

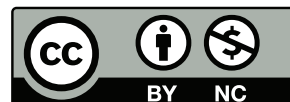
# MAT246: Concepts in Abstract Mathematics:

Lecture 0101 Notes

Tianyu Du

October 29, 2018

This work is licensed under a Creative Commons  
“Attribution-NonCommercial 4.0 International” li-  
cense.



## Contents

<b>1</b>	<b>Lecture 1 Sep. 7 2018</b>	<b>3</b>
<b>2</b>	<b>Lecture 2 Sep. 10 2018</b>	<b>3</b>
<b>3</b>	<b>Lecture 3 Sep. 12 2018</b>	<b>4</b>
<b>4</b>	<b>Lecture 4 Sep. 14 2018</b>	<b>5</b>
<b>5</b>	<b>Lecture 5 Sep. 17 2018</b>	<b>6</b>
<b>6</b>	<b>Lecture 6 Sep. 19 2018</b>	<b>8</b>
<b>7</b>	<b>Lecture 7 Sep. 21 2018</b>	<b>9</b>
<b>8</b>	<b>Lecture 8 Sep. 24 2018</b>	<b>10</b>
<b>9</b>	<b>Lecture 9 Sep. 26 2018</b>	<b>12</b>
<b>10</b>	<b>Lecture 10 Sep. 28 2018</b>	<b>14</b>
<b>11</b>	<b>Lecture 11 Oct. 1 2018</b>	<b>16</b>
	11.1 Rational and Irrational Numbers . . . . .	16
<b>12</b>	<b>Lecture 12 Oct. 3 2018</b>	<b>18</b>

<b>13 Lecture 13 Oct. 5 2018</b>	<b>20</b>
<b>14 Lecture 14 Oct. 10 2018 Euclidean Algorithm</b>	<b>21</b>
14.1 Conventional Method . . . . .	21
14.2 Euclidean Algorithm . . . . .	22
<b>15 Lecture 15 Oct. 12 2018 Public Key Cryptography, RSA Public Key</b>	<b>22</b>
15.1 RSA Public Key Procedures . . . . .	23
<b>16 Lecture 16 Oct. 15 2018 RSA Cryptography Examples</b>	<b>24</b>
16.1 Recall . . . . .	24
16.2 More Examples . . . . .	24
<b>17 Lecture 17 Oct. 17 2018</b>	<b>25</b>
17.1 Chinese Remainder Theorem . . . . .	25
17.2 Complex Numbers . . . . .	27
<b>18 Lecture 18 Oct. 19 2018 Complex Numbers</b>	<b>28</b>
18.1 The Geometric Representation of Complex Numbers . . . . .	28
18.2 Polar Coordinates . . . . .	29
18.2.1 Coordinate Conversion . . . . .	29
18.2.2 Multiplication in Polar Coordinates . . . . .	29
<b>19 Lecture 19 Oct. 22 2018</b>	<b>30</b>
19.1 De Moivre's Theorem . . . . .	30
19.2 Roots of Unity . . . . .	31
<b>20 Lecture 20. Oct 24 2018</b>	<b>33</b>
<b>21 Lecture 21. Oct 29 2018</b>	<b>34</b>
21.1 Proof of the Fundamental Theorem of Algebra . . . . .	35

## 1 Lecture 1 Sep. 7 2018

**Definition 1.1.** Let  $\mathbb{N} := \{1, 2, 3, \dots\}$  be the set of **natural numbers**.

**Theorem 1.1** (Principle of Mathematical Induction). Suppose  $S$  is a set of natural numbers,  $S \subseteq \mathbb{N}$ . If

1.  $1 \in S$
2.  $k \in S \implies k + 1 \in S, \forall k \in \mathbb{N}$

then,  $S = \mathbb{N}$

**Example 1.1.** Show that

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}$$

*Proof.* ■

## 2 Lecture 2 Sep. 10 2018

**Theorem 2.1** (Extended Principle of Mathematical Induction). Suppose set  $S \subseteq \mathbb{N}$  and let  $n_0 \in \mathbb{N}$  fixed, if

1.  $n_0 \in S$
2.  $\forall k \geq n_0, k \in S \implies k + 1 \in S$

then  $\{n_0, n_0 + 1, n_0 + 2, \dots\} \subseteq S$

**Example 2.1.** Show that

$$n! \geq 3^n \quad \forall n \geq 7$$

*Proof.* ■

**Theorem 2.2** (Well-Ordering Principle). Every non-empty subset of natural number has a smallest element.

*Proof.* (Principle of Mathematical Induction)

Let  $S \subseteq \mathbb{N}$

Suppose  $1 \in S \wedge (k \in S \implies k + 1 \in S, \forall k \in \mathbb{N})$

Show:  $S = \mathbb{N}$

Let  $T = \mathbb{N} \setminus S$

Suppose  $T \neq \emptyset$

By Well-Ordering Principle, there exists a smallest element of  $T$ , denoted as  $t_0 \in \mathbb{N}$ .  
 Since  $1 \in S$ , therefore  $t_0 \neq 1$ .  
 Therefore  $t_0 > 2$ .  
 Thus  $t_0 - 1 \in \mathbb{N}$  and since  $t_0 = \min T$ ,  $t_0 - 1 \notin T$   
 Therefore  $t_0 - 1 \in S$ , then,  $t_0 - 1 + 1 = t_0 \in S$ ,  
 Contradict the assumption that  $t_0 \in T$ .  
 Thus  $T = \emptyset$  and  $S = \mathbb{N}$ . ■

**Remark 2.1.** We can use principle of Mathematical Induction to prove Well-Ordering Principle as well.

### 3 Lecture 3 Sep. 12 2018

**Definition 3.1.** Let  $a, b \in \mathbb{N}$  and  $a$  **divides**  $b$ , written as  $a|b$  if

$$\exists c \in \mathbb{N} \text{ s.t. } b = ac$$

And  $a$  is a **divisor** of  $b$ .

**Definition 3.2.** A natural number  $p$  (except 1) is called **prime** if the only divisors of  $p$  are 1 and  $p$ .

**Lemma 3.1** (Prime numbers are building blocks of natural numbers). Every natural number other than 1 is a *product*<sup>1</sup> of prime numbers.

**Theorem 3.1** (Principle of Complete Induction). Suppose  $S \subseteq \mathbb{N}$  and if

1.  $n_0 \in S$
2.  $n_0, n_0 + 1, \dots, k \in S \implies k + 1 \in S, \forall k \geq n_0$

then

$$\{n_0, n_0 + 1, \dots\} \subseteq S$$

*Proof of Lemma.* Let  $S \subseteq \mathbb{N}$  for which the lemma is true,

Want to show:  $S = \mathbb{N} \setminus \{1\}$

(Base Case) For 2 it's a product of prime. Thus  $2 \in S$

(Inductive Step) Suppose  $\{2, 3, \dots, k\} \subseteq S$

Consider  $k + 1$ , if  $k + 1$  is a prime then  $k + 1$  can be written as a product of itself, as a product of one single prime.

---

<sup>1</sup>Product could mean the product of a single number.

Else, if  $k + 1$  is not a prime, then  $\exists 1 < m, n < k + 1$  s.t.  $k + 1 = mn$ .

By induction hypothesis of strong induction,  $m, n$  can both be written as product of primes.

$m = \prod_{i=1}^{\ell} p_i$ ,  $n = \prod_{i=1}^t q_i$  where  $p_i, q_i$  are all primes.

and  $k + 1 = \prod_{i=1}^t q_i \prod_{i=1}^{\ell} p_i$

thus  $k + 1 \in S$

by principle of strong induction,  $\{2, 3, \dots\} \subseteq S$ . ■

**Theorem 3.2.** There is no largest prime number.

*Proof.* (By contradiction)

Assume there is a largest prime  $p$ ,

then  $\{2, 3, 5, \dots, p\}$  is the set of all primes

Let  $M := (2 * 3 * 5 * \dots * p) + 1 \in \mathbb{N}$

$M$  is either prime or not.

Suppose  $M$  is not a prime, then by Lemma 3.1,  $\exists p'$  dividing  $M$ .

Obviously  $\forall i \in \{2 * 3 * 5 * \dots * p\}$ ,  $i \nmid M$ .

There is no prime dividing  $M$ , which contradict Lemma 3.1

Thus  $M$  is a prime, and  $M > p$ , which contradicts assumption

Therefore there is no largest prime. ■

## 4 Lecture 4 Sep. 14 2018

**Theorem 4.1** (the Fundamental Theorem of Arithmetic). Every natural (except 1) is a product of prime(s), and the prime(s) in the product are unique including multiplicity except for the order.

*Proof.* We have already proven that the existential parts of this theorem in Lemma 3.1.

(Proof for the uniqueness part) Suppose there exists natural number (not 1) has 2 different prime factorizations.

By well ordering principle, there is a smallest  $n$ , which has two distinct prime factorizations.

Say  $n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_{\ell}$  where  $p_i, q_i$  are all primes.

Notice that  $p_i \neq q_j$  for any combination of  $(i, j)$  since if so  $\frac{n}{p_i} = \frac{n}{q_j}$  is a natural number smaller than  $n$  having 2 distinct prime factorization, which contradicts our assumption above.

Specifically,  $p_1 \neq q_1$ .

(Case 1:  $p_1 < q_1$ )

Let  $m := n - p_1 q_2 \dots q_{\ell} \in \mathbb{N}$

Notice  $m = p_1(p_2p_3 \dots p_k - q_2q_3 \dots q_\ell)$   
 Also  $m = (q_1 - p_1)(q_2q_3 \dots q_\ell)$   
 $\implies m = p_1 \dots p_k = q_2q_3 \dots q_\ell(q_1 - p_1)$   
 $\implies p_1 | m$  also notices that  $p_1 \nmid q_2q_3 \dots q_\ell$   
 $\implies p_1 | (q_1 - p_1) \implies p_1 | q_1 \implies p_1 = q_1$   
 Contradicts the assumption that  $p_1 < q_1$   
 The other case goes a similar proof. ■

**Definition 4.1.** A natural number  $n$  is called **composite** if it's not 1 or a prime number.

**Remark 4.1.** Natural numbers are partitioned into 3 categories, 1, prime and composite numbers.

**Example 4.1.** Find 20 consecutive composite numbers.

$$(21!) + 2, (21!) + 3, \dots, (21!) + 21$$

**Example 4.2.** Find  $k$  consecutive composite numbers.

$$(k + 1!) + 2, (k + 1!) + 3, \dots, (k + 1!) + k + 1$$

## 5 Lecture 5 Sep. 17 2018

**Definition 5.1.** Let  $a, b \in \mathbb{Z}$ , and let  $m \in \mathbb{N}$ . If  $m | a - b$  then we say " $a$  and  $b$  are congruent modulo  $m$ "

**Remark 5.1.** Regular Induction  $\iff$  Complete Induction  $\iff$  Well-Ordering Principle

*Proof.* (WTS: Complete Induction  $\implies$  Well-Ordering Principle)

Let  $S \subseteq \mathbb{N}$  and  $S \neq \emptyset$

(WTS,  $S$  has the smallest element)

Assume  $S$  does not have the smallest element.

Let  $T := S^c$

Clearly  $1 \in T$  (prop 1)

Since other wise 1 could be the smallest element of  $S$ .

Let  $k \in \mathbb{N}$ .

Suppose  $1, 2, 3, \dots, k \in T$ , if  $k + 1 \notin T$ , then  $k + 1 \in S$  and  $k + 1$  becomes the smallest element of  $S$  and contradicts our assumption above.

Therefore  $1, 2, 3, \dots, k \in T \implies k + 1 \in T$ .

By principle of strong induction,  $T = \mathbb{N}$ .

Thus,  $S = \emptyset$ , and contradicts our definition of  $S$ .

Therefore  $\forall S \subseteq \mathbb{N}$  s.t.  $S \neq \emptyset$ ,  $S$  has the smallest element (Well-Ordering Principle). ■

**Example 5.1** (Application 2). Is  $2^{29} + 3$  divisible by 7?

*Solution.* Notice  $2^2 \equiv 4 \pmod{7}$  and  $2^3 \equiv 1 \pmod{7}$ .

$$\implies (2^3)^9 \equiv 1^9 \pmod{7}$$

$$\implies 2^{27} \equiv 1 \pmod{7}$$

$$\implies 2^{29} \equiv 4 \pmod{7}$$

$$\text{Also } 3 \equiv 3 \pmod{7}$$

$$\implies 2^{29} + 3 \equiv 4 + 3 \pmod{7}$$

$$\implies 2^{29} + 3 \equiv 7 \pmod{7}$$

$$\implies 7 | 2^{29} + 3. \quad \blacksquare$$

**Theorem 5.1** (Rules on computing congruence). Let  $a, b, c, d \in \mathbb{Z}$  and  $m \in \mathbb{N}$ .

$$1. a \equiv b \pmod{m} \wedge c \equiv d \pmod{m} \implies a + c \equiv b + d \pmod{m}$$

$$2. a \equiv b \pmod{m} \wedge c \equiv d \pmod{m} \implies ac \equiv bd \pmod{m}$$

*Proof.* Let  $a, b, c, d \in \mathbb{Z}$  and  $m \in \mathbb{N}$ ,

suppose  $a \equiv b \pmod{m} \wedge c \equiv d \pmod{m}$

by definition of congruence,  $\exists p, q \in \mathbb{Z}$  s.t.  $(a - b) = pm \wedge (c - d) = qm$

$$\implies (a + c - b - d) = (p + q)m, (p + q) \in \mathbb{Z}$$

$$\implies a + c \equiv b + d \pmod{m}$$

$$\text{And } a = b + pm \wedge c = d + qm$$

$$ac - bd = (b + pm)(d + qm) - bd$$

$$= bd + dpm + qbm + pqm^2 - bd$$

$$= (dp + qb + pqm)m$$

$$\implies m | ac - bd$$

$$\implies ac \equiv bd \pmod{m} \quad \blacksquare$$

**Proposition 5.1** (Corollary from theorem 5.1).

$$a \equiv b \pmod{m} \implies a + c \equiv b + c \pmod{m}$$

and

$$a \equiv b \pmod{m} \implies a^k \equiv b^k \pmod{m}, \forall k \in \mathbb{Z}_{\geq 0}$$

## 6 Lecture 6 Sep. 19 2018

**Theorem 6.1.** Let  $a, b \in \mathbb{Z}$ ,

$$a = b \implies a \equiv b \pmod{m} \forall m \in \mathbb{N}$$

**Example 6.1.** What is the remainder when  $3^{202} + 5^9$  is divided by 8

*Solution.* Notice  $3^2 \equiv 1 \pmod{8}$

Therefore,  $(3^2)^{101} \equiv 1^{101} \pmod{8}$

That's,  $3^{202} \equiv 1 \pmod{8}$

Also  $5^2 \equiv 1 \pmod{8}$

$\implies (5^2)^4 \equiv 1^4 \pmod{8}$

$\implies 5^9 \equiv 5 \pmod{8}$

$\implies 3^{202} + 5^9 \equiv 5 + 1 \pmod{8}$

$\implies$  the remainder is 6.

(Notice that  $3^{202} + 5^9 \equiv 6 \equiv 14 \equiv 22 \equiv \dots \pmod{8}$ , and the remainder is the smallest integer satisfying above relation.) ■

**Theorem 6.2.** Let  $M \in \mathbb{Z}$  and  $M = d_N \dots d_2 d_1 d_0$ ,  $d_i \in \{0, 1, \dots, 9\}$ <sup>2</sup>, then

$$3|M \iff 3 \mid \sum_{i=0}^N d_i$$

*Proof.* Notice  $10 \equiv 1 \pmod{3}$ ,  $100 \equiv 1 \pmod{3}$  and so on,

(Fact)  $10^k \equiv 1 \pmod{3}$ ,  $\forall k \in \mathbb{Z}_{\geq 0}$

Then  $d_i 10^i \equiv d_i \pmod{3}$ ,  $\forall i$

Therefore,  $\sum_{i=0}^N 10^i d_i \equiv \sum_{i=0}^N d_i \pmod{3}$

Therefore  $\sum_{i=0}^N 10^i d_i \equiv 0 \pmod{3} \iff \sum_{i=0}^N d_i \equiv 0 \pmod{3}$  ■

**Theorem 6.3.** Let  $M \in \mathbb{Z}$  and  $M = d_N \dots d_2 d_1 d_0$ ,  $d_i \in \{0, 1, \dots, 9\}$ , then

$$11|M \iff 11 \mid \sum_{i=0}^N (-1)^i d_i$$

*Proof.* Notice  $10^i \equiv (-1)^i \pmod{11}$

Therefore  $10^i d_i \equiv (-1)^i d_i$

Thus,  $\sum_{i=0}^N 10^i d_i \equiv \sum_{i=0}^N (-1)^i d_i \pmod{11}$

Then,  $\sum_{i=0}^N 10^i d_i \equiv 0 \pmod{11} \iff \sum_{i=0}^N (-1)^i d_i \equiv 0 \pmod{11}$  ■

---

<sup>2</sup>This means the integer  $M$  is constructed from digits  $d_i$ . For example,  $M = 256$ , then  $d_0 = 6$ ,  $d_1 = 5$ ,  $d_2 = 2$



## 7 Lecture 7 Sep. 21 2018

**Theorem 7.1.** Suppose  $p$  is a prime and  $a, b \in \mathbb{N}$ , if  $p|ab$  then  $p|a \vee p|b$ .

*Proof.* If  $a = 1 \vee b = 1$ , then done. And for the case  $a = b = 1$ , the proposition is vacuously true.

Let  $a, b > 1$ ,

By the fundamental theorem of arithmetic, we can write  $a, b$  as their unique prime factorization

$$a = p_1^{\alpha_1} \dots p_k^{\alpha_k}, \alpha_j \geq 1 \text{ and } b = q_1^{\beta_1} \dots q_\ell^{\beta_\ell}, \beta_j \geq 1$$

then  $ab = p_1^{\alpha_1} \dots p_k^{\alpha_k} q_1^{\beta_1} \dots q_\ell^{\beta_\ell}$  is the unique prime factorization of  $ab$ .

Since  $p \in \mathbb{P}$ , therefore,  $p = p_j \vee p = q_j \implies p|a \vee p|b$

■

**Remark 7.1.** We have shown that  $a \equiv b \pmod{m} \implies ca \equiv cb \pmod{m}$ . But notice that

$$ca \equiv cb \pmod{m} \not\Rightarrow a \equiv b \pmod{m}$$

**Definition 7.1.** Let  $a, b \in \mathbb{Z}$ , then we say  $a$  and  $b$  are **relatively prime** if they have no prime factor in common.

**Theorem 7.2.** Suppose  $p$  is a prime and  $a \in \mathbb{Z}$  and  $p \nmid a$ , then  $ax \equiv ay \pmod{p} \implies x \equiv y \pmod{p}$ .

*Proof.* Let  $x, y, a \in \mathbb{N}$  and  $p \in \mathbb{P}$ .

Suppose  $ax \equiv ay \pmod{p}$

Then  $p|a(x - y)$

By theorem 7.1,  $p|a \vee p|(x - y)$

But by our assumption,  $p \nmid a$ , therefore  $p|(x - y)$

Thus  $x \equiv y \pmod{p}$

■

**Theorem 7.3** (Generalization of Theorem 7.2). Let  $m \in \mathbb{N}$  and  $a \in \mathbb{Z}$  and  $a$  and  $m$  are relatively prime. Then

$$ax \equiv ay \pmod{m} \implies x \equiv y \pmod{m}$$

*Proof.* Suppose  $ax \equiv ay \pmod{m}$

Then  $m|a(x - y)$

Therefore  $m|a \vee m|(x - y)$

For  $m$  to divide  $a$ , all of  $m$ 's prime factors have to be in the prime factorization of  $|a|$ .

But  $m$  and  $a$  are relatively prime, therefore  $m \nmid a$ .

Therefore  $m \mid (x - y)$  and that's  $x \equiv y \pmod{m}$

■

**Theorem 7.4.** Any integer  $a$  is congruent to mod  $m$  to exactly one of  $\{0, 1, \dots, m - 1\}$ .

**Theorem 7.5** (Fermat's Little Theorem). If  $p$  is a prime and  $p \nmid a$  (i.e.  $a$  and  $p$  are relatively prime), then

$$a^{p-1} \equiv 1 \pmod{p}$$

*Proof.* Let  $S := \{a1, a2, \dots, a(p-1)\}$

Notice that if  $ax_i \equiv ax_j \pmod{p}$ , since  $p \nmid a$ ,  $x_1 \equiv x_2 \pmod{p}$ .

Since  $1 \leq x_i, x_j \leq p - 1$ , then  $x_i = x_j$ .

Therefore all elements in  $S$  are distinct with mod  $p$

i.e.  $x_i \not\equiv x_j \pmod{p}$ ,  $\forall (i, j) \in \mathbb{Z}^2$ .

Since  $p \nmid a \wedge p \nmid m$ ,  $\forall m \in \{1, 2, \dots, (p-1)\}$

So no element in  $S$  is congruent to  $0 \pmod{p}$ .

Thus,  $S$  contains  $p - 1$  numbers and no two of them are congruent mod  $p$ .

Also none of them are congruent to  $0 \pmod{p}$ .

By theorem 7.4, each element in  $S$  is congruent to one corresponding element in set  $\{1, 2, \dots, p - 1\}$ .

Therefore  $(a1)(a2) \dots (a(p-1)) \equiv 1 * 2 * \dots * (p-1) \pmod{p}$

That's  $a^{p-1}(1 * 2 * \dots * (p-1)) \equiv 1 * 2 * \dots * (p-1) \pmod{p}$

Clearly  $p \nmid (1 * 2 * \dots * (p-1))$ , since if a prime divides a product of natural numbers, the prime must divide at least one of elements in the product.

Therefore  $a^{p-1} \equiv 1 \pmod{p}$

■

## 8 Lecture 8 Sep. 24 2018

**Definition 8.1.** Let  $p \in \mathbb{N}$  and  $a \in \mathbb{Z}$ . The **multiplicative inverse** mod  $p$  of  $a$  is an integer  $b$  such that

$$ab \equiv 1 \pmod{p}$$

**Remark 8.1.** Notice that the multiplicative inverse is generally not unique but unique up to  $\pmod{p}$ .

**Corollary 8.1.** Let  $p \in \mathbb{P}$ ,  $a \in \mathbb{N}$  and  $p \nmid a$ . Then

$$\exists b \in \mathbb{Z}, \text{ s.t. } ba \equiv 1 \pmod{p}$$

*Proof.* Let  $p \in \mathbb{Z}$  and  $a \in \mathbb{Z}$

Suppose  $p \nmid a$ , then by Fermat's little theorem,

$$a^{p-1} \equiv 1 \pmod{p} \implies a^{p-2}a \equiv 1 \pmod{p}$$

Take  $b = a^{p-2} \in \mathbb{Z}$  and  $ab \equiv 1 \pmod{p}$  ■

**Example 8.1.** Let  $a = 8$  and  $p = 5$ . Obviously  $p \nmid a$ . By corollary above,

$$\exists b \in \mathbb{Z}, \text{ s.t. } 8b \equiv 1 \pmod{5}$$

Notice  $b = 2$  satisfies above equation.

**Remark 8.2.** Corollary 8.1 requires  $p$  to be a prime.

**Corollary 8.2** (Generalization). Let  $a$  and  $m \in \mathbb{N}$  and  $a$  and  $m$  are relatively prime, then

$$\exists b \in \mathbb{Z}, \text{ s.t. } ab \equiv 1 \pmod{m}$$

**Theorem 8.1** (Wilson's Theorem). Let  $p \in \mathbb{P}$  then

$$(p-1)! \equiv -1 \pmod{p}$$

*Proof.* Let  $p \in \mathbb{P}$

if  $p = 2 \vee p = 3$ , then  $1! \equiv -1 \pmod{2}$  and  $2! \equiv -1 \pmod{3}$ .

Otherwise, suppose  $p > 3$ ,

Consider, let  $S := \{2, 3, 4, \dots, p-2\}$

Notice that none of  $S$  is divisible by  $p$ .

Therefore  $p$  is relatively prime to all elements in  $S$ .

Then by Corollary 8.1,  $\exists b_i \in \mathbb{Z}$  s.t.  $b_i s_i \equiv 1 \pmod{p}$ ,  $\forall s_i \in S$ .

Notice that 0 has no multiplicative inverse and

$$(p-1)(p-1) = p^2 - 2p + 1 \equiv 1 \pmod{p}$$

That's, 1 and  $(p-1)$  have themselves as their multiplicative inverse.

Also notice that for any  $s_i \in S$ ,  $s_i$  does not have itself as its multiplicative inverse.

If  $a \in S$  has itself as its multiplicative inverse, then

$$\begin{aligned} a^2 &\equiv 1 \pmod{p} \\ \implies a^2 - 1 &\equiv 0 \pmod{p} \\ \implies (a+1)(a-1) &\equiv 0 \pmod{p} \\ \implies p &\mid (a+1)(a-1) \end{aligned}$$

Notice that at last one of  $(a + 1)$  and  $(a - 1)$  is in set  $S$  since  $p > 3 \implies S \neq \emptyset$ .  
This contradicts what we argued above, *none of  $S$  is divisible by  $p$* .

That's

$$s_i s_i \not\equiv 1 \pmod{p}, \forall s_i \in S$$

*Note that if  $y$  is a multiplicative inverse of  $x$ , then  $x$  is a multiplicative inverse of  $y$ .*

Notice that for any  $s_i \in S$ , by Corollary 8.1,

there exists an integer  $b_i$  s.t.  $s_i b_i \equiv 1 \pmod{p}$

And the multiplicative inverse is unique up to  $\pmod{p}$ ,

Thus  $s_i(b_i \pmod{p}) \equiv 1 \pmod{p}$  and  $(b_i \pmod{p}) \in S$ .

And for all elements in  $S$  has one of their multiplicative inverse in  $S$ ,

That's

$$s_i s_j \equiv 1 \pmod{p}, i \neq j$$

Notice  $p > 3$  implies  $p$  is odd, so  $|S|$  is even.

Match every pair of multiplicative inverses in  $S$  and they collapse to  $1 \pmod{p}$

Therefore

$$\begin{aligned} 2 \cdot 3 \cdot 4 \cdots (p-2) &\equiv 1 \pmod{p} \\ \implies 2 \cdot 3 \cdot 4 \cdots (p-2) \cdot (p-1) &\equiv (p-1) \pmod{p} \\ \implies (p-1)! &\equiv -1 \pmod{p} \end{aligned}$$

■

## 9 Lecture 9 Sep. 26 2018

**Remark 9.1.** Recall that an integer  $n$  is even iff  $n \equiv 0 \pmod{2}$  and is odd iff  $n \equiv 1 \pmod{2}$ .

**Theorem 9.1.** There are infinitely many primes of the form  $4k + 3$ , where  $k \in \mathbb{Z}$ .

*Proof.* Note that odd numbers  $n$  can be classified as  $n \equiv 1 \pmod{4}$  and  $n \equiv 3 \equiv -1 \pmod{4}$

(Suppose 1) there are only finitely many primes in the form  $4k + 3$ .

Let finite set  $S := \{p_1, p_2, \dots, p_m\}$  denotes the collection of them.

And notice that  $p_i \equiv -1 \pmod{4}, \forall p_i \in S$ .

Let

$$M := (p_1 \cdot p_2 \cdots p_m)^2 + 2$$

and  $M \equiv 1 + 2 \equiv 3 \equiv -1 \pmod{4}$ .

Therefore  $M$  is an odd natural number.

By the Fundamental Theorem of Arithmetic,  $M$  can be factorized into product of

primes.

$$M = \prod_{i=1}^{\ell} q_i$$

and since  $M$  is odd,  $q_i \neq 2 \forall i$ . Thus all  $q_i$  are odd.

(Suppose 2) All  $q_i \equiv 1 \pmod{4}$ .

Then  $M \equiv 1 \pmod{4}$ .

Contradict the fact that  $M \equiv -1 \pmod{4}$ . Thus (Suppose 2) is false.

Therefore  $\exists i$ , s.t.  $q_i \equiv -1 \pmod{4}$ .

From (Suppose 1),  $S$  is the collection of all primes that  $\equiv -1 \pmod{4}$ .

Therefore  $q_i = p_j$  for some  $j$ .

Therefore  $p_j | M$ .

Also note that  $p_j | (p_1 \cdot p_2 \cdots p_m) \implies p_j | (p_1 \cdot p_2 \cdots p_m)^2$   
 $\implies p_j | 2 \implies p_j = 2$  contradicts the fact that  $p_j$  is odd.

Therefore (Suppose 1) is false, there are infinitely many primes taking the form  $4k + 3$ . ■

**Example 9.1.** Find  $7^{20^{30}} \pmod{5}$ .

*Solution.* Let  $n := 20^{30}$ .

Notice that  $7^4 \equiv 1 \pmod{5}$ .

And if  $n \equiv r \pmod{4}$  where  $r \in \mathbb{Z}$ ,

$n = 4k + r$  and  $7^n \equiv 7^{4k+r} \equiv (7^4)^k \times 7^r \equiv 1^k \times 7^r \equiv 7^r \pmod{5}$ .

Notice that  $20 \equiv 0 \pmod{4} \implies 20^{30} \equiv 0 \pmod{4}$ .

Thus  $r = 0$ .

Therefore  $7^n \equiv 7^0 \equiv 1 \pmod{5}$ .

Thus  $7^{20^{30}} \pmod{5} = 1$ . ■

**Example 9.2.** Find  $10^{3^{30}} \pmod{7}$ .

*Solution.* Notice that  $10^6 \equiv 1 \pmod{7}$ .

And  $3 \equiv 3 \pmod{6}, 3^2 \equiv 3 \pmod{6}, 3^3 \equiv 3 \pmod{6} \dots$

Using induction, we can show that

$$3^k \equiv 3 \pmod{6}, \forall k \in \mathbb{Z}_{\geq 0}$$

Therefore  $3^{30} \equiv 3 \pmod{6}$ .

That's  $3^{30} = 6k + 3$  for some  $k$ .

Thus  $10^{3^{30}} \equiv (10^6)^k \times 10^3 \equiv (1)^k \times 10^3 \equiv -1 \equiv 6 \pmod{7}$ .

So  $10^{3^{30}} \pmod{7} = 6$ . ■

## 10 Lecture 10 Sep. 28 2018

**Example 10.1.** Find  $8^{9^{10^{11}}} \bmod 5$ .

*Solution.* Let  $n := 9^{10^{11}}$

And notices that  $8^4 \equiv 1 \pmod{5}$ .

Then find  $n \bmod 4$

Note that  $9 \equiv 1 \pmod{4} \implies 9^{10^{11}} \equiv 1 \pmod{4}$ .

Thus  $n = 4k + 1$ .

Therefore  $8^{9^{10^{11}}} \equiv (8^4)^k \cdot 8 \equiv 1 \cdot 3 \pmod{5}$ .

That's  $8^{9^{10^{11}}} \bmod 5 = 3$ . ■

**Definition 10.1** (Euler  $\phi$ -function). Let  $m \in \mathbb{N}$  and  $\phi(m) : \mathbb{N} \rightarrow \mathbb{N}$  is defined as *the number of elements in  $\{1, 2, \dots, m-1\}$  that are relatively prime to  $m$ .*

**Example 10.2.** For  $m = 8$ , note that  $\{1, 3, 5, 7\} \subset \{1, 2, \dots, 7\}$  are relatively prime with 8, therefore  $\phi(8) = 4$ .

And for  $m = 11$ , since  $m$  is a prime, then every integer between 1 and  $m-1$  are relatively prime with 11. Therefore  $\phi(11) = 10$ .

And notice that  $\phi(p) = p-1$  if  $p \in \mathbb{P}$ . (Fermat's Little Theorem)

**Proposition 10.1.** Let  $p, q$  be two distinct primes, then

$$\phi(pq) = (p-1)(q-1)$$

*Proof.* Let  $S := \{1, 2, \dots, pq-1\}$ .

WLOG, assume  $p < q$ .

We need find all elements in  $S$  that with either  $p$  or  $q$  in their prime factorization to find elements in  $S$  that are not relatively prime to  $pq$ .

And those elements are multiples of  $p$  and multiples of  $q$ .

And since  $pq \notin S$ , the largest multiple of  $p$  in  $S$  is  $(q-1)p$  and the largest multiple of  $q$  in  $S$  is  $p(q-1)$ .

And since there is no multiple of both  $p$  and  $q$  in set  $S$ , therefore there's no overlapping between multiples of  $p$  and multiples of  $q$ .

Therefore exists  $(p-1) + (q-1)$  elements that are not relatively prime to  $pq$ .

Therefore  $\phi(pq) = (pq-1) - (p-1) - (q-1)$

$$= pq - p - q + 1$$

$$= (p-1)(q-1)$$
 ■

**Proposition 10.2.** For any natural number  $m \in \mathbb{N}$ . Therefore  $m$  can be expressed as

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Then

$$\phi(m) = \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_k^{\alpha_k})$$

And

$$\phi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^{\alpha-1}(p - 1)$$

Therefore

$$\phi(m) = (p_1^{\alpha_1} - p_1^{\alpha_1-1})(p_2^{\alpha_2} - p_2^{\alpha_2-1}) \cdots (p_k^{\alpha_k} - p_k^{\alpha_k-1})$$

**Example 10.3.**

$$\begin{aligned} \phi(6) &= \phi(2^1 3^1) \\ &= \phi(2^1)\phi(3^1) \\ &= (2^1 - 2^0)(3^1 - 3^0) \\ &= (2 - 1)(3 - 1) = 2 \end{aligned}$$

**Example 10.4.**

$$\begin{aligned} \phi(8) &= \phi(2^3) \\ &= (2^3 - 2^2) = 4 \end{aligned}$$

**Theorem 10.1** (Euler's Theorem). Suppose  $m \in \mathbb{N} \setminus \{1\}$ . And  $a \in \mathbb{N}$ <sup>3</sup> Assume  $a$  and  $m$  are relatively prime, then

$$a^{\phi(m)} \equiv 1 \pmod{m}$$

**Remark 10.1.** This theorem is a generalization of Fermat's Little Theorem. When  $m \in \mathbb{P}$ , it becomes Fermat's Little Theorem.

*Proof.* Let  $S := \{r_1, r_2, \dots, r_{\phi(m)}\}$  be the set of all elements in  $\{1, 2, \dots, m - 1\}$  that are relatively prime to  $m$ .

Let  $T := \{ar_1, ar_2, \dots, ar_{\phi(m)}\}$ .

(Observation 1) that no two elements in  $S$  are congruent to each other  $\pmod{m}$ . Since all elements are in the range  $[1, m - 1]$  and they are the remainder while  $r_i$  is divided by  $m$ .

---

<sup>3</sup>Also true for  $a \in \mathbb{Z}$

Also notice that elements in  $T$  are not congruent to each other  $\pmod m$ .

Since, suppose

$$ar_i \equiv ar_j \pmod m$$

for some  $(i, j)$ .

Since  $a$  and  $m$  are relatively prime, therefore we could use cancellation law.

$$r_i \equiv r_j \pmod m$$

This would contradict our observation 1

(Observation 2) elements in  $T$  are not congruent to each other  $\pmod m$ .

Therefore elements in  $S$  are congruent to elements in  $T \pmod m$  in some order.

Therefore

$$r_1 r_2 r_3 \cdots r_{\phi(m)} \equiv a^{\phi(m)} r_1 r_2 \cdots r_{\phi(m)} \pmod m$$

And notice  $r_1 r_2 r_3 \cdots r_{\phi(m)}$  is a product of natural numbers relatively prime to  $m$ .

Therefore  $r_1 r_2 r_3 \cdots r_{\phi(m)}$  is relatively prime to  $m$ .

And by cancellation law, we have

$$a^{\phi(m)} \equiv 1 \pmod m$$



## 11 Lecture 11 Oct. 1 2018

### 11.1 Rational and Irrational Numbers

**Definition 11.1.** A **rational number** is an expression in form

$$\frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0$$

**Definition 11.2.** Two rational numbers  $\frac{m_1}{n_1}, \frac{m_2}{n_2} \in \mathbb{Q}$  are **equal** if and only if  $m_1 n_2 = m_2 n_1$ .

**Definition 11.3.** Arithmetic on  $\mathbb{Q}$  are defined as

- **Addition**  $+$  :  $\frac{m_1}{n_1} + \frac{m_2}{n_2} := \frac{m_1 n_2 + m_2 n_1}{n_1 n_2}$
- **Multiplication**  $\times$  :  $\frac{m_1}{n_1} \times \frac{m_2}{n_2} := \frac{m_1 m_2}{n_1 n_2}$
- **Subtraction**  $-$  :  $\frac{m_1}{n_1} - \frac{m_2}{n_2} := \frac{m_1 n_2 - m_2 n_1}{n_1 n_2}$



- **Division**  $\div : \frac{\frac{m_1}{n_1}}{\frac{m_2}{n_2}} := \frac{m_1 n_2}{n_1 m_2}$ , defined only if  $m_2 \neq 0$ .

**Definition 11.4.** The **multiplicative inverse** of a non-zero rational number  $x \neq 0$  is a rational number  $y$  such that  $xy = 1$ .

**Remark 11.1.** Let  $x = \frac{m}{n} \neq 0$ , then the multiplicative inverse  $y = \frac{n}{m}$ .

**Example 11.1.** Claim:  $\sqrt{2}$  is not rational.

*Proof.* Assume  $\sqrt{2}$  is rational,  
by definition of rational numbers,  $\sqrt{2} = \frac{m}{n}$  where  $m, n \in \mathbb{Z}, n \neq 0$ .  
Divide numerator and denominator by their common prime factors (if any).  
Assume  $m$  and  $n$  have been reduced so that they are relatively prime.

$$\begin{aligned} \Rightarrow 2 &= \frac{m^2}{n^2} \\ \Leftrightarrow 2n^2 &= m^2 \\ \Rightarrow 2|m^2 \end{aligned}$$

Consider if  $2 \nmid m$ , then  $m$  is odd, then  $2 \nmid m^2$ .  
Take the contraposition,  $2|m^2 \Rightarrow 2|m$ .

$$\begin{aligned} \Rightarrow 2|m \\ \Rightarrow m &= 2q, q \in \mathbb{Z} \\ \Rightarrow 2n^2 &= 4q^2 \\ \Rightarrow n^2 &= 2q^2 \\ \Rightarrow 2|n^2 \\ \Rightarrow 2|n \end{aligned}$$

That's  $2|m \wedge 2|n$ , which contradicts our assumption that  $m$  and  $n$  are relatively prime.  
Therefore  $\sqrt{2}$  cannot be rational. ■

**Definition 11.5** (non-rigorous definition). **Real numbers**, denoted as  $\mathbb{R}$ , are numbers representing distance of points on a line from 0.

**Definition 11.6.** **Irrational numbers** are real numbers which are not rational.  
( $\mathbb{R} \setminus \mathbb{Q}$ )

**Proposition 11.1.** Let  $p \in \mathbb{P}$  and  $m \in \mathbb{Z}$ , then

$$p|m^2 \implies p|m$$

*Proof.* Let  $m = q_1 q_2 \dots q_\ell$  be the unique prime factorization.

Suppose  $p \nmid m$ , then  $p \notin \{q_1, q_2, \dots, q_\ell\}$ .

Obviously,  $m^2 = q_1^2 q_2^2 \dots q_\ell^2$  as it's prime factorization.

Then  $p \nmid m^2$ . ■

**Example 11.2.**  $\sqrt{p} \notin \mathbb{Q}$ ,  $\forall p \in \mathbb{P}$ .

*Proof.* Let  $p \in \mathbb{P}$ , Suppose  $\sqrt{p} \in \mathbb{Q}$ .

Therefore  $\sqrt{p} = \frac{m}{n}$  where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ .

Assume  $\frac{m}{n}$  has been reduced such that  $m$  and  $n$  are relatively prime.

$$\begin{aligned} \implies pn^2 &= m^2 \\ \implies p|m^2 \\ \implies p|m \\ \implies m &= pr, r \in \mathbb{Z}. \\ \implies pn^2 &= p^2 r^2 \\ \implies n^2 &= pr^2 \\ \implies p|n^2 \\ \implies p|n \end{aligned}$$

Contradicts the assumption that  $m$  and  $n$  are relatively prime. ■

## 12 Lecture 12 Oct. 3 2018

**Definition 12.1.** A natural number (other than 1) is called a **perfect square** if it is the square of some natural number.

**Theorem 12.1.** A natural number  $m$  is a perfect square if and only if every prime factor occurs with an even power in its prime decomposition.

*Proof.* ( $\implies$ ) Suppose  $m$  is a perfect square,

Then  $m = n^2$ ,  $n \in \mathbb{N}$ .

Let  $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$  be the prime decomposition.

Then  $m = p_1^{2\alpha_1} \dots p_k^{2\alpha_k}$ .

Obviously all prime factors in the prime factorization occurs with an even power.

( $\Leftarrow$ ) Suppose  $m = p_1^{2\alpha_1} \dots p_k^{2\alpha_k}$  as its prime decomposition.  
Then  $m = (p_1^{\alpha_1} \dots p_k^{\alpha_k})^2$  and  $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} \in \mathbb{N}$ .  
Therefore  $m$  is a perfect square. ■

**Theorem 12.2** (Generalization). Let  $n \in \mathbb{N}$  other than 1, then <sup>4</sup>

$$\sqrt{n} \in \mathbb{Q} \iff n \text{ is a perfect square}$$

*Proof.* ( $\Leftarrow$ ) if  $n$  is perfect square, then  $\sqrt{n} \in \mathbb{N}$ .

Obviously a natural number is rational.

( $\Rightarrow$ ) Suppose  $\sqrt{n} \in \mathbb{Q}$ .

Then

$$\sqrt{n} = \frac{m}{l} \in \mathbb{Q}$$

where  $m, l \in \mathbb{Z}$  and  $l \neq 0$ .

Since  $\sqrt{n} > 0$ , WLOG, assume  $m, l \geq 0$ .

Suppose  $m, l$  are relatively prime. (Otherwise, factorize the fraction so that  $m$  and  $l$  are relatively prime.)

Then

$$m^2 = nl^2$$

(Suppose 1)  $l > 1$  and  $p$  is a prime in the prime decomposition of  $m$ , i.e.  $p|l$ ,

Thus  $p|l^2$  and therefore  $p|m^2$ .

By proposition 11.1 (previous lecture),  $p|m$

And we have  $p|l \wedge p|m$  which contradicts our assumption that  $m, l$  are relatively prime.

Therefore (Suppose 1) is false and  $l \leq 1$  (so that  $l$  has no prime factor).

Also notice that  $l \in \mathbb{Z}$  and  $l \geq 0$ . therefore  $l = 1$ .

Therefore  $n = m^2$  and  $n$  is a perfect square. ■

**Example 12.1.** Claim  $\sqrt[3]{4}$  is irrational.

*Proof.* Suppose  $\sqrt[3]{4}$  is rational and

$$\sqrt[3]{4} = \frac{m}{n} \implies 4 = \frac{m^3}{n^3} \implies 2^2 n^3 = m^3$$

Suppose

$$\begin{aligned} n &= p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ m &= q_1^{\beta_1} \dots q_\ell^{\beta_\ell} \end{aligned}$$

---

<sup>4</sup>The square root here denotes the positive square root.

The prime factor 2 has power of 2 or  $2 + 3\alpha_j$  on the left hand side.

And have power of  $3\beta_i$  on the right hand side.

The left hand side power is congruent to 2 mod 3 and the right hand side is congruent to 0 mod 3.

It's impossible for them to be equal. Thus, contradicts the uniqueness of prime decomposition.

Therefore  $\sqrt[3]{4}$  cannot be rational. ■

### 13 Lecture 13 Oct. 5 2018

**Example 13.1.**  $\sqrt{3} + \sqrt{5}$  is irrational.

*Proof.* Suppose  $\sqrt{3} + \sqrt{5}$  are rational then  $\sqrt{3} + \sqrt{5} = \frac{m}{n}$ .

$$\begin{aligned} \Rightarrow \sqrt{5} &= \frac{m}{n} - \sqrt{3} \\ \Rightarrow 5 &= \left(\frac{m}{n} - \sqrt{3}\right)^2 = \frac{m^2}{n^2} - \frac{2m\sqrt{3}}{n} + 3 \\ \Rightarrow \sqrt{3} &= \frac{5 - 3 - \frac{m^2}{n^2}}{-\frac{2n}{n}} \end{aligned}$$

Obviously the right hand side is rational, leads to contradiction.

Therefore  $\sqrt{3} + \sqrt{5} \notin \mathbb{Q}$ . ■

**Example 13.2.** Are there two irrational numbers  $x, y$  such that  $x^y \in \mathbb{Q}$ ?

*Solution.* Consider  $\sqrt{3}^{\sqrt{2}}$ .

**case 1:**  $\sqrt{3}^{\sqrt{2}} \in \mathbb{Q}$ , then take  $x = \sqrt{3}$  and  $y = \sqrt{2}$ .

**case 2:**  $\sqrt{3}^{\sqrt{2}} \notin \mathbb{Q}$ , then take  $x = \sqrt{3}^{\sqrt{2}}$ ,  $y = \sqrt{2}$ .

And  $x^y = \sqrt{3}^{\sqrt{2}\sqrt{2}} = \sqrt{3}^2 = 3 \in \mathbb{Q}$ . ■

**Remark 13.1.** Basic arithmetic operations on integers preserve rationality.

**Theorem 13.1** (Rational Root Theorem). Consider a polynomial with integer coefficients,

$$a_0 + a_1x + a_2x^2 + \cdots + a_kx^k, \quad a_i \in \mathbb{Z}$$

If  $\frac{m}{n}$  is a rational root for the polynomial and  $m$  and  $n$  are relatively prime. Then

$$m|a_0 \wedge n|a_k$$

*Proof.* Suppose  $\frac{m}{n}$  is a rational root for the polynomial,  
then

$$a_0 + a_1 \frac{m}{n} + a_2 \frac{m^2}{n^2} + \cdots + a_k \frac{m^k}{n^k} = 0$$

Thus

$$a_0 n^k + a_1 m n^{k-1} + a_2 m^2 n^{k-2} + \cdots + a_k m^k = 0$$

And

$$-a_0 n^k = a_1 m n^{k-1} + a_2 m^2 n^{k-2} + \cdots + a_k m^k$$

Therefore  $m | a_0 n^k$  and since  $m \nmid n$ , thus  $m | a_0$ .

Similarly,

$$-a_k m^k = a_0 n^k + a_1 m n^{k-1} + a_2 m^2 n^{k-2} + \cdots + a_{k-1} m^{k-1} n$$

Which implies  $n | a_k m^k$  and since  $n \nmid m$ , thus  $n | a_k$ . ■

## 14 Lecture 14 Oct. 10 2018 Euclidean Algorithm

**Definition 14.1.** The **greatest common divisor**(gcd) of  $m, n \in \mathbb{N}$  is denoted as  $\gcd(m, n)$  or  $(m; n)$  is the largest natural number that divides both  $m$  and  $n$ .

**Example 14.1.**

$$\gcd(27, 15) = 3$$

$$\gcd(36, 48) = 12$$

$$\gcd(7, 21) = 7$$

### 14.1 Conventional Method

Factorize  $m$  and  $n$  into primes and in general,

$$m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

$$n = p_1^{\beta_1} \cdots p_k^{\beta_k}$$

where  $\{p_1, \dots, p_k\} = \{\text{prime factors of } m\} \cup \{\text{prime factors of } n\}$ . And  $\alpha_i, \beta_i \geq 0$ .  
and gcd could be found by

$$\gcd(m, n) = p_1^{\min\{\alpha_1, \beta_1\}} \cdots p_k^{\min\{\alpha_k, \beta_k\}}$$

## 14.2 Euclidean Algorithm

Notice that  $r|a, b \implies r|\gcd(a, b)$ .

For  $a, b \in \mathbb{N}$ , WLOG, assuming  $a \geq b$ .

$$a = q_0b + r_0 \quad r_0, q_0 \in \mathbb{N}, 0 \leq r_0 < b$$

$$b = q_1r_0 + r_1 \quad r_1, q_1 \in \mathbb{N}, 0 \leq r_1 < r_0$$

$$r_0 = q_2r_1 + r_2$$

$r_i$  is strictly decreasing, and it's guaranteed to be 0 after certain iterations.

...

$$r_{k-2} = q_k r_{k-1} + r_k$$

$$r_{k-1} = q_{k+1} r_k + 0$$

and then  $r_k$  is the greatest common divisor of  $a$  and  $b$ .

*Proof.* WTS:  $r_k = \gcd(a, b)$

Obviously  $r_k|r_{k-1}$

Then,  $r_k|r_{k-1} \wedge r_k|r_k \implies r_k|r_{k-2}$

Similarly, tracing upwards through the Euclidean Algorithm,

we have  $r_k|b$  and  $r_k|a$ .

so  $r_k \leq \gcd(a, b)$  since  $r_k$  is a common divisor of  $a$  and  $b$ .

Since  $\gcd|a \wedge \gcd|b$ ,

Therefore  $\gcd|r_0$ ,

similarly, tracing downwards in the Euclidean Algorithm,

$\gcd|r_k$ , so  $\gcd \leq r_k$ .

Therefore  $r_k = \gcd(a, b)$  ■

**Theorem 14.1.** Given natural numbers  $a$  and  $b$  with the greatest common divisor  $d$ , there exists integers  $x$  and  $y$  such that

$$d = ax + by$$

*Proof.* This can be seen by working upwards in the sequence of equations that constitute the Euclidean Algorithm. ■

## 15 Lecture 15 Oct. 12 2018 Public Key Cryptography, RSA Public Key

**Lemma 15.1.** Let  $N = pq$  where  $p \neq q$  are distinct primes, and let  $n$  and  $M$  be integers.

Then

$$n \equiv 1 \pmod{\phi(N)} \implies M^n \equiv M \pmod{N}$$

*Proof.* Note that  $\phi(N) = \phi(pq) = (p-1)(q-1)$ .

And suppose  $\phi(N) \mid (n-1)$ ,

Then  $k\phi(N) = n-1$  for some  $k$ .

That's  $n = 1 + k\phi(N)$ .

Therefore  $M^n = M^{1+k\phi(N)} = (M^{\phi(N)})^k \cdot M$

It's sufficient to show  $M^n \equiv M \pmod{N}$

by showing  $M^n \equiv M \pmod{p}$  and  $M^n \equiv M \pmod{q}$

To show  $M^n \equiv M \pmod{p}$ ,

Case 1:  $p \mid M$ , then  $0^n \equiv 0 \pmod{p}$ , done.

Case 2:  $p \nmid M$ , then  $M^n = (M^{\phi(N)})^k \cdot M = (M^{(p-1)})^{(q-1)k} \cdot M$

By Fermat's Little Theorem,  $M^{p-1} \equiv 1 \pmod{p}$ .

Therefore  $M^n \equiv 1^{(q-1)k} \times M \pmod{p}$ . ■

## 15.1 RSA Public Key Procedures

Procedures:

1. **Receiver:** pick two large distinct primes  $p \neq q$  and calculate  $N = p \times q$ .
2. **Receiver:** calculate  $\phi(N) = (p-1)(q-1)$  and pick  $e$  relatively prime to  $\phi(N)$ .
3. **Receiver:** announce  $N$  and  $e$ .
4. **Sender:** choose message  $M \in \mathbb{N}$  satisfies  $M < N$  (if  $M \geq N$ , break  $M$  into pieces.)
5. **Sender:** find  $M^e \equiv R \pmod{N}$ .
6. **Sender:** announce the encoded message  $R$ .
7. **Receiver:** pick  $d \geq 0$  s.t.  $de + k\phi(N) = 1$  as the decoder. Such z-linear combination is guaranteed to exist.
8. **Receiver:** the original message  $M$  can be found by  $R^d \equiv M \pmod{N}$

*Proof.*  $R^d \equiv (M^e)^d \equiv M^{ed} \pmod{N}$

Since  $ed \equiv 1 \pmod{\phi(N)}$

By lemma 15.1,  $M^{ed} \equiv M \pmod{N}$ . ■

## 16 Lecture 16 Oct. 15 2018 RSA Cryptography Examples

### 16.1 Recall

1. **Receiver:** choose  $p, q \in \mathbb{P}$  and computes  $N = pq$ ,  $\phi(N) = (p-1)(q-1)$  and choose  $e$  s.t.  $\gcd(e, \phi(N)) = 1$ . Then announces  $e, N$ .
2. **Sender:** choose  $0 \leq M < N$  and calculate  $R$  such that  $R = M^e \pmod{N}$ .
3. **Receiver:** compute decoder  $d$  s.t.  $de + k\phi(N) = 1$ . And decode message  $M^* = R^d \pmod{N}$

### 16.2 More Examples

**Example 16.1. Receiver:** pick  $p = 11, q = 7$ . Calculate  $N = 77$  and  $\phi(N) = 10 * 6 = 60$ . Pick  $e = 13$  which is relatively prime to  $\phi(N)$ .

**Receiver:** announces  $N = 77$  and  $e = 13$  to sender.

**Sender:** pick message  $M = 71 < N$  and *encodes* message by computing  $71^{13} \equiv R \pmod{77}$ .

$$\begin{aligned}71 &\equiv -6 \pmod{77} \\71^3 &\equiv (-6)^3 \equiv 216 \equiv 15 \pmod{77} \\(71)^6 &\equiv (71^3)^2 \equiv 15^2 \equiv 225 \equiv -6 \pmod{77} \\(71)^{12} &\equiv (71^6)^2 \equiv (-6)^2 \equiv 36 \pmod{77} \\(71)^{13} &\equiv 36 \times (-6) \equiv -216 \equiv 15 \pmod{77}\end{aligned}$$

And calculate  $R = 15$  satisfies  $71^{13} \equiv 15 \pmod{77}$ .

**Sender:** announces  $R = 15$  to the rest of world.

**Receiver:** find  $d \geq 0$  satisfying  $d \times e + k \times \phi(N) = 1$ . as the *decoder*. And find that  $d = 37, k = -8$ .

**Receiver:** compute  $R^d \pmod{77}$

$$\begin{aligned}15^2 &\equiv 225 \equiv -6 \pmod{77} \\15^6 &\equiv -216 \equiv 15 \pmod{77} \\15^{12} &\equiv 15^2 \equiv -6 \pmod{77} \\15^{24} &\equiv 36 \pmod{77} \\15^{36} &\equiv -216 \equiv 15 \pmod{77} \\15^{37} &\equiv 15^2 \equiv 225 \equiv -6 \equiv 71 \pmod{77} \\ \implies M^* &= 15^{37} \pmod{77} = 71\end{aligned}$$



**Security of RSA** For anyone knowing  $N$  but does not know  $\phi(N)$ . To compute the decoder  $d$ ,  $\phi(N)$  needs to be calculated.

1. **Method 1:** Use definition and iterating through  $\{1, 2, \dots, N\}$  and compute  $\phi(N)$ .
2. **Method 2:** Factorize  $N$  and find  $p$  and  $q$ , then calculate  $\phi(N) = (p-1)(q-1)$ .

Both brute force methods are impractical in terms of run-time.

**Example 16.2. Receiver:** pick  $p = 11, q = 7$  then  $N = pq = 77$  and  $\phi(N) = (p-1)(q-1) = 60$  and choose  $e = 13$ .

**Receiver:** announce  $N = 77$  and  $e = 13$ .

**Sender:** pick  $M = 76 < 77$  and  $M^{13} \equiv (-1)^{13} \equiv -1 \equiv 76 \pmod{77}$ . Announce  $R = 76$ .

**Receiver:** find decoder  $d \geq 0$  s.t.  $d \times 13 + k \times 60 = 1$ . Found  $d = 13$ .

**Receiver:** Compute  $R^d \pmod{77}$ .

$$\begin{aligned} R &= 76 \equiv -1 \pmod{77} \\ R^{13} &\equiv (-1)^{13} \equiv -1 \equiv 76 \pmod{77} \\ \implies M^* &= R^{13} \pmod{77} = 76 \end{aligned}$$

## 17 Lecture 17 Oct. 17 2018

**Remark 17.1.** In RSA, picking the decoder  $d \geq 0$  s.t.  $de + \phi(N)k = 1$  is equivalent to pick  $d$  such that

$$de \equiv 1 \pmod{\phi(N)}$$

### 17.1 Chinese Remainder Theorem

**Theorem 17.1** (Chinese Remainder Theorem (CRT)). Solve system of congruent equations, where  $m_1$  and  $m_2$  are relatively prime,

$$\begin{cases} x \equiv a \pmod{m_1} \\ x \equiv b \pmod{m_2} \end{cases}$$

The solution is given by

$$x = ax_2m_2 + bx_1m_1$$

where  $x_1$  and  $x_2$  satisfy

$$\begin{cases} x_1m_1 \equiv 1 \pmod{m_2} \\ x_2m_2 \equiv 1 \pmod{m_1} \end{cases}$$

The general solution  $x$  is the superposition of two specific solutions.

*Proof.* If  $m_1$  and  $m_2$  are relatively prime, by Theorem 14.1,

$$\exists x_1, x_2 \in \mathbb{Z} \text{ s.t. } x_1 m_1 + x_2 m_2 = 1$$

Taking congruence with respect to mod  $m_1$  and  $m_2$  gives

$$\begin{cases} 1 \equiv x_2 m_2 \pmod{m_1} \\ 1 \equiv x_1 m_1 \pmod{m_2} \end{cases}$$

Consider

$$x = ax_2 m_2 + bx_1 m_1$$

Clearly

$$\begin{aligned} x - a &= a(x_2 m_2 - 1) + bx_1 m_1 \\ m_1 | (x_2 m_2 - 1) \wedge m_1 | bx_1 m_1 &\implies m_1 | x - a \\ &\implies x \equiv a \pmod{m_1} \end{aligned}$$

Similarly, we can show  $x \equiv b \pmod{m_2}$ .

Thus  $x$  is the solution to system of equations

$$\begin{cases} x \equiv a \pmod{m_1} \\ x \equiv b \pmod{m_2} \end{cases}$$

■

**Example 17.1.** Solve

$$\begin{cases} x \equiv 5 \pmod{7} \\ x \equiv 13 \pmod{8} \end{cases}$$

*Solution.* Solve

$$\begin{cases} x_1 \times 7 \equiv 1 \pmod{8} \\ x_2 \times 8 \equiv 1 \pmod{7} \end{cases}$$

Solve  $x_1 = 7$  and  $x_2 = 1$ . And one solution is given by

$$x = ax_2 m_2 + bx_1 m_1 = 5 \times 8 \times 1 + 13 \times 7 \times 7 = 677$$

■

## 17.2 Complex Numbers

**Definition 17.1** (9.1.3). A **complex number** is an expression of the form  $a + bi$  where  $a$  and  $b$  are real numbers. The real number  $a$  is called the *real part* of  $a + bi$ , denoted as  $\Re(a + bi)$ . And the real number  $b$  is called the *imaginary part* of  $a + bi$ , denoted as  $\Im(a + bi)$ .

**Definition 17.2.**

$$i^2 = -1$$

**Remark 17.2.** Shorthands for complex numbers

- $a + i0 = a$
- $0 + ib = ib$
- $0 + i0 = 0$

**Remark 17.3.**

$$\mathbb{C} \subset \mathbb{R}$$

**Definition 17.3.** Arithmetic on complex numbers are defined as following,

- **Addition**( $+$  :  $\mathbb{C}^2 \rightarrow \mathbb{C}$ ) is defined as

$$(a + ib) + (c + id) := (a + c) + i(b + d)$$

- **Multiplication**( $\times$  :  $\mathbb{C}^2 \rightarrow \mathbb{C}$ ) is defined as

$$(a + ib) \times (c + id) := (ac - bd) + i(ad + bc)$$

**Proposition 17.1.** Let  $z = a + ib \in \mathbb{C}$  be a complex number, the **multiplication inverse** of  $z$  is given by

$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2}$$

*Proof.*

$$\frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} = \frac{a - ib}{(a^2 + b^2) + i(ab - ab)} = \frac{a - ib}{a^2 + b^2}$$

■

## 18 Lecture 18 Oct. 19 2018 Complex Numbers

**Definition 18.1.** The **division** on complex numbers is equivalent to multiplying a complex number by its inverse, and is defined as

$$\frac{c + id}{a + id} = (c + id) \frac{a - ib}{a^2 + b^2}$$

**Notation 18.1.** The set of all complex numbers is denoted as

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i^2 = -1\}$$

**Remark 18.1.** Notice that  $\mathbb{C}$  is closed under the 4 basic operations of arithmetics. Anything like this is called a **field**.

**Example 18.1.** The irrational set  $\mathbb{Q}^c$  is *not* a field.

**Definition 18.2.** The **complex conjugate** of a complex number  $a + ib$  is defined as

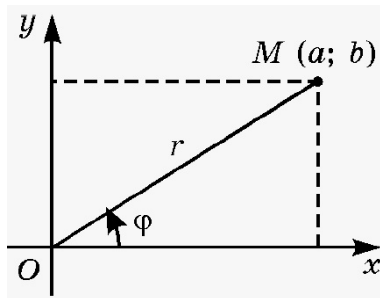
$$a - ib$$

**Definition 18.3.** The **modulus** of a complex number  $a + ib$  is defined as

$$|a + ib| = \sqrt{a^2 + b^2} \in \mathbb{R}_{\geq 0}$$

### 18.1 The Geometric Representation of Complex Numbers

Any complex could be represented as a vector in a 2-dimensional coordinate, with real line on the  $x$  axis and imaginary line on the  $y$  axis.



**Remark 18.2** (Geometrical Interpretation). The **modulus** is the *distance* from the point to the origin.

**Remark 18.3** (Geometrical Interpretation). The **conjugate** is the *reflection* of the point about the real ( $x$ ) axis.

## 18.2 Polar Coordinates

### 18.2.1 Coordinate Conversion

Consider a complex number represented by  $(a, b)$  in Cartesian coordinate and  $(r, \theta)$  in polar coordinate.

Cartesian	Polar
$(a, b)$	$(\sqrt{a^2 + b^2}, \arctan(\frac{b}{a}))$
$(r \cos(\theta), r \sin(\theta))$	$(r, \theta)$

**Remark 18.4.**  $\arctan(\frac{b}{a})$  gives multiple solutions due to the periodicity of  $\tan$ . We need to use the signs of real and imaginary values to determine which value of  $\arctan \frac{b}{a}$  to take.

**Example 18.2.** Consider  $1 + i$ , it could be represented as  $(1, 1)$  in Cartesian coordinate. Converting it into polar coordinates gives  $(\sqrt{2}, \frac{\pi}{4})$ . Converting back gives

$$\begin{aligned} 1 + i &= \sqrt{2}(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})) \\ &= \sqrt{2}(\cos(\frac{\pi}{4} + 2\pi k) + i \sin(\frac{\pi}{4} + 2\pi k)), \quad \forall k \in \mathbb{Z} \end{aligned}$$

### 18.2.2 Multiplication in Polar Coordinates

Consider the product of two complex numbers  $r_1(\cos(\theta_1) + i \sin(\theta_1))$  and  $r_2(\cos(\theta_2) + i \sin(\theta_2))$ :

$$\begin{aligned} &r_1(\cos(\theta_1) + i \sin(\theta_1)) \times r_2(\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 \left[ (\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + i (\cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2)) \right] \end{aligned}$$

By triangle inequality

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

**Example 18.3.** Every complex number has a square root.

*Proof.* Let  $z = r(\cos(\theta) + i \sin(\theta)) \in \mathbb{C}$ .

Consider  $w = \sqrt{r}(\cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2}))$

By above result, we could easily verify that  $w^2 = z$ .

Notice that,  $w = \sqrt{r}(\cos(\frac{\theta}{2} + \pi) + i \sin(\frac{\theta}{2} + \pi))$  is also a square root of  $z$ . ■

## 19 Lecture 19 Oct. 22 2018

### 19.1 De Moivre's Theorem

**Theorem 19.1.** (De Moivre's Theorem) Let  $z = r[\cos(\theta) + i \sin(\theta)] \in \mathbb{C}$ , and the  $n^{\text{th}}$  power of  $z$  is given by

$$(r[\cos(\theta) + i \sin(\theta)])^n = r^n[\cos(n\theta) + i \sin(n\theta)], \quad \forall n \in \mathbb{N}$$

*Proof.* (By induction)

**Base Case** for  $n = 1$ , obviously  $z^1 = z$

**Inductive Step** let  $k \in \mathbb{N}$ ,

suppose  $z^k = r^k[\cos(k\theta) + i \sin(k\theta)]$

Consider  $z^{k+1}$ ,

$$\begin{aligned} r^{k+1} &= r^k[\cos(k\theta) + i \sin(k\theta)] \times r[\cos(\theta) + i \sin(\theta)] \\ &= r^{k+1}[(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\cos(k\theta)\sin(\theta) + \sin(k\theta)\cos(\theta))] \\ &\quad \text{By Triangle Identity} \\ &= r^{k+1}[\cos((k+1)\theta) + i \sin((k+1)\theta)] \end{aligned}$$

We could then conclude what the theorem stated by principle of mathematical induction. ■

**Example 19.1.** Calculate  $(1 + i)^8$ .

*Solution.*  $1 + i$  can be written as  $(1, 1)$  in Cartesian coordinate.

Then it can be converted into Polar coordinate as

$$\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$

Then by De Moivre's theorem,

$$\begin{aligned} &\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)\right)^8 \\ &= (\sqrt{2})^8 \left(\cos\left(\frac{\pi}{4} \times 8\right) + i \sin\left(\frac{\pi}{4} \times 8\right)\right) \\ &= 16 \times (\cos(2\pi) + i \sin(2\pi)) \\ &= 16(\cos(0) + i \sin(0)) \\ &= 16 \end{aligned}$$

■

*Geometrically Interpretation:* rotates the vector anti-clockwise by  $(n - 1)\theta$  and enlarge the magnitude by factor of  $n$ .

## 19.2 Roots of Unity

**Example 19.2.** Find all roots of  $z^2 = 1$ , where  $z \in \mathbb{C}$ .

*Solution.* In polar coordinates, let  $z = r(\cos \theta + i \sin \theta)$ .  
Thus by De Moivre's Theorem,  $z^2 = r^2(\cos(2\theta) + i \sin(2\theta))$ .  
And  $1 = 1(\cos(0 + 2k\pi) + i \sin(0 + 2k\pi))$ ,  $k \in \mathbb{Z}$  in polar coordinate.  
Solving the equation  $z^2 = 1$  gives

$$\begin{cases} r^2 = 1 \\ 2\theta = 2k\pi, k \in \mathbb{Z} \end{cases}$$

We can conclude that  $r = 1$  since it represents a *distance* and  $r \in \mathbb{R}_{\geq 0}$ .

- $k = 0$  :  $r = 1, \theta = 0 \rightarrow 1(\cos(0) + i \sin(0)) = 1$
- $k = 1$  :  $r = 1, \theta = \pi \rightarrow 1(\cos(\pi) + i \sin(\pi)) = -1$
- $k = 2$  :  $r = 1, \theta = 2\pi \rightarrow 1(\cos(2\pi) + i \sin(2\pi)) = 1$

■

From the repeating pattern we can conclude that  $\forall k \in \mathbb{Z}$  <sup>5</sup>

$$z = 1(\cos(\pi k) + i \sin(\pi k)) = \pm 1$$

**Example 19.3.** Find all roots of  $z^n = 1$

*Solution.*

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

$$1 = 1(\cos(2k\pi) + i \sin(2k\pi))$$

$$\implies r = 1$$

$$\wedge n\theta = 2k\pi \iff \theta = k \frac{2\pi}{n}$$

Consider cases

- $k = 0$  :  $r = 1, \theta = 0$
- $k = 1$  :  $r = 1, \theta = \frac{2\pi}{n}$

---

<sup>5</sup>The case  $k < 0$  is covered by symmetry.

- $k = 2$  :  $r = 1, \theta = 2\frac{2\pi}{n}$
- $k = 3$  :  $r = 1, \theta = 3\frac{2\pi}{n}$

Until  $k = n$ , we have  $r = 1 \wedge \theta = n\frac{2\pi}{n} = 2\pi$ , where  $z|_{k=n} = z|_{k=0}$  and the root starts repeating.

There are  $n$  roots in total,

$$z = \cos(k\frac{2\pi}{n}) + i\sin(k\frac{2\pi}{n}), k \in \{0, 1, \dots, n-1\}$$

■

**Example 19.4.** Solve  $z^3 = 1$

**Example 19.5.** Solve  $z^4 = 1$

**Geometrically Interpretation** Divides the unit ball into  $n$  equal slices.

**Example 19.6.** Solve  $z^3 = 2 + 2i$

*Solution.* In polar coordinate,

$$2 + 2i = \sqrt{8}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)$$

We have to solve

$$\begin{aligned} 3\theta &= \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z} \\ \implies \theta &= \frac{\pi}{12} + k\frac{2\pi}{3}, k \in \mathbb{Z} \end{aligned}$$

And clearly  $r = \sqrt{2}$ . And roots are found by plugging in  $k$  with 0, 1, 2.

$$\begin{cases} z_1 = \sqrt{2}(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12})) \\ z_2 = \sqrt{2}(\cos(\frac{\pi}{12} + \frac{2\pi}{3}) + i\sin(\frac{\pi}{12} + \frac{2\pi}{3})) \\ z_3 = \sqrt{2}(\cos(\frac{\pi}{12} + \frac{4\pi}{3}) + i\sin(\frac{\pi}{12} + \frac{4\pi}{3})) \end{cases}$$

■



## 20 Lecture 20. Oct 24 2018

**Theorem 20.1** (The Fundamental Theorem of Algebra). Every *non-constant* polynomial (with complex coefficients) has a complex root. i.e. for

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_i \in \mathbb{C}, \quad n \geq 1$$

there exists  $r \in \mathbb{C}$  such that  $p(r) = 0$ .

**Example 20.1.** Let  $p(x) = x^3 - 3x^2 - 9x + 27$ ,  $p(x) = (x - 3)^2(x + 3)$ .

**Interpretation** *Linear polynomials (polynomials of degree 1) are building blocks of all polynomials.*

**Theorem 20.2** (Long Division). Suppose  $p(z)$  is a non-constant polynomial and  $r \in \mathbb{C}$ , there exists a polynomial  $q(z) \in \mathcal{P}$  and  $c \in \mathbb{C}$  such that

$$p(z) = q(z)(z - r) + c$$

where  $c$  is the *remainder* of long division.

**Definition 20.1.** A polynomial  $f(z)$  is a **factor** of another polynomial  $p(z)$  if

$$\exists q(z) \in \mathcal{P}, \text{ s.t. } p(z) = f(z)q(z)$$

**Theorem 20.3** (Factor Theorem). A complex number  $r$  is a root of a polynomial  $p(z)$  if and only if  $(z - r)$  is a factor of  $p(z)$ .

*Proof.* (  $\Leftarrow$  ) Suppose  $(z - r)$  is a factor.

By definition of factor,  $\exists q(z) \in \mathcal{P}$  such that  $p(z) = q(z)(z - r)$ .

Plugging in  $r$  gives  $p(r) = q(r)(r - r) = 0$  and suggests  $r$  is a root of  $p(z)$ .

(  $\Rightarrow$  ) Suppose  $r$  is a root of  $p(z)$ .

By the theorem of long division,  $\exists q(z) \in \mathcal{P}$  and  $c \in \mathbb{C}$  satisfying

$$p(z) = q(z)(z - r) + c.$$

Plugging in  $z = r$  gives  $p(r) = q(r)(r - r) + c = 0$ , which implies  $c = 0$ .

That's  $p(z) = q(z)(z - r)$ . ■

**Theorem 20.4** (Extended Fundamental Theorem of Algebra). A non-zero<sup>6</sup> polynomial of degree  $n$  has exactly  $n$  roots, counting multiplicities.

*Proof.* Let  $p(z) \in \mathcal{P}$  with degree  $n \geq 0$  and suppose  $p(z)$  is non-zero.

Case 1:  $n = 0$ , then  $p(z)$  has 0 roots.

Case 2:  $n \geq 1$ , by the fundamental theorem of algebra,

---

<sup>6</sup>For zero polynomial, it has infinitely many roots.

$p(z)$  has a root  $r_1 \in \mathbb{C}$ .

By factor theorem,  $\exists q_1(z) \in \mathcal{P}(\mathbb{C})$ , s.t.  $p(z) = (z - r_1)q_1(z)$ .

Note that  $q_1(z)$  has degree of  $n - 1$ .

If  $n - 1 \geq 1$ , repeating above argument and we have

$\exists r_2 \in \mathbb{C}$ ,  $\exists q_2(z) \in \mathcal{P}(\mathbb{C})$ , s.t.  $q_1(z) = (z - r_2)q_2(z)$ .

Note that  $q_2(z)$  has degree of  $n - 2$ .

Equivalently  $p(z) = (z - r_1)(z - r_2)q_2(z)$ .

Iterating till  $q_i(z)$  has degree 0 (i.e. constant), this will be achieved after exactly  $n$  iterations.

Aggregately, we can factorize  $p(z)$  into

$$p(z) = (z - r_1)(z - r_2) \dots (z - r_n)q_n$$

where  $q_n$  is a constant.

Obviously there are  $n$  (possibly repeating) roots, namely  $r_1, r_2, \dots, r_n$ . ■

## 21 Lecture 21. Oct 29 2018

**Lemma 21.1** (Triangle Inequality).

$$|z_1 + z_2| \leq |z_1| + |z_2|, \forall z_1, z_2 \in \mathbb{C}$$

**Lemma 21.2** (Extended version of triangle inequality).

$$\left| \sum_{i=1}^n z_i \right| \leq \sum_{i=1}^n |z_i|, \forall (z_i) \in \mathbb{C}^n$$

**Definition 21.1.** A **closed curve in the plane** is a continuous function mapping from  $[0, 2\pi]$  to  $\mathbb{C}$  such that its values at 0 and  $2\pi$  are the same.

**Definition 21.2.** If  $\phi(t) : [0, 2\pi] \rightarrow \mathbb{C}$  is a closed curve that **does not go through the origin**, its **winding number** is the number of times a vector from the origin to a point on the curve winds around the origin as  $t$  goes from 0 to  $2\pi$ .

**Example 21.1.** Consider

$$\phi(t) = f(t) + i(g)$$

where  $f, g : [0, 2\pi] \rightarrow \mathbb{R}$  are continuous. Then  $\phi(t)$  is continuous.

**Example 21.2.** Consider

$$\phi(t) = \cos(t) + i \sin(t)$$

the function above is a closed curve with winding number +1.

**Remark 21.1.** If points on the curve go around the origin *anti-clockwise* as  $t$  goes from 0 to  $2\pi$ , then we consider the winding number to be *negative*.

**Example 21.3.** Curve  $\phi(t) = \cos(3t) + i \sin(3t)$  has winding number +3.

**Example 21.4.** Curve  $\phi(t) = 27 \cos(4t) + 27i \sin(4t)$  has winding number +4.

**Example 21.5.** Curve  $\phi(t) = \sin(t) + i \cos(t)$  has winding number  $-1$ .

**Example 21.6.** A non-zero constant (e.g.  $\phi(t) = 3 + 4i$ ) is closed and not passing the origin, it has winding number 0.

**Remark 21.2.** The notation of winding number only apply to closed curves that do not passes the origin.

## 21.1 Proof of the Fundamental Theorem of Algebra

*Proof.* Idea: prove by contradiction.

Suppose  $p(z)$  is a non-constant polynomial with no roots.  
i.e.

$$p(z) \neq 0, \forall z \in \mathbb{C}$$

and degree of  $p(z) = n > 0$ .

For each radius  $R > 0$  define

$$\phi_R(t) := R(\cos(t) + i \sin(t))$$

Then for each  $R > 0$ , let

$$p_R(t) := p(\phi_R(t))$$

note that  $p_R(t) : [0, 2\pi] \rightarrow \mathbb{C}$  and it's a closed curve.

Also note that since  $p_R(t) \neq 0 \forall t \in [0, 2\pi]$ ,  $p_R(t)$  does not go through the origin.

We will show that

1. If  $R$  is large enough then the winding number of  $p_R(t) = \deg(p(z))$ .
2. If  $R$  is small enough then the winding number of  $p_R(t) = 0$ .

But the winding number of  $p_R(t)$  is a continuous function of  $R$  and it has co-domain of integers. Then it must be the case that  $p_R(t)$  is constant, but this contradicts our assumption that  $\deg(p(z)) > 0$ . ■