Introduction to Real Analysis

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1 The Axiom of Completeness

1.1 Preliminaries

Definition 1.1. A set $A \subset \mathbb{R}$ is **bounded above** if

$$\exists u \in \mathbb{R} \ s.t. \ \forall a \in A, \ u \geq a \tag{1.1}$$

It is said to be **bounded below** if

$$\exists l \in \mathbb{R} \ s.t. \ \forall a \in A, \ l \leq a \tag{1.2}$$

Example 1.1. The set of integers, $\mathbb{Z}$, is neither bounded from above nor below. Sets $\{1, 2, 3\}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$ are bounded from both above and below.

Notation 1.1. Let $A \subset \mathbb{R}$, we use $A^{\uparrow}$ and $A^{\downarrow}$ to denote collections of upper bounds of A and lower bounds of A. Note that $A^{\uparrow}$ and $A^{\downarrow}$ are potentially empty.

Definition 1.2. A real number $s \in \mathbb{R}$ is the **least upper bound(supremum)** for a set $A \subset \mathbb{R}$ if $s \in A^{\uparrow}$ and $\forall u \in A^{\uparrow}, \ s \leq u$. Such s is denoted as $s := \sup A$.

Definition 1.3. A real number $f \in \mathbb{R}$ is the **greatest lower bound (infimum)** for A if $f \in A^{\downarrow}$ and $\forall l \in A^{\downarrow}, \ l \leq f$. Such f is often written as $f := \inf A$.

Axiom 1.1 (The Axiom of Completeness/Least Upper Bounded Property)**.** $\forall \varnothing \neq A \subset \mathbb{R}$ such that $A^{\uparrow} \neq \varnothing$, $\exists \sup A$.

Definition 1.4. Let $\varnothing \neq A \subset \mathbb{R}$, $a_0 \in A$ is the **maximum** of A if $\forall a \in A, a_0 \geq a$; $a_1 \in A$ is the **minimum** of A if $\forall a \in A, a_1 \leq a$.

Example 1.2. $\mathbb{Q} \subset \mathbb{R}$ does not satisfy the axiom of completeness.

Proposition 1.1. Let $\varnothing \neq A \subset \mathbb{R}$ bounded above, and $c \in \mathbb{R}$. Define $c + A := \{a + c : a \in A\}$. Then

$$\sup(c + A) = c + \sup A \tag{1.3}$$

Proof. Step 1: Show $c + \sup A \in (c + A)^{\uparrow}$*:*
Let $x \in c+A$, $\exists \ a \in A \ s.t. \ x = c+a$. Then, $x = c+a \leq c+\sup A$. Therefore, $x \leq c+\sup A \ \forall x \in A$, which implies what desired.
Step 2: Show $\forall u \in (c + A)^{\uparrow}, \ c + \sup A \leq u$*:*
Let $u \in (c + A)^{\uparrow}$, then $u \geq c + a \ \forall a \in A \implies u - c \geq a \ \forall a \in A \implies u - c \in A \uparrow \implies u - c \geq \sup A \implies u \geq c + \sup A$.
Hence, $\sup(c + A) = c + \sup A$. ■

Lemma 1.1 (Alternative Definition of Supremum)**.** Let $s \in A^{\uparrow}$ for some nonempty $A \subset \mathbb{R}$. The following statements are equivalent:

(i) $s = \sup A$;

(ii) $\forall \varepsilon, \exists a \in A,\ s.t.\ a > s - \varepsilon$ (i.e. $s - \varepsilon \notin A^{\uparrow}$).

Proof. Immediately. ■

Theorem 1.1 (Nested Interval Property)**.** Let $(I_n)_n$ be a sequence of closed intervals $I_n := [a_n, b_n]$ such that these intervals are *nested* in a sense that

$$I_{n+1} \subset I_n \ \forall n \in \mathbb{N} \tag{1.4}$$

Then,

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing \tag{1.5}$$

Proof. Note that the sequence $(a_n)_{n \in \mathbb{N}}$ is bounded above by any b_k, by the completeness axiom, there exists $a^* := \sup_{n \in \mathbb{N}} a_n$. Since $a^* \in (a_n)^{\uparrow}$, $a^* \geq a_n\ \forall n \in \mathbb{N}$. Further, because a^* is the *least* upper bound, then for every upper bound b_n, it must be $a^* \leq b_n\ \forall n \in \mathbb{N}$. Therefore, $x^* \in [a_n, b_n]\ \forall n \in \mathbb{N}$. That is, $x^* \in \bigcap_{n \in \mathbb{N}} I_n$. ■

Note that NIP requires all intervals to be closed. One instance when this fails to hold: $\bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Theorem 1.2 (Archimedean Property)**.**

(i) $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N}\ s.t.\ n > x$;

(ii) $\forall y \in \mathbb{R}_{++},\ \exists n \in \mathbb{N}\ s.t. \frac{1}{n} < y$.

Archimedean property *of natural numbers* can be interpreted as *there is no real number that bounds* $\mathbb{N}$. This interpretation can be seen by considering the negations of above statements:

(i) $\exists x \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N},\ n \leq x$;

(ii) $\exists y \in \mathbb{R}_{++}\ s.t.\ \forall n \in \mathbb{N},\ y \leq \frac{1}{n}$.

Proof of (i) by Contradiction. Suppose the negated statement (i) is true, $\mathbb{N}$ is bounded above. By the completeness axiom, there exists $a^* := \sup \mathbb{N}$. $\exists n \in \mathbb{N}\ s.t.\ a^* - 1 < n$. In this case, $a^* < n+1 \in \mathbb{N}$, which means $a^* \notin \mathbb{N}^{\uparrow}$ and leads to a contradiction. ■

Proof of (ii). Let $y^* \in \mathbb{R}_{++}$, take $x = \frac{1}{y}$. By statement (i), there exists $n^* \in \mathbb{N}$ such that $n > \frac{1}{y}$. Because $y > 0$, $\frac{1}{n} < y$. ■

1.2 Density of Rational Numbers

Theorem 1.3. For every $a, b \in \mathbb{R}$ such that $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

The above theorem says $\mathbb{Q}$ is in fact **dense** in $\mathbb{R}$. More generally, one says a set $A \subset X$ is dense whenever the closure of A, $\overline{A} = X$.

Proof. Step 1: Since $b - a > 0$, by the first Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$. Such natural number satisfies $\frac{1}{n} < b - a$.
Step 2: Let m be smallest integer such that $m > an$. That is, $m - 1 \leq an < m$. Obviously, $a < \frac{m}{n}$ since $n > 0$. Further, since $m \leq an + 1$, with results from step (i), $m < bn - 1 + 1 = bn$, and $\frac{m}{n} < b$. Therefore $\frac{m}{n} \in (a, b)$. ■

Theorem 1.4. $\exists \alpha \in \mathbb{R}$ *s.t.* $\alpha^2 = 2$.

Proof. Let $\Omega := \{t \in \mathbb{R} : t^2 < 2\}$, which is obviously a set in $\mathbb{R}$ bounded from above. By the completeness axiom, Ω possesses a supremum, and we claim $\alpha := \sup \Omega$ satisfies $\alpha^2 = 2$. Suppose $\alpha^2 > 2$, then there exists $\varepsilon > 0$ such that $\alpha^2 - 2\alpha\varepsilon + \varepsilon^2 > 2$. Therefore, $\alpha > \alpha - \varepsilon \in \Omega^{\uparrow}$, which contradicts the fact that α is the least upper bound. Suppose $\alpha^2 < 2$, then there exists some $\varepsilon > 0$ such that $\alpha + \varepsilon \in \Omega$, which contradicts the assumption that α is an upper bound. Hence, it must be the case that $\alpha^2 = 2$. ■

2 Sequences

Theorem 2.1 (Triangle Inequality)**.** Let $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 2.1. Let $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a - b| \tag{2.1}$$

Proof. Note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$. And $|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$, which implies $|b| - |a| \leq |a - b|$. Therefore, by taking the absolute value, $||a| - |b|| \leq |a - b|$. ■

Definition 2.1. A sequence $(a_n) \subset \mathbb{R}$ **converges** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geq N \implies |a_n - a| < \varepsilon \tag{2.2}$$

Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the open ball centred at a with radius ε is denoted as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} \tag{2.3}$$

Theorem 2.2. The limit of any convergent sequence is unique.

Proof. Let (a_n) be a convergent sequence, assume, for contradiction, that $(a_n) \to L_1$ and $(a_n) \to L_2$ such that $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{3}$, because $(a_n) \to L_1$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L_1| < \frac{|L_1 - L_2|}{3}$. Therefore, for every $n \geq N$,

$$\begin{aligned} |a_n - L_2| &= |a_n - L_1 - (L_2 - L_1)| && (2.4) \\ &\geq ||a_n - L_1| - |L_2 - L_1|| && (2.5) \\ &= ||L_1 - L_2| - |a_n - L_1|| && (2.6) \\ &= 3\varepsilon - |a_n - L_1| && (2.7) \\ &> 2\varepsilon && (2.8) \end{aligned}$$

Therefore, there does not exist any $N' \in \mathbb{N}$ such that $|a_n - L_2| < \varepsilon$ for every $n \geq \mathbb{N}$. ∎

Definition 2.2. A sequence (a_n) is **divergent** if it does not converge.

Example 2.1. The sequence $(a_n) := (1, -1/2, 1/3, 1/4, -1/5, 1/5, -1/5, 1/5, \cdots)$ is divergent.

Proof. Let $\varepsilon := \frac{2}{5 \times 3}$, assume, for contradiction, that $(a_n) \to L$ for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - L| < \frac{2}{15}$. Since the sequence is alternating, it must be the case that $\left|L - \frac{1}{5}\right| < \frac{2}{15}$. Similarly,

$$\begin{aligned} \left|-\frac{1}{5} - L\right| &= \left|\frac{1}{5} + L\right| && (2.9) \\ &= \left|\frac{1}{5} + L - \frac{1}{5} + \frac{1}{5}\right| && (2.10) \\ &= \left|(L - \frac{1}{5}) - (-\frac{2}{5})\right| && (2.11) \\ &\geq \left|\left|L - \frac{1}{5}\right| - \frac{6}{15}\right| && (2.12) \\ &= \frac{6}{15} - \left|L - \frac{1}{5}\right| && (2.13) \\ &> \frac{4}{15} && (2.14) \\ &> \varepsilon && (2.15) \end{aligned}$$

the strict inequality suggests there cannot be a $M \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for every $n \geq M$. ∎

Alternative Proof. If (a_n) is convergent, then all of its subsequences must converge to the same limit. Obviously, there are subsequences of (a_n) converging to $\frac{1}{5}$ and $-\frac{1}{5}$ respectively, this leads to a contradiction. ∎

Definition 2.3. A sequence is **bounded** if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N},\ |a_n| < M$.

Theorem 2.3. Every convergent sequence is bounded.

Proof. Let $(a_n) \to L$, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for every $n > N$. Note that $|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$, which implies $|a_n| < |L| + 1$. Let $Q := \max_{n<N} a_n$, take $M := \max\{Q, |L| + 1\}$, then M bounds (a_n). ■

Theorem 2.4 (Algebraic Limit Theorem)**.** Let $(a_n) \to a, (b_n) \to b$ be convergent sequences, and $c \in \mathbb{R}$, then

(i) $(ca_n) \to ca$;

(ii) $(a_n + b_n) \to a + b$;

(iii) $(a_n b_n) \to ab$;

(iv) $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$, provided $b_n, b \neq 0$.

Proof (i). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \frac{\varepsilon}{|c|}$. Then, for every $n \geq N$, $|ca_n - ca| = |c||a_n - a| < \varepsilon$. ■

Proof (ii). Let $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3}$ $\forall n \geq N_1$ and $|b_n - b| < \frac{\varepsilon}{3}$ $\forall n \geq N_2$. Take $N := \max\{N_1, N_2\}$, let $n \geq N$,

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{2\varepsilon}{3} < \varepsilon \tag{2.16}$$

■

Proof (iii). Note that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n + a_n b - a_n b - ab| && (2.17) \\ &\leq |a_n b_n - a_n b| + |a_n b - ab| && (2.18) \\ &\leq |a_n||b_n - b| + |b||a_n - a| && (2.19) \end{aligned}$$

Let $N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}$ for every $n \geq N_1$. Because (a_n) is convergent, let M denote its bound such that $|a_n| < M$ $\forall n \in \mathbb{N}$. Let $N_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2M}$. Then for every $n \geq N_3 := \max\{N_1, N_2\}$, $|a_n b_n - ab| < \varepsilon$. ■

Proof (iv). **Claim i:** when n is sufficiently larger, $|b_n| > 0$ is bounded away from zero by M. Let $\varepsilon = \frac{|b|}{10}$, then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|b_n - b| < \frac{|b|}{10}$. Note that for every such n,

$$\begin{aligned} |b_n| &= |b_n - b - (-b)| && (2.20) \\ &\geq ||b_n - b| - |b|| && (2.21) \\ &\geq |b| - |b_n - b| && (2.22) \\ &> \frac{9|b|}{10} && (2.23) \end{aligned}$$

Claim ii: $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$. Let $\varepsilon > 0$, note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b}{b_n b} - \frac{b_n}{b_n b}\right| \tag{2.24}$$

$$= \frac{1}{|b_n||b|}|b_n - b| \tag{2.25}$$

from the first claim, $\frac{1}{|b_n|} < \frac{10}{9|b|}$ for every $n \geq N_1$. Since $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$, $|b_n - b| < \frac{10\varepsilon}{9|b|^2}$. Consequently, for every $n \geq N_3 := \max\{N_1, N_2\}$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Then the result is immediate from property (iii) in the algebraic limit theorem. ■

Theorem 2.5 (Order Limit Theorem)**.** Let $(a_n) \to a$ and $(b_n) \to b$, then

(i) $a_n \geq 0 \ \forall n \in \mathbb{N} \implies a \geq 0$;

(ii) $a_n \leq b_n \ \forall n \in \mathbb{N} \implies a \leq b$;

(iii) $\exists c \in \mathbb{R} \ s.t. \ c \leq b_n \ \forall n \in \mathbb{N} \implies c \leq b$;

(iv) $\exists c \in \mathbb{R} \ s.t. \ a_n \leq c \ \forall n \in \mathbb{N} \implies a \leq c$.

Proof. (i) Assume, for contradiction, $a < 0$. Take $\varepsilon = \frac{|a|}{2}$, then for some $N \in \mathbb{N}$, for every $n \geq N$ $a_n \in V_\varepsilon(a)$. However, this contradicts the fact that $a_n \geq 0$.
(ii) Consider sequence $(b_n - a_n)$ in which $b_n - a_n \geq 0$ for every $n \in \mathbb{N}$. $(b_n - a_n) \to (b - a)$ by the algebraic limit theorem. By property (i), $b - a \geq 0$.
(iii) and (iv) Consider constant sequence defined as (c_n) such that $c_n = c$ for every $n \in \mathbb{N}$, the results are immediate by applying (ii). ■

Theorem 2.6 (Squeeze Theorem)**.** Let $(x_n) \to L$ and $(z_n) \to \ell$. If for every $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$, then $(y_n) \to \ell$.

Proof. Let $\varepsilon > 0$, because both $(x_n) \to \ell$ and $(y_n) \to \ell$,

$$\exists N_1 \ s.t. \ n \geq N_1 \implies |x_n - \ell| < \varepsilon \implies x_n > \ell - \varepsilon \tag{2.26}$$

$$\exists N_2 \ s.t. \ n \geq N_2 \implies |z_n - \ell| < \varepsilon \implies z_n < \ell + \varepsilon \tag{2.27}$$

Take $N_3 := \max\{N_1, N_2\}$, then for every $n \geq N_3$,

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \tag{2.28}$$

$$\implies y_n \in V_\varepsilon(\ell) \tag{2.29}$$

therefore $(y_n) \to \ell$ by definition. ■

2.1 Monotone Convergence Theorem

Definition 2.4. A sequence (a_n) is said to be **monotone** if it is either increasing ($a_{n+1} \geq a_n \ \forall n \in \mathbb{N}$) or decreasing ($a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$).

Theorem 2.7 (Monotone Convergence Theorem)**.** If a sequence (a_n) is bounded, then it converges.

Proof. WLOG, assume (a_n) is increasing, let $\Gamma := \{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$, because Γ is bounded, $s := \sup_n \Gamma$ is well-defined by the completeness of real numbers.
Claim: $(a_n) \to s$. Let $\varepsilon > 0$, by the definition of supremum, $\exists N \in \mathbb{N}$ such that $a_N > s - \varepsilon$. Because the sequence is increasing and $s + \varepsilon \in \Gamma^{\uparrow}$, $n \geq N \implies s - \varepsilon < a_n < s + \varepsilon$. $(a_n) \to s$ by definition. ■

2.2 Series

Definition 2.5. Let (a_i) be a sequence, then the n-th **partial sum** is defined as $s_n := \sum_{i=1}^n a_i$. And the **infinite sum/series** of (a_n) is defined as

$$\sum_{i=1}^{\infty} a_i = \begin{cases} s & \text{if } (s_n) \to s \\ \text{undefined/diverges} & \text{otherwise} \end{cases} \tag{2.30}$$

Example 2.2. $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

Proof. Obviously the corresponding partial sums are increasing because the sequence $(\frac{1}{i^2})$ is positive.
Claim: (s_n) is bounded from above. Let $n \in \mathbb{N}$, observe

$$\sum_{i=1}^{n} \frac{1}{i^2} = 1 + \frac{1}{2 \times 2} + \frac{1}{3 \times 3} + \cdots + \frac{1}{n \times n} \tag{2.31}$$

$$\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(n-1) \times n} \tag{2.32}$$

$$= 2 - \frac{1}{n} \leq 2 \tag{2.33}$$

The result is immediate by the monotone convergence theorem. ■

Example 2.3 (Harmonic Series)**.** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. **Claim:** there exists a subsequence of (s_n) diverges, so (s_n) cannot be convergent. Consider the subsequence (s_k) constructed by defining $s_k := s_{2^k}$. Note that

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \tag{2.34}$$

$$> 1 + \frac{1}{2}k \tag{2.35}$$

Clearly, the subsequence is unbounded, and therefore cannot be convergent. ■

Definition 2.6. Let (a_n) be a sequence, then for every strictly increasing sequence $(n_i)_i$ in $\mathbb{N}$, (a_{n_i}) is a **subsequence** of (a_n).

Theorem 2.8. All subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let $(a_n) \to \ell$, let (a_{n_k}) be a subsequence of (a_n). Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\varepsilon(\ell)$. By the definition of subsequences, there exists some $K \in \mathbb{N}$ such that $n_K = N$. Take such K, then for every $k \geq K$, it must be $n_k \geq N$. Therefore $a_{n_k} \in V_\varepsilon(\ell)$ for every $k \geq K$, and $(a_{n_k}) \to \ell$ by definition. ■

Corollary 2.2. A sequence (a_n) must be divergent if there exists two subsequences of it converge to two different limits.

Proof. Immediate by taking the contrapositive form of above theorem. ■

Theorem 2.9 (Bolzano–Weierstrass)**.** Every bounded sequence contains a convergent subsequence.

Proof. Suppose (a_n) is bounded by certain $M > 0$, that's, for every $n \in \mathbb{N}$, $-M < a_n < M$. Consider the split $I_1^\ell := [-M, 0]$ and $I_1^u := [0, M]$. At least one of above closed intervals contain an infinitely many elements of (a_n). Define the interval as I_2. At each I_n, one can split it evenly into two closed intervals such that at least one of these sub-intervals contain infinitely many element in the sequence, and I_{n+1} is defined to be such sequence. Note that the sequence of closed intervals constructed from above recursive procedure is in fact nested. Obviously $\lim_{n\to\infty} |I_n| = 0$. Further, by the nested interval property, one can show that $\cap_{n\in\mathbb{N}} I_n \neq \varnothing$. Then $\cap_{n\in\mathbb{N}} I_n$ must be a singleton with a in it. One can construct such that $a_{n_k} \in I_k$. Note that $|I_n| = \frac{1}{2^{n-1}}$, therefore, for every $\varepsilon > 0$, one can take $N \geq \log_2\left(\frac{1}{\varepsilon}\right) + 1$, so that for every $k \geq N$, by definition of subsequences, $n_k \geq n$, so that $a_{n_k}, a \in I_N$. This implies $a_{n_k} \in V_\varepsilon(a)$ and $(a_{n_k}) \to a$. ■

2.3 Cauchy Criterion

Definition 2.7. A sequence (a_n) is a **Cauchy** sequence if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; m, n \geq N \implies |a_n - a_m| < \varepsilon \tag{2.36}$$

Proposition 2.1. Every convergent sequence is Cauchy.

Proof. Let $(a_n) \to \ell$, let $\varepsilon > 0$. By the convergence of sequence, $\exists N \in \mathbb{N}$ such that for every $n \geq N$, $|a_n - \ell| < \frac{\varepsilon}{2}$, which turns out to imply $a_n, a_m \in V_\varepsilon(\ell)$. ■

Lemma 2.1. Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence, take $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for every $m, n \geq N$, $|a_n - a_m| < 1$. In particular, take $m = N$, for every $n \geq N$, $|a_n - a_N| < 1$, and $|a_n| \leq |a_N| + 1$. Then (a_n) is clearly bounded by:

$$M := \max\{|a_n| : n \leq N\} \cup \{|a_N| + 1\} \tag{2.37}$$

■

Theorem 2.10 (Cauchy Criterion)**.** A sequence *of real numbers* is convergent if and only if it's Cauchy.

Proof. ($\Longleftarrow$) Suppose (a_n) is Cauchy, by the lemma established above, (a_n) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence $(a_{n_k}) \to \ell$.
Claim: $(a_n) \to \ell$. Let $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $n_k, n \geq N_1$, $|a_{n_k} - a_n| < \frac{\varepsilon}{2}$. And there exists another $N_2 \in \mathbb{N}$ such that for every $n_k \geq N_2$, $|a_{n_k} - \ell| < \frac{\varepsilon}{2}$. Take $N_3 := \max\{N_1, N_2\}$. Note that for every $n \geq N_3$, one can choose some $n_k \geq n$ and derive

$$\begin{aligned} |a_n - \ell| &= |a_n - a_{n_k} + a_{n_k} - \ell| && (2.38) \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - \ell| && (2.39) \\ &< \varepsilon && (2.40) \end{aligned}$$

($\Longrightarrow$) Already shown in previous proposition. ■

2.4 Convergence Test for Series

Theorem 2.11 (n-th term test; necessary condition for convergent series)**.** Series $\sum_{i=1}^{\infty} a_n$ converges $\Longrightarrow \lim_{n\to\infty} a_n = 0$.

Proof. Suppose the partial sums converges to ℓ, by the definition of partial sums, $a_n = s_{n+1} - s_n$. Further, the convergence of partial sums guarantees the convergence of (a_n). By taking limit on both sides of above identity, it can be shown $\lim_{n\to\infty} a_n = 0$. ■