Lecture Notes
MATH205A: Real Analysis I (Autumn 2020)
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1 Measures

1.1 Motivation

Motivation of this course is to define a notion of *length* on subsets of $\mathbb{R}$ such that

1. $length([a, b]) = b - a$.
2. (countable additivity) $length(\bigcup^\infty A_i) = \sum^\infty length(A_i)$ where A_i's are disjoint.
3. (translation invariance) for all $a \in \mathbb{R}$, $length(A + a) = length(A)$.

Fact 1.1. it is impossible to construct such length for all subsets of $\mathbb{R}$.

Proof. This proof shows it is impossible to construct a notion of length on $[0, 1]$ with desired properties.

For $x, y \in [0, 1]$, define an equivalence relation as $x \sim y \iff x - y \in \mathbb{Q}$. By the axiom of choice, we may construct a set A containing exactly one element from each equivalence class of $x \in [0, 1]$. Obviously, $A \subseteq [0, 1]$.

For each $r \in [-1, 1] \cap \mathbb{Q}$, let $A_r := A + r$, and $A_r \subseteq [-1, 2]$. By translation invariance, $length(A_r) = length(A)$. Note that for any $y \in [0, 1]$, there exists some $x \in A$ such that $x \sim y$, therefore, $y \in A_{y-x} \subseteq \bigcup_r A_r$. Hence, $[0, 1] \subseteq \bigcup_r A_r$.

If the notion of length satisfies countable additivity, $length(\bigcup_r A_r)$ is either zero or infinity, which leads to a contradiction. ■

Lebesgue's Resolution: we only defines length for a subset of $\mathcal{P}(\mathbb{R})$, which contains *everything that may ever arrive in practice*, i.e., σ-algebras.

1.2 Algebras and σ-algebra

Definition 1.1. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called an **algebra** if

1. $X \in \mathcal{A}$,

2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

3. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Consequently: (1) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$; (2) $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_i A_i, \bigcap_i A_i \in \mathcal{A}$ (easily shown by induction); (3) $\varnothing \in \mathcal{A}$.

Definition 1.2. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called a σ-**algebra** if

1. $X \in \mathcal{A}$,

2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

3. $A_1, A_2 \ldots, \in \mathcal{A}, \implies \bigcup_i^\infty A_i \in \mathcal{A}$.

Example 1.1 (trivial examples)**.** The power set of X is a σ-algebra on X; $\{\varnothing, X\}$ is a σ-algebra on X.

Example 1.2 (finite/co-finite algebra)**.** Let X be an infinite set and $\mathcal{A}$ be the collection of subsets A such that either A is finite or A^c is finite. $\mathcal{A}$ is an algebra.

Proof. $X \in \mathcal{A}$ since $X^c = \varnothing$ is finite. For a $X \in \mathcal{A}$, if X is finite, then $X^c \in \mathcal{A}$. If X is infinite, X^c is finite and $X^c \in \mathcal{A}$. Let $A, B \in \mathcal{A}$, if both A and B are finite, $A \cup B$ is finite and in $\mathcal{A}$. If A is finite and B is co-finite, then $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. If both A and B are co-finite, $(A \cup B)^c$ is finite so that $A \cup B \in \mathcal{A}$. ∎

Note the $\mathcal{A}$ is <u>not</u> a σ-algebra if X is infinite: take distinct points $x_1, x_2, \cdots \in \mathcal{A}$, then the union of them is neither finite or co-finite, and therefore not in $\mathcal{A}$.

Example 1.3 (countable/co-countable σ-algebra)**.** The collection of subsets $A \subseteq X$, such that either A is countable or A^c is countable, forms a σ-algebra.

Example 1.4. Let $X = \mathbb{R}$ and $\mathcal{A}$ be the collection of all <u>finite</u> <u>disjoint</u> unions of half-open intervals (i.e., sets like $(a, b], (-\infty, b], (a, \infty)$), $\mathcal{A}$ is an algebra. (Not working for open intervals).

Example 1.5 (counter example)**.** Let X be an infinite set, $\mathcal{A}$ be the collection of finite subsets of X. Then, $\mathcal{A}$ is <u>not</u> an algebra.

Proposition 1.1. Let X be a set and $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ be an arbitrary (not necessarily countable) collection of σ-algebras, then $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i$ is a σ-algebra.

Proof. Since $X \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ ∎

Corollary 1.1. Let X be a set, and $\mathcal{P}$ is an arbitrary collection of subsets of X, then $\exists!$ smallest σ-algebra $\mathcal{A}$ containing $\mathcal{P}$. That is, for any σ-algebra $\mathcal{B} \supseteq \mathcal{P}$, $\mathcal{A} \subseteq \mathcal{B}$. $\mathcal{A}$ is defined as the σ-algebra **generated by** $\mathcal{P}$, denoted as $\sigma(\mathcal{P})$.

Proof. For any $\mathcal{P}$, the power set of X is obviously a σ-algebra containing $\mathcal{P}$. Then we can take $\mathcal{A}$ as the intersection of all σ-algebras containing $\mathcal{P}$. ∎

1.3 Borel σ-algebra

Definition 1.3. The **Borel σ-algebra** of $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the set of open intervals in $\mathbb{R}$.

Fact 1.2. $\mathcal{B}(\mathbb{R})$ is generated by the collection of all closed intervals as well.

Proof. Let $\mathcal{F}$ denote the σ-algebra generated by all closed intervals. Any open interval can be written as a countable union of closed sets: $(a, b) = \bigcup_{n=1}^{\infty}[a + 1/n, b - 1/n]$, therefore $(a, b) \in \mathcal{F}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

Similarly, $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$, hence $\mathcal{B}(\mathbb{R})$ is a σ-algebra contains all closed sets. Therefore, $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$. ∎

Fact 1.3. $\mathcal{B}(\mathbb{R})$ is generated by

1. all open sets,
2. all closed sets,
3. all half-open intervals.

Example 1.6 (counter example)**.** $\mathcal{B}(\mathbb{R})$ is not generated by the collection of singletons.

Proof. ∎

Definition 1.4. The Borel algebra of $\mathbb{R}^d$, $\mathcal{B}(\mathbb{R}^d)$, is the σ-algebra generated by

1. all open sets in $\mathbb{R}^d$,
2. all closed sets in $\mathbb{R}^d$,
3. all closed cubes (regions) in $\mathbb{R}^d$: $\prod_{i=1}^{d}[a_i, b_i]$.

1.4 Measures

Definition 1.5. For a set X and a σ-algebra $\mathcal{A}$ of X, the pair $(X, \mathcal{A})$ is called a **measurable space**.

Definition 1.6. A **measure** μ on a measurable space $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$,
2. $\mu(\cup_i^{\infty} A_i) = \sum_i^{\infty} \mu(A_i)$ for disjoint sequence (A_i)

For now, we don't require the translation invariance property.

The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Example 1.7 (counting measure)**.**

Example 1.8 (point-mass measure)**.**

Proposition 1.2. A measure μ possesses the following basic properties:

1. (Monotonicity) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2. (Sub-additivity) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
3. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing set, let $\cup_{i=1}^{\infty} A_i$ denoted $A_i \nearrow A$, $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.
4. If $A_1 \searrow A \equiv \cap_{i=1}^{\infty} A_i$, and there exists $\mu(A_i) < \infty$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Proof. ∎

Example 1.9 (counter example)**.** Let $X = \mathbb{Z}$, $\mathcal{A} = 2^{\mathbb{Z}}$ and μ be the counting measure. Define $A_i = \{i, i+1, \dots\}$, then $A_i \searrow A = \varnothing$, but $\lim_{n\to\infty} \mu(A_n) = \infty \neq \mu(\varnothing)$.

1.5 Outer Measure

Definition 1.7. Let X be a set, $\mu^* : 2^X \to [0, \infty]$ is an **outer measure** if

1. $\mu^*(\varnothing) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B$.
3. (countable sub-additivity) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Key difference between outer measure and measure:

1. Outer measure does not require countable additivity,
2. outer measure is defined on 2^X instead of a σ-algebra .

Example 1.10.

1.6 Lebesgue Measure on $\mathbb{R}$

Definition 1.8. Let $A \subseteq \mathbb{R}$, define the **Lebesgue outer measure**:

$$\lambda^*(A) = \inf \left\{ \sum_{i\in\mathbb{N}} b_i - a_i : A \subseteq \bigcup_{i\in\mathbb{N}} (a_i, b_i) \right\} \tag{1}$$

The Lebesgue outer measure of a set A is simply in the infimum of total lengths (the conventional notion of length) of open intervals cover A.

Proposition 1.3. The Lebesgue measure satisfies the following properties:

1. λ^* is an outer measure on $\mathbb{R}$,
2. $\lambda^*([a, b]) = b - a$ for all $a < b$.

Proof. (1.1) $\lambda^*(\varnothing) = 0$ since $(-\varepsilon, \varepsilon)$ covers $\varnothing$ for arbitrarily small ε.

(1.2) Let $A \subseteq B$, Ω_A and Ω_B be collection of sequences of open intervals covering A and B respectively. Then, any cover of B must be a cover of A, that is, $\Omega_A \subseteq \Omega_B$. Therefore, $\lambda^*(A) \leq \lambda^*(B)$.

(1.3) Let $A_1, A_2, \cdots \subseteq \mathbb{R}$ and $A = \bigcup_{i=1}^{\infty} A_i$. For all i, we may find (a_{ij}, b_{ij}) covers A_i such that

$$\sum_{j=1}^{\infty}(b_{ij} - b_{ij}) \leq \lambda^*(A_i) + \varepsilon 2^{-i} \tag{2}$$

Also, $\{(a_{ij}, b_{ij})\}_{i,j}$ is a countable union of open intervals that covers A.

$$\lambda^*(A) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(b_{ij} - a_{ij}) \tag{3}$$

$$\leq \sum_{i=1}^{\infty}(\lambda^*(A_i) + \varepsilon 2^{-i}) \tag{4}$$

$$= \sum_{i=1}^{\infty}\lambda^*(A_i) + \varepsilon \tag{5}$$

Therefore, $\lambda^*(A) \leq \sum_{i=1}^{\infty}\lambda^*(A_i)$.

(2) Note that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$ for all $\varepsilon > 0$. Therefore,

$$\lambda^*([a, b]) \leq \inf_{\varepsilon > 0} \lambda^*(a - \varepsilon, b + \varepsilon) = b - a \tag{6}$$

Now show $\lambda^*([a, b]) \geq b - a$. We want to show that $\sum_{i=1}^{\infty}(b_i - a_i) \geq b - a$ for all possible covering of $[a, b]$, which implies the infimum of them is at least $b - a$.

Take an arbitrary covering $\{(a_i, b_i)\}_i$ of $[a, b]$. Since $[a, b]$ is compact, there exists a finite covering $[a, b] \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$ (reindexed), it suffices to show the finite sum $\sum_{i=1}^{\infty}(b_i - a_i) \geq b - a$.

(1) We firstly define an *interval* to be any open, closed or half-open intervals. The *length* of an interval is the difference between two end points.

Note that if an interval I contains a finite collection of disjoint sub-intervals, then the length of I is at least the sum of lengths of sub-intervals. The equality holds when I is exactly finite union of disjoint sub-intervals.

(2) Suppose $[a, b] \subseteq \bigcup_{i=1}^{n}(a_i, b_i)$, let $I_i = [a, b] \cap (a_i, b_i)$. Easy to verify that the length of $I_i \leq$ length of $(a_i, b_i) = b_i - a_i$. Moreover, $\bigcup_{i=1}^{n} I_i = [a, b] \cup \bigcup_{i=1}^{n}(a_i, b_i) = [a, b]$.

(3) For all i, define $I_i' = I_i \backslash (I_1 \cup I_2 \cup \cdots \cup I_{i-1})$. This procedure allows us to express $[a, b]$ as a finite union of disjoint sub-intervals: $[a, b] = \bigcup_{i=1}^{n} I_i'$. Each I_i' is a finite union of disjoint intervals as well, the conventional notion of I_i' is well-defined. Then $b - a =$ sum of lengths of I_i'.

However, $\ell(I_i') \leq \ell(I_i) \leq b_i - a_i$ and sum of lengths of $I_i' \leq$ sum of lengths of $I_i \leq \sum_{i=1}^{n} b_i - a_i$. Therefore, $b - a \leq \sum_{i=1}^{n} b_i - a_i \leq \sum_{i=1}^{\infty} b_i - a_i$. Hence, $b - a = \sum_{i=1}^{\infty} b_i - a_i$ and $\lambda^*[a, b] = b - a$ consequently. ■

1.7 Construct Lebesgue Measure

Definition 1.9. Let X be a set with outer measure μ^*. A set $B \subseteq X$ is μ^*-**measurable** if

$$\forall A \subseteq X, \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \tag{7}$$

Theorem 1.1. For any set X with outer measure μ^* on it, let $\mathcal{M}_{\mu^*}$ denote the set of all μ^*-**measurable** sets. Then, $\mathcal{M}_{\mu^*}$ is a σ-algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ (μ^* restricted to $\mathcal{M}_{\mu^*}$) is a measure.

Proof. To show B is μ^*-measurable, it suffices to show that $\forall A \subseteq X, \mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$, because the opposite inequality always holds by sub-additivity.

(1.1) Let $A \subseteq X$, $\mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^c) = \mu^*(A \cap \varnothing^c) = \mu^*(A)$, therefore, $\varnothing \in \mathcal{M}_{\mu^*}$.

(1.2) Let $A \subseteq X$ and $B \in \mathcal{M}_{\mu^*}$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap (B^c)^c) + \mu^*(A \cap B^c)$. Hence, $B^c \in \mathcal{M}_{\mu^*}$.

(1.3.1) Let $B_1, B_2 \in \mathcal{M}_{\mu^*}$, we are going to show $B_1 \cup B_2 \in \mathcal{M}_{\mu^*}$. Fix any $A \subseteq X$,

$$\begin{aligned} \mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c) && (8) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) && (9) \end{aligned}$$

Moreover,

$$\mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1^c \cap B_2^c) \tag{10}$$

Therefore,

$$\begin{aligned} \mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) && (11) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \text{ since } B_2 \in \mathcal{M}_{\mu^*} && (12) \\ &= \mu^*(A) \text{ since } B_1 \in \mathcal{M}_{\mu^*} && (13) \end{aligned}$$

Therefore, $\mathcal{M}_{\mu^*}$ is an algebra.

(1.3.2) Now show that $\mathcal{M}_{\mu^*}$ is a σ-algebra . For any sequence of sets $A_i \in \mathcal{M}_{\mu^*}$, we can define $B_n := A_n \setminus \cup_{j=1}^{i-1} A_j$ such that $\cup B_i = \cup A_i$. Therefore, it is suffices to show $\mathcal{M}_{\mu^*}$ is closed under countable disjoint unions.

We are going to show the union $\cup B_i$ is μ^*-measurable for any disjoint sequence of μ^*-measurable B_i's.

Claim: let $A \subseteq X$, $\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^n B_i)^c)$. The claim can be proved by induction on n.

When $n = 1$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ because B_1 is μ^*-measurable.

Suppose the claim holds for n, then

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap (\cup_{i=1}^n B_i)^c \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^n B_i)^c \cap B_{n+1}^c) \tag{14}$$

because $B_{n+1} \in \mathcal{M}_{\mu^*}$. Moreover, since all B_i's are disjoint, $B_{n+1} \subseteq B_i^c$ for all i. Hence,

$$B_{n+1} \subseteq \cap_{i=1}^n B_i^c = (\cup_{i=1}^n B_i)^c \tag{15}$$

Also,

$$(\cup_{i=1}^n B_i)^c \cap B_{n+1}^c = \cap_{i=1}^{n+1} B_i^c \tag{16}$$

Consequently,

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^{n+1} B_i)^c) \tag{17}$$

Hence,

$$\begin{align}
\mu^*(A) &= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^n B_i^c)) \tag{18}\\
&\geq \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^\infty B_i^c)) \tag{19}\\
&= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \tag{20}
\end{align}$$

Take $n \to \infty$

$$\begin{align}
\mu^*(A) &\geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \tag{21}\\
&\geq \mu^*(A \cap \cup_{i=1}^\infty B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \tag{22}
\end{align}$$

Therefore, $\cup_{i=1}^\infty B_i$ is μ^*-measurable.

(2) Let $B_1, B_2, \dots$ be a sequence of disjoint sets from $\mathcal{M}_{\mu^*}$. Using the above fact and take $A = \cup_{i=1}^\infty B_i$,

$$\mu^*(A) \geq \mu^*(\cup_{i=1}^\infty B_i) + \mu^*(\varnothing) = \mu^*(\cup_{i=1}^\infty B_i) \tag{23}$$

The opposite inequality holds by sub-additivity. Therefore, μ^* is a measure on $\mathcal{M}_{\mu^*}$. ∎

Definition 1.10. Let λ^* be the Lebesgue outer measure on $\mathbb{R}$, then the collection $\mathcal{M}_{\lambda^*}$ of λ^*-measurable sets is called the **Lebesgue σ-algebra** . The restriction $\lambda = \lambda^*|_{\mathcal{M}_{\lambda^*}}$, which is a measure on $\mathcal{M}_{\lambda^*}$, is called the **Lebesgue measure**. Any set in $\mathcal{M}_{\lambda^*}$ is called a **Lebesgue measurable set**.

Theorem 1.2. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$.

Proof. Note that $\{(-\infty, b] : b \in \mathbb{R}\}$ generates $\mathcal{B}(\mathbb{R})$, it suffices to show $\{(-\infty, b] : b \in \mathbb{R}\} \subseteq \mathcal{M}_{\lambda^*}$.

Let $B = (-\infty, b]$, we are going to show B is λ^*-measurable. Let $A \subseteq \mathbb{R}$ and (a_n, b_n) be a sequence of open intervals covers A. For every $n \in \mathbb{N}$,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n) \cap (-\infty, b]) + \lambda^*((a_n, b_n) \cap (b, \infty)) \tag{24}$$

Three cases follow:

1. $b > b_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.
2. $b_n > b > a_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b]) + \lambda^*((b, b_n]) = b_n - a_n$.
3. $a_n > b$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.

Therefore,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = b_n - a_n \tag{25}$$

By monotonicity and sub-additivity:

$$\begin{aligned} \lambda^*(A \cap B) + \lambda^*(A \cap B^c) &\leq \lambda^*(\cup(a_n, b_n) \cap B) + \lambda^*(\cup(a_n, b_n) \cap B^c) && (26) \\ &\leq \sum \lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) && (27) \\ &= \sum_{n=1}^{\infty} b_n - a_n && (28) \end{aligned}$$

Take the infimum of all such covering, we can show

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A) \tag{29}$$

Therefore, B is μ^*-measurable and $\mathcal{M}_{\lambda^*}$ is a σ-algebra containing all such intervals and $\mathcal{B}(\mathbb{R})$ $\subseteq \mathcal{M}_{\lambda^*}$. ■

1.8 Lebesgue Measure on $\mathbb{R}^d$

Definition 1.11. Steps to construct Lebesgue measure on $\mathbb{R}^d$:

1. Define open cubes on $\mathbb{R}^d$ as a Cartesian product of open intervals: $Q := \prod_{i=1}^d (a_i, b_i)$. Define Lebesgue outer measure:

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \prod_{i=1}^{d} (b_{ni} - a_{ni}) : A \subseteq \bigcup_{n=1}^{\infty} Q_n \right\} \tag{30}$$

2. Show λ^* is an outer measure and $\lambda^*(Q) = \prod_{i=1}^d (b_i - a_i)$.
3. $\mathcal{M}_{\lambda^*}$ is the Lebesgue σ-algebra on $\mathbb{R}^d$. Restricting λ^* on $\mathcal{M}_{\lambda^*}$ defines the Lebesgue measure.
4. Show that any Borel set in $\mathbb{R}^d$ is Lebesgue measurable by showing that there is a generating set of $\mathcal{B}(\mathbb{R}^d)$ is in $\mathcal{M}_{\lambda^*}$.

1.9 Uniqueness of the Lebesgue Measure

The next goal is to prove the uniqueness of Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ subject to the criterion that the measure of any interval (cube) is the volume in the usual sense (product of side lengths).

Theorem 1.3. Let λ be the Lebesgue measure on $\mathbb{R}^d$, then for any Lebesgue measurable set A,

1. $\lambda(A) = \inf\{\lambda(U) : \text{open } U \supseteq A\}$,
2. $\lambda(A) = \sup\{\lambda(K) : \text{compact } U \subseteq A\}$.

Proof. (1.1) WLOG $\lambda(A) < \infty$, by monotonicity, $\lambda(A) \leq \lambda(U)$ for any open cover, $\lambda(A) \leq \inf\{..\}$.

(1.2)Let $\varepsilon > 0$, $\exists$ a sequence of open intervals (R_i) such that

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon \tag{31}$$

Let $U := \cup R_i$ open, hence $\inf\{..\} \leq \lambda(U) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon$. Since this ε can be arbitrarily small, we conclude $\inf\{..\} \leq \lambda(A)$.

(2.1) let A be a Lebesgue measurable set, assume A is bounded, so that $\lambda(A) < \infty$. Then there exists a compact $C \supseteq A$. $C \backslash A$ is Lebesgue measurable as well.

By conclusion of part (1), there exists a open set $U \supseteq C\backslash A$ such that

$$\lambda(C\backslash A) \leq \lambda(U) \leq \lambda(C\backslash A) + \varepsilon \tag{32}$$

Let $K = C\backslash U$, K is compact. Moreover, let $a \in K$, then $a \in C$ and $a \notin U$. Therefore, $a \notin C\backslash A$, it must be $x \in A$. Hence, $K \subseteq A$.

$$\begin{aligned}
\lambda(K) &= \lambda(C\backslash U) && (33)\\
&\geq \lambda(C) - \lambda(U) && (34)\\
&\geq \lambda(C) - (\lambda(C\backslash A) + \varepsilon) && (35)\\
&= \lambda(C) - \lambda(C) + \lambda(A) - \varepsilon && (36)\\
&= \lambda(A) - \varepsilon && (37)
\end{aligned}$$

Take $\varepsilon \to 0$ and $\lambda(A) \leq \sup\{..\}$. By monotonicity, $\lambda(A) \geq \sup\{..\}$.

(2.2) Other cases: suppose A is unbounded and $\lambda(A) > 0$. Take an arbitrary $b < \lambda(A)$. We will show that $\sup\{..\} \geq b$, this will prove that $\lambda(A) \leq \sup\{...\}$.

To show $\sup\{..\} \geq b$, it suffices to show that there exists a compact set $K \subseteq A$ such that $\lambda(K) \geq b$.

Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of compact sets increasing to $\mathbb{R}^d$.

Then $A \cap C_j \uparrow A$ and $\lambda(A \cap C_1) < \infty$, which implies $\lambda(A) = \lim_{j\to\infty} \uparrow \lambda(A \cap C_j)$. Since $b < \lambda(A)$, there exists j such that $\lambda(A \cap C_j) \geq b$, where $A \cap C_j$ is compact. Hence, $b \leq \sup\{..\}$ and $\lambda(A) \leq \sup\{..\}$. $\lambda(A) \geq \sup\{..\}$ holds by monotonicity.

When $\lambda(A) = 0$, $0 \leq \lambda(K)$ for all K so that $0 \leq \sup\{..\}$. The opposite inequality holds by monotonicity. ∎

2 Integrations