

Introduction to Real Analysis

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1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subseteq \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

Example 1.1. The set of integers, \mathbb{Z} , is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subseteq \mathbb{R}$, we use A^\uparrow and A^\downarrow to denote collections of upper bounds of A and lower bounds of A . When A is bounded, either A^\uparrow or A^\downarrow is empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound (supremum)** for a set $A \subseteq \mathbb{R}$ if

- (i) $s \in A^\uparrow$;
- (ii) and $\forall u \in A^\uparrow, s \leq u$.

Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if

- (i) $f \in A^\downarrow$;
- (ii) and $\forall l \in A^\downarrow, l \leq f$.

Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property). $\forall \emptyset \neq A \subseteq \mathbb{R}$ such that $A^\uparrow \neq \emptyset$, $\exists \mathbb{R} \ni u = \sup A$.

Definition 1.4. Let $\emptyset \neq A \subseteq \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subseteq \mathbb{R}$ does not satisfy the axiom of completeness. Let $A = \{r \in \mathbb{Q} : r < \sqrt{2}\}$, clearly A is bounded above, but for every $r' \in \mathbb{Q} \cap A^\uparrow$, there exists $r'' \in (\sqrt{2}, r') \cap A^\uparrow$.

Proposition 1.1. Let $\emptyset \neq A \subseteq \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

Proof. Step 1: Show $c + \sup A \in (c + A)^\uparrow$:

Let $x \in c + A$, $\exists a \in A$ s.t. $x = c + a$. Then, $x = c + a \leq c + \sup A$. Therefore, $x \leq c + \sup A \forall x \in A$, which implies what desired.

Step 2: Show $\forall u \in (c + A)^\uparrow$, $c + \sup A \leq u$:

Let $u \in (c + A)^\uparrow$, then $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.

Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum). Let $s \in A^\uparrow$ for some nonempty $A \subseteq \mathbb{R}$. The following statements are equivalent:

- (i) $s = \sup A$;
- (ii) $\forall \varepsilon, \exists a \in A$, s.t. $a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^\uparrow$).

Proof. The proof is immediate by the definition of supremum as the least upper bound. ■

Theorem 1.1 (Nested Interval Property). Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \tag{1.5}$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k .

By the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$.

Since $a^* \in (a_n)^\uparrow$, $a^* \geq a_n \forall n \in \mathbb{N}$.

Further, because a^* is the *least* upper bound, then for every upper bound b_n , it must be $a^* \leq b_n \forall n \in \mathbb{N}$. Therefore, $a^* \in [a_n, b_n] \forall n \in \mathbb{N}$. That is, $a^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Remark 1.1. Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$.

Theorem 1.2 (Archimedean Property).

- (i) $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > x$;
- (ii) $\forall y \in \mathbb{R}_{++}$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Archimedean property of natural numbers can be interpreted as *there is no real number that bounds \mathbb{N}* . This interpretation can be seen by considering the negations of above statements:

- (i) $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $n \leq x$;
- (ii) $\exists y \in \mathbb{R}_{++}$ s.t. $\forall n \in \mathbb{N}$, $y \leq \frac{1}{n}$ (i.e. $n \leq \frac{1}{y}$).

Proof of (i). Suppose, for contradiction, (i) is not true, then \mathbb{N} is bounded above in \mathbb{R} .

By the completeness axiom, there exists $a^* := \sup \mathbb{N}$.

Therefore, $\exists n \in \mathbb{N}$ s.t. $a^* - 1 < n$.

In this case, $a^* < n + 1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^\uparrow$ and leads to a contradiction. ■

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ■

Remark 1.2. The two statements of Archimedean property are equivalent.

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Remark 1.3. The above theorem says \mathbb{Q} is in fact **dense** in \mathbb{R} . More generally, one says a set $A \subseteq X$ is dense whenever the closure of A , $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.

Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ■

Theorem 1.4. $\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in \mathbb{R} bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^\uparrow$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ■

2 Sequences

2.1 Definitions

Theorem 2.1 (Triangle Inequality). Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.1. Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \quad (2.1)$$

Proof. Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$.

Similarly, $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$.

Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ■

Definition 2.1. A sequence $(a_n) \subseteq \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies a_n \in V_\varepsilon(a) \quad (2.2)$$

Definition 2.2. Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at a with radius ε is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \quad (2.3)$$

Theorem 2.2. The limit of any convergent sequence is unique.

Proof. Let (a_n) be a convergent sequence, assume, for contradiction, that $(a_n) \rightarrow L_1$ and $(a_n) \rightarrow L_2$ such that $L_1 \neq L_2$.

Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \rightarrow L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$|a_n - L_2| = |a_n - L_1 - (L_2 - L_1)| \quad (2.4)$$

$$\geq ||a_n - L_1| - |L_2 - L_1|| \quad (2.5)$$

$$= ||L_1 - L_2| - |a_n - L_1|| \quad (2.6)$$

$$= 3\varepsilon - |a_n - L_1| \quad (2.7)$$

$$> 2\varepsilon \quad (2.8)$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq N'$. ■

Definition 2.3. A sequence (a_n) is **divergent** if it does not converge.

Example 2.1. The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \dots)$ is divergent.

Proof. Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \rightarrow L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $|L - \frac{1}{5}| < \frac{2}{15}$. Similarly,

$$\left| -\frac{1}{5} - L \right| = \left| \frac{1}{5} + L \right| \quad (2.9)$$

$$= \left| \frac{1}{5} + L - \frac{1}{5} + \frac{1}{5} \right| \quad (2.10)$$

$$= \left| \left(L - \frac{1}{5} \right) - \left(-\frac{2}{5} \right) \right| \quad (2.11)$$

$$\geq \left| \left| L - \frac{1}{5} \right| - \frac{6}{15} \right| \quad (2.12)$$

$$= \frac{6}{15} - \left| L - \frac{1}{5} \right| \quad (2.13)$$

$$> \frac{4}{15} \quad (2.14)$$

$$> \varepsilon \quad (2.15)$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ■

Alternative Proof. If (a_n) is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of (a_n) converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ■

Definition 2.4. A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |a_n| < M$.

Theorem 2.3. Every convergent sequence is bounded.

Proof. Let $(a_n) \rightarrow L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n < N} a_n$, take $M := \max\{Q, |L| + 1\}$, then M bounds (a_n) . ■

2.2 Limit Theorems

Theorem 2.4 (Algebraic Limit Theorem). Let $(a_n) \rightarrow a, (b_n) \rightarrow b$ be convergent sequences, and $c \in \mathbb{R}$, then

- (i) $(ca_n) \rightarrow ca$;
- (ii) $(a_n + b_n) \rightarrow a + b$;
- (iii) $(a_nb_n) \rightarrow ab$;
- (iv) $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$, provided $(b_n), b \neq 0$.

Proof (i). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

Proof (ii). Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3} \forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3} \forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \quad (2.16)$$

■

Proof (iii). Note that

$$|a_nb_n - ab| = |a_nb_n + a_nb - a_nb - ab| \quad (2.17)$$

$$\leq |a_nb_n - a_nb| + |a_nb - ab| \quad (2.18)$$

$$\leq |a_n||b_n - b| + |b||a_n - a| \quad (2.19)$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because (a_n) is convergent, let M denote its bound such that $|a_n| < M \forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_nb_n - ab| < \varepsilon$. ■

Proof (iv). *Claim i:* when n is sufficiently larger, $|b_n| > 0$ is bounded away from zero by M .

Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such n ,

$$|b_n| = |b_n - b - (-b)| \quad (2.20)$$

$$\geq ||b_n - b| - |b|| \quad (2.21)$$

$$\geq |b| - |b_n - b| \quad (2.22)$$

$$> \frac{9|b|}{10} \quad (2.23)$$

Claim ii: $\left(\frac{1}{b_n}\right) \rightarrow \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_nb} - \frac{b_n}{b_nb}\right| \quad (2.24)$$

$$= \frac{1}{|b_n||b|}|b_n - b| \quad (2.25)$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \rightarrow b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

Theorem 2.5 (Order Limit Theorem). Let $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$, then

- (i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;
- (ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;
- (iii) $\exists c \in \mathbb{R} \text{ s.t. } c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;
- (iv) $\exists c \in \mathbb{R} \text{ s.t. } a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

Proof. (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.

(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \rightarrow (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.

(iii) and (iv) Consider constant sequence defined as (c_n) such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

Theorem 2.6 (Squeeze Theorem). Let $(x_n) \rightarrow L$ and $(z_n) \rightarrow \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \rightarrow \ell$.

Remark: squeeze theorem does not impose the prior that (y_n) is convergent.

Proof. Let $\varepsilon > 0$, because both $(x_n) \rightarrow \ell$ and $(y_n) \rightarrow \ell$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \quad (2.26)$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \quad (2.27)$$

Take $N_3 := \max\{N_1, N_2\}$, then for every $n \geq N_3$,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \quad (2.28)$$

$$\implies y_n \in V_\varepsilon(\ell) \quad (2.29)$$

therefore $(y_n) \rightarrow \ell$ by definition. ■

2.3 Monotone Convergence Theorem

Definition 2.5. A sequence (a_n) is said to be **monotone** if it is either increasing ($a_{n+1} \geq a_n \forall n \in \mathbb{N}$) or decreasing ($a_{n+1} \leq a_n \forall n \in \mathbb{N}$).

Theorem 2.7 (Monotone Convergence Theorem). If a monotone sequence (a_n) is bounded, then it converges.

Proof. WLOG, assume (a_n) is increasing, let $\Gamma := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, because Γ is bounded, $s := \sup_n \Gamma$ is well-defined by the completeness of real numbers.

Claim: $(a_n) \rightarrow s$. Let $\varepsilon > 0$, by the definition of supremum, $\exists N \in \mathbb{N}$ such that $a_N > s - \varepsilon$. Because the sequence is increasing and $s + \varepsilon \in \Gamma^\uparrow$, $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$. $(a_n) \rightarrow s$ by definition. ■

2.4 Series

Definition 2.6. Let (a_i) be a sequence, then the n -th **partial sum** is defined as $s_n := \sum_{i=1}^n a_i$. And the **infinite sum/series** of (a_n) is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \rightarrow s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \quad (2.30)$$

Example 2.2. $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

Proof. Obviously the corresponding partial sums are increasing because the sequence $(\frac{1}{i^2})$ is positive.

Claim: (s_n) is bounded from above. Let $n \in \mathbb{N}$, observe

$$\sum_{i=1}^n \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \quad (2.31)$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \quad (2.32)$$

$$= 2 - \frac{1}{n} \leq 2 \quad (2.33)$$

The result is immediate by the monotone convergence theorem. ■

Example 2.3 (Harmonic Series). $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. Claim: there exists a subsequence of (s_n) diverges, so (s_n) cannot be convergent. Consider the subsequence (s_k) constructed by defining $s_k := s_{2^k}$. Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \quad (2.34)$$

$$> 1 + \frac{1}{2}k \quad (2.35)$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. Therefore, the original sequence of partial sums cannot be convergent. ■

Definition 2.7. Let (a_n) be a sequence, then for every strictly increasing sequence $(n_i)_i$ in \mathbb{N} , (a_{n_i}) is a **subsequence** of (a_n) .

Theorem 2.8. All subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \rightarrow \ell$, let (a_{n_k}) be a subsequence of (a_n) . Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\varepsilon(\ell)$. By the definition of subsequences, there exists some $K \in \mathbb{N}$ such that $n_K \geq N$. Take such K , then for every $k \geq K$, it must be $n_k \geq N$. Therefore $a_{n_k} \in V_\varepsilon(\ell)$ for every $k \geq K$, and $(a_{n_k}) \rightarrow \ell$ by definition. ■

Remark 2.1. Note the implication of above theorem is two-fold:

- (i) Every subsequence of a convergent sequence is convergent;
- (ii) All subsequences converge to the same limit.

Corollary 2.2. A sequence (a_n) must be divergent if there exists two subsequences of it converge to two different limits.

Proof. Immediate by taking the contrapositive form of above theorem. ■

Theorem 2.9 (Bolzano–Weierstrass). Every bounded sequence contains a convergent subsequence.

Proof. Suppose (a_n) is bounded by certain $M > 0$, that's, for every $n \in \mathbb{N}$, $-M < a_n < M$. Consider the split $I_1^\ell := [-M, 0]$ and $I_1^u := [0, M]$. At least one of above closed intervals contain an infinitely many elements of (a_n) .

Define the interval as I_2 . At each I_n , one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and I_{n+1} is defined to be such closed interval containing infinitely many elements.

Note that the sequence (I_n) is nested by construction. By the nested interval property, one can show that $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Also, $\lim_{n \rightarrow \infty} |I_n| = 0$. Then $\cap_{n \in \mathbb{N}} I_n$ must be a singleton with a in it. One can construct such that $a_{n_k} \in I_k$. Note that $|I_n| = \frac{1}{2^{n-1}}$, therefore, for every $\varepsilon > 0$, one can take $N \geq \log_2 \left(\frac{1}{\varepsilon}\right) + 1$, so that for every $k \geq N$, by definition of subsequences, $n_k \geq n$, so that $a_{n_k}, a \in I_N$. This implies $a_{n_k} \in V_\varepsilon(a)$ and $(a_{n_k}) \rightarrow a$. ■

2.5 Cauchy Criterion

Definition 2.8. A sequence (a_n) is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n \geq N \implies |a_n - a_m| < \varepsilon \quad (2.36)$$

Proposition 2.1. Every convergent sequence is Cauchy.

Proof. Let $(a_n) \rightarrow \ell$, let $\varepsilon > 0$. By the convergence of sequence, $\exists N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - \ell| < \frac{\varepsilon}{2}$, which turns out to imply $a_n, a_m \in V_\varepsilon(\ell)$. ■

Lemma 2.1. Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, $|a_n - a_m| < 1$. In particular, take $m = N$, for every $n \geq N$, $|a_n - a_N| < 1$, and $|a_n| \leq |a_N| + 1$. Then (a_n) is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \quad (2.37)$$

■

Theorem 2.10 (Cauchy Criterion). A sequence in \mathbb{R} is convergent if and only if it's Cauchy.

Proof. (\Leftarrow) Suppose (a_n) is Cauchy, by the lemma established above, (a_n) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence $(a_{n_k}) \rightarrow \ell$.

Claim: $(a_n) \rightarrow \ell$. Let $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $n_k, n \geq N_1$, $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$. And there exists another $N_2 \in \mathbb{N}$ such that for every $n_k \geq N_2$, $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$.

Take $N_3 := \max\{N_1, N_2\}$.

Note that for every $n \geq N_3$, one can choose some $n_k \geq n$ as leverage and derive

$$|a_n - \ell| = |a_n - a_{n_k} + a_{n_k} - \ell| \quad (2.38)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| \quad (2.39)$$

$$< \varepsilon \quad (2.40)$$

(\Rightarrow) Already shown in previous proposition. ■

2.6 Convergence Test for Series

Theorem 2.11 (n -th term test).

$$\sum_{i=1}^{\infty} a_n \text{ converges} \implies \lim_{n \rightarrow \infty} a_n = 0 \quad (2.41)$$

Remark: this theorem is only a necessary condition for convergence of series.

Proof. Suppose the partial sums converges to ℓ , by the definition of partial sums, $a_n = s_{n+1} - s_n$. Further, the convergence of partial sums guarantees the convergence of (a_n) . By taking limit on both sides of above identity, it can be shown $\lim_{n \rightarrow \infty} a_n = 0$. ■

Theorem 2.12 (Cauchy Criterion for Series). For series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:

- (i) Series converges;
- (ii) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, \left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$ (i.e. *tail* sum sequence converges);
- (iii) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m > n \geq N, \left| \sum_{k=n+1}^m a_k \right| < \varepsilon$. (i.e. partial sum is Cauchy)

Proof. (i) \implies (ii): Suppose (S_n) converges, let $\varepsilon > 0, \exists N$ s.t. $\forall n \geq N, |S_n - L| < \varepsilon$. Note that

$$L - S_n = \lim_{m \rightarrow \infty} \sum_{k=1}^m a_k - S_n \quad (2.42)$$

$$= \lim_{m \rightarrow \infty} \left[\sum_{k=1}^m a_k - S_n \right] \quad (2.43)$$

$$= \lim_{m \rightarrow \infty} \sum_{k=n+1}^m a_k \quad (2.44)$$

which implies the convergence of tail sums.

(ii) \implies (iii): Suppose the tail sum converges, let $\varepsilon > 0$, note that

$$\left| \sum_{k=n+1}^m a_k \right| = \left| \sum_{k=m+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k \right| \quad (2.45)$$

$$\leq \left| \sum_{k=m+1}^{\infty} a_k \right| + \left| \sum_{k=n+1}^{\infty} a_k \right| \quad (2.46)$$

Both terms can be made arbitrarily small by (ii), specifically, one can choose N_1 and N_2 such that both terms are strictly bounded by $\frac{\varepsilon}{2}$, and $N_3 := \max\{N_1, N_2\}$ is the desired value.

(iii) \implies (i): Since the partial sum is a Cauchy sequence in a complete space, it must converges, so the series is well-defined. ■

2.6.1 The Comparison Test

Definition 2.9. A sequence (a_n) is a **geometric sequence** with coefficient r if $a_{n+1} = ra_n$.

Proposition 2.2. Geometric sequences whenever $r \in (-1, 1)$. Note that when $r = -1$, the sequence becomes an alternating sequence, and the convergence property is indefinite.

Proposition 2.3. Let (a_n) be a geometric sequence with coefficient r , then for every $m \in \mathbb{N}$,

$$rS_m^a = ra_0 + r^2a_0 + \cdots + r^{n+1}a_0 \quad (2.47)$$

$$\implies (r-1)S_m^a = r^{n+1}a_0 - a_0 \quad (2.48)$$

$$\implies S_m^a = a_0 \frac{1 - r^{m+1}}{1 - r} \quad (2.49)$$

Theorem 2.13 (The Comparison Test). Let (a_n) and (b_n) be two sequences satisfy $|a_n| \leq b_n$ for every $n \in \mathbb{N}$. Then

- (i) $\sum_{i=1}^{\infty} b_n$ converges $\implies \sum_{i=1}^{\infty} a_n$ converges;
- (ii) $\sum_{i=1}^{\infty} a_n$ diverges $\implies \sum_{i=1}^{\infty} b_n$.

Proof. Part 1: Suppose (b_n) converges, it is therefore Cauchy. Let $\varepsilon > 0$. Note that for every $m > n$:

$$|S_m^a - S_n^a| = \left| \sum_{k=n+1}^m a_k \right| \quad (2.50)$$

$$\leq \sum_{k=n+1}^m |a_k| \quad (2.51)$$

$$\leq \sum_{k=n+1}^m b_k \quad (2.52)$$

Therefore exists $N \in \mathbb{N}$ such that $\sum_{k=n+1}^m b_k \leq \left| \sum_{k=n+1}^m b_k \right| < \varepsilon$ for every $m, n \geq N$. Taking such N provides the cutoff needed for (S_n^a) to be Cauchy. Because $(S_n^a) \subseteq \mathbb{R}$, it converges.

Part 2: The result is immediate by taking the contrapositive form of the previous statement. ■

2.6.2 The Root Test

Definition 2.10. Let (a_n) be a bounded sequence, then

$$\limsup(a_n) := \sup_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.53)$$

$$\liminf(a_n) := \inf_{n \rightarrow \infty} \{a_k : k \geq n\} \quad (2.54)$$

$$(2.55)$$

Theorem 2.14 (The Root Test). Let (a_n) be a sequence in which $a_n \geq 0$ for every $n \in \mathbb{N}$, let $\ell = \limsup a_n^{1/n}$, then

(i) If $\ell < 1$, then (S_n^a) converges;

(ii) If $\ell > 1$, then (S_n^a) diverges;

(iii) If $\ell = 0$, inconclusive.

Proof. Part 1:(Idea: compare with geometric series with $r < 1$) Suppose $\ell < 1$, pick $r \in (\ell, 1)$, and let $\varepsilon = r - \ell$. By the convergence of supremum, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\left| \sup_{k \geq n} a_k^{1/k} - \ell \right| < \varepsilon \quad (2.56)$$

$$\implies a_n^{1/n} \leq \sup_{k \geq n} a_k^{1/k} < \ell + \varepsilon =: r \quad (2.57)$$

Therefore, for every $n \geq N$, $a_n < r^n$. Because (a_n) is assumed to be a non-negative sequence, then $|a_n| < r^n$. Construct new sequences:

$$b_k = \begin{cases} a_k & \forall k < N \\ r^k & \forall k \geq N \end{cases} \quad (2.58)$$

Then, clearly $|a_n| \leq b_k$ for every $k \in \mathbb{N}$. And (b_n) is a sequence with geometric tails (which has coefficient less than one). So $\sum^\infty b_k$ converges, which implies $\sum^\infty a_k$ converges by the comparison test.

Part 2: Suppose $\ell > 1$.

Note that the necessary condition for $\sum a_n^{1/n}$ to converge is $\lim_{n \rightarrow \infty} a_n^{1/n} = 0$, which implies every subsequence of $(a_n^{1/n})$ converges to zero. We are going to prove the divergence of series by constructing a subsequence of $(a_n^{1/n})$ does not converge to zero.

Take $\varepsilon = \ell - 1 > 0$, there exists N such that for every $n \geq N$:

$$\ell - \varepsilon < \sup_{k \geq n} a_k^{1/k} \quad (2.59)$$

$$\implies 1 < \sup_{k \geq n} a_k^{1/k} \quad (2.60)$$

By definition of supremum, there exists $n_1 \geq n$ such that

$$a_{n_1}^{1/n_1} > 1 \quad (2.61)$$

For every $n \geq N$, we can construct a subsequence of $(a_n^{1/n})$ such that every term in it is strictly greater than 1, which means it cannot converge to 0. Therefore, series diverges. ■

2.6.3 Other Tests

Theorem 2.15 (Limit Comparison Test). Let $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ satisfy:

- (i) $b_n \geq 0$;
- (ii) $\limsup \frac{|a_n|}{b_n} < \infty$;
- (iii) $\sum_{n=1}^\infty b_n$ converges.

Then $\sum_{n=1}^\infty a_n$ converges as well.

Theorem 2.16 (Ratio Test). Given sequence $(a_n)_{n=1}^\infty$ such that $a_n \geq 0$, then

- 1. If $\limsup \frac{a_{n+1}}{a_n} < 1$, $\sum_{n=1}^\infty a_n$ converges;
- 2. If $\limsup \frac{a_{n+1}}{a_n} > 1$, $\sum_{n=1}^\infty a_n$ diverges.

Theorem 2.17 (Integral Test). Let $f(x)$ be a *positive* and *monotone decreasing* function on $[1, \infty)$. Consider $(f(x_n))$, then

$$\sum_{n=1}^{\infty} f(n) \text{ convergent} \iff \int_1^{\infty} f(x) dx < \infty \quad (2.62)$$

Theorem 2.18 (Alternating Series Test). For an alternating sequence $\sum_{n=1}^{\infty} (-1)^n a_n$, if $(a_n) \searrow 0$, then the series converges.

Proof. **TODO** ■

2.7 Absolute and Conditional Convergence

Corollary 2.3 (Corollary of Comparison Test). If $\sum_{i=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition 2.11. For any series $\sum_{n=1}^{\infty} a_n$, if

1. $\sum_{i=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ **converges absolutely**;
2. $\sum_{i=1}^{\infty} |a_n|$ does not converge, then $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 2.4. Alternating harmonic series converges conditionally.

However, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

Definition 2.12. $\sum_{n=1}^{\infty} b_n$ is called a **rearrangement** of series $\sum_{n=1}^{\infty} a_n$ if there exists $f : \mathbb{N} \rightarrow \mathbb{N}$ such that f is a bijection and $b_{f(k)} = a_k$ for every $k \in \mathbb{N}$.

Theorem 2.19 (Riemann Series Theorem). If series $\sum_{n=1}^{\infty} a_n$ converges conditionally, for every $\alpha \in \mathbb{R}$, there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ converges to α .

Proof. The proof is non-trivial and omitted. ■

Theorem 2.20. If series $\sum_{n=1}^{\infty} a_n$ converges absolutely to some value $A \in \mathbb{R}$, then every rearrangement $\sum_{n=1}^{\infty} b_n$ converges to A .

Proof. Define partial sum sequences

$$S_n := \sum_{k=1}^n a_k \quad T_m := \sum_{k=1}^m b_k \quad (2.63)$$

Suppose $(S_n) \rightarrow A$, want to show: $(T_n) \rightarrow A$.

Let $\varepsilon > 0$ fixed.

By convergence of (S_n) , there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |S_n - A| < \frac{\varepsilon}{2} \quad (2.64)$$

Because $\sum_{n=1}^{\infty} a_n$ converges absolutely, by the Cauchy criterion for convergent series (i.e. partial sum sequence is Cauchy), there exists $N_2 \in \mathbb{N}$ such that

$$n > m \geq N_2 \implies \sum_{k=n+1}^m |a_k| < \frac{\varepsilon}{2} \quad (2.65)$$

Define $N := \max\{N_1, N_2\}$, $M := \max\{f(k) : 1 \leq k \leq N\}$.

$$|T_m - S_N| = |b_1 + \cdots + b_m - a_1 - \cdots - a_N| \quad (2.66)$$

$$= |b_1 + \cdots + b_m - b_{f(1)} - \cdots - b_{f(N)}| \quad (2.67)$$

Note that for every $m \geq M$, by construction, $\{b_{f(1)}, \dots, b_{f(N)}\} \subseteq \{b_1, \dots, b_m\}$.

Note that for each $b_{f(k)} \in \{b_1, \dots, b_m\}$, either $k > N$ or $k \leq N$. But all $b_{f(k)}$ with $k \leq N$ were subtracted, so $b_{f(k)}$ elements left are all from $\{a_k : k \geq N+1\}$.

$$\dots = \left| \sum_{k \in \mathcal{I} \geq N+1} a_k \right| \quad (2.68)$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \quad (2.69)$$

Therefore, for all $m \geq M$,

$$|T_m - A| = |T_m - S_n + S_n - A| \quad (2.70)$$

$$\leq |T_m - S_n| + |S_n - A| \quad (2.71)$$

$$< \varepsilon \quad (2.72)$$

The desired result is immediate. ■

3 Topology in \mathbb{R}

3.1 Definitions

Definition 3.1. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** if

$$\forall x \in \mathcal{O} \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(x) \subseteq \mathcal{O} \quad (3.1)$$

Theorem 3.1. Arbitrary union of open sets is open; Any finite intersection of open sets is open.

Proof. Let \mathcal{O}_α open for all $\alpha \in \mathcal{A}$. Let $\mathcal{O} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha$. If $x \in \mathcal{O}$, there exists some $\alpha \in \mathcal{A}$ such that $x \in \mathcal{O}_\alpha$. There exists $V_\varepsilon(x) \subseteq \mathcal{O}_\alpha \subseteq \mathcal{O}$. Hence \mathcal{O} is open.

Let $\{\mathcal{O}_i : 1 \leq i \leq n\}$ be a collection of open sets, let $\mathcal{O} := \bigcap_{i=1}^{\infty} \mathcal{O}_i$. If $x \in \mathcal{O}$, there exists $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(x) \subseteq \mathcal{O}_i$ for every i . Take $\varepsilon := \min\{\varepsilon_i\}$, which exists and is strictly positive by finiteness of index set. Therefore $V_\varepsilon(x) \subseteq \mathcal{O}_i$ for every i , and therefore in \mathcal{O} . ■

Definition 3.2. x is a **limit point** of A if $\forall \varepsilon > 0$,

$$V_\varepsilon(x) \cap A - \{x\} \neq \emptyset \quad (3.2)$$

Remark: this definition does not require x to be an element of A .

Theorem 3.2. x is a limit point of A if and only if there exists a sequence $(a_n)_{n=1}^\infty \subseteq A$ such that $a_n \neq x \forall n \in \mathbb{N}$ and $(a_n)_{n=1}^\infty \rightarrow x$.

Proof. (\implies) Let x be a limit point, take $\varepsilon = \frac{1}{n}$, immediate by the definition of limit point.

(\impliedby) Trivially by definition of sequential convergence. ■

Definition 3.3. $X \subseteq \mathbb{R}$ is **closed** if it contains all its limit points.

Definition 3.4. $x \in A$ is an **isolated point** if it is not a limit point of A .

Definition 3.5. The **closure** of A , denoted as \overline{A} , is defined to be the union of A and all limit points of A .

Definition 3.6. $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Theorem 3.3. Let $x \in \mathbb{R}$, there exists a sequence $(q_n)_{n=1}^\infty \subseteq \mathbb{Q}$ such that $(q_n)_{n=1}^\infty \rightarrow x$.

Proof. Let $x \in \mathbb{R}$. Note that $\forall u < v \in \mathbb{R}$, there exists $q \in (u, v) \cap \mathbb{Q}$. Hence, for every $n \in \mathbb{N}$, $\exists q_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < q_n < x + \frac{1}{n}$. It is evident that $(q_n)_{n=1}^\infty \rightarrow x$. ■

Lemma 3.1. \overline{A} is the smallest closed set containing A .

Proof. It is evident that \overline{A} is a closed set containing A .

Now show the closure is in fact the smallest closed set. Let $B \subsetneq \overline{A}$ be a proper subset of the closure, we are going to show that B is not closed. Let $x \in \overline{A} - B \neq \emptyset$.

Note that $\overline{A} \equiv A \cup A'$, then either $x \in A$ or $x \in A'$. If $x \in A$, then B does not contain A . If $x \in A'$, then B does not contain all limit points of A , so it is not closed. ■

Theorem 3.4. Equivalent definitions of openness and closedness:

(i) \mathcal{O} is open if and only if \mathcal{O}^c is closed;

(ii) \mathcal{F} is closed if and only if \mathcal{F}^c is open.

Proof. (\implies) Let \mathcal{O} be an open set, let $(x_n) \rightarrow x$ be a convergent sequence in \mathcal{O}^c . It is evident that if $x \in \mathcal{O}$, infinitely many elements in the tail of (x_n) would be in $V_\varepsilon(x) \subseteq \mathcal{O}$, which leads to a contradiction. Therefore \mathcal{O}^c contains all of its limit points, and \mathcal{O}^c is therefore closed.

(\impliedby) Let \mathcal{F}^c be a closed set, suppose \mathcal{F} is not open, there exists $x \in \mathcal{F}$ such that for all $\varepsilon > 0$, $V_\varepsilon(x) \cap \mathcal{F}^c \neq \emptyset$. Then we can construct a sequence in \mathcal{F}^c converge to x , which leads to a contradiction that there is a limit point of a sequence in \mathcal{F}^c not contained by \mathcal{F}^c .

The second part is immediate. ■

Theorem 3.5. Any intersection of closed sets is closed; any finite union of closed sets is closed.

Proof. Direct result from De Morgan's law and the previous theorem. ■

Remark: Limit points and boundary points are completely different. Example: let $\Omega = [1, 2] \cup 3$, then 3 is a boundary point but not a limit point (i.e. it is isolated). And 0.5 is a limit point but not a boundary point.

3.2 Compactness

Definition 3.7. A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a convergent subsequence converges to some limit $x \in K$.

Theorem 3.6. A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. (\implies) Suppose $K \subseteq \mathbb{R}$ is compact.

Show K is bounded: suppose, for contradiction, K is unbounded, then for every $N \in \mathbb{N}$, one can construct a sequence as following: $a_1 \in K$ and $a_{n+1} > \max\{a_n, n\}$. Such sequence diverges to positive infinity, and every subsequence of it converges to infinity as well (easy to verify). This leads to a contradiction to the compactness of K .

Show K is closed: Suppose, for contradiction, K is not closed, then there exists some limit point of K say $x \notin K$. Consider the sequence $(x_n) \rightarrow x$ in K , because every subsequence of such convergent sequence converges to the same limit $x \notin K$, which leads to a contradiction of compactness.

(\impliedby) Let $(x_n) \subseteq K$, then (x_n) is bounded and therefore possesses a convergent subsequence by Bolzano-Weierstrass Theorem. Further, because K is closed, then the limit point must be in K . ■

Theorem 3.7 (Nested Compact Set Property). Let $\mathbb{R}^n \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$, where $K_n \neq \emptyset$ are all compact sets, then

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset \quad (3.3)$$

Proof. Construct a sequence such that $x_n \in K_n$ for every $n \in \mathbb{N}$. In particular, $(x_n) \subseteq K_1$. Because K_1 is compact, it has a convergent subsequence $(x_{n_k}) \rightarrow x \in K_1$. Then every subsequence of (x_{n_k}) converges to the same limit x .

Note that by dropping out the first element of the subsequence, the resulted sequence starts with x_{n_2} . By the definition of subsequences, $n_2 \geq 2$, therefore, the truncated subsequence is contained in K_2 because of the compactness of K_2 . As a result, $x \in K_2$. Applying the same argument on all natural numbers, it is immediate that $x \in K_n \forall n \in \mathbb{N}$. So $x \in \bigcap_{n \in \mathbb{N}} K_n$. ■

Proof. (Cantor's Argument). Suppose, for contradiction, the intersection is empty. Define $U_n := K_1 \setminus K_n$. Note that $U_n = K_1 \cap K_n^c = K_n^c$, which is open. Further, $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} K_1 \cap K_n^c = K_1 \cap (\bigcup_{n \in \mathbb{N}} K_n^c) = K_1 \cap (\bigcap_{n \in \mathbb{N}} K_n)^c = K_1 \setminus \bigcap_{n \in \mathbb{N}} K_n = K_1$. Therefore, $\mathcal{C} = \{U_n : n \in \mathbb{N}\}$ is an open cover of K_1 . Because K_1 is compact, there exists a finite subcover of \mathcal{C} . Take n^* to be the greatest index in this finite subcover, then for every $x' \in K_{n^*+1} \subseteq K_1$, x' is not in the union of the constructed subcover, which leads to a contradiction. ■

Example 3.1. Note that the closedness itself is not sufficient for the nest compact set property to hold. For instance, the following sequence of closed sets are nested: $F_n := [n, \infty)$, but indeed, for every $x \in \mathbb{R}$, there exists a natural number $n > x$, so that $x \notin \bigcap_{n \in \mathbb{N}} F_n$. Therefore, $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$.

Definition 3.8. Let $A \subseteq \mathbb{R}$, an **open cover** for A is a collection of open sets $\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ such that $A \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$.

Theorem 3.8 (Heine-Borel). Let $K \subseteq \mathbb{R}$, then the following are equivalent:

- (i) K is (sequentially) compact;
- (ii) K is closed and bounded;
- (iii) Every open cover of K has a finite subcover.

Proof. The equivalence of (i) and (ii) has been proven previously.

Show (iii) \implies (ii): suppose every open cover of K has a finite subcover, consider the following cover of K : $\mathcal{C} = \{[-n, n] : n \in \mathbb{N}\}$. Let M be the greatest index in the finite subcover \mathcal{C} , and obviously K is bounded by M .

Suppose, for contradiction, that K is not closed. Let y be a limit point of K but $y \notin K$. Then, for every $\varepsilon > 0$, $V_\varepsilon^o(y) \cap K \neq \emptyset$. We've shown that K is bounded, take $M \in \mathbb{R}$ such that $(-M, M) \supset K$. Define the following cover:

$$\mathcal{C} := \left\{ (-M, M) \setminus \overline{V_\varepsilon(y)} : \varepsilon \in \mathbb{R}_{++} \right\} \quad (3.4)$$

Because K is compact, there exists a finite subcover of \mathcal{C} , which is clearly a contradiction.

Show (ii) \implies (iii): Suppose K is closed and bounded, because of the transitivity of covering, it is sufficient to show that for every $M \in \mathbb{R}_+$, every open cover of $[-M, M]$ has a finite subcover.

Let $M \in \mathbb{R}_+$, and $\mathcal{C} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ is an open cover of $[-M, M]$. Suppose, for contradiction, there is no finite subcover. Then either $[-M, 0]$ or $[0, M]$ does not have a finite subcover from \mathcal{C} . Define such interval as I_1 . Interval I_n is defined inductively from I_{n-1} by firstly bisecting I_{n-1} into two closed intervals and then taking the partition that cannot be covered by any finite subcover of \mathcal{C} . Note that (I_n) is a sequence of nested compact sets, by Cantor's intersection theorem, there intersection is nonempty. Further, because the length of interval shrinks to zero as $n \rightarrow \infty$, the intersection must be a singleton. Let $\{x\} = \bigcap_{n \in \mathbb{N}} I_n$, there exists some $\lambda \in \Lambda$, such that $x \in \mathcal{O}_\lambda$. Because \mathcal{O}_λ is open, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$. Take $k \in \mathbb{N}$ such that $|I_k| < 2\varepsilon$, clearly $I_k \subseteq V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$. Then \mathcal{O}_λ is a finite subcover of I_k , which leads to a contradiction. ■

3.3 Connected Sets

Definition 3.9. $\emptyset \neq A, B \subseteq \mathbb{R}$ are **separated** if and only if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Definition 3.10. $E \subseteq \mathbb{R}$ is **disconnected** if $E = A \cup B$ where A, B are nonempty separated sets.

Proposition 3.1 (Equivalent Definiton). $E \subseteq \mathbb{R}$ is disconnected if and only if it can be expressed as the union of two *nonempty disjoint open* (open in E) sets.

Theorem 3.9. A set $E \subseteq \mathbb{R}$ is connected if for every nonempty disjoint sets A, B such that $E = A \cup B$, then there exists a sequence $(a_n) \subseteq A$ converges to some point $a \in B$, or a sequence $(b_n) \subseteq B$ converges to some point $b \in A$.

Remark: Essentially, a set E is connected if, no matter how it is partitioned into two nonempty disjoint sets, it is always possible to show that at least one of the sets contains a limit point of the other.

Proof. Suppose E is connected, then for any partition $E = A \cup B$, either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. WLOG, $\overline{A} \cap B \neq \emptyset$, take $a \in \overline{A} \cap B \subseteq B$, and there exists a convergent sequence in A converging to $a \in B$. ■

Theorem 3.10. Let $E \subseteq \mathbb{R}$, the following are equivalent:

- (i) E is connected;
- (ii) For every $a < c < b$, $a, b \in E \implies c \in E$ (intervals).

Remark: the collection of connected subsets of \mathbb{R} is exactly the collection of all intervals.

Proof. (\implies) Suppose E is connected, considering the following sets

$$A := (-\infty, c) \cap E \tag{3.5}$$

$$B := (c, \infty) \cap E \tag{3.6}$$

Note that $a \in A$ and $b \in B$, so both of them are nonempty. And A and B are separated. Suppose, for contradiction, $c \notin E$, $E = A \cup B$, which leads to a contradiction to the assumption that E is connected.

(\impliedby) Suppose (ii), show E is connected. Let A and B be two nonempty set such that $A \cup B = E$ and $A \cap B = \emptyset$. We are going to show that A and B must be separated in this case. Let $a_0 \in A$ and $b_0 \in B$, WLOG, suppose $a_0 < b_0$. By (ii), the entire interval $[a_0, b_0] \subseteq E$. Split $[a_0, b_0]$ into two half intervals $[\alpha, \beta]$ and $[\beta, \gamma]$. Note that it is impossible for $\{\beta\}$ to be the only point intersect both A and B , because in this case A and B cannot be disjoint. Take the one intersects both A and B , denoted as $[a_1, b_1]$.

One can construct a sequence of closed intervals inductively, such that every I_n intersects both A and B . Also, previous result shows that $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$, and is in fact a singleton. Let $x \in \cap_{n \in \mathbb{N}} I_n$, if $x \in A$, then there exists $(b_n) \subseteq B$ such that $(b_n) \rightarrow x$. Similarly, if $x \in B$, there exists $(a_n) \subseteq A$ such that $(a_n) \rightarrow x$. As a result, either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Therefore, E is connected. ■

3.4 Cantor Set

Definition 3.11. Define sequence of sets

$$S_0 = [0, 1] \tag{3.7}$$

$$S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{3.8}$$

inductively, where S_n is defined by removing the mid-one-third of elements from each component of S_{n-1} . The **Cantor set** is defined as

$$\mathcal{C} := \bigcap_{n \in \mathbb{N}} S_n \neq \emptyset \quad (3.9)$$

\mathcal{C} is nonempty because each S_n is a finite union of closed set. Altogether with the fact that each of S_n is bounded, so \mathcal{C} is an intersection of nested compact sets. Therefore, \mathcal{C} is nonempty by Cantor's intersection theorem.

Definition 3.12. A set is called **perfect** if it is closed and has no isolated point.

Proposition 3.2. \mathcal{C} has measure zero.

Proof. Note that on while constructing S_n , intervals with total length of $\frac{2^n}{3^{n+1}}$ are removed from S_{n-1} . To construct a Cantor set, the total length of intervals from $[0, 1]$ equals

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 \quad (3.10)$$

Therefore the length left for Cantor set is zero. ■

Proposition 3.3. $\mathcal{C}^{int} = \emptyset$.

Proof. Note that for any set to have nonempty interior, it must contains some open intervals. Claim: for every open interval (a, b) , it cannot be contained in \mathcal{C} . Let $a < b$, note that for every partition of S_n has length $\frac{1}{3^n}$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{3^n} < b - a$. Therefore, $(b - a) \not\subseteq S_n$ for such n . So that \mathcal{C} cannot contain any open interval. ■

Proposition 3.4. \mathcal{C} is closed.

Proof. \mathcal{C} is the intersection of infinitely many closed sets, so it is closed. ■

Proposition 3.5. \mathcal{C} is compact.

Proof. \mathcal{C} is bounded by $[0, 1]$ and closed by previous proposition. Therefore, $\mathcal{C} \subseteq \mathbb{R}$ is compact. ■

Proposition 3.6. \mathcal{C} is perfect.

Proof. We are going to show that every point $x \in \mathcal{C}$ is the limit of some sequence in \mathcal{C} .

Case 1: x is not the right endpoint of any intervals in S_n for any $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, let x_n be the right endpoint of the interval in S_n containing x . Obviously, $(x_n) \rightarrow x$.

Case 2: x is the right endpoint of some closed interval in some S_n (the implication of this observation is that we cannot simply take the right endpoint, so take the left endpoint). For every $n \in \mathbb{N}$, take x_n to be the left end of S_n containing x . Clearly, $(x_n) \rightarrow x$. ■

Theorem 3.11. Any nonempty perfect set P is uncountable.

Proof. Note that P is obviously not finite. Suppose, for contradiction, P , then there exists an enumeration of $P = \{x_1, x_2, \dots, x_n, \dots\}$. Construct a sequence of compact sets as following: take $\varepsilon > 0$, there exists $y_1 \neq x_1$ such that $y_1 \in P \cap [x_1 - \varepsilon, x_1 + \varepsilon]$. Let $\delta_1 := \frac{|y_1 - x_1|}{2}$, and take $K_1 := [y_1 - \delta_1, y_1 + \delta_1] \cap P$. **TODO: Show K_1 is compact.** Note that $x_1 \notin K_1$.

Apply the same argument on K_1 to construct K_2 such that $x_2 \notin K_2$, so that $P \supset K_1 \supset K_2 \supset \dots$. By construction, no points in P is in the intersection $\bigcap_{n \in \mathbb{N}} K_n$. However, the intersection is nonempty and the element belongs to the intersection is clearly in P , which is a contradiction. ■

Proposition 3.7. \mathcal{C} is uncountable.

Proof. \mathcal{C} is a nonempty perfect set, so it is uncountable. ■

Another Direct Proof. Consider the ternary expansions of all numbers in \mathcal{C} , it is easy to observe that for every $x \in \mathcal{C}$, if the ternary expansion of x contains 1 somewhere, x cannot be in \mathcal{C} .

Therefore, \mathcal{C} has the same cardinality as the collection of all binary expansions, which has the same cardinality of \mathbb{R} (refer to the construction of real numbers). ■

4 Functional Limits and Continuity

Definition 4.1. Let $f : A \rightarrow \mathbb{R}$ be a function, let c be a limit point of domain A , then $\lim_{x \rightarrow c} f(x) = L$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } x \in V_\delta^o(c) \implies f(x) \in V_\varepsilon(L) \quad (4.1)$$

Remark: The definition of continuity is stated in terms of punctuated ball, however, it is often easier to argue $x \in V_\delta(c) \implies f(x) \in V_\varepsilon(L)$.

Example 4.1. Let $g(x) = x^2$, show that $\lim_{x \rightarrow 2} g(x) = 4$.

Proof. Let $\varepsilon > 0$, note that for all $\delta < 1$, for all $x \in V_\delta^o(2)$,

$$|x^2 - 4| = |x - 2| |x + 2| \quad (4.2)$$

$$|x| = |x - 2 + 2| \leq |x - 2| + 2 < 3 \quad (4.3)$$

$$|x + 2| \leq |x| + 2 < 5 \quad (4.4)$$

$$\implies |x^2 - 4| < 5\delta \quad (4.5)$$

Take $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{5}\}$, both inequality reasoning (because $\delta < 1$) and ε requirement are valid. ■

Theorem 4.1 (Sequential Criterion for Functional Limits). Given a function $f : A \rightarrow \mathbb{R}$ and $c \in A'$, then the following are equivalent:

- (i) $\lim_{x \rightarrow c} f(x) = L$;
- (ii) $\forall (x_n) \subseteq A \setminus \{c\}$ such that $(x_n) \rightarrow c$, $(f(x_n)) \rightarrow L$.

Proof. (i) \implies (ii): assume $f(x) \rightarrow L$, let $(x_n) \subseteq A \setminus \{c\}$ be an arbitrary convergent sequence with limit c .

Let $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in V_\delta^0(c)$, $f(x) \in V_\varepsilon(L)$.

Consider such δ , by the convergence of sequence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V_\delta(c)$.

Moreover, note that $x_n \neq c \ \forall n \in \mathbb{N}$, therefore $n \geq N \implies x_n \in V_\delta^o(c)$, which further implies $f(x_n) \in V_\varepsilon(L)$ by the limit property of f .

(ii) \implies (i): assume, for contradiction, $\lim_{x \rightarrow c} f(x) \neq L$.

Negating the definition of functional limit gives

$$\exists \varepsilon^* > 0 \text{ s.t. } \forall \delta > 0 \ \exists x_\delta \in V_\delta^o(c) \text{ s.t. } f(x_\delta) \notin V_{\varepsilon^*}(L) \quad (4.6)$$

For every $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$, and define $x_n := x_\delta$ from above statement.

Clearly, $(x_n) \rightarrow c$ by construction, but $(f(x_n))$ is bounded away from L by $\varepsilon^* > 0$. This leads to a contradiction of (ii). \blacksquare

Theorem 4.2 (Convergence Criterion for Functional Limits). Let $f : A \rightarrow \mathbb{R}$ and $c \in A'$. If there exists two sequences $(x_n), (y_n) \subseteq A \setminus \{c\}$ converging to c , but $\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n)$, then $\lim_{x \rightarrow c} f(x)$ does not exist.

Proof. In the previous theorem, the negation of (ii) proposes exactly the existence of two convergent sequences in $A \setminus \{c\}$ converging to the same limit c but their image sequences does not converge to the same limit. The result is immediate by taking the contraposition of (i) \implies (ii) part. \blacksquare

Example 4.2. Limit of $f(x) := \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at 0 does not exist.

Example 4.3 (Dirichlet Function). Limit of $f(x) := \mathbb{1}\{x \in \mathbb{Q}\}$ does not exist anywhere in \mathbb{R} .

Example 4.4. Limit of $f(x) := x\mathbb{1}\{x \in \mathbb{Q}\}$ only exists at $x = 0$.

Theorem 4.3 (Characterizations of Continuity: Alternative Notations). Let $f : A \rightarrow \mathbb{R}$, $c \in A$, then f is continuous at c if and only if one of the following holds:

- (i) $\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon;$
- (ii) $\forall V_\varepsilon(f(c)) \ \exists V_\delta(c) \text{ s.t. } x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(c));$
- (iii) $\forall A \supseteq (x_n) \rightarrow c \in A' \ (f(x_n)) \rightarrow f(c).$

Proposition 4.1 (Criterion of Discontinuity). Let $f : A \rightarrow \mathbb{R}$, $c \in A'$, if there exists sequence $(x_n) \subseteq A$ converges to c but $(f(x_n)) \not\rightarrow f(c)$, then f is not continuous at c .

Example 4.5 (Thomae's Function). Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \wedge \gcd(p, q) = 1 \\ 1 & \text{if } x = 0 \end{cases} \quad (4.7)$$

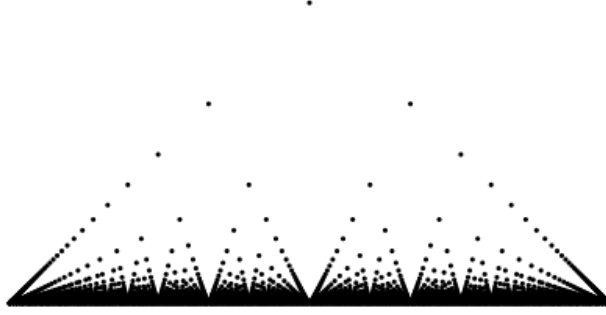


Figure 1: Thomae's Function in the Unit Interval

Proposition 4.2. For every $a \in \mathbb{R}$, $\lim_{x \rightarrow a} f(x) = 0$. As a result, $D_f = \mathbb{Q}$.

Proof. WLOG, consider the domain $a \in (0, 1)$ only, show:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in V_\delta^o(a), f(x) \in V_\varepsilon(0) \quad (4.8)$$

Fix $\varepsilon > 0$.

Note that there exists $N \in \mathbb{N}$ such that $n \geq N \implies \left| \frac{1}{n} \right| < \varepsilon$.

Because \mathbb{Q} is countable, define finite set L following Cantor's diagonal order

$$L := \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{N-2}{N-1} \right\} \setminus \{a\} \quad (4.9)$$

That is, L contains every rational number such that its denominator in the lowest form is less than N excluding a (if a is rational).

Define $m := \min_{q_i \in L} |a - q_i|$, which is well-defined because L is finite.

Take $\delta := \frac{m}{2}$, note that $V_\delta(a) \cap \mathbb{Q} \cap L = \emptyset$ by construction.

Let $x \in V_\delta^o(a)$, either

- (i) $x \in \mathbb{Q} \implies x \notin L \implies x = \frac{p}{q}$ where $q \geq N$, which implies $f(x) = \frac{1}{q} < \varepsilon$;
- (ii) or $x \notin \mathbb{Q} \implies f(x) = 0 < \varepsilon$.

Either case implies the limit to be zero.

Therefore f is discontinuous on \mathbb{Q} . ■

Theorem 4.4. Composition of continuous functions is continuous.

Given $f : A \rightarrow \mathbb{R}$, $g : B \rightarrow \mathbb{R}$ such that the range $f(A) \subseteq B$. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$. Then $g \circ f$ is continuous at c .

Proof. Let $\varepsilon > 0$, and $g(x)$ is continuous at $f(c)$.

$$\exists \tilde{\delta} \text{ s.t. } f(x) \in V_{\tilde{\delta}}(f(c)) \implies g \circ f(x) \in V_\varepsilon(g \circ f(c)) \quad (4.10)$$

$$\exists \delta > 0 \text{ s.t. } x \in V_\delta(c) \implies f(x) \in V_{\tilde{\delta}}(f(c)) \quad (4.11)$$

■

4.1 Continuous Functions on Compact Sets

Theorem 4.5. Let K be a compact set in \mathbb{R} , and $f : K \rightarrow \mathbb{R}$ is a continuous function, then $f(K)$ is compact.

Proof. Let $(y_n) \subseteq K$. Consider $f^{-1}(y_n) \neq \emptyset$, take $x_n \in f^{-1}(y_n)$ to construct a sequence. Because K is compact, there exists a subsequence of (x_n) converges to $x \in K$. Since f is continuous, $f(x_n)$ converges to $f(x) \in f(K)$. ■

Theorem 4.6 (Extreme Value Theorem). If $f : K \rightarrow \mathbb{R}$ is continuous on a compact set $K \subset \mathbb{R}$. Then f attains a maximum and minimum value. That is,

$$\exists x_0, x_1 \in K, \text{ s.t. } f(x_0) \leq f(x) \leq f(x_1) \forall x \in K \quad (4.12)$$

Proof. $f(K)$ is compact, in particular it is bounded in \mathbb{R} . So it possesses a supremum $\sup f(K)$. Then there exists a sequence converging to the supremum. Because $f(K)$ is bounded as well, its limit can be attained in K . Therefore the maximum is attainable (minimum case is similar). ■

4.2 Uniform Continuity

Example 4.6 (Uniformly Continuous). $f(x) = 3x + 1$ is continuous for every $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$ gives the definition of continuity. Note that δ does not depend on particular c . ■

Example 4.7 (Non-uniformly Continuous). $y = x^2$ is continuous for every $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$,

Note that $|x + c| = |x - c + 2c| \leq |x - c| + 2|c|$.

Suppose $\delta \leq 1$ (the following argument works for $x \in V_1(c)$ only):

$$|x^2 - c^2| = |x - c| |x + c| \quad (4.13)$$

$$\leq \delta |x + c| \quad (4.14)$$

$$\leq \delta(|x - c| + 2|c|) \quad (4.15)$$

$$\leq \delta(1 + 2|c|) < \varepsilon \quad (4.16)$$

Take $\delta := \min \left\{ 1, \frac{1}{1+2|c|} \right\}$. Note that δ depends on particular realization c . ■

Definition 4.2. A function f is **uniformly continuous** on $A \subset \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x, y \in \mathbb{R} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad (4.17)$$

Theorem 4.7 (Sequential Criterion for Absence of Uniform Continuity). Let $f : A \rightarrow \mathbb{R}$, the following are equivalent:

- (i) f fails to be uniformly continuous on A ;

- (ii) $\exists \varepsilon_0 > 0$ and two sequences $(x_n), (y_n) \subseteq A$ such that $|x_n - y_n| \rightarrow 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for every $n \in \mathbb{N}$.

Proof. $f : A \rightarrow \mathbb{R}$ is not uniformly continuous,

iff $\exists \varepsilon > 0$ s.t. $\forall \delta > 0 \exists x, y \in \mathbb{R}$ s.t. $|x_\delta - y_\delta| < \delta \wedge |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$.

For each $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$ and construct two sequences. ■

Example 4.8. Let $f(x) = \sin(x^{-1})$ defined on $A = (0, 1)$.

Take $x_n := \frac{1}{2n + \frac{\pi}{2}}$ and $y_n = \frac{1}{2n + \frac{3\pi}{2}}$. It is evident that $f(x_n) = 1$ and $f(y_n) = -1$ for every $n \in \mathbb{N}$. By taking $\varepsilon_0 = 2$, the proof is complete.

Example 4.9. $f(x) = x^2$ is not uniformly continuous.

Consider two sequences $(x_n) = (n)$ and $(y_n) = (n + \frac{1}{n})$.

Note that $|x_n - y_n| = |\frac{1}{n}| \rightarrow 0$.

Moreover, $|f(x_n) - f(y_n)| = |2 + \frac{1}{n^2}| > 2$.

Therefore f is not uniformly continuous.

Theorem 4.8 (Uniform Continuity on Compact Set). A continuous function f on a compact K is uniformly continuous.

Proof. Suppose, for contradiction, f is continuous but not uniformly continuous.

Then there exists $\varepsilon_0 > 0$ and $(x_n), (y_n) \subseteq K$ such that $|x_n - y_n| \rightarrow 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Because K is compact, there exists subsequences of (x_n) and (y_n) converging to $x, y \in K$.

Suppose $x \neq y$, it contradicts $|x_n - y_n| \rightarrow 0$ because this property holds for subsequences as well.

Therefore, (x_{n_k}) and (y_{n_k}) converge to the same limit.

Because f is continuous, $\lim_{k \rightarrow \infty} [f(x_{n_k}) - f(y_{n_k})] = 0 < \varepsilon_0$, contradiction. ■

4.3 Intermediate Value Theorem

Theorem 4.9. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, if $L \in \mathbb{R}$ such that either

(i) $L \in (f(a), f(b))$,

(ii) or $L \in (f(b), f(a))$.

Then $\exists c \in (a, b)$ such that $f(c) = L$.

Lemma 4.1. All the connected subsets of \mathbb{R} are intervals.

Theorem 4.10. Continuous map preserves connectedness. That is, for some $f : G \rightarrow \mathbb{R}$ continuous with $G \subseteq \mathbb{R}$, then $E \subseteq G$ connected $\implies f(E)$ connected.

Proof. Suppose $f(E) = A \cup B$, where $A, B \neq \emptyset$ and disjoint.

Note that $E = f^{-1}(A) \cup f^{-1}(B)$ with $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ (evident using contradiction).

Because E is connected, either there exists a convergent sequence in $f^{-1}(A)$ converges to somewhere in $f^{-1}(B)$ or a convergent sequence in $f^{-1}(B)$ converges to somewhere in $f^{-1}(A)$.

WLOG, suppose there exists a sequence $(x_n) \in f^{-1}(A)$ converges to some point $x \in f^{-1}(B)$.

Because f is continuous, $\lim_{n \rightarrow \infty} f(x_n) = f(x) \in B$.

That is, there exists sequence $(f(x_n)) \subseteq A$ converges to $f(x) \in B$.

E is connected. ■

Proof of IVT.. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function.

WLOG, suppose $f(a) < f(b)$.

Let $L \in (f(a), f(b))$.

Note that $f([a, b])$ is connected by previous theorem, therefore $f([a, b])$ is an interval by previous lemma.

hence, $f([a, b])$ contains every point between $f(a)$ and $f(b)$, in particular L .

Therefore, $\exists c \in [a, b]$ with $f(c) = L$.

Note that $f(a), f(b) \neq L$, so $c \neq a, b$.

Therefore, the chosen $c \in (a, b)$. ■

Remark 4.1. EVT says the continuous functions' image of a compact set is compact.

IVT says the continuous functions' image of a connected set is connected.

Definition 4.3. A function f possesses the **intermediate value property** on closed interval $[a, b]$ if for every $x < y$ in $[a, b]$ and $f(x) < L < f(y)$, it is always possible to find $c \in (a, b)$ such that $f(c) = L$.

Theorem 4.11 (IVT restated). f is continuous $\implies f$ satisfies IVP.

Remark 4.2. The converse of IVT is not true.

Consider

$$g(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (4.18)$$

$g(x) : [-1, 1] \rightarrow [-1, 1]$ satisfies IVP, but $g(x)$ is not continuous on $[-1, 1]$, in particular, $g(x)$ is discontinuous at 0.

Proposition 4.3. If a sequence of functions (f_n) converges uniformly to f , then (f_n) converges point-wise to f as well.

Example 4.10. $f_n(x) - \frac{(x^2 + nx)}{n}$ has point-wise limit to $f(x) = x$ but it does not converge uniformly.

5 Sequences and Series of Functions

5.1 Sequences of Functions

Definition 5.1. For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$. The sequence $(f_n)_{n \in \mathbb{N}}$ **converges point-wise** on A to a function $f : A \rightarrow \mathbb{R}$ if

$$\forall x \in A, (f_n(x)) \rightarrow f(x) \quad (5.1)$$

That is,

$$\forall x \in A, \forall \varepsilon > 0 \exists N_x \in \mathbb{N} \text{ s.t. } \forall n \geq N_x, f_n(x) \in V_\varepsilon(f(x)) \quad (5.2)$$

That is, the sequence induced at each $x \in A$ is convergent, and the value of N_x can depend on specific x .

Definition 5.2. Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$. Then (f_n) **converges uniformly** on A to $f : A \rightarrow \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall x \in A \forall n \geq N, f_n(x) \in V_\varepsilon(f(x)) \quad (5.3)$$

The uniform convergence requires a single N to work for every $x \in A$.

Theorem 5.1 (Cauchy Criterion). Given a sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$, the following are equivalent:

- (i) (f_n) converges uniformly on A ;
- (ii) (**Uniformly Cauchy**) $\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |f_n(x) - f_m(x)| < \varepsilon \forall m, n \geq N, \forall x \in A$.

Proof. (\implies)

The sufficient condition is immediate by triangle inequality.

Suppose $(f_n) \xrightarrow{\text{unif}} f$, let $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\forall n \geq N, |f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for every $x \in A$.

Take the same N and let $m, n \geq N$, let $x \in A$,

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \quad (5.4)$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \quad (5.5)$$

$$< \varepsilon \quad (5.6)$$

(\impliedby)

Assume (f_n) is uniformly Cauchy.

Let $\varepsilon > 0$.

$\forall x \in A, (f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} .

By the completeness of \mathbb{R} , $(f_n(x)) \rightarrow f(x)$ for some point-wise limit $f(x)$.

We are going to show (f_n) converges uniformly to the point-wise limit.

By uniform Cauchy, $\exists N \in \mathbb{N}$ such that $\forall m \geq N, n = m + k \geq N$, and $\forall x \in A$,

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2} \quad (5.7)$$

Fix $x \in A$,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (5.8)$$

$$\implies |f_m(x) - f(x)| = \lim_{n \rightarrow \infty} |f_m(x) - f_n(x)| \quad (5.9)$$

$$= \lim_{k \rightarrow \infty} |f_m(x) - f_{m+k}(x)| \quad (5.10)$$

$$\leq \frac{\varepsilon}{2} < \varepsilon \text{ (order limit theorem)} \quad (5.11)$$

■

Theorem 5.2 (Continuous Limit Theorem). Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f . If each f_n is continuous at $c \in A$, then f is continuous at c .

Proof. Let (f_n) be a sequence of continuous functions, let $c \in A$, note that for arbitrary $N \in \mathbb{N}$,

$$|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \quad (5.12)$$

$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \quad (5.13)$$

Where the last term can be made arbitrarily small given the uniform convergence property of (f_n) . And the first two terms can be made arbitrarily small as well given the continuity properties.

This concludes the continuity of f . ■

Theorem 5.3 (Differentiable Limit Theorem). Let $(f_n) \rightarrow f$ pointwise on closed interval $[a, b]$, and assume that each f_n is differentiable. If $(f'_n) \rightarrow g$ uniformly on $[a, b]$, then the limit f is differentiable and $f' = g$.

Proof. Let $c \in [a, b]$, note that

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad (5.14)$$

The following holds for every $n \in \mathbb{N}$:

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \quad (5.15)$$

$$= \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) + f'_n(c) - g(c) \right| \quad (5.16)$$

$$\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| + |f'_n(c) - g(c)| \quad (5.17)$$

In particular, one may choose $N_1 \in \mathbb{N}$ such that

$$|f'_n(c) - g(c)| < \frac{\varepsilon}{3} \quad (5.18)$$

for every $n \geq N_1$.

Moreover, given the Cauchy property of uniform convergence, there exists $N_2 \in \mathbb{N}$ such that $\forall m, n \geq N_2, |f'_m(x) - f'_n(x)| < \frac{\varepsilon}{3}$ for every $x \in A$.

Define $N = \max\{N_1, N_2\}$.

Note that $\exists \delta > 0$ such that

$$x \in V_\delta^o(c) \implies \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\varepsilon}{3} \quad (5.19)$$

Let $m \geq N$, for an arbitrary $x \in V_\delta^o(c)$, WLOG, suppose $x < c$. Applying the mean value theorem on $f_m - f_N$ gives

$$f'_m(\eta) - f'_N(\eta) = \frac{f_m(c) - f_N(c) - [f_m(x) - f_N(x)]}{c - x} \quad (5.20)$$

$$\implies \frac{\varepsilon}{3} > |f'_m(\eta) - f'_N(\eta)| = \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| \quad (5.21)$$

$$\implies \lim_{m \rightarrow \infty} \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| = \left| \frac{f(c) - f(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| \quad (5.22)$$

$$\leq \frac{\varepsilon}{3} \text{ (order limit theorem)} \quad (5.23)$$

Therefore, for every $x \in V_\delta^0(c)$,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon \quad (5.24)$$

■

Theorem 5.4. Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f'_n) converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ converges, then (f_n) converges uniformly on $[a, b]$.

Theorem 5.5 (Stronger Version). Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f'_n) converges uniformly to a function g on $[a, b]$.

If there exists a point $x_0 \in [a, b]$ for which $f_n(x_0)$ converges, then (f_n) converges uniformly,

Furthermore, the limit function f is differentiable and satisfies $f' = g$.

5.2 Series of Functions

Definition 5.3. A series of function $\sum_{i=1}^{\infty} f_n(x)$ converges point-wise/uniformly if $\sum_{i=1}^k f_n(x)$ converges point-wise/uniformly.

Example 5.1. Note that $\sum_{i=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly.

$$\left| \sum_{i=1}^k \frac{\sin(nx)}{n^2} - \sum_{i=1}^{\ell} \frac{\sin(nx)}{n^2} \right| \leq \sum_{n=\ell+1}^k \frac{1}{n^2} < \varepsilon \quad (5.25)$$

Hence, uniformly Cauchy.

Theorem 5.6. If $f_n : S \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^m$ continuous, then if $\sum_{n=1}^{\infty} f_n \rightarrow f$ uniformly, then f is also continuous.

Definition 5.4. Let $f_k : S \rightarrow \mathbb{R}^m$, then (f_k) is **uniformly Cauchy** on S if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \sup_{x \in S} \left\| \sum_{i=k+1}^{\ell} f_i(x) \right\| \leq \varepsilon \quad \forall \ell > k \geq N \quad (5.26)$$

Theorem 5.7. (f_k) is uniformly Cauchy $\iff \sum f_k \rightarrow f$ uniformly.

This is the same as the theorem mentioned in previous section. Simply define $g_n := \sum_{i=1}^n s f_i(x)$, and (g_n) is uniformly Cauchy if and only if it converges to $g_{\infty} \equiv \sum_{n=1}^{\infty} f_n(x)$ uniformly.

Proof. (\Leftarrow)

Let $g_m := \sum_{k=1}^m f_k(x)$, suppose $g_k \rightarrow f$ uniformly. Then,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |g_k - f| < \frac{\varepsilon}{2} \quad \forall k \geq N, \forall x \in S \quad (5.27)$$

For every $k > \ell \geq N$,

$$|g_k - g_{\ell}| \leq |g_k - f| + |f - g_{\ell}| < \varepsilon \quad \forall x \in S \quad (5.28)$$

Therefore, (f_k) is uniformly Cauchy.

(\Rightarrow)

Suppose (f_k) is uniformly Cauchy. Then for every fixed $x \in S$, $(g_k(x))$ is Cauchy.

So $f(x) = \lim_{k \rightarrow \infty} g_k(x)$ exists.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } |g_k(x) - g_{\ell}(x)| < \varepsilon \quad \forall k > \ell \geq N, \forall x \in S \quad (5.29)$$

$$\implies |f(x) - g_k(x)| = \lim_{\ell \rightarrow \infty} |g_{\ell}(x) - g_k(x)| \leq \varepsilon \quad \forall x \in S \quad (5.30)$$

$$\implies g_k \rightarrow f \text{ uniformly} \quad (5.31)$$

■

Theorem 5.8 (Term-by-term Continuity Theorem). Let f_n be continuous functions defined on a set $A \subseteq \mathbb{R}$, suppose $\sum_{n=1}^{\infty} f_n \rightarrow f$ uniformly, then f is continuous on A .

Theorem 5.9 (Term-by-term Differentiability Theorem). Let f_n be continuous functions defined on $A \subseteq \mathbb{R}$, and suppose

(i) $\sum_{n=1}^{\infty} f'_n \rightarrow g$ uniformly;

(ii) There exists a point x_0 such that the series converges at x_0 .

Then, $\sum_{n=1}^{\infty} f_n \rightarrow f$ converges uniformly to a differentiable function f with $f'(x) = g(x)$.

That is

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (5.32)$$

$$f'(x) = \sum_{n=1}^{\infty} f'_n(x) \quad (5.33)$$

Theorem 5.10 (Weierstrass M-test). Let $a_n : S \rightarrow \mathbb{R}^m$, and sequence $M_n \in \mathbb{R}$. If

$$\exists N \in \mathbb{N} \forall k \geq N \sup_{x \in S} |a_n(x)| \leq M_n \quad (5.34)$$

then $\sum_{n=1}^{\infty} M_n$ converges implies $\sum_{n=1}^{\infty} a_n$ converges uniformly.

Proof. The desired result can be shown by showing the sequence of partial sums is Cauchy.

Let $\varepsilon > 0$, show that there exists $k \in \mathbb{N}$ such that

$$\forall m, n \geq k, \quad |S_m(x) - S_n(x)| < \varepsilon \quad (5.35)$$

WLOG, suppose $m > n$, then

$$|S_m(x) - S_n(x)| = |a_{n+1}(x) + \cdots a_m(x)| \quad (5.36)$$

$$\leq |M_{n+1} + \cdots + M_m| \quad (5.37)$$

■

Proof.

$$\forall x \in S \sum_{n=1}^{\infty} |a_n(x)| \leq \sum_{n=1}^{\infty} \sup_{x \in S} |a_n(x)| \quad (5.38)$$

$$\leq \sum_{n=1}^N \sup_{x \in S} |a_n(x)| + \sum_{n=N+1}^{\infty} M_n < \infty \quad (5.39)$$

Therefore, $f(x) = \sum_{n=1}^{\infty} a_n(x)$ exists.

Note that $\forall x \in S, \forall \ell \geq N$:

$$\left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| = \left| \sum_{k=\ell+1}^{\infty} a_k(x) \right| \quad (5.40)$$

$$\leq \sum_{k=\ell+1}^{\infty} |a_k(x)| \quad (5.41)$$

$$\leq \sum_{k=\ell+1}^{\infty} M_k < \infty \quad (5.42)$$

$$\Rightarrow \lim_{\ell \rightarrow \infty} \sup_{x \in S} \left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| \leq \lim_{\ell \rightarrow \infty} \sum_{k=\ell+1}^{\infty} M_k = 0 \quad (5.43)$$

Therefore $\sum a_k \rightarrow f$ uniformly. ■

Example 5.2. Consider $\sum \frac{x^n}{n!}$ defined on $x \in [-A, A]$. Note that

$$\sup_{x \in S} \left| \frac{x^n}{n!} \right| \leq \frac{A^n}{n!} =: M_n \quad (5.44)$$

By ratio test, $\sum M_n$ converges.

By M-test $\sum \frac{x^n}{n!}$ converges uniformly on $[-A, A]$.

Therefore, $\sum \frac{x^n}{n!}$ is continuous on $[-A, A]$.

Example 5.3 (Geometric Series).

$$\sum_{n=1}^{\infty} (-x^2)^n \quad (5.45)$$

By ratio test, the series converges only on $(-1, 1)$. In particular, it converges to $\frac{1}{1+x^2}$.

For every $0 < r < 1$, for $x \in [-r, r]$, then

$$\sup_{x \in [-r, r]} |(-x^2)^n| = r^{2n} \quad (5.46)$$

and $\sum_{n=1}^{\infty} r^{2n} = \frac{1}{1+r^2}$ implies $\sum_{n=1}^{\infty} (-x^2)^n$ converges uniformly on $[-r, r]$.

Example 5.4 (Weierstrass Function). Define

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2^n} \cos(10^n \pi x)}_{f_n(x)} \quad (5.47)$$

Note that $\sup_{x \in S} |f_n(x)| \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow 1$. Therefore, $\sum f_n \rightarrow f$ uniformly on \mathbb{R} , so that f is continuous.

Proposition 5.1. f is nowhere differentiable.

Proof. Let $x \in \mathbb{R}$ with decimal expansion $x_0 \cdot x_1 x_2 \dots$.

Construct sequence $(z_n) \rightarrow x$ as following:

Fix n , let $y_0 = x_0.x_1 \cdot x_n$ and $y_1 = y_0 + \frac{1}{10^n}$.

Note that $10^n \pi y_i$ is an integer multiplied by π , which implies $f(y_0) = \frac{(-1)^{x_n}}{2^n}$, $f(y_1) = \frac{(-1)^{x_{n+1}}}{2^n}$ such that

$$\left| \frac{f(z_n) - f(x)}{z_n - x} \right| \rightarrow \infty \quad (5.48)$$

■

Example 5.5. Let

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \equiv \sum_{n=1}^{\infty} h_n(x) \quad (5.49)$$

Claim: $h(x)$ is continuous and differentiable on \mathbb{R} .

Continuity. Note that each $h_n(x)$ is continuous, so is each partial sum $S_k := \sum_{n=1}^k h_n(x)$.

By Weierstrass M-test, each $|h_n(x)| \leq \frac{1}{n^2} \equiv M_n$, the series converges to $h(x)$ uniformly.

Therefore, by the definition of series convergence, the sequence of partial sums, in which all elements are continuous, converges uniformly to $h(x)$, hence $h(x)$ is continuous as well. ■

Differentiability. We've already shown the sequence of partial sum converges uniformly to $h(x)$.

All we need to show is the derivative of partial sums converges uniformly to some function.

Equivalently, we are showing the uniform convergence of series of derivatives.

In particular,

$$h'_n(x) = -\frac{2x}{(x^2 + n^2)^2} \quad (5.50)$$

$$S'_k(x) = \sum_{n=1}^k h'_n(x) = -\sum_{n=1}^k \frac{2x}{(x^2 + n^2)^2} \quad (5.51)$$

Apply M-Test again,

$$|h'_n(x)| = \left| -\frac{2x}{(x^2 + n^2)^2} \right| \quad (5.52)$$

$$= \frac{2}{\frac{x^2 + n^2}{x}(x^2 + n^2)} \quad (5.53)$$

$$= \frac{2}{(x + \frac{n^2}{x})(x^2 + n^2)} \quad (5.54)$$

Note that

$$\left(\sqrt{x} - \frac{n}{\sqrt{x}}\right)^2 = x - 2n + \frac{n^2}{x} \geq 0 \quad (5.55)$$

$$\implies \left(x + \frac{n^2}{x}\right) \geq 2n \quad (5.56)$$

Therefore,

$$|h'_n(x)| \leq \frac{2}{2n(x^2 + n^2)} \quad (5.57)$$

$$\leq \frac{1}{n^3} \quad (5.58)$$

Therefore,

$$\sum_{n=1}^{\infty} h'_n(x) \rightarrow g(x) \text{ uniformly} \quad (5.59)$$

Hence, $h(x)$ is differentiable, in particular, $h'(x) = g(x)$. ■

5.3 Power Series

Definition 5.5. A **power series** takes the form of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5.60)$$

in which $f(0) < \infty$ trivially.

Theorem 5.11. If a power series converges at some point $x_0 \in \mathbb{R}$, then it it converges absolutely for any x such that $|x| \leq |x_0|$. TODO: verify the inequality, strictly?

Proof. The result can be easily established via comparison test.

Suppose $f(x_0)$ converges, let $x \in \overline{V}_{|x_0|}(0)$.

By the convergence of $f(x_0)$, $(a_n x_0^n) \rightarrow 0$, so that $|a_n x_0^n| \leq M \forall n \in \mathbb{N}$.

$$|a_n x^n| = |a_n| \left| \frac{x^n}{x_0^n} x_0^n \right| \quad (5.61)$$

$$= |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \quad (5.62)$$

$$\leq M \left| \frac{x}{x_0} \right|^n \quad (5.63)$$

In which $\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$ converges as a power series with ratio less than 1. Therefore, $\sum_{n=0}^{\infty} a_n x^n$ is convergent by comparison test. ■

Definition 5.6. Radius of convergence and interval of convergence **TODO:** *Do we need to define this?*

Theorem 5.12 (Abel's Theorem). Suppose $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converge somewhere $x = R > 0$, then $g(x)$ converges uniformly (as a series of functions) on $[0, R]$.

The same holds for $-R$: if $g(-R)$ converges, then $g(x)$ converges uniformly on $[-R, 0]$.

Theorem 5.13. If a power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges point-wisely on $A \subseteq \mathbb{R}$, then it converges uniformly on any compact subset $K \subseteq A$.

Proof. **TODO:** *Complete this proof.* ■

6 Appendix I: Midterms

Proposition 6.1 (Midterm 1, Q7). Let (a_n) be a bounded sequence, let $S := \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}$. Then $\exists(a_{n_k}) \rightarrow s := \sup S$.

Lemma 6.1. S is bounded above, therefore $s := \sup S$ is well-defined.

Proof. Every element in S is bounded from above by infinitely many elements from (a_n) , which are themselves bounded. Therefore S is bounded from above.

So s is well-defined by the completeness axiom. ■

Proof. Case 1. Suppose there are infinitely many $a_n > s$.

Then either $\forall k \in \mathbb{N} \exists a_{n_k} \in [s, s + \frac{1}{k})$, such subsequence $(a_{n_k}) \rightarrow s$.

Or $\exists k \in \mathbb{N}$ such that no a_n in $[s, s + \frac{1}{k})$, this leads to a contradiction to the assumption that s is the least upper bound. ■

Proof. Case 2. Suppose there are only finitely many $a_n > s$.

Consider $(s - \frac{1}{k}, s]$, by the definition of supremum, $\forall k \in \mathbb{N}, \exists s_k \in (s - \frac{1}{k}, s] \cap S$.

But $s_k < a_n$ for infinitely many a_n by definition of S .

In particular, because there are only finitely many $a_n > s$, there must be infinitely many a_n clustered in $(s - \frac{1}{k}, s]$ for every $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, one can pick $a_{n_k} \in (s - \frac{1}{k}, s]$ and $(a_{n_k}) \rightarrow s$. ■

Proposition 6.2 (Another Midterm Question). The open clopen set in \mathbb{R} is \emptyset and \mathbb{R} .

Proof. Suppose, for contradiction, there exists $U, V \neq \emptyset$ and $V = U^c$, further U is clopen.

By definition of closedness and openness, V is clopen as well.

Pick $a \in U$ and $b \in V$. WLOG, assume $a < b$.

Construct $X \subseteq U$ as

$$X := \{x \in U : x < b\} \tag{6.1}$$

X is nonempty because $a \in X$.

X is certainly bounded above by b , hence, by the completeness axiom, $\exists \alpha = \sup X$.

Calim: $\alpha \in \partial U$.

By the definition of supremum, for every $\varepsilon > 0$,

$$\forall \varepsilon > 0, V_\varepsilon(\alpha) \cap X \neq \emptyset \tag{6.2}$$

$$\implies V_\varepsilon(\alpha) \cap U \neq \emptyset \tag{6.3}$$

Case 1:

If $V_\varepsilon(\alpha) \cap V = \emptyset$ for some $\varepsilon > 0$, then $V_\varepsilon(\alpha) \subseteq U$.

Subcase 1:

If $\alpha + \frac{\varepsilon}{2} \leq b$, then $\alpha \neq \sup X$ (contradiction);

Subcase 2:

If $\alpha + \frac{\varepsilon}{2} > b$, then $b \in V_{\varepsilon/2}(\alpha) \cap V \subseteq V_\varepsilon(\alpha) \cap V = \emptyset$ (contradiction).

Case 2: $\forall \varepsilon > 0$, $V_\varepsilon(\alpha) \cap V \neq \emptyset$, then by definition, $\alpha \in \partial U \subseteq U$ and $\alpha \in \partial U \subseteq U$ because U and V are closed.

This leads to a contradiction that $U \cap V = \emptyset$. ■

Proposition 6.3 (Sample Midterm 2). Let $\mathcal{K} = \{K_\lambda : \lambda \in \Lambda\}$ be a collection of compact subsets of \mathbb{R} such that the intersection of every finite sub-collection is nonempty. Then $\bigcap_{\lambda \in \Lambda} K_\lambda \neq \emptyset$.

Proof. Suppose, for contradiction, $\bigcap_{\lambda \in \Lambda} K_\lambda = \emptyset$.

Take an arbitrary $K_0 \in \mathcal{K}$, which is obviously non-empty.

Then,

$$K_0 = K_0 \cap \emptyset^c \tag{6.4}$$

$$= K_0 \cap \left(\bigcap_{\lambda \in \Lambda} K_\lambda \right)^c \tag{6.5}$$

$$= K_0 \cap \bigcup_{\lambda \in \Lambda} K_\lambda^c \tag{6.6}$$

$$= \bigcup_{\lambda \in \Lambda} K_0 \cap K_\lambda^c \tag{6.7}$$

For an arbitrary $\lambda \in \Lambda$, let $x \in K_0 \cap K_\lambda^c \neq \emptyset$.

Because K_λ^c is open in \mathbb{R} , $\exists V_\varepsilon(x) \subseteq K_\lambda^c$.

Note that because K_λ^c is open in \mathbb{R} , therefore $K_0 \cap K_\lambda^c$ is open in K_0 .

Hence, $\bigcup_{\lambda \in \Lambda} K_0 \cap K_\lambda^c$ is an open cover of K_0 . The compactness of K_0 suggests it has a finite subcover, say $\bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c$. That is,

$$K_0 = \bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c \tag{6.8}$$

$$= K_0 \cap \bigcup_{\lambda=1}^n K_\lambda^c \tag{6.9}$$

$$= K_0 \setminus \bigcap_{\lambda=1}^n K_\lambda \tag{6.10}$$

$$\implies \bigcap_{\lambda=1}^n K_\lambda = \emptyset \tag{6.11}$$

$\Rightarrow \Leftarrow$ ■

Theorem 6.1. If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges for all $x \in (-R, R)$ as well.

Further, the differentiated series converges uniformly for every compact $K \subseteq (-R, R)$.

Moreover, provided the uniform convergence of differentiated series, the limit function $\sum_{n=0}^{\infty} a_n x^n$

is differentiable. Therefore,

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad (6.12)$$

Remark: Differentiating a power series does not change its radius of convergence.

Proof. Prove using comparison test. ■

7 Taylor's Series

Idea Given $f(x) \in C^\infty$, we are trying express $f(x)$ as a series $\sum_{n=0}^{\infty} a_n x^n$ such that the series converges somewhere else other than $x = 0$. If such series exists, it must satisfy

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (7.1)$$

Example 7.1.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (7.2)$$

This series has infinity radius of convergence, specifically,

$$\left| \frac{x^{n+1}}{(n+1)!} / \frac{x^n}{n!} \right| = \left| \frac{x}{n+1} \right| \quad (7.3)$$

$$\implies \limsup \left| \frac{x}{n+1} \right| = 0 < 1 \quad (7.4)$$

Theorem 7.1 (Lagrangian Remainder Theorem). Let f be differentiable for $N + 1$ times on $(-R, R)$, let

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \in \{0, \dots, N\} \quad (7.5)$$

And the **Taylor's polynomial** of order N is defined as

$$S_N(x) := \sum_{n=0}^N a_n x^n \quad (7.6)$$

For every $x \neq 0$ in $(-R, R)$, define the **error term** associated with the Taylor's polynomial of order N :

$$E_N(x) := f(x) - S_N(x) \quad (7.7)$$

Then there exists $c \in V_x(0)$ such that

$$E_N(x) := \frac{f^{N+1}(c)}{(N+1)!} x^{N+1} \quad (7.8)$$

Example 7.2 (Bound Estimation Error). Consider $f(x) = \sin(x)$ and its Taylor's polynomial of order 5 in $V_2(0)$:

$$S_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \quad (7.9)$$

$$|E_5(x)| = \left| \sin(x) - x + \frac{x^3}{3} - \frac{x^5}{5} \right| \quad (7.10)$$

$$= \left| \frac{\frac{d^6 \sin(x)}{dx^6}}{6!} x^6 \right| \text{ for some } c \in V_2(0) \quad (7.11)$$

$$\leq \frac{|x^6|}{6!} \approx 0.089 \quad (7.12)$$

In general, for every fixed $N \in \mathbb{N}$, the tail of Taylor's series

$$|\sin(x) - S_N(x)| = \frac{\left| \frac{d^N \sin(x)}{dx^N} \right|_{x=c} |x|^N}{(N+1)!} \quad (7.13)$$

$$\leq \frac{R^N}{(N+1)!} \rightarrow 0 \text{ uniformly by Cauchy's criterion} \quad (7.14)$$

Theorem 7.2 (Cauchy Mean Value Theorem). Let f and g be two continuous functions defined on $[a, b]$, and differentiable on (a, b) . Then, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (7.15)$$

Proof. Define

$$h(x) := [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x) \quad (7.16)$$

Apply the Lagrange's mean value theorem on $h(x)$, there exists $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} \quad (7.17)$$

$$= \frac{f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) - f(b)g(a) + f(a)g(a) + g(b)f(a) - g(a)f(a)}{b - a} = 0 \quad (7.18)$$

Therefore,

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0 \quad (7.19)$$

$$\implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad (7.20)$$

■

Proof of Lagrange's Remainder Theorem. Apply Cauchy MVT on $E_N(x)$ and x^{N+1} on $[0, x]$, there exists $x_1 \in (0, x)$ such that

$$\frac{E_N(x) - E_N(0)}{x^{N+1} - 0} = \frac{E_N(x)}{x^{N+1}} \quad (7.21)$$

$$= \frac{E'_N(x_1)}{(N+1)x_1^N} \quad (7.22)$$

Apply the GMT again on $[0, x_1]$, there exists $x_2 \in (0, x_1)$ such that

$$\frac{E'_N(x_1) - E'_N(0)}{(N+1)x_1^N - 0} = \frac{E'_N(x_1)}{(N+1)x_1^N} \quad (7.23)$$

$$= \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \quad (7.24)$$

where the first equality holds because

$$E_N^{(k)}(0) = f^{(k)} - k!a_k = 0 \quad (7.25)$$

Applying the GMT iteratively suggests that there exists $c \in (0, x)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \quad (7.26)$$

$$= \frac{E_N^{(3)}(x_3)}{(N-1)N(N+1)x_3^{N-2}} \quad (7.27)$$

$$= \dots \quad (7.28)$$

$$= \frac{E_N^{(N+1)}(c)}{(N+1)!} \quad (7.29)$$

$$\implies E_N(x) = \frac{E_N^{(N+1)}(c)}{(N+1)!} x^{N+1} \quad (7.30)$$

Further, observe that $E_N(x) = f(x) - S_N(x)$ and $E_N^{(N+1)}(x) = f^{(N+1)}(x)$.
Therefore,

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \quad (7.31)$$

■

Remark 7.1 (Motivation for Taylor's Series). For function $f(\cdot)$, we want a power series converges to $f(\cdot)$ on some interval.

As mentioned before, it must be the case

$$a_n = \frac{f^{(n)}(0)}{n!} \quad (7.32)$$

Let $f(\cdot) \in C^\infty$, can we always find a suitable power series?

Potential problems are

- (i) The found series converges at $\{0\}$;
- (ii) Converges to some function $g(\cdot) \neq f(\cdot)$.

Example 7.3.

$$g(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (7.33)$$

observe that $\eta_0 = g(0) = 0$, and

$$\eta_1 = g'(0) = \lim_{h \rightarrow 0} \frac{\exp(-1/h^2)}{h} = 0 \quad (7.34)$$

The same argument is applicable for higher order terms η_k , therefore,

$$S_\infty(x) = \sum_{k=1}^{\infty} \frac{\eta_k}{k!} f^{(k)}(0) x^k = 0 \quad (7.35)$$

The Taylor's series converges to zero function, which is clearly not g .

8 Integration

Remark 8.1. In this section, we are assuming f to be bounded unless otherwise mentioned.

8.1 Riemann Integrable Functions

Definition 8.1. A **partition** P of interval $[a, b]$ is a finite set of points that has a, b as its elements, and

$$a = x_0 < x_1 < \cdots < x_n = b \quad (8.1)$$

Notation 8.1. Given a function partition P , for every $x_k \in P$, define

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad (8.2)$$

$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad (8.3)$$

Definition 8.2. The **lower sum** and **upper sum** of a function f and a partition P on $[a, b]$ are defined

$$L(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) m_k \quad (8.4)$$

$$U(f, P) = \sum_{k=1}^n (x_k - x_{k-1}) M_k \quad (8.5)$$

Example 8.1. For every partition P of every interval $I \subseteq \mathbb{R}$,

$$U(\mathbf{1}\{x \in \mathbb{Q}\}, P) = 1 \quad (8.6)$$

$$L(\mathbf{1}\{x \in \mathbb{Q}\}, P) = 0 \quad (8.7)$$

Definition 8.3. A partition Q is a **refinement** of another partition P if

$$P \subseteq Q \quad (8.8)$$

Lemma 8.1. If $P \subseteq Q$, then for every function f ,

$$L(f, P) \leq L(f, Q) \quad (8.9)$$

$$U(f, P) \geq U(f, Q) \quad (8.10)$$

Proof. Suppose Q contains one extra split $z \in [x_{k-1}, x_k]$, then

$$m_k(x_k - x_{k-1}) = m_k(x_k - z + z - x_{k-1}) \quad (8.11)$$

$$= m_k(x_k - z) + m_k(z - x_{k-1}) \quad (8.12)$$

$$\geq (z - x_{k-1}) \inf_{x \in [x_{k-1}, z]} f(x) + (x_k - z) \inf_{x \in [z, x_k]} f(x) \quad (8.13)$$

■

Lemma 8.2. Let P_1, P_2 be any two partitions,

$$L(f, P_1) \leq U(f, P_2) \quad (8.14)$$

Therefore, lower sums are bounded above and upper sums are bounded from below.

Proof.

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \quad (8.15)$$

$$\leq U(f, P_1 \cup P_2) \quad (8.16)$$

$$\leq U(f, P_2) \quad (8.17)$$

■

Definition 8.4. Define the **upper** and **lower integrals** as

$$U(f) := \inf_P U(f, P) \quad (8.18)$$

$$L(f) := \sup_P L(f, P) \quad (8.19)$$

note that the infimum and supremum over partitions are well-defined by the previous lemma.
 f is **(Riemann) integrable** if $U(f) = L(f)$

Lemma 8.3. For any bounded function $f : [a, b] \rightarrow \mathbb{R}$,

$$U(f) \geq L(f) \quad (8.20)$$

Proof. Fix an arbitrary partition P .

For every partition P'

$$U(f, P') \geq L(f, P) \quad (8.21)$$

$$\implies \sup_{P'} L(f, P') \leq U(f, P) \quad (8.22)$$

$$\iff L(f) \leq U(f, P) \quad (8.23)$$

The last inequality holds for every partition P , hence $L(f) \in \{U(f, P) : P\}^\downarrow$ and

$$L(f) \leq \inf_P U(f, P) \quad (8.24)$$

$$\iff L(f) = U(f) \quad (8.25)$$

■

Theorem 8.1 (Criterion of Integrability). A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is integrable if and only if

$$\forall \varepsilon > 0 \exists P_\varepsilon \text{ s.t. } U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \quad (8.26)$$

Proof. (\implies) Suppose f is integrable, let $\varepsilon > 0$.

Because $U(f)$ is the infimum of upper sums,

$$\exists P_1 \text{ s.t. } U(f) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \quad (8.27)$$

And $L(f)$ is the the supremum of lower sums,

$$\exists P_2 \text{ s.t. } L(f) - \frac{\varepsilon}{2} < L(f, P_2) \leq L(f) \quad (8.28)$$

Take $P = P_1 \cup P_2$,

$$L(f) - \frac{\varepsilon}{2} < L(f, P_2) \leq L(f, P) \leq U(f, P) \leq U(f, P_1) < U(f) + \frac{\varepsilon}{2} \quad (8.29)$$

$$\implies U(f, P) - L(f, P) < \varepsilon \quad (8.30)$$

(\Leftarrow) Suppose, for contradiction, f is not integrable:

$$L(f) < U(f) \quad (8.31)$$

Take

$$\varepsilon = \frac{U(f) - L(f)}{2} \quad (8.32)$$

In this case, for every partition P ,

$$L(f, P) \leq L(f) < U(f) \leq U(f, P) \quad (8.33)$$

$$\implies U(f, P) - L(f, P) \geq U(f) - L(f) > \varepsilon \quad (8.34)$$

Contradiction. ■

Theorem 8.2. If f is continuous on $[a, b]$, then it is integrable.

Proof. Because $[a, b]$ is compact, $f([a, b])$ is bounded and f is uniformly continuous on $[a, b]$.

Let $\varepsilon > 0$, by the definition of continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, b], |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a} \quad (8.35)$$

Construct partition P using by splitting $[a, b]$ into intervals of length $\delta/2$: $I_k := [a + k\frac{\delta}{2}, a + (k+1)\frac{\delta}{2}]$ so that there are $\frac{2(b-a)}{\delta}$ intervals in total.

$$U(f, P) - L(f, P) = \sum_k [\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)] |I_k| \quad (8.36)$$

$$= \sum_k [\sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x)] \frac{\delta}{2} \quad (8.37)$$

$$< \frac{\varepsilon}{b - a} \frac{2(b - a)}{\delta} \frac{\delta}{2} = \varepsilon \quad (8.38)$$

By the criterion of integrability, f is integrable on $[a, b]$. ■

Example 8.2 (Functions with Point Discontinuity). Define f on $[0, 2]$ as

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \quad (8.39)$$

Let $\varepsilon > 0$, define partition

$$P_\varepsilon = \left\{ 0, 1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}, 1 \right\} \quad (8.40)$$

Note that $U(f, P_\varepsilon) = 2$ and

$$L(f, P_\varepsilon) = \left(1 - \frac{\varepsilon}{4}\right) * 1 + \frac{\varepsilon}{2} * 0 + \left(1 - \frac{\varepsilon}{4}\right) = 2 - \frac{\varepsilon}{2} \quad (8.41)$$

and

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \frac{\varepsilon}{2} < \varepsilon \quad (8.42)$$

Theorem 8.3. If f is bounded on $[a, b]$ and integrable on $[c, d]$ for every $c \in (a, b)$.

Proof. Suppose $|f(x)| \leq M \forall x \in [a, b]$, let $\varepsilon > 0$.

Take

$$c = a + \frac{\varepsilon}{4M} \quad (8.43)$$

Note that because f is integrable on $[c, b]$, there exists another partition P' of $[c, b]$ such that

$$U(f, P') - L(f, P') < \frac{\varepsilon}{2} \quad (8.44)$$

Construct P_ε as

$$P_\varepsilon = \{a\} \cup P' \quad (8.45)$$

so that P_ε is a partition of $[a, b]$, and

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \left[\sup_{x \in [a, c]} f(x) - \inf_{x \in [a, c]} f(x) \right] \frac{\varepsilon}{4M} + U(f, P') - L(f, P') \quad (8.46)$$

$$< 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} \quad (8.47)$$

$$< \varepsilon \quad (8.48)$$

Therefore, by the criterion of integrability, f is integrable on $[a, b]$. ■

8.2 Properties of the Integral

Theorem 8.4. Assume $f : [a, b] \rightarrow \mathbb{R}$ is bounded and let $c \in (a, b)$. Then f is integrable on $[a, b]$ iff f is integrable on $[a, c]$ and $[c, b]$.

Proof. (\implies) Suppose f is integrable on $[a, b]$.

Let $\varepsilon > 0$, there exists P'_ε of $[a, b]$ such that

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon \quad (8.49)$$

Define $P_\varepsilon := P'_\varepsilon \cup \{a, c, b\}$ so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \varepsilon \quad (8.50)$$

one can obtain partitions of $[a, c]$ and $[c, b]$, P_ε^1 and P_ε^2 , following

$$P_\varepsilon^1 := \{x \in P_\varepsilon : x \leq c\} \quad (8.51)$$

$$P_\varepsilon^2 := \{x \in P_\varepsilon : x \geq c\} \quad (8.52)$$

In the case,

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = U(f, P_\varepsilon^1) - L(f, P_\varepsilon^1) + U(f, P_\varepsilon^2) - L(f, P_\varepsilon^2) < \varepsilon \quad (8.53)$$

Because $U(f, P_\varepsilon^i) - L(f, P_\varepsilon^i) \geq 0$ for $i \in \{1, 2\}$,

$$U(f, P_\varepsilon^i) - L(f, P_\varepsilon^i) \leq U(f, P_\varepsilon^1) - L(f, P_\varepsilon^1) + U(f, P_\varepsilon^2) - L(f, P_\varepsilon^2) < \varepsilon \quad \forall i \in \{1, 2\} \quad (8.54)$$

(\Leftarrow) Assume f is integrable on both $[a, c]$ and $[c, b]$.

Let $\varepsilon > 0$ be given, $\exists P'_\varepsilon$ of $[a, c]$ and P''_ε of $[c, b]$ such that

$$U(f, P'_\varepsilon) - L(f, P'_\varepsilon) < \frac{\varepsilon}{2} \quad (8.55)$$

$$U(f, P''_\varepsilon) - L(f, P''_\varepsilon) < \frac{\varepsilon}{2} \quad (8.56)$$

Take $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ so that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \quad (8.57)$$

Lemma 8.4.

$$\int_a^b f = \int_a^c f + \int_c^b f \quad (8.58)$$

Proof. Recall

$$\int_a^b f = U(f) = L(f) \quad (8.59)$$

Given any $\varepsilon > 0$, there exists partition P such that

$$\int_a^b f < L(f, P) + \varepsilon \quad (8.60)$$

$$= L(f, P_1) + L(f, P_2) + \varepsilon \quad (8.61)$$

$$\leq \int_a^c f + \int_a^b f + \varepsilon \quad (8.62)$$

$$\implies \int_a^b f \leq \int_a^c f + \int_c^b f \quad (8.63)$$

Further, note that

$$\int_a^c f + \int_c^b f < L(f, P_1) + L(f, P_2) + \varepsilon \quad (8.64)$$

$$= L(f, P) + \varepsilon \quad (8.65)$$

$$\leq \int_a^b f + \varepsilon \quad (8.66)$$

$$\implies \int_a^c f + \int_c^b f \leq \int_a^b f \quad (8.67)$$

■

Hence, for any $\varepsilon > 0$, $\exists P_1, P_2$ of $[a, c]$ and $[c, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad (8.68)$$

$$U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2} \quad (8.69)$$

■

Theorem 8.5. Assume f and g are integrable on $[a, b]$,

(i) $f + g$ is integrable on $[a, b]$ and

$$\int_a^b f + g = \int_a^b f + \int_a^b g \quad (8.70)$$

(ii) For any $k \in \mathbb{R}$, kf is integrable and

$$\int_a^b kf = k \int_a^b f \quad (8.71)$$

(iii) If $m \leq f \leq M$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a) \quad (8.72)$$

(iv) If $f(x) \leq g(x) \forall x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g \quad (8.73)$$

(v)

$$\int_a^b |f| \geq \left| \int_a^b f \right| \quad (8.74)$$

Proof (i).

Let $\varepsilon > 0$, by the criterion of integrability, there exists P_1 and P_2 on $[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2 \quad (8.75)$$

$$U(g, P_2) - L(g, P_2) < \varepsilon/2 \quad (8.76)$$

Take $P = P_1 \cup P_2$,

$$U(f, P) - L(f, P) < \varepsilon/2 \quad (8.77)$$

$$U(g, P) - L(g, P) < \varepsilon/2 \quad (8.78)$$

Notice that

$$U(f, P) + U(g, P) = U(f + g, P) \quad (8.79)$$

$$L(f, P) + L(g, P) = L(f + g, P) \quad (8.80)$$

which implies

$$U(f + g, P) - L(f + g, P) < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (8.81)$$

Therefore $f + g$ is integrable.

The value of integration is simply from the additivity of lower and upper integrals as summations. ■

Theorem 8.6 (Sequential Criterion of Integrability). Let f be a bounded function defined on $[a, b]$, the the following are equivalent:

(i) f is integrable on $[a, b]$;

(ii) There exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ such that

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0 \quad (8.82)$$

and in this case

$$\int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n) \quad (8.83)$$

Proof.

(\implies) Suppose f is integrable, for every $n \in \mathbb{N}$, take $\varepsilon = 1/n$.

By the criterion of integrability, there exists P_n such that

$$U(f, P_n) - L(f, P_n) < \varepsilon = 1/n \quad (8.84)$$

Take above P_n for each $n \in \mathbb{N}$ and the sequence $(P_n)_{n=1}^\infty$ is what was desired.

(\impliedby) Suppose there exists such sequence, then for every $\varepsilon > 0$, simply take any natural number $n \geq \frac{1}{\varepsilon}$, and P_n satisfies

$$U(f, P_n) - L(f, P_n) < \varepsilon \quad (8.85)$$

Therefore, f is integrable. ■

Proof (ii).

Case 1 For $k \geq 0$,

$$U(kf, P) = kU(f, P) \quad (8.86)$$

$$L(kf, P) = kL(f, P) \quad (8.87)$$

for any partition P .

Because f is integrable, there exists (P_n) satisfying the sequential criterion of integrability:

$$(U(kf, P_n) - L(kf, P_n))_{n \in \mathbb{N}} = (kU(f, P_n) - kL(f, P_n))_{n \in \mathbb{N}} \quad (8.88)$$

$$= k(U(f, P_n) - L(f, P_n))_{n \in \mathbb{N}} \quad (8.89)$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty \quad (8.90)$$

Therefore kf is integrable.

Case 2 For $k < 0$,

$$\sup(kf) = \inf(-kf) = (-k) \inf f \quad (8.91)$$

$$\inf(kf) = \sup(-kf) = (-k) \sup f \quad (8.92)$$

In this case,

$$U(kf, P) = (-k)L(f, P) \quad (8.93)$$

$$L(kf, P) = (-k)U(f, P) \quad (8.94)$$

for any partition P .

The same sequence (P_n) satisfying the sequential criterion of integrability of f works for kf as well. Therefore, kf is integrable. ■

Proof (iii). Suppose $m \leq f \leq M$ on $[a, b]$, take partition $P = \{a, b\}$:

$$(b-a)m = L(f, P) \leq \int_a^b f \leq U(f, P) = (b-a)M \quad (8.95)$$

■

Proof (iv). Suppose $f \leq g$ on $[a, b]$, define $h := g - f \geq 0$. By theorem (i) and (iii),

$$0 = (b-a)0 \leq \int_a^b h = \int_a^b (g - f) = \int_a^b g - \int_a^b f \quad (8.96)$$

$$\implies \int_a^b f \leq \int_a^b g \quad (8.97)$$

■

Proof (v).

$$-|f| \leq f \leq |f| \quad (8.98)$$

$$\implies -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \quad (8.99)$$

$$\implies \left| \int_a^b f \right| \leq \int_a^b |f| \quad (8.100)$$

■

Definition 8.5. Let $a < b$, let f be an integrable function defined on $[a, b]$:

$$\int_b^a f := -\int_a^b f \quad (8.101)$$

8.3 Convergence and Integrability

Motivation Is there a sequence of integrable functions that converges to a function that is not integrable?

Example 8.3. Consider the sequence of functions $(f_n) \rightarrow f$ in which each $f_n(x)$ is defined on $[a, b]$.

$$f_n(x) := \begin{cases} n & \text{if } 0 < x < s \\ 0 & \text{otherwise} \end{cases} \quad (8.102)$$

The (point-wise) limit function $f \equiv 0$.

For every n ,

$$\int_0^1 f_n(x) dx = 1 \quad (8.103)$$

But

$$\int_0^1 f(x) dx = 0 \quad (8.104)$$

Therefore, *the value of integral is not necessarily preserved under point-wise convergence.*

Theorem 8.7 (Integrable Limit Theorem). Assume $(f_n) \rightarrow f$ uniformly on $[a, b]$ and each f_n is integrable for all $n \in \mathbb{N}$, then f is integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f \quad (8.105)$$

Proof. Step 1: Show f is integrable

Because $f_n \rightarrow f$ uniformly, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$ for every $x \in [a, b]$.

Since f_N is integrable, there exists P_ε such that $U(f_N, P_\varepsilon) - L(f_N, P_\varepsilon) < \frac{\varepsilon}{3}$.

Consider

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = |U(f, P_\varepsilon) - L(f, P_\varepsilon)| \quad (8.106)$$

$$= |U(f, P_\varepsilon) - U(f_N, P_\varepsilon) + U(f_N, P_\varepsilon) - L(f, P_\varepsilon) + L(f_N, P_\varepsilon) - L(f_N, P_\varepsilon)| \quad (8.107)$$

$$\leq |U(f, P_\varepsilon) - U(f_N, P_\varepsilon)| + |U(f_N, P_\varepsilon) - L(f, P_\varepsilon)| + |L(f_N, P_\varepsilon) - L(f_N, P_\varepsilon)| \quad (8.108)$$

$$< \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3} \quad (8.109)$$

$$= \varepsilon \quad (8.110)$$

TODO: Verify this proof. ■

Theorem 8.8 (Fundamental Theorem of Calculus).

(i) If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $F : [a, b] \rightarrow \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f = F(b) - F(a) \quad (8.111)$$

(ii) Let $g : [a, b]$ be integrable, define $G(x) = \int_a^x g$. Then $G(x)$ is continuous on $[a, b]$. Moreover, if g is continuous at $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

Proof. Part I. Let P be some partition of $[a, b]$, in which each segment is denoted as $[x_{k-1}, x_k]$. In every segment, by the mean value theorem, there exists t_k such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(t_k) = f(t_k) \quad (8.112)$$

Within each segment, $m_k \leq f(t_k) \leq M_k$,

$$L(f, P) \leq \sum_k \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} (x_k - x_{k-1}) \leq U(f, P) \quad (8.113)$$

$$\implies L(f, P) \leq \sum_k F(x_k) - F(x_{k-1}) \leq U(f, P) \quad (8.114)$$

$$\implies L(f, P) \leq F(b) - F(a) \leq U(f, P) \quad (8.115)$$

$$\implies L(f) \leq F(b) - F(a) \leq U(f) \quad (8.116)$$

$$\implies \int_a^b f = F(b) - F(a) \quad (8.117)$$

■

Proof. Part II. Note that g is integrable implies g is bounded by some $M \in \mathbb{N}$.

Take $x, y \in [a, b]$, WLOG, assume $x > y$.

Then

$$G(x) - G(y) = \int_a^x g - \int_a^y g = \int_y^x g \quad (8.118)$$

$$\implies |G(x) - G(y)| = \left| \int_y^x g \right| \leq \int_y^x |g| \leq M(x - y) \quad (8.119)$$

Lemma 8.5. Lipschitz continuous implies $G(x)$ is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given, take $\delta = \frac{\varepsilon}{M}$:

$$|x - y| < \delta \implies |G(x) - G(y)| < M \frac{\varepsilon}{M} = \varepsilon \quad (8.120)$$

TODO: Complete definitions of Lipschitz continuity

■

Let $c \in [a, b]$ be given, and suppose g is continuous at c .

Want to show $G'(c) = g(c)$, need to show

$$\lim_{x \rightarrow c} \frac{G(x) - G(c)}{x - c} = g(c) \quad (8.121)$$

Need to show

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \implies \left| \frac{G(x) - G(c)}{x - c} - g(c) \right| < \varepsilon \quad (\dagger) \quad (8.122)$$

Note that

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x - c} \int_c^x g(t) dt \quad (8.123)$$

$$g(c) = \frac{1}{x - c} \int_c^x g(c) dt \quad (8.124)$$

Substitute into (†)

$$\left| \frac{G(x) - G(c)}{x - c} - g(c) \right| = \left| \frac{1}{x - c} \int_c^x [g(t) - g(c)] dt \right| \quad (8.125)$$

Because g is continuous at c , then

$$\exists \delta > 0, \text{ s.t. } |t - c| < \delta \implies |g(t) - g(c)| < \varepsilon \quad (8.126)$$

Further, since $t \in [x, c]$, then $|x - c| < \delta \implies |t - c| < \delta$.

Therefore,

$$|x - c| < \delta \implies \left| \frac{1}{x - c} \int_c^x [g(t) - g(c)] dt \right| \leq \frac{1}{x - c} \int_c^x |g(t) - g(c)| dt < \frac{1}{x - c} \int_c^x \varepsilon dt = \varepsilon \quad (8.127)$$

■

8.4 Lebesgue Criterion for Reimann Integrability

Example 8.4. Consider the Thomae's function:

$$t(x) := \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \wedge \gcd(m, n) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \quad (8.128)$$

$t(x)$ is integrable on $[0, 1]$ because the set of discontinuity is $\mathbb{Q} \cap [0, 1]$ which is countable and has measure zero.

$$\int_0^1 t(x) dx = 0 \quad (8.129)$$

Proof. **TODO:** revise this proof Because \mathbb{Q} is dense in $[0, 1]$, therefore, $L(t, P) = 0$.

Now show

$$\forall \varepsilon > 0, \exists P_\varepsilon, \text{ s.t. } U(t, P_\varepsilon) < \varepsilon \quad (8.130)$$

Let $\varepsilon > 0$ fixed, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{1}{N} < \frac{\varepsilon}{2}$.

Take the smallest natural number and denote it as N .

Consider the set of K fractions:

$$\{p_1, \dots, p_K\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots, \frac{N-2}{N-1}\right\} \quad (8.131)$$

take $\delta = \frac{\varepsilon}{4K}$, construct partition $[p_k - \delta, p_k + \delta]$.

$$U(t, P) = \sum_{i=1}^n M_i \Delta x_i = \sum_{i \in \mathcal{A}} M_i \Delta x_i + \sum_{i \notin \mathcal{A}} M_i \Delta x_i \quad (8.132)$$

where \mathcal{A} denotes all the intervals contains x_i such that $t(x_i) \geq \frac{\varepsilon}{2}$.

Notice that

$$\sum_{i \in \mathcal{A}} M_i \Delta x_i \leq \sum_{i \in \mathcal{A}} \Delta x_i \leq \frac{2\varepsilon}{4k} k = \frac{\varepsilon}{2} \quad (8.133)$$

$$\sum_{i \notin \mathcal{A}} M_i \Delta x_i < \sum_{i \notin \mathcal{A}} \frac{\varepsilon}{2} \Delta x_i < \frac{\varepsilon}{2} \quad (8.134)$$

Therefore, with the provided partition P , $U(t, P) < \varepsilon$. ■

Definition 8.6. A set $A \subseteq \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$, there exists a countable collection of open intervals $\mathcal{O}_n = [a_n, b_n]$ such that

$$A \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{O}_n \wedge \sum_{n \in \mathbb{N}} |\mathcal{O}_n| \leq \varepsilon \quad (8.135)$$

where $|\mathcal{O}_n| := |b_n - a_n|$.

Proposition 8.1. Finite sets have measure zero.

Proof. Let $A = \{x_n\}_{n=1}^N$. Let $\varepsilon > 0$ fixed.

For every $n \in [N]$, define $\mathcal{O}_n := (x_n - \frac{\varepsilon}{2N}, x_n + \frac{\varepsilon}{2N})$.

The result is immediate. ■

Proposition 8.2. Countable sets have measure zero.

Proof. Let $A = \{x_1, x_2, \dots\}$. Let $\varepsilon > 0$ fixed.

For every $n \in \mathbb{N}$, define

$$\mathcal{O}_n := \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right) \quad (8.136)$$

Obviously, A is covered by $\{\mathcal{O}_n : n \in \mathbb{N}\}$.

$$\sum_{n \in \mathbb{N}} |\mathcal{O}_n| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \quad (8.137)$$

$$= \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} \quad (8.138)$$

$$= \frac{\varepsilon}{2} \sum_{n=0}^{\infty} \frac{\varepsilon}{2^n} \quad (8.139)$$

$$= \frac{\varepsilon}{2} 2 = \varepsilon \quad (8.140)$$

■

Remark 8.2. Measure zero $\not\Rightarrow$ countable.

Counter example: Cantor set is uncountable but has measure zero.

Theorem 8.9. Countable union of sets of measure zero has measure zero.

Proof. Let $A = \bigcup_{n \in \mathbb{N}} A_n$, in which every A_n has measure zero.

For each $n \in \mathbb{N}$, because A_n has measure zero, there exists a countable collection of open intervals $\{\mathcal{O}_j^n : j \in \mathbb{N}\}$ covers it and satisfies $\sum_{j \in \mathbb{N}} |\mathcal{O}_j^n| \leq \frac{\varepsilon}{2^n}$.

Clearly, the union $\bigcup_{n \in \mathbb{N}} \{\mathcal{O}_j^n : j \in \mathbb{N}\}$ is consisting of open intervals and covers A .

Moreover

$$\sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{N}} |\mathcal{O}_j^n| \leq \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^n} = \frac{\varepsilon}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} = \varepsilon \quad (8.141)$$

■

Definition 8.7. The **oscillation** of a function f on **set** A is defined as

$$\text{osc}(f, A) := \sup \{|f(x) - f(y)| : x, y \in A \cap \text{Dom}(f)\} \quad (8.142)$$

Definition 8.8. The **oscillation** of f at a **point** x is defined as

$$\text{osc}(f; x) := \lim_{h \rightarrow 0^+} \left(\sup \{|f(x') - f(x'')| : x', x'' \in V_h(x) \cap \text{Dom}(f)\} \right) \quad (8.143)$$

$$= \lim_{h \rightarrow 0^+} \text{osc}(f, V_h(x)) \quad (8.144)$$

Theorem 8.10. A function f defined on $A \subseteq \mathbb{R}$ is continuous at $x \in A$ if and only if $\text{osc}(f; x) = 0$.

Proof.

(\Leftarrow) Suppose $\text{osc}(f; x) = 0$, for every $\varepsilon > 0$, there exists δ such that (by the definition of limits)

$$0 < h < \delta \implies \sup \{|f(x') - f(x'')| : x', x'' \in V_h(x) \cap \text{Dom}(f)\} < \varepsilon \quad (8.145)$$

Take the same δ , let $z \in V_\delta(x)$, then $|f(z) - f(x)| < \varepsilon$. (\implies) Suppose f is continuous at x , then for every $\varepsilon > 0$ fixed, there exists $\delta > 0$ such that

$$\forall z \in V_\delta(x), |f(z) - f(x)| < \frac{\varepsilon}{2} \quad (8.146)$$

Take the same $\delta > 0$, and for every $0 < h < \delta$, for every $x', x'' \in V_{\varepsilon/2}(x)$

$$|f(x') - f(x'')| = |f(x') - f(x) + f(x) - f(x'')| \quad (8.147)$$

$$\leq |f(x') - f(x)| + |f(x) - f(x'')| \quad (8.148)$$

$$< 2\frac{\varepsilon}{2} = \varepsilon \quad (8.149)$$

$$\implies \sup |f(x') - f(x'')| < \varepsilon \quad (8.150)$$

■

Theorem 8.11 (The Riemann Lebesgue Theorem). Let f be a function defined on $[a, b]$, then the following statements are equivalent:

- (i) f is integrable on $[a, b]$;
- (ii) The set of discontinuity of f on $[a, b]$, $D_f \cap [a, b]$, has measure zero.

Proof.

(\implies) Define $A := D_f \cap [a, b] = \{x \in [a, b] : \text{osc}(f; x) > 0\}$. WTS: A has measure zero.

Suppose f is Riemann integrable on $[a, b]$, construct the subsets of A as

$$A_k := \left\{ x \in [a, b] : \text{osc}(f; x) \geq \frac{1}{k} \right\} \quad (8.151)$$

Note that $A = \bigcup_{k \in \mathbb{N}} A_k$.

Let $\varepsilon > 0$ fixed, then there exists P_ε of $[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \frac{\varepsilon}{k} \quad (8.152)$$

For a given i , $x \in A_k \cup (x_{i-1}, x_i)$

$$[M_i(f) - m_i(f)]\Delta_i \geq \frac{1}{k}\Delta_i \quad \because \text{osc}(f; x) \geq \frac{1}{k} \quad (8.153)$$

$$\implies \frac{\varepsilon}{k} > U(f, P_\varepsilon) - L(f, P_\varepsilon) \quad (8.154)$$

$$= \sum_{i=1}^m [M_i(f) - m_i(f)]\Delta_i \quad (8.155)$$

$$\geq \sum_{\text{over intervals s.t. } [(x_{i-1}, x_i) \cap A_k] \neq \emptyset} [M_i(f) - m_i(f)]\Delta_i \quad (8.156)$$

$$\geq \frac{1}{k} \sum \Delta_i \quad (8.157)$$

$$\implies \sum \Delta_i < \varepsilon \quad (8.158)$$

Therefore the partition covers A_k and has measure zero. Moreover, A is a countable union of measure zero sets, so A has measure zero.

(\Leftarrow) Suppose $|D_f \cap [a, b]| = 0$.

Idea: construct partitions in which high oscillations are found in intervals with small Δx and low oscillations are found in high Δ .

Let $A := D_f \cap [a, b]$, and $|A| = 0$.

By the defining property of oscillations,

$$A = \{x \in [a, b] : \text{osc}(f; x) > 0\} \quad (8.159)$$

For every $s > 0$, define

$$A_s = \{x \in [a, b] : \text{osc}(f; x) \geq s\} \subseteq A \quad (8.160)$$

Let $\varepsilon > 0$ fixed, take $\ell = \frac{\varepsilon}{2(b-a)}$.

Since $|A_\ell| = 0$, there exists a countable collection of open intervals (I_i) that covers A_ℓ . Moreover,

$$\sum_{i=1}^{\infty} m(I_i) < \frac{\varepsilon}{2(M-m)} \quad (8.161)$$

Since A_ℓ is compact, there exists a finite subcover I_1, \dots, I_n , then

$$Z := [a, b] \cap (\cup_i^n I_i)^c \quad (8.162)$$

is still compact.

And $\text{osc}(f; x) < \frac{\varepsilon}{2(b-a)}$ for every $x \in Z$.

For every $x \in Z$, $\exists \delta_x > 0$ such that $x', x'' \in (x - \delta_x, x + \delta_x)$.

Exists a finite subcover U_1, \dots, U_k .

Let P_ε be the partition such that $[x_{i-1}, x_i)$ in either U_j or I_k , then

$$\sum (M_i - m_i) \Delta x_i = \sum_{C'} (M_i - m_i) \Delta x_i + \sum_{C''} (M_i - m_i) \Delta x_i < \varepsilon \quad (8.163)$$

■