Lecture Notes
MATH205A: Real Analysis I (Autumn 2020)
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1 Measures

1.1 Motivation

Motivation of this course is to define a notion of *length* on subsets of $\mathbb{R}$ such that

1. $length([a, b]) = b - a$.
2. (countable additivity) $length(\bigcup^\infty A_i) = \sum^\infty length(A_i)$ where A_i's are disjoint.
3. (translation invariance) for all $a \in \mathbb{R}$, $length(A + a) = length(A)$.

Fact 1.1. it is impossible to construct such length for all subsets of $\mathbb{R}$.

Proof. This proof shows it is impossible to construct a notion of length on $[0, 1]$ with desired properties.

For $x, y \in [0, 1]$, define an equivalence relation as $x \sim y \iff x - y \in \mathbb{Q}$. By the axiom of choice, we may construct a set A containing exactly one element from each equivalence class of $x \in [0, 1]$. Obviously, $A \subseteq [0, 1]$.

For each $r \in [-1, 1] \cap \mathbb{Q}$, let $A_r := A + r$, and $A_r \subseteq [-1, 2]$. By translation invariance, $length(A_r) = length(A)$. Note that for any $y \in [0, 1]$, there exists some $x \in A$ such that $x \sim y$, therefore, $y \in A_{y-x} \subseteq \bigcup_r A_r$. Hence, $[0, 1] \subseteq \bigcup_r A_r$.

If the notion of length satisfies countable additivity, $length(\bigcup_r A_r)$ is either zero or infinity, which leads to a contradiction. ■

Lebesgue's Resolution: we only defines length for a subset of $\mathcal{P}(\mathbb{R})$, which contains *everything that may ever arrive in practice*, i.e., σ-algebras.

1.2 Algebras and σ-algebra

Definition 1.1. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called an **algebra** if

1. $X \in \mathcal{A}$,

2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

3. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Consequently: (1) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$; (2) $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_i A_i, \bigcap_i A_i \in \mathcal{A}$ (easily shown by induction); (3) $\varnothing \in \mathcal{A}$.

Definition 1.2. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called a σ-**algebra** if

1. $X \in \mathcal{A}$,

2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

3. $A_1, A_2 \ldots, \in \mathcal{A}, \implies \bigcup_i^\infty A_i \in \mathcal{A}$.

Example 1.1 (trivial examples)**.** The power set of X is a σ-algebra on X; $\{\varnothing, X\}$ is a σ-algebra on X.

Example 1.2 (finite/co-finite algebra)**.** Let X be an infinite set and $\mathcal{A}$ be the collection of subsets A such that either A is finite or A^c is finite. $\mathcal{A}$ is an algebra.

Proof. $X \in \mathcal{A}$ since $X^c = \varnothing$ is finite. For a $X \in \mathcal{A}$, if X is finite, then $X^c \in \mathcal{A}$. If X is infinite, X^c is finite and $X^c \in \mathcal{A}$. Let $A, B \in \mathcal{A}$, if both A and B are finite, $A \cup B$ is finite and in $\mathcal{A}$. If A is finite and B is co-finite, then $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. If both A and B are co-finite, $(A \cup B)^c$ is finite so that $A \cup B \in \mathcal{A}$. ■

Note the $\mathcal{A}$ is not a σ-algebra if X is infinite: take distinct points $x_1, x_2, \cdots \in \mathcal{A}$, then the union of them is neither finite or co-finite, and therefore not in $\mathcal{A}$.

Example 1.3 (countable/co-countable σ-algebra)**.** The collection of subsets $A \subseteq X$, such that either A is countable or A^c is countable, forms a σ-algebra.

Example 1.4. Let $X = \mathbb{R}$ and $\mathcal{A}$ be the collection of all finite disjoint unions of half-open intervals (i.e., sets like $(a, b], (-\infty, b], (a, \infty)$), $\mathcal{A}$ is an algebra. (Not working for open intervals).

Example 1.5 (counter example)**.** Let X be an infinite set, $\mathcal{A}$ be the collection of finite subsets of X. Then, $\mathcal{A}$ is not an algebra.

Proposition 1.1. Let X be a set and $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ be an arbitrary (not necessarily countable) collection of σ-algebras, then $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i$ is a σ-algebra.

Proof. Since $X \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ ■

Corollary 1.1. Let X be a set, and $\mathcal{P}$ is an arbitrary collection of subsets of X, then $\exists!$ smallest σ-algebra $\mathcal{A}$ containing $\mathcal{P}$. That is, for any σ-algebra $\mathcal{B} \supseteq \mathcal{P}$, $\mathcal{A} \subseteq \mathcal{B}$. $\mathcal{A}$ is defined as the σ-algebra **generated by** $\mathcal{P}$, denoted as $\sigma(\mathcal{P})$.

Proof. For any $\mathcal{P}$, the power set of X is obviously a σ-algebra containing $\mathcal{P}$. Then we can take $\mathcal{A}$ as the intersection of all σ-algebras containing $\mathcal{P}$. ■

1.3 Borel σ-algebra

Definition 1.3. The **Borel σ-algebra** of $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the set of open intervals in $\mathbb{R}$.

Fact 1.2. $\mathcal{B}(\mathbb{R})$ is generated by the collection of all closed intervals as well.

Proof. Let $\mathcal{F}$ denote the σ-algebra generated by all closed intervals. Any open interval can be written as a countable union of closed sets: $(a, b) = \bigcup_{n=1}^{\infty}[a + 1/n, b - 1/n]$, therefore $(a, b) \in \mathcal{F}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

Similarly, $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$, hence $\mathcal{B}(\mathbb{R})$ is a σ-algebra contains all closed sets. Therefore, $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$. ∎

Fact 1.3. $\mathcal{B}(\mathbb{R})$ is generated by

1. all open sets,
2. all closed sets,
3. all half-open intervals.

Example 1.6 (counter example)**.** $\mathcal{B}(\mathbb{R})$ is not generated by the collection of singletons.

Proof. ∎

Definition 1.4. The Borel algebra of $\mathbb{R}^d$, $\mathcal{B}(\mathbb{R}^d)$, is the σ-algebra generated by

1. all open sets in $\mathbb{R}^d$,
2. all closed sets in $\mathbb{R}^d$,
3. all closed cubes (regions) in $\mathbb{R}^d$: $\prod_{i=1}^{d}[a_i, b_i]$.

1.4 Measures

Definition 1.5. For a set X and a σ-algebra $\mathcal{A}$ of X, the pair $(X, \mathcal{A})$ is called a **measurable space**.

Definition 1.6. A **measure** μ on a measurable space $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$,
2. $\mu(\cup_i^{\infty} A_i) = \sum_i^{\infty} \mu(A_i)$ for disjoint sequence (A_i)

For now, we don't require the translation invariance property.

The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Example 1.7 (counting measure)**.**

Example 1.8 (point-mass measure)**.**

Proposition 1.2. A measure μ possesses the following basic properties:

1. (Monotonicity) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2. (Sub-additivity) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
3. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing set, let $\cup_{i=1}^{\infty} A_i$ denoted $A_i \nearrow A$, $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.
4. If $A_1 \searrow A \equiv \cap_{i=1}^{\infty} A_i$, and there exists $\mu(A_i) < \infty$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Proof. ■

Example 1.9 (counter example)**.** Let $X = \mathbb{Z}$, $\mathcal{A} = 2^{\mathbb{Z}}$ and μ be the counting measure. Define $A_i = \{i, i+1, \dots\}$, then $A_i \searrow A = \varnothing$, but $\lim_{n\to\infty} \mu(A_n) = \infty \neq \mu(\varnothing)$.

1.5 Outer Measure

Definition 1.7. Let X be a set, $\mu^* : 2^X \to [0, \infty]$ is an **outer measure** if

1. $\mu^*(\varnothing) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B$.
3. (countable sub-additivity) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Key difference between outer measure and measure:

1. Outer measure does not require countable additivity,
2. outer measure is defined on 2^X instead of a σ-algebra .

Example 1.10.

1.6 Lebesgue Measure on $\mathbb{R}$

Definition 1.8. Let $A \subseteq \mathbb{R}$, define the **Lebesgue outer measure**:

$$\lambda^*(A) = \inf \left\{ \sum_{i\in\mathbb{N}} b_i - a_i : A \subseteq \bigcup_{i\in\mathbb{N}} (a_i, b_i) \right\} \tag{1}$$

The Lebesgue outer measure of a set A is simply in the infimum of total lengths (the conventional notion of length) of open intervals cover A.

Proposition 1.3. The Lebesgue measure satisfies the following properties:

1. λ^* is an outer measure on $\mathbb{R}$,
2. $\lambda^*([a, b]) = b - a$ for all $a < b$.

Proof. (1.1) $\lambda^*(\varnothing) = 0$ since $(-\varepsilon, \varepsilon)$ covers $\varnothing$ for arbitrarily small ε.

(1.2) Let $A \subseteq B$, Ω_A and Ω_B be collection of sequences of open intervals covering A and B respectively. Then, any cover of B must be a cover of A, that is, $\Omega_A \subseteq \Omega_B$. Therefore, $\lambda^*(A) \leq \lambda^*(B)$.

(1.3) Let $A_1, A_2, \cdots \subseteq \mathbb{R}$ and $A = \bigcup_{i=1}^{\infty} A_i$. For all i, we may find (a_{ij}, b_{ij}) covers A_i such that

$$\sum_{j=1}^{\infty}(b_{ij} - b_{ij}) \leq \lambda^*(A_i) + \varepsilon 2^{-i} \tag{2}$$

Also, $\{(a_{ij}, b_{ij})\}_{i,j}$ is a countable union of open intervals that covers A.

$$\lambda^*(A) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(b_{ij} - a_{ij}) \tag{3}$$

$$\leq \sum_{i=1}^{\infty}(\lambda^*(A_i) + \varepsilon 2^{-i}) \tag{4}$$

$$= \sum_{i=1}^{\infty}\lambda^*(A_i) + \varepsilon \tag{5}$$

Therefore, $\lambda^*(A) \leq \sum_{i=1}^{\infty}\lambda^*(A_i)$.

(2) Note that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$ for all $\varepsilon > 0$. Therefore,

$$\lambda^*([a, b]) \leq \inf_{\varepsilon > 0} \lambda^*(a - \varepsilon, b + \varepsilon) = b - a \tag{6}$$

Now show $\lambda^*([a, b]) \geq b - a$. We want to show that $\sum_{i=1}^{\infty}(b_i - a_i) \geq b - a$ for all possible covering of $[a, b]$, which implies the infimum of them is at least $b - a$.

Take an arbitrary covering $\{(a_i, b_i)\}_i$ of $[a, b]$. Since $[a, b]$ is compact, there exists a finite covering $[a, b] \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$ (reindexed), it suffices to show the finite sum $\sum_{i=1}^{\infty}(b_i - a_i) \geq b - a$.

(1) We firstly define an *interval* to be any open, closed or half-open intervals. The *length* of an interval is the difference between two end points.

Note that if an interval I contains a finite collection of disjoint sub-intervals, then the length of I is at least the sum of lengths of sub-intervals. The equality holds when I is exactly finite union of disjoint sub-intervals.

(2) Suppose $[a, b] \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$, let $I_i = [a, b] \cap (a_i, b_i)$. Easy to verify that the length of $I_i \leq$ length of $(a_i, b_i) = b_i - a_i$. Moreover, $\bigcup_{i=1}^{n} I_i = [a, b] \cup \bigcup_{i=1}^{n}(a_i, b_i) = [a, b]$.

(3) For all i, define $I_i' = I_i \backslash (I_1 \cup I_2 \cup \cdots \cup I_{i-1})$. This procedure allows us to express $[a, b]$ as a finite union of disjoint sub-intervals: $[a, b] = \bigcup_{i=1}^{n} I_i'$. Each I_i' is a finite union of disjoint intervals as well, the conventional notion of I_i' is well-defined. Then $b - a =$ sum of lengths of I_i'.

However, $\ell(I_i') \leq \ell(I_i) \leq b_i - a_i$ and sum of lengths of $I_i' \leq$ sum of lengths of $I_i \leq \sum_{i=1}^{n} b_i - a_i$. Therefore, $b - a \leq \sum_{i=1}^{n} b_i - a_i \leq \sum_{i=1}^{\infty} b_i - a_i$. Hence, $b - a = \sum_{i=1}^{\infty} b_i - a_i$ and $\lambda^*[a, b] = b - a$ consequently. ■

1.7 Construct Lebesgue Measure

Definition 1.9. Let X be a set with outer measure μ^*. A set $B \subseteq X$ is μ^*-**measurable** if

$$\forall A \subseteq X, \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \tag{7}$$

Theorem 1.1. For any set X with outer measure μ^* on it, let $\mathcal{M}_{\mu^*}$ denote the set of all μ^*-**measurable** sets. Then, $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ (μ^* restricted to $\mathcal{M}_{\mu^*}$) is a measure.

Proof. To show B is μ^*-measurable, it suffices to show that $\forall A \subseteq X, \mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$, because the opposite inequality always holds by sub-additivity.

(1.1) Let $A \subseteq X$, $\mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^c) = \mu^*(A \cap \varnothing^c) = \mu^*(A)$, therefore, $\varnothing \in \mathcal{M}_{\mu^*}$.

(1.2) Let $A \subseteq X$ and $B \in \mathcal{M}_{\mu^*}$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap (B^c)^c) + \mu^*(A \cap B^c)$. Hence, $B^c \in \mathcal{M}_{\mu^*}$.

(1.3.1) Let $B_1, B_2 \in \mathcal{M}_{\mu^*}$, we are going to show $B_1 \cup B_2 \in \mathcal{M}_{\mu^*}$. Fix any $A \subseteq X$,

$$\begin{aligned} \mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c) && (8) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) && (9) \end{aligned}$$

Moreover,

$$\mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1^c \cap B_2^c) \tag{10}$$

Therefore,

$$\begin{aligned} \mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) && (11) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \text{ since } B_2 \in \mathcal{M}_{\mu^*} && (12) \\ &= \mu^*(A) \text{ since } B_1 \in \mathcal{M}_{\mu^*} && (13) \end{aligned}$$

Therefore, $\mathcal{M}_{\mu^*}$ is an algebra.

(1.3.2) Now show that $\mathcal{M}_{\mu^*}$ is a σ-algebra . For any sequence of sets $A_i \in \mathcal{M}_{\mu^*}$, we can define $B_n := A_n \setminus \cup_{j=1}^{i-1} A_j$ such that $\cup B_i = \cup A_i$. Therefore, it is suffices to show $\mathcal{M}_{\mu^*}$ is closed under countable disjoint unions.

We are going to show the union $\cup B_i$ is μ^*-measurable for any disjoint sequence of μ^*-measurable B_i's.

Claim: let $A \subseteq X$, $\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^n B_i)^c)$. The claim can be proved by induction on n.

When $n = 1$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ because B_1 is μ^*-measurable.

Suppose the claim holds for n, then

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap (\cup_{i=1}^n B_i)^c \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^n B_i)^c \cap B_{n+1}^c) \tag{14}$$

because $B_{n+1} \in \mathcal{M}_{\mu^*}$. Moreover, since all B_i's are disjoint, $B_{n+1} \subseteq B_i^c$ for all i. Hence,

$$B_{n+1} \subseteq \cap_{i=1}^n B_i^c = (\cup_{i=1}^n B_i)^c \tag{15}$$

Also,

$$(\cup_{i=1}^n B_i)^c \cap B_{n+1}^c = \cap_{i=1}^{n+1} B_i^c \tag{16}$$

Consequently,

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^{n+1} B_i)^c) \tag{17}$$

Hence,

$$\begin{aligned} \mu^*(A) &= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^n B_i^c)) && (18) \\ &\geq \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^\infty B_i^c)) && (19) \\ &= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) && (20) \end{aligned}$$

Take $n \to \infty$

$$\begin{aligned} \mu^*(A) &\geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) && (21) \\ &\geq \mu^*(A \cap \cup_{i=1}^\infty B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) && (22) \end{aligned}$$

Therefore, $\cup_{i=1}^\infty B_i$ is μ^*-measurable.

(2) Let $B_1, B_2, \ldots$ be a sequence of disjoint sets from $\mathcal{M}_{\mu^*}$. Using the above fact and take $A = \cup_{i=1}^\infty B_i$,

$$\mu^*(A) \geq \mu^*(\cup_{i=1}^\infty B_i) + \mu^*(\varnothing) = \mu^*(\cup_{i=1}^\infty B_i) \tag{23}$$

The opposite inequality holds by sub-additivity. Therefore, μ^* is a measure on $\mathcal{M}_{\mu^*}$. ■

Definition 1.10. Let λ^* be the Lebesgue outer measure on $\mathbb{R}$, then the collection $\mathcal{M}_{\lambda^*}$ of λ^*-measurable sets is called the **Lebesgue σ-algebra** . The restriction $\lambda = \lambda^*|_{\mathcal{M}_{\lambda^*}}$, which is a measure on $\mathcal{M}_{\lambda^*}$, is called the **Lebesgue measure**. Any set in $\mathcal{M}_{\lambda^*}$ is called a **Lebesgue measurable set**.

Theorem 1.2. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$.

Proof. Note that $\{(-\infty, b] : b \in \mathbb{R}\}$ generates $\mathcal{B}(\mathbb{R})$, it suffices to show $\{(-\infty, b] : b \in \mathbb{R}\} \subseteq \mathcal{M}_{\lambda^*}$.

Let $B = (-\infty, b]$, we are going to show B is λ^*-measurable. Let $A \subseteq \mathbb{R}$ and (a_n, b_n) be a sequence of open intervals covers A. For every $n \in \mathbb{N}$,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n) \cap (-\infty, b]) + \lambda^*((a_n, b_n) \cap (b, \infty)) \tag{24}$$

Three cases follow:

1. $b > b_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.

2. $b_n > b > a_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b]) + \lambda^*((b, b_n]) = b_n - a_n$.

3. $a_n > b$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.

Therefore,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = b_n - a_n \tag{25}$$

By monotonicity and sub-additivity:

$$\begin{aligned}
\lambda^*(A \cap B) + \lambda^*(A \cap B^c) &\leq \lambda^*(\cup(a_n, b_n) \cap B) + \lambda^*(\cup(a_n, b_n) \cap B^c) && (26)\\
&\leq \sum \lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) && (27)\\
&= \sum_{n=1}^{\infty} b_n - a_n && (28)
\end{aligned}$$

Take the infimum of all such covering, we can show

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A) \tag{29}$$

Therefore, B is μ^*-measurable and $\mathcal{M}_{\lambda^*}$ is a σ-algebra containing all such intervals and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$. ■

1.8 Lebesgue Measure on $\mathbb{R}^d$

Definition 1.11. Steps to construct Lebesgue measure on $\mathbb{R}^d$:

1. Define open cubes on $\mathbb{R}^d$ as a Cartesian product of open intervals: $Q := \prod_{i=1}^{d}(a_i, b_i)$. Define Lebesgue outer measure:

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \prod_{i=1}^{d} (b_{ni} - a_{ni}) : A \subseteq \bigcup_{n=1}^{\infty} Q_n \right\} \tag{30}$$

2. Show λ^* is an outer measure and $\lambda^*(Q) = \prod_{i=1}^{d}(b_i - a_i)$.

3. $\mathcal{M}_{\lambda^*}$ is the Lebesgue σ-algebra on $\mathbb{R}^d$. Restricting λ^* on $\mathcal{M}_{\lambda^*}$ defines the Lebesgue measure.

4. Show that any Borel set in $\mathbb{R}^d$ is Lebesgue measurable by showing that there is a generating set of $\mathcal{B}(\mathbb{R}^d)$ is in $\mathcal{M}_{\lambda^*}$.

1.9 Uniqueness of the Lebesgue Measure

The next goal is to prove the uniqueness of Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ subject to the criterion that the measure of any interval (cube) is the volume in the usual sense (product of side lengths).

Theorem 1.3. Let λ be the Lebesgue measure on $\mathbb{R}^d$, then for any Lebesgue measurable set A,

1. $\lambda(A) = \inf\{\lambda(U) : \text{open } U \supseteq A\}$,
2. $\lambda(A) = \sup\{\lambda(K) : \text{compact } K \subseteq A\}$.

Proof. (1.1) WLOG $\lambda(A) < \infty$, by monotonicity, $\lambda(A) \leq \lambda(U)$ for any open cover, $\lambda(A) \leq \inf\{..\}$.

(1.2)Let $\varepsilon > 0$, $\exists$ a sequence of open intervals (R_i) such that

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon \tag{31}$$

Let $U := \cup R_i$ open, hence $\inf\{..\} \leq \lambda(U) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon$. Since this ε can be arbitrarily small, we conclude $\inf\{..\} \leq \lambda(A)$.

(2.1) let A be a Lebesgue measurable set, assume A is bounded, so that $\lambda(A) < \infty$. Then there exists a compact $C \supseteq A$. $C \backslash A$ is Lebesgue measurable as well.

By conclusion of part (1), there exists a open set $U \supseteq C \backslash A$ such that

$$\lambda(C \backslash A) \leq \lambda(U) \leq \lambda(C \backslash A) + \varepsilon \tag{32}$$

Let $K = C \backslash U$, K is compact. Moreover, let $a \in K$, then $a \in C$ and $a \notin U$. Therefore, $a \notin C \backslash A$, it must be $x \in A$. Hence, $K \subseteq A$.

$$\begin{aligned}
\lambda(K) &= \lambda(C \backslash U) && (33)\\
&\geq \lambda(C) - \lambda(U) && (34)\\
&\geq \lambda(C) - (\lambda(C \backslash A) + \varepsilon) && (35)\\
&= \lambda(C) - \lambda(C) + \lambda(A) - \varepsilon && (36)\\
&= \lambda(A) - \varepsilon && (37)
\end{aligned}$$

Take $\varepsilon \to 0$ and $\lambda(A) \leq \sup\{..\}$. By monotonicity, $\lambda(A) \geq \sup\{..\}$.

(2.2) Other cases: suppose A is unbounded and $\lambda(A) > 0$. Take an arbitrary $b < \lambda(A)$. We will show that $\sup\{..\} \geq b$, this will prove that $\lambda(A) \leq \sup\{...\}$.

To show $\sup\{..\} \geq b$, it suffices to show that there exists a compact set $K \subseteq A$ such that $\lambda(K) \geq b$.

Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of compact sets increasing to $\mathbb{R}^d$.

Then $A \cap C_j \uparrow A$ and $\lambda(A \cap C_1) < \infty$, which implies $\lambda(A) = \lim_{j \to \infty} \uparrow \lambda(A \cap C_j)$. Since $b < \lambda(A)$, there exists j such that $\lambda(A \cap C_j) \geq b$, where $A \cap C_j$ is compact. Hence, $b \leq \sup\{..\}$ and $\lambda(A) \leq \sup\{..\}$. $\lambda(A) \geq \sup\{..\}$ holds by monotonicity.

When $\lambda(A) = 0$, $0 \leq \lambda(K)$ for all K so that $0 \leq \sup\{..\}$. The opposite inequality holds by monotonicity. ∎

Lemma 1.1. For each $k \in \mathbb{Z}$, define **dyadic cubes** in $\mathbb{R}^d$ as set in the following form:

$$\prod_{i=1}^{d} [j_i 2^{-k}, (j_i + 1)2^{-k}) \tag{38}$$

where $j_i \in \mathbb{Z}$ for every i. Let $\mathcal{D}$ denote the collection of dyadic cubes.

Then, any open set $U \subseteq \mathbb{R}^d$ can be expressed as a countable union of some members of $\mathcal{D}$.

A dyadic cube of side length 2^{-k} has a unique parent of side length 2^{-k+1} and a unique grandparent with side length 2^{-k+2}.

Proof. Given open set U, let $\mathcal{D}_U$ denote the set of all dyadic half open cubes D such that $D \subseteq U$ but the parent of U does not fully contain U.

Claim 1: $U = \bigcup_{D \in \mathcal{D}_U} D$. Obviously, $\bigcup_{D \in \mathcal{D}_U} \subseteq U$. To show the converse, take any $x \in U$, since U is open, there exists $D \in \mathcal{D}_U$ such that $x \in D \subseteq U$.

Let D_0 be the earliest ancestor of D such that $x \in D_0 \subseteq U$. Obviously, $D_0 \in \mathcal{D}_U$. Therefore, $U \subseteq \bigcup_{D \in \mathcal{D}_U} D$.

Claim 2: Two dyadic cubes can overlap if and only if one is the ancestor of the other. By construction, dyadic cubes in $\mathcal{D}_U$ are disjoint.

Claim 3: $\mathcal{D}_U$ is countable because $\mathcal{D}$ is itself countable. ■

Proposition 1.4. Lebesgue measure is the only measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which assigns the *correct volume* to any d-dimensional intervals or even any d-dimensional dyadic cube.

Proof. Let λ denote the Lebesgue measure, let μ be another measure satisfying the desired property.

By lemma, for all open set U, $\mu(U) = \sum_{j=1}^{\infty} \mu(D_j) = \sum_{j=1}^{\infty} \lambda(D_j) = \lambda(U)$, where $\{D_j\}$ is a collection of disjoint dyadic cubes contains with union U. Therefore, $\lambda(A) = \mu(A)$ for all open Borel set A.

Let $A \in \mathcal{B}(\mathbb{R}^d)$, let open $U \supseteq A$, then $\mu(A) \leq \mu(U) = \lambda(U)$ for all U. Taking the infimum over all $U \supseteq A$, we conclude $\mu(A) \leq \lambda(A)$ for all Borel set A.

Next, take any bounded Borel set A, let V be a bounded open set containing A. Then,

$$\mu(V) = \mu(A) + \mu(V \backslash A) \tag{39}$$

$$\leq \lambda(A) + \lambda(V \backslash A) \tag{40}$$

$$= \lambda(V) \tag{41}$$

But we also know that $\mu(V) = \lambda(V)$ since V is open, the inequality holds as equality. Moreover, the previous conclusion implies $\mu(A) \leq \lambda(A)$ and $\mu(V \backslash A) \leq \lambda(V \backslash A)$, it must be $\mu(A) = \lambda(A)$ and $\mu(V \backslash A) = \lambda(V \backslash A)$. Therefore, $\mu(A) = \lambda(A)$ for all bounded Borel set A.

Lastly, any Borel set can be written as a a countable disjoint union of bounded Borel set, therefore, $\mu(A) = \lambda(A)$ for all Borel set A. ■

Proposition 1.5. The Lebesgue outer measure on $\mathbb{R}^d$ is translation invariant. In particular, Lebesgue measure is translation invariant and any translation of Lebesgue measurable set is Lebesgue measurable.

Proof. $\lambda^*(A+x) = \lambda^*(A)$ follows the definition of λ^*: translate all covering intervals by $+x$ and the volumes of these intervals stay the same. Since λ is simply the restriction of λ^* on Lebesgue measurable sets, λ is translation invariant as well.

Now take Lebesgue measurable B, for all$A \subseteq \mathbb{R}^d$:

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c) \tag{42}$$
$$\implies \lambda^*(A - x) = \lambda^*((A - x) \cap B) + \lambda^*((A - x) \cap B^c) \tag{43}$$

Note that

$$(A - x) + x = A \tag{44}$$
$$(A - x) \cap B + x = A \cap (B + x) \tag{45}$$
$$(A - x) \cap B^c + x = A \cap (B + x)^c \tag{46}$$

By translational invariance of λ^*,

$$\lambda^*(A) = \lambda^*(A \cap (B + x)) + \lambda^*(A \cap (B + x)^c) \tag{47}$$

Therefore, $B + x$ is Lebesgue measurable as well. ■

Theorem 1.4. Let μ be a non-zero measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is finite on bounded Borel sets and translation invariant. Then, $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$, where λ is the Lebesgue measure.

Remark 1.1. Borel σ-algebra is closed under translation.

Proof. Let $c = \mu([0,1)^d) \in (0, \infty)$. Then $[0,1)^d$ is the disjoint union of 2^{dk} half-open dyadic intervals with side length 2^{-k}. All of these sub-intervals have the same μ since μ is translation invariant. Therefore, for every dyadic sub-interval with side length 2^{-k}, $\mu(D) = 2^{-dk}c$.

Let $\nu(A) = \frac{1}{c}\mu(A)$, then ν is a measure that is finite on bounded sets and agrees with λ on all half-open dyadic cubes. By the previous proposition, λ is the only measure assign correct volumes to dyadic cubes, therefore, $\nu = \lambda$. ■

Theorem 1.5. Under the axiom of choice, there exists a non-Lebesgue subset of $\mathbb{R}$.

Proof. Todo. ■

2 Functions

2.1 Measurable Functions

Definition 2.1. A function $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is **measurable** if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

In this course, we mainly consider functions with extended-$\mathbb{R}$ as codomain: $Y = [-\infty, \infty]$, denoted as $\mathbb{R}^*$.

Definition 2.2. The σ-algebra on $\mathbb{R}^*$ is defined to be the σ-algebra generated by $\mathcal{B}(\mathbb{R}) \cup \{-\infty\} \cup \{\infty\}$.

Proposition 2.1.

$$\begin{aligned}\sigma(\mathcal{B}(\mathbb{R}) \cup \{-\infty\} \cup \{\infty\}) = \mathcal{B}(\mathbb{R}) &\cup \{B \cup \{\infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (1)\\ &\cup \{B \cup \{-\infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (2)\\ &\cup \{B \cup \{-\infty, \infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (3)\end{aligned}$$

Proposition 2.2. Equivalently, f is measurable if for every $t \in \mathbb{R}$,

$$\{x \in X : f(x) \leq t\} \in \mathcal{A} \quad (4)$$
$$\{x \in X : f(x) < t\} \in \mathcal{A} \quad (5)$$
$$\{x \in X : f(x) \geq t\} \in \mathcal{A} \quad (6)$$
$$\{x \in X : f(x) > t\} \in \mathcal{A} \quad (7)$$

More generally, to determine the measurability of $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$, we only need to check whether $f^{-1}(C) \in \mathcal{A}$ for all C in a generating collection $\mathcal{C}$ of $\mathcal{B}$. The converse holds true trivially.

Proof. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be two measurable spaces, let $\mathcal{C}$ be a collection of subsets of Y generates $\mathcal{B}$.

($\Longrightarrow$) Let f be a measurable function, then for every $C \in \mathcal{C} \subseteq \mathcal{B}$. Obviously, $f^{-1}(C) \in \mathcal{A}$ by definition.

($\Longleftarrow$) Suppose $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$. Define

$$\mathcal{B}_0 := \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{A}\} \supseteq \mathcal{C} \quad (8)$$

It's easy to check $\mathcal{B}_0$ is in fact a σ-algebra : $f^{-1}(\varnothing) = \varnothing \in \mathcal{A}$, $f^{-1}(B^c) = (f^{-1}(B))^c$, and $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$. Therefore, $\mathcal{B} \subseteq \mathcal{B}_0$ and all $B \in \mathcal{B}$ satisfies $f^{-1}(B) \in \mathcal{A}$. ∎

Example 2.1. $f(x) = \mathbb{1}\{x \in \mathbb{Q}\}$ is measurable.

2.2 Simple Functions

Definition 2.3. A function $f : (X, \mathcal{A}) \to (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$ is called **simple** if there exists finitely many disjoint sets $A_1, \ldots, A_n$ and real numbers $a_1, \ldots, a_n$ such that

$$f(x) = \begin{cases} a_i & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i \forall i \in [n] \end{cases} \quad (9)$$

Let $\mathbb{S}$ denote the set of simple functions, and $\mathbb{S}^+$ denote the set of non-negative simple functions.

Proposition 2.3. All simple functions are measurable.

Proof. For any subset of $\mathbb{R}^*$, the pre-image is either X or a union of some (potentially none) A_i's. ■

2.3 Properties of Measurable Functions

Example 2.2. Let $f : \mathbb{R}^d \to \mathbb{R}$, then all of the following functions are measurable:

$$f(x, y) = x + y \tag{10}$$

$$f(x, y) = \max\{x, y\} \equiv x \vee y \tag{11}$$

$$f(x, y) = \min\{x, y\} \equiv x \wedge y \tag{12}$$

$$f(x, y) = x - y \tag{13}$$

$$f(x, y) = \alpha x \quad \alpha \in \mathbb{R} \tag{14}$$

Proposition 2.4 (Component-wise Measurable Functions)**.** Let $f, g : (X, \mathcal{A}) \to (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$ be measurable, let $h(x) = (f(x), g(x)) \in \mathbb{R}^{*2}$, then f is measurable.

Proof.

$$h^{-1}([-\infty, t] \times [-\infty, s]) = f^{-1}([-\infty, t]) \cap g^{-1}([-\infty, s]) \in \mathcal{A} \tag{15}$$

And, $\mathcal{B}(\mathbb{R}^*)$ can be generated by sets with forms $[-\infty, t] \times [-\infty, s]$. ■

Proposition 2.5 (Composite of Measurable Functions)**.** Let $(X, \mathcal{A}), (Y, \mathcal{B}), (Z, \mathcal{C})$ be measurable spaces, let $f : X \to Y$ and $g : Y \to Z$ be measurable functions. Then, the composite $g \circ f : X \to Z$ is measurable.

Corollary 2.1. Let $f, g : X \to \mathbb{R}$ be measurable functions, then $f + g$, $f - g$, $\max\{f, g\}$, and $\min\{f, g\}$ are all measurable.

Proof. $f+g$ and $f-g$ can be written as the composition of $h_1(x) = (f(x), g(x))$ and $h_2(x, y) = x \pm y$, which are all measurable.

$f \vee g$ and $f \wedge g$ are measurable as special cases of next proposition. ■

Proposition 2.6. Let $f_1, f_2, \ldots$ be a sequence of measurable maps from $(X, \mathcal{A}) \to (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$, then $\sup_n f_n$ and $\inf_n f_n$ are measurable.

Proof. Note $\{x \in X : \sup_n f_n \leq t\} = \bigcup_{n=1}^{\infty} \{x \in X : f_n \leq t\} \in \mathcal{A}$ for every t, therefore the supremum is measurable. ■

Corollary 2.2. $\limsup f_n$ and $\liminf f_n$ are measurable.

Proof. Let $g_k = \sup_{n \geq k} f_n$, g_k is measurable. $\limsup f_n = \inf_k g_k$ is measurable as well. Similar proof for the measurability of $\liminf f_n$. ■

Proposition 2.7. Let f and g be $\mathbb{R}^*$-valued measurable functions. Then sets

$$\{x \in A : f(x) < g(x)\}, \{x \in A : f(x) \leq g(x)\} \tag{16}$$

are measurable.

Proof.

$$\{x \in A : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} (\{x \in A : f(x) < r\} \cap \{x \in A : r < g(x)\}) \tag{17}$$

■

Corollary 2.3. Let $u, v : X \to \mathbb{R}^*$ be a measurable functions, then $\{x \in X : u(x) = v(x)\}$ is measurable.

Proof. Note that $\{x \in X : u(x) = v(x)\} = \{x \in X : u(x) \leq v(x)\} \cap \{x \in X : u(x) \geq v(x)\}$. ■

Corollary 2.4. Let $\{f_n\}$ be a sequence of measurable functions from $(X, \mathcal{A}) \to (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$. Then,

$$\{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\} \tag{18}$$

is measurable.

Proof. Note that $\{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\} = \{x \in X : \liminf f_n(x) = \limsup f_n(x)\}$, the result follows from previous lemma. ■

Corollary 2.5. If $\{f_n\}$ is a sequence of measurable functions such that $\lim f_n(x)$ exists for all $x \in X$, then $\lim f_n$ is a measurable function on $(X, \mathcal{A})$.

Proof. In this case, $\{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\} = X$, and $\lim f_n = \liminf f_n$ on X. ■

Corollary 2.6. If $\{f_n\}$ is a sequence of measurable function from X to $[0, \infty]$, then $\sum_{n=1}^{\infty} f_n$ is measurable.

Proof. Follows the previous corollary directly: define $g_k = \sum_{n=1}^{k} f_n$ and $\lim_{k \to \infty} g_k = \sum_{n=1}^{\infty} f_n$. ■

3 Integrals

3.1 Integrating Simple Functions

Definition 3.1. Let $f \in \mathbb{S}^+$ with representation $\{(A_i, a_i)\}_{i=1}^{n}$. WLOG, $\bigcup_{i=1}^{n} A_i = X$. Then, define

$$\int_X f \, d\mu := \sum_{i=1}^{n} a_i \mu(A_i) \tag{1}$$

Proposition 3.1. The notion of integral on simple functions is well defined. Specifically, let $\{(A_i, a_i)\}_{i=1}^{n}$ and $\{(B_j, b_j)\}_{j=1}^{m}$ be any two representations of f, $\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j)$.

Proof. First note that $\{A_i \cap B_j\}_{i,j}$ are disjoint sets with union X. Moreover, for any i, j, if $A_i \cap B_j \neq \varnothing$, take some $x \in A_i \cap B_j$, $f(x) = a_i = b_j$. Therefore, $a_i\mu(A_i \cap B_j) = b_i\mu(A_i \cap B_j)$ since either $a_i = b_j$ or $\mu(A_i \cap B_j) = \mu(\varnothing) = 0$.

$$\sum_{i=1}^{n} a_i\mu(A_i) = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(A_i \cap B_j) \tag{2}$$

$$= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(A_i \cap B_j) \tag{3}$$

$$= \sum_{j=1}^{m} b_j\mu(B_j) \tag{4}$$

■

3.2 Integrating Measurable Functions

Definition 3.2. For a non-negative measurable function $f : X \to [0, \infty]$, define its Lebesgue integral as

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \text{ is a non-negative simple function such that } g \leq f \right\} \tag{5}$$

For any measurable $f : X \to [-\infty, \infty]$, let

$$f^+(x) = \max\{f(x), 0\} \tag{6}$$

$$f^-(x) = -\min\{f(x), 0\} \tag{7}$$

So that $f = f^+ - f^-$, and f is measurable if and only if both f^+ and f^- are measurable.

If at least one of $\int f^+ \, d\mu$, $\int f^- \, d\mu$ is finite, the integral of f exists (well-defined) and is defined as

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu \tag{8}$$

If both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite, f is said to be **integrable**.

3.3 Properties of Integral of Non-negative Simple Functions

Proposition 3.2 (Linearity)**.** If f, g are non-negative simple functions, then

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu \tag{9}$$

Moreover, for any $\alpha \geq 0$,

$$\int \alpha f \, d\mu = \alpha \int f \, d\mu \tag{10}$$

Proof. Let f and g be simple functions represented by $\{(A_i, a_i)\}_{i=1}^n$ and $\{(B_j, b_j)\}_{j=1}^m$. WLOG, $\cup A_i = \cup B_j = X$. Then $f+g$ is a simple function with representation $\{(A_i \cap B_j, a_i + b_j)\}_{i,j}$, where $\cup_{i,j} A_i \cap B_j = X$. ■

Proposition 3.3. Let f, g be non-negative simple functions with $f \geq g$ everywhere. Then $\int f \, d\mu \geq \int g \, d\mu$.

Proof. Let f and g be simple functions represented by $\{(A_i, a_i)\}_{i=1}^n$ and $\{(B_j, b_j)\}_{j=1}^m$.

Claim: $a_i \mu(A_i \cap B_j) \geq b_j \mu(A_i \cap B_j)$ for every (i, j). If $A_i \cap B_j \neq \varnothing$, then taking some $x \in A_i \cap B_j$ implies $a_i \geq b_j$. If $A_i \cap B_j = \varnothing$, the equality holds trivially.

Note that $\int f$ and $\int g$ can be written as $\sum_{i,j} a_i \mu(A_i \cap B_j)$ and $\sum_{i,j} b_j \mu(A_i \cap B_j)$ respectively, therefore $\int f \geq \int g$ by the previous claim. ■

Proposition 3.4 (Approximation using Simple Functions)**.** Let $f : X \to [0, \infty]$ be a measurable function. Then there exists an increasing sequence of non-negative simple functions f_n such that $f_n \leq f_{n+1}$ and

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{11}$$

for all x.

Proof. For each n and $1 \leq k \leq n2^n$, let

$$A_{n,k} = \left\{ x \in X : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\} \tag{12}$$

Define

$$f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } x \in A_{n,k} \\ n & \text{otherwise} \end{cases} \tag{13}$$

That is, for a $x \in X$, if $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$ for some k, we take $f_n(x) = \frac{k-1}{2^n}$; if $f(x) \geq n$, we define $f_n(x) = n$. Clearly, each f_n is a simple function.

Claim 1: $f_n \leq f_{n+1}$. Easy to verify.

Claim 2: $\lim_{n\to\infty} f_n(x) = f(x)$. Let $x \in X$, (i) if $f(x) = \infty$, then $f_n(x) = n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = \infty = f(x)$.

(ii) if $f(x) < \infty$, then $\exists n_0$ such that $f(x) < n_0$. For every $n \geq n_0$, $x \in A_{n,k}$ for some k such that $f_n(x) = \frac{k-1}{2^n}$ and $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$. Therefore, for all $n \geq n_0$, $|f_n(x) - f(x)| < \frac{1}{2^n}$, which implies $\lim_{n\to\infty} f_n(x) = f(x)$. ■

Proposition 3.5 (Monotone Convergence 1: $\mathbb{S}_+ \uparrow \mathbb{S}_+$)**.** Let f_n be a sequence of non-negative simple functions that increase to another non-negative simple function f at each point, then

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu \tag{14}$$

Proof. By monotonicity, $f_n \le f$ for all n and $\int f \, d\mu \ge \lim \int f_n \, d\mu$.

Fix $0 < \varepsilon < 1$ and define $g = (1-\varepsilon)f$. Suppose f is represented by (A_i, a_i). Then for every n, i, define

$$A_{n,i} = \{x \in A_i : f_n(x) \ge (1-\varepsilon)a_i\} \tag{15}$$

Define

$$g_n(x) = \begin{cases} (1-\varepsilon)a_i & \text{if } x \in A_{n_i} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

In order to show $\int f \, d\mu \le \lim \int f_n \, d\mu$, we are constructing this g_n satisfying

$$(1-\varepsilon)\int f \, d\mu \le \lim \int g_n \, d\mu \le \lim \int f_n \, d\mu \le \int f \, d\mu \tag{17}$$

where the last equality has been shown above. The equality can then be shown by taking $\varepsilon \to 0$ and using Squeeze theorem. Note that $(1-\varepsilon)\int f \, d\mu \not\le \int g_n \, d\mu$, only the limit does.

By construction, $g_n \le f_n$ and $\int g_n \, d\mu \le \int f_n \, d\mu$ as a result.

$$\begin{aligned}
\lim_n \int f_n \, d\mu &\ge \lim_n g_n \, d\mu && (18)\\
&= \lim_n \sum_{i=1}^{K} (1-\varepsilon)a_i \mu(A_{n,i}) && (19)\\
&= \sum_{i=1}^{K} (1-\varepsilon)a_i \lim_n \mu(A_{n,i}) && (20)\\
&= \sum_{i=1}^{K} (1-\varepsilon)a_i \mu(A_i) \text{ Since for all } i,\ A_{n,i} \uparrow A_i \text{ as } n \to \infty. && (21)\\
&= (1-\varepsilon)\int f \, d\mu && (22)
\end{aligned}$$

Taking $\varepsilon \to 0$ completes the proof. ∎

Proposition 3.6 (Monotone Convergence 2: $\mathbb{S}_+ \uparrow$ Measurable)**.** Let $f : X \to [0, \infty]$ be a measurable function. Let f_n be a sequence of non-negative simple functions such that $f_n \uparrow f$ point-wise. Then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu \tag{23}$$

Proof. The proof follows the previous proposition and the definition of $\int f \, d\mu$. Since $f_n \uparrow f$, $f_n \le f$ and $\int f_n \le \int f$ for all n. $\int f_n$ is a bounded monotone sequence, therefore $\lim \int f_n$ exists and $\le \int f$.

To show the other equality, it suffices to prove $\lim \int f_n \ge \int g$ for any non-negative simple functions $g \le f$.

Define $g_n = \min\{g, f_n\}$, easy to show that $g_n(x) \leq g_{n+1}(x)$.

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \min\{g, f_n\} \tag{24}$$
$$= \min\{g(x), f(x)\} \tag{25}$$
$$= g(x) \tag{26}$$

since $f_n \uparrow f$ and $g \leq f$.

By the previous proposition, $\int g\ d\mu = \lim \int g_n\ d\mu$ since g_n and g are non-negative simple functions. Since $g_n \leq f_n$ everywhere, so $\int g_n\ d\mu \leq \int f_n\ d\mu$. Taking limit on both sides implies $\int g \leq \lim \int f_n$. ■

Proposition 3.7 (Vector Space Properties for Non-negative Integrable Functions)**.** Let $f, g : X \in [0, \infty]$ be integrable (of course, measurable as well) functions and $\alpha \geq 0$. Then

1. $\int f + g\ d\mu = \int f\ d\mu + \int g\ d\mu$.
2. $\int \alpha f\ d\mu = \alpha \int f\ d\mu$.
3. If $f \geq g$ everywhere, then $\int f\ d\mu \geq \int g\ d\mu$.

Proof. We know that there exists sequences of non-negative simple functions f_n and g_n such that $f_n \uparrow f$ and $g_n \uparrow g$. Note that $f_n + g_n$ is a sequence of simple functions increases to $f + g$. Therefore,

$$\int (f+g) d\mu = \lim_{n\to\infty} \int (f_n + g_n)\ d\mu \tag{27}$$
$$= \lim_{n\to\infty} \left(\int f_n\ d\mu + \int g_n\ d\mu \right) \tag{28}$$
$$= \lim_{n\to\infty} \int f_n\ d\mu + \lim_{n\to\infty} \int g_n\ d\mu \tag{29}$$
$$= \int f\ d\mu + \int g\ d\mu \tag{30}$$

Similarly, taking $\alpha f_n \uparrow \alpha f$ leads to the second result.

Finally, if $f \geq g$ everywhere, then

$$\{h \in \mathbb{S}_+ \text{ and } h \leq g\} \subseteq \{h \in \mathbb{S}_+ \text{ and } h \leq f\} \tag{31}$$

Therefore, the supremum of integrals of functions from a larger collection is larger. ■

3.4 Linearity of Lebesgue Integral for Arbitrary Integrable Functions

Theorem 3.1 (Vector Space Property of Integral Functions)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g : X \to \mathbb{R}^*$ be integrable functions, let $\alpha \in \mathbb{R}$. Then, $f + g$ and αf are integrable, and

$$\int f + g d\mu = \int f d\mu + \int g d\mu \tag{32}$$

$$\int \alpha f d\mu = \alpha \int f d\mu \tag{33}$$

Proof. It's easy to check that $(f + g)^+ \leq f^+ + g^+$ and $(f + g)^- \leq f^- + g^-$. By monotonicity, $\int (f + g)^+ d\mu, \int (f + g)^- d\mu < \infty$. Therefore, $f + g$ is integrable.

Moreover, $f + g = f^+ - f^- + g^+ - g^- \iff f + g + f^- + g^- = f^+ + g^+$. We can apply the linearity of non-negative integrable functions to derive the result.

When $\alpha \geq 0$, $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. The proof for cases with $\alpha < 0$ is similar. ■

Corollary 3.1. Let f, g be integrable functions such that $f \geq g$, then $\int f \; d\mu \geq \int g \; d\mu$.

Proof. Let $h = f - g = f + (-1)g \geq 0$, which is integrable by the previous theorem. And $\int h \; d\mu \geq 0$ since its the supremum of integrals for simple functions less than h, which includes the zero function (has zero integral). ■

Lemma 3.1. A function f is integrable <u>if and only if</u> $|f|$ is integrable.

Proof. Note that $|f| = f^+ + f^-$, and $\int f^+ + f^- \; d\mu < \infty$ by the integrability of f. Therefore, $|f|$ is integrable.

Moreover, $|f|^+ = f^+ + f^-$, therefore, the integrability of $|f|$ implies both $\int f^+ \; d\mu$ and $\int f^- \; d\mu$ are finite. ■

Proposition 3.8. All integrable function f satisfies the triangle inequality

$$\left| \int f \; d\mu \right| \leq \int |f| \; d\mu \tag{34}$$

Proof.

$$\begin{aligned}
\left| \int f \; d\mu \right| &= \left| \int f^+ - f^- \; d\mu \right| &(35)\\
&= \left| \int f^+ \; d\mu - \int f^- \; d\mu \right| &(36)\\
&\leq \left| \int f^+ \; d\mu \right| + \left| \int f^- \; d\mu \right| &(37)\\
&= \int f^+ \; d\mu + \int f^- \; d\mu &(38)\\
&= \int |f| \; d\mu &(39)
\end{aligned}$$

■

4 Limit Theorems (i.e., when we can exchange limits and integrals)

Theorem 4.1 (Monotone Convergence Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_n : X \to [0, \infty]$ be a non-decreasing sequence of measurable functions converge to f. Then,

$$\int f \; d\mu = \lim_{n\to\infty} \int f_n \; d\mu \tag{1}$$

Proof. f measurable since $f = \lim_n f_n = \liminf_n f_n$. Moreover, $\int f_n \; d\mu$ is a non-decreasing sequence to the limit $\int f \; d\mu$, therefore $\int f \; d\mu \geq \lim_n \int f_n \; d\mu$.

For each $n \in \mathbb{N}$, there exists a non-decreasing sequence of non-negative simple functions $g_{n,k}$ converges to f_n. Define

$$h_n = \max\{g_{1,n}, g_{2,n}, \ldots, g_{n,n}\} \tag{2}$$

Note that h_n is a non-decreasing sequence since

$$\begin{aligned} h_{n+1} &= \max\{g_{1,n+1}, g_{2,n+1}, \ldots, g_{n+1,n+1}\} &(3)\\ &\geq \max\{g_{1,n+1}, g_{2,n+1}, \ldots, g_{n,n+1}\} &(4)\\ &\geq \max\{g_{1,n}, g_{2,n}, \ldots, g_{n,n}\} = h_n &(5)\end{aligned}$$

Moreover, for any $m \in \mathbb{N}$, for any $n \geq m$,

$$h_n(x) = \max\{g_{1,n}, g_{2,n}, \ldots, g_{n,n}\} \geq g_{m,n} \tag{6}$$

Therefore, by taking the limit $n \to \infty$,

$$\lim_{n\to\infty} h_n(x) \geq \lim_{n\to\infty} g_{m,n} = f_m \tag{7}$$

Taking limit $m \to \infty$ on both sides

$$\begin{aligned} \lim_n h_n(x) = \lim_m \lim_n h_n(x) &\geq \lim_m f_m = f &(8)\\ \implies \int \lim_n h_n(x) \; d\mu &\geq \int f \; d\mu &(9)\end{aligned}$$

Note that, by construction,

$$h_n = \max\{g_{1,n}, g_{2,n}, \ldots, g_{n,n}\} \leq \max\{f_1, \ldots, f_n\} = f_n \tag{10}$$

Therefore,

$$\int \lim_{n\to\infty} f_n(x) \; d\mu \geq \int f \; d\mu \tag{11}$$

■

Corollary 4.1. Let (f_n) be a sequence (not necessarily increasing) non-negative measurable functions, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu \tag{12}$$

Theorem 4.2 (Fatou's Lemma)**.** Let f_n be a sequence of non-negative measurable functions, then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu \tag{13}$$

Proof. Define $g_n = \inf_{k\geq n} f_k$, then g_n is an increasing sequence of non-negative functions. By construction, $\int g_n \, d\mu \leq \inf_{k\geq n} \int f_k \, d\mu$. By MCT,

$$\int \liminf_{n\to\infty} f_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu \tag{14}$$

$$= \lim_{n\to\infty} \int g_n \, d\mu \tag{15}$$

$$\leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k \, d\mu \tag{16}$$

$$= \liminf_{n\to\infty} \int f_n \, d\mu \tag{17}$$

■

Theorem 4.3 (Lebesgue's Dominated Convergence Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let f and f_n be $\mathbb{R}^*$-valued measurable functions on X such that $f_n \to f$ point-wise. If there exists a non-negative integrable function g such that $|f_n| \leq g$ for all n, then, all f and f_n are integrable, moreover,

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu \tag{18}$$

Proof. Since $|f_n| \leq g$, all f_n are integrable. The limit f also satisfies $|f| \leq g$ and is integrable.

For now, assume f_n are $\mathbb{R}$-valued instead of $\mathbb{R}^*$-valued.

Note that $f + g = \lim_{n\to\infty} f_n + g$ is non-negative (because of the dominance) and integrable, by Fatou's lemma

$$\int f + g \, d\mu = \int \liminf f + g \, d\mu \leq \liminf \int f_n + g \, d\mu \tag{19}$$

$$= \liminf \int f_n \, d\mu + \int g \, d\mu \tag{20}$$

$$\implies \int f \, d\mu \leq \liminf \int f_n \, d\mu \tag{21}$$

Similarly, $g - f = \lim_{n\to\infty} g - f_n$ is non-negative and integrable as well, by Fatou's lemma

$$\int g - f \; d\mu = \int \liminf g - f_n \; d\mu \leq \liminf \int g - f_n \; d\mu \tag{22}$$

$$\implies -\int f \; d\mu \leq -\liminf \int f_n \; d\mu \tag{23}$$

$$\implies \int f \; d\mu \geq \limsup \int f_n \; d\mu \tag{24}$$

Also, $\liminf \int f_n \; d\mu \leq \limsup \int f_n \; d\mu$, therefore,

$$\liminf \int f_n \; d\mu \geq \int f \; d\mu \geq \limsup \int f_n \; d\mu \geq \liminf \int f_n \; d\mu \tag{25}$$

$$\implies \int f \; d\mu = \lim \int f_n \; d\mu \tag{26}$$

■

Proposition 4.1 (A Stronger Result)**.** Given assumptions of the dominated convergence theorem, f_n L^1-converges to f.

$$\lim_{n\to\infty} \int |f_n - f| \; d\mu = 0 \tag{27}$$

Proof. Note that $|f_n - f| \to 0$ point-wise, and $|f_n - f| \leq 2g$. The dominated convergence theorem suggests $\lim_{n\to\infty} \int |f_n - f| \; d\mu = \int 0 \; d\mu = 0$. ■

4.1 The Notion of Almost Everywhere

Definition 4.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, a set $N \subseteq X$ (not necessarily measurable) is called **negligible w.r.t.** μ if $N \subseteq A$ for some $A \in \mathcal{A}$ with $\mu(A) = 0$.

A property is said to hold **almost everywhere** w.r.t. μ (denoted as μ-a.e.) if the set on which this property fails is negligible.

Proposition 4.2. Let $f : X \to [0, \infty]$ be an integrable function, then f is finite μ-a.e.

Proof. Let $A := f^{-1}(\infty)$, define $h_n(x) := n\mathbb{1}\{x \in A\}$. Clearly, h_n is a simple function $\leq f$ for every n, by monotonicity, $\int f \; d\mu \leq \int h_n \; d\mu = n\mu(A)$. If $\mu(A) > 0$, taking $n \to \infty$ leads to a contradiction. ■

Alternative Proof. Note: this intuitive proof is non-rigorous. Since $f \geq 0$, let $A := f^{-1}(\infty)$, $\int f \; d\mu \geq \int_A f \; d\mu = \infty\mu(A)$, $\mu(A)$ must be zero. ■

Corollary 4.2. If $f : X \to \mathbb{R}^*$ is integrable w.r.t. μ, then $|f| < \infty$ μ-a.e.

Proof. f is integrable implies both $\int f^+ \; d\mu, \int f^- \; d\mu < \infty$. Then, by the previous proposition, $f^+ < \infty$ except for a negligible set A, and $f^- < \infty$ expect for a negligible set B. Therefore, $|f| = \infty$ on set $A \cup B$, which is negligible as well. ■

Proposition 4.3. Let $f : X \to [0, \infty]$ be measurable, then

$$\int f \; d\mu = 0 \iff f = 0 \; \mu - a.e. \tag{28}$$

Proof. ($\Longleftarrow$) Suppose $f = 0$ a.e., for every simple function $g \leq f$, let (a_i, A_i) be the representation of g.

Suppose $a_i > 0$ for some A_i, then $f(x) \geq a_i > 0$ for all $x \in A_i$, since $f = 0$ a.e., $\mu(A_i) = 0$. Therefore, $\int g \; d\mu = \sum_i a_i \mu(A_i) = 0$, so is the integral of f.

($\Longrightarrow$) Suppose $\int f \; d\mu = 0$, note that

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > 1/n\} \tag{29}$$

Define $A_n = \{x : f(x) > 1/n\}$, then A_n is an increasing sequence of sets.

Suppose, for contradiction, there exists some A_n with $\mu(A_n) > 0$, define $g(x) = \frac{1}{n}\mathbb{1}\{x \in A_n\}$. $f \geq g$ by construction, so that $\int f \; d\mu \geq \int g \; d\mu = \frac{1}{n}\mu(A_n) > 0$. This leads to a contradiction, so all $\mu(A_n) = 0$, and $\mu(\{x : f(x) > 0\}) = \lim_n \mu(A_n) = 0$. ■

Corollary 4.3. Let $f : X \to \mathbb{R}^*$ be a measurable function,

$$f = 0 \; a.e. \implies \int f \; d\mu = 0 \tag{30}$$

Proof. $f = 0$ a.e. implies $f^+, f^- = 0$ a.e., apply the previous proposition, $\int f^+ \; d\mu = \int f^- \; d\mu = 0$, so is $\int f \; d\mu$.

Note the converse is not true, obviously one may take $f^+ = f^-$ so that $f^+ \; d\mu = \int f^- \; d\mu \neq 0$ and $\int f \; d\mu = 0$. ■

Corollary 4.4. Let $f, g : X \to \mathbb{R}^*$ be integrable functions, then

$$f = 0 \; a.e. \implies \int f \; d\mu = \int g \; d\mu \tag{31}$$

Proof. Let $\tilde{f} = f(x)\mathbb{1}\{x \in \mathbb{R}\}$ and $\tilde{g} = g(x)\mathbb{1}\{x \in \mathbb{R}\}$, we are doing this to avoid subtracting infinity from infinity.

Both $|\tilde{f}|$ and $|\tilde{g}|$ are bounded by $|f|$ and $|g|$ and are integrable. Moreover, $f = \tilde{f} = g = \tilde{g}$ a.e. by construction. Lastly, since $|\tilde{f}|, |\tilde{g}| < \infty$, $\tilde{f} - \tilde{g}$ is integrable and we can write

$$\int \tilde{f} - \tilde{g} \; d\mu = \int \tilde{f} \; d\mu - \int \tilde{g} \; d\mu = 0 \tag{32}$$

$$\implies \int f \; d\mu = \int \tilde{f} \; d\mu = \int g \; d\mu = \int \tilde{g} \; d\mu \tag{33}$$

■

Proposition 4.4. Monotone convergence theorem and dominated convergence theorem holds even if $f_n \to f$ a.e. In DCT, we can also have $|f_n| \leq g$ a.e.

Proof for MCT. Suppose $f_n \geq 0$ a.e.

$$A = \{x : f_n(x) \geq 0 \ \forall n \wedge \lim_{n\to\infty} f_n(x) = f(x)\} \tag{34}$$

Therefore, $A^c = \bigcup_n \{x : f_n(x) < 0\} \cup \{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}$, which is a countable union of measure zero sets, hence $\mu(A^c) = 0$.

Define $\tilde{f}_n = \mathbb{1}_A f_n$ and $\tilde{f} = \mathbb{1}_A f$, apply the original version of MCT on $\tilde{f}_n$ and f, then use the fact that $\int \tilde{f}_n \, d\mu = \int f_n \, d\mu$ and $\int \tilde{f} \, d\mu = \int f \, d\mu$. ■

Proof for DCT. The proof is similar, we can construct sets on which the desired properties holds denoted as A. Define $\tilde{f}(x) := f(x)\mathbb{1}\{x \in A\}$ and apply the original DCT. Lastly, use the fact that modifying f on a measure zero set A^c does not change the value of integral. ■

5 Integral of Complex-Valued Functions

Definition 5.1. A function $f : X \to \mathbb{C}$ is called **measurable** if both $\Re(f)$ and $\Im(f)$ (both are $\mathbb{R}$-valued functions by construction of $\mathbb{C}$) are measurable. Similarly, f is **integrable** if both its real and imaginary parts are integrable. Define

$$\int f \, d\mu = \int \Re(f) \, d\mu + i \int \Im(f) \, d\mu \in \mathbb{C} \tag{1}$$

Proposition 5.1 (Linearity of Integral of Complex-Valued Functions)**.** Let f, g be integrable complex-valued functions, then

1. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.
2. for all $\alpha \in \mathbb{C}$, $\int (\alpha f) \, d\mu = \alpha \int f \, d\mu$.

Proposition 5.2 (Triangle Inequality)**.** Let $f : X \to \mathbb{C}$ be an integrable function, then

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu \tag{2}$$

Proof. Note that there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$\left| \int f \, d\mu \right| = \alpha \int f \, d\mu \tag{3}$$

To see this point, let $z = re^{i\theta} \in \mathbb{C}$ so that $|z| = r$, let $\alpha = e^{-i\theta}$, which satisfies $\alpha z = r = |z|$.

Therefore,

$$\begin{aligned} \left| \int f \, d\mu \right| &= \alpha \int f \, d\mu & (4) \\ &= \int (\alpha f) \, d\mu & (5) \\ &= \int \Re(\alpha f) \, d\mu + i \int \Im(\alpha f) \, d\mu & (6) \\ \implies \int \Im(\alpha f) \, d\mu &= 0 & (7) \end{aligned}$$

Therefore,

$$\left| \int f \, d\mu \right| = \int \Re(\alpha f) \, d\mu \leq \int |\alpha f| \, d\mu = \int |f| \, d\mu \tag{8}$$

where the last step holds because $|\alpha| = 1$. ■

6 Convergence of Measurable Functions

Definition 6.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\{f_n\}_n$ be a sequence of real-valued measurable functions on X, let $f : X \to \mathbb{R}$ be a measurable function. Then, $f_n \to f$ **in measure** if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0 \tag{1}$$

Note: this definition is a generalization of convergence in probability.

Remark 6.1. Convergence almost everywhere does not imply convergence in measure.

Counter-example. Take $\mu = \lambda$, and define $f_n(x) = \mathbb{1}\{x \in [n, \infty)\}$, then $f_n \to 0$ everywhere. However, $\lambda(\{x : |f_n(x)| > 1/2\}) = \lambda([n, \infty)) = \infty \not\to 0$. ■

Remark 6.2. Convergence in measure does not imply convergence almost everywhere (even if we are considering a finite measure).

Counter-example. Define

$$\begin{aligned}
f_1(x) &= 1 && (2)\\
f_2(x) &= \mathbb{1}\{x \in [0, 1/2]\} && (3)\\
f_3(x) &= \mathbb{1}\{x \in [1/2, 1]\} && (4)\\
f_4(x) &= \mathbb{1}\{x \in [0, 1/4]\} && (5)\\
f_5(x) &= \mathbb{1}\{x \in [1/4, 1/2]\} && (6)\\
f_6(x) &= \mathbb{1}\{x \in [1/2, 3/4]\} && (7)\\
f_7(x) &= \mathbb{1}\{x \in [3/4, 1]\} && (8)\\
f_8(x) &= \mathbb{1}\{x \in [1/8, 1/4]\} && (9)
\end{aligned}$$

and so on. in general, $\{x : f_n(x) = 1\}$ shrinks exponentially as $n \to \infty$, hence $f_n \to 0$ in Lebesgue measure. However, for any fixed $x \in [0, 1]$, there are infinitely many n such that $f_n(x) = 1$, therefore, f_n does not converge to 0 pointwise. ■

Proposition 6.1. Let μ be a finite measure, then convergence a.e. implies convergence in measure.

Proof. Suppose $f \to f_n$ a.e. Let $\varepsilon > 0$. Note that if there exists x such that $|f_n - f(x)| \geq \varepsilon$ for infinitely many n, then $f_n \not\to f$ at x. That is,

$$\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\} \subseteq \{x : f_n(x) \not\to f(x)\} \tag{10}$$

By monotonicity,

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\}) \leq \mu(\{x : f_n(x) \not\to f(x)\}) = 0 \tag{11}$$

Further, note that

$$\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} \{x : |f_k(x) - f(x)| > \varepsilon\}}_{B_n} \tag{12}$$

Where $x \in B_n$ indicates there exists a $k \geq n$ such that $|f_k(x) - f(x)| > \varepsilon$. If we take the intersection of all B_n, it means for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $|f_k(x) - f(x)| > \varepsilon$, which is equivalent to saying there are infinitely many k such that $|f_k(x) - f(x)| > \varepsilon$.

Clearly $B_1 \supseteq B_2 \supseteq \dots$, there must exist some B_i such that $\mu(B_i)$ since μ is a finite measure. Therefore,

$$0 = \mu(\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\}) = \lim_{n\to\infty} \mu(B_n) \tag{13}$$

Hence, $\lim_{n\to\infty} \mu(B_n) = 0$. However, $B_n \supseteq \{x : |f_n(x) - f(x)| > \varepsilon\}$, therefore,

$$\lim_{n\to\infty} \{x : |f_n(x) - f(x)| > \varepsilon\} = 0 \tag{14}$$

■

Proposition 6.2. Let f_n be a sequence of measurable real-valued functions converge to a measurable f in measure, then there exists a subsequence of f_n converges to f a.e.

Proof. Suppose $f_n \to f$ in measure, take $\varepsilon = 1$, there exists infinitely many n_1 such that

$$\mu(\{x : |f_{n_1} - f(x)| > 1\}) < 2^{-1} \tag{15}$$

Then for every k, we can choose $n_k > n_{k-1}$ such that

$$\mu(\underbrace{\{x : |f_{n_k} - f(x)| > \frac{1}{k}\}}_{A_k}) < 2^{-k} \tag{16}$$

Let $B = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$, define $B_j = \bigcup_{k=j}^{\infty} A_k$. Note that for all j, $B \subseteq B_j$, therefore,

$$\mu(B) \leq \mu(B_j) = \mu(\bigcup_{k=j}^{\infty} A_k) \leq \sum_{k=j}^{\infty} \mu(A_k) < \sum_{k=j}^{\infty} 2^{-j+1} \tag{17}$$

Take $j \to \infty$, $\mu(B) = 0$. If $x \notin B$, $x \in B^c = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} A_k^c$, which means $\exists j$ such that $x \in A_k^c$ for all $k \geq j$. That is

$$\exists j \ s.t. \ \forall k \geq j \ \ |f_{n_k} - f(x)| \leq \frac{1}{k} \tag{18}$$

Therefore, this subsequence n_k converges t o $f(x)$ a.e. ■

Lemma 6.1 (Borel-Cantelli Lemma)**.** If $A_1, A_2, \cdots,$ is a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(A_k) < \infty \tag{19}$$

then

$$\mu\left(\{x : x \in \text{ infinitely many } A_k\}\right) = 0 \tag{20}$$

Proof. Define

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \tag{21}$$

Easy to verify that $x \in B$ if and only if $x \in$ infinitely many A_k. For every j,

$$B \subseteq \bigcup_{k=j}^{\infty} A_k \tag{22}$$

Hence

$$\mu(B) \leq \mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq \sum_{k=j}^{\infty} \mu(A_k) \to 0 \text{ as } j \to \infty \tag{23}$$

Therefore, $\mu(B) = 0$. ■

Theorem 6.1 (Egorov's Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f_n be a sequence of measurable $\mathbb{R}$-valued functions converging a.e. to a $\mathbb{R}$-valued function f.

Then for all $\varepsilon > 0$, $\exists$ a set $B \in \mathcal{A}$ such that

1. $\mu(B^c) < \varepsilon$,
2. and $f_n \to f$ uniformly on B.

On a finite measure space, convergence a.e. implies convergence uniformly on a slightly smaller set.

Proof. Let $\varepsilon > 0$.

For all $n \in \mathbb{N}$, define

$$g_n(x) := \sup_{k \geq n} |f_k(x) - f(x)| \tag{24}$$

since $f_n \to f$ a.e., $g_n(x)$ is finite a.e. Moreover, $g_n(x) \to 0$ a.e. as $n \to \infty$ (both holds where $f_n \to f$).

Since $\mu(X) < \infty$, $g_n(x) \to 0$ in measure by previous results. Then, for every $k \in \mathbb{N}$, there exists n_k such that

$$\mu\left(\left\{x : g_{n_k}(x) > \frac{1}{k}\right\}\right) < \frac{\varepsilon}{2^k} \tag{25}$$

Since there are infinitely many n_k to choose, we may choose an increasing sequence of n_k's. Define

$$B^c = \left\{x : g_{n_k}(x) > \frac{1}{k} \text{ for some } k\right\} \tag{26}$$

Then,

$$\mu(B^c) = \mu\left(\bigcup_{k=1}^{\infty}\left\{x : g_{n_k}(x) > \frac{1}{k}\right\}\right) \tag{27}$$

$$\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \tag{28}$$

Lastly, we can show that $f_n \to f$ uniformly on B. Note that for every $\delta > 0$, take $k_\delta \geq \frac{1}{\delta}$, if $x \in B$, then $g_{n_{k_\delta}}(x) \leq \frac{1}{k_\delta} < \delta$. Therefore, $\sup_{n \geq n_{k_\delta}} |f_n(x) - f(x)| < \delta$.

Therefore, $\forall x \in B, n \geq n_{n_\delta}$, $|f_n(x) - f(x)| < \delta$ and $f_n \to f$ uniformly on B. ∎

Definition 6.2. A sequence of measurable $\mathbb{R}$-valued functions f_n converges to a $\mathbb{R}$-valued measurable function f in L^1 if

$$\lim_{n\to\infty} \int |f_n - f| \; d\mu = 0 \tag{29}$$

Proposition 6.3 (Markov Inequality)**.** If $g \geq 0$, then for all $t \geq 0$,

$$\mu\left(\{x : g(x) \geq t\}\right) \leq \frac{\int g \; d\mu}{t} \tag{30}$$

In probabilistic notations:

$$P(g \geq t) \leq \frac{\mathbb{E}[g]}{t} \tag{31}$$

Proof. Define $h(x) := t\mathbb{1}\{g \geq t\}$, obviously, $h \leq g$.

$$\int h \; d\mu = t\mu(\{x : g(x) \geq t\}) \leq \int g \; d\mu \tag{32}$$

The result follows. ∎

Proposition 6.4. $f_n \xrightarrow{L^1} f \implies f_n \xrightarrow{\mu} f$.

Proof. Let $\varepsilon > 0$, apply Markov inequality on every $|f_n - f|$:

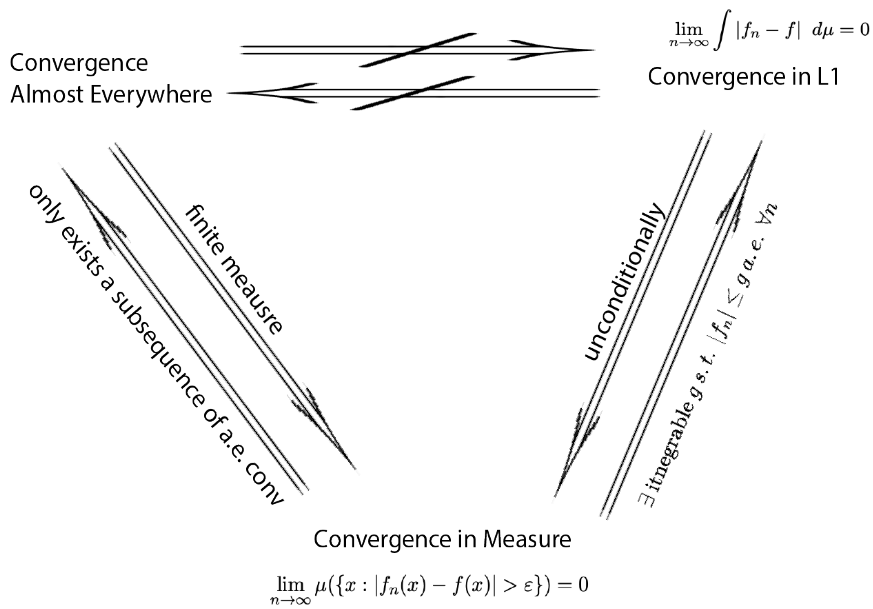
$$\mu\left(\{x : |f_n(x) - f(x)| \geq \varepsilon\}\right) \leq \frac{\int |f_n - f| \ d\mu}{\varepsilon} \to 0 \text{ as } n \to \infty \tag{33}$$

Therefore, $f_n \stackrel{\mu}{\to} f$. ■

Remark 6.3.

1. $f_n \stackrel{a.e.}{\to} f \not\Longrightarrow f_n \stackrel{L^1}{\to} f$.
2. $f_n \stackrel{L^1}{\to} f \not\Longrightarrow f_n \stackrel{a.e.}{\to} f$.
3. $f_n \stackrel{\mu}{\to} f \not\Longrightarrow f_n \stackrel{a.e.}{\to} f$.

Figure 1: Modes of Convergences



Proposition 6.5 (Dominated Convergence Theorem II)**.** Suppose $f_n \stackrel{\mu}{\to} f$, and $\exists$ integrable g such that $|f_n| \leq g$ a.e. for all n. Then, $f_n \stackrel{L^1}{\to} f$ (in particular, $\int f_n \ d\mu \to \int f \ d\mu$).

The convergence in measure version of the dominated convergence theorem.

Proof. Suppose, for contradiction, $f_n \not\to f$ in L^1. Equivalently, there exists ε and a subsequence f_{n_k} such that for all k:

$$\int |f_{n_k} - f| \ d\mu \geq \varepsilon \quad (\dagger) \tag{34}$$

But the convergence in measure implies $f_{n_k} \to f$ in measure as well. Then there exists a subsequence n_{k_ℓ} such that $f_{n_{k_\ell}} \to f$ almost everywhere.

By the previous dominated convergence theorem, $\lim_{\ell\to\infty}\int \left| f_{n_{k_\ell}} - f \right| \, d\mu = 0$, contradicts (†).

■

7 Normed Space

Definition 7.1. Let V be a vector space over $\mathbb{R}$ (over $\mathbb{C}$), a **norm** on V is a map $||\cdot|| : V \to \mathbb{R}$ satisfies the following properties:

1. (Non-negativity) $||x|| \geq 0 \ \forall x \in V$,
2. $||x|| = 0 \iff x = 0$,
3. (Linearity) $||ax|| = |a| \, ||x||$ for all $a \in \mathbb{R}(\in \mathbb{C})$,
4. (Triangle Inequality) $||x + y|| \leq ||x|| + ||y|| \ \forall x, y \in V$.

Example 7.1. For $V = \mathbb{R}^n$, for every $p \geq 1$, the ℓ^p **norm** is defined as

$$||x||_{L^p} = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{1}$$

Note: we only define L^p norm for $p \geq 1$, since for $p < 1$, the triangle inequality fails.

For $p = \infty$,

$$||x||_{\ell^\infty} = \max_{1 \leq i \leq n} |x_i| \tag{2}$$

Example 7.2. Let $C[a, b]$ denote the space of continuous functions map from $[a, b]$ to $\mathbb{R}$, where $[a, b]$ is a compact interval. The **sup-norm** is defined as

$$||f||_\infty = \sup_{x \in [a,b]} |f(x)| \tag{3}$$

This supremum is finite since $|f|$ is continuous and $|f|([a, b])$ is compact.

The **1-norm** is defined as

$$||f||_1 = \int_{[a,b]} |f| \ d\lambda \tag{4}$$

Definition 7.2. Let S be a set, a **metric** d on S is a function $d : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 7.3. A norm on a vector induces a metric, the **metric d induced by norm $||\cdot||$** is defined as

$$d(x, y) := ||x - y|| \tag{5}$$

Note: the converse is false, i.e., there are metrics not induced by any norm. For example, $d(x, y) := \mathbb{1}\{x = y\}$ *is in general not induced by any norm.*

Definition 7.4. Let S be a set with a metric d, a sequence of points $\{x_n\}_{n=1}^{\infty}$ **converges** to $x \in S$ if

$$\lim_{n\to\infty} d(x_n, x) = 0 \tag{6}$$

A sequence is **Cauchy** with respect to d if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ s.t. \ \forall m, n \geq n_0, d(x_m, x_n) < \varepsilon \tag{7}$$

Definition 7.5. A metric space w.r.t d is **complete** if every Cauchy sequence w.r.t. d converges to somewhere in the space.

Remark 7.1. In order to show the completeness of a metric space, take an arbitrary Cauchy sequence in this space, and show

1. construct the limit, in cases of functional spaces, we usually define the limit f as the point wise limit,
2. show this sequence converges to the proposed limit,
3. show the proposed limit is in the metric space.

Example 7.3. $C[a, b]$ with the supremum norm is complete.

Example 7.4. $C[a, b]$ with L^1 norm is not complete.

Proof. Using counter-example: for $[a, b] = [-1, 1]$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ nx & \text{if } x \in (0, 1/n) \\ 1 & \text{if } x \in [1/n, 1] \end{cases} \tag{8}$$

The sequence of f_n is Cauchy but converges to $f = \mathbb{1}\{x \geq 0\} \notin C[a, b]$. ■

Proposition 7.1. $C[a, b]$ under sup-norm is complete.

Proof. Suppose f_n is a Cauchy sequence in $C[a, b]$ under supremum norm. For all $x \in [a, b]$,

$$f_n(x) - f_m(x) \leq ||f_n - f_m||_\infty \to 0 \tag{9}$$

since f_n is Cauchy. Therefore, $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$ and $\lim_{n\to\infty} f_n(x)$ exists. Define f to be the point-wise limit of f_n.

Claim: $f \in C[a, b]$ and $f_n \to f$ in sup-norm.

For all $\varepsilon > 0$, there exists N, such that for all $m, n \geq N$,

$$||f_m - f_n||_\infty < \varepsilon \tag{10}$$

Therefore, for all $x \in [a, b]$, $|f_n(x) - f_m(x)| < ||f_m - f_n||_\infty < \varepsilon$.

Fixing n, take $m \to \infty$, this shows for all $n \geq N$, for all $x \in [a, b]$

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \varepsilon \tag{11}$$

Therefore, for all $n \geq N$, $||f - f_n||_\infty \leq \varepsilon$. Hence $f \to f_n$ in sup-norm.

Now show the continuity of f: take $x_0 \in [a, b]$, given $\varepsilon > 0$, since $f_n \to f$ in sup-norm, there exists N such that for all $n \geq N$,

$$||f - f_n||_\infty \leq \frac{\varepsilon}{3} \tag{12}$$

In particular, $||f - f_N||_\infty \leq \frac{\varepsilon}{3}$.

Moreover, since f_N is continuous, $\exists\delta > 0$ such that $|x - x_0| < \delta \implies |f_N(x) - f_N(x)| < \varepsilon/3$ for every x. Take any $x \in \mathcal{B}_\delta(x_0)$, by triangle inequality,

$$\begin{align}
|f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \tag{13}\\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \tag{14}
\end{align}$$

Hence, $f \in C[a, b]$. ■

8 Functional Analysis: L^p Spaces

8.1 An Auxiliary Construction: the $\mathcal{L}^p$ Spaces

We will firstly define $\mathcal{L}^p$ spaces, which is a little simpler than L^p spaces. The main difference is $\mathcal{L}^p$ spaces are simply spaces of functions, while L^p does not distinguish functions that are equal almost everywhere. In fact, L^p spaces are spaces of equivalence classes of functions, an element $f \in L^p$ actually denote the set of all functions that equal f almost everywhere.

Definition 8.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, for every $1 \leq p < \infty$, the $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{R})$ space is the collection of all measurable functions $f : X \to \mathbb{R}$ such that

$$\int |f|^p \; d\mu < \infty \tag{1}$$

Similarly, $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{C})$ denotes the collection of all measurable functions $f : X \to \mathbb{C}$ such that

$$\int |f|^p \; d\mu < \infty \tag{2}$$

Thought out this notes, we use $\mathcal{L}^p$ to denote $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{R})$ or $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{C})$, unless specified otherwise, statements about $\mathcal{L}^p$ hold for both spaces.

Proposition 8.1. $\mathcal{L}^p$ space is a vector space.

Proof.

1. Note that $0 \in \mathcal{L}^p$.
2. If $f \in \mathcal{L}^p$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$, then

   $$\int |\alpha f|^p \; d\mu = |\alpha|^p \int |f|^p \; d\mu < \infty \tag{3}$$

   Therefore, $\alpha f \in \mathcal{L}^p$.
3. For all $x \in X$,

   $$\begin{aligned} |f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p && (4)\\ &\leq (2\max\{|f(x)|, |g(x)|\})^2 && (5)\\ &\leq 2^p \max\{|f(x)|^p, |g(x)|^p\} && (6)\\ &\leq 2^p(|f(x)|^p + |g(x)|^p) && (7) \end{aligned}$$

   Thus,

   $$\begin{aligned} &\int |f+g|^p \; d\mu < \infty && (8)\\ \implies\ & f + g \in \mathcal{L}^p && (9) \end{aligned}$$

Hence, $\mathcal{L}^p$ is a vector space. ■

Definition 8.2. $\mathcal{L}^\infty(X, \mathcal{A}, \mu, \mathbb{R}/\mathbb{C})$ is defined to be the set of all bounded measurable $f : X \to \mathbb{R}/\mathbb{C}$.

Definition 8.3. For $f \in \mathcal{L}^p$ with $p < \infty$, define

$$||f||_p = \left(\int |f|^p \ d\mu\right)^{\frac{1}{p}} \tag{10}$$

for $p = \infty$, $||f||_\infty$'s definition is a little bit more complicated, for continuous functions, it collides with the sup-norm. However, it's not the same as sup-norm for discontinuous functions.

Definition 8.4. Given a measure space $(X, \mathcal{A}, \mu)$, a set B is called **μ-null/negligible** if $B \subseteq A$ for some $A \in \mathcal{A}$ with $\mu(A) = 0$ (note that B is not necessarily measurable).

A subset $N \subseteq X$ is called **locally μ-null** if $\forall A \in \mathcal{A}$ with $\mu(A) < \infty$, $A \cap N$ is μ-null. *A locally μ-null set N shrinks any measurable set to μ-null set by taking intersection.*

A property of elements of X is said to hold **locally a.e.** if the set on which it fails is locally μ-null.

We use this notion of locally null to circumvent non-sigma finite cases.

Definition 8.5. For $f \in \mathcal{L}^\infty$, define

$$||f||_\infty = \inf \{M \geq 0 : \{x : |f(x)| > M\} \text{ is locally } \mu\text{-null.}\} \tag{11}$$

this is called the **essential supremum** of $|f|$. *Equivalently, $||f||_\infty$ is the least (locally a.e.) upper bound of $|f|$.*

Note that $||f||_\infty$ is only a semi-norm, we may modify a function on a measure-zero set without changing the value of $||f||_\infty$.

Our previous definitions of semi-norms on $\mathcal{L}^p$ spaces satisfy

$$||f||_p = 0 \iff \int |f|^p \ d\mu = 0 \iff |f|^p = 0 \ a.e. \iff f = 0 \ a.e. \tag{12}$$

This definition of semi-norm on $\mathcal{L}^\infty$ ensures $||f||_\infty = 0 \iff f = 0 \ a.e.$.

Example 8.1. Take $X = [0, 1]$ and $\mu = \lambda$,

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2} \\ 2 & \text{otherwise} \end{cases} \tag{13}$$

Then $||f||_\infty = 1$ but $\sup f = 2$. To see this, note that $\{x : |f(x)| > 1\} = \{1/2\}$ has zero measure. However, for any $M < 1$, the same has non-zero Lebesgue measure.

Lemma 8.1. Countable union of locally μ-null sets is locally μ-null.

Proof. Let $B_1, B_2 \dots$ be μ-null, then for any $A \in \mathcal{A}$,

$$\mu\left(A \cap \bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A \cap B_i\right) \leq \sum_{i=1}^{\infty} \mu(A \cap B_i) = 0 \tag{14}$$

■

Proposition 8.2.

$$\mu\left(\{x : |f(x)| > ||f||_\infty\}\right) \text{ is locally } \mu\text{-null.} \tag{15}$$

$$\mu\left(\{x : |f(x)| > c\}\right) \text{ is not locally } \mu\text{-null } \forall c < ||f||_\infty \tag{16}$$

Proof. First, note that by definition of $||f||_\infty$, it follows that $\{x : |f(x)| > c\}$ is not locally μ-null for any $c < ||f||_\infty$, which is the infimum. Moreover,

$$\{x : |f(x)| > ||f||_\infty\} = \bigcup_{n=1}^{\infty} \{x : |f(x)| > ||f||_\infty + 1/n\} \tag{17}$$

By the previous lemma, the result follows. ■

Proposition 8.3. $||f||_p$ and $||f||_\infty$ are semi-norms.

Proof. $||f||_p$ or $||f||_\infty = 0$ only implies $f = 0$ almost everywhere but not everywhere, this fact makes them semi-norms.

Later in L^p spaces, we will define the zero vector to be the collection of functions that are zero almost everywhere, this modification guarantees $||\cdot||$ to be a norm on L^p. ■

Definition 8.6. Given $p \in (1, \infty)$, the **conjugate exponent** q is defined as

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{18}$$

That is,

$$q = \frac{p}{p-1} \tag{19}$$

For $p = \infty$, $q = 1$.

Lemma 8.2 (Young's Inequality)**.** Take $p \in (1, \infty)$, let q be the conjugate exponent of p, then for all $x, y \geq 0$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \tag{20}$$

Proof. ■

Theorem 8.1 (Hölder's Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, take $1 \leq p \leq \infty$, and q be it's conjugate exponent. Take $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$, then the product

$$fg \in \mathcal{L}^1 \tag{21}$$

and

$$||fg||_1 \leq ||f||_p ||g||_q \tag{22}$$

Proof. $p \in (1, \infty)$. For all x, and for any function f and g, by Young's inequality,

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \tag{23}$$

Integrating both sides,

$$||fg||_1 \leq \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \tag{24}$$

If $||f||_p = ||g||_q = 1$, then

$$||fg||_1 \leq \frac{1}{p} + \frac{1}{q} = 1 \tag{25}$$

Now take arbitrary $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, if $||f||_p = 0$ or $||g||_q = 0$, then $fg = 0$ a.e. and there is nothing to prove.

So assume $||f||_p > 0$ and $||g||_q > 0$, let

$$\tilde{f} = \frac{f}{||f||_p} \quad \tilde{g} = \frac{g}{||g||_q} \tag{26}$$

By construction, $||\tilde{f}||_p = 1 = ||\tilde{g}||_q$. By Equation (25) , $||\tilde{f}\tilde{g}|| \leq 1$, but $||\tilde{f}\tilde{g}|| = \frac{||fg||_1}{||f||_p ||g||_q}$. This proves the Hölder's inequality when $p \in (1, \infty)$. ■

Proof. $p = 1$ *and* $q = \infty$. Let $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$. Claim:

$$\{x : |f(x)g(x)| > ||g||_\infty |f(x)|\} \tag{27}$$

is μ-null.

Proof of the Claim. Note that

$$\{x : |f(x)g(x)| > ||g||_\infty |f(x)|\} = \bigcup_{n=1}^{\infty} (\{x : |f(x)| > 1/n\} \cap \{x : |g(x)| > ||g||_\infty\}) \tag{28}$$

By Markov ineuqliaty,

$$\mu(\{x : |f(x)| > 1/n\}) \leq \frac{\int |f| \; d\mu}{1/n} < \infty \tag{29}$$

The intersection of a locally μ-null set with a set of finite measure is μ-null, moreover, the countable union of μ-null sets is μ-null. ■

By the claimed property,

$$||fg||_1 = \int |fg| \; d\mu \leq \int ||g||_\infty |f| \; d\mu = ||g||_\infty ||f||_1 \tag{30}$$

This shows the Hölder's inequality. ■

Example 8.2. Take $X = \{x_1, \cdots, x_n\}$ and μ to be the counting measure on X. Let $p = q = 2$ and $f, g \in \mathcal{L}^2$. Define $v = (f(x_1), \ldots, f(x_n)) \in \mathbb{R}^n$ and $u = (g(x_1), \ldots, g(x_n)) \in \mathbb{R}^n$.

$$||fg||_1 = \sum_{i=1}^{n} \mu(\{x_i\}) \, |f(x_i)g(x_i)| = \sum_{i=1}^{n} |f(x_i)g(x_i)| \tag{31}$$

Therefore,

$$|\langle v, u\rangle| = \left| \sum_{i=1}^{n} f(x_i)g(x_i) \right| \leq ||fg||_1 \tag{32}$$

In this finite dimensional case with counting measure,

$$||f||_2 = \sqrt{\sum_{i=1}^{n} \mu(\{x_i\}) f(x_i)^2} = \sqrt{\sum_{i=1}^{n} f(x_i)^2} = ||v||_2 \tag{33}$$

The same holds for g, in this case Hölder's inequality induces the Cauchy-Switchz inequality.

Theorem 8.2 (Minkowski's Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Take $1 \leq p \leq \infty$. If $f, g \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, then $f + g \in \mathcal{L}^p$ and

$$||f + g||_p \leq ||f||_p + ||g||_p \tag{34}$$

Proof. First, suppose that $p \in (1, \infty)$. Let q be the conjugate exponent of p. We have already shown that $\mathcal{L}^p$ is a vector space, so $f + g \in \mathcal{L}^p$.

Note that

$$1/p + 1/q = 1 \implies (p+q)/(pq) = 1 \implies p + q = pq \implies p = (p-1)q \tag{35}$$

Therefore,

$$\int (|f+g|^{p-1})^q \; d\mu = \int |f+g|^p \; d\mu < \infty \tag{36}$$

Therefore, $|f+g|^{p-1} \in \mathcal{L}^q$. By Hölder's inequality,

$$\begin{aligned}
\int |f+g|^p \; d\mu &= \int |f+g|\,|f+g|^{p-1} \; d\mu && (37)\\
&\leq \int |f|\,|f+g|^{p-1} \; d\mu + \int |g|\,|f+g|^{p-1} \; d\mu && (38)\\
&\leq ||f||_p |||f+g|^{p-1}||_q + ||g||_p |||f+g|^{p-1}||_q && (39)
\end{aligned}$$

where

$$|||f+g|^{p-1}||_q = \left(\int (|f+g|^{p-1})^q\right)^{1/q} = \left(\int |f+g|^p\right)^{1/q} \tag{40}$$

If $||f+g||_p = 0$, we are done. Suppose not, divide $(\int |f+g|^p \; d\mu)^{1/q}$ on both sides,

$$\frac{\int |f+g|^p \; d\mu}{(\int |f+g|^p \; d\mu)^{1/q}} \leq ||f||_p + ||g||_p \tag{41}$$

$$\implies (\int |f+g|^p \; d\mu)^{1-1/q} = (\int |f+g|^p \; d\mu)^{1/p} = ||f+g||_p \leq ||f||_p + ||g||_p \tag{42}$$

When $p = 1$,

$$||f+g||_1 = \int |f+g| \; d\mu \leq \int (|f| + |g|) \; d\mu = ||f||_1 + ||g||_1 \tag{43}$$

When $p = \infty$, define

$$N_1 = \{x : |f(x)| > ||f||_\infty\} \tag{44}$$

$$N_2 = \{x : |g(x)| > ||g||_\infty\} \tag{45}$$

Then N_1 and N_2 are locally μ-null, so is $N_1 \cup N_2$. For $x \notin N_1 \cup N_2$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty \tag{46}$$

■

Thus, we have shown $||\cdot||_p$ on $\mathcal{L}^p$ satisfies

1. If $f = 0$, then $||f||_p = 0$,

2. $||\alpha f||_p = |\alpha| ||f||_p$ for any scalar α,

3. $||f + g||_p \leq ||f||_p + ||g||_p$.

Thus $||\cdot||_p$ satisfies all conditions of being a norm except that $||f||_p = 0 \not\Rightarrow f = 0$, thus $||\cdot||_p$ is a semi-norm on $\mathcal{L}^p$.

8.2 L^p Spaces

Note that $||\cdot||_p$ is a **semi-norm** on $\mathcal{L}^p$, to make it a norm, we introduce the L^p space.

Definition 8.7. For $1 \leq p < \infty$, define the class of zero vectors

$$\mathcal{N}^p := \{f \in \mathcal{L}^p : f \text{ is measurable and } f = 0 \ a.e.\} \tag{47}$$

which is a subspace of $\mathcal{L}^p$. Define L^p to be the quotient space:

$$L^p(X, \mathcal{A}, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu)/\mathcal{N}^p(X, \mathcal{A}, \mu) \tag{48}$$

That is, an element $[f] \in L^p$ (an equivalence class) is the collection of all $g \in \mathcal{L}^p$ such that $f - g = 0$ almost everywhere:

$$[f] := \{g \in \mathcal{L}^p : f - g \in \mathcal{N}^p\} \tag{49}$$

Then L^p is a vector space over $\mathbb{R}$ or $\mathbb{C}$, and $||\cdot||_p$ is well-defined: for any f, for all $g \in [f]$, $||f||_p = ||g||_p$ since $f = g$ almost everywhere so their integrals are the same. Most importantly, $||\cdot||_p$ is a norm on L^p. For $p = \infty$, we define

$$\mathcal{N}^\infty := \{f : f \text{ is bounded, measure and } f = 0 \ a.e.\} \tag{50}$$

Then $L^\infty := \mathcal{L}^p/\mathcal{N}^p$.

Note that L^p for $1 \leq p \leq \infty$ is also a vector space with equivalence relations. In general, we treat L^p as a space of functions instead of a space of classes of functions.

Proposition 8.4. Convergence in L^p $(1 \leq p < \infty)$ implies convergence in measure.

Proof. By Markov's inequality,

$$\begin{aligned} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) &= \mu(\{x : |f_n(x) - f(x)|^p > \varepsilon^p\}) \qquad (51)\\ &\leq \frac{\int |f_n - f|^p \ d\mu}{\varepsilon^p} \to 0 \text{ as } n \to \infty \qquad (52) \end{aligned}$$

■

Corollary 8.1. Let $f_n \to f$ in L^p with $1 \leq p < \infty$, then there exists a subsequence $f_{n_k} \to f$ a.e.

Proof. As convergence in L^p implies convergence in measure, which further implies existence of a.e. converging subsequences. ■

Theorem 8.3. For any $1 \le p \le \infty$, the $||\cdot||_p$ norm on L^p is complete.

Proof. For $1 \le p < \infty$, let (f_n) be a Cauchy sequence in L^p.

Step 1: Find a subsequence (f_{n_k}) *such that* $||f_{n_k} - f_{n_{k+1}}||_p \le 2^{-k}$ *for all* k. By Cauchy property, we may find n_1 such that $||f_{n_1} - f_n|| \le 2^{-1}$ for all $n \ge n_1$. Also, find a $n_2 \ge n_1$ such that $||f_{n_2} - f_n|| \le 2^{-2}$ for all $n \ge n_2$, etc.

Step 2: construct the limit Define

$$A_k := \{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k/2}\} \tag{53}$$

Then, by Markov's inequality,

$$\mu(A_k) \le \frac{\int |f_{n_k} - f_{n_{k+1}}|^p \ d\mu}{(2^{-k/2})^p} \tag{54}$$

$$\le \frac{2^{-kp}}{(2^{-k/2})^p} = 2^{-kp/2} \tag{55}$$

Thus, $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Define

$$B := \{x : x \in \text{ infinitely many} A_k\} = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j \tag{56}$$

By Borel-Cantelli lemma, $\mu(B) = 0$. Take any $x \notin B$, then for sufficiently large k,

$$\left| f_{n_k}(x) - f_{n_{k+1}} \right| \le 2^{-k/2} \tag{57}$$

This shows for all $x \notin B$, the constructed $(f_{n_k}(x))$ is a Cauchy sequence in $\mathbb{R}$, therefore, it's convergent.

Define the almost point-wise limit

$$f(x) := \begin{cases} \lim_{k\to\infty} f_{n_k}(x) & \text{if } x \notin B \\ 0 & \text{if } x \in B \end{cases} \tag{58}$$

Step 3: Show $f_n \to f$ *in* L^p. Note that $f_{n_k} \to f$ almost everywhere, so that $|f|^p \to |f_{n_k}|^p$. By Fatou's lemma,

$$\int |f|^p \ d\mu \le \liminf_{k\to\infty} \int |f_{n_k}|^p \ d\mu \tag{59}$$

But the Cauchy property of f_n implies that $\sup_n ||f_n||_p < \infty$ (find n such that $||f_n - f_m||_p \le 1$ for all $m \ge n$. Thus, $\forall m \ge n$, $||f_m||_p \le ||f_n - f_m||_p + ||f_n||_p \le 1 + ||f_n||_p$. Therefore, $||f||_p < \infty$.

For any $\varepsilon > 0$, we can find N so large that $||f_n - f_m||_p < \varepsilon$ for all $n, m \ge N$ since f_n is Cauchy.

By Fatou's lemma, for all $n \geq N$,

$$\int |f_n - f|^p \ d\mu \leq \liminf_{n\to\infty} \int |f_n - f|^p \ d\mu \tag{60}$$

But when k is so large that $n_K \geq N$, we have

$$\int |f_n - f_{n_k}|^p \ d\mu = ||f_n - f_{n_k}||_p^p \leq \varepsilon^p \tag{61}$$

Thus, fo all $n \geq N$, $||f - f_n||_p \leq \varepsilon$. ∎

Proof. for $p = \infty$ case. Let f_n be Cauchy in L^∞, as before, find a subsequence f_{n_k} such that

$$||f_{n_k} - f_{n_{k+1}}||_\infty \leq 2^{-k} \quad \forall k \tag{62}$$

Then for all k, there exists a locally μ-null set N_k such that for all $x \notin N_k$.

$$\left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| \leq 2^{-k} \tag{63}$$

Let $N = \bigcup_{k=1}^\infty N_k$, so that N is locally μ-null as well. Then for all $x \notin N$, $f_{n_k}(x)$ is a Cauchy sequence of real numbers, define $f(x) = \lim_k f_{n_k}(x)$ outside N and $f(x) = 0$ on N.

Claim: $f_n \to f$ in L^∞. Note that for all $x \notin N$, for all k,

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} \left|f_{n_j}(x) - f_{n_{j+1}}(x)\right| \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1} \tag{64}$$

Thus, $||f - f_{n_k}||_\infty \leq 2^{-k+1}$.

Take any $\varepsilon > 0$, find N so large that $\forall m, n \geq N$, $||f_m - f_n||_\infty \leq \varepsilon$. Then find k so large that $n_k \geq N$ and $2^{-k+1} \leq \varepsilon$. Then for all $n \geq N$,

$$||f - f_n||_\infty \leq ||f - f_{n_k}||_\infty + +||f_{n_k} - f_n|| \leq 2\varepsilon \tag{65}$$

Taking $\varepsilon' = \varepsilon/2$ concludes $f_n \to f$ in L^∞. ∎

9 Signed and Complex Measures

Definition 9.1. Let $(X, \mathcal{A})$ be a measurable space, let $\mu : \mathcal{A} \to [-\infty, \infty]$ be a function. We say that μ is a **signed measure** if

1. $\mu(\varnothing) = 0$,
2. and μ is countable additive: for all disjoint $A_1, A_2, \cdots \in \mathcal{A}$, $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

From now on, we use **measure** to denote the conventional notion of measure, that is, $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\varnothing) = 0$ and countable additivity. The term **signed measure** denotes functions $\mu : \mathcal{A} \to [-\infty, \infty]$ with above properties.

Remark 9.1. Note that the countable additivity does not change if we permute A_i's, thus, implies $\sum_{i=1}^{\infty} \mu(A_i)$ should now change under any rearrangement of the terms. This implies that if $\mu(\cup_{i=1}^{\infty} A_i)$ is finite, $\sum_{i=1}^{\infty} |\mu(A_i)| < \infty$.

Proposition 9.1. If μ is a signed measure, then μ cannot be both ∞ and $-\infty$.

Proof. **Case 1**: if $\mu(X) \in \mathbb{R}$, then for any A, $\mu(X) = \mu(A) + \mu(A^c)$, both of $\mu(A)$ and $\mu(A^c)$ must be finite.

Case 2: if $\mu(X) = \infty$, then $\mu(A) + \mu(A^c) = \mu(X) = \infty$, none of $\mu(A)$ or $\mu(A^c)$ can be $-\infty$.

Case 3: if $\mu(X) = -\infty$, then $\mu(A) + \mu(A^c) = \mu(X) = -\infty$, none of $\mu(A)$ or $\mu(A^c)$ can be ∞. ■

Proposition 9.2 (Weak Monotonicity)**.** If $\mu(A)$ is finite (i.e., in $\mathbb{R}$), then $\mu(B)$ is finite for any $B \subseteq A$, $B \in \mathcal{A}$.

Proof. $\mu(A) = \mu(B) + \mu(A \backslash B) \in \mathbb{R}$, both $\mu(B)$ and $\mu(A \backslash B)$ must be finite. ■

Definition 9.2. A signed measure is called **finite** if $\mu(A)$ is finite for all $A \in \mathcal{A}$.

Example 9.1 (Relationship between integrable function and measure)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \in L^1$, define $\nu(A) = \int_A f \, d\mu$, then ν is a signed measure.

Example 9.2 (Construction of signed measure)**.** If ν_1 and ν_2 are measures and at least one of them if finite, then $\nu_1 - \nu_2$ is a signed measure.

9.1 Hahn Decomposition Theorem

Let $(X, \mathcal{A})$ be a measurable space and let μ be a signed measure on $(X, \mathcal{A})$.

Definition 9.3. A set $A \in \mathcal{A}$ is called a **positive set for** μ if $\mu(B) \geq 0$ for all $B \subseteq A, B \in \mathcal{A}$. Similarly, a set $A \in \mathcal{A}$ is called a **negative set for** μ if $\mu(B) \leq 0$ for all $B \subseteq A, B \in \mathcal{A}$.

Lemma 9.1. If $A \in \mathcal{A}$ satisfies $-\infty < \mu(A) < 0$, then there exists a negative set $B \subseteq A$ such that $\mu(B) \leq \mu(A)$.

Proof. Let $\delta_1 = \sup\{\mu(E) : E \in \mathcal{A}, E \subseteq A\}$, note that $\delta_1 \geq 0$ since $\mu(\varnothing) = 0$.

By the definition of δ_1 we can find $A_1 \subseteq A$ such that $\mu(A_1) \geq \delta_1/2$ if $\delta_1 < \infty$, or $\mu(A_1) \geq 1$ if $\delta_1 = \infty$. Thus, $\mu(A_1) \geq \min\{\delta_1/2, 1\}$.

Let $\delta_2 = \sup\{\mu(E) : E \in \mathcal{A}, E \subseteq A \backslash A_1\}$, similarly, we can choose $A_2 \subseteq A \backslash A_1$ and $A_2 \in \mathcal{A}$ such that $\mu(A_2) \geq \min\{\delta_2/2, 1\}$.

Similarly, choose $A_n \in \mathcal{A}$, $A_n \subseteq A \backslash (A_1 \cup \ldots A_{n-1})$, such that $\mu(A_n) \geq \min\{\delta_n/2, 1\}$. Then by definition, $A_1, A_2, \ldots$ are disjoint, they are all contained in A.

Let $B = A \backslash (\bigcup_{i=1}^{\infty} A_i)$.

Claim: this B is a negative set such that $\mu(B) \leq \mu(A)$.

Note that $\mu(A) \in \mathbb{R} \implies \mu(B) \in \mathbb{R}$. Thus, $\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A) - \mu(B) \in \mathbb{R}$.
But $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ since $A_n's$ are disjoint. Therefore, $\mu(A_i) \to 0$ as $i \to \infty$. However, $\mu(A_i) \geq \min\{\delta_i/2, 1\} \geq 0$. It must be $\delta_i \to 0$ as $i \to 0$.
Take any $E \subseteq B$ such that $E \in \mathcal{A}$. Then $E \subseteq B \subseteq A \backslash (A_1 \cup \ldots A_{n-1})$ for all $n \in \mathbb{N}$. So by definition of δ_n, we have $\mu(E) \leq \delta_n$, thus $\mu(E) \leq 0$ as we take $n \to \infty$. Hence B is a negative set.
Finally, since $\mu(A_i) \to 0$, $\mu(B) = \mu(A) - \sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A)$.

■

Theorem 9.1 (Hahn Decomposition Theorem)**.** Let $(X, \mathcal{A})$ be a measurable space and μ a signed measure on $(X, \mathcal{A})$. Then, there exists disjoint $P \cup N$ in $\mathcal{A}$ such that $X = P \cup N$ such that P is a positive set for μ and N is a negative set for μ.

Proof. Since μ is a signed measure, we know that it cannot take value at both ∞ and $-\infty$. WLOG, suppose μ never takes value $-\infty$. Let

$$L = \inf\{\mu(A) : A \in \mathcal{A} \ s.t. \ A \text{ is negative}\} \tag{1}$$

Then there exists a sequence of negative sets A_n such that $\mu(A_n) \to L$. Define $B = \cup_{n=1}^{\infty} A_n$. For sure, $B \in \mathcal{A}$.

Claim: B is a negative set.
Take and $E \subseteq B$ such that $E \in \mathcal{A}$, then

$$E = E \cap B = \bigcup_{i=1}^{\infty} E \cap A_i = \bigcup_{i=1}^{\infty} E \cap (A_i \backslash (A_1 \cup \cdots \cup A_{i-1})) \tag{2}$$

where the last step holds because we only consider the net incremental at each step. Moreover, $\{E \cap (A_i \backslash (A_1 \cup \cdots \cup A_{i-1}))\}_i$ are disjoint.
Thus,

$$\mu(E) = \sum_{i=1}^{\infty} \mu(\underbrace{E \cap (A_i \backslash (A_1 \cup \cdots \cup A_{i-1}))}_{\subseteq A_i}) \tag{3}$$

Since A_i's are all negative sets, we must have $\mu(E) \leq 0$ and B is a negative set.

Claim: $\mu(B) = L$.
Since $A_n \subseteq B$,

$$\mu(B) = \mu(A_n) + \mu(B \backslash A_n) \tag{4}$$

But B is a negative set, so $\mu(B \backslash A_n) \leq 0$. Thus,

$$\mu(B) \leq \mu(A_n) \quad \forall n \in \mathbb{N} \tag{5}$$

Thus, $\mu(B) \leq \lim_n \mu(A_n) = L$. But B itself is a negative set, and L is the infimum, so $L \leq \mu(B)$.

In particular, we've shown that $L > -\infty$ since μ never takes value at $-\infty$.
Let $N = B$ and $P = N^c$. Since $B \in \mathcal{A}$, both $N, P \in \mathcal{A}$.

Claim: P is a positive set.
Suppose not, then $\exists A \subseteq P$ such that $A \in \mathcal{A}$ and $-\infty < \mu(A) < 0$.
By the lemma, there exists a negative set $D \subseteq A$ and $\mu(D) \leq \mu(A) < 0$. Note that $D \subseteq A \subseteq P$, but then $N \cup D$ is a negative set as a union of negative sets. Then,

$$\mu(N \cup D) = \mu(N) + \mu(D) = L + \mu(D) < L \tag{6}$$

which leads to a contradiction.

Consequently, this $X = N \cup P$ is a Hahn decomposition. ■

Theorem 9.2 (Jordan Decomposition Theorem)**.** Every signed measure is the difference of two measures, at least one of which is finite.

$$\mu = \mu^+ - \mu^- \tag{7}$$

Proof. Let μ be a signed measure, let (N, P) be a Hahn decomposition of X.

For every $A \in \mathcal{A}$, define

$$\mu^+(A) = \mu(A \cap P) \tag{8}$$
$$\mu^-(A) = -\mu(A \cap N) \tag{9}$$

Since P is a positive set, $\mu^+(A) \geq 0$, similarly, since N is negative, $\mu^-(A) \geq 0$ as well.

Let $A_1, A_2, \ldots$ be disjoint sets in $\mathcal{A}$, then

$$\begin{aligned}
\mu^+(\cup_i A_i) &= \mu(P \cap (\cup_i A_i)) && (10)\\
&= \mu(\cup_i (P \cap A_i)) && (11)\\
&= \sum_i \mu(P \cap A_i) && (12)\\
&= \sum_i \mu^+(A_i) && (13)
\end{aligned}$$

So μ^+ is a measure. Similarly, μ^- is a measure as well.

$$\mu^+(A) - \mu^-(A) = \mu(A \cap P) + \mu(A \cap N) = \mu(A) \tag{14}$$

Therefore, $\mu = \mu^+ - \mu^-$. Lastly, note that $\mu(X) = \mu(P) + \mu(N) = \mu^+(P) - \mu^-(N)$, we need at least one of them to be finish in order to avoid subtracting infinity from infinity. ■

Proposition 9.3. Let (μ^+, μ^-) be the decomposition of a signed measure from Hahn decomposition (P, N), that is, $\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$ for any $A \in \mathcal{A}$. Then,

$$\mu^+(A) = \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\} \tag{15}$$
$$\mu^-(A) = \sup\{-\mu(B) : B \subseteq A, B \in \mathcal{A}\} \tag{16}$$

Proof. Take any $A \in \mathcal{A}$, take any $B \subseteq A$ such that $B \in \mathcal{A}$. Then

$$\begin{aligned}
\mu(B) &= \mu^+(B) - \mu^-(B) && (17)\\
&\leq \mu^+(B) \because \mu^-(B) \geq 0 && (18)\\
&\leq \mu^+(A) \because \mu^+ \text{ is a measure} && (19)
\end{aligned}$$

Therefore, $\mu^+(A) \geq \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\}$.

On the other hand, $\mu^+(A) = \mu(A \cap P)$ by definition, take $B = A \cap P \subseteq A$, which satisfies $A \cap P \in \mathcal{A}$. Then $\mu^+(A) \leq \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\}$.

The similar logic works for μ^-. ■

Definition 9.4. The pair of (μ^+, μ^-) defined above is called the **Jordan decomposition** of the signed measure μ, where μ^+ and μ^- are called the **positive and negative parts of** μ.

Definition 9.5. The **variation** of μ is defined to be the measure $|\mu| = \mu^+ + \mu^-$. The **total variation** of μ is the number $||\mu|| = |\mu|(X)$.

9.2 Complex Measures

Definition 9.6. Let $(X, \mathcal{A})$ be a measurable space, $\mu : \mathcal{A} \to \mathbb{C}$ is called a **complex measure** if for all disjoint $A_1, A_2, \dots \in \mathcal{A}$, $\mu(\cup_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mu(A_i)$ and $\mu(\varnothing) = 0$. In particular, this implies the sum is absolutely converged.

Any complex measure μ can be written uniquely as

$$\mu = \mu' + i\mu'' \tag{20}$$

where

$$\mu'(A) = \Re(\mu(A)) \tag{21}$$

$$\mu''(A) = \Im(\mu(A)) \tag{22}$$

Let $\mu' = \mu_1 - \mu_2$ and $\mu'' = \mu_3 - \mu_4$ be Jordan compositions of μ' and μ'' respectively. Then

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4 \tag{23}$$

is called the **Jordan decomposition** of complex measure μ.

Definition 9.7. The **variation** of a complex measure μ is defined as

$$|\mu|(A) := \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| : A_1, \dots, A_n \in \mathcal{A} \text{ disjoint s.t. } \bigcup_{i=1}^{n} A_i = A \right\} \tag{24}$$

Note that the supremum is taken over all *finite partitions of* A. It is easy to check that if μ is a finite signed measure, this definition of variation is the same as the previous one.

Lemma 9.2. Suppose $\mu : \mathcal{A} \to [0, \infty]$ is a function such that (i) $\mu(\varnothing) = 0$ and (ii) is finite additivity (that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint A and B). Moreover, if $\lim_{n\to\infty} \mu(A_n) = 0$ for all $A_n \searrow \varnothing$, then μ is a measure.

Proof. It suffices to check the countable additivity of μ, let $B_1, B_2, \dots$ be a disjoint sequence of sets in $\mathcal{A}$.

Let $B = \bigcup_i B_i$ and define $A_n := B \backslash \bigcup_{i=1}^{n-1} B_i$. Easy to check $A_n \searrow \varnothing$. Therefore, by finite additivity of μ: $\mu(A_n) = \mu(B) - \sum_{i=1}^{n-1} \mu(B_i) \to 0$. Taking $n \to \infty$ implies $\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$. ■

Proposition 9.4. Let μ be a complex measure, then $|\mu|$ is a measure.

Proof. Obviously, $|\mu|(\varnothing) = 0$.

Take any disjoint $A, B \in \mathcal{A}$. Now show the finite additivity of $|\mu|$: let $C_1, \ldots, C_n$ be a measurable disjoint partition of $A \cup B$, so $(C_i \cap A)$ and $(C_i \cap B)$ are partitions of A and B respectively.

$$\begin{aligned}
|\mu|(A) + |\mu|(B) &\geq \sum |\mu(C_i \cap A)| + \sum |\mu(C_i \cap B)| && (25)\\
&\geq \sum |\mu(C_i \cap A) + \mu(C_i \cap B)| && (26)\\
&= \sum |\mu(C_i)| \because C_i \subseteq A \cup B && (27)\\
&\geq |\mu|(A \cup B) && (28)
\end{aligned}$$

Conversely, let $C_1, \ldots, C_n$ be a partition of A and $D_1, \ldots, D_m$ be a partition of B, then $C_1, \ldots, C_n, D_1, \ldots, D_m$ is a partition of $A \cup B$.

$$|\mu|(A \cup B) \geq \sum_{i=1}^{n} |\mu(C_i)| + \sum_{i=1}^{m} |\mu(D_i)| \tag{29}$$

Taking supremum of partitions (C_i) for A and (D_i) for B,

$$|\mu|(A \cup B) \geq |\mu|(A) + |\mu|(B) \tag{30}$$

Therefore, $|\mu|$ is finitely additive.

Now take a $A_n \searrow \varnothing$ in $\mathcal{A}$, using the Jordan decomposition: $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where μ_i are measures. By triangle inequality in $\mathbb{C}$,

$$|\mu(A)| \leq \sum_{i=1}^{4} \mu_i(A) \tag{31}$$

then for all measurable partitions $A_1, \ldots, A_n$ of A,

$$\sum_{j=1}^{n} |\mu(A_j)| \leq \sum_{i=1}^{4} \sum_{j=1}^{n} \mu_i(A_j) = \sum_{i=1}^{4} \mu_i(A) \tag{32}$$

Taking supremum of all such partitions,

$$|\mu|(A) \leq \sum_{i=1}^{4} \mu_i(A) \tag{33}$$

Since $A_n \searrow \varnothing$) implies $\mu_i(A_n) \to 0$ as μ_i's are finite measures (there is no ∞ in $\mathbb{C}$), $|\mu|(A_n) \to 0$. By Previous lemma, $|\mu|$ is a measure. ■

Proposition 9.5 (Completeness of Total Variation)**.** The total variation is a norm on the space of finite signed/complex measures.

Proof. Obviously, $||\mu||$ is a norm. Now show the completeness.

Let $\{\mu_n\}$ be a Cauchy (in total variation) sequence of measures, for all $A \in \mathcal{A}$, $|\mu(A)| \leq |\mu|(A)$ since A is a trivial partition of A.

For any $m, n \in \mathbb{N}, A \in \mathcal{A}$, $\mu_m - \mu_n$ is a signed measure,

$$\begin{aligned} |\mu_m(A) - \mu_n(A)| &\leq |\mu_m - \mu_n|(A) && (34) \\ &\leq ||\mu_m - \mu_n|| && (35) \end{aligned}$$

Therefore, $\{\mu_n(A)\}$ is a Cauchy sequence in $\mathbb{R}$ for all $A \in \mathcal{A}$. Define μ as the "set-wise" limit of μ_n:

$$\mu(A) := \lim_{n\to\infty} \mu_n(A) \tag{36}$$

Now show μ is a measure: observe that $\mu_n \to \mu(A)$ uniformly over all $A \in \mathcal{A}$ by Equation (35). The finite additivity of μ follows its definition.

Fix arbitrary $A_n \searrow \varnothing$, show that $\mu(A_n) \to 0$. Take any $\varepsilon > 0$, find N so large that $|\mu_N(A) - \mu(A)| < \varepsilon$ for all A by uniform convergence.

Find j_0 so large such that for all $j \geq j_0$, $|\mu_N(A_j)| < \varepsilon/2$. For all $j \geq j_0$,

$$|\mu(A_j)| \leq |\mu(A_j) - \mu_N(A_j)| + |\mu_N(A_j)| < \varepsilon \tag{37}$$

Lastly, we show $||\mu_n - \mu|| \to 0$. Take any partition $A_n, \dots, A_k$ of X, take any $\varepsilon > 0$, the Cauchy property of $\{\mu_n\}$ provides a N so large that for all $m, n \geq N$, $||\mu_m - \mu_n|| < \varepsilon$.

$$\sum_{j=1}^{k} |\mu_m(A_j) - \mu_n(A_j)| \leq ||\mu_m - \mu_n|| < \varepsilon \tag{38}$$

Take $m \to \infty$,

$$\sum_{j=1}^{k} |\mu(A_j) - \mu_n(A_j)| \leq \varepsilon \tag{39}$$

Since above inequality holds for all partitions of X, $||\mu - \mu_m|| < \varepsilon$. ■

9.3 Integration w.r.t. Signed and Complex Measures

Definition 9.8. Let $\mu = \mu^+ - \mu^-$ be a signed measure and its corresponding Jordan decomposition, define

$$\int f \; d\mu = \int f \; d(\mu^+ - \mu^-) = \int f \; d\mu^+ - \int f \; d\mu^- \tag{40}$$

Easy to check that $f \mapsto \int f \; d\mu$ and $\mu \mapsto \int f \; d\mu$ are linear maps.

When μ is a complex measure: $\mu = \mu' + i\mu''$, define

$$\int f \; d\mu = \int f \; d\mu' + i \int f \; d\mu'' \tag{41}$$

10 Radon-Nikodym Theorem

Definition 10.1. Let $(X, \mathcal{A})$ be a measurable space, let μ, ν be two measures on this space, ν is **absolutely continuous** w.r.t. μ if for every $A \in \mathcal{A}$:

$$\mu(A) = 0 \implies \nu(A) = 0 \tag{1}$$

denoted as $\nu \ll \mu$.

Theorem 10.1 (Radon-Nikodym)**.** Let $(X, \mathcal{A})$ be a measurable space, let μ, ν be two σ-finite measures. Suppose ν is absolutely continuous w.r.t. μ, then there exists a measurable map $g : X \to [0, \infty)$ such that

$$\nu(A) = \int_A g \; d\mu \tag{2}$$

for every $A \in \mathcal{A}$.

The map g is defined as the **Radon-Nikodym derivative**, denoted as $\frac{d\nu}{d\mu}$, g is unique up to μ-a.e. equivalence.

Interpretations Let χ_A denote the indicator function of set A, recall that $\int_A f \; d\mu \equiv \int f \chi_A \; d\mu$. Then, $\nu(A) = \int_A 1 \; d\nu = \int \chi_A \; d\nu = \int g \chi_A \; d\mu$ for all A. Moreover, for any integrable f,

$$\int f \; d\nu = \int fg \; d\mu \tag{3}$$

This allows us to denote g as $\frac{d\nu}{d\mu}$.

Example 10.1. Suppose $(X, \mathcal{A})$ is a metric space (take $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ here), suppose g is continuous w.r.t. the metric, let $A = B(x, \varepsilon)$ be the ε-open ball around $x \in X$, then for sufficiently small ε:

$$\nu(A) = \nu(B(x, \varepsilon)) \tag{4}$$

$$\int_A g \; d\mu \approx g(x) \int_A \; d\mu = g(x)\mu(B(x, \varepsilon)) \tag{5}$$

That is,

$$\frac{d\nu}{d\mu} = g(x) \approx \frac{\nu(B(x, \varepsilon))}{\mu(B(x, \varepsilon))} \tag{6}$$

Actually,

$$g(x) = \lim_{\varepsilon \to 0} \frac{\nu(B(x, \varepsilon))}{\mu(B(x, \varepsilon))} \tag{7}$$

Therefore, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ captures the relative growth rate of ν to μ when we initially apply them on a small ball and expand the radius of this ball.

Lemma 10.1. Let $(X, \mathcal{A})$ be a measurable space, let ν be a measure on it, let ν be a finite measure. Then, $\nu \ll \mu$ if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ s.t.\ \mu(A) < \delta \implies \nu(A) < \varepsilon \ \forall A \in \mathcal{A} \tag{8}$$

Recall the definition of uniform continuity and $\frac{df(x)}{dx}$.

Proof. ($\Longleftarrow$) Suppose $\mu(A) = 0$, $\nu(A) < \varepsilon$ for all $\varepsilon > 0$, it must be $\nu(A) = 0$.

($\Longrightarrow$) Suppose $\nu \ll \mu$, suppose the condition fails, $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists A$ with $\mu(A) < \delta$ but $\nu(A) \geq \varepsilon$.

We can find a sequence $A_1, A_2, \ldots$ such that $\mu(A_j) < \delta_j = 2^{-j}$ for all j and $\nu(A_j) \geq \varepsilon$. It follows $\sum \mu(A_j) < \infty$. By Borel-Cantelli lemma,

$$\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k\right) = 0 \tag{9}$$

Define $B_j = \bigcup_{k=j}^{\infty} A_k$ and $B = \bigcap_{j=1}^{\infty} B_j$. Since $B_j \searrow B$ and ν is a finite measure, $\nu(B) = \lim_j \nu(B_j)$. But for any j, $\nu(B_j) \geq \nu(A_j) \geq \varepsilon$. Therefore, $\nu(B) \geq \varepsilon$, which contradicts $\nu \ll \mu$. ∎

Proof of Radon-Nikodym Theorem. Let ν, μ be finite measures, let

$$\mathcal{F} := \left\{ f : X \to [0, \infty] : f \text{ measurable and } \int_A f \, d\mu \leq \nu(A) \ \forall A \in \mathcal{A} \right\} \tag{10}$$

We are choosing the largest $g \in \mathcal{F}$ *as* $\frac{d\nu}{d\mu}$.

Claim: $f, g \in \mathcal{F} \implies f \vee g \equiv \max\{f, g\} \in \mathcal{F}$.

Proof. Let $B := \{x : f(x) \geq g(x)\}$, for any $A \in \mathcal{A}$,

$$\begin{aligned} \int_A f \vee g \, d\mu &= \int_{A \cap B} f \vee g \, d\mu + \int_{A \cap B^c} f \vee g \, d\mu & (11) \\ &= \int_{A \cap B} f \, d\mu + \int_{A \cap B^c} g \, d\mu \leq \nu(A \cap B) + \nu(A \cap B^c) = \nu(A) & (12) \end{aligned}$$

∎

Let $(f_n) \in \mathcal{F}$ be a sequence such that

$$\lim_{n \to \infty} \int f_n \, d\mu = \sup\{\int f \, d\mu : f \in \mathcal{F}\} \tag{13}$$

For every $n \in \mathbb{N}$, take $g_n(x) = \max_{j \leq n} f_j(x)$, $g_n \in \mathcal{F}$ by previous claim. Moreover, $g_n(x) \uparrow$ for all $x \in X$.

$$\int f_n \, d\mu \leq \int g_n \, d\mu \leq \sup\{\int f \, d\mu : f \in \mathcal{F}\} \tag{14}$$

By squeeze theorem, $\lim_{n\to\infty} \int g_n \; d\mu = \sup\{\int f \; d\mu : f \in \mathcal{F}\}$.

Define $g(x) = \lim_{n\to\infty} g_n(x)$, which alway exists but is potentially infinity. By MCT,

$$\int g \; d\mu = \lim_{n\to\infty} \int g_n \; d\mu = \sup\{\int f \; d\mu : f \in \mathcal{F}\} \tag{15}$$

Note that $\forall A \in \mathcal{A}$,

$$\int_A g \; d\mu = \lim_{n\to\infty} \int_A g_n \; d\mu \leq \nu(A) \tag{16}$$

So $g \in \mathcal{F}$ and attains the supremum, in terms of integral, over $\mathcal{F}$.

Claim: $\forall A \in \mathcal{A}$, $\int_A g \; d\mu = \nu(A)$.

Proof. Define $\nu_0(A) = \nu(A) - \int_A g\ d\mu$. Since ν is a measure and $A \mapsto \int_A g\ d\mu$ is also a finite measure. Therefore, ν_0 is a finite signed measure. Moreover, since $g \in \mathcal{F}$, $\nu_0(A) \geq 0$ for all $A \in \mathcal{A}$.

Suppose, for contradiction, $\nu_0(A) > 0$ for some $A \in \mathcal{A}$. It must be $\nu_0(X) > 0$. But $\mu(X) < \infty$, there exists $\varepsilon > 0$ such that $\nu_0(X) > \varepsilon\mu(X)$. Note that $\nu_0 - \varepsilon\mu$ is a finite signed measure, let (P, N) be the Hahn decomposition of $\nu_0 - \varepsilon\mu$. Then for any $A \in \mathcal{A}$,

$$\begin{aligned}
\nu(A) &= \int_A g\ d\mu + \nu_0(A) && (17)\\
&\geq \int_A g\ d\mu + \nu_0(A \cap P) && (18)\\
&= \int_A g\ d\mu + \underbrace{\nu_0(A \cap P) - \varepsilon\mu(A \cap P)}_{\geq 0} + \varepsilon\mu(A \cap P) && (19)\\
&\geq \int_A g\ d\mu + \varepsilon\mu(A \cap P) && (20)\\
&= \int_A g + \varepsilon\chi_{A \cap P}\ d\mu && (21)
\end{aligned}$$

Therefore, $g + \varepsilon\chi_{A \cap P} \in \mathcal{F}$. Take $A = X$:

$$\int g + \varepsilon\chi_{A \cap P}\ d\mu = \int g\ d\mu + \varepsilon\mu(P \cap A) \geq \int g\ d\mu \tag{22}$$

Suppose, for contradiction, $\mu(P) \leq 0$, it must be $\mu(P) = 0$, by absolute continuity, $\nu \ll \mu$, $\nu(P) = 0$ as well. Then, since $\int_P g\ d\mu$ is bounded on a measure zero set, it must be zero,

$$\nu_0(P) = \nu(P) - \int_P g\ d\mu = 0 \tag{23}$$

Thus

$$\begin{aligned}
& (\nu_0 - \varepsilon\mu)(P) = 0 && (24)\\
\implies\ & (\nu_0 - \varepsilon\mu)(X) = (\nu_0 - \varepsilon\mu)(P) + (\nu_0 - \varepsilon\mu)(N) \leq 0 && (25)
\end{aligned}$$

Contradicts $\nu_0(X) \geq \varepsilon\mu(X)$, therefore, $\mu(P) > 0$.

This leads to a contradiction since $g + \varepsilon\chi_{A \cap P}$ has strictly larger integral than g. Therefore, $\nu_0 = 0$. ■

Suppose μ and ν are σ-finite. Let $B_1, B_2, \cdots \in \mathcal{A}$ be a partition pf X such that $\mu(B_n), \nu(B_n)$ are finite. Moreover, define $\mu_n(A) := \mu(A \cap B_n)$ and $\nu_n(A) := \nu(A \cap B_n)$, both of μ_n and ν_n are finite on X (in particular, on B_n) and $\nu_n \ll \mu_n$.

For every $n \in \mathbb{N}$, there exists measurable $g_n : X \to [0, \infty]$ such that

$$\nu_n(A) = \int_A g_n\ d\mu \tag{26}$$

Therefore,

$$\nu(A \cap B_n) = \int g_n \chi_{A \cap B_n} \, d\mu \tag{27}$$
$$= \int g_n \chi_{B_n} \chi_A \, d\mu \tag{28}$$
$$= \int_A g_n \chi_{B_n} \, d\mu \tag{29}$$

Let $g = \sum_{n=1}^{\infty} g_n \chi_{B_n}$, then

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A \cap B_n) \tag{30}$$
$$= \sum_{n=1}^{\infty} \int g_n \chi_{B_n} \chi_A \, d\mu \tag{31}$$
$$= \sum_{n=1}^{\infty} \chi_A \int g_n \chi_{B_n} \, d\mu \tag{32}$$
$$= \int \chi_A \sum_{n=1}^{\infty} g_n \chi_{B_n} \, d\mu \tag{33}$$
$$= \int_A g \, d\mu \tag{34}$$
$$\tag{35}$$

Since $g_n < \infty$ everywhere for all n, so is g. ■

Remark 10.1 (Uniqueness of Radon-Nikodym Derivative)**.** Let g and h be two Radon-Nikodym derivatives of ν w.r.t. μ.

Case 1: suppose $\nu(X) < \infty$, then for all $A \in \mathcal{A}$, by definitoin,

$$\int_A g \, d\mu = \nu(A) = \int_A h \, d\mu \tag{36}$$

Let $B := \{x, g(x) > h(x)\}$, $(g-h)\chi_A \geq 0$ and $(g-h)\chi_A = 0$ a.e. on A. Similarly, $(h-g)\chi_{A^c} \geq 0$ and $(h-g)\chi_{A^c} = 0$ a.e. on A^c. Add them together, $g - h = 0$ a.e. on X.

Case 2: suppose ν is σ-finite, let $B_1, B_2, \ldots$ be disjoint measurable sets such that $\nu(B_n) < \infty$ and $\cup_n B_n = X$. Since $g = h$ a.e. on every B_n as shown before, $g = h$ a.e. on X.

Theorem 10.2 (Radon-Nikodym Theorem for Finite Signed and Complex Measures)**.** Let $(X, \mathcal{A})$ be a measurable space, let μ be a $\underline{\sigma\text{-finite}}$ measure on X. Let ν be a $\underline{\text{finite signed or complex}}$ measure on X. Suppose that $|\nu| \ll \mu$ (in this case, we simply say $\nu \ll \mu$). Then there exists $g \in \mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R})$ or $\mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{C})$ such that

$$\nu(A) = \int_A g \, d\mu \quad \forall A \in \mathcal{A} \tag{37}$$

Moreover, g is unique up to μ-a.e. equivalence.

Proof. $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ where $\nu_1, \nu_2, \nu_3, \nu_4$ are finite. $|\nu| \ll \mu \implies \nu_i \ll \mu$ for $i = 1, 2, 3, 4$. Let $g_i = \frac{d\nu_i}{d\mu}$, then $g = g_1 - g_2 + ig_3 - ig_4$ is the Radon-Nikodym derivative of ν w.r.t. μ. ■ Check

11 Lebesgue Decomposition Theorem

Definition 11.1. Let $(X, \mathcal{A})$ be a measurable space, let μ be a measure on $(X, \mathcal{A})$, then μ is **concentrated** on a set $E \in \mathcal{A}$ if $\mu(E^c) = 0$.

We say that a signed or complex measure μ is concentrated on E if the measure $|\mu|$ is concentrated on E.

Definition 11.2. Two measures / signed measures / complex measures μ and ν on measurable space $(X, \mathcal{A})$ are **mutually singular** if $\exists E \in \mathcal{A}$ such that μ is concentrated on E and ν is concentrated on E^c.

$$\mu \perp \nu \tag{1}$$

Example 11.1. Any measure on $\mathbb{R}$ that is concentrated on $\mathbb{Z}$ is mutually singular to the Lebesgue measure, which is concentrated on $\mathbb{Z}^c$.

Theorem 11.1 (Lebesgue Decomposition)**.** Let $(X, \mathcal{A})$ be a measurable space, let μ be a measure (the reference measure) on it. Let ν be a finite signed, complex measure, or σ-finite measure on $(X, \mathcal{A})$, then there is a unique decomposition

$$\nu = \nu_a + \nu_s \tag{2}$$

such that

$$\nu_a \ll \mu \tag{3}$$

$$\nu_s \perp \mu \tag{4}$$

Proof. **Case 1: suppose ν is a finite measure.** Define $\mathcal{N}$ to be the collection of μ-negligible sets:

$$\mathcal{N} := \{B \in \mathcal{A} : \mu(B) = 0\} \tag{5}$$

Let

$$S := \sup\{\nu(B) : B \in \mathcal{N}\} < \infty \text{ since } \nu \text{ is finite.} \tag{6}$$

Then there exists a sequence of sets $B_n \in \mathcal{N}$ such that $S = \lim_{n\to\infty} \nu(B_n)$, define

$$N = \bigcup_{n=1}^{\infty} B_n \tag{7}$$

Easy to verify that $\mu(N) \leq \sum_{n=1}^{\infty} \mu(B_n) = 0$, so $N \in \mathcal{N}$. Obviously, $\nu(N) \leq S$ since $N \in \mathcal{N}$. Moreover, since $\nu(N) \geq \nu(B_n)$ for every $n \in \mathbb{N}$, $\nu(N) \geq \lim_n \nu(B_n) = S$. Thus, $\nu(N) = S$, so that N is the ν-maximal set in $\mathcal{N}$.

For every $A \in \mathcal{A}$, define

$$\nu_a(A) = \nu(A \cap N^c) \tag{8}$$
$$\nu_s(A) = \nu(A \cap N) \tag{9}$$

So that $\nu = \nu_a + \nu_s$.

Claim: $\nu_s \perp \mu$.
Easy to verify that $\mu(N) = 0$ and $\nu_s(N^c) = \nu(N^c \cap N) = 0$.

Claim: $\nu_a \ll \mu$.
Take any $B \in \mathcal{A}$ such that $\mu(B)$. Suppose, for contradiction, $\nu_a(B) \neq 0$, that is, $\nu(B \cap N^c) \neq 0$. Since we assumed ν is a finite measure (not signed), $\nu(B \cap N^c) > 0$. Thus,

$$\nu(N \cup B) = \nu(N) + \nu(B \cap N^c) > \nu(N) = S \tag{10}$$

but $N \cup B \in \mathcal{N}$, this leads to a contradiction.

Case 2: suppose ν is a finite signed or complex measure, we can find N as above for $|\nu|$ and define $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$.

Case 3: if ν is a σ-finite measure, we can firstly express X as a disjoint union $D_1, D_2, \ldots$ with finite ν measure, and then find $N_i \subseteq D_i$ as the ν-maximal element among all μ-zero subsets of D_i. Lastly, define $N = \bigcup_{i=1}^{\infty} N_i$ and follow the construction before.

Uniqueness: suppose

$$\nu = \nu_a + \nu_s = \nu_a' + \nu_s' \tag{11}$$

Assume ν is a finite / finite signed / complex measure, then

$$\nu_a - \nu_a' = \nu_s' - \nu_s \tag{12}$$

The left hand side is absolutely continuous and the right hand side is singular to μ by the following lemma, hence, they must be both zero.

Lemma 11.1. The notion of absolute continuity and singularity are closed under linear combinations.

Proof. TODO ■

Lemma 11.2. If a measure is both absolutely continuous and singular with respect to μ, then it must be zero.

Proof. TODO ■

■

12 Product Measure and Fubini's Theorem

12.1 Dynkin's π-λ System

Definition 12.1. Let X be a set, a collection $\mathcal{P}$ of subsets of X is called a **π-system** if $A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$.

Definition 12.2. Let X be a set, a collection $\mathcal{D}$ of subsets of X is called a **λ-system / Dynkin class** if

1. $X \in \mathcal{D}$,
2. $A, B \in \mathcal{D}$, $A \subseteq B \implies B \backslash A \in \mathcal{D}$,
3. if $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Remark 12.1 (An equivalent definiton)**.** The third requirement of λ-system may be replaced by closure under countable disjoint union.

Proof. TODO ■

Remark 12.2. A σ-algebra is always a λ-system but not converse.

Example 12.1. Take any two probability measures μ and ν on $\mathbb{R}$, let

$$\mathcal{D} = \{A \in \mathcal{B}(\mathbb{R}) : \mu(A) = \nu(A)\} \tag{1}$$

Then $\mathcal{D}$ is always a λ-system but not necessarily a σ-algebra.

Theorem 12.1 (Dynkin's π-λ theorem)**.** Let X be a set, let $\mathcal{P}$ be a π-system and $\mathcal{D}$ be a λ-system. Then

$$\mathcal{P} \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \mathcal{D} \tag{2}$$

Proof. Note that an arbitrary intersection of λ-system is a λ-system. Let $\mathcal{D}$ be the smallest λ-system contains $\mathcal{P}$. Define

$$\mathcal{D}_1 = \{\mathcal{A} \in \mathcal{D} : A \cap B \in \mathcal{D} \quad \forall B \in \mathcal{P}\} \tag{3}$$

Since $\mathcal{P}$ is a π-system, $A \cap B \in \mathcal{P} \subseteq \mathcal{D}$ for all $A, B \in \mathcal{P} \subseteq \mathcal{D}$, thus, $\mathcal{P} \subseteq \mathcal{D}_1$. Note that

1. $A \cap B = B \in \mathcal{D}$ for all $B \in \mathcal{P}$, therefore, $X \in \mathcal{D}_1$.
2. Let $A, B \in \mathcal{D}_1$, for every $C \in \mathcal{P}$, $A \cap B, B \cap C \in \mathcal{D}$. Therefore,

$$(B \cap C) \backslash (A \cap C) = (B \backslash A) \cap C \in \mathcal{D} \tag{4}$$

Hence, $B \backslash A \in \mathcal{D}_1$.

3. If $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{D}_1$, then for all $B \in \mathcal{P}$,

$$(\bigcup A_i) \cap B = \bigcup (A_i \cap B) \in \mathcal{D} \tag{5}$$

Therefore, $\bigcup A_i \in \mathcal{D}_1$, so $\mathcal{D}_1$ is a λ-system.

Since $\mathcal{D}_1$ is a λ-system contains $\mathcal{P}$, hence $\mathcal{D} \subseteq \mathcal{D}_1$. Therefore, $\mathcal{D}_1 = \mathcal{D}$. This shows $\forall A \in \mathcal{D} = \mathcal{D}_1, B \in \mathcal{P}$, $A \cap B \in \mathcal{D}$.

Define

$$\mathcal{D}_2 = \{\mathcal{A} \in \mathcal{D} : A \cap B \in \mathcal{D} \quad \forall B \in \mathcal{D}\} \tag{6}$$

By the previous result, $\mathcal{P} \subseteq \mathcal{D}_2$. Also, $\mathcal{D}_2$ is a λ-system, so $\mathcal{D}_2 = \mathcal{D}$.

Moreover, for all $A, B \in \mathcal{D}$, $A \cap B \in \mathcal{D}$, so that $\mathcal{D}$ is also a π-system.

Lemma 12.1. A collection of sets is both π and λ if and only if its a σ-algebra.

Proof. For a λ-system $\mathcal{D}$, it contains $X^c = \varnothing$ and is closed under complement (take one of two sets to be X).
Let $A_1, A_2, \dots \in \mathcal{D}$, we may define $B_n = A_1 \cup \dots \cup A_n$, so that $\bigcup A_n = \bigcup B_n$ and B_n is an increasing sequence. In particular, since $\mathcal{D}$ is closed under complement (as a λ-system) and finite intersection (as a π-system), $B_n \in \mathcal{D}$ as well. By definition of λ-system, $\bigcup A_n \in \mathcal{D}$. ∎

Therefore, $\mathcal{D}$ is a σ-algebra containing $\mathcal{P}$, it follows $\sigma(\mathcal{P}) \subseteq \mathcal{D}$. ∎

Corollary 12.1. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{A})$. If μ and ν agree on a π-system $\mathcal{P}$ that generate $\mathcal{A}$, then $\mu = \nu$ on $\mathcal{A}$.

Corollary 12.2. Let μ and ν be measures on a measurable space $(X, \mathcal{A})$. Let $\mathcal{P}$ be a π-system on X such that

1. $\sigma(\mathcal{P}) = \mathcal{A}$,
2. $\forall A \in \mathcal{P}$, $\mu(A) = \nu(A) < \infty$,
3. $\exists$ a sequence $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{P}$ such that $\bigcup A_i = X$ and $\mu(A_i) = \nu(A_i) < \infty$ for all i.

Then $\mu = \nu$.

Intuition: for a π-system that approximates the entire space X via an ascending sequence and generates $\mathcal{A}$, then it suffices to show $\mu = \nu$ on the π-system in order to show $\mu = \nu$.

Proof. Case 1: finite measures. Suppose μ and ν are finite measures, let

This proof needs to be revised.

$$\mathcal{D} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\} \tag{7}$$

we are going to show $\mathcal{D} = \mathcal{A}$ indeed.

Firstly, we show $\mathcal{D}$ is a λ-system.

(1) Using the last assumption, we may construct a sequence in $\mathcal{P}$ increasing to X, taking the limit shows $\mu(X) = \nu(X)$ and $X \in \mathcal{D}$ as a result.

(2) Let $A, B \in \mathcal{D}$ such that $A \subseteq B$, since μ and ν are finite on $\mathcal{P}$,

$$\mu(B \backslash A) = \mu(B) - \mu(A) \tag{8}$$
$$= \nu(B) - \nu(A) \tag{9}$$
$$= \nu(B \backslash A) \tag{10}$$

(3) If $A_1 \subseteq A_2 \cdots \in \mathcal{D}$, then

$$\mu(\cup A_i) = \lim \mu(A_i) = \lim \nu(A_i) = \nu(\cup A_i) \tag{11}$$

Since $\mathcal{P} \subseteq \mathcal{D}$, the π-λ theorem implies $\mathcal{A} = \sigma(\mathcal{P}) \subseteq \mathcal{D}$. Thus, $\mathcal{A} = \mathcal{D}$. ■

Proof. The general case. There exists $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{P}$ such that $\bigcup A_i = X$. By assumption (3), $\mu(A_i) = \nu(A_i) < \infty$ for every i. Define

$$\mathcal{D}_i = \{\mathcal{B} \in \mathcal{A} : \mu(B \cap A_i) = \nu(B \cap A_i)\} \tag{12}$$

$\mathcal{D}_i$ isa λ-system containing $\mathcal{P}$, so that $\mathcal{A} = \sigma(\mathcal{P}) \subseteq \mathcal{D}_i$. Hence, $\mathcal{D}_i = \mathcal{A}$.

For every $B \in \mathcal{A}$, $\mu(B \cap A_i) = \nu(B \cap A_i)$ for all i. But

$$\mu(B) = \lim \mu(B \cap A_i) = \lim \nu(B \cap A_i) = \nu(B) \tag{13}$$

Thus, $\mu = \nu$. ■

12.2 Product Measures

Definition 12.3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces, suppose μ and ν are σ-finite. Let $X \times Y$ be the Cartesian product of X and Y

$$X \times Y := \{(x, y) : x \in X, y \in Y\} \tag{14}$$

The **product σ-algebra** , $\mathcal{A} \times \mathcal{B}$, is the σ-algebra generated by all sets of the form $A \times B$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$, such sets are called **rectangles**.

Theorem 12.2. There exists a unique measure $\mu \times \nu$ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ that satisfies $\forall A \in \mathcal{A}, B \in \mathcal{B}$,

$$\mu \times \nu(A \times B) = \mu(A)\nu(B) \tag{15}$$

Here, we only require the product measure to be well behave on rectangles but not other sets in $\mathcal{A}$.

Proof. Uniqueness. Observe that the set of all rectangles $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ is a π-system:

$$\begin{aligned}(A \times B) \cap (A' \times B') &= \{(a, b) : a \in A, b \in B, a \in A', b \in B'\} && (16)\\ &= \{(a, b) : a \in A \cap A', b \in B \cap B'\} && (17)\\ &= \underbrace{(A \cap A')}_{\in \mathcal{A}} \times \underbrace{(B \cap B')}_{\in \mathcal{B}} && (18)\end{aligned}$$

Since μ and ν are σ-finite, there exists $A_1 \cup A_2 \cup \cdots \in \mathcal{A}$ such that $\bigcup A_i = X$ and $\mu(A_i) < \infty$ for all i. Similarly, there is another such sequence such that $\nu(B_i) < \infty$. Combing these two sequences,

$$A_1 \times B_1 \subseteq A_2 \times B_2 \subseteq \dots \tag{19}$$

Such that $\mu \times \nu(A_i \times B_i) = \mu(A_i)\nu(B_i) < \infty$ for all i and $\bigcup(A_i \times Y_i) = X \times Y$.

If γ_1 and γ_2 are two candidates for $\mu \times \nu$, then there exists sequence of rectangles $R_1 \subseteq R_2 \subseteq \dots$ such that $\bigcup R_i = X \times Y$, where $\gamma_1(R_i) = \gamma_2(R_i)$ for all i. By the previous corollary, $\gamma_1 = \gamma_2$ on the entire $\mathcal{A}$. ■

Proof. Existence. $\forall E \in \mathcal{A} \times \mathcal{B}$ and $\forall x \in X, y \in Y$, define

$$\begin{aligned}E_x &= \{y \in Y : (x, y) \in E\} && (20)\\ E_y &= \{x \in X : (x, y) \in E\} && (21)\end{aligned}$$

Similarly, for any measurable $f : X \times Y \to \mathbb{R}^*$, define

$$\begin{aligned}&f_x : Y \to \mathbb{R}^* \quad f_x(y) = f(x, y) && (22)\\ &f_y : X \to \mathbb{R}^* \quad f_y(x) = f(x, y) && (23)\end{aligned}$$

Lemma 12.2. $\forall E \in \mathcal{A} \times \mathcal{B}$, $\forall x \in X$, $E_x \in \mathcal{B}$; $\forall y \in Y$, $E_y \in \mathcal{A}$.

Proof. Take any $x \in X$, let

$$\mathcal{F} = \{E \in \mathcal{A} \times \mathcal{B} : E_x \in \mathcal{B}\} \tag{24}$$

we show that $\mathcal{F} = \mathcal{A} \times \mathcal{B}$.

Note that $\forall x \in X$, for every rectangle, $A \in \mathcal{A}$, $B \in \mathcal{B}$, $(A \times B)_x = B \in \mathcal{B}$. Thus $\mathcal{F}$ contains all rectangles.

(i) $\varnothing \in \mathcal{F}$.

(ii) Let $E \in \mathcal{F}$, then $(E^c)_x = (E_x)^c \in \mathcal{B}$, therefore, $E^c \in \mathcal{F}$.

(iii) Let $E_1, E_2, \dots \in \mathcal{F}$, then $(\cup E_i)_x = \cup \underbrace{(E_i)_x}_{\in \mathcal{B}} \in \mathcal{F}$.

Therefore, $\mathcal{F}$ is a σ-algebra containing all rectangles, thus $\mathcal{F} \supseteq \sigma(\text{Rectangles}) = \mathcal{A} \times \mathcal{B}$. Hence, $\mathcal{F} = \mathcal{A} \times \mathcal{B}$.

The same proof works for E_y. ■

Take any measurable $f : X \times Y \to \mathbb{R}^*$, for all $B \in \mathcal{B}(\mathbb{R}^*)$, let $x \in X$,

$$f_x^{-1} = \{y : f_x(y) \in B\} \tag{25}$$
$$= \{y : f(x, y) \in B\} \tag{26}$$
$$= \{y : (x, y) \in f^{-1}(B)\} \tag{27}$$
$$= (f^{-1}(B))_x \in \mathcal{B} \tag{28}$$

This shows f_x is measurable, a similar argument works for f_y.

Proposition 12.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. $\forall E \in \mathcal{A} \times \mathcal{B}$,

$$x \mapsto \nu(E_x) \tag{29}$$
$$y \mapsto \mu(E_y) \tag{30}$$

are measurable with respect to $\mathcal{A}$ and $\mathcal{B}$.
Intuitively, $x \mapsto \nu(E_x)$ computes the side length at a particular level of x.

Proof. First, suppose μ and ν are finite measures, let

$$\mathcal{D} = \{E \in \mathcal{A} \times \mathcal{B} : x \mapsto \nu(E_x) \text{ is } \mathcal{B} \text{ measurable}\} \tag{31}$$

TODO ■

Finally, for every $E \in \mathcal{A} \times \mathcal{B}$, define

$$\gamma_1(E) = \int_X \nu(E_x) \, d\mu(x) \tag{32}$$
$$\gamma_2(E) = \int_Y \nu(E_y) \, d\nu(y) \tag{33}$$

Intuitively, for γ_1, at each $x \in X$, we compute the length of vertical cross-section E_x, then we sum all these lengths across different locations in X. Obviously, $\gamma_1(\varnothing) = 0$ since $(\varnothing)_x = 0$. The same holds for γ_2.

Let $E_1, E_2 \cdots \in \mathcal{A} \times \mathcal{B}$ be a sequence of disjoint sets, $(E_i)_x$ for all $x \in X$. By MCT,

$$\sum \gamma_1(E_i) = \sum \int_X \nu((E_i)_x) \, d\mu(x) \tag{34}$$
$$= \int_X \sum \nu((E_i)_x) \, d\mu(x) \tag{35}$$
$$= \int_X \nu(\cup (E_i)_x) \, d\mu(x) \tag{36}$$
$$= \gamma_1((\cup E_i)_x) \tag{37}$$

Therefore, both γ_1 and γ_2 are measures.

Now, for any rectangle $A \times B$, $(A \times B)_x = B$ if $x \in A$ and equals $\varnothing$ otherwise.

$$\gamma_1(A \times B) = \int \nu((A \times B)_x)\ d\mu(x) \tag{38}$$

$$= \int \nu(B)\chi_A(x)\ d\mu(x) \tag{39}$$

$$= \nu(B)\mu(A) \tag{40}$$

Similarly, we can show that $\gamma_2(A \times B) = \mu(A)\nu(B)$. By the uniqueness of product measure, $\gamma_1 = \gamma_2$. ∎

Definition 12.4. We define the **product measure** as $\mu \times \nu = \gamma_1 = \gamma_2$,

$$\gamma_1(E) = \int_X \nu(E_x)\ d\mu(x) \tag{41}$$

$$\gamma_2(E) = \int_Y \nu(E_y)\ d\nu(y) \tag{42}$$

12.3 Fubini's Theorem

Theorem 12.3 (Tonelli's Theorem)**.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, let $f : X \times Y \to [0, \infty]$ be a measurable function (not necessarily $(\mu \times \nu)$-integrable). Then,

1. The map $x \mapsto \int_Y f_x\ d\nu$ is $\mathcal{A}$-measurable, and $y \mapsto \int_X f_y\ d\mu$ is $\mathcal{B}$-measurable.

2. the following iterated formula holds:

$$\int_{X \times Y} f\ d(\mu \times \nu) = \int_X \left(\int_Y f_x\ d\nu \right)\ d\mu \tag{43}$$

$$= \int_Y \left(\int_X f_y\ d\mu \right) d\nu \tag{44}$$

Proof. If $f = \chi_E$ for some set $E \in \mathcal{A}$, then we have the first conclusion since

$$\int_Y f_x\ d\nu = \int_Y (\chi_E)_x\ d\nu = \nu(E_x) \tag{45}$$

The same holds for f_y, we have shown this kind of maps are measurable.

The second part follows from the construction of product measure:

$$\int_{X \times Y} \chi_E\ d(\mu \times \nu) = (\mu \times \nu)(E) \tag{46}$$

$$= \int_X \nu(E_x)\ d\mu(x) \tag{47}$$

$$= \int_X \int_Y f_x\ d\nu\ d\mu(x) \tag{48}$$

where the last equality holds because of Equation (45).

So by linearity the theorem holds for any non-negative simple function f, and by MCT, for any non-negative measurable function f. ■

Theorem 12.4 (Fubini's Theorem)**.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces, let $f : X \times Y \to [-\infty, \infty]$ be a measurable mapping and $(\mu \times \nu)$-integrable, then

1. For μ-a.e. x, f_x is ν-integrable; for ν-a.e. y, f_y is μ-integrable.
2. Define

$$I_f(x) := \begin{cases} \int_Y f_x \, d\mu & \text{if } f_x \text{ is } \mu\text{-integrable} \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

$$J_f(y) := \begin{cases} \int_X f_y \, d\nu & \text{if } f_y \text{ is } \nu\text{-integrable} \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

then

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X I_f \, d\mu = \int_Y J_f \, d\nu \tag{51}$$

Proof. TODO. ■

13 Riesz Representation Theorem

13.1 Locally Compact Hausdorff Spaces

Definition 13.1. A **Hausdorff space** is a topological space where for any two distinct points in it, there exists neighbourhoods of each which are disjoint.

Distinct points are separated.

Definition 13.2. A Hausdorff space is called **locally compact** if every point has an open neighbourhood whose closure is compact.

Lemma 13.1. Let X be a Hausdorff space, let K and L be disjoint compact subsets of X. Then, there exists disjoint open sets U, V such that $K \subseteq U$ and $L \subseteq V$.

Disjoint compact sets are separated by open sets.

Proof. This proof needs revising! WLOG, assume $K, L \neq \varnothing$, suppose K consists of single a single point x.

$\forall y \in L$, $\exists$ disjoint open sets, $U_x \ni x$ and $V_y \ni y$ since $K \cap L = \varnothing$ and X is Hausdorff. Then $\{V_y\}_{y \in L}$ is an open cover for L. By compactness of L, there exists a finite sub-cover $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$.

Let

$$U = \bigcap_{i=1}^{n} U_{y_i} \tag{1}$$

$$V = \bigcup_{i=1}^{n} V_{y_i} \tag{2}$$

Then U and V are open disjoint and $x \in U$, $L \subseteq V$. Le K be an arbitrary compact set. $\forall y \in L$, $\exists$ disjoint open sets $U_y \supseteq K$ and $V_y \ni y$.

Again, $\{V_y\}_{y \in K}$ is an open cover for L, there exists a finite sub-cover, and take $U = \bigcap_{i=1}^n U_{y_i}$ and $V = \bigcup_{i=1}^n V_{y_i}$. ■

Lemma 13.2. Let X be a locally compact Hausdorff space, take $x \in X$ and an open neighbourhood U of x. Then, there exists open set V such that $x \in V \subseteq \overline{V} \subseteq U$, and $\overline{V}$ is compact.

U is locally compact as well.

Proof. The local compactness implies there exists open $W \ni x$ such that $\overline{W}$ is compact. Let

$$W_1 = W \cap U \tag{3}$$

then W_1 is open and $x \in W_1$. Also $\overline{W_1}$ is a closed subset of compact set is also compact thus $\overline{W_1}$ is compact.

Let $K = \overline{W_1} \backslash W_1 = \overline{W_1} \cap W_1^c$. K (the boundary) is a closed set contained in compact $\overline{W_1}$. So K is compact. So by Lemma 1, there exists disjoint open sets V_1, V_2 such that $K \subseteq V_1$ and $x \in V_2$.

Let $V = V_2 \cap W$, note that

1. $x \in V$,
2. V is open,
3. $V \subseteq U$,
4. $\overline{V}$ is a closed subset of the compact set $\overline{W_1}$.
5. $\overline{V} \subseteq U$: $V \subseteq W$ and V is separated from the boundary of W_1 by an open set. From this, it is not hard to see that $\overline{V} \subseteq W_1$ thus $\overline{V} \subseteq U$.

■

Lemma 13.3. Let X be a locally compact Hausdorff space, let K be a compact subset of X, suppose there exists an open U such that $K \subseteq U$. Then, there exists open V such that $K \subseteq V \subseteq \overline{V} \subseteq U$, moreover, $\overline{V}$ is compact.

Proof. For each $x \in K$, find an open set V_x such that

$$x \in V_x \subseteq \overline{V_x} \subseteq U \tag{4}$$

and $\overline{V_x}$ is compact. $\{V_x\}_{x\in K}$ is an open cover for K, thus take a finite sub-cover of it: $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$. Let $V = \bigcup_{i=1}^{n} V_{x_i}$, then V is open, contains K and $\overline{V} \subseteq U$. $\overline{V}$ being a closed subset of compact set $\bigcup \overline{V_x}$ is also compact. ■

Definition 13.3. A topological space is called **normal** if it is Hausdorff and any pair of disjoint closed sets can be separated by disjoint open sets.

Lemma 13.4. Any compact Hausdorff space is normal.

Theorem 13.1 (Urysohn's Lemma)**.** Let X be a normal topological space, let E and F be disjoint closed subsets of X. Then, $\exists$ a continuous function $f : X \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.

Proof. Let D be the set of Dyadic rationals in $(0,1)$, i.e., all numbers of the form $\frac{k}{2^n}$. We will individually construct a family of open sets $\{U_r\}_{r\in D}$. First note that E and F being closed sets and X normal, then there exists disjoint open sets $U \supseteq E$ and $V \supseteq F$ such that

$$U \subseteq V^c \tag{5}$$

where V^c is closed, $E \cap U^c = \varnothing$, and $\overline{U} \cap F = \varnothing$. Moreover, $\subseteq U \subseteq \overline{U} \subseteq F^c$.

Let $U_{1/2} = U$, applying the same argument on (E, U^c) and get $U_{1/4} \subseteq U = U_{1/2}$. Same for $\overline{U}, F)$, get $U_{3/4}$ such that

$$U = U_{1/2} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq F^c \tag{6}$$

Continuous by induction, we find $\{U_r\}_{r\in D}$ such that

1. $E \subseteq U_r$, $\overline{U_r} \subseteq F^c$ for all $r \in D$.
2. For all $r < s$, $\overline{U_r} \subseteq U_s$.

Define

$$f(x) = \begin{cases} 1 & \text{if } x \notin \bigcup_{r\in D} U_r \\ \inf\{r : x \in U_r\} & \text{otherwise} \end{cases} \tag{7}$$

To show the continuity, since f is real-valued, it suffices to show that $f^{-1}((r,s))$ is open for any Dyadic rational (r,s) since all intervals of this form generates the Euclidean topology on the real line.

First, suppose the $0 < r < s < 1$, $x \in f^{-1}((r,s))$ if and only if $r < f(x) < s$, then

1. $x \notin \overline{U_q}$ for some $q > r$: $f(x) > r$ if and only if $f(x) > q' > q > r$ for some $q', q \in D$ implies $f(x) \notin U_{q'}$, but $\overline{U_q} \subseteq U_{q'}$, so $f \notin \overline{U_q}$.
2. $x \in U_p$ for some $p < s$.

if and only if

$$x \in \left(\bigcup_{q>r} \overline{U_q^c} \right) \cap \left(\bigcup_{p<s} U_p \right) \tag{8}$$

So $f^{-1}((r,s))$ is open for all $r, s \in (0,1)$. Similar arguments work for $r \leq 0 < s < 1$, $0 < r < 1 \leq s$ and other cases. ■