STA447: Stochastic Processes

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Contents

1 Markov Chain Probabilities

Definition 1.1. A **discrete-time, discrete-space, and time-homogenous Markov chain** is a triple of (S, v, p) in which

(i) S represents the *state space*, which is nonempty and countable;

(ii) *initial probability* v, which is a distribution on S;

(iii) and *transition probability* p_{ij}.

Definition 1.2. A Markov chain satisfies the **time-homogenous property** if

$$P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i) = p_{ij} \quad \forall n \in \mathbb{N} \tag{1.1}$$

Definition 1.3. A Markov chain satisfies the **Markov property** if

$$P(X_{n+1} = j | X_n = i_n, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_{n+1} = j | X_n = i_n) \tag{1.2}$$

That is, the chain is *memoryless*.

Proposition 1.1 (Multistep Arrival Probability)**.** Let $m = |S|$ and $\mu_i^{(n)} := P(X_n = i)$ denote the probability that the state ends up at i after n step. By the law of total expectation,

$$\begin{align} P(X_n = i) &= \sum_{j \in S} P(X_n = i, X_{n-1} = j) \tag{1.3} \\ &= \sum_{j \in S} P(X_n = i | X_{n-1} = j) P(X_{n-1} = j) \tag{1.4} \\ &= \sum_{j \in S} P(X_{n-1} = j) p_{ij} \tag{1.5} \\ &= \sum_{j \in S} \mu_j^{(n)} p_{ij} \tag{1.6} \end{align}$$

Let $\mu^{(n)} := \left[\mu_1^{(n)}, \mu_2^{(n)}, \cdots, \mu_m^{(n)}\right] \in \mathbb{R}^{1 \times m}$ and $P = [p_{ij}] \in \mathbb{R}^{m \times m}$. In matrix notation:

$$\mu^{(n)} = \mu^{(n-1)} P \tag{1.7}$$

where $\mu^{(0)} = v = [v_1, v_2, \cdots, v_m]$. Define $P^0 = I_m$, then

$$\mu^{(n)} = vP^n \tag{1.8}$$

Proposition 1.2 (Multistep Transition Probability)**.** Define $p_{ij}^{(n)} := P(X_{m+n} = j | X_m = i)$ to be the probability of arriving state j after n steps, starting from state i. By the time-homogenous property,

$$p_{ij}^{(n)} = P(X_{m+n} = j | X_m = i) \quad \forall m \in \mathbb{N} \tag{1.9}$$

Let $P^{(n)} := [p_{ij}^{(n)}] \in \mathbb{R}^{m \times m}$.
Initial Step: for $n = 1$, $P^{(1)} = P$ by definition.

Inductive Step: for $n \in \mathbb{N}$,

$$\begin{aligned}
p_{ij}^{(n+1)} &= P(X_{n+1} = j | X_0 = i) && (1.10)\\
&= \sum_{k \in S} P(X_{n+1} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i) && (1.11)\\
&= \sum_{k \in S} P(X_{n+1} = j | X_n = k) p_{ik}^{(n)} && (1.12)\\
&= \sum_{k \in S} p_{ik}^{(n)} p_{kj} && (1.13)\\
&= [P^{(n)} P]_{ij} && (1.14)
\end{aligned}$$

Therefore,

$$P^{(n+1)} = P^{(n)} P \tag{1.15}$$

and

$$P^{(n)} = P^n \tag{1.16}$$

Theorem 1.1 (Chapman-Kolmogorov Equations)**.** Let $n = (n_1, n_2, \cdots, n_k)$ be a multi-set of non-negative integers, then

$$P^{(\sum_{i=1}^k n_i)} = \prod_{i=1}^k P^{(n_i)} \quad (\dagger) \tag{1.17}$$

Proof. Prove by induction on the size of multi-set:
Base case is trivial for $k = 1$.
Inductive step for $k > 1$, suppose ($\dagger$) holds for every set of length k, consider another multi-set with length $k+1$: $n' = (n_1, n_2, \cdots, n_k, n_{k+1})$. Let $\delta := \sum_{i=1}^k n_i$.

$$\begin{aligned}
P_{ij}^{(\delta+n_{k+1})} &= P(X_{\delta+n_{k+1}} = j | X_0 = i) && (1.18)\\
&= \sum_{k \in S} P(X_{\delta+n_{k+1}} = j | X_\delta = k, X_0 = i) P(X_\delta | X_0 = i) && (1.19)\\
&= \sum_{k \in S} P(X_{\delta+n_{k+1}} = j | X_\delta = k) P(X_\delta | X_0 = i) && (1.20)\\
&= \sum_{k \in S} P(X_{n_{k+1}} = j | X_0 = k) P(X_\delta = k | X_0 = i) && (1.21)\\
&= \sum_{k \in S} p_{kj}^{n_{k+1}} p_{ik}^{(\delta)} && (1.22)\\
&= [P^{(\delta)} P^{(n_{k+1})}]_{ij} && (1.23)\\
\implies P^{(\delta+n_{k+1})} &= P^{(\delta)} P^{(n_{k+1})} && (1.24)
\end{aligned}$$

■

Corollary 1.1 (Chapman-Kolmogorov Inequality)**.** For every $k \in S$,

$$p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)} \tag{1.25}$$

For $k, \ell \in S$,

$$p_{ij}^{(m+s+n)} \geq p_{ik}^{(m)} p_{k\ell}^{(s)} p_{\ell j}^{(n)} \tag{1.26}$$

Notation 1.1. Let $N(i) := |\{n \geq 1 : X_n = i\}|$ denote the number of arrivals to state i of the chain.

Definition 1.4. Define the **return probability** from state i to j, f_{ij}, as the probability of arriving state j starting from state i. That is,

$$\begin{align} f_{ij} &= P(\exists n \geq 1 \; s.t. \; X_n = j | X_0 = i) \tag{1.27} \\ &= P(N(j) \geq 1 | X_0 = i) \tag{1.28} \end{align}$$

Proposition 1.3. The probability of firstly arriving j, then arriving k (denoted as event E) starting from i equals

$$P(E|X_0 = i) = f_{ij} f_{jk} \tag{1.29}$$

Proof. The proof follows the time-homogenous property. ■

Corollary 1.2.

$$\begin{align} P(N(i) \geq k | X_0 = i) &= (f_{ii})^k \tag{1.30} \\ P(N(j) \geq k | X_0 = i) &= f_{ij} (f_{jj})^{k-1} \tag{1.31} \end{align}$$

Definition 1.5. A state i in a Markov chain is **recurrent** if $f_{ii} = 1$. Otherwise, this state is **transient**.

Theorem 1.2 (Recurrent State Theorem)**.** The following statements are equivalent:

(i) State i is recurrent (i.e., $f_{ii} = 1$);

(ii) $P(N(i) = \infty | X_0 = i) = 1$, that is, state i will be visited infinitely often;

(iii) $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.

The following statements are equivalent:

(a) State i is transient;

(b) $P(N(i) = \infty | X_0 = i) = 0$, that is, state i will only be visited finitely many times;

(c) $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

Proof. We only show the equivalence of $(i) \sim (iii)$, $(a) \sim (c)$ are simply the negation of previous statements. $(i) \iff (ii)$:

$$\begin{align} P(N(i) = \infty | X_0 = i) &= P(\lim_{k \to \infty} N(i) \geq k | X_0 = i) \tag{1.32} \\ &= \lim_{k \to \infty} P(N(i) \geq k | X_0 = i) \tag{1.33} \\ &= \lim_{k \to \infty} (f_{ii})^k = 1 \text{ if and only if } f_{ii} = 1 \tag{1.34} \end{align}$$

$(i) \iff (iii)$:

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} P(X_n = i | X_0 = i) \tag{1.35}$$

$$= \sum_{n=1}^{\infty} \mathbb{E}(1_{X_n=i} | X_0 = i) \tag{1.36}$$

$$= \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{X_n=i} \middle| X_0 = i \right) \tag{1.37}$$

$$= \mathbb{E}(N(i) | X_0 = i) \tag{1.38}$$

$$= \sum_{n=k}^{\infty} kP(N(i) = k | X_0 = i) \tag{1.39}$$

$$= \sum_{n=k}^{\infty} P(N(i) \geq k | X_0 = i) \tag{1.40}$$

$$= \sum_{n=k}^{\infty} (f_{ii})^k \tag{1.41}$$

$$= \infty \text{ if and only if } f_{ii} = 1 \tag{1.42}$$

■

Lemma 1.1 (Stirling's Approximation)**.**

$$n! \approx (n/e)^n \sqrt{2\pi n} \tag{1.43}$$

Proposition 1.4. For simple random walk, if $p = 1/2$, then $f_{ii} = 1$ $\forall i \in S$. Otherwise, all states are transient.

$$\forall i \in S,\ f_{ii} = 1 \iff p = \frac{1}{2} \tag{1.44}$$

Proof. For simplicity, consider state 0 and the series $\sum_{n=1}^{\infty} p_{00}^{(n)}$.
Note that for odd n's, $p_{00}^{(n)} = 0$.

For all even n's such that $n = 2k$,

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{k=1}^{\infty} p_{00}^{(2k)} \tag{1.45}$$

$$= \sum_{k=1}^{\infty} \binom{2k}{k} p^k (1-p)^k \tag{1.46}$$

$$= \sum_{k=1}^{\infty} \frac{2k!}{(k!)^2} p^k (1-p)^k \tag{1.47}$$

$$\approx \sum_{k=1}^{\infty} \frac{(2k/e)^{2k}\sqrt{4\pi k}}{(k^k e^{-k}\sqrt{2\pi k})^2} p^k (1-p)^k \tag{1.48}$$

$$= \sum_{k=1}^{\infty} \frac{2^{2k} k^{2k} e^{-2k} 2\sqrt{\pi k}}{k^{2k} e^{-2k} 2\pi k} p^k (1-p)^k \tag{1.49}$$

$$= \sum_{k=1}^{\infty} \frac{2^{2k}}{\sqrt{\pi k}} p^k (1-p)^k \tag{1.50}$$

$$= \sum_{k=1}^{\infty} \frac{4^k}{\sqrt{\pi k}} p^k (1-p)^k \tag{1.51}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi k}} [4p(1-p)]^k \tag{1.52}$$

When $p = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} k^{-1/2} \tag{1.53}$$

$$= \infty \tag{1.54}$$

When $p \neq \frac{1}{2}$,

$$\sum_{n=1}^{\infty} p_{00}^{(n)} < \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} [4\pi(1-p)]^k \tag{1.55}$$

$$< \infty \tag{1.56}$$

By the recurrent state theorem, $f_{ii} = 1 \iff p = 1/2$.
For other $i \neq 0$, the prove is similar. ■

Theorem 1.3 (f-Expansion)**.**

$$f_{ij} = p_{ij} + \sum_{k \in S \setminus \{j\}} p_{ik} f_{kj} \tag{1.57}$$

Proof.

$$
\begin{aligned}
f_{ij} &= P(\exists n \in \mathbb{Z}_{++} \ s.t. \ X_n = j | X_0 = i) && (1.58)\\
&= \sum_{k \in S} P(\exists n \in \mathbb{Z}_{++} \ s.t. \ X_n = j | X_0 = i, X_1 = k) P(X_1 = k | X_0 = i) && (1.59)\\
&= \sum_{k \in S} P(\exists n \in \mathbb{Z}_{++} \ s.t. \ X_n = j | X_1 = k) P(X_1 = k | X_0 = i) \text{ (Markov Property)} && (1.60)\\
&= \underbrace{P(\exists n \in \mathbb{Z}_{++} \ s.t. \ X_n = j | X_1 = j)}_{=1} P(X_1 = j | X_0 = i) + \sum_{k \neq j} f_{kj} P(X_1 = k | X_0 = i) && (1.61)\\
&= p_{ij} + \sum_{k \neq j} f_{kj} p_{ik} && (1.62)
\end{aligned}
$$

■

Definition 1.6. State i is said to **communicate** with state j, denoted as $i \to j$, if $f_{ij} > 0$.

Proposition 1.5 (Alternative Defintion)**.** The following statements are equivalent:

(i) $i \to j$;

(ii) $\exists m \geq 1, \ s.t. \ p_{ij}^{(m)} > 0$.

Proof. TODO: *Proof.* ■

Definition 1.7. A Markov chain s **irreducible** if $i \to j \ \forall i, j \in S$.

1.1 Recurrence and Transience Equivalence Theorem

Theorem 1.4 (Sum Lemma)**.** If

(i) $i \to k$;

(ii) $\ell \to j$;

(iii) $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$.

Then, $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

Proof. Suppose $i \to k$ and $\ell \to j$, then there exists m and r such that $p_{ik}^{(m)} > 0$ and $p_{\ell j}^{(r)} > 0$.
By the Chapman-Kolmogorov inequality, $p_{ij}^{(m+n+r)} \geq p_{ik}^{(m)} p_{k\ell}^{(n)} p_{\ell j}^{(r)}$.
Then,

$$
\begin{aligned}
\sum_{n=1}^{\infty} p_{ij}^{(n)} &\geq \sum_{n=m+r+1}^{\infty} p_{ij}^{(n)} && (1.63)\\
&= \sum_{s=1}^{\infty} p_{ij}^{(m+s+r)} && (1.64)\\
&\geq \sum_{s=1}^{\infty} p_{ik}^{(m)} p_{k\ell}^{(s)} p_{\ell j}^{(r)} && (1.65)\\
&= p_{ik}^{(m)} p_{\ell j}^{(r)} \sum_{s=1}^{\infty} p_{k\ell}^{(s)} = \infty && (1.66)
\end{aligned}
$$

■

Theorem 1.5. If $i \leftrightarrow k$, then

$$f_{ii} = 1 \iff f_{kk} = 1 \tag{1.67}$$

Proof. TODO: *Proof.* ■

Theorem 1.6 (Case Theorem)**.** For an *irreducible* Markov chain, it is either

(a) a **recurrent** Markov chain: $\forall i \in S,\ f_{ii} = 1$ and $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty\ \forall i, j \in S$;

(b) or a **transient** Markov chain: $\forall i \in S,\ f_{ii} < 1$ and $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty\ \forall i, j \in S$.

Proof. TODO: *Proof.* ■

Theorem 1.7 (Finite Space Theorem)**.** An *irreducible* Markov chain on a *finite* state space is always recurrent.

Proof. Let $i \in S$ (u.i.),

$$\sum_{j \in S} \sum_{n=1}^{\infty} p_{ij}^{(n)} = \sum_{n=1}^{\infty} \sum_{j \in S} p_{ij}^{(n)} \tag{1.68}$$

$$= \sum_{n=1}^{\infty} 1 = \infty \tag{1.69}$$

Because S is finite, $\exists k \in S$ such that $\sum_{n=1}^{\infty} p_{ik}^{(n)} = \infty$. Therefore, all states are recurrent. ■

Theorem 1.8 (Hit-Lemma)**.** Define H_{ij} as the event in which the chain starts from j and visits i without firstly returning to j (*direct path from j to i*) [1]:

$$H_{ij} := \{\exists n \in \mathbb{N}\ s.t.\ X_n = i \wedge X_m \neq j\ \forall m < n\} \tag{1.70}$$

If $j \to i$ with $j \neq i$, then $P(H_{ij} | X_0 = j) > 0$.

Theorem 1.9 (f-Lemma)**.** For all $i, j \in S$, if $j \to i$ and $f_{jj} = 1$, then $f_{ij} = 1$.

Proof. For $i = j$, trivial.
Suppose $i \neq j$, since $j \to i$, then $P(H_{ij} | X_0 = j) > 0$.
Further,

$$P(X_n \neq j\ \forall n \in \mathbb{Z}_{++} | X_0 = j) \geq P(H_{ij} | X_0 = j) P(X_n \neq j\ \forall n \in \mathbb{Z}_{++} | X_0 = i) \tag{1.71}$$

$$\implies 0 = 1 - f_{jj} \geq P(H_{ij} | X_0 = j)(1 - f_{ij}) \tag{1.72}$$

$$\implies f_{ij} = 1 \tag{1.73}$$

■

[1] Notation abuse: H_{ij} describes the event starting from j and ending at i, instead of the other way round.

Theorem 1.10 (Infinite Returns Lemma)**.** For an *irreducible* Markov chain,

(i) if this chain is recurrent, then $P(N(j) = \infty | X_0 = i) = 1 \; \forall i, j \in S$;

(ii) if this chain is transient, then $P(N(j) = \infty | X_0 = i) = 0 \; \forall i, j \in S$.

Proof. Let $i, j \in S$.
Suppose the chain is irreducible and recurrent, if $i = j$, then $f_{ii} = f_{jj} = 1$.
Otherwise, $i \neq j$. Since $j \to i$, by the f-Lemma, $f_{jj} = f_{ii} = f_{ij} = f_{ji} = 1$.

$$P(N(j) = \infty | X_0 = i) = \lim_{k \to \infty} P(N(j) \geq k | X_0 = i) \tag{1.74}$$
$$= \lim_{k \to \infty} f_{ij} f_{jj}^{k-1} \tag{1.75}$$
$$= 1 \tag{1.76}$$

When the chain is transient, $f_{jj} < 1$, and $\lim_{k \to \infty} f_{ij} f_{jj}^{k-1} = 0$. ∎

Theorem 1.11 (Recurrent Equivalences Theorem)**.** For a irreducible Markov chain (so that $i \to j$ for all $i, j \in S$), the following statements are equivalent:

(1) $\exists k, \ell \in S$ such that $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} = \infty$;

(2) $\forall i, j \in S, \; \sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$;

(3) $\exists k \in S \; s.t. \; f_{kk} = 1$;

(4) $\forall j \in S, \; f_{jj} = 1$;

(5) $\forall i, j \in S, \; f_{ij} = 1$;

(6) $\exists k, \ell \in S$ such that $P(N(\ell) = \infty | X_0 = k) = 1$;

(7) $\forall i, j \in S, \; P(N(j) = \infty | X_0 = i)$.