ECO426H1 Market Design: Auctions and Matching Markets

Tianyu Du

January 27, 2020

Contents

Chapter 1

Private Value Auctions

1.1 Standard Auctions

Definition 1.1.1. An **auction** is an *informational environment* consisting of

(i) a valuation structure for the bidders;

(ii) a distribution of information available to the bidders.

Assumption 1.1.1. In this chapter, we shall impose the following assumption on bidders' valuations:

(i) Each bidder's valuation is independently and identically distributed on some interval $[0, \omega]$ according to a distribution function F:

$$V_i \overset{i.i.d.}{\sim} F \; s.t. \; \text{supp}(F) = \mathbb{R}_+ \tag{1.1.1}$$

(ii) F belongs to the common knowledge in this system;

(iii) Bidders' valuations have finite expectations:

$$\mathbb{E}[V_i] < \infty \tag{1.1.2}$$

Assumption 1.1.2. Moreover, we assume bidders' behaviours to satisfy the following properties:

(i) Bidders are risk neutral, they are maximizing expected profits;

(ii) Each bidder it both willing and able to pay up to his or her value.

Definition 1.1.2. A **strategy** of a bidder is a mapping from the space of his/her valuation to a bid:

$$s : [0, \omega] \to \mathbb{R}_+ \tag{1.1.3}$$

Definition 1.1.3. An equilibrium of auction is **symmetric** if all bidders are following the same bidding strategy s.

Proposition 1.1.1. In a symmetric equilibrium of the second-price auction, $s(v) = v$ is a weakly dominant strategy.

Proof. For a fixed valuation $v_i \in [0, \omega]$ of bidder i.
Let $p := \max_{j \neq i} b_j$ be highest bidding price by other bidders.
Let $\pi_i(b, p)$ denote bidder i's profit when bidding b given the highest price from other bidders to be p.
Part 1: consider another bidding $z_i < v_i$, the following cases are possible:

(i) $v_i < p \implies z_i < v_i < p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i losses anyway).

(ii) $v_i = p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i is indifferent).

(iii) $v_i > p$:

  (a) $v_i > z_i > p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = v_i - p$;

  (b) $v_i > z_i = p \implies \pi_i(v_i, p) \geq \pi_i(z_i, p)$;

  (c) $v_i > p > z_i \implies \pi_i(v_i, p) > \pi_i(z_i, p)$.

Hence, bidding v_i weakly dominates bidding any value below it.
Part 2: for $z_i > v_i$, the following cases are possible:

(i) TODO:

Therefore, bidding v_i weakly dominates bidding any other values. ■

Proposition 1.1.2. In a symmetric equilibrium of the first-price auction, equilibrium bidding strategies are given by

$$s(v_i) = \mathbb{E}[\max_{j \neq i} v_j | v_j \leq v_i] \tag{1.1.4}$$

which is the *expected second highest valuation conditional on v_i being the highest valuation.*

Proof. Let $s(v)$ denote an equilibrium strategy.

Lemma 1.1.1. For any agent, bidding more than $s(\omega)$ can never be optimal. Bidding $b > s(\omega)$ makes this agent win for sure. In such case, bidding $b' \in (s(\omega), b)$ strictly dominates bidding b.

Lemma 1.1.2. For any agent, $s(0) = 0$. Bidding any positive number would cause negative payoff with positive probability, and therefore, leads to a negative expected profit.

Lemma 1.1.3. Because s is monotonically increasing, therefore,

$$\max_{j \neq i} s(v_j) = s(\max_{j \neq i} v_j) \tag{1.1.5}$$

Let p denote the highest price among all other $N-1$ bidders and let $F^{(N-1)}(x)$ denote the distribution of p. The expected profit of bidder i by bidding an arbitrary $b \in \mathbb{R}_+$ is

$$\pi_i(b, v_i) = P(b > p)(v_i - s(v_i)) + P(b = p)(v_i - s(v_i)) + P(b < p)0 \tag{1.1.6}$$

Note that $b > p = s(\max_{j \neq i} v_j)$ if and only if $s^{-1}(b) > \max_{j \neq i} v_j$. It follows

$$P(b > p) = P(\max_{j \neq i} v_j < s^{-1}(b)) = F^{(N-1)}(s^{-1}(b)) \tag{1.1.7}$$

Therefore,

$$\pi_i(b, v_i) = F^{(N-1)}(s^{-1}(b))(v_i - b) \tag{1.1.8}$$

The first order condition implies

$$\frac{\partial \pi_i}{\partial b}\pi_i(b, v_i) = \frac{\partial \pi_i}{\partial b}F^{N-1}(s^{-1}(b))v_i - F^{N-1}(s^{-1}(b))b \tag{1.1.9}$$

$$= f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{1.1.10}$$

For a symmetric equilibrium, all other bidders are following the same strategy s so that $s(v_i) = b$, therefore,

$$f^{(N-1)}(s^{-1}(b))\frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \tag{1.1.11}$$

$$\implies f^{(N-1)}(s^{-1}(b))(v_i - b) - F^{(N-1)}(s^{-1}(b))s'(v_i) = 0 \tag{1.1.12}$$

$$\implies f^{(N-1)}(s^{-1}(b))v_i = F^{(N-1)}(s^{-1}(b))s'(v_i) + f^{(N-1)}(s^{-1}(b))s(v_i) \tag{1.1.13}$$

$$\implies f^{(N-1)}(v_i)v_i = \frac{d}{dv_i}\left[F^{(N-1)}(v_i)s(v_i)\right] \tag{1.1.14}$$

$$\implies \int_0^{v_i} f^{(N-1)}(y)y\ dy = F^{(N-1)}(v_i)s(v_i) - F^{(N-1)}(0)s(0) \tag{1.1.15}$$

$$\implies F^{(N-1)}(v_i)s(v_i) = \int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{1.1.16}$$

$$\implies s(v_i) = \frac{1}{F^{(N-1)}(v_i)}\int_0^{v_i} f^{(N-1)}(y)y\ dy \tag{1.1.17}$$

$$\implies s(v_i) = \mathbb{E}\left[\max_{j\neq i} v_j \,\middle|\, \max_{j\neq i} v_j < v_i\right] \tag{1.1.18}$$

$\blacksquare$