MAT237: Lecture Notes

Advanced Calculus

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1.1 The Geometry of Euclidean Space

Example 1.1. Consider $(1, 2) \in \mathbb{R}^2$ as a point or a vector.

Remark 1.1. All vectors in this course are considered as column vectors. Reasoning: suppose a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, then the transformation can be implemented as

$$f(\vec{x}) = A\vec{x},\ A \in mn$$

if $\vec{x}$ is a column vector.

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$, the **dot product** $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as,

$$ab = \sum_i a_i b_i$$

Let $\vec{a} \in \mathbb{R}^n$, the **Euclidean norm** $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$||\vec{a}|| = \sqrt{ab}$$

Interpretation the Euclidean norm of $\vec{a}$, $||\vec{a}||$ is the length of $\vec{a}$, or the distance of $\vec{a}$ from the origin. And $||\vec{a} - \vec{b}||$ is the distance from $\vec{a}$ to $\vec{b}$.

Two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ is **orthogonal** if and only if

$$ab = 0$$

Theorem 1.1. (Cauchy Schwarz inequality)

$$|ab| \leq ||\vec{a}||||\vec{b}||$$

Theorem 1.2. (Triangle inequality)

$$||\vec{a} + \vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$$

Theorem 1.3.

$$ab = ||\vec{a}||||\vec{b}|| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

If $\vec{u} \in$ is a **unit vector** if

$$||\vec{u}|| = 1$$

The **projection** of $\vec{a}$ onto the line through $\vec{u}$ is defined as

$$(ua)\vec{u}$$

1.2 Subspaces of $\mathbb{R}^n$

A subspace V if $\mathbb{R}^n$ is a <u>subset</u> of $\mathbb{R}^n$ such that

$$\vec{a}, \vec{b} \in V \wedge c_1, c_2 \in \mathbb{R} \implies c_1\vec{a} + c_2\vec{b} \in V$$

Example 1.2. Suppose

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \\ -1 & 0 \end{pmatrix}$$

And consider

$$V = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

V is a subspace with dimension 2.

Theorem 1.4. Let $A \in mn$ with $m > n$ and columns are independent then $V = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ is a n-dimensional subspace of $\mathbb{R}^n$.

Example 1.3. Consider

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 9 & -2 \end{pmatrix}$$

and

$$V = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$$

Then V is a 1-dimensional subspace of $\mathbb{R}^3$.

Theorem 1.5. $A \in mn$ and $m < n$ and rows are linearly independent, then $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$ is a $(n - m)$ dimensional subspace.

1.3 Cross Product

(*Only available in* $\mathbb{R}^3$) is a way to multiplying two vectors in $\mathbb{R}^3$ to get another vector in $\mathbb{R}^3$. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ then the cross product $\times : \mathbb{R}^6 \to \mathbb{R}^3$ is defined as

$$\vec{a} \times \vec{b} := det(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix})$$

$$\text{where } \vec{i} = (1, 0, 0),\ \vec{j} = (0, 1, 0),\ \vec{k} = (0, 0, 1)$$

Remark 1.2. $\vec{a} \times \vec{b}$ is the vector such that

1. orthogonal to both $\vec{a}$ and $\vec{b}$.
2. it's length is $||\vec{a}||||\vec{b}|| \sin\theta$.

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$, then

1. $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{a} = \vec{0}$
3. $(c_1\vec{a_1} + c_2\vec{a_2}) \times \vec{b} = c_1(\vec{a_1} \times \vec{b_1}) + c_2(\vec{a_2} \times \vec{b_2})$
4. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

1.4 Functions of Several Variables

Remark 1.3. Idea of differential calculus: more general general functions can then be approximated by linear functions.

Consider function $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is

$$\{(x, y, z) : z = f(x, y)\} \subseteq \mathbb{R}^3$$