{k=n+1}^m b_k \quad (2.51)$$

Therefore exists  $N \in \mathbb{N}$  such that  $\sum_{k=n+1}^m b_k \leq \left| \sum_{k=n+1}^m b_k \right| < \varepsilon$  for every  $m, n \geq N$ . Taking such  $N$  provides the cutoff needed for  $(S_n^a)$  to be Cauchy. Because  $(S_n^a) \subset \mathbb{R}$ , it converges.

*Part 2:* The result is immediate by taking the contrapositive form of the previous statement.  $\blacksquare$

### 2.4.2 The Root Test

**Definition 2.9.** Let  $(a_n)$  be a bounded sequence, then

$$\limsup(a_n) := \sup_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.52)$$

$$\liminf(a_n) := \inf_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.53)$$

$$(2.54)$$

**Theorem 2.14** (The Root Test). Let  $(a_n)$  be a sequence in which  $a_n \geq 0$  for every  $n \in \mathbb{N}$ , let  $\ell = \limsup \sqrt[n]{a_n}$ , then

- (i) If  $\ell < 1$ , then  $(S_n^a)$  converges;
- (ii) If  $\ell > 1$ , then  $(S_n^a)$  diverges;
- (iii) If  $\ell = 1$ , inconclusive.

*Proof. Part 1:* (Idea: compare with geometric series with  $r < 1$ ) Suppose  $\ell < 1$ , pick  $r \in (\ell, 1)$ , and let  $\varepsilon = r - \ell$ . ■