Elements of Real Analysis

Based on Lecture Notes for MAT337: Introduction to Real Analysis (2019Winter)

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- GitHub: `https://github.com/TianyuDu/Spikey_UofT_Notes`
- Website: `TianyuDu.com/notes`

TO-DO

1. Add Dedekind cut to section 1.
2. Refine subsection titles.

Contents

1 Real Numbers

1.1 Definitions

Definition 1.1. Reals are proper initial segments of $\mathbb{Q}$ with no maximum.

Definition 1.2. A subset $A \subset \mathbb{Q}$ is an **initial segment** if

$$y \in A, x \in \mathbb{Q}, x < y \implies x \in A \tag{1.1}$$

Definition 1.3. A is **proper** if $A \neq \mathbb{Q}$.

Definition 1.4. A has no maximal elements if

$$\forall x \in A,\ \exists y \in A \ s.t.\ y > x \tag{1.2}$$

Example 1.1.

$$\sqrt{2} \approx A_{\sqrt{2}} := \{q \in \mathbb{Q} : q < \sqrt{2}\} \equiv \{q \in \mathbb{Q} : q \leq 0 \vee q^2 < 2\} \tag{1.3}$$

$$x \approx A_x := \{q \in \mathbb{Q} : q < x\} \tag{1.4}$$

1.2 The Axiom of Completeness

Axiom 1.1 (Axiom of Completeness)**.** Every non-empty subset $B \subset \mathbb{R}$ that is bounded has a supremum (i.e. the least upper bound). That's

$$\forall B \subset \mathbb{R},\ s.t.\ B \neq \varnothing\ \exists b \in \mathbb{R}\ s.t.\ \begin{cases} \forall x \in B,\ x \leq b \text{ (upper bound)} \\ \forall c \in \mathbb{R} (\forall x \in B, x \leq c) \implies b \leq c \text{ (least upper bound)} \end{cases} \tag{1.5}$$

Theorem 1.1. $\mathbb{Q}$ is *dense* in $\mathbb{R}$, that's

$$\forall x < y \in \mathbb{R},\ \exists q \in \mathbb{Q}\ s.t.\ x < q < y \tag{1.6}$$

Theorem 1.2 (Cardinality)**.** Let A, B be non-empty subsets of $\mathbb{R}$, then the following statements are equivalent:

(i) $\exists h : A \to B$ such that h is bijective;

(ii) $\exists f : A \to B$ and $g : B \to A$ such that both f and g are injective.

Proof. (i) is the definition for sets A and B to have the same cardinality. And the existence of injection from A to B implies the cardinality of A cannot be greater than the cardinality of B. Similarly, the existence of injection from B to A implies the cardinality of B cannot be greater than the cardinality of A. Therefore A and B share the same cardinality. ∎

Theorem 1.3 (Nested Intervals)**.** Let (I_n) be a sequence of closed and non-empty intervals in $\mathbb{R}$ such that

$$I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \tag{1.7}$$

then

$$\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing \tag{1.8}$$

Proof. Claim:

$$x := \sup\{\min(I_n) : n \in \mathbb{N}\} \in \bigcap_{n\in\mathbb{N}} I_n \tag{1.9}$$

Let $n \in \mathbb{N}$, then $x \geq \min(I_n)$. Now show $x \leq \max(I_n)\ \forall n \in \mathbb{N}$. Suppose not, then $\exists k \in \mathbb{N}$ such that $x > \max(I_k)$. Then by the definition of supremum, there exists $j \in \mathbb{N}$ such that $\max(I_k) < \min(I_j)$. Note that if $k = j$, this leads to a contradiction. If $k < j$, then because $I_k \supset I_j$, $\max(I_k) \geq \max(I_j) \geq \min(I_j) \geq \min(I_k)$, this leads to a contradiction. If $k > j$, then $I_k \subset I_j$, thus $\min(I_j) \leq \min(I_k) \leq \max(I_k) \leq \max(I_j)$, which also leads to a contradiction.
Therefore we conclude

$$\min(I_n) \leq x \leq \max(I_n)\ \forall n \in \mathbb{N} \tag{1.10}$$

therefore $x \in I_n\ \forall n \in \mathbb{N}$, so $x \in \bigcap_{n\in\mathbb{N}} I_n$. ■

Theorem 1.4. There exists no injection from $\mathbb{R}$ to $\mathbb{N}$.

Proof. $\mathbb{R}$ has cardinality c but $\mathbb{N}$ has cardinality $\aleph_0$. ■

2 Sequences and Series

Definition 2.1. A sequence $(a_n)_{n=1}^{\infty}$ of real numbers **converges** to a real number a if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ s.t.\ \forall n > N\ |a_n - a| < \varepsilon \tag{2.1}$$

If there does not exist such a, we conclude $(a_n)_{n=1}^{\infty}$ is **divergent**.

Theorem 2.1. Every convergent sequence is bounded.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$ with limit point a. Then take $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1$. Take

$$M := \max\{\max_{n\leq N}\{|a_n|\}, |a| + 1\} \tag{2.2}$$

and the sequence is bounded by M. ■

Definition 2.2. Let $(a_n)_{n=1}^{\infty}$ be a sequence, then a sub-sequence of (a_n) is any sequence in the form $(a_{n_k})_{k=1}^{\infty}$ such that $n_1 < n_2 < \cdots < n_k < \cdots$.

Remark 2.1. A sub-sequence can be generated with a strictly increasing function defined on $\mathbb{N}$ and a sequence (a_n).

Theorem 2.2 (Bolzano-Weierstrass)**.** Every bounded sequence has a convergent sub-sequence.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence bounded by $M > 0$. Define

$$I_0 := [-M, M] \tag{2.3}$$
$$J^0 := [-M, 0] \tag{2.4}$$
$$J^1 := [0, M] \tag{2.5}$$
$$X^0 := \{n \in \mathbb{N} : a_n \in J^0\} \tag{2.6}$$
$$X^1 := \{n \in \mathbb{N} : a_n \in J^1\} \tag{2.7}$$

therefore $\mathbb{N} = X^0 \cup X^1$. Thus at least one of X^0 and X^1 is infinite. If X^0 is infinite, define $I_1 := J^0$, otherwise, define $I_1 := J^1$. Let

$$A := \{x \in \mathbb{R} : \{n \in \mathbb{N} : x < a_n\} \text{ is infinite}\} \tag{2.8}$$

which is the lower bound of selected infinite half intervals. And define $a := \inf(A)$. we can construct a sub-sequence, for each $n \in \mathbb{N}$, take $a_n \in I_n$. And by the nested interval theorem, the intersection of all those selected intervals is non-empty. And a is the limit point of the constructed sequence. So a convergent sub-sequence exists. ∎

Definition 2.3. A sequence $(a_n)_{n=1}^{\infty}$ is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; \forall m, n > N, \; |a_n - a_m| < \varepsilon \tag{2.9}$$

Theorem 2.3 (Convergent $\implies$ Cauchy)**.** Every convergent sequence is a Cauchy sequence.

Proof. Let (a_n) be a convergent sequence, fix $\varepsilon > 0$. Suppose $(a_n) \to a$, take $\varepsilon^* = \varepsilon/2$. Thus, there exists $N \in \mathbb{N}$ such that $\forall n > N, |a_n - a| < \varepsilon^* = \varepsilon/2$. By taking such N, $\forall n, m > N$, both $|a_n - a|$ and $|a_m - a| < \varepsilon/2$. By triangle inequality, $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, we've shown that for an arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall m, n > N, |a_n - a_m| < \varepsilon$. Therefore (a_n) is Cauchy. ∎

Theorem 2.4 (Cauchy $\implies$ Convergent)**.** Every Cauchy sequence is convergent.

Proof. Let (a_n) be a Cauchy sequence.
Claim: (a_n) is bounded.

Proof. Bounded. Take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall m, n > N, |a_n - a_m| < 1$. Take $m = N + 1$ and define $a^* := a_m$. Then we have $\forall n > N$, $|a_n - a^*| < 1$, which implies $|a_n| < |a^*| + 1$. Define

$$M := \max\{\max\{a_n : n \leq N\}, |a^*| + 1\} \tag{2.10}$$

So (a_n) is bounded by M. ∎

Then by Bolzano-Weierstrass Theorem, there exists a sub-sequence $(a_{n_k})_{k=1}^{\infty}$ converges to some limit point $a \in \mathbb{R}$. We are going to show $(a_n) \to a$. Fix $\varepsilon > 0$, by the convergence of the sub-sequence

$$\exists N_1 \in \mathbb{N}\ s.t.\ \forall n \geq N_1,\ |a_{n_k} - a| < \frac{\varepsilon}{2} \tag{2.11}$$

Also since the sequence itself is Cauchy,

$$\exists N_2 \in \mathbb{N},\ s.t.\ \forall m, n \geq N_2,\ |a_n - a| < \frac{\varepsilon}{2} \tag{2.12}$$

Take $N^* := \max\{N_1, N2_\}$. Show $|a_n - a| < \varepsilon\ \forall n \geq N^*$. Note that

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \tag{2.13}$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \tag{2.14}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{2.15}$$

since $n_k \geq n$ by the definition of sub-sequences. ■

Corollary 2.1. A sequence is Cauchy if and only if it is convergent.

Proof. Let (a_n) be a Cauchy sequence.
Claim: (a_n) is bounded.

Proof. Take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that

$$\forall m, n > N\ |a_n - a_m| < 1 \tag{2.16}$$

take $m := N + 1$, define $a^* := a_{m+1}$, then

$$\forall n > N,\ |a_n - a^*| < 1 \implies |a_n| \leq |a^*| + 1 \tag{2.17}$$

Define $M := \max\{\max_{n \leq N}\{a_n\}, |a^*| + 1\}$, and (a_n) is bounded by M. ■

By the Bolzano-Weierstrass Theorem, there exists a sub-sequence $(a_{n_k})_{k=1}^{\infty}$ converges to some limit point $a \in \mathbb{R}$. Show $(a_n) \to a$ as well.
Let $\varepsilon > 0$, by convergence of the sub-sequence,

$$\exists N_1 \in \mathbb{N},\ s.t.\ \forall n \geq N_1,\ |a_{n_k} - a| < \varepsilon/2 \tag{2.18}$$

By the Cauchy property of (a_n),

$$\exists N_2 \in \mathbb{N},\ s.t.\ \forall m, n \geq N_2,\ |a_n - a_m| < \varepsilon/2 \tag{2.19}$$

take $N^* := \max\{N_1, N_2\}$. Let $n \geq N^*$ and note that $n_k \geq n \geq N^*$

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \tag{2.20}$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \tag{2.21}$$

$$\varepsilon/2 + \varepsilon/2 = \varepsilon \tag{2.22}$$

then take such N^* for the fixed $\varepsilon > 0$. Convergence of (a_n) shown. ∎

Theorem 2.5 (the Uniqueness of the Limit Point)**.** If $(a_n) \to a$ and $(a_n) \to b$, then $a = b$.

Proof. Suppose $a \neq b$, define $s := |a - b| > 0$. Take $\varepsilon = \frac{s}{2}$, there does not exist such $N \in \mathbb{N}$ satisfying

$$\forall n \geq N, \begin{cases} |a_n - a| < \varepsilon \\ |a_n - b| < \varepsilon \end{cases} \tag{2.23}$$

above notion indicates that the sequence is converging to two separate limit points simultaneously. ∎

Theorem 2.6 (Properties of Limits)**.** If $(a_n) \to a$, $(b_n) \to b$, and $c \in \mathbb{R}$, then

(i) $(c \cdot a_n) \to c \cdot a$;

(ii) $(a_n + c) \to a + c$;

(iii) $(a_n + b_n) \to a + b$;

(iv) $(a_n \cdot b_n) \to a \cdot b$.

3 Convergence of Series

Definition 3.1. A series $\sum_{n=1}^{\infty} a_n$ is **convergent** if

$$\exists a \in \mathbb{R} \ s.t. \ \sum_{n=1}^{\infty} a_n = a \tag{3.1}$$

Definition 3.2 (Alternative Definition)**.** Let $(S_n) := (\sum_{i=1}^{n} a_i)_{n=1}^{\infty}$ denote the *sequence of partial sums* associated with series $\sum_{n=1}^{\infty} a_n$, then the series is convergent if and only if its partial sum converges to a real number.

Theorem 3.1 (Cauchy Criterion)**.** A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \ s.t. \ \forall m \geq n \geq N, \ |\sum_{i=n}^{m} a_i| < \varepsilon \tag{3.2}$$

That's, the partial sum sequence is Cauchy.

Corollary 3.1. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Corollary 3.2 (Absolute Convergence Test)**.** If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Corollary 3.3. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, and, let $f : \mathbb{N} \to \mathbb{N}$ be a bijection, then

$$\sum_{n=1}^{\infty} a_{f(n)} \tag{3.3}$$

is convergent.

Given the absolute convergence, the rearrangement of sequence does not affect the convergence of series.

Theorem 3.2. Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $a_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Theorem 3.3. If $\sum_{n=1}^{\infty}|a_n|$ is convergent, let $f : \mathbb{N} \to \mathbb{N}$ be a bijection, then $\sum_{n=1}^{\infty}|a_{f(n)}|$ is also convergent.

Theorem 3.4. Suppose $\sum_{n=1}^{\infty}|a_n|$ is convergent, let $f, g : \mathbb{N} \to \mathbb{N}$ be two bijections, then

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_{g(n)} \tag{3.4}$$

Theorem 3.5 (Monotone Convergence)**.** Every monotone sequence, which is bounded, is convergent.

Corollary 3.4. Given sequence $(a_n) \subset \mathbb{R}_{++}$ and series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums is therefore a monotonically increasing sequence, so the partial sum (S_n) is convergent if it is bounded.

Example 3.1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. Let $m \in \mathbb{N}$, so

$$\begin{aligned} S_m &= 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{m \times m} && (3.5)\\ &< 1 + \frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \cdots + \frac{1}{m \times (m-1)} && (3.6)\\ &= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) && (3.7)\\ &= 2 - \frac{1}{m} < 2 && (3.8) \end{aligned}$$

therefore (S_m) is non-decreasing and bounded above by 2. So (S_m) is convergent, so is $\sum_{n=1}^{\infty} \frac{1}{n^2}$. ■

4 Order and Converging Sequences

Proposition 4.1. If $(a_n) \geq 0$ is convergent to $a \in \mathbb{R}$, then $a \geq 0$.

Proof. By contradiction. ■

Proposition 4.2. If $(a_n) \leq (b_n)$ are convergent to a and b, respectively, then $a \leq b$.

Proof. Construct sequence $(b_n - a_n) \geq 0$ and apply the previous proposition. ■

Definition 4.1 (limsup)**.** Let (a_n) be a bounded sequence, for each $m \in \mathbb{N}$, define

$$b_m := \sup_{n \geq m} a_n \tag{4.1}$$

For any $m_0 \leq m_1 \in \mathbb{N}$, it by the definition of supremum, it must be the case $b_{m_0} \geq b_{m_1}$. Therefore, (b_m) is a monotonically non-decreasing sequence. Also since (a_n) is bounded, (b_m) is bounded as well. Thus, according to the monotone sequence theorem, (b_m) converges to some limit $b \in \mathbb{R}$. Define

$$\limsup_{n \to \infty} a_n := b \tag{4.2}$$

Definition 4.2 (liminf)**.** Let (a_n) be a bounded sequence, for each $m \in \mathbb{N}$, define

$$b_m := \inf_{n \geq m} a_n \tag{4.3}$$

For any $m_0 \leq m_1 \in \mathbb{N}$, it by the definition of infimum, it must be the case $b_{m_0} \leq b_{m_1}$. Therefore, (b_m) is a monotonically non-increasing sequence. Also since (a_n) is bounded, (b_m) is bounded as well. Thus, according to the monotone sequence theorem, (b_m) converges to some limit $b \in \mathbb{R}$. Define

$$\liminf_{n \to \infty} a_n := b \tag{4.4}$$

Theorem 4.1.

$$\lim_{n \to \infty} = a \iff \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n = a \tag{4.5}$$

Proof. ($\implies$) Suppose $(a_n) \to a$, for each $m \in \mathbb{N}$, define $b_m := \sup_{n \geq m} a_n$ and $c_m := \inf_{n \geq m} a_n$. By the definitions of infimum, supremum, and, the convergence of sequence. For each $\varepsilon > 0$, for large enough $m \in \mathbb{N}$, for every $n \geq m$, we can bound a_n in the range $(a - \varepsilon, a + \varepsilon)$, so are the supremum and infimum.

$$\forall \varepsilon > 0,\ \exists m \in \mathbb{N},\ s.t. \begin{cases} b_m < a + \varepsilon \\ c_m > a - \varepsilon \end{cases} \tag{4.6}$$

Also, by the convergence of (a_n), there exists $N^* \in \mathbb{N}$ such that $\forall n \geq N^*$, $|a_n - a| < \frac{\varepsilon}{2}$, which means $a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$. Therefore,

$$a - \frac{\varepsilon}{2} \leq \underbrace{\inf_{n \geq \mathbb{N}} a_n}_{c_N} \leq \underbrace{\sup_{n \geq \mathbb{N}} a_n}_{b_N} \leq a + \frac{\varepsilon}{2} \tag{4.7}$$

so, since c_N is increasing, and b_N is decreasing, $(c_N) \to a$ and $(b_N) \to a$. ■

Definition 4.3 (Double Index Sequence)**.** A sequence is said to be in **double index form** (i.e. indexed by $\mathbb{N}^2$, which is also countable) if it can be written as

$$(a_{m,n}),\ m, n \in \mathbb{N} \tag{4.8}$$

and $\lim_{m\to\infty, n\to\infty} a_{m,n} = r$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N},\ s.t.\ \forall m, n \geq N,\ |a_{m,n} - r| < \varepsilon \tag{4.9}$$

Theorem 4.2. Suppose, for sequence $(a_{m,n})$,

$$\lim_{m\to\infty} (\lim_{n\to\infty} a_{m,n}) = a \tag{4.10}$$

$$\lim_{n\to\infty} (\lim_{m\to\infty} a_{m,n}) = b \tag{4.11}$$

$$\lim_{m\to\infty, n\to\infty} a_{m,n} = r \tag{4.12}$$

if a, b, r all exist, then $a = b = r$.

Remark 4.1. The theorem extends to sequence with countably many indices.