

# Notes on MAT137 Video Playlist 3

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## Info.

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## 1 Video Playlist 1

## 2 Video Playlist 2

## 3 Video Playlist 3

### 3.1 Define Derivate As Slope

**Definition** Let  $a \in \mathbb{R}$ , and  $f(x)$  is defined on  $(a - \delta, a + \delta)$ , then the **derivative** of  $f(x)$  at  $a$  is,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

**Definition** If function is **differentiable** at point  $x = a$ , if and only if, there exists,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Interpretation**  $f'(a)$  is the slope of tangent line at  $x = a$ .

### 3.2 Calculate $f'(x)$ by definition

**Example**  $f(x) = 4x - x^2$ , find  $f'(1)$ :

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{4(h+1) - (h+1)^2 - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{4h + 4 - 3 - h^2 - 2h - 1}{h} = \lim_{h \rightarrow 0} \frac{-h^2 + 2h}{h} \\ &= \lim_{h \rightarrow 0} -h + 2 = 2 \end{aligned}$$

### 3.3 Rate of Change

**Definition** Define derivative as rate of change. Let  $x = f(t)$ , then  $f'(x)$  can be represented as,

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = f'(t) = \frac{dx}{dt}$$

### 3.4 The Product Rule (Formal Version)

Let  $a \in \mathbb{R}$ ,  $f$  and  $g$  are functions defined at  $(a - \delta, a + \delta)$ , let  $h(x) = f(x)g(x)$ . Then, if  $f(x), g(x)$  are differentiable at  $a$ , we have,

$$h'(a) = f'(a)g(a) + f(a)g'(a)$$

### 3.5 Differentiable $\implies$ Continuous

**Recall**  $f(x)$  is **differentiable** at  $a$ :

$$\exists \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1)$$

**Recall**  $f(x)$  is **continuous** at  $a$ :

$$\lim_{x \rightarrow a} f(x) = f(a) \quad (2)$$

**Proof.**

Since  $f(x)$  is differentiable at  $a$

$$(1) \iff \exists \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$\text{And } \lim_{x \rightarrow a} (x - a) = 0$$

$$\implies \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \lim_{x \rightarrow a} (x - a) = 0$$

$$\implies \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} (x - a) = 0$$

$$\implies \lim_{x \rightarrow a} (f(x) - f(a)) = 0$$

$$\implies \lim_{x \rightarrow a} f(x) = f(a)$$

■

### 3.6 Proof of product rule for derivative.

$(fg)' = f'g + fg'$ , see above for a formal definition.

$$\begin{aligned}
 & \text{Let } h = fg \\
 h'(a) &= \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{f(x)g(x) + f(a)g(x) - f(a)g(x) - f(a)g(a)}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{g(x)(f(x) - f(a)) + f(a)(g(x) - g(a))}{x - a} \\
 &= \lim_{x \rightarrow a} g(x) \frac{f(x) - f(a)}{x - a} + \lim_{x \rightarrow a} f(a) \frac{g(x) - g(a)}{x - a} \\
 &= g(a) \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \\
 &= g(a)f'(a) + f(a)g'(a)
 \end{aligned}$$

■

### 3.7 Partial proof of differentiation rule

**WTS**  $\frac{d}{dx}x^c = cx^{c-1}$ ,  $\forall c \in \mathbb{R}$

Here we only prove statements is true  $\forall c \in \mathbb{Z}^+$

**Proof.**

**Base: c = 1**

$$\begin{aligned}
 f(x) &= x \\
 f'(x) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\
 &= \lim_{x \rightarrow a} 1 = 1
 \end{aligned}$$

**Induction step**

$$\text{Assume } \frac{d}{dx}[x^k] = kx^{k-1}|_{x=a}$$

$$\text{For } f(x) = x^{k+1}$$

$$\begin{aligned}
 f'(x) &= \frac{d}{dx}[x * x^k] \\
 &= x^k + xkx^{k-1} \\
 &= (k+1)x^k
 \end{aligned}$$

■

### 3.8 Higher Order Derivatives: Notations

Original function:  $f(x)$

- **Lagrange** notation:  $f^{(n)}$
- **Leibnitz** notation:  $\frac{d^n f}{dx^n}$

### 3.9 Continuous But Not differentiable

**Definition** Function  $f(x)$  is **non-differentiable** at  $a$ .

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ DNE}$$

**Example 1** **Corner/Kink**  $f(x) = |x|$  at 0.

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} &= \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \\ \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} &= \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1 \\ \lim_{x \rightarrow 0^-} &\neq \lim_{x \rightarrow 0^+} \\ \implies \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} &\text{ DNE} \end{aligned}$$

**Example 2** **Vertical Tangent Line**  $g(x) = x^{\frac{1}{3}}$  at 0,

$$g'(0) = \lim_{x \rightarrow 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{\frac{2}{3}}} = \infty (\text{DNE})$$

**Caution** Difference between **vertical asymptote** and **vertical tangent line**

- Vertical asymptote:  $f(a) = \infty$  ( $f(a)$  is not defined)
- Vertical tangent line:  $f(a)$  is defined,  $f'(a)$  is undefined.

### 3.10 Chain Rule

#### Derivation

$$\begin{aligned}(g \circ f)'(a) &= \lim_{x \rightarrow a} \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \lim_{x \rightarrow a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}\end{aligned}$$

**Attention:** we could only apply the operation above if  $f(x) \neq f(a)$  during the process of  $x \rightarrow a$ .

This holds for majority of functions we operate in calculus.

$$\begin{aligned}&= \lim_{f(x) \rightarrow f(a)} \frac{g(f(x)) - g(f(a))}{x - a} f'(a) \\ &= g'(f(a)) \cdot f'(a)\end{aligned}$$

■

**Formal Theorem of Chain Rule** Let  $a \in \mathbb{R}$ , let  $f$  and  $g$  be functions. If  $f$  is differentiable at  $a$  and  $g$  is differentiable at  $f(a)$ , then,  $(g \circ f)$  is differentiable at  $a$ ,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

### 3.11 Derivatives of Trig Functions

#### Basic 6 results

1.  $\frac{d}{dx} \sin(x) = \cos(x)$
2.  $\frac{d}{dx} \cos(x) = -\sin(x)$
3.  $\frac{d}{dx} \tan(x) = \sec^2(x)$
4.  $\frac{d}{dx} \cot(x) = -\csc^2(x)$
5.  $\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$
6.  $\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$

**Proof.** Prove (i) and (ii) and use (i), (ii) and quotient rule to derive (iii), (iv), (v) and (vi).



**Proof. (i)** WTS  $f(x) = \sin(x)$ , then  $f'(x) = \cos(x)$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
 &= \lim_{h \rightarrow 0} \cos(x) \frac{\sin(h)}{h} \\
 &= \cos(x)
 \end{aligned}
 \quad \blacksquare \quad (3)$$

**Proof. (ii)** WTS  $f(x) = \cos(x)$ , then  $f'(x) = -\sin(x)$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(h)\sin(x) - \cos(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\cos(h) - 1)\cos(x) - \sin(h)\sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} -\frac{\sin(h)}{h} \sin(x) \\
 &= -\sin(x)
 \end{aligned}
 \quad \blacksquare \quad (4)$$

**Recall** Compound angle formula:

1.  $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$
2.  $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$
3.  $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
4.  $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

### 3.12 Implicit Differentiation

**Key** Use chain rule.

### 3.13 Derivative of Exponential Functions

Let  $f(x) = a^x$  ( $a > 0$ ), find  $f'(x)$ , by definition,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a^h - 1)a^x}{h} \end{aligned}$$

By property of limit,  $a^x$  is the only variable, so that  $a^x$  is a constant

$$= a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

(5)

Equivalently,  $\frac{d}{dx}a^x = L_a a^x$

**Definition**  $e$  is the only positive number, such that,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

So that,  $\frac{d}{dx}e^x = e^x$

### 3.14 Properties of logarithms

**Definition** Let  $a > 0, a \neq 1, x > 0, y \in \mathbb{R}$ ,

$$\log_a x = y \iff a^y = x$$

#### Properties

1.  $\log_a 1 = 0$
2.  $\log_a a = 1$
3.  $\log_a x = \frac{\log_b x}{\log_b a}$
4.  $\log_a xy = \log_a x + \log_a y$
5.  $\log_a \frac{x}{y} = \log_a x - \log_a y$
6.  $\log_a x^r = r \log_a x$

**Proof.** (i) let  $a > 0, a \neq 1, \text{let } x, y > 0$ , **WTS**  $\log_a xy = \log_a x + \log_a y$

$$\text{Let } p = \log_a x \iff a^p = x$$

$$\text{Let } q = \log_a y \iff a^q = y$$

$$\text{We have } a^p a^q = xy$$

$$\iff a^{p+q} = xy$$

$$\iff \log_a xy = p + q = \log_a x + \log_a y$$

■

### 3.15 The derivatives of logarithm functions

**For**  $\ln x \quad \frac{d}{dx} \ln x = \frac{1}{x}$

$$e^{\ln x} = x$$

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$

$$\frac{d}{d \ln x} e^{\ln x} \cdot \frac{d}{dx} \ln x = 1$$

$$x \frac{d \ln x}{dx} = 1$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

■

### 3.16 Derivative of other exponentials

**WTS**  $\frac{d}{dx} a^x = \ln a \cdot a^x$ ,

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$

$$= \frac{d}{dx} e^{x \ln a} \cdot \frac{d}{dx} \ln a$$

$$= e^{x \ln a} \ln a$$

$$= \ln a \cdot a^x$$

■

### 3.17 The power rule, complete proof

**WTS**  $x^c = cx^{c-1}$

$$\begin{aligned}
 x^c &= (e^{\ln x})^c = e^{c \ln x} \\
 \text{So that } \frac{d}{dx} x^c &= \frac{d}{dx} e^{c \ln x} \\
 &= \frac{de^{c \ln x}}{d \ln xc} \cdot \frac{\ln xc}{d \ln x} \cdot \frac{d \ln x}{dx} \\
 &= e^{c \ln x} \cdot c \cdot \frac{1}{x} \\
 &= c \cdot x^c \cdot \frac{1}{x} \\
 &= cx^{c-1}
 \end{aligned}$$

■

### 3.18 Logarithmic Differentiation

**Example**  $f(x) = \cos(x)^{\sin(x)}(\star)$ , find  $f'(x)$

**Step1.** Take  $\ln$  on both sides of  $(\star)$

$$\ln f(x) = \ln \cos(x)^{\sin(x)} = \sin(x) \ln \cos(x)$$

**Step2.** Take derivative.

$$\frac{f'(x)}{f(x)} = \cos(x) \ln \cos(x) - \sin^2(x) \frac{1}{\cos(x)}$$

**Step3.** Solve for  $f'(x)$

$$f'(x) = \cos(x)^{\sin(x)} (\cos(x) \ln \cos(x) - \sin^2(x) \frac{1}{\cos(x)})$$

## 4 Video Playlist 4

### 4.1 Functions

**In calculus** We assume the domain is the largest subset of  $\mathbb{R}$  that makes sense. And assume the codomain is always  $\mathbb{R}$ .

Notations	Math	Computer Science
	Domain	Domain
	Codomain	Range
	Range	Image

## 4.2 Inverse Functions

**Definition** Let  $f : A \rightarrow B$  be a function. Function  $f^{-1} : B \rightarrow A$  is the **inverse function** if and only if

$$\forall x \in A, \forall y \in B, x = f^{-1}(y) \iff y = f(x)$$

### Properties

- $\forall x \in A, f^{-1}(f(x)) = x$
- $\forall y \in B, f(f^{-1}(y)) = y$

**Pre-condition** Function  $f$  has inverse function  $f^{-1}$  if and only if  $f$  is **injective/one-to-one** function.

## 4.3 Surjective Functions

**Why function don't have an inverse: Part 1.**

**Definition** Function  $f(x)$  is **surjective/onto** if  $\text{codomain}(f(x)) = \text{range}(f(x))$ .

**Problem** If  $f(x)$  is not surjective, then some points in codomain has no corresponding point in domain, then  $f^{-1}$  is not a function.

**Solution** Shrink the codomain to range.

**Example** Let  $f(x) = e^x$ ,  $g(x) = \ln x$ , then we have,

- $\text{Domain}(f(x)) = \mathbb{R}$   
 $\text{Codomain}(f(x)) = \mathbb{R}$   
 $\text{Range}(f(x)) = (0, \infty)$
- $\text{Domain}(g(x)) = (0, \infty)$   
 $\text{Codomain}(g(x)) = \mathbb{R}$   
 $\text{Range}(g(x)) = \mathbb{R}$

**Definition** Definition of inverse in calculus (*simplified, we don't consider codomain here.*)

Let  $f(x)$  be a function, and  $f^{-1}(x)$  be the **inverse** of it. Then,

- $\text{Domain}(f^{-1}(x)) = \text{Range}(f(x))$
- $\text{Range}(f^{-1}(x)) = \text{Domain}(f(x))$

also,

$$\forall x \in \text{Domain}(f(x)), \forall y \in \text{Range}(f(x)), x = f^{-1}(y) \iff y = f(x)$$

and,

$$\forall x \in \text{Domain}(f(x)), f^{-1}(f(x)) = x$$

$$\forall y \in \text{Range}(f(x)), f(f^{-1}(y)) = y$$

#### 4.4 Injective function

**Definition** Let  $f(x)$  be a function, with  $\text{Domain}(f(x)) = A$ , we say  $f(x)$  is **injective/one-to-one** when,

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

equivalently (contrapositive)

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

**Theorem** Function  $f$  has an inverse if and only if  $f$  is **injective**.

**Example**  $f(x) = x^2$  has no inverse, but we could take it's inverse by shrinking the domain.

- Take domain =  $[0, \infty)$ ,  $f^{-1}(x) = \sqrt{x}$
- Take domain =  $(-\infty, 0]$ ,  $f^{-1}(x) = -\sqrt{x}$

#### 4.5 Some theorems

Let  $f(x)$  be a function with domain  $I$ .

**Theorem 1** Function  $f$  has an inverse function  $f^{-1}$  if and only if  $f$  is injective.

**Theorem 2** For function  $f$ , if

1.  $f$  is **continuous** (*This means,  $f$  is continuous on its domain.*).
2.  $I$  is an **interval**.

then,  $f^{-1}(x)$  is continuous.

**Theorem 3** If

1.  $f$  is **differentiable**.
2.  $\forall x \in I, f'(x) \neq 0$  (*This ensures the inverse function does not have a vertical tangent line, which causes non-differentiability.*).

then,  $f^{-1}(x)$  is differentiable.

**Theorem 4**  $\forall x \in I$  with  $y = f(x)$ , we have

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

**Proof.**

$$\begin{aligned} f(f^{-1}(y)) &= y \\ \frac{d}{dy} f(f^{-1}(y)) &= \frac{d}{dy} y \\ \frac{d}{dy} f(f^{-1}(y)) &= 1 \\ f'(f^{-1}(y)) \cdot (f^{-1})'(y) &= 1 \\ f'(x) \cdot (f^{-1})'(y) &= 1 \\ (f^{-1})'(y) &= \frac{1}{f'(x)} \end{aligned}$$

■

## 4.6 ArcSin

**Note**  $\text{ArcSin}$  is **NOT** the inverse of  $\text{Sin}$ .  $y = \sin(x)$  has *domain*  $= \mathbb{R}$  and *range*  $= [-1, 1]$ , so that, it is **not injective**.

**Definition**  $\text{ArcSin}$  is the inverse function to the **restriction** of  $\sin$  to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . So that,  $\text{Domain}(\text{ArcSin}) = \text{Range}(\text{Sin}) = [-1, 1]$ , and,  $\text{Range}(\text{ArcSin}) = \text{Domain}(\text{Sin}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

**Meaning**  $\text{ArcSin}(\frac{1}{2}) = t$  means:

$$\begin{cases} \sin(t) = \frac{1}{2} \\ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \end{cases}$$

**Composite**

$$\begin{aligned} \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \text{ArcSin}(\text{Sin}(x)) &= x \\ \forall y \in [-1, 1], \text{Sin}(\text{ArcSin}(y)) &= y \end{aligned}$$

## 4.7 Derivative of ArcSin

**Result**

$$\frac{d\text{ArcSin}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

**Derive.**

$$\begin{aligned}
 & \forall x \in [-1, 1] \\
 & \sin(\arcsin(x)) = x \\
 & \frac{d}{dx} \sin(\arcsin(x)) = \frac{d}{dx} x \\
 & \cos(\arcsin(x)) \cdot \frac{d}{dx} \arcsin(x) = 1 \\
 & \frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} \\
 & \text{Let } \theta = \arcsin(x) \\
 & \cos^2(\theta) = 1 - \sin^2(\theta) \\
 & \cos(\theta) = \pm \sqrt{1 - x^2} \\
 & \text{Since } \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \sin(\theta) \geq 0 \\
 & \implies \cos(\theta) = +\sqrt{1 - x^2} \\
 & \implies \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

■

## 4.8 Other inverse trig functions

**4.8.1**  $y = \cos(x)$

**Definition**  $\arccos$  is the inverse function to the restriction of  $\cos(x)$  to  $[0, \pi]$ , and,

$$\forall x \in [-1, 1], \forall y \in [0, \pi], x = \arccos(y) \iff \cos(y) = x$$

**Result**

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

**4.8.2**  $y = \tan(x)$

**Definition**  $\arctan(x)$  is the inverse function to the restriction of  $\tan(x)$  to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , and,

$$\forall y \in \mathbb{R}, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], x = \arctan(y) \iff \tan(x) = y$$

## 5 Video Playlist 5

### 5.1 Usage of MVT

**Theorem** Let  $I$  be an open interval. Let  $f$  be a function defined on  $I$ . If  $\forall x \in I, f'(x) = 0$  then  $f$  is a constant function.



*If we want to prove this theorem, we need mean value theorem*

## 5.2 Local Extreme Theorem

**Definition** Let  $f$  be a function with domain  $I$ , let  $c \in I$ .

- $f$  takes **maximum** at  $c$  if  $\forall x \in I, f(x) \leq f(c)$ .
- $f$  takes **local maximum** at  $c$  if  $\exists \delta > 0, \text{ s.t. } |x - c| < \delta \implies f(x) \leq f(c)$ .

**Definition** Let  $f$  be a function with domain  $I$ , let  $c \in I$ .

- $f$  takes **minimum** at  $c$  if  $\forall x \in I, f(x) \geq f(c)$ .
- $f$  takes **local minimum** at  $c$  if  $\exists \delta > 0, \text{ s.t. } |x - c| < \delta \implies f(x) \geq f(c)$ .

*End-point cannot be a local extremum since the definition of local extremum requires a open interval at both left and right sides around point  $c$ .*

**Theorem (Local EVT)** Let  $f$  be a function with domain  $I$  as an interval. Let  $c \in I$ , then if,

1.  $f(c)$  is an extremum.
2.  $c$  is an interior point.

then,  $f'(c) = 0$  or DNE.

**Definition** Point  $c \in I$  for function  $f$  is a **critical point** if  $f'(c) = 0$  or it does not exist.

**Proof. (Local EVT)** Proof is in two parts: (1)  $f$  has maximum at  $c$ , (2)  $f$  has minimum at  $c$ .

Part1:  $f(c)$  is a maximum

Take left and right side limits

$$\text{As } x \rightarrow c^+, x - c > 0$$

$$\text{As } x \rightarrow c^-, x - c < 0$$

$$\text{By definition of maximum } f(x) - f(c) \leq 0$$

Left limit

$$x - c < 0 \wedge f(x) - f(c) \leq 0$$

$$\implies \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

Right limit

$$x - c > 0 \wedge f(x) - f(c) \leq 0$$

$$\implies \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

For limit to exist

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \wedge \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$\implies \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0$$

$$\iff f'(c) = 0$$

Part2:  $f(c)$  is a minimum

Take left and right side limits

$$\text{As } x \rightarrow c^+, x - c > 0$$

$$\text{As } x \rightarrow c^-, x - c < 0$$

$$\text{By definition of minimum } f(x) - f(c) \geq 0$$

Left limit

$$x - c < 0 \wedge f(x) - f(c) \geq 0$$

$$\implies \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

Right limit

$$x - c > 0 \wedge f(x) - f(c) \geq 0$$

$$\implies \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

For limit to exist

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0 \wedge \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

$$\implies \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0$$

$$\iff f'(c) = 0$$

### 5.3 Find Extremum

**Example** find extremum of function  $f(x) = x^3 - 3x^2 - 9x + 3$  for  $I = [-4, 4]$   
**Steps**

1. Ensure existence of extremum.  $f$  is polynomial and therefore continuous, and  $[-4, 4]$  is a compact set. By EVT, extremum exist.
2. Find all *critical points* and *end-points*.
3. Compare values at candidate points.

### 5.4 Rolle's Theorem

**Theorem** let  $a < b$ , let  $f$  be a function defined on a closed interval  $[a, b]$  (Compact set). Then, if,

1.  $f(x)$  is continuous on  $[a, b]$ .
2.  $(\wedge) f(x)$  is differentiable on  $(a, b)$ .
3.  $(\wedge) f(a) = f(b)$ .

then,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

**Proof.**

By EVT,  $f(x)$  has extremum in  $[a, b]$ .

**Case1** Interior Extremum Point. ( $c \in (a, b)$ )

By Local EVT,  $f'(c) = 0 \vee f'(c) DNE$

By (ii)  $f'(c) = 0$

**Case2** End-point Extremum

Since (iii)  $f(a) = f(b)$

$\forall x \in (a, b)$

$f(x) \leq \max(f(a), f(b))$

$f(x) \geq \min(f(a), f(b))$

$\implies f(x)$  is constant.

$\implies \forall c \in (a, b), f'(c) = 0$

■

### 5.5 Application of Rolle's Theorem

**Application** How many zeros does a function have.

**Step 1** Use IVT to prove it has *at least*  $n$  zeros.

**Step 2** Use Rolle's theorem to prove it has *at most*  $n$  zeros.

**Example**

$$g(x) = x^6 + x^2 + x - 2$$

**IVT Applied**

$$g(-2) = 64$$

$$g(0) = -2$$

$$g(1) = 1$$

So that,  $g(x)$  has at least 2 zeros.

**Rolle's theorem applied** Assume  $f(x_1) = f(x_2) = 0$ , by Rolle's theorem, there must exist a  $a \in (x_1, x_2)$  such that  $f'(a) = 0$

**Conclusion 1** Between any two zeros of  $f$  there must be *at least* one zero of  $f'$ .

**Conclusion 2**  $\#$  of zeros of  $f' \geq \#$  of zeros of  $f - 1$

**Conclusion 2'**  $\#$  of zeros of  $f \leq \#$  of zeros of  $f' + 1$

$$g'(x) = 6x^5 + 2x + 1$$

$$g''(x) = 30x^4 + 2$$

$$g''(x) \text{ has no zeros}$$

## 5.6 (Lagrange)Mean Value Theorem

**Theorem** Let  $a < b$ , let  $f$  be a function defined on  $[a, b]$ , if,

1.  $f$  is continuous on  $[a, b]$ .
2.  $f$  is differentiable on  $(a, b)$ .

then,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

### 5.7 Proof. of MVT

$$\text{Let } m = \frac{f(b) - f(a)}{b - a}$$

$$\text{Let } g(x) = f(x) - f(a) - m(x - a)$$

$$\text{Satisfies } g(a) = f(a) - f(a) - m(a - a) = 0$$

$$\wedge g(b) = f(b) - f(a) - m(b - a) = 0$$

By Rolle's Theorem

$$g(a) = g(b) = 0$$

$$\exists c \in (a, b) \text{ s.t. } g'(c) = 0$$

$$\implies \frac{d}{dx}[f(x) - f(a) - m(x - a)] = 0$$

$$\implies f'(c) = \frac{f(b) - f(a)}{b - a}$$

■

### 5.8 Zero-derivative implies constant

**Theorem** Let  $a < b$ . Let  $f$  be a function defined on  $[a, b]$ , then,

$$\forall x \in (a, b), f'(x) = 0 \wedge f \text{ is continuous on } [a, b] \implies f \text{ is constant on } [a, b].$$

**proof.**

$$\text{Let } x_1, x_2 \in [a, b] \wedge x_1 < x_2$$

$$\text{By MVT, } \exists c \in (x_1, x_2), \text{ s.t.}$$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\because f'(c) = 0$$

$$\therefore f(x_1) = f(x_2)$$

### 5.9 Monotonicity of functions

**Definition** Let  $f$  be a function defined on an interval  $I$ .

- $f$  is **increasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2)$$

- $f$  is **non-decreasing on I** when

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

**Theorem** Let  $a < b$ . Let  $f$  be a function defined on  $(a, b)$ . Then,

$$\forall x \in (a, b), f'(x) > 0 \implies f \text{ is increasing on } (a, b)$$

**Theorem** Let  $a < b$ . Let  $f$  be a function defined on  $[a, b]$ . Then,

$$\forall x \in (a, b), f'(x) > 0 \wedge f \text{ is continuous on } [a, b] \implies f \text{ is increasing on } [a, b]$$

**Short summary** On an open interval

- $f' = 0 \implies f$  constant.
- $f' > 0 \implies f$  increasing.
- $f' < 0 \implies f$  decreasing.

## 6 Video Playlist 6

**Note** This chapter focus on *optimization applications*, and there's no video for this topic.

## 7 Video Playlist 7

### 7.1 Integral

**Integral** Let  $a < b$ , let  $f$  be a positive function, then *integral of  $f$  from  $a$  to  $b$*  is denoted as:

$$\int_a^b f(x) dx$$

this is represented as the area of region under function  $f$  from  $x = a$  to  $x = b$ .

### 7.2 Sigma Notation

**Sigma Notation** The sigma notation, with **index**  $i$ , could be represented in the following form:

$$\sum_{i=1}^N a_i = a_1 + a_2 + \cdots + a_N$$

### 7.3 Supremum and Infimum

**Definitions** Let  $A \subseteq \mathbb{R}$ , let  $a \in \mathbb{R}$ :

- **Upper bound:**  $a$  is a upper bound of  $A$  means  $\forall x \in A, x \leq a$ .
- **Least upper bound(l.u.b) / Supremum:**  $a$  is the least upper bound or supremum(sup) of  $A$  iff  $a$  is an upper bound of  $A$  and  $\forall b \in \{\text{upper bound of } A\}, a \leq b$ .

- **Maximum:** if supremum of  $A \in A$ , it's maximum of  $A$ .
- **Bounded above:**  $A$  is bounded above if  $A$  has (at least) one upper bound.

**Definitions (counter-part)** Let  $A \subseteq \mathbb{R}$ , let  $a \in \mathbb{R}$ :

- **Lower bound:**  $a$  is a lower bound of  $A$  means  $\forall x \in A, x \geq a$ .
- **Greatest lower bound(g.l.b) / Infimum:**  $a$  is the greatest lower bound (g.l.b) or infimum(inf) of  $A$  iff  $a$  is a lower bound of  $A$  and  $\forall b \in \{\text{Lower bound of } A\}, a \geq b$ .
- **Minimum:** if infimum of  $A \in A$ , it's the minimum of  $A$ .
- **Bounded below:**  $A$  is bounded below if  $A$  has (at least) one lower bound.

**Theorem: The l.u.b. principle** Let  $A \subseteq \mathbb{R}$ , if  $A$  is bounded above and  $A \neq \emptyset$ , then,  $A$  has a least upper bound(supremum).

**Theorem: The g.l.b principle** Let  $A \subseteq \mathbb{R}$ , if  $A$  is bounded below and  $A \neq \emptyset$ , then,  $A$  has a greatest lower bound(infimum).

## 7.4 Supremum and Infimum of a function

**Definition** Supremum of a function  $f$  on a domain  $I$  is defined as:

$$\sup_{x \in I} f(x) = \sup\{f(x) \mid x \in I\}$$

**Theorem** Let  $f$  be a function defined on domain  $I \neq \emptyset$ , if  $f$  is bounded above, then  $\exists \sup_{x \in I} f(x)$ . Similarly, if  $f$  is bounded below, then  $\exists \inf_{x \in I} f(x)$ .

**Theorem(EVT)** Let  $a < b$ , let  $f$  defined on  $[a, b]$ , if  $f$  is continuous on  $[a, b]$ , then  $f$  has a maximum and a minimum on  $[a, b]$ .

## 7.5 Definition of Integral (i)

**Definition** A **partition** of the interval  $[a, b]$  is a finite set  $P$ , s.t.  $\{a, b\} \subseteq P$ .

**Notation**  $P = \{x_0, x_1, \dots, x_N\}$  on  $[a, b]$ . Implicitly,  $x_i$  are ordered, such that,  $a = x_0 < x_1 < \dots < x_N = b$ .

Let  $f$  be bounded on  $[a, b]$ , let  $P = \{x_0, x_1, \dots, x_N\}$ , let  $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ , and  $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ , and  $\Delta x_i = x_i - x_{i-1}$ .

**Definition** P-Lower sum of  $f$  is defined as:

$$L_P(f) = \sum_{i=1}^N (m_i \Delta x_i)$$

**Definition** P-Upper sum of  $f$  is defined as:

$$U_P(f) = \sum_{i=1}^N (M_i \Delta x_i)$$

**Property** For all partition  $P$  on interval  $[a, b]$ , the lower sum and upper sum satisfy the following inequality,

$$L_P(f) \leq \int_a^b f(x) dx \leq U_P(f)$$

## 7.6 Definition of Integral (ii): Properties of $U_P(f)$ and $L_P(f)$

Let  $f$  be a bounded function on  $[a, b]$ , let  $P$  and  $Q$  be partitions of  $[a, b]$ , the lower sums and upper sums have the following properties.

1. (Always)  $L_P(f) \leq U_P(f)$ .
2. If  $P \subseteq Q$  ( $Q$  is a finer partition), then  $L_P(f) \leq L_Q(f) \wedge U_P(f) \geq U_Q(f)$ .
3. (Always)  $L_P(f) \leq U_Q(f)$

*Proof*

Let  $R = P \cup Q$ ,

so that,  $P \subseteq R \wedge Q \subseteq R$ . ( $R$  is finer than both  $P$  and  $Q$ )

$$\begin{aligned} L_P(f) &\leq L_R(f) \leq U_R(f) \leq U_Q(f) \\ \implies L_P(f) &\leq U_Q(f) \end{aligned}$$

■

## 7.7 Definition of Integral (iii): Upper Integral and Lower Integral

**Definition** Let  $f$  be a bounded function on  $[a, b]$ , then, lower integral of  $f$  from  $a$  to  $b$  is defined as,

$$\underline{I}_a^b(f) = \sup\{\text{lower sums of } f\}$$

and the upper integral of  $f$  from  $a$  to  $b$  is defined as,

$$\overline{I}_a^b(f) = \inf\{\text{upper sums of } f\}$$

Then if  $\underline{I}_a^b(f) < \overline{I}_a^b(f)$ , then  $f$  is **non-integrable** on  $[a, b]$ .



### 7.8 An example of integrable function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{on } [-1, 1]$$

### 7.9 An example of non-integrable function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \quad \text{on } [-1, 1]$$

### 7.10 Integrals as limits

**Definition** Let  $P = \{x_0, x_1, \dots, x_N\}$  be a partition of  $[a, b]$ , the **norm** of  $P$  is defined as:

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_N\}$$

**Theorem - Lower Integrals** For lower integrals, we have,

$$\underline{I}_a^b(f) = \lim_{\|P\| \rightarrow 0} L_P(f) = \sup\{\text{lower sums of } f\}$$

alternatively, using  $\delta - \epsilon$  expression,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall P \text{ over } [a, b], \|P\| < \delta \implies |L_P(f) - \underline{I}_a^b(f)| < \epsilon$$

**theorem - Upper Integrals** For upper integrals, we have,

$$\overline{I}_a^b(f) = \lim_{\|P\| \rightarrow 0} U_P(f)$$

### 7.11 Riemann Sums

**Definition** Fix a partition  $P$  on  $[a, b]$ ,  $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ ,  $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ , pick  $x_i^* \in [x_{i-1}, x_i]$ , so that,

$$\begin{aligned} m_i &\leq f(x_i^*) \leq M_i \\ \implies m_i \Delta x_i &\leq f(x_i^*) \Delta x_i \leq M_i \Delta x_i \\ \implies L_P(f) &= \sum_{i=1}^N (m_i \Delta x_i) \leq \sum_{i=1}^N (f(x_i^*) \Delta x_i) \leq \sum_{i=1}^N (M_i \Delta x_i) = U_P(f) \end{aligned}$$

where the term  $\sum_{i=1}^N (f(x_i^*) \Delta x_i)$  is called a **Riemann sum**.

**Definition** Let  $f$  be a bounded function on  $[a, b]$ , let  $P = \{x_0, x_1, \dots, x_N\}$  be a partition on  $[a, b]$ , for each  $i$ , pick **any** point  $x_i^* \in [x_{i-1}, x_i]$ . then,

$$S_P^*(f) = \sum_{i=1}^N (f(x_i^*) * \Delta x_i)$$

is a **Riemann sum** for  $f$  and  $P$ . (There are infinitely many Riemann sum).

In general, we have,

$$L_P(f) \leq S_P^*(f) \leq U_P(f)$$

and also,

$$\lim_{\|P\| \rightarrow 0} L_P(f) = \underline{I_a^b(f)}$$

$$\lim_{\|P\| \rightarrow 0} U_P(f) = \overline{I_a^b(f)}$$

and if  $f$  is **integrable**, then

$$\lim_{\|P\| \rightarrow 0} L_P(f) = \lim_{\|P\| \rightarrow 0} U_P(f) = \int_a^b f(x) dx$$

By Squeeze Theorem,

$$\lim_{\|P\| \rightarrow 0} S_P^*(f) = \int_a^b f(x) dx$$

## 7.12 Properties of the integral

### Property 1

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

### Property 2

$$\int_a^b [cf(x)] dx = c \int_a^b f(x) dx$$

**Property 3** If  $f$  is bounded on  $[a, c]$ , and  $f$  is integrable on  $[a, b]$  and integrable on  $[b, c]$ , then,

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

### Property 4: Backward Integrals

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$

**Negative function  $f$**  Integral for negative function is the negative area.

$$\int_a^b f(x) dx$$

## 8 Video Playlist 8

### 8.1 Anti-derivatives

**Notations**

- **Definite integral**  $\int_a^b f(x) dx$
- **Indefinite integral**  $\int f(x) dx$

**Definition** Let  $f$  be a function defined on an interval, *an anti-derivative* of  $f$  is any function  $F$  that

$$F' = f$$

**Note** As a consequence of MVT, if two functions have same derivative on an interval, then they differ by a constant.

### 8.2 Functions Defined as Integrals

Consider integrable function  $f$ , define function  $F$  as the definite integral from  $a$ , a fixed point in domain of  $f$ , to another point  $x$  in domain of  $f$ , that's,

$$F(x) = \int_a^x f(t) dt$$

**Methodology** Let  $I$  be an interval, let  $a \in I$  and let  $f$  be a function integrable on  $I$ , then for each  $x \in I$ , compute  $F(x) = \int_a^x f(t) dt$  as a number.

### 8.3 The Fundamental Theorem of Calculus: Part 1

*This provides connections between definite integrals and anti-derivatives*

**Theorem: FTC(part 1)**

- Let  $I$  be an interval,
- Let  $a \in I$ ,
- Let  $f$  be a function on  $I$ .

Define  $F(x)$  as

$$F(x) = \int_a^x f(t) dt$$

If  $f$  is continuous, then  $F$  is differentiable and  $F' = f$ , that's,

$$F'(x) = f(x) \quad \forall x \in I$$

### 8.4 A Proof of Part 1 of the FTC

**Proof.**

$$\begin{aligned}
 & \text{Let (fix) } x \in I \\
 & \text{WTS. } F'(x) = f(x) \\
 F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} (F(x+h) - F(x)) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} \left( \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} \int_x^{x+h} f(t) dt \right]
 \end{aligned}$$

Consider  $h > 0$  (for negative  $h$ , the proof would be similar)

$$\text{Let } M_h = \sup_{[x, x+h]} (f)$$

$$\text{Let } m_h = \inf_{[x, x+h]} (f)$$

Then we have, by definition of infimum and supremum,

$$m_h \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq M_h$$

Since  $f$  is continuous on  $[x, x+h]$ , by EVT, it has maximum and minimum on this interval.

$$\exists c_h \in [x, x+h] \text{ s.t. } M_h = f(c_h)$$

$$\exists d_h \in [x, x+h] \text{ s.t. } m_h = f(d_h)$$

$$\therefore \lim_{h \rightarrow 0} c_h = x \wedge \lim_{h \rightarrow 0} d_h = x$$

$$\therefore \lim_{h \rightarrow 0} M_h = \lim_{h \rightarrow 0, c_h \rightarrow x} f(c_h) = f(x) \text{ (since } f \text{ is continuous.)}$$

$$\text{Similarly, } \lim_{h \rightarrow 0} m_h = \lim_{h \rightarrow 0, d_h \rightarrow x} f(d_h) = f(x)$$

$$\text{By Squeeze Theorem, } \lim_{h \rightarrow 0} \left[ \frac{1}{h} \int_x^{x+h} f(t) dt \right] = f(x)$$

$$\therefore F'(x) = f(x) \forall x \in I$$

■

### 8.5 The Fundamental Theorem of Calculus: Part 2

*This provides a quick way to compute definite integrals.*

**Theorem: FTC(part 2)**

- Let  $a < b \in \mathbb{R}$ ,
- let  $f$  be continuous on  $[a, b]$ ,

then,

$$\int_a^b f(x) \, dx = G(b) - G(a)$$

where  $G$  is any anti-derivative of  $f$ .

**Notation**

$$G(b) - G(a) = G(x)|_{x=a}^{x=b} = G(x)|_a^b$$

## 8.6 A Proof of Part 2 of the FTC

**Proof.**

We know that, from the first part of FTC,  $G' = f$ ,

$$\text{WTS. } \int_a^b f(x) \, dx = G(b) - G(a)$$

$$\text{Define } F(x) = \int_a^x f(t) \, dt$$

$$\text{WTS. } F(b) = G(b) - G(a)$$

$$\text{Since } f \text{ is continuous, } F' = f$$

By the consequence of MVT,

$$F' = G' \implies \exists C \in \mathbb{R} \text{ s.t. } F - G = C \forall x \in [a, b]$$

$$\text{at } x = a, F(a) = 0 \implies C = -G(a)$$

$$\implies \forall x \in [a, b] F(x) = G(x) - G(a)$$

$$\text{at } x = b, F(b) = G(b) - G(a)$$

■

## 8.7 Summary: Definite and indefinite integrals, notation, definitions and theorems.

### 8.7.1 Definite Integral.

$$\int_a^b f(x) \, dx$$

**Theorem (Formal definite)** if  $\overline{I}_a^b(f) = \underline{I}_a^b(f)$  then  $\int_a^b f(x) \, dx = \overline{I}_a^b(f) = \underline{I}_a^b(f)$ .

**Theorem (FTC: part 2)** Choose one anti-derivative  $G(x)$  of  $f(x)$ , then compute the definite integral as  $\int_a^b f(x) dx = G(b) - G(a)$ .

### 8.7.2 Indefinite Integral

$$\int f(x) dx \text{ A collection of functions.}$$

**Find indefinite integral** Find  $G(x)$  as one anti-derivative, by the consequence of MVT, then the indefinite integral of  $f$  could be constructed as,

$$F(x) = \{G(x) + C \mid C \in \mathbb{R}\}$$

### 8.7.3 Function Defined by an Integral.

$$F(x) = \int_a^x f(t) dt \text{ This is one function with fixed value of } a.$$

**Theorem (FTC: part 1)** if  $f$  is continuous, then  $F'(x) = f(x)$

## 9 Video Playlist 9

### 9.1 Integration By Substitution: derivation of the formula

*Backwards usage of chain rule.*

If  $\int f(x) dx = F(x)$  is the anti-derivative of  $f(x)$ , then

$$F(g(x)) = \int f(g(x))g'(x) dx = F(g(x))$$

### 9.2 Example 2

### 9.3 Example 3

### 9.4 Example 4

**Theorem** Let  $a < b$ , let  $f$  be a continuous function, let  $g$  be a function with continuous derivative in  $[a, b]$ , assume the range of  $g$  on  $[a, b]$  is contained in the domain of  $f$ . Then,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

### 9.5 Integration by parts

*Backwards product rule*

Let  $f$  and  $g$  be two differentiable function, by product rule of differentiation, we have,

$$\begin{aligned} f'(x)g(x) + f(x)g'(x) &= \frac{d}{dx} f(x)g(x) \\ \implies \int f'(x)g(x) + f(x)g'(x) dx &= f(x)g(x) + C \\ \implies \int f'(x)g(x) dx + \int f(x)g'(x) dx &= f(x)g(x) + C \\ \implies \int f'(x)g(x) dx &= f(x)g(x) - \int f(x)g'(x) dx \end{aligned}$$

*The integral constant is implicitly contained in the integral term.*

## 9.6 Examples

**Example 1**

$$\int x^2 e^x dx$$

**Example 2**

$$\int e^x \sin x dx$$

Use integration by parts twice.

**Example 3**

$$\int \arctan x dx$$

Consider the form  $1 \times f(x)$  as partition method.

## 9.7 Integration of products of trigonometric functions

**Types**

$$\int \sin^n x \cos^m x dx$$

$$\int \sec^n x \tan^m x dx$$

**Keys**

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sec^2(x) = 1 + \tan^2(x)$$

**Summary I** Consider the integral in the following form

$$\int \sin^n x \cos^m x \, dx$$

- If **m is odd** then try  $u = \sin(x)$ , then  $du = \cos(x)dx$
- If **n is odd** then try  $u = \cos(x)$ , then  $du = -\sin(x)dx$

## 10 Video Playlist 10

**Note** This chapter focus on *volumes*, and there's no video for this topic.

## 11 Video Playlist 11

### 11.1 What Is a Sequence

**Definition** A **sequence** is a function with domain  $\mathbb{N}$ .

#### 11.1.1 Conventions

**Functions** function with domain interval.

- $x$  as variable.
- $f(x)$  as value at  $x$ .

**Sequence** function with domain  $\mathbb{N}$ .

- $n$  as variable.
- $a_n$  as value at  $n$ .

*A sequence is not a set.*

#### 11.1.2 Describe sequences

**Equation**  $a_n = \frac{2^n n!}{n+1}$

**First few values**  $\{1, 2, 4, 8, 16, \dots\}$

**Words**  $p_n = n$ -th prime.

**Recurrence relation** e.g. Fibonacci Sequence.

$$\{F_n\}_{n=0}^{\infty} : F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2} \, \forall n \geq 2$$



**A general definition** A sequence is a function with domain  $\{n \in \mathbb{Z} \mid n \geq n_0\}$  for some fixed  $n_0 \in \mathbb{Z}$ .

## 11.2 The Limit of a Sequence

**Example**

$$\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty} \quad \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

**Definition(Limit)** We say that the sequence  $\{a_n\}_{n=0}^{\infty}$  converges to the number  $L \in \mathbb{R}$  when

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq n_0 \implies |L - a_n| < \epsilon$$

denoted as

$$\lim_{n \rightarrow \infty} a_n = L \text{ or } a_n \rightarrow L$$

*Tail: all terms of the sequence after the first few terms.*

*Every interval centred at  $L$  contains a tail of the sequence.*

**Definition** A sequence is **convergent** if it has a limit. This sequence is **divergent** if it does not have a limit.

## 11.3 Properties of Limits of Sequences

**Properties from the limit of functions**

- Limit laws: Yes
- Squeeze theorem: Yes
- *L'Hôpital's Rule*: No

### 11.3.1 Sequence from a function

Let  $c \in \mathbb{Z}$  and function  $f$  defined on  $[c, \infty)$ , and define the sequence  $\{a_n\}_{n=c}^{\infty}$  as

$$a_n = f(n)$$

We have if  $\lim_{n \rightarrow \infty} f(n) = L$  then  $\lim_{n \rightarrow \infty} a_n = L$ . If  $\lim_{n \rightarrow \infty} f(n)$  DNE, then  $\lim_{n \rightarrow \infty} a_n$  may or may not exist.

### 11.3.2 Composite of sequence and function

**Theorem** If  $a_n \rightarrow L$  and  $f$  is continuous at  $L$  then

$$f(a_n) \rightarrow f(L)$$

## 11.4 Monotonic and Bounded Sequences

### 11.4.1 Monotonic Sequences

**Definition** We say  $\{a_n\}_{n=0}^{\infty}$  is **increasing** if

$$\forall n, m \in \mathbb{N}, n < m \implies a_n < a_m$$

Also, we say this sequence is **non-decreasing** if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, n < m \implies a_n \leq a_m$$

**Definition** We say  $\{a_n\}_{n=0}^{\infty}$  is **decreasing** if

$$\forall n, m \in \mathbb{N}, n < m \implies a_n > a_m$$

Also, if the inequality is in the weak form as

$$\forall n, m \in \mathbb{N}, n < m \implies a_n \geq a_m$$

we say this sequence is **non-increasing**.

**Definition** We say a sequence  $\{a_n\}_{n=0}^{\infty}$  is **monotonic** if it has any of the four properties above.

**Definition**  $\{a_n\}_{n=0}^{\infty}$  is **eventually decreasing** if

$$\exists n_0 \in \mathbb{N}, \text{ s.t. } \forall n \in \mathbb{N}, n \geq n_0 \implies a_n > a_{n+1}$$

### 11.4.2 Bounded Sequences

**Definition** We say a sequence  $\{a_n\}_{n=0}^{\infty}$  is **bounded below** if

$$\exists A \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, A \leq a_n$$

Similarly, the sequence is **bounded above** if

$$\exists B \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, B \geq a_n$$

**Definition** We say a sequence is **bounded** if and only if it is both bounded above and below.

**Theorem** If a sequence is convergent then it is bounded.

**Theorem 2A(The monotone convergence theorem for sequence)** If a sequence is eventually increasing and bounded above, then it is convergent

**Theorem** If a sequence is eventually increasing and not bounded above then it divergent to  $\infty$ .

**Remark** for a sequence:

$$\text{Sequence} \begin{cases} \text{Convergent} \\ \text{Divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{Oscillating} \end{cases} \end{cases}$$

### 11.5 Proof: Every convergent sequence is bounded

**Theorem** Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence, if  $\{a_n\}_{n=0}^{\infty}$  is convergent then the sequence is bounded. Equivalently,

*Proof.*

Assume sequence  $\{a_n\}_{n=0}^{\infty}$  is convergent.

Let  $L$  be the limit.

By the definition of limit, choose  $\epsilon = 10$

So that,  $\exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$

Take  $A = \min\{a_0, \dots, a_{n_0-1}, L - 10\}$

Take  $B = \max\{a_0, \dots, a_{n_0-1}, L + 10\}$

By definition of max and min, let  $n \in \mathbb{N}$

case  $1n > n_0 \implies A \leq a_n \leq B$

case  $2n \geq n_0 \implies L - 10 \leq a_n \leq L + 10$

Since  $A \leq L - 10 \wedge B \geq L + 10$

$\implies A \leq a_n \leq B \forall n \in \mathbb{N}$

$\therefore \{a_n\}_{n=0}^{\infty}$  is bounded. ■

### 11.6 The monotone convergence theorem of sequences

**(General) Theorem** If a sequence is (eventually) monotonic and bounded then it is convergent.

**(Particular Case) Theorem 1** If a sequence is increasing and bounded above the it's convergent.

*Proof.*

Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence that's increasing and bounded above.

Consider  $A = \{a_n \mid n \in \mathbb{N}\} \neq \emptyset$

By least upper bound principle, there exists a supremum of set  $A$

Take  $L = \sup\{A\}$

Let  $\epsilon > 0$

By definition of supremum,

$\exists a_{n_0} \in A$  s.t.  $a_{n_0} > L - \epsilon$

Take this value  $n_0$

Since sequence is increasing,

$\forall n \geq n_0$   $a_n > L - \epsilon$

Also, by definition of supremum,  $a_n \leq L$

$\implies a_n \leq L + \epsilon$

Therefore,  $\forall n \in \mathbb{N}, n \geq n_0 \implies L - \epsilon < a_n < L + \epsilon$

Therefore,  $\lim_{n \rightarrow \infty} \{a_n\}_{n=0}^{\infty} = L$

Therefore,  $\{a_n\}_{n=0}^{\infty}$  is convergent. ■

## 11.7 the Big theorem of sequences

**Definition** (for positive sequences only) Let  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  be positive sequences.

$$a_n \ll b_n \iff \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

say  $\{a_n\}$  is **much smaller than**  $\{b_n\}$ .

**Theorem** for every  $a > 0$  and  $c > 1$

$$\ln n \ll n^a \ll c^n \ll n! \ll n^n$$

## 12 Video Playlist 12

### 12.1 Improper Integral

#### 12.1.1 Improper integral "type 1" (Unbounded domain)

**Definition** Let  $a \in \mathbb{R}$  and  $f$  continuous on  $[a, \infty]$  the integral of  $f$  from  $a$  to  $\infty$ , denoted as

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

assuming the limit exists. If the limit exists, the integral is called **convergent**, otherwise, it's called **divergent**.

## 12.2 The most important family of improper integrals

Let  $p \in \mathbb{R}$  consider

$$\int_1^\infty \frac{1}{x^p} dx$$

**Summary**

$$\int_1^\infty \frac{1}{x^p} dx \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent to } \infty & \text{if } p \leq 1 \end{cases}$$

## 12.3 Example

$$\int_0^\infty \sin(x) dx$$

## 12.4 The most important family of improper integral

**Consider**

$$I = \int_1^\infty \frac{1}{x^p} dx$$

**Summary**

1.  $p > 1 \iff I$  converges.
2.  $p \leq 1 \iff I$  diverges to  $\infty$ .

## 12.5 Example

Vertical asymptote improper.

$$\int_0^1 \ln(x) dx$$

## 12.6 Doubly improper integrals

**General Strategy** Assume  $A$  has **multiple** improper.

1. Break  $A$  into pieces with **single** improper at their endpoints.
2. If each piece convergent **separately**, then  $A$  converges.
3. Else,  $A$  diverges, it's *not a number*.

## 12.7 Basic Comparison Test

**Theorem** Let  $a \in \mathbb{R}$ ,

Let  $f$  and  $g$  be *continuous* functions on  $[a, \infty)$ , and

$$\forall x \geq a, 0 \leq f(x) \leq g(x)$$

we have,

1.  $\int_a^\infty g(x) dx < \infty \implies \int_a^\infty f(x) dx < \infty$
2.  $\int_a^\infty f(x) dx = \infty \implies \int_a^\infty g(x) dx = \infty$

## 12.8 Examples

## 12.9 Limit Comparison Test

**Theorem** Let  $a \in \mathbb{R}$ ,  $f$  and  $g$  are *positive* and *continuous* functions on  $[a, \infty)$ .

And the following limit exists,

$$L = \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \in \mathbb{R}$$

Then,  $\int_a^\infty f(x) dx$  and  $\int_a^\infty g(x) dx$  are **both** convergent or **both** divergent.

## 12.10 Proof of LCT

Omitted

# 13 Video Playlist 13

## 13.1 Infinite Sums

Nothing.

## 13.2 Definition of series

**Definition** Series  $\sum_{n=1}^\infty a_n$  is defined as

$$\lim_{k \rightarrow \infty} S_k$$

where  $S_k = \sum_{n=1}^k a_n$  as finite sum. If the above limit exist, we say series  $\sum_{n=1}^\infty a_n$  is convergent (it's a *number*), else series is divergent and it's not a number.

## 13.3 Example

$$\sum_{n=1}^\infty \frac{1}{n^2 + n} = \sum_{n=1}^\infty \frac{1}{n} - \frac{1}{n+1} = \lim_{k \rightarrow \infty} 1 - \frac{1}{k+1} = 1$$

### 13.4 Divergent Series Examples

$$S = \sum_{n=1}^{\infty} 1 = \lim_{k \rightarrow \infty} k = \infty$$

$$S = \sum_{n=1}^{\infty} (-1)^n = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases} \quad \text{divergent due to oscillation.}$$

### 13.5 Geometric Series

Let  $x \in \mathbb{R}$

$$S = \sum_{n=0}^{\infty} x^n = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} x^n$$

Consider

$$\begin{aligned} S_k &= 1 + x + x^2 + \cdots + x^k \\ xS_k &= x + x^2 + x^3 + \cdots + x^{k+1} \\ S_k - xS_k &= 1 - x^{k+1} \end{aligned}$$

If  $x = 1$ , the series is simply divergent to  $\infty$ .

$$S_k = \frac{1 - x^{k+1}}{1 - x}, \quad x \neq 1$$

$$S = \lim_{k \rightarrow \infty} \frac{1 - x^{k+1}}{1 - x} = \frac{1 - \lim_{k \rightarrow \infty} (x^{k+1})}{1 - x} = \begin{cases} \frac{1}{1-x} & \iff x \in (-1, 1) \\ \text{Divergent} & \begin{cases} \infty & \iff x > 1 \\ \text{Oscillating} & \iff x \leq -1 \end{cases} \end{cases}$$

### 13.6 Linearity of series

Simple form of fact

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} c b_n = \sum_{n=0}^{\infty} a_n + c \sum_{n=0}^{\infty} b_n, \quad \forall c \in \mathbb{R}$$

**Theorem** If series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  are both convergent, then  $\sum_{n=0}^{\infty} a_n + b_n$  is also convergent and

$$\sum_{n=0}^{\infty} a_n + b_n = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

*Proof.*

$$\text{Let } \sum_{n=0}^{\infty} a_n = \lim_{k \rightarrow \infty} S_k$$

$$\text{Let } \sum_{n=0}^{\infty} b_n = \lim_{k \rightarrow \infty} T_k$$

$$\text{Let } \sum_{n=0}^{\infty} a_n + b_n = \lim_{k \rightarrow \infty} R_k$$

$$\text{Where } S_k = \sum_{n=0}^k a_n$$

$$T_k = \sum_{n=0}^k b_n$$

$$R_k = \sum_{n=0}^k a_n + b_n$$

$$\text{Since } R_k = S_k + T_k, \forall k \in \mathbb{N}$$

$$\text{By limit laws, } \lim_{k \rightarrow \infty} S_k + \lim_{k \rightarrow \infty} T_k = \lim_{k \rightarrow \infty} R_k$$

$$\implies \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n$$

■

**Theorem** If series  $\sum_{n=0}^{\infty} a_n$  is convergent, then for any  $c \in \mathbb{R}$ , series  $\sum_{n=0}^{\infty} c a_n$  is also convergent and

$$\sum_{n=0}^{\infty} c a_n = c \sum_{n=0}^{\infty} a_n$$

### General proof procedure

1. Write series as limit of partial sums.
2. Manipulate partial sums (finite).
3. Manipulate limits.

## 13.7 the Tail of a series

**Fact** Consider two series

$$\sum_{n=0}^{\infty} a_n \text{ convergent} \iff \sum_{n=1}^{\infty} a_n \text{ convergent}$$

And  $\sum_{n=1}^{\infty} a_n$  is a tail of series  $\sum_{n=0}^{\infty} a_n$ .



**Notation** We say  $\sum_n^\infty a_n$  is convergent or divergent without specifying the starting index of the series.

**Specific form of theorem** If  $\forall n \in \mathbb{N}$  [Condition(s)] then  $\sum_n^\infty a_n$  is convergent or divergent.

**General form of theorem** If  $\exists n_0 \in \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}, n \geq n_0 \implies$  [Condition(s)] then  $\sum_n^\infty a_n$  is convergent.

### 13.8 A necessary condition for convergence of series

**Fact** Series  $\sum_{n=0}^\infty a_n$  is convergent if and only if the sequence of its partial sums  $\{S_n\}_{n=0}^k$  is convergent.

**Theorem** If  $\sum_{n=0}^\infty a_n$  is convergent then

$$\lim_{n \rightarrow \infty} a_n = 0$$

**Note** The above theorem is often used as it's contrapositive form

$$\lim_{n \rightarrow \infty} a_n \neq 0 \implies \sum_{n=0}^\infty a_n \text{ is divergent}$$

*Proof.*

$$\text{Let } S = \sum_{n=0}^\infty a_n \text{ be convergent}$$

$$S = \lim_{k \rightarrow \infty} S_k, \quad S_k = \sum_{n=0}^k a_n$$

$$S = \lim_{k-1 \rightarrow \infty} S_{k-1}$$

$$\implies \lim_{k \rightarrow \infty} S_k - \lim_{k-1 \rightarrow \infty} S_{k-1} = 0$$

By the convergence assumption, those two limits above exist.

$$\implies \lim_{k \rightarrow \infty} S_k - S_{k-1} = 0$$

$$\implies \lim_{k \rightarrow \infty} a_k = 0$$

$$\implies \{a_n\}_{n=1}^\infty \rightarrow 0$$

■

### 13.9 Positive series

**Definition** A series  $\sum_{n=0}^\infty a_n$  is positive when  $\forall n \in \mathbb{N}, a_n > 0$ . And a series is positive means it could never diverge to  $-\infty$  or *oscillating*.

**Notation** (For positive series only)

1.  $\sum_n^\infty a_n = \infty \iff$  divergent.
2.  $\sum_n^\infty a_n < \infty \iff$  convergent.

### 13.10 The Integral Test

**Theorem** Let  $a \in \mathbb{R}$ , let  $f$  be a continuous, positive and decreasing function on  $[a, \infty)$ , then

$$\int_a^\infty f(x) dx < \infty \iff \sum_n^\infty f(n) < \infty$$

That's the improper integral and series have the same convergence/divergence feature. Note as

$$\int_a^\infty f(x) dx \sim \sum_n^\infty f(n)$$

### 13.11 Examples

**p-series** consider for what values of  $p$  the following series is convergent.

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

**Example2**

$$\sum_{n=2}^\infty \frac{1}{n \ln n}$$

### 13.12 Comparison Tests for Series

*Works exactly the same as basic and limit comparison tests for improper integrals.*

**Basic Comparison Test** Consider series  $\sum_{n=1}^\infty a_n$  and  $\sum_{n=1}^\infty b_n$ , assume<sup>1</sup>

$$\forall n \in \mathbb{N}, 0 \leq a_n \leq b_n$$

then

$$\begin{aligned} \sum_{n=1}^\infty b_n < \infty &\implies \sum_{n=1}^\infty c_n < \infty \\ \sum_{n=1}^\infty a_n = \infty &\implies \sum_{n=1}^\infty b_n = \infty \end{aligned}$$

---

<sup>1</sup>Notice that *for large natural number n* would be sufficient here also.

**Limit Comparison Test** Consider series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$ , assume existing limit

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} > 0$$

Then both series convergent or both of them divergent.