

MAT237: Multivariable Calculus

Tianyu Du

February 1, 2019

Contents

1	Limits, continuity, and related topics	2
2	Differentiation and related topics	2
2.1	2
2.2	2
2.3	2
2.4	2
2.5	2
2.6	2
2.7	2
2.8	Optimization	2
3	The Implicit and Inverse Function Theorems	3
3.1	The Implicit Function Theorem I	3
3.2	Geometric content of the Implicit Function Theorem	3
3.3	Transformations, and the Inverse Function Theorem	4
4	Integration	5
4.1	Basics	5
4.2	Integration on Higher Dimensions	5
4.3	Iterated Integrals	7

1 Limits, continuity, and related topics

2 Differentiation and related topics

2.1

2.2

2.3

2.4

2.5

2.6

2.7

2.8 Optimization

Theorem 2.8.1. Let $S \subset \mathbb{R}^n$ be an open set and $f, g : S \rightarrow \mathbb{R}$ be C^1 functions. If \mathbf{x} is a *local extremal* satisfying $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{x}) \neq \mathbf{0}$, then

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = 0 \end{cases} \quad (2.8.1)$$

Lemma 2.8.1. $\nabla g(\mathbf{x})$ is orthogonal to the constraint set $g^{-1}(0)$.

Proposition 2.8.1. Equations (2.8.1) $\implies \nabla f(\mathbf{x}) \perp g^{-1}(0)$ at \mathbf{x} .

Theorem 2.8.2. Let $S \subseteq \mathbb{R}^n$ be an open set, and $f, \{g_i\}_{i=1}^k : S \rightarrow \mathbb{R}$ be C^1 functions. Define $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_k(\mathbf{x}))$.

If $\mathbf{x} \in S$ is a *local extremal* of f such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, and $\{\nabla g_i(\mathbf{x})\}$ are linearly independent (i.e. $\text{rank}(D\mathbf{g}(\mathbf{x})) = k$), then

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^k \text{ s.t. } \begin{cases} \nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases} \quad (2.8.2)$$

Remark 2.8.1. Procedure of optimization on *open sets*:

- (i) Find all critical points.
- (ii) Find optimizers among critical points.

Remark 2.8.2. Procedure of optimization with *inequality constraints*:

- (i) Find critical points without the constraints.
- (ii) Find critical points on the constraints.
- (iii) Find optimizers among candidates.

3 The Implicit and Inverse Function Theorems

3.1 The Implicit Function Theorem I

Theorem 3.1.1 (Implicit Function Theorem). Let $S \subseteq \mathbb{R}^{n+k}$ be an open set, and function $F : S \rightarrow \mathbb{R}^k$ be a C^1 function. Suppose there exists point $\mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^k$ such that

$$F(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad (3.1.1)$$

If

$$\det(D_{\mathbf{y}}(F(\mathbf{a}, \mathbf{b}))) \neq 0 \quad (3.1.2)$$

then there exists $r_0, r_1 > 0$ and a C^1 function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$\forall \mathbf{x} \in \mathcal{B}(r_0, \mathbf{a}), \mathbf{f}(\mathbf{x}) \in \mathcal{B}(r_1, \mathbf{b}) \wedge F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0} \quad (3.1.3)$$

and define $\mathbf{y} \equiv \mathbf{f}(\mathbf{x})$, the derivative of \mathbf{f} can be found as

$$D\mathbf{f}(\mathbf{x}) = -[D_{\mathbf{y}}F(\mathbf{x}, \mathbf{y})]^{-1}D_{\mathbf{x}}F(\mathbf{x}, \mathbf{y}) \quad (3.1.4)$$

Remark 3.1.1. Procedure to prove solvability of non-linear equations

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (3.1.5)$$

near (\mathbf{a}, \mathbf{b}) .

(i) Verify $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

(ii) Assert

$$\det(D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})) \neq 0 \quad (3.1.6)$$

(iii) Approximate solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ using

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) \approx \mathbf{a} + D\mathbf{f}(\mathbf{a})\mathbf{h} \quad (3.1.7)$$

$$= \mathbf{a} - [D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \quad (3.1.8)$$

3.2 Geometric content of the Implicit Function Theorem

Definition 3.1. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a} \in S$. S is **singular** at \mathbf{a} if

$$\forall r > 0 \ S \cap \mathcal{B}(r, \mathbf{a}) \text{ cannot be represented as a } C^1 \text{ graph.} \quad (3.2.1)$$

S is **regular** at \mathbf{a} if it is not singular there.

Theorem 3.2.1 (k dimensional manifold as level set). Let $U \subseteq \mathbb{R}^n$ and let $\mathbf{F} : U \rightarrow \mathbb{R}^{n-k}$ be a C^1 function.

$$S \equiv \mathbf{F}^{-1}(\mathbf{0}) \quad (3.2.2)$$

Let $\mathbf{a} \in U$, if

$$\text{rank}(D\mathbf{F}(\mathbf{a})) = n - k \quad (3.2.3)$$

then $\exists r > 0$ such that the *level set* of \mathbf{F} near \mathbf{a}

$$\mathcal{B}(r, \mathbf{a}) \cap S \quad (3.2.4)$$

can be represented as a C^1 graph.

Theorem 3.2.2 (k dimensional manifold as parameterization). Let $T \subseteq \mathbb{R}^k$ and let $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be a C^1 function.

$$S \equiv \mathbf{f}(T) \quad (3.2.5)$$

Let $\mathbf{t} \in T$, if

$$\text{rank}(\mathbf{f}(\mathbf{t})) = k \quad (3.2.6)$$

then $\exists r > 0$ such that the *parameterization of \mathbf{f} near \mathbf{t}*

$$\mathbf{f}(T \cap \mathcal{B}(r, \mathbf{t})) \quad (3.2.7)$$

can be represented as a C^1 graph.

3.3 Transformations, and the Inverse Function Theorem

Example 3.3.1 (Polar coordinate in \mathbb{R}^2). Let

$$U \equiv \{(r, \theta) : r > 0 \wedge \theta \in (-\pi, \pi)\} \quad (3.3.1)$$

$$V \equiv \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \quad (3.3.2)$$

Define $\mathbf{f} : U \rightarrow V$ as

$$\mathbf{f}(r, \theta) \equiv \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} \quad (3.3.3)$$

Example 3.3.2 (Spherical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, \varphi) = \begin{pmatrix} r \cos(\theta) \sin(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\theta) \end{pmatrix} \quad (3.3.4)$$

Example 3.3.3 (Cylindrical coordinate in \mathbb{R}^3). Define

$$\mathbf{f}(r, \theta, z) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \\ z \end{pmatrix} \quad (3.3.5)$$

Theorem 3.3.1 (Inverse Function Theorem). Let U and V be open subsets in \mathbb{R}^n , and $\mathbf{f} : U \rightarrow V$. Let $\mathbf{a} \in U$ and define $\mathbf{b} \equiv \mathbf{f}(\mathbf{a}) \in V$. If

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0 \quad (3.3.6)$$

then there exists $M \subseteq U$ and $N \subseteq V$ such that

- (i) $\mathbf{a} \in M$ and $\mathbf{b} \in N$,
- (ii) \mathbf{f} is bijective between M and N ,
- (iii) $\mathbf{f}^{-1} : N \rightarrow M$ is C^1 ,

and **for all $\mathbf{x} \in M$** such $\mathbf{y} \equiv \mathbf{f}(\mathbf{x}) \in N$,

$$D\mathbf{f}^{-1}(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1} \quad (3.3.7)$$

4 Integration

4.1 Basics

Theorem 4.1.1 (**Properties of infimum and supremum**). Let $A \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, and $f, g : A \rightarrow \mathbb{R}$ are bounded functions. Let m and M denote the infimum and supremum respectively, then

- (i) $m_A f + m_A g \leq m_A(f + g) \leq M_A(f + g) \leq M_A f + M_A g$
- (ii) If $A' \subseteq A$, then $m_A f \leq m_{A'} f \leq M_{A'} f \leq M_A f$
- (iii) If $f(\mathbf{x}) \leq g(\mathbf{x}) \forall \mathbf{x} \in A$, then $m_A f \leq m_A g$ and $M_A f \leq M_A g$
- (iv) $|M_A f| \leq M_A |f|$
- (v) $M_A |f| - m_A |f| \leq M_A f - m_A f$
- (vi) $\forall c \in \mathbb{R}, M_A(cf) - m_A(cf) = |c|(M_A f - m_A f)$
- (vii) $M_A f - m_A f = \sup\{f(x) - f(y) : x, y \in A\}$

4.2 Integration on Higher Dimensions

Definition 4.1. A **rectangle** $\mathcal{R} \subseteq \mathbb{R}^n$ is defined as

$$\mathcal{R} \equiv \prod_{i=1}^n [a_i, b_i] \quad (4.2.1)$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.

Definition 4.2. A **partition** P of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ is a list of n **finite** and increasing list of real numbers

$$P = \{L_1, L_2, \dots, L_n\} \quad (4.2.2)$$

where $L_i = \{e_j\}_{j=0}^{T_i}$ such that

$$a_i = e_0 < e_1 < \dots < e_{T_i} = b_i \quad (4.2.3)$$

and such partition induces a set of rectangles(boxes) $\mathcal{B}(P) \equiv \{B_j\}_{j=1}^J \subseteq \mathcal{R}$.

Definition 4.3. Let P and P' be two partitions of \mathcal{R} . Then P' is a **refinement** of P if

$$\forall B_j \in \mathcal{B}(P), B'_j \in \mathcal{B}(P') \quad B'_j \subseteq B_j \vee B'_j{}^{int} \cap B_j{}^{int} = \emptyset \quad (4.2.4)$$

Definition 4.4. Define the **volume** of rectangle $\mathcal{R} = \prod_{i=1}^n [a_i, b_i]$ as

$$V^n(\mathcal{R}) \equiv \prod_{i=1}^n (b_i - a_i) \quad (4.2.5)$$

Definition 4.5. The **lower Riemann sum** of f with partition P on \mathcal{R} is defined as

$$L_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \inf_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.6)$$

and the **upper Riemann sum** is defined as

$$U_P f \equiv \sum_{B_j \in \mathcal{B}(P)} \sup_{\mathbf{x} \in B_j} f(\mathbf{x}) V^n(B_j) \quad (4.2.7)$$

Definition 4.6. The **upper integral** and **lower integral** of f on \mathcal{R} are defined as

$$\bar{I}_{\mathcal{R}}f \equiv \inf_P U_P f \quad (4.2.8)$$

$$\underline{I}_{\mathcal{R}}f \equiv \sup_P L_P f \quad (4.2.9)$$

Definition 4.7. A bounded real-valued function f defined on \mathcal{R} is **integrable** if

$$\underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.10)$$

and the integral is defined as

$$\int \cdots \int_{\mathcal{R}} f \, dV^n \equiv \underline{I}_{\mathcal{R}}f = \bar{I}_{\mathcal{R}}f \quad (4.2.11)$$

Lemma 4.2.1. Let f be a bounded real-valued function defined on \mathcal{R} , f is integrable if and only if $\forall \epsilon > 0$, there exists a partition P of \mathcal{R} such that

$$U_P f - L_P f < \epsilon \quad (4.2.12)$$

Theorem 4.2.1. Let f and g be two integrable functions on $\mathcal{R} \subseteq \mathbb{R}^n$, let $c \in \mathbb{R}$,

- (i) $f + g : \mathcal{R} \rightarrow \mathbb{R}$ is integrable and $\int_{\mathcal{R}}(f + g) = \int_{\mathcal{R}} f + \int_{\mathcal{R}} g$
- (ii) $c \cdot f$ is integrable and $\int_{\mathcal{R}} c \cdot f = c \int_{\mathcal{R}} f$
- (iii) $f(\mathbf{x}) \geq g(\mathbf{x}) \, \forall \mathbf{x} \in \mathcal{R} \implies \int_{\mathcal{R}} f \geq \int_{\mathcal{R}} g$
- (iv) $|f|$ is integrable and $|\int_{\mathcal{R}} f| \leq \int_{\mathcal{R}} |f|$

Definition 4.8. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S , the **indicator function** of S is $\chi_S : \mathcal{R} \rightarrow \{0, 1\}$, defined as

$$\chi_S(\mathbf{x}) \equiv \mathbb{I}(\mathbf{x} \in S) \quad (4.2.13)$$

Definition 4.9. Let $S \subseteq \mathbb{R}^n$ be a bounded set, and there exists rectangle \mathcal{R} covers S . Let $f : \mathcal{R} \rightarrow \mathbb{R}$ be a bounded function, then f is **integrable on S** if $\chi_S f$ is integrable on \mathcal{R} . And

$$\int \cdots \int_S f \, dV^n \equiv \int \cdots \int_{\mathcal{R}} \chi_S f \, dV^n \quad (4.2.14)$$

Definition 4.10. Let $Z \subseteq \mathbb{R}^n$, Z has **zero content** if for all $\epsilon > 0$, there exists a finite set of rectangles $\{R_\ell\}_{\ell=1}^L$ covers Z and

$$\sum_{\ell=1}^L V^n(R_\ell) < \epsilon \quad (4.2.15)$$

Proposition 4.2.1. Let $Z \subseteq \mathbb{R}^n$ has zero content, then

- (i) For any $Z' \subseteq Z$, Z' has zero content.
- (ii) Finite union of content zero sets has zero content.
- (iii) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function, it's graph $\{(x, f(x)) : x \in [a, b]\}$ has zero content.
- (iv) Let $\mathbf{f} : [a, b] \rightarrow \mathbb{R}^2$ be a C^1 function, the parameterization $\mathbf{f}([a, b])$ has zero content.

Theorem 4.2.2. Let \mathcal{R} be a rectangle in \mathbb{R}^n and f is integrable on \mathcal{R} if

$$\{\mathbf{x} \in \mathcal{R} : f \text{ is discontinuous at } \mathbf{x}\} \quad (4.2.16)$$

has zero content.

Proposition 4.2.2 (Folland 4.22). Suppose $Z \subseteq \mathbb{R}^n$ has zero content. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f \, dV^n = 0$.

4.3 Iterated Integrals

Theorem 4.3.1 (Fubini's Theorem). Let $\mathcal{R} = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \mathcal{R} \rightarrow \mathbb{R}$ is bounded. Assuming that

- (i) f is integrable on \mathcal{R} .
- (ii) for each $y \in [c, d]$, the function $f_y(x) \equiv f(x, y)$ is integrable on $[a, b]$.
- (iii) Define $g(y) \equiv \int_a^b f(x, y) dy$ is integrable on $[c, d]$.

Then

$$\iint_{\mathcal{R}} f \, dA = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy \quad (4.3.1)$$