

Elements of Real Analysis

Based on Lecture Notes for MAT337: Introduction to Real Analysis (2019Winter)

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- GitHub: https://github.com/TianyuDu/Spikey_UofT_Notes
- Website: TianyuDu.com/notes

TO-DO

1. Add Dedekind cut to section 1.
2. Refine subsection titles.

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1 Real Numbers

1.1 Definitions

Definition 1.1. Reals are proper initial segments of \mathbb{Q} with no maximum.

Definition 1.2. A subset $A \subset \mathbb{Q}$ is an **initial segment** if

$$y \in A, x \in \mathbb{Q}, x < y \implies x \in A \quad (1.1)$$

Definition 1.3. A is **proper** if $A \neq \mathbb{Q}$.

Definition 1.4. A has no maximal elements if

$$\forall x \in A, \exists y \in A \text{ s.t. } y > x \quad (1.2)$$

Example 1.1.

$$\sqrt{2} \approx A_{\sqrt{2}} := \{q \in \mathbb{Q} : q < \sqrt{2}\} \equiv \{q \in \mathbb{Q} : q \leq 0 \vee q^2 < 2\} \quad (1.3)$$

$$x \approx A_x := \{q \in \mathbb{Q} : q < x\} \quad (1.4)$$

1.2 The Axiom of Completeness

Axiom 1.1 (Axiom of Completeness). Every non-empty subset $B \subset \mathbb{R}$ that is bounded has a supremum (i.e. the least upper bound). That's

$$\forall B \subset \mathbb{R}, \text{ s.t. } B \neq \emptyset \exists b \in \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in B, x \leq b \text{ (upper bound)} \\ \forall c \in \mathbb{R} (\forall x \in B, x \leq c) \implies b \leq c \text{ (least upper bound)} \end{cases} \quad (1.5)$$

Theorem 1.1. \mathbb{Q} is *dense* in \mathbb{R} , that's

$$\forall x < y \in \mathbb{R}, \exists q \in \mathbb{Q} \text{ s.t. } x < q < y \quad (1.6)$$

Theorem 1.2 (Cardinality). Let A, B be non-empty subsets of \mathbb{R} , then the following statements are equivalent:

- (i) $\exists h : A \rightarrow B$ such that h is bijective;
- (ii) $\exists f : A \rightarrow B$ and $g : B \rightarrow A$ such that both f and g are injective.

Proof. (i) is the definition for sets A and B to have the same cardinality. And the existence of injection from A to B implies the cardinality of A cannot be greater than the cardinality of B . Similarly, the existence of injection from B to A implies the cardinality of B cannot be greater than the cardinality of A . Therefore A and B share the same cardinality. ■

Theorem 1.3 (Nested Intervals). Let (I_n) be a sequence of closed and non-empty intervals in \mathbb{R} such that

$$I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \quad (1.7)$$

then

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset \quad (1.8)$$

Proof. Claim:

$$x := \sup\{\min(I_n) : n \in \mathbb{N}\} \in \bigcap_{n \in \mathbb{N}} I_n \quad (1.9)$$

Let $n \in \mathbb{N}$, then $x \geq \min(I_n)$. Now show $x \leq \max(I_n) \forall n \in \mathbb{N}$. Suppose not, then $\exists k \in \mathbb{N}$ such that $x > \max(I_k)$. Then by the definition of supremum, there exists $j \in \mathbb{N}$ such that $\max(I_k) < \min(I_j)$. Note that if $k = j$, this leads to a contradiction. If $k < j$, then because $I_k \supset I_j$, $\max(I_k) \geq \max(I_j) \geq \min(I_j) \geq \min(I_k)$, this leads to a contradiction. If $k > j$, then $I_k \subset I_j$, thus $\min(I_j) \leq \min(I_k) \leq \max(I_k) \leq \max(I_j)$, which also leads to a contradiction. Therefore we conclude

$$\min(I_n) \leq x \leq \max(I_n) \forall n \in \mathbb{N} \quad (1.10)$$

therefore $x \in I_n \forall n \in \mathbb{N}$, so $x \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Theorem 1.4. There exists no injection from \mathbb{R} to \mathbb{N} .

Proof. \mathbb{R} has cardinality c but \mathbb{N} has cardinality \aleph_0 . ■

2 Sequences and Series

Definition 2.1. A sequence $(a_n)_{n=1}^{\infty}$ of real numbers **converges** to a real number a if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n > N |a_n - a| < \varepsilon \quad (2.1)$$

If there does not exist such a , we conclude $(a_n)_{n=1}^{\infty}$ is **divergent**.

Theorem 2.1. Every convergent sequence is bounded.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence in \mathbb{R} with limit point a . Then take $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n > N \implies |a_n - a| < 1 \implies |a_n| < |a| + 1$. Take

$$M := \max\{\max_{n \leq N} |a_n|, |a| + 1\} \quad (2.2)$$

and the sequence is bounded by M . ■

Definition 2.2. Let $(a_n)_{n=1}^{\infty}$ be a sequence, then a sub-sequence of (a_n) is any sequence in the form $(a_{n_k})_{k=1}^{\infty}$ such that $n_1 < n_2 < \dots < n_k < \dots$.

Remark 2.1. A sub-sequence can be generated with a strictly increasing function defined on \mathbb{N} and a sequence (a_n) .

Theorem 2.2 (Bolzano-Weierstrass). Every bounded sequence has a convergent sub-sequence.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence bounded by $M > 0$. Define

$$I_0 := [-M, M] \quad (2.3)$$

$$J^0 := [-M, 0] \quad (2.4)$$

$$J^1 := [0, M] \quad (2.5)$$

$$X^0 := \{n \in \mathbb{N} : a_n \in J^0\} \quad (2.6)$$

$$X^1 := \{n \in \mathbb{N} : a_n \in J^1\} \quad (2.7)$$

therefore $\mathbb{N} = X^0 \cup X^1$. Thus at least one of X^0 and X^1 is infinite. If X^0 is infinite, define $I_1 := J^0$, otherwise, define $I_1 := J^1$. Let

$$A := \{x \in \mathbb{R} : \{n \in \mathbb{N} : x < a_n\} \text{ is infinite}\} \quad (2.8)$$

which is the lower bound of selected infinite half intervals. And define $a := \inf(A)$. we can construct a sub-sequence, for each $n \in \mathbb{N}$, take $a_n \in I_n$. And by the nested interval theorem, the intersection of all those selected intervals is non-empty. And a is the limit point of the constructed sequence. So a convergent sub-sequence exists. ■

Definition 2.3. A sequence $(a_n)_{n=1}^\infty$ is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N, |a_n - a_m| < \varepsilon \quad (2.9)$$

Theorem 2.3 (Convergent \implies Cauchy). Every convergent sequence is a Cauchy sequence.

Proof. Let (a_n) be a convergent sequence, fix $\varepsilon > 0$. Suppose $(a_n) \rightarrow a$, take $\varepsilon^* = \varepsilon/2$. Thus, there exists $N \in \mathbb{N}$ such that $\forall n > N, |a_n - a| < \varepsilon^* = \varepsilon/2$. By taking such $N, \forall n, m > N$, both $|a_n - a|$ and $|a_m - a| < \varepsilon/2$. By triangle inequality, $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, we've shown that for an arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall m, n > N, |a_n - a_m| < \varepsilon$. Therefore (a_n) is Cauchy. ■

Theorem 2.4 (Cauchy \implies Convergent). Every Cauchy sequence is convergent.

Proof. Let (a_n) be a Cauchy sequence.

Claim: (a_n) is bounded.

Proof. Bounded. Take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall m, n > N, |a_n - a_m| < 1$. Take $m = N + 1$ and define $a^* := a_m$. Then we have $\forall n > N, |a_n - a^*| < 1$, which implies $|a_n| < |a^*| + 1$. Define

$$M := \max\{\max\{a_n : n \leq N\}, |a^*| + 1\} \quad (2.10)$$

So (a_n) is bounded by M . ■

Then by Bolzano-Weierstrass Theorem, there exists a sub-sequence $(a_{n_k})_{k=1}^\infty$ converges to some limit point $a \in \mathbb{R}$. We are going to show $(a_n) \rightarrow a$. Fix $\varepsilon > 0$, by the convergence of the sub-sequence

$$\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq N_1, |a_{n_k} - a| < \frac{\varepsilon}{2} \quad (2.11)$$

Also since the sequence itself is Cauchy,

$$\exists N_2 \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N_2, |a_n - a_m| < \frac{\varepsilon}{2} \quad (2.12)$$

Take $N^* := \max\{N_1, N_2\}$. Show $|a_n - a| < \varepsilon \forall n \geq N^*$. Note that

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \quad (2.13)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \quad (2.14)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (2.15)$$

since $n_k \geq n$ by the definition of sub-sequences. ■

Corollary 2.1. A sequence is Cauchy if and only if it is convergent.

Proof. Let (a_n) be a Cauchy sequence.

Claim: (a_n) is bounded.

Proof. Take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that

$$\forall m, n > N \quad |a_n - a_m| < 1 \quad (2.16)$$

take $m := N + 1$, define $a^* := a_{m+1}$, then

$$\forall n > N, \quad |a_n - a^*| < 1 \implies |a_n| \leq |a^*| + 1 \quad (2.17)$$

Define $M := \max\{\max_{n \leq N}\{a_n\}, |a^*| + 1\}$, and (a_n) is bounded by M . ■

By the Bolzano-Weierstrass Theorem, there exists a sub-sequence $(a_{n_k})_{k=1}^{\infty}$ converges to some limit point $a \in \mathbb{R}$. Show $(a_n) \rightarrow a$ as well.

Let $\varepsilon > 0$, by convergence of the sub-sequence,

$$\exists N_1 \in \mathbb{N}, \text{ s.t. } \forall n \geq N_1, \quad |a_{n_k} - a| < \varepsilon/2 \quad (2.18)$$

By the Cauchy property of (a_n) ,

$$\exists N_2 \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N_2, \quad |a_n - a_m| < \varepsilon/2 \quad (2.19)$$

take $N^* := \max\{N_1, N_2\}$. Let $n \geq N^*$ and note that $n_k \geq n \geq N^*$

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \quad (2.20)$$

$$\leq |a_n - a_{n_k}| + |a_{n_k} - a| \quad (2.21)$$

$$\varepsilon/2 + \varepsilon/2 = \varepsilon \quad (2.22)$$

then take such N^* for the fixed $\varepsilon > 0$. Convergence of (a_n) shown. ■

Theorem 2.5 (the Uniqueness of the Limit Point). If $(a_n) \rightarrow a$ and $(a_n) \rightarrow b$, then $a = b$.

Proof. Suppose $a \neq b$, define $s := |a - b| > 0$. Take $\varepsilon = \frac{s}{2}$, there does not exist such $N \in \mathbb{N}$ satisfying

$$\forall n \geq N, \quad \begin{cases} |a_n - a| < \varepsilon \\ |a_n - b| < \varepsilon \end{cases} \quad (2.23)$$

above notion indicates that the sequence is converging to two separate limit points simultaneously. ■

Theorem 2.6 (Properties of Limits). If $(a_n) \rightarrow a$, $(b_n) \rightarrow b$, and $c \in \mathbb{R}$, then

- (i) $(c \cdot a_n) \rightarrow c \cdot a$;
- (ii) $(a_n + c) \rightarrow a + c$;
- (iii) $(a_n + b_n) \rightarrow a + b$;
- (iv) $(a_n \cdot b_n) \rightarrow a \cdot b$.

3 Convergence of Series

Definition 3.1. A series $\sum_{n=1}^{\infty} a_n$ is **convergent** if

$$\exists a \in \mathbb{R} \text{ s.t. } \sum_{n=1}^{\infty} a_n = a \quad (3.1)$$

Definition 3.2 (Alternative Definition). Let $(S_n) := (\sum_{i=1}^n a_i)_{n=1}^{\infty}$ denote the *sequence of partial sums* associated with series $\sum_{n=1}^{\infty} a_n$, then the series is convergent if and only if its partial sum converges to a real number.

Theorem 3.1 (Cauchy Criterion). A series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall m \geq n \geq N, \left| \sum_{i=n}^m a_i \right| < \varepsilon \quad (3.2)$$

That's, the partial sum sequence is Cauchy.

Corollary 3.1. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$

Corollary 3.2 (Absolute Convergence Test). If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

Corollary 3.3. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, and, let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection, then

$$\sum_{n=1}^{\infty} a_{f(n)} \quad (3.3)$$

is convergent.

Given the absolute convergence, the rearrangement of sequence does not affect the convergence of series.

Theorem 3.2. Suppose $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq 0$ and $a_n \rightarrow 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Theorem 3.3. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection, then $\sum_{n=1}^{\infty} |a_{f(n)}|$ is also convergent.

Theorem 3.4. Suppose $\sum_{n=1}^{\infty} |a_n|$ is convergent, let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be two bijections, then

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_{g(n)} \quad (3.4)$$

Theorem 3.5 (Monotone Convergence). Every monotone sequence, which is bounded, is convergent.

Corollary 3.4. Given sequence $(a_n) \subset \mathbb{R}_{++}$ and series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums is therefore a monotonically increasing sequence, so the partial sum (S_n) is convergent if it is bounded.

Example 3.1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. Let $m \in \mathbb{N}$, so

$$S_m = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{m \times m} \quad (3.5)$$

$$< 1 + \frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \cdots + \frac{1}{m \times (m-1)} \quad (3.6)$$

$$= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) \quad (3.7)$$

$$= 2 - \frac{1}{m} < 2 \quad (3.8)$$

therefore (S_m) is non-decreasing and bounded above by 2. So (S_m) is convergent, so is $\sum_{n=1}^{\infty} \frac{1}{n^2}$. ■

4 Order and Converging Sequences

Proposition 4.1. If $(a_n) \geq 0$ is convergent to $a \in \mathbb{R}$, then $a \geq 0$.

Proof. By contradiction. ■

Proposition 4.2. If $(a_n) \leq (b_n)$ are convergent to a and b , respectively, then $a \leq b$.

Proof. Construct sequence $(b_n - a_n) \geq 0$ and apply the previous proposition. ■

Definition 4.1 (limsup). Let (a_n) be a bounded sequence, for each $m \in \mathbb{N}$, define

$$b_m := \sup_{n \geq m} a_n \quad (4.1)$$

For any $m_0 \leq m_1 \in \mathbb{N}$, it by the definition of supremum, it must be the case $b_{m_0} \geq b_{m_1}$. Therefore, (b_m) is a monotonically non-decreasing sequence. Also since (a_n) is bounded, (b_m) is bounded as well. Thus, according to the monotone sequence theorem, (b_m) converges to some limit $b \in \mathbb{R}$. Define

$$\limsup_{n \rightarrow \infty} a_n := b \quad (4.2)$$

Definition 4.2 (liminf). Let (a_n) be a bounded sequence, for each $m \in \mathbb{N}$, define

$$b_m := \inf_{n \geq m} a_n \quad (4.3)$$

For any $m_0 \leq m_1 \in \mathbb{N}$, it by the definition of infimum, it must be the case $b_{m_0} \leq b_{m_1}$. Therefore, (b_m) is a monotonically non-increasing sequence. Also since (a_n) is bounded, (b_m) is bounded as well. Thus, according to the monotone sequence theorem, (b_m) converges to some limit $b \in \mathbb{R}$. Define

$$\liminf_{n \rightarrow \infty} a_n := b \quad (4.4)$$

Theorem 4.1.

$$\lim_{n \rightarrow \infty} a_n = a \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = a \quad (4.5)$$

Proof. (\implies) Suppose $(a_n) \rightarrow a$, for each $m \in \mathbb{N}$, define $b_m := \sup_{n \geq m} a_n$ and $c_m := \inf_{n \geq m} a_n$. By the definitions of infimum, supremum, and, the convergence of sequence. For each $\varepsilon > 0$, for

large enough $m \in \mathbb{N}$, for every $n \geq m$, we can bound a_n in the range $(a - \varepsilon, a + \varepsilon)$, so are the supremum and infimum.

$$\forall \varepsilon > 0, \exists m \in \mathbb{N}, \text{ s.t. } \begin{cases} b_m < a + \varepsilon \\ c_m > a - \varepsilon \end{cases} \quad (4.6)$$

Also, by the convergence of (a_n) , there exists $N^* \in \mathbb{N}$ such that $\forall n \geq N^*, |a_n - a| < \frac{\varepsilon}{2}$, which means $a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$. Therefore,

$$a - \frac{\varepsilon}{2} \leq \underbrace{\inf_{n \geq \mathbb{N}} a_n}_{c_N} \leq \underbrace{\sup_{n \geq \mathbb{N}} a_n}_{b_N} \leq a + \frac{\varepsilon}{2} \quad (4.7)$$

so, since c_N is increasing, and b_N is decreasing, $(c_N) \rightarrow a$ and $(b_N) \rightarrow a$. ■

Definition 4.3 (Double Index Sequence). A sequence is said to be in **double index form** (i.e. indexed by \mathbb{N}^2 , which is also countable) if it can be written as

$$(a_{m,n}), \quad m, n \in \mathbb{N} \quad (4.8)$$

and $\lim_{m \rightarrow \infty, n \rightarrow \infty} a_{m,n} = r$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m, n \geq N, |a_{m,n} - r| < \varepsilon \quad (4.9)$$

Theorem 4.2. Suppose, for sequence $(a_{m,n})$,

$$\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} a_{m,n}) = a \quad (4.10)$$

$$\lim_{n \rightarrow \infty} (\lim_{m \rightarrow \infty} a_{m,n}) = b \quad (4.11)$$

$$\lim_{m \rightarrow \infty, n \rightarrow \infty} a_{m,n} = r \quad (4.12)$$

if a, b, r all exist, then $a = b = r$.

Remark 4.1. The theorem extends to sequence with countably many indices.