MATH205A: Measure Theory

September 14, 2020

1 Lecture 1

1.1 Motivation

Motivation of this course is to define a notion of *length* on subsets of $\mathbb{R}$ such that

1. $length([a,b]) = b - a$.
2. (countable additivity) $length(\bigcup^{\infty} A_i) = \sum^{\infty} length(A_i)$ where A_i's are disjoint.
3. (translation invariance) for all $a \in \mathbb{R}$, $length(A + a) = length(A)$.

Fact 1.1. it is impossible to construct such length for all subsets of $\mathbb{R}$.

Proof. This proof shows it is impossible to construct a notion of length on $[0,1]$ with desired properties.

For $x, y \in [0,1]$, define an equivalence relation as $x \sim y \iff x - y \in \mathbb{Q}$. By the axiom of choice, we may construct a set A containing exactly one element from each equivalence class of $x \in [0,1]$. Obviously, $A \subseteq [0,1]$.

For each $r \in [-1,1] \cap \mathbb{Q}$, let $A_r := A + r$, and $A_r \subseteq [-1,2]$. By translation invariance, $length(A_r) = length(A)$. Note that for any $y \in [0,1]$, there exists some $x \in A$ such that $x \sim y$, therefore, $y \in A_{y-x} \subseteq \bigcup_r A_r$. Hence, $[0,1] \subseteq \bigcup_r A_r$.

If the notion of length satisfies countable additivity, $length(\bigcup_r A_r)$ is either zero or infinity, which leads to a contradiction. ■

Lebesgue's Resolution: we only defines length for a subset of $\mathcal{P}(\mathbb{R})$, which contains *everything that may ever arrive in practice*, i.e., σ-algebras.

1.2 Algebras and σ-algebra

Definition 1.1. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called an **algebra** if

1. $X \in \mathcal{A}$,
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,
3. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Consequently: (1) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$; (2) $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_i A_i, \bigcap_i A_i \in \mathcal{A}$ (easily shown by induction); (3) $\varnothing \in \mathcal{A}$.

Definition 1.2. Let X be a set, a collection $\mathcal{A}$ of subsets of X is called a σ-**algebra** if

1. $X \in \mathcal{A}$,
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,
3. $A_1, A_2 \ldots, \in \mathcal{A}, \implies \bigcup_i^\infty A_i \in \mathcal{A}$.

Example 1.1 (trivial examples)**.** The power set of X is a σ-algebra on X; $\{\varnothing, X\}$ is a σ-algebra on X.

Example 1.2 (finite/co-finite algebra)**.** Let X be an infinite set and $\mathcal{A}$ be the collection of subsets A such that either A is finite or A^c is finite. $\mathcal{A}$ is an algebra.

Proof. $X \in \mathcal{A}$ since $X^c = \varnothing$ is finite. For a $X \in \mathcal{A}$, if X is finite, then $X^c \in \mathcal{A}$. If X is infinite, X^c is finite and $X^c \in \mathcal{A}$. Let $A, B \in \mathcal{A}$, if both A and B are finite, $A \cup B$ is finite and in $\mathcal{A}$. If A is finite and B is co-finite, then $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. If both A and B are co-finite, $(A \cup B)^c$ is finite so that $A \cup B \in \mathcal{A}$. ■

Note the $\mathcal{A}$ is not a σ-algebra if X is infinite: take distinct points $x_1, x_2, \dots \in \mathcal{A}$, then the union of them is neither finite or co-finite, and therefore not in $\mathcal{A}$.

Example 1.3 (countable/co-countable σ-algebra)**.** The collection of subsets $A \subseteq X$, such that either A is countable or A^c is countable, forms a σ-algebra.

Example 1.4. Let $X = \mathbb{R}$ and $\mathcal{A}$ be the collection of all finite disjoint unions of half-open intervals (i.e., sets like $(a, b], (-\infty, b], (a, \infty)$), $\mathcal{A}$ is an algebra. (Not working for open intervals).

Example 1.5 (counter example)**.** Let X be an infinite set, $\mathcal{A}$ be the collection of finite subsets of X. Then, $\mathcal{A}$ is not an algebra.

Proposition 1.1. Let X be a set and $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ be an arbitrary (not necessarily countable) collection of σ-algebras, then $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i$ is a σ-algebra.

Proof. Since $X \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ ■

Corollary 1.1. Let X be a set, and $\mathcal{P}$ is an arbitrary collection of subsets of X, then $\exists!$ smallest σ-algebra $\mathcal{A}$ containing $\mathcal{P}$. That is, for any σ-algebra $\mathcal{B} \supseteq \mathcal{P}$, $\mathcal{A} \subseteq \mathcal{B}$. $\mathcal{A}$ is defined as the σ-algebra **generated by** $\mathcal{P}$, denoted as $\sigma(\mathcal{P})$.

Proof. For any $\mathcal{P}$, the power set of X is obviously a σ-algebra containing $\mathcal{P}$. Then we can take $\mathcal{A}$ as the intersection of all σ-algebras containing $\mathcal{P}$. ■

1.3 Borel σ-algebra

Definition 1.3. The **Borel σ-algebra** of $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$, is the σ-algebragenerated by the set of <u>open intervals</u> in $\mathbb{R}$.

Fact 1.2. $\mathcal{B}(\mathbb{R})$ is generated by the collection of all closed intervals as well.

Proof. Let $\mathcal{F}$ denote the σ-algebragenerated by all closed intervals. Any open interval can be written as a countable union of closed sets: $(a, b) = \bigcup_{n=1}^{\infty}[a + 1/n, b - 1/n]$, therefore $(a, b) \in \mathcal{F}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

Similarly, $[a, b] = \bigcap_{n=1}^{\infty}(a - 1/n, b + 1/n)$, hence $\mathcal{B}(\mathbb{R})$ is a σ-algebracontains all closed sets. Therefore, $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$. ■

Fact 1.3. $\mathcal{B}(\mathbb{R})$ is generated by

1. all open sets,
2. all closed sets,
3. all half-open intervals.

Example 1.6 (counter example)**.** $\mathcal{B}(\mathbb{R})$is <u>not</u> generated by the collection of singletons.

Proof. ■

Definition 1.4. The Borel algebra of $\mathbb{R}^d$, $\mathcal{B}(\mathbb{R}^d)$, is the σ-algebragenerated by

1. all open sets in $\mathbb{R}^d$,
2. all closed sets in $\mathbb{R}^d$,
3. all closed cubes (regions) in $\mathbb{R}^d$: $\prod_{i=1}^{d}[a_i, b_i]$.

1.4 Measures

Definition 1.5. For a set X and a σ-algebra$\mathcal{A}$ of X, the pair $(X, \mathcal{A})$ is called a **measurable space**.

Definition 1.6. A **measure** μ on a measurable space $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$,
2. $\mu(\cup_i^{\infty} A_i) = \sum_i^{\infty} \mu(A_i)$ for disjoint sequence (A_i)

For now, we don't require the translation invariance property.

The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Example 1.7 (counting measure)**.**

Example 1.8 (point-mass measure)**.**

Proposition 1.2. A measure μ possesses the following basic properties:

1. (Monotonicity) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2. (Sub-additivity) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Proof. ∎