ECO326 Advanced Microeconomic Theory

A Course in Game Theory

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`Github Page` `https://github.com/TianyuDu/Spikey_UofT_Notes`
`Note Page` `TianyuDu.com/notes`

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory*.

Contents

1 Lecture 1. Games and Dominant Strategies

Assumption 1.1 (pg.4)**.** Assume that each decision-maker is *rational* in the sense that he is aware of his alternatives, forms expectation about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization.

Definition 1.1 (pg.4)**.** A model of **rational choice** consists

- A set A of *actions.*
- A set C of *consequences.*
- A *consequence function* $g : A \to C$.
- A *preference relation* $\succsim$ on C.

Definition 1.2 (pg.7)**.** A **preference relation** is a complete reflexive and transitive binary relation.

Definition 1.3 (11.1)**.** A **strategic game** consists of

- a finite set of **players** N.
- for each player $i \in N$, an **actions** $A_i \neq \emptyset$.
- for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i) \rangle$.

Definition 1.4 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2 Lecture 2. Iterated Elimination and Rationalizability

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (60.2)**.** The set $X \subseteq A$ of outcomes of a finite strategic game $\langle N, (A_i), (u_i) \rangle$ **survives iterated elimination of strictly dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \ldots, T-1$.
- For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\langle N, (X_i^t), (u_i^t) \rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.
- No action in X_t^T is strictly dominated in game $\langle N, (X_i^T), (u_i^T) \rangle$.

Proposition 2.1 (61.2)**.** If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\langle N, (A_i), (u_i) \rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

2.2 Rationalizability

Definition 2.2 (pg.54)**.** A **belief** of player i (about the actions of the other players) is a probability measure, μ_i, on $A_{-i} = \times_{j \in N \backslash \{i\}} A_j$.
μ_i is a mapping such that

- $\mu_i : A_{-i} \to [0, 1]$.
- $\mu_i(A_{-i}) = 1$.
- For all countable piece-wise disjoint collection $\{E_i\}_{i \in I}$, it satisfies the *countable additivity property*:

$$\mu_i(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu_i(E_i)$$

Definition 2.3 (lec.2)**.** For a player $i \in N$, $a_i^* \in A_i$ is the **best response to belief** μ_i in a strategic game $\langle N, (A_i), (u_i) \rangle$ if and only if

$$\forall a_i \in A_i, \quad \sum_{a_{-i} \in A_{-i}} u_i(a_i^*, a_{-i}) \mu_i(a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \mu_i(a_{-i})$$

Equivalently,

$$\forall a_i \in A_i, \ \mathbb{E}[u_i(a_i^*, a_{-i})] \geq \mathbb{E}[u_i(a_i, a_{-i})]$$

Definition 2.4 (59.1)**.** An action of player i in a strategic game is a **never best response** if it is not a best response to any belief of player i.

3 Lecture 3. Nash Equilibrium

Definition 3.1 (14.1)**.** A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $a^* \in A$ of actions with property that for every player $i \in N$

$$(a_i^*, a_{-i}^*) \succsim_i (a_i, a_{-i}^*) \ \forall a_i \in A_i$$

Definition 3.2 (pg.15)**.** The **best-response function** for a player i is defined as

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i}) \ \forall a_i' \in A_i\}$$

Remark 3.1. The best-response of a_{-i} can be written as

$$B_i(a_{-i}) = \bigcap_{a_i' \in A_i} \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i})\}$$

where each of them is the upper contour set of a_i'.
Thus, if $\succsim_i$ is quasi-concave, then $B_i(a_{-i})$ is an intersection of convex sets and therefore itself convex.

Remark 3.2 (pg.15)**.** So a Nash equilibrium is a profile $a^* \in A$ such that

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

Lemma 3.1 (pg.19)**.** A strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if equivalent to the following statement:
Define set-valued function $B : A \to A$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and there exists $a^* \in A$ such that $a^* \in B(a^*)$.

Lemma 3.2 (20.1 Kakutani's fixed point theorem)**.** Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- for all $x \in X$ the set $f(x)$ is non-empty and convex.
- the graph of f is closed. *(i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$ then $y \in f(x)$)*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Definition 3.3 (pg.20)**.** A preference relation $\succsim_i$ over A is quasi-concave on A_i if for every $a^* \in A$ the upper contour set over a_i^*, given other players' strategies

$$\{a_i \in A_i : (a_{-i}^*, a_i) \succsim_i a^*\}$$

is convex.

Proposition 3.1 (20.3)**.** The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if for all $i \in N$,

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous
- quasi-concave on A_i.

Proof. Let $B : A \to A$ be a correspondence defined as

$$B(a) := \times_{i\in N} B_i(a_{-i})$$

Note that for each $a \in A$ and for each $i \in N$,
$B_i(a_{-i}) \neq \emptyset$ since preference $\succsim_i$ is continuous and A_i is compact (EVT).
Also $B_i(a_{-i})$ is convex since it's basically an intersection of upper contour sets and each of those upper contour is convex since $\succsim_i$ is quasi-concave.
So the Cartesian product of the finite collection of B_i is non-empty and convex.
Also the graph B is closed since $\succsim_i$ is continuous.
So there exists $a^* \in A$ such that $a^* \in B(a^*)$.
So Nash equilibrium presents. ∎

4 Lecture 6. Extensive Form Games and Subgame Perfection

4.1 Extensive Form Game

Definition 4.1 (89.1)**.** An **extensive game with perfect information** has the following components.

- A set N of **players**.
- A set H of sequences (finite or infinite) of **histories** with properties:

- $\emptyset \in H$.
- For all $L < K$, $(a^k)_{k=1,2,\ldots,K} \in H \implies (a^k)_{k=1,2,\ldots,L} \in H$.
- For infinite sequence $(a^k)_{k=1}^{\infty}$,
  $(a^k)_{k=1,2,\ldots,L} \in H,\ \forall L \in \mathbb{Z}_{++} \implies (a^k)_{k=1}^{\infty} \in H$.

And each component of history $h \in H$ is an **action** taken by a player.

- A function $P : H \backslash Z \to N$, where for $h \in H$, $P(h) \in N$ is defined by the player who takes an action after the history h.
- For each player $i \in N$ a **preference relation** $\succsim_i$ defined on Z.

Notation 4.1 (pg.90)**.** An extensive game with perfect information can be represented by a 4-tuple, $\langle N, H, P, (\succsim_i)\rangle$. *Sometimes it is convenient to specify the structure of an extensive game without specifying the players' preference, as* $\langle N, H, P\rangle$.

Definition 4.2 (pg.90)**.** A history $(a^k)_{k=1,2,\ldots,K} \in H$ is **terminal** if

1. it is infinite,
2. or (i.e. it cannot be extended to another valid history sequence)

$$\forall a^{K+1},\ (a^k)_{k=1,2,\ldots,K+1} \notin H$$

The set of terminal histories is denoted by Z.

Notation 4.2 (pg.90, the action set)**.** After any nonterminal history, $h \in H \backslash Z$, the player $P(h)$ chooses an action from set

$$A(h) = \{a : (h, a) \in H\}$$

Remark 4.1. Note that all player function, action set and player preference relation are defined on H. Thus, unlike a normal form game, which was *player oriented*, we'd better consider an extensive form game as *history oriented*.

Definition 4.3 (pg.90)**.** We refer to the empty set, which is required to be an element of H, as the **initial history**.

Definition 4.4 (92.1)**.** A **strategy of player** $i \in N$, s_i, in an extensive game with perfect information $\langle N, H, P, (\succsim_i)\rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Remark 4.2 (pg.92)**.** A strategy specifies the action chosen by a player for *every* history after which it is his turn to move, *even for histories that is, if the strategy is followed, are never reached.*

Definition 4.5 (pg.93)**.** For each strategy profile $s = (s_i)_{i \in N}$ in the extensive game $\langle N, H, P, (\succsim_i) \rangle$, the **outcome** of s, $O(s)$, is defined as the terminal history that results when each player $i \in N$ follows the precepts of s_i. That is, $O(s)$ is the (possibly infinite) history

$$(a^1, \ldots, a^K) \in Z$$

such that

$$\forall k \in \{0, 1, \ldots K-1\},\ s_{P(a^1, \ldots, a^k)}(a^1, \ldots, a^k) = a^{k+1}$$

Definition 4.6 (93.1)**.** A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$\forall s_i \in S_i,\ O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

Definition 4.7 (94.1)**.** The **strategic form of the extensive game with perfect information**, $\Gamma = \langle N, H, P, (\succsim_i) \rangle$, is the strategic game $\langle N, (S_i), (\succsim'_i) \rangle$ in which for each player $i \in N$

- S_i is the **set of strategies** of player i in Γ.
- $\succsim'_i$ is defined on $\times_{i \in N} S_i$ and defined by

$$\forall s, s' \in \times_{i \in N} S_i,\ s \succsim'_i s' \iff O(s) \succsim_i O(s')$$

Definition 4.8 (pg.94)**.** A **reduced strategy** of player i is defined to be a function f_i whose domain is a *subset* of $\{h \in H : P(h) = i\}$ and has the following properties

1. it associates with every history h in the domain of f_i an action in $A(h)$.
2. a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i. (i.e., for any $h = (a^k)$ and for any $h' = (a^k)_{k=1}^{L}$ as a subsequence of h such that $P(h') = i$, $f_i(h') = a^{L+1}$.)

Remark 4.3 (pg.94)**.** Each **reduced strategy** of player i corresponds to a set of strategies of player i, such that for each vector of strategies of the other players each strategy in this set yields the same outcome. (strategies in the same set are **outcome-equivalent**.)
That's, for each strategy $s_i \in S_i$, its reduced strategy can be defined with an outcome equivalence class, $[s_i]$,

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j,\ O(s_{-i}, s_i) = O(s_{-i}, s_i')\}$$

But in some other game, the definition of outcome-equivalence is more general and defined by generating the same payoff (through possibly difference outcomes), then the reduced strategy is defined as

$$[s_i] \equiv \{s_i' \in S_i : \forall s_{-i} \in \times_{j \in N \setminus \{i\}} S_j,\ \forall j \in N,\ O(s_{-i}, s_i) \sim_j O(s_{-i}, s_i')\}$$

Definition 4.9 (95.1.1)**.** Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim_i') \rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i, s_i' \in S_i$ to be **equivalent** if

$$\forall s_{-i} \in S_{-i},\ \forall j \in N,\ (s_{-i}, s_i) \sim_j' (s_{-i}, s_i')$$

Definition 4.10 (95.1.2)**.** The **reduced strategic form of** Γ is the strategic game $\langle N, (S_i'), (\succsim_i'') \rangle$ in which for each $i \in N$ each set S_i' contains one member of each set of equivalent strategies in S_i and $\succsim_i''$ is the preference ordering over $\times_{j \in N} S_j'$ induced by $\succsim_i'$.

4.2 Subgame Perfection

Definition 4.11 (97.1)**.** The **subgame of extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h) \rangle$ where

- $H|_h$ is the set of sequences h' such that $(h, h') \in H$.
- $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$.
- $\succsim_i |_h$ is defined by $h' \succsim_i |_h h'' \iff (h, h') \succsim_i (h, h'') \in Z$.

Notation 4.3 (pg.97)**.** Given strategy $s_i \in S_i$ and $h \in H \in \Gamma$, $s_i|_h$ represents the **strategy that** s_i **induces in the subgame** $\Gamma(h)$. That's, for each $h' \in H_h$

$$s_i|_h(h') \equiv s_i(h, h')$$

Notation 4.4. Let O_h denote the **outcome function of** $\Gamma(h)$, that's, for all $h' \in H|_h$,

$$O_h(h') \equiv O(h, h')$$

Definition 4.12 (97.2)**.** A **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i |_h O_h(s^*_{-i}|_h, s_i|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.

Definition 4.13 (pg.97)**.** Equivalently, define SPNE to be a strategy profile s^* in Γ for which for any history $h \in H$ the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Remark 4.4 (pg. 97)**.** The notion of SPNE requires the action prescribed by each player's strategy to be optimal, given other players' strategies, after *every* history.

Proposition 4.1 (99.2)**.** Every finite extensive game with perfect information has a subgame perfect equilibrium.