ECO325: Lecture Notes

Advanced Economic Theory: Macro

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Notes

Github `https://github.com/TianyuDu/Spikey_UofT_Notes`

Direct Download `TianyuDu.com/notes`

Color notations

- Important equations for model setup.
- Important equations as results from model.
- Implications of model result.

Revisions

- Revise October 2. 2018. Midterm 1. Lec1-Lec4
- Revise October 21. 2018. Lec5-Lec6

Contents

1 Lecture 1. September 6. 2018

Definition 1.1. A **growth miracle** are episodes where thee growth in a country far exceeds the world average over a extended period of time. Result the country experiencing the miracle moves up the wold income distribution.

Definition 1.2. A **growth disaster** is an episodes where the growth in a country falls short at the world average for an extended period of time. Result the country moves down in the world income distribution.

Facts (with corresponding result from Solow growth model.)

1. Real output(Y) grows at a (more or less) constant rate $(n+g)$.
2. Stock of real capital(K) grows at a (more or less) constant rate$(n+g)$ (but it grows faster than labor input$(L)(n)$.
3. Growth rates of real output and the stock of capital are about the same. (both $n+g$)
4. The rate of growth of output per capita$(\frac{Y}{L})$ varies greatly across countries. (g varies across countries)

1.1 Solow Growth Model (continuous time version)

Intro. Solow growth model decomposes the growth in output per capita into portions accounted for by increase in inputs and the portion contributed to increases in productivity.

Notations In the baseline model we denote K as capital, L as labor and A as technology.

1.1.1 Production Function

Remark 1.1. ~~Harrod-neutral technology here, refer to Uzawa's theorem.~~

Definition 1.3. The **effective labor input**(total units off effective labor) is defined as $A(t)L(t)$

Definition 1.4. The production function is defined as a real-valued mapping from input factor space to an output level.

$$Y(t) = F(K(t), A(t)L(t)) \tag{1}$$

Example 1.1. Cobb-Douglas form of production function.

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha},\ \alpha \in (0,1)$$

Assumption 1.1. The production function is assumed to be constant return to scale in K and AL.

$$Y(cK, cAL) = cY(K, AL),\ \forall c \geq 0$$

This CRS assumption is the result of two separate assumptions.

1. *The economy is big enough that the gains from specialization have been exhausted.* $\implies$ There is **no** increasing return to scale.
2. *Inputs other than capital, labor, and the effectiveness of labor are relatively unimportant.* $\implies$ There is **no** decreasing return to scale.

Definition 1.5. Define $c := \frac{1}{AL}$, the **intensive form** of production function is

$$y(t) = \frac{Y(t)}{A(t)L(t)} = f(k(t))$$

where $y := \frac{Y}{AL}$ denotes the **output per unit of effective labor** and $k := \frac{K}{AL}$ denote the capital stock per unit of effective labor.

2 Lecture 2 September 13. 2018

2.1 Solow Growth Model: Setup

Definition 2.1. Production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ maps input factors:

- $K(t) :=$ aggregate capital stock at time t.
- $L(t) :=$ aggregate labor supple at time t.
- $A(t) :=$ labor argument technology[1] (effectiveness of labor) at time t.

to output values ($Y(t) :=$ aggregate output at time t.) The production function takes the form of

$$Y(t) = F(K(t), A(t)L(t))$$

Assumption 2.1 (Assumptions on Production Function)**.** The production function are assumed to be constant return to scale in $A(t)L(t)$ and $K(t)$.

$$cF(K(t), A(t)L(t)) = F(cK(t), cA(t)L(t)),\ \forall c > 0$$

Definition 2.2. The **intensive form of production function** is defined as the output per unit of effective labor.
Let

$$f(t) := \frac{Y(t)}{A(t)L(t)}$$

[1] Harrod-neutral technology

and

$$k(t) := \frac{K(t)}{A(t)L(t)}$$

denote the output and capital per unit of effective labor respectively. By the assumption of *CRS* on *aggregate* production function, take $c = \frac{1}{A(t)L(t)}$. The *intensive form* production function can be expressed as

$$y(t) = f(k(t)) \tag{1}$$

Assumption 2.2 (Assumptions on Intensive Form Production Function)**.** the function $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy *Inada Conditions.*

1. $f(0) = 0$: capital is necessary for production.
2. $f'(k) > 0,\ \forall k \in \mathbb{R}_+$: the marginal return of capital per effective unit of labor is positive.
3. $f''(k) < 0,\ \forall k \in \mathbb{R}_+$: capital per effective unit of labor is experiencing diminishing marginal return.
4. $\lim_{k\to 0} f'(k) = \infty$
5. $\lim_{k\to\infty} f''(k) = 0$

Remark 2.1. The role of assumption 2.2 is to ensure that the path of the economy does not diverge.

Example 2.1 (Cobb-Douglas Production Function)**.** Consider the Cobb-Douglas production function

$$Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha},\ \alpha \in (0, 1)$$

Check. Let $c \in \mathbb{R}_+$,

$$\begin{aligned} F(cK, cAL) &= (cK)^\alpha (cAL)^{1-\alpha} \\ &= c^\alpha c^{1-\alpha} K^\alpha AL^{1-\alpha} \\ &= cK^\alpha AL^{1-\alpha} = cF(K, AL) \end{aligned}$$

CRS on aggregate form is shown.
Notice that $f(k) = k^\alpha$
And

1. $f(0) = 0^\alpha = 0$
2. $f'(k) = \alpha k^{\alpha-1} > 0, \forall k \in \mathbb{R}_+$
3. $f''(k) = (\alpha - 1)\alpha k^{\alpha-2} < 0,\ \forall k \in \mathbb{R}_+$
4. $\lim_{k\to 0} \alpha \frac{1}{k^{1-\alpha}} = \infty$

5. $\lim_{k\to\infty} \alpha \frac{1}{k^{1-\alpha}} = 0$

Inada conditions on intensive form are shown.

■

Assumption 2.3 (Assumptions on the Economy)**.** Assume the initial values of K, A, L are given and strictly positive. Labor and Knowledge are assumed to grow at an exogenously given constant rate, denoted as n, g respective.

$$\dot{L(t)} = nL(t), \ \ n > 0 \tag{2}$$

$$\dot{A(t)} = gA(t), \ \ g > 0 \tag{3}$$

Proposition 2.1. Notice the growth rate of variable $X(t)$ is given by

$$g_X := \frac{\dot{X(t)}}{X(t)} = \frac{\partial \ln X(t)}{\partial t}$$

Proof.

$$\frac{\partial \ln X(t)}{\partial t} = \frac{\partial \ln X(t)}{\partial X(t)} \frac{\partial X(t)}{\partial t}$$

$$= \frac{1}{X(t)} \dot{X(t)} = \frac{\dot{X(t)}}{X(t)} = g_X$$

■

Proposition 2.2. The functional form of technology and labor at time t can be found by solving ODEs

$$L(t) = e^{nt} L(0) \tag{4}$$

$$A(t) = e^{gt} A(0) \tag{5}$$

Assume there is *no government* and the Solow economy is a *closed economy.* The output is divided between *consumption* and *investment* as

$$Y(t) = C(t) + I(t)$$

And given δ as depreciation rate of capital, in discrete time (let $\Delta t = 1$)we have

$$K(t+1) = (1-\delta)K(t) + I(t)$$

$$\iff I(t) = K(t+1) - K(t) + \delta K(t)$$

$$\text{As } \Delta \to 0 \text{ (convert to continuous time)}$$

$$I(t) = \dot{K}(t) + \delta K(t)$$

Assumption 2.4. Assume investment equals saving and a constant friction $s \in [0, 1]$ of output is saved at each epoch. The marginal propensity to save, s is given exogenously.

Therefore,

$$\begin{aligned} I(t) = sY(t) &\implies \dot{K}(t) + \delta K(t) = sY(t) \\ &\implies \dot{K}(t) = sY(t) - \delta(K(t)) \end{aligned}$$

2.2 Dynamics of $k(t)$

For simplicity, assuming $n, g, \delta > 0$ and the dynamics of capital per effective unit of labor follows:

$$\begin{aligned} \dot{k}(t) := \frac{\partial k(t)}{\partial t} &= \frac{\partial}{\partial t}\frac{K(t)}{A(t)L(t)} \\ &= \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2} \\ &= \frac{\dot{K}}{AL} - \frac{K\dot{A}L}{(AL)^2} - \frac{KA\dot{L}}{(AL)^2} \\ &= \frac{sY - \delta K}{AL} - \frac{\dot{A}}{A}\frac{K}{AL} - \frac{\dot{L}}{L}\frac{K}{AL} \\ &= sy(t) - (n + g + \delta)k(t) \end{aligned}$$

Where $sy(t)$ is the **actual investment** per unit of effective labor and $(n + g + \delta)k(t)$ is the **break-even investment** per unit of effective labor.

Remark 2.2. The **convergence speed** is <u>inversely</u> correlated with the value of $|k(t) - k^*|$, where k^* denotes the steady state level of capital stock per effective unit of labor.

Remark 2.3. With <u>convex</u> production function ($f''(k) > 0$), then $k(t) < k^* \implies \dot{k} < 0$ and $k(t) > k^* \implies \dot{k} > 0$. The steady state value k^* is <u>steady but not stable</u> (with $k(t) \neq k^*$, k does not automatically converge to k^*).

3 Lecture 3 September 20. 2018

3.1 Dynamic Transitions

Remark 3.1. For the dynamic transition function of capital per unit of effective labor:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t) \tag{1}$$

And dynamic transition and phase diagram can be expressed as

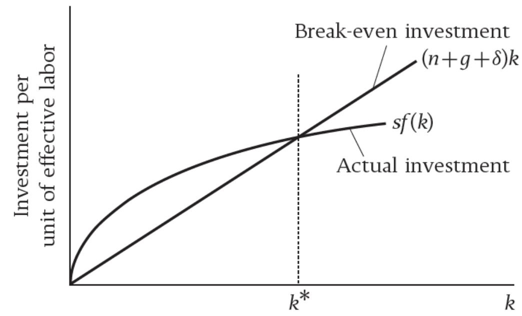


Figure 1: Dynamic Transition of Capital Per Unit of Effective Labor

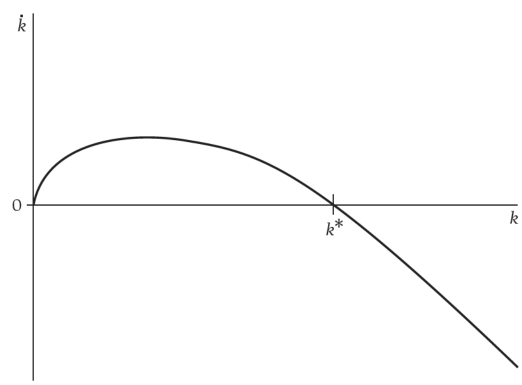


Figure 2: Phase Diagram of Capital Per Unit of Effective Labor

Definition 3.1. Steady level of capital per unit of effective labor(k^*) is defined as the level of capital per unit of effective labor that equates break-even investment per unit of effective labor and actual investment per unit of effective labor. So that k does not deviate from k^*.[2]

$$k^* := \{k \in \mathbb{R}_+ : sf(k) = (n + g + \delta)k\}$$

Remark 3.2. The values of other endogenous variables at steady state are derived from k^*.

Example 3.1. Find the steady state growth rate of investment, consumption and output per unit of effective labor.

$$y^* = f(k^*) \tag{2}$$

$$i^* = sf(k^*) = (n + g + \delta)k^* \tag{3}$$

$$c^* = y^* - i^* = f(k^*) - (n + g + \delta)k^* = (1 - s)f(k^*) \tag{4}$$

[2]The definition can also be expressed as $k^* := \{k \in \mathbb{R}_+ : \dot{k}(k) = 0\}$

For the growth rate of each endogenous variable (per unit of effective labor).

$$\frac{\partial i(t)}{\partial t}|_{k=k^*} = 0 \tag{5}$$

$$\frac{\partial c(t)}{\partial t}|_{k=k^*} = 0 \tag{6}$$

$$\frac{\partial y(t)}{\partial t}|_{k=k^*} = 0 \tag{7}$$

above relations are equivalent to

$$\text{On steady state} \begin{cases} \dot{i}(t) = 0 \\ \dot{c}(t) = 0 \\ \dot{y}(t) = 0 \end{cases} \tag{8}$$

Proof. By definition of consumption per unit of effective labor,
$c(\cdot) = (1-s)f(k(t))$
$\implies \dot{c}(t) := \frac{\partial c(\cdot)}{\partial t} = (1-s)f'(k(t))\dot{k(t)}$ by chain rule
Since $\dot{k}|_{k=k^*} = 0$ and $(1-s)f'(k(t)) < \infty$
Thus $\dot{c}(t)|_{k=k^*} = 0$
And $i(t) = sf(k(t))$, which is constant at $sf(k^*)$ at steady state.

■

3.2 Balanced Growth Path

Definition 3.2. A **balanced growth path** is a situation where each variable in the model are all growing at a constant rate. [3] [4]

3.2.1 Growth Rates on Balanced Growth Path

Population and Technology By definition of population and technological progress,

$$g_A := \frac{\dot{A}}{A} = g \tag{9}$$

$$g_L := \frac{\dot{L}}{L} = n \tag{10}$$

Capital per person Since $\frac{K(t)}{L(t)} = \frac{k(t)A(t)L(t)}{L(t)} = k(t)A(t)$, and the growth rate of $x(t)$ can be found as $\frac{\partial \ln x(t)}{\partial t}$. Then

[3]Variables are not required to grow at the same rate by this definition.
[4]Variables remaining fixed are also considered as growing at a constant rate ($g = 0$).

Solution.

$$
\begin{aligned}
\frac{\partial \ln \frac{K(t)}{L(t)}}{\partial t} &= \frac{\partial k(t)A(t)}{\partial t} \\
&= \frac{\partial \ln k(t)}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\
&= \frac{\dot{k}(t)}{k(t)} + g
\end{aligned}
$$

And at the steady state, by definition, $\dot{k}(t)|_{k-k^*} = 0$, therefore

$$
g^*_{\frac{K}{L}} = g \tag{11}
$$

■

Output and Consumption per person Similarly,

Solution.

$$
\begin{aligned}
\frac{Y(t)}{L(t)} &= y(t)A(t) \\
g_{\frac{Y}{L}} &= \frac{\partial \ln y(t) + \ln A(t)}{\partial t} \\
&= \frac{\partial \ln y}{\partial t} + \frac{\partial \ln A(t)}{\partial t} \\
&= g + \frac{\dot{y}}{y}
\end{aligned}
$$

and for consumption per person,

$$
\begin{aligned}
\frac{C(t)}{L(t)} &= c(t)A(t) \\
g_{\frac{C}{L}} &= g + \frac{\dot{c}}{c}
\end{aligned}
$$

Thus, on the balanced growth path, [5]

$$
g^*_{\frac{Y}{L}} = g \tag{12}
$$

$$
g^*_{\frac{C}{L}} = g \tag{13}
$$

■

Proposition 3.1. Along the balanced growth path, consumption and output per person also grow at rate g.

[5] g^*_X denotes the growth rate of variable X on the balanced growth path.

Proposition 3.2. Along the balanced growth path, aggregate variables, $Y(t), I(t), C(t)$ are all growing at a rate $n+g$.

$$g_Y^* = g_C^* = g_I^* = n + g \tag{14}$$

Proof.

$$\begin{aligned} g_K &= \frac{\partial \ln K(t)}{\partial t} \\ &= \frac{\partial \ln A(t)L(t)k(t)}{\partial t} \\ &= \frac{\partial \ln A(t)}{\partial t} + \frac{\partial \ln L(t)}{\partial t} + \frac{\partial \ln k(t)}{\partial t} \\ &= g + n + \frac{\dot{k}}{k} \end{aligned}$$

and at balanced growth path, $\frac{\dot{k}}{k}|_{k=k^*} = 0$, therefore

$$g_K^* = n + g \tag{15}$$

and proof for $C(t)$ and $I(t)$ follows the same path. ■

Definition 3.3. The **golden rule level of capital per unit of effective labor** (k_G) is the steady state level of capital per unit of effective labor that maximizes steady state consumption per unit of effective labor.

$$k_G = argmax_{k^* \in \mathbf{k}^*(\Theta)} \{c^* = f(k^*) - (n + g + \delta)k^*\}$$

Definition 3.4. The **golden rule level of saving rate** s_G is the saving rate such that the golden role level of capital per unit of effective labor is achieved.

Proof. (First Order Necessary Condition for k_G).

$$\begin{aligned} &\frac{\partial c^*(k^*)}{\partial k^*} = 0 \\ \Longrightarrow \quad &\frac{\partial f(k^*) - (n + g + \delta)k^*}{\partial k^*} = 0 \\ \Longrightarrow \quad &f'(k^*) = (n + g + \delta) \end{aligned}$$

Thus, golden rule level of capital stock per unit of effective labor k_G can be expressed as [6]

$$k_G = \{k \in \mathbb{R}_+ : f'(k) = (n + g + \delta)\} \tag{16}$$

■

[6]Notice that the zero solution, $k^* = 0$ is a trivial steady state and we ignore this case during this course.

3.3 Experiment

3.3.1 Impact of Change in the Saving Rate ($s_1 > s_0$)

Suppose at time t_0, the saving rate parameter increases discretely: $s_0 \to s_1$.

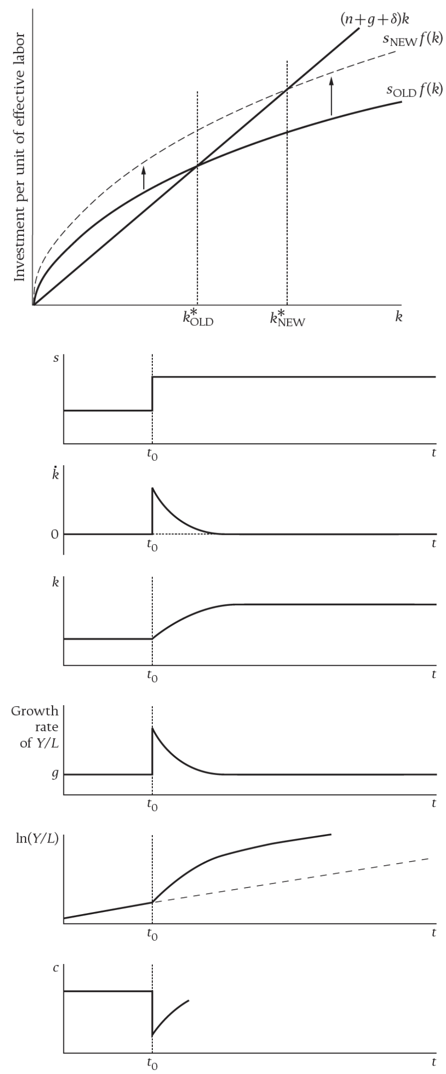


Figure 3: Effect of an Increase in Saving Rate.

Remark 3.3. The relation of c_0^* and c_1^* depends on the relative position of s_1 and the golden rule level of saving rate s_G.

3.3.2 Derive the Effect of Change in s Mathematically

Goal Find $\frac{\partial k^*}{\partial s}$. And notice that $k^*(n+g+\delta) = sf(k^*)$ for any steady state capital level k^*. And the steady state level of capital per unit of effective labor can be written as a function of parameters, as $k^*(n, g, \delta, s)$.

Impact on k^*

Solution. At any steady state level, k^* satisfies

$$sf(k^*(n, g, \delta, s)) = (n+g+\delta)k^*(n, g, \delta, s)$$

Differentiate both sides with respect to s,
We have

$$sf'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n+g+\delta)\frac{\partial k^*}{\partial s}$$

Rearrange and get

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)}$$

Notice that the slope of break-even investment is greater than the slope of the actual investment at the steady state, therefore

$$\frac{\partial k^*}{\partial s} > 0$$

■

Impact on y^*

Solution. Using chain rule we have

$$\begin{aligned} \frac{\partial y^*}{\partial s} &= \frac{\partial f(k^*)}{\partial s} \\ &= \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial s} > 0, \ \forall k^* \in \mathbf{k}^*(\Theta) \end{aligned}$$

■

To get a sense on how much y^* changes wit respect to change in s, we could look at the elasticity.

$$\eta = \frac{\partial y^*}{\partial s}\frac{s}{y^*} = f'(k^*)\frac{\partial k^*}{\partial s}\frac{s}{f(k^*)} = \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)}$$

Recall that $(n+g+\delta) = \frac{sf(k^*)}{k^*}$ and rearrange the elasticity

$$\begin{aligned}\eta &= \frac{\partial y^*}{\partial s}\frac{s}{y^*} \\ &= \frac{f'(k^*)s}{(n+g+\delta) - sf'(k^*)} \\ &= \frac{sf'(k^*)}{\frac{sf(k^*)}{k^*} - sf'(k^*)} \\ &= \frac{f'(k^*)}{\frac{f(k^*)}{k^*} - f'(k^*)} \\ &= \frac{f'(k^*)\frac{k^*}{f(k^*)}}{1 - f'(k^*)\frac{k^*}{f(k^*)}} \\ &= \frac{\alpha_K}{1-\alpha_K}\end{aligned}$$

Remark 3.4. α_K denotes the elasticity of output per unit of effective unit labor with respect to capital stock per unit of effective labor, along the balanced growth path. And

$$\alpha_K \approx \frac{1}{3}$$

Remark 3.5. If the production function is in the Cobb-Douglas form, then $\alpha_K = \alpha$.

Example 3.2. If $\alpha_K \approx \frac{1}{3}$ then

$$\eta = \frac{\partial y^*}{\partial s}\frac{s}{y^*} \approx \frac{1}{2}$$

Impact on c^* Notice that on the balanced growth path $c^* = y^* - i^*$.

$$c^* = f(k^*) - (n+g+\delta)k^* \tag{17}$$

and differentiate with respect to s

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}$$

And notice that the sign of $\frac{\partial c^*}{\partial s}$ depends on the relative slope of production function and break-even investment. By the first order condition of golden rule level of capital per unit of effective labor, $(n+g+\delta) = f'(k_G)$

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - f'(k_G)]\frac{\partial k^*}{\partial s}$$

And

$$\begin{cases} k^* = k_G \implies f'(k^*) = f'(k_G) \implies \frac{\partial c^*}{\partial s} = 0 \\ k^* < k_G \implies f'(k^*) > f'(k_G) \implies \frac{\partial c^*}{\partial s} > 0 \\ k^* > k_G \implies f'(k^*) < f'(k_G) \implies \frac{\partial c^*}{\partial s} < 0 \end{cases}$$

4 Lecture 4 September 27. 2018

4.1 Speed of Convergence

Methodology Look at the change in k and linearize using first order Taylor's expansion.

Recall $\dot{k}(t)$ is a function of $k(t)$ since

$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t) \tag{1}$$

And the first order Taylor series approximation of a function $f(x)$ around the point $x = x_0$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Then

$$\begin{aligned}\dot{k}(k) &\approx \dot{k}(k^*) + \frac{\partial \dot{k}(k)}{\partial k}|_{k=k^*}(k-k^*)\\ &= 0 + \frac{\partial \dot{k}(k)}{\partial k}|_{k=k^*}(k-k^*)\end{aligned}$$

Differentiating the both sides of equation (1) with respect to k.

$$\begin{aligned}\frac{\partial \dot{k}(k)}{\partial k}|_{k=k^*} &= sf'(k^*) - (n+g+\delta)\\ &= \frac{(n+g+\delta)k^*}{f(k^*)}f'(k^*) - (n+g+\delta)\\ &= (n+g+\delta)\Big[\frac{f'(k^*)k^*}{f(k^*)} - 1\Big]\\ &= (n+g+\delta)(\alpha(k^*) - 1)\end{aligned}$$

where

$$\alpha_k(k^*) = f'(k^*)\frac{k^*}{f(k^*)} \tag{2}$$

denotes the elasticity of y with respect to k *at steady state*. So

$$\begin{aligned}\dot{k}(k(t)) &\approx (n+g+\delta)(\alpha(k^*)-1)(k(t)-k^*)\\ \implies \frac{\partial(k - k^*(k))}{\partial t} &= \dot{k}(k(t)) \approx (n+g+\delta)(\alpha(k^*)-1)(k(t)-k^*)\end{aligned}$$

Let $\lambda := (n+g+\delta)(1-\alpha(k^*))$ then

$$k(t) - k^* \approx e^{-\lambda t}(k(0) - k^*) \tag{3}$$

Remark 4.1. *Derive.* (above equation)
Let $X(t) := k(t) - k^*$

And since $\frac{\partial k(t)}{\partial t} = \frac{\partial (k(t)-k^*)}{\partial t}$
Therefore $\dot{X}(t) = \dot{k}(t) \approx -\lambda X(t)$
$\implies X(t) \approx X(0)e^{-\lambda t}$
$\iff k(t) - k^* \approx (k(0) - k^*)e^{-\lambda t}$ ■

Then note that 3

$$
\begin{gathered}
y(t) = f(k(t)) \\
\implies \dot{y}(t) = f'(k(t))\dot{k}(t) \\
\text{(Take the first order Taylor series approximation around } k = k^*) \\
\implies y(t) \approx f(k^*) + f'(k^*)(k(t) - k^*) \\
\implies y(t) - y^* \approx f'(k^*)(k(t) - k^*) \\
\implies \frac{\dot{y}(t)}{y(t) - y^*} = \frac{f'(k^*)\dot{k}(t)}{f'(k^*)(k(t) - k^*)} = \frac{\dot{k}(t)}{k(t) - k^*} \approx -\lambda \\
\implies y(t) - y^* \approx e^{-\lambda t}(y(0) - y^*)
\end{gathered}
$$

Example 4.1. How long does it take to move 1/2 way to the balance growth path. Assuming population growth rate is 2%, growth in output per worker is 2% and depreciation is 2% and $\alpha_K = \frac{1}{3}$.

Solution. $\lambda = (1 - \alpha_K)(n + g + \delta)$ Since we know along the balanced growth path,

$$
\begin{gathered}
\frac{Y(t)}{L(t)} = y^* A(t) \\
\implies \frac{\partial \ln \frac{Y(t)}{L(t)}}{\partial t} = g
\end{gathered}
$$

Therefore $g = 0.02$ and therefore $\lambda = 0.04$.
To find the date where we have moved half way we need to solve

$$
\begin{gathered}
\frac{y(\tilde{t}) - y^*}{y(0) - y^*} = 0.5 \approx e^{-\lambda t} \\
\implies \ln(0.5) \approx -\lambda \tilde{t} \\
\implies \tilde{t} = \frac{-\ln(0.5)}{0.04} \approx 17.33
\end{gathered}
$$

■

4.2 General Statements

Solow growth model identifies 2 sources of output per worker,

1. Differences in the among of capital per worker.
2. Differences in the effectiveness of productivity of labor A.

Notice that the output per worker

$$\frac{Y(t)}{L(t)} = \frac{F(K(t), A(t)L(t))}{L(t)} = F(\frac{K(t)}{L(t)}, A(t))$$

Notice that in the long run balanced growth path

$$\frac{K(t)}{L(t)} = k(t)A(t)$$

$$\implies \frac{\partial \ln(\frac{K(t)}{L(t)})}{\partial t} = \frac{\dot{k}(t)}{k(t)} + \frac{\dot{A}(t)}{A(t)}$$

Therefore along the balanced growth path $\dot{k}(t) = 0$ so only the growth in A matters.

4.3 Growth Accounting

Consider the growth rate of aggregate output $Y(t)$, take the total differential and get

$$\dot{Y}(t) = \frac{\partial Y(t)}{\partial K(t)}\frac{\partial K(t)}{\partial t} + \frac{\partial Y(t)}{\partial L(t)}\frac{\partial L(t)}{\partial t} + \frac{\partial Y(t)}{\partial A(t)}\frac{\partial A(t)}{\partial t} \tag{4}$$

$$\implies \frac{\dot{Y}(t)}{Y(t)} = \frac{\partial Y(t)}{\partial K(t)}\frac{1}{Y(t)}\frac{\partial K(t)}{\partial t} + \frac{\partial Y(t)}{\partial L(t)}\frac{1}{Y(t)}\frac{\partial L(t)}{\partial t} + \frac{\partial Y(t)}{\partial A(t)}\frac{1}{Y(t)}\frac{\partial A(t)}{\partial t} \tag{5}$$

Then express the equation in terms of growth rates in K, L, A variables,

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\partial Y(t)}{\partial K(t)}\frac{K(t)}{Y(t)}\frac{\frac{\partial K(t)}{\partial t}}{K(t)} + \frac{\partial Y(t)}{\partial L(t)}\frac{L(t)}{Y(t)}\frac{\frac{\partial L(t)}{\partial t}}{L(t)} + \frac{\partial Y(t)}{\partial A(t)}\frac{A(t)}{Y(t)}\frac{\frac{\partial A(t)}{\partial t}}{A(t)} \tag{6}$$

$$= \alpha_K\frac{\dot{K}(t)}{K(t)} + \alpha_L\frac{\dot{L}(t)}{L(t)} + R(t) \tag{7}$$

where $R(t)$ is the **Solow Residual** and

$$R(t) = \frac{\partial Y(t)}{\partial A(t)}\frac{A(t)}{Y(t)}\frac{\dot{A}(t)}{A(t)} \tag{8}$$

And $\alpha_K(t)$ and $\alpha_L(t)$ denote the elasticity of output with respect to capital and labor respectively.

Example 4.2. Assume the output growth is 40% and capital growth is 20% and labor growth is 30%. If $\alpha_K = 0.3$ and $\alpha_L = 0.7$. What's the contribution to output growth of capital?

$$\alpha_K\frac{\dot{K}(t)}{K(t)} = 0.3 \times 20\% = 0.06$$

and the contribution from labor is

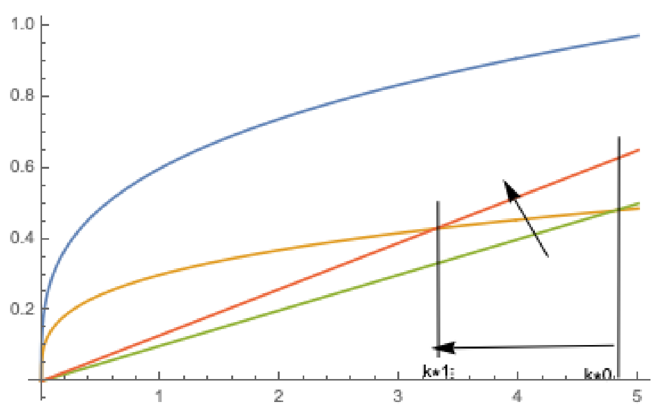
$$\alpha_L \frac{\dot{L}(t)}{L(t)} = 0.7 \times 30\% = 0.21$$

and the Solow residual is

$$R(t) = g_Y - 6\% - 21\% = 0.4 - 0.06 - 0.21 = 0.13$$

Example 4.3. Let's assume the economy is on its balanced growth path. Assume that a change in medicine increase survival rate during child birth.
What would the effect of this be on steady state k^*, y^*, c^*, i^*.
First show the growth that despite break-even investment and actual investment. Label the steady state values.

Solution. **The effect would be an increase in n**
Suppose $n_0 \to n_1$ with $n_0 < n_1$.
Therefore k^* falls, and y^* falls.
Since consumption and actual investment are constant frictions of y^*,
Therefore both c^*, i^* falls. ■



5 Lecture 5 October 11. 2018

5.1 Economy Setup

- Infinite horizon continuous time model.
- Exogenous growth rates of technology/productivity and population.

$$A(t) = A(0)e^{gt} \tag{1}$$

$$L(t) = L(0)e^{nt} \tag{2}$$

- (Endogenous saving decisions) A key difference of endogenous growth model from the Solow model is that the saving decision and the capital stock are determined by the interaction of utility/profit maximizing households and firms.

5.2 Households

5.2.1 Assumptions and Behaviours

Assumption 5.1. (Setup)

- There are a large numbers of identical households in the economy (H). *To ensure that no household has market power.*
- And the size of each household are assumed to grow at rate n.
- Each household has $\frac{L(t)}{H}$ members in it at time t.
- Each household has *initial* capital holding of $\frac{K(0)}{H}$.
- Each member of the household supplies 1 unit of labor at each point in time inelastically. (*No uncertainty*)
- For simplicity, there's no depreciation of capital stock. ($\delta = 0$)
- Capital is rented to firms at rate $r(t)$. (Note that investment is covered by the capital accumulation and interest rate.)
- Labor is hired at wage rate $W(t) = w(t)A(t)$. Where $W(t)$ is the wage per unit of labor and $w(t)$ is the wage per unit of effective labor.

5.2.2 Objective Functions

Household's *objective function* is given by the **lifetime utility function**

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H}\, dt, \;\; \rho > 0 \tag{3}$$

where ρ is the **discount rate** and $C(t)$ is the **consumption *per person***.

Remark 5.1. ρ measures consumers' attitude between present and future utilities. When $\rho = 0$, household values utility in the future equally. When $\rho > 0$, household values future utility less than present utility. The greater ρ is, the less value household puts on future consumption/utility (*i.e. consumers are less patient*). Notice that ρ cannot be negative in this model, since with $\rho < 0$ household values utility at $t = \infty$ infinitely more than current utility, the lifetime utility does not converge. For instance, with $\rho > 0$, $e^{-\rho t}$ is increasing in t, then infinite-horizon household could attain infinite utility by allocating consumption at $t = \infty$. [7]

[7]Intuitively, if $\rho < 0$, households could save everything and spend their saving when $t > \infty$. This scenario is certainly unrealistic.

5.2.3 Household Budget Constraints

Household's budget is given in terms of present discounted value.
Since the household has $\frac{L(t)}{H}$ members, labor income at time t is

$$A(t)w(t)\frac{L(t)}{H} = W(t)\frac{L(t)}{H}$$

The amount of consumption for the household at the time t is

$$C(t)\frac{L(t)}{H}$$

And let $R(t)$ denote the **continuously compounding interest rate**:

$$R(t) := \int_{\tau=0}^{t} r(\tau)\ d\tau \tag{4}$$

Remark 5.2. We know under continuously compounding interest, one unit of output invested at $\tau = 0$ is worth $e^{R(t)}$ units at $\tau = t$. Conversely, 1 unit of output at $\tau = t$ is worth $e^{-R(t)}$ at $\tau = 0$.

Remark 5.3. The budget constraint implies that the present discounted value of lifetime consumption must not exceed the present discounted value of labor income plus initial wealth/capital.

And the household lifetime budget constraint is given by

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}\ dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)\frac{A(t)L(t)}{H}\ dt \tag{5}$$

Then normalize budget constraint (5) into *per unit of effective labor variables* to get an equivalent form.

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}\ dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)\frac{A(t)L(t)}{H}\ dt \\
\iff &\int_{t=0}^{\infty} e^{-R(t)}c(t)\frac{A(t)L(t)}{H}\ dt \leq \frac{k(0)A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)\frac{A(t)L(t)}{H}\ dt \\
\iff &\int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}\frac{A(0)L(0)}{H}\ dt \leq \frac{k(0)A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}\frac{A(0)L(0)}{H}\ dt \\
\iff &\int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)\ dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}w(t)\ dt
\end{aligned}$$

The **assets** owned by household on date $t = s$, $\frac{K(s)}{H}$, is given by *reverse dis-*

counting saving and initial capital to $\tau = s$.

$$
\begin{aligned}
\frac{K(s)}{H} &= e^{R(s)}\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)+R(s)}\left\{\frac{w(t)A(t)L(t)}{H} - \frac{c(t)A(t)L(t)}{H}\right\} dt \\
&= e^{R(s)}\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)+R(s)}\{w(t)-c(t)\}\frac{A(t)L(t)}{H}dt \\
\implies e^{-R(s)}\frac{K(s)}{H} &:= \frac{K(0)}{H} + \int_{t=0}^{s} e^{R(t)}\{w(t)-c(t)\}\frac{A(t)L(t)}{H}\ dt
\end{aligned}
$$

Substitute to the budget constraint,

$$
\begin{aligned}
&\frac{K(0)}{H} + \int_{t=0}^{\infty}\frac{w(t)A(t)L(t)}{H}\ dt - \int_{t=0}^{\infty}\frac{c(t)A(t)L(t)}{H}\ dt \geq 0 \\
\iff &\frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}\{w(t)-c(t)\}\frac{A(t)L(t)}{H}\ dt \geq 0 \\
\iff &\lim_{s\to\infty}\left\{\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)}\{w(t)-c(t)\}\frac{A(t)L(t)}{H}\ dt \geq 0\right\}
\end{aligned}
$$

$$
\implies \lim_{s\to\infty} e^{-R(s)}\frac{K(s)}{H} \geq 0 \tag{6}
$$

These implies that the household's asset holding cannot be negative at the limit. And this is a **no-Ponzi game** condition.

Definition 5.1. A **Ponzi game** is a scheme in which someone issues debt and rolls it over forever. In other words, the person always pays off debt by issuing more debt.

Remark 5.4. If households could run a Ponzi scheme, then the present value of lifetime consumption can exceed the present value of lifetime resources.

5.3 Instantaneous Utility Function

Assumption 5.2. The household **instantaneous utility function** is assumed to be in the form of constant relative risk aversion (CRRA) as

$$
u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \tag{7}
$$

And lifetime utility function becomes

$$
\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t}\frac{(A(t)c(t))^{1-\theta}}{1-\theta}e^{nt}\frac{L(0)}{H}\ dt \\
&\int_{t=0}^{\infty} e^{-\rho t}\frac{c(t)^{1-\theta}}{1-\theta}A(0)^{1-\theta}e^{(1-\theta)gt}e^{nt}\frac{L(0)}{H}\ dt \\
&= A(0)^{1-\theta}\frac{L(0)}{H}\int_{t=0}^{\infty} e^{-\rho t+nt+(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}\ dt \\
&= A(0)^{1-\theta}\frac{L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n-(1-\theta)g)t}\frac{c(t)^{1-\theta}}{1-\theta}\ dt
\end{aligned}
$$

$$B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta}\, dt \tag{8}$$

where

$$B := A(0)^{1-\theta} \frac{L(0)}{H} \text{ and } \beta := \rho - n - (1-\theta)g \tag{9}$$

Assumption 5.3. Assuming $\theta > 0$ and $\beta = \rho - n - (1-\theta)g > 0$.

Remark 5.5. If $\beta \leq 0$ then the integral would diverge and the maximization problem does not have a well-defined solution.

6 Lecture 6 Oct 18. 2018

6.1 Firm Setup

Assumption 6.1. Firms

- are all *identical* and
- employ stocks of capital and labor
- perfectly competitive in factor markets (no control over wage and rent)
- maximize profits
- each firm has access to the production function $Y = F(K, AL)$

Notation 6.1. Let

- $w(t)$ denote the *real* wage per unit of effective labor
- $r(t)$ denote the rental rate of capital
- $A(t)$ denotes the labor augmenting technology/ knowledge

Remark 6.1. The profit is given by

$$\pi(t) = F(K(t), A(t)L(t)) - w(t)A(t)L(t) - r(t)K(t)$$

Assumption 6.2. Technology in this model, $A(t)$, is given exogenously so it isn't a choice for firms.

6.2 Firm's Optimization

Maximization [8]

$$\max_{K(t),L(t)} \pi(t) = F(K(t), A(t)L(t)) - w(t)A(t)L(t) - r(t)K(t)$$

[8] Note since $A(t)$ is given exogenously, the arguments of maximization could also be written as $\max_{K(t),A(t)L(t)}$

First Order Condition

$$K(t) : \frac{\partial F(K(t), A(t)L(t))}{\partial K(t)} - r(t) = 0$$
$$L(t) : \frac{\partial F(K(t), A(t)L(t))}{\partial L(t)} - w(t)A(t) = 0$$

Example 6.1 (Cobb-Douglas Form)**.** Consider

$$F(K(t), A(t)L(t)) = K^{\alpha}(AL)^{1-\alpha}$$

$$K(t) : \alpha K^{\alpha-1}(AL)^{1-\alpha} = r(t)$$
$$L(t) : (1-\alpha)K^{\alpha}(AL)^{-\alpha}A(t) = A(t)w(t)$$

In per unit of effective form

$$\alpha k(t)^{\alpha-1} = r(t)$$
$$(1-\alpha)k(t)^{\alpha} = w(t)$$

In general we could have the following first order necessary conditions in terms of the per unit of effective labor variables.

$$f'(k(t)) = r(t)$$
$$f(k(t)) - k(t)f'(k(t)) = w(t)$$

Proof.

$$r(t) = \frac{\partial F(\cdot)}{\partial K} = \frac{\partial ALf(\frac{K}{AL})}{\partial K}$$
$$= AL\frac{\partial f(k)}{\partial k}\frac{\partial \frac{K}{AL}}{\partial K} = \frac{\partial f(k)}{\partial k} = f'(k)$$

$\blacksquare$

Proof.

$$A(t)w(t) = \frac{\partial F(K, AL)}{\partial L} = \frac{\partial ALf(\frac{K}{AL})}{\partial L}$$
$$= Af(k) + AL\frac{\partial f(\frac{K}{AL})}{\partial AL}\frac{\partial AL}{\partial L}$$
$$= Af(k) + AL\frac{\partial f(\frac{K}{AL})}{\partial k}\frac{\partial \frac{K}{AL}}{\partial AL}\frac{\partial AL}{\partial L}$$
$$= Af(k) + A^2Lf'(k)\frac{-K}{(AL)^2}$$
$$= Af(k) - kf'(k)\frac{A^2L}{AL}$$
$$= Af(k) - kf'(k)A$$
$$\implies w(t) = f(k) - f'(k)$$

$\blacksquare$

6.3 Household Behaviour

Household are going to chose a function $\mathcal{C}(t) : [0, \infty) \to \mathbb{R}$ to maximize lifetime utility subjected to their lifetime budget constraint.

$$\max_{\{c(t)\}_{t=0}^{\infty}} B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta}\, dt$$
$$s.t.\ 0 \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t))\, dt$$

Setup Lagrangian at each time point,

$$\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta}\, dt + \lambda \Big[k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t))\, dt \Big] \quad (1)$$

First Order Necessary Conditions On an arbitrary time period t,

$$c(t) : Be^{-\beta t} c(t)^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0 \quad (2)$$
$$\lambda : k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} (w(t) - c(t))\, dt = 0 \quad (3)$$

6.4 The Behaviour of Consumption

The the log of both size of (2) gives

$$\ln(B) + \ln(e^{-\beta t}) + \ln(c(t)^{-\theta}) = \ln(\lambda) + \ln(e^{-R(t)}) + \ln(e^{(n+g)t})$$
$$\iff \ln(B) - \beta t - \theta \ln(c(t)) = \ln(\lambda) - R(t) + (n+g)t$$

Since above equation holds for all t, take the derivative with respect to t

$$\implies -\beta - \theta \frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$
$$\iff \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \beta - (n+g)}{\theta}$$
$$\because \beta = \rho - n - (1-\theta)g$$
$$\iff \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

Remark 6.2. The growth rate of consumption per person is given by

$$\frac{\partial \ln(C(t))}{\partial t} = \frac{r(t) - \rho}{\theta} \quad (4)$$

Remark 6.3. Go backwards by solving the ODE

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{\partial \ln(c(t))}{\partial t}$$
$$\implies c(t) = c(0) e^{\frac{R(t) - (\rho + \theta g)t}{\theta}}$$

Using the budget constraint to solve for $c(0)$.

$$
\begin{aligned}
k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t)\ dt &= \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t)\ dt \\
&= \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(0) e^{\frac{R(t)-(\rho+\theta g)t}{\theta}}\ dt \\
&= c(0) \int_{t=0}^{\infty} e^{(\frac{1}{\theta}-1)R(t)} e^{(n-\frac{\rho}{\theta})t}\ dt \\
\implies c(0) &= \frac{k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t)\ dt}{\int_{t=0}^{\infty} e^{(\frac{1}{\theta}-1)R(t)} e^{(n-\frac{\rho}{\theta})t}\ dt}
\end{aligned}
$$

6.5 Dynamics of the Economy

6.5.1 Dynamic of c

From the firms problem, $r(t) = f'(k(t))$, so at equilibrium,

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

In the long run, $\dot{c}(t) = 0$. This happens if and only if

$$f'(k^*) = \rho + \theta g$$

Therefore

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - f(k^*)}{\theta} \tag{5}$$

That's, $k(t) < k^* \implies f'(k(t)) > f(k^*) \implies \dot{c}(t) > 0$.

Figure 4: Phase diagram of $\dot{c}$ and $\dot{k}$

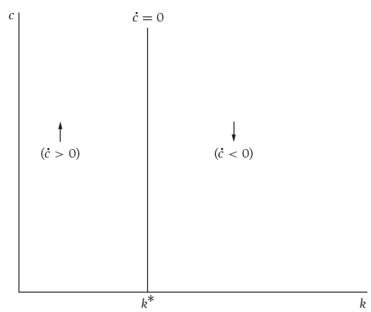


FIGURE 2.1 The dynamics of c

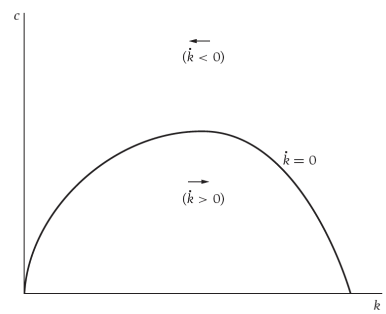


FIGURE 2.2 The dynamics of k

6.5.2 Dynamic of k

Assuming there's no depreciation ($\delta = 0$)

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t) \tag{6}$$

Below the $\dot{k} = 0$ curve, $\dot{k} > 0$ since actual investment per unit of effective labor exceeds the break-even level of investment per unit of effective labor.
Putting two phase diagrams together, The **steady state** is given at point E,

Figure 5: Combined phase diagram

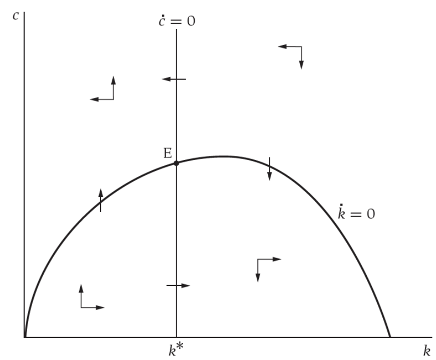


FIGURE 2.3 The dynamics of c and k

where $\dot{c} = \dot{k} = 0$.

6.5.3 Initial Point Problem (Saddle Path)

For any given value of $k(0)$, there exists an unique value of $c(0)$, so that one economy starts with $k(0), c(0)$ would evolve and converge to the steady state E. Such path of convergence is called **saddle path**.

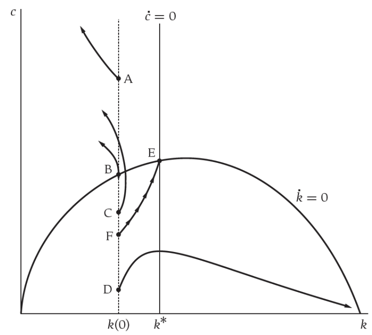


FIGURE 2.4 The behavior of c and k for various initial values of c