ECO426H1 Market Design: Auctions and Matching Markets

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Contents

1 Preliminary: Auctions

Definition 1.1. An **auction** is an informational environment consisting of

(i) **Bidding format rules**: the form of the bids, which can be price only, multi-attribute, price and quantity, or quantity only;

(ii) **Bidding process rules**: Closing/timing rules, available information, rules for bid improvements/counter-bids, closing conditions;

(iii) **Price and allocation rules**: final prices, quantities, winners.

Auctions are commonly referred to as a market mechanism as well as a price discovery mechanism

Definition 1.2. A **market mechanism** uses prices to determine allocations.

Definition 1.3. An auction is a **private value** auction if agents' valuations do not dependent on other buyers' valuations. Otherwise, the auction is called a **interdependent / common value** auction.

Assumption 1.1. In a private value auction, we shall impose the following assumption on bidders' valuations:

(i) Each bidder's valuation is independently and identically distributed on some interval $[0, \omega]$ according to a distribution function F:

$$V_i \overset{i.i.d.}{\sim} F \ s.t. \ \text{supp}(F) = \mathbb{R}_+ \tag{1.1}$$

(ii) F belongs to the common knowledge in this system;

(iii) Bidders' valuations have finite expectations:

$$\mathbb{E}\left[V_i\right] < \infty \tag{1.2}$$

Assumption 1.2. Moreover, we assume bidders' behaviours to satisfy the following properties:

(i) Bidders are risk neutral, they are maximizing expected profits;

(ii) Each bidder it both willing and able to pay up to his or her value.

Definition 1.4. A **strategy** of a bidder is a mapping from the space of his/her valuation to a bid:

$$s : [0, \omega] \to \mathbb{R}_+ \tag{1.3}$$

Definition 1.5. An equilibrium of auction is **symmetric** if all bidders are following the same bidding strategy s.

Definition 1.6. A bidder is **bidding sincerely / truthfully** if he bids his true value.

Definition 1.7. An asymmetric game where played have private information is said to be **strategy proof** if it is a weakly-dominant strategy for every player to reveal his/her private information.

Definition 1.8. An auction selling one item is a **standard auction** if the bidder with highest value is always the winner. That is, a standard auction maximizes social value.

2 Ascending Auctions: Extensive Form Games

Definition 2.1. In an **English outcry auction**, bidders announce the prices,

(i) Bidders announce prices,

(ii) with minimum increment between two bids (i.e., the ticking price).

(iii) The auction ends when there's no further bid or when a time limit is reached.

(iv) The winner is the one bidding the highest price.

(v) The winner pays his bid.

Remark 2.1. Bidding speed matters in English outcry auctions: two bidders cannot announce the same bid at the same time.

Definition 2.2. In an **English auction / Japanese button auction**,

(i) The auctioneer announces prices, the price goes up by the ticking price each round;

(ii) in each round, bidders who feel this price is acceptable remain active, other bidders become inactive;

(iii) bidders cannot be reactivated.

(iv) the auction ends when there's no active bidder.

(v) the winner is the last bidder becomes inactive, if there's a tie, winner is randomly chosen.

(vi) the price paid is the last announce price (the price corresponds to no active bidder).

Remark 2.2. In an English auction, the winner is the one with the highest valuation, but the price is that of the second highest valuation plus the ticking price.

Remark 2.3. In the English auction, the auctioneer learns (at the end) the valuations of all bidders except the valuation of the highest bidder.

3 Second Price Sealed-Bid Auction with Private Values

Definition 3.1. In the **Vickrey auction / second price sealed-bid auction**,

(i) Buyers submit a sealed-bid;

(ii) The winner is the one with the highest bid,

(iii) the winner pays the 2nd highest bid.

Remark 3.1. The second price sealed-bid auction and an English auction with negligible ticking price generate the same outcome.
However, second price auction is a strategic for game, but English auction is an extensive form game. They are not exactly identical.

Proposition 3.1. In a symmetric equilibrium of the second-price auction, $s(v) = v$ is a weakly dominant strategy.

Proof. For a fixed valuation $v_i \in [0, \omega]$ of bidder i.
Let $p := \max_{j \neq i} b_j$ be highest bidding price by other bidders.
Let $\pi_i(b, p)$ denote bidder i's profit when bidding b given the highest price from other bidders to be p.
Part 1: consider another bidding $z_i < v_i$, the following cases are possible:

(i) $v_i < p \implies z_i < v_i < p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i losses anyway).

(ii) $v_i = p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i is indifferent).

(iii) $v_i > p$:

   (a) $v_i > z_i > p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = v_i - p$;
   (b) $v_i > z_i = p \implies \pi_i(v_i, p) \geq \pi_i(z_i, p)$;
   (c) $v_i > p > z_i \implies \pi_i(v_i, p) > \pi_i(z_i, p)$.

Hence, bidding v_i weakly dominates bidding any value below it.
Part 2: for $z_i > v_i$, the argument is similar.
Therefore, bidding v_i weakly dominates bidding any other values. ■

Remark 3.2. Refer to the general k^{th} price sealed-bid auction with private values for an alternative proof to this proposition.

4 First Price Sealed-Bid Auction with Private Values

Notation 4.1. Let $\beta^K(v)$ denote the symmetric equilibrium strategy in a k-th price auction.

Remark 4.1. For every continuous distribution F, the probability for a tie to happen is zero. Therefore, we ignore the tie for now.

Definition 4.1 (First Price Auction)**.** Let N denote the set of bidders such that $|N| = n$. For each bidder $i \in N$, his valuation of the auctioned item V_i follows some distribution F. Further assume that $V_i \perp V_j$ for every $i \neq j$.
Let $W(b, v_i)$ denote the event that player i, who has valuation v_i, wins by bidding $b \in \mathbb{R}_+$, define

$$W(b, v_i) \iff b > \max_{j \neq i} b_j \tag{4.1}$$

The payoff (utility) of bidder i, who has valuation v_i, is

$$U(b, v_i) = \begin{cases} v_i - b & \text{if } W(b, v_i) \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

4.1 Symmetric Equilibrium Behaviour

Consider a symmetric environment such that all bidders are using the same <u>strictly increasing</u> strategy $s(\cdot)$ such that $s(\cdot)$ is invertible.

Equilibrium Strategy

Proposition 4.1. In a symmetric equilibrium of the first-price auction, equilibrium bidding strategies are given by

$$s(v_i) = \mathbb{E}\left[\max_{j \neq i} v_j | v_j \leq v_i\right] \tag{4.3}$$

which is the expected second highest valuation conditional on v_i being the highest valuation.

Proof. Let $s(v)$ denote an equilibrium strategy.

Lemma 4.1. For any agent, bidding more than $s(\omega)$ can never be optimal. Bidding $b > s(\omega)$ makes this agent win for sure. In such case, bidding $b' \in (s(\omega), b)$ strictly dominates bidding b.

Lemma 4.2. For any agent, $s(0) = 0$. Bidding any positive number would cause negative payoff with positive probability, and therefore, leads to a negative expected profit.

Lemma 4.3. Because s is monotonically increasing, therefore,

$$\max_{j \neq i} s(v_j) = s(\max_{j \neq i} v_j) \tag{4.4}$$

Let p denote the highest price among all other $N-1$ bidders and let $F^{(N-1)}(x)$ denote the distribution of p.
The expected profit of bidder i by bidding an arbitrary $b \in \mathbb{R}_+$ is

$$\pi_i(b, v_i) = P(b > p)(v_i - s(v_i)) + P(b = p)(v_i - s(v_i)) + P(b < p)0 \tag{4.5}$$

Note that $b > p = s(\max_{j \neq i} v_j)$ if and only if $s^{-1}(b) > \max_{j \neq i} v_j$. It follows

$$P(b > p) = P(\max_{j \neq i} v_j < s^{-1}(b)) = F^{(N-1)}(s^{-1}(b)) \tag{4.6}$$

Therefore,

$$\pi_i(b, v_i) = F^{(N-1)}(s^{-1}(b))(v_i - b) \tag{4.7}$$

The first order condition implies

$$\frac{\partial \pi_i}{\partial b}\pi_i(b, v_i) = \frac{\partial \pi_i}{\partial b} F^{N-1}(s^{-1}(b))v_i - F^{N-1}(s^{-1}(b))b \tag{4.8}$$

$$= f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{4.9}$$

For a symmetric equilibrium, all other bidders are following the same strategy s so that $s(v_i) = b$, therefore,

$$f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{4.10}$$

$$\implies f^{(N-1)}(s^{-1}(b))(v_i - b) - F^{(N-1)}(s^{-1}(b))s'(v_i) = 0 \tag{4.11}$$

$$\implies f^{(N-1)}(s^{-1}(b))v_i = F^{(N-1)}(s^{-1}(b))s'(v_i) + f^{(N-1)}(s^{-1}(b))s(v_i) \tag{4.12}$$

$$\implies f^{(N-1)}(v_i)v_i = \frac{d}{dv_i}\left[F^{(N-1)}(v_i)s(v_i)\right] \tag{4.13}$$

$$\implies \int_0^{v_i} f^{(N-1)}(y)y\ dy = F^{(N-1)}(v_i)s(v_i) - F^{(N-1)}(0)s(0) \tag{4.14}$$

$$\implies F^{(N-1)}(v_i)s(v_i) = \int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{4.15}$$

$$\implies s(v_i) = \frac{1}{F^{(N-1)}(v_i)}\int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{4.16}$$

$$\implies s(v_i) = \mathbb{E}\left[\max_{j\neq i} v_j \,\middle|\, \max_{j\neq i} v_j < v_i\right] \tag{4.17}$$

■

When $F = Unif(0,1)$.

$$\beta^I(v) = \frac{n-1}{n}v \tag{4.18}$$

Probability of Winning

$$P(W(b, v_i)) = P(b > \max_{j\neq i} s(v_j)) \tag{4.19}$$

$$= P(b > s(\max_{j\neq i} v_j)) \tag{4.20}$$

$$= P(\max_{j\neq i} v_j \leq s^{-1}(b))) \tag{4.21}$$

$$= F(s^{-1}(b))^{n-1} \tag{4.22}$$

$$= F(v_i)^{n-1} \text{ because } b = s(v_i) \tag{4.23}$$

When $F = Unif(0,1)$,

$$P(W(b, v_i)) = v_i^{n-1} \tag{4.24}$$

Expected Payment from Bidder i with v_i Conditioned on Winning Suppose bidder i is following strategy $s(\cdot)$. Then,

$$\mathbb{E}\left[Payment_i | v_i, W(b, v_i)\right] = b = s(v_i) \tag{4.25}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Payment_i | v_i, W(b, v_i)\right] = \frac{n-1}{n}v_i \tag{4.26}$$

Unconditional Payment from Bidder i with v_i

$$\begin{aligned}
\mathbb{E}\left[Payment_i|v_i\right] &= P(W(b,v_i))\mathbb{E}\left[Payment_i|v_i, W(b,v_i)\right] + P(Loss)\times 0 && (4.27)\\
&= P(W(b,v_i))\mathbb{E}\left[Payment_i|v_i, W(b,v_i)\right] && (4.28)\\
&= F(v_i)^{n-1}s(v_i) && (4.29)
\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Payment_i|v_i\right] = \frac{n-1}{n}v_i^n \tag{4.30}$$

Expected Payoff of Bidder i with v_i

$$\begin{aligned}
\mathbb{E}\left[U|v_i\right] &= P(W(s(v_i),v_i))v_i - \mathbb{E}\left[Payment_i|v_i\right] && (4.31)\\
&= F(v_i)^{n-1}v_i - F(v_i)^{n-1}s(v_i) && (4.32)\\
&= F(v_i)^{n-1}[v_i - s(v_i)] && (4.33)
\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[U|v_i\right] = \frac{v_i^n}{n} \tag{4.34}$$

Unconditional Payment from Bidder i This is the same as the expected revenue from bidder i:

$$\begin{aligned}
\mathbb{E}[Payment_i] &= \int_{\mathbb{R}_+} \mathbb{E}\left[Payment_i|v_i\right]dF && (4.35)\\
&= \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f(v_i)\ dv_i && (4.36)
\end{aligned}$$

When $F = Unif(0,1)$,

$$\begin{aligned}
\mathbb{E}[Payment_i] &= \int_0^1 \frac{n-1}{n}v_i^n\ dv_i && (4.37)\\
&= \frac{n-1}{n(n+1)} && (4.38)
\end{aligned}$$

Auctioneer's Expected Revenue Since all bidders are the same,

$$\begin{aligned}
\mathbb{E}\left[Revenue\right] &= n\ \mathbb{E}[Payment_i] && (4.39)\\
&= n\ \int_{\mathbb{R}_+} F(v_i)^{n-1}s(v_i)f_i\ dv_i && (4.40)
\end{aligned}$$

When $F = Unif(0,1)$,

$$\mathbb{E}\left[Revenue\right] = \frac{n-1}{n+1} \tag{4.41}$$

5 Generalization: k^{th} Price Sealed Bid Private Value Auction

5.1 Uniform Values

In a k^{th} price auction, the bidder with the highest bidding wins, and pays the k^{th} highest bid. Let n denote the number of bidders.

Proposition 5.1. Assume $v_i \overset{i.i.d.}{\sim} Unif(0,1)$, the following strategy forms a symmetric equilibrium in k^{th} price auction:

$$\beta^k(v) = \frac{n-1}{n-k+1}v \tag{5.1}$$

Proof. We are going to verify the proposed strategy indeed forms an equilibrium.
Assume the optimal strategy is linear in v, say αv with $\alpha \in [0,1]$, and all bidders other than i are following this strategy.
The expected payoff of bidder i with value v_i from bidding b is

$$\begin{aligned}
U(b, v_i) &= \mathbb{E}P(W(b, v_i))(v_i - b_{n:k}) && (5.2)\\
&= \mathbb{E}P(b \geq \alpha v_j \ \forall j \neq i)(v_i - b_{n:k}) && (5.3)\\
&= \mathbb{E}P\left(v_j \leq \frac{b}{\alpha}\right)^{n-1}(v_i - b_{n:k}) && (5.4)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} \mathbb{E}\left[b_{n:k} | v_j \leq \frac{b}{\alpha} \ \forall j \neq i\right] && (5.5)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} \alpha\mathbb{E}\left[v_{n-1:k-1} | v_j \leq \frac{b}{\alpha} \ \forall j \neq i\right] && (5.6)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} b\mathbb{E}\left[v_{n-1:k-1} | v_j \leq 1 \ \forall j \neq i\right] && (5.7)\\
&= \left(\frac{b}{\alpha}\right)^{n-1} v_i - \left(\frac{b}{\alpha}\right)^{n-1} b\mathbb{E}\left[v_{n-1:k-1}\right] && (5.8)\\
& && (5.9)
\end{aligned}$$

Note that each individual $v_j \sim Unif(0,1)$ for every j.
The density function of $v_{n-1:k-1}$ is

$$\begin{aligned}
f_{v_{n-1:k-1}}(x) &= (n-1)\binom{n-2}{k-2}F(x)^{n-k}(1-F(x))^{k-2}f(x) && (5.10)\\
&= (n-1)\binom{n-2}{k-2}x^{n-k}(1-x)^{k-2} && (5.11)
\end{aligned}$$

Taking the expectation

$$\mathbb{E}\left[v_{n-1:k-1}\right] = \int_0^1 x f_{v_{n-1:k-1}}(x)\ dx \tag{5.12}$$
$$= (n-1)\binom{n-2}{k-2}\int_0^1 x^{n-k+1}(1-x)^{k-2}\ dx \tag{5.13}$$
$$= (n-1)\binom{n-2}{k-2}\frac{\Gamma(k-1)\Gamma(-k+n+2)}{\Gamma(n+1)} \tag{5.14}$$
$$= \frac{-k+n+1}{n} \tag{5.15}$$

Therefore,

$$U(b, v_i) = \left(\frac{b}{\alpha}\right)^{n-1}\left(v_i - b\frac{-k+n+1}{n}\right) \tag{5.16}$$

Proposition 5.2. Let $X_i \overset{i.i.d.}{\sim} Unif(0,1)$ for $i = 1, \cdots n$, then

$$\mathbb{E}\left[X_{n:k}\right] = \frac{n-k+1}{n+1} \tag{5.17}$$

Taking the first order condition,

$$\frac{\partial}{\partial b}U(b, v_i) = \frac{1}{\alpha^{n-1}}\frac{\partial}{\partial b}\left(b^{n-1}\left(v_i - b\frac{-k+n+1}{n}\right)\right) = 0 \tag{5.18}$$
$$\implies (n-1)b^{n-2}\left(v_i - b\frac{-k+n+1}{n}\right) - \frac{-k+n+1}{n}b^{n-1} = 0 \tag{5.19}$$
$$\implies (n-1)\left(v_i - b\frac{-k+n+1}{n}\right) - \frac{-k+n+1}{n}b = 0 \tag{5.20}$$
$$\implies (n-1)v_i - (n-1)b\frac{-k+n+1}{n} - \frac{-k+n+1}{n}b = 0 \tag{5.21}$$
$$\implies (n-1)v_i = (n-k+1)b \tag{5.22}$$
$$\implies \beta^K(v_i) = \frac{n-1}{n-k+1} \tag{5.23}$$

$\blacksquare$

Probability of Winning Given v_i

$$P(W(s(v_i), v_i)) = v_i^{n-1} \tag{5.24}$$

Probability of Payment Given v_i Conditioned on Winning Given that v_i is the highest value, the payment conditioned on winning is

$$\mathbb{E}\left[s(v_{n-1:k-1}) | v_j \leq v_i\ \forall j \neq i\right] \tag{5.25}$$

6 Dutch/Descending Price Auction: An Extensive Form Game

Definition 6.1. The **Dutch/descending price** auction is a first-price auction:

(i) There is a price clock: displays a price that is decreasing.

(ii) The auction stops as soon as someone accepts the price.

(iii) The first bidder accepts is the winner, and the price paid is exactly the last price displayed.

Remark 6.1. Dutch auction and first-price auction are equivalent, bidders use the same strategy, they have the same payoffs, and the auctioneer gets the same revenue.

7 Revenue Equivalence Theorem

Theorem 7.1. For an auction with n bidders. Suppose that values are independently and identically distributed and all bidders are risk-neutral. Then any symmetric and increasing equilibrium of any auction such that

(i) The winner is always the bidder with the highest valuation (i.e., standard auction);

(ii) the bidder with the lowest valuation, v_* has the same expected payoff $U_i(v_*)$ for all i,

yields the same expected revenue for the seller, and the same expected price for any bidder in equilibrium.

Proof. The expected payoff in equilibrium of someone with value v_i is

$$\begin{aligned} U_i(v_i) &= v_i P_w(v_i) - \mathbb{E}\left[Payment_i\right] \quad (7.1) \\ &= v_i P(v_i \geq v_j \ \forall j \neq i) - \mathbb{E}\left[Payment_i\right] \quad (7.2) \end{aligned}$$

Note that $v_i P(v_i \geq v_j \ \forall j \neq i)$ is independent from the auction format.
Consider the case when the bidder is bidding $\beta(\tilde{v})$ instead of $\beta(v_i)$, that is, bidder i is pretending to have another valuation,

$$\begin{aligned} U_i(\tilde{v}, v_i) &= v_i P_w(\tilde{v}) - \mathbb{E}\left[Payment_i\right] \quad (7.3) \\ &= v_i P_w(\tilde{v}) - \mathbb{E}\left[Payment_i\right] + \tilde{v} P_w(\tilde{v}) - \tilde{v} P_w(\tilde{v}) \quad (7.4) \\ &= U_i(\tilde{v}) + P_w(\tilde{v})(v_i - \tilde{v}) \quad (7.5) \end{aligned}$$

Therefore,

$$\begin{aligned} U_i(v_i) &\geq U_i(\tilde{v}, v_i) \quad (7.6) \\ &= U_i(\tilde{v}) + P_w(\tilde{v})(v_i - \tilde{v}) \quad (7.7) \\ \implies P_w(\tilde{v}) &\leq \frac{U_i(v_i) - U_i(\tilde{v})}{v_i - \tilde{v}} \quad (7.8) \end{aligned}$$

Similarly, the same argument holds: someone with value $\tilde{v}$ won't deviate to behave like another v_i value bidder.

$$\begin{aligned} U_i(\tilde{v}) &\geq U_i(v_i, \tilde{v}) \quad (7.9) \\ &= U_i(v_i) + P_w(v_i)(\tilde{v} - v_i) \quad (7.10) \\ \implies P_w(v_i) &\geq \frac{U_i(v_i) - U_i(\tilde{v})}{v_i - \tilde{v}} \quad (7.11) \end{aligned}$$

Hence, by taking the limit $\tilde{v} \to v_i$:

$$\frac{dU_i(v)}{dv} = P_w(v) \tag{7.12}$$

$$\implies U_i(v_i) = U_i(v_*) + \int_{v_*}^{v_i} P_w(v)\ dv \tag{7.13}$$

Since $U_i(v_i)$ is independent from the auction format as well, therefore, the expected payment from bidder i is independent from the auction format. Hence, the expected revenue of the auctioneer is

$$n \times \mathbb{E}\left[Payment_i\right] \tag{7.14}$$

which is independent from the auction format as well. ■

8 Reserve Price & Optimal Auctions

Let r denote the reserve price, and b_1, b_2 denote the highest and second highest bid respectively. Then

$$\text{Seller's revenue } = \begin{cases} 0 & \text{if } b_1 < r \\ r & \text{if } b_2 \leq r \leq b_1 \\ b_2 & \text{if } r < b_2 \end{cases} \tag{8.1}$$

Definition 8.1. An **optimal auction** is an auction that maximizes the seller's revenue.

Theorem 8.1 (Myerson (1981), Riley & Samuelson (1981))**.** If bidders' valuations are drawn independently from a distribution F, then a seller's optimal auction is a second-price auction with reserve price r such that

$$\psi(r) \equiv r - \frac{1 - F(r)}{f(r)} = 0 \tag{8.2}$$

Proof. Note that this proof is extremely informal.
The expected revenue can be written as (as if there is only one bidder):

$$\mathbb{E}\left[Revenue\right] = P(v \leq r)0 + P(v > r)r \tag{8.3}$$

$$= (1 - F(r))r \tag{8.4}$$

Taking the first order condition

$$(1 - F(r)) - rf(r) = 0 \tag{8.5}$$

$$\implies r - \frac{1 - F(r)}{f(r)} = 0 \tag{8.6}$$

■

Remark 8.1. The theorem is applicable only if the auctioneer knows the exact distribution of bidders' values. In general, the auctioneer may know bidders' values are independently and identically distributed following some distribution, but not necessarily knows the exact form of this distribution.

Theorem 8.2 (Bulow & Klemperer, 1996)**.** When valuations are private and independent, a second-price auction with $n + 1$ bidders gives a higher expected revenue than an optimal mechanism with n bidders.

Remark 8.2. The auctioneer is better off by bringing one additional bidder than by setting an optimal reserve price following the previous theorem.

Definition 8.2. A **Kirkegaard auction** is a two step auction:

(i) Run a second-price auction with n bidders and optimal reserve price. If the object isn't sold go to step 2.

(ii) Sell the object for \$0 to $n+1$-th bidder.

Obviously, this auction with $n+1$ bidders generates the same expected revenue as optimal n-auction (by setting the optimal reserve price).

Definition 8.3. An auction is **constrained optimal** if it maximizes the revenue among all auctions such that the object sold with certainty.

Proof. to Bulow & Klemperer, 1996. Note that English auction with $n+1$ bidders is constrained optimal, therefore

$$\begin{aligned}\mathbb{E}\left[\text{Optimal } n \text{ auction}\right] &= \mathbb{E}\left[\text{Kirkegaard } n+1\right] && (8.7)\\ &\leq \mathbb{E}\left[\text{English } n+1\right] && (8.8)\end{aligned}$$

■

Corollary 8.1. The seller's expected revenue is the same for the English, Dutch, second-price and first-price auctions.

9 Common Value Auction

Definition 9.1. In a **common value** auction, all bidders' valuation are identical. However, bidders are receive private signals s_i provides information on the actual value v. Each bidder chooses a strategy $\beta_i(s_i) \in \mathbb{R}_+$ for all possible types of signals could be received.
Specifically, bidders share a prior belief on the common value v

$$v \sim p(v) \tag{9.1}$$

and private signals are from the same distribution depends on the actual value v

$$s_i \overset{i.i.d.}{\sim} p(s|v) \tag{9.2}$$

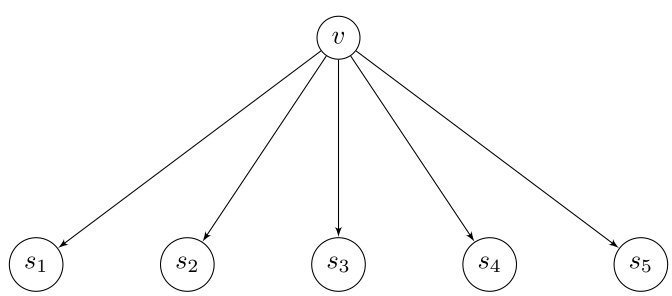

Figure 9.1: DAG representation of common value auction with 5 bidders

Example 9.1. Possible types of signals are

$$s_i \overset{i.i.d.}{\sim} F \tag{9.3}$$

$$v = \sum_{i=1}^{n} s_i \tag{9.4}$$

or

$$s_i = v + \varepsilon_i \tag{9.5}$$

$$\varepsilon_i \overset{i.i.d.}{\sim} Q \ s.t. \ \mathbb{E}\left[\varepsilon_i\right] = 0 \tag{9.6}$$

In most cases, v is positively correlated with the signal s_i received by each bidder. That is,

$$\frac{\partial}{\partial ds_j}\mathbb{E}\left[v|s_1, \cdots, s_n\right] \geq 0 \ \forall j \in N \tag{9.7}$$

Hence, we can assume bidding strategy β to be an increasing function of signal s.

In a common value auction, all bidders know the underlying random process generating v. Given s_i, bidder i computes his **own estimation** of the common value v:

$$v_i := \mathbb{E}\left[v|s_i\right] = \iint_{s_{-i}} p(v|s_{-i})p(s_{-i}) \tag{9.8}$$

Formally let

$$\Omega_i := \{s_j \ s.t. \ j \neq i | s_j \leq s_i \ \forall j \neq i\} \subseteq \mathbb{R}^{n-1} \tag{9.9}$$

denote the collection of realizations of signal generation such that bidder i wins the auction.
When bidder i is winning the bid, the expected value is

$$\mathbb{E}[v|s_j \leq s_i \ \forall j \neq i, s_i] = \frac{1}{P(\Omega_i)} \iint_{\Omega_i} p(s_{-i})\mathbb{E}\left[v|s_i, s_{-i}\right] \ ds_{-i} \tag{9.10}$$

$$\leq \iint_{\mathbb{R}^{n-1}} p(s_{-i})\mathbb{E}\left[v|s_i, s_{-i}\right] \ ds_{-i} \tag{9.11}$$

$$= \mathbb{E}\left[v|s_i\right] = v_i \tag{9.12}$$

where the last inequality holds because $\mathbb{E}\left[v|s_i, s_{-i}\right]$ is increasing in each s.

Definition 9.2. Winner's curse winning the object reduces how much you think it is worth. Similarly, for **loser's curse** losing the auction may increase how much you think it is worth.

Proposition 9.1. Bidding one's own estimation $\mathbb{E}\left[v|s_i\right]$ is not dominant in common value auction.

Proposition 9.2. Symmetric bidding strategies in a second price common value auction are given by

$$\beta^*(s_i) = \mathbb{E}\left[v|s_i, s_{-i} = s_i\right] \tag{9.13}$$

In which the bidder assumes all other bidders are receiving the same signal.

Proof. ■

10 Combinatorial Auction: The VCG Mechanism

Definition 10.1. A Vickrey-Clarke-Groves (VCG) auction consists of a set of items to be sold X. Each bidder $i \in N$ has a private **value** for each possible bundle of items:

$$v_i : \mathcal{P}(X) \to \mathbb{R} \tag{10.1}$$

Each bidder submits a (sealed) **bidding** for every possible bundle of items:

$$b_i : \mathcal{P}(X) \to \mathbb{R} \tag{10.2}$$

An **assignment** characterize the allocation of items to bidders:

$$\mu : N \to \mathcal{P}(X) \tag{10.3}$$

such that no item is shared between two bidders:

$$\mu(i) \cap \mu(j) = \varnothing \ \forall i \neq j \tag{10.4}$$

The **outcome** assignment seeks to maximize the social value.
Note that the auctioneer does not know v_j's, this social value is computed based on biddings b_j instead of bidders' actual values.

$$\mu^* = \underset{\mu}{\operatorname{argmax}} \sum_{i \in N} b_i(\mu(i)) \tag{10.5}$$

The **price** paid by bidder i is the externality this bidder imposes on other bidders.:

$$p_i = \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) - \sum_{j \neq i} b_j(\mu^*(j)) \tag{10.6}$$

Remark 10.1. The auctioneer does not have to allocate all items in X, that is, μ is not necessary a partition of X. $\bigcup_{i \in N} \mu(i)$ is not necessary X.

Remark 10.2. When $|X| = 1$, VCG mechanism is the second price auction.

Proposition 10.1. Submitting one's true valuation function (i.e., $b_i = v_i$) is a dominate strategy in the VCG auction, that is, VCG auction is strategy proof.

Proof. Suppose all other bidders are bidding b_j.
Let μ^* be the allocation when bidder i bid $b_i = v_i$ while all other bidders bid b_j:

$$\mu^* = \underset{\mu}{\operatorname{argmax}}\, v_i(\mu(i)) + \sum_{j \neq i} b_j(\mu(j)) \quad (\dagger) \tag{10.7}$$

Then, for bidder i, the payoff by bidding v_i is

$$v_i(\mu^*(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\mu^*(j)) \tag{10.8}$$

Alternatively, bidder i could bid $b_i \neq v_i$, let

$$\hat{\mu} = \underset{\mu}{\operatorname{argmax}} \sum_{i \in N} b_i(\mu_i(i)) \tag{10.9}$$

The payoff from bidding b_i instead is

$$v_i(\hat{\mu}(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \tag{10.10}$$

Take the difference between two payoffs:

$$v_i(\mu^*(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\mu^*(j)) - \left(v_i(\hat{\mu}(i)) - \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \right) \tag{10.11}$$

$$= v_i(\mu^*(i)) + \sum_{j \neq i} b_j(\mu^*(j)) - \left(v_i(\hat{\mu}(i)) + \sum_{j \neq i} b_j(\hat{\mu}(j)) \right) \tag{10.12}$$

$$\geq 0 \text{ by } (\dagger) \tag{10.13}$$

Therefore, bidding one's own value function is dominant. ■

Proposition 10.2. The price paid by any bidder in VCG auctions is non-negative.

Proof.

$$p_i = \max_{\mu} \sum_{j \neq i} b_j(\mu(j)) - \sum_{j \neq i} b_j(\mu^*(j)) \geq 0 \tag{10.14}$$

■

11 Keyword Auctions

12 Two side, One-to-One Matching: Marriage Market

Definition 12.1 (Marriage Market)**.** Suppose there are a finite set of **women**: $W = \{w_1, w_2, \cdots\}$, and a finite set of **men**: $M = \{m_1, m_2, \cdots\}$. Each man $m \in M$ has a strict preference relation P_m over the women and the option of remaining single, that is, P_m is defined over $W \cup \{m\}$. Similarly, each $w \in W$ has a preference relation p_w over $M \cup \{w\}$.

Notation 12.1. The strict preference P_v is often written as $\succ_v$. And the weak preference (i.e., preferred or indifferent to) is often written as R_v or $\succsim_v$.

Definition 12.2. A man $m \in M$ is **unacceptable** for woman w if and only if

$$w P_w m \tag{12.1}$$

Definition 12.3. A (one-to-one) **matching** is a mapping $\mu : W \cup M \to W \cup M$ such that

- $\forall m \in M,\ \mu(m) \in W \cup \{m\}$;
- $\forall w \in W,\ \mu(w) \in M \cup \{w\}$;
- $\mu(w) = m \iff \mu(m) = w$.

Remark 12.1. Someone $v \in M \cup W$ is **unmatched** if $\mu(v) = v$.

Remark 12.2. There's no price in a matching problem, so we can't really talk about equilibrium.

Definition 12.4. A matching μ is **individually rational** if for each $v \in M \cup W$,

$$\mu(v) \succsim_v v \tag{12.2}$$

Definition 12.5. A pair (m, w) **blocks** a matching μ if

- $\mu(m) \neq w$;
- $w \succsim_m \mu(m)$;
- $m \succsim_w \mu(w)$.

Definition 12.6. A matching μ is **stable** if

- it is individually rational;
- and there is no (m, w) pair blocks μ.

Theorem 12.1 (David Gale & Lloyd Shapley, 1962)**.** For any set of preferences, there always exists at least one stable matching.

Algorithm 12.1 (Deferred Acceptance Algorithm)**.** Man-proposing version:

- Step 1
  - Each man proposes to his most preferred, acceptable woman (if a man finds no women acceptable he remains single);
  - Each woman who received at least one offer (proposed):
    - temporarily holds the offer from the most preferred man among those who made an offer to her and are acceptable,
    - rejects all other offers.
- Step $k \geq 2$, each man whose offer has been rejected in the previous step proposes to his most preferred woman among the acceptable women he has not yet proposed to (if there is no such woman he remains single).

The algorithm stops when no rejection happens, and women are matched with their on-hold men.

Theorem 12.2 (David Gale & Lloyd Shapley, 1962)**.** For any set of preferences, there always exists at least one stable matching.

Theorem 12.3. The Deferred Acceptance algorithm (DA) produces a stable matching that is

- The <u>most preferred</u> stable matching for the <u>proposing</u> side;
- The <u>least preferred</u> stable matching for the <u>receiving</u> side.

Proposition 12.1. A pair of man m and women m is **achievable** if there exists a **stable matching** μ such that $\mu(m) = w$.

Proposition 12.2. Under man-proposing DA, no man can be rejected by an achievable woman.

Proposition 12.3. Let μ and μ' be two stable matchings. Suppose all men weakly prefer μ to μ'. Then all women must weakly prefer μ' to μ.

Theorem 12.4. A matching mechanism that uses the Deferred Acceptance algorithm is strategyproof for the proposing side.

Theorem 12.5. There is no matching mechanism that satisfies, for any matching problem, the following two properties at the same time:

(a) The matching is stable with respect to the submitted preference lists;

(b) The mechanism is strategyproof for all individuals.

13 Two side: Many-to-One Matching: Medical Matching

Definition 13.1 (Medical Matching)**.** The agents in this problem consist of a finite set of **doctors**(residents) and a finite set of **hospitals**:

$$D = \{d_1, d_2, \cdots\} \tag{13.1}$$

$$H = \{h_1, h_2, \cdots\} \tag{13.2}$$

For each $h \in H$, there is a **capacity** q_h denotes the maximum number of residents h can hire. And each $h \in H$ has a strict preference over the set of doctors, $P_h^\sharp$. Note that $P_h^\sharp$ is only defined on sets with sizes up to q_h.
For each $d \in D$, d has a strict preference, P_d, over hospitals.

Definition 13.2. A preference relation $P_h^\sharp$ over $\mathcal{P}(D)$ is **responsive** if there exists another preference relation P_h over D such that: for every $S \subseteq D$ such that $|S| < q_h$, and any $d, d' \notin S$,

$$S \cup \{d\} P_h^\sharp S \cup \{d'\} \iff dP_h d' \tag{13.3}$$

and

$$S \cup \{d\} P_h^\sharp S \iff d \text{ is acceptable to } h \tag{13.4}$$

Definition 13.3. A **matching** is a function $\mu : U \cup D \to H \cup D$ such that

- $\forall d \in D,\ \mu(d) \in H \cup \{d\}$;
- $\forall h \in H$,
  - $|\mu(h)| \leq q_h$ (within capacity);
  - $\mu(h) \subseteq D$ (only matched to doctors).
- $\mu(d) = h \iff d \in \mu(h)$.

Definition 13.4. A matching μ is **individually rational** if

- For each doctor $d \in D$, $\mu(d) \succsim_d d$.
- For each hospital $h \in H$, $\forall d \in \mu(h)$, $d \succsim_h \varnothing$.

Definition 13.5. A pair (d, h) **blocks** a matching μ if

- $d \notin \mu(h)$;
- $hP_d\mu(d)$;

- dP_hd' for some $d' \in \mu(h)$.

Equivalently, with responsive preference:

$$\mu(h) \cup \{d\} \backslash \{d'\} P_h^\sharp \mu(h) \tag{13.5}$$

That is, hospital h would like to replace d' with d.

Definition 13.6. A matching μ is **non-wasteful** if

$$hP_d\mu(d) \implies |\mu(h)| = q_h \vee \varnothing P_h d \tag{13.6}$$

If d prefers a hospital to her match then that hospital has filled its capacity (or d is not acceptable to h).

Definition 13.7. A matching μ is **stable** if

- it is individually rational;
- there is no blocking pair;
- it is non-wasteful.

Algorithm 13.1 (Deferred Acceptance Algorithm for Many-to-One Matchings)**.** Hospital-proposing version:

- Step 1
  - Each hospital proposes to its most preferred set of doctors;
  - Each doctor accepts the proposal/oer from their most preferred acceptable hospital, and rejects all other others.
- Step $k \geq 2$, each hospital that had one or more rejections at the previous step proposes to its most preferred set of doctors such that
  - The set must contain all doctors that have accepted the hospital's offer at the previous step;
  - Any additional doctor in the set must be a doctor to whom the hospital has not proposed to before.

  Each doctor accepts the proposal/oer from their most preferred acceptable hospital, and rejects all other others.

The algorithm stops when no more offers are rejected.

Proposition 13.1. Doctor proposing DA yields the doctor-optimal matching, Hospital proposing DA yields the hospital-optimal matching.

Proposition 13.2. The doctor-proposing DA is strategyproof for doctors (but not for hospitals). However, the hospital proposing DA is not strategyproof for hospitals.

14 Kidney Exchange

15 Appendix A: Order Statistics

Definition 15.1. Let $(X_1, \cdots, X_n)$ be n random variables on the probability space $(\Omega, \mathcal{F}, P)$, further assume they are iid following distribution function $F(\cdot)$. For each $\omega \in \Omega$, realizations of above random variables can be sorted as

$$X_{(n)}(\omega) \leq X_{(n-1)}(\omega) \leq \cdots \leq X_{(1)}(\omega) \tag{15.1}$$

For each ω, the random variable $X_{n:k}$ is defined such that $X_{n:k}(\omega)$ equals the k-th largest value, $X_{(k)}(\omega)$.

Distribution Function Let $x \in X(\Omega)$, then

$$X_{n:k} \leq x \iff (\text{no } X_i > x) \bigcup (\text{exactly } 1 \ X_i > x) \bigcup \cdots \bigcup (\text{exactly } k-1 \ X_i > x) \tag{15.2}$$

$$\iff (X_i \leq x \ \forall i) \bigcup (\text{exactly } n-1 \ X_i \leq x) \bigcup \cdots \bigcup (\text{exactly } n-k+1 \ X_i \leq x) \tag{15.3}$$

$$\iff \bigcup_{j=n-k+1}^{n} (\text{exactly } j \ X_i \leq x) \tag{15.4}$$

Note that events in the union are mutually exclusive, therefore,

$$F_{n:k}(x) = P(X_{n:k} \leq x) = \sum_{j=n-k+1}^{n} P\left(\text{exactly } j \ X_i \leq x\right) \tag{15.5}$$

$$= \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \tag{15.6}$$

Density Function

$$
\begin{aligned}
f_{n:k}(x) &= \frac{d}{dx} F_{n:k}(x) && (15.7)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} && (15.8)\\
&= \frac{d}{dx} \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} F(x)^j (1-F(x))^{n-j} && (15.9)\\
&= \sum_{j=n-k+1}^{n} \left[\frac{n!}{j!(n-j)!} j F(x)^{j-1} (1-F(x))^{n-j} - \frac{n!}{j!(n-j)!} (n-j) F(x)^j (1-F(x))^{n-j-1} \right] f(x) && (15.10)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{j!(n-j)!} j F(x)^{j-1} (1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j)!} (n-j) F(x)^j (1-F(x))^{n-j-1} f(x) && (15.11)\\
&= \sum_{j=n-k+1}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j (1-F(x))^{n-j-1} f(x) && (15.12)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) && (15.13)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x)\\
&\quad - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j (1-F(x))^{n-j-1} f(x)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) && (15.14)\\
&\quad + \sum_{j=n-k+2}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x)\\
&\quad - \sum_{i=n-k+2}^{n} \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1} (1-F(x))^{n-i} f(x) \text{ (substitute } j = i-1)\\
&= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) && (15.15)\\
&= n \frac{(n-1)!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) && (15.16)\\
&= n \binom{n-1}{k-1} F(x)^{n-k} (1-F(x))^{k-1} f(x) && (15.17)
\end{aligned}
$$