MAT237: Lecture Notes

Advanced Calculus

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1.1 The Geometry of Euclidean Space

Example 1.1. Consider $(1, 2) \in \mathbb{R}^2$ as a point or a vector.

Remark 1.1. All vectors in this course are considered as column vectors. Reasoning: suppose a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, then the transformation can be implemented as

$$f(\vec{x}) = \mathbf{A}\vec{x},\ \mathbf{A} \in M_{m\times n}(\mathbb{R})$$

if $\vec{x}$ is a column vector.

Definition 1.1. Let $\vec{a}, \vec{b} \in \mathbb{R}^n$, the **dot product** $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as,

$$\vec{a} \cdot \vec{b} = \sum_i a_i b_i$$

Definition 1.2. Let $\vec{a} \in \mathbb{R}^n$, the **Euclidean norm** $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{b}}$$

Interpretation the Euclidean norm of $\vec{a}$, $||\vec{a}||$ is the length of $\vec{a}$, or the distance of $\vec{a}$ from the origin. And $||\vec{a} - \vec{b}||$ is the distance from $\vec{a}$ to $\vec{b}$.

Definition 1.3. Two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ is **orthogonal** if and only if

$$\vec{a} \cdot \vec{b} = 0$$

Theorem 1.1. (Cauchy Schwarz inequality)

$$|\vec{a} \cdot \vec{b}| \leq ||\vec{a}||||\vec{b}||$$

Theorem 1.2. (Triangle inequality)

$$||\vec{a} + \vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$$

Theorem 1.3.

$$\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

Definition 1.4. If $\vec{u} \in \mathbb{R}^n$ is a **unit vector** if

$$||\vec{u}|| = 1$$

Definition 1.5. The **projection** of $\vec{a}$ onto the line through $\vec{u}$ is defined as

$$(\vec{u} \cdot \vec{a})\vec{u}$$

1.2 Subspaces of $\mathbb{R}^n$

Definition 1.6. A subspace V if $\mathbb{R}^n$ is a <u>subset</u> of $\mathbb{R}^n$ such that

$$\vec{a}, \vec{b} \in V \wedge c_1, c_2 \in \mathbb{R} \implies c_1\vec{a} + c_2\vec{b} \in V$$

Example 1.2. Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \\ -1 & 0 \end{pmatrix}$$

And consider

$$V = \{\mathbf{A}\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

V is a subspace with dimension 2.

Theorem 1.4. Let $\mathbf{A} \in M_{m\times n}(\mathbb{R})$ with $m > n$ and columns are independent then $V = \{\mathbf{A}\vec{x} : \vec{x} \in \mathbb{R}^n\}$ is a n-dimensional subspace of $\mathbb{R}^n$.

Example 1.3. Consider

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 9 & -2 \end{pmatrix}$$

and

$$V = \{\vec{x} \in \mathbb{R}^3 : \mathbf{A}\vec{x} = \vec{0}\}$$

Then V is a 1-dimensional subspace of $\mathbb{R}^3$.

Theorem 1.5. $\mathbf{A} \in M_{m\times n}(\mathbb{R})$ and $m < n$ and rows are linearly independent, then $\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \vec{0}\}$ is a $(n - m)$ dimensional subspace.

1.3 Cross Product

(*Only available in* $\mathbb{R}^3$) is a way to multiplying two vectors in $\mathbb{R}^3$ to get another vector in $\mathbb{R}^3$.

Definition 1.7. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ then the **cross product** $\times : \mathbb{R}^6 \to \mathbb{R}^3$ is defined as

$$\vec{a} \times \vec{b} := det(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix})$$
$$\text{where } \vec{i} = (1, 0, 0),\ \vec{j} = (0, 1, 0),\ \vec{k} = (0, 0, 1)$$

Remark 1.2. $\vec{a} \times \vec{b}$ is the vector such that

1. orthogonal to both $\vec{a}$ and $\vec{b}$.
2. it's length is $||\vec{a}||||\vec{b}|| \sin\theta$.

Proposition 1.1. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$, then

1. $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{a} = \vec{0}$
3. $(c_1\vec{a_1} + c_2\vec{a_2}) \times \vec{b} = c_1(\vec{a_1} \times \vec{b_1}) + c_2(\vec{a_2} \times \vec{b_2})$
4. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

1.4 Functions of Several Variables

Remark 1.3. Idea of differential calculus: more general general functions can then be approximated by linear functions.

Definition 1.8. Consider function $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is

$$\{(x, y, z) : z = f(x, y)\} \subseteq \mathbb{R}^3$$

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2.1 Visualize function with two variables

Definition 2.1. Given $f : \mathbb{R}^2 \to \mathbb{R}$ define **graph** of

$$\mathcal{G}(f) := \{(x, y, z) : z = f(x, y)\}$$

and the **level set** of f is the set $\{(x, y) : f(x, y) = c\}$, with several values of c, it's called **contour plot**.

Example 2.1. $f(x, y) = \frac{x^2}{4} - \frac{y^2}{9}$.

Definition 2.2. Consider function $f : \mathbb{R}^3 \to \mathbb{R}$ we still define the graph of it as

$$\mathcal{G}(f) := \{(x, y, z, w) : w = f(x, y, z)\} \subseteq \mathbb{R}^4$$

and the **level sets** (*level surfaces*) of f are defined as

$$\{(x, y, z) : f(x, y, z) = c\} \subseteq \mathbb{R}^3$$

Definition 2.3. Consider real value function $f : \mathbb{R}^n \to \mathbb{R}$, it's graph is a subset of $\mathbb{R}^{n+1}$ and the contour is a subset of $\mathbb{R}^n$.

2.2 Subsets of $\mathbb{R}^n$

Definition 2.4. Given $r > 0$ and $\vec{a} \in \mathbb{R}^n$, the **open ball** of radius r centred at $\vec{a}$ is defined as

$$\mathcal{B}(r, \vec{a}) := \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{a}|| < r\}$$

Definition 2.5. The **sphere** of radius r centred at $\vec{a}$ is defined as

$$\{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{a}|| = r\}$$

Definition 2.6. Let $S \subseteq \mathbb{R}^n$, then S is **bounded** if and only if

$$\exists r > 0 \ s.t. \ S \subseteq \mathcal{B}(r, \vec{0})$$

Example 2.2.

$$S_1 = \{(x,y,z) : x^2 + y^2 - \cos e^{e^z} \leq 5\} \text{ Unbounded}$$
$$S_2 = \{(x,y,z) : x^2 + y^2 + z^2 - \cos e^{e^z} \leq 5\} \text{ Bounded}$$
$$S_3 = \{(x,y) : xy = -1\} \text{ Unbounded}$$

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Definition 3.1. Let $S \subseteq \mathbb{R}^n$, the **complement** of S in $\mathbb{R}^n$ denoted as S^c is defined as

$$S^c := \{\vec{x} \in \mathbb{R}^n : \vec{x} \notin S\}$$

Definition 3.2. A point $\vec{x} \in \mathbb{R}^n$ and let $S \subseteq \mathbb{R}^n$ then $\vec{x}$ is in the **interior** of S, denoted as $\vec{x} \in S^{int}$ if

$$\exists \epsilon > 0 \ s.t. \ \mathcal{B}(\epsilon, \vec{x}) \subseteq S$$

Definition 3.3. $\vec{x}$ is in the **boundary** of S, denoted as $\vec{x} \in \partial S$, if

$$\forall \epsilon > 0, \ \mathcal{B}(\epsilon, \vec{x}) \cap S \neq \emptyset \wedge \mathcal{B}(\epsilon, \vec{x}) \cap S^c \neq \emptyset$$

Definition 3.4. $\vec{x}$ is in the **closure** of S, denoted as $\vec{x} \in \overline{S}$

$$\forall \epsilon > 0, \ \mathcal{B}(\epsilon, \vec{x}) \cap S \neq \emptyset$$

Theorem 3.1. Notice that

$$\overline{S} = \partial S \cup S^{int}$$

Remark 3.1. Every point of S is either an interior point or a boundary point.

Example 3.1.

$$S = \mathcal{B}(r, \vec{a}) = \{\vec{x} : ||\vec{x} - \vec{a}|| < r\}$$

Claim (true):

1. $S^{int} = S$
2. $\partial S = \{\vec{x} : ||\vec{x} - \vec{a}|| = r\}$
3. $\overline{S} = \{\vec{x} : ||\vec{x} - \vec{a}|| \leq r\}$

Example 3.2. Consider

$$S = \{x \in (0,1) : x \in \mathbb{Q}\} \subseteq \mathbb{R}$$

Claim (true):

1. $S^{int} = \emptyset$
2. $\partial S = [0, 1]$
3. $\overline{S} = [0, 1]$

Theorem 3.2. For all set $S \subseteq \mathbb{R}^n$,

$$S^{int} \subseteq S \subseteq \overline{S}$$

Proof. Let $\vec{x} \in S^{int}$, by definition of interior points, $\exists \epsilon > 0 \ s.t. \ \mathcal{B}(\epsilon, \vec{x}) \subseteq S$,
Since $\vec{x} \in \mathcal{B}(\epsilon, \vec{x})$ by definition of open ball $\implies \vec{x} \in S \ \forall \vec{x} \in S^{int} \implies S^{int} \subseteq S$
Since $\overline{S} = S^{int} \cup \partial S$, therefore $S^{int} \subseteq \overline{S}$. ■

Theorem 3.3. For all $S \subseteq \mathbb{R}^n$,

$$\partial S = \partial(S^c)$$

Proof. Let $\vec{x} \in \partial(S^c)$
$\iff \forall \epsilon > 0, \ \mathcal{B}(\epsilon, \vec{x}) \cap S \neq \emptyset \wedge \mathcal{B}(\epsilon, \vec{x}) \cap S^c \neq \emptyset$
$\iff \forall \epsilon > 0, \ \mathcal{B}(\epsilon, \vec{x}) \cap (S^c)^c \neq \emptyset \wedge \mathcal{B}(\epsilon, \vec{x}) \cap S^c \neq \emptyset$
$\iff \vec{x} \in \partial S$ ■

Definition 3.5. A set $S \subseteq \mathbb{R}^n$ is **open** if $S = S^{int}$ and is **closed** if $S = \overline{S}$.

Remark 3.2. A set S can be both open and closed or neither open or closed.

Example 3.3. Consider set $S = \mathbb{R}^n$, $\partial S = \emptyset$ then $S = S^{int} = \overline{S}$ and S is <u>both</u> open and closed.

Example 3.4. Consider $S = \mathcal{B}(r, \vec{a}) \subseteq \mathbb{R}^n$, and $S = S^{int} \implies S$ is open.

Example 3.5. Consider $S = \emptyset$, $S = S^{int} = \partial S = \overline{S} = \emptyset$ and S is <u>both</u> open and closed.

Example 3.6. Consider $S = \mathbb{Q}$, $S^{int} = \emptyset$ and $\partial S = \mathbb{R}$, S is <u>neither</u> open or closed.

Remark 3.3. Most sets are neither open or closed.

Theorem 3.4. Let $S \subseteq \mathbb{R}^n$ be a set, the following statements are equivalent,

1. $S \subseteq \mathbb{R}^n$ is an open set.
2. $S \subseteq S^{int}$
3. $\forall \vec{x} \in S, \ \exists \ \epsilon > 0, \ s.t. \ \mathcal{B}(\epsilon, \vec{x}) \subseteq S$

Theorem 3.5. Let $T \subseteq \mathbb{R}^n$, the following statements are equivalent,

1. T is a closed set.
2. $\partial T \subseteq T$

3. T^c is open.

Proof. Let T be a closed set, by definition, $\partial T \subseteq T$.
By theorem 3.3, $\partial(T^c) \subseteq T$,
$\iff \partial T^c \nsubseteq T^c$
$\iff$ no points in T^c is boundary point
$\iff \forall \vec{x} \in T^c,\ \neg(\forall \epsilon > 0,\ \mathcal{B}(\epsilon, \vec{x}) \cap T^c \neq \emptyset \wedge \mathcal{B}(\epsilon, \vec{x}) \cap T \neq \emptyset)$
$\iff \forall \vec{x} \in T^c, \exists \epsilon > 0,\ s.t.\ \mathcal{B}(\epsilon, \vec{x}) \cap T^c = \emptyset \vee \mathcal{B}(\epsilon, \vec{x}) \cap T = \emptyset$
Clearly, since $\vec{x} \in T^c$, $\mathcal{B}(\epsilon, \vec{x}) \cap T^c \neq \emptyset$
$\iff \forall \vec{x} \in T^c, \exists \epsilon > 0,\ s.t.\ \mathcal{B}(\epsilon, \vec{x}) \cap (T^c)^c = \emptyset$,
By definition of open set, T^c is open. ■