

# MAT237: Multivariable Calculus

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# 1 Limits, continuity, and related topics

## 2 Differentiation and related topics

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### 2.8 Optimization

**Theorem 2.8.1.** Let  $S \subset \mathbb{R}^n$  be an open set and  $f, g : S \rightarrow \mathbb{R}$  be  $C^1$  functions. If  $\mathbf{x}$  is a *local extremal* satisfying  $g(\mathbf{x}) = 0$ , and  $\nabla g(\mathbf{x}) \neq \mathbf{0}$ , then

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \quad (2.8.1)$$

**Lemma 2.8.1.**  $\nabla g(\mathbf{x})$  is orthogonal to the constraint set  $g^{-1}(0)$ .

**Proposition 2.8.1.** Equations (2.8.1)  $\implies \nabla f(\mathbf{x}) \perp g^{-1}(0)$  at  $\mathbf{x}$ .

**Theorem 2.8.2.** Let  $S \subseteq \mathbb{R}^n$  be an open set, and  $f, \{g_i\}_{i=1}^k : S \rightarrow \mathbb{R}$  be  $C^1$  functions. Define  $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_k(\mathbf{x}))$ .

If  $\mathbf{x} \in S$  is a *local extremal* of  $f$  such that  $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ , and  $\{\nabla g_i(\mathbf{x})\}$  are linearly independent (i.e.  $\text{rank}(D\mathbf{g}(\mathbf{x})) = k$ ), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \quad (2.8.2)$$

**Remark 2.8.1.** Procedure of optimization on *open sets*:

- (i) Find all critical points.
- (ii) Find optimizers among critical points.

**Remark 2.8.2.** Procedure of optimization with *inequality constraints*:

- (i) Find critical points without the constraints.
- (ii) Find critical points on the constraints.
- (iii) Find optimizers among candidates.

### 3 The Implicit and Inverse Function Theorems

#### 3.1 The Implicit Function Theorem I

**Theorem 3.1.1** (Implicit Function Theorem). Let  $S \subseteq \mathbb{R}^{n+k}$  be an open set, and function  $F : S \rightarrow \mathbb{R}^k$  be a  $C^1$  function. Suppose there exists point  $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$  such that

$$F(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad (3.1.1)$$

If

$$\det(D_{\mathbf{y}}(F(\mathbf{a}, \mathbf{b}))) \neq 0 \quad (3.1.2)$$

then there exists  $r_0, r_1 > 0$  and a  $C^1$  function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$  such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}), \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \quad (3.1.3)$$

and define  $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$ , the derivative of  $\mathbf{f}$  can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}}F(\mathbf{x}, \mathbf{y})]^{-1}D_{\mathbf{x}}F(\mathbf{x}, \mathbf{y}) \quad (3.1.4)$$

**Remark 3.1.1.** Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (3.1.5)$$

near  $(\mathbf{a}, \mathbf{b})$ .

(i) Verify  $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ .

(ii) Assert

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.6)$$

(iii) Approximate solution  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  using

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \quad (3.1.7)$$

$$= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \quad (3.1.8)$$

#### 3.2 Geometric content of the Implicit Function Theorem

**Definition 3.2.1.** Let  $S \subseteq \mathbb{R}^n$  and  $\mathbf{a} \in S$ .  $S$  is **singular** at  $\mathbf{a}$  if

$$\forall r > 0 \ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \quad (3.2.1)$$

$S$  is **regular** at  $\mathbf{a}$  if it is not singular there.

**Theorem 3.2.1** ( $k$  dimensional manifold as level set). Let  $U \subseteq \mathbb{R}^n$  and let  $\mathbf{F} : U \rightarrow \mathbb{R}^{n-k}$  be a  $C^1$  function.

$$S \equiv \mathbf{F}^{-1}(\mathbf{0}) \quad (3.2.2)$$

Let  $\mathbf{a} \in U$ , if

$$\text{rank}(D\mathbf{F}(\mathbf{a})) = n - k \quad (3.2.3)$$

then  $\exists r > 0$  such that the *level set of  $\mathbf{F}$  near  $\mathbf{a}$*

$$\mathcal{B}(r, \mathbf{a}) \cap S \quad (3.2.4)$$

can be represented as a  $C^1$  graph.

**Theorem 3.2.2** ( $k$  dimensional manifold as parameterization). Let  $T \subseteq \mathbb{R}^k$  and let  $\mathbf{f} : U \rightarrow \mathbb{R}^n$  be a  $C^1$  function.

$$S \equiv \mathbf{f}(T) \quad (3.2.5)$$

Let  $\mathbf{t} \in T$ , if

$$\text{rank}(\mathbf{f}(\mathbf{t})) = k \quad (3.2.6)$$

then  $\exists r > 0$  such that the *parameterization of  $\mathbf{f}$  near  $\mathbf{t}$*

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \quad (3.2.7)$$

can be represented as a  $C^1$  graph.

### 3.3 Transformations, and the Inverse Function Theorem

**Example 3.3.1** (Polar coordinate in  $\mathbb{R}^2$ ). Let

$$U \equiv \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \quad (3.3.1)$$

$$V \equiv \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \quad (3.3.2)$$

Define  $\mathbf{f} : U \rightarrow V$  as

$$\mathbf{f}(r, \theta) \equiv \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} \quad (3.3.3)$$

**Example 3.3.2** (Spherical coordinate in  $\mathbb{R}^3$ ). Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\varphi) \end{pmatrix} \quad (3.3.4)$$

**Example 3.3.3** (Cylindrical coordinate in  $\mathbb{R}^3$ ). Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{pmatrix} \quad (3.3.5)$$

**Theorem 3.3.1** (Inverse Function Theorem). Let  $U$  and  $V$  be open subsets in  $\mathbb{R}^n$ , and  $\mathbf{f} : U \rightarrow V$ . Let  $\mathbf{a} \in U$  and define  $\mathbf{b} \equiv \mathbf{f}(\mathbf{a}) \in V$ . If

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0 \quad (3.3.6)$$

then there exists  $M \subseteq U$  and  $N \subseteq V$  such that

- (i)  $\mathbf{a} \in M$  and  $\mathbf{b} \in N$ ,
- (ii)  $\mathbf{f}$  is bijective between  $M$  and  $N$ ,
- (iii)  $\mathbf{f}^{-1} : N \rightarrow M$  is  $C^1$ ,

and **for all  $\mathbf{x} \in M$**  such  $\mathbf{y} \equiv \mathbf{f}(\mathbf{x}) \in N$ ,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \quad (3.3.7)$$

## 4 Integration

### 4.1 Basics

**Theorem 4.1.1** (**Properties of infimum and supremum**). Let  $A \subseteq \mathbb{R}^n$  and  $A \neq \emptyset$ , and  $f, g : A \rightarrow \mathbb{R}$  are bounded functions. Let  $m$  and  $M$  denote the infimum and supremum respectively, then

- (i)  $m_A f + m_A g \leq m_A(f + g) \leq M_A(f + g) \leq M_A f + M_A g$
- (ii) If  $A' \subseteq A$ , then  $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$
- (iii) If  $f(\mathbf{x}) \leq g(\mathbf{x}) \forall \mathbf{x} \in A$ , then  $m_A f \leq m_A g$  and  $M_A f \leq M_A g$
- (iv)  $|M_A f| \leq M_A |f|$
- (v)  $M_A |f| - m_A |f| \leq M_A f - m_A f$
- (vi)  $\forall c \in \mathbb{R}, M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$
- (vii)  $M_A f - m_A f = \sup\{f(x) - f(y) : x, y \in A\}$

### 4.2 Integration on Higher Dimensions

**Definition 4.2.1.** A rectangle  $\mathcal{R} \subseteq \mathbb{R}^n$  is defined as

$$\mathcal{R} \equiv \prod_{i=1}^n [a_i, b_i] \quad (4.2.1)$$

where  $a_i, b_i \in \mathbb{R}$  and  $a_i < b_i$ .

**Definition 4.2.2.** A **partition**  $P$  of rectangle  $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$  is a list of  $n$  **finite** and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \quad (4.2.2)$$

where  $L_i = \{e_j\}_{j=0}^{T_i}$  such that

$$a_i = e_0 < e_1 < \dots < e_{T_i} = b_i \quad (4.2.3)$$

and such partition induces a set of rectangles(boxes)  $\mathcal{B}(P) \equiv \{B_j\}_{j=1}^J \subseteq \mathcal{R}$ .

**Definition 4.2.3.** Let  $P$  and  $P'$  be two partitions of  $\mathcal{R}$ . Then  $P'$  is a **refinement** of  $P$  if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee B'_j \cap B_j = \emptyset \quad (4.2.4)$$

**Definition 4.2.4.** Define the **volume** of rectangle  $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$  as

$$V^n(\mathcal{R}) \equiv \prod_{i=1}^n (b_i - a_i) \quad (4.2.5)$$

**Definition 4.2.5.** The **lower Riemann sum** of  $f$  with partition  $P$  on  $\mathcal{R}$  is defined as

$$L_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.6)$$

and the **upper Riemann sum** is defined as

$$U_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.7)$$

**Definition 4.2.6.** The **upper integral** and **lower integral** of  $f$  on  $\mathcal{R}$  are defined as

$$\bar{I}_{\mathcal{R}}f \equiv \inf_P U_P f \quad (4.2.8)$$

$$\underline{I}_{\mathcal{R}}f \equiv \sup_P L_P f \quad (4.2.9)$$

**Definition 4.2.7.** A bounded real-valued function  $f$  defined on  $\mathcal{R}$  is **integrable** if

$$\underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.10)$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \, dV^n \equiv \underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.11)$$

**Lemma 4.2.1.** Let  $f$  be a bounded real-valued function defined on  $\mathcal{R}$ ,  $f$  is integrable if and only if  $\forall \epsilon > 0$ , there exists a partition  $P$  of  $\mathcal{R}$  such that

$$U_P f - L_P f < \epsilon \quad (4.2.12)$$

**Theorem 4.2.1.** Let  $f$  and  $g$  be two integrable functions on  $\mathcal{R} \subseteq \mathbb{R}^n$ , let  $c \in \mathbb{R}$ ,

- (i)  $f + g : \mathcal{R} \rightarrow \mathbb{R}$  is integrable and  $\int_{\mathcal{R}}(f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$
- (ii)  $c \cdot f$  is integrable and  $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$
- (iii)  $f(\mathbf{x}) \geq g(\mathbf{x}) \, \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$
- (iv)  $|f|$  is integrable and  $|\int_{\mathcal{R}} f| \leq \int_{\mathcal{R}} |f|$

**Definition 4.2.8.** Let  $S \subseteq \mathbb{R}^n$  be a bounded set, and there exists rectangle  $\mathcal{R}$  covers  $S$ , the **indicator function** of  $S$  is  $\chi_S : \mathcal{R} \rightarrow \{0, 1\}$ , defined as

$$\chi_S(\mathbf{x}) \equiv \mathbb{I}(\mathbf{x} \in S) \quad (4.2.13)$$

**Definition 4.2.9.** Let  $S \subseteq \mathbb{R}^n$  be a bounded set, and there exists rectangle  $\mathcal{R}$  covers  $S$ . Let  $f : \mathcal{R} \rightarrow \mathbb{R}$  be a bounded function, then  $f$  is **integrable on  $S$**  if  $\chi_S f$  is integrable on  $\mathcal{R}$ . And

$$\int \cdots \int_S f \, dV^n \equiv \int \cdots \int_{\mathcal{R}} \chi_S f \, dV^n \quad (4.2.14)$$

**Definition 4.2.10.** Let  $Z \subseteq \mathbb{R}^n$ ,  $Z$  has **zero content** if for all  $\epsilon > 0$ , there exists a finite set of rectangles  $\{R_\ell\}_{\ell=1}^L$  covers  $Z$  and

$$\sum_{\ell=1}^L V^n(R_\ell) < \epsilon \quad (4.2.15)$$

**Proposition 4.2.1.** Let  $Z \subseteq \mathbb{R}^n$  has zero content, then

- (i) For any  $Z' \subseteq Z$ ,  $Z'$  has zero content.
- (ii) Finite union of content zero sets has zero content.
- (iii) Let  $f : [a, b] \rightarrow \mathbb{R}$  be an integrable function, it's graph  $\{(x, f(x)) : x \in [a, b]\}$  has zero content.
- (iv) Let  $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^2$  be a  $C^1$  function, the parameterization  $\mathbf{f}([a, b])$  has zero content.

**Theorem 4.2.2.** Let  $\mathcal{R}$  be a rectangle in  $\mathbb{R}^n$  and  $f$  is integrable on  $\mathcal{R}$  if

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \quad (4.2.16)$$

has zero content.

**Proposition 4.2.2** (Folland 4.22). Suppose  $Z \subseteq \mathbb{R}^n$  has zero content. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded, then  $f$  is integrable on  $Z$  and  $\int_Z f \, dV^n = 0$ .

### 4.3 Iterated Integrals

**Theorem 4.3.1** (Fubini's Theorem). Let  $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$  and  $f : \mathcal{R} \rightarrow \mathbb{R}$  is bounded. Assuming that

- (i)  $f$  is integrable on  $\mathcal{R}$ .
- (ii) for each  $y \in [c, d]$ , the function  $f_y(x) \equiv f(x, y)$  is integrable on  $[a, b]$ .
- (iii) Define  $g(y) \equiv \int_a^b f(x, y) dy$  is integrable on  $[c, d]$ .

Then

$$\iint_{\mathcal{R}} f \, dA = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy \quad (4.3.1)$$

**Proposition 4.3.1.** Let  $S \subseteq \mathbb{R}^n$  be an unbounded set, and  $f : S \rightarrow \mathbb{R}$ . Then improper integral  $\int \cdots \int_S f \, d^n \mathbf{x}$  is absolutely convergent on  $\mathbb{R}^n$  if and only if  $\int \cdots \int_{\mathbb{R}^n} \chi_S f \, d^n \mathbf{x}$  is absolutely convergent.

### 4.4 Change of Variables

**Theorem 4.4.1** (Change of Variable). Let  $U$  and  $V$  be two open subsets of  $\mathbb{R}^n$ , and let  $\mathbf{G} : U \rightarrow V$  be a  $C^1$  bijection. Let  $T \subset U$  and  $S \subset V$ . Suppose  $\mathbf{G}(T) = S$ , then

$$\int \cdots \int_S f \, d\Omega = \int \cdots \int_T f \circ \mathbf{G} \, |\det D\mathbf{G}| \, d\Theta \quad (4.4.1)$$

**Corollary 4.4.1.** Let  $S$  be a region in  $\mathbb{R}^n$ , suppose  $S$  can be parameterized by  $\mathbf{G} : T \rightarrow S$ . By the change of variable formula, consider the special case  $f(\mathbf{x}) = 1$ ,

$$|S| = \int \cdots \int_S 1 \, d\Omega = \int \cdots \int_T 1 \, |\det D\mathbf{G}(\mathbf{u})| \, d\Theta \quad (4.4.2)$$

**Example 4.4.1** (Polar Coordinate). Define the coordinate transformation mapping from polar to Cartesian,

$$\mathbf{P}(r, \theta) \equiv (x, y) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad (4.4.3)$$

and  $|\det D\mathbf{P}(r, \theta)| = r$ .

**Example 4.4.2** (Cylindrical Coordinate). Define the coordinate transformation mapping from cylindrical to Cartesian as

$$\mathbf{C}(r, \theta, z) \equiv (x, y, z) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad r \in \mathbb{R}_+ \quad z \in \mathbb{R} \quad (4.4.4)$$

and  $|\det D\mathbf{C}(r, \theta, z)| = r$ .

**Example 4.4.3** (Spherical Coordinate). Define the coordinate transformation mapping from spherical to Cartesian as

$$\mathbf{S}(r, \theta, \varphi) = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} \quad (4.4.5)$$

and  $|\det D\mathbf{S}(r, \theta, \varphi)| = r^2 \sin \varphi$

## 5 Vector Calculus

### 5.1 Line Integrals

#### 5.1.1 Arc Length

**Definition 5.1.1.** Let  $C$  be a smooth curve in  $\mathbb{R}^n$  parameterized by  $C^1$  function  $\mathbf{g}$  such that  $\mathbf{g}'(t) \neq \mathbf{0}$  for every appropriate  $t$ .

$$C \equiv \{\mathbf{g}(t) : t \in [a, b]\} \quad (5.1.1)$$

and the **arc length** of  $C$  is defined as

$$\int_C d^n \mathbf{x} \equiv \int_C ds \equiv \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.2)$$

**Proposition 5.1.1.** The arc length of a curve  $C$  is an intrinsic property of the geometric object  $C$  and should not depend on the particular parameterization we use.

*Proof.* Let  $\varphi : [c, d] \rightarrow [a, b]$  be a bijection, so that  $\mathbf{h} \equiv \mathbf{g} \circ \varphi$  is also a valid parameterization of  $C$  such that

$$C \equiv \{\mathbf{h}(u) : u \in [c, d]\} \quad (5.1.3)$$

The arc length of  $C$  can be computed using

$$\int_C ds = \int_c^d \|\mathbf{h}'(u)\| du \quad (5.1.4)$$

$$= \int_c^d \|\mathbf{g}'(\varphi(u))\| \times \|\varphi'(u)\| du \quad (5.1.5)$$

$$= \int_a^b \|\mathbf{g}'(t)\| dt \text{ by change of variable formula.} \quad (5.1.6)$$

■

**Remark 5.1.1** (Interpretations). Suppose  $\mathbf{g}$  is a parameterization of  $C$ .

- (i)  $\int_a^b \mathbf{g}'(t) dt = \mathbf{g}(b) - \mathbf{g}(a)$  measures the distance between two endpoints of  $C$ .
- (ii) Choosing a parameterization is effectively choosing an **orientation** for the curve  $C$ .

**Definition 5.1.2.** A function  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  is called **piecewise smooth** if

- (i) it's *continuous*, and
- (ii) it's derivate exists and is continuous except at finitely many points  $t_j$ , at which the one-sided limits exists.

#### 5.1.2 Line Integrals of Scalar Functions

**Definition 5.1.3.** Let smooth curve  $C \subseteq \mathbb{R}^n$ ,  $f : C \rightarrow \mathbb{R}$  and  $\mathbf{g}$  be a parameterization of  $C$ , then

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \quad (5.1.7)$$

**Remark 5.1.2.** The line integrals of scalar functions are also independent from the choices of parameterizations.

**Definition 5.1.4.**

$$\text{Average of } f \text{ over } C \equiv \frac{\int_C f ds}{\int_C ds} \quad (5.1.8)$$



### 5.1.3 Line Integrals of Vector Fields

**Definition 5.1.5.** Let smooth  $C \in \mathbb{R}^n$  with parameterization  $\mathbf{g}$  and  $\mathbf{F} : C \rightarrow \mathbb{R}^n$  defined on it, the **line integral** of  $\mathbf{F}$  over  $C$  is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt \quad (5.1.9)$$

**Proposition 5.1.2.** The line integral  $\int_C \mathbf{F} \cdot d\mathbf{x}$  is independent of the parameterization *as long as the orientation is unchanged*.

### 5.1.4 Rectifiable Curves

**Remark 5.1.3.** Let  $C$  be a curve in  $\mathbb{R}^n$  parameterized by injection  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  such that  $\mathbf{g}'(t) \neq \mathbf{0}$ . Let  $P$  be a partition of  $[a, b]$ . Denote

$$L_P(C) \equiv \sum_j \|\mathbf{g}(t_j) - \mathbf{g}(t_{j-})\| \quad (5.1.10)$$

**Definition 5.1.6.** A curve  $C$  is **rectifiable** if the set  $\{L_P(C) : P\}$  is bounded. And the arc length of  $C$  is defined as

$$L(C) \equiv \sup\{L_P(C) : P\} \quad (5.1.11)$$

**Theorem 5.1.1.** The supremum found above,  $L(C)$  is the precisely the arc length of  $C$ :

$$L(C) = \int_a^b \|\mathbf{g}'(t)\| dt \quad (5.1.12)$$

## 5.2 Green's Theorem

### 5.2.1 Preliminary Definitions

**Definition 5.2.1.** A **simple closed curve** is a curve with parameterization  $\mathbf{g} : [a, b] \rightarrow \mathbb{R}^n$  where

- (i)  $\mathbf{g}$  is continuous;
- (ii)  $\mathbf{g}(a) = \mathbf{g}(b)$ ;
- (iii)  $\mathbf{g}$  is injective with its domain restricted to  $(a, b)$ .

**Definition 5.2.2.** A *simple closed curve* is **piecewise smooth** if it has a parameterization  $\mathbf{g}$  such that

- (i)  $\mathbf{g}$  is continuously differentiable with  $\mathbf{g}'(t) \neq \mathbf{0}$  except finitely many breakpoints;
- (ii)  $\mathbf{g}'(t)$  is *one side continuous* at breakpoints of the curve.

**Definition 5.2.3.** A **regular region**  $S \subseteq \mathbb{R}^n$  is a set satisfying both

- (i)  $S$  is compact;
- (ii)  $\overline{S^{int}} = S$ .

**Definition 5.2.4.** Let  $S \subseteq \mathbb{R}^2$ ,  $S$  has **piecewise smooth boundary** if  $\partial S$  consists of one or more *disjoint, piecewise smooth, simple closed curve*.

**Definition 5.2.5.** Let  $S \subseteq \mathbb{R}^2$ , then **positive orientation** on  $\partial S$  is the orientation on each of the closed curves that make up the boundary such that the region is on the *left* with respect to the positive direction on the curve.

**Theorem 5.2.1** (Green's Theorem). Suppose  $S \subseteq \mathbb{R}^2$  is a regular region with piecewise smooth region  $\partial S$ . Suppose  $\mathbf{F}$  is a  $C^1$  vector field defined on  $\overline{S}$ , then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left( \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA \quad (5.2.1)$$

**Corollary 5.2.1.** Suppose  $S$  is a regular region in  $\mathbb{R}^2$  with piecewise smooth boundary  $\partial S$ , and let  $\mathbf{n}(\mathbf{x})$  be the *unit outward normal* vector to  $\partial S$  at  $\mathbf{x} \in \partial S$ . Suppose also that  $\mathbf{F}$  is a vector field defined on  $\overline{S}$ , then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \left( \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dA \quad (5.2.2)$$

### 5.3 Surface Integrals

#### 5.3.1 Surface Areas and Surface Integrals

**Definition 5.3.1.** Suppose  $S$  is a surface in  $\mathbb{R}^3$  and parameterized by

$$\mathbf{G}(\mathbf{u}) : R \rightarrow S \quad (5.3.1)$$

where  $\text{rank}(D\mathbf{G}(\mathbf{u})) = 2$  for every  $\mathbf{u} \in R \setminus Z$  where  $Z$  is a probably empty set with zero content. If  $\left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\|$  is integrable, then

$$\text{Area}(S) \equiv \iint_R \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.2)$$

**Definition 5.3.2.** Let  $f : S \rightarrow \mathbb{R}$  be a real-valued continuous function defined on a super set of  $S$ , the **integral of a real-valued function on a surface** is defined as

$$\iint_S f(\mathbf{x}) \, dA \equiv \iint_R f(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.3)$$

**Definition 5.3.3.** Let  $\mathbf{F} : S \rightarrow \mathbb{R}^3$  be a continuous vector field defined on a super set of  $S$ , the **integral of vector field on a surface** is defined as

$$\iint_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} \, dA \equiv \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (5.3.4)$$

**Remark 5.3.1.** Surface integrals of **real-valued functions** are independent of the choice of parametrization.

**Remark 5.3.2.** But the choice of parameterization can change the sign of surface integrals of **vector fields**. We need to **choose the direction of the normal,  $\mathbf{n}$** .

**Definition 5.3.4.** Let  $S \subseteq \mathbb{R}^3$  be a two dimensional sub-manifold, and  $f$  is a real-valued function defined on a super set of  $S$ . Define the **average of  $f$  over  $S$**  as

$$\text{aver of } f \text{ over } S \equiv \frac{\iint_S f \, dA}{\iint_S 1 \, dA} \quad (5.3.5)$$

**Remark 5.3.3.** A note on the relation between integrals of a vector field and a real-valued function. The surface of vector field  $\mathbf{F}$  on  $S$  is defined by *reducing  $\mathbf{F}$  to a real-valued function  $\mathbf{F} \cdot \mathbf{n}$*  and then follow the definition of ordinary real-valued function on  $S$ . Define  $h \equiv \mathbf{F} \cdot \mathbf{n}$ ,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_S h \, dA \quad (5.3.6)$$

$$\equiv \iint_R h(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.7)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \mathbf{n}(\mathbf{G}(\mathbf{u})) \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.8)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \frac{\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v}}{\left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\|} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta \quad (5.3.9)$$

$$= \iint_R \mathbf{F}(\mathbf{G}(\mathbf{u})) \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \quad (5.3.10)$$

### 5.3.2 An invariance property

**Remark 5.3.4.** As mentioned above, given  $\mathbf{n}(\mathbf{x})$  fixed, we can define the surface integral of vector field as the surface integral of a real-valued function defined as  $h(\mathbf{x}) \equiv \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ . And as argued before, one  $\mathbf{n}$  is fixed (i.e. orientation is fixed), the value of integral is deterministic. Therefore we can conclude **the integral of a vector field  $\mathbf{F}$  over a surface  $S$  depends on the orientation of  $S$  but otherwise independent of the parameterization.**

**Remark 5.3.5.** Let  $S \subseteq \mathbb{R}^2$  be a two dimensional sub-manifold parameterized by  $\mathbf{G} : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that  $\text{rank}(\mathbf{G}(\mathbf{u})) = 2$  for all but zero-content sets on its domain.

Let  $\varphi : W \subseteq \mathbb{R}^2 \rightarrow R$  be a bijection such that  $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$  is another parameterization of  $S$ .

Now consider the integral of vector field  $\mathbf{F}$  under parameterization  $\mathbf{H}$ ,

$$\iint_S \mathbf{F} \cdot \mathbf{u} \, dA = \iint_W \mathbf{F}(\mathbf{H}) \cdot \left( \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right) d\Theta \quad (5.3.11)$$

$$= \iint_W \mathbf{F} \circ \mathbf{G} \circ \varphi \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) \text{det } D\varphi \, d\Theta \quad (5.3.12)$$

$$= \pm \iint_R \mathbf{F} \circ \mathbf{G} \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta \text{ (change of variable)} \quad (5.3.13)$$

**Theorem 5.3.1** (Invariance). Let  $\mathbf{G} : R \rightarrow \mathbb{R}^3$  and  $\mathbf{H} \equiv \mathbf{G} \circ \varphi : W \rightarrow \mathbb{R}^3$  be two parameterizations of  $S$ , then

$$\iint_R f \circ \mathbf{G} \left\| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right\| d\Theta = \iint_W f \circ \mathbf{H} \left\| \frac{\partial \mathbf{H}}{\partial s} \times \frac{\partial \mathbf{H}}{\partial t} \right\| d\Theta \quad (5.3.14)$$

and

$$\iint_R \mathbf{F} \circ \mathbf{G} \cdot \left( \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) d\Theta = \pm \iint_W \mathbf{F} \circ \mathbf{H} \cdot \left( \frac{\partial \mathbf{H}}{\partial u} \times \frac{\partial \mathbf{H}}{\partial v} \right) d\Theta \quad (5.3.15)$$

### 5.3.3 Volume and Area

**Theorem 5.3.2.** Let  $R$  be an arbitrary regular region in  $\mathbb{R}^3$ , and let  $S$  be the boundary surface of  $R$ , define

$$S_h \equiv \{\mathbf{x} + \delta \mathbf{n} : \mathbf{x} \in S \wedge \delta \in [0, h]\} \quad (5.3.16)$$

where  $S_h$  can be interpreted as a *shell of region  $R$  with thickness  $h$* . Then the surface area of  $S$  is

$$\text{area}(S) = \lim_{h \rightarrow 0} \frac{|S_h|}{h} \quad (5.3.17)$$

## 5.4 Divergence, Gradient and Curl

**Definition 5.4.1.** Let  $U \subseteq \mathbb{R}^n$  be an open set, and define real-valued function  $f : U \rightarrow \mathbb{R}$  and vector field  $\mathbf{F} : U \rightarrow \mathbb{R}^n$ . Then we define

1. The **gradient** of  $f$  as  $\nabla f$ ;
2. The **divergence** of  $\mathbf{F}$  as  $\nabla \cdot \mathbf{F}$ ;
3. The **curl** of  $\mathbf{F}$  as  $\nabla \times \mathbf{F}$ .

**Definition 5.4.2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^1$  real-valued function, define the **Laplacian** of  $f$  as a mapping from *real-valued functional space* to *real-valued functional space* defined as

$$\text{div}(\text{grad})f \equiv \sum_j \partial_j^2 f = \Delta f = \nabla^2 f \quad (5.4.1)$$

**Theorem 5.4.1.** For every  $C^2$  real valued function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,

$$\text{curl}(\text{grad}f) = \mathbf{0} \quad (5.4.2)$$

For every  $C^2$  vector field defined in  $\mathbb{R}^3$  or a subset of it,

$$\text{div}(\text{curl}\mathbf{F}) = 0 \quad (5.4.3)$$

Note that the domain of  $f$  and  $\mathbf{F}$  must be  $\mathbb{R}^3$  or a subset of it, otherwise the curl operation is not well-defined.

**Theorem 5.4.2** (Product rules).

$$\text{grad}(fg) = f\text{grad}g + g\text{grad}f \quad (5.4.4)$$

$$\text{div}(f\mathbf{G}) = f\text{div}\mathbf{G} + \text{grad}f \cdot \mathbf{G} \quad (5.4.5)$$

$$\text{curl}(f\mathbf{G}) = f\text{curl}\mathbf{G} + \text{grad}f \times \mathbf{G} \quad (5.4.6)$$