ECO326 Advanced Microeconomic Theory

A Course in Game Theory

Tianyu Du

February 5, 2019

`Github Page https://github.com/TianyuDu/Spikey_UofT_Notes`
`Note Page TianyuDu.com/notes`

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory*.

Contents

1 Lecture 1. Jan. 7 2019
Games and Dominant Strategies

Game Theory Choice environment where individual choices impact others.

	W	S
W	$(1-c, 1-c)$	$(1-c, 1)$
S	$(1, 1-c)$	$(0, 0)$

Figure 1.1: Payoff Matrix for Example 1

Example 1.1.

Suppose $c \in (0, 1)$. In this game,

i $N = \{i, j\}$,

ii $A_i = A_j = \{W, S\}$,

Definition 1.1 (pg.7)**.** A **preference relation** is a complete reflexive and transitive binary relation.

Definition 1.2 (11.1, lec.1)**.** A **(strategic) game** consists of

i a finite set of **players** N, with $|N| \geq 2$.

ii for each player $i \in N$, an **actions** $A_i \neq \emptyset$.

iii for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i) \rangle$.

Definition 1.3 (Equivalent definition of game)**.** For each player $i \in N$ we can define a utility function, $u_i : A \to \mathbb{R}$ such that

$$\forall (a_i), (a_i)' \in A,\ u_i((a_i)) \geq u_i((a_i)') \iff (a_i) \succsim_i (a_i)' \tag{1.1}$$

So the game can be defined as a triple $\langle N, (A_i), (u_i) \rangle$.

Definition 1.4 (lec.1)**.** An **action profile** is a n-tuple of actions $a_i \in A_i$ for each player $i \in N$ and denoted as

$$(a_i)_{i \in N} \text{ or } (a_i)$$

The **action profile space** is defined as

$$A \equiv \times_{i \in N} A_i$$

Definition 1.5 (lec.1)**.** Action $a_i \in A_i$ is **strictly dominated** by action $\tilde{a}_i \in A_i$ if

$$\forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) < u_i(\tilde{a}_i, a_{-i})$$

And a_i is **weakly dominated** by $\tilde{a}_i$ if

$$\forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) \leq u_i(\tilde{a}_i, a_{-i})$$

and

$$\exists a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) < u_i(\tilde{a}_i, a_{-i})$$

Corollary 1.1 (Consequence of RCT)**.** It is irrational to play strictly dominated actions. So rational choice theory suggests a player would never play strictly dominated strategies.

Definition 1.6. Action $a_i \in A_i$ is **strictly dominant** if it strictly dominates all other actions.

Definition 1.7. Action $a_i \in A_i$ is **weakly dominant** if it weakly dominates all other actions.

Definition 1.8. Action $a_i \in A_i$ is **weakly/strictly dominated** if there exists another strategy weakly/strictly dominates a_i.

	S	C
S	(-1, -1)	(-10, 0)
C	(0, -10)	(-5, -5)

Figure 1.2: Payoff matrix for example 2

Example 1.2 (Prisoner Dilemma)**.** Note that S is strictly dominated by C. Therefore C is strictly dominant for both players.

	L	C	R
U	(2, 2)	(5, 0)	(3, 0)
M	(2, 7)	(2, 5)	(2, 6)
D	(5, 3)	(4, 2)	(3, 1)

Figure 1.3: Payoff matrix for example 2

Example 1.3. So in this game, for player 2, L is strictly dominant.
For player 1, M is strictly dominated by D. And M is weakly dominated by U.

Example 1.4. There are three candidates, $\{A, B, C\}$. And there are 50 players (voters, note that $\emptyset \notin A_i$ since they must vote). And

$$\forall i \in N,\ A_i = \{A, B, C\}$$

Each individual has strictly preference over A, B, C. If tie is encountered, randomization would be taken.

i $A \succ B \succ C$,

ii $A \succ AC_{tie} \succ C$

Claim 1: There are no weakly or strictly dominant actions.

Proof. Let $a_i \in \{V_A, V_B, V_C\}$ denote the action taken by player $i \in N$,
Note that weak dominance is a necessary condition for strict dominance,
So above claim is reduced to *there are no weakly dominant actions.*
The reduced claim is equivalent to the following statement,

$$\forall a_i \in A_i,\ \exists \tilde{a}_i \in A_i \ s.t.\ a_i \neq \tilde{a}_i$$
$$s.t.\ \exists a_{-i} \in A_{-i} \ s.t.\ u_i(a_i, a_{-i}) > u_i(\tilde{a}_i, a_{-i}) \lor \forall a_{-i} \in A_{-i},\ u_i(a_i, a_{-i}) = u_i(\tilde{a}_i, a_{-i})$$

Let n_{-i}^{j} denote the number of voters other than i voting for candidate j. Clearly each $a_{-i} \in A_{-i}$ would induce an outcome as a triple $(n_{-i}^{A}, n_{-i}^{B}, n_{-i}^{C})$.
Consider action V_A, and a_{-i} induces

$$(n_{-i}^{A}, n_{-i}^{B}, n_{-i}^{C}) = (1, 24, 24)$$

then

$$(V_B, a_{-i}) \succ_i (V_A, a_{-i})$$

So V_A failed to be a dominant strategy of any kind.
Similarly, consider action V_B, if a_{-i} induces

$$(n_{-i}^{A}, n_{-i}^{B}, n_{-i}^{C}) = (24, 1, 24)$$

then

$$(V_A, a_{-i}) \succsim_i (V_B, a_{-i})$$

So V_B failed to be a dominant strategy.
Similarly, consider action V_C, if a_{-i} induces

$$(n_{-i}^{A}, n_{-i}^{B}, n_{-i}^{C}) = (24, 24, 1)$$

then

$$(V_A, a_{-i}) \succsim_i (V_C, a_{-i})$$

So V_B failed to be a dominant strategy. ∎

Claim 2: Only voting for your least preferred candidate is weakly dominated.

Proof. We are going to show there exists a strategy (voting for B) weakly dominates voting for C.

Vote A	Cases	Vote C
A	$n_{-i}^{A} > n_{-i}^{B}, n_{-i}^{C}$	A, AC
B	$n_{-i}^{B} > n_{-i}^{A}, n_{-i}^{C}$	B, BC
C, BC	$n_{-i}^{C} > n_{-i}^{A}, n_{-i}^{B}$	C
B	$n_{-i}^{A} = n_{-i}^{B} > n_{-i}^{C}$	AB
A	$n_{-i}^{A} = n_{-i}^{C} > n_{-i}^{B}$	C
BC	$n_{-i}^{C} = n_{-i}^{B} > n_{-i}^{A}$	C

Figure 1.4: Voting for A versus Voting for C

∎

Definition 1.9 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2 Lecture 2. Jan. 14 2019
Iterated Elimination and Rationalizability

Example 2.1 (Bubble Game)**.** Consider a player game

$$\langle N, (A_i), (u_i) \rangle \tag{2.1}$$

where

$$A_i = \{0, \ldots, 100\}, \ \forall i \tag{2.2}$$

and

$$u_i(a_i, a_{-i}) = a_i - penalty_i(a_i, a_{-i}) \tag{2.3}$$

$$penalty_i = \begin{cases} 0 \text{ if } a_i < \max_{j \neq i} a_j - 1 \\ 10(a_i - \max_{j \neq i} a_j + 1) \text{ if } a_i \geq \max_{j \neq i} -1 \end{cases} \tag{2.4}$$

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (IESD)**.** Given the initial game,

$$G_0 = \langle N, (A_i^0), (u_i) \rangle$$

At stage $k \in \mathbb{N}$,

$$G_k = \langle N, (A_i^k), (u_i) \rangle$$

In stage k, for all $i \in N$, find the set of <u>strictly</u> dominated actions, $D_i^k \subsetneq A_i^k$.

i) If $\forall i \in N \ s.t. \ D_i^k = \emptyset$, conclude the profile

$$(A_i^k)$$

to be the set of action profiles survive from IESD.

ii) If $\exists i \in N \ s.t. \ D_i^k \neq \emptyset$, define

$$\forall i \in N, \ A_i^{k+1} := A_i^k \backslash D_i^k$$

Example 2.2. Action profile (M, R) survives the IESD.

Proof.

$$k = 0, \ A_1^0 = \{U, M, D\}, \ A_2^0 = \{L, R\}$$
$$k = 1, \ A_1^1 = \{U, M\}, \ A_2^1 = \{L, R\}$$
$$k = 2, \ A_1^2 = \{U, M\}, \ A_2^2 = \{R\}$$
$$k = 3, \ A_1^3 = \{M\}, \ A_2^3 = \{R\}$$

■

Example 2.3 (Hotelling Model of Politics)**.** Players maximize their votes by choosing where to stand along a natural number line.

	L	R
U	4,0	2,2
M	1, 2	5,3
D	0,5	1,4

Figure 2.1: Game for Example 2.1

- Player $N = \{1, 2\}$
- Action set $A_i = \{1, \ldots, M\}$, with $2 \nmid M$ and $M > 3$.
- Payoff

$$u_i(a_i, a_{-i}) = \begin{cases} a_i + \frac{1}{2}(a_{-i} - a_i - 1) \text{ if } a_i < a_{-i} \\ \frac{M}{2} \text{ if } a_i = a_{-i} \\ M - [a_{-i} + \frac{1}{2}(a_i - a_{-i} - 1)] \text{ if } a_i > a_{-i} \end{cases} \tag{2.5}$$

Claim i. $a_i = 1$ is strictly dominated by $a_i = 2$.

Proof.

$$u_i(a_i = 1, a_{-i}) = \begin{cases} \frac{M}{2} \text{ if } a_{-i} = 1 \\ \frac{a_{-i}}{2} \text{ if } a_{-i} > 1 \end{cases} \tag{2.6}$$

$$u_i(a_i = 2, a_{-i}) = \begin{cases} M - 1 \text{ if } a_{-i} = 1 \\ \frac{M}{2} \text{ if } a_{-i} = 2 \\ \frac{a_{-i}}{2} + \frac{1}{2} \text{ if } a_{-i} > 2 \end{cases} \tag{2.7}$$

■

Claim ii. $\lfloor \frac{n}{2} \rfloor + 1$ is the only action survives.

Proof. Similarly, we can eliminate all edge-values iteratively. ■

Definition 2.2. For each $i \in N$, the **best-response function** of this player is a correspondence $B_i : A_{-i} \rightrightarrows A_i$ defined as

$$B_i(a_{-i}) := \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \ \forall a_i' \in A_i\} \tag{2.8}$$

Definition 2.3. A **belief** of player i (about the actions of the other players) is a probability measure, α_i, on $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$.
α_i is a mapping such that

- $\alpha_i : A_{-i} \to [0, 1]$.
- $\alpha_i(A_{-i}) = 1$.
- For all countable piece-wise disjoint collection

$$\{E_j\}_{j \in \mathcal{J}} \in \mathcal{P}(A_{-i})$$

α_i satisfies the *countable additivity property*:

$$\alpha_i(\bigcup_{i \in I} E_i) = \sum_{i \in I} \alpha_i(E_i)$$

Definition 2.4. a_i is a **best response** to the belief α_i if

$$\forall a_i' \in A_i,\ \sum_{a_{-i}} u_i(a_i, a_{-i})\alpha_i(a_{-i}) \geq \sum_{a_{-i}} u_i(a_i', a_{-i})\alpha_i(a_{-i}) \tag{2.9}$$

or, more generally,

$$\forall a_i' \in A_i, \mathbb{E}[u_i(a_i, a_{-i})|\alpha_i] \geq \mathbb{E}[u_i(a_i', a_{-i})|\alpha_i] \tag{2.10}$$

Definition 2.5. $a_i \in A_i$ is a **never best response** if it is not a best response given any belief α_i.

Corollary 2.1. *Iterative Elimination of Never Best Response*: same procedures but D_i^k is the set of never best responses for player i at game G^k.

Example 2.4. For player 1, D is not strictly dominated, but it is a never best response.

Proof. Let α be a probability measure on $\{L, R\}$ such that $\alpha(L) = p \in [0, 1]$.

$$\mathbb{E}[u_1|U, \alpha] = 10p \tag{2.11}$$
$$\mathbb{E}[u_1|M, \alpha] = 10 - 10p \tag{2.12}$$
$$\mathbb{E}[u_1|D, \alpha] = 1 \tag{2.13}$$

Case i

$$p \geq 0.5 \implies \mathbb{E}[u_1|U, \alpha] \geq 5 \tag{2.14}$$

Case ii

$$p < 0.5 \implies \mathbb{E}[u_1|M, \alpha] > 5 \tag{2.15}$$

Therefore, for any belief α, D cannot be a best response. So D is a never best response. ■

	L	R
U	10,0	0,0
M	0,0	10,0
D	1,0	1,0

Definition 2.6. An action $a_i \in A_i$ is **rationalizable** if it survives *iterative elimination of never best responses.*

Lemma 2.1 (i385.3)**.** In a two player game, a_i is strictly dominated if and only if it is a never best response.

Assumption 2.1 (Common knowledge rationality)**.** We assume our players of game all acknowledge that other players are playing the game in rational ways.

3 Lecture 3. Jan. 21 2019

Definition 3.1. A **pure strategy Nash equilibrium** is a strategy profile (a_i) such that

$$\forall i \in N,\ a_i \in Br_i(a_{-i}) \tag{3.1}$$

Remark 3.1. That's, a NE is a situation that if player i knows what other players do, the action given by the NE profile is still a best response.

Remark 3.2 (Interpretations)**.** A Nash equilibrium is

i) An action profile induces a stable outcome,

ii) A creditable agreement, such that no player has incentive to break the agreement.

Example 3.1 (Cournot Duopoly)**.** Consider a game with

i) Player $N = \{1, 2\}$

ii) Action set $A_i = [0, \infty)\ \forall i \in N$

And revenue R_i defined by

$$R_i = p_i q_i \tag{3.2}$$

where price is linear in quantity supplied,

$$p_i = \alpha - (q_i + q_{-i}),\ \ \alpha \in \mathbb{R} \tag{3.3}$$

and firms face fixed cost $c \in \mathbb{R}$, with the assumption that $\alpha > c$. So the cost function is given by

$$C_i(q_i) = cq_i \tag{3.4}$$

The profit function is given by

$$\Pi_i(q_i, q_{-i}) = (\alpha - (q_i + q_{-i}) - c)q_i \tag{3.5}$$

$$= (\alpha - c - q_{-i})q_i - q_i^2 \tag{3.6}$$

Given $q_{-i} \in A_{-i}$, the best response is given by

$$Br_i(q_{-i}) = \text{argmax}_{q_i \in [0,\infty)} \Pi_i(q_i, q_{-i}) \tag{3.7}$$

$$= \max\{0, \frac{\alpha - c - q_{-i}}{2}\}\ \forall i \in N \tag{3.8}$$

Considering the case that both players are producing positive quantities, we can solve q_i^* by

$$Br_i \circ Br_{-i}(q_i) = \frac{\alpha - c - \frac{\alpha - c - q_i}{2}}{2} = q_i \tag{3.9}$$

$$\implies 2q_i - \frac{q_i}{2} = \frac{\alpha - c}{2} \tag{3.10}$$

$$\implies q_i^* = \frac{\alpha - c}{3} \tag{3.11}$$

Remark 3.3. If fixed cost presents, even if the game is symmetric, the Nash equilibrium could be asymmetric. (e.g. one firm is out of market and the other firm produces the monopoly amount)

Remark 3.4. Nash equilibrium induces an *individual level optimality* instead of the common wealth optimality.

Example 3.2 (Continue Cournot Duopoly)**.** Note that in the Cournot Duopoly game, for each $i \in N$, the Nash equilibrium profit is

$$\Pi^*_{NE} = (\alpha - c - \frac{2(\alpha - c)}{3})\frac{\alpha - c}{3} \tag{3.12}$$
$$= \frac{(\alpha - c)^2}{9} \tag{3.13}$$

So the total profit for the two firms is $\frac{2(\alpha-c)^2}{9}$.
Now considering if the two firms form a Cartel, the *aggregate* quantity produced is

$$Q^* = \text{argmax}_{Q \in \mathbb{R}_{\geq 0}}(\alpha - c - Q)Q \tag{3.14}$$
$$= \frac{\alpha - c}{2} \tag{3.15}$$
$$\implies \Pi^*_{Cartel} = (\alpha - c - \frac{\alpha - c}{2})\frac{\alpha - c}{2} \tag{3.16}$$
$$= \frac{(\alpha - c)^2}{4} > \frac{2(\alpha - c)^2}{9} \tag{3.17}$$

The fact $\Pi^*_{Cartel} > 2 \times \Pi^*_{NE}$ suggests the Nash equilibrium action profile did not induce the optimal common wealth outcome. However, the Cartel action profile is not a stable outcome since every player has incentive to increase their production level.

Example 3.3 (Prisoner's Dilemma)**.** The Nash equilibrium(Confess, Confess) is not the best outcome for the two players as a group. The optimal action profile for a group is (Silent, Silent).

Proposition 3.1. No strategy that is eliminated during iterated elimination of *never best response* can be played in a Nash equilibrium.

4 Lecture 4. Jan. 28. 2019

Example 4.1 (From last lecture)**.** Consider the payoff matrix Both (A, A) and (B, B) are Nash

	A	B
A	(1,1)	(0,0)
B	(0,0)	(0,0)

equilibria. But in the former NE, for each $i \in N$,

$$Br_i(a_{-i} = A) = \{A\} \tag{4.1}$$

which is a singleton.
In the second NE, for each $i \in N$,

$$Br_i(a_{-i} = B) = \{A, B\} \tag{4.2}$$

We call Nash equilibria of the first type *strict Nash equilibria* and the later one *weak Nash equilibria*. More formal definitions of these two types of Nash equilibria are given below.

Definition 4.1. A **strict Nash equilibrium** is an action profile (a_i) such that

$$\forall i \in N,\ |Br_i(a_{-i})| = 1$$

Definition 4.2. A **weak Nash equilibrium** is a Nash equilibrium that is not strict. That's, a weak Nash equilibrium is an action profile (a_i) such that

$$\exists i \in N,\ |Br_i(a_{-i})| > 1$$

Example 4.2 (Cournot with n firms)**.** Consider the game

$$\langle N, (\mathbb{R}_{\geq 0}), (\pi_i)\rangle \tag{4.3}$$

Where $|N| = n$ and each firm picks a quantity $q_i \in A_i \equiv \mathbb{R}_{\geq 0}$.
Define

$$Q \equiv \sum_j q_j \tag{4.4}$$

For each $i \in N$, define

$$Q_{-i} \equiv \sum_{j \neq i} q_j \tag{4.5}$$

And the market has *linear demand curve*

$$P(\{q_i\}_{i \in N}) = \alpha - \sum_j q_j = \alpha - Q \tag{4.6}$$

And firms face fixed cost

$$\forall i \in N,\ C(q_i) = cq_i \text{ where } 0 < c < \alpha \tag{4.7}$$

The profit function is

$$\forall i \in N,\ \pi_i(q_i, Q_{-i}) = (\alpha - c - Q_{-i})q_i - q_i^2 \tag{4.8}$$

Question What are the Nash equilibria in this environment?
For each firm $i \in N$, the best response correspondence is

$$Br_i(Q_{-i}) = \max\{0, \operatorname*{argmax}_{q_i \in \mathbb{R}_{\geq 0}} \pi_i(q_i, Q_{-i})\} \tag{4.9}$$

$$= \max\{0, \frac{\alpha - c - Q_{-i}}{2}\} \tag{4.10}$$

Assume We have a symmetric Nash equilibrium,

$$\forall i \in N,\ q_i^* = q^* = \frac{Q}{n} \tag{4.11}$$

$$\implies q^* = Br_i(\frac{n-1}{n}Q) \tag{4.12}$$

$$\implies 2q^* = \alpha - c - (n-1)q^* \tag{4.13}$$

$$\implies q^* = \frac{\alpha - c}{n+1} \tag{4.14}$$

Check Then check the validity of symmetric Nash equilibrium by asserting for every player, if all other players are playing the action suggested by the symmetric Nash equilibrium, then this player should also play it. That's

$$
\begin{aligned}
q^* &= Br_i(\{q_j\}_{j\neq i}) && (4.15)\\
&= \frac{\alpha - c}{2} - \frac{1}{2}\frac{n-1}{n+1}(\alpha - c) && (4.16)\\
&= \frac{1}{2}((\alpha - c) - \frac{n-1}{n+1}(\alpha - c)) && (4.17)\\
&= \frac{1}{2}\frac{2}{n+1}(\alpha - c) && (4.18)\\
&= \frac{\alpha - c}{n+1} = q^* && (4.19)
\end{aligned}
$$

Uniqueness We are going to show the symmetric Nash equilibrium is the only possible equilibrium action profile.

Proof. Suppose there exists some non-symmetric Nash equilibrium.
For concreteness, assuming there exists $\epsilon > 0$ such that

$$\exists i, j \in N,\ q_i = q_j + \epsilon \tag{4.20}$$

Define $Q_{-ij} \equiv Q - q_i - q_j$.
For firm i,

$$
\begin{aligned}
q_i = Br_i(Q_{-i}) &= \frac{1}{2}(\alpha - c - Q_{-i}) && (4.21)\\
&= \frac{1}{2}(\alpha - c - Q_{-ij} - q_j) && (4.22)\\
&= \frac{1}{2}(\alpha - c - Q_{-ij} - q_i + \epsilon) && (4.23)\\
\implies 3q_i &= \alpha - c - Q_{-ij} + \epsilon && (4.24)\\
\implies 3q_i &= \alpha - c - Q_{-j} + q_i + \epsilon && (4.25)\\
\implies 2q_i - \epsilon &= \alpha - c - Q_{-j} && (4.26)\\
\implies q_i - \frac{\epsilon}{2} &= Br(Q_{-j}) = q_j && (4.27)\\
&= q_i - \epsilon && (4.28)\\
\implies \epsilon &= 2\epsilon && (4.29)
\end{aligned}
$$

which contradicts the assumption that $\epsilon > 0$.
Therefore we conclude that the symmetric Nash equilibrium is the only Nash equilibrium in this environment. ■

Example 4.3 (Cournot duopoly with fixed cost)**.** Consider the game

$$\mathcal{G} = \langle N = (1, 2), (A_i = \mathbb{R}_{\geq 0}), (\pi_i)\rangle \tag{4.30}$$

Where the cost function is defined as

$$C_i(q_i) = \begin{cases} cq_i + f & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases} \tag{4.31}$$

where $\alpha > c > 0$.
For each firm $i \in N$, the best response function, conditioned on $q_i > 0$, is

$$q_i = Br_i(q_{-i}) = \frac{\alpha - c - q_{-i}}{2} \tag{4.32}$$

We have to **check the profit** to assert the profit is non-negative while firm i is producing above quantity. Because, otherwise, this firm could always derivate to $q_i = 0$ to avoid loss (earning zero profit).

$$\begin{aligned}
\pi_i(Br_{q_{-i}}, q_{-i}) &= (\alpha - q_{-i} - \frac{\alpha - c - q_{-i}}{2} - c)\frac{\alpha - c - q_{-i}}{2} - f && (4.33)\\
&= \frac{(\alpha - c - q_{-i})^2}{4} - f && (4.34)\\
\pi_i(q_{-i}) \geq 0 &\iff \alpha - c - q_{-i} \geq 2\sqrt{f} && (4.35)\\
&\iff q_{-i} \leq \alpha - c - 2\sqrt{f} && (4.36)
\end{aligned}$$

Therefore we can modify our **best response correspondence** to

$$Br_i(q_{-i}) = \begin{cases} \frac{\alpha - c - q_{-i}}{2} & \text{if } q_{-i} \leq \alpha - c - 2\sqrt{f} \\ 0 & \text{if } q_{-i} \geq \alpha - c - 2\sqrt{f} \end{cases} \tag{4.37}$$

Monopoly: the monopoly amount, conditioned on the firm decides to produce, is $\frac{\alpha - c}{2}$. In the monopoly case, suppose the NE action profile is (the opposite case can be shown by symmetry)

$$(\frac{\alpha - c}{2}, 0) \tag{4.38}$$

We have to assert both

$$\begin{cases} \frac{\alpha - c}{2} \in Br_1(0) \\ 0 \in Br_2(\frac{\alpha - c}{2}) \end{cases} \tag{4.39}$$

$$\implies \begin{cases} 0 \leq \alpha - c - 2\sqrt{f} & \text{for firm 1 to produce positive amount.} \\ \frac{\alpha - c}{2} \geq \alpha - c - 2\sqrt{f} & \text{for firm 2 to produce zero.} \end{cases} \tag{4.40}$$

$$\begin{aligned}
&\implies 0 \leq \alpha - c - 2\sqrt{f} \leq \frac{\alpha - c}{2} && (4.41)\\
&\implies -(\alpha - c) \leq -2\sqrt{f} \leq -\frac{\alpha - c}{2} && (4.42)\\
&\implies \sqrt{f} \in [\frac{\alpha - c}{4}, \frac{\alpha - c}{2}] && (4.43)
\end{aligned}$$

Positive symmetric equilibrium: we've proven, in the general case, it's impossible for both firms to produce positive but different amounts. Therefore we have to assert

$$\begin{aligned}
&\frac{\alpha - c}{3} \in Br_i(\frac{\alpha - c}{3}) \ \forall i \in N && (4.44)\\
&\implies \frac{\alpha - c}{3} \leq \alpha - c - 2\sqrt{f} && (4.45)\\
&\implies \sqrt{f} \leq \frac{1}{3}(\alpha - c) && (4.46)
\end{aligned}$$

Zero symmetric equilibrium: we have to assert

$$
\begin{aligned}
&0 \in Br_i(0)\ \forall i \in N && (4.47)\\
\implies &0 \geq \alpha - c - 2\sqrt{f} && (4.48)\\
\implies &\sqrt{f} \geq \frac{\alpha - c}{2} && (4.49)
\end{aligned}
$$

Example 4.4 (Discrete price Bertrand duopoly)**.** Consider the following game

$$\mathcal{G} = \langle N = \{1, 2\}, (A_i = \{k\epsilon : k \in \mathbb{Z}_{\geq 0}\}), (\pi_i)\rangle \tag{4.50}$$

The profit function can be derived as

$$\pi_i(p_i, p_{-i}) = \begin{cases} (\alpha - p_i)(p_i - c) & \text{if } p_i < p_{-i} \\ \frac{\alpha - p_i}{2}(p_i - c) & \text{if } p_i = p_{-i} \\ 0 & \text{if } p_i > p_{-i} \end{cases} \tag{4.51}$$

Claim the only Nash equilibria are (c, c) and $(c + \epsilon, c + \epsilon)$.
Justify (c, c), consider any firm $i \in N$, currently $\pi_i = 0$

$$
\begin{aligned}
\uparrow p_i &\implies \pi_i \leftarrow \pi_i' = 0 ✗ && (4.52)\\
\downarrow p_i &\implies \pi_i \leftarrow \pi_i' < 0 ✗ && (4.53)
\end{aligned}
$$

So no firm has incentive to derivate from this action profile, so (c, c) is a Nash equilibrium by definition.
Consider the action profile $(c + \epsilon, c + \epsilon)$, both firms are earning a positive profit $\pi_i > 0$. For any firm $i \in N$,

$$
\begin{aligned}
\uparrow p_i &\implies \pi_i \leftarrow \pi_i' = 0 ✗ && (4.54)\\
\downarrow p_i &\implies \pi_i \leftarrow \pi_i' = 0 ✗ && (4.55)
\end{aligned}
$$

So $(c + \epsilon, c + \epsilon)$ is a Nash equilibrium by definition.
No other Nash equilibrium
Claim: (p_1, p_2) cannot be a Nash equilibrium if any $p_i < c$.
Obviously, set $p_i < c$ would induce negative profit and firm i do better off by setting $p_i \leftarrow c$.
Claim: the symmetric profit (p, p) with $p > c + \epsilon$ cannot be a Nash equilibrium. For both firms, the current profit is

$$\pi_1 = \pi_2 = \frac{1}{2}(\alpha - c - k\epsilon)k\epsilon \tag{4.56}$$

And reducing price by ϵ leads to a profit of

$$\pi_i' = (\alpha - c - k\epsilon + \epsilon)(k - 1)\epsilon > \pi_i \tag{4.57}$$

so such action profile cannot be Nash equilibrium.
Claim (p_1, p_2) with $p_1 \neq p_2$ and $p_1, p_2 > c$ cannot be a Nash equilibrium. The firm charges higher price can always reduce it's price to the price charged by the other firm and gain a positive profit.
Claim (c, p_2) with $p_2 \geq c + \epsilon$ cannot be a Nash equilibrium. The firm charging $p = c$ can always increase its price to $c + \epsilon$ to earn positive profit.
Therefore there's no Nash equilibrium other than (c, c) and $(c + \epsilon, c + \epsilon)$.

Example 4.5 (Bertrand duopoly with differentiated products)**.** Consumers are uniformly distributed on a preference line $[0, 1]$.
For a firm $i \in N$, let $x_i \in [0, 1]$ measure consumer's preference towards firm i's products.
Define

$$x_{-i} \equiv 1 - x_i \sim Unif(0, 1) \tag{4.58}$$

Consumer buy product i if

$$x_i - p_i \geq x_{-i} - p_{-i} \tag{4.59}$$

and purchases product 1 if

$$x_i - p_i \leq x_{-i} - p_{-i} \tag{4.60}$$

Then solve the boundary $x^* \in [0, 1]$ such that consumers at x^* are indifferent between two products.

$$x + p_i = (1 - x) + p_{-i} \tag{4.61}$$

$$\implies x^* = \frac{1 - p_i + p_{-i}}{2} \tag{4.62}$$

So any consumer with $x_i > x^*$ would choose firm i's product, also because consumers are uniformly distributed on $x_i \in [0, 1]$. So the portion of consumers buying firm i's product is

$$1 - x_i = \frac{1 + p_i - p_{-i}}{2} \tag{4.63}$$

And, clearly, The demand function for firm i is

$$D_i(q_i, q_{-i}) = \begin{cases} 0 & \text{if } p_i \geq p_{-i} + 1 \\ \frac{1+p_i-p_{-i}}{2} & \text{if } p_i \in [p_{-i} - 1, p_{-i} + 1] \\ 1 & \text{if } p_i \leq p_{-i} - 1 \end{cases} \tag{4.64}$$

Consider the case where $|p_i - p_{-i}| \leq 1$,

$$\pi_i = D_i(p_i)(p_i - c) = \frac{1 + p_i - p_{-i}}{2}(p_i - c) \tag{4.65}$$

Take the first order condition

$$\frac{\partial \pi_i}{\partial p_i} = 0 \tag{4.66}$$

$$\implies p_i = \frac{p_{-i} + c - 1}{2} \tag{4.67}$$

5 Lecture 5 Feb. 7 2019

Example 5.1. Matching pennies: an example of game without pure strategy Nash equilibrium.

	H	T
H	(1,-1)	(-1,1)
L	(-1,1)	(1,-1)

Definition 5.1. Suppose player i has a *finite* set of pure strategies A_i, then a **mixed strategy** $\sigma_i \in \Delta(A_i)$ is a probability distribution over A_i.

Definition 5.2. The **support** of σ_i is defined as

$$\mathcal{S}(\sigma_i) \equiv \{a \in A_i : \sigma_i(a) > 0\} \tag{5.1}$$

Notation 5.1. Given action set $A_i \equiv \{a_i\}$, a mixed strategy can be denoted as A_i^α where α is a multi-index.

Remark 5.1. A pure strategy $a_i \in A_i$ is a mixed strategy with

$$\sigma_i(a_i) = 1 \tag{5.2}$$

So mixed strategy is a generalization of pure strategy.

Proposition 5.1. In a finite game, given the independence of randomization, the probability of the action profile $a = (a_i)$ to be realized given mixed strategy profile (σ_i) is

$$\sigma(a) \equiv \mathbb{P}[(a_i)|(\sigma_i)] = \prod_{i \in N} \sigma_i(a_i) \tag{5.3}$$

and for player i, the **expected payoff** from mixed strategy profile (σ_i) is

$$U_i((\sigma_i)) = \sum_{a \in A} \big[\prod_{j \in N} \sigma_j(a_j)\big] u_i(a) = \mathbb{E}[u_i(a)|(\sigma_i)] \tag{5.4}$$

Proposition 5.2. The **expected payoff** from mixed strategy profile $(\sigma_i) \equiv (\sigma_i, \sigma_{-i})$ is

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}[u_i(a)|(\sigma_i)] = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_{-i}(a_{-i})\sigma_i(a_i) \tag{5.5}$$

Definition 5.3. A pure strategy $a_i \in A_i$ is **strictly dominated** by a mixed strategy $\sigma_i \in \Delta(A_i)$ if

$$\forall a_{-i} \in A_{-i} \ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i}) \tag{5.6}$$

Proposition 5.3. We can also define the strict dominance by replacing $\forall a_{-i} \in A_{-i}$ with $\forall \sigma_{-i} \in \Delta A_{-i}$ since

$$\forall a_{-i} \in A_{-i} \ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i}) \iff \forall \sigma_{-i} \in \Delta(A_{-i}) \ U_i(a_i, \sigma_{-i}) < U_i(\sigma_i, \sigma_{-i}) \tag{5.7}$$

Proof. ($\Longrightarrow$) Suppose

$$\forall a_{-i} \in A_{-i} \ u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i}) \tag{5.8}$$

Let $\sigma_{-i} \in \Delta(A_{-i})$,

$$U_i(a_i, \sigma_{-i}) = \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_{-i}(a_{-i}) \tag{5.9}$$

$$< \sum_{a_{-i} \in A_{-i}} U_i(\sigma_i, a_{-i})\sigma_{-i}(a_{-i}) \tag{5.10}$$

$$= U_i(\sigma_i, \sigma_{-i}) \tag{5.11}$$

Thus

$$\forall \sigma_{-i} \in \Delta(A_{-i})\ U_i(a_i, \sigma_{-i}) < U_i(\sigma_i, \sigma_{-i}) \tag{5.12}$$

($\Longleftarrow$) Suppose

$$\forall \sigma_{-i} \in \Delta(A_{-i})\ U_i(a_i, \sigma_{-i}) < U_i(\sigma_i, \sigma_{-i}) \tag{5.13}$$

Let $a_{-i} \in A_{-i}$, consider σ_{-i} defined as $\sigma^*_{-i}(x) \equiv \mathbb{I}(x = a_{-i})$ so that

$$U_i(a_i, \sigma^*_{-i}) < U_i(\sigma_i, \sigma^*_{-i}) \tag{5.14}$$

$$\implies u_i(a_i, a_{-i}) < U_i(\sigma_i, a_{-i}) \tag{5.15}$$

The equivalence is shown. ∎

Definition 5.4. The **best response** to a mixed strategy $\sigma_{-i} \in \Delta A_{-i}$ is defined as

$$Br_i(\sigma_{-i}) \equiv \{\sigma_i \in \Delta(A_i) : \forall \tilde{\sigma}_i \in \Delta(A_i),\ U_i(\sigma_i, \sigma_{-i}) \geq U_i(\tilde{\sigma}_i, \sigma_{-i})\} \tag{5.16}$$

Proposition 5.4. Let $(\sigma_i, \sigma_{-i}) \in \Delta(A)$ then $\sigma_i \in Br_i(\sigma_{-i})$ *if and only if*

1. $\forall a_j, a_k \in \mathcal{S}(\sigma_i),\ a_j \sim_i a_k$,
2. and $\forall a_j \in \mathcal{S}(\sigma_i)\ a_k \notin \mathcal{S}(\sigma_i),\ a_j \succsim_i a_k$

Proof. ($\implies$) case 1: suppose (1) is false, for concreteness, assume

$$\exists a_j, a_k \in \mathcal{S}(\sigma_i)\ s.t.\ U_i(a_j, \sigma_{-i}) > U_i(a_k, \sigma_{-i}) \tag{5.17}$$

Define

$$\mathcal{U}_j \equiv U_i(a_j, \sigma_{-i}) = \sum_{a_{-i} \in A_{-i}} u_i(a_j, a_{-i})\sigma_{-i}(a_{-i}) \tag{5.18}$$

$$\mathcal{U}_k \equiv U_i(a_k, \sigma_{-i}) = \sum_{a_{-i} \in A_{-i}} u_i(a_k, a_{-i})\sigma_{-i}(a_{-i}) \tag{5.19}$$

And $\mathcal{U}_j > \mathcal{U}_k$ by assumption. Consider σ'_i defined with

$$\sigma'_i(a) = \begin{cases} \sigma_i(a_j) + \sigma_i(a_k) & \text{if } a = a_j \\ 0 & \text{if } a = a_k \\ \sigma_i(a) & \text{otherwise} \end{cases} \tag{5.20}$$

Therefore

$$U_i(\sigma'_i, \sigma_{-i}) \equiv \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma'_i(a_i)\sigma_{-i}(a_{-i}) \tag{5.21}$$

$$= \sum_{a_i \in A_i \backslash \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma'_i(a_i)\sigma_{-i}(a_{-i}) + \sigma'_i(a_j)\mathcal{U}_j + \sigma'_i(a_k)\mathcal{U}_k \tag{5.22}$$

$$= \sum_{a_i \in A_i \backslash \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) + \sigma_i(a_j)\mathcal{U}_j + \sigma_i(a_k)\mathcal{U}_j \tag{5.23}$$

$$> \sum_{a_i \in A_i \backslash \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) + \sigma_i(a_j)\mathcal{U}_j + \sigma_i(a_k)\mathcal{U}_k \tag{5.24}$$

$$= \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) \equiv U_i(\sigma_i, \sigma_{-i}) \tag{5.25}$$

So that σ_i' strictly dominates σ_i, so $\sigma_i \notin Br_i(\sigma_{-i})$.
case 2: suppose (2) is false, for concreteness, assume

$$\exists a_j \in \mathcal{S}(\sigma_i),\ a_k \notin \mathcal{S}(\sigma_i)\ s.t.\ U_i(a_k, \sigma_{-i}) > U_i(a_j, \sigma_{-i}) \tag{5.26}$$

Consider mixed strategy σ_i' defined as

$$\sigma_i'(a) = \begin{cases} 0 & \text{if } a = \sigma_j \\ \sigma_i(a_j) & \text{if } a = \sigma_k \\ \sigma_i(a) & \text{otherwise} \end{cases} \tag{5.27}$$

By previous definitions, $\mathcal{U}_k > \mathcal{U}_j$.
Then

$$U_i(\sigma_i', \sigma_{-i}) \equiv \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i'(a_i)\sigma_{-i}(a_{-i}) \tag{5.28}$$

$$= \sum_{a_i \in A_i \setminus \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i'(a_i)\sigma_{-i}(a_{-i}) + \sigma_i'(a_j)\mathcal{U}_j + \sigma_i'(a_k)\mathcal{U}_k \tag{5.29}$$

$$= \sum_{a_i \in A_i \setminus \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) + 0\ \mathcal{U}_j + \sigma_i(a_j)\mathcal{U}_k \tag{5.30}$$

$$> \sum_{a_i \in A_i \setminus \{a_j, a_k\}} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma_i(a_i)\sigma_{-i}(a_{-i}) + \sigma_i(a_j)\mathcal{U}_j \tag{5.31}$$

$$= U_i(\sigma_i, \sigma_{-i}) \tag{5.32}$$

Therefore $\sigma_i \notin Br_i(\sigma_{-i})$, *modus tollens.*
($\Longleftarrow$) suppose $\sigma_i \notin Br_i(\sigma_{-i})$, that's

$$\exists \sigma_i' \in \Delta(A_i)\ s.t.\ U_i(\sigma_i', \sigma_{-i}) > U_i(\sigma_i, \sigma_{-i}) \tag{5.33}$$

Therefore

$$\exists \bar{a}_i, \underline{a}_i \in \mathcal{S}(\sigma_i)\ s.t.\ U_i(\bar{a}_i, \sigma_{-i}) \geq U_i(\sigma_i, \sigma_{-i}) \geq U_i(\underline{a}_i, \sigma_{-i}) \tag{5.34}$$

$$\wedge \exists \bar{a}_i', \underline{a}_i' \in \mathcal{S}(\sigma_i')\ s.t.\ U_i(\bar{a}_i', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \geq U_i(\underline{a}_i', \sigma_{-i}) \tag{5.35}$$

$$\Longrightarrow U_i(\bar{a}_i', \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) > U_i(a_i, \sigma_{-i}) \geq U_i(\underline{a}_i, \sigma_{-i}) \tag{5.36}$$

$$\Longrightarrow U_i(\bar{a}_i', \sigma_{-i}) > U_i(\underline{a}_i, \sigma_{-i}) \tag{5.37}$$

$$\Longrightarrow \bar{a}_i' \succ_i \underline{a}_i \in \mathcal{S}(\sigma_i) \tag{5.38}$$

In both cases, $\overline{a}_i' \in \mathcal{S}(\sigma_i)$ and $\overline{a}_i' \notin \mathcal{S}(\sigma_i)$, the two requirements are falsified, *modus tollens.* ■

Remark 5.2. Procedure to solve Nash equilibria for two player finite games,

(i) Solve for pure strategy Nash equilibria.

(ii) Eliminate strictly dominated strategies.

(iii) For player i, solve a mixed strategy scheme σ_j so that player i is indifferent between actions in a non-singleton subset of his/her action set, A_i^*.

(iv) Player i would play any mixed strategy with support A_i^*.

(v) Repeat (iii) and (iv) for player j.