Notes on Probability Theory
18.175

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1 Preliminaries

Definition 1.1. A **probability space** is a triple $(\Omega, \mathcal{F}, P)$ where Ω is the **sample space**, $\mathcal{F}$ is a σ-algebra of Ω (**events**) and $P : \mathcal{F} \to [0, 1]$ is the **probability function**.

Remark 1.1. $(\Omega, \mathcal{F})$ is a **measurable space** or **Borel space**.

Definition 1.2. A **algebra**, $\mathcal{A}$, of set X is a collection of subsets of X closed under complementation and *finite* union.

Definition 1.3. A **σ-algebra** of set X is a collection of subsets of X closed under complementation and *countable* union.

Definition 1.4. A **semi-algebra** $\mathcal{S}$ is a collection of sets closed under intersection such that $S \in \mathcal{S}$ implies that S^c is a *finite disjoint* union of sets in $\mathcal{S}$.

Lemma 1.1. If $\mathcal{S}$ is a semi-algebra, then the set S^c of *finite disjoint* unions of sets in $\mathcal{S}$ is an algebra, called the **algebra generated by** $\mathcal{S}$.

Definition 1.5. A **measure** on algebra is a function $\mu : \mathcal{A} \to \mathbb{R}$ such that

(i) $\mu(A) \geq \mu(\emptyset) = 0\ \forall A \in \mathcal{A}$,

(ii) and countably additive for *disjoint* set $\{A_i\}_i$

$$\mu(\cup_i A_i) = \sum_i \mu(A_i) \tag{1.1}$$

Definition 1.6. A measure μ on $\mathcal{F}$ is a **probability measure** if $\mu(\Omega) = 1$.

Definition 1.7. The **Borel σ-algebra** $\mathcal{B}$ on a topological space is the smallest σ-algebra *containing all open sets.*

Theorem 1.1. For each *right continuous, non-decreasing* function F such that $\lim_{x\to-\infty} F = 0$ and $\lim_{x\to\infty} F = 1$, there is an *unique* measure defined on the Borel sets of $\mathbb{R}$ with

$$P((a,b]) \equiv F(b) - F(a) \tag{1.2}$$

Definition 1.8. A collection $\mathcal{P}$ of sets is a π**-system** is it's closed under intersection.

Definition 1.9. A collection of sets $\mathcal{L}$ is a λ**-system** if

2 Random Variables

Definition 2.1. A **measurable space** is a tuple (S, Σ) where Σ is a σ-algebra on S.

Definition 2.2. Let (X, Σ) and (Y, Π) be two measurable spaces, and function $f : X \to Y$ is a **measurable function** if

$$\forall \mathcal{E} \in \Pi,\ f^{-1}(\mathcal{E}) \in \Sigma$$

Denoted as $f : (X, \Sigma) \to (Y, \Pi)$.

Definition 2.3. A **random variable** is a measurable function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$. We say X is $\mathcal{F}$ measurable.