

ECO421: Topics in Economic of Information

Games with Incomplete Information

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1 Knowledge

1.1 Defining an Information Structure

Notation 1.1. Let Ω denote the all possible states of world, and let A denote the set of all agents. Let $\omega^* \in \Omega$ denote the true state of world.

Definition 1.1. Every $E \subseteq \Omega$ is called an **event** or a type/piece of **information**.

Definition 1.2. Let $E \subseteq \Omega$ be a piece of information and $i \in A$, and we say agent i **knows** E in state $\omega \in \Omega$ if this agent knows

- (i) The true state of world $\omega^* \in E$;
- (ii) but, all $\omega \in E$ are considered to be possible.

Remark: the agent is certain about the true state ω^* if and only if $E = \{\omega^*\}$.

Definition 1.3. For an agent $i \in A$, the **information structure** of this agent, \mathcal{T}_i , is a partition of Ω into types of information.

Notation 1.2. Each element of the partition corresponds to one piece of information (an event).

(Information encoding) One may define a mapping $T_i(\cdot) : \Omega \rightarrow \mathcal{T}_i$, such that for every $\omega \in \Omega$, let $T_i(\omega) \in \mathcal{T}_i$ denote the piece of information known by this agent in state ω .

Equivalently, $T_i(\omega) \subseteq \Omega$ denotes the set of all states that agent i considers possible given her information in state ω .

Remark 1.1 (Interpretation). At some state of world, ω^* , agent i receives information $T_i(\omega^*)$ but not ω^* . This agent only knows $\omega^* \in T_i(\omega^*)$. The information encoding mechanism is a kind of blurring system preventing the agent from identifying the exact ω^* .

Definition 1.4. A **knowledge-based type space** is defined to be a tuple

$$(\Omega, (\mathcal{T}_i)_{i \in A}) \tag{1.1}$$

where \mathcal{T}_i is an information structure of agent i .

Definition 1.5. An event $E \subseteq \Omega$ is **true** in state $\omega^* \in \Omega$ if $\omega^* \in E$.

Definition 1.6 (Equivalent Definition). An agent i is said to **know** event E in state ω^* if and only if

$$T_i(\omega^*) \subseteq E \tag{1.2}$$

That is, the true state $\omega^* \in T_i(\omega) \subseteq E$, so that agent i is certain that E is true.

Definition 1.7. Define $K_i(E)$ to be the set of states in which agent i knows E to be true, that is,

$$K_i(E) := \{\omega : T_i(\omega) \subseteq E\} \subseteq \Omega \tag{1.3}$$

1.2 Learning

Definition 1.8. Given a piece of information $F \subseteq \Omega$ characterizing that $\omega^* \in F$, suppose F is known by all agents, the new **information structure updated by F** is defined to be

$$(\Omega^F, (T_i^F(\cdot))) \quad (1.4)$$

where

$$\Omega^F = \Omega \cap F \quad (1.5)$$

$$T_i^F(\omega) = T_i(\omega) \cap F \quad \forall \omega \in \Omega^F \quad (1.6)$$

Definition 1.9. The set of states where $E \subseteq \Omega^F$ is known given F is

$$K_i(E|F) := \{\omega \in \Omega^F : T_i^F(\omega) \subseteq E\} \quad (1.7)$$

$$= \{\omega \in \Omega^F : T_i(\omega) \cap F \subseteq E\} \quad (1.8)$$

Notation 1.3. For $i, j \in A$, the states in which i knows that player j knows E can be denoted as

$$K_i(K_j(E)) \quad (1.9)$$

1.3 Knowledge Hierarchies and Common knowledges

Definition 1.10. An event $E \subseteq \Omega$ is a **common knowledge** in state ω if everyone knows E and knows everyone else possesses the same information. That is:

$$\omega \in \bigcap_{i \in A} K_i(E) \quad (1.10)$$

$$\omega \in \bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \quad (1.11)$$

$$\omega \in \bigcap_{k \in A} K_k \left[\bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \right] \quad (1.12)$$

$$\vdots \quad (1.13)$$

2 Beliefs

3 Adverse Selection

3.1 Market for Lemons

Two types of cars

1. high quality $P(h) = \lambda$
2. low quality $P(l) = 1 - \lambda$

Assuming sellers know the quality of car but buyers don't.

1. b_θ : value of car with type $\theta \in \{h, l\}$ for the buyer.
2. s_θ value for the seller.

Assume

1. $b_h > b_l$ and $s_h > s_l$ (defining what high quality is);
2. $b_h > s_h$ and $b_l > s_l$ (benefit to trade).

Therefore, if the car quality is observed, then it is beneficial to trade, and two prices exist such that

$$b_\theta > p_\theta > s_\theta \quad \forall \theta \in \{l, h\} \quad (3.1)$$

If the quality is not observed, only one price p exists.

Seller type h sells if and only if $p > s_h$ and type l if and only if $p > s_l$.

Seller behaviours:

- (a) Both types sell: $p > s_h$;
- (b) Only type h sells: $s_h > p > s_l$;
- (c) Nobody wants to sell: $s_l > p$.

Buyer behaviours:

- (a) Both types sell: buyers buy if average quality $\lambda b_h + (1 - \lambda)b_l > p$;
- (b) Only type l sells, buyers buy if

$$\exists p \in \mathbb{R}_+ \text{ s.t. } b_l > p > s_l \quad (3.2)$$

Aggregate conditions

- (a) Trade with both types if and only if

$$\exists p \in \mathbb{R}_+ \text{ s.t. } \lambda b_h + (1 - \lambda)b_l > p > s_h \quad (3.3)$$

That is,

$$\lambda b_h + (1 - \lambda)b_l > s_h \quad (\dagger) \quad (3.4)$$

- (b) The second case (low quality trading) is always satisfied given assumption $b_l > s_l$.

The **market failure** occurs because high quality cars are withdrawn from the market.

3.2 Market for Insurances

- p denote the price;
- C denote the compensation (paid out if claim);
- D denote the damage;
- Δ denote value of peace of mind;
- $\pi_h > \pi_l$ probability of damage;
- $\lambda = P(h)$.

Buyer behaviours for type $\theta \in \{l, h\}$:

1. Buy: $\Delta - p + \pi_\theta(C - D)$;
2. Don't buy: $\pi_\theta(-D)$.

Buy if and only if

$$\Delta - p + \pi_\theta(C - D) \geq \pi_\theta(-D) \quad (3.5)$$

$$\implies p \leq \Delta + \underbrace{\pi_\theta C}_{\text{expected compensation}} \quad (\dagger\dagger) \quad (3.6)$$

Not that $(\dagger\dagger)$ is independent from D .

Seller behaviour

- (a) If both types buy insurance:

$$\pi = \lambda\pi_h + (1 - \lambda)\pi_l \quad (3.7)$$

Sellers sell if

$$p - \pi C \geq 0 \quad (3.8)$$

Aggregated conditions:

$$\exists p \in \mathbb{R}_+ \text{ s.t. } \Delta + \pi_l C \geq p \geq \pi C \quad (3.9)$$

$$\iff \Delta + \pi_l C > \lambda\pi_h + (1 - \lambda)\pi_l C \quad (3.10)$$

$$\iff \Delta > \lambda(\pi_h - \pi_l)C \quad (3.11)$$

$$\iff \pi_h - \pi_l < \frac{\Delta}{\lambda C} \quad (\dagger) \quad (3.12)$$

(\dagger) says both types of buyers participate in the market and sellers sell insurances if the difference between two types is insignificant.

If (\dagger) is not satisfied, then there is no trade with both type (only one type trades).

- (b) (Price is high enough so that low type does not want to buy) Suppose only type h trades, insurer sells if and only if

$$\pi_h C < p \quad (3.13)$$

And buyer type h wants

$$\Delta + \pi_h C > p \quad (3.14)$$

Given $\Delta > 0$, there would always be trading between sellers and type h buyers.

3.3 Monopoly Under Adverse Selection

Sellers

1. Decision (p, q) where p denotes price and q denotes quality;
2. Profit $\pi(p, q) = p - 1/2q^2$.

Buyers

1. $u_\theta(p, q) = \theta q - p$;
2. θ denotes the taste for quality: $\theta_h > \theta_l$;
3. $P(\theta_h) = \lambda$.

Case 1: 1th degree price discrimination Monopolist knows type θ for each consumer, and design good such that

$$\max_{(p,q)} p - 1/2q^2 \quad (3.15)$$

$$s.t. \theta q - p \geq 0 \text{ (Individual Rationality)} \quad (3.16)$$

Solution:

$$q^* = \theta^*, \quad p^* = \theta^{*2} \quad (3.17)$$

Case 2: 2nd degree price discrimination Monopolist does not the θ .
Monopolist design a *menu* of products

$$(p_1, q_1), (p_2, q_2), \dots, (p_j, q_j) \quad (3.18)$$

Provided there are only two types, it is sufficient to construct a menu with two types:

$$(p_l, q_l), (p_h, q_h) \quad (3.19)$$

Note that it is possible that $(p_l, q_l) = (p_h, q_h)$. Further, it is allowed to exclude certain type of consumers by setting $(p_i, q_i) = (0, 0)$.

Monopolist's problem designing an optimal menu such that both types are willing to purchase and each type only buy the bundle designed for them.

$$\max_{(p_l, q_l) \in \mathbb{R}_+^2, (p_h, q_h) \in \mathbb{R}_+^2} \lambda(p_h - 1/2q_h^2) + (1 - \lambda)(p_l - 1/2q_l^2) \quad (3.20)$$

$$\theta_l q_l - p_l \geq 0 \text{ (Individual rationality for low type)} \quad (3.21)$$

$$\theta_h q_h - p_h \geq 0 \text{ (Individual rationality for high type)} \quad (3.22)$$

$$\theta_h q_h - p_h \geq \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \quad (3.23)$$

$$\theta_l q_l - p_l \geq \theta_l q_h - p_h \text{ (Incentive compatibility for low type)} \quad (3.24)$$

Proposition 3.1 (Step 0).

$$IC_h \wedge IC_l \implies q_h \geq q_l \quad (3.25)$$

Proof.

$$IC_h \iff \theta_h(q_h - q_l) \geq p_h - p_l \quad (3.26)$$

$$IC_l \iff \theta_l(q_l - q_h) \geq p_l - p_h \quad (3.27)$$

Summing two conditions:

$$(q_h - q_l)(\theta_h - \theta_l) \geq 0 \quad (3.28)$$

Provided that $\theta_h > \theta_l$,

$$q_h \geq q_l \quad (3.29)$$

■

Proposition 3.2 (Step 1).

$$\text{IC}_h \wedge \text{IR}_l \implies \text{IR}_h \quad (3.30)$$

Proof.

$$\text{IC}_h \iff \theta_h q_h - p_h \geq \theta_h q_l - p_l \quad (3.31)$$

$$\implies \theta_h q_h - p_h \geq \theta_h q_l - p_l \geq \theta_l q_l - p_l \geq 0 \text{ (By IR for low type)} \quad (3.32)$$

$$\implies \theta_h q_h - p_h \geq 0 \iff \text{IR}_h \quad (3.33)$$

■

Proposition 3.3 (Step 2). Given step 1, IC_h constrain is binding.

Proposition 3.4 (Step 3). IC_h is binding and $q_h \geq q_l$ imply IC_l .

Proof.

$$\text{binding IC}_h \iff \theta_h(q_h - q_l) = p_h - p_l \quad (3.34)$$

$$\implies \theta_l(q_h - q_l) \leq \theta_h(q_h - q_l) = p_h - p_l \quad (3.35)$$

$$\implies \theta_l(q_h - q_l) \leq p_h - p_l \quad (3.36)$$

$$\iff \theta_l q_l - p_l \geq \theta_l q_h - p_h \quad (3.37)$$

$$\iff \text{IC}_l \quad (3.38)$$

■

Proposition 3.5 (Step 4). IR_l is binding.

Solving the reduced problem

$$\max_{(p_l, q_l) \in \mathbb{R}_+^2, (p_h, q_h) \in \mathbb{R}_+^2} \lambda(p_h - 1/2q_h^2) + (1 - \lambda)(p_l - 1/2q_l^2) \quad (3.39)$$

$$\theta_l q_l - p_l = 0 \text{ (Individual rationality for low type)} \quad (3.40)$$

$$\theta_h q_h - p_h = \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \quad (3.41)$$

The problem reduced to

$$\max_{q_l, q_h \in \mathbb{R}_+} \dots \quad (3.42)$$

Solving the problem gives

$$q_h = \theta_h \quad (3.43)$$

$$q_l = \theta_l - \frac{\lambda}{1 - \lambda}(\theta_h - \theta_l) \quad (3.44)$$