ECO210H1 Mathematical Methods for Economic Theory: Notes

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Contents

1 Quadratic Forms

Definition 1.1 (Quadratic Form)**.**

$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$$

where $\mathbf{x} \in \mathbb{R}^n$ and symmetric $\mathbf{A} \in \mathbb{M}_{n\times n}$.

Definition 1.2 (Definiteness of Quadratic Form)**.** Let $\mathcal{Q}$ be a quadratic form

1. Positive definite.
2. Negative definite
3. Positive semidefinite.
4. Negative semidefinite.
5. Indefinite.

Definition 1.3 (k^{th} Order Leading Principal Minor)**.**

Proposition 1.1. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let D_k where $k = 1, \dots, n$ be its leading principal minors. Then

1. $\mathbf{A}$ is positive definite $\iff D_k > 0\ \forall k$.
2. $\mathbf{A}$ is negative definite $\iff (-1)^k D_k > 0\ \forall k$.

Definition 1.4 (k^{th} Order Principal Minor)**.**

Proposition 1.2. Let $\mathbf{A}$ be a $n \times n$ symmetric matrix. Then

1. $\mathbf{A}$ is positive semidefinite $\iff$ all its principal minors are non-negative.
2. $\mathbf{A}$ is negative semidefinite $\iff$ all its k^{th} order principal minors are non-positive for k odd and non-negative for k even.

2 Concavity and Convexity

Definition 2.1 (Convex Sets in $\mathbb{R}^n$)**.** A set $S \subseteq \mathbb{R}^n$ is convex if for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$, we have

$$t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in S$$

for all $t \in [0, 1]$.

Definition 2.2 (Convex Combination)**.** Given a finite number of points $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$ in a real vector space, a convex combination of these points is

$$\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{x}_i$$

where

$$\sum_{i=1}^{n} \alpha_i = 1 \wedge a_i \geq 0 \; \forall i$$

Definition 2.3 (Convex Hull (of a finite point set))**.** Let S be a finite set of vectors from a real vector space,

$$Conv(S) = \Big\{ \sum_{i=1}^{|S|} \alpha_i x_i \Big| (\alpha_i \geq 0 \; \forall i) \wedge \sum_{i=1}^{|S|} \alpha_i = 1 \Big\}$$

Definition 2.4 (Affine Combination)**.**

$$\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{x}_i$$

where

$$\sum_{i=1}^{n} \alpha_i = 1$$

Proposition 2.1. The intersection of convex sets is convex.

Definition 2.5 (Concave Function)**.** Let $D \subseteq \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$ is **concave** if for all $\mathbf{x}_1, \mathbf{x}_2 \in D$

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \geq t f(\mathbf{x}_1) + (1-t) f(\mathbf{x}_2) \quad \forall t \in [0, 1]$$

Definition 2.6 (Convex Function)**.** Let $D \subseteq \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$ is **convex** if for all $\mathbf{x}_1, \mathbf{x}_2 \in D$

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2) \quad \forall t \in [0,1]$$

Definition 2.7 (Strictly Concave Function)**.** $f : D \to \mathbb{R}$ is **strictly concave** if and only if, for all $\mathbf{x}_1 \neq \mathbf{x}_2 \in D$,

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) > tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2) \quad \forall t \in (0,1)$$

Theorem 2.1. $f : D \to \mathbb{R}$ defined on a convex set is

1. Concave if and only if set

$$A \equiv \{(\mathbf{x}, y) : \mathbf{x} \in S, y \leq f(\mathbf{x})\}$$

   is convex.

2. Convex if and only if set

$$A \equiv \{(\mathbf{x}, y) : \mathbf{x} \in S, y \geq f(\mathbf{x})\}$$

   is convex.

Theorem 2.2 (Jensen's Inequality)**.** Let $f : D \to \mathbb{R}$ defined on a convex set is concave if and only if for all $n \geq 2$,

$$f(\sum_{i=1}^{n} \lambda_i \mathbf{x}_i) \geq \sum_{i=1}^{n} \lambda_i f(\mathbf{x}_i)$$

where

$$(\forall i : \lambda_i \geq 0) \wedge \sum_{i=1}^{n} \lambda_i = 1$$

and f is convex if and only if for all $n \geq 2$,

$$f(\sum_{i=1}^{n} \lambda_i \mathbf{x}_i) \leq \sum_{i=1}^{n} \lambda_i f(\mathbf{x}_i)$$

where

$$(\forall i : \lambda_i \geq 0) \wedge \sum_{i=1}^{n} \lambda_i = 1$$

3 Optimization: definitions