Discrete Mathematics Recitation Class

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Contents

Schröder-Bernstein Theorem

Theorem

Let A and B be sets. If there exists $f : A \to B$ that is injective and $g : B \to A$ that is injective, then there exists a bijection $h : A \to B$.

Proof.

Let $f : A \to B$ and $g : B \to A$ be injective functions. We know that $(\mathcal{P}(A), \subseteq)$ is a complete lattice. Define $F : \mathcal{P}(A) \to \mathcal{P}(A)$ by $F(X) = A \setminus g''(B \setminus f''X)$. $F(X)$ is the complement of points in A mapped to be g from the points that are not in the range of f restricted to X.

Claim: F is order-preserving. To see this, let $Y \subseteq Z \subseteq A$. So $f''Y \subseteq f''Z$ and $B \setminus f''Z \subseteq B \setminus f''Y$. Therefore $g''(B \setminus f''Z) \subseteq g''(B \setminus f''Y)$. And so $F(Y) = A \setminus g''(B \setminus f''Y) \subseteq A \setminus g''(B \setminus f''Z)$.

Schröder-Bernstein Theorem

Proof(continue).

By TK Theorem, F has a fixed point. Let $X \subseteq A$ be such that $F(X) = X$. Let $C = rang$. So $g^{-1} : C \to B$ is an injection and $A \setminus X \subseteq C$. Define

$$h = (f \restriction X) \cup (g^{-1} \restriction (A \setminus X))$$

Now, $domh = A$. We have $ran(g^{-1} \restriction (A \setminus X)) = B \setminus f''X$, so $ranh = B$. Therefore $h : A \to B$ is a bijection. $\square$

Corollary

If $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$, *then* $|A| = |B|$.

A Flawed Definition of $\mathbb{N}$

Let V be the set of all sets (does such V really exist?) and let L be the set of all sets that have $\emptyset$ as a member:

$$L = \{x \in V | \emptyset \in x\}$$

$(L, \subset)$ is a complete lattice. (why?) Define the **successor operation** $S : V \to V$ by

$$S(x) = x \cup \{x\} \text{ for all } x \in V$$

Define $F : L \to L$ such that for all $A \in L$,

$$F(A) = A \cup S''A$$

A Flawed Definition of $\mathbb{N}$

For all $A, B \in L$, if $A \subseteq B$, then $S''A \subseteq S''B$. So F is an order-preserving function on the complete lattice $(L, \subseteq)$. Therefore, by TK Theorem, F has a least fixed point.
Let $\mathbb{N}_{def}$ be the least fixed point of F. $\mathbb{N}_{def}$ is the $\subset$-least set X such that

$$\emptyset \in X, S(\emptyset) \in X, S(S(\emptyset)) \in X, \cdots$$

By defining

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= S(\emptyset) = \{\emptyset\} \\ 2 &:= S(S(\emptyset)) = \{\emptyset, \{\emptyset\}\} \\ &\vdots \end{aligned}$$

The set $\mathbb{N}_{def}$ interprets the natural numbers.

A Flawed Definition of $\mathbb{N}$

$$\mathbb{N}_{def} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \cdots\}$$

Set n has n elements (cardinality), thus we can define addition($+$) and multiplication($\cdot$) in $\mathbb{N}$:
Let $m = \{0, 1\}, k = \{2, 3\}$

$$\begin{aligned}
\text{Addition}(+): & |m| = 2, |k| = 2, m \cap k = \emptyset \\
& |m \cup k| = |\{0, 1, 2, 3\}| = |m| + |k| = 4 = |n| \\
& n = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} \\
\text{Multiplication}(\cdot): & |k \times m| = |\{(0, 2), (2, 3), (1, 2), (1, 3)\}| = 4 = |n|
\end{aligned}$$

$$\forall n \in \mathbb{N}_{def}, S(n) = n + 1$$

Properties of $\mathbb{N}_{def}$

- $+$ & $\cdot$: commutativity, associativity, distributivity, identity
- $\leqslant$: a well ordering of $\mathbb{N}_{def}$
- Every $n \in \mathbb{N}_{def}$ except 0 is the successor of some $k \in \mathbb{N}_{def}$, i.e. $n = k + 1$.
- $\mathbb{N}_{def}$ satisfies the principle of induction. If a peoperty $P(x)$ is such that $P(0)$ holds, and $\forall n \in \mathbb{N}_{def}$, if $P(n)$ holds, then $P(n+1)$ holds, then $\forall n \in \mathbb{N}_{def}$, $P(n)$ holds. (proof by contradiction)