STA347: Probability

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Contents

1 Preliminaries

Definition 1.1. A **standard uniform** is defined to be $\mathcal{U} \sim unif[0,1]$ if and only if

$$P(\mathcal{U} \leq u) = u \ \forall u \in [0,1] \tag{1.1}$$

Definition 1.2. $Z \sim unif\{0, \cdots, p-1\}$ if and only if

$$P(Z = i) = P(Z = j) \quad \forall i, j \in \{0, \cdots, p-1\} \tag{1.2}$$

Theorem 1.1. If $U = \sum_{n=1}^{\infty} Z_i p^{-i}$, then the following are equivalent:

(i) $U \sim unif[0,1]$;

(ii) $Z_i \overset{i.i.d.}{\sim} Z \overset{d}{=} unif\{0, \cdots, p-1\}$.

Definition 1.3. Two random processes X, Y on a common sample space $\mathcal{X}$ are **identically distributed**, $X \overset{d}{=} Y$ if and only if

$$\mathbb{E}[g(X)] = \mathbb{E}[g(Y)] \quad \forall g : \mathcal{X} \to \mathbb{R} \tag{1.3}$$

Proposition 1.1. Specifically, for $A \overset{d}{=} B$, take $g = I_A$ where $A \subset \mathcal{X}$. It is evident that for every such subset, the probability **probability** as

$$\mathbb{P}[X \in A] = \mathbb{E}[I_A(X)] = \mathbb{E}[I_A(Y)] = \mathbb{P}[Y \in A] \tag{1.4}$$

Theorem 1.2 (Invariance)**.** If $X \overset{d}{=} Y$, then

$$\varphi(X) \overset{d}{=} \varphi(Y) \quad \forall \varphi : \mathcal{X} \to \mathcal{Y} \tag{1.5}$$

Proof.

$$\mathbb{E}[h \circ \varphi(X)] = \mathbb{E}[h \circ \varphi(Y)] \quad \forall h : \mathcal{Y} \to \mathbb{R} \tag{1.6}$$

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Definition 1.4. The **expectation** operator

$$\mathbb{E} : \mathcal{R} \to \mathbb{R} \cup \{\pm\infty\} \cup \{\text{DNE}\} \tag{1.7}$$

where $\mathcal{R}$ is the space of *real-valued* random processes.

Proposition 1.2. Let $W \sim unif\{1, \cdots, n\}$, then

$$
\begin{aligned}
& n + 1 - W \overset{d}{=} W && (1.8)\\
\implies & (n+1-W)^2 \overset{d}{=} W^2 && (1.9)\\
\implies & (n+1)^2 - 2(n+1)W + W^2 \overset{d}{=} W^2 && (1.10)\\
\implies & \mathbb{E}[(n+1)^2 - 2(n+1)W + W^2] = \mathbb{E}[W^2] && (1.11)\\
\implies & \mathbb{E}[W] = \frac{n+1}{2} && (1.12)
\end{aligned}
$$

Proposition 1.3.

$$(n+1-W)^3 \overset{d}{=} W^3 \tag{1.13}$$

$$\implies 2\mathbb{E}[W^3] = (n+1)^3 - 3(n+1)^2\mathbb{E}[W] + 3(n+1)\mathbb{E}[W^2] \tag{1.14}$$

$$\implies 2\mathbb{E}[W^3] = (n+1)^3 - 3(n+1)^2\frac{n+1}{2} + 3(n+1)\mathbb{E}[W^2] \tag{1.15}$$

$$\implies 2\mathbb{E}[W^3] = -\frac{(n+1)^2}{2} + 3(n+1)\mathbb{E}[W^2] \tag{1.16}$$

$$\implies \mathbb{E}[W^3] = n(\mathbb{E}[W])^2 \tag{1.17}$$

Proposition 1.4. $\mathbb{E}[W^4]$. TODO

Definition 1.5. $W \sim unif\{1, \cdots, n\}$, then the *distance between* W^2 and $\mathbb{E}[W^2]$ is defined as

$$d(W^2, \mathbb{E}[W^2]) := \sqrt{\mathbb{E}[W^2 - \mathbb{E}[W])^2]^2} = \sqrt{\mathbb{V}[W^2]} = \sigma_{W^2} \tag{1.18}$$

Corollary 1.1 (Corollary of Jensen's Inequality)**.**

$$\mathbb{E}[W^2] \geq (\mathbb{E}[W])^2 \tag{1.19}$$

and equality holds <u>if and only if</u>

$$\mathbb{E}[(W - \mathbb{E}[W])^2] = 0 \tag{1.20}$$

which is equivalent to

$$P(W = \mathbb{E}[W]) = 1 \tag{1.21}$$

Proof.

$$\mathbb{V}[W] = \mathbb{E}[(W - \mathbb{E}[W])^2] \geq 0 \tag{1.22}$$

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Lemma 1.1. $u = \sum_{i=1}^{\infty} z_i p^{-i}$, and let $z = (z_i : i \in \mathbb{N}) \in \dot{p}^{\infty}$