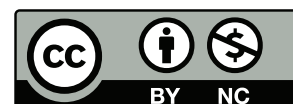


# Introduction to Real Analysis

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# 1 The Axiom of Completeness

**Definition 1.1.** A set  $A \subset \mathbb{R}$  is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

**Example 1.1.** The set of integers,  $\mathbb{Z}$ , is neither bounded from above nor below. Sets  $\{1, 2, 3\}$  and  $\{\frac{1}{n} : n \in \mathbb{N}\}$  are bounded from both above and below.

**Notation 1.1.** Let  $A \subset \mathbb{R}$ , we use  $A^\uparrow$  and  $A^\downarrow$  to denote collections of upper bounds of  $A$  and lower bounds of  $A$ . Note that  $A^\uparrow$  and  $A^\downarrow$  are potentially empty.

**Definition 1.2.** A real number  $s \in \mathbb{R}$  is the **least upper bound (supremum)** for a set  $A \subset \mathbb{R}$  if  $s \in A^\uparrow$  and  $\forall u \in A^\uparrow, s \leq u$ . Such  $s$  is denoted as  $s := \sup A$ .

**Definition 1.3.** A real number  $f \in \mathbb{R}$  is the **greatest lower bound (infimum)** for  $A$  if  $f \in A^\downarrow$  and  $\forall l \in A^\downarrow, l \leq f$ . Such  $f$  is often written as  $f := \inf A$ .

**Axiom 1.1** (The Axiom of Completeness/Least Upper Bounded Property).  $\forall \emptyset \neq A \subset \mathbb{R}$  such that  $A^\uparrow \neq \emptyset$ ,  $\exists \sup A$ .

**Definition 1.4.** Let  $\emptyset \neq A \subset \mathbb{R}$ ,  $a_0 \in A$  is the **maximum** of  $A$  if  $\forall a \in A, a_0 \geq a$ ;  $a_1 \in A$  is the **minimum** of  $A$  if  $\forall a \in A, a_1 \leq a$ .

**Example 1.2.**  $\mathbb{Q} \subset \mathbb{R}$  does not satisfy the axiom of completeness.

**Proposition 1.1.** Let  $\emptyset \neq A \subset \mathbb{R}$  bounded above, and  $c \in \mathbb{R}$ . Define  $c + A := \{a + c : a \in A\}$ . Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

*Proof. Step 1: Show  $c + \sup A \in (c + A)^\uparrow$ :*

Let  $x \in c + A$ ,  $\exists a \in A$  s.t.  $x = c + a$ . Then,  $x = c + a \leq c + \sup A$ . Therefore,  $x \leq c + \sup A \forall x \in c + A$ , which implies what desired.

*Step 2: Show  $\forall u \in (c + A)^\uparrow, c + \sup A \leq u$ :*

Let  $u \in (c + A)^\uparrow$ , then  $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$ .

Hence,  $\sup(c + A) = c + \sup A$ . ■

**Lemma 1.1** (Alternative Definition of Supremum). Let  $s \in A^\uparrow$  for some nonempty  $A \subset \mathbb{R}$ . The following statements are equivalent:

- (i)  $s = \sup A$ ;

(ii)  $\forall \varepsilon, \exists a \in A, \text{ s.t. } a > s - \varepsilon$  (i.e.  $s - \varepsilon \notin A^\uparrow$ ).

*Proof.* Immediately. ■

**Theorem 1.1** (Nested Interval Property). Let  $(I_n)_n$  be a sequence of closed intervals  $I_n := [a_n, b_n]$  such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \quad \forall n \in \mathbb{N} \quad (1.4)$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.5)$$

*Proof.* Note that the sequence  $(a_n)_{n \in \mathbb{N}}$  is bounded above by any  $b_k$ , by the completeness axiom, there exists  $a^* := \sup_{n \in \mathbb{N}} a_n$ . Since  $a^* \in (a_n)^\uparrow$ ,  $a^* \geq a_n \quad \forall n \in \mathbb{N}$ . Further, because  $a^*$  is the *least* upper bound, then for every upper bound  $b_n$ , it must be  $a^* \leq b_n \quad \forall n \in \mathbb{N}$ . Therefore,  $x^* \in [a_n, b_n] \quad \forall n \in \mathbb{N}$ . That is,  $x^* \in \bigcap_{n \in \mathbb{N}} I_n$ . ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold:  $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$ .

**Theorem 1.2** (Archimedean Property).

- (i)  $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$ ;
- (ii)  $\forall y \in \mathbb{R}_{++}, \exists \mathbb{N} \text{ s.t. } \frac{1}{n} < y$ .