Notes on Measure Theory

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Theorem 0.1. There is no measure μ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ satisfying the following two conditions: (i) $\mu((a,b]) = b - a$ for every $a < b$ and (ii) $\mu(x + A) = \mu(A)$ for all $a \in \mathbb{R}$ and $A \in \mathcal{P}(\mathbb{R})$.

Proof. Suppose, for contradiction, there exists such a measure μ, then $\mu((0,1]) = 1 < \infty$.

Claim: the only measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ satisfying $\mu((0,1]) < \infty$ and $\mu(x + A) = \mu(A)$ is the zero measure.

To prove the claim, let $I := (0,1]$ and defien the following equivalence relation on I:

$$x \sim y \iff x - y \in \mathbb{Q} \tag{1}$$

the corresponding equivalence class of x on I can be written as

$$[x] = \{x + r : r \in \mathbb{Q} \wedge x + r \in I\} \tag{2}$$

The collection of all such equivalence classes, $\mathcal{A}$, is a disjoint decomposition of I. (for every $x \in I$, $[x]$ must in $\mathcal{A}$ and $x \in [x]$ trivially. If there exists different $[x] \neq [y]$ but $[x] \cap [y] \neq \varnothing$, take $z \in [x] \cap [y]$, by the transitivity of equivalence relation, $x \sim z \sim y$. Therefore, $[x] = [y]$, contradiction.)

For each $[x] \in \mathcal{A}$, take exactly one $a_x \in [x]$ and define set $A := \{a_x : [x] \in \mathcal{A}\}$. As a result, set A satisfies the following two properties:

1. $\forall x \in I, \exists a_x \in A \ s.t. \ a_x \in [x]$.
2. $\forall x, y \in A$, $x \sim y \implies x = y$.

Since $\mathbb{Q} \cup (-1, 1]$ is countable, let $(r_n)_{n \in \mathbb{N}}$ be an enumeration of all elements in it.

For each $n \in \mathbb{N}$, define $A_n := r_n + A$.

Note that for any m, n such that $A_m \cap A_n \neq \varnothing$, take $x \in A_m \cap A_n$. By definition,

$$x = r_n + a_n \tag{3}$$

$$x = r_m + a_m \tag{4}$$

where $a_n, a_m \in A$ and $r_n, r_m \in \mathbb{Q}$. Consequently,

$$a_n - a_m = r_m - r_n \in \mathbb{Q} \tag{5}$$

Therefore, $a_n \sim a_m$. By the second property of A, $a_n = a_m$. Thus, $r_m = r_n$ and $m = n$.

Take the counterposition of what we just proved, $m \neq n \implies A_m \cap A_n = \varnothing$.

Let $z \in (0, 1]$, there exists some $a \in A$ such that $z \in [x]$. That is, $z = x + r$ for some $r \in \mathbb{Q} \cap (-1, 1]$. There must exist some $m \in \mathbb{N}$ such that $r_m = r$, and consequently, $z \in A_m$.

Therefore, $(0, 1] \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq (-1, 2]$ (the second relation is obvious). Moreover,

$$\mu((0,1]) \leq \mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \mu((-1,2]) = \mu((-1,0]) + \mu((0,1]) + \mu((1,2]) = 3\mu((0,1]) \tag{6}$$

Note that we just proved $\bigcup_{n\in\mathbb{N}} A_n$ is a disjoint union, hence,

$$\mu((0,1]) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3\mu((0,1]) \tag{7}$$

$$\implies ((0,1])\mu \leq \sum_{n=1}^{\infty} \mu(A + r_n) \leq 3\mu((0,1]) \tag{8}$$

$$\implies \mu((0,1]) \leq \sum_{n=1}^{\infty} \mu(A) \leq 3\mu((0,1]) \tag{9}$$

Since $\mu((0, 1])$ is finite, the only value $\mu(A)$ can take is zero, and $\mu(I) = 0$ as well. Consequently, for any set $S \in \mathcal{P}(\mathbb{R})$, if $S \subseteq I$, then $\mu(S) \leq \mu(I)$ and $\mu(S) = 0$. Otherwise, let $l = \lfloor \inf(S) \rfloor$ and $u = \lceil \sup(S) \rceil$.

$$I \subseteq S \subseteq \bigcup_{n=l}^{u} (n, n+1] \tag{10}$$

Therefore,

$$0 \leq \mu(S) \leq \sum_{n=l}^{u} \mu(n + (0, 1]) = \sum_{n=l}^{u} \mu((0, 1]) = 0 \tag{11}$$

It's shown that $\mu(S) = 0$ for every $S \subseteq \mathcal{P}(\mathbb{R})$.

This leads to a contradiction to the first property required ($\mu((a, b]) = b - a$). $\blacksquare$