Notes on Probability Theory
18.175

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Contents

1 Preliminaries

Definition 1.1. A **probability space** is a triple $(\Omega, \mathcal{F}, P)$ where Ω is the **sample space**, $\mathcal{F}$ is a σ-algebra of Ω (**events**) and $P : \mathcal{F} \to [0, 1]$ is the **probability function**.

Remark 1.1. $(\Omega, \mathcal{F})$ is a **measurable space** or **Borel space**.

Definition 1.2. A **algebra**, $\mathcal{A}$, of set X is a collection of subsets of X closed under complementation and *finite* union.

Definition 1.3. A **σ-algebra** of set X is a collection of subsets of X closed under complementation and *countable* union.

Remark 1.2. We can also define *algebra* and *σ-algebra* using closures under complementation and *finite/countable intersection.*

Proof. Use DeMorgan's Law. ∎

Definition 1.4. A measure μ on $\mathcal{A}$ is **σ-finite** if there exists *countable* collection $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$ and $\cup A_n = \Omega$.

Definition 1.5. A **semi-algebra** $\mathcal{S}$ is a collection of sets closed under intersection such that $S \in \mathcal{S}$ implies that S^c is a *finite disjoint* union of sets in $\mathcal{S}$.

Lemma 1.1. Let $\mathcal{S}$ be a semi-algebra, then

$$\overline{\mathcal{S}} = \text{all } \textit{finite disjoint unions} \text{ of sets in } \mathcal{S} \tag{1.1}$$

is an algebra, called the **algebra generated by** $\mathcal{S}$.

Proof. We are going to show the equivalent definition of algebra, that's, $\bar{\mathcal{S}}$ is closed under complementation and finite intersection.
Intersection: Let $A, B \in \bar{\mathcal{S}}$, then by definition of $\bar{\mathcal{S}}$,

$$A = \cup_i A_i \;\; A_i \in \mathcal{S} \tag{1.2}$$
$$B = \cup_j B_j \;\; B_j \in \mathcal{S} \tag{1.3}$$

Then by definition of semi-algebra, $A_i \cap B_j \in \mathcal{S}$. Then

$$A \cap B = (\cup_i A_i) \cap (\cup_j B_j) \tag{1.4}$$
$$= \cup_{i,j} A_i \cap B_j \in \bar{\mathcal{S}} \tag{1.5}$$

By an inductive argument, we've shown that $\bar{\mathcal{S}}$ is closed under intersection.
Complementation: Let $A \in \bar{\mathcal{S}}$, by definition

$$A = \cup_i A_i \;\; A_i \in \mathcal{S} \tag{1.6}$$

Therefore, by DeMorgan's Law, $A^c = \cap_i A_i^c$ and by definition of semi-algebra, for each A_i^c, it's a finite union of disjoint sets in $\mathcal{S}$.
By definition of $\bar{\mathcal{S}}$, each $A_i^c \in \bar{\mathcal{S}}$. And as shown above, $\bar{\mathcal{S}}$ is closed under finite intersection.
Therefore $A^c \in \bar{\mathcal{S}}$.
So $\bar{\mathcal{S}}$ is closed under complementation.
Therefore $\bar{\mathcal{S}}$ is an algebra. ∎

Definition 1.6. A **measure** on algebra is a function $\mu : \mathcal{A} \to \mathbb{R}$ such that

(i) $\mu(A) \geq \mu(\emptyset) = 0 \; \forall A \in \mathcal{A}$,

(ii) and countably additive for *disjoint* set $\{A_i\}_i$

$$\mu(\cup_i A_i) = \sum_i \mu(A_i) \tag{1.7}$$

Definition 1.7. A measure μ on $\mathcal{F}$ is a **probability measure** if $\mu(\Omega) = 1$.

Definition 1.8. The **Borel σ-algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra *containing all open sets.*

Theorem 1.1. For each *right continuous, non-decreasing* function F such that $\lim_{x\to-\infty} F = 0$ and $\lim_{x\to\infty} F = 1$, there is an *unique* measure defined on the Borel sets of $\mathbb{R}$ with

$$P((a, b]) \equiv F(b) - F(a) \tag{1.8}$$

Definition 1.9. A collection $\mathcal{P}$ of sets is a **π-system** is it's closed under intersection.

Definition 1.10. A collection $\mathcal{L}$ of subsets of Ω is a **λ-system**(Dynkin system) if

(i) $\Omega \in \mathcal{L}$.

(ii) (*Closed under set difference*) If $A, B \in \mathcal{L} \wedge A \subseteq B \implies B \backslash A \in \mathcal{L}$.

(iii) (*Contain set sequence limit*) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$, then $A \in \mathcal{L}$.

Theorem 1.2. If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system containing $\mathcal{P}$, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$, where $\sigma(\mathcal{A})$ denotes the smallest σ-algebra containing $\mathcal{A}$.

Theorem 1.3 (Caratheodory Extension Theorem)**.** If μ is a a σ-finite measure on an algebra $\mathcal{A}$, then μ has a *unique* extension to the σ-algebra generated by $\mathcal{A}$.

2 Random Variables

Definition 2.1. A **measurable space** is a tuple (S, Σ) where Σ is a σ-algebra on S.

Remark 2.1. The definition of *measurable spaces* does not require a specific measure.

Definition 2.2. Let (X, Σ) and (Y, Π) be two measurable spaces, and function $f : X \to Y$ is a **measurable function** if

$$\forall \mathcal{E} \in \Pi,\ f^{-1}(\mathcal{E}) \in \Sigma$$

Denoted as $f : (X, \Sigma) \to (Y, \Pi)$.

Definition 2.3. A **random variable** is a measurable function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$. We say X is $\mathcal{F}$-measurable.

Theorem 2.1. If $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{A}$ and $\mathcal{A}$ generates $\mathcal{S}$, then X is a measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$.

Definition 2.4. Let $F_X(x) \equiv P(X \leq x)$ be the **distribution function** for X. And write $f = f_X = F_X'$ for the **density function** of X. The distribution function must be

(i) Non-decreasing

(ii) Right-continuous

(iii) $\lim_{x\to\infty} F(x) = 1$

(iv) $\lim_{x\to-\infty} F(x) = 0$