ECO421: Topics in Economic of Information
Games with Incomplete Information

Tianyu Du

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Contents

1   Knowledge

Notation 1.1. Let Ω denote the all possible states of world, and let A denote the set of all agents.

Definition 1.1. Every $E \subseteq \Omega$ is called an **event** or a type/piece of **information**.

Definition 1.2. Let $E \subseteq \Omega$ be a piece of information and $i \in A$, and we say agent i **knows** E in state $\omega \in \Omega$ if this agent knows

 (i) The true state of world $\omega^* \in E$;

 (ii) but, all $\omega \in E$ are considered to be possible.

Remark: the agent is certain about the true state ω^* if and only if $E = \{\omega^*\}$.

Definition 1.3. For an agent $i \in A$, the **information structure** of this agent, $\mathcal{T}_i$, is a *partition* of Ω.

Notation 1.2. Each element of the partition corresponds to one piece of information.
One may define a mapping $T_i(\cdot) : \Omega \to \mathcal{T}_i$, such that for every $\omega \in \Omega$, let $T_i(\omega) \in \mathcal{T}_i$ denote the piece of information known by this agent in state ω.
Equivalently, $T_i(\omega) \subseteq \Omega$ denotes the set of all states that i considers possible given her information in state ω.

Definition 1.4. A **knowledge-based type space** is defined to be a tuple

$$(\Omega, (\mathcal{T}_i)_{i \in A}) \tag{1.1}$$

Definition 1.5 (Equivalent Definition)**.** An agent i is said to **know** information E in state ω if and only if

$$T_i(\omega) \subseteq E \tag{1.2}$$

That is, the true state $\omega^* \in T_i(\omega) \subseteq E$, so that i believes E to be certain.

Definition 1.6. Define $K_i(E)$ to be the set of states in which agent i knows E, that is,

$$K_i(E) := \{\omega \in \Omega : T_i(\omega) \subseteq E\} \tag{1.3}$$

Definition 1.7. Given a piece of information $F \subseteq \Omega$ characterizing that $\omega^* \in F$, the new **information structure updated by** F is defined to be

$$(\Omega^F, (T_i^F(\cdot))) \tag{1.4}$$

where

$$\Omega^F = \Omega \cap F \tag{1.5}$$
$$T_i^F(\omega) = T_i(\omega) \cap F \quad \forall \omega \in \Omega^F \tag{1.6}$$

Definition 1.8. The set of states where E is known given F is

$$K_i(E|F) := \left\{\omega \in \Omega^F : T_i^F(\omega) \subseteq E\right\} \tag{1.7}$$

Notation 1.3. For $i, j \in A$, the following are equivalent:

1. states in which i knows that player j knows E;

2. states in which i knows $K_j(E)$.

This set of states can be denoted as

$$K_i(K_j(E)) \tag{1.8}$$

Definition 1.9. An event $E \subseteq \Omega$ is a **common knowledge** in state ω if

$$\omega \in \bigcap_{i \in A} K_i(E) \tag{1.9}$$

$$\omega \in \bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \tag{1.10}$$

$$\omega \in \bigcap_{k \in A} K_k \left[\bigcap_{j \in A} K_j \left(\bigcap_{i \in A} K_i(E) \right) \right] \tag{1.11}$$

2   Beliefs

3   Adverse Selection

3.1   Market for Lemons

Two types of cars

1. high quality $P(h) = \lambda$

2. low quality $P(l) = 1 - \lambda$

Assuming sellers know the quality of car but buyers don't.

1. b_θ: value of car with type $\theta \in \{h, l\}$ for the buyer.

2. s_θ value for the seller.

Assume

1. $b_h > b_l$ and $s_h > s_l$ (defining what high quality is);

2. $b_h > s_h$ and $b_l > s_l$ (benefit to trade).

Therefore, if the car quality is observed, then it is beneficial to trade, and two prices exist such that

$$b_\theta > p_\theta > s_\theta \ \forall \theta \in \{l, h\} \tag{3.1}$$

If the quality is not observed, only one price p exists.
Seller type h sells if and only if $p > s_h$ and type l if and only if $p > s_l$.
Seller behaviours:

(a) Both types sell: $p > s_h$;

(b) Only type h sells: $s_h > p > s_l$;

(c) Nobody wants to sell: $s_l > p$.

Buyer behaviours:

(a) Both types sell: buyers buy if average quality $\lambda b_+ (1 - \lambda)b_l > p$;

(b) Only type l sells, buyers buy if

$$\exists\, p \in \mathbb{R}_+ \ s.t.\ b_l > p > s_l \tag{3.2}$$

Aggregate conditions

(a) Trade with both types if and only if

$$\exists\, p \in \mathbb{R}_+ \ s.t. \lambda b_h + (1 - \lambda)b_l > p > s_h \tag{3.3}$$

That is,

$$\lambda b_h + (1 - \lambda)b_l > s_h \quad (\dagger) \tag{3.4}$$

(b) The second case (low quality trading) is always satisfied given assumption $b_l > s_l$.

The **market failure** occurs because high quality cars are withdrawn from the market.

3.2   Market for Insurances

- p denote the price;

- C denote the compensation (paid out if claim);

- D denote the damage;

- Δ denote value of peace of mind;

- $\pi_h > \pi_l$ probability of damage;

- $\lambda = P(h)$.

Buyer behaviours   for type $\theta \in \{l, h\}$:

1. Buy: $\Delta - p + \pi_\theta(C - D)$;

2. Don't buy: $\pi_\theta(-D)$.

Buy if and only if

$$\Delta - p + \pi_\theta(C - D) \geq \pi_\theta(-D) \tag{3.5}$$

$$\implies p \leq \Delta + \underbrace{\pi_\theta C}_{\text{expected compensation}} \quad (\dagger\dagger) \tag{3.6}$$

Not that ($\dagger\dagger$) is independent from D.

Seller behaviour

(a) If both types buy insurance:

$$\pi = \lambda\pi_h + (1-\lambda)\pi_l \qquad (3.7)$$

Sellers sell if

$$p - \pi C \geq 0 \qquad (3.8)$$

Aggregated conditions:

$$\exists\, p \in \mathbb{R}_+ \ s.t. \ \Delta + \pi_l C \geq p \geq \pi C \qquad (3.9)$$

$$\iff \Delta + \pi_l C > \lambda\pi_h + (1-\lambda)\pi_l C \qquad (3.10)$$

$$\iff \Delta > \lambda(\pi_h - \pi_l)C \qquad (3.11)$$

$$\iff \pi_h - \pi_l < \frac{\Delta}{\lambda C} \quad (\dagger) \qquad (3.12)$$

($\dagger$) says both types of buyers participate in the market and sellers sell insurances if <u>the difference between two types is insignificant</u>.
If ($\dagger$) is not satisfied, then there is <u>no trade with both type (only one type trades)</u>.

(b) (<u>Price is high enough so that low type does not want to buy</u>) Suppose only type h trades, insurer sells if and only if

$$\pi_h C < p \qquad (3.13)$$

And buyer type h wants

$$\Delta + \pi_h C > p \qquad (3.14)$$

Given $\Delta > 0$, there would always be trading between sellers and type h buyers.

3.3   Monopoly Under Adverse Selection

Sellers

1. Decision (p, q) where p denotes price and q denotes quality;

2. Profit $\pi(p, q) = p - 1/2q^2$.

Buyers

1. $u_\theta(p, q) = \theta q - p$;

2. θ denotes the taste for quality: $\theta_h > \theta_l$;

3. $P(\theta_h) = \lambda$.

Case 1: 1th degree price discrimination Monopolist knows type θ for each consumer, and design good such that

$$\max_{(p,q)} p - 1/2q^2 \tag{3.15}$$

$$s.t. \ \theta q - p \geq 0 \ \text{(Individual Rationality)} \tag{3.16}$$

Solution:

$$q^* = \theta^*, \ p^* = \theta^{*2} \tag{3.17}$$

Case 2: 2nd degree price discrimination Monopolist does not the θ.
Monopolist design a *menu* of products

$$(p_1, q_1), (p_2, q_2), \cdots, (p_j, q_j) \tag{3.18}$$

Provided there are only two types, it is sufficient to construct a menu with two types:

$$(p_l, q_l), (p_h, q_h) \tag{3.19}$$

Note that it is possible that $(p_l, q_l) = (p_h, q_h)$. Further, it is allowed to exclude certain type of consumers by setting $(p_i, q_i) = (0, 0)$.
Monopolist's problem designing an optimal menu such that both types are willing to purchase and each type only buy the bundle designed for them.

$$\max_{(p_l,q_l) \in \mathbb{R}_+^2, (p_h,q_h) \in \mathbb{R}_+^2} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.20}$$

$$\theta_l q_l - p_l \geq 0 \ \text{(Individual rationality for low type)} \tag{3.21}$$

$$\theta_h q_h - p_h \geq 0 \ \text{(Individual rationality for high type)} \tag{3.22}$$

$$\theta_h q_h - p_h \geq \theta_h q_l - p_l \ \text{(Incentive compatibility for high type)} \tag{3.23}$$

$$\theta_l q_l - p_l \geq \theta_l q_h - p_h \ \text{(Incentive compatibility for low type)} \tag{3.24}$$

Proposition 3.1 (Step 0).

$$\text{IC}_h \wedge \text{IC}_l \implies q_h \geq q_l \tag{3.25}$$

Proof.

$$\text{IC}_h \iff \theta_h(q_h - q_l) \geq p_h - p_l \tag{3.26}$$

$$\text{IC}_l \iff \theta_l(q_l - q_h) \geq p_l - p_h \tag{3.27}$$

Summing two conditions:

$$(q_h - q_l)(\theta_h - \theta_l) \geq 0 \tag{3.28}$$

Provided that $\theta_h > \theta_l$,

$$q_h \geq q_l \tag{3.29}$$

∎

Proposition 3.2 (Step 1).

$$\text{IC}_h \wedge \text{IR}_l \implies \text{IR}_h \tag{3.30}$$

Proof.

$$\text{IC}_h \iff \theta_h q_h - p_h \geq \theta_h q_l - p_l \tag{3.31}$$

$$\implies \theta_h q_h - p_h \geq \theta_h q_l - p_l \geq \theta_l q_l - p_l \geq 0 \text{ (By IR for low type)} \tag{3.32}$$

$$\implies \theta_h q_h - p_h \geq 0 \iff \text{IR}_h \tag{3.33}$$

∎

Proposition 3.3 (Step 2). Given step 1, IC_h constrain is binding.

Proposition 3.4 (Step 3). IC_h is binding and $q_h \geq q_l$ imply IC_l.

Proof.

$$\text{binding IC}_h \iff \theta_h(q_h - q_l) = p_h - p_l \tag{3.34}$$

$$\implies \theta_l(q_h - q_l) \leq \theta_h(q_h - q_l) = p_h - p_l \tag{3.35}$$

$$\implies \theta_l(q_h - q_l) \leq p_h - p_l \tag{3.36}$$

$$\iff \theta_l q_l - p_l \geq \theta_l q_h - p_h \tag{3.37}$$

$$\iff \text{IC}_l \tag{3.38}$$

∎

Proposition 3.5 (Step 4). IR_l is binding.

Solving the reduced problem

$$\max_{(p_l,q_l)\in\mathbb{R}^2_+,(p_h,q_h)\in\mathbb{R}^2_+} \lambda(p_h - 1/2q_h^2) + (1-\lambda)(p_l - 1/2q_l^2) \tag{3.39}$$

$$\theta_l q_l - p_l = 0 \text{ (Individual rationality for low type)} \tag{3.40}$$

$$\theta_h q_h - p_h = \theta_h q_l - p_l \text{ (Incentive compatibility for high type)} \tag{3.41}$$

The problem reduced to

$$\max_{q_l,q_h\in\mathbb{R}_+} \dots \tag{3.42}$$

Solving the problem gives

$$q_h = \theta_h \tag{3.43}$$

$$q_l = \theta_l - \frac{\lambda}{1-\lambda}(\theta_h - \theta_l) \tag{3.44}$$