Discrete Mathematics Recitation Class

Tianyu Qiu

University of Michigan - Shanghai Jiaotong University

Joint Institute

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Contents

The Natural Numbers (P4)

Definitions

1. *greater than (or equal to)*
2. *exact division*
3. *odd & even*
4. *prime*

Question

Prove that every natural number is either even or odd and not both.

Uniqueness of Odevity (P4)

Proof.
Suppose that $n \in \mathbb{N}$,

1. prove that
   if n is odd, n cannot be even
2. prove that
   if n is even, n cannot be odd
3. prove that *n cannot be neither odd nor even* (**Do not forget!**)

$\square$

Proof.
Suppose that $n \in \mathbb{N}$,

1. prove that
   if n is not odd, n must be even
2. prove that
   if n is not even, n must be odd
3. prove that *n cannot be both odd and even* (**Do not forget!**)

$\square$

Propositions

Definition (P2)

- Declarative
- One can tell true (T) or false (F) immediately

Logic operations

- unary operation: $\neg$(negation)
- binary operations: $\vee$(disjunction), $\wedge$(conjunction), $\Rightarrow$(implication), $\Leftrightarrow$(biconditonal/equivalence)

Negation (P7)

Definition (by truth table)
Suppose A is a proposition:

A	¬A
T	F
F	T

Conjunction (P8)

Definition (by truth table)
Suppose A, B are propositions:

A	B	A∧B
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (P9)

Definition (by truth table)
Suppose A, B are propositions:

A	B	A∨B
T	T	T
T	F	T
F	T	T
F	F	F

Implication (P11)

Definition (by truth table)
Suppose A, B are propositions:

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (P13)

Definition (by truth table)
Suppose A, B are propositions:

A	B	A⇔B
T	T	T
T	F	F
F	T	F
F	F	T

Precedence of Logical Operators (DMA P11)

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Logical Equivalence

Definitions

1. *tautology* (P10)

   A compound expression that is always true.
2. *contradiction* (P10)

   A compound expression that is always false.
3. *logical equivalence* (P14)

   The biconditional of two propositions is tautology.

Logical Equivalence Examples

1. De Morgan Rules (P14)

$$\neg(A \vee B) \equiv (\neg A) \wedge (\neg B) \qquad \neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$

2. Contraposition (P15)

A	B	¬A	¬B	A⇒B	¬B⇒¬A	A⇒B⇔¬B⇒¬A
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

3. Other Logical Equivalences (P18-P20)

Tips for Logic Operations

Usages

1. Express some operation with other operations
   e.g. Prove that $\neg(A \Rightarrow B)$ is equivalent to $A \land \neg B$
2. Prove tautologies/contradictions without truth tables
   e.g. Prove that $(A \land B) \Rightarrow (A \lor B)$ is a tautology.

Comparison

- Truth table
  1. High effiency for no more than 2 prositional variables
  2. Low effiency when it comes to 3 or more propositional variables
- Logical equivalence transformation
  1. Practice & familiarity required

Arguments

Definitions (P27)

1. *arguments*
2. *premise*
3. *conclusion*

Rules of Inference (P34-P38)

1. Hypothetical Syllogisms
2. Disjunctive and Conjunctive Syllogisms
3. Simple Arguments

Validity & Soundness (P39)

Definitions

1. *validity*: True premises leads to true conclusion. (Conclusion does not need to be true if premises are not true.)
2. *soundness*: All premises are true. (Only deals with truth of premises.)

e.g. Example 1.22 (P39)

When deal with practical proving questions

1. Valid & Sound
2. Contraposition

Predicate

$\boxed{\text{For any natural number } n,}$ $\boxed{n^3 > 10.}$

Quantifier Predicate

Definitions

1. *predicate*: Declarative sentence involving variables.
2. *arity*: number of distinct variables in the predicate.

Difference between Predicates and Propositions:

e.g.

- "$x > 5$" is not a proposition. (we cannot tell the true or false immediately because the value of x is unknown)
- "$x + y < 5$" is a binary predicate.

Predicate

$$\text{predicate logic}\begin{cases}\text{basic predicate variables}: A, B, P, Q, \cdots \\ \text{variables}: x, y, z, \cdots \\ \text{constants}: a, b, c, \cdots \\ \text{domain of discourse (domain)}\end{cases}$$

$$\text{Predicates} \xrightarrow[\text{replace variables by constants}]{\text{bound with quantifiers}} \text{Statements}$$

Logical Quantifiers

Definitions (P22)

1. $\forall$: for all$\cdots$
2. $\exists$: there exists$\cdots$

Hanging Quantifiers (P23)

Appearing at the back is equivalent to appearing just before the expression.

Contraposition of Quantifiers (P24)

$$\neg((\exists x \in M)A(x)) \equiv (\forall x \in M)\neg A(x)$$

$$\neg((\forall x \in M)A(x)) \equiv (\exists x \in M)\neg A(x)$$

Passing negation symbol in quantifier statements from the very left to the very right. (P28)

Vacuous Truth (P25)

Definition

- Apply only for universal quantifier $\forall$
- Domain $M = \emptyset$
- Regardless of $A(x)$ being true or false

A universal statement is true unless there is a counterexample to prove it false.

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Nesting Quantifiers (DMA P60)

TABLE 1 Quantifications of Two Variables.

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Rules of Inference

- 1. Universal Instantiation
  2. Universal Generalization
  3. Existential Instantiation
  4. Existential Generalization
- All other rules of inference that are valid in propositional logic

Proof by Contradiction

Recall the definition of validity of arguments: If all premises are true, then the conclusion is also true.

Question

Prove that there are infinitely many prime numbers.

Proof.

Suppose that there are not infinitely many primes. Therefore there must be finitely many primes: $p_1 < p_2 < \cdots < p_n$ and hence a largest prime p_n. Let

$$m = \prod_{1 \leqslant i \leqslant n} p_i + 1$$

By our assumption, m is not prime. So there exists p_j such that $p_j | m$, i.e. $m = p_j k$ for some integer k. But now,

$$1 = p_j \left(k - \prod_{1 \leqslant i \leqslant n} p_i \right)$$

which shows that $p_j | 1$. This is a contradiction. Therefore, we can conclude that there are infinitely many primes. $\square$

Sets

Definition (P48,P51)

- collection of objects
- ignore order
- ignore repetitions

Express Sets with Predicates: $X = \{x | P(x)\}$ *i.e.* $x \in X$ iff $P(x)$
e.g.
Suppose $A = \{x | A(x)\}, B = \{x | B(x)\}, C$ are sets,

- $\forall x \in A(A(x) \land B(x)) \Leftrightarrow A \subseteq B \Leftrightarrow (A(x) \Rightarrow B(x))$
- $\exists x \in A(A(x) \land B(x)) \Leftrightarrow A \cap B \neq \emptyset$
- $C = \{x | \neg A(x)\} = A^c$

The examples above may help you understand the relation between sets and predicates better.

Sets

Definitions

1. *empty set (P49)*
2. *equality of two sets (P50)*
3. *(proper) subset (P50)*
4. *cardinality (P52)*
5. *powerset (P52)*

- Empty set ($\emptyset$) is subset of any set.
- $A \subseteq X \equiv A \in \mathcal{P}(X)$
- For any finite set X, $|\mathcal{P}(X)| = 2^{|X|}$

e.g.

Let $A = \{\emptyset, \{\{\emptyset\}\}\}$, $B = \{\emptyset\}$ and $C = \{\{\emptyset\}\}$. $B \subseteq A$, but $B \notin A$, and $C \in A$, but $C \nsubseteq A$.

Operations on Sets (P53-P54)

Definitions

1. *union*
2. *intersection*
3. *difference*
4. *complement*
5. *disjoint*

Complex Operations on Sets

Suppose A, B, C are sets,

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cup C) = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cap C) = (A \setminus C) \cup (B \setminus C)$
- $A \setminus B = B^c \cap A$
- $(A \setminus B)^c = A^c \cup B$

Use Venn Diagrams to help understand.

Relationship between Sets & Predicates

Suppose $A(x), B(x)$ are predicates, $A = \{x | A(x) \text{ (is true)}\}$, $B = \{x | B(x) \text{ (is true)}\}$ are sets. $A, B \subseteq M$, which is the domain of discourse. Then we have the following relationships:

- $A \cap B$: a set in which all elements makes both $A(x)$ and $B(x)$ true: $A(x) \wedge B(x)$.
- $A \cup B$: a set in which all elements makes either $A(x)$ or $B(x)$ true: $A(x) \wedge B(x)$.
- $A \setminus B (A \cap B^c)$: a set in which all elements makes $A(x)$ true and makes $B(x)$ false: $A(x) \wedge \neg B(x)$
- $A \subseteq B$: Any element in A (that makes $A(x)$ true) is in B (makes $B(x)$ true): $A(x) \Rightarrow B(x)$.
- $A = B$: Any element in A is in B, any element not in A is not in B: $A(x) \Leftrightarrow B(x)$.

Operations on Sets (P55)

e.g. For a finite number $n \in \mathbb{N}$ of sets $A_0, A_1, \cdots, A_n$:

$$\bigcup_{k=0}^{n} A_k := A_0 \cup A_1 \cup \cdots \cup A_n \qquad \bigcap_{k=0}^{n} A_k := A_0 \cap A_1 \cap \cdots \cap A_n$$

If n extends to ∞,

$$x \in \bigcup_{k=0}^{\infty} A_k :\Leftrightarrow \underset{k\in\mathbb{N}}{\exists}\ x \in A_k \qquad x \in \bigcap_{k=0}^{\infty} A_k :\Leftrightarrow \underset{k\in\mathbb{N}}{\forall}\ x \in A_k$$

Ordered Pairs & Cartesian Products (P57-P58)

Definitions

1. $(a, b) := \{\{a\}, \{a, b\}\}$
2. $(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d)$
3. Cartesian product of two sets $A \times B := \{(a, b) | a \in A \wedge b \in B\}$.
4. ordered $n-$tuple & $n-$fold Cartesian product

Question

How to express ordered n-tuple with sets?

Paradoxes (P59-P63)

- Epimenides Paradox
- Barber Paradox (Russell's Paradox)

Russell's Paradox:

Theorem

The set of all sets that are not members of themselves is not a set. i.e.

$$R := \{x | x \notin x\} \text{ is not a set.}$$

Proof.

The proof is by contradiction. Suppose that R is a set. If $R \in R$, then $R \notin R$ by the definition of R, which is a contradiction. If $R \notin R$, then $R \in R$ by the definition of R, which is also a contradiction. $\square$

Exercises (DMA P35)

1. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.
2. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.
3. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.
4. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

Solutions

- 1.

$$
\begin{aligned}
&(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q \\
&\equiv \neg(\neg p \wedge (\neg p \vee q)) \vee \neg q \\
&\equiv \neg(\neg p \wedge (\neg p \vee q) \wedge q) \\
&\equiv \neg(\neg p \wedge q) \equiv \neg q \vee p
\end{aligned}
$$

  Thus this expression is not a tautology.

- 2.

$$
\begin{aligned}
&(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p \\
&\equiv \neg(\neg q \wedge (\neg p \vee q)) \vee \neg p \\
&\equiv \neg(\neg q \wedge (\neg p \vee q) \wedge p) \\
&\equiv \neg(\neg q \wedge \neg p \wedge p) \\
&\equiv \neg F \equiv T
\end{aligned}
$$

  Thus this expression is a tautology.

Solutions

- 3. Assume that this expression is not a tautology (i.e. for some truth value for p, q, r, the truth value of this expression will be F.) We can find that p must be T and r must be F. In this way, $p \to r = F$. However, not matter q is T or F, we will always find that

$$(p \to q) \wedge (q \to r) = F,$$

which means that the whole expression is equivalent to $F \to F$, which is always true(tautology). This result contradicts with our assumption.

- 4. The method is just the same as question 3.

Summary for Questions

When dealing with questions involving tautology:

- To show something is a tautology:
    1. Manipulate the expression with logical equivalence to obtain T (Question 2).
    2. Make use of Proof by Contradiction (Question 3&4).
- To show something is not a tautology:
    1. After Manipulation, the expression still contains propositional variable(s) (Question 1).
    2. Give propositional variables specific truth values to show that the expression is not a tautology (counterexample).
- Truth table is always reliable and sometimes not so dull as you expected.