MAT237 Lecture Notes

A Compact Version of Notes by Tyler Holden

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1 The Topology of $\mathbb{R}^n$

2 Sets and Notataion

Definition 2.1. A **set** is any collection of distinct objects.

Definition 2.2. Let S be a set and A and $B \subseteq S$, the binary operator **union** is defined as

$$A \cup B = \{x \in S : x \in A \vee x \in B\}$$

Definition 2.3. Let S be a set and A and $B \subseteq S$, the binary operator **intersection** is defined as

$$A \cap B = \{x \in S : x \in A \wedge x \in B\}$$

Definition 2.4. Let S be a set and $A \subseteq S$, then the **complement** of A with respect to S is defined as

$$A^c = \{x \in S : x \notin A\}$$

Definition 2.5. The **Cartesian Product** of two sets A and B is the collection of ordered pairs, one from A and one from B, denoted as

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

Definition 2.6. Let $f : A \to B$ be a function, then

1. If $U \subseteq A$ then we define the **image** of U as

$$f(U) = \{y \in B : \exists x \in U s.t. \ f(x) = y\} = \{f(x) : x \in U\}$$

2. If $V \subseteq B$ then we define the **pre-image** of V as

$$f^{-1}(V) = \{x \in A : f(x) \in V\}$$

Definition 2.7. Let $f : A \to B$ be a function, we say that

1. f is **injective** if and only if

$$f(x) = f(y) \implies x = y, \ \forall x, y \in A$$

2. f is **surjective** if and only if

$$\forall y \in B, \exists x \in A \ s.t. \ f(x) = y.$$

3. f is **bijective** if and only if it is both injective and surjective.

2.1 Structures on $\mathbb{R}^n$

Definition 2.8. The **Euclidean inner product**, also know as **dot product**. Given two vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ we write

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} := \sum_{i=1}^{n} x_i y_i$$

Proposition 2.1. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then the inner product satisfies

1. **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.
2. **Non-negative:** $\langle \vec{x}, \vec{x} \rangle \geq 0$ and $\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$.
3. **Linearity:** $\langle c\vec{x} + \vec{y}, \vec{z} \rangle = c\langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

Theorem 2.1. (**Cauchy-Schwarz inequality**) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ then

$$|\langle \vec{x}, \vec{y} \rangle| \leq ||\vec{x}|| ||\vec{y}||$$

Proposition 2.2. Let $\vec{x}, \vec{y} \in R^n$ and $c \in \mathbb{R}$, the norm $||\cdot||$ satisfies the following properties:

1. **Non-degeneracy:** $||\vec{x}|| \geq 0$ and $||\vec{x}|| = 0 \iff \vec{x} = \vec{0}$.
2. **Normality:** $||c\vec{x}|| = |c|||\vec{x}||$.
3. **Triangle Inequality:** $||\vec{x}|| + ||\vec{y}|| \geq ||\vec{x} + \vec{y}||$

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