Notes on MAT137 Video Playlist 3

Tianyu Du

Saturday 14th April, 2018

Info.
Created: September, 2017
Last modified: Saturday 14th April, 2018
Revision: 00:40 Saturday 14th April, 2018

Contents

1   Video Playlist 1

2   Video Playlist 2

3   Video Playlist 3

3.1   Define Derivate As Slope

Definition   Let $a \in \mathbb{R}$, and f(x) is defined on $(a-\delta, a+\delta)$, then the **derivative** of f(x) at a is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Definition   If function is **differentiable** at point x = a, if and only if, there exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Interpretation $f'(a)$ is the slope of tangent line a x = a.

3.2   Calculate f'(x) by definition

Example $f(x) = 4x - x^2$, find $f'(1)$:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{4(h+1) - (h+1)^2 - 3}{h}$$
$$= \lim_{h \to 0} \frac{4h + 4 - 3 - h^2 - 2h - 1}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h}$$
$$= \lim_{h \to 0} -h + 2 = 2$$

3.3   Rate of Change

Definition   Define derivative as rate of change. Let $x = f(t)$, then $f'(x)$ can be represented as,

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = f'(t) = \frac{dx}{dt}$$

3.4   The Product Rule (Formal Version)

Let $a \in \mathbb{R}$, f and g are functions defined at $(a - \delta, a + \delta)$, let $h(x) = f(x)g(x)$. Then, if $f(x), g(x)$ are differentiable at a, we have,

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

3.5   Differentiable $\implies$ Continuous

Recall   f(x) is **differentiable** at a:

$$\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

Recall   f(x) is **continuous** at a:

$$\lim_{x \to a} f(x) = f(a) \tag{2}$$

Proof.

$$\text{Since f(x) is differentiable at a}$$

$$(1) \iff \exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{And } \lim_{x \to a}(x - a) = 0$$

$$\implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} x - a = 0$$

$$\implies \lim_{x \to a} \frac{f(x) - f(a)}{x - a} x - a = 0$$

$$\implies \lim_{x \to a} f(x) - f(a) = 0$$

$$\implies \lim_{x \to a} f(x) = f(a)$$

$$\blacksquare$$

3.6   Proof of product rule for derivative.

$(fg)' = f'g + fg'$, see above for a formal definition.

$$\text{Let } h = fg$$

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)g(x) + f(a)g(x) - f(a)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))}{x - a}$$

$$= \lim_{x \to a} g(x)\frac{f(x) - f(a)}{x - a} + \lim_{x \to a} f(a)\frac{g(x) - g(a)}{x - a}$$

$$= g(a)\lim_{x \to a}\frac{f(x) - f(a)}{x - a} + f(a)\lim_{x \to a}\frac{g(x)g(a)}{x - a}$$

$$= g(a)f'(a) + f(a)g'(a)$$

$$\blacksquare$$

3.7   Partial proof of differentiation rule

WTS $\frac{d}{dx}x^c = cx^{c-1}, \forall c \in \mathbb{R}$

Here we only prove statements is true $\forall c \in \mathbb{Z}^+$

Proof.

$$\textbf{Base: c = 1}$$

$$f(x) = x$$

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} 1 = 1$$

$$\textbf{Induction step}$$

$$\text{Assume } \frac{d}{dx}[x^k] = kx^{k-1}|_{x=a}$$

$$\text{For } f(x) = x^{k+1}$$

$$f'(x) = \frac{d}{dx}[x * x^k]$$

$$= x^k + xkx^{k-1}$$

$$= (k+1)x^k$$

$$\blacksquare$$

3.8   Higher Order Derivatives: Notations

Original function: $f(x)$

- **Lagrange** notation: $f^{(n)}$
- **Leibnitz** notation: $\frac{d^n f}{dx^n}$

3.9   Continuous But Not differentiable

Definition   Function $f(x)$ is **non-differentiable** at a.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \textbf{ DNE}$$

Example 1   **Corner/Kink** $f(x) = |x|$ at 0.

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

$$\lim_{x \to 0^-} \neq \lim_{x \to 0^+}$$

$$\implies \lim_{x \to 0} \frac{f(x) - f(0)}{x} \textbf{ DNE}$$

Example 2   Vertical Tangent Line $g(x) = x^{\frac{1}{3}}$ at 0,

$$g'(0) = \lim_{x \to 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty(\textbf{DNE})$$

Caution   Difference between **vertical asymptote** and **vertical tangent line**

- Vertical asymptote: $f(a) = \infty$ (f(a) is not defined)

- Vertical tangent line: $f(a)$ is defined, $f'(a)$ is undefined.

3.10   Chain Rule

Derivation

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$$

$$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$$

Attention: we could only apply the operation above if $f(x) \neq f(a)$ during the process of $x \to a$.
This holds for majority of functions we operate in calculus.

$$= \lim_{f(x) \to f(a)} \frac{g(f(x)) - g(f(a))}{x - a} f'(a)$$

$$= g'(f(a)) \cdot f'(a)$$

$$\blacksquare$$

Formal Theorem of Chain Rule   Let $a \in \mathbb{R}$, let f and g be functions. If f is differentiable at a and g is differentiable at $f(a)$, then, $(g \circ f)$ is differentiable at a,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

3.11   Derivatives of Trig Functions

Basic 6 results

1. $\frac{d}{dx} sin(x) = cos(x)$

2. $\frac{d}{dx} cos(x) = -sin(x)$

3. $\frac{d}{dx} tan(x) = sec^2(x)$

4. $\frac{d}{dx} cot(x) = -csc^2(x)$

5. $\frac{d}{dx} sec(x) = sec(x)tan(x)$

6. $\frac{d}{dx} csc(x) = -csc(x)cot(x)$

Proof. Prove (i) and (ii) and use (i), (ii) and quotient rule to derive (iii), (iv), (v) and (vi).

Proof. (i) **WTS** $f(x) = sin(x)$, then $f'(x) = cos(x)$

$$f'(x) = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h}$$
$$= \lim_{h \to 0} \frac{sin(x)cos(h) + cos(x)sin(h) - sin(x)}{h}$$
$$= \lim_{h \to 0} \frac{sin(x)(cos(h) - 1) + cos(x)sin(x)}{h}$$
$$= \lim_{h \to 0} cos(x)\frac{sin(h)}{h}$$
$$= cos(x)$$

$\blacksquare$ (3)

Proof. (ii) **WTS** $f(x) = cos(x)$, then $f'(x) = -sin(x)$

$$f'(x) = \lim_{h \to 0} \frac{cos(x+h) - cos(x)}{h}$$
$$= \lim_{h \to 0} \frac{cos(x)cos(h) - sin(h)sin(x) - cos(x)}{h}$$
$$= \lim_{h \to 0} \frac{(cos(h) - 1)cos(x) - sin(h)sin(x)}{h}$$
$$= \lim_{h \to 0} -\frac{sin(h)}{h}sin(x)$$
$$= -sin(x)$$

$\blacksquare$ (4)

Recall Compound angle formula:

1. $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + sin(\beta)cos(\alpha)$

2. $sin(\alpha - \beta) = sin(\alpha)cos(\beta) - sin(\beta)cos(\alpha)$

3. $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$

4. $cos(\alpha - \beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$

3.12   Implicit Differentiation

Key Use chain rule.

3.13   Derivative of Exponential Functions

Let $f(x) = a^x$ $(a > 0)$, find $f'(x)$, by definition,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h \to 0} \frac{(a^n - 1)a^x}{h}
\end{aligned}
$$

By property of limit, h is the only variable, so that a^x is a constant

$$
= a^x \lim_{h \to 0} \frac{a^h - 1}{h}
$$

$$\tag{5}$$

Equivalently, $\frac{d}{dx} a^x = L_a a^x$

Definition e is the only positive number, such that,

$$
\lim_{h \to 0} \frac{e^h - 1}{h} = 1
$$

So that, $\frac{d}{dx} e^x = e^x$

3.14   Properties of logarithms

Definition   Let $a > 0, a \neq 1, x > 0, y \in \mathbb{R}$,

$$
\log_a x = y \iff a^y = x
$$

Properties

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a x = \frac{\log_b x}{\log_b a}$

4. $\log_a xy = \log_a x + \log_a y$

5. $\log_a \frac{x}{y} = \log_a x - \log_a y$

6. $\log_a x^r = r \log_a x$

Proof. (i)   let $a > 0, a \neq 1, let x, y > 0$, **WTS** $\log_a xy = \log_a x + \log_a y$

$$\text{Let } p = \log_a x \iff a^p = x$$
$$\text{Let } q = \log_a y \iff a^q = y$$
$$\text{We have } a^p a^q = xy$$
$$\iff a^{p+q} = xy$$
$$\iff \log_a xy = p + q = \log_a x + \log_a y$$
$$\blacksquare$$

3.15   The derivatives of logarithm functions

For $\ln x \quad \frac{d}{dx} \ln x = \frac{1}{x}$

$$e^{\ln x} = x$$
$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$
$$\frac{d}{d \ln x} e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$$
$$x \frac{d \ln x}{dx} = 1$$
$$\frac{d}{dx} \ln x = \frac{1}{x}$$
$$\blacksquare$$

3.16   Derivative of other exponentials

WTS $\quad \frac{d}{dx} a^x = \ln a \cdot a^x,$

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$
$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$
$$= \frac{d}{dx} e^{x \ln a} \cdot \frac{d}{dx} \ln a$$
$$= e^{x \ln a} \ln a$$
$$= \ln a \cdot a^x$$
$$\blacksquare$$

3.17   The power rule, complete proof

WTS $x^c = cx^{c-1}$

$$x^c = (e^{\ln x})^c = e^{c\ln x}$$

$$\text{So that } \frac{d}{dx}x^c = \frac{d}{dx}e^{c\ln x}$$

$$= \frac{de^{c\ln x}}{d\ln xc} \cdot \frac{\ln xc}{d\ln x} \cdot \frac{d\ln x}{dx}$$

$$= e^{c\ln x} \cdot c \cdot \frac{1}{x}$$

$$= c \cdot x^c \cdot \frac{1}{x}$$

$$= cx^{c-1}$$

$$\blacksquare$$

3.18   Logarithmic Differentiation

Example $f(x) = cos(x)^{sin(x)}(\star)$, find $f'(x)$
   Step1. Take ln on both sides of $(\star)$

$$\ln f(x) = \ln cos(x)^{sin(x)} = sin(x)\ln cos(x)$$

Step2. Take derivative.

$$\frac{f'(x)}{f(x)} = cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)}$$

   Step3. Solve for $f'(x)$

$$f'(x) = cos(x)^{sin(x)}(cos(x)\ln cos(x) - sin^2(x)\frac{1}{cos(x)})$$

4   Video Playlist 4

4.1   Functions

In calculus   We assume the domain is the largest subset of $\mathbb{R}$ that makes sense. And assume the codomain is always $\mathbb{R}$.

	Math	Computer Science
Notations	Domain	Domain
	Codomain	Range
	Range	Image

4.2   Inverse Functions

Definition   Let $f : A \to B$ be a function. Function $f^{-1} : B \to A$ is the **inverse function** is and only if

$$\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$$

Properties

- $\forall x \in A, f^{-1}(f(x)) = x$

- $\forall y \in B, f(f^{-1}(y)) = y$

Pre-condition   Function f has inverse function f^{-1} if and only if f is **injective/one-to-one** function.

4.3   Surjective Functions

Why function don't have an inverse: Part 1.

Definition   Function $f(x)$ is **surjective/onto** if $codomain(f(x)) = range(f(x))$.

Problem   If $f(x)$ is not surjective, then some points in codomain has no corresponding point in domain, then f^{-1} is not a function.

Solution   **Shrink** the codomain to range.

Example   Let $f(x) = e^x$, $g(x) = \ln x$, then we have,

-
  - $Domain(f(x)) = \mathbb{R}$
  - $Codomain(f(x)) = \mathbb{R}$
  - $Range(f(x)) = (0, \infty)$
-
  - $Domaing(x) = (0, \infty)$
  - $Codomaing(x) = \mathbb{R}$
  - $Rangeg(x) = \mathbb{R}$

Definition   Definition of inverse in calculus (*simplified, we don't consider codomain here.*)
    Let $f(x)$ be a function, and $f^{-1}(x)$ be the **inverse** of it. Then,

- $Domain(f^{-1}(x)) = Range(f(x))$

- $Range(f^{-1}(x)) = Domain(f(x))$

also,

$$\forall x \in Domain(f(x)), \forall y \in Range(f(x)), x = f^{-1}(y) \iff y = f(x)$$

and,

$$\forall x \in Domain(f(x)), f^{-1}(f(x)) = x$$
$$\forall y \in Range(f(x)), f(f^{-1}(y)) = y$$

4.4   Injective function

Definition   Let $f(x)$ be a function, with $Domain(f(x)) = A$, we say $f(x)$ is **injective/one-to-one** when,

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

equivalently (contrapositive)

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Theorem   Function f has an inverse if and only if f is **injective**.

Example $f(x) = x^2$ has no inverse, but we could take it's inverse by <u>shrinking the domain</u>.

- Take domain $= [0, \infty)$, $f^{-1}(x) = \sqrt{x}$
- Take domain $= (-\infty, 0]$, $f^{-1}(x) = -\sqrt{x}$

4.5   Some theorems

Let $f(x)$ be a function with domain I.

Theorem 1   Function f has an inverse function f^{-1} if and only if f is injective.

Theorem 2   For function f, if

1. f is **continuous** (*This means, f is continuous on its domain.*).

2. I is an **interval**.

then, $f^{-1}(x)$ is continuous.

Theorem 3   If

1. f is **differentiable**.

2. $\forall x \in I, f'(x) \neq 0$ (*This ensures the inverse function does not have a vertical tangent line, which causes non-differentiability*).

then, $f^{-1}(x)$ is differentiable.

Theorem 4 $\forall x \in I$ with $y = f(x)$, we have

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

   Proof.

$$f(f^{-1}(y)) = y$$
$$\frac{d}{dy}f(f^{-1}(y)) = \frac{d}{dy}y$$
$$\frac{d}{dy}f(f^{-1}(y)) = 1$$
$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$
$$f'(x) \cdot (f^{-1})'(y) = 1$$
$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$$\blacksquare$$

4.6   ArcSin

Note $ArcSin$ is **NOT** the inverse of Sin. $y = sin(x)$ has $domain = \mathbb{R}$ and $range = [-1, 1]$, so that, it is **not injective**.

Definition $ArcSin$ is the inverse function to the **restriction** of sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So that, $Domain(ArcSin) = Range(Sin) = [-1, 1]$, and, $Range(ArcSin) = Domain(Sin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Meaning $ArcSin(\frac{1}{2}) = t$ means:

$$\begin{cases} sin(t) = \frac{1}{2} \\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \end{cases}$$

Composite
$$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], ArcSin(Sin(x)) = x$$
$$\forall y \in [-1, 1], Sin(ArcSin(y)) = y$$

4.7   Derivative of ArcSin

Result
$$\frac{dArcSin(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Derive.

$$\forall x \in [-1, 1]$$
$$Sin(ArcSin(x)) = x$$
$$\frac{d}{dx}Sin(ArcSin(x)) = \frac{d}{dx}x$$
$$Cos(ArcSin(x)) \cdot \frac{d}{dx}ArcSin(x) = 1$$
$$\frac{d}{dx}ArcSin(x) = \frac{1}{Cos(ArcSin(x))}$$
$$\text{Let } \theta = ArcSin(x)$$
$$Cos^2(\theta) = 1 - Sin^2(\theta)$$
$$Cos(\theta) = \pm\sqrt{(1 - x^2)}$$
$$\text{Since } \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], Sin(\theta) \geq 0$$
$$\implies Cos(\theta) = +\sqrt{1 - x^2}$$
$$\implies \frac{d}{dx}ArcSin(x) = \frac{1}{\sqrt{1 - x^2}}$$

$\blacksquare$

4.8   Other inverse trig functions

4.8.1 $y = Cos(x)$

Definition $ArcCos$ is the inverse function to the restriction of $Cos(x)$ to $[0, \pi]$, and,
$$\forall x \in [-1, 1], \forall y \in [0, \pi], x = ArcCos(y) \iff Cos(y) = x$$

Result
$$\frac{d}{dx}ArcCos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

4.8.2 $y = Tan(x)$

Definition $ArcTan(x)$ is the inverse function to the restriction of $Tan(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and,
$$\forall y \in \mathbb{R}, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], x = ArcTan(y) \iff Tan(x) = y$$

5   Video Playlist 5

5.1   Usage of MVT

Theorem   Let I be an open interval. Let f be a function defined on I. If $\forall x \in I, f'(x) = 0$ then f is a constant function.

If we want to prove this theorem, we need mean value theorem

5.2   Local Extreme Theorem

Definition   Let f be a function with domain I, let $c \in I$.

- f takes **maximum** at c if $\forall x \in I, f(x) \leq f(c)$.

- f takes **local maximum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \leq f(c)$.

Definition   Let f be a function with domain I, let $c \in I$.

- f takes **minimum** at c if $\forall x \in I, f(x) \geq f(c)$.

- f takes **local minimum** at c if $\exists \delta > 0, s.t. |x - c| < \delta \implies f(x) \geq f(c)$.

End-point cannot be a local extremum since the definition of local extremum requires a open interval at both left and right sides around point c.

Theorem (Local EVT)   Let f be a function with domain I as an interval. Let $c \in I$, then if,

1. $f(c)$ is an extremum.

2. c is an interior point.

then, $f'(c) = 0$ or DNE.

Definition   Point $c \in I$ for function f is a **critical point** if $f'(c) = 0$ or it does not exist.

Proof. (Local EVT)   Proof is in two parts: (1) f has maximum at c, (2) f has minimum at c.

$$\text{Part1: } f(c) \text{ is a maximum}$$

$$\text{Take left and right side limits}$$

$$\text{As} x \to c^+, x - c > 0$$

$$\text{As} x \to c^-, x - c < 0$$

$$\text{By definition of maximum} f(x) - f(c) \le 0$$

$$\text{Left limit}$$

$$x - c < 0 \wedge f(x) - f(c) \le 0$$

$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \ge 0$$

$$\text{Right limit}$$

$$x - c > 0 \wedge f(x) - f(c) \le 0$$

$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \le 0$$

$$\text{For limit to exist}$$

$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \le 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \ge 0$$

$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$

$$\iff f'(c) = 0$$

$$\text{Part2: } f(c) \text{ is a minimum}$$

$$\text{Take left and right side limits}$$

$$\text{As} x \to c^+, x - c > 0$$

$$\text{As} x \to c^-, x - c < 0$$

$$\text{By definition of maximum} f(x) - f(c) \ge 0$$

$$\text{Left limit}$$

$$x - c < 0 \wedge f(x) - f(c) \ge 0$$

$$\implies \lim x \to c^- \frac{f(x) - f(c)}{x - c} \le 0$$

$$\text{Right limit}$$

$$x - c > 0 \wedge f(x) - f(c) \ge 0$$

$$\implies \lim x \to c^+ \frac{f(x) - f(c)}{x - c} \ge 0$$

$$\text{For limit to exist}$$

$$\lim x \to c^+ \frac{f(x) - f(c)}{x - c} \ge 0 \wedge \lim x \to c^- \frac{f(x) - f(c)}{x - c} \le 0$$

$$\implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$

$$\iff f'(c) = 0$$

$$\blacksquare$$

5.3   Find Extremum

Example   find extremum of function $f(x) = x^3 - 3x^2 - 9x + 3$ for $I = [-4, 4]$
  Steps

1. Ensure existence of extremum. f is polynomial and therefore continuous, and $[-4, 4]$ is a compact set. By EVT, extremum exist.

2. Find all *critical points* and *end-points*.

3. Compare values at candidate points.

5.4   Rolle's Theorem

Theorem   let $a < b$, let f be a function defined on a closed interval $[a, b]$ (Compact set). Then, if,

1. $f(x)$ is continuous on $[a, b]$.

2. $(\wedge)$ $f(x)$ is differentiable on (a, b).

3. $(\wedge)$ $f(a) = f(b)$.

then,
$$\exists c \in (a, b) s.t. f'(c) = 0$$

  Proof.

$$\text{By EVT, } f(x) \text{ has extremum in } [a, b].$$
$$\textbf{Case1} \text{Interior Extremum Point.} (c \in (a, b))$$
$$\text{By Local EVT, } f'(c) = 0 \vee f'(c) DNE$$
$$\text{By (ii)} f'(c) = 0$$
$$\textbf{Case2} \text{End-point Extremum}$$
$$\text{Since (iii)} f(a) = f(b)$$
$$\forall x \in (a, b)$$
$$f(x) \le max(f(a), f(b))$$
$$f(x) \ge min(f(a), f(b))$$
$$\implies f(x) \text{ is constant.}$$
$$\implies \forall c \in (a, b), f(c) = 0$$
$$\blacksquare$$

5.5   Application of Rolle's Theorem

Application   How many zeros does a function have.

Step 1   Use IVT to prove it has *at least* n zeros.

Step 2 Use Rolle's theorem to prove it has *at most* n zeros.

Example
$$g(x) = x^6 + x^2 + x - 2$$

IVT Applied

$$g(-2) = 64$$
$$g(0) = -2$$
$$g(1) = 1$$

So that, $g(x)$ has at least 2 zeros.

Rolle's theorem applied Assume $f(x_1) = f(x_2) = 0$, by Rolle's theorem, there must exits a $a \in (x_1, x_2)$ such that $f'(a) = 0$
    Conclusion 1 Between any two zeros of f there must be *at least* one zero of f'.
    Conclusion 2 $\sharp$ of zeros of $f' \geq \sharp$ of zeros of f - 1
    Conclusion 2' $\sharp$ of zeros of f $\leq \sharp$ of zeros of $f' + 1$

$$g'(x) = 6x^5 + 2x + 1$$
$$g''(x) = 30x^4 + 2$$
$$g''(x) \text{ has no zeros}$$

5.6   (Lagrange)Mean Value Theorem

Theorem Let $a < b$, let f be a function defined on $[a, b]$, if,

1. f is continuous on $[a, b]$.

2. f is differentiable on (a, b).

then,
$$\exists c \in (a, b) s.t. f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.7   Proof. of MVT

$$\text{Let } m = \frac{f(b) - f(a)}{b - a}$$
$$\text{Let } g(x) = f(x) - f(a) - m(x - a)$$
$$\text{Satisfies } g(a) = f(a) - f(a) - m(a - a) = 0$$
$$\wedge g(b) = f(b) - f(a) - m(b - a) = 0$$
$$\text{By Rolle's Theorem}$$
$$g(a) = g(b) = 0$$
$$\exists c \in (a, b) s.t. g'(c) = 0$$
$$\implies \frac{d}{x}[f(x) - f(a) - m(x - a)] = 0$$
$$\implies f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\blacksquare$$

5.8   Zero-derivative implies constant

Theorem   Let $a < b$. Let f be a function defined on $[a, b]$, then,

$\forall x \in (a, b), f'(x) = 0 \wedge\ f$ is continuous on $[a, b] \implies\ f$ is constant on [a,b].

proof.

$$\text{Let } x_1, x_2 \in [a, b] \wedge x_1 < x_2$$
$$\text{By MVT, } \exists c \in (x_1, x_2), s.t.$$
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$\because f'(c) = 0$$
$$\therefore f(x_1) = f(x_2)$$

5.9   Monotonicity of functions

Definition   Let f be a function defined on an interval I.

- f is **increasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2)$$

- f is **non-decreasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

Theorem   Let $a < b$. Let f be a function defined on (a, b). Then,

$$\forall x \in (a, b), f'(x) > 0 \implies f \text{ is increasing on (a,b)}$$

Theorem   Let $a < b$. Let f be a function defined on $[a, b]$. Then,

$$\forall x \in (a, b), f'(x) > 0 \wedge f \text{ is continuous on } [a, b] \implies f \text{ is increasing on } [a, b]$$

Short summary   On an <u>open interval</u>

- $f' = 0 \implies f$ constant.

- $f' > 0 \implies f$ increasing.

- $f' < 0 \implies f$ decreasing.

6   Video Playlist 6

Note   This chapter focus on *optimization applications*, and there's no video for this topic.

7   Video Playlist 7

7.1   Integral

Integral   Let $a < b$, let f be a <u>positive</u> function, then *integral of f from a to b* is denoted as:

$$\int_a^b f(x) \; dx$$

this is represented as the area of region under function f from $x = a$ to $x = b$.

7.2   Sigma Notation

Sigma Notation   The sigma notation, with **index** i, could be represented in the following form:

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_N$$

7.3   Supremum and Infimum

Definitions   Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Upper bound**: a is a <u>upper bound</u> of A means $\forall x \in A, \; x \leq a$.

- **Least upper bound(l.u.b) / Supremum**: a is the <u>least upper bound</u> or <u>supremum(sup)</u> of A iff a is an upper bound of A and $\forall b \in \{\text{upper bound of A}\}, \; a \leq b$.

- **Maximum**: if supremum of $A \in A$, it's <u>maximum</u> of A.

- **Bounded above**: A is <u>bounded above</u> if A has (at least) one upper bound.

Definitions (counter-part)   Let $A \subseteq \mathbb{R}$, let $a \in \mathbb{R}$:

- **Lower bound**: a is a <u>lower bound</u> of A means $\forall x \in A, \; x \geq a$.

- **Greatest lower bound(g.l.b) / Infimum**: a is the <u>greatest lower bound (g.l.b)</u> or <u>infimum(inf)</u> of A iff a is a lower bound of A and $\forall b \in \{$Lower bound of A$\}$, $a \geq b$.

- **Minimum**: if infimum of $A \in A$, it's the <u>minimum</u> of A.

- **Bounded below**: A is <u>bounded below</u> if A has (at least) one lower bound.

Theorem: The l.u.b. principle   Let $A \subseteq \mathbb{R}$, if A is <u>bounded above</u> and $A \neq \emptyset$, then, A has a least upper bound(supremum).

Theorem: The g.l.b principle   Let $A \subseteq \mathbb{R}$, if A is <u>bounded below</u> and $A \neq \emptyset$, then, A has a greatest lower bound(infimum).

7.4   Supremum and Infimum of a function

Definition   Supremum of a function f on a domain I is defined as:

$$\sup_{x \in I} f(x) = \sup\{f(x) \mid x \in I\}$$

Theorem   Let f be a function defined on domain $I \neq \emptyset$, if f is <u>bounded above</u>, then $\exists \sup_{x \in I} f(x)$. Similarly, if f is <u>bounded below</u>, then $\exists \inf_{x \in I} f(x)$.

Theorem(EVT)   Let $a < b$, let f defined on $[a, b]$, if f is <u>continuous</u> on $[a, b]$, then f has a maximum and a minimum on $[a, b]$.

7.5   Definition of Integral (i)

Definition   A **partition** of the interval $[a, b]$ is a finite set P, s.t. $\{a, b\} \subseteq P$.

Notation $P = \{x_0, x_1, \ldots x_N\}$ on $[a, b]$. Implicitly, x_i are <u>ordered</u>, such that, $a = x_0 < x_1 < \cdots < x_N = b$.

Let f be bounded on $[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$, let $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, and $\Delta x_i = x_i - x_{i-1}$.

Definition P-Lower sum of f is defined as:

$$L_P(f) = \sum_{i=1}^{N} (m_i \Delta x_i)$$

Definition P-Upper sum of f is defined as:

$$U_P(f) = \sum_{i=1}^{N} (M_i \Delta x_i)$$

Property For all partition P on interval $[a, b]$, the lower sum and upper sum satisfy the following inequality,

$$L_P(f) \leq \int_a^b f(x) \, dx \leq U_P(f)$$

7.6 Definition of Integral (ii): Properties of $U_P(f)$ and $L_P(f)$

Let f be a <u>bounded</u> function on $[a, b]$, let P and Q be partitions of $[a, b]$, the lower sums and upper sums have the following properties.

1. (Always) $L_P(f) \leq U_P(f)$.

2. If $P \subseteq Q$ (Q is a *finer* partition), then $L_P(f) \leq L_Q(f) \wedge U_P(f) \geq U_Q(f)$.

3. (Always) $L_P(f) \leq U_Q(f)$

    Proof
$$\text{Let } R = P \cup Q,$$
$$\text{so that, } P \subseteq R \wedge Q \subseteq R. \text{ (R is finer than both P and Q)}$$
$$L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f)$$
$$\implies L_P(f) \leq U_Q(f)$$

$\blacksquare$

7.7 Definition of Integral (iii): Upper Integral and Lower Integral

Definition Let f be a <u>bounded</u> function on $[a, b]$, then, <u>lower integral of f from a to b</u> is defined as,
$$\underline{I_a^b(f)} = \sup\{\text{lower sums of } f\}$$
and the <u>upper integral of f from a to b</u> is defined as,
$$\overline{I_a^b(f)} = \inf\{\text{upper sums of } f\}$$

Then if $\underline{I_a^b(f)} < \overline{I_a^b(f)}$, then f is **non-integrable** on $[a, b]$.

7.8   An example of integrable function

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \quad \text{on } [-1, 1]$$

7.9   An example of non-integrable function

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases} \quad \text{on } [-1, 1]$$

7.10   Integrals as limits

Definition   Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$, the **norm** of P is defined as:

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_N\}$$

Theorem - Lower Integrals   For lower integrals, we have,

$$\underline{I_a^b(f)} = \lim_{\|P\| \to 0} L_P(f) = \sup\{\text{lower sums of } f\}$$

alternatively, using $\delta - \epsilon$ expression,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall P \text{ over } [a, b], \|P\| < \delta \implies |L_P(f) - \underline{I_a^b(f)}| < \epsilon$$

theorem - Upper Integrals   For upper integrals, we have,

$$\overline{I_a^b(f)} = \lim_{\|P\| \to 0} U_P(f)$$

7.11   Riemann Sums

Definition   Fix a partition P on $[a, b]$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, pick $x_i^\star \in [x_{i-1}, x_i]$, so that,

$$m_i \leq f(x_i^\star) \leq M_i$$
$$\implies m_i \Delta x_i \leq f(x_i^\star) \Delta x_i \leq M_i \Delta x_i$$
$$\implies L_P(f) = \sum_{i=1}^{N}(m_i \Delta x_i) \leq \sum_{i=1}^{N}(f(x_i^\star) * \Delta x_i) \leq \sum_{i=1}^{N}(M_i \Delta x_i) = U_P(f)$$

where the term $\sum_{i=1}^{N}(f(x_i^\star) \Delta x_i)$ is called a **Riemann sum**.

Definition   Let f be a <u>bounded</u> function on $[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition on $[a, b]$, for each i, pick **any** point $x_i^\star \in [x_{i-1}, x_i]$. then,

$$S_P^\star(f) = \sum_{i=1}^{N} (f(x_i^\star) * \Delta x_i)$$

is <u>a</u> **Riemann sum** for f and P. *(There are infinitely many Riemann sum).*

In general, we have,

$$L_P(f) \leq S_P^\star(f) \leq U_P(f)$$

and also,

$$\lim_{\|P\| \to 0} L_P(f) = \underline{I_a^b(f)}$$

$$\lim_{\|P\| \to 0} U_P(f) = \overline{I_a^b(f)}$$

and if f is **integrable**, then

$$\lim_{\|P\| \to 0} L_P(f) = \lim_{\|P\| \to 0} U_P(f) = \int_a^b f(x) \ dx$$

By <u>Squeeze Theorem</u>,

$$\lim_{\|P\| \to 0} S_P^\star(f) = \int_a^b f(x) \ dx$$

7.12   Properties of the integral

Property 1

$$\int_a^b [f(x) + g(x)] \ dx = \int_a^b f(x) \ dx + \int_a^b g(x) \ dx$$

Property 2

$$\int_a^b [cf(x)] \ dx = c \int_a^b f(x) \ dx$$

Property 3   If f is bounded on $[a, c]$, and f is integrable on $[a, b]$ and integrable on $[b, c]$, then,

$$\int_a^c f(x) \ dx = \int_a^b f(x) \ dx + \int_b^c f(x) \ dx$$

Property 4: Backward Integrals

$$\int_b^a f(x) \ dx = - \int_a^b f(x) \ dx$$

Negative function f Integral for negative function is the negative area.

$$\int_a^b f(x) \ dx$$

8   Video Playlist 8

8.1   Anti-dervatives

Notations

- **Definite integral** $\int_a^b f(x) \ dx$
- **Indefinite integral** $\int f(x) \ dx$

Definition   Let f be a function defined on an interval, *an* **anti-dervative** of f is any function F that
$$F' = f$$

Note   As a consequence of MVT, if two functions have same dervative on an interval, then they differ by a constant.

8.2   Functions Defined as Integrals

Consider integrable function f, define function F as the definite integral from a, a fixed point in domain of f, to another point x in domain of f, that's,

$$F(x) = \int_a^x f(t) \ dt$$

Methodology   Let I be an interval, let $a \in I$ and let f be a function integrable on I, then for each $x \in I$, compute $F(x) = \int_a^x f(t) \ dt$ as a number.

8.3   The Fundamental Theorem of Calculus: Part 1

This provides connections between definite integrals and anti-dervatives

Theorem: FTC(part 1)

- Let I be an interval,
- Let $a \in I$,
- Let f be a function on I.

Define $F(x)$ as

$$F(x) = \int_a^x f(t) \ dt$$

If f is continuous, then F is differentiable and $F' = f$, that's,

$$F'(x) = f(x) \quad \forall x \in I$$

8.4   A Proof of Part 1 of the FTC

Proof.

$$\text{Let(fix) } x \in I$$

$$\text{WTS. } F'(x) = f(x)$$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} [\frac{1}{h}(F(x+h) - F(x))]$$

$$= \lim_{h \to 0} [\frac{1}{h}(\int_a^{x+h} f(t) \ dt - \int_a^x f(t) \ dt)]$$

$$= \lim_{h \to 0} [\frac{1}{h} \int_x^{x+h} f(t) \ dt]$$

Consider $h > 0$ (for negative h, the proof would be similar)

$$\text{Let } M_h = \sup_{[x,x+h]} (f)$$

$$\text{Let } m_h = \inf_{[x,x+h]} (f)$$

Then we have, by definition of infimum and supremum,

$$m_h \leq \frac{1}{h} \int_x^{x+h} f(t) \ dt \leq M_h$$

Since f is continuous on $[x, x+h]$, by EVT, it has maximum and minimum on this interval.

$$\exists c_h \in [x, x+h] \ s.t. \ M_h = f(c_h)$$

$$\exists d_h \in [x, x+h] \ s.t. \ m_h = f(d_h)$$

$$\because \lim_{h \to 0} c_h = x \wedge \lim_{h \to 0} d_h = x$$

$$\therefore \lim_{h \to 0} M_h = \lim_{h \to 0, c_h \to x} f(c_h) = f(x) \text{ (since } f \text{ is continuous.)}$$

$$\text{Similarly, } \lim_{h \to 0} m_h = \lim_{h \to 0, d_h \to x} f(d_h) = f(x)$$

$$\text{By Squeeze Theorem, } \lim_{h \to 0} [\frac{1}{h} \int_x^{x+h} f(t) \ dt] = f(x)$$

$$\therefore F'(x) = f(x) \ \forall x \in I$$

∎

8.5   The Fundamental Theorem of Calculus: Part 2

This provides a quick way to compute definite integrals.

Theorem: FTC(part 2)

- Let $a < b \in \mathbb{R}$,

- let f be continuous on $[a, b]$,

then,

$$\int_a^b f(x) \ dx = G(b) - G(a)$$

where G is any anti-dervative of f.

Notation

$$G(b) - G(a) = G(x)|_{x=a}^{x=b} = G(x)|_a^b$$

8.6   A Proof of Part 2 of the FTC

Proof.

We know that, from the first part of FTC, $G' = f$,

$$\text{WTS. } \int_a^b f(x) = G(b) - G(a)$$

$$\text{Define } F(x) = \int_a^x f(t) \ dt$$

$$\text{WTS. } F(b) = G(b) - G(a)$$

$$\text{Since } f \text{ is continuous}, F' = f$$

$$\text{By the consequence of MVT,}$$

$$F' = G' \implies \exists C \in \mathbb{R} s.t. F - G = C \forall x \in [a, b]$$

$$\text{at } x = a, F(a) = 0 \implies C = -G(a)$$

$$\implies \forall x \in [a, b] F(x) = G(x) - G(a)$$

$$\text{at } x = b, F(b) = G(b) - G(a)$$

∎

8.7   Summary: Definite and indefinite integrals, notation, definitions and theorems.

8.7.1   Definite Integral.

$$\int_a^b f(x) \ dx$$

Theorem (Formal definite)   if $\overline{I_a^b}(f) = \underline{I_a^b}(f)$ then $\int_a^b f(x) \ dx = \overline{I_a^b}(f) = \underline{I_a^b}(f)$.

Theorem (FTC: part 2) Choose <u>one</u> anti-dervative $G(x)$ of $f(x)$, then compute the definite integral as $\int_a^b f(x)\ dx = G(b) - G(a)$.

8.7.2   Indefinite Integral

$$\int f(x)\ dx \text{ A \underline{collection} of functions.}$$

Find indefinite integral Find $G(x)$ as <u>one</u> anti-dervative, by the consequence of MVT, then the indefinite integral of f could be constructed as,

$$F(x) = \{G(x) + C \mid C \in \mathbb{R}\}$$

8.7.3   Function Defined by an Integral.

$$F(x) = \int_a^x f(t)\ dt \text{ This is \underline{one} function with fixed value of } a.$$

Theorem (FTC: part 1) if f is continuous, then $F'(x) = f(x)$

9   Video Playlist 9

9.1   Integration By Substitution: derivation of the formula

Backwards usage of chain rule.

If $\int f(x)\ dx = F(x)$ is the anti-derivative of $f(x)$, then

$$F(g(x)) = \int f(g(x))g'(x)\ dx = F(g(x))$$

9.2   Example 2

9.3   Example 3

9.4   Example 4

Theorem Let $a < b$, let f be a continuous function, let g be a function with <u>continuous derivative</u> in $[a, b]$, assume the range of g on $[a, b]$ is contained in the domain of f. Then,

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

9.5   Integration by parts

Backwards product rule

Let f and g be two differentiable function, by product rule of differentiation, we have,

$$f'(x)g(x) + f(x)g'(x) = \frac{d}{dx}f(x)g(x)$$

$$\implies \int f'(x)g(x) + f(x)g'(x) \ dx = f(x)g(x) + C$$

$$\implies \int f'(x)g(x) \ dx + \int f(x)g'(x) \ dx = f(x) + g(x) + C$$

$$\implies \int f'(x)g(x) \ dx = f(x) + g(x) - \int f(x)g'(x) \ dx$$

The integral constant is implicitly contained in the integral term.

9.6   Examples

Example 1
$$\int x^2 e^2 \ dx$$

Example 2
$$\int e^2 \sin x \ dx$$

Use integration by parts twice.

Example 3
$$\int \arctan x \ dx$$

Consider the form $1 \times f(x)$ as partition method.

9.7   Integration of products of trigonometric functions

Types
$$\int sin^n x \ cos^m x \ dx$$

$$\int sec^n x \ tan^m x \ dx$$

Keys
$$sin^2(x) + cos^2(x) = 1$$
$$sec^2(x) = 1 + tan^2(x)$$

Summary I   Consider the integral in the following form

$$\int \sin^n x \, \cos^m x \, dx$$

- If **m is odd** then try $u = \sin(x)$, then $du = \cos(x)dx$

- If **n is odd** then try $u = \cos(x)$, then $du = -\sin(x)dx$

10   Video Playlist 10

Note   This chapter focus on *volumes*, and there's no video for this topic.

11   Video Playlist 11

11.1   What Is a Sequence

Definition   A **sequence** is a <u>function</u> with domain $\mathbb{N}$.

11.1.1   Conventions

Functions   function with domain <u>interval</u>.

- x as variable.

- $f(x)$ as value at x.

Sequence   function with domain $\mathbb{N}$.

- n as variable.

- a_n as value at n.

  A sequence is not a set.

11.1.2   Describe sequences

Equation $a_n = \frac{2^n n!}{n+1}$

First few values $\{1, 2, 4, 8, 16, \dots\}$

Words $p_n = $ n-th prime.

Recurrence relation   e.g. Fibonacci Sequence.

$$\{F_n\}_{n=0}^{\infty} : F_0 = F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \ \forall n \geq 2$$

A general definition   A sequence is a function with domain $\{n \in \mathbb{Z} \mid n \geq n_0\}$ for some fixed $n_0 \in \mathbb{Z}$.

11.2   The Limit of a Sequence

Example

$$\{\frac{n}{n+1}\}_{n=0}^{\infty} \quad \lim_{n \to \infty} \frac{n}{n+1} = 1$$

Definition(Limit)   We say that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to the number $L \in \mathbb{R}$ when

$$\forall \epsilon > 0, \ \exists n_0 \in \mathbb{N} s.t. \ \forall n \in \mathbb{N}, \ n \geq n_0 \implies |L - a_n| < \epsilon$$

denoted as

$$\lim_{n \to \infty} a_n = L \text{ or } a_n \to L$$

Tail: all terms of the sequence after the first few terms.
Every interval centred at L contains a tail of the sequence.

Definition   A sequence is **convergent** if it has a limit.   This sequence is **divergent** if it does not have a limit.

11.3   Properties of Limits of Sequences

Properties from the limit of functions

- Limit laws: Yes

- Squeeze theorem: Yes

- *$L'H\hat{o}pital's \ Rule$*: No

11.3.1   Sequence from a function

Let $c \in \mathbb{Z}$ and function f defined on $[c, \infty)$, and define the seuqnce $\{a_n\}_{n=c}^{\infty}$ as

$$a_n = f(n)$$

We have if $\lim_{n \to \infty} f(n) = L$ then $\lim_{n \to \infty} a_n = L$. If $\lim_{n \to \infty} f(n)$ DNE, then $\lim_{n \to \infty} a_n$ <u>may or may not exist</u>.

11.3.2   Composite of sequence and function

Theorem   If $a_n \to L$ and f is continuous at L then

$$f(a_n) \to f(L)$$

11.4   Monotonic and Bounded Sequences

11.4.1   Monotonic Sequences

Definition   We say $\{a_n\}_{n=0}^{\infty}$ is **increasing** if

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n < a_m$$

Also, we say this sequence is **non-decreasing** if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n \leq a_m$$

Definition   We say $\{a_n\}_{n=0}^{\infty}$ is **decreasing** if

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n > a_m$$

Also, if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, \ n < m \implies a_n \geq a_m$$

we say this sequence is **non-increasing**.

Definition   We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **monotonic** is if is has any of the four properties above.

Definition $\{a_n\}_{n=0}^{\infty}$ is **eventually decreasing** if

$$\exists n_0 \in \mathbb{N}, \ s.t. \forall n \in \mathbb{N}, n \geq n_0 \implies a_n > a_{n+1}$$

11.4.2   Bounded Sequences

Definition   We say a sequence $\{a_n\}_{n=0}^{\infty}$ is **bounded below** if

$$\exists A \in \mathbb{R} s.t. \forall n \in \mathbb{N}, \ A \leq a_n$$

Similarly, the sequence is **bounded above** if

$$\exists B \in \mathbb{R}.s.t. \forall n \in \mathbb{N}, \ B \geq a_n$$

Definition   We say a sequence is **bounded** if and only if it is <u>both</u> bounded above and below.

Theorem   If a sequence is <u>convergent</u> then it is <u>bounded</u>.

Theorem 2A(The monotone convergence theorem for sequence)   If a sequence is <u>eventually increasing</u> and <u>bounded above</u>, then it is <u>convergent</u>

Theorem   If a sequence is <u>eventually increasing</u> and <u>not bounded above</u> then it <u>divergent to ∞</u>.

Remark   for a sequence:

$$\text{Sequence} \begin{cases} \text{Convergent} \\ \text{Divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ Oscillating \end{cases} \end{cases}$$

11.5   Proof: Every convergent sequence is bounded

Theorem   Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, if $\{a_n\}_{n=0}^{\infty}$ is <u>convergent</u> then the sequence is <u>bounded</u>. Equivalently,

Proof.

$$\text{Assume sequence } \{a_n\}_{n=0}^{\infty} \text{ is convergent.}$$
$$\text{Let } L \text{ be the limit.}$$
$$\text{By the definition of limit, choose } \epsilon = 10$$
$$\text{So that, } \exists n_0 \in \mathbb{N} s.t. \forall n \in \mathbb{N}, n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$$
$$\text{Take } A = min\{a_0, \dots, a_{n_0 - 1}, L - 10\}$$
$$\text{Take } B = max\{a_0, \dots, a_{n_0 - 1}, L + 10\}$$
$$\text{By definition of max and min, let } n \in \mathbb{N}$$
$$\text{case } 1 n > n_0 \implies A \leq a_n \leq B$$
$$\text{case } 2 n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$$
$$\text{Since } A \leq L - 10 \wedge B \geq L + 10$$
$$\implies A \leq a_n \leq B \forall n \in \mathbb{N}$$
$$\therefore \{a_n\}_{n=0}^{\infty} \text{ is bounded.}$$

$\blacksquare$

11.6   The monotone convergence theorem of sequences

(General) Theorem   If a sequence is (eventually) <u>monotonic</u> and <u>bounded</u> then it is <u>convergent</u>.

(Particular Case) Theorem 1   If a sequence is <u>increasing</u> and <u>bounded above</u> the it's <u>convergent</u>.

Proof.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence that's increasing and bounded above.

Consider $A = \{a_n \mid n \in \mathbb{N}\} \neq \emptyset$

By least upper bound principle, there exists a supremum of set A

Take $L = sup\{A\}$

Let $\epsilon > 0$

By definition of supremum,

$\exists a_{n0} \in A \ s.t. \ a_{n0} > L - \epsilon$

Take this value n_0

Since sequence is increasing,

$\forall n \geq n_0 \ a_n > L - \epsilon$

Also, by definition of supremum, $a_n \leq L$

$\implies a_n \leq L + \epsilon$

Therefore, $\forall n \in \mathbb{N}, n \geq n_0 \implies L - \epsilon < a_n < L + \epsilon$

Therefore, $\lim_{n \to \infty} \{a_n\}_{n=0}^{\infty} = L$

Therefore, $\{a_n\}_{n=0}^{\infty}$ is convergent.

∎

11.7   the Big theorem of sequences

Definition   (for positive sequences only) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be positive sequences.

$$a_n << b_n \iff \lim_{n \to \infty} \frac{a_n}{b_n} = 0$$

say $\{a_n\}$ is **much smaller than** $\{b_n\}$.

Theorem   for every $a > 0$ and $c > 1$

$$\ln n << n^a << c^n << n! << n^n$$

12   Video Playlist 12

12.1   Improper Integral

12.1.1   Improper integral "type 1" (Unbounded domain)

Definition   Let $a \in \mathbb{R}$ and f continuous on $[a, \infty]$ the integral of f from a to ∞, denoted as

$$\int_a^{\infty} f(x) \ dx = \lim_{b \to \infty} \int_a^b f(x) \ dx$$

assuming the limit exists. If the limit exists, the integral is called **convergent**, otherwise, it's called **divergent**.

12.2   The most important family if the improper integrals

Let $p \in \mathbb{R}$ consider

$$\int_1^\infty \frac{1}{x^p} \ dx$$

Summary

$$\int_1^\infty \frac{1}{x^p} \ dx \text{ is } \begin{cases} \textbf{convergent} \text{ if } p > 1 \\ \textbf{divergent to } \infty \text{ if } p \leq 1 \end{cases}$$

12.3   Example

$$\int_0^\infty sin(x) \ dx$$

12.4   The most important family of improper integral

Consider

$$I = \int_1^\infty \frac{1}{x^p} \ dx$$

Summary

1. $p > 1 \iff I$ converges.

2. $p \leq 1 \iff I$ diverges to ∞.

12.5   Example

Vertical asymptote improper.

$$\int_0^1 ln(x) \ dx$$

12.6   Doubly improper integrals

General Strategy   Assume A has **multiple** improper.

1. Break A into pieces with **single** improper at their endpoints.

2. If each piece convergent **seperately**, then A converges.

3. Else, A diverges, it's *not a number*.

12.7   Basic Comparison Test

Theorem   Let $a \in \mathbb{R}$,
Let f and g be *continuous* functions one $[a, \infty)$, and

$$\forall x \geq a, 0 \leq f(x) \leq g(x)$$

we have,

1. $\int_a^\infty g(x) \, dx < \infty \implies \int_a^\infty f(x) \, dx < 0$

2. $\int_a^\infty f(x) \, dx = \infty \implies \int_a^\infty g(x) \, dx = \infty$

12.8   Examples

12.9   Limit Comparison Test

Theorem   Let $a \in \mathbb{R}$, f and g are *positive* and *continuous* functions on $[a, \infty)$. And the following limit exists,

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)} \in \mathbb{R}$$

Then, $\int_a^\infty f(x) \, dx$ and $\int_a^\infty g(x) \, dx$ are **both** convergent or **both** divergent.

12.10   Proof of LCT

<u>Omitted</u>

13   Video Playlist 13

13.1   Infinite Sums

Nothing.

13.2   Definition of series

Definition   Series $\sum_{n=1}^\infty a_n$ is defined as

$$\lim_{k \to \infty} S_k$$

where $S_k = \sum_{n=1}^k a_n$ as finite sum. If the above limit exist, we say series $\sum_{n=1}^\infty a_n$ is convergent (it's a *number*), else series is divergent and it's not a number.

13.3   Example

$$\sum_{n=1}^\infty \frac{1}{n^2 + n} = \sum_{n=1}^\infty \frac{1}{n} - \frac{1}{n+1} = \lim_{k \to \infty} 1 - \frac{1}{k+1} = 1$$

13.4   Divergent Series Examples

$$S = \sum_{n=1}^{\infty} 1 = \lim_{k \to \infty} k = \infty$$

$$S = \sum_{n=1}^{\infty} (-1)^n = \begin{cases} 0 \text{ if k is odd} \\ 1 \text{ if k is even} \end{cases} \quad \text{divergent due to osciallation.}$$

13.5   Geometric Series

Let $x \in \mathbb{R}$

$$S = \sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{n=0}^{\infty} x^n$$

Consider

$$S_k = 1 + x + x^2 + \cdots + x^k$$
$$xS_k = x + x^2 + x^3 + \cdots + x^{k+1}$$
$$S_k - xS_k = 1 - x^{k+1}$$

If x = 1, the series is simply divergent to ∞.

$$S_k = \frac{1 - x^{k+1}}{1 - x}, \ x \neq 1$$

$$S = \lim_{k \to \infty} 1 - x^{k+1} = \frac{1 - \lim_{k \to \infty}(x^{(}k+1))}{1 - x} = \begin{cases} \frac{1}{1-x} \iff x \in (-1, 1) \\ \text{Divergent} \begin{cases} \infty \iff x > 1 \\ \text{Oscillating} \iff x \leq -1 \end{cases} \end{cases}$$

13.6   Linearity of series

Simple form of fact

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} c \, b_n = \sum_{n=0}^{\infty} a_n + c \, b_n, \ \forall c \in \mathbb{R}$$

Theorem   If series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent, then $\sum_{n=0}^{\infty} a_n + b_n$ is also convergent and

$$\sum_{n=0}^{\infty} a_n + b_n = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

Proof.

$$\text{Let } \sum_{n=0}^{\infty} a_n = \lim_{k \to \infty} S_k$$

$$\text{Let } \sum_{n=0}^{\infty} b_n = \lim_{k \to \infty} T_k$$

$$\text{Let } \sum_{n=0}^{\infty} a_n + b_n = \lim_{k \to \infty} R_k$$

$$\text{Where } S_k = \sum_{n=0}^{k} a_n$$

$$T_k = \sum_{n=0}^{k} b_n$$

$$R_k = \sum_{n=0}^{k} a_n + b_n$$

$$\text{Since } R_k = S_k + T_k, \ \forall k \in \mathbb{N}$$

$$\text{By limit laws, } \lim_{k \to \infty} S_k + \lim_{k \to \infty} T_k = \lim_{k \to \infty} R_k$$

$$\implies \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n$$

$$\blacksquare$$

Theorem   If series $\sum_{n=0}^{\infty} a_n$ is convergent, then for any $c \in \mathbb{R}$, series $\sum_{n=0}^{\infty} c\, a_n$ is also convergent and

$$\sum_{n=0}^{\infty} c\, a_n = c \sum_{n=0}^{\infty} a_n$$

General proof procedure

1. Write series as limit of partial sums.

2. Manipulate partial sums (finite).

3. Manipulate limits.

13.7   the Tail of a series

Fact   Consider two series

$$\sum_{n=0}^{\infty} a_n \text{ convergent} \iff \sum_{n=1}^{\infty} a_n \text{ convergent}$$

And $\sum_{n=1}^{\infty} a_n$ is a tail of series $\sum_{n=0}^{\infty} a_n$.

Notation   We say $\sum_n^\infty a_n$ is convergent or divergent without specifying the starting index of the series.

Specific form of theorem   If $\forall n \in \mathbb{N}$ [Condition(s)] then $\sum_n^\infty a_n$ is convergent or divergent.

General form of theorem   If $\exists\, n_0 \in \mathbb{N}\ s.t.\ \forall n \in N,\ n \geq n_0 \implies$ [Condition(s)] then $\sum_n^\infty a_n$ is convergent.

13.8   A necessary condition for convergence of series

Fact   Series $\sum_{n=0}^\infty a_n$ is convergent if and only if the sequence of its partial sums $\{S_n\}_{n=0}^k$ is convergent.

Theorem   If $\sum_{n=0}^\infty a_n$ is convergent then

$$\lim_{n \to \infty} a_n = 0$$

Note   The above theorem is often used as it's contrapositive form

$$\lim_{n \to \infty} a_n \neq 0 \implies \sum_{n=0}^\infty a_n \text{ is divergent}$$

Proof.

$$\text{Let } S = \sum_{n=0}^\infty a_n \text{ be convergent}$$

$$S = \lim_{k \to \infty} S_k,\ S_k = \sum_{n=0}^k a_n$$

$$S = \lim_{k-1 \to \infty} S_{k-1}$$

$$\implies \lim_{k \to \infty} S_k - \lim_{k-1 \to \infty} S_{k-1} = 0$$

By the convergence assumption, those two limits above exist.

$$\implies \lim_{k \to \infty} S_k - S_{k-1} = 0$$

$$\implies \lim_{k \to \infty} a_k = 0$$

$$\implies \{a_n\}_{n=1}^\infty \to 0$$

$\blacksquare$

13.9   Positive series

Definition   A series $\sum_{n=0}^\infty a_n$ is positive when $\forall n \in \mathbb{N},\ a_n > 0$. And a series is positive means it could never diverge to $-\infty$ or *oscillating*.

Notation   (For positive series only)

1. $\sum_n^\infty a_n = \infty \iff$ divergent.

2. $\sum_n^\infty a_n < \infty \iff$ convergent.

13.10   The Integral Test

Theorem   Let $a \in \mathbb{R}$, let f be a continuous, positive and decreasing function on $[a, \infty)$, then

$$\int_a^\infty f(x) \; dx < \infty \iff \sum_n^\infty f(n) < \infty$$

That's the improper integral and series have the same convergence/divergence feature. Note as

$$\int_a^\infty f(x) \; dx \sim \sum_n^\infty f(n)$$

13.11   Examples

p-series   consider for what values of p the following series is convergent.

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

Example2

$$\sum_{n=2}^\infty \frac{1}{n \ln n}$$

13.12   Comparison Tests for Series

Works exactly the same as basic and limit comparison tests for improper integrals.

Basic Comparison Test   Consider series $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$, assume[1]

$$\forall n \in \mathbb{N}, \; 0 \le a_n \le b_n$$

then

$$\sum_{n=1}^\infty b_n < \infty \implies \sum_{n=1}^\infty c_n < \infty$$

$$\sum_{n=1}^\infty a_n = \infty \implies \sum_{n=1}^\infty b_n = \infty$$

---

[1]Notice that *for large natural number n* would be sufficient here also.

Limit Comparison Test   Consider series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, assume existing limit

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} > 0$$

Then both series convergent or both of them divergent.

14   Video Playlist 14

14.1   Power Series: Example

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

Domain of $g(x)$ is defined as

$$\{x \in \mathbb{R} \mid g(x) \text{ is convergent}\}$$

The above series convergent when $x \in [-3, 3)$, and $[-3, 3)$ is the **interval of convergence** and 3 is the **radius of convergent**.

14.2   Main Theorem

Definition   Let $a \in \mathbb{R}$, a power series centred at a is a function f defined by a equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

Main Theorem   Let $f(x)$ be a power series, then

1. Domain of f is an interval centred at a, with radius of convergence R, $0 \leq \mathbb{R} \leq \infty$

2. In the *interior* of the interval of convergence, the series is *absolutely convergent*, in the *exterior* of IC, the series is *divergent*, and at the boundaries this theorem is *inconclusive*.

3. In the *interior* of the IC, power series can be treated like polynomial, <u>without change of radius of convergence</u>.

14.3   Taylor Polynomial Definition 1

Definition   Let $f(x)$ and $g(x)$ be two functions that are continuous at a, let $n > 0$ and <u>g is a approximation for f near a of order n</u> when

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x - a)^n} = 0$$

Definition   Let $a \in \mathbb{R}$, let f be a continuous function defined at a, let $n \in \mathbb{N}$ then the nth **Taylor Polynomial** for f at a is the polynomial P_n of *smallest* possible degree is an approximating for f near a of order n, that's,

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x - a)^n} = 0$$

14.4   Taylor Polynomial Definition 2

Definition   Let $a \in \mathbb{R}$, $n \in \mathbb{N}$ and let f be a C^n function at a, the nth **Taylor Polynomial** for f at a is a polynomial P_n s.t.

$$P_n(a) = f(a), \ P_n'(a) = f'(a), \ \ldots P_n^{(n)}(a) = f^{(n)}(a)$$

with smallest possible degree.

14.5   Taylor Polynomial Definition 3

Definition   Let $a \in \mathbb{R}$, let f be a C^∞ function at a, the **Taylor's series** for f at a is the power series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

with the fact that

$$\forall k \in \mathbb{N}, S^{(k)}(a) = f^{(k)}(a)$$