

MAT224 Notes

Tianyu Du

January 2018

Info.

Created: January. 9 2018

Last modified: January 30, 2018

This work is licensed under a Creative Commons “Attribution-NonCommercial-ShareAlike 3.0 Unported” license.



Contents

1	Lecture1 Jan.9 2018	2
1.1	Vector spaces	2
1.2	Examples of vector spaces	3
1.3	Some properties of vector spaces	3
2	Lecture2 Jan.10 2018	5
2.1	Some properties of vector spaces-Cont'd	5
2.2	Subspaces	5
2.3	Examples of subspaces	6
2.4	Recall from MAT223	7
3	Lecture3 Jan.16 2018	7
3.1	Linear Combination	7
3.2	Combination of subspaces	8
4	Lecture4 Jan.17 2018	11
4.1	Cont'd	11
4.2	Linear Independence	12

5	Lecture5 Jan.23 2018	13
5.1	Linear independence, recall definitions	13
5.1.1	Alternative definitions of linear independency	13
5.2	Basis	13
5.3	Dimensions	16
5.3.1	Consequences of fundamental theorem	17
5.3.2	Use dimension to prove facts about linearly (in)dependent sets and subspaces	17
6	Lecture6 Jan.24 2018	18
6.1	Basis and Dimension	18
7	Lecture7 Jan.30. 2018	21
7.1	Linear Transformations	21
7.2	Properties of linear transformations	22
7.3	Definitions	24

1 Lecture1 Jan.9 2018

1.1 Vector spaces

Definition A real¹ **vector space** is a set V together with two vector operations vector addition and scalar multiplication such that

1. **AC** Additive Closure: $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$
2. **C** Commutative: $\forall \vec{v}, \vec{y} \in V, \vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. **AA** Additive Associative: $\forall \vec{x}, \vec{y}, \vec{z} \in V, (\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. **Z** Zero Vector: $\exists \vec{0} \in V s.t. \forall \vec{x} \in V, \vec{x} + \vec{0} = \vec{x}$
5. **AI** Additive Inverse: $\forall \vec{x} \in V, \exists -\vec{x} \in V s.t. \vec{x} + (-\vec{x}) = \vec{0}$
6. **SC** Scalar Closure: $\forall \vec{x}, c \in \mathbb{R}, c\vec{x} \in V$
7. **DVA** Distributive Vector Additions: $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{R}, c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. **DSA** Distributive Scalar Additions: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (c+d)\vec{x} = c\vec{x} + d\vec{x}$

¹A vector space is real if scalar which defines scalar multiplication is real.

9. **SMA** Scalar Multiplication Associative: $\forall \vec{x} \in V, c, d \in \mathbb{R}, (cd)\vec{x} = c(d\vec{x})$

10. **O** One: $\forall \vec{x} \in V, 1\vec{x} = \vec{x}$

Note For V to be a vector space, need to know or be given operations of vector additions multiplication and check all 10 properties hold.

1.2 Examples of vector spaces

Example 1 \mathbb{R}^n w.r.t. usual component-wise addition and scalar multiplication.

Example 2 $M_{m \times n}(\mathbb{R})$ set of all $m \times n$ matrices with real entry. w.r.t. usual entry-wise addition and scalar multiplication.

Example 3 $\mathbb{P}_n(\mathbb{R})$ set of polynomials with real coefficients, of degree less or equal to n , w.r.t. usual degree-wise polynomial addition and scalar multiplication.

Note If define $\mathbb{P}_n^*(\mathbb{R})$ as set of all polynomials of degree exactly equal to n w.r.t. normal degree-wise multiplication and addition.

Then it is **NOT** a vector space.

Explanation: $(1 + x^n), (1 - x^n) \in \mathbb{P}_n^*(\mathbb{R})$ but $(1 + x^n) + (1 - x^n) = 2 \notin \mathbb{P}_n^*(\mathbb{R})$

Example 4 Something unusual, define V as

$$V = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$$

with vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$$

and scalar multiplication

$$c(x_1, x_2) = (cx_1 + c - 1, cx_2 + c - 1)$$

This is a vector space.

1.3 Some properties of vector spaces

Suppose V is a vector space, then it has the following properties.

Property 1 The zero vector is unique.

proof.

Assume $\vec{0}, \vec{0}^*$ are two zero vectors in V

WTS: $\vec{0} = \vec{0}^*$

Since $\vec{0}$ is the zero vector, by Z $\vec{0}^* + \vec{0} = \vec{0}^*$

Similarly, $\vec{0} + \vec{0}^* = \vec{0}$

Also, $\vec{0} + \vec{0}^* = \vec{0}^* + \vec{0}$ by commutative vector addition.

So, $\vec{0}^* = \vec{0}$

■

Property 2 $\forall \vec{x} \in V$, the additive inverse $-\vec{x}$ is unique.

proof.

Exercise.

Property 3 $\forall \vec{x} \in V, 0\vec{x} = \vec{0}$.

proof.

By property of number 0: $0\vec{x} = (0 + 0)\vec{x}$

By DSA: $0\vec{x} = 0\vec{x} + 0\vec{x}$

By AI, $\exists(-0\vec{x})$ s.t.

$0\vec{x} + (-0\vec{x}) = 0\vec{x} + 0\vec{x} + (-0\vec{x})$

By AA

$\implies 0\vec{x} = \vec{0}$

Property 4 $\forall c \in \mathbb{R}, c\vec{0} = \vec{0}$

proof.

Exercise.

2 Lecture2 Jan.10 2018

2.1 Some properties of vector spaces-Cont'd

Property 5 For a vector space V , $\forall \vec{x} \in V$, $(-1)\vec{x} = (-\vec{x})$. (we could use this property to find the additive inverse with scalar multiplication with (-1)).

proof.

$$\begin{aligned}
 (-\vec{x}) &= (-\vec{x}) + \vec{0} \quad \text{By property of zero vector} \\
 &= (-\vec{x}) + 0\vec{x} \quad \text{By property 3} \\
 &= (-\vec{x}) + (1 + (-1))\vec{x} \quad \text{By property of zero as real number} \\
 &= (-\vec{x}) + 1\vec{x} + (-1)\vec{x} \\
 &= \vec{0} + (-1)\vec{x} \\
 &= (-1)\vec{x}
 \end{aligned}$$

■

Property 6 For a vector space V , let $\vec{x} \in V$ and $c \in \mathbb{R}$, then,

$$c\vec{x} = \vec{0} \implies c = 0 \vee \vec{x} = \vec{0}$$

proof.

Exercise.

2.2 Subspaces

Loosely A subspace is a space contained within a vector space.

Definition Let V be a vector space and $W \subseteq V$, W is a **subspace** of V if W is itself a vector space w.r.t. operations of vector addition and scalar multiplication from V .

Theorem Let V be a vector space, and $W \subseteq V$, W has the same operations of vector addition and scalar multiplication as in V . Then, W is a subspace of V iff:

1. W is non-empty. $W \neq \emptyset$.
2. W is closed under addition. $\forall \vec{x}, \vec{y} \in W$, $\vec{x} + \vec{y} \in W$.

3. W is closed under scalar multiplication. $\forall \vec{x} \in W, c \in \mathbb{R}, c\vec{x} \in W$.

Proof.

Forward:

If W is a subspace

$$\implies \vec{0} \in W$$

$$\implies W \neq \emptyset$$

Also, additive and scalar multiplication closures $\implies (ii), (iii)$

Backward:

Let $W \neq \emptyset \wedge (ii) \wedge (iii)$

WTS. 10 axioms in definition of vector space hold

$(ii) \implies$ Additive Closure

$(iii) \implies$ Scalar Multiplication Closure

Because $W \subseteq V$, and V is a vector space, so properties hold $\forall \vec{w} \in W$.

Additive inverse: by property 5 and scalar multiplication closure,

$$\forall \vec{x} \in W, -\vec{x} = (-1)\vec{x} \in W.$$

Also, existence of additive identity: $(-\vec{x}) + \vec{x} = \vec{0} \in W$.

2.3 Examples of subspaces

Example 1 Let $V = \mathbb{M}_{n \times n}(\mathbb{R})$, V is a subspace.

Example 2 Define W as

$$W = \{A \in \mathbb{M}_{n \times n}(\mathbb{R}) \mid A \text{ is not symmetric}\}$$

Explanation: Let $A_1 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $A_1, A_2 \in W$ but

$$A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W.$$

Example 3 Let $V = \mathbb{P}_2(\mathbb{R})$, is W defined as following,

$$W = \{p(x) \in V \mid p(1) = 0\}$$

a subspace of V ?

proof.

WTS: (i)

Let $z(x) = 0$ or $z(x) = x^2 - 1, \forall x \in \mathbb{R}$

$\implies W \neq \emptyset$

WTS: (ii)

Let $p_1, p_2 \in W$, which means $p_1(1) = p_2(1) = 0$

$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0 + 0 = 0$

$\implies p_1 + p_2 \in W$

$\implies W$ is closed under addition.

WTS: (iii) Let $p \in W$ and $c \in \mathbb{R}$

$\implies p(1) = 0$

Since $(c * p)(x) = c * p(x)$, we have $(c * p)(1) = c * p(1) = c * 0 = 0$

$\implies cp \in W$.

So W is a subspace of V .



2.4 Recall from MAT223

Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$, then $Nul(A)$ is a subspace of \mathbb{R}^n and $Col(A)$ is a subspace of \mathbb{R}^m .

3 Lecture3 Jan.16 2018

3.1 Linear Combination

Definition Let V be a vector space, $\vec{v}_1, \dots, \vec{v}_n \in V$, $a_1, \dots, a_n \in \mathbb{R}$ the expression

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

is called a **linear combination** of $\vec{v}_1, \dots, \vec{v}_n$.

Theorem Let V be a vector space, W is a subspace of V , $\forall \vec{w}_1, \dots, \vec{w}_k \in W$, $c_1, \dots, c_k \in \mathbb{R}$, we have

$$c_1 \vec{w}_1 + \dots + c_k \vec{w}_k \in W$$

Subspaces are closed under linear combinations, since subspaces are closed under scalar multiplication and vector addition.

Theorem Let V be a vector space, let $\vec{v}_1, \dots, \vec{v}_k \in V$ then the set of all linear combination of $\vec{v}_1, \dots, \vec{v}_k$

$$W = \left\{ \sum_{i=1}^k c_i \vec{v}_i \mid c_i \in \mathbb{R} \forall i \right\}$$

is a subspace of V .

proof.

Consider $\vec{0} \in W$

So, $W \neq \emptyset$

Let $c \in \mathbb{R}$, Let $\vec{x} \in W \wedge \vec{y} \in W$

By definition of span, we have,

$$\vec{x} = \sum_{i=1}^k a_i \vec{v}_i, \quad \vec{y} = \sum_{i=1}^k b_i \vec{v}_i$$

Consider, $\vec{x} + c\vec{y}$

$$\vec{x} + c\vec{y} = \sum_{i=1}^k a_i \vec{v}_i + c \sum_{i=1}^k b_i \vec{v}_i = \sum_{i=1}^k (a_i + cb_i) \vec{v}_i \in W$$

■

Definition Let V be a vector space, $\vec{v}_1, \dots, \vec{v}_k \in V$, **span** of the set of vectors $\{\vec{v}_i\}_{i=1}^k$ is defined as the collection of all possible linear combinations of $\{\vec{v}_i\}_{i=1}^k$. By pervious theorem, span is a subspace.

3.2 Combination of subspaces

Definition Let W_1, W_2 be two sets, then the **union** of W_1, W_2 is defined as:

$$W_1 \cup W_2 = \{\vec{w} \mid \vec{w} \in W_1 \wedge \vec{w} \in W_2\}$$

the **intersection** of W_1, W_2 is defined as:

$$W_1 \cap W_2 = \{\vec{w} \mid \vec{w} \in W_1 \vee \vec{w} \in W_2\}$$

Now consider W_1, W_2 to be two subspaces of vector space V , then we have,

1. $W_1 \cup W_2$ is **not** a subspace.

2. $W_1 \cap W_2$ is a subspace.

proof.

Falsify the statement by providing counter-example:

Consider,

$$W_1 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 = 0\}$$

$$W_2 = \{(x_1, x_2) \mid x_2 \in \mathbb{R}, x_1 = 0\}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$$

$$\text{But, } \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$$

■

proof.

Because W_1 and W_2 are both subspaces, so

$$\vec{0} \in W_1 \cap W_2 \implies W_1 \cap W_2 \neq \emptyset$$

$$\text{Let } \vec{x}, \vec{y} \in W_1 \cap W_2, c \in \mathbb{R}$$

$$\text{Consider, } \vec{x} + c\vec{y}$$

Sine W_1, W_2 are subspaces,

$$\vec{x} + c\vec{y} \in W_1 \wedge \vec{x} + c\vec{y} \in W_2$$

$$\implies \vec{x} + c\vec{y} \in W_1 \cap W_2$$

So, $W_1 \cap W_2$ is a subspace.

■

Definition Let W_1, W_2 be subspaces of vector space V , define the **sum** of two subspaces as:

$$W_1 + W_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in W_1 \wedge \vec{y} \in W_2\}$$

Note Let $\vec{x} = \vec{0} \in W_1, \forall \vec{y} \in W_2, \vec{y} \in W_1 + W_2$ so that, $W_2 \subseteq W_1 + W_2$. Similarly, let $\vec{y} = \vec{0} \in W_2, \forall \vec{x} \in W_1, \vec{x} \in W_1 + W_2$. so that, $W_1 \subseteq W_1 + W_2$. So we have $\forall \vec{v} \in W_1 \cap W_2, \vec{v} \in W_1 + W_2$. So that,

$$W_1 \cap W_2 \subseteq W_1 + W_2$$

Note $W_1 + W_2$ is a subspace of V .
proof.

Let $\vec{x}_1, \vec{x}_2 \in W_1, \vec{y}_1, \vec{y}_2 \in W_2$

By properties of subspaces,

$$\forall c \in \mathbb{R}, \vec{x}_1 + c\vec{x}_1 \in W_1 \wedge \vec{y}_2 + c\vec{y}_2 \in W_2$$

Consider, $\vec{x}_1 + \vec{y}_1 \in W_1 + W_2, \vec{x}_2 + \vec{y}_2 \in W_1 + W_2$

$$\begin{aligned} & (\vec{x}_1 + \vec{y}_1) + c(\vec{x}_2 + \vec{y}_2) \\ &= (\vec{x}_1 + c\vec{x}_2) + (\vec{y}_1 + c\vec{y}_2) \in W_1 + W_2 \end{aligned}$$

■

Definition Let W_1, W_2 be subspaces of vector space V , say V is **direct sum** of W_1 and W_2 , written as $V = W_1 \oplus W_2$, if every $\vec{x} \in V$ can be written uniquely as $\vec{x} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$.

Equivalently Let W_1 and W_2 be subspaces of V , $V = W_1 \oplus W_2 \iff V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$.

4 Lecture4 Jan.17 2018

4.1 Cont'd

Cont'd Proof of Theorem

proof.

(Forward direction) Suppose $V = W_1 \oplus W_2$

WTS. $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

Let $V = W_1 \oplus W_2$

$\implies \forall \vec{x} \in V$, can be written uniquely as

$$\vec{x} = \vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$$

$\implies V = W_1 + W_2$ by definition of *sum*.

Let $\vec{x} \in W_1 \cap W_2$

Decomposition, let $\vec{z} \in W_1 \cap W_2$

$$\vec{z} = \vec{z} + \vec{0}, \vec{z} \in W_1, \vec{0} \in W_2$$

$$\vec{z} = \vec{0} + \vec{z}, \vec{0} \in W_1, \vec{z} \in W_2$$

Since decomposition is unique, $\vec{z} = \vec{0}$

So, $W_1 \cap W_2 = \{\vec{0}\}$

(Backward direction) Suppose $V = W_1 + W_2 \wedge W_1 \cap W_2 = \{\vec{0}\}$

WTS. $V = W_1 \oplus W_2$

Assume $\vec{x} = \vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2$

$$\vec{x} = \vec{w}'_1 + \vec{w}'_2, \vec{w}'_1 \in W_1, \vec{w}'_2 \in W_2$$

$$\implies \vec{w}_1 + \vec{w}_2 = \vec{w}'_1 + \vec{w}'_2$$

$$\implies \vec{w}_1 - \vec{w}'_1 = \vec{w}'_2 - \vec{w}_2$$

Where, by definition of subspace, $\vec{w}_1 - \vec{w}'_1 \in W_1 \wedge \vec{w}'_2 - \vec{w}_2 \in W_2$

So, $\vec{w}_1 - \vec{w}'_1, \vec{w}'_2 - \vec{w}_2 \in W_1 \cap W_2$

Since $W_1 \cap W_2 = \{\vec{0}\}$

$$\implies \vec{w}_1 = \vec{w}'_1 \wedge \vec{w}_2 = \vec{w}'_2$$

So the decomposition is unique.

■

4.2 Linear Independence

Theorem (Redundancy theorem) Let V be a vector space, $\{\vec{x}_1, \dots, \vec{x}_n\}$, let $\vec{x} \in \{\vec{x}_1, \dots, \vec{x}_n\}$, then

$$\text{span}\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}\} = \text{span}\{\vec{x}_1, \dots, \vec{x}_n\}$$

we say \vec{x} is the **redundant** vector that contributes nothing to the span.
proof.

$$\begin{aligned} \text{let } \vec{x} &\in \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} \\ \vec{x} &= \sum_{i=1}^n c_i \vec{x}_i \text{ for } c_i \in \mathbb{R} \forall i \\ \text{So, } \text{span}\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}\} &= \left\{ \sum_{i=1}^n a_i \vec{x}_i + d\vec{x} \mid a_i, d \in \mathbb{R} \forall i \right\} \\ &= \left\{ \sum_{i=1}^n a_i \vec{x}_i + \sum_{i=1}^n c_i \vec{x}_i \mid a_i, c_i \in \mathbb{R} \forall i \right\} \\ &= \left\{ \sum_{i=1}^n (a_i + c_i) \vec{x}_i \mid a_i, c_i \in \mathbb{R} \forall i \right\} \\ \text{Let } d_i &= a_i + c_i \in \mathbb{R} \\ &= \left\{ \sum_{i=1}^n d_i \vec{x}_i \mid d_i \in \mathbb{R} \forall i \right\} \\ &= \text{span}\{\vec{x}_1, \dots, \vec{x}_n\} \end{aligned}$$

■

Definition Let V be a vector space, let $\{\vec{x}_1, \dots, \vec{x}_n\} \in V$, we say $\{v_i\}_{i=1}^n$ is **linearly independent** if the only set of scalars $\{c_1, \dots, c_n\}$ that satisfies,

$$\sum_{i=1}^n c_i \vec{x}_i = 0$$

is $\{0, \dots, 0\}$.

Definition In contrast, we say a set of vector, with size n , is **linearly dependent** if

$$\exists \vec{c} \neq \vec{0} \in \mathbb{R}^n, \text{ s.t. } \sum_{i=1}^n c_i \vec{v}_i = 0$$

Theorem Let V be a vector space, $\{\vec{v}_i\}_{i=1}^n \in V$ is *linearly dependent* if and only if,

$$\exists \vec{x}_j \in \{\vec{v}_i\}_{i=1}^n \text{ s.t. } \vec{x}_j \in \text{span}\{\{\vec{v}_i\}_{i=1}^n \setminus \{\vec{v}_j\}\}$$

Theorem Let V be a vector space, $\{\vec{v}_i\}_{i=1}^n \in V$ is *linearly independent* if and only if,

$$\forall \vec{x}_j \in \{\vec{v}_i\}_{i=1}^n, \vec{x}_j \notin \text{span}\{\{\vec{v}_i\}_{i=1}^n \setminus \{\vec{v}_j\}\}$$

5 Lecture5 Jan.23 2018

5.1 Linear independence, recall definitions

Acknowledgment: special thanks to Frank Zhao.

Definition Let $\{\vec{x}_1, \dots, \vec{x}_k\}$ is **linearly independent** if only scalars $c_1 \dots c_k$ s.t.

$$\sum_{i=1}^k c_i \vec{x}_i = 0(\star)$$

are $c_1 = \dots = c_k = 0$

linearly dependent means at least one $c_i \neq 0$, (\star) still holds.

5.1.1 Alternative definitions of linear independency

Definition(Alternative.1) $\{\vec{x}_1 \dots \vec{x}_k\}$ is **linearly independent** iff none of them can be written as a linear combination of the remaining $k-1$ vectors.

Definition(Alternative.2) $\{\vec{x}_1 \dots \vec{x}_k\}$ is **linearly dependent** iff at least one of them can be written as a linear combination of the remaining $k-1$ vectors.

5.2 Basis

Definition Let V be a vector space, a non-empty set S of vectors from V is a **basis** for V if

1. $V = \text{span}\{S\}$
2. S is linearly independent.

Theorem Characterization of basis, non-empty subset $S = \{\vec{x}_i\}_{i=1}^n$ of vector space V is basis for V iff every $\vec{x} \in V$ can be written uniquely as

linear combination for vectors in S .

proof.

Forwards

Suppose S is a basis for V

So every $\vec{x} \in V$ can be written as a linear combination of vectors in S

To prove the uniqueness, assume two expressions of $\vec{x} \in V$

$$\vec{x} = \begin{cases} c_1\vec{x}_1 + \cdots + c_k\vec{x}_k \\ b_1\vec{x}_1 + \cdots + d_k\vec{x}_k \end{cases}$$

Consider,

$$c_1\vec{x}_1 + \cdots + c_k\vec{x}_k - (b_1\vec{x}_1 + \cdots + d_k\vec{x}_k) = \vec{0}$$

$$\iff \sum_{i=1}^k (c_i - b_i)\vec{x}_i = \vec{0}$$

Since vectors in basis S are linear independent,

$$c_i = b_i \forall i \in \mathbb{Z} \cap [1, k]$$

So the representation is unique.

Backwards

Suppose every $\vec{x} \in V$ can be written uniquely as linear combination of vectors in S .

WTS: $V = \text{span}\{S\} \wedge S$ is linearly independent

By the assumption, spanning set is shown.

All we need to show is linear independence.

Consider,

$$\sum_{i=1}^n c_i \vec{x}_i = \vec{0}$$

Also, we know

$$\sum_{i=1}^n 0\vec{x}_i = \vec{0}$$

By the uniqueness of representation

$$\text{We have identical expression } \sum_{i=1}^n c_i \vec{x}_i = \sum_{i=1}^n 0\vec{x}_i$$

$$\therefore c_i = 0 \forall i \in \mathbb{Z} \cap [1, n]$$

■

Example

$$\begin{aligned}
 V &= \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \\
 (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1 + 1, x_2 + y_2 + 1) \\
 c(x_1, x_2) &= (cx_1 + c - 1, cx_2 + c - 1)
 \end{aligned}$$

Show that $\{(1, 0), (6, 3)\}$ is a basis of V .

By theorem, $\{(1, 0), (6, 3)\}$ is basis if every $(a, b) \in V$ can be written uniquely as linear combination of $\{(1, 0), (6, 3)\}$.

$$\exists \text{ unique scalars } c_1, c_2 \in \mathbb{R} \text{ s.t. } c_1(1, 0) + c_2(6, 3) = (a, b)$$

proof.

By definition of scalar multiplication and vector addition in this space,

$$\begin{aligned}
 \text{Consider } (a, b) &= c_1(1, 0) + c_2(6, 3) = (2c_1 - 1, c_1 - 1) + (7c_2 - 1, 4c_2 - 1) \\
 &= (2c_1 + 7c_2 - 1, c_1 + 4c_2 - 1)
 \end{aligned}$$

Consider the coefficients of variables

$$\begin{cases} 2c_1 + 7c_2 - 1 = a \\ c_1 + 4c_2 - 1 = b \end{cases}$$

WTS, the above system of linear equations has unique solution for all a, b

The system has a unique solution $\forall a, b \in \mathbb{R}$

Since the coefficient matrix has rank 2

$$\text{rank}\left(\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}\right) = 2$$

Since obviously the columns are linearly independent.

■

5.3 Dimensions

Definition For a vector space V , the **dimension** of V is the minimum number of vectors required to span V .

Fundamental Theorem if V vector space is spanned by m vectors, then any set of more than m vectors from V must be linearly dependent.

Fundamental Theorem (Alternative) If V is vector space spanned by m vectors, then any linearly independent set in V must contain less or equal to m vectors.

5.3.1 Consequences of fundamental theorem

Theorem if $S = \{\vec{v}_i\}_{i=1}^k$ and $T = \{\vec{w}_i\}_{i=1}^l$ are two bases of vector space V then $l = k$.

proof.

Since S spans V and T is linearly independent

$$\therefore l \leq k$$

(flip) Since T spans V and S is linearly independent

$$\therefore k \leq l$$

$$\implies l \leq k \wedge k \leq l$$

$$\implies k = l$$

■

Definition So we can define the **dimension** of V , as $\dim(V)$ as the number vectors in any basis for V . For special case $V = \{\vec{0}\}$, $\dim(V) = 0$.

Example

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{P}_n(\mathbb{R})) = n + 1$
- $\dim(\mathbb{M}_{m \times n}(\mathbb{R})) = m \times n$

5.3.2 Use dimension to prove facts about linearly (in)dependent sets and subspaces

Theorem If V is a vector space, $\dim(V) = n$, $S = \{\vec{x}_k\}_{i=1}^k$ is subset of V , if $k > n$ then S is linearly dependent.

Note $k \leq n \nRightarrow S$ is linear dependent.

Theorem If W is subspace of vector space V , then

1. $\dim(W) \leq \dim(V)$
2. $\dim(W) = \dim(V) \iff W = V$

proof.

(1) Suppose $\dim(V) = n, \dim(W) = k$

WTS, $k \leq n$

Any basis for W is a linearly independent set of k vectors from V .

Since V is spanned by n vectors, since $\dim(V) = n$

By fundamental theorem, $k \leq n$

$$\iff \dim(W) \leq \dim(V)$$

(2) By contradiction, assume $\dim(V) = \dim(W) = n$ but $V \neq W$

Then $\exists \vec{x} \in V \wedge \vec{x} \notin W$

Take S as a basis of W , then $\vec{x} \notin \text{span}\{S\}$

Then $S \cup \vec{x}$ is linearly independent

$\implies S \cup \{\vec{x}\}$ is linearly independent in V containing $n + 1$ vectors

This contradicts the assumption by fundamental theorem since $\dim(V) = n$ so it could not contain more than n linearly independent vectors

■

6 Lecture6 Jan.24 2018

6.1 Basis and Dimension

Theorem Let V be a vector space, S is a spanning set of V , and I is a linearly independent subset of V , s.t. $I \subseteq S$, then \exists basis B for V s.t. $I \subseteq B \subseteq S$.

Explaining

1. Any spanning set for V can be **reduced** to basis for V by removing the linearly dependent(redundant) vector in the spanning set, using redundancy theorem to get a linearly independent spanning set.
2. Linear independent set can be **enlarged** to a basis for V .

proof.

omitted.

■

Corollary Let V be a vector space and $\dim(V) = n$, any set of n linearly independent vectors from V is a basis for V .

proof. If n linearly independent vectors did not span V , then could be enlarged to a basis of V by previous theorem, but then have a basis containing more than n vectors from V , which is impossible by the fundamental theorem since we given the $\dim(V) = n$, proven by contradiction.

Example Let $V = P_2(\mathbb{R})$, $p_1(x) = 2 - 5x$, $p_2(x) = 2 - 5x + 4x^2$, find $p_3 \in P_2(\mathbb{R})$ s.t. $\{p_1(x), p_2(x), p_3(x)\}$ is basis for $P_2(\mathbb{R})$

Note Since $\dim(P_2(\mathbb{R})) = 3$ so any 3 linearly independent vectors from $P_2(\mathbb{R})$ will be a basis for $P_2(\mathbb{R})$.

Solutions e.g. constant function $p_3(x) = 1$, since $1 \notin \text{span}\{p_1(x), p_2(x)\}$, so $\{p_1(x), p_2(x), p_3(x)\}$ is a basis of $P_2(\mathbb{R})$. e.g. $p_3(x) = x$, since $x \notin \text{span}\{p_1(x), p_2(x)\}$

Theorem Let U and W be subspaces of vector space V , then we have

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

proof.

Let $\{\vec{v}_i\}_1^k$ be basis for $U \cap W$

$$\implies \dim(U \cap W) = k$$

Since $\{\vec{v}_i\}_1^k$ is basis for $U \cap W$ then it's a linearly independent subset of U

So it could be enlarged to basis for U , $\{\vec{v}_1, \dots, \vec{v}_k, \vec{y}_1, \dots, \vec{y}_r\}$

$$\text{So } \dim(U) = k + r$$

We also could enlarge a basis for W $\{\vec{v}_1, \dots, \vec{v}_k, \vec{z}_1, \dots, \vec{z}_s\}$

$$\implies \dim(W) = k + s$$

WTS. $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{y}_1, \dots, \vec{y}_r, \vec{z}_1, \dots, \vec{z}_s\}$ is a basis for $U + W$

(If we could show this) $\dim(U + W) = k + r + s = (k + r) + (k + s) - k$

$$= \dim(U) + \dim(W) - \dim(U \cap W)$$

WTS. $\{\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{y}_1, \dots, \vec{y}_r, \vec{z}_1, \dots, \vec{z}_s\}$ is linearly independent

Consider $a_1\vec{v}_1 + \dots + a_k\vec{v}_k + b_1\vec{y}_1 + \dots + b_r\vec{y}_r + c_1\vec{z}_1 + \dots + c_s\vec{z}_s = \vec{0}$ (*)

$$\text{From (*) } \implies \sum (c_i\vec{z}_i) = -\sum (a_i\vec{v}_i) - \sum b_i\vec{y}_i$$

$$\implies \sum (c_i\vec{z}_i) \in U \wedge \sum (c_i\vec{z}_i) \in W$$

$$\iff \sum (c_i\vec{z}_i) \in U \cap W$$

Since $\{\vec{v}_i\}$ is a basis for $U \cap W$

$$\implies \sum (c_i\vec{z}_i) = \sum (d_i\vec{v}_i)$$

$$\implies c_i = d_i = 0 \text{ since } \{\vec{z}_i, \vec{v}_i\} \text{ is a basis}$$

Rewrite (*)

$$\sum (a_i\vec{v}_i) + \sum b_i\vec{y}_i = 0$$

$$\implies a_i = b_i = 0 \text{ since } \{\vec{v}_i, \vec{y}_i\} \text{ is a basis for } U$$

■

Corollary For direct sum, since the intersection is $\{\vec{0}\}$

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

Example Let U, W are subspaces of \mathbb{R}^3 such that $\dim(U) = \dim(W) = 2$, why is $U \cap W \neq \{\vec{0}\}$

Solutions Geometrically, U and W are planes through origin then the

intersection would be a line through origin ($U \neq W$) or a plane through origin. ($U = W$), so shown.

Question V is a vector space, $\dim(V) = n$, $U \neq W$ are subspaces of V but $\dim(U) = \dim(W) = (n - 1)$, proof:

1. $V = U + W$
2. $\dim(U \cap W) = (n - 2)$

7 Lecture7 Jan.30. 2018

7.1 Linear Transformations

Definition Let V, W be vector spaces, a function $T : V \rightarrow W$ is a **linear transformation**² if

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) \quad \forall \vec{x}, \vec{y} \in V$
2. $T(c\vec{x}) = cT(\vec{x}) \quad \forall \vec{x} \in V, c \in \mathbb{R}$

Linear transformation preserves the natural operations on vector spaces.

Theorem(Alternative definition) Transformation $T : V \rightarrow W$ is linear if and only if

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y}), \quad \forall \vec{x}, \vec{y} \in V, c, d \in \mathbb{R}$$

Linear transformations preserves linear combinations.

Example (from 223) Rotation through angle θ about the origin in \mathbb{R}^2 .

Example (from 223) Matrix transformation, let $A \in M_{m \times n}(\mathbb{R})$, transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as

$$T(\vec{x}) = A\vec{x}$$

is linear.

²In some textbooks, this is annotated as **linear mapping**.

³Notice that the vector additions on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

⁴Notice that the scalar multiplication on the left and right sides of the equation are defined in different vector spaces, in V and W respectively.

Example $T : P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$ defined by

$$T(\vec{p}(x)) = \vec{p}'(x)$$

Example $T : M_{m \times n}(\mathbb{R}) \rightarrow M_{n \times m}(\mathbb{R})$ defined by

$$T(A) = A^T$$

7.2 Properties of linear transformations

Property(i) Linear transformation $T : V \rightarrow W$ are uniquely defined by their values on any basis for V .

proof.

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be any basis for V

Every vector $\vec{x} \in V$ can be uniquely written as some linear combination of the $\{\vec{v}_i\}_{i=1}^k$

$$\vec{x} = \sum_{i=1}^k c_i \vec{v}_i, \quad c_i \in \mathbb{R}, \quad \text{and } c_i \text{ are uniquely determined } \forall \vec{x} \in V$$

$$\implies T(\vec{x}) = T\left(\sum_{i=1}^k c_i \vec{v}_i\right)$$

$$= \sum_{i=1}^k c_i T(\vec{v}_i) \text{ since the transformation } T \text{ is linear.}$$

Since c_i s are unique,

so the value of $T(\vec{x})$ is uniquely determined by its value on basis vectors $T(\vec{v}_i)$. ■

Property(ii) Let $T : V \rightarrow W$ be a linear transformation, let A be a subspace of vector space V , then the **image** $T(A)$ defined as

$$T(A) = \{T(\vec{x}) \mid \vec{x} \in A\}$$

called the image of A under linear transformation T is a subspace of W .
Linear transformation maps subspaces of W to subspaces of W .

proof.

Since A is a subspace so it's non-empty, therefore $\exists T(\vec{x}), \vec{x} \in A$

So $T(A) \neq \emptyset$

Let $\vec{w}_1, \vec{w}_2 \in T(A)$

$\implies \vec{w}_1 = T(\vec{x}_1), \vec{w}_2 = T(\vec{x}_2), \vec{x}_1, \vec{x}_2 \in A$

$\implies \vec{w}_1 + \vec{w}_2 = T(\vec{x}_1) + T(\vec{x}_2) = T(\vec{x}_1 + \vec{x}_2)$ since T is linear.

Since $\vec{x}_1 + \vec{x}_2 \in A$ by the definition of subspaces.

$\implies \vec{w}_1 + \vec{w}_2 \in T(A)$

So $T(A)$ is closed under vector addition.

Let $\vec{w} \in T(A)$

$\implies \vec{w} = T(\vec{x}), \vec{x} \in A$

Let $c \in \mathbb{R}$

Consider $c\vec{w} = cT(\vec{x}) = T(c\vec{x})$

Since $c\vec{x} \in A$

So $c\vec{w} \in T(A)$

So $T(A)$ is closed under scalar multiplication. ■

Property(derived from the definition) For all linear transformation $T : V \rightarrow W$, we have ⁵

$$T(\vec{0}) = \vec{0}$$

Property(iii) Let transformation $T : V \rightarrow W$ be linear, let B be a subspace of W , then its **pre-image** defined as

$$T^{-1}(B) = \{\vec{x} \in V \mid T(\vec{x}) \in B\}$$

is a subspace of V . ⁶

⁵In the equation, clearly, the zero vector on the left side of the equation is in space V and the zero vector on the right side is in space W .

⁶The pre-image and inverse share the same notation, but in this case, transformation T is not necessarily invertible.

proof.

$$\begin{aligned}
 & \text{Let } \vec{w}_1, \vec{w}_2 \in T^{-1}(B) \\
 & \implies T(\vec{w}_1), T(\vec{w}_2) \in B \\
 & \implies aT(\vec{w}_1) + bT(\vec{w}_2) \in B, \forall a, b \in \mathbb{R} \text{ since } B \text{ is a subspace.} \\
 & \implies T(a\vec{w}_1 + b\vec{w}_2) \in B \\
 & \implies a\vec{w}_1 + b\vec{w}_2 \in T^{-1}(B)
 \end{aligned}$$

So $T^{-1}(B)$ is closed under both vector addition and scalar multiplication,

So $T^{-1}(B)$ is a subspace. ■

7.3 Definitions

Let $T : V \rightarrow W$ to be a linear transformation,

Definition the **Image** of transformation T is defined as

$$Im(T) = T(V) = \{T(\vec{x}) \mid \vec{x} \in V\}$$

Definition the **Rank** of transformation T is defined as

$$Rank(T) = dim(Im(T))$$

Definition the **Kernel** of transformation T is defined as

$$Ker(T) = T^{-1}(\{\vec{0}\}) = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$$

Definition the **Nullity** of transformation T is defined as

$$Nullity(T) = dim(ker(T))$$

Example $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ is linear defined by

$$T(\vec{p}(x)) = \vec{p}(2x + 1) - 8\vec{p}(x)$$

find $Ker(T)$.

Theorem Let $T : V \rightarrow W$ be a linear transformation, let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be the spanning set of V ⁷, then $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ spans $Im(T)$

proof.

Let $\vec{w} \in Im(T)$

Since $V = span\{\vec{v}_1, \dots, \vec{v}_k\}$

For any $\vec{x} \in V$ can be written as

$$\vec{x} = \sum_{i=1}^k c_i \vec{v}_i, \quad c_i \in \mathbb{R}$$

$$\implies \vec{w} = T(\vec{x}) = T\left(\sum_{i=1}^k c_i \vec{v}_i\right)$$

$$= \sum_{i=1}^k c_i T(\vec{v}_i)$$

as a linear combination of $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$

So $Im(T) = span\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$

■

⁷The set is only the spanning set of V , it's not necessarily to be a basis of V .