

Notes on Probability Theory

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1 Preliminaries

Definition 1.1. A **probability space** is a triple (Ω, \mathcal{F}, P) where Ω is the **sample space**, \mathcal{F} is a σ -algebra of Ω (**events**) and $P : \mathcal{F} \rightarrow [0, 1]$ is the **probability function**.

Remark 1.1. (Ω, \mathcal{F}) is a **measurable space** or **Borel space**.

Definition 1.2. A **algebra**, \mathcal{A} , of set X is a collection of subsets of X closed under complementation and *finite* union.

Definition 1.3. A **σ -algebra** of set X is a collection of subsets of X closed under complementation and *countable* union.

Remark 1.2. We can also define *algebra* and *σ -algebra* using closures under complementation and *finite/countable intersection*.

Proof. Use DeMorgan's Law. ■

Definition 1.4. A measure μ on \mathcal{A} is **σ -finite** if there exists *countable* collection $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$ and $\cup A_n = \Omega$.

Definition 1.5. A **semi-algebra** \mathcal{S} is a collection of sets closed under intersection such that $S \in \mathcal{S}$ implies that S^c is a *finite disjoint* union of sets in \mathcal{S} .

Lemma 1.1. Let \mathcal{S} be a semi-algebra, then

$$\overline{\mathcal{S}} = \text{all finite disjoint unions of sets in } \mathcal{S} \tag{1.1}$$

is an algebra, called the **algebra generated by \mathcal{S}** .

Proof. We are going to show the equivalent definition of algebra, that's, $\bar{\mathcal{S}}$ is closed under complementation and finite intersection.

Intersection: Let $A, B \in \bar{\mathcal{S}}$, then by definition of $\bar{\mathcal{S}}$,

$$A = \cup_i A_i \quad A_i \in \mathcal{S} \quad (1.2)$$

$$B = \cup_j B_j \quad B_j \in \mathcal{S} \quad (1.3)$$

Then by definition of semi-algebra, $A_i \cap B_j \in \mathcal{S}$. Then

$$A \cap B = (\cup_i A_i) \cap (\cup_j B_j) \quad (1.4)$$

$$= \cup_{i,j} A_i \cap B_j \in \bar{\mathcal{S}} \quad (1.5)$$

By an inductive argument, we've shown that $\bar{\mathcal{S}}$ is closed under intersection.

Complementation: Let $A \in \bar{\mathcal{S}}$, by definition

$$A = \cup_i A_i \quad A_i \in \mathcal{S} \quad (1.6)$$

Therefore, by DeMorgan's Law, $A^c = \cap_i A_i^c$ and by definition of semi-algebra, for each A_i^c , it's a finite union of disjoint sets in \mathcal{S} .

By definition of $\bar{\mathcal{S}}$, each $A_i^c \in \bar{\mathcal{S}}$. And as shown above, $\bar{\mathcal{S}}$ is closed under finite intersection.

Therefore $A^c \in \bar{\mathcal{S}}$.

So $\bar{\mathcal{S}}$ is closed under complementation.

Therefore $\bar{\mathcal{S}}$ is an algebra. ■

Definition 1.6. A **measure** on algebra is a function $\mu : \mathcal{A} \rightarrow \mathbb{R}$ such that

$$(i) \quad \mu(A) \geq \mu(\emptyset) = 0 \quad \forall A \in \mathcal{A},$$

$$(ii) \quad \text{and countably additive for } \textit{disjoint} \text{ set } \{A_i\}_i$$

$$\mu(\cup_i A_i) = \sum_i \mu(A_i) \quad (1.7)$$

Definition 1.7. A measure μ on \mathcal{F} is a **probability measure** if $\mu(\Omega) = 1$.

Definition 1.8. The **Borel σ -algebra** \mathcal{B} on a topological space is the smallest σ -algebra *containing all open sets*.

Theorem 1.1. For each *right continuous, non-decreasing* function F such that $\lim_{x \rightarrow -\infty} F = 0$ and $\lim_{x \rightarrow \infty} F = 1$, there is an *unique* measure defined on the Borel sets of \mathbb{R} with

$$P((a, b]) \equiv F(b) - F(a) \quad (1.8)$$

Definition 1.9. A collection \mathcal{P} of sets is a **π -system** if it's closed under intersection.

Definition 1.10. A collection \mathcal{L} of subsets of Ω is a **λ -system**(Dynkin system) if

$$(i) \quad \Omega \in \mathcal{L}.$$

$$(ii) \quad (\textit{Closed under set difference}) \text{ If } A, B \in \mathcal{L} \wedge A \subseteq B \implies B \setminus A \in \mathcal{L}.$$

$$(iii) \quad (\textit{Contain set sequence limit}) \text{ If } A_n \in \mathcal{L} \text{ and } A_n \uparrow A, \text{ then } A \in \mathcal{L}.$$

Theorem 1.2. If \mathcal{P} is a π -system and \mathcal{L} is a λ -system containing \mathcal{P} , then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$, where $\sigma(\mathcal{A})$ denotes the smallest σ -algebra containing \mathcal{A} .

Theorem 1.3 (Caratheodory Extension Theorem). If μ is a σ -finite measure on an algebra \mathcal{A} , then μ has a *unique* extension to the σ -algebra generated by \mathcal{A} .

2 Random Variables

Definition 2.1. A **measurable space** is a tuple (S, Σ) where Σ is a σ -algebra on S .

Remark 2.1. The definition of *measurable spaces* does not require a specific measure.

Definition 2.2. Let (X, Σ) and (Y, Π) be two measurable spaces, and function $f : X \rightarrow Y$ is a **measurable function** if

$$\forall \mathcal{E} \in \Pi, f^{-1}(\mathcal{E}) \in \Sigma$$

Denoted as $f : (X, \Sigma) \rightarrow (Y, \Pi)$.

Definition 2.3. A **random variable** is a measurable function $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B})$. We say X is \mathcal{F} -measurable.

Theorem 2.1. If $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{A}$ and \mathcal{A} generates \mathcal{S} , then X is a measurable map from (Ω, \mathcal{F}) to (S, \mathcal{S}) .

Definition 2.4. Let $F_X(x) \equiv P(X \leq x)$ be the **distribution function** for X . And write $f = f_X = F'_X$ for the **density function** of X . The distribution function must be

- (i) Non-decreasing
- (ii) Right-continuous
- (iii) $\lim_{x \rightarrow \infty} F(x) = 1$
- (iv) $\lim_{x \rightarrow -\infty} F(x) = 0$