Notes on Measure Theory

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June 8, 2020

1 Sigma Algebra

Definition 1.1. For a set X, a set $\mathcal{A} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if it satisfies the following properties:

1. $\varnothing, X \in \mathcal{A}$;
2. for all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$ as well;
3. for a sequence in $\mathcal{A}$, $\{A_i\}_{i \in \mathbb{N}}$, the union $\bigcup_{i \in \mathbb{N}} \in \mathcal{A}$ as well.

An element $A \in \mathcal{A}$ is called a **$\mathcal{A}$-measurable set**.

Remark 1.1. It's easy to show that the largest σ-algebra of set X is the power set $\mathcal{P}(X)$, and the smallest σ-algebra is $\{\varnothing, X\}$.

Theorem 1.1. Let $\{\mathcal{A}_i\}_{i \in I}$ be the collection of all σ-algebra on X. Then, $\bigcap_{i \in I} \mathcal{A}_i$ is also a σ-algebra on X.

Proof. ■

Definition 1.2. For $\mathcal{M} \subseteq \mathcal{P}(X)$ ($\mathcal{A}$ is not necessarily a σ-algebra), the smallest σ-algebra (by taking intersections) containing $\mathcal{M}$ is defined as the **σ-algebra generated by** $\mathcal{M}$. The generated σ-algebra is simply the intersection of all σ-algebras that are supersets of $\mathcal{M}$.

$$\sigma(\mathcal{M}) = \bigcap_{\mathcal{A} \supseteq \mathcal{M} \ s.t. \ \mathcal{A} \text{ is } \sigma\text{-algebra}} \mathcal{A} \tag{1}$$

Definition 1.3. Let (X, τ) be a topological space, then the **Borel algebra** is σ-algebra generated by the collection of open sets τ.

$$\mathcal{B}(X) := \sigma(\tau) \tag{2}$$

Remark 1.2. We do not use the entire power set for analysis because it's too large to construct a sensible measure on (see Theorem 1.2).

Theorem 1.2. There is no measure μ on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ satisfying the following two conditions: (i) $\mu((a,b]) = b - a$ for every $a < b$ and (ii) $\mu(x + A) = \mu(A)$ for all $a \in \mathbb{R}$ and $A \in \mathcal{P}(\mathbb{R})$.

Proof. Suppose, for contradiction, there exists such a measure μ, then $\mu((0,1]) = 1 < \infty$.

Claim: the only measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ satisfying $\mu((0,1]) < \infty$ and $\mu(x+A) = \mu(A)$ is the zero measure.

To prove the claim, let $I := (0,1]$ and defien the following equivalence relation on I:

$$x \sim y \iff x - y \in \mathbb{Q} \tag{3}$$

the corresponding equivalence class of x on I can be written as

$$[x] = \{x + r : r \in \mathbb{Q} \wedge x + r \in I\} \tag{4}$$

The collection of all such equivalence classes, $\mathcal{A}$, is a disjoint decomposition of I. (for every $x \in I$, $[x]$ must in $\mathcal{A}$ and $x \in [x]$ trivially. If there exists different $[x] \neq [y]$ but $[x] \cap [y] \neq \varnothing$, take $z \in [x] \cap [y]$, by the transitivity of equivalence relation, $x \sim z \sim y$. Therefore, $[x] = [y]$, contradiction.)

For each $[x] \in \mathcal{A}$, take exactly one $a_x \in [x]$ and define set $A := \{a_x : [x] \in \mathcal{A}\}$. As a result, set A satisfies the following two properties:

1. $\forall x \in I, \exists a_x \in A \ s.t. \ a_x \in [x]$.
2. $\forall x, y \in A$, $x \sim y \implies x = y$.

Since $\mathbb{Q} \cup (-1,1]$ is countable, let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of all elements in it.

For each $n \in \mathbb{N}$, define $A_n := r_n + A$.

Note that for any m, n such that $A_m \cap A_n \neq \varnothing$, take $x \in A_m \cap A_n$. By definition,

$$x = r_n + a_n \tag{5}$$

$$x = r_m + a_m \tag{6}$$

where $a_n, a_m \in A$ and $r_n, r_m \in \mathbb{Q}$. Consequently,

$$a_n - a_m = r_m - r_n \in \mathbb{Q} \tag{7}$$

Therefore, $a_n \sim a_m$. By the second property of A, $a_n = a_m$. Thus, $r_m = r_n$ and $m = n$.

Take the counterposition of what we just proved, $m \neq n \implies A_m \cap A_n = \varnothing$.

Let $z \in (0,1]$, there exists some $a \in A$ such that $z \in [x]$. That is, $z = x + r$ for some $r \in \mathbb{Q} \cap (-1,1]$. There must exist some $m \in \mathbb{N}$ such that $r_m = r$, and consequently, $z \in A_m$.

Therefore, $(0,1] \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq (-1,2]$ (the second relation is obvious). Moreover,

$$\mu((0,1]) \leq \mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \mu((-1,2]) = \mu((-1,0]) + \mu((0,1]) + \mu((1,2]) = 3\mu((0,1]) \tag{8}$$

Note that we just proved $\bigcup_{n\in\mathbb{N}} A_n$ is a disjoint union, hence,

$$\mu((0,1]) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq 3\mu((0,1]) \tag{9}$$

$$\implies ((0,1])\mu \leq \sum_{n=1}^{\infty} \mu(A + r_n) \leq 3\mu((0,1]) \tag{10}$$

$$\implies \mu((0,1]) \leq \sum_{n=1}^{\infty} \mu(A) \leq 3\mu((0,1]) \tag{11}$$

Since $\mu((0,1])$ is finite, the only value $\mu(A)$ can take is zero, and $\mu(I) = 0$ as well. Consequently, for any set $S \in \mathcal{P}(\mathbb{R})$, if $S \subseteq I$, then $\mu(S) \leq \mu(I)$ and $\mu(S) = 0$. Otherwise, let $l = \lfloor \inf(S) \rfloor$ and $u = \lceil \sup(S) \rceil$.

$$I \subseteq S \subseteq \bigcup_{n=l}^{u} (n, n+1] \tag{12}$$

Therefore,

$$0 \leq \mu(S) \leq \sum_{n=l}^{u} \mu(n + (0,1]) = \sum_{n=l}^{u} \mu((0,1]) = 0 \tag{13}$$

It's shown that $\mu(S) = 0$ for every $S \subseteq \mathcal{P}(\mathbb{R})$.

This leads to a contradiction to the first property required ($\mu((a,b]) = b - a$). ■

2 Measurable Spaces and Measurable Maps

Definition 2.1. Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be two measurable spaces. A function $f : X_1 \to X_2$ is a **measurable map** with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$ if

$$f^{-1}(A_2) \in \mathcal{A}_1 \quad \forall A_2 \in \mathcal{A}_2 \tag{14}$$

That is, the pre-image of every set in $\mathcal{A}_2$ is an element in $\mathcal{A}_1$ as well.

Theorem 2.1. Let $(\Omega, \mathcal{A})$ be a measurable space, then the indicator function for any $A \in \mathcal{A}$, $\mathcal{X}_A : \Omega \to \mathbb{R}$, is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbb{R})$.

$$\mathcal{X}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Proof. ■

Theorem 2.2. The composition of measurable maps is measurable.

Proof. ■

Theorem 2.3. For measurable spaces $(X, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and measurable maps $f, g : \Omega \to \mathbb{R}$, $f + g$, $f - g$ and $|f|$ are measurable.

Proof. ■

3 Lebesgue Measures