# Forecasting and Time Series Econometrics
# ECO374 Winter 2019

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# 1 Introduction and Statistics Review

**Definition 1.1.** Given random variable $X$, the $k^{th}$ **non-central moment** is defined as

$$\mathbb{E}[X^k] \tag{1.1}$$

**Definition 1.2.** Given random variable $X$, the $k^{th}$ **central moment** is defined as

$$\mathbb{E}[(X - \mathbb{E}[X])^k] \tag{1.2}$$

**Remark 1.1.** Moments of order higher than a certain $k$ may not exist for certain distribution.

**Definition 1.3.** Given the **joint density** $f(X, Y)$ of two *continuous* random variables, the **conditional density** of random $Y$ conditioned on $X$ is

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y, x)}{f_X(x)} \tag{1.3}$$

**Definition 1.4.** Given *discrete* random variables $X$ and $Y$, the **conditional density** of $Y$ conditioned on $X$ is defined as

$$P(Y = y|X = x) = \frac{P(Y = y \wedge X = x)}{P(X = x)} \tag{1.4}$$

**Assumption 1.1.** Assumptions on linear regression on time series data:

(i) **Linearity**

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + u \tag{1.5}$$

(ii) **Zero Conditional Mean**

$$\mathbb{E}[u|X_1, X_2, \ldots, X_k] = 0 \tag{1.6}$$

(iii) **Homoscedasitcity**

$$\mathbb{V}[u|X_1, X_2, \ldots, X_k] = \sigma_u^2 \tag{1.7}$$

(iv) **No Serial Correlation**

$$Cov(u_t, u_s) = 0 \ \forall t \neq s \in \mathbb{Z} \tag{1.8}$$

(v) **No Perfect Collinearity**

(vi) **Sample Variation in Regressors**

$$\mathbb{V}[X_j] > 0 \ \forall j \tag{1.9}$$

**Theorem 1.1** (Gauss-Markov Theorem)**.** Under assumptions 1.1, the OLS estimators $\hat{\beta}_j$ are *best linear unbiased estimators* of the unknown population regression coefficients $\beta_j$.

**Remark 1.2.** The *no serial correlation* assumption is typically not satisfied for time series data. And the *linearity* assumption is also too restrictive for time series featuring complex dynamics. Hence, for time series data we typically use other models than linear regression with OLS.

# 2 Statistics and Time Series

## 2.1 Stochastic Processes

**Definition 2.1.** A **stochastic process** (or **time series process**) is a family (collection) random variables indexed by $t \in \mathcal{T}$ and defined on some given probability space $(\Omega, \mathcal{F}, P)$.

$$\{Y_t\} = Y_1, \ldots, Y_T \tag{2.1}$$

**Definition 2.2.** The function from $\mathcal{T}$ to $\mathbb{R}$ which assigns to each point in time $t \in \mathcal{T}$ the realization of the random variable $Y_t$, $y_t$ is called a **realization** or a **trajectory** or an **outcome** of the stochastic process.

$$\{y_t\} = y_1, \ldots, y_T \tag{2.2}$$

Such realization is called is a **time series**.

**Definition 2.3.** A **time series model** or a **model** for the observations, $\{y_t\}$, is a specification of the *joint distribution* of $\{Y_t\}$ for which $\{y_t\}$ is a realization.

**Assumption 2.1.** The **ergodicity** assumption requires the observations cover in principle all possible events.

**Definition 2.4.** A stochastic process $\{Y_t\}$ is **first order strongly stationary** if all random variables $Y_t \in \{Y_t\}$ has the *same probability density function.*

**Definition 2.5** (1.7)**.** A stochastic process $\{Y_t\}$ is **strictly stationary** if for all $h, n \geq 1$, $(Y_1, \ldots, Y_n)$ and $(Y_{1+h}, \ldots, Y_{n+h})$ have the same distribution.

**Definition 2.6.** A stochastic process $\{Y_t\}$ is **first order weakly stationary** if

$$\forall t \in \mathcal{T},\ \mu_{Y_t} \equiv \mathbb{E}[Y_t] = \bar{\mu} \tag{2.3}$$

**Definition 2.7.** A stochastic process $\{Y_t\}$ is **second order weakly stationary**, or **covariance stationary** if all random variables $\{Y_t\}$ have the same mean and variance. And the covariances do not depend on $t$. That's, for all $t \in \mathcal{T}$,

(i) $\mathbb{E}[Y_t] = \mu\ \forall t$

(ii) $\mathbb{V}[Y_t] = \sigma^2 < \infty\ \forall t$

(iii) $Cov(Y_t, Y_s) = Cov(Y_{t+r}, Y_{s+r})\ \forall t, s, r \in \mathbb{Z}$

## 2.2 Auto-correlations

**Definition 2.8.** Let $\{Y_t\}$ be a stochastic process with $\mathbb{V}[Y_t] < \infty\ \forall t \in \mathcal{T}$, the **auto-covariance function** is defined as

$$\gamma_Y(t, s) \equiv Cov(Y_t, Y_s) \tag{2.4}$$
$$= \mathbb{E}[(Y_t - \mathbb{E}[Y_t])(Y_s - \mathbb{E}[Y_s])] \tag{2.5}$$
$$= \mathbb{E}[Y_t Y_s] - \mathbb{E}[Y_t]\mathbb{E}[Y_s] \tag{2.6}$$

**Lemma 2.1.** If $\{Y_t\}$ is stationary, then the auto-covariance function does not depend on specific time point $t$. We can write the $h \in \mathbb{Z}$ degree auto-covariance as

$$\gamma_Y(h) \equiv \gamma_X(t, t+h)\ \forall t \in \mathcal{T} \tag{2.7}$$

**Proposition 2.1.** By the symmetry of covariance,

$$\gamma_Y(h) = \gamma_Y(-h) \tag{2.8}$$

**Definition 2.9.** The **auto-correlation coefficient** of order $k$ is given by

$$\rho_{Y_t,Y_{t-k}} = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{\mathbb{V}[Y_t]}\sqrt{\mathbb{V}[Y_{t-k}]}} \tag{2.9}$$

**Definition 2.10.** Let $\{Y_t\}$ be a *covariance stationary process* and the **auto-correlation function** (ACF) is a mapping from *order* of auto-correlation coefficient to the coefficient $\rho_Y : k \to \rho_{Y_t,Y_{t-k}}$, defined as

$$\rho_Y(k) \equiv \frac{\gamma(k)}{\gamma(0)} = corr(Y_{t+k}, Y_t) \tag{2.10}$$

notice the choice of $t$ does not matter, by definition of covariance stationary process.

**Proposition 2.2.** Note that

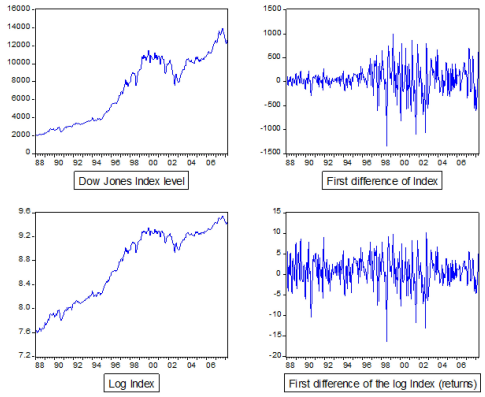
$$\rho_k = \rho_{-k} = \rho_{|k|} \tag{2.11}$$

so the ACF for stationary process can be simplified to a mapping

$$\rho : k \to \rho_{|k|} \tag{2.12}$$

**Remark 2.1.** Strong stationarity is difficult to test so we will focus on weak(covariance) stationarity only.

**Proposition 2.3.** For a non-stationary stochastic process $\{Y_t\}$, $\{\Delta Y_t\}$ becomes *first order weakly stationary (mean stationary)* and $\{\Delta \log(Y_t)\}$ becomes *second order weakly stationary (covariance stationary)*.




**Definition 2.11** (1.8)**.** A stochastic process $\{Y_t\}$ is called a **Gaussian process** if all *finite* dimensional distribution from the process are multivariate normally distributed. That's

$$\forall n \in \mathbb{Z}_{>0},\ \forall (t_1, \ldots, t_n) \in \mathcal{T}^n, (Y_{t_1}, \ldots, Y_{t_n}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \tag{2.13}$$

**Notation 2.1.** Consider the problem of forecasting $Y_{T+1}$ from observations $\{Y_t\}_{t=1}^T$, the *best linear predictor* is denoted as

$$\mathbb{P}_T Y_{T+1} = \sum_{i=1}^{T} a_i L^i Y_{T+1} \tag{2.14}$$

And $Y_{T+1}$ can be expressed as

$$Y_{T+1} = \mathbb{P}_T Y_{T+1} + \varepsilon_{T+1} \tag{2.15}$$

where $\varepsilon_{T+1}$ denotes the forecast error which is assumed to be *uncorrelated* with $Y_T, \ldots, Y_1$.

**Definition 2.12** (3.3)**.** The **partial auto-correlation function** (PACF) $\alpha(h)$ with $h \in \mathbb{Z}_{\geq 0}$ of a *stationary* process is defined as

$$\alpha(0) = 1 \tag{2.16}$$

$$\alpha(1) = corr(Y_2, Y_1) = \rho(1) \tag{2.17}$$

$$\alpha(h) = corr\Big(Y_{h+1} - \mathbb{P}(Y_{h+1}|1, Y_2, \ldots, Y_h), X_1 - \mathbb{P}(Y_1|1, Y_2, \ldots, Y_h)\Big) \tag{2.18}$$

**Remark 2.2** (Interpretation of PACF)**.** partial auto-correlation $r_k$ only measures correlation between two variables $Y_t$ and $Y_{t+k}$ while controlling $(Y_{t+1}, \ldots, Y_{t+k-1})$.

**Remark 2.3.** Properties of ACF and PACF

| processes | ACF | PACF |
|---|---|---|
| AR($p$) | Declines exponentially (monotonic or oscillating) to zero | $\alpha(h) = 0 \ \forall h > p$ |
| MA($q$) | $\rho(h) = 0 \ \forall h > q$ | Declines exponentially (monotonic or oscillating) to zero |

## 2.3 Test for Auto-correlation

To test single auto-correlation with

$$H_0 : \rho_k = 0 \tag{2.19}$$

we can use usual t-statistic.
While testing the joint hypothesis

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_k = 0 \tag{2.20}$$

we are using the **Ljung-Box Q-statistic**:

$$Q_k = T(T+1) \sum_{j=1}^{k} \frac{\hat{\rho}_j^2}{T-j} \sim \chi_k^2 \tag{2.21}$$

## 2.4 Causality and Inevitability

**Definition 2.13** (Causality)**.** An $ARMA(p, q)$ process $\{Y_t\}$ with

$$\Phi(L) Y_t = \Theta(L) \varepsilon_t \tag{2.22}$$

is called **causal with respect to** $\{\varepsilon_t\}$ if there exists a sequence $\{\psi_j\}$ with the property $\sum_j^\infty |\psi_j| < \infty$ such that

$$Y_t = \Psi(L) \varepsilon_t \ \text{ with } \psi_0 = 1 \tag{2.23}$$

where $\Psi(L) \equiv \sum_{j=0}^\infty \psi_j L^j$. The above equation is referred to as the **causal representation** of $\{Y_t\}$ with respect to $\{\varepsilon_t\}$.

**Proposition 2.4.** By the definition of causality, a pure MA process (which is stationary) is naturally a causal representation with respect to it's own error term $\{\varepsilon_t\}$.

**Theorem 2.1.** Let $\{Y_t\}$ be an $ARMA(p,q)$ process with

$$\Phi(L)Y_t = \Theta(L)\varepsilon_t \tag{2.24}$$

such that polynomials $\Phi(z)$ and $\Theta(z)$ have no common roots.
Then $\{Y_t\}$ is causal with respect to $\{\varepsilon_t\}$ if and only if all roots $\Phi(z)$ are outside the unit circle. The coefficients $\Psi$ are then uniquely defined by identity

$$\Psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\Theta(z)}{\Phi(z)} \tag{2.25}$$

**Definition 2.14** (Invertibility)**.** An $ARMA(p,q)$ process for $\{Y_t\}$ satisfying

$$\Phi(L)Y_t = \Theta(L)\varepsilon_t \tag{2.26}$$

is called **invertible** with respect to $\{\varepsilon\}$ if and only if there exists a sequence $\Pi$ with the property $\sum_{j=0}^{\infty} |\pi_j| < \infty$ such that

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_j L^j Y_t \tag{2.27}$$

**Proposition 2.5.** By the definition of invertibility, a stationary auto-regressive process is naturally invertible with respect to it's own error term $\{\varepsilon_t\}$.

**Theorem 2.2.** Let $\{Y_t\}$ be an $ARMA(p,q)$ process with

$$\Phi(L)Y_t = \Theta(L)\varepsilon_t \tag{2.28}$$

Then $\{Y_t\}$ is invertible with respect to $\{\varepsilon_t\}$ if and only if all roots of $\Theta(z)$ are outside the unit circle. And the coefficients of $\Pi$ are then uniquely determined by the relation

$$\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\Phi(z)}{\Theta(z)} \tag{2.29}$$

# 3 Forecasting Tools

## 3.1 Information Set

**Definition 3.1.** For stochastic process $\{Y_t\}$, the **information set** $I_t$ is the *known time series* up to time $t$.

**Definition 3.2.** A **forecast** $f_{t,h}$ the image of a *time series model* $g$ under given information set $I_t$. Specifically,

$$f_{t,h} = g(I_t) \in \Omega \tag{3.1}$$

## 3.2 Forecast Horizon

**Remark 3.1.** *Covariance stationary* processes are **short-memory processes**. More recent observation contains information far more relevant for the future than older information.

**Remark 3.2.** *Non-stationary* processes are **long-memory processes** and older information is as relevant for the forecast as more recent information.

### 3.2.1 Forecasting Environments: *Recursive*

(i) *Updating* with *flexible* information set.

(ii) Advantageous if model is *stable over time*.

(iii) Not robust to *structural break*.

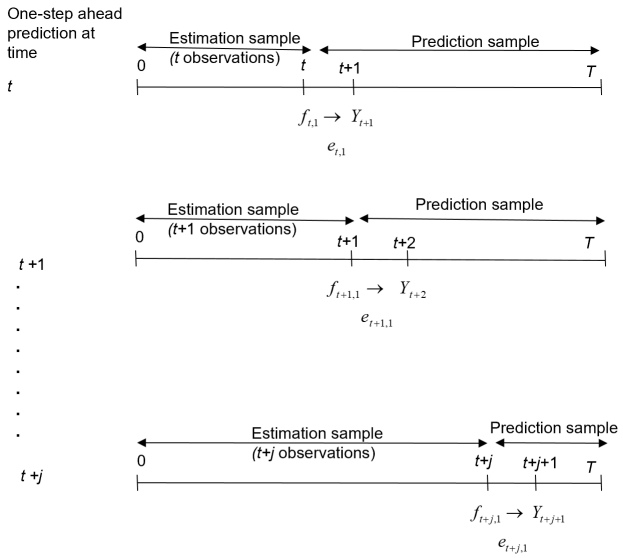



Figure 3.1: Recursive Forecasting Scheme

### 3.2.2 Forecasting Environments: *Rolling*

(i) *Updating* with *fixed-size* information set.

(ii) Robust against *structural breaks*.

(iii) Not fully exploit information available.

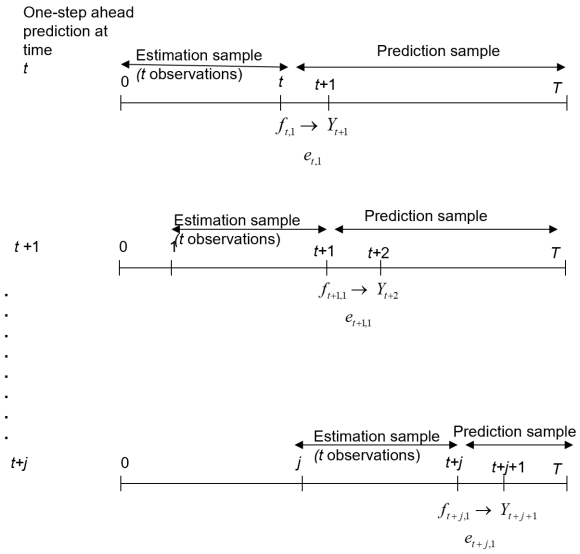



Figure 3.2: Rolling Forecasting Scheme

### 3.2.3 Forecasting Environments: *Fixed*

(i) *One estimation* and forecast with *fixed-size but updated* information set.

(ii) Computationally cheap.

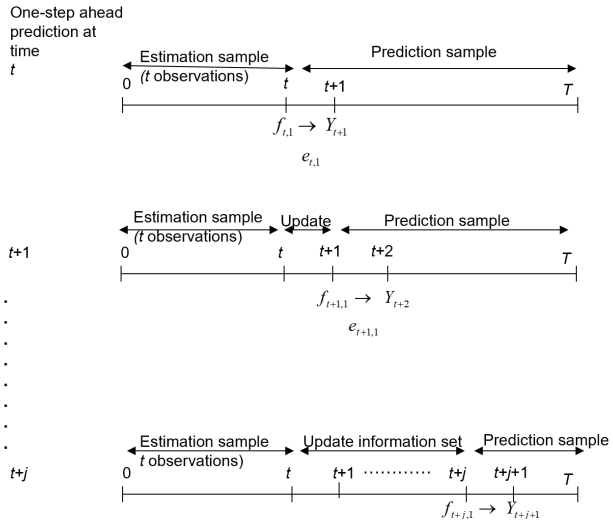



Figure 3.3: Fixed Forecasting Scheme

## 3.3 Loss Function

**Definition 3.3.** A loss function $L(e)$ is a real-valued function defined on the space of *forecast errors*, $\mathcal{E}$, and satisfies the following properties

(i) $L(e) = 0 \iff ||e|| = 0$;

(ii) $\forall e \in \mathcal{E}, L(e) \geq 0$[1];

(iii) $L$ is *monotonically increasing* in the norm of forecast error.

**Example 3.1** (Symmetric Loss Functions with $\mathcal{E} = \mathbb{R}$)**.**

$$L(e) = ae^2, \ a > 0 \tag{3.2}$$
$$L(e) = a|e|, \ a > 0 \tag{3.3}$$

**Example 3.2** (Asymmetric Loss Functions with $\mathcal{E} = \mathbb{R}$)**.**

$$L(e) = exp(ae) - ae - 1, \ a > 0 \text{ Linex Function} \tag{3.4}$$
$$L(e) = a \ |e| \ \mathbb{I}(e \geq 0) + b \ |e| \ \mathbb{I}(e < 0) \text{ Lin-lin Function} \tag{3.5}$$

### 3.4 Optimal Forecast

**Definition 3.4.** Based on information set $I_t$, the optimal forecast for future value $y_{t+h}$ is the $f^*_{t,h}$ minimize the expected loss function

$$\mathbb{E}[L|I_t] = \int L(y_{t+h} - f_{t,h}) f(y_{t+h}) \ dy_{t+h} \tag{3.6}$$

**Assumption 3.1.** Assuming the forecast $f(y_{t+h}|I_t)$ follows

$$f(y_{t+h}|I_t) \sim \mathcal{N}(\mathbb{E}[Y_{t+h}|I_t], \mathbb{V}[Y_{t+h}|I_t]) \tag{3.7}$$

**Proposition 3.1.** Given symmetric quadratic $L$, the optimal forecast $f^*_{t,h}$ is

$$\mu_{t+h|t} \equiv \mathbb{E}[Y_{t+h}|I_t] \tag{3.8}$$

*Proof.*

$$\min_{f_{t,h} \in \mathbb{R}} L \equiv \int (y_{t+h} - f_{t,h})^2 f(y_{t+h}|I_t) \ dy_{t+h} \tag{3.9}$$
$$\frac{\partial L}{\partial f_{t,h}} = -2 \int (y_{t+h} - f_{t,h}) f(y_{t+h}|I_t) \ dy_{t+h} = 0 \tag{3.10}$$
$$\implies \int (y_{t+h} - f_{t,h}) f(y_{t+h}|I_t) \ dy_{t+h} = 0 \tag{3.11}$$
$$\implies \int y_{t+h} f(y_{t+h}|I_t) \ dy_{t+h} = f_{t,h} \int f(y_{t+h}|I_t) \ dy_{t+h} \tag{3.12}$$
$$\implies f_{t,h} = \mathbb{E}[y_{t+h}|I_t] \equiv \mu_{t+h|t} \tag{3.13}$$

■

[1]Since forecasting here can be considered as an optimization process, with $L$ as the objective function. It's fine for $L$ not satisfying the non-negativity condition. However, by convention, we assume $L$ to be non-negative.

# 4 Moving Average Process

## 4.1 Wold Decomposition Theorem

**Theorem 4.1** (Wold Decomposition Theorem)**.** Every covariance stationary stochastic process $\{Y_t\}$ with mean zero and finite positive variance can be *uniquely* represented as

$$Y_t = V_t + \sum_{j=0}^{\infty} \psi_j L^j \varepsilon_t = V_t + \Psi(L)\varepsilon_t \tag{4.1}$$

where

(i) $\{V_t\}$ is a *deterministic component* (e.g. trend or cycle);

(ii) $\varepsilon_t \sim \text{WN}(0, \sigma^2)$ is the *stochastic component*;

(iii) $\psi_0 = 1$[2] and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$;

(iv) $\mathbb{E}[\varepsilon_t, V_s] = 0 \ \forall t, s \in \mathcal{T}$.

**Definition 4.1.** The stochastic component $\{\varepsilon_t\}$ in the decomposition is called **random shocks** or **innovations**.

**Lemma 4.1.** Given $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, then for all $\varepsilon > 0$ there exists a natural number $J$ such that

$$\sum_{j=J}^{\infty} \psi_j^2 < \varepsilon \tag{4.2}$$

**Corollary 4.1.** By above lemma, assuming $V_t = 0$, we can approximate the decomposition by a linear combination of finite innovations.

$$Y_t \approx \hat{Y}_t = \sum_{j=0}^{n} \psi_j L^j \varepsilon_t \tag{4.3}$$

and the approximation is *accurate in Euclidean norm*, that's,

$$\mathbb{E}[Y_t - \sum_{j=0}^{n} \psi_j L^j \varepsilon_t]^2 \to 0 \text{ as } n \to 0 \tag{4.4}$$

## 4.2 Moving Average Process

**Definition 4.2.** The **Moving Average** of order $q$ with deterministic trend, MA($q$), process is defined by the following stochastic difference equation

$$Y_t = \mu + \Theta(L)\varepsilon_t = \mu + \theta_0\varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q} \text{ where } \theta_0 = 1, \theta_q \neq 0 \tag{4.5}$$

where $\{\varepsilon_t\}$ is innovations.

**Lemma 4.2.** The infinite lag polynomial in $\Psi(L)$ can be approximated by

$$\Psi(L) \approx \frac{\Theta_q(L)}{\Phi_p(L)} \tag{4.6}$$

**Definition 4.3.** MA(1) process takes the form of

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} \tag{4.7}$$

[2]We can always normalize $\psi_0$ to 1.

**Unconditional Moments of MA(1)**

$$\mathbb{E}[Y_t] = \mathbb{E}[\mu + \varepsilon_t + \theta\varepsilon_{t-1}] = \mu \tag{4.8}$$

$$\begin{aligned}
\mathbb{V}[Y_t] &= \mathbb{E}[(\varepsilon_t + \theta\varepsilon_{t-1})^2] && (4.9)\\
&= \mathbb{V}[\varepsilon_t] + \theta^2\mathbb{V}[\varepsilon_{t-1}] && (4.10)\\
&= (1+\theta^2)\sigma_\varepsilon^2 && (4.11)
\end{aligned}$$

**Auto-covariance**

$$\begin{aligned}
\gamma_0 &= \mathbb{V}[Y_t] = (1+\theta^2)\sigma_\varepsilon^2 && (4.12)\\
\gamma_1 &= \mathbb{E}[(Y_t - \mu)(Y_{t-1} - \mu)] && (4.13)\\
&= \mathbb{E}[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] && (4.14)\\
&= \theta\sigma_\varepsilon^2 && (4.15)\\
\gamma_k &= 0 \ \forall k > 1 && (4.16)
\end{aligned}$$

**Auto-correlation**

$$\begin{aligned}
\rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2} && (4.17)\\
\rho_k &= 0 \ \forall k > 1 && (4.18)
\end{aligned}$$

**Definition 4.4.** A MA(1) process is **invertible** if $|\theta| < 1$, so that it can be written as an AR($\infty$) process.

*Inverting.* Let

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t+1} \tag{4.19}$$

where $|\theta| < 1$. Then,

$$\begin{aligned}
Y_t &= \mu + \varepsilon_t + \theta\varepsilon_{t+1} && (4.20)\\
\implies Y_t - \mu &= (1+\theta L)\varepsilon_t && (4.21)\\
\implies \frac{Y_t - \mu}{1-(-\theta L)} &= \varepsilon_t && (4.22)\\
\implies \varepsilon_t &= (Y_t - \mu)\sum_{j=0}^{\infty}(-\theta L)^j && (4.23)
\end{aligned}$$

■

**Equivalence** note that for MA(1) process,

$$r_1 = \rho_1 = \frac{\theta}{1+\theta^2} \tag{4.24}$$

and for any $\theta$, $\frac{1}{\theta}$ will generate the same auto-correlation. We always choose the invertible MA representation with $|\theta| < 1$.

## 4.3 Forecasting with MA(1)

### 4.3.1 Forecasting with Horizon $h = 1$

**Point estimate**

$$f_{t,1} = \mathbb{E}[Y_{t+1}|I_t] \tag{4.25}$$
$$= \mathbb{E}[\mu + \theta\varepsilon_t + \varepsilon_{t+1}|I_t] \tag{4.26}$$
$$= \mu + \theta\varepsilon_t \tag{4.27}$$

**Forecasting error**

$$e_{t,1} = Y_{t+1} - f_{t,1} = \varepsilon_{t+1} \tag{4.28}$$

**Forecasting uncertainty**

$$\sigma^2_{t+1|t} = \mathbb{V}[Y_{t+1}|I_t] \tag{4.29}$$
$$= \mathbb{E}[\varepsilon^2_{t+1}|I_t] \tag{4.30}$$
$$= \sigma^2_\varepsilon \tag{4.31}$$

**Density forecast** assuming *normality* of $\varepsilon$

$$\mathcal{F} \equiv \mathcal{N}(\mu_{t+1|t}, \sigma^2_{t+1|t}) \tag{4.32}$$
$$\mu_{t+1|t} = \mu + \theta\varepsilon_t \tag{4.33}$$
$$\sigma^2_{t+1|t} = \sigma^2_\varepsilon \tag{4.34}$$

### 4.3.2 Forecasting with Horizon $h = 2$

**Point estimate**

$$f_{t,2} = \mathbb{E}[Y_{t+2}|I_t] \tag{4.35}$$
$$= \mathbb{E}[\mu + \theta\varepsilon_{t+1} + \varepsilon_{t+2}|I_t] \tag{4.36}$$
$$= \mu \tag{4.37}$$

**Forecasting Error**

$$Y_{t+2} - ft, 2 = \theta\varepsilon_{t+1} + \varepsilon_{t+2} \tag{4.38}$$

**Forecasting Uncertainty**

$$\sigma^2_{t+2|t} \equiv \mathbb{V}[Y_{t+2}|I_t] \tag{4.39}$$
$$= (1 + \theta^2)\sigma^2_\varepsilon \tag{4.40}$$

**Density forecast**

$$\mathcal{F} = \mathcal{N}(\mu, (1 + \theta^2)\sigma^2_\varepsilon) \tag{4.41}$$

**Remark 4.1.** Since for any $h > 1$, MA(1) only generates the unconditional mean $\mu$ as the point estimate, we say MA(1) process is **short memory**.

### 4.4 Properties of MA(2) Process

**Model**

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \tag{4.42}$$

**Unconditional Moments**

$$\mathbb{E}[Y_t] = \mu \tag{4.43}$$
$$\mathbb{V}[Y_t] = (1 + \theta_1^2 + \theta_2^2)\sigma_\varepsilon^2 \tag{4.44}$$

**Auto-covariance**

$$\gamma_0 = \mathbb{V}[Y_t] = (1 + \theta_1^2 + \theta_2^2)\sigma_\varepsilon^2 \tag{4.45}$$
$$\gamma_1 = (\theta_1 + \theta_1\theta_2)\sigma_\varepsilon^2 \tag{4.46}$$
$$\gamma_2 = \theta_2 \sigma_\varepsilon^2 \tag{4.47}$$

**Auto-correlation**

$$\rho_1 \equiv \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \tag{4.48}$$
$$\rho_2 \equiv \frac{\gamma_2}{\gamma_0} = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} \tag{4.49}$$

**Optimal Forecasting**

$$f_{t,1} = \mu + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} \tag{4.50}$$
$$f_{t,2} = \mu + \theta_2 \varepsilon_t \tag{4.51}$$
$$f_{t,h} = \mu \ \forall h > 2 \tag{4.52}$$

# 5 Auto-Regression Process and Seasonality

## 5.1 AR Process

**Definition 5.1.** An **auto-regressive** model of order $p$ is taken in the form of

$$Y_t = c + \sum_{j=1}^{p} \phi_j L^j Y_t + \varepsilon_t \tag{5.1}$$

or equivalently

$$\Phi_p(L) Y_t = c + \varepsilon_t \tag{5.2}$$

**Definition 5.2.** An **auto-regressive** process of order 1 takes the form of stochastic difference equation

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t \tag{5.3}$$

where $\phi$ is called the **persistence parameter**.

**Proposition 5.1.** AR(1) process is stationary if and only if $|\phi| < 1$.

**ACF and PACF**

$$\rho_1 = r_1 = \phi \tag{5.4}$$
$$r_k = 0 \ \forall k > 1 \tag{5.5}$$

### 5.1.1 Forecasting with AR(1) and $h = 1$

[3] **Point Estimate**

$$f_{t,1} = \mathbb{E}[Y_{t+1}|I_t] \tag{5.6}$$
$$= c + \phi Y_t \tag{5.7}$$

**Forecast Variance(Uncertainty)**

$$\mathbb{V}[Y_{t+1}|I_t] = \mathbb{V}[c + \phi Y_t + \varepsilon_{t+1}|I_t] = \sigma_\varepsilon^2 \tag{5.8}$$

**Density Forecast**

$$\mathcal{F} = \mathcal{N}(c + \phi Y_t, \sigma_\varepsilon^2) \tag{5.9}$$

### 5.1.2 Forecasting with AR(1) and $h = s > 1$

**Point Estimate**

$$f_{t,s} = \mathbb{E}[Y_{t+s}|I_t] \tag{5.10}$$
$$= c + \mathbb{E}[\phi Y_{t+s-1}|I_t] \tag{5.11}$$
$$= (1 + \phi + \cdots + \phi^{s-1})c + \phi^s Y_t \tag{5.12}$$

**Forecasting Uncertainty**

$$\mathbb{V}[Y_{t+s}|I_t] = \mathbb{V}[\varepsilon_{t+s} + \phi\varepsilon_{t+s-1} + \cdots + \phi^{s-1}\varepsilon_{t+1}|I_t] \tag{5.13}$$
$$= \sum_{j=0}^{s-1} \phi^{2j}\sigma_\varepsilon^2 \tag{5.14}$$

### 5.1.3 Forecasting with AR(1) and $h \to \infty$

**Assumption 5.1.** For this subsection, assuming the AR(1) process is stationary, that's, $|\phi| < 1$.

**Point Estimate**

$$\lim_{h\to\infty} f_{t,h} = \frac{c}{1-\phi} \tag{5.15}$$

**Forecasting Uncertainty**

$$\lim_{h\to\infty} \mathbb{V}[Y_{t+h}|I_t] = \frac{\sigma_\varepsilon^2}{1-\phi^2} \tag{5.16}$$

**Remark 5.1.** The convergences demonstrated above suggest auto-regressive process is still a **short memory** process.

[3] While examining the optimal forecast in this section, we are assuming the loss function is *symmetric*.

### 5.1.4 Forecasting with AR(2) process

**Definition 5.3.** AR(2) process

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \tag{5.17}$$

**Unconditional Moments**

$$\mathbb{E}[Y_t] = c + \phi_1 \mathbb{E}[Y_{t-1}] + \phi_2 \mathbb{E}[Y_{t-2}] \tag{5.18}$$

$$\implies \mu_Y = \frac{c}{1 - \phi_1 - \phi_2} \tag{5.19}$$

**Auto-covariance and Auto-correlation**

$$\rho_1 = r_1 \tag{5.20}$$

$$r_2 = \phi_2 + \text{ sampling error} \tag{5.21}$$

**Optimal Forecasts** $h = 1$

$$f_{t,1} = \mathbb{E}[Y_{t+1}|I_t] \tag{5.22}$$

$$= \mathbb{E}[c + \phi_1 Y_t + \phi_2 Y_{t-1} + \varepsilon_{t+1}|I_t] \tag{5.23}$$

$$= c + \phi_1 Y_t + \phi_2 Y_{t-1} \tag{5.24}$$

$$e_{t,1} = \varepsilon_{t+1} \tag{5.25}$$

$$\sigma^2_{t+1|t} = \mathbb{V}[Y_{t+1}|I_t] = \sigma^2_\varepsilon \tag{5.26}$$

**Optimal Forecasts** $h = 2$

$$f_{t,2} = \mathbb{E}[Y_{t+2}|I_t] \tag{5.27}$$

$$= \mathbb{E}[c + \phi_1 Y_{t+1} + \phi_2 Y_t + \varepsilon_{t+2}|I_t] \tag{5.28}$$

$$= c + \phi_1 f_{t,1} + \phi_2 Y_t \tag{5.29}$$

$$e_{t,2} = Y_{t+2} - f_{t,2} \tag{5.30}$$

$$= \phi_1(Y_{t+1} - f_{t,1}) + \varepsilon_{t+2} \tag{5.31}$$

$$= \phi_1 e_{t,1} + \varepsilon_{t+2} \tag{5.32}$$

$$\sigma^2_{t+2|t} = \mathbb{V}[Y_{t+2}|I + t] \tag{5.33}$$

$$= \phi_1^2 \sigma^2_{t+1|t} + \sigma^2_\varepsilon \tag{5.34}$$

$$= (1 + \phi_1^2)\sigma^2_\varepsilon \tag{5.35}$$

**Optimal Forecasts** $h = s > 2$

$$f_{t,s} = \mathbb{E}[Y_{t+s}|I_t] \tag{5.36}$$

$$= c + \phi_1 f_{t,s-1} + \phi_2 f_{t,s-2} \tag{5.37}$$

$$e_{t,s} = \phi_1 e_{t,s-1} + \phi_2 e_{t,s-2} + \varepsilon_{t+s} \tag{5.38}$$

$$\sigma^2_{t+s|t} = \mathbb{V}[Y_{t+s}|I_t] \tag{5.39}$$
$$= \mathbb{V}[e_{t,s}|I_t] \tag{5.40}$$
$$= \phi_1 \sigma^2_{t+s-1|t} + \phi_2 \sigma^2_{t+s-2|t} + \sigma^2_\varepsilon \tag{5.41}$$

**Remark 5.2.** AR(2) is still classified as *short memory processes* as

$$\lim_{s\to\infty} f_{t,s} = \mu \tag{5.42}$$
$$\lim_{s\to\infty} \sigma^2_{t+2|t} = \sigma^2_Y \tag{5.43}$$

### 5.1.5 AP($p$) process

**Definition 5.4.** Let $\Phi(L)Y_t = c + \varepsilon_t$ be an AR($p$) process with trend $c$, then $\Phi(\cdot)$ is called the **characteristic polynomial** of this stochastic process.

**Theorem 5.1.** An autoregressive process is stationary if and only if all roots of its characteristic polynomial are outside the unit circle on $\mathbb{C}$.

**Forecasting with AR($p$) Process** we apply a recursive scheme, or **chain rule of forecasting**, in which the we use the *forecasted* values to make prediction on even further values.

**Example 5.1** (AP($p$) chain rule of forecasting)**.**

$$f_{t,1} = c + \sum_{j=1}^{p} \phi_j L^j Y_{t+1} \tag{5.44}$$

$$f_{t,2} = c + \phi_1 f_{t,1} + \sum_{j=2}^{p} \phi_j L^j Y_{t+2} \tag{5.45}$$

$$f_{t,3} = c + \phi_1 f_{t,2} + \phi_2 f_{t,1} + \sum_{j=3}^{p} \phi_j L^j Y_{t+3} \tag{5.46}$$

$$f_{t,s} = c + \sum_{j=1}^{p} \phi_j f_{t,s-j} \ \forall s > p \tag{5.47}$$

## 5.2 Seasonality

### 5.2.1 Deterministic Seasonality

**Definition 5.5.** The seasonality is **deterministic** if the regressors indicating seasons are always exactly predictable.

### 5.2.2 Stochastic Seasonality

**Definition 5.6.** The seasonality os **stochastic** if the seasonal component is driven by random variables.

**Definition 5.7.** A seasonal AP($p$) model, S-AR($p$), is defined by

$$Y_t = c + \phi_s Y_{t-s} + \phi_{2s} + Y_{t-2s} + \cdots + \phi_{ps} Y_{t-ps} + \varepsilon_t \tag{5.48}$$

$$\Phi_p(L^s) Y_t = c + \varepsilon_t \tag{5.49}$$

where $s$ refers to the **data frequency**. Such model seeks to explain the **dynamics across seasons**.

**Definition 5.8.** Characteristics of realizations from S-AR($p$) process:

(i) ACF decays slowly with spikes at multiples of $s$.

(ii) PACF only spikes at multiples of $s$.

**Definition 5.9.** A seasonal MA($q$) model, S-MA($q$), is given by

$$Y_t = \mu + \Theta_q(L^s)\varepsilon_t \tag{5.50}$$

**Remark 5.3.** Characteristics of realizations from S-MA($q$) process:

(i) ACF only spikes at multiples of $s$.

(ii) PACF decays slowly with spikes at multiples of $s$.

**Proposition 5.2** (Combing ARMA and S-ARMA)**.** Given ARMA

$$\Phi_p(L) Y_t = c + \Theta_q(L)\varepsilon_t \tag{5.51}$$

and S-ARMA

$$\Phi'_p(L^{s_1}) Y_t = c + \Theta'_q(L^{s_2})\varepsilon_t \tag{5.52}$$

The combined model is given by multiplying the lag polynomials

$$\Phi_p(L)\Phi'_p(L^{s_1}) Y_t = c + \Theta_q(L)\Theta'_q(L^{s_2})\varepsilon_t \tag{5.53}$$

# 6 Model Assessment and Asymmetric Loss

## 6.1 Model Assessment

**Definition 6.1. Akaike information criterion**(AIC) of a model with $k$ parameters is defined as

$$AIC \equiv -2\ln(\mathcal{L}) + 2k \tag{6.1}$$

**Definition 6.2. Bayes information criterion**(BIC)/**Schwarz information criterion**(SIC) of a model with $k$ parameters and fitted on the sample with size $N$ is defined as

$$BIC \equiv -2\ln(\mathcal{L}) + 2\ln(N)k \tag{6.2}$$

**Definition 6.3.** Given time series data sample with size $T$, and use the *recursive scheme* starting from $t < T$, with forecasting horizon $h$, given sequence of *ground truth*

$$\mathcal{Y} = (y_j)_{j=t+h}^{T} \tag{6.3}$$

we can construct a sequence of forecast

$$\mathcal{F} = (f_{t,h}, f_{t+1,h}, f_{t+2,h}, \dots, f_{T-h,h}) \tag{6.4}$$

and a sequence of forecasting errors

$$\mathcal{E} = (e_{j,t})_{j=t+h}^{T} \tag{6.5}$$

then,

$$\text{MSE} \equiv \frac{1}{|\mathcal{F}|} \sum_{e \in \mathcal{E}} e^2 \tag{6.6}$$

$$\text{MAE} \equiv \frac{1}{|\mathcal{F}|} \sum_{e \in \mathcal{E}} ||e|| \tag{6.7}$$

$$\text{MAPE} \equiv \frac{1}{|\mathcal{F}|} \sum_{(y,e) \in (\mathcal{Y},\mathcal{E})} ||\frac{e}{y}|| \tag{6.8}$$

## 6.2 Asymmetric Loss

**Definition 6.4** (Log-Normal Distribution)**.** Let $X$ be a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. Define $Y \equiv \exp(X)$, then $Y$ follows **log-normal distribution**, with

$$\mathbb{E}[Y] = \exp(\mu + \frac{\sigma^2}{2}) \tag{6.9}$$

**Example 6.1.** Consider the Lin-ex loss function

$$L(e) = \exp(ae) - ae - 1 \tag{6.10}$$

then the expected loss for $h$ step forecasting made at $t$ is

$$\begin{aligned}
&\mathbb{E}[L(e_{t,h})|I_t] && (6.11)\\
&= \mathbb{E}[\exp(a(y_{t+h} - f_{t,h})) - a(y_{t+h} - f_{t,h}) - 1|I_t] && (6.12)\\
&= \mathbb{E}[\exp(ay_{t+h}) \exp(-af_{t,h})|I_t] - \mathbb{E}[ay_{t+h}|I_t] + af_{t,h} - 1 && (6.13)\\
&= \exp(-af_{t,h})\mathbb{E}[\exp(ay_{t+h})|I_t] - a\mathbb{E}[y_{t+h}|I_t] + af_{t,h} - 1 && (6.14)\\
& && (6.15)
\end{aligned}$$

To find the optimal forecasting, take the FOC

$$\begin{aligned}
&\frac{\partial \mathbb{E}[L(e_{t,h})|I_t]}{\partial f_{t,h}} = 0 && (6.16)\\
\implies &-a \exp(-af_{t,h})\mathbb{E}[\exp(ay_{t+h})|I_t] + a = 0 && (6.17)\\
\implies &\exp(-af_{t,h})\mathbb{E}[\exp(ay_{t+h})|I_t] = 1 && (6.18)\\
\implies &-af_{t,h} + \log(\mathbb{E}[\exp(ay_{t+h})|I_t]) = 0 && (6.19)\\
\implies &f_{t,h} = \frac{1}{a} \log(\mathbb{E}[\exp(ay_{t+h})|I_t]) && (6.20)
\end{aligned}$$

Assuming $y_{t+h} \sim \mathcal{N}(\mu_{t+h|t}, \sigma^2_{t+h|t})$,

$$\implies f_{t,h} = \frac{1}{a}\log(\exp(a\mathbb{E}[y_{t+h}|I_t] + \frac{a^2\sigma^2_{t+h|t}}{2})) \tag{6.21}$$

$$\implies f_{t,h} = \mathbb{E}[y_{t+h}|I_t] + \frac{a\sigma^2_{t+h|t}}{2} \tag{6.22}$$

So if $a < 0$, the penalty on making negative error is higher than positive error, and the optimal forecast would be *pushed down* to less than the conditional mean.

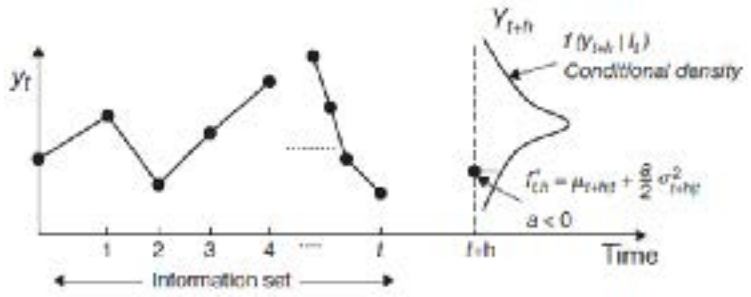



Figure 6.1: Illustration of the Optimal Forecast with Lin-Ex loss and $a < 0$

**Forecasting** Assuming AR(1) process.
**Error with** $h = 1$

$$e_{t+1,t} = c + \phi Y_t + \varepsilon_{t+1} - f_{t,1} \tag{6.23}$$

$$= \varepsilon_{t+1} - \frac{a\sigma^2_{t+1|t}}{2} \tag{6.24}$$

$$\mathbb{E}[e_{t+1,t}] = -\frac{a\sigma^2_{t+1|t}}{2} \tag{6.25}$$

$$\mathbb{V}[e_{t+1,t}|I_t] = \sigma^2_\varepsilon \tag{6.26}$$

**Error with** $h = 2$

$$e_{t+2,t} = Y_{t+2} - f_{t,2} \tag{6.27}$$

$$= c + \phi(Y_{t+1}) + \varepsilon_{t+2} - f_{t,2} \tag{6.28}$$

$$= c + \phi c + \phi\varepsilon_{t+1} + \phi Y_t + \varepsilon_{t+2} - f_{t,2} \tag{6.29}$$

$$= c + \phi c + \phi\varepsilon_{t+1} + \phi Y_t + \varepsilon_{t+2} - \mathbb{E}[Y_{t+2}|I_t] - \frac{a\sigma^2_{t+2|t}}{2} \tag{6.30}$$

$$= \phi\varepsilon_{t+1} + \varepsilon_{t+2} - \frac{a\sigma^2_{t+2|t}}{2} \tag{6.31}$$

Note that for every $h > 0$, the term $\sigma_{t+h|t}$ is constant conditioned on $I_t$.

$$\mathbb{E}[e_{t+2,t}|I_t] = -\frac{a\sigma^2_{t+2|t}}{2} \tag{6.32}$$

$$\mathbb{V}[e_{t+2,t}|I_t] = (1 + \phi^2)\sigma^2_\varepsilon \tag{6.33}$$