

Lecture Notes
MATH205A: Real Analysis I (Autumn 2020)
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September 23, 2020

1 Measures

1.1 Motivation

Motivation of this course is to define a notion of *length* on subsets of \mathbb{R} such that

1. $length([a, b]) = b - a$.
2. (countable additivity) $length(\bigcup^\infty A_i) = \sum^\infty length(A_i)$ where A_i 's are disjoint.
3. (translation invariance) for all $a \in \mathbb{R}$, $length(A + a) = length(A)$.

Fact 1.1. it is impossible to construct such length for all subsets of \mathbb{R} .

Proof. This proof shows it is impossible to construct a notion of length on $[0, 1]$ with desired properties.

For $x, y \in [0, 1]$, define an equivalence relation as $x \sim y \iff x - y \in \mathbb{Q}$. By the axiom of choice, we may construct a set A containing exactly one element from each equivalence class of $x \in [0, 1]$. Obviously, $A \subseteq [0, 1]$.

For each $r \in [-1, 1] \cap \mathbb{Q}$, let $A_r := A + r$, and $A_r \subseteq [-1, 2]$. By translation invariance, $length(A_r) = length(A)$. Note that for any $y \in [0, 1]$, there exists some $x \in A$ such that $x \sim y$, therefore, $y \in A_{y-x} \subseteq \bigcup_r A_r$. Hence, $[0, 1] \subseteq \bigcup_r A_r$.

If the notion of length satisfies countable additivity, $length(\bigcup_r A_r)$ is either zero or infinity, which leads to a contradiction. ■

Lebesgue's Resolution: we only defines length for a subset of $\mathcal{P}(\mathbb{R})$, which contains *everything that may ever arrive in practice*, i.e., σ -algebras.

1.2 Algebras and σ -algebra

Definition 1.1. Let X be a set, a collection \mathcal{A} of subsets of X is called an **algebra** if

1. $X \in \mathcal{A}$,

$$2. A \in \mathcal{A} \implies A^c \in \mathcal{A},$$

$$3. A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}.$$

Consequently: (1) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$; (2) $A_1, \dots, A_n \in \mathcal{A} \implies \bigcup_i A_i, \bigcap_i A_i \in \mathcal{A}$ (easily shown by induction); (3) $\emptyset \in \mathcal{A}$.

Definition 1.2. Let X be a set, a collection \mathcal{A} of subsets of X is called a σ -algebra if

$$1. X \in \mathcal{A},$$

$$2. A \in \mathcal{A} \implies A^c \in \mathcal{A},$$

$$3. A_1, A_2, \dots \in \mathcal{A} \implies \bigcup_i^\infty A_i \in \mathcal{A}.$$

Example 1.1 (trivial examples). The power set of X is a σ -algebra on X ; $\{\emptyset, X\}$ is a σ -algebra on X .

Example 1.2 (finite/co-finite algebra). Let X be an infinite set and \mathcal{A} be the collection of subsets A such that either A is finite or A^c is finite. \mathcal{A} is an algebra.

Proof. $X \in \mathcal{A}$ since $X^c = \emptyset$ is finite. For a $X \in \mathcal{A}$, if X is finite, then $X^c \in \mathcal{A}$. If X is infinite, X^c is finite and $X^c \in \mathcal{A}$. Let $A, B \in \mathcal{A}$, if both A and B are finite, $A \cup B$ is finite and in \mathcal{A} . If A is finite and B is co-finite, then $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. If both A and B are co-finite, $(A \cup B)^c$ is finite so that $A \cup B \in \mathcal{A}$. ■

Note the \mathcal{A} is not a σ -algebra if X is infinite: take distinct points $x_1, x_2, \dots \in \mathcal{A}$, then the union of them is neither finite or co-finite, and therefore not in \mathcal{A} .

Example 1.3 (countable/co-countable σ -algebra). The collection of subsets $A \subseteq X$, such that either A is countable or A^c is countable, forms a σ -algebra.

Example 1.4. Let $X = \mathbb{R}$ and \mathcal{A} be the collection of all finite disjoint unions of half-open intervals (i.e., sets like $(a, b], (-\infty, b], (a, \infty)$), \mathcal{A} is an algebra. (Not working for open intervals).

Example 1.5 (counter example). Let X be an infinite set, \mathcal{A} be the collection of finite subsets of X . Then, \mathcal{A} is not an algebra.

Proposition 1.1. Let X be a set and $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ be an arbitrary (not necessarily countable) collection of σ -algebras, then $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i$ is a σ -algebra.

Proof. Since $X \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ ■

Corollary 1.1. Let X be a set, and \mathcal{P} is an arbitrary collection of subsets of X , then $\exists!$ smallest σ -algebra \mathcal{A} containing \mathcal{P} . That is, for any σ -algebra $\mathcal{B} \supseteq \mathcal{P}$, $\mathcal{A} \subseteq \mathcal{B}$. \mathcal{A} is defined as the σ -algebra **generated by** \mathcal{P} , denoted as $\sigma(\mathcal{P})$.

Proof. For any \mathcal{P} , the power set of X is obviously a σ -algebra containing \mathcal{P} . Then we can take \mathcal{A} as the intersection of all σ -algebras containing \mathcal{P} . ■

1.3 Borel σ -algebra

Definition 1.3. The **Borel σ -algebra** of \mathbb{R} , denoted as $\mathcal{B}(\mathbb{R})$, is the σ -algebra generated by the set of open intervals in \mathbb{R} .

Fact 1.2. $\mathcal{B}(\mathbb{R})$ is generated by the collection of all closed intervals as well.

Proof. Let \mathcal{F} denote the σ -algebra generated by all closed intervals. Any open interval can be written as a countable union of closed sets: $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n]$, therefore $(a, b) \in \mathcal{F}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

Similarly, $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n)$, hence $\mathcal{B}(\mathbb{R})$ is a σ -algebra contains all closed sets. Therefore, $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$. ■

Fact 1.3. $\mathcal{B}(\mathbb{R})$ is generated by

1. all open sets,
2. all closed sets,
3. all half-open intervals.

Example 1.6 (counter example). $\mathcal{B}(\mathbb{R})$ is not generated by the collection of singletons.

Proof. ■

Definition 1.4. The Borel algebra of \mathbb{R}^d , $\mathcal{B}(\mathbb{R}^d)$, is the σ -algebra generated by

1. all open sets in \mathbb{R}^d ,
2. all closed sets in \mathbb{R}^d ,
3. all closed cubes (regions) in \mathbb{R}^d : $\prod_{i=1}^d [a_i, b_i]$.

1.4 Measures

Definition 1.5. For a set X and a σ -algebra \mathcal{A} of X , the pair (X, \mathcal{A}) is called a **measurable space**.

Definition 1.6. A **measure** μ on a measurable space (X, \mathcal{A}) is a map $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(\bigcup_i^{\infty} A_i) = \sum_i^{\infty} \mu(A_i)$ for disjoint sequence (A_i)

For now, we don't require the translation invariance property.

The triple (X, \mathcal{A}, μ) is called a **measure space**.

Example 1.7 (counting measure).

Example 1.8 (point-mass measure).

Proposition 1.2. A measure μ possesses the following basic properties:

1. (Monotonicity) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2. (Sub-additivity) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
3. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing set, let $\bigcup_{i=1}^{\infty} A_i$ denoted $A_i \nearrow A$, $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.
4. If $A_1 \searrow A \equiv \bigcap_{i=1}^{\infty} A_i$, and **there exists** $\mu(A_i) < \infty$, then $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.

Proof. ■

Example 1.9 (counter example). Let $X = \mathbb{Z}$, $\mathcal{A} = 2^{\mathbb{Z}}$ and μ be the counting measure. Define $A_i = \{i, i+1, \dots\}$, then $A_i \searrow A = \emptyset$, but $\lim_{n \rightarrow \infty} \mu(A_n) = \infty \neq \mu(\emptyset)$.

1.5 Outer Measure

Definition 1.7. Let X be a set, $\mu^* : 2^X \rightarrow [0, \infty]$ is an **outer measure** if

1. $\mu^*(\emptyset) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B$.
3. (countable sub-additivity) $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Key difference between outer measure and measure:

1. Outer measure does not require countable additivity,
2. outer measure is defined on 2^X instead of a σ -algebra .

Example 1.10.

1.6 Lebesgue Measure on \mathbb{R}

Definition 1.8. Let $A \subseteq \mathbb{R}$, define the **Lebesgue outer measure**:

$$\lambda^*(A) = \inf \left\{ \sum_{i \in \mathbb{N}} b_i - a_i : A \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i) \right\} \quad (1)$$

The Lebesgue outer measure of a set A is simply in the infimum of total lengths (the conventional notion of length) of open intervals cover A .

Proposition 1.3. The Lebesgue measure satisfies the following properties:

1. λ^* is an outer measure on \mathbb{R} ,
2. $\lambda^*([a, b]) = b - a$ for all $a < b$.

Proof. (1.1) $\lambda^*(\emptyset) = 0$ since $(-\varepsilon, \varepsilon)$ covers \emptyset for arbitrarily small ε .

(1.2) Let $A \subseteq B$, Ω_A and Ω_B be collection of sequences of open intervals covering A and B respectively. Then, any cover of B must be a cover of A , that is, $\Omega_A \subseteq \Omega_B$. Therefore, $\lambda^*(A) \leq \lambda^*(B)$.

(1.3) Let $A_1, A_2, \dots \subseteq \mathbb{R}$ and $A = \bigcup_{i=1}^{\infty} A_i$. For all i , we may find (a_{ij}, b_{ij}) covers A_i such that

$$\sum_{j=1}^{\infty} (b_{ij} - a_{ij}) \leq \lambda^*(A_i) + \varepsilon 2^{-i} \quad (2)$$

Also, $\{(a_{ij}, b_{ij})\}_{i,j}$ is a countable union of open intervals that covers A .

$$\lambda^*(A) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (b_{ij} - a_{ij}) \quad (3)$$

$$\leq \sum_{i=1}^{\infty} (\lambda^*(A_i) + \varepsilon 2^{-i}) \quad (4)$$

$$= \sum_{i=1}^{\infty} \lambda^*(A_i) + \varepsilon \quad (5)$$

Therefore, $\lambda^*(A) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$.

(2) Note that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$ for all $\varepsilon > 0$. Therefore,

$$\lambda^*([a, b]) \leq \inf_{\varepsilon > 0} \lambda^*(a - \varepsilon, b + \varepsilon) = b - a \quad (6)$$

Now show $\lambda^*([a, b]) \geq b - a$. We want to show that $\sum_{i=1}^{\infty} (b_i - a_i) \geq b - a$ for all possible covering of $[a, b]$, which implies the infimum of them is at least $b - a$.

Take an arbitrary covering $\{(a_i, b_i)\}_i$ of $[a, b]$. Since $[a, b]$ is compact, there exists a finite covering $[a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i)$ (reindexed), it suffices to show the finite sum $\sum_{i=1}^n (b_i - a_i) \geq b - a$.

(1) We firstly define an *interval* to be any open, closed or half-open intervals. The *length* of an interval is the difference between two end points.

Note that if an interval I contains a finite collection of disjoint sub-intervals, then the length of I is at least the sum of lengths of sub-intervals. The equality holds when I is exactly finite union of disjoint sub-intervals.

(2) Suppose $[a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i)$, let $I_i = [a, b] \cap (a_i, b_i)$. Easy to verify that the length of $I_i \leq$ length of $(a_i, b_i) = b_i - a_i$. Moreover, $\bigcup_{i=1}^n I_i = [a, b] \cup \bigcup_{i=1}^n (a_i, b_i) = [a, b]$.

(3) For all i , define $I'_i = I_i \setminus (I_1 \cup I_2 \cup \dots \cup I_{i-1})$. This procedure allows us to express $[a, b]$ as a finite union of disjoint sub-intervals: $[a, b] = \bigcup_{i=1}^n I'_i$. Each I'_i is a finite union of disjoint intervals as well, the conventional notion of I'_i is well-defined. Then $b - a =$ sum of lengths of I'_i .

However, $\ell(I'_i) \leq \ell(I_i) \leq b_i - a_i$ and sum of lengths of $I'_i \leq$ sum of lengths of $I_i \leq \sum_{i=1}^n b_i - a_i$. Therefore, $b - a \leq \sum_{i=1}^n b_i - a_i \leq \sum_{i=1}^{\infty} b_i - a_i$. Hence, $b - a = \sum_{i=1}^{\infty} b_i - a_i$ and $\lambda^*[a, b] = b - a$ consequently. ■

1.7 Construct Lebesgue Measure

Definition 1.9. Let X be a set with outer measure μ^* . A set $B \subseteq X$ is μ^* -**measurable** if

$$\forall A \subseteq X, \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \quad (7)$$

Theorem 1.1. For any set X with outer measure μ^* on it, let \mathcal{M}_{μ^*} denote the set of all μ^* -**measurable** sets. Then, \mathcal{M}_{μ^*} is a σ -algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ (μ^* restricted to \mathcal{M}_{μ^*}) is a measure.

Proof. To show B is μ^* -measurable, it suffices to show that $\forall A \subseteq X, \mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$, because the opposite inequality always holds by sub-additivity.

(1.1) Let $A \subseteq X$, $\mu^*(A \cap \emptyset) + \mu^*(A \cap \emptyset^c) = \mu^*(A \cap \emptyset^c) = \mu^*(A)$, therefore, $\emptyset \in \mathcal{M}_{\mu^*}$.

(1.2) Let $A \subseteq X$ and $B \in \mathcal{M}_{\mu^*}$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap (B^c)^c) + \mu^*(A \cap B^c)$.

Hence, $B^c \in \mathcal{M}_{\mu^*}$.

(1.3.1) Let $B_1, B_2 \in \mathcal{M}_{\mu^*}$, we are going to show $B_1 \cup B_2 \in \mathcal{M}_{\mu^*}$. Fix any $A \subseteq X$,

$$\mu^*(A \cap (B_1 \cup B_2)) = \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c) \quad (8)$$

$$= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) \quad (9)$$

Moreover,

$$\mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1^c \cap B_2^c) \quad (10)$$

Therefore,

$$\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) \quad (11)$$

$$= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \text{ since } B_2 \in \mathcal{M}_{\mu^*} \quad (12)$$

$$= \mu^*(A) \text{ since } B_1 \in \mathcal{M}_{\mu^*} \quad (13)$$

Therefore, \mathcal{M}_{μ^*} is an algebra.

(1.3.2) Now show that \mathcal{M}_{μ^*} is a σ -algebra. For any sequence of sets $A_i \in \mathcal{M}_{\mu^*}$, we can define $B_n := A_n \setminus \bigcup_{j=1}^{n-1} A_j$ such that $\bigcup B_i = \bigcup A_i$. Therefore, it suffices to show \mathcal{M}_{μ^*} is closed under countable disjoint unions.

We are going to show the union $\bigcup B_i$ is μ^* -measurable for any disjoint sequence of μ^* -measurable B_i 's.

Claim: let $A \subseteq X$, $\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c)$. The claim can be proved by induction on n .

When $n = 1$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ because B_1 is μ^* -measurable.

Suppose the claim holds for n , then

$$\mu^*(A \cap (\bigcup_{i=1}^n B_i)^c) = \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c \cap B_{n+1}) + \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c \cap B_{n+1}^c) \quad (14)$$

because $B_{n+1} \in \mathcal{M}_{\mu^*}$. Moreover, since all B_i 's are disjoint, $B_{n+1} \subseteq B_i^c$ for all i . Hence,

$$B_{n+1} \subseteq \cap_{i=1}^n B_i^c = (\cup_{i=1}^n B_i)^c \quad (15)$$

Also,

$$(\cup_{i=1}^n B_i)^c \cap B_{n+1}^c = \cap_{i=1}^{n+1} B_i^c \quad (16)$$

Consequently,

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^{n+1} B_i)^c) \quad (17)$$

Hence,

$$\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^n B_i^c)) \quad (18)$$

$$\geq \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^\infty B_i^c)) \quad (19)$$

$$= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (20)$$

Take $n \rightarrow \infty$

$$\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (21)$$

$$\geq \mu^*(A \cap \cup_{i=1}^\infty B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (22)$$

Therefore, $\cup_{i=1}^\infty B_i$ is μ^* -measurable.

(2) Let B_1, B_2, \dots be a sequence of disjoint sets from \mathcal{M}_{μ^*} . Using the above fact and take $A = \cup_{i=1}^\infty B_i$,

$$\mu^*(A) \geq \mu^*(\cup_{i=1}^\infty B_i) + \mu^*(\emptyset) = \mu^*(\cup_{i=1}^\infty B_i) \quad (23)$$

The opposite inequality holds by sub-additivity. Therefore, μ^* is a measure on \mathcal{M}_{μ^*} . ■

Definition 1.10. Let λ^* be the Lebesgue outer measure on \mathbb{R} , then the collection \mathcal{M}_{λ^*} of λ^* -measurable sets is called the **Lebesgue σ -algebra**. The restriction $\lambda = \lambda^*|_{\mathcal{M}_{\lambda^*}}$, which is a measure on \mathcal{M}_{λ^*} , is called the **Lebesgue measure**. Any set in \mathcal{M}_{λ^*} is called a **Lebesgue measurable set**.

Theorem 1.2. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$.

Proof. Note that $\{(-\infty, b] : b \in \mathbb{R}\}$ generates $\mathcal{B}(\mathbb{R})$, it suffices to show $\{(-\infty, b] : b \in \mathbb{R}\} \subseteq \mathcal{M}_{\lambda^*}$.

Let $B = (-\infty, b]$, we are going to show B is λ^* -measurable. Let $A \subseteq \mathbb{R}$ and (a_n, b_n) be a

sequence of open intervals covers A . For every $n \in \mathbb{N}$,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n) \cap (-\infty, b]) + \lambda^*((a_n, b_n) \cap (b, \infty)) \quad (24)$$

Three cases follow:

1. $b > b_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.
2. $b_n > b > a_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b]) + \lambda^*((b, b_n]) = b_n - a_n$.
3. $a_n > b$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.

Therefore,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = b_n - a_n \quad (25)$$

By monotonicity and sub-additivity:

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(\cup(a_n, b_n) \cap B) + \lambda^*(\cup(a_n, b_n) \cap B^c) \quad (26)$$

$$\leq \sum \lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) \quad (27)$$

$$= \sum_{n=1}^{\infty} b_n - a_n \quad (28)$$

Take the infimum of all such covering, we can show

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A) \quad (29)$$

Therefore, B is μ^* -measurable and \mathcal{M}_{λ^*} is a σ -algebra containing all such intervals and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$. ■

1.8 Lebesgue Measure on \mathbb{R}^d

Definition 1.11. Steps to construct Lebesgue measure on \mathbb{R}^d :

1. Define open cubes on \mathbb{R}^d as a Cartesian product of open intervals: $Q := \prod_{i=1}^d (a_i, b_i)$. Define Lebesgue outer measure:

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \prod_{i=1}^d (b_{ni} - a_{ni}) : A \subseteq \bigcup_{n=1}^{\infty} Q_n \right\} \quad (30)$$

2. Show λ^* is an outer measure and $\lambda^*(Q) = \prod_{i=1}^d (b_i - a_i)$.
3. \mathcal{M}_{λ^*} is the Lebesgue σ -algebra on \mathbb{R}^d . Restricting λ^* on \mathcal{M}_{λ^*} defines the Lebesgue measure.
4. Show that any Borel set in \mathbb{R}^d is Lebesgue measurable by showing that there is a generating set of $\mathcal{B}(\mathbb{R}^d)$ is in \mathcal{M}_{λ^*} .

1.9 Uniqueness of the Lebesgue Measure

The next goal is to prove the uniqueness of Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ subject to the criterion that the measure of any interval (cube) is the volume in the usual sense (product of side lengths).

Theorem 1.3. Let λ be the Lebesgue measure on \mathbb{R}^d , then for any Lebesgue measurable set A ,

1. $\lambda(A) = \inf\{\lambda(U) : \text{open } U \supseteq A\},$
2. $\lambda(A) = \sup\{\lambda(K) : \text{compact } K \subseteq A\}.$

Proof. (1.1) WLOG $\lambda(A) < \infty$, by monotonicity, $\lambda(A) \leq \lambda(U)$ for any open cover, $\lambda(A) \leq \inf\{.. \}$.

(1.2) Let $\varepsilon > 0$, \exists a sequence of open intervals (R_i) such that

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon \quad (31)$$

Let $U := \cup R_i$ open, hence $\inf\{.. \} \leq \lambda(U) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon$. Since this ε can be arbitrarily small, we conclude $\inf\{.. \} \leq \lambda(A)$.

(2.1) let A be a Lebesgue measurable set, assume A is bounded, so that $\lambda(A) < \infty$. Then there exists a compact $C \supseteq A$. $C \setminus A$ is Lebesgue measurable as well.

By conclusion of part (1), there exists a open set $U \supseteq C \setminus A$ such that

$$\lambda(C \setminus A) \leq \lambda(U) \leq \lambda(C \setminus A) + \varepsilon \quad (32)$$

Let $K = C \setminus U$, K is compact. Moreover, let $a \in K$, then $a \in C$ and $a \notin U$. Therefore, $a \notin C \setminus A$, it must be $x \in A$. Hence, $K \subseteq A$.

$$\lambda(K) = \lambda(C \setminus U) \quad (33)$$

$$\geq \lambda(C) - \lambda(U) \quad (34)$$

$$\geq \lambda(C) - (\lambda(C \setminus A) + \varepsilon) \quad (35)$$

$$= \lambda(C) - \lambda(C) + \lambda(A) - \varepsilon \quad (36)$$

$$= \lambda(A) - \varepsilon \quad (37)$$

Take $\varepsilon \rightarrow 0$ and $\lambda(A) \leq \sup\{.. \}$. By monotonicity, $\lambda(A) \geq \sup\{.. \}$.

(2.2) Other cases: suppose A is unbounded and $\lambda(A) > 0$. Take an arbitrary $b < \lambda(A)$. We will show that $\sup\{.. \} \geq b$, this will prove that $\lambda(A) \leq \sup\{.. \}$.

To show $\sup\{.. \} \geq b$, it suffices to show that there exists a compact set $K \subseteq A$ such that $\lambda(K) \geq b$.

Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of compact sets increasing to \mathbb{R}^d .

Then $A \cap C_j \uparrow A$ and $\lambda(A \cap C_1) < \infty$, which implies $\lambda(A) = \lim_{j \rightarrow \infty} \lambda(A \cap C_j)$. Since $b < \lambda(A)$, there exists j such that $\lambda(A \cap C_j) \geq b$, where $A \cap C_j$ is compact. Hence, $b \leq \sup\{.. \}$ and $\lambda(A) \leq \sup\{.. \}$. $\lambda(A) \geq \sup\{.. \}$ holds by monotonicity.

When $\lambda(A) = 0$, $0 \leq \lambda(K)$ for all K so that $0 \leq \sup\{.. \}$. The opposite inequality holds by monotonicity. ■

Lemma 1.1. For each $k \in \mathbb{Z}$, define **dyadic cubes** in \mathbb{R}^d as set in the following form:

$$\prod_{i=1}^d [j_i 2^{-k}, (j_i + 1) 2^{-k}) \quad (38)$$

where $j_i \in \mathbb{Z}$ for every i . Let \mathcal{D} denote the collection of dyadic cubes.

Then, any open set $U \subseteq \mathbb{R}^d$ can be expressed as a countable union of some members of \mathcal{D} .

A dyadic cube of side length 2^{-k} has a unique parent of side length 2^{-k+1} and a unique grandparent with side length 2^{-k+2} .

Proof. Given open set U , let \mathcal{D}_U denote the set of all dyadic half open cubes D such that $D \subseteq U$ but the parent of U does not fully contain U .

Claim 1: $U = \bigcup_{D \in \mathcal{D}_U} D$. Obviously, $\bigcup_{D \in \mathcal{D}_U} D \subseteq U$. To show the converse, take any $x \in U$, since U is open, there exists $D \in \mathcal{D}_U$ such that $x \in D \subseteq U$.

Let D_0 be the earliest ancestor of D such that $x \in D_0 \subseteq U$. Obviously, $D_0 \in \mathcal{D}_U$. Therefore, $U \subseteq \bigcup_{D \in \mathcal{D}_U} D$.

Claim 2: Two dyadic cubes can overlap if and only if one is the ancestor of the other. By construction, dyadic cubes in \mathcal{D}_U are disjoint.

Claim 3: \mathcal{D}_U is countable because \mathcal{D} is itself countable. ■

Proposition 1.4. Lebesgue measure is the only measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which assigns the *correct volume* to any d -dimensional intervals or even any d -dimensional dyadic cube.

Proof. Let λ denote the Lebesgue measure, let μ be another measure satisfying the desired property.

By lemma, for all open set U , $\mu(U) = \sum_{j=1}^{\infty} \mu(D_j) = \sum_{j=1}^{\infty} \lambda(D_j) = \lambda(U)$, where $\{D_j\}$ is a collection of disjoint dyadic cubes contains with union U . Therefore, $\lambda(A) = \mu(A)$ for all open Borel set A .

Let $A \in \mathcal{B}(\mathbb{R}^d)$, let open $U \supseteq A$, then $\mu(A) \leq \mu(U) = \lambda(U)$ for all U . Taking the infimum over all $U \supseteq A$, we conclude $\mu(A) \leq \lambda(A)$ for all Borel set A .

Next, take any bounded Borel set A , let V be a bounded open set containing A . Then,

$$\mu(V) = \mu(A) + \mu(V \setminus A) \quad (39)$$

$$\leq \lambda(A) + \lambda(V \setminus A) \quad (40)$$

$$= \lambda(V) \quad (41)$$

But we also know that $\mu(V) = \lambda(V)$ since V is open, the inequality holds as equality. Moreover, the previous conclusion implies $\mu(A) \leq \lambda(A)$ and $\mu(V \setminus A) \leq \lambda(V \setminus A)$, it must be $\mu(A) = \lambda(A)$ and $\mu(V \setminus A) = \lambda(V \setminus A)$. Therefore, $\mu(A) = \lambda(A)$ for all bounded Borel set A .

Lastly, any Borel set can be written as a countable disjoint union of bounded Borel set, therefore, $\mu(A) = \lambda(A)$ for all Borel set A . ■

Proposition 1.5. The Lebesgue outer measure on \mathbb{R}^d is translation invariant. In particular, Lebesgue measure is translation invariant and any translation of Lebesgue measurable set is Lebesgue measurable.

Proof. $\lambda^*(A+x) = \lambda^*(A)$ follows the definition of λ^* : translate all covering intervals by $+x$ and the volumes of these intervals stay the same. Since λ is simply the restriction of λ^* on Lebesgue measurable sets, λ is translation invariant as well.

Now take Lebesgue measurable B , for all $A \subseteq \mathbb{R}^d$:

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c) \quad (42)$$

$$\implies \lambda^*(A-x) = \lambda^*((A-x) \cap B) + \lambda^*((A-x) \cap B^c) \quad (43)$$

Note that

$$(A-x) + x = A \quad (44)$$

$$(A-x) \cap B + x = A \cap (B+x) \quad (45)$$

$$(A-x) \cap B^c + x = A \cap (B+x)^c \quad (46)$$

By translational invariance of λ^* ,

$$\lambda^*(A) = \lambda^*(A \cap (B+x)) + \lambda^*(A \cap (B+x)^c) \quad (47)$$

Therefore, $B+x$ is Lebesgue measurable as well. ■

Theorem 1.4. Let μ be a non-zero measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is finite on bounded Borel sets and translation invariant. Then, $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$, where λ is the Lebesgue measure.

Remark 1.1. Borel σ -algebra is closed under translation.

Proof. Let $c = \mu([0,1]^d) \in (0, \infty)$. Then $[0,1]^d$ is the disjoint union of 2^{dk} half-open dyadic intervals with side length 2^{-k} . All of these sub-intervals have the same μ since μ is translation invariant. Therefore, for every dyadic sub-interval with side length 2^{-k} , $\mu(D) = 2^{-dk}c$.

Let $\nu(A) = \frac{1}{c}\mu(A)$, then ν is a measure that is finite on bounded sets and agrees with λ on all half-open dyadic cubes. By the previous proposition, λ is the only measure assign correct volumes to dyadic cubes, therefore, $\nu = \lambda$. ■

Theorem 1.5. Under the axiom of choice, there exists a non-Lebesgue subset of \mathbb{R} .

Proof. Todo. ■

2 Integrations