ECO326 Advanced Microeconomic Theory

A Course in Game Theory

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`Github Page` `https://github.com/TianyuDu/Spikey_UofT_Notes`
`Note Page` `TianyuDu.com/notes`

Readme this note is based on the course content of *ECO326 Advanced Microeconomics - Game Theory*, this note contains all materials covered during lectures and mentioned in the course syllabus. However, notations, statements of theorems and proofs are following the book *A Course in Game Theory* by Osborne and Rubinstein, so they might be, to some extent, more mathematical than the required text for ECO326, *An Introduction to Game Theory.*

Contents

1 Lecture 1. Games and Dominant Strategies

Assumption 1.1 (pg.4)**.** Assume that each decision-maker is *rational* in the sense that he is aware of his alternatives, forms expectation about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization.

Definition 1.1 (pg.4)**.** A model of **rational choice** consists

- A set A of *actions.*
- A set C of *consequences.*
- A *consequence function* $g : A \to C$.
- A *preference relation* $\succsim$ on C.

Definition 1.2 (pg.7)**.** A **preference relation** is a complete reflexive and transitive binary relation.

Definition 1.3 (11.1)**.** A **strategic game** consists of

- a finite set of **players** N.
- for each player $i \in N$, an **actions** $A_i \neq \emptyset$.
- for each player $i \in N$, a **preference relation** $\succsim_i$ defined on $A \equiv \times_{i \in N} A_i$.

and can be written as a triple $\langle N, (A_i), (\succsim_i) \rangle$.

Definition 1.4 (pg.11)**.** A strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is **finite** if

$$|A_i| < \aleph_0 \ \forall i \in N$$

2 Lecture 2. Iterated Elimination and Rationalizability

2.1 Iterated Elimination of Strictly Dominated Strategies(Actions)

Definition 2.1 (60.2)**.** The set $X \subseteq A$ of outcomes of a finite strategic game $\langle N, (A_i), (u_i) \rangle$ **survives iterated elimination of strictly dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $\overline{((X_j^t)_{j \in N})_{t=0}^T}$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \ldots, T-1$.
- For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\langle N, (X_i^t), (u_i^t) \rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.
- No action in X_t^T is strictly dominated in game $\langle N, (X_i^T), (u_i^T) \rangle$.

Proposition 2.1 (61.2)**.** If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\langle N, (A_i), (u_i) \rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

2.2 Rationalizability

Definition 2.2 (pg. 54)**.** A **belief** of player i (about the actions of the other players) is a probability measure, μ_i, on $A_{-i} = \times_{j \in N \backslash \{i\}} A_j$.

Definition 2.3 (59.1)**.** An action of player i in a strategic game is a **never best response** if it is not a best response to any belief of player i.

Definition 2.4 (59.2)**.** The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **strictly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

3 Lecture 3. Nash Equilibrium

Definition 3.1 (14.1)**.** A **Nash equilibrium of a strategic game** $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $a^* \in A$ of actions with property that for every player $i \in N$

$$(a_i^*, a_{-i}^*) \succsim_i (a_i, a_{-i}^*) \ \forall a_i \in A_i$$

Definition 3.2 (pg.15)**.** The **best-response function** for a player i is defined as

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a_i', a_{-i}) \ \forall a_i' \in A_i\}$$

Remark 3.1. The best-response of a_{-i} can be written as

$$B_i(a_{-i}) = \bigcap_{a'_i \in A_i} \{a_i \in A_i : (a_i, a_{-i}) \succsim_i (a'_i, a_{-i})\}$$

where each of them is the upper contour set of a'_i.
Thus, if $\succsim_i$ is quasi-concave, then $B_i(a_{-i})$ is an intersection of convex sets and therefore itself convex.

Remark 3.2 (pg.15)**.** So a Nash equilibrium is a profile $a^* \in A$ such that

$$a^*_i \in B_i(a^*_{-i}) \ \forall i \in N$$

Lemma 3.1 (pg.19)**.** A strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if equivalent to the following statement:
Define set-valued function $B : A \to A$ by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and there exists $a^* \in A$ such that $a^* \in B(a^*)$.

Lemma 3.2 (20.1 Kakutani's fixed point theorem)**.** Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which

- for all $x \in X$ the set $f(x)$ is non-empty and convex.
- the graph of f is closed. *(i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$ then $y \in f(x)$)*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Definition 3.3 (pg.20)**.** A preference relation $\succsim_i$ over A is quasi-concave on A_i if for every $a^* \in A$ the upper contour set over a^*_i, given other players' strategies

$$\{a_i \in A_i : (a^*_{-i}, a_i) \succsim_i a^*\}$$

is convex.

Proposition 3.1 (20.3)**.** The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a Nash equilibrium if for all $i \in N$,

- the set A_i of actions of player i is a nonempty compact convex subset of a Euclidian space

and the preference relation $\succsim_i$ is

- continuous
- quasi-concave on A_i.

Proof. Let $B : A \to A$ be a correspondence defined as

$$B(a) := \times_{i \in N} B_i(a_{-i})$$

Note that for each $a \in A$ and for each $i \in N$,
$B_i(a_{-i}) \neq \emptyset$ since preference $\succsim_i$ is continuous and A_i is compact (EVT).
Also $B_i(a_{-i})$ is convex since it's basically an intersection of upper contour sets and each of those upper contour is convex since $\succsim_i$ is quasi-concave.
So the Cartesian product of the finite collection of B_i is non-empty and convex.
Also the graph B is closed since $\succsim_i$ is continuous.
So there exists $a^* \in A$ such that $a^* \in B(a^*)$.
So Nash equilibrium presents. ■

4 Lecture 6. Extensive Form Games and Subgame Perfection

4.1 Extensive Form Game

Definition 4.1 (89.1)**.** An **extensive game with perfect information** has the following components.

- A set N of **players**.
- A set H of sequences (finite or infinite) of **histories** with properties:
  - $\emptyset \in H$.
  - For all $L < K$, $(a^k)_{k=1,2,\ldots,K} \in H \implies (a^k)_{k=1,2,\ldots,L} \in H$.
  - For infinite sequence $(a^k)_{k=1}^{\infty}$,
    $(a^k)_{k=1,2,\ldots,L} \in H,\ \forall L \in \mathbb{Z}_{++} \implies (a^k)_{k=1}^{\infty} \in H$.

  And each component of history $h \in H$ is an **action** taken by a player.
- A function $P : H \backslash Z \to N$, where for $h \in H$, $P(h) \in N$ is defined by the player who takes an action <u>after</u> the history h.
- For each player $i \in N$ a **preference relation** $\succsim_i$ <u>defined on Z</u>.

Notation 4.1 (pg.90)**.** An extensive game with perfect information can be represented by a 4-tuple, $\langle N, H, P, (\succsim_i)\rangle$. *Sometimes it is convenient to specify the structure of an extensive game without specifying the players' preference, as* $\langle N, H, P\rangle$.

Definition 4.2 (pg.90)**.** A history $(a^k)_{k=1,2,\ldots,K} \in H$ is **terminal** if

1. it is infinite,
2. or (i.e. it cannot be extended to another valid history sequence)

$$\forall a^{K+1},\ (a^k)_{k=1,2,\ldots,K+1} \notin H$$

The set of terminal histories is denoted by Z.

Notation 4.2 (pg.90, the action set)**.** After any nonterminal history, $h \in H \backslash Z$, the player $P(h)$ chooses an action from set

$$A(h) = \{a : (h, a) \in H\}$$

Remark 4.1. Note that all player function, action set and player preference relation are defined on H. Thus, unlike a normal form game, which was *player oriented*, we'd better consider an extensive form game as *history oriented*.

Definition 4.3 (pg.90)**.** We refer to the empty set, which is required to be an element of H, as the **initial history**.

Definition 4.4 (92.1)**.** A **strategy of player** $i \in N$, s_i, in an extensive game with perfect information $\langle N, H, P, (\succsim_i)\rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Remark 4.2 (pg.92)**.** A strategy specifies the action chosen by a player for *every* history after which it is his turn to move, *even for histories that is, if the strategy is followed, are never reached.*

Definition 4.5 (pg.93)**.** For each strategy profile $s = (s_i)_{i\in N}$ in the extensive game $\langle N, H, P, (\succsim_i)\rangle$, the **outcome** of s, $O(s)$, is defined as the terminal history that results when each player $i \in N$ follows the precepts of s_i. That is, $O(s)$ is the (possibly infinite) history

$$(a^1, \ldots, a^K) \in Z$$

such that

$$\forall k \in \{0, 1, \ldots K-1\},\ s_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k) = a^{k+1}$$

Definition 4.6 (93.1)**.** A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$\forall s_i \in S_i,\ O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

Definition 4.7 (94.1)**.** The **strategic form of the extensive game with perfect information**, $\Gamma = \langle N, H, P, (\succsim_i)\rangle$, is the strategic game $\langle N, (S_i), (\succsim'_i)\rangle$ in which for each player $i \in N$

- S_i is the **set of strategies** of player i in Γ.
- $\succsim'_i$ is defined on $\times_{i\in N} S_i$ and defined by

$$\forall s, s' \in \times_{i\in N} S_i,\ s \succsim'_i s' \iff O(s) \succsim_i O(s')$$

Definition 4.8 (pg.94)**.** A **reduced strategy** of player i is defined to be a function f_i whose domain is a *subset* of $\{h \in H : P(h) = i\}$ and has the following properties

1. it associates with every history h in the domain of f_i an action in $A(h)$.
2. a history h with $P(h) = i$ is in the domain of f_i if and only if all the actions of player i in h are those dictated by f_i. (i.e., for any $h = (a^k)$ and for any $h' = (a^k)_{k=1}^{L}$ as a subsequence of h such that $P(h') = i$, $f_i(h') = a^{L+1}$.)

Remark 4.3 (pg.94)**.** Each **reduced strategy** of player i corresponds to a <u>set of strategies of player i</u>, such that for each vector of strategies of the <u>other players</u> each strategy in this set yields the <u>same outcome</u>. (strategies in the same set are **outcome-equivalent**.)
That's, for each strategy $s_i \in S_i$, its reduced strategy can be defined with an outcome equivalence class, $[s_i]$,

$$[s_i] \equiv \{s'_i \in S_i : \forall s_{-i} \in \times_{j\in N\setminus\{i\}} S_j,\ O(s_{-i}, s_i) = O(s_{-i}, s'_i)\}$$

But in some other game, the definition of outcome-equivalence is more general and defined by <u>generating the same payoff</u> (through possibly difference outcomes), then the reduced strategy is defined as

$$[s_i] \equiv \{s'_i \in S_i : \forall s_{-i} \in \times_{j\in N\setminus\{i\}} S_j,\ \forall j \in N,\ O(s_{-i}, s_i) \sim_j O(s_{-i}, s'_i)\}$$

Definition 4.9 (95.1.1)**.** Let $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim'_i)\rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i, s'_i \in S_i$ to be **equivalent** if

$$\forall s_{-i} \in S_{-i},\ \forall j \in N,\ (s_{-i}, s_i) \sim'_j (s_{-i}, s'_i)$$

Definition 4.10 (95.1.2)**.** The **reduced strategic form of** Γ is the strategic game $\langle N, (S'_i), (\succsim''_i)\rangle$ in which for each $i \in N$ each set S'_i contains one member of each set of equivalent strategies in S_i and $\succsim''_i$ is the preference ordering over $\times_{j\in N} S'_j$ induced by $\succsim'_i$.

4.2 Subgame Perfection

Definition 4.11 (97.1)**.** The **subgame of extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i |_h)\rangle$ where

- $H|_h$ is the set of sequences h' such that $(h, h') \in H$.
- $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$.
- $\succsim_i |_h$ is defined by $h' \succsim_i |_h h'' \iff (h, h') \succsim_i (h, h'') \in Z$.

Notation 4.3 (pg.97)**.** Given strategy $s_i \in S_i$ and $h \in H \in \Gamma$, $s_i|_h$ represents the **strategy that** s_i **induces in the subgame** $\Gamma(h)$. That's, for each $h' \in H_h$

$$s_i|_h(h') \equiv s_i(h, h')$$

Notation 4.4. Let O_h denote the **outcome function of** $\Gamma(h)$, that's, for all $h' \in H|_h$,

$$O_h(h') \equiv O(h, h')$$

Definition 4.12 (97.2)**.** A **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i |_h O_h(s^*_{-i}|_h, s_i|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.

Definition 4.13 (pg.97)**.** Equivalently, define SPNE to be a strategy profile s^* in Γ for which for any history $h \in H$ the strategy profile $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Remark 4.4 (pg. 97)**.** The notion of SPNE requires the action prescribed by each player's strategy to be optimal, given other players' strategies, after *every* history.

Proposition 4.1 (99.2)**.** Every finite extensive game with perfect information has a subgame perfect equilibrium.