MAT223 Linear Algebra Tophat Chapter 4

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1 The Rank Theorems

1.1 Subspaces of $\mathbb{R}^n$

Theorem 1 (Subspace Theorem)**.** *Let S be a subspace of $\mathbb{R}^n$. If S is spanned by m vectors, and contains k linearly independent vectors, then $k \leq m$.*

1.2 Bases

Theorem 2. *Let $S \neq \{\vec{0}\}$ be a subspace pf $\mathbb{R}^n$. Then there's a basis for S.*

Theorem 3. *A basis for a subspace $S \subseteq \mathbb{R}^n$ can only have one size.*

Definition 1 (Dimension)**.** *The **dimension** $dimS$ of a nonzero subspace v is $\sharp B$ for any basis B of S.*

1.3 Expansions and Orthogonalization

Theorem 4 (Gram-Schmidt Orthonormalization Procedure)**.** *Let $\vec{b}_1, \ldots, \vec{b}_k \subseteq \mathbb{R}^n$ be a basis for a subspace S. Define*

$$\vec{w_1} = \hat{b_1}$$
$$\vec{w_2} = \hat{x_2}, \vec{x_2} = \vec{b_2} - proj_{\vec{w_1}}\vec{b_2}$$
$$\vec{w_k} = \hat{x_k}, \vec{x_k} = \vec{b_k} - \sum_{i=1}^{k-1} proj_{\vec{w_i}}\vec{b_k}$$

*The vectors $\{\vec{w}_i\}_1^k$ produced as above are an **orthonormal basis** for S.*

1.4 Rank Unification

Definition 2 (Rank)**.** *The* ***rank*** *of a matrix A, denoted $rank(A)$ is the number of pivots in A.*

Theorem 5 (Rank Theorem)**.** *Let A be an $m \times n$ matrix with rank r. Then*

$$dim(col(A)) = dim(row(A)) = r$$

Moreover, if A R where R is in row-echelon form then

1. *The r nonzero rows of R form a basis for row(A).*
2. *The r pivot columns of A form a basis for col(A).*

Theorem 6 (Rank-Nullity Theorem)**.** *Let A be a $m \times n$ matrix. Then*

$$rank(A) + dim(ker(A)) = n$$

We call $dim(ker(A))$ as ***nullity*** *of A.*

Note The *rank-nullity theorem* is a way to quantitatively characterize how far a given matrix might be from having $A\vec{x} = \vec{b}$ be uniquely solvable.

Note To find a basis for kernel space, we write all basic variables of system $A\vec{x} = \vec{0}$ in terms of free variables.

Theorem 7 (Rank Inequalities)**.** *Let A be a $m \times n$ matrix, we have*

$$rank(A) \leq min(m, n)$$

Definition 3 (Maximal Rank)**.** *A $m \times n$ matrix has* ***full rank*** *or* ***maximal rank*** *when $rank(A) = min(m, n)$*

Note A square matrix with full rank must be invertible.

Theorem 8. *Let A, B and C be matrices such that the products below are well-defined. Then*

1. $col(AB) \subseteq col(A)$
2. $row(CA) \subseteq row(A)$

In the above $\subseteq$ will simply be $=$ when B or C respectively is invertible.

Theorem 9. *If A and B are two matrices whose product is defined then*

$$rank(AB) \leq min(rank(A), rank(B))$$

1.5 Maximal Rank

Note If A were square, then A having full rank ensures that system $A\vec{x} = \vec{b}$ is always *uniquely solvable.*

Theorem 10 (Rank $= \sharp$ of Columns)**.** *Let A be a $m \times n$ matrix, the following statements are equivalent.*

1. *$rank(A) = n$.*
2. *$row(A) = \mathbb{R}^n$.*
3. *Columns of A are linearly independent.*
4. *A^T A is invertible.*
5. *$\exists C \in \mathbb{M}_{n\times m}$ such that $C * A = I_n$.*
6. *$A\vec{x} = \vec{0}$ has only trivial solution.*

Theorem 11 (Rank $= \sharp$ of rows)**.** *Let A be $m \times n$. The following are equivalent statements.*

1. *$rank(A) = m$.*
2. *$col(A) = \mathbb{R}^m$.*
3. *Rows of A are linearly independent.*
4. *AA^T is invertible.*
5. *$\exists D \in \mathbb{M}_{n\times m}$ such that $A * D = I_m$.*
6. *$A\vec{x} = \vec{b}$ holds for all $\vec{b} \in \mathbb{R}^m$.*

2 The Fundamental Theorem of Linear Algebra

2.1 Prelude: Orthogonal Complements

Definition 4. *Let $S \subseteq \mathbb{R}^n$ be a subspace, define $S^\perp \in \mathbb{R}^n$ as*

$$S^\perp = \{\vec{u} \in \mathbb{R}^n | \vec{u} \cdot \vec{v} = 0, \forall \vec{v} \in S\}$$

$S^\perp$ is the the ***orthogonal complement*** *of S in $\mathbb{R}^n$.*

Theorem 12. *Let $\vec{v} \in \mathbb{R}^n$, and let $S \subseteq \mathbb{R}^n$ be a subspace. Then there are vectors*

$$\vec{s} \in S$$
$$\vec{s_\perp} \in S^\perp$$

such that,

$$\vec{v} = \vec{s} + \vec{s_\perp}$$

Explanation Vectors can be expressed in terms of pieces in orthogonal space.

Note This fact is expressed as $\mathbb{R}^n = S \bigoplus S^\perp$ (*direct sum*).

Proof. Since $S \in \mathbb{R}^n$ is a subspace, and every subspace has basis.
Let B_1 be a basis of S.
Let $B = \{\vec{s_i}\}_1^{dimS}$ be the orthonormal basis of S generated from B_1 via GSO.

So that, $\vec{s_i} \cdot \vec{s_j} = \begin{cases} 1, i = j \in \mathbb{Z}_1^{dimS} \\ 0, i \neq j \in \mathbb{Z}_1^{dimS} \end{cases}$

For a vector $\vec{v} \in \mathbb{R}^n$

Let $\vec{s} = \sum_{i=1}^{dimS} (\vec{v} \cdot \vec{s_i})\vec{s_i}$. $\vec{s}$ is a linear combination of vectors in the orthonormal basis B of space S, so obviously, $\vec{s} \in S$.

Define $\vec{s_\perp} = \vec{v} - \vec{s}$.

$\forall j \in \mathbb{Z}_1^{dimS}$, Consider $\vec{s_\perp} \cdot \vec{s_j}$.

$\vec{s_\perp} \cdot \vec{s_j} = (\vec{v} - \vec{s}) \cdot \vec{s_j} = \vec{v} \cdot \vec{s_j} - \sum_{i=1}^{dimS} (\vec{v} \cdot \vec{s_i} \cdot \vec{s_i} \cdot \vec{s_j}) = \vec{v} \cdot \vec{s_j} - \vec{v} \cdot \vec{s_j} = 0$

So that, $\vec{s_\perp} \in S^\perp$.
By definition of $\vec{s_\perp}$ above, $\vec{v} = \vec{s} + \vec{s_\perp} \in \mathbb{R}^n$, where $\vec{s} \in S \wedge \vec{s_\perp} \in S^\perp$. □

2.2 The Fundamental Theorem of Linear Algebra

Proposition 1. *Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$. Then,*

$$Col(A^T)^\perp = Ker(A)$$

Proof. **Part1**

Let $\vec{v} \in Ker(A)$, we have $A\vec{v} = \vec{0}$

So that, $\begin{bmatrix} Row_1(A)\vec{v} \\ Row_2(A)\vec{v} \\ \vdots \\ Row_m(A)\vec{v} \end{bmatrix} = \vec{0}$

So that, $\vec{v}$ is orthogonal to all rows of A.

So that, $\vec{v} \in Row(A)^{\perp} \wedge Row(A) = Col(A^T)$

$\vec{v} \in Col(A^T)^{\perp}$

We have, $\vec{v} \in Ker(A) \implies \vec{v} \in Col(A^T)^{\perp}$

So that, $Ker(A) \subseteq Col(A^T)^{\perp}$

Part2

Let $\vec{v} \in Col(A^T)^{\perp}$

Since $Col(A^T) = Row(A)$, we have $\vec{v} \in Row(A)^{\perp}$

So that, $Row_j(A) \cdot \vec{v} = 0, \forall j \in \mathbb{Z}_1^m$

So that, $A\vec{v} = \vec{0}$, which implies $\vec{v} \in Ker(A)$.

We have, $\vec{v} \in Col(A^T)^{\perp} \implies \vec{v} \in Ker(A)$

Equivalently, $Col(A^T)^{\perp} \subseteq Ker(A)$

Now we have $Ker(A) \subseteq Col(A^T)^{\perp} \wedge Col(A^T)^{\perp} \subseteq Ker(A) \iff Ker(A) = Col(A^T)^{\perp} \iff Ker(A)^{\perp} = Col(A^T)$

$\square$

Theorem 13 (The Fundamental Theorem of Linear Algebra)**.** *Let* $A \in \mathbb{M}_{m\times n}(\mathbb{R})$

Then, (i) $Col(A^T) = Ker(A)^{\perp}$

(ii) $\mathbb{R}^n = Col(A^T) \bigoplus Ker(A)$

And, $\forall \vec{v} \in Col(A)$*, we have,* $A\vec{x} = \vec{b}$ *solved by* $\vec{x} = \vec{p} + \vec{v_h}$*, where* $\vec{p} \in Row(A)$ *and* $\vec{v_h} \in Ker(A)$*.*

Explanation (ii): **Orthogonal decomposition** of $\mathbb{R}^n$ into the *null space* and the *row space* of matrix A.

Note For the counter part of this theorem over $\mathbb{R}^m$, consider matrix $B = A^T$ and proof via the same vein.

2.3 The Diagrams

Let matrix $A \in \mathbb{M}_{m\times n}(\mathbb{R})$.

2.3.1 Decomposition of $\mathbb{R}^n$

Representation $\mathbb{R}^n = Row(A) \bigoplus Ker(A)$.

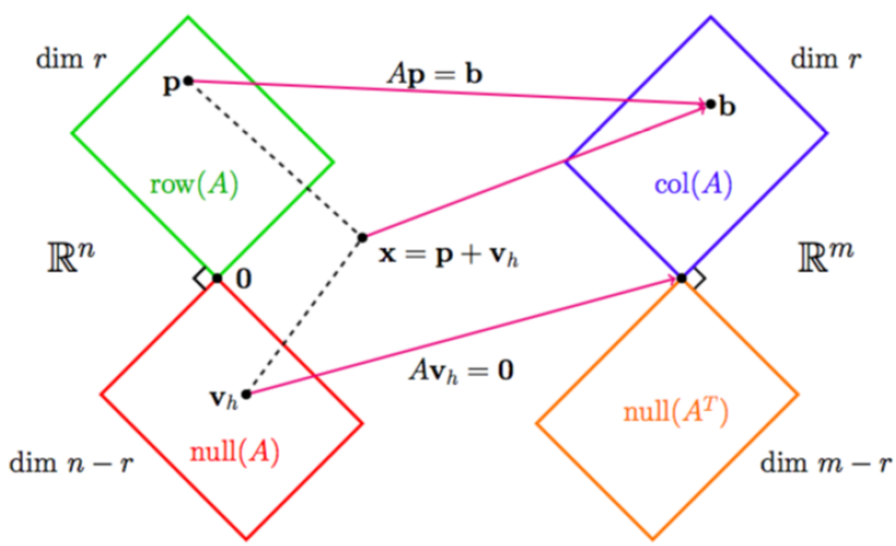

Figure 1: The decomposition of $\mathbb{R}^n$

2.3.2 Decomposition of $\mathbb{R}^m$

Representation $\mathbb{R}^m = Col(A) \bigoplus Ker(A^T)$.

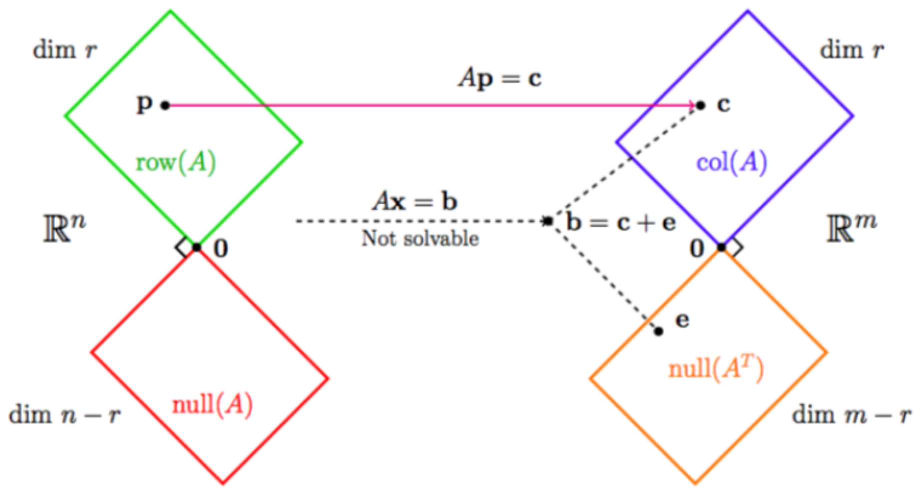

Figure 2: The decomposition of $\mathbb{R}^m$