Introduction to Real Analysis

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Contents

1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subseteq \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \ s.t. \ \forall a \in A, \ u \geq a \tag{1.1}$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \ s.t. \ \forall a \in A, \ l \leq a \tag{1.2}$$

Example 1.1. The set of integers, $\mathbb{Z}$, is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subseteq \mathbb{R}$, we use $A^{\uparrow}$ and $A^{\downarrow}$ to denote collections of upper bounds of A and lower bounds of A. When A is bounded, either $A^{\uparrow}$ or $A^{\downarrow}$ is empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound (supremum)** for a set $A \subseteq \mathbb{R}$ if

(i) $s \in A^{\uparrow}$;

(ii) and $\forall u \in A^{\uparrow}, \ s \leq u$.

Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if

(i) $f \in A^{\downarrow}$;

(ii) and $\forall l \in A^{\downarrow}, \ l \leq f$.

Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property)**.** $\forall \varnothing \neq A \subseteq \mathbb{R}$ such that $A^{\uparrow} \neq \varnothing$, $\exists \ \mathbb{R} \ni u = \sup A$.

Definition 1.4. Let $\varnothing \neq A \subseteq \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subseteq \mathbb{R}$ does not satisfy the axiom of completeness. Let $A = \{r \in \mathbb{Q} : r < \sqrt{2}\}$, clearly A is bounded above, but for every $r' \in \mathbb{Q} \cap A^{\uparrow}$, there exists $r'' \in (\sqrt{2}, r') \cap A^{\uparrow}$.

Proposition 1.1. Let $\varnothing \neq A \subseteq \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \tag{1.3}$$

Proof. Step 1: Show $c + \sup A \in (c + A)^{\uparrow}$*:*
Let $x \in c+A$, $\exists\, a \in A$ *s.t.* $x = c+a$. Then, $x = c+a \leq c+\sup A$. Therefore, $x \leq c+\sup A \;\forall x \in A$, which implies what desired.
Step 2: Show $\forall u \in (c + A)^{\uparrow},\ c + \sup A \leq u$*:*
Let $u \in (c + A)^{\uparrow}$, then $u \geq c + a\ \forall a \in A \implies u - c \geq a\ \forall a \in A \implies u - c \in A \uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.
Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum)**.** Let $s \in A^{\uparrow}$ for some nonempty $A \subseteq \mathbb{R}$. The following statements are equivalent:

(i) $s = \sup A$;

(ii) $\forall \varepsilon, \exists a \in A,\ s.t.\ a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^{\uparrow}$).

Proof. The proof is immediate by the definition of supremum as the least upper bound. ■

Theorem 1.1 (Nested Interval Property)**.** Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subseteq I_n\ \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing \tag{1.5}$$

Proof. Note that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded above by any b_k.
By the completeness axiom, there exists $a^* := \sup_{n\in\mathbb{N}} a_n$.
Since $a^* \in (a_n)^{\uparrow}$, $a^* \geq a_n\ \forall n \in \mathbb{N}$.
Further, because a^* is the *least* upper bound, then for every upper bound b_n, it must be $a^* \leq b_n\ \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n]\ \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n\in\mathbb{N}} I_n$. ■

Remark 1.1. Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Theorem 1.2 (Archimedean Property)**.**

(i) $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}\ s.t.\ n > x$;

(ii) $\forall y \in \mathbb{R}_{++},\ \exists n \in \mathbb{N}\ s.t. \frac{1}{n} < y$.

Archimedean property *of natural numbers* can be interpreted as *there is no real number that bounds* $\mathbb{N}$. This interpretation can be seen by considering the negations of above statements:

(i) $\exists x \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N},\ n \leq x$;

(ii) $\exists y \in \mathbb{R}_{++}\ s.t.\ \forall n \in \mathbb{N},\ y \leq \frac{1}{n}$ (i.e. $n \leq \frac{1}{y}$).

Proof of (i). Suppose, for contradiction, (i) is not true, then $\mathbb{N}$ is bounded above in $\mathbb{R}$.
By the completeness axiom, there exists $a^* := \sup \mathbb{N}$.
Therefore, $\exists n \in \mathbb{N}$ *s.t.* $a^* - 1 < n$.
In this case, $a^* < n + 1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^{\uparrow}$ and leads to a contradiction. ∎

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ∎

Remark 1.2. The two statements of Archimedean property are equivalent.

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Remark 1.3. The above theorem says $\mathbb{Q}$ is in fact **dense** in $\mathbb{R}$. More generally, one says a set $A \subseteq X$ is dense whenever the closure of A, $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.
Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ∎

Theorem 1.4. $\exists \alpha \in \mathbb{R}$ *s.t.* $\alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in $\mathbb{R}$ bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^{\uparrow}$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ∎

2 Sequences

2.1 Definitions

Theorem 2.1 (Triangle Inequality)**.** Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.1. Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \tag{2.1}$$

Proof. Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$.
Similarly, $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$.
Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ∎

Definition 2.1. A sequence $(a_n) \subseteq \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geq N \implies a_n \in V_\varepsilon(a) \tag{2.2}$$

Definition 2.2. Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at a with radius ε is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{2.3}$$

Theorem 2.2. The limit of any convergent sequence is unique.

Proof. Let (a_n) be a convergent sequence, assume, for contradiction, that $(a_n) \to L_1$ and $(a_n) \to L_2$ such that $L_1 \neq L_2$.
Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \to L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$\begin{aligned}
|a_n - L_2| &= |a_n - L_1 - (L_2 - L_1)| && (2.4)\\
&\geq ||a_n - L_1| - |L_2 - L_1|| && (2.5)\\
&= ||L_1 - L_2| - |a_n - L_1|| && (2.6)\\
&= 3\varepsilon - |a_n - L_1| && (2.7)\\
&> 2\varepsilon && (2.8)
\end{aligned}$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq N'$. ■

Definition 2.3. A sequence (a_n) is **divergent** if it does not converge.

Example 2.1. The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \cdots)$ is divergent.

Proof. Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \to L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $\left|L - \frac{1}{5}\right| < \frac{2}{15}$. Similarly,

$$\begin{aligned}
\left|-\frac{1}{5} - L\right| &= \left|\frac{1}{5} + L\right| && (2.9)\\
&= \left|\frac{1}{5} + L - \frac{1}{5} + \frac{1}{5}\right| && (2.10)\\
&= \left|(L - \frac{1}{5}) - (-\frac{2}{5})\right| && (2.11)\\
&\geq \left|\left|L - \frac{1}{5}\right| - \frac{6}{15}\right| && (2.12)\\
&= \frac{6}{15} - \left|L - \frac{1}{5}\right| && (2.13)\\
&> \frac{4}{15} && (2.14)\\
&> \varepsilon && (2.15)
\end{aligned}$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ■

Alternative Proof. If (a_n) is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of (a_n) converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ■

Definition 2.4. A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N},\ |a_n| < M$.

Theorem 2.3. Every convergent sequence is bounded.

Proof. Let $(a_n) \to L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n<N} a_n$, take $M := \max\{Q, |L| + 1\}$, then M bounds (a_n). ■

2.2 Limit Theorems

Theorem 2.4 (Algebraic Limit Theorem)**.** Let $(a_n) \to a, (b_n) \to b$ be convergent sequences, and $c \in \mathbb{R}$, then

(i) $(ca_n) \to ca$;

(ii) $(a_n + b_n) \to a + b$;

(iii) $(a_n b_n) \to ab$;

(iv) $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$, provided $(b_n), b \neq 0$.

Proof (i). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

Proof (ii). Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3}\ \forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3}\ \forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \tag{2.16}$$

■

Proof (iii). Note that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n + a_n b - a_n b - ab| && (2.17)\\ &\leq |a_n b_n - a_n b| + |a_n b - ab| && (2.18)\\ &\leq |a_n||b_n - b| + |b||a_n - a| && (2.19) \end{aligned}$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because (a_n) is convergent, let M denote its bound such that $|a_n| < M\ \forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_n b_n - ab| < \varepsilon$. ■

Proof (iv). Claim i: when n is sufficiently larger, $|b_n| > 0$ is bounded away from zero by M.
Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such n,

$$
\begin{aligned}
|b_n| &= |b_n - b - (-b)| && (2.20)\\
&\geq ||b_n - b| - |b|| && (2.21)\\
&\geq |b| - |b_n - b| && (2.22)\\
&> \frac{9|b|}{10} && (2.23)
\end{aligned}
$$

Claim ii: $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$
\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b}{b_n b} - \frac{b_n}{b_n b}\right| && (2.24)\\
&= \frac{1}{|b_n||b|}|b_n - b| && (2.25)
\end{aligned}
$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

Theorem 2.5 (Order Limit Theorem). Let $(a_n) \to a$ and $(b_n) \to b$, then

(i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;

(ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;

(iii) $\exists c \in \mathbb{R}\ s.t.\ c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;

(iv) $\exists c \in \mathbb{R}\ s.t.\ a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

Proof. (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.
(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \to (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.
(iii) and (iv) Consider constant sequence defined as (c_n) such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

Theorem 2.6 (Squeeze Theorem). Let $(x_n) \to L$ and $(z_n) \to \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \to \ell$.
Remark: squeeze theorem does not impose the prior that (y_n) is convergent.

Proof. Let $\varepsilon > 0$, because both $(x_n) \to \ell$ and $(y_n) \to \ell$,

$$
\begin{aligned}
&\exists N_1 \in \mathbb{N}\ s.t.\ n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon && (2.26)\\
&\exists N_2 \in \mathbb{N}\ s.t.\ n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon && (2.27)
\end{aligned}
$$

Take $N_3 := \max\{N_1, N_2\}$, then for every $n \geq N_3$,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \tag{2.28}$$
$$\implies y_n \in V_\varepsilon(\ell) \tag{2.29}$$

therefore $(y_n) \to \ell$ by definition. ■

2.3 Monotone Convergence Theorem

Definition 2.5. A sequence (a_n) is said to be **monotone** if it is either increasing ($a_{n+1} \geq a_n\ \forall n \in \mathbb{N}$) or decreasing ($a_{n+1} \leq a_n\ \forall n \in \mathbb{N}$).

Theorem 2.7 (Monotone Convergence Theorem)**.** If a monotone sequence (a_n) is bounded, then it converges.

Proof. WLOG, assume (a_n) is increasing, let $\Gamma := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, because Γ is bounded, $s := \sup_n \Gamma$ is well-defined by the completeness of real numbers.
Claim: $(a_n) \to s$. Let $\varepsilon > 0$, by the definition of supremum, $\exists N \in \mathbb{N}$ such that $a_N > s - \varepsilon$. Because the sequence is increasing and $s + \varepsilon \in \Gamma^{\uparrow}$, $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$. $(a_n) \to s$ by definition. ■

2.4 Series

Definition 2.6. Let (a_i) be a sequence, then the n-th **partial sum** is defined as $s_n := \sum_{i=1}^{n} a_i$. And the **infinite sum/series** of (a_n) is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \to s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \tag{2.30}$$

Example 2.2. $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

Proof. Obviously the corresponding partial sums are increasing because the sequence $(\frac{1}{i^2})$ is positive.
Claim: (s_n) is bounded from above. Let $n \in \mathbb{N}$, observe

$$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \tag{2.31}$$
$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \tag{2.32}$$
$$= 2 - \frac{1}{n} \leq 2 \tag{2.33}$$

The result is immediate by the monotone convergence theorem. ■

Example 2.3 (Harmonic Series)**.** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. Claim: there exists a subsequence of (s_n) diverges, so (s_n) cannot be convergent. Consider the subsequence (s_k) constructed by defining $s_k := s_{2^k}$. Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \tag{2.34}$$

$$> 1 + \frac{1}{2}k \tag{2.35}$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. Therefore, the original sequence of partial sums cannot be convergent. ■

Definition 2.7. Let (a_n) be a sequence, then for every strictly increasing sequence $(n_i)_i$ in $\mathbb{N}$, (a_{n_i}) is a **subsequence** of (a_n).

Theorem 2.8. All subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \to \ell$, let (a_{n_k}) be a subsequence of (a_n). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\varepsilon(\ell)$. By the definition of subsequences, there exists some $K \in \mathbb{N}$ such that $n_K \geq N$. Take such K, then for every $k \geq K$, it must be $n_k \geq N$. Therefore $a_{n_k} \in V_\varepsilon(\ell)$ for every $k \geq K$, and $(a_{n_k}) \to \ell$ by definition. ■

Remark 2.1. Note the implication of above theorem is two-fold:

(i) Every subsequence of a convergent sequence is convergent;

(ii) All subsequences converge to the same limit.

Corollary 2.2. A sequence (a_n) must be divergent if there exists two subsequences of it converge to two different limits.

Proof. Immediate by taking the contrapositive form of above theorem. ■

Theorem 2.9 (Bolzano–Weierstrass)**.** Every bounded sequence contains a convergent subsequence.

Proof. Suppose (a_n) is bounded by certain $M > 0$, that's, for every $n \in \mathbb{N}$, $-M < a_n < M$. Consider the split $I_1^\ell := [-M, 0]$ and $I_1^u := [0, M]$. At least one of above closed intervals contain an infinitely many elements of (a_n).
Define the interval as I_2. At each I_n, one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and I_{n+1} is defined to be such closed interval containing infinitely many elements.
Note that the sequence (I_n) is nested by construction. By the nested interval property, one can show that $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$.
Also, $\lim_{n\to\infty} |I_n| = 0$. Then $\cap_{n\in\mathbb{N}} I_n$ must be a singleton with a in it. One can construct such that $a_{n_k} \in I_k$. Note that $|I_n| = \frac{1}{2^{n-1}}$, therefore, for every $\varepsilon > 0$, one can take $N \geq \log_2\left(\frac{1}{\varepsilon}\right) + 1$, so that for every $k \geq N$, by definition of subsequences, $n_k \geq n$, so that $a_{n_k}, a \in I_N$. This implies $a_{n_k} \in V_\varepsilon(a)$ and $(a_{n_k}) \to a$. ■

2.5 Cauchy Criterion

Definition 2.8. A sequence (a_n) is a **Cauchy** sequence if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ m, n \geq N \implies |a_n - a_m| < \varepsilon \tag{2.36}$$

Proposition 2.1. Every convergent sequence is Cauchy.

Proof. Let $(a_n) \to \ell$, let $\varepsilon > 0$. By the convergence of sequence, $\exists N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - \ell| < \frac{\varepsilon}{2}$, which turns out to imply $a_n, a_m \in V_\varepsilon(\ell)$. ■

Lemma 2.1. Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, $|a_n - a_m| < 1$. In particular, take $m = N$, for every $n \geq N$, $|a_n - a_N| < 1$, and $|a_n| \leq |a_N| + 1$. Then (a_n) is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \tag{2.37}$$

■

Theorem 2.10 (Cauchy Criterion)**.** A sequence in $\mathbb{R}$ is convergent if and only if it's Cauchy.

Proof. ($\Longleftarrow$) Suppose (a_n) is Cauchy, by the lemma established above, (a_n) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence $(a_{n_k}) \to \ell$.
Claim: $(a_n) \to \ell$. Let $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $n_k, n \geq N_1$, $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$.
And there exists another $N_2 \in \mathbb{N}$ such that for every $n_k \geq N_2$, $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$.
Take $N_3 := \max\{N_1, N_2\}$.
Note that for every $n \geq N_3$, one can choose some $n_k \geq n$ as leverage and derive

$$\begin{aligned} |a_n - \ell| &= |a_n - a_{n_k} + a_{n_k} - \ell| && (2.38) \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| && (2.39) \\ &< \varepsilon && (2.40) \end{aligned}$$

($\Longrightarrow$) Already shown in previous proposition. ■

2.6 Convergence Test for Series

Theorem 2.11 (n-th term test)**.**

$$\sum_{i=1}^{\infty} a_n \text{ converges} \implies \lim_{n \to \infty} a_n = 0 \tag{2.41}$$

Remark: this theorem is only a necessary condition for convergence of series.

Proof. Suppose the partial sums converges to ℓ, by the definition of partial sums, $a_n = s_{n+1} - s_n$. Further, the convergence of partial sums guarantees the convergence of (a_n). By taking limit on both sides of above identity, it can be shown $\lim_{n \to \infty} a_n = 0$. ■

Theorem 2.12 (Cauchy Criterion for Series)**.** For series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:

(i) Series converges;

(ii) $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ s.t.\ \forall n \geq N,\ \left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$ (i.e. *tail* sum sequence converges);

(iii) $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ s.t.\ \forall m > n \geq N,\ \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$. (i.e. partial sum is Cauchy)

Proof. $(i) \implies (ii)$: Suppose (S_n) converges, let $\varepsilon > 0$, $\exists N\ s.t.\ \forall n \geq N, |S_n - L| < \varepsilon$. Note that

$$L - S_n = \lim_{m\to\infty} \sum_{k=1}^{m} a_k - S_n \tag{2.42}$$

$$= \lim_{m\to\infty} \left[\sum_{k=1}^{m} a_k - S_n\right] \tag{2.43}$$

$$= \lim_{m\to\infty} \sum_{k=n+1}^{m} a_k \tag{2.44}$$

which implies the convergence of tail sums.
$(ii) \implies (iii)$: Suppose the tail sum converges, let $\varepsilon > 0$, note that

$$\left|\sum_{k=n+1}^{m} a_k\right| = \left|\sum_{k=m+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k\right| \tag{2.45}$$

$$\leq \left|\sum_{k=m+1}^{\infty} a_k\right| + \left|\sum_{k=n+1}^{\infty} a_k\right| \tag{2.46}$$

Both terms can be made arbitrarily small by (ii), specifically, one can choose N_1 and N_2 such that both terms are strictly bounded by $\frac{\varepsilon}{2}$, and $N_3 := \max\{N_1, N_2\}$ is the desired value.
$(iii) \implies (i)$: Since the partial sum is a Cauchy sequence in a complete space, it must converges, so the series is well-defined. ■

2.6.1 The Comparison Test

Definition 2.9. A sequence (a_n) is a **geometric sequence** with coefficient r if $a_{n+1} = ra_n$.

Proposition 2.2. Geometric sequences whenever $r \in (-1, 1)$. Note that when $r = -1$, the sequence becomes an alternating sequence, and the convergence property is indefinite.

Proposition 2.3. Let (a_n) be a geometric sequence with coefficient r, then for every $m \in \mathbb{N}$,

$$rS_m^a = ra_0 + r^2 a_0 + \cdots + r^{n+1} a_0 \tag{2.47}$$

$$\implies (r-1)S_m^a = r^{n+1} a_0 - a_0 \tag{2.48}$$

$$\implies S_m^a = a_0 \frac{1 - r^{m+1}}{1 - r} \tag{2.49}$$

Theorem 2.13 (The Comparison Test)**.** Let (a_n) and (b_n) be two sequences satisfy $|a_n| \leq b_n$ for every $n \in \mathbb{N}$. Then

(i) $\sum_{i=1}^{\infty} b_n$ converges $\implies$ $\sum_{i=1}^{\infty} a_n$ converges;

(ii) $\sum_{i=1}^{\infty} a_n$ diverges $\implies$ $\sum_{i=1}^{\infty} b_n$.

Proof. Part 1: Suppose (b_n) converges, it is therefore Cauchy. Let $\varepsilon > 0$. Note that for every $m > n$:

$$|S_m^a - S_n^a| = \left| \sum_{k=n+1}^{m} a_k \right| \tag{2.50}$$

$$\leq \sum_{k=n+1}^{m} |a_k| \tag{2.51}$$

$$\leq \sum_{k=n+1}^{m} b_k \tag{2.52}$$

Therefore exists $N \in \mathbb{N}$ such that $\sum_{k=n+1}^{m} b_k \leq \left|\sum_{k=n+1}^{m} b_k\right| < \varepsilon$ for every $m, n \geq N$. Taking such N provides the cutoff needed for (S_n^a) to be Cauchy. Because $(S_n^a) \subseteq \mathbb{R}$, it converges.
Part 2: The result is immediate by taking the contrapositive form of the previous statement. ■

2.6.2 The Root Test

Definition 2.10. Let (a_n) be a bounded sequence, then

$$\limsup(a_n) := \sup_{n\to\infty} \{a_k : k \geq n\} \tag{2.53}$$

$$\liminf(a_n) := \inf_{n\to\infty} \{a_k : k \geq n\} \tag{2.54}$$

$$\tag{2.55}$$

Theorem 2.14 (The Root Test)**.** Let (a_n) be a sequence in which $a_n \geq 0$ for every $n \in \mathbb{N}$, let $\ell = \limsup a_n^{1/n}$, then

(i) If $\ell < 1$, then (S_n^a) converges;

(ii) If $\ell > 1$, then (S_n^a) diverges;

(iii) If $\ell = 0$, inconclusive.

Proof. Part 1:(Idea: compare with geometric series with $r < 1$) Suppose $\ell < 1$, pick $r \in (\ell, 1)$, and let $\varepsilon = r - \ell$. By the convergence of supremum, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\left|\sup_{k\geq n} a_k^{1/k} - \ell\right| < \varepsilon \tag{2.56}$$

$$\implies a_n^{1/n} \leq \sup_{k\geq n} a_k^{1/k} < \ell + \varepsilon =: r \tag{2.57}$$

Therefore, for every $n \geq N$, $a_n < r^n$. Because (a_n) is assumed to be a non-negative sequence, then $|a_n| < r^n$. Construct new sequences:

$$b_k = \begin{cases} a_k \ \forall k < N \\ r^k \ \forall k \geq N \end{cases} \tag{2.58}$$

Then, clearly $|a_n| \leq b_k$ for every $k \in \mathbb{N}$. And (b_n) is a sequence with geometric tails (which has coefficient less than one). So $\sum^\infty b_k$ converges, which implies $\sum^\infty a_k$ converges by the comparison test.

Part 2: Suppose $\ell > 1$.

Note that the necessary condition for $\sum a_n^{1/n}$ to converge is $\lim_{n\to\infty} a_n^{1/n} = 0$, which implies every subsequence of $(a_n^{1/n})$ converges to zero. We are going to prove the divergence of series by constructing a subsequence of $(a_n^{1/n})$ does not converge to zero.

Take $\varepsilon = \ell - 1 > 0$, there exists N such that for every $n \geq N$:

$$\ell - \varepsilon < \sup_{k\geq n} a_k^{1/k} \tag{2.59}$$

$$\implies 1 < \sup_{k\geq n} a_k^{1/k} \tag{2.60}$$

By definition of supremum, there exists $n_1 \geq n$ such that

$$a_{n_1}^{1/n_1} > 1 \tag{2.61}$$

For every $n \geq \mathbb{N}$, we can construct a subsequence of $(a_n^{1/n})$ such that every term in it is strictly greater than 1, which means it cannot converge to 0. Therefore, series diverges. ■

2.6.3 Other Tests

Theorem 2.15 (Limit Comparison Test)**.** Let $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ satisfy:

(i) $b_n \geq 0$;

(ii) $\limsup \frac{|a_n|}{b_n} < \infty$;

(iii) $\sum_{n=1}^\infty b_n$ converges.

Then $\sum_{n=1}^\infty a_n$ converges as well.

Theorem 2.16 (Ratio Test)**.** Given sequence $(a_n)_{n=1}^\infty$ such that $a_n \geq 0$, then

1. If $\limsup \frac{a_{n+1}}{a_n} < 1$, $\sum_{n=1}^\infty a_n$ converges;
2. If $\limsup \frac{a_{n+1}}{a_n} > 1$, $\sum_{n=1}^\infty a_n$ diverges.

Theorem 2.17 (Integral Test)**.** Let $f(x)$ be a *positive* and *monotone decreasing* function on $[1, \infty)$. Consider $(f(x_n))$, then

$$\sum_{n=1}^{\infty} f(n) \text{ convergent } \iff \int_1^{\infty} f(x)\, dx < \infty \tag{2.62}$$

Theorem 2.18 (Alternating Series Test)**.** For an alternating sequence $\sum_{n=1}^{\infty}(-1)^n a_n$, if $(a_n) \searrow 0$, then the series converges.

Proof. TODO ■

2.7 Absolute and Conditional Convergence

Corollary 2.3 (Corollary of Comparison Test)**.** If $\sum_{i=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition 2.11. For any series $\sum_{n=1}^{\infty} a_n$, if

1. $\sum_{i=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ **converges absolutely**;
2. $\sum_{i=1}^{\infty} |a_n|$ does not converge, then $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 2.4. Alternating harmonic series converges conditionally.
However, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

Definition 2.12. $\sum_{n=1}^{\infty} b_n$ is called a **rearrangement** of series $\sum_{n=1}^{\infty} a_n$ if there exists $f : \mathbb{N} \to \mathbb{N}$ such that f is a bijection and $b_{f(k)} = a_k$ for every $k \in \mathbb{N}$.

Theorem 2.19 (Riemann Series Theorem)**.** If series $\sum_{n=1}^{\infty} a_n$ converges conditionally, for every $\alpha \in \mathbb{R}$, there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ converges to α.

Proof. The proof is non-trivial and omitted. ■

Theorem 2.20. If series $\sum_{n=1}^{\infty} a_n$ converges absolutely to some value $A \in \mathbb{R}$, then every rearrangement $\sum_{n=1}^{\infty} b_n$ converges to A.

Proof. Define partial sum sequences

$$S_n := \sum_{k=1}^{n} a_k \quad T_m := \sum_{k=1}^{m} b_k \tag{2.63}$$

Suppose $(S_n) \to A$, want to show: $(T_n) \to A$.
Let $\varepsilon > 0$ fixed.
By convergence of (S_n), there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |S_n - A| < \frac{\varepsilon}{2} \tag{2.64}$$

Because $\sum_{n=1}^{\infty} a_n$ converges absolutely, by the Cauchy criterion for convergent series (i.e. partial sum sequence is Cauchy), there exists $N_2 \in \mathbb{N}$ such that

$$n > m \geq N_2 \implies \sum_{k=n+1}^{m} |a_k| < \frac{\varepsilon}{2} \tag{2.65}$$

Define $N := \max\{N_1, N_2\}$, $M := \max\{f(k) : 1 \leq k \leq N\}$.

$$|T_m - S_N| = |b_1 + \cdots + b_m - a_1 - \cdots - a_N| \tag{2.66}$$

$$= \left|b_1 + \cdots + b_m - b_{f(1)} - \cdots - b_{f(N)}\right| \tag{2.67}$$

Note that for every $m \geq M$, by construction, $\{b_{f(1)}, \cdots b_{f(N)}\} \subseteq \{b_1 \cdots, b_m\}$.
Note that for each $b_{f(k)} \in \{b_1 \cdots b_m\}$, either $k > N$ or $k \leq N$. But all $b_{f(k)}$ with $k \leq N$ were subtracted, so $b_{f(k)}$ elements left are all from $\{a_k : k \geq N+1\}$.

$$... = \left| \sum_{k \in \mathcal{I} \geq N+1} a_k \right| \tag{2.68}$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \tag{2.69}$$

Therefore, for all $m \geq M$,

$$|T_m - A| = |T_M - S_n + S_n - A| \tag{2.70}$$

$$\leq |T_M - S_n| + |S_n - A| \tag{2.71}$$

$$< \varepsilon \tag{2.72}$$

The desired result is immediate. ∎

3 Topology in $\mathbb{R}$

3.1 Definitions

Definition 3.1. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** if

$$\forall x \in \mathcal{O} \ \exists \ \varepsilon > 0 \ s.t. \ V_\varepsilon(x) \ s.t. \ V_\varepsilon(x) \subseteq \mathcal{O} \tag{3.1}$$

Theorem 3.1. Arbitrary union of open sets is open; Any finite intersection of open sets is open.

Proof. Let $\mathcal{O}_\alpha$ open for all $\alpha \in \mathcal{A}$. Let $\mathcal{O} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha$. If $x \in \mathcal{O}$, there exists some $\alpha \in \mathcal{A}$ such that $x \in \mathcal{O}_\alpha$. There exists $V_\varepsilon(x) \subseteq \mathcal{O}_\alpha \subseteq \mathcal{O}$. Hence $\mathcal{O}$ is open.
Let $\{\mathcal{O}_i : 1 \leq i \leq n\}$ be a collection of open sets, let $\mathcal{O} := \bigcap_{i=1}^{\infty} \mathcal{O}_i$. If $x \in \mathcal{O}$, there exists $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(x) \subseteq \mathcal{O}_i$ for every i. Take $\varepsilon := \max\{\varepsilon_i\}$, which exists and is strictly positive by finiteness of index set. Therefore $V_\varepsilon(x) \subseteq \mathcal{O}_i$ for every i, and therefore in $\mathcal{O}$. ∎

Definition 3.2. x is a **limit point** of A if $\forall \varepsilon > 0$,

$$V_\varepsilon(x) \cap A - \{x\} \neq \varnothing \tag{3.2}$$

Remark: this definition does not require x to be an element of A.

Theorem 3.2. x is a limit point A if and only if there exists a sequence $(a_n)_{n=1}^\infty \subseteq A$ such that $\underline{a_n \neq x\ \forall n \in \mathbb{N}}$ and $(a_n)_{n=1}^\infty \to x$.

Proof. ($\Longrightarrow$) Let x be a limit point, take $\varepsilon = \frac{1}{n}$, immediate by the definition of limit point.
($\Longleftarrow$) Trivially by definition of sequential convergence. ∎

Definition 3.3. $X \subseteq \mathbb{R}$ is **closed** if it contains all its limit points.

Definition 3.4. $x \in A$ is an **isolated point** is it is not a limit point of A.

Definition 3.5. The **closure** of A, denoted as $\overline{A}$, is defined to be the union of A and all limit points of A.

Definition 3.6. $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Theorem 3.3. Let $x \in \mathbb{R}$, there exists a sequence $(q_n)_{n=1}^\infty \subseteq \mathbb{Q}$ such that $(q_n)_{n=1}^\infty \to x$.

Proof. Let $x \in \mathbb{R}$. Note that $\forall u < v \in \mathbb{R}$, there exists $q \in (u, v) \cap \mathbb{Q}$. Hence, for every $n \in \mathbb{N}$, $\exists q_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < q_n < x + \frac{1}{n}$. It is evident that $(q_n)_{n=1}^\infty \to x$. ∎

Lemma 3.1. $\overline{A}$ is the smallest closed set containing A.

Proof. It is evident that $\overline{A}$ is a closed set containing A.
Now show the closure is in fact the smallest closed set. Let $B \subsetneq \overline{A}$ be a proper subset of the closure, we are going to show that B is not closed. Let $x \in \overline{A} - B \neq \varnothing$.
Note that $\overline{A} \equiv A \cup A'$, then either $x \in A$ or $x \in A'$. If $x \in A$, then B does not contain A. If $x \in A'$, then B does not contain all limit points of A, so it is not closed. ∎

Theorem 3.4. Equivalent definitions of openness and closedness:

(i) $\mathcal{O}$ is open if and only if $\mathcal{O}^c$ is closed;

(ii) $\mathcal{F}$ is closed if and only if $\mathcal{F}^c$ is open.

Proof. ($\Longrightarrow$) Let $\mathcal{O}$ be an open set, let $(x_n) \to x$ be a convergent sequence in $\mathcal{O}^c$. It is evident that if $x \in \mathcal{O}$, infinitely many elements in the tail of (x_n) would be in $V_\varepsilon(x) \subseteq \mathcal{O}$, which leads to a contradiction. Therefore $\mathcal{O}^c$ contains all of its limit points, and $\mathcal{O}^c$ is therefore closed.
($\Longleftarrow$) Let $\mathcal{F}^c$ be a closed set, suppose $\mathcal{F}$ is not open, there exists $x \in \mathcal{F}$ such that for all $\varepsilon > 0$, $V_\varepsilon(x) \cap \mathcal{F}^c \neq \varnothing$. Then we can construct a sequence in $\mathcal{F}^c$ converge to x, which leads to a contradiction that there is a limit point of a sequence in $\mathcal{F}^c$ not contained by $\mathcal{F}^c$.
The second part is immediate. ∎

Theorem 3.5. Any intersection of closed sets is closed; any finite union of closed sets is closed.

Proof. Direct result from De Morgan's law and the previous theorem. ■

Remark: Limit points and boundary points are completely different. Example: let $\Omega = [1,2] \cup 3$, then 3 is a boundary point but not a limit point (i.e. it is isolated). And 0.5 is a limit point but not a boundary point.

3.2 Compactness

Definition 3.7. A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a convergent subsequence converges to some limit $x \in K$.

Theorem 3.6. A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. ($\Longrightarrow$) Suppose $K \subseteq \mathbb{R}$ is compact.
Show K is bounded: suppose, for contradiction, K is unbounded, then for every $N \in \mathbb{N}$, one can construct a sequence as following: $a_1 \in K$ and $a_{n+1} > \max\{a_n, n\}$. Such sequence diverges to positive infinity, and every subsequence of it converges to infinity as well (easy to verify). This leads to a contradiction to the compactness of K.
Show K is closed: Suppose, for contradiction, K is not closed, then there exists some limit point of K say $x \notin K$. Consider the sequence $(x_n) \to x$ in K, because every subsequence of such convergent sequence converges to the same limit $x \notin K$, which leads to a contradiction of compactness.
($\Longleftarrow$) Let $(x_n) \subseteq K$, then (x_n) is bounded and therefore possesses a convergent subsequence by Bolzano-Weierstrass Theorem. Further, because K is closed, then the limit point must be in K. ■

Theorem 3.7 (Nested Compact Set Property)**.** Let $\mathbb{R}^n \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$, where $K_n \neq \varnothing$ are all compact sets, then

$$\bigcap_{n \in \mathbb{N}} K_n \neq \varnothing \tag{3.3}$$

Proof. Construct a sequence such that $x_n \in K_n$ for every $n \in \mathbb{N}$. In particular, $(x_n) \subseteq K_1$. Because K_1 is compact, it has a convergent subsequence $(x_{n_k}) \to x \in K_1$. Then every subsequence of (x_{n_k}) converges to the same limit x.
Note that by dropping out the first element of the subsequence, the resulted sequence starts with x_{n_2}. By the definition of subsequences, $n_2 \geq 2$, therefore, the truncated subsequence is contained in K_2 because of the compactness of K_2. As a result, $x \in K_2$. Applying the same argument on all natural numbers, it is immediate that $x \in K_n \ \forall n \in \mathbb{N}$. So $x \in \bigcap_{n \in \mathbb{N}} K_n$. ■

Proof. (Cantor's Argument). Suppose, for contradiction, the intersection is empty. Define $U_n := K_1 \backslash K_n$. Note that $U_n = K_1 \cap K_n^c = K_n^c$, which is open. Further, $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} K_1 \cap K_n^c = K_1 \cap (\bigcup_{n \in \mathbb{N}} K_n^c) = K_1 \cap (\bigcap_{n \in \mathbb{N}} K_n)^c = K_1 \backslash \bigcap_{n \in \mathbb{N}} K_n = K_1$. Therefore, $\mathcal{C} = \{U_n : n \in \mathbb{N}\}$ is an open cover of K_1. Because K_1 is compact, there exists a finite subcover of $\mathcal{C}$. Take n^* to be the greatest index in this finite subcover, then for every $x' \in K_{n^*+1} \subseteq K_1$, x' is not in the union of the constructed subcover, which leads to a contradiction. ■

Example 3.1. Note that the closedness itself is not sufficient for the nest compact set property to hold. For instance, the following sequence of closed sets are nested: $F_n := [n, \infty)$, but indeed, for every $x \in \mathbb{R}$, there exists a natural number $n > x$, so that $x \notin \bigcap_{n\in\mathbb{N}} F_n$. Therefore, $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$.

Definition 3.8. Let $A \subseteq \mathbb{R}$, an **open cover** for A is a collection of open sets $\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ such that $A \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$.

Theorem 3.8 (Heine-Borel)**.** Let $K \subseteq \mathbb{R}$, then the following are equivalent:

(i) K is (sequentially) compact;

(ii) K is closed and bounded;

(iii) Every open cover of K has a finite subcover.

Proof. The equivalence of (i) and (ii) has been proven previously.
Show $(iii) \implies (ii)$: suppose every open cover of K has a finite subcover, consider the following cover of K: $\mathcal{C} = \{[-n, n] : n \in \mathbb{N}\}$. Let M be the greatest index in the finite subcover $\mathcal{C}$, and obviously K is bounded by M.
Suppose, for contradiction, that K is not closed. Let y be a limit point of K but $y \notin K$. Then, for every $\varepsilon > 0$, $V_\varepsilon^o(y) \cap K \neq \varnothing$. We've shown that K is bounded, take $M \in \mathbb{R}$ such that $(-M, M) \supset K$. Define the following cover:

$$\mathcal{C} := \left\{(-M, M) \setminus \overline{V_\varepsilon(y)} : \varepsilon \in \mathbb{R}_{++}\right\} \tag{3.4}$$

Because K is compact, there exists a finite subcover of $\mathcal{C}$, which is clearly a contradiction.
Show $(ii) \implies (iii)$: Suppose K is closed and bounded, because of the transitivity of covering, it it sufficient to show that for every $M \in \mathbb{R}_+$, every open cover of $[-M, M]$ has a finite subcover.
Let $M \in \mathbb{R}_+$, and $\mathcal{C} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ is an open cover of $[-M, M]$. Suppose, for contradiction, there is no finite subcover. Then either $[-M, 0]$ or $[0, M]$ does not have a finite subcover from $\mathcal{C}$. Define such interval as I_1. Interval I_n is defined inductively from I_{n-1} by firstly bisecting I_{n-1} into two closed intervals and then taking the partition that cannot be covered by any finite subcover of $\mathcal{C}$. Note that (I_n) is a sequence of nested compact sets, by Cantor's intersection theorem, there intersection is nonempty. Further, because the length of interval shrinks to zero as $n \to \infty$, the intersection must be a singleton. Let $\{x\} = \bigcap_{n\in\mathbb{N}} I_n$, there exists some $\lambda \in \Lambda$, such that $x \in \mathcal{O}_\lambda$. Because $\mathcal{O}_\lambda$ is open, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$. Take $k \in \mathbb{N}$ such that $|I_k| < 2\varepsilon$, clearly $I_k \subseteq V_\varepsilon(x) \subseteq \mathcal{O}_\lambda$. Then $\mathcal{O}_\lambda$ is a finite subcover of I_k, which leads to a contradiction. ∎

3.3 Connected Sets

Definition 3.9. $\varnothing \neq A, B \subseteq \mathbb{R}$ are **separated** if and only if $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.

Definition 3.10. $E \subseteq \mathbb{R}$ is **disconnected** if $E = A \cup B$ where A, B are nonempty separated sets.

Proposition 3.1 (Equivalent Definiton)**.** $E \subseteq \mathbb{R}$ is disconnected if and only if it can be expressed as the union of two *nonempty disjoint open* (open in E) sets.

Theorem 3.9. A set $E \subseteq \mathbb{R}$ is connected if for every nonempty disjoint sets A, B such that $E = A \cup B$, then there exists a sequence $(a_n) \subseteq A$ converges to some point $a \in B$, or a sequence $(b_n) \subseteq B$ converges to some point $b \in A$.
Remark: Essentially, a set E is connected if, no matter how it is partitioned into two nonempty disjoint sets, it is always possible to show that at least one of the sets contains a limit point of the other.

Proof. Suppose E is connected, then for any partition $E = A \cup B$, either $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$. WLOG, $\overline{A} \cap B \neq \varnothing$, take $a \in A' \cap B \subseteq B$, and there exists a convergent sequence in A converging to $a \in B$. ■

Theorem 3.10. Let $E \subseteq \mathbb{R}$, the following are equivalent:

(i) E is connected;

(ii) For every $a < c < b$, $a, b \in E \implies c \in E$ (intervals).

Remark: the collection of connected subsets of $\mathbb{R}$ is exactly the collection of all intervals.

Proof. ($\implies$) Suppose E is connected, considering the following sets

$$A := (-\infty, c) \cap E \tag{3.5}$$

$$B := (c, \infty) \cap E \tag{3.6}$$

Note that $a \in A$ and $b \in B$, so both of them are nonempty. And A and B are separated. Suppose, for contradiction, $c \notin E$, $E = A \cup B$, which leads to a contradiction to the assumption that E is connected.
($\impliedby$) Suppose (ii), show E is connected. Let A and B be two nonempty set such that $A \cup B = E$ and $A \cap B = \varnothing$. We are going to show that A and B must be separated in this case. Let $a_0 \in A$ and $b_0 \in B$, WLOG, suppose $a_0 < b_0$. By (ii), the entire interval $[a_0, b_0] \subseteq E$. Split $[a_0, b_0]$ into two half intervals $[\alpha, \beta]$ and $[\beta, \gamma]$. Note that it is impossible for $\{\beta\}$ to be the only point intersect both A B, because in this case A and B cannot be disjoint. Take the one intersects both A and B, denoted as $[a_1, b_1]$.
One can construct a sequence of closed intervals inductively, such that every I_n intersects both A and B. Also, previous result shows that $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$, and is in fact a singleton. Let $x \in \cap_{n\in\mathbb{N}} I_n$, if $x \in A$, then there exists $(b_n) \subseteq B$ such that $(b_n) \to x$. Similarly, if $x \in B$, there exists $(a_n) \subseteq A$ such that $(a_n) \to x$. As a result, either $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$. Therefore, E is connected. ■

3.4 Cantor Set

Definition 3.11. Define sequence of sets

$$S_0 = [0, 1] \tag{3.7}$$

$$S_1 = [0, \frac{1}{3}] \cup [\frac{3}{2}, 1] \tag{3.8}$$

inductively, where S_n is defined by removing the mid-one-third of elements from each component of S_{n-1}. The **Cantor set** is defined as

$$\mathcal{C} := \bigcap_{n\in\mathbb{N}} S_n \neq \varnothing \tag{3.9}$$

$\mathcal{C}$ is nonempty because each S_n is a finite union of closed set. Altogether with the fact that each of S_n is bounded, so $\mathcal{C}$ is an intersection of nested compact sets. Therefore, $\mathcal{C}$ is nonempty by Cantor's intersection theorem.

Definition 3.12. A set is called **perfect** if it is closed and has no isolated point.

Proposition 3.2. $\mathcal{C}$ has measure zero.

Proof. Note that on while constructing S_n, intervals with total length of $\frac{2^n}{3^{n+1}}$ are removed from S_{n-1}. To construct a Cantor set, the total length of intervals from $[0,1]$ equals

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 \tag{3.10}$$

Therefore the length left for Cantor set is zero. ∎

Proposition 3.3. $\mathcal{C}^{int} = \varnothing$.

Proof. Note that for any set to have nonempty interior, it must contains some open intervals. Claim: for every open interval (a,b), it cannot be contained in $\mathcal{C}$. Let $a < b$, note that for every partition of S_n has length $\frac{1}{3^n}$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{3^n} < b-a$. Therefore, $(b-a) \nsubseteq S_n$ for such n. So that $\mathcal{C}$ cannot contain any open interval. ∎

Proposition 3.4. $\mathcal{C}$ is closed.

Proof. $\mathcal{C}$ is the intersection of infinitely many closed sets, so it is closed. ∎

Proposition 3.5. $\mathcal{C}$ is compact.

Proof. $\mathcal{C}$ is bounded by $[0,1]$ and closed by previous proposition. Therefore, $\mathcal{C} \subseteq \mathbb{R}$ is compact. ∎

Proposition 3.6. $\mathcal{C}$ is perfect.

Proof. We are going to show that every point $x \in \mathcal{C}$ is the limit of some sequence in $\mathcal{C}$.
Case 1: x is not the right endpoint of any intervals in S_n for any $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, let x_n be the right endpoint of the interval in S_n containing x. Obviously, $(x_n) \to x$.
Case 2: x is the right endpoint of some closed interval in some S_n (the implication of this observation is that we cannot simply take the right endpoint, so take the left endpoint). For every $n \in \mathbb{N}$, take x_n to be the left end of S_n containing x. Clearly, $(x_n) \to x$. ∎

Theorem 3.11. Any nonempty perfect set P is uncountable.

Proof. Note that P is obviously not finite. Suppose, for contradiction, P, then there exists an enumeration of $P = \{x_1, x_2, \cdots, x_n, \cdots\}$. Construct a sequence of compact sets as following: take $\varepsilon > 0$, there exists $y_1 \neq x_1$ such that $y_1 \in P \cap [x_1 - \varepsilon, x_1 + \varepsilon]$. Let $\delta_1 := \frac{|y_1 - x_1|}{2}$, and take $K_1 := [y_1 - \delta_1, y_1 + \delta_1] \cap P$. TODO: Show K_1 is compact. Note that $x_1 \notin K_1$.
Apply the same argument on K_1 to construct K_2 such that $x_2 \notin K_2$, so that $P \supset K_1 \supset K_2 \supset \cdots$. By construction, no points in P is in the intersection $\bigcap_{n\in\mathbb{N}} K_n$. However, the intersection is nonempty and the element belongs to the intersection is clearly in P, which is a contradiction. ■

Proposition 3.7. $\mathcal{C}$ is uncountable.

Proof. $\mathcal{C}$ is a nonempty perfect set, so it is uncountable. ■

Another Direct Proof. Consider the ternary expansions of all numbers in $\mathcal{C}$, it is easy to observe that for every $x \in \mathcal{C}$, if the ternary expansion of x contains 1 somewhere, x cannot be in $\mathcal{C}$.
Therefore, $\mathcal{C}$ has the same cardinality as the collection of all binary expansions, which has the same cardinality of $\mathbb{R}$ (refer to the construction of real numbers). ■

4 Functional Limits and Continuity

Definition 4.1. Let $f : A \to \mathbb{R}$ be a function, let c be a limit point of domain A, then $\lim_{x\to c} f(x) = L$ if

$$\forall\varepsilon > 0\ \exists\delta > 0\ s.t.\ x \in V_\delta^o(c) \implies f(x) \in V_\varepsilon(L) \tag{4.1}$$

Remark: The definition of continuity is stated in terms of punctuated ball, however, it is often easier to argue $x \in V_\delta(c) \implies f(x) \in V_\varepsilon(L)$.

Example 4.1. Let $g(x) = x^2$, show that $\lim_{x\to 2} g(x) = 4$.

Proof. Let $\varepsilon > 0$, note that for all $\delta < 1$, for all $x \in V_\delta^o(2)$,

$$\left|x^2 - 4\right| = |x-2|\,|x+2| \tag{4.2}$$
$$|x| = |x - 2 + 2| \le |x-2| + 2 < 3 \tag{4.3}$$
$$|x+2| \le |x| + 2 < 5 \tag{4.4}$$
$$\implies \left|x^2 - 4\right| < 5\delta \tag{4.5}$$

Take $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{5}\}$, both inequality reasoning (because $\delta < 1$) and ε requirement are valid. ■

Theorem 4.1 (Sequential Criterion for Functional Limits)**.** Given a function $f : A \to \mathbb{R}$ and $c \in A'$, then the following are equivalent:

(i) $\lim_{x\to c} f(x) = L$;

(ii) $\forall (x_n) \subseteq A\backslash\{c\}$ such that $(x_n) \to c$, $(f(x_n)) \to L$.

Proof. $(i) \implies (ii)$: assume $f(x) \to L$, let $(x_n) \subseteq A \backslash \{c\}$ be an arbitrary convergent sequence with limit c.
Let $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in V_\delta^0(c)$, $f(x) \in V_\varepsilon(L)$.
Consider such δ, by the convergence of sequence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V_\delta(c)$.
Moreover, note that $x_n \neq c \ \forall n \in \mathbb{N}$, therefore $n \geq N \implies x_n \in V_\delta^o(c)$, which further implies $f(x_n) \in V_\varepsilon(L)$ by the limit property of f.
$(ii) \implies (i)$: assume, for contradiction, $\lim_{x \to c} f(x) \neq L$.
Negating the definition of functional limit gives

$$\exists \ \varepsilon^* > 0 \ s.t. \ \forall \delta > 0 \ \exists x_\delta \in V_\delta^o(c) \ s.t. \ f(x_\delta) \notin V_{\varepsilon^*}(L) \tag{4.6}$$

For every $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$, and define $x_n := x_\delta$ from above statement.
Clearly, $(x_n) \to c$ by construction, but $(f(x_n))$ is bounded away from L by $\varepsilon^* > 0$. This leads to a contradiction of (ii). ■

Theorem 4.2 (Convergence Criterion for Functional Limits)**.** Let $f : A \to \mathbb{R}$ and $c \in A'$. If there exists two sequences $(x_n), (y_n) \subseteq A \backslash \{c\}$ converging to c, but $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$, then $\lim_{x \to c} f(x)$ does not exist.

Proof. In the previous theorem, the negation of (ii) proposes exactly the existence of two convergent sequences in $A \backslash \{c\}$ converging to the same limit c but their image sequences does not converge to the same limit. The result is immediate by taking the contraposition of $(i) \implies (ii)$ part. ■

Example 4.2. Limit of $f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at 0 does not exist.

Example 4.3 (Dirichlet Function)**.** Limit of $f(x) := \mathbb{1}\{x \in \mathbb{Q}\}$ does not exist anywhere in $\mathbb{R}$.

Example 4.4. Limit of $f(x) := x\mathbb{1}\{x \in \mathbb{Q}\}$ only exists at $x = 0$.

Theorem 4.3 (Characterizations of Continuity: Alternative Notations)**.** Let $f : A \to \mathbb{R}$, $c \in A$, then f is continuous at c <u>if and only if</u> one of the following holds:

(i) $\forall \varepsilon > 0 \ \exists \ \delta > 0 \ s.t. \ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$;

(ii) $\forall V_\varepsilon(f(c)) \ \exists \ V_\delta(c) \ s.t. \ x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(x))$;

(iii) $\forall A \supseteq (x_n) \to c \in A' \ (f(x_n)) \to f(c)$.

Proposition 4.1 (Criterion of Discontinuity)**.** Let $f : A \to \mathbb{R}$, $c \in A'$, if there exists sequence $(x_n) \subseteq A$ converges to c but $(f(x_n)) \not\to f(c)$, then f is not continuous at c.

Example 4.5 (Thomae's Function)**.** Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \wedge \gcd(p, q) = 1 \\ 1 & \text{if } x = 0 \end{cases} \tag{4.7}$$

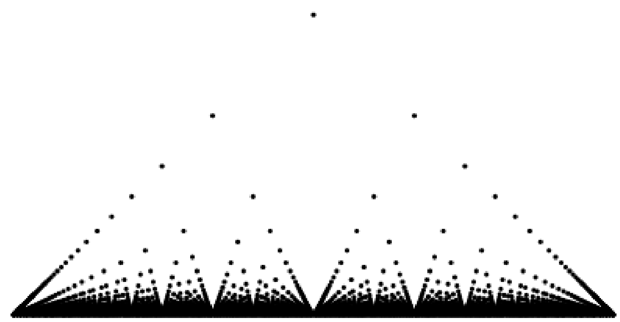

Figure 1: Thomae's Function in the Unit Interval

Proposition 4.2. For every $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = 0$. As a result, $D_f = \mathbb{Q}$.

Proof. WLOG, consider the domain $a \in (0,1)$ only, show:

$$\forall \varepsilon > 0 \ \exists\ \delta > 0 \ s.t.\ \forall x \in V^o_\delta(a),\ f(x) \in V_\varepsilon(0) \tag{4.8}$$

Fix $\varepsilon > 0$.
Note that there exists $N \in \mathbb{N}$ such that $n \geq N \implies \left|\frac{1}{n}\right| < \varepsilon$.
Because $\mathbb{Q}$ is countable, define finite set L following Cantor's diagonal order

$$L := \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \cdots, \frac{N-2}{N-1}\right\} \setminus \{a\} \tag{4.9}$$

That is, L contains every rational number such that its denominator in the lowest form is less than N excluding a (if a is rational).
Define $m := \min_{q_i \in L} |a - q_i|$, which is well-defined because L is finite.
Take $\delta := \frac{m}{2}$, note that $V_\delta(a) \cap \mathbb{Q} \cap L = \varnothing$ by construction.
Let $x \in V^o_\delta(a)$, either
(i) $x \in \mathbb{Q} \implies x \notin L \implies x = \frac{p}{q}$ where $q \geq N$, which implies $f(x) = \frac{1}{q} < \varepsilon$;
(ii) or $x \notin \mathbb{Q} \implies f(x) = 0 < \varepsilon$.
Either case implies the limit to be zero.
Therefore f is discontinuous on $\mathbb{Q}$. ■

Theorem 4.4. Composition of continuous functions is continuous.
Given $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ such that the range $f(A) \subseteq B$. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$. Then $g \circ f$ is continuous at c.

Proof. Let $\varepsilon > 0$, and $g(x)$ is continuous at $f(c)$.

$$\exists\ \tilde{\delta}\ s.t.\ f(x) \in V_{\tilde{\delta}}(f(c)) \implies g \circ f(x) \in V_\varepsilon(g \circ f(x)) \tag{4.10}$$

$$\exists\ \delta > 0\ s.t.\ x \in V_\delta(c) \implies f(x) \in V_{\tilde{\varepsilon}=\tilde{\delta}}(f(c)) \tag{4.11}$$

■

4.1 Continuous Functions on Compact Sets

Theorem 4.5. Let K be a compact set in $\mathbb{R}$, and $f : K \to \mathbb{R}$ is a continuous function, then $f(K)$ is compact.

Proof. Let $(y_n) \subseteq K$. Consider $f^{-1}(y_n) \neq \varnothing$, take $x_n \in f^{-1}(y_n)$ to construct a sequence.
Because K is compact, there exists a subsequence of (x_n) converges to $x \in K$.
Since f is continuous, $f(x_n)$ converges to $f(x) \in f(K)$. ■

Theorem 4.6 (Extreme Value Theorem)**.** If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subset \mathbb{R}$. Then f attains a maximum and minimum value. That is,

$$\exists\ x_0, x_1 \in K,\ s.t.\ f(x_0) \leq f(x) \leq f(x_1)\ \forall x \in K \tag{4.12}$$

Proof. $f(K)$ is compact, in particular it is bounded in $\mathbb{R}$. So it possesses a supremum $\sup f(K)$. Then there exists a sequence converging to the supremum. Because $f(K)$ is bounded as well, its limit can be attained in K. Therefore the maximum is attainable (minimum case is similar). ■

4.2 Uniform Continuity

Example 4.6 (Uniformly Continuous)**.** $f(x) = 3x + 1$ is continuous for every $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$ gives the definition of continuity. Note that δ does not depend on particular c. ■

Example 4.7 (Non-uniformly Continuous)**.** $y = x^2$ is continuous for every $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$,
Note that $|x + c| = |x - c + 2c| \leq |x - c| + 2\,|c|$.
Suppose $\delta \leq 1$ (the following argument works for $x \in V_1(c)$ only):

$$\begin{aligned}
\left|x^2 - c^2\right| &= |x - c|\,|x + c| && (4.13)\\
&\leq \delta\,|x + c| && (4.14)\\
&\leq \delta(|x - c| + 2\,|c|) && (4.15)\\
&\leq \delta(1 + 2\,|c|) < \varepsilon && (4.16)
\end{aligned}$$

Take $\delta := \min\left\{1, \frac{1}{1+2|c|}\right\}$. Note that δ depends on particular realization c. ■

Definition 4.2. A function f is **uniformly continuous** on $A \subset \mathbb{R}$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ \forall x, y \in \mathbb{R}\ \ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \tag{4.17}$$

Theorem 4.7 (Sequential Criterion for Absence of Uniform Continuity)**.** Let $f : A \to \mathbb{R}$, the following are equivalent:

(i) f fails to be uniformly continuous on A;

(ii) $\exists \varepsilon_0 > 0$ and two sequences $(x_n), (y_n) \subseteq A$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for every $n \in \mathbb{N}$.

Proof. $f : A \to \mathbb{R}$ is not uniformly continuous,
$\underline{\text{iff}}$ $\exists \varepsilon > 0$ *s.t.* $\forall \delta > 0$ $\exists x, y \in \mathbb{R}$ *s.t.* $|x_\delta - y_\delta| < \delta \wedge |f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$.
For each $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$ and construct two sequences. ∎

Example 4.8. Let $f(x) = \sin(x^{-1})$ defined on $A = (0, 1)$.
Take $x_n := \frac{1}{2n+\frac{\pi}{2}}$ and $y_n = \frac{1}{2n+\frac{3\pi}{2}}$. It is evident that $f(x_n) = 1$ and $f(y_n) = -1$ for every $n \in \mathbb{N}$.
By taking $\varepsilon_0 = 2$, the proof is complete.

Example 4.9. $f(x) = x^2$ is not uniformly continuous.
Consider two sequences $(x_n) = (n)$ and $(y_n) = \left(n + \frac{1}{n}\right)$.
Note that $|x_n - y_n| = \left|\frac{1}{n}\right| \to 0$.
Moreover, $|f(x_n) - f(y_n)| = \left|2 + \frac{1}{n^2}\right| > 2$.
Therefore f is not uniformly continuous.

Theorem 4.8 (Uniform Continuity on Compact Set)**.** A continuous function fo f on a compact K is uniformly continuous.

Proof. Suppose, for contradiction, f is continuous but not uniformly continuous.
Then there exists $\varepsilon_0 > 0$ and $(x_n), (y_n) \subseteq K$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.
Because K is compact, there exists subsequences of (x_n) and (y_n) converging to $x, y \in K$.
Suppose $x \neq y$, it contradicts $|x_n - y_n| \to 0$ because this property holds for subsequences as well.
Therefore, (x_{n_k}) and (y_{n_k}) converge to the same limit.
Because f is continuous, $\lim_{k\to\infty}[f(x_{n_k}) - f(y_{n_k})] = 0 < \varepsilon_0$, contradiction. ∎

4.3 Intermediate Value Theorem

Theorem 4.9. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, if $L \in \mathbb{R}$ such that either

(i) $L \in (f(a), f(b))$,

(ii) or $L \in (f(b), f(a))$.

Then $\exists\ c \in (a, b)$ such that $f(c) = L$.

Lemma 4.1. All the connected subsets of $\mathbb{R}$ are intervals.

Theorem 4.10. Continuous map preserves connectedness. That is, for some $f : G \to \mathbb{R}$ continuous with $G \subseteq \mathbb{R}$, then $E \subseteq G$ connected $\implies$ $f(E)$ connected.

Proof. Suppose $f(E) = A \cup B$, where $A, B \neq \varnothing$ and disjoint.
Note that $E = f^{-1}(A) \cup f^{-1}(B)$ with $f^{-1}(A) \cap f^{-1}(B) = \varnothing$ (evident using contradiction).
Because E is connected, either there exists a convergent sequence in $f^{-1}(A)$ converges to somewhere in $f^{-1}(B)$ or a convergent sequence in $f^{-1}(B)$ converges to somewhere in $f^{-1}(A)$.
WLOG, suppose there exists a sequence $(x_n) \in f^{-1}(A)$ converges to some point $x \in f^{-1}(B)$.

Because f is continuous, $\lim_{n\to\infty} f(x_n) = f(x) \in B$.
That is, there exists sequence $(f(x_n)) \subseteq A$ converges to $f(x) \in B$.
E is connected. ■

Proof of IVT.. Let $f : [a,b] \to \mathbb{R}$ be a continuous function.
WLOG, suppose $f(a) < f(b)$.
Let $L \in (f(a), f(b))$.
Note that $f([a,b])$ is connected by previous theorem, therefore $f([a,b])$ is an interval by previous lemma.
hence, $f([a,b])$ contains every point between $f(a)$ and $f(b)$, in particular L.
Therefore, $\exists\ c \in [a,b]$ with $f(c) = L$.
Note that $f(a), f(b) \neq L$, so $c \neq a, b$.
Therefore, the chosen $c \in (a,b)$. ■

Remark 4.1. EVT says the continuous functions' image of a compact set is compact.
IVT says the continuous functions' image of a connected set is connected.

Definition 4.3. A function f possesses the **intermediate value property** on closed interval $[a,b]$ if for every $x < y$ in $[a,b]$ and $f(x) < L < f(y)$, it is always possible to find $c \in (a,b)$ such that $f(c) = L$.

Theorem 4.11 (IVT restated)**.** f is continuous $\implies$ f satisfies IVP.

Remark 4.2. The converse of IVT is not true.
Consider

$$g(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{4.18}$$

$g(x) : [-1,1] \to [-1,1]$ satisfies IVP, but $g(x)$ is not continuous on $[-1,1]$, in particular, $g(x)$ is discontinuous at 0.

Proposition 4.3. If a sequence of functions (f_n) converges uniformly to f, then (f_n) converges point-wise to f as well.

Example 4.10. $f_n(x) - \frac{(x^2+nx)}{n}$ has point-wise limit to $f(x) = x$ but it does not converge uniformly.

5 Sequences and Series of Functions

5.1 Sequences of Functions

Definition 5.1. For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$. The sequence $(f_n)_{n\in\mathbb{N}}$ **converges point-wise** on A to a function $f : A \to \mathbb{R}$ if

$$\forall x \in A, (f_n(x)) \to f(x) \tag{5.1}$$

That is,

$$\forall x \in A, \forall \varepsilon > 0\ \exists N_x \in \mathbb{N}\ s.t.\ \forall n \geq N_x,\ f_n(x) \in V_\varepsilon(f(x)) \tag{5.2}$$

That is, the sequence induced at each $x \in A$ is convergent, and the value of N_x can depend on specific x.

Definition 5.2. Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$. Then (f_n) **converges uniformly** on A to $f : A \to \mathbb{R}$ if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ \forall x \in A\ \forall n \geq N,\ f_n(x) \in V_\varepsilon(f(x)) \tag{5.3}$$

The uniform convergence requires a single N to work for every $x \in A$.

Theorem 5.1 (Cauchy Criterion)**.** Given a sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$, the following are equivalent:

(i) (f_n) converges uniformly on A;

(ii) (**Uniformly Cauchy**)$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ |f_n(x) - f_m(x)| < \varepsilon\ \forall m, n \geq N, \forall x \in A$.

Proof. ($\implies$)
The sufficient condition is immediate by triangle inequality.
Suppose $(f_n) \overset{unif}{\to} f$, let $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\forall n \geq N, |f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for every $x \in A$.
Take the same N and let $m, n \geq N$, let $x \in A$,

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| && (5.4)\\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| && (5.5)\\ &< \varepsilon && (5.6) \end{aligned}$$

($\impliedby$)
Assume (f_n) is uniformly Cauchy.
Let $\varepsilon > 0$.
$\forall x \in A$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.
By the completeness of $\mathbb{R}$, $(f_n(x)) \to f(x)$ for some point-wise limit $f(x)$.
We are going to show (f_n) converges uniformly to the point-wise limit.
By uniform Cauchy, $\exists N \in \mathbb{N}$ such that $\forall m \geq N$, $n = m + k \geq N$, and $\forall x \in A$,

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2} \tag{5.7}$$

Fix $x \in A$,

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty} f_n(x) && (5.8)\\
\implies |f_m(x) - f(x)| &= \lim_{n\to\infty} |f_m(x) - f_n(x)| && (5.9)\\
&= \lim_{k\to\infty} |f_m(x) - f_{m+k}(x)| && (5.10)\\
&\leq \frac{\varepsilon}{2} < \varepsilon \text{ (order limit theorem)} && (5.11)
\end{aligned}
$$

■

Theorem 5.2 (Continuous Limit Theorem)**.** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let (f_n) be a sequence of continuous functions, let $c \in A$, note that for arbitrary $N \in \mathbb{N}$,

$$
\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| && (5.12)\\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| && (5.13)
\end{aligned}
$$

Where the last term can be made arbitrarily small given the uniform convergence property of (f_n). And the first two terms can be made arbitrarily small as well given the continuity properties. This concludes the continuity of f. ■

Theorem 5.3 (Differentiable Limit Theorem)**.** Let $(f_n) \to f$ pointwise on closed interval $[a, b]$, and assume that each f_n is differentiable. If $(f'_n) \to g$ uniformly on $[a, b]$, then the limit f is differentiable and $f' = g$.

Proof. Let $c \in [a, b]$, note that

$$
f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \tag{5.14}
$$

The following holds for every $n \in \mathbb{N}$:

$$
\begin{aligned}
&\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| && (5.15)\\
&= \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) + f'_n(c) - g(c) \right| && (5.16)\\
&\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| + \left| f'_n(c) - g(c) \right| && (5.17)
\end{aligned}
$$

In particular, one may choose $N_1 \in \mathbb{N}$ such that

$$
\left| f'_n(c) - g(c) \right| < \frac{\varepsilon}{3} \tag{5.18}
$$

for every $n \geq N_1$.
Moreover, given the Cauchy property of uniform convergence, there exists $N_2 \in \mathbb{N}$ such that $\forall m, n \geq N_2$, $|f'_m(x) - f'_n(x)| < \frac{\varepsilon}{3}$ for every $x \in A$.
Define $N = \max\{N_1, N_2\}$.
Note that $\exists\ \delta > 0$ such that

$$x \in V^o_\delta(c) \implies \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\varepsilon}{3} \tag{5.19}$$

Let $m \geq N$, for an arbitrary $x \in V^o_\delta(c)$, WLOG, suppose $x < c$. Applying the mean value theorem on $f_m - f_N$ gives

$$\begin{aligned} f'_m(\eta) - f'_N(\eta) &= \frac{f_m(c) - f_N(c) - [f_m(x) - f_N(x)]}{c - x} && (5.20)\\ \implies \frac{\varepsilon}{3} > \left|f'_m(\eta) - f'_N(\eta)\right| &= \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| && (5.21)\\ \implies \lim_{m \to \infty} \left| \frac{f_m(c) - f_m(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| &= \left| \frac{f(c) - f(x)}{c - x} - \frac{f_N(c) - f_N(x)}{c - x} \right| && (5.22)\\ &\leq \frac{\varepsilon}{3} \text{ (order limit theorem)} && (5.23) \end{aligned}$$

Therefore, for every $x \in V^0_\delta(c)$,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon \tag{5.24}$$

■

Theorem 5.4. Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f'_n) converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_n)$ converges, then (f_n) converges uniformly on $[a, b]$.

Theorem 5.5 (Stronger Version)**.** Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f'_n) converges uniformly to a function g on $[a, b]$.
If there exists a point $x_0 \in [a, b]$ for which $f_n(x_0)$ converges, then (f_n) converges uniformly,
Furthermore, the limit function f is differentiable and satisfies $f' = g$.

5.2 Series of Functions

Definition 5.3. A series of function $\sum_{i=1}^{\infty} f_n(x)$ converges point-wise/uniformly if $\sum_{i=1}^{k} f_n(x)$ converges point-wise/uniformly.

Example 5.1. Note that $\sum_{i=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly.

$$\left| \sum_{i=1}^{k} \frac{\sin(nx)}{n^2} - \sum_{i=1}^{\ell} \frac{\sin(nx)}{n^2} \right| \leq \sum_{n=\ell+1}^{k} \frac{1}{n^2} < \varepsilon \tag{5.25}$$

Hence, uniformly Cauchy.

Theorem 5.6. If $f_n : S \subseteq \mathbb{R}^k \to \mathbb{R}^m$ continuous, then if $\sum_{n=1}^{\infty} f_n \to f$ uniformly, then f is also continuous.

Definition 5.4. Let $f_k : S \to \mathbb{R}^m$, then (f_k) is **uniformly Cauchy** on S if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; \sup_{x \in S} || \sum_{i=k+1}^{\ell} f_i(x) || \leq \varepsilon \; \forall \ell > k \geq N \tag{5.26}$$

Theorem 5.7. (f_k) is uniformly Cauchy $\iff \sum f_k \to f$ uniformly.
This is the same as the theorem mentioned in previous section. Simply define $g_n := \sum_{i=1}^{n} s f_i(x)$, and (g_n) is uniformly Cauchy if and only if it converges to $g_\infty \equiv \sum_{n=1}^{\infty} f_n(x)$ uniformly.

Proof. ($\Longleftarrow$)
Let $g_m := \sum_{k=1}^{m} f_k(x)$, suppose $g_k \to f$ uniformly. Then,

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; |g_k - f| < \frac{\varepsilon}{2} \; \forall k \geq N, \forall x \in S \tag{5.27}$$

For every $k > \ell \geq N$,

$$|g_k - g_\ell| \leq |g_k - f| + |f - g_\ell| < \varepsilon \; \forall x \in S \tag{5.28}$$

Therefore, (f_k) is uniformly Cauchy.
($\Longrightarrow$)
Suppose $f(f_k)$ is uniformly Cauchy. Then for every fixed $x \in S$, $(g_k(x))$ is Cauchy.
So $f(x) = \lim_{k \to \infty} g_k(x)$ exists.

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; |g_k(x) - g_\ell(x)| < \varepsilon \; \forall k > \ell \geq N, \forall x \in S \tag{5.29}$$

$$\Longrightarrow \; |f(x) - g_k(x)| = \lim_{\ell \to \infty} |g_\ell(x) - g_k(x)| \leq \varepsilon \; \forall x \in S \tag{5.30}$$

$$\Longrightarrow \; g_k \to f \text{ uniformly} \tag{5.31}$$

■

Theorem 5.8 (Term-by-term Continuity Theorem)**.** Let f_n be continuous functions defined on a set $A \subseteq \mathbb{R}$, suppose $\sum_{n=1}^{\infty} f_n \to f$ uniformly, then f is continuous on A.

Theorem 5.9 (Term-by-term Differentiability Theorem)**.** Let f_n be continuous functions defined on $A \subseteq \mathbb{R}$, and suppose

(i) $\sum_{n=1}^{\infty} f_n' \to g$ uniformly;

(ii) There exists a point x_0 such that the series converges at x_0.

Then, $\sum_{n=1}^{\infty} f_n \to f$ converges uniformly to a differentiable function f with $f'(x) = g(x)$.

That is

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{5.32}$$

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \tag{5.33}$$

Theorem 5.10 (Weierstrass M-test)**.** Let $a_n : S \to \mathbb{R}^m$, and sequence $M_n \in \mathbb{R}$. If

$$\exists N \in \mathbb{N}\ \forall k \geq N\ \sup_{x \in S} |a_n(x)| \leq M_n \tag{5.34}$$

then $\sum_{n=1}^{\infty} M_n$ converges implies $\sum_{n=1}^{\infty} a_n$ converges uniformly.

Proof. The desired result can be shown by showing the sequence of partial sums is Cauchy. Let $\varepsilon > 0$, show that there exists $k \in \mathbb{N}$ such that

$$\forall m, n \geq k, \quad |S_m(x) - S_n(x)| < \varepsilon \tag{5.35}$$

WLOG, suppose $m > n$, then

$$|S_m(x) - S_n(x)| = |a_{n+1}(x) + \cdots a_m(x)| \tag{5.36}$$

$$\leq |M_{n+1} + \cdots + M_n| \tag{5.37}$$

■

Proof.

$$\forall x \in S \sum_{n=1}^{\infty} |a_n(x)| \leq \sum_{n=1}^{\infty} \sup_{x \in S} |a_n(x)| \tag{5.38}$$

$$\leq \sum_{n=1}^{N} \sup_{x \in S} |a_n(x)| + \sum_{n=N+1}^{\infty} M_n < \infty \tag{5.39}$$

Therefore, $f(x) = \sum_{n=1}^{\infty} a_n(x)$ exists.

Note that $\forall x \in S, \forall \ell \geq N$:

$$\left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| = \left| \sum_{k=\ell+1}^{\infty} a_k(x) \right| \tag{5.40}$$

$$\leq \sum_{k=\ell+1}^{\infty} |a_n(x)| \tag{5.41}$$

$$\leq \sum_{k=\ell+1}^{\infty} M_k < \infty \tag{5.42}$$

$$\implies \lim_{\ell\to\infty} \sup_{x\in S} \left| f(x) - \sum_{k=1}^{\ell} a_k(x) \right| \leq \lim_{\ell\to\infty} \sum_{k=\ell+1}^{\infty} M_n = 0 \tag{5.43}$$

Therefore $\sum a_k \to f$ uniformly. ∎

Example 5.2. Consider $\sum \frac{x^n}{n!}$ defined on $x \in [-A, A]$. Note that

$$\sup_{x\in S} \left| \frac{x^n}{n!} \right| \leq \frac{A^n}{n!} =: M_n \tag{5.44}$$

By ratio test, $\sum M_n$ converges.
By M-test $\sum \frac{x^n}{n!}$ converges uniformly on $[-A, A]$.
Therefore, $\sum \frac{x^n}{n!}$ is continuous on $[-A, A]$.

Example 5.3 (Geometric Series)**.**

$$\sum_{n=1}^{\infty} (-x^2)^n \tag{5.45}$$

By ratio test, the series converges only on $(-1, 1)$. In particular, it converges to $\frac{1}{1+x^2}$.
For every $0 < r < 1$, for $x \in [-r, r]$, then

$$\sup_{x\in[-r,r]} \left|(-x^2)^n\right| = r^{2n} \tag{5.46}$$

and $\sum_{n=1}^{\infty} r^{2n} = \frac{1}{1+r^2}$ implies $\sum_{n=1}^{\infty}(-x^2)^n$ converges uniformly on $[-r, r]$.

Example 5.4 (Weierstrass Function)**.** Define

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2^n} \cos(10^n \pi x)}_{f_n(x)} \tag{5.47}$$

Note that $\sup_{x\in S} |f_n(x)| \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n} \to 1$. Therefore, $\sum f_n \to f$ uniformly on $\mathbb{R}$, so that f is continuous.

Proposition 5.1. f is nowhere differentiable.

Proof. Let $x \in \mathbb{R}$ with decimal expansion $x_0 \cdot x_1 x_2 \cdots$.
Construct sequence $(z_n) \to x$ as following:
Fix n, let $y_0 = x_0.x_1 \cdot x_n$ and $y_1 = y_0 + \frac{1}{10^n}$.
Note that $10^n \pi y_i$ is an integer multiplied by π, which implies $f(y_0) = \frac{(-1)^{x_n}}{2^n}$, $f(y_1) = \frac{(-1)^{x_n+1}}{2^n}$ such that

$$\left| \frac{f(z_n) - f(x)}{z_n - x} \right| \to \infty \tag{5.48}$$

■

Example 5.5. Let

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \equiv \sum_{n=1}^{\infty} h_n(x) \tag{5.49}$$

Claim: $h(x)$ is continuous and differentiable on $\mathbb{R}$.

Continuity. Note that each $h_n(x)$ is continuous, so is each partial sum $S_k := \sum_{n=1}^{k} h_n(x)$.
By Weierstrass M-test, each $|h_n(x)| \leq \frac{1}{n^2} \equiv M_n$, the series converges to $h(x)$ uniformly.
Therefore, by the definition of series convergence, the sequence of partial sums, in which all elements are continuous, converges uniformly to $h(x)$, hence $h(x)$ is continuous as well. ■

Differentiability. We've already shown the sequence of partial sum converges uniformly to $h(x)$.
All we need to show is the derivative of partial sums converges uniformly to some function.
Equivalently, we are showing the uniform convergence of series of derivatives.
In particular,

$$h_n'(x) = -\frac{2x}{(x^2 + n^2)^2} \tag{5.50}$$

$$S_k'(x) = \sum_{n=1}^{k} h_n'(x) = -\sum_{n=1}^{k} \frac{2x}{(x^2 + n^2)^2} \tag{5.51}$$

Apply M-Test again,

$$\left|h_n'(x)\right| = \left| -\frac{2x}{(x^2 + n^2)^2} \right| \tag{5.52}$$

$$= \frac{2}{\frac{x^2+n^2}{x}(x^2 + n^2)} \tag{5.53}$$

$$= \frac{2}{(x + \frac{n^2}{x})(x^2 + n^2)} \tag{5.54}$$

Note that

$$\left(\sqrt{x} - \frac{n}{\sqrt{x}}\right)^2 = x - 2n + \frac{n^2}{x} \geq 0 \tag{5.55}$$

$$\implies \left(x + \frac{n^2}{x}\right) \geq 2n \tag{5.56}$$

Therefore,

$$\left|h'_n(x)\right| \leq \frac{2}{2n(x^2+n^2)} \tag{5.57}$$

$$\leq \frac{1}{n^3} \tag{5.58}$$

Therefore,

$$\sum_{n=1}^{\infty} h'_n(x) \to g(x) \text{ uniformly} \tag{5.59}$$

Hence, $h(x)$ is differentiable, in particular, $h'(x) = g(x)$. ■

5.3 Power Series

Definition 5.5. A **power series** takes the form of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5.60}$$

in which $f(0) < \infty$ trivially.

Theorem 5.11. If a power series converges at some point $x_0 \in \mathbb{R}$, then it it converges absolutely for any x such that $|x| \leq |x_0|$. TODO: *verify the inequality, strictly?*

Proof. The result can be easily established via comparison test.
Suppose $f(x_0)$ converges, let $x \in \overline{V}_{|x_0|}(0)$.
By the convergence of $f(x_0)$, $(a_n x_0^n) \to 0$, so that $|a_n x_0^n| \leq M \ \forall n \in \mathbb{N}$.

$$|a_n x^n| = |a_n| \left|\frac{x^n}{x_0^n} x_0^n\right| \tag{5.61}$$

$$= |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \tag{5.62}$$

$$\leq M \left|\frac{x}{x_0}\right|^n \tag{5.63}$$

In which $\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$ converges as a power series with ratio less than 1.
Therefore, $\sum_{n=0}^{\infty} a_n x^n$ is convergent by comparison test. ■

Definition 5.6. **Radius of convergence** and **interval of convergence** TODO: *Do we need to define this?*

Theorem 5.12 (Abel's Theorem)**.** Suppose $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converge somewhere $x = R > 0$, then $g(x)$ converges uniformly (as a series of functions) on $[0, R]$.
The same holds for $-R$: if $g(-R)$ converges, then $g(x)$ converges uniformly on $[-R, 0]$.

Theorem 5.13. If a power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges point-wisely on $A \subseteq \mathbb{R}$, then it converges uniformly on any compact subset $K \subseteq A$.

Proof. TODO: *Complete this proof.* ■

6 Appendix: Miscellaneous

Proposition 6.1 (Midterm 1, Q7)**.** Let (a_n) be a bounded sequence,
let $S := \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}$.
Then $\exists (a_{n_k}) \to s := \sup S$.

Lemma 6.1. S is bounded above, therefore $s := \sup S$ is well-defined.

Proof. Every element in S is bounded from above by infinitely many elements from (a_n), which are themselves bounded. Therefore S is bounded from above.
So s is well-defined by the completeness axiom. ■

Proof. Case 1. Suppose there are infinitely many $a_n > s$.
Then either $\forall k \in \mathbb{N}\ \exists a_{n_k} \in [s, s + \frac{1}{k})$, such subsequence $(a_{n_k}) \to s$.
Or $\exists k \in \mathbb{N}$ such that no a_n in $[s, s + \frac{1}{k})$, this leads to a contradiction to the assumption that s is the least upper bound. ■

Proof. Case 2. Suppose there are only finitely many $a_n > s$.
Consider $(s - \frac{1}{k}, s]$, by the definition of supremum, $\forall k \in \mathbb{N},\ \exists s_k \in (s - \frac{1}{k}, s] \cap S$.
But $s_k < a_n$ for infinitely many a_n by definition of S.
In particular, because there are only finitely many $a_n > s$, there must be infinitely many a_n clustered in $(s - \frac{1}{k}, s]$ for every $k \in N$.
For each $k \in \mathbb{N}$, one can pick $a_{n_k} \in (s - \frac{1}{k}, s]$ and $(a_{n_k}) \to s$. ■

Proposition 6.2 (Another Midterm Question)**.** The open clopen set in $\mathbb{R}$ is $\varnothing$ and $\mathbb{R}$.

Proof. Suppose, for contradiction, there exists $U, V \neq \varnothing$ and $V = U^c$, further U is clopen.
By definition of closedness and openness, V is clopen as well.
Pick $a \in U$ and $b \in V$. WLOG, assume $a < b$.
Construct $X \subseteq U$ as

$$X := \{x \in U : x < b\} \tag{6.1}$$

X is nonempty because $a \in X$.
X is certainly bounded above by b, hence, by the completeness axiom, $\exists\ \alpha = \sup X$.
Calim: $\alpha \in \partial U$.
By the definition of supremum, for every $\varepsilon > 0$,

$$\forall \varepsilon > 0,\ V_\varepsilon(\alpha) \cap X \neq \varnothing \tag{6.2}$$

$$\implies V_\varepsilon(\alpha) \cap U \neq \varnothing \tag{6.3}$$

Case 1:
If $V_\varepsilon(\alpha) \cap V = \varnothing$ for some $\varepsilon > 0$, then $V_\varepsilon(\alpha) \subseteq U$.
Subcase 1:
If $\alpha + \frac{\varepsilon}{2} \leq b$, then $\alpha \neq \sup X$ (contradiction);

Subcase 2:
If $\alpha + \frac{\varepsilon}{2} > b$, then $b \in V_{\varepsilon/2}(\alpha) \cap V \subseteq V_\varepsilon(\alpha) \cap V = \varnothing$ (contradiction).
Case 2: $\forall \varepsilon > 0,\ V_\varepsilon(\alpha) \cap V \neq \varnothing$, then by definition, $\alpha \in \partial U \subseteq U$ and $\alpha \in \partial U \subseteq U$ because U and V are closed.
This leads to a contradiction that $U \cap V = \varnothing$. ■

Proposition 6.3 (Sample Midterm 2)**.** Let $\mathcal{K} = \{K_\lambda : \lambda \in \Lambda\}$ be a collection of compact subsets of $\mathbb{R}$ such that the intersection of every finite sub-collection is nonempty. Then $\bigcap_{\lambda\in\Lambda} K_\lambda \neq \varnothing$.

Proof. Suppose, for contradiction, $\bigcap_{\lambda\in\Lambda} K_\lambda = \varnothing$.
Take an arbitrary $K_0 \in \mathcal{K}$, which is obviously non-empty.
Then,

$$\begin{aligned} K_0 &= K_0 \cap \varnothing^c && (6.4)\\ &= K_0 \cap \left(\bigcap_{\lambda\in\Lambda} K_\lambda\right)^c && (6.5)\\ &= K_0 \cap \bigcup_{\lambda\in\Lambda} K_\lambda^c && (6.6)\\ &= \bigcup_{\lambda\in\Lambda} K_0 \cap K_\lambda^c && (6.7)\end{aligned}$$

For an arbitrary $\lambda \in \Lambda$, let $x \in K_0 \cap K_\lambda^c \neq \varnothing$.
Because K_λ^c is open in $\mathbb{R}$, $\exists\ V_\varepsilon(x) \subseteq K_\lambda^c$.
Note that because K_λ^c is open in $\mathbb{R}$, therefore $K_0 \cap K_\lambda^c$ is open in K_0.
Hence, $\bigcup_{\lambda\in\Lambda} K_0 \cap K_\lambda^c$ is an open cover of K_0. The compactness of K_0 suggests it has a finite subcover, say $\bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c$. That is,

$$\begin{aligned} K_0 &= \bigcup_{\lambda=1}^n K_0 \cap K_\lambda^c && (6.8)\\ &= K_0 \cap \bigcup_{\lambda=1}^n K_\lambda^c && (6.9)\\ &= K_0 \backslash \bigcap_{\lambda=1}^n K_\lambda && (6.10)\\ \implies \bigcap_{\lambda=1}^n K_\lambda &= \varnothing && (6.11)\end{aligned}$$

$\Rightarrow\Leftarrow$ ■

Theorem 6.1. If $\sum_{n=0}^\infty a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=0}^\infty n a_n x^{n-1}$ converges for all $x \in (-R, R)$ as well.
Further, the differentiated series converges uniformly for every compact $K \subseteq (-R, R)$.
Moreover, provided the uniform convergence of differentiated series, the limit function $\sum_{n=0}^\infty a_n x^n$

is differentiable. Therefore,

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{6.12}$$

Remark: Differentiating a power series does not change its radius of convergence.

Proof. Prove using comparison test. ■

6.1 Taylor's Series

Idea Given $f(x) \in C^{\infty}$, we are trying express $f(x)$ as a series $\sum_{n=0}^{\infty} a_n x^n$ such that the series converges somewhere else other than $x = 0$. If such series exists, it must satisfy

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{6.13}$$

Example 6.1.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{6.14}$$

This series has infinity radius of convergence, specifically,

$$\left| \frac{x^{n+1}}{(n+1)!} / \frac{x^n}{n!} \right| = \left| \frac{x}{n+1} \right| \tag{6.15}$$

$$\implies \limsup \left| \frac{x}{n+1} \right| = 0 < 1 \tag{6.16}$$

Theorem 6.2 (Lagrangian Remainder Theorem)**.** Let f be differentiable for $N+1$ times on $(-R, R)$, let

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \in \{0, \cdots, N\} \tag{6.17}$$

And the **Taylor's polynomial** of order N is defined as

$$S_N(x) := \sum_{n=0}^{N} a_n x^n \tag{6.18}$$

For every $x \neq 0$ in $(-R, R)$, define the **error term** associated with the Taylor's polynomial of order N:

$$E_N(x) := f(x) - S_N(x) \tag{6.19}$$

Then there exists $c \in V_x(0)$ such that

$$E_N(x) := \frac{f^{N+1}(c)}{(N+1)!}x^{N+1} \tag{6.20}$$

Example 6.2 (Bound Estimation Error)**.** Consider $f(x) = \sin(x0$ and its Taylor's polynomial of order 5 in $V_2(0)$:

$$\begin{aligned}
S_5(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} && (6.21)\\
|E_5(x)| &= \left|\sin(x) - x + \frac{x^3}{3} - \frac{x^5}{5}\right| && (6.22)\\
&= \left|\frac{\frac{d^6 \sin(x)}{dx^6}}{6!}x^6\right| \text{ for some } c \in V_2(0) && (6.23)\\
&\leq \frac{|x^6|}{6!} \approx 0.089 && (6.24)
\end{aligned}$$

In general, for every fixed $N \in \mathbb{N}$, the tail of Taylor's series

$$\begin{aligned}
|\sin(x) - S_N(x)| &= \frac{\left.\frac{d^n \sin(x)}{dx^n}\right|_{x=c} |x|^N}{(N+1)!} && (6.25)\\
&\leq \frac{R^N}{(N+1)!} \to 0 \text{ uniformly by Cauchy's criterion} && (6.26)
\end{aligned}$$

Theorem 6.3 (Cauchy Mean Value Theorem)**.** Let f and g be two continuous functions defined on $[a, b]$, and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{6.27}$$

Proof. Define

$$h(x) := [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x) \tag{6.28}$$

Apply the Lagrange's mean value theorem on $h(x)$, there exists $c \in (a, b)$ such that

$$\begin{aligned}
h'(c) &= \frac{h(b) - h(a)}{b - a} && (6.29)\\
&= \frac{f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) - f(b)g(a) + f(a)g(a) + g(b)f(a) - g(a)f(a)}{b-a} = 0 && (6.30)
\end{aligned}$$

Therefore,

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0 \tag{6.31}$$

$$\implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \tag{6.32}$$

■

Proof of Lagrange's Reminder Theorem. Apply Cauchy MVT on $E_N(x)$ and x^{N+1} on $[0, x]$, there exists $x_1 \in (0, x)$ such that

$$\frac{E_N(x) - E_N(0)}{x^{N+1} - 0} = \frac{E_N(x)}{x^{N+1}} \tag{6.33}$$

$$= \frac{E'_N(x_1)}{(N+1)x_1^N} \tag{6.34}$$

Apply the GMT again on $[0, x_1]$, there exists $x_2 \in (0, x_1)$ such that

$$\frac{E'_N(x_1) - E'_N(0)}{(N+1)x_1^N - 0} = \frac{E'_N(x_1)}{(N+1)x_1^N} \tag{6.35}$$

$$= \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \tag{6.36}$$

where the first equality holds because

$$E_N^{(k)}(0) = f^{(k)} - k!a_k = 0 \tag{6.37}$$

Applying the GMT iteratively suggests that there exists $c \in (0, x)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E''_N(x_2)}{N(N+1)x_2^{N-1}} \tag{6.38}$$

$$= \frac{E_N^{(3)}(x_3)}{(N-1)N(N+1)x_3^{N-2}} \tag{6.39}$$

$$= \cdots \tag{6.40}$$

$$= \frac{E_N^{(N+1)}(c)}{(N+1)!} \tag{6.41}$$

$$\implies E_N(x) = \frac{E_N^{(N+1)}(c)}{(N+1)!} x^{N+1} \tag{6.42}$$

Further, observe that $E_N(x) = f(x) - S_N(x)$ and $E_N^{(N+1)}(x) = f^{(N+1)}(x)$.
Therefore,

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \tag{6.43}$$

■

Remark 6.1 (Motivation for Taylor's Series)**.** For function $f(\cdot)$, we want a power series converges to $f(\cdot)$ on some interval.
As mentioned before, it must be the case

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{6.44}$$

Let $f(\cdot) \in C^\infty$, *can we always find a suitable power series?*
Potential problems are

(i) The found series converges at $\{0\}$;

(ii) Converges to some function $g(\cdot) \neq f(\cdot)$.

Example 6.3.

$$g(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{6.45}$$

observe that $\eta_0 = g(0) = 0$, and

$$\eta_1 = g'(0) = \lim_{h \to 0} \frac{\exp(-1/h^2)}{h} = 0 \tag{6.46}$$

The same argument is applicable for higher order terms η_k, therefore,

$$S_\infty(x) = \sum_{k=1}^{\infty} \frac{\eta_k}{k!} f^{(k)}(0) x^k = 0 \tag{6.47}$$

The Taylor's series converges to zero function, which is clearly not g.

7 Integration

Remark 7.1. In this section, we are assuming f to be bounded unless otherwise mentioned.

7.1 Riemann Integrable Functions

Definition 7.1. A **partition** P of interval $[a, b]$ is a finite set of points that has a, b as its elements, and

$$a = x_0 < x_1 < \cdots < x_n = b \tag{7.1}$$

Notation 7.1. Given a function partition P, for every $x_k \in P$, define

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\} \tag{7.2}$$

$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\} \tag{7.3}$$

Definition 7.2. The **lower sum** and **upper sum** of a function f and a partition P on $[a, b]$ are defined

$$L(f, P) = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k \tag{7.4}$$

$$U(f, P) = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k \tag{7.5}$$

Example 7.1. For every partition P of every interval $I \subseteq \mathbb{R}$,

$$U(\mathbb{1}\{x \in \mathbb{Q}\}, P) = 1 \tag{7.6}$$

$$L(\mathbb{1}\{x \in \mathbb{Q}\}, P) = 0 \tag{7.7}$$

Definition 7.3. A partition Q is a **refinement** of another partition P if

$$P \subseteq Q \tag{7.8}$$

Lemma 7.1. If $P \subseteq Q$, then for every function f,

$$L(f, P) \leq L(f, Q) \tag{7.9}$$

$$U(f, P) \geq U(f, Q) \tag{7.10}$$

Proof. Suppose Q contains one extra split $z \in [x_{k-1}, x_k]$, then

$$m_k(x_k - x_{k-1}) = m_k(x_k - z + z - x_{k-1}) \tag{7.11}$$

$$= m_k(x_k - z) + m_k(z - x_{k-1}) \tag{7.12}$$

$$\geq (z - x_{k-1}) \inf_{x \in [x_{k-1}, z]} f(x) + (x_k - z) \inf_{x \in [z, x_k]} f(x) \tag{7.13}$$

■

Lemma 7.2. Let P_1, P_2 be any two partitions,

$$L(f, P_1) \leq U(f, P_2) \tag{7.14}$$

Therefore, lower sums are bounded above and upper sums are bounded from below.

Proof.

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \tag{7.15}$$

$$\leq U(f, P_1 \cup P_2) \tag{7.16}$$

$$\leq U(f, P_2) \tag{7.17}$$

■

Definition 7.4. Define the **upper** and **lower integrals** as

$$U(f) := \inf_P U(f, P) \tag{7.18}$$

$$L(f) := \sup_P L(f, P) \tag{7.19}$$

note that the infimum and supremum over partitions are well-defined by the previous lemma.
f is **(Riemann) integrable** if $U(f) = L(f)$