

ECO426H1 Market Design: Auctions and Matching Markets

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February 24, 2020

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1 Auctions

Definition 1.1. An **auction** is an informational environment consisting of

- (i) **Bidding format rules:** the form of the bids, which can be price only, multi-attribute, price and quantity, or quantity only;
- (ii) **Bidding process rules:** Closing/timing rules, available information, rules for bid improvements/counter-bids, closing conditions;
- (iii) **Price and allocation rules:** final prices, quantities, winners.

Auctions are commonly referred to as a market mechanism as well as a price discovery mechanism

Definition 1.2. A **market mechanism** uses prices to determine allocations.

Definition 1.3. An auction is a **private value** auction if agents' valuations do not depend on other buyers' valuations. Otherwise, the auction is called a **interdependent / common value** auction.

1.1 Private Value Auctions

Assumption 1.1. In this chapter, we shall impose the following assumption on bidders' valuations:

- (i) Each bidder's valuation is independently and identically distributed on some interval $[0, \omega]$ according to a distribution function F :

$$V_i \stackrel{i.i.d.}{\sim} F \text{ s.t. } \text{supp}(F) = \mathbb{R}_+ \quad (1.1)$$

- (ii) F belongs to the common knowledge in this system;
- (iii) Bidders' valuations have finite expectations:

$$\mathbb{E}[V_i] < \infty \quad (1.2)$$

Assumption 1.2. Moreover, we assume bidders' behaviours to satisfy the following properties:

- (i) Bidders are risk neutral, they are maximizing expected profits;
- (ii) Each bidder is both willing and able to pay up to his or her value.

Definition 1.4. A **strategy** of a bidder is a mapping from the space of his/her valuation to a bid:

$$s : [0, \omega] \rightarrow \mathbb{R}_+ \quad (1.3)$$

Definition 1.5. An equilibrium of auction is **symmetric** if all bidders are following the same bidding strategy s .

Definition 1.6. A bidder is **bidding sincerely / truthfully** if he bids his true value.

Proposition 1.1. In a symmetric equilibrium of the second-price auction, $s(v) = v$ is a weakly dominant strategy.

Proof. For a fixed valuation $v_i \in [0, \omega]$ of bidder i .

Let $p := \max_{j \neq i} b_j$ be highest bidding price by other bidders.

Let $\pi_i(b, p)$ denote bidder i 's profit when bidding b given the highest price from other bidders to be p .

Part 1: consider another bidding $z_i < v_i$, the following cases are possible:

(i) $v_i < p \implies z_i < v_i < p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i losses anyway).

(ii) $v_i = p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = 0$ (bidder i is indifferent).

(iii) $v_i > p$:

(a) $v_i > z_i > p \implies \pi_i(v_i, p) = \pi_i(z_i, p) = v_i - p$;

(b) $v_i > z_i = p \implies \pi_i(v_i, p) \geq \pi_i(z_i, p)$;

(c) $v_i > p > z_i \implies \pi_i(v_i, p) > \pi_i(z_i, p)$.

Hence, bidding v_i weakly dominates bidding any value below it.

Part 2: for $z_i > v_i$, the following cases are possible:

(i)

Therefore, bidding v_i weakly dominates bidding any other values. ■

Proposition 1.2. In a symmetric equilibrium of the first-price auction, equilibrium bidding strategies are given by

$$s(v_i) = \mathbb{E}[\max_{j \neq i} v_j | v_j \leq v_i] \quad (1.4)$$

which is the *expected second highest valuation conditional on v_i being the highest valuation*.

Proof. Let $s(v)$ denote an equilibrium strategy.

Lemma 1.1. For any agent, bidding more than $s(\omega)$ can never be optimal. Bidding $b > s(\omega)$ makes this agent win for sure. In such case, bidding $b' \in (s(\omega), b)$ strictly dominates bidding b .

Lemma 1.2. For any agent, $s(0) = 0$. Bidding any positive number would cause negative payoff with positive probability, and therefore, leads to a negative expected profit.

Lemma 1.3. Because s is monotonically increasing, therefore,

$$\max_{j \neq i} s(v_j) = s(\max_{j \neq i} v_j) \quad (1.5)$$

Let p denote the highest price among all other $N - 1$ bidders and let $F^{(N-1)}(x)$ denote the distribution of p .

The expected profit of bidder i by bidding an arbitrary $b \in \mathbb{R}_+$ is

$$\pi_i(b, v_i) = P(b > p)(v_i - s(v_i)) + P(b = p)(v_i - s(v_i)) + P(b < p)0 \quad (1.6)$$

Note that $b > p = s(\max_{j \neq i} v_j)$ if and only if $s^{-1}(b) > \max_{j \neq i} v_j$. It follows

$$P(b > p) = P(\max_{j \neq i} v_j < s^{-1}(b)) = F^{(N-1)}(s^{-1}(b)) \quad (1.7)$$

Therefore,

$$\pi_i(b, v_i) = F^{(N-1)}(s^{-1}(b))(v_i - b) \quad (1.8)$$

The first order condition implies

$$\frac{\partial \pi_i}{\partial b} \pi_i(b, v_i) = \frac{\partial \pi_i}{\partial b} F^{N-1}(s^{-1}(b))v_i - F^{N-1}(s^{-1}(b))b \quad (1.9)$$

$$= f^{(N-1)}(s^{-1}(b)) \frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \quad (1.10)$$

For a symmetric equilibrium, all other bidders are following the same strategy s so that $s(v_i) = b$, therefore,

$$f^{(N-1)}(s^{-1}(b)) \frac{v_i - b}{s'(v_i)} - F^{(N-1)}(s^{-1}(b)) = 0 \quad (1.11)$$

$$\implies f^{(N-1)}(s^{-1}(b))(v_i - b) - F^{(N-1)}(s^{-1}(b))s'(v_i) = 0 \quad (1.12)$$

$$\implies f^{(N-1)}(s^{-1}(b))v_i = F^{(N-1)}(s^{-1}(b))s'(v_i) + f^{(N-1)}(s^{-1}(b))s(v_i) \quad (1.13)$$

$$\implies f^{(N-1)}(v_i)v_i = \frac{d}{dv_i} \left[F^{(N-1)}(v_i)s(v_i) \right] \quad (1.14)$$

$$\implies \int_0^{v_i} f^{(N-1)}(y)y \, dy = F^{(N-1)}(v_i)s(v_i) - F^{(N-1)}(0)s(0) \quad (1.15)$$

$$\implies F^{(N-1)}(v_i)s(v_i) = \int_0^{v_i} f^{(N-1)}(y)y \, dy \quad (1.16)$$

$$\implies s(v_i) = \frac{1}{F^{(N-1)}(v_i)} \int_0^{v_i} f^{(N-1)}(y)y \, dy \quad (1.17)$$

$$\implies s(v_i) = \mathbb{E} \left[\max_{j \neq i} v_j \mid \max_{j \neq i} v_j < v_i \right] \quad (1.18)$$

■

2 First Price Private Value Auction

Remark 2.1. For every continuous distribution F , the probability for a tie to happen is zero. Therefore, we ignore the tie for now.

Problem Setup Let N denote the set of bidders such that $|N| = n$. For each bidder $i \in N$, his valuation of the auctioned item V_i follows some distribution F . Further assume that $V_i \perp V_j$ for every $i \neq j$.

Let $W(b, v_i)$ denote the event that player i , who has valuation v_i , wins by bidding $b \in \mathbb{R}_+$, define

$$W(b, v_i) \iff b > \max_{j \neq i} b_j \quad (2.1)$$

The payoff (utility) of bidder i , who has valuation v_i , is

$$U(b, v_i) = \begin{cases} v_i - b & \text{if } W(b, v_i) \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

2.1 Symmetric Equilibrium Behaviour

Consider a symmetric environment such that all bidders are using the same strictly increasing strategy $s(\cdot)$ such that $s(\cdot)$ is invertible.

Equilibrium Strategy when $F = Unif(0, 1)$.

$$\beta^I(v) = \frac{n-1}{n}v \quad (2.3)$$

Probability of Winning

$$P(W(b, v_i)) = P(b > \max_{j \neq i} s(v_j)) \quad (2.4)$$

$$= P(b > s(\max_{j \neq i} v_j)) \quad (2.5)$$

$$= P(\max_{j \neq i} v_j \leq s^{-1}(b)) \quad (2.6)$$

$$= F(s^{-1}(b))^{n-1} \quad (2.7)$$

$$= F(v_i)^{n-1} \text{ because } b = s(v_i) \quad (2.8)$$

When $F = Unif(0, 1)$,

$$P(W(b, v_i)) = v_i^{n-1} \quad (2.9)$$

Expected Payment from Bidder i with v_i Conditioned on Winning Suppose bidder i is following strategy $s(\cdot)$. Then,

$$\mathbb{E}[Payment_i | v_i, W(b, v_i)] = b = s(v_i) \quad (2.10)$$

When $F = Unif(0, 1)$,

$$\mathbb{E}[Payment_i | v_i, W(b, v_i)] = \frac{n-1}{n}v_i \quad (2.11)$$

Unconditional Payment from Bidder i with v_i

$$\mathbb{E}[Payment_i | v_i] = P(W(b, v_i))\mathbb{E}[Payment_i | v_i, W(b, v_i)] + P(Loss) \times 0 \quad (2.12)$$

$$= P(W(b, v_i))\mathbb{E}[Payment_i | v_i, W(b, v_i)] \quad (2.13)$$

$$= F(v_i)^{n-1}s(v_i) \quad (2.14)$$

When $F = Unif(0, 1)$,

$$\mathbb{E}[Payment_i | v_i] = \frac{n-1}{n}v_i^n \quad (2.15)$$

Expected Payoff of Bidder i with v_i

$$\mathbb{E}[U | v_i] = P(W(s(v_i), v_i))v_i - \mathbb{E}[Payment_i | v_i] \quad (2.16)$$

$$= F(v_i)^{n-1}v_i - F(v_i)^{n-1}s(v_i) \quad (2.17)$$

$$= F(v_i)^{n-1}[v_i - s(v_i)] \quad (2.18)$$

When $F = Unif(0, 1)$,

$$\mathbb{E}[U | v_i] = \frac{v_i^n}{n} \quad (2.19)$$

Unconditional Payment from Bidder i This is the same as the expected revenue from bidder i :

$$\mathbb{E}[Payment_i] = \int_{\mathbb{R}_+} \mathbb{E}[Payment_i|v_i]dF \quad (2.20)$$

$$= \int_{\mathbb{R}_+} F(v_i)^{n-1} s(v_i) f(v_i) dv_i \quad (2.21)$$

When $F = Unif(0, 1)$,

$$\mathbb{E}[Payment_i] = \int_0^1 \frac{n-1}{n} v_i^n dv_i \quad (2.22)$$

$$= \frac{n-1}{n(n+1)} \quad (2.23)$$

Auctioneer's Expected Revenue Since all bidders are the same,

$$\mathbb{E}[Revenue] = n \mathbb{E}[Payment_i] \quad (2.24)$$

$$= n \int_{\mathbb{R}_+} F(v_i)^{n-1} s(v_i) f_i dv_i \quad (2.25)$$

When $F = Unif(0, 1)$,

$$\mathbb{E}[Revenue] = \frac{n-1}{n+1} \quad (2.26)$$

3 Second Price Private Value Auction

4 The General Case: k^{th} Price Private Value Auction

5 Common Value Auction

6 Combinatorial Auction: The VCG Mechanism

7 Matching Market

8 Appendix A: Order Statistics

Definition 8.1. Let (X_1, \dots, X_n) be n random variables on the probability space (Ω, \mathcal{F}, P) , further assume they are iid following distribution function $F(\cdot)$. For each $\omega \in \Omega$, realizations of above random variables can be sorted as

$$X_{(n)}(\omega) \leq X_{(n-1)}(\omega) \leq \dots \leq X_{(1)}(\omega) \quad (8.1)$$

For each ω , the random variable $X_{n:k}$ is defined such that $X_{n:k}(\omega)$ equals the k -th largest value, $X_{(k)}(\omega)$.

Distribution Function Let $x \in X(\Omega)$, then

$$X_{n:k} \leq x \iff (\text{no } X_i > x) \bigcup (\text{exactly } 1 \text{ } X_i > x) \bigcup \cdots \bigcup (\text{exactly } k-1 \text{ } X_i > x) \quad (8.2)$$

$$\iff (X_i \leq x \ \forall i) \bigcup (\text{exactly } n-1 \text{ } X_i \leq x) \bigcup \cdots \bigcup (\text{exactly } n-k+1 \text{ } X_i \leq x) \quad (8.3)$$

$$\iff \bigcup_{j=n-k+1}^n (\text{exactly } j \text{ } X_i \leq x) \quad (8.4)$$

Note that events in the union are mutually exclusive, therefore,

$$F_{n:k}(x) = P(X_{n:k} \leq x) = \sum_{j=n-k+1}^n P(\text{exactly } j \text{ } X_i \leq x) \quad (8.5)$$

$$= \sum_{j=n-k+1}^n \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \quad (8.6)$$

Density Function

$$f_{n:k}(x) = \frac{d}{dx} F_{n:k}(x) \quad (8.7)$$

$$= \frac{d}{dx} \sum_{j=n-k+1}^n \binom{n}{j} F(x)^j (1-F(x))^{n-j} \quad (8.8)$$

$$= \frac{d}{dx} \sum_{j=n-k+1}^n \frac{n!}{j!(n-j)!} F(x)^j (1-F(x))^{n-j} \quad (8.9)$$

$$= \sum_{j=n-k+1}^n \left[\frac{n!}{j!(n-j)!} j F(x)^{j-1} (1-F(x))^{n-j} - \frac{n!}{j!(n-j)!} (n-j) F(x)^j (1-F(x))^{n-j-1} \right] f(x) \quad (8.10)$$

$$= \sum_{j=n-k+1}^n \frac{n!}{j!(n-j)!} j F(x)^{j-1} (1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j)!} (n-j) F(x)^j (1-F(x))^{n-j-1} f(x) \quad (8.11)$$

$$= \sum_{j=n-k+1}^n \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x) - \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j (1-F(x))^{n-j-1} f(x) \quad (8.12)$$

$$= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) \quad (8.13)$$

$$\begin{aligned} &+ \sum_{j=n-k+2}^n \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x) \\ &- \sum_{j=n-k+1}^{n-1} \frac{n!}{j!(n-j-1)!} F(x)^j (1-F(x))^{n-j-1} f(x) \\ &= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) \quad (8.14) \end{aligned}$$

$$\begin{aligned} &+ \sum_{j=n-k+2}^n \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} f(x) \\ &- \sum_{i=n-k+2}^n \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1} (1-F(x))^{n-i} f(x) \text{ (substitute } j=i-1) \\ &= \frac{n!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) \quad (8.15) \end{aligned}$$

$$= n \frac{(n-1)!}{(n-k)!(k-1)!} F(x)^{n-k} (1-F(x))^{k-1} f(x) \quad (8.16)$$

$$= n \binom{n-1}{k-1} F(x)^{n-k} (1-F(x))^{k-1} f(x) \quad (8.17)$$