

Lecture Notes
MATH205A: Real Analysis I (Autumn 2020)
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1 Measures

1.1 Motivation

Motivation of this course is to define a notion of *length* on subsets of \mathbb{R} such that

1. $length([a, b]) = b - a$.
2. (countable additivity) $length(\bigcup^\infty A_i) = \sum^\infty length(A_i)$ where A_i 's are disjoint.
3. (translation invariance) for all $a \in \mathbb{R}$, $length(A + a) = length(A)$.

Fact 1.1. it is impossible to construct such length for all subsets of \mathbb{R} .

Proof. This proof shows it is impossible to construct a notion of length on $[0, 1]$ with desired properties.

For $x, y \in [0, 1]$, define an equivalence relation as $x \sim y \iff x - y \in \mathbb{Q}$. By the axiom of choice, we may construct a set A containing exactly one element from each equivalence class of $x \in [0, 1]$. Obviously, $A \subseteq [0, 1]$.

For each $r \in [-1, 1] \cap \mathbb{Q}$, let $A_r := A + r$, and $A_r \subseteq [-1, 2]$. By translation invariance, $length(A_r) = length(A)$. Note that for any $y \in [0, 1]$, there exists some $x \in A$ such that $x \sim y$, therefore, $y \in A_{y-x} \subseteq \bigcup_r A_r$. Hence, $[0, 1] \subseteq \bigcup_r A_r$.

If the notion of length satisfies countable additivity, $length(\bigcup_r A_r)$ is either zero or infinity, which leads to a contradiction. ■

Lebesgue's Resolution: we only defines length for a subset of $\mathcal{P}(\mathbb{R})$, which contains *everything that may ever arrive in practice*, i.e., σ -algebras.

1.2 Algebras and σ -algebra

Definition 1.1. Let X be a set, a collection \mathcal{A} of subsets of X is called an **algebra** if

1. $X \in \mathcal{A}$,

$$2. A \in \mathcal{A} \implies A^c \in \mathcal{A},$$

$$3. A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}.$$

Consequently: (1) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$; (2) $A_1, \dots, A_n \in \mathcal{A} \implies \bigcup_i A_i, \bigcap_i A_i \in \mathcal{A}$ (easily shown by induction); (3) $\emptyset \in \mathcal{A}$.

Definition 1.2. Let X be a set, a collection \mathcal{A} of subsets of X is called a σ -**algebra** if

$$1. X \in \mathcal{A},$$

$$2. A \in \mathcal{A} \implies A^c \in \mathcal{A},$$

$$3. A_1, A_2, \dots \in \mathcal{A} \implies \bigcup_i^\infty A_i \in \mathcal{A}.$$

Example 1.1 (trivial examples). The power set of X is a σ -algebra on X ; $\{\emptyset, X\}$ is a σ -algebra on X .

Example 1.2 (finite/co-finite algebra). Let X be an infinite set and \mathcal{A} be the collection of subsets A such that either A is finite or A^c is finite. \mathcal{A} is an algebra.

Proof. $X \in \mathcal{A}$ since $X^c = \emptyset$ is finite. For a $X \in \mathcal{A}$, if X is finite, then $X^c \in \mathcal{A}$. If X is infinite, X^c is finite and $X^c \in \mathcal{A}$. Let $A, B \in \mathcal{A}$, if both A and B are finite, $A \cup B$ is finite and in \mathcal{A} . If A is finite and B is co-finite, then $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. If both A and B are co-finite, $(A \cup B)^c$ is finite so that $A \cup B \in \mathcal{A}$. ■

Note the \mathcal{A} is not a σ -algebra if X is infinite: take distinct points $x_1, x_2, \dots \in \mathcal{A}$, then the union of them is neither finite or co-finite, and therefore not in \mathcal{A} .

Example 1.3 (countable/co-countable σ -algebra). The collection of subsets $A \subseteq X$, such that either A is countable or A^c is countable, forms a σ -algebra.

Example 1.4. Let $X = \mathbb{R}$ and \mathcal{A} be the collection of all finite disjoint unions of half-open intervals (i.e., sets like $(a, b], (-\infty, b], (a, \infty)$), \mathcal{A} is an algebra. (Not working for open intervals).

Example 1.5 (counter example). Let X be an infinite set, \mathcal{A} be the collection of finite subsets of X . Then, \mathcal{A} is not an algebra.

Proposition 1.1. Let X be a set and $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ be an arbitrary (not necessarily countable) collection of σ -algebras, then $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i$ is a σ -algebra.

Proof. Since $X \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ ■

Corollary 1.1. Let X be a set, and \mathcal{P} is an arbitrary collection of subsets of X , then $\exists!$ smallest σ -algebra \mathcal{A} containing \mathcal{P} . That is, for any σ -algebra $\mathcal{B} \supseteq \mathcal{P}$, $\mathcal{A} \subseteq \mathcal{B}$. \mathcal{A} is defined as the σ -algebra **generated by** \mathcal{P} , denoted as $\sigma(\mathcal{P})$.

Proof. For any \mathcal{P} , the power set of X is obviously a σ -algebra containing \mathcal{P} . Then we can take \mathcal{A} as the intersection of all σ -algebras containing \mathcal{P} . ■

1.3 Borel σ -algebra

Definition 1.3. The **Borel σ -algebra** of \mathbb{R} , denoted as $\mathcal{B}(\mathbb{R})$, is the σ -algebra generated by the set of open intervals in \mathbb{R} .

Fact 1.2. $\mathcal{B}(\mathbb{R})$ is generated by the collection of all closed intervals as well.

Proof. Let \mathcal{F} denote the σ -algebra generated by all closed intervals. Any open interval can be written as a countable union of closed sets: $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n]$, therefore $(a, b) \in \mathcal{F}$ and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

Similarly, $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n)$, hence $\mathcal{B}(\mathbb{R})$ is a σ -algebra contains all closed sets. Therefore, $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$. ■

Fact 1.3. $\mathcal{B}(\mathbb{R})$ is generated by

1. all open sets,
2. all closed sets,
3. all half-open intervals.

Example 1.6 (counter example). $\mathcal{B}(\mathbb{R})$ is not generated by the collection of singletons.

Proof. ■

Definition 1.4. The Borel algebra of \mathbb{R}^d , $\mathcal{B}(\mathbb{R}^d)$, is the σ -algebra generated by

1. all open sets in \mathbb{R}^d ,
2. all closed sets in \mathbb{R}^d ,
3. all closed cubes (regions) in \mathbb{R}^d : $\prod_{i=1}^d [a_i, b_i]$.

1.4 Measures

Definition 1.5. For a set X and a σ -algebra \mathcal{A} of X , the pair (X, \mathcal{A}) is called a **measurable space**.

Definition 1.6. A **measure** μ on a measurable space (X, \mathcal{A}) is a map $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(\bigcup_i^{\infty} A_i) = \sum_i^{\infty} \mu(A_i)$ for disjoint sequence (A_i)

For now, we don't require the translation invariance property.

The triple (X, \mathcal{A}, μ) is called a **measure space**.

Example 1.7 (counting measure).

Example 1.8 (point-mass measure).

Proposition 1.2. A measure μ possesses the following basic properties:

1. (Monotonicity) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
2. (Sub-additivity) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
3. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing set, let $\bigcup_{i=1}^{\infty} A_i$ denoted $A_i \nearrow A$, $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.
4. If $A_1 \searrow A \equiv \bigcap_{i=1}^{\infty} A_i$, and **there exists** $\mu(A_i) < \infty$, then $\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$.

Proof. ■

Example 1.9 (counter example). Let $X = \mathbb{Z}$, $\mathcal{A} = 2^{\mathbb{Z}}$ and μ be the counting measure. Define $A_i = \{i, i+1, \dots\}$, then $A_i \searrow A = \emptyset$, but $\lim_{n \rightarrow \infty} \mu(A_n) = \infty \neq \mu(\emptyset)$.

1.5 Outer Measure

Definition 1.7. Let X be a set, $\mu^* : 2^X \rightarrow [0, \infty]$ is an **outer measure** if

1. $\mu^*(\emptyset) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B$.
3. (countable sub-additivity) $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Key difference between outer measure and measure:

1. Outer measure does not require countable additivity,
2. outer measure is defined on 2^X instead of a σ -algebra .

Example 1.10.

1.6 Lebesgue Measure on \mathbb{R}

Definition 1.8. Let $A \subseteq \mathbb{R}$, define the **Lebesgue outer measure**:

$$\lambda^*(A) = \inf \left\{ \sum_{i \in \mathbb{N}} b_i - a_i : A \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i) \right\} \quad (1)$$

The Lebesgue outer measure of a set A is simply in the infimum of total lengths (the conventional notion of length) of open intervals cover A .

Proposition 1.3. The Lebesgue measure satisfies the following properties:

1. λ^* is an outer measure on \mathbb{R} ,
2. $\lambda^*([a, b]) = b - a$ for all $a < b$.

Proof. (1.1) $\lambda^*(\emptyset) = 0$ since $(-\varepsilon, \varepsilon)$ covers \emptyset for arbitrarily small ε .

(1.2) Let $A \subseteq B$, Ω_A and Ω_B be collection of sequences of open intervals covering A and B respectively. Then, any cover of B must be a cover of A , that is, $\Omega_A \subseteq \Omega_B$. Therefore, $\lambda^*(A) \leq \lambda^*(B)$.

(1.3) Let $A_1, A_2, \dots \subseteq \mathbb{R}$ and $A = \bigcup_{i=1}^{\infty} A_i$. For all i , we may find (a_{ij}, b_{ij}) covers A_i such that

$$\sum_{j=1}^{\infty} (b_{ij} - a_{ij}) \leq \lambda^*(A_i) + \varepsilon 2^{-i} \quad (2)$$

Also, $\{(a_{ij}, b_{ij})\}_{i,j}$ is a countable union of open intervals that covers A .

$$\lambda^*(A) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (b_{ij} - a_{ij}) \quad (3)$$

$$\leq \sum_{i=1}^{\infty} (\lambda^*(A_i) + \varepsilon 2^{-i}) \quad (4)$$

$$= \sum_{i=1}^{\infty} \lambda^*(A_i) + \varepsilon \quad (5)$$

Therefore, $\lambda^*(A) \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$.

(2) Note that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$ for all $\varepsilon > 0$. Therefore,

$$\lambda^*([a, b]) \leq \inf_{\varepsilon > 0} \lambda^*(a - \varepsilon, b + \varepsilon) = b - a \quad (6)$$

Now show $\lambda^*([a, b]) \geq b - a$. We want to show that $\sum_{i=1}^{\infty} (b_i - a_i) \geq b - a$ for all possible covering of $[a, b]$, which implies the infimum of them is at least $b - a$.

Take an arbitrary covering $\{(a_i, b_i)\}_i$ of $[a, b]$. Since $[a, b]$ is compact, there exists a finite covering $[a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i)$ (reindexed), it suffices to show the finite sum $\sum_{i=1}^n (b_i - a_i) \geq b - a$.

(1) We firstly define an *interval* to be any open, closed or half-open intervals. The *length* of an interval is the difference between two end points.

Note that if an interval I contains a finite collection of disjoint sub-intervals, then the length of I is at least the sum of lengths of sub-intervals. The equality holds when I is exactly finite union of disjoint sub-intervals.

(2) Suppose $[a, b] \subseteq \bigcup_{i=1}^n (a_i, b_i)$, let $I_i = [a, b] \cap (a_i, b_i)$. Easy to verify that the length of $I_i \leq$ length of $(a_i, b_i) = b_i - a_i$. Moreover, $\bigcup_{i=1}^n I_i = [a, b] \cup \bigcup_{i=1}^n (a_i, b_i) = [a, b]$.

(3) For all i , define $I'_i = I_i \setminus (I_1 \cup I_2 \cup \dots \cup I_{i-1})$. This procedure allows us to express $[a, b]$ as a finite union of disjoint sub-intervals: $[a, b] = \bigcup_{i=1}^n I'_i$. Each I'_i is a finite union of disjoint intervals as well, the conventional notion of I'_i is well-defined. Then $b - a =$ sum of lengths of I'_i .

However, $\ell(I'_i) \leq \ell(I_i) \leq b_i - a_i$ and sum of lengths of $I'_i \leq$ sum of lengths of $I_i \leq \sum_{i=1}^n b_i - a_i$. Therefore, $b - a \leq \sum_{i=1}^n b_i - a_i \leq \sum_{i=1}^{\infty} b_i - a_i$. Hence, $b - a = \sum_{i=1}^{\infty} b_i - a_i$ and $\lambda^*[a, b] = b - a$ consequently. ■

1.7 Construct Lebesgue Measure

Definition 1.9. Let X be a set with outer measure μ^* . A set $B \subseteq X$ is μ^* -**measurable** if

$$\forall A \subseteq X, \mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \quad (7)$$

Theorem 1.1. For any set X with outer measure μ^* on it, let \mathcal{M}_{μ^*} denote the set of all μ^* -**measurable** sets. Then, \mathcal{M}_{μ^*} is a σ -algebra and $\mu^*|_{\mathcal{M}_{\mu^*}}$ (μ^* restricted to \mathcal{M}_{μ^*}) is a measure.

Proof. To show B is μ^* -measurable, it suffices to show that $\forall A \subseteq X, \mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$, because the opposite inequality always holds by sub-additivity.

(1.1) Let $A \subseteq X$, $\mu^*(A \cap \emptyset) + \mu^*(A \cap \emptyset^c) = \mu^*(A \cap \emptyset^c) = \mu^*(A)$, therefore, $\emptyset \in \mathcal{M}_{\mu^*}$.

(1.2) Let $A \subseteq X$ and $B \in \mathcal{M}_{\mu^*}$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) = \mu^*(A \cap (B^c)^c) + \mu^*(A \cap B^c)$.

Hence, $B^c \in \mathcal{M}_{\mu^*}$.

(1.3.1) Let $B_1, B_2 \in \mathcal{M}_{\mu^*}$, we are going to show $B_1 \cup B_2 \in \mathcal{M}_{\mu^*}$. Fix any $A \subseteq X$,

$$\mu^*(A \cap (B_1 \cup B_2)) = \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c) \quad (8)$$

$$= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) \quad (9)$$

Moreover,

$$\mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1^c \cap B_2^c) \quad (10)$$

Therefore,

$$\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) \quad (11)$$

$$= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \text{ since } B_2 \in \mathcal{M}_{\mu^*} \quad (12)$$

$$= \mu^*(A) \text{ since } B_1 \in \mathcal{M}_{\mu^*} \quad (13)$$

Therefore, \mathcal{M}_{μ^*} is an algebra.

(1.3.2) Now show that \mathcal{M}_{μ^*} is a σ -algebra. For any sequence of sets $A_i \in \mathcal{M}_{\mu^*}$, we can define $B_n := A_n \setminus \bigcup_{j=1}^{n-1} A_j$ such that $\bigcup B_i = \bigcup A_i$. Therefore, it suffices to show \mathcal{M}_{μ^*} is closed under countable disjoint unions.

We are going to show the union $\bigcup B_i$ is μ^* -measurable for any disjoint sequence of μ^* -measurable B_i 's.

Claim: let $A \subseteq X$, $\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c)$. The claim can be proved by induction on n .

When $n = 1$, $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ because B_1 is μ^* -measurable.

Suppose the claim holds for n , then

$$\mu^*(A \cap (\bigcup_{i=1}^n B_i)^c) = \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c \cap B_{n+1}) + \mu^*(A \cap (\bigcup_{i=1}^n B_i)^c \cap B_{n+1}^c) \quad (14)$$

because $B_{n+1} \in \mathcal{M}_{\mu^*}$. Moreover, since all B_i 's are disjoint, $B_{n+1} \subseteq B_i^c$ for all i . Hence,

$$B_{n+1} \subseteq \cap_{i=1}^n B_i^c = (\cup_{i=1}^n B_i)^c \quad (15)$$

Also,

$$(\cup_{i=1}^n B_i)^c \cap B_{n+1}^c = \cap_{i=1}^{n+1} B_i^c \quad (16)$$

Consequently,

$$\mu^*(A \cap (\cup_{i=1}^n B_i)^c) = \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\cup_{i=1}^{n+1} B_i)^c) \quad (17)$$

Hence,

$$\mu^*(A) = \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^n B_i^c)) \quad (18)$$

$$\geq \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^\infty B_i^c)) \quad (19)$$

$$= \sum_{i=1}^n \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (20)$$

Take $n \rightarrow \infty$

$$\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (21)$$

$$\geq \mu^*(A \cap \cup_{i=1}^\infty B_i) + \mu^*(A \cap (\cup_{i=1}^\infty B_i)^c) \quad (22)$$

Therefore, $\cup_{i=1}^\infty B_i$ is μ^* -measurable.

(2) Let B_1, B_2, \dots be a sequence of disjoint sets from \mathcal{M}_{μ^*} . Using the above fact and take $A = \cup_{i=1}^\infty B_i$,

$$\mu^*(A) \geq \mu^*(\cup_{i=1}^\infty B_i) + \mu^*(\emptyset) = \mu^*(\cup_{i=1}^\infty B_i) \quad (23)$$

The opposite inequality holds by sub-additivity. Therefore, μ^* is a measure on \mathcal{M}_{μ^*} . ■

Definition 1.10. Let λ^* be the Lebesgue outer measure on \mathbb{R} , then the collection \mathcal{M}_{λ^*} of λ^* -measurable sets is called the **Lebesgue σ -algebra**. The restriction $\lambda = \lambda^*|_{\mathcal{M}_{\lambda^*}}$, which is a measure on \mathcal{M}_{λ^*} , is called the **Lebesgue measure**. Any set in \mathcal{M}_{λ^*} is called a **Lebesgue measurable set**.

Theorem 1.2. $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$.

Proof. Note that $\{(-\infty, b] : b \in \mathbb{R}\}$ generates $\mathcal{B}(\mathbb{R})$, it suffices to show $\{(-\infty, b] : b \in \mathbb{R}\} \subseteq \mathcal{M}_{\lambda^*}$.

Let $B = (-\infty, b]$, we are going to show B is λ^* -measurable. Let $A \subseteq \mathbb{R}$ and (a_n, b_n) be a

sequence of open intervals covers A . For every $n \in \mathbb{N}$,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n) \cap (-\infty, b]) + \lambda^*((a_n, b_n) \cap (b, \infty)) \quad (24)$$

Three cases follow:

1. $b > b_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.
2. $b_n > b > a_n$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b]) + \lambda^*((b, b_n]) = b_n - a_n$.
3. $a_n > b$: $\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = \lambda^*((a_n, b_n)) = b_n - a_n$.

Therefore,

$$\lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) = b_n - a_n \quad (25)$$

By monotonicity and sub-additivity:

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(\cup(a_n, b_n) \cap B) + \lambda^*(\cup(a_n, b_n) \cap B^c) \quad (26)$$

$$\leq \sum \lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n) \cap B^c) \quad (27)$$

$$= \sum_{n=1}^{\infty} b_n - a_n \quad (28)$$

Take the infimum of all such covering, we can show

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A) \quad (29)$$

Therefore, B is μ^* -measurable and \mathcal{M}_{λ^*} is a σ -algebra containing all such intervals and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\lambda^*}$. ■

1.8 Lebesgue Measure on \mathbb{R}^d

Definition 1.11. Steps to construct Lebesgue measure on \mathbb{R}^d :

1. Define open cubes on \mathbb{R}^d as a Cartesian product of open intervals: $Q := \prod_{i=1}^d (a_i, b_i)$. Define Lebesgue outer measure:

$$\lambda^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \prod_{i=1}^d (b_{ni} - a_{ni}) : A \subseteq \bigcup_{n=1}^{\infty} Q_n \right\} \quad (30)$$

2. Show λ^* is an outer measure and $\lambda^*(Q) = \prod_{i=1}^d (b_i - a_i)$.
3. \mathcal{M}_{λ^*} is the Lebesgue σ -algebra on \mathbb{R}^d . Restricting λ^* on \mathcal{M}_{λ^*} defines the Lebesgue measure.
4. Show that any Borel set in \mathbb{R}^d is Lebesgue measurable by showing that there is a generating set of $\mathcal{B}(\mathbb{R}^d)$ is in \mathcal{M}_{λ^*} .

1.9 Uniqueness of the Lebesgue Measure

The next goal is to prove the uniqueness of Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ subject to the criterion that the measure of any interval (cube) is the volume in the usual sense (product of side lengths).

Theorem 1.3. Let λ be the Lebesgue measure on \mathbb{R}^d , then for any Lebesgue measurable set A ,

1. $\lambda(A) = \inf\{\lambda(U) : \text{open } U \supseteq A\},$
2. $\lambda(A) = \sup\{\lambda(K) : \text{compact } K \subseteq A\}.$

Proof. (1.1) WLOG $\lambda(A) < \infty$, by monotonicity, $\lambda(A) \leq \lambda(U)$ for any open cover, $\lambda(A) \leq \inf\{.. \}$.

(1.2) Let $\varepsilon > 0$, \exists a sequence of open intervals (R_i) such that

$$\lambda(A) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon \quad (31)$$

Let $U := \cup R_i$ open, hence $\inf\{.. \} \leq \lambda(U) \leq \sum_{i=1}^{\infty} \lambda(R_i) \leq \lambda(A) + \varepsilon$. Since this ε can be arbitrarily small, we conclude $\inf\{.. \} \leq \lambda(A)$.

(2.1) let A be a Lebesgue measurable set, assume A is bounded, so that $\lambda(A) < \infty$. Then there exists a compact $C \supseteq A$. $C \setminus A$ is Lebesgue measurable as well.

By conclusion of part (1), there exists a open set $U \supseteq C \setminus A$ such that

$$\lambda(C \setminus A) \leq \lambda(U) \leq \lambda(C \setminus A) + \varepsilon \quad (32)$$

Let $K = C \setminus U$, K is compact. Moreover, let $a \in K$, then $a \in C$ and $a \notin U$. Therefore, $a \notin C \setminus A$, it must be $a \in A$. Hence, $K \subseteq A$.

$$\lambda(K) = \lambda(C \setminus U) \quad (33)$$

$$\geq \lambda(C) - \lambda(U) \quad (34)$$

$$\geq \lambda(C) - (\lambda(C \setminus A) + \varepsilon) \quad (35)$$

$$= \lambda(C) - \lambda(C) + \lambda(A) - \varepsilon \quad (36)$$

$$= \lambda(A) - \varepsilon \quad (37)$$

Take $\varepsilon \rightarrow 0$ and $\lambda(A) \leq \sup\{.. \}$. By monotonicity, $\lambda(A) \geq \sup\{.. \}$.

(2.2) Other cases: suppose A is unbounded and $\lambda(A) > 0$. Take an arbitrary $b < \lambda(A)$. We will show that $\sup\{.. \} \geq b$, this will prove that $\lambda(A) \leq \sup\{.. \}$.

To show $\sup\{.. \} \geq b$, it suffices to show that there exists a compact set $K \subseteq A$ such that $\lambda(K) \geq b$.

Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of compact sets increasing to \mathbb{R}^d .

Then $A \cap C_j \uparrow A$ and $\lambda(A \cap C_1) < \infty$, which implies $\lambda(A) = \lim_{j \rightarrow \infty} \lambda(A \cap C_j)$. Since $b < \lambda(A)$, there exists j such that $\lambda(A \cap C_j) \geq b$, where $A \cap C_j$ is compact. Hence, $b \leq \sup\{.. \}$ and $\lambda(A) \leq \sup\{.. \}$. $\lambda(A) \geq \sup\{.. \}$ holds by monotonicity.

When $\lambda(A) = 0$, $0 \leq \lambda(K)$ for all K so that $0 \leq \sup\{.. \}$. The opposite inequality holds by monotonicity. ■

Lemma 1.1. For each $k \in \mathbb{Z}$, define **dyadic cubes** in \mathbb{R}^d as set in the following form:

$$\prod_{i=1}^d [j_i 2^{-k}, (j_i + 1) 2^{-k}) \quad (38)$$

where $j_i \in \mathbb{Z}$ for every i . Let \mathcal{D} denote the collection of dyadic cubes.

Then, any open set $U \subseteq \mathbb{R}^d$ can be expressed as a countable union of some members of \mathcal{D} .

A dyadic cube of side length 2^{-k} has a unique parent of side length 2^{-k+1} and a unique grandparent with side length 2^{-k+2} .

Proof. Given open set U , let \mathcal{D}_U denote the set of all dyadic half open cubes D such that $D \subseteq U$ but the parent of U does not fully contain U .

Claim 1: $U = \bigcup_{D \in \mathcal{D}_U} D$. Obviously, $\bigcup_{D \in \mathcal{D}_U} D \subseteq U$. To show the converse, take any $x \in U$, since U is open, there exists $D \in \mathcal{D}_U$ such that $x \in D \subseteq U$.

Let D_0 be the earliest ancestor of D such that $x \in D_0 \subseteq U$. Obviously, $D_0 \in \mathcal{D}_U$. Therefore, $U \subseteq \bigcup_{D \in \mathcal{D}_U} D$.

Claim 2: Two dyadic cubes can overlap if and only if one is the ancestor of the other. By construction, dyadic cubes in \mathcal{D}_U are disjoint.

Claim 3: \mathcal{D}_U is countable because \mathcal{D} is itself countable. ■

Proposition 1.4. Lebesgue measure is the only measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ which assigns the *correct volume* to any d -dimensional intervals or even any d -dimensional dyadic cube.

Proof. Let λ denote the Lebesgue measure, let μ be another measure satisfying the desired property.

By lemma, for all open set U , $\mu(U) = \sum_{j=1}^{\infty} \mu(D_j) = \sum_{j=1}^{\infty} \lambda(D_j) = \lambda(U)$, where $\{D_j\}$ is a collection of disjoint dyadic cubes contains with union U . Therefore, $\lambda(A) = \mu(A)$ for all open Borel set A .

Let $A \in \mathcal{B}(\mathbb{R}^d)$, let open $U \supseteq A$, then $\mu(A) \leq \mu(U) = \lambda(U)$ for all U . Taking the infimum over all $U \supseteq A$, we conclude $\mu(A) \leq \lambda(A)$ for all Borel set A .

Next, take any bounded Borel set A , let V be a bounded open set containing A . Then,

$$\mu(V) = \mu(A) + \mu(V \setminus A) \quad (39)$$

$$\leq \lambda(A) + \lambda(V \setminus A) \quad (40)$$

$$= \lambda(V) \quad (41)$$

But we also know that $\mu(V) = \lambda(V)$ since V is open, the inequality holds as equality. Moreover, the previous conclusion implies $\mu(A) \leq \lambda(A)$ and $\mu(V \setminus A) \leq \lambda(V \setminus A)$, it must be $\mu(A) = \lambda(A)$ and $\mu(V \setminus A) = \lambda(V \setminus A)$. Therefore, $\mu(A) = \lambda(A)$ for all bounded Borel set A .

Lastly, any Borel set can be written as a countable disjoint union of bounded Borel set, therefore, $\mu(A) = \lambda(A)$ for all Borel set A . ■

Proposition 1.5. The Lebesgue outer measure on \mathbb{R}^d is translation invariant. In particular, Lebesgue measure is translation invariant and any translation of Lebesgue measurable set is Lebesgue measurable.

Proof. $\lambda^*(A+x) = \lambda^*(A)$ follows the definition of λ^* : translate all covering intervals by $+x$ and the volumes of these intervals stay the same. Since λ is simply the restriction of λ^* on Lebesgue measurable sets, λ is translation invariant as well.

Now take Lebesgue measurable B , for all $A \subseteq \mathbb{R}^d$:

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c) \quad (42)$$

$$\implies \lambda^*(A-x) = \lambda^*((A-x) \cap B) + \lambda^*((A-x) \cap B^c) \quad (43)$$

Note that

$$(A-x) + x = A \quad (44)$$

$$(A-x) \cap B + x = A \cap (B+x) \quad (45)$$

$$(A-x) \cap B^c + x = A \cap (B+x)^c \quad (46)$$

By translational invariance of λ^* ,

$$\lambda^*(A) = \lambda^*(A \cap (B+x)) + \lambda^*(A \cap (B+x)^c) \quad (47)$$

Therefore, $B+x$ is Lebesgue measurable as well. ■

Theorem 1.4. Let μ be a non-zero measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is finite on bounded Borel sets and translation invariant. Then, $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$, where λ is the Lebesgue measure.

Remark 1.1. Borel σ -algebra is closed under translation.

Proof. Let $c = \mu([0,1]^d) \in (0, \infty)$. Then $[0,1]^d$ is the disjoint union of 2^{dk} half-open dyadic intervals with side length 2^{-k} . All of these sub-intervals have the same μ since μ is translation invariant. Therefore, for every dyadic sub-interval with side length 2^{-k} , $\mu(D) = 2^{-dk}c$.

Let $\nu(A) = \frac{1}{c}\mu(A)$, then ν is a measure that is finite on bounded sets and agrees with λ on all half-open dyadic cubes. By the previous proposition, λ is the only measure assign correct volumes to dyadic cubes, therefore, $\nu = \lambda$. ■

Theorem 1.5. Under the axiom of choice, there exists a non-Lebesgue subset of \mathbb{R} .

Proof. Todo. ■

2 Functions

2.1 Measurable Functions

Definition 2.1. A function $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is **measurable** if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

In this course, we mainly consider functions with extended- \mathbb{R} as codomain: $Y = [-\infty, \infty]$, denoted as \mathbb{R}^* .

Definition 2.2. The σ -algebra on \mathbb{R}^* is defined to be the σ -algebra generated by $\mathcal{B}(\mathbb{R}) \cup \{-\infty\} \cup \{\infty\}$.

Proposition 2.1.

$$\sigma(\mathcal{B}(\mathbb{R}) \cup \{-\infty\} \cup \{\infty\}) = \mathcal{B}(\mathbb{R}) \cup \{B \cup \{\infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (1)$$

$$\cup \{B \cup \{-\infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (2)$$

$$\cup \{B \cup \{-\infty, \infty\} : B \in \mathcal{B}(\mathbb{R})\} \quad (3)$$

Proposition 2.2. Equivalently, f is measurable if for every $t \in \mathbb{R}$,

$$\{x \in X : f(x) \leq t\} \in \mathcal{A} \quad (4)$$

$$\{x \in X : f(x) < t\} \in \mathcal{A} \quad (5)$$

$$\{x \in X : f(x) \geq t\} \in \mathcal{A} \quad (6)$$

$$\{x \in X : f(x) > t\} \in \mathcal{A} \quad (7)$$

More generally, to determine the measurability of $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$, we only need to check whether $f^{-1}(C) \in \mathcal{A}$ for all C in a generating collection \mathcal{C} of \mathcal{B} . The converse holds true trivially.

Proof. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be two measurable spaces, let \mathcal{C} be a collection of subsets of Y generates \mathcal{B} .

(\implies) Let f be a measurable function, then for every $C \in \mathcal{C} \subseteq \mathcal{B}$. Obviously, $f^{-1}(C) \in \mathcal{A}$ by definition.

(\impliedby) Suppose $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$. Define

$$\mathcal{B}_0 := \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{A}\} \supseteq \mathcal{C} \quad (8)$$

It's easy to check \mathcal{B}_0 is in fact a σ -algebra : $f^{-1}(\emptyset) = \emptyset \in \mathcal{A}$, $f^{-1}(B^c) = (f^{-1}(B))^c$, and $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$. Therefore, $\mathcal{B} \subseteq \mathcal{B}_0$ and all $B \in \mathcal{B}$ satisfies $f^{-1}(B) \in \mathcal{A}$. ■

Example 2.1. $f(x) = \mathbb{1}\{x \in \mathbb{Q}\}$ is measurable.

2.2 Simple Functions

Definition 2.3. A function $f : (X, \mathcal{A}) \rightarrow (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$ is called **simple** if there exists finitely many disjoint sets A_1, \dots, A_n and real numbers a_1, \dots, a_n such that

$$f(x) = \begin{cases} a_i & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i \forall i \in [n] \end{cases} \quad (9)$$

Let \mathcal{S} denote the set of simple functions, and \mathcal{S}^+ denote the set of non-negative simple functions.

Proposition 2.3. All simple functions are measurable.

Proof. For any subset of \mathbb{R}^* , the pre-image is either X or a union of some (potentially none) A_i 's. ■

2.3 Properties of Measurable Functions

Example 2.2. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$, then all of the following functions are measurable:

$$f(x, y) = x + y \quad (10)$$

$$f(x, y) = \max\{x, y\} \equiv x \vee y \quad (11)$$

$$f(x, y) = \min\{x, y\} \equiv x \wedge y \quad (12)$$

$$f(x, y) = x - y \quad (13)$$

$$f(x, y) = \alpha x \quad \alpha \in \mathbb{R} \quad (14)$$

Proposition 2.4 (Component-wise Measurable Functions). Let $f, g : (X, \mathcal{A}) \rightarrow (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$ be measurable, let $h(x) = (f(x), g(x)) \in \mathbb{R}^{*2}$, then f is measurable.

Proof.

$$h^{-1}([-\infty, t] \times [-\infty, s]) = f^{-1}([-\infty, t]) \cap g^{-1}([-\infty, s]) \in \mathcal{A} \quad (15)$$

And, $\mathcal{B}(\mathbb{R}^*)$ can be generated by sets with forms $[-\infty, t] \times [-\infty, s]$. ■

Proposition 2.5 (Composite of Measurable Functions). Let $(X, \mathcal{A}), (Y, \mathcal{B}), (Z, \mathcal{C})$ be measurable spaces, let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be measurable functions. Then, the composite $g \circ f : X \rightarrow Z$ is measurable.

Corollary 2.1. Let $f, g : X \rightarrow \mathbb{R}$ be measurable functions, then $f + g, f - g, \max\{f, g\}$, and $\min\{f, g\}$ are all measurable.

Proof. $f + g$ and $f - g$ can be written as the composition of $h_1(x) = (f(x), g(x))$ and $h_2(x, y) = x \pm y$, which are all measurable.

$f \vee g$ and $f \wedge g$ are measurable as special cases of next proposition. ■

Proposition 2.6. Let f_1, f_2, \dots be a sequence of measurable maps from $(X, \mathcal{A}) \rightarrow (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$, then $\sup_n f_n$ and $\inf_n f_n$ are measurable.

Proof. Note $\{x \in X : \sup_n f_n \leq t\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n \leq t\} \in \mathcal{A}$ for every t , therefore the supremum is measurable. ■

Corollary 2.2. $\limsup f_n$ and $\liminf f_n$ are measurable.

Proof. Let $g_k = \sup_{n \geq k} f_n$, g_k is measurable. $\limsup f_n = \inf_k g_k$ is measurable as well. Similar proof for the measurability of $\liminf f_n$. ■

Proposition 2.7. Let f and g be \mathbb{R}^* -valued measurable functions. Then sets

$$\{x \in A : f(x) < g(x)\}, \{x \in A : f(x) \leq g(x)\} \quad (16)$$

are measurable.

Proof.

$$\{x \in A : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} (\{x \in A : f(x) < r\} \cap \{x \in A : r < g(x)\}) \quad (17)$$

■

Corollary 2.3. Let $u, v : X \rightarrow \mathbb{R}^*$ be measurable functions, then $\{x \in X : u(x) = v(x)\}$ is measurable.

Proof. Note that $\{x \in X : u(x) = v(x)\} = \{x \in X : u(x) \leq v(x)\} \cap \{x \in X : u(x) \geq v(x)\}$. ■

Corollary 2.4. Let $\{f_n\}$ be a sequence of measurable functions from $(X, \mathcal{A}) \rightarrow (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*))$. Then,

$$\{x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\} \quad (18)$$

is measurable.

Proof. Note that $\{x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\} = \{x \in X : \liminf f_n(x) = \limsup f_n(x)\}$, the result follows from previous lemma. ■

Corollary 2.5. If $\{f_n\}$ is a sequence of measurable functions such that $\lim f_n(x)$ exists for all $x \in X$, then $\lim f_n$ is a measurable function on (X, \mathcal{A}) .

Proof. In this case, $\{x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\} = X$, and $\lim f_n = \liminf f_n$ on X . ■

Corollary 2.6. If $\{f_n\}$ is a sequence of measurable function from X to $[0, \infty]$, then $\sum_{n=1}^{\infty} f_n$ is measurable.

Proof. Follows the previous corollary directly: define $g_k = \sum_{n=1}^k f_n$ and $\lim_{k \rightarrow \infty} g_k = \sum_{n=1}^{\infty} f_n$. ■

3 Integrals

3.1 Integrating Simple Functions

Definition 3.1. Let $f \in \mathbb{S}^+$ with representation $\{(A_i, a_i)\}_{i=1}^n$. WLOG, $\bigcup_{i=1}^n A_i = X$. Then, define

$$\int_X f \, d\mu := \sum_{i=1}^n a_i \mu(A_i) \quad (1)$$

Proposition 3.1. The notion of integral on simple functions is well defined. Specifically, let $\{(A_i, a_i)\}_{i=1}^n$ and $\{(B_j, b_j)\}_{j=1}^m$ be any two representations of f , $\sum_{i=1}^n a_i \mu(A_i) = \sum_{j=1}^m b_j \mu(B_j)$.

Proof. First note that $\{A_i \cap B_j\}_{i,j}$ are disjoint sets with union X . Moreover, for any i, j , if $A_i \cap B_j \neq \emptyset$, take some $x \in A_i \cap B_j$, $f(x) = a_i = b_j$. Therefore, $a_i \mu(A_i \cap B_j) = b_j \mu(A_i \cap B_j)$ since either $a_i = b_j$ or $\mu(A_i \cap B_j) = \mu(\emptyset) = 0$.

$$\sum_{i=1}^n a_i \mu(A_i) = \sum_{i=1}^n a_i \sum_{j=1}^m \mu(A_i \cap B_j) \quad (2)$$

$$= \sum_{j=1}^m b_j \sum_{i=1}^n \mu(A_i \cap B_j) \quad (3)$$

$$= \sum_{j=1}^m b_j \mu(B_j) \quad (4)$$

■

3.2 Integrating Measurable Functions

Definition 3.2. For a non-negative measurable function $f : X \rightarrow [0, \infty]$, define its Lebesgue integral as

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \text{ is a non-negative simple function such that } g \leq f \right\} \quad (5)$$

For any measurable $f : X \rightarrow [-\infty, \infty]$, let

$$f^+(x) = \max\{f(x), 0\} \quad (6)$$

$$f^-(x) = -\min\{f(x), 0\} \quad (7)$$

So that $f = f^+ - f^-$, and f is measurable if and only if both f^+ and f^- are measurable.

If at least one of $\int f^+ \, d\mu$, $\int f^- \, d\mu$ is finite, the integral of f exists (well-defined) and is defined as

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu \quad (8)$$

If both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite, f is said to be **integrable**.

3.3 Properties of Integral of Non-negative Simple Functions

Proposition 3.2 (Linearity). If f, g are non-negative simple functions, then

$$\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu \quad (9)$$

Moreover, for any $\alpha \geq 0$,

$$\int \alpha f \, d\mu = \alpha \int f \, d\mu \quad (10)$$

Proof. Let f and g be simple functions represented by $\{(A_i, a_i)\}_{i=1}^n$ and $\{(B_j, b_j)\}_{j=1}^m$. WLOG, $\cup A_i = \cup B_j = X$. Then $f + g$ is a simple function with representation $\{(A_i \cap B_j, a_i + b_j)\}_{i,j}$, where $\cup_{i,j} A_i \cap B_j = X$. ■

Proposition 3.3. Let f, g be non-negative simple functions with $f \geq g$ everywhere. Then $\int f d\mu \geq \int g d\mu$.

Proof. Let f and g be simple functions represented by $\{(A_i, a_i)\}_{i=1}^n$ and $\{(B_j, b_j)\}_{j=1}^m$.

Claim: $a_i \mu(A_i \cap B_j) \geq b_j \mu(A_i \cap B_j)$ for every (i, j) . If $A_i \cap B_j \neq \emptyset$, then taking some $x \in A_i \cap B_j$ implies $a_i \geq b_j$. If $A_i \cap B_j = \emptyset$, the equality holds trivially.

Note that $\int f$ and $\int g$ can be written as $\sum_{i,j} a_i \mu(A_i \cap B_j)$ and $\sum_{i,j} b_j \mu(A_i \cap B_j)$ respectively, therefore $\int f \geq \int g$ by the previous claim. ■

Proposition 3.4 (Approximation using Simple Functions). Let $f : X \rightarrow [0, \infty]$ be a measurable function. Then there exists an increasing sequence of non-negative simple functions f_n such that $f_n \leq f_{n+1}$ and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (11)$$

for all x .

Proof. For each n and $1 \leq k \leq n2^n$, let

$$A_{n,k} = \left\{ x \in X : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\} \quad (12)$$

Define

$$f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } x \in A_{n,k} \\ n & \text{otherwise} \end{cases} \quad (13)$$

That is, for a $x \in X$, if $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$ for some k , we take $f_n(x) = \frac{k-1}{2^n}$; if $f(x) \geq n$, we define $f_n(x) = n$. Clearly, each f_n is a simple function.

Claim 1: $f_n \leq f_{n+1}$. Easy to verify.

Claim 2: $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. Let $x \in X$, (i) if $f(x) = \infty$, then $f_n(x) = n$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} f_n(x) = \infty = f(x)$.

(ii) if $f(x) < \infty$, then $\exists n_0$ such that $f(x) < n_0$. For every $n \geq n_0$, $x \in A_{n,k}$ for some k such that $f_n(x) = \frac{k-1}{2^n}$ and $\frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}$. Therefore, for all $n \geq n_0$, $|f_n(x) - f(x)| < \frac{1}{2^n}$, which implies $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. ■

Proposition 3.5 (Monotone Convergence 1: $\mathbb{S}_+ \uparrow \mathbb{S}_+$). Let f_n be a sequence of non-negative simple functions that increase to another non-negative simple function f at each point, then

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \quad (14)$$

Proof. By monotonicity, $f_n \leq f$ for all n and $\int f \, d\mu \geq \lim \int f_n \, d\mu$.

Fix $0 < \varepsilon < 1$ and define $g = (1 - \varepsilon)f$. Suppose f is represented by (A_i, a_i) . Then for every n, i , define

$$A_{n,i} = \{x \in A_i : f_n(x) \geq (1 - \varepsilon)a_i\} \quad (15)$$

Define

$$g_n(x) = \begin{cases} (1 - \varepsilon)a_i & \text{if } x \in A_{n,i} \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

In order to show $\int f \, d\mu \leq \lim \int f_n \, d\mu$, we are constructing this g_n satisfying

$$(1 - \varepsilon) \int f \, d\mu \leq \lim \int g_n \, d\mu \leq \lim \int f_n \, d\mu \leq \int f \, d\mu \quad (17)$$

where the last equality has been shown above. The equality can then be shown by taking $\varepsilon \rightarrow 0$ and using Squeeze theorem. Note that $(1 - \varepsilon) \int f \, d\mu \not\leq \int g_n \, d\mu$, only the limit does.

By construction, $g_n \leq f_n$ and $\int g_n \, d\mu \leq \int f_n \, d\mu$ as a result.

$$\lim_n \int f_n \, d\mu \geq \lim_n \int g_n \, d\mu \quad (18)$$

$$= \lim_n \sum_{i=1}^K (1 - \varepsilon)a_i \mu(A_{n,i}) \quad (19)$$

$$= \sum_{i=1}^K (1 - \varepsilon)a_i \lim_n \mu(A_{n,i}) \quad (20)$$

$$= \sum_{i=1}^K (1 - \varepsilon)a_i \mu(A_i) \text{ Since for all } i, A_{n,i} \uparrow A_i \text{ as } n \rightarrow \infty. \quad (21)$$

$$= (1 - \varepsilon) \int f \, d\mu \quad (22)$$

Taking $\varepsilon \rightarrow 0$ completes the proof. ■

Proposition 3.6 (Monotone Convergence 2: $\mathbb{S}_+ \uparrow$ Measurable). Let $f : X \rightarrow [0, \infty]$ be a measurable function. Let f_n be a sequence of non-negative simple functions such that $f_n \uparrow f$ point-wise. Then

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (23)$$

Proof. The proof follows the previous proposition and the definition of $\int f \, d\mu$. Since $f_n \uparrow f$, $f_n \leq f$ and $\int f_n \leq \int f$ for all n . $\int f_n$ is a bounded monotone sequence, therefore $\lim \int f_n$ exists and $\leq \int f$.

To show the other equality, it suffices to prove $\lim \int f_n \geq \int g$ for any non-negative simple functions $g \leq f$.

Define $g_n = \min\{g, f_n\}$, easy to show that $g_n(x) \leq g_{n+1}(x)$.

$$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} \min\{g, f_n\} \quad (24)$$

$$= \min\{g(x), f(x)\} \quad (25)$$

$$= g(x) \quad (26)$$

since $f_n \uparrow f$ and $g \leq f$.

By the previous proposition, $\int g \, d\mu = \lim \int g_n \, d\mu$ since g_n and g are non-negative simple functions. Since $g_n \leq f_n$ everywhere, so $\int g_n \, d\mu \leq \int f_n \, d\mu$. Taking limit on both sides implies $\int g \leq \lim \int f_n$. ■

Proposition 3.7 (Vector Space Properties for Non-negative Integrable Functions). Let $f, g : X \in [0, \infty]$ be integrable (of course, measurable as well) functions and $\alpha \geq 0$. Then

$$1. \int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

$$2. \int \alpha f \, d\mu = \alpha \int f \, d\mu.$$

$$3. \text{ If } f \geq g \text{ everywhere, then } \int f \, d\mu \geq \int g \, d\mu.$$

Proof. We know that there exists sequences of non-negative simple functions f_n and g_n such that $f_n \uparrow f$ and $g_n \uparrow g$. Note that $f_n + g_n$ is a sequence of simple functions increases to $f + g$. Therefore,

$$\int (f + g) d\mu = \lim_{n \rightarrow \infty} \int (f_n + g_n) \, d\mu \quad (27)$$

$$= \lim_{n \rightarrow \infty} \left(\int f_n \, d\mu + \int g_n \, d\mu \right) \quad (28)$$

$$= \lim_{n \rightarrow \infty} \int f_n \, d\mu + \lim_{n \rightarrow \infty} \int g_n \, d\mu \quad (29)$$

$$= \int f \, d\mu + \int g \, d\mu \quad (30)$$

Similarly, taking $\alpha f_n \uparrow \alpha f$ leads to the second result.

Finally, if $f \geq g$ everywhere, then

$$\{h \in \mathbb{S}_+ \text{ and } h \leq g\} \subseteq \{h \in \mathbb{S}_+ \text{ and } h \leq f\} \quad (31)$$

Therefore, the supremum of integrals of functions from a larger collection is larger. ■

3.4 Linearity of Lebesgue Integral for Arbitrary Integrable Functions

Theorem 3.1 (Vector Space Property of Integral Functions). Let (X, \mathcal{A}, μ) be a measure space, let $f, g : X \rightarrow \mathbb{R}^*$ be integrable functions, let $\alpha \in \mathbb{R}$. Then, $f + g$ and αf are integrable, and

$$\int f + g d\mu = \int f d\mu + \int g d\mu \quad (32)$$

$$\int \alpha f d\mu = \alpha \int f d\mu \quad (33)$$

Proof. It's easy to check that $(f + g)^+ \leq f^+ + g^+$ and $(f + g)^- \leq f^- + g^-$. By monotonicity, $\int (f + g)^+ d\mu, \int (f + g)^- d\mu < \infty$. Therefore, $f + g$ is integrable.

Moreover, $f + g = f^+ - f^- + g^+ - g^- \iff f + g + f^- + g^- = f^+ + g^+$. We can apply the linearity of non-negative integrable functions to derive the result.

When $\alpha \geq 0$, $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. The proof for cases with $\alpha < 0$ is similar. ■

Corollary 3.1. Let f, g be integrable functions such that $f \geq g$, then $\int f d\mu \geq \int g d\mu$.

Proof. Let $h = f - g = f + (-1)g \geq 0$, which is integrable by the previous theorem. And $\int h d\mu \geq 0$ since it's the supremum of integrals for simple functions less than h , which includes the zero function (has zero integral). ■

Lemma 3.1. A function f is integrable if and only if $|f|$ is integrable.

Proof. Note that $|f| = f^+ + f^-$, and $\int f^+ + f^- d\mu < \infty$ by the integrability of f . Therefore, $|f|$ is integrable.

Moreover, $|f|^+ = f^+ + f^-$, therefore, the integrability of $|f|$ implies both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite. ■

Proposition 3.8. All integrable function f satisfies the triangle inequality

$$\left| \int f d\mu \right| \leq \int |f| d\mu \quad (34)$$

Proof.

$$\left| \int f d\mu \right| = \left| \int f^+ - f^- d\mu \right| \quad (35)$$

$$= \left| \int f^+ d\mu - \int f^- d\mu \right| \quad (36)$$

$$\leq \left| \int f^+ d\mu \right| + \left| \int f^- d\mu \right| \quad (37)$$

$$= \int f^+ d\mu + \int f^- d\mu \quad (38)$$

$$= \int |f| d\mu \quad (39)$$

■

4 Limit Theorems (i.e., when we can exchange limits and integrals)

Theorem 4.1 (Monotone Convergence Theorem). Let (X, \mathcal{A}, μ) be a measure space, let $f_n : X \rightarrow [0, \infty]$ be a non-decreasing sequence of measurable functions converge to f . Then,

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (1)$$

Proof. f measurable since $f = \lim_n f_n = \liminf_n f_n$. Moreover, $\int f_n \, d\mu$ is a non-decreasing sequence to the limit $\int f \, d\mu$, therefore $\int f \, d\mu \geq \lim_n \int f_n \, d\mu$.

For each $n \in \mathbb{N}$, there exists a non-decreasing sequence of non-negative simple functions $g_{n,k}$ converges to f_n . Define

$$h_n = \max\{g_{1,n}, g_{2,n}, \dots, g_{n,n}\} \quad (2)$$

Note that h_n is a non-decreasing sequence since

$$h_{n+1} = \max\{g_{1,n+1}, g_{2,n+1}, \dots, g_{n+1,n+1}\} \quad (3)$$

$$\geq \max\{g_{1,n+1}, g_{2,n+1}, \dots, g_{n,n+1}\} \quad (4)$$

$$\geq \max\{g_{1,n}, g_{2,n}, \dots, g_{n,n}\} = h_n \quad (5)$$

Moreover, for any $m \in \mathbb{N}$, for any $n \geq m$,

$$h_n(x) = \max\{g_{1,n}, g_{2,n}, \dots, g_{n,n}\} \geq g_{m,n} \quad (6)$$

Therefore, by taking the limit $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} h_n(x) \geq \lim_{n \rightarrow \infty} g_{m,n} = f_m \quad (7)$$

Taking limit $m \rightarrow \infty$ on both sides

$$\lim_n h_n(x) = \lim_m \lim_n h_n(x) \geq \lim_m f_m = f \quad (8)$$

$$\implies \int \lim_n h_n(x) \, d\mu \geq \int f \, d\mu \quad (9)$$

Note that, by construction,

$$h_n = \max\{g_{1,n}, g_{2,n}, \dots, g_{n,n}\} \leq \max\{f_1, \dots, f_n\} = f_n \quad (10)$$

Therefore,

$$\int \lim_{n \rightarrow \infty} f_n(x) \, d\mu \geq \int f \, d\mu \quad (11)$$

■

Corollary 4.1. Let (f_n) be a sequence (not necessarily increasing) non-negative measurable functions, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu \quad (12)$$

Theorem 4.2 (Fatou's Lemma). Let f_n be a sequence of non-negative measurable functions, then

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu \quad (13)$$

Proof. Define $g_n = \inf_{k \geq n} f_k$, then g_n is an increasing sequence of non-negative functions. By construction, $\int g_n \, d\mu \leq \inf_{k \geq n} \int f_k \, d\mu$. By MCT,

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu = \int \lim_{n \rightarrow \infty} g_n \, d\mu \quad (14)$$

$$= \lim_{n \rightarrow \infty} \int g_n \, d\mu \quad (15)$$

$$\leq \lim_{n \rightarrow \infty} \inf_{k \geq n} \int f_k \, d\mu \quad (16)$$

$$= \liminf_{n \rightarrow \infty} \int f_n \, d\mu \quad (17)$$

■

Theorem 4.3 (Lebesgue's Dominated Convergence Theorem). Let (X, \mathcal{A}, μ) be a measure space, let f and f_n be \mathbb{R}^* -valued measurable functions on X such that $f_n \rightarrow f$ point-wise. If there exists a non-negative integrable function g such that $|f_n| \leq g$ for all n , then, all f and f_n are integrable, moreover,

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu \quad (18)$$

Proof. Since $|f_n| \leq g$, all f_n are integrable. The limit f also satisfies $|f| \leq g$ and is integrable.

For now, assume f_n are \mathbb{R} -valued instead of \mathbb{R}^* -valued.

Note that $f + g = \lim_{n \rightarrow \infty} f_n + g$ is non-negative (because of the dominance) and integrable, by Fatou's lemma

$$\int f + g \, d\mu = \int \liminf_{n \rightarrow \infty} f + g \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n + g \, d\mu \quad (19)$$

$$= \liminf_{n \rightarrow \infty} \int f_n \, d\mu + \int g \, d\mu \quad (20)$$

$$\implies \int f \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu \quad (21)$$

Similarly, $g - f = \lim_{n \rightarrow \infty} g - f_n$ is non-negative and integrable as well, by Fatou's lemma

$$\int g - f \, d\mu = \int \liminf g - f_n \, d\mu \leq \liminf \int g - f_n \, d\mu \quad (22)$$

$$\implies - \int f \, d\mu \leq - \liminf \int f_n \, d\mu \quad (23)$$

$$\implies \int f \, d\mu \geq \limsup \int f_n \, d\mu \quad (24)$$

Also, $\liminf \int f_n \, d\mu \leq \limsup \int f_n \, d\mu$, therefore,

$$\liminf \int f_n \, d\mu \geq \int f \, d\mu \geq \limsup \int f_n \, d\mu \geq \liminf \int f_n \, d\mu \quad (25)$$

$$\implies \int f \, d\mu = \lim \int f_n \, d\mu \quad (26)$$

■

Proposition 4.1 (A Stronger Result). Given assumptions of the dominated convergence theorem, f_n L^1 -converges to f .

$$\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu = 0 \quad (27)$$

Proof. Note that $|f_n - f| \rightarrow 0$ point-wise, and $|f_n - f| \leq 2g$. The dominated convergence theorem suggests $\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu = \int 0 \, d\mu = 0$. ■

4.1 The Notion of Almost Everywhere

Definition 4.1. Let (X, \mathcal{A}, μ) be a measure space, a set $N \subseteq X$ (not necessarily measurable) is called "negligible w.r.t. μ " if $N \subseteq A$ for some $A \in \mathcal{A}$ with $\mu(A) = 0$.

A property is said to hold **almost everywhere** w.r.t. μ (denoted as μ -a.e.) if the set on which this property fails is negligible.

Proposition 4.2. Let $f : X \rightarrow [0, \infty]$ be an integrable function, then f is finite μ -a.e.

Proof. Let $A := f^{-1}(\infty)$, define $h_n(x) := n \mathbb{1}\{x \in A\}$. Clearly, h_n is a simple function $\leq f$ for every n , by monotonicity, $\int f \, d\mu \leq \int h_n \, d\mu = n\mu(A)$. Taking $n \rightarrow \infty$ leads to a contradiction. ■

Corollary 4.2. If $f : X \rightarrow \mathbb{R}^*$ is integrable w.r.t. μ , then $|f| < \infty$ μ -a.e.

Proof. f is integrable implies $\int f^+ \, d\mu, \int f^- \, d\mu < \infty$. Then, by the previous proposition, $f^+ < \infty$ except for a negligible set A , and $f^- < \infty$ except for a negligible set B . Therefore, $|f| = \infty$ on set $A \cup B$, which is negligible as well. ■

Proposition 4.3. Let $f : X \rightarrow [0, \infty]$ be measurable, then

$$\int f \, d\mu = 0 \iff f = 0 \, \mu - a.e. \quad (28)$$

Proof. (\implies) Suppose $f = 0$ a.e., for every simple function $g \leq f$, let (a_i, A_i) be the representation of g , then $\int g \, d\mu = 0$ by definition. Suppose $a_i > 0$ for some A_i , then $f(x) \geq a_i > 0$ for all $x \in A_i$, since $f = 0$ a.e., $\mu(A_i) = 0$. Therefore, $\int g \, d\mu = 0$, so is the integral of f .

(\impliedby), suppose $\int f \, d\mu = 0$, note that

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > 1/n\} \quad (29)$$

Define $A_n = \{x : f(x) > 1/n\}$, then A_n is an increasing sequence of sets, therefore, suppose there exists some A_n with $\mu(A_n) > 0$, define $g(x) = \frac{1}{n} \mathbb{1}\{x \in A_n\}$. $f \geq g$ by construction, so that $\int f \, d\mu \geq \int g \, d\mu = \frac{1}{n} \mu(A_n) > 0$. This leads to a contradiction, so all $\mu(A_n) = 0$, and $\mu(\{x : f(x) > 0\}) = \lim_n \mu(A_n) = 0$. ■

Corollary 4.3. Let $f : X \rightarrow \mathbb{R}^*$ be a measurable function,

$$f = 0 \, \mu - a.e. \implies \int f \, d\mu = 0 \quad (30)$$

Proof. $f = 0$ a.e. implies $f^+, f^- = 0$ a.e., apply the previous proposition, $\int f^+ \, d\mu = \int f^- \, d\mu = 0$, so is $\int f \, d\mu$.

Note the converse is not true, it is possible that $\int f^+ \, d\mu = \int f^- \, d\mu \neq 0$ such that $\int f \, d\mu = 0$. ■

Corollary 4.4. Let $f, g : X \rightarrow \mathbb{R}^*$ be integrable functions, then

$$f = 0 \, \mu - a.e. \implies \int f \, d\mu = \int g \, d\mu \quad (31)$$

Proof. Let $\tilde{f} = f(x) \mathbb{1}\{x \in \mathbb{R}\}$ and $\tilde{g} = g(x) \mathbb{1}\{x \in \mathbb{R}\}$, we are doing this to avoid subtracting infinity from infinity. $|\tilde{f}|$ and $|\tilde{g}|$ are bounded by $|f|$ and $|g|$ and are integrable. Moreover, $f = \tilde{f} = g = \tilde{g}$ a.e. by construction. Lastly, since $|\tilde{f}|, |\tilde{g}| < \infty$, we can write

$$\int \tilde{f} - \tilde{g} \, d\mu = \int \tilde{f} \, d\mu - \int \tilde{g} \, d\mu = 0 \quad (32)$$

$$\implies \int f \, d\mu = \int \tilde{f} \, d\mu = \int g \, d\mu = \int \tilde{g} \, d\mu \quad (33)$$

■

Proposition 4.4. Monotone convergence theorem and dominated convergence theorem holds even if $f_n \rightarrow f$ a.e. In DCT, we can also have $|f_n| \leq g$ a.e.

Proof for MCT. Suppose $f_n \geq 0$ a.e.

$$A = \{x : f_n(x) \geq 0 \, \forall n \wedge \lim_{n \rightarrow \infty} f_n(x) = f(x)\} \quad (34)$$

Therefore, $A^c = \bigcup_n \{x : f_n(x) < 0\} \cup \{x : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}$, which is a countable union of measure zero sets, hence $\mu(A^c) = 0$.

Define $\tilde{f}_n = \mathbb{1}_A f_n$ and $\tilde{f} = \mathbb{1}_A f$, apply the original version of MCT on \tilde{f}_n and \tilde{f} , then exert the fact that $\int \tilde{f}_n d\mu = \int f_n d\mu$ and $\int \tilde{f} d\mu = \int f d\mu$. ■

5 Integral of Complex-Valued Functions

Definition 5.1. A function $f : X \rightarrow \mathbb{C}$ is called **measurable** if both $\Re(f)$ and $\Im(f)$ (both are real-valued functions) are measurable. Similarly, f is **integrable** if both its real and imaginary parts are integrable.

Define

$$\int f d\mu = \int \Re(f) d\mu + i \int \Im(f) d\mu \quad (1)$$

Proposition 5.1. Let f, g be integrable complex-valued functions, then

1. $\int (f + g) d\mu = \int f d\mu + \int g d\mu$.
2. for all $\alpha \in \mathbb{C}$, $\int (\alpha f) d\mu = \alpha \int f d\mu$.

Proposition 5.2 (Triangle Inequality). Let $f : X \rightarrow \mathbb{C}$ be an integrable function, then

$$\left| \int f d\mu \right| \leq \int |f| d\mu \quad (2)$$

Proof. Note that there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$\left| \int f d\mu \right| = \alpha \int f d\mu \quad (3)$$

To see this point, let $z = re^{i\theta} \in \mathbb{C}$ so that $|z| = r$, let $\alpha = e^{-i\theta}$, which satisfies $\alpha z = r = |z|$.

Therefore,

$$\left| \int f d\mu \right| = \alpha \int f d\mu \quad (4)$$

$$= \int (\alpha f) d\mu \quad (5)$$

$$= \int \Re(\alpha f) d\mu + i \int \Im(\alpha f) d\mu \quad (6)$$

$$\implies \int \Im(\alpha f) d\mu = 0 \quad (7)$$

Therefore,

$$\left| \int f d\mu \right| = \int \Re(\alpha f) d\mu \leq \int |\alpha f| d\mu = \int |f| d\mu \quad (8)$$

where the last step holds because $|\alpha| = 1$. ■

6 Convergence of Measurable Functions

Definition 6.1. Let (X, \mathcal{A}, μ) be a measure space, let $\{f_n\}_n$ be a sequence of real-valued measurable functions on X , let $f : X \rightarrow \mathbb{R}$ be a measurable function.

Then $f_n \rightarrow f$ **in measure** if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0 \quad (1)$$

Remark 6.1. Convergence almost everywhere does not imply convergence in measure.

Counter-example. ■

Remark 6.2. Convergence in measure does not imply convergence almost everywhere (even if we are considering a finite measure).

Counter-example. ■

Proposition 6.1. Let μ be a finite measure, then convergence a.e. implies convergence in measure.

Proof. Suppose $f \rightarrow f_n$ a.e. Let $\varepsilon > 0$. Note that if there exists x such that $|f_n - f(x)| \geq \varepsilon$ for infinitely many n , then $f_n \not\rightarrow f$ at x . Therefore,

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\}) \leq \mu(\{x : f_n(x) \not\rightarrow f(x)\}) = 0 \quad (2)$$

Further, note that

$$\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} \{x : |f_k(x) - f(x)| > \varepsilon\}}_{B_n} \quad (3)$$

Where $x \in B_n$ indicates there exists a $k \geq n$ such that $|f_k(x) - f(x)| > \varepsilon$. If we take the intersection of all B_n , it means for all $n \in \mathbb{N}$, there exists $k \geq n$ such that $|f_k(x) - f(x)| > \varepsilon$, which is equivalent to saying there are infinitely many k such that $|f_k(x) - f(x)| > \varepsilon$.

Clearly $B_1 \supseteq B_2 \supseteq \dots$, there must exist some B_i such that $\mu(B_i) < \infty$ since μ is a finite measure.

Therefore,

$$0 = \mu(\{x : |f_n(x) - f(x)| > \varepsilon \text{ for infinitely many } n\}) = \lim_{n \rightarrow \infty} \mu(B_n) \quad (4)$$

Hence, $\lim_{n \rightarrow \infty} \mu(B_n) = 0$. However, $B_n \supseteq \{x : |f_n(x) - f(x)| > \varepsilon\}$, therefore,

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0 \quad (5)$$

■

Proposition 6.2. Let f_n be a sequence of measurable real-valued functions converge to a measurable f in measure, then there exists a subsequence of f_n converges to f a.e.

Proof. Suppose $f_n \rightarrow f$ in measure, take $\varepsilon = 1$, there exists infinitely many n_1 such that

$$\mu(\{x : |f_{n_1} - f(x)| > 1\}) < 2^{-1} \quad (6)$$

Then for every k , we can choose $n_k > n_{k-1}$ such that

$$\underbrace{\mu(\{x : |f_{n_k} - f(x)| > \frac{1}{k}\})}_{A_k} < 2^{-k} \quad (7)$$

Let $B = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$, define $B_j = \bigcup_{k=j}^{\infty} A_k$. Note that for all j , $B \subseteq B_j$, therefore,

$$\mu(B) \leq \mu(B_j) = \mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq \sum_{k=j}^{\infty} \mu(A_k) < \sum_{k=j}^{\infty} 2^{-j+1} \quad (8)$$

Take $j \rightarrow \infty$, $\mu(B) = 0$. If $x \notin B$, $x \in B^c = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} A_k^c$, which means $\exists j$ such that $x \in A_k^c$ for all $k \geq j$. That is

$$\exists j \text{ s.t. } \forall k \geq j \quad |f_{n_k} - f(x)| \leq \frac{1}{k} \quad (9)$$

Therefore, this subsequence n_k converges to $f(x)$ a.e. ■

Lemma 6.1 (Borel-Cantelli Lemma). If A_1, A_2, \dots , is a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(A_k) < \infty \quad (10)$$

then

$$\mu(\{x : x \in \text{infinitely many } A_k\}) = 0 \quad (11)$$

Proof. Define

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad (12)$$

Easy to verify that $x \in B$ if and only if $x \in$ infinitely many A_k . For every j ,

$$B \subseteq \bigcup_{k=j}^{\infty} A_k \quad (13)$$

Hence

$$\mu(B) \leq \mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq \sum_{k=j}^{\infty} \mu(A_k) \rightarrow 0 \text{ as } j \rightarrow \infty \quad (14)$$

Therefore, $\mu(B) = 0$. ■

Theorem 6.1 (Egorov's Theorem). Let (X, \mathcal{A}, μ) be a measure space with $\mu(X) < \infty$. Let f_n be a sequence of measurable \mathbb{R} -valued functions converging a.e. to a \mathbb{R} -valued function f .

Then for all $\varepsilon > 0$, \exists a set $B \in \mathcal{A}$ such that

1. $\mu(B^c) < \varepsilon$,
2. and $f_n \rightarrow f$ uniformly on B .

Proof. Let $\varepsilon > 0$.

For all $n \in \mathbb{N}$, define

$$g_n(x) := \sup_{k \geq n} |f_k(x) - f(x)| \quad (15)$$

since $f_n \rightarrow f$ a.e., $g_n(x)$ is finite a.e. Moreover, $g_n(x) \rightarrow 0$ a.e. as $n \rightarrow \infty$ (both holds where $f_n \rightarrow f$).

Since $\mu(X) < \infty$, $g_n(x) \rightarrow 0$ in measure by previous results. Then, for every $k \in \mathbb{N}$, there exists n_k such that

$$\mu \left(\left\{ x : g_{n_k}(x) > \frac{1}{k} \right\} \right) < \frac{\varepsilon}{2^k} \quad (16)$$

Since there are infinitely many n_k to choose, we may choose an increasing sequence of n_k 's. Define

$$B^c = \left\{ x : g_{n_k}(x) > \frac{1}{k} \text{ for some } k \right\} \quad (17)$$

Then,

$$\mu(B^c) = \mu \left(\bigcup_{k=1}^{\infty} \left\{ x : g_{n_k}(x) > \frac{1}{k} \right\} \right) \quad (18)$$

$$\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \quad (19)$$

Lastly, we can show that $f_n \rightarrow f$ uniformly on B . Note that for every $\delta > 0$, take $k_\delta \geq \frac{1}{\delta}$, if $x \in B$, then $g_{n_{k_\delta}}(x) \leq \frac{1}{k_\delta} < \delta$. Therefore, $\sup_{n \geq n_{k_\delta}} |f_n(x) - f(x)| < \delta$.

Therefore, $\forall x \in B, n \geq n_{k_\delta}, |f_n(x) - f(x)| < \delta$ and $f_n \rightarrow f$ uniformly on B . ■

Definition 6.2. A sequence of measurable \mathbb{R} -valued functions f_n converges to a \mathbb{R} -valued measurable function f in L^1 if

$$\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu = 0 \quad (20)$$

Proposition 6.3 (Markov Inequality). If $g \geq 0$, then for all $t \geq 0$,

$$\mu(\{x : g(x) \geq t\}) \leq \frac{\int g \, d\mu}{t} \quad (21)$$

In probabilistic notations:

$$P(g \geq t) \leq \frac{\mathbb{E}[g]}{t} \quad (22)$$

Proof. Define $h(x) := t\mathbb{1}\{g \geq t\}$, obviously, $h \leq g$.

$$\int h \, d\mu = t\mu(\{x : g(x) \geq t\}) \leq \int g \, d\mu \quad (23)$$

The result follows. ■

Proposition 6.4. $f_n \xrightarrow{L^1} f \implies f_n \xrightarrow{\mu} f$.

Proof. Let $\varepsilon > 0$, apply Markov inequality on every $|f_n - f|$:

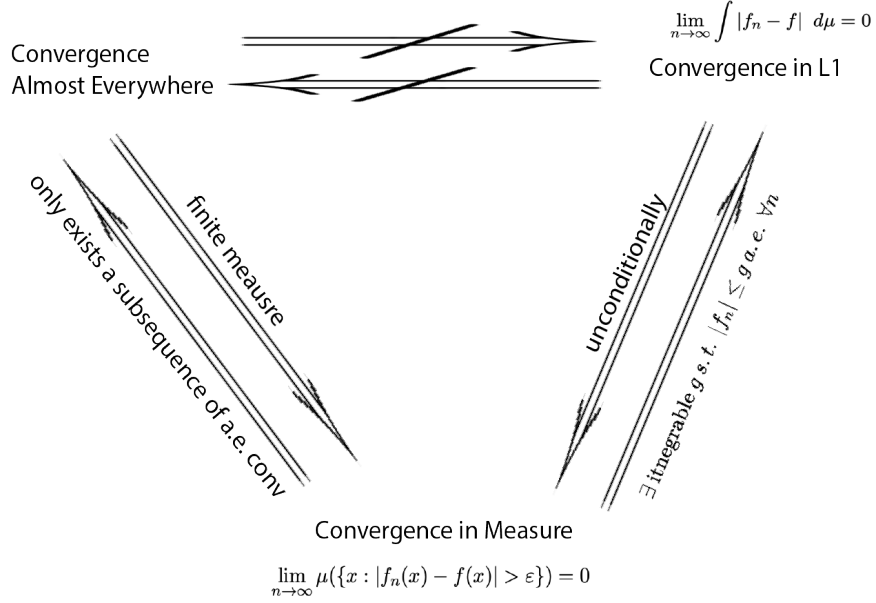
$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{\int |f_n - f| \, d\mu}{\varepsilon} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (24)$$

Therefore, $f_n \xrightarrow{\mu} f$. ■

Remark 6.3.

1. $f_n \xrightarrow{a.e.} f \not\Rightarrow f_n \xrightarrow{L^1} f$.
2. $f_n \xrightarrow{L^1} f \not\Rightarrow f_n \xrightarrow{a.e.} f$.
3. $f_n \xrightarrow{\mu} f \not\Rightarrow f_n \xrightarrow{a.e.} f$.

Figure 1: Modes of Convergences



Proposition 6.5 (Dominated Convergence Theorem II). Suppose $f_n \xrightarrow{\mu} f$, and \exists integrable g such that $|f_n| \leq g$ a.e. for all n . Then, $f_n \xrightarrow{L^1} f$, in particular, $\int f_n d\mu \rightarrow \int f d\mu$.

Proof. Suppose, for contradiction, $f_n \not\xrightarrow{L^1} f$. Equivalently, there exists ε and a subsequence f_{n_k} such that for all k :

$$\int |f_{n_k} - f| d\mu \geq \varepsilon \quad (\dagger) \quad (25)$$

But the convergence in measure implies $f_{n_k} \rightarrow f$ in measure as well. Then there exists a subsequence n_{k_ℓ} such that $f_{n_{k_\ell}} \rightarrow f$ almost everywhere.

By the previous dominated convergence theorem, $\lim_{\ell \rightarrow \infty} \int |f_{n_{k_\ell}} - f| d\mu = 0$, contradicts (\dagger) . ■

7 Normed Space

Definition 7.1. Let V be a vector space over \mathbb{R} (over \mathbb{C}), a **norm** on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

1. $\|x\| \geq 0 \quad \forall x \in V$,
2. $\|x\| = 0 \iff x = 0$,
3. $\|ax\| = |a| \|x\|$ for all $a \in \mathbb{R}(\in \mathbb{C})$,
4. (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

Example 7.1. For $V = \mathbb{R}^n$, for every $p \geq 1$, the L^p norm is defined as

$$\|x\|_{L^p} = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \quad (1)$$

Note: for $p < 1$, the triangle inequality fails.

Example 7.2. Let $C[a, b]$ denote the collection of continuous functions from $[a, b]$ to \mathbb{R} , where $[a, b]$ is a compact interval.

The sup norm is defined as

$$\|f\|_{\infty} = \sup_{x \in [a, b]} |f(x)| \quad (2)$$

The 1-norm is defined as

$$\|f\|_1 = \int_{[a, b]} |f| \, d\lambda \quad (3)$$

Definition 7.2. Let S be a set, a **metric** d on S is a function $d : S \times S \rightarrow \mathbb{R}$ such that for all $x, y, z \in S$:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) \leq d(x, z) + d(y, z)$.

Proposition 7.1. A norm induces a metric: $d(x, y) := \|x - y\|$.

Note: the converse is false, i.e., there are metrics not induced by any norm. For example, $d(x, y) := \mathbf{1}\{x = y\}$ is in general not induced by any norm.

Definition 7.3. Let S be a set with a metric d , a sequence of points x_n converges to $x \in S$ if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 \quad (4)$$

A sequence is **Cauchy** with respect to d if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall m, n \geq n_0, d(x_m, x_n) < \varepsilon$.

Definition 7.4. A metric space w.r.t d is **complete** if every Cauchy sequence w.r.t. d converges to somewhere in the space.

Example 7.3. $C[a, b]$ with the supremum norm is complete.

Example 7.4. $C[a, b]$ with L^1 norm is not complete.

Proof. Using counter-example: for $[a, b] = [-1, 1]$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ nx & \text{if } x \in (0, 1/n) \\ 1 & \text{if } x \in [1/n, 1] \end{cases} \quad (5)$$

The sequence of f_n is Cauchy but converges to $f = \mathbf{1}\{x \geq 0\} \notin C[a, b]$. ■

Proposition 7.2. $C[a, b]$ under sup-norm is complete.

Proof. Suppose f_n is a Cauchy sequence in $C[a, b]$ under supremum norm. For all $x \in [a, b]$,

$$f_n(x) - f_m(x) \leq \|f_n - f_m\|_\infty \rightarrow 0 \quad (6)$$

since f_n is Cauchy. Therefore, $f_n(x)$ is a Cauchy sequence in \mathbb{R} and $\lim_{n \rightarrow \infty} f_n(x)$ exists. Define f to be the point-wise limit of f_n .

Claim: $f \in C[a, b]$ and $f_n \rightarrow f$ in sup-norm.

For all $\varepsilon > 0$, there exists N , such that for all $m, n \geq N$,

$$\|f_m - f_n\|_\infty < \varepsilon \quad (7)$$

Therefore, for all $x \in [a, b]$, $|f_n(x) - f_m(x)| < \|f_m - f_n\|_\infty < \varepsilon$.

Fixing n , take $m \rightarrow \infty$, this shows for all $n \geq N$, for all $x \in [a, b]$

$$|f(x) - f_n(x)| = \lim_{m \rightarrow \infty} |f_m(x) - f_n(x)| \leq \varepsilon \quad (8)$$

Therefore, for all $n \geq N$, $\|f - f_n\|_\infty \leq \varepsilon$. Hence $f \rightarrow f_n$ in sup-norm.

Now show the continuity of f : take $x_0 \in [a, b]$, given $\varepsilon > 0$, since $f_n \rightarrow f$ in sup-norm, there exists N such that for all $n \geq N$,

$$\|f - f_n\|_\infty \leq \frac{\varepsilon}{3} \quad (9)$$

In particular, $\|f - f_N\|_\infty \leq \frac{\varepsilon}{3}$.

Moreover, since f_N is continuous, $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \varepsilon/3$ for every x . Take any $x \in \mathcal{B}_\delta(x_0)$, by triangle inequality,

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \quad (10)$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad (11)$$

Hence, $f \in C[a, b]$. ■

8 Functional Analysis: L^p Spaces

We will firstly define \mathcal{L}^p spaces, which is a little simpler than L^p spaces.

Definition 8.1. Let (X, \mathcal{A}, μ) be a measure space, for every $1 \leq p < \infty$, the $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{R})$ space is the collection of all measurable functions $f : X \rightarrow \mathbb{R}$ such that

$$\int |f|^p d\mu < \infty \quad (1)$$

Similarly, $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{C})$ denotes the collection of all measurable functions $f : X \rightarrow \mathbb{C}$ such that $\int |f|^p d\mu < \infty$.

Thought out this notes, we use \mathcal{L}^p to denote $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{R})$ or $\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{C})$.

Proposition 8.1 (\mathcal{L}^p space is a vector space). Note that $0 \in \mathcal{L}^p$, and if $f \in \mathcal{L}^p$ and $\alpha \in \mathbb{R}/\mathbb{C}$, then

$$\int |\alpha f|^p d\mu = |\alpha|^p \int |f|^p d\mu < \infty \quad (2)$$

Therefore, $\alpha f \in \mathcal{L}^p$.

For all $x \in X$,

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \quad (3)$$

$$\leq (2 \max\{|f(x)|, |g(x)|\})^p \quad (4)$$

$$\leq 2^p \max\{|f(x)|^p, |g(x)|^p\} \quad (5)$$

$$\leq 2^p(|f(x)|^p + |g(x)|^p) \quad (6)$$

$$\implies \int |f + g|^p d\mu < \infty \quad (7)$$

$$\implies f + g \in \mathcal{L}^p \quad (8)$$

Hence, \mathcal{L}^p is a vector space.

Definition 8.2. Let $\mathcal{L}^\infty(X, \mathcal{A}, \mu, \mathbb{R}/\mathbb{C})$ be the set of all bounded measurable $f : X \rightarrow \mathbb{R}/\mathbb{C}$.

Definition 8.3. For $f \in \mathcal{L}^p$ with $p < \infty$, define

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \quad (9)$$

for $p = \infty$, $\|f\|_\infty$'s definition is a little bit more complicated, for continuous functions, it collides with the sup-norm. However, it's not the same as sup-norm for discontinuous functions.

Definition 8.4. Given a measure space (X, \mathcal{A}, μ) , a set B is called **μ -null/negligible** if $B \subseteq A$ for some $A \in \mathcal{A}$ with $\mu(A) = 0$.

A subset $N \subseteq X$ is called **locally μ -null** if $\forall A \in \mathcal{A}$ with $\mu(A) < \infty$, $A \cap N$ is μ -null.

A property of elements of X is said to hold **locally a.e.** if the set on which it fails is locally μ -null.

We use this notion of locally null to circumvent non-sigma finite cases.

Definition 8.5. For $f \in \mathcal{L}^\infty$, define

$$\|f\|_\infty = \inf \{M \geq 0 : \text{the set of all } x \text{ with } |f(x)| > M \text{ is locally } \mu\text{-null.}\} \quad (10)$$

this is called the **essential supremum** of $|f|$. Equivalently, $\|f\|_\infty$ is the infimum of M such that $|f(x)| \leq M$ locally a.e.

Note that $\|f\|_\infty$ is only a semi-norm, we may modify a function on a measure-zero set without changing the value of $\|f\|_\infty$.

Our previous definitions of semi-norms on \mathcal{L}^p spaces satisfy

$$\|f\|_p = 0 \iff \int |f|^p d\mu = 0 \iff |f|^p = 0 \text{ a.e.} \iff f = 0 \text{ a.e.} \quad (11)$$

This definition of semi-norm on \mathcal{L}^∞ ensures $\|f\|_\infty = 0 \iff f = 0 \text{ a.e.}$.

Example 8.1. Take $X = [0, 1]$ and $\mu = \lambda$,

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2} \\ 2 & \text{otherwise} \end{cases} \quad (12)$$

Then $\|f\|_\infty = 1$ but $\sup f = 2$. To see this, note that $\{x : |f(x)| > 1\} = \{1/2\}$ has zero measure. However, for any $M < 1$, the same has non-zero Lebesgue measure.

Proposition 8.2.

$$\mu(\{x : |f(x)| > \|f\|_\infty\}) \text{ is locally } \mu\text{-null.} \quad (13)$$

$$\mu(\{x : |f(x)| > c\}) \text{ is not locally } \mu\text{-null } \forall c < \|f\|_\infty \quad (14)$$

Lemma 8.1. Countable union of locally μ -null sets is locally μ -null.

Proposition 8.3. $\|f\|_p$ and $\|f\|_\infty$ are semi-norms.

Definition 8.6. Given $p \in (1, \infty)$, the **conjugate exponent** q is defined as

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (15)$$

That is,

$$q = \frac{p}{p-1} \quad (16)$$

For $p = \infty$, $q = 1$.

Lemma 8.2 (Young's Inequality). Take $p \in (1, \infty)$, let q be the conjugate exponent of p , then for all $x, y \geq 0$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad (17)$$

Proof. ■

Theorem 8.1 (Hölder's Inequality). Let (X, \mathcal{A}, μ) be a measure space, take $1 \leq p \leq \infty$, and q be its conjugate exponent. Take $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$, then

$$fg \in \mathcal{L}^1 \quad (18)$$

and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (19)$$

Example 8.2. Take $X = \{x_1, \dots, x_n\}$ and μ to be the counting measure on X . Let $p = q = 2$ and $f, g \in \mathcal{L}^2$. Define $v = (f(x_1), \dots, f(x_n)) \in \mathbb{R}^n$ and $u = (g(x_1), \dots, g(x_n)) \in \mathbb{R}^n$.

$$\|fg\|_1 = \sum_{i=1}^n \mu(\{x_i\}) |f(x_i)g(x_i)| = \sum_{i=1}^n |f(x_i)g(x_i)| \quad (20)$$

Therefore,

$$|\langle v, u \rangle| = \left| \sum_{i=1}^n f(x_i)g(x_i) \right| \leq \|fg\|_1 \quad (21)$$

In this finite dimensional case with counting measure,

$$\|f\|_2 = \sqrt{\sum_{i=1}^n \mu(\{x_i\}) f(x_i)^2} = \sqrt{\sum_{i=1}^n f(x_i)^2} = \|v\|_2 \quad (22)$$

The same holds for g , in this case Hölder's inequality reduces to cauchy-Switchz inequality.

Proof. ■

Theorem 8.2 (Minkowski's Inequality). Let (X, \mathcal{A}, μ) be a measure space. Take $1 \leq p \leq \infty$. If $f, g \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, then $f + g \in \mathcal{L}^p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. First, suppose that $p \in (1, \infty)$. Let q be the conjugate exponent of p . We have already shown that \mathcal{L}^p is a vector space, so $f + g \in \mathcal{L}^p$.

Note that

$$1/p + 1/q = 1 \implies (p + q)/(pq) = 1 \implies p + q = pq \implies p = (p - 1)q \quad (23)$$

Therefore,

$$\int (|f + g|^{p-1})^q d\mu = \int |f + g|^p d\mu < \infty \quad (24)$$

Therefore, $|f + g|^{p-1} \in \mathcal{L}^q$. By Hölder's inequality,

$$\int |f + g|^p d\mu = \int |f + g| |f + g|^{p-1} d\mu \quad (25)$$

$$\leq \int |f| |f + g|^{p-1} d\mu + \int |g| |f + g|^{p-1} d\mu \quad (26)$$

$$\leq \|f\|_p \|f + g|^{p-1}\|_q + \|g\|_p \|f + g|^{p-1}\|_q \quad (27)$$

where

$$\|f + g|^{p-1}\|_q = \left(\int (|f + g|^{p-1})^q d\mu \right)^{1/q} = \left(\int |f + g|^p d\mu \right)^{1/q} \quad (28)$$

If $\|f + g\|_p = 0$, we are done. Suppose not, divide $(\int |f + g|^p d\mu)^{1/q}$ on both sides,

$$\frac{\int |f + g|^p d\mu}{(\int |f + g|^p d\mu)^{1/q}} \leq \|f\|_p + \|g\|_p \quad (29)$$

$$\implies (\int |f + g|^p d\mu)^{1-1/q} = (\int |f + g|^p d\mu)^{1/p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (30)$$

When $p = 1$,

$$\|f + g\|_1 = \int |f + g| d\mu \leq \int (|f| + |g|) d\mu = \|f\|_1 + \|g\|_1 \quad (31)$$

When $p = \infty$, define

$$N_1 = \{x : |f(x)| > \|f\|_\infty\} \quad (32)$$

$$N_2 = \{x : |g(x)| > \|g\|_\infty\} \quad (33)$$

Then N_1 and N_2 are locally μ -null, so is $N_1 \cup N_2$. For $x \notin N_1 \cup N_2$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty \quad (34)$$

■

Note that $\|\cdot\|_p$ is a **semi-norm** on \mathcal{L}^p , to make it a norm, we introduce the L^p space.

Definition 8.7. For $1 \leq p < \infty$, define the class of zero vectors

$$\mathcal{N}^p := \{f \in \mathcal{L}^p : f \text{ is measurable and } f = 0 \text{ a.e.}\} \quad (35)$$

which is a subspace of \mathcal{L}^p . Define L^p to be the quotient space:

$$L^p(X, \mathcal{A}, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu) / \mathcal{N}^p(X, \mathcal{A}, \mu) \quad (36)$$

That is, an element $[f] \in L^p$ (an equivalent class) is the collection of all $g \in \mathcal{L}^p$ such that $f - g = 0$

almost everywhere:

$$L^p \ni [f] := \{g \in \mathcal{L}^p : f - g \in \mathcal{N}^p\} \quad (37)$$

Then L^p is a vector space over \mathbb{R} or \mathbb{C} , and $\|\cdot\|_p$ is well-defined: for any f , for all $g \in [f]$, $\|f\|_p = \|g\|_p$ since $f = g$ almost everywhere so their integrals are the same. Most importantly, $\|\cdot\|_p$ is a norm on L^p .

For $p = \infty$, we define

$$\mathcal{N}^\infty := \{f : f \text{ is bounded, measure and } f = 0 \text{ a.e.}\} \quad (38)$$

Then $L^\infty := \mathcal{L}^p / \mathcal{N}^p$.

Note that L^p for $1 \leq p \leq \infty$ is also a vector space with equivalence relations. In general, we treat L^p as a space of functions instead of a space of classes of functions.

Proposition 8.4. Convergence in L^p ($1 \leq p < \infty$) implies convergence in measure.

Proof. By Markov's inequality,

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = \mu(\{x : |f_n(x) - f(x)|^p > \varepsilon^p\}) \quad (39)$$

$$\leq \frac{\int |f_n - f|^p d\mu}{\varepsilon^p} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (40)$$

■

Corollary 8.1. Let $f_n \rightarrow f$ in L^p with $1 \leq p < \infty$, then there exists a subsequence $f_{n_k} \rightarrow f$ a.e.

Proof. As convergence in L^p implies convergence in measure, which further implies existence of a.e. converging subsequences. ■

Theorem 8.3. For any $1 \leq p \leq \infty$, the $\|\cdot\|_p$ norm on L^p is complete.

Proof. For $1 \leq p < \infty$, let (f_n) be a Cauchy sequence in L^p .

Step 1: Find a subsequence (f_{n_k}) such that $\|f_{n_k} - f_{n_{k+1}}\|_p \leq 2^{-k}$ for all k . By Cauchy property, we may find n_1 such that $\|f_{n_1} - f_n\| \leq 2^{-1}$ for all $n \geq n_1$. Also, find a $n_2 \geq n_1$ such that $\|f_{n_2} - f_n\| \leq 2^{-2}$ for all $n \geq n_2$, etc.

Step 2: construct the limit Define

$$A_k := \{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k/2}\} \quad (41)$$

Then, by Markov's inequality,

$$\mu(A_k) \leq \frac{\int |f_{n_k} - f_{n_{k+1}}|^p d\mu}{(2^{-k/2})^p} \quad (42)$$

$$\leq \frac{2^{-kp}}{(2^{-k/2})^p} = 2^{-kp/2} \quad (43)$$

Thus, $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Define

$$B := \{x : x \in \text{infinitely many } A_k\} = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j \quad (44)$$

By Borel-Cantelli lemma, $\mu(B) = 0$. Take any $x \notin B$, then for sufficiently large k ,

$$|f_{n_k}(x) - f_{n_{k+1}}(x)| \leq 2^{-k/2} \quad (45)$$

This shows for all $x \notin B$, the constructed $(f_{n_k}(x))$ is a Cauchy sequence in \mathbb{R} , therefore, it's convergent.

Define the almost point-wise limit

$$f(x) := \begin{cases} \lim_{k \rightarrow \infty} f_{n_k}(x) & \text{if } x \notin B \\ 0 & \text{if } x \in B \end{cases} \quad (46)$$

Step 3: Show $f_n \rightarrow f$ in L^p . Note that $f_{n_k} \rightarrow f$ almost everywhere, so that $|f|^p \rightarrow |f_{n_k}|^p$. By Fatou's lemma,

$$\int |f|^p d\mu \leq \liminf_{k \rightarrow \infty} \int |f_{n_k}|^p d\mu \quad (47)$$

But the Cauchy property of f_n implies that $\sup_n \|f_n\|_p < \infty$ (find n such that $\|f_n - f_m\|_p \leq 1$ for all $m \geq n$. Thus, $\forall m \geq n$, $\|f_m\|_p \leq \|f_n - f_m\|_p + \|f_n\|_p \leq 1 + \|f_n\|_p$. Therefore, $\|f\|_p < \infty$).

For any $\varepsilon > 0$, we can find N so large that $\|f_n - f_m\|_p < \varepsilon$ for all $n, m \geq N$ since f_n is Cauchy. By Fatou's lemma, for all $n \geq N$,

$$\int |f_n - f|^p d\mu \leq \liminf_{n \rightarrow \infty} \int |f_n - f|^p d\mu \quad (48)$$

But when k is so large that $n_K \geq N$, we have

$$\int |f_n - f_{n_k}|^p d\mu = \|f_n - f_{n_k}\|_p^p \leq \varepsilon^p \quad (49)$$

Thus, for all $n \geq N$, $\|f - f_n\|_p \leq \varepsilon$. ■

Proof. for $p = \infty$ case. Let f_n be Cauchy in L^∞ , as before, find a subsequence f_{n_k} such that

$$\|f_{n_k} - f_{n_{k+1}}\|_\infty \leq 2^{-k} \quad \forall k \quad (50)$$

Then for all k , there exists a locally μ -null set N_k such that for all $x \notin N_k$.

$$|f_{n_k}(x) - f_{n_{k+1}}(x)| \leq 2^{-k} \quad (51)$$

Let $N = \bigcup_{k=1}^{\infty} N_k$, so that N is locally μ -null as well. Then for all $x \notin N$, $f_{n_k}(x)$ is a Cauchy sequence of real numbers, define $f(x) = \lim_k f_{n_k}(x)$ outside N and $f(x) = 0$ on N .

Claim: $f_n \rightarrow f$ in L^∞ . Note that for all $x \notin N$, for all k ,

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} |f_{n_j}(x) - f_{n_{j+1}}(x)| \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1} \quad (52)$$

Thus, $\|f - f_{n_k}\|_\infty \leq 2^{-k+1}$.

Take any $\varepsilon > 0$, find N so large that $\forall m, n \geq N$, $\|f_m - f_n\|_\infty \leq \varepsilon$. Then find k so large that $n_k \geq N$ and $2^{-k+1} \leq \varepsilon$. Then for all $n \geq N$,

$$\|f - f_n\|_\infty \leq \|f - f_{n_k}\|_\infty + \|f_{n_k} - f_n\|_\infty \leq 2\varepsilon \quad (53)$$

Taking $\varepsilon' = \varepsilon/2$ concludes $f_n \rightarrow f$ in L^∞ . ■

9 Signed and Complex Measures

Definition 9.1. Let (X, \mathcal{A}) be a measurable space, let $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ be a function. We say that μ is a **signed measure** if

1. $\mu(\emptyset) = 0$,
2. and μ is countable additive: for all disjoint $A_1, A_2, \dots \in \mathcal{A}$, $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

From now on, we use **measure** to denote the conventional notion of measure, that is, $\mu : \mathcal{A} \rightarrow [0, \infty]$ with $\mu(\emptyset) = 0$ and countable additivity. The term **signed measure** denotes functions $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ with above properties.

Remark 9.1. Note that the countable additivity does not change if we permute A_i 's, thus, implies $\sum_{i=1}^{\infty} \mu(A_i)$ should now change under any rearrangement of the terms. This implies that if $\mu(\cup_{i=1}^{\infty} A_i)$ is finite, $\sum_{i=1}^{\infty} |\mu(A_i)| < \infty$.

Proposition 9.1. If μ is a signed measure, then μ cannot be both ∞ and $-\infty$.

Proof. **Case 1:** if $\mu(X) \in \mathbb{R}$, then for any A , $\mu(X) = \mu(A) + \mu(A^c)$, both of $\mu(A)$ and $\mu(A^c)$ must be finite.

Case 2: if $\mu(X) = \infty$, then $\mu(A) + \mu(A^c) = \mu(X) = \infty$, none of $\mu(A)$ or $\mu(A^c)$ can be $-\infty$.

Case 3: if $\mu(X) = -\infty$, then $\mu(A) + \mu(A^c) = \mu(X) = -\infty$, none of $\mu(A)$ or $\mu(A^c)$ can be ∞ . ■

Proposition 9.2. If $\mu(A)$ is finite (i.e., in \mathbb{R}), then $\mu(B) \in \mathbb{R}$ for any $B \subseteq A$, $B \in \mathcal{A}$.

Proof. $\mu(A) = \mu(B) + \mu(A \setminus B) \in \mathbb{R}$, both $\mu(B)$ and $\mu(A \setminus B)$ must be finite. ■

Definition 9.2. A signed measure is called **finite** if $\mu(A) \in \mathbb{R}$ for all $A \in \mathcal{A}$.

9.1 Construction of Signed Measures

Example 9.1 (Relationship between integrable function and measure). Let (X, \mathcal{A}, μ) be a measure space, let $f \in L^1$, define $\nu(A) = \int_A f d\mu$, then ν is a signed measure.

Example 9.2 (Construction of signed measure). If ν_1 and ν_2 are measures and at least one of them is finite, then $\nu_1 - \nu_2$ is a signed measure.

9.2 Hahn Decomposition Theorem

Let (X, \mathcal{A}) be a measurable space and let μ be a signed measure on (X, \mathcal{A}) .

Definition 9.3. A set $A \in \mathcal{A}$ is called a **positive set for μ** if $\mu(B) \geq 0$ for all $B \subseteq A, B \in \mathcal{A}$. Similarly, a set $A \in \mathcal{A}$ is called a **negative set for μ** if $\mu(B) \leq 0$ for all $B \subseteq A, B \in \mathcal{A}$.

Lemma 9.1. If $A \in \mathcal{A}$ satisfies $-\infty < \mu(A) < 0$, then there exists a negative set $B \subseteq A$ such that $\mu(B) \leq \mu(A) < 0$.

Proof. Let $\delta_1 = \sup\{\mu(E) : E \in \mathcal{A}, E \subseteq A\}$, note that $\delta_1 \geq 0$ since $\mu(\emptyset) = 0$.

By the definition of δ_1 we can find $A_1 \subseteq A$ such that $\mu(A_1) \geq \delta_1/2$ if $\delta_1 < \infty$, or $\mu(A_1) \geq 1$ if $\delta_1 = \infty$. Thus, $\mu(A_1) \geq \min\{\delta_1/2, 1\}$.

Let $\delta_2 = \sup\{\mu(E) : E \in \mathcal{A}, E \subseteq A \setminus A_1\}$, similarly, we can choose $A_2 \subseteq A \setminus A_1$ and $A_2 \in \mathcal{A}$ such that $\mu(A_2) \geq \min\{\delta_2/2, 1\}$.

Similarly, choose $A_n \in \mathcal{A}$, $A_n \subseteq A \setminus (A_1 \cup \dots \cup A_{n-1})$, such that $\mu(A_n) \geq \min\{\delta_n/2, 1\}$. Then by definition, A_1, A_2, \dots are disjoint, they are all contained in A .

Let $B = A \setminus (\bigcup_{i=1}^{\infty} A_i)$.

Claim: this B is a negative set such that $\mu(B) \leq \mu(A)$.

Note that $\mu(A) \in \mathbb{R} \implies \mu(B) \in \mathbb{R}$. Thus, $\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A) - \mu(B) \in \mathbb{R}$.

But $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ since A_i 's are disjoint. Therefore, $\mu(A_i) \rightarrow 0$ as $i \rightarrow \infty$. However, $\mu(A_i) \geq \min\{\delta_i/2, 1\} \geq 0$. It must be $\delta_i \rightarrow 0$ as $i \rightarrow \infty$.

Take any $E \subseteq B$ such that $E \in \mathcal{A}$. Then $E \subseteq B \subseteq A \setminus (A_1 \cup \dots \cup A_{n-1})$ for all $n \in \mathbb{N}$. So by definition of δ_n , we have $\mu(E) \leq \delta_n$, thus $\mu(E) \leq 0$ as we take $n \rightarrow \infty$. Hence B is a negative set.

Finally, since $\mu(A_i) \rightarrow 0$, $\mu(B) = \mu(A) - \sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A)$.

■

Theorem 9.1 (Hahn Decomposition Theorem). Let (X, \mathcal{A}) be a measurable space and μ a signed measure on (X, \mathcal{A}) . Then, there exists disjoint $P \cup N$ in \mathcal{A} such that $X = P \cup N$ such that P is a positive set for μ and N is a negative set for μ .

Proof. Since μ is a signed measure, we know that it cannot take value at both ∞ and $-\infty$. WLOG, suppose μ never takes value $-\infty$. Let

$$L = \inf\{\mu(A) : A \in \mathcal{A} \text{ s.t. } A \text{ is negative}\} \quad (1)$$

Then there exists a sequence of negative sets A_n such that $\mu(A_n) \rightarrow L$. Define $B = \bigcup_{n=1}^{\infty} A_n$. For sure, $B \in \mathcal{A}$.

Claim: B is a negative set.

Take and $E \subseteq B$ such that $E \in \mathcal{A}$, then

$$E = E \cap B = \bigcup_{i=1}^{\infty} E \cap A_i = \bigcup_{i=1}^{\infty} E \cap (A_i \setminus (A_1 \cup \dots \cup A_{i-1})) \quad (2)$$

where the last step holds because we only consider the net incremental at each step. Moreover, $\{E \cap (A_i \setminus (A_1 \cup \dots \cup A_{i-1}))\}_i$ are disjoint.

Thus,

$$\mu(E) = \sum_{i=1}^{\infty} \underbrace{\mu(E \cap (A_i \setminus (A_1 \cup \dots \cup A_{i-1})))}_{\subseteq A_i} \quad (3)$$

Since A_i 's are all negative sets, we must have $\mu(E) \leq 0$ and B is a negative set.

Claim: $\mu(B) = L$.

Since $A_n \subseteq B$,

$$\mu(B) = \mu(A_n) + \mu(B \setminus A_n) \quad (4)$$

But B is a negative set, so $\mu(B \setminus A_n) \leq 0$. Thus,

$$\mu(B) \leq \mu(A_n) \quad \forall n \in \mathbb{N} \quad (5)$$

Thus, $\mu(B) \leq \lim_n \mu(A_n) = L$. But B itself is a negative set, and L is the infimum, so $L \leq \mu(B)$.

In particular, we've shown that $L > -\infty$ since μ never takes value at $-\infty$.

Let $N = B$ and $P = N^c$. Since $B \in \mathcal{A}$, both $N, P \in \mathcal{A}$.

Claim: P is a positive set.

Suppose not, then $\exists A \subseteq P$ such that $A \in \mathcal{A}$ and $-\infty < \mu(A) < 0$.

By the lemma, there exists a negative set $D \subseteq A$ and $\mu(D) \leq \mu(A) < 0$. Note that $D \subseteq A \subseteq P$, but then $N \cup D$ is a negative set as a union of negative sets. Then,

$$\mu(N \cup D) = \mu(N) + \mu(D) = L + \mu(D) < L \quad (6)$$

which leads to a contradiction.

Consequently, this $X = N \cup P$ is a Hahn decomposition. ■

Theorem 9.2 (Jordan Decomposition Theorem). Every signed measure is the difference of two

measures, at least one of which is finite.

$$\mu = \mu^+ - \mu^- \quad (7)$$

Proof. Let μ be a signed measure, let (N, P) be a Hahn decomposition of X .

For every $A \in \mathcal{A}$, define

$$\mu^+(A) = \mu(A \cap P) \quad (8)$$

$$\mu^-(A) = -\mu(A \cap N) \quad (9)$$

Since P is a positive set, $\mu^+(A) \geq 0$, similarly, since N is negative, $\mu^-(A) \geq 0$ as well.

Let A_1, A_2, \dots be disjoint sets in \mathcal{A} , then

$$\mu^+(\cup_i A_i) = \mu(P \cap (\cup_i A_i)) \quad (10)$$

$$= \mu(\cup_i (P \cap A_i)) \quad (11)$$

$$= \sum_i \mu(P \cap A_i) \quad (12)$$

$$= \sum_i \mu^+(A_i) \quad (13)$$

So μ^+ is a measure. Similarly, μ^- is a measure as well.

$$\mu^+(A) - \mu^-(A) = \mu(A \cap P) + \mu(A \cap N) = \mu(A) \quad (14)$$

Therefore, $\mu = \mu^+ - \mu^-$. Lastly, note that $\mu(X) = \mu(P) + \mu(N) = \mu^+(P) - \mu^-(N)$, we need at least one of them to be finite in order to avoid subtracting infinity from infinity. ■

Proposition 9.3. Let (μ^+, μ^-) be the decomposition of a signed measure from Hahn decomposition (P, N) , that is, $\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$ for any $A \in \mathcal{A}$. Then,

$$\mu^+(A) = \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\} \quad (15)$$

$$\mu^-(A) = \sup\{-\mu(B) : B \subseteq A, B \in \mathcal{A}\} \quad (16)$$

Proof. Take any $A \in \mathcal{A}$, take any $B \subseteq A$ such that $B \in \mathcal{A}$. Then

$$\mu(B) = \mu^+(B) - \mu^-(B) \quad (17)$$

$$\leq \mu^+(B) \because \mu^-(B) \geq 0 \quad (18)$$

$$\leq \mu^+(A) \because \mu^+ \text{ is a measure} \quad (19)$$

Therefore, $\mu^+(A) \geq \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\}$.

On the other hand, $\mu^+(A) = \mu(A \cap P)$ by definition, take $B = A \cap P \subseteq A$, which satisfies $A \cap P \in \mathcal{A}$. Then $\mu^+(A) \leq \sup\{\mu(B) : B \subseteq A, B \in \mathcal{A}\}$.

The similar logic works for μ^- . ■

Definition 9.4. The pair of (μ^+, μ^-) defined above is called the **Jordan decomposition** of the signed measure μ , where μ^+ and μ^- are called the **positive and negative parts of μ** . The **variation** of μ is defined to be the measure $|\mu| = \mu^+ + \mu^-$. The **total variation** of μ is the number $\|\mu\| = |\mu|(X)$.

9.3 Complex Measures

Definition 9.5. Let (X, \mathcal{A}) be a measurable space, $\mu : \mathcal{A} \rightarrow \mathbb{C}$ is called a **complex measure** if for all disjoint $A_1, A_2, \dots \in \mathcal{A}$, $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ and $\mu(\emptyset) = 0$. In particular, this implies the sum is absolutely converged.

Any complex measure μ can be written uniquely as

$$\mu = \mu' + i\mu'' \quad (20)$$

where

$$\mu'(A) = \Re(\mu(A)) \quad (21)$$

$$\mu''(A) = \Im(\mu(A)) \quad (22)$$

Let $\mu' = \mu_1 - \mu_2$ and $\mu'' = \mu_3 - \mu_4$ be Jordan compositions of μ' and μ'' respectively. Then

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4 \quad (23)$$

is called the **Jordan decomposition** of complex measure μ .

Definition 9.6. The **variation** of a complex measure μ is defined as

$$|\mu|(A) := \sup \left\{ \sum_{i=1}^n |\mu(A_i)| : A_1, \dots, A_n \in \mathcal{A} \text{ disjoint s.t. } \bigcup_{i=1}^n A_i = A \right\} \quad (24)$$

Note that the supremum is taken over all *finite partitions of A* . It is easy to check that if μ is a finite signed measure, this definition of variation is the same as the previous one.

Lemma 9.2. Suppose $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a function such that (i) $\mu(\emptyset) = 0$ and (ii) is finite additivity (that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint A and B). Moreover, if $\lim_{n \rightarrow \infty} \mu(A_n) = 0$ for all $A_n \searrow \emptyset$, then μ is a measure.

Proof. It suffices to check the countable additivity of μ , let B_1, B_2, \dots be a disjoint sequence of sets in \mathcal{A} .

Let $B = \bigcup_i B_i$ and define $A_n := B \setminus \bigcup_{i=1}^{n-1} B_i$. Easy to check $A_n \searrow \emptyset$. Therefore, by finite additivity of μ : $\mu(A_n) = \mu(B) - \sum_{i=1}^{n-1} \mu(B_i) \rightarrow 0$. Taking $n \rightarrow \infty$ implies $\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$. ■

Proposition 9.4. Let μ be a complex measure, then $|\mu|$ is a measure.

Proof. Obviously, $|\mu|(\emptyset) = 0$.

Take any disjoint $A, B \in \mathcal{A}$. Now show the finite additivity of $|\mu|$: let C_1, \dots, C_n be a measurable disjoint partition of $A \cup B$, so $(C_i \cap A)$ and $(C_i \cap B)$ are partitions of A and B respectively.

$$|\mu|(A) + |\mu|(B) \geq \sum |\mu(C_i \cap A)| + \sum |\mu(C_i \cap B)| \quad (25)$$

$$\geq \sum |\mu(C_i \cap A) + \mu(C_i \cap B)| \quad (26)$$

$$= \sum |\mu(C_i)| \because C_i \subseteq A \cup B \quad (27)$$

$$\geq |\mu|(A \cup B) \quad (28)$$

Conversely, let C_1, \dots, C_n be a partition of A and D_1, \dots, D_m be a partition of B , then $C_1, \dots, C_n, D_1, \dots, D_m$ is a partition of $A \cup B$.

$$|\mu|(A \cup B) \geq \sum_{i=1}^n |\mu(C_i)| + \sum_{i=1}^m |\mu(D_i)| \quad (29)$$

Taking supremum of partitions (C_i) for A and (D_i) for B ,

$$|\mu|(A \cup B) \geq |\mu|(A) + |\mu|(B) \quad (30)$$

Therefore, $|\mu|$ is finitely additive.

Now take a $A_n \searrow \emptyset$ in \mathcal{A} , using the Jordan decomposition: $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where μ_i are measures. By triangle inequality in \mathbb{C} ,

$$|\mu(A)| \leq \sum_{i=1}^4 \mu_i(A) \quad (31)$$

then for all measurable partitions A_1, \dots, A_n of A ,

$$\sum_{j=1}^n |\mu(A_j)| \leq \sum_{i=1}^4 \sum_{j=1}^n \mu_i(A_j) = \sum_{i=1}^4 \mu_i(A) \quad (32)$$

Taking supremum of all such partitions,

$$|\mu|(A) \leq \sum_{i=1}^4 \mu_i(A) \quad (33)$$

Since $A_n \searrow \emptyset$ implies $\mu_i(A_n) \rightarrow 0$ as μ_i 's are finite measures (there is no ∞ in \mathbb{C}), $|\mu|(A_n) \rightarrow 0$. By Previous lemma, $|\mu|$ is a measure. ■

Proposition 9.5 (Completeness of Total Variation). The total variation is a norm on the space of finite signed/complex measures.

Proof. Obviously, $\|\mu\|$ is a norm. Now show the completeness.

Let $\{\mu_n\}$ be a Cauchy (in total variation) sequence of measures, for all $A \in \mathcal{A}$, $|\mu(A)| \leq |\mu|(A)$ since A is a trivial partition of A .

For any $m, n \in \mathbb{N}$, $A \in \mathcal{A}$, $\mu_m - \mu_n$ is a signed measure,

$$|\mu_m(A) - \mu_n(A)| \leq |\mu_m - \mu_n|(A) \quad (34)$$

$$\leq \|\mu_m - \mu_n\| \quad (35)$$

Therefore, $\{\mu_n(A)\}$ is a Cauchy sequence in \mathbb{R} for all $A \in \mathcal{A}$. Define μ as the "set-wise" limit of μ_n :

$$\mu(A) := \lim_{n \rightarrow \infty} \mu_n(A) \quad (36)$$

Now show μ is a measure: observe that $\mu_n \rightarrow \mu(A)$ uniformly over all $A \in \mathcal{A}$ by Equation (35). The finite additivity of μ follows its definition.

Fix arbitrary $A_n \searrow \emptyset$, show that $\mu(A_n) \rightarrow 0$. Take any $\varepsilon > 0$, find N so large that $|\mu_N(A) - \mu(A)| < \varepsilon$ for all A by uniform convergence.

Find j_0 so large such that for all $j \geq j_0$, $|\mu_N(A_j)| < \varepsilon/2$. For all $j \geq j_0$,

$$|\mu(A_j)| \leq |\mu(A_j) - \mu_N(A_j)| + |\mu_N(A_j)| < \varepsilon \quad (37)$$

Lastly, we show $\|\mu_n - \mu\| \rightarrow 0$. Take any partition A_1, \dots, A_k of X , take any $\varepsilon > 0$, the Cauchy property of $\{\mu_n\}$ provides a N so large that for all $m, n \geq N$, $\|\mu_m - \mu_n\| < \varepsilon$.

$$\sum_{j=1}^k |\mu_m(A_j) - \mu_n(A_j)| \leq \|\mu_m - \mu_n\| < \varepsilon \quad (38)$$

Take $m \rightarrow \infty$,

$$\sum_{j=1}^k |\mu(A_j) - \mu_n(A_j)| \leq \varepsilon \quad (39)$$

Since above inequality holds for all partitions of X , $\|\mu - \mu_n\| < \varepsilon$. ■

9.4 Integration w.r.t. Signed and Complex Measures

Definition 9.7. Let $\mu = \mu^+ - \mu^-$ be a signed measure and its corresponding Jordan decomposition, define

$$\int f d\mu = \int f d(\mu^+ - \mu^-) = \int f d\mu^+ - \int f d\mu^- \quad (40)$$

Easy to check that $f \mapsto \int f d\mu$ and $\mu \mapsto \int f d\mu$ are linear maps.

When μ is a complex measure: $\mu = \mu' + i\mu''$, define

$$\int f d\mu = \int f d\mu' + i \int f d\mu'' \quad (41)$$

10 Radon-Nikodym Theorem

Definition 10.1. Let (X, \mathcal{A}) be a measurable space, let μ, ν be two measures on this space, ν is **absolutely continuous** w.r.t. μ if for every $A \in \mathcal{A}$:

$$\mu(A) = 0 \implies \nu(A) = 0 \quad (1)$$

denoted as $\nu \ll \mu$.

Theorem 10.1 (Radon-Nikodym). Let (X, \mathcal{A}) be a measurable space, let μ, ν be two σ -finite measures. Suppose ν is absolutely continuous w.r.t. μ , then there exists a measurable map $g : X \rightarrow [0, \infty)$ such that

$$\nu(A) = \int_A g \, d\mu \quad (2)$$

for every $A \in \mathcal{A}$.

Interpretations Let χ_A denote the indicator function of set A , recall that $\int_A f \, d\mu \equiv \int f \chi_A \, d\mu$. Then, $\nu(A) = \int_A 1 \, d\nu = \int \chi_A \, d\nu = \int g \chi_A \, d\mu$ for all A . Moreover, for any integrable f ,

$$\int f \, d\nu = \int f g \, d\mu \quad (3)$$

This allows us to denote g as $\frac{d\nu}{d\mu}$.

Example 10.1. Suppose (X, \mathcal{A}) is a metric space (take $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ here), suppose g is continuous w.r.t. the metric, let $A = B(x, \varepsilon)$ be the ε -open ball around $x \in X$, then for sufficiently small ε :

$$\nu(A) = \nu(B(x, \varepsilon)) \quad (4)$$

$$\int_A g \, d\mu \approx g(x) \int_A d\mu = g(x) \mu(B(x, \varepsilon)) \quad (5)$$

That is,

$$\frac{d\nu}{d\mu} = g(x) \approx \frac{\nu(B(x, \varepsilon))}{\mu(B(x, \varepsilon))} \quad (6)$$

Actually,

$$g(x) = \lim_{\varepsilon \rightarrow 0} \frac{\nu(B(x, \varepsilon))}{\mu(B(x, \varepsilon))} \quad (7)$$

Therefore, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ captures the relative growth rate of ν to μ when we initially apply them on a small ball and expand the radius of this ball.

Lemma 10.1. Let (X, \mathcal{A}) be a measurable space, let ν be a measure on it, let μ be a finite measure.

Then, $\nu \ll \mu$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \mu(A) < \delta \implies \nu(A) < \varepsilon \forall A \in \mathcal{A} \quad (8)$$

Recall the definition of uniform continuity and $\frac{df(x)}{dx}$.

Proof. (\Leftarrow) Suppose $\mu(A) = 0$, $\nu(A) < \varepsilon$ for all $\varepsilon > 0$, it must be $\nu(A) = 0$.

(\Rightarrow) Suppose $\nu \ll \mu$, suppose the condition fails, $\exists \varepsilon > 0$ such that $\forall \delta > 0, \exists A$ with $\mu(A) < \delta$ but $\nu(A) \geq \varepsilon$.

We can find a sequence A_1, A_2, \dots such that $\mu(A_j) < \delta_j = 2^{-j}$ for all j and $\nu(A_j) \geq \varepsilon$. It follows $\sum \mu(A_j) < \infty$. By Borel-Cantelli lemma,

$$\mu \left(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k \right) = 0 \quad (9)$$

Define $B_j = \bigcup_{k=j}^{\infty} A_k$ and $B = \bigcap_{j=1}^{\infty} B_j$. Since $B_j \searrow B$ and ν is a finite measure, $\nu(B) = \lim_j \nu(B_j)$. But for any j , $\nu(B_j) \geq \nu(A_j) \geq \varepsilon$. Therefore, $\nu(B) \geq \varepsilon$, which contradicts $\nu \ll \mu$. ■

Proof of Radon-Nikodym Theorem. Let ν, μ be finite measures, let

$$\mathcal{F} := \left\{ f : X \rightarrow [0, \infty] : f \text{ measurable and } \int_A f d\mu \leq \nu(A) \forall A \in \mathcal{A} \right\} \quad (10)$$

We are choosing the largest $g \in \mathcal{F}$ as $\frac{d\nu}{d\mu}$.

Claim: $f, g \in \mathcal{F} \implies f \vee g \equiv \max\{f, g\} \in \mathcal{F}$.

Proof. Let $B := \{x : f(x) \geq g(x)\}$, for any $A \in \mathcal{A}$,

$$\int_A f \vee g d\mu = \int_{A \cap B} f \vee g d\mu + \int_{A \cap B^c} f \vee g d\mu \quad (11)$$

$$= \int_{A \cap B} f d\mu + \int_{A \cap B^c} g d\mu \leq \nu(A \cap B) + \nu(A \cap B^c) = \nu(A) \quad (12)$$

■

Let $(f_n) \in \mathcal{F}$ be a sequence such that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup \left\{ \int f d\mu : f \in \mathcal{F} \right\} \quad (13)$$

For every $n \in \mathbb{N}$, take $g_n(x) = \max_{j \leq n} f_j(x)$, $g_n \in \mathcal{F}$ by previous claim. Moreover, $g_n(x) \uparrow$ for all $x \in X$.

$$\int f_n d\mu \leq \int g_n d\mu \leq \sup \left\{ \int f d\mu : f \in \mathcal{F} \right\} \quad (14)$$

By squeeze theorem, $\lim_{n \rightarrow \infty} \int g_n d\mu = \sup \left\{ \int f d\mu : f \in \mathcal{F} \right\}$.

Define $g(x) = \lim_{n \rightarrow \infty} g_n(x)$, which always exists but is potentially infinity. By MCT,

$$\int g \, d\mu = \lim_{n \rightarrow \infty} \int g_n \, d\mu = \sup\left\{\int f \, d\mu : f \in \mathcal{F}\right\} \quad (15)$$

Note that $\forall A \in \mathcal{A}$,

$$\int_A g \, d\mu = \lim_{n \rightarrow \infty} \int_A g_n \, d\mu \leq \nu(A) \quad (16)$$

So $g \in \mathcal{F}$ and attains the supremum, in terms of integral, over \mathcal{F} .

Claim: $\forall A \in \mathcal{A}$, $\int_A g \, d\mu = \nu(A)$.

Proof. Define $\nu_0(A) = \nu(A) - \int_A g \, d\mu$. Since ν is a measure and $A \mapsto \int_A g \, d\mu$ is also a finite measure. Therefore, ν_0 is a finite signed measure. Moreover, since $g \in \mathcal{F}$, $\nu_0(A) \geq 0$ for all $A \in \mathcal{A}$.

Suppose, for contradiction, $\nu_0(A) > 0$ for some $A \in \mathcal{A}$. It must be $\nu_0(X) > 0$. But $\mu(X) < \infty$, there exists $\varepsilon > 0$ such that $\nu_0(X) > \varepsilon\mu(X)$. Note that $\nu_0 - \varepsilon\mu$ is a finite signed measure, let (P, N) be the Hahn decomposition of $\nu_0 - \varepsilon\mu$. Then for any $A \in \mathcal{A}$,

$$\nu(A) = \int_A g \, d\mu + \nu_0(A) \quad (17)$$

$$\geq \int_A g \, d\mu + \nu_0(A \cap P) \quad (18)$$

$$= \int_A g \, d\mu + \underbrace{\nu_0(A \cap P) - \varepsilon\mu(A \cap P)}_{\geq 0} + \varepsilon\mu(A \cap P) \quad (19)$$

$$\geq \int_A g \, d\mu + \varepsilon\mu(A \cap P) \quad (20)$$

$$= \int_A g + \varepsilon\chi_{A \cap P} \, d\mu \quad (21)$$

Therefore, $g + \varepsilon\chi_{A \cap P} \in \mathcal{F}$. Take $A = X$:

$$\int g + \varepsilon\chi_{A \cap P} \, d\mu = \int g \, d\mu + \varepsilon\mu(P \cap A) \geq \int g \, d\mu \quad (22)$$

Suppose, for contradiction, $\mu(P) \leq 0$, it must be $\mu(P) = 0$, by absolute continuity, $\nu \ll \mu$, $\nu(P) = 0$ as well. Then, since $\int_P g \, d\mu$ is bounded on a measure zero set, it must be zero,

$$\nu_0(P) = \nu(P) - \int_P g \, d\mu = 0 \quad (23)$$

Thus

$$(\nu_0 - \varepsilon\mu)(P) = 0 \quad (24)$$

$$\implies (\nu_0 - \varepsilon\mu)(X) = (\nu_0 - \varepsilon\mu)(P) + (\nu_0 - \varepsilon\mu)(N) \leq 0 \quad (25)$$

Contradicts $\nu_0(X) \geq \varepsilon\mu(X)$, therefore, $\mu(P) > 0$.

This leads to a contradiction since $g + \varepsilon\chi_{A \cap P}$ has strictly larger integral than g . Therefore, $\nu_0 = 0$. ■

Suppose μ and ν are σ -finite. Let $B_1, B_2, \dots \in \mathcal{A}$ be a partition of X such that $\mu(B_n), \nu(B_n)$ are finite. Moreover, define $\mu_n(A) := \mu(A \cap B_n)$ and $\nu_n(A) := \nu(A \cap B_n)$, both of μ_n and ν_n are finite on X (in particular, on B_n) and $\nu_n \ll \mu_n$.

For every $n \in \mathbb{N}$, there exists measurable $g_n : X \rightarrow [0, \infty]$ such that

$$\nu_n(A) = \int_A g_n \, d\mu \quad (26)$$

Therefore,

$$\nu(A \cap B_n) = \int g_n \chi_{A \cap B_n} d\mu \quad (27)$$

$$= \int g_n \chi_{B_n} \chi_A d\mu \quad (28)$$

$$= \int_A g_n \chi_{B_n} d\mu \quad (29)$$

Let $g = \sum_{n=1}^{\infty} g_n \chi_{B_n}$, then

$$\nu(A) = \sum_{n=1}^{\infty} \nu(A \cap B_n) \quad (30)$$

$$= \sum_{n=1}^{\infty} \int g_n \chi_{B_n} \chi_A d\mu \quad (31)$$

$$= \sum_{n=1}^{\infty} \chi_A \int g_n \chi_{B_n} d\mu \quad (32)$$

$$= \int \chi_A \sum_{n=1}^{\infty} g_n \chi_{B_n} d\mu \quad (33)$$

$$= \int_A g d\mu \quad (34)$$

$$(35)$$

Since $g_n < \infty$ everywhere for all n , so is g . ■

Remark 10.1 (Uniqueness of Radon-Nikodym Derivative). Let g and h be two Radon-Nikodym derivatives of ν w.r.t. μ .

Case 1: suppose $\nu(X) < \infty$, then for all $A \in \mathcal{A}$, by definitoin,

$$\int_A g d\mu = \nu(A) = \int_A h d\mu \quad (36)$$

Let $B := \{x, g(x) > h(x)\}$, $(g - h)\chi_A \geq 0$ and $(g - h)\chi_A = 0$ a.e. on A . Similarly, $(h - g)\chi_{A^c} \geq 0$ and $(h - g)\chi_{A^c} = 0$ a.e. on A^c . Add them together, $g - h = 0$ a.e. on X .

Case 2: suppose ν is σ -finite, let B_1, B_2, \dots be disjoint measurable sets such that $\nu(B_n) < \infty$ and $\cup_n B_n = X$. Since $g = h$ a.e. on every B_n as shown before, $g = h$ a.e. on X .

Theorem 10.2 (Radon-Nikodym Theorem for Complex Measures). TODO

Proof. ■

10.1 Lebesgue Decomposition Theorem

Definition 10.2. Let (X, \mathcal{A}) be a measurable space, let μ be a measure on (X, \mathcal{A}) , then μ is **concentrated** on a set $E \in \mathcal{A}$ if $\mu(E^c) = 0$.

We say that a signed complex measure μ is concentrated on E if $|\mu|$ is concentrated on E .

Definition 10.3. Two measures / signed measures / complex measures μ and ν on measurable space (X, \mathcal{A}) are **mutually singular** if $\exists E \in \mathcal{A}$ such that μ is concentrated on E and ν is concentrated on E^c .

$$\mu \perp \nu \tag{37}$$

Example 10.2. Any measure on \mathbb{R} that is concentrated on \mathbb{Z} is mutually singular to the Lebesgue measure, which is concentrated on \mathbb{Z}^c .

Theorem 10.3 (Lebesgue Decomposition). Let (X, \mathcal{A}) be a measurable space, let μ be a measure (the reference measure) on it. Let ν be a finite signed, complex measure, or σ -finite measure on (X, \mathcal{A}) , then there is a unique decomposition

$$\nu = \nu_a + \nu_s \tag{38}$$

such that

$$\nu_a \ll \mu \tag{39}$$

$$\nu_s \perp \mu \tag{40}$$

Proof. Case 1: suppose ν is a finite measure. Define \mathcal{N} to be the collection of μ -negligible sets:

$$\mathcal{N} := \{B \in \mathcal{A} : \mu(B) = 0\} \tag{41}$$

Let

$$S := \sup\{\nu(B) : B \in \mathcal{N}\} < \infty \text{ since } \nu \text{ is finite.} \tag{42}$$

Then there exists a sequence of sets $B_n \in \mathcal{N}$ such that $S = \lim_{n \rightarrow \infty} \nu(B_n)$, define

$$N = \bigcup_{n=1}^{\infty} B_n \tag{43}$$

Easy to verify that $\mu(N) \leq \sum_{n=1}^{\infty} \mu(B_n) = 0$, so $N \in \mathcal{N}$. Obviously, $\nu(N) \leq S$ since $N \in \mathcal{N}$. Moreover, since $\nu(N) \geq \nu(B_n)$ for every $n \in \mathbb{N}$, $\nu(N) \geq \lim_n \nu(B_n) = S$. Thus, $\nu(N) = S$, so that N is the ν -maximal set in \mathcal{N} .

For every $A \in \mathcal{A}$, define

$$\nu_a(A) = \nu(A \cap N^c) \tag{44}$$

$$\nu_s(A) = \nu(A \cap N) \tag{45}$$

So that $\nu = \nu_a + \nu_s$.

Claim: $\nu_s \perp \mu$.

Easy to verify that $\mu(N) = 0$ and $\nu_s(N^c) = \nu(N^c \cap N) = 0$.

Claim: $\nu_a \ll \mu$.

Take any $B \in \mathcal{A}$ such that $\mu(B) > 0$. Suppose, for contradiction, $\nu_a(B) \neq 0$, that is, $\nu(B \cap N^c) \neq 0$. Since we assumed ν is a finite measure (not signed), $\nu(B \cap N^c) > 0$. Thus,

$$\nu(N \cup B) = \nu(N) + \nu(B \cap N^c) > \nu(N) = S \quad (46)$$

but $N \cup B \in \mathcal{N}$, this leads to a contradiction.

Case 2: suppose ν is a finite signed or complex measure, we can find N as above for $|\nu|$ and define $\nu_a(A) = \nu(A \cap N^c)$ and $\nu_s(A) = \nu(A \cap N)$.

Case 3: if ν is a σ -finite measure, we can firstly express X as a disjoint union D_1, D_2, \dots with finite ν measure, and then find $N_i \subseteq D_i$ as the ν -maximal element among all μ -zero subsets of D_i . Lastly, define $N = \bigcup_{i=1}^{\infty} N_i$ and follow the construction before.

Uniqueness: suppose

$$\nu = \nu_a + \nu_s = \nu'_a + \nu'_s \quad (47)$$

Assume ν is a finite / finite signed / complex measure, then

$$\nu_a - \nu'_a = \nu'_s - \nu_s \quad (48)$$

The left hand side is absolutely continuous and the right hand side is singular to μ by the following lemma, hence, they must be both zero.

Lemma 10.2. The notion of absolute continuity and singularity are closed under linear combinations.

Proof. **TODO** ■

Lemma 10.3. If a measure is both absolutely continuous and singular with respect to μ , then it must be zero.

Proof. **TODO** ■

■

11 Product Measure and Fubini's Theorem

11.1 Dynkin's π - λ System

Definition 11.1. Let X be a set, a collection \mathcal{P} of subsets of X is called a π -**system** if $A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$.

Definition 11.2. Let X be a set, a collection \mathcal{D} of subsets of X is called a λ -**system** / **Dynkin class** if

1. $X \in \mathcal{D}$,
2. $A, B \in \mathcal{D}, A \subseteq B \implies B \setminus A \in \mathcal{D}$,
3. if $A_1 \subseteq A_2 \subseteq \dots \in \mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Remark 11.1 (An equivalent definition). The third requirement of λ -system may be replaced by closure under countable disjoint union.

Proof. **TODO** ■

Remark 11.2. A σ -algebra is always a λ -system but not converse.

Example 11.1. Take any two probability measures μ and ν on \mathbb{R} , let

$$\mathcal{D} = \{A \in \mathcal{B}(\mathbb{R}) : \mu(A) = \nu(A)\} \quad (1)$$

Then \mathcal{D} is always a λ -system but not necessarily a σ -algebra.

Theorem 11.1 (Dynkin's π - λ theorem). Let X be a set, let \mathcal{P} be a π -system and \mathcal{D} be a λ -system. Then

$$\mathcal{P} \subseteq \mathcal{D} \implies \sigma(\mathcal{P}) \subseteq \mathcal{D} \quad (2)$$

Proof. Note that an arbitrary intersection of λ -system is a λ -system. Let \mathcal{D} be the smallest λ -system contains \mathcal{P} . Define

$$\mathcal{D}_1 = \{A \in \mathcal{D} : A \cap B \in \mathcal{D} \quad \forall B \in \mathcal{P}\} \quad (3)$$

Since \mathcal{P} is a π -system, $A \cap B \in \mathcal{P} \subseteq \mathcal{D}$ for all $A, B \in \mathcal{P} \subseteq \mathcal{D}$, thus, $\mathcal{P} \subseteq \mathcal{D}_1$. Note that

1. $A \cap B = B \in \mathcal{D}$ for all $B \in \mathcal{P}$, therefore, $X \in \mathcal{D}_1$.
2. Let $A, B \in \mathcal{D}_1$, for every $C \in \mathcal{P}$, $A \cap B, B \cap C \in \mathcal{D}$. Therefore,

$$(B \cap C) \setminus (A \cap C) = (B \setminus A) \cap C \in \mathcal{D} \quad (4)$$

Hence, $B \setminus A \in \mathcal{D}_1$.

3. If $A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{D}_1$, then for all $B \in \mathcal{P}$,

$$(\bigcup A_i) \cap B = \bigcup (A_i \cap B) \in \mathcal{D} \quad (5)$$

Therefore, $\bigcup A_i \in \mathcal{D}_1$, so \mathcal{D}_1 is a λ -system.

Since \mathcal{D}_1 is a λ -system contains \mathcal{P} , hence $\mathcal{D} \subseteq \mathcal{D}_1$. Therefore, $\mathcal{D}_1 = \mathcal{D}$. This shows $\forall A \in \mathcal{D} = \mathcal{D}_1, B \in \mathcal{P}, A \cap B \in \mathcal{D}$.

Define

$$\mathcal{D}_2 = \{A \in \mathcal{D} : A \cap B \in \mathcal{D} \quad \forall B \in \mathcal{P}\} \quad (6)$$

By the previous result, $\mathcal{P} \subseteq \mathcal{D}_2$. Also, \mathcal{D}_2 is a λ -system, so $\mathcal{D}_2 = \mathcal{D}$.

Moreover, for all $A, B \in \mathcal{D}$, $A \cap B \in \mathcal{D}$, so that \mathcal{D} is also a π -system.

Lemma 11.1. A collection of sets is both π and λ if and only if its a σ -algebra.

Proof. For a λ -system \mathcal{D} , it contains $X^c = \emptyset$ and is closed under complement (take one of two sets to be X).

Let $A_1, A_2, \dots \in \mathcal{D}$, we may define $B_n = A_1 \cup \dots \cup A_n$, so that $\bigcup A_n = \bigcup B_n$ and B_n is an increasing sequence. In particular, since \mathcal{D} is closed under complement (as a λ -system) and finite intersection (as a π -system), $B_n \in \mathcal{D}$ as well. By definition of λ -system, $\bigcup A_n \in \mathcal{D}$. ■

Therefore, \mathcal{D} is a σ -algebra containing \mathcal{P} , it follows $\sigma(\mathcal{P}) \subseteq \mathcal{D}$. ■

Corollary 11.1. Let μ and ν be σ -finite measures on a measurable space (X, \mathcal{A}) . If μ and ν agree on a π -system \mathcal{P} that generate \mathcal{A} , then $\mu = \nu$ on \mathcal{A} .

Corollary 11.2. Let μ and ν be measures on a measurable space (X, \mathcal{A}) . Let \mathcal{P} be a π -system on X such that

1. $\sigma(\mathcal{P}) = \mathcal{A}$,
2. $\forall A \in \mathcal{P}, \mu(A) = \nu(A) < \infty$,
3. \exists a sequence $A_1 \subseteq A_2 \subseteq \dots \in \mathcal{P}$ such that $\bigcup A_i = X$ and $\mu(A_i) = \nu(A_i) < \infty$ for all i .

Then $\mu = \nu$.

Intuition: for a π -system that approximates the entire space X via an ascending sequence and generates \mathcal{A} , then it suffices to show $\mu = \nu$ on the π -system in order to show $\mu = \nu$.

Proof. Case 1: finite measures. Suppose μ and ν are finite measures, let

$$\mathcal{D} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\} \quad (7)$$

we are going to show $\mathcal{D} = \mathcal{A}$ indeed.

Firstly, we show \mathcal{D} is a λ -system.

This proof needs to be revised.

(1) Using the last assumption, we may construct a sequence in \mathcal{P} increasing to X , taking the limit shows $\mu(X) = \nu(X)$ and $X \in \mathcal{D}$ as a result.

(2) Let $A, B \in \mathcal{D}$ such that $A \subseteq B$, since μ and ν are finite on \mathcal{P} ,

$$\mu(B \setminus A) = \mu(B) - \mu(A) \quad (8)$$

$$= \nu(B) - \nu(A) \quad (9)$$

$$= \nu(B \setminus A) \quad (10)$$

(3) If $A_1 \subseteq A_2 \subseteq \dots \in \mathcal{D}$, then

$$\mu(\cup A_i) = \lim \mu(A_i) = \lim \nu(A_i) = \nu(\cup A_i) \quad (11)$$

Since $\mathcal{P} \subseteq \mathcal{D}$, the π - λ theorem implies $\mathcal{A} = \sigma(\mathcal{P}) \subseteq \mathcal{D}$. Thus, $\mathcal{A} = \mathcal{D}$. ■

Proof. The general case. There exists $A_1 \subseteq A_2 \subseteq \dots \in \mathcal{P}$ such that $\bigcup A_i = X$. By assumption (3), $\mu(A_i) = \nu(A_i) < \infty$ for every i . Define

$$\mathcal{D}_i = \{B \in \mathcal{A} : \mu(B \cap A_i) = \nu(B \cap A_i)\} \quad (12)$$

\mathcal{D}_i is a λ -system containing \mathcal{P} , so that $\mathcal{A} = \sigma(\mathcal{P}) \subseteq \mathcal{D}_i$. Hence, $\mathcal{D}_i = \mathcal{A}$.

For every $B \in \mathcal{A}$, $\mu(B \cap A_i) = \nu(B \cap A_i)$ for all i . But

$$\mu(B) = \lim \mu(B \cap A_i) = \lim \nu(B \cap A_i) = \nu(B) \quad (13)$$

Thus, $\mu = \nu$. ■

12 Product Measures

Definition 12.1. Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be two measure spaces, suppose μ and ν are σ -finite. Let $X \times Y$ be the Cartesian product of X and Y

$$X \times Y := \{(x, y) : x \in X, y \in Y\} \quad (1)$$

The **product σ -algebra**, $\mathcal{A} \times \mathcal{B}$, is the σ -algebra generated by all sets of the form $A \times B$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$, such sets are called **rectangles**.

Theorem 12.1. There exists a unique measure $\mu \times \nu$ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ that satisfies $\forall A \in \mathcal{A}, B \in \mathcal{B}$,

$$\mu \times \nu(A \times B) = \mu(A)\nu(B) \quad (2)$$

Here, we only require the product measure to be well behaved on rectangles but not other sets in \mathcal{A} .

Proof. Uniqueness. Observe that the set of all rectangles $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ is a π -system:

$$(A \times B) \cap (A' \times B') = \{(a, b) : a \in A, b \in B, a \in A', b \in B'\} \quad (3)$$

$$= \{(a, b) : a \in A \cap A', b \in B \cap B'\} \quad (4)$$

$$= \underbrace{(A \cap A')}_{\in \mathcal{A}} \times \underbrace{(B \cap B')}_{\in \mathcal{B}} \quad (5)$$

Since μ and ν are σ -finite, there exists $A_1 \cup A_2 \cup \dots \in \mathcal{A}$ such that $\bigcup A_i = X$ and $\mu(A_i) < \infty$ for all i . Similarly, there is another such sequence such that $\nu(B_i) < \infty$. Combining these two sequences,

$$A_1 \times B_1 \subseteq A_2 \times B_2 \subseteq \dots \quad (6)$$

Such that $\mu \times \nu(A_i \times B_i) = \mu(A_i)\nu(B_i) < \infty$ for all i and $\bigcup(A_i \times B_i) = X \times Y$.

If γ_1 and γ_2 are two candidates for $\mu \times \nu$, then there exists sequence of rectangles $R_1 \subseteq R_2 \subseteq \dots$ such that $\bigcup R_i = X \times Y$, where $\gamma_1(R_i) = \gamma_2(R_i)$ for all i . By the previous corollary, $\gamma_1 = \gamma_2$ on the entire \mathcal{A} . ■

Proof. Existence. $\forall E \in \mathcal{A} \times \mathcal{B}$ and $\forall x \in X, y \in Y$, define

$$E_x = \{y \in Y : (x, y) \in E\} \quad (7)$$

$$E_y = \{x \in X : (x, y) \in E\} \quad (8)$$

Similarly, for any measurable $f : X \times Y \rightarrow \mathbb{R}^*$, define

$$f_x : Y \rightarrow \mathbb{R}^* \quad f_x(y) = f(x, y) \quad (9)$$

$$f_y : X \rightarrow \mathbb{R}^* \quad f_y(x) = f(x, y) \quad (10)$$

Lemma 12.1. $\forall E \in \mathcal{A} \times \mathcal{B}, \forall x \in X, E_x \in \mathcal{B}; \forall y \in Y, E_y \in \mathcal{A}$.

Proof. Take any $x \in X$, let

$$\mathcal{F} = \{E \in \mathcal{A} \times \mathcal{B} : E_x \in \mathcal{B}\} \quad (11)$$

we show that $\mathcal{F} = \mathcal{A} \times \mathcal{B}$.

Note that $\forall x \in X$, for every rectangle, $A \in \mathcal{A}, B \in \mathcal{B}, (A \times B)_x = B \in \mathcal{B}$. Thus \mathcal{F} contains all rectangles.

(i) $\emptyset \in \mathcal{F}$.

(ii) Let $E \in \mathcal{F}$, then $(E^c)_x = (E_x)^c \in \mathcal{B}$, therefore, $E^c \in \mathcal{F}$.

(iii) Let $E_1, E_2, \dots \in \mathcal{F}$, then $(\bigcup E_i)_x = \bigcup \underbrace{(E_i)_x}_{\in \mathcal{B}} \in \mathcal{B}$.

Therefore, \mathcal{F} is a σ -algebra containing all rectangles, thus $\mathcal{F} \supseteq \sigma(\text{Rectangles}) = \mathcal{A} \times \mathcal{B}$.

Hence, $\mathcal{F} = \mathcal{A} \times \mathcal{B}$.

The same proof works for E_y . ■

Take any measurable $f : X \times Y \rightarrow \mathbb{R}^*$, for all $B \in \mathcal{B}(\mathbb{R}^*)$, let $x \in X$,

$$f_x^{-1} = \{y : f_x(y) \in B\} \quad (12)$$

$$= \{y : f(x, y) \in B\} \quad (13)$$

$$= \{y : (x, y) \in f^{-1}(B)\} \quad (14)$$

$$= (f^{-1}(B))_x \in \mathcal{B} \quad (15)$$

This shows f_x is measurable, a similar argument works for f_y .

Proposition 12.1. Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure spaces. $\forall E \in \mathcal{A} \times \mathcal{B}$,

$$x \mapsto \nu(E_x) \quad (16)$$

$$y \mapsto \mu(E_y) \quad (17)$$

are measurable with respect to \mathcal{A} and \mathcal{B} .

Intuitively, $x \mapsto \nu(E_x)$ computes the side length at a particular level of x .

Proof. First, suppose μ and ν are finite measures, let

$$\mathcal{D} = \{E \in \mathcal{A} \times \mathcal{B} : x \mapsto \nu(E_x) \text{ is } \mathcal{B} \text{ measurable}\} \quad (18)$$

TODO ■

Finally, for every $E \in \mathcal{A} \times \mathcal{B}$, define

$$\gamma_1(E) = \int_X \nu(E_x) d\mu(x) \quad (19)$$

$$\gamma_2(E) = \int_Y \mu(E_y) d\nu(y) \quad (20)$$

Intuitively, for γ_1 , at each $x \in X$, we compute the length of vertical cross-section E_x , then we sum all these lengths across different locations in X . Obviously, $\gamma_1(\emptyset) = 0$ since $(\emptyset)_x = \emptyset$. The same holds for γ_2 .

Let $E_1, E_2 \dots \in \mathcal{A} \times \mathcal{B}$ be a sequence of disjoint sets, $(E_i)_x$ for all $x \in X$. By MCT,

$$\sum \gamma_1(E_i) = \sum \int_X \nu((E_i)_x) d\mu(x) \quad (21)$$

$$= \int_X \sum \nu((E_i)_x) d\mu(x) \quad (22)$$

$$= \int_X \nu(\cup (E_i)_x) d\mu(x) \quad (23)$$

$$= \gamma_1((\cup E_i)_x) \quad (24)$$

Therefore, both γ_1 and γ_2 are measures.

Now, for any rectangle $A \times B$, $(A \times B)_x = B$ if $x \in A$ and equals \emptyset otherwise.

$$\gamma_1(A \times B) = \int \nu((A \times B)_x) d\mu(x) \quad (25)$$

$$= \int \nu(B) \chi_A(x) d\mu(x) \quad (26)$$

$$= \nu(B) \mu(A) \quad (27)$$

Similarly, we can show that $\gamma_2(A \times B) = \mu(A) \nu(B)$. By the uniqueness of product measure, $\gamma_1 = \gamma_2$. ■

Definition 12.2. We define the **product measure** as $\mu \times \nu = \gamma_1 = \gamma_2$,

$$\gamma_1(E) = \int_X \nu(E_x) d\mu(x) \quad (28)$$

$$\gamma_2(E) = \int_Y \nu(E_y) d\nu(y) \quad (29)$$

13 Fubini's Theorem

Theorem 13.1 (Tonelli's Theorem). Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure spaces, let $f : X \times Y \rightarrow [0, \infty]$ be a measurable function (not necessarily $(\mu \times \nu)$ -integrable). Then,

1. The map $x \mapsto \int_Y f_x d\nu$ is \mathcal{A} -measurable, and $y \mapsto \int_X f_y d\mu$ is \mathcal{B} -measurable.
2. the following iterated formula holds:

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left(\int_Y f_x d\nu \right) d\mu \quad (1)$$

$$= \int_Y \left(\int_X f_y d\mu \right) d\nu \quad (2)$$

Theorem 13.2 (Fubini's Theorem). Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be two σ -finite measure spaces, let $f : X \times Y \rightarrow [-\infty, \infty]$ be a measurable mapping and $(\mu \times \nu)$ -integrable, then

1. For μ -a.e. x , f_x is ν -integrable; for ν -a.e. y , f_y is μ -integrable.
2. Define

$$I_f(x) := \begin{cases} \int_Y f_x d\nu & \text{if } f_x \text{ is } \nu\text{-integrable} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$$J_f(y) := \begin{cases} \int_X f_y d\mu & \text{if } f_y \text{ is } \mu\text{-integrable} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

then

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X I_f \, d\mu = \int_Y J_f \, d\nu \quad (5)$$