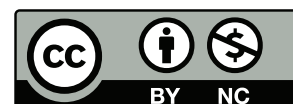


Introduction to Real Analysis

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Tuesday 17th September, 2019

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Contents

1	The Axiom of Completeness	2
1.1	Preliminaries	2
1.2	Density of Rational Numbers	4
2	Sequences	4

1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \text{ s.t. } \forall a \in A, u \geq a \quad (1.1)$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \text{ s.t. } \forall a \in A, l \leq a \quad (1.2)$$

Example 1.1. The set of integers, \mathbb{Z} , is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subset \mathbb{R}$, we use A^\uparrow and A^\downarrow to denote collections of upper bounds of A and lower bounds of A . Note that A^\uparrow and A^\downarrow are potentially empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound (supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^\uparrow$ and $\forall u \in A^\uparrow, s \leq u$. Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if $f \in A^\downarrow$ and $\forall l \in A^\downarrow, l \leq f$. Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property). $\forall \emptyset \neq A \subset \mathbb{R}$ such that $A^\uparrow \neq \emptyset, \exists \sup A$.

Definition 1.4. Let $\emptyset \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

Proposition 1.1. Let $\emptyset \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \quad (1.3)$$

Proof. Step 1: Show $c + \sup A \in (c + A)^\uparrow$:

Let $x \in c + A, \exists a \in A \text{ s.t. } x = c + a$. Then, $x = c + a \leq c + \sup A$. Therefore, $x \leq c + \sup A \forall x \in c + A$, which implies what desired.

Step 2: Show $\forall u \in (c + A)^\uparrow, c + \sup A \leq u$:

Let $u \in (c + A)^\uparrow$, then $u \geq c + a \forall a \in A \implies u - c \geq a \forall a \in A \implies u - c \in A^\uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.

Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum). Let $s \in A^\uparrow$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

- (i) $s = \sup A$;
- (ii) $\forall \varepsilon, \exists a \in A, \text{ s.t. } a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^\uparrow$).

Proof. Immediately. ■

Theorem 1.1 (Nested Interval Property). Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \quad \forall n \in \mathbb{N} \quad (1.4)$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.5)$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k , by the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$. Since $a^* \in (a_n)^\uparrow$, $a^* \geq a_n \quad \forall n \in \mathbb{N}$. Further, because a^* is the *least* upper bound, then for every upper bound b_n , it must be $a^* \leq b_n \quad \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n] \quad \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$.

Theorem 1.2 (Archimedean Property).

- (i) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x$;
- (ii) $\forall y \in \mathbb{R}_{++}, \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < y$.

Archimedean property of *natural numbers* can be interpreted as *there is no real number that bounds \mathbb{N}* . This interpretation can be seen by considering the negations of above statements:

- (i) $\exists x \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, n \leq x$;
- (ii) $\exists y \in \mathbb{R}_{++} \text{ s.t. } \forall n \in \mathbb{N}, y \leq \frac{1}{n}$.

Proof of (i) by Contradiction. Suppose the negated statement (i) is true, \mathbb{N} is bounded above. By the completeness axiom, there exists $a^* := \sup \mathbb{N}$. $\exists n \in \mathbb{N} \text{ s.t. } a^* - 1 < n$. In this case, $a^* < n+1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^\uparrow$ and leads to a contradiction. ■

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ■

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The above theorem says \mathbb{Q} is in fact **dense** in \mathbb{R} . More generally, one says a set $A \subset X$ is dense whenever the closure of A , $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.

Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ■

Theorem 1.4. $\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in \mathbb{R} bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^\uparrow$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ■

2 Sequences

Theorem 2.1 (Triangle Inequality). Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.1. Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \quad (2.1)$$

Proof. Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$. And $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$. Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ■

Definition 2.1. A sequence $(a_n) \subset \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |a_n - a| < \varepsilon \quad (2.2)$$

Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at a with radius ε is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \quad (2.3)$$

Theorem 2.2. The limit of any convergent sequence is unique.

Proof. Let (a_n) be a convergent sequence, assume, for contradiction, that $(a_n) \rightarrow L_1$ and $(a_n) \rightarrow L_2$ such that $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \rightarrow L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$|a_n - L_2| = |a_n - L_1 - (L_2 - L_1)| \quad (2.4)$$

$$\geq ||a_n - L_1| - |L_2 - L_1|| \quad (2.5)$$

$$= ||L_1 - L_2| - |a_n - L_1|| \quad (2.6)$$

$$= 3\varepsilon - |a_n - L_1| \quad (2.7)$$

$$> 2\varepsilon \quad (2.8)$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq N$. ■

Definition 2.2. A sequence (a_n) is **divergent** if it does not converge.

Example 2.1. The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \dots)$ is divergent.

Proof. Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \rightarrow L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $|L - \frac{1}{5}| < \frac{2}{15}$. Similarly,

$$\left| -\frac{1}{5} - L \right| = \left| \frac{1}{5} + L \right| \quad (2.9)$$

$$= \left| \frac{1}{5} + L - \frac{1}{5} + \frac{1}{5} \right| \quad (2.10)$$

$$= \left| \left(L - \frac{1}{5} \right) - \left(-\frac{2}{5} \right) \right| \quad (2.11)$$

$$\geq \left| \left| L - \frac{1}{5} \right| - \frac{6}{15} \right| \quad (2.12)$$

$$= \frac{6}{15} - \left| L - \frac{1}{5} \right| \quad (2.13)$$

$$> \frac{4}{15} \quad (2.14)$$

$$> \varepsilon \quad (2.15)$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ■

Alternative Proof. If (a_n) is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of (a_n) converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ■

Definition 2.3. A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, |a_n| < M$.

Theorem 2.3. Every convergent sequence is bounded.

Proof. Let $(a_n) \rightarrow L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n < N} a_n$, take $M := \max\{Q, |L| + 1\}$, then M bounds (a_n) . ■

Theorem 2.4 (Algebraic Limit Theorem). Let $(a_n) \rightarrow a, (b_n) \rightarrow b$ be convergent sequences, and $c \in \mathbb{R}$, then

- (i) $(ca_n) \rightarrow ca$;
- (ii) $(a_n + b_n) \rightarrow a + b$;
- (iii) $(a_nb_n) \rightarrow ab$;
- (iv) $\left(\frac{a_n}{b_n}\right) \rightarrow \frac{a}{b}$, provided $b_n, b \neq 0$.

Proof (i). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

Proof (ii). Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3} \forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3} \forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \quad (2.16)$$

■

Proof (iii). Note that

$$|a_nb_n - ab| = |a_nb_n + a_nb - a_nb - ab| \quad (2.17)$$

$$\leq |a_nb_n - a_nb| + |a_nb - ab| \quad (2.18)$$

$$\leq |a_n||b_n - b| + |b||a_n - a| \quad (2.19)$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because (a_n) is convergent, let M denote its bound such that $|a_n| < M \forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_nb_n - ab| < \varepsilon$. ■

Proof (iv). **Claim i:** when n is sufficiently larger, $|b_n| > 0$ is bounded away from zero by M . Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such n ,

$$|b_n| = |b_n - b - (-b)| \quad (2.20)$$

$$\geq ||b_n - b| - |b|| \quad (2.21)$$

$$\geq |b| - |b_n - b| \quad (2.22)$$

$$> \frac{9|b|}{10} \quad (2.23)$$

Claim ii: $\left(\frac{1}{b_n}\right) \rightarrow \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_nb} - \frac{b_n}{b_nb}\right| \quad (2.24)$$

$$= \frac{1}{|b_n||b|}|b_n - b| \quad (2.25)$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \rightarrow b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

Theorem 2.5 (Order Limit Theorem). Let $(a_n) \rightarrow a$ and $(b_n) \rightarrow b$, then

- (i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;
- (ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;
- (iii) $\exists c \in \mathbb{R} \text{ s.t. } c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;
- (iv) $\exists c \in \mathbb{R} \text{ s.t. } a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

Proof. (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.

(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \rightarrow (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.

(iii) and (iv) Consider constant sequence defined as (c_n) such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

Theorem 2.6 (Squeeze Theorem). Let $(x_n) \rightarrow L$ and $(z_n) \rightarrow \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \rightarrow \ell$.

Proof. Suppose $(y_n) \not\rightarrow \ell$, then there exists $\varepsilon > 0$ such that there are infinitely many $y_n \notin V_\varepsilon(\ell)$. Take the same $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n, z_n \in V_\varepsilon(\ell)$. Clearly, this contradicts $x_n \leq y_n \leq z_n$. ■