Probabilistic Graphical Models

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1 Graphical Representations

1.1 Factors

Definition 1.1. Let $X_1, X_2, \cdots, X_k$ be a set of random variables, then a **factor** ϕ is a mapping from values of these random variables to $\mathbb{R}$.

$$\phi : Val(X_1, X_2, \cdots, X_k) \to \mathbb{R} \tag{1}$$

The set of random variables $\{X_1, X_2, \cdots, X_k\}$ is defined as the **scope** of ϕ.

Definition 1.2. Let ϕ_1 and ϕ_2 be two factors with scopes $\{A, B\}$ and $\{B, C\}$. Then the **factor product** $\phi_1 \times \phi_2$ is a factor with scope $\{A, B, C\}$ defined as

$$\phi_1 \cdot \phi_2(a, b, c) = \phi_1(a, b) \cdot \phi_2(b, c) \tag{2}$$

Definition 1.3. Let ϕ be a factor with scope $\{A, B, C\}$, then **marginalizing** C **from** ϕ results in a factor ϕ' with scope $\{A, B\}$ defined as the following:

$$\phi'(a, b) = \sum_{c \in Val(C)} \phi(a, b, c) \tag{3}$$

Definition 1.4. The **factor reduction** operation restricts $\phi(A, B, C)$ to take only a specific value of $C = c$, and results in a factor ϕ' with scope $\{A, B\}$.

$$\phi'(a, b) = \phi(a, b, c) \tag{4}$$

1.2 Semantics and Factorization

Definition 1.5. A **Bayesian network** consists of (i) a directed acyclic graph (DAG) G whose nodes correspond to random variables $X_1, \cdots, X_n$ (ii) and a conditional probability distribution $P(X_i | Par_G(X_i))$ for each node X_i. The joint distribution is defined as the factorization

$$P(X_1, \cdots, X_n) = \prod_{i=1}^{n} P(X_i | Par_G(X_i)) \tag{5}$$

Definition 1.6. Let G be a graph over $X_1, \cdots, X_n$, then the joint probability P **factorizes** over G if and only if

$$P(X_1, \cdots, X_n) = \prod_{i=1}^{n} P(X_i | Par_G(X_i)) \tag{6}$$

1.3 Pass of Influences in Bayesian Networks

Definition 1.7. A path $X_1 - \cdots - X_k$ in Bayesian network G is **active** if there is no explaining-away structure $X_{i-1} \to X_i \leftarrow X_{i+1}$ in it.

Definition 1.8. Let $Z \subseteq V_G$ be a set of random variables in the Bayesian network, then a path $X_1 - \cdots - X_k$ in G is **active conditioned on** Z if

1. for all explaining-away structure $X_{i-1} \to X_i \leftarrow X_{i+1}$ in the path, X_i or some decedents of X_i are in Z,
2. and no other node in the path is in Z.

Definition 1.9. Let $X, Y, Z \subseteq V_G$, if there is no path from X to Y is active conditioned on Z, then X and Y are **d-separated** by Z in graph G denoted as $X \perp_G Y|Z$.