Discrete Mathematics Recitation Class

Tianyu Qiu

University of Michigan - Shanghai Jiaotong University

Joint Institute

Summer Term 2020

Contents

Time Complexity for Common Algorithms

	$T(n)$
Linear Search	$\Theta(n)$
Binary Search	$\Theta(logn)$
Insertion Sort	$\Theta(n^2)$
Selection Sort	$\Theta(n^2)$
Bubble Sort	$\Theta(n^2)$
Merge Sort	$\Theta(nlogn)$
Quick Sort	$\Theta(nlogn)$

Theorem

A sorting algorithm based on comparisons of pairs of elements needs $\Omega(nlogn)$ comparisons to sort a list of n elements.

Representation of Integers

Definitions:

1. base b expansion of a
2. digits
3. base conversion
4. integer addition in base 2 ($T(n) = O(n)$)
5. integer multiplication in base 2($T(n) = O(n^2)$)

Examples for Some Algorithms

1. The Division Algorithm (use $O(qloga)$ bit opearions)
2. Modular Exponentiation (use $O((logm)^2 logn)$ bit operations to find b^n mod m)
3. Euclidean Algorithm

Theorem (Lamé's Theorem)

Let $a, b \in \mathbb{N} \setminus \{0\}$ with $a \geq b$. Then the number of divisions used by Euclidean Algorithm to find $gcd(a, b)$ is not greater than five times the number of decimal digits of b.

Recurrence Relations

Definition:
Let $f : \mathbb{N} \times \mathbb{C}^k \longrightarrow \mathbb{C}$ and let $a_0, \ldots, a_{k-1} \in \mathbb{C}$. A function $g : \mathbb{N} \longrightarrow \mathbb{C}$ that satisfies:

$$
\begin{aligned}
g(n) &= a_n & 0 \leqslant n < k \\
g(n) &= f(n, g(n-1), \cdots, g(n-k)) & n \geqslant k
\end{aligned}
$$

is said to satisfy recurrence relation defined by f with initial conditions $a_0, \cdots, a_{k-1}$. (P306)

Theorem

Let $f : \mathbb{N} \times \mathbb{C}^k \longrightarrow \mathbb{C}$ and let $a_0, \ldots, a_{k-1} \in \mathbb{C}$. Then there exists a unique $g : \mathbb{N} \longrightarrow \mathbb{C}$ that satisfies the recurrence relation define by f with initial conditions $a_0, \ldots, a_{k-1}$. (P308)

Linear Recurrence Relations

Definition:
linear recurrence relation (P316):

1. degree k
2. homogeneous & inhomogeneous

Theorem

Let (a_n) and (b_n) satisfy the homogeneous linear recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} \tag{1}$$

Then for all $A, B \in \mathbb{C}$, the sequence $(Aa_n + Bb_n)$ also satisfies (1).

Characteristic Polynomial

Definition:
characteristic polynomial: If $\alpha \in \mathbb{C}$ and the sequence (a_n) defined by $a_n = \alpha^n$ satisfies the homogeneous linear recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} \tag{2}$$

Then $\alpha^n = c_1 \alpha^{n-1} + \cdots + c_k \alpha^{n-k}$. So, if $\alpha \neq 0$, then α is a root of the polynomial

$$\lambda^k - c_1 \lambda^{k-1} - \cdots - c_k \tag{3}$$

(3) is the characteristic polynomial of the recurrence relation (2).

Characteristic Polynomial

Theorem

If $\alpha_1, \dots, \alpha_k$ are roots of the characteristic polynomial of the linear recurrence relation (2) then for all $A_1, \dots, A_k \in \mathbb{C}$, the sequence (a_n) defined by

$$a_n = A_1\alpha_1^n + \dots + A_k\alpha_k^n$$

satisfies (2).

Homogenous Linear Recurrence Relations

Theorem

Let $a_0, \ldots, a_{k-1} \in \mathbb{C}$. *Let* $\alpha_1, \ldots, \alpha_k$ *be k distinct roots of the characteristic polymial of the recurrence relation*

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} \tag{4}$$

Then there exists a sequence (a_n) *in the form*

$$a_n = q_1 \alpha_1^n + \cdots + q_k \alpha_k^n$$

that satisfies (4) with initial conditions $a_0, \ldots, a_{k-1}$.

Homogenous Linear Recurrence Relations

Theorem

Let $a_0, \dots, a_{k-1} \in \mathbb{C}$. Let $\alpha_1, \dots, \alpha_t$ be roots of the characteristic polymial of the recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} \tag{5}$$

with multiplicities $m_1, \dots, m_t$, respectively. Then there exists a sequence (a_n) in the form

$$a_n = Q_1 \alpha_1^n + \cdots + Q_t \alpha_t^n$$

$$\text{with } Q_i = \sum_{j=0}^{m_i - 1} q_{i,j} n^j \text{ for } 1 \leq i \leq t$$

that satisfies (5) with initial conditions $a_0, \dots, a_{k-1}$

Inhomogeneous Linear Recurrence Relations

Suppose that the sequences (a_n) and (b_n) both satisfy the recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + f(n) \tag{6}$$

$$\text{So } a_n - b_n = c_1 (a_{n-1} - b_{n-1}) + \cdots + c_k (a_{n-k} - b_{n-k})$$

And $(a_n - b_n)$ satisfies the recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} \tag{7}$$

Theorem

Let (a_n) satisfy the recurrence relation (6). If (b_n) satisfies the recurrence relation (6) then (b_n) is of the form

$$b_n = c_n + a_n$$

where (c_n) satisfies the recurrence relation (7).

Inhomogeneous Linear Recurrence Relations

This means that by finding a single sequence (a_n) satisfying

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + f(n) \tag{8}$$

we can determine a sequence (b_n) satisfying (8) with any prescribed initial conditions.

Inhomogeneous Linear Recurrence Relations

Theorem

Let $c_1, \ldots, c_k \in \mathbb{R}$ and consider the inhomogenoeous recurrence relation

$$x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + f(n) \text{ with } f(n) = \left(\sum_{i=0}^{t} b_i n^i \right) s^n \quad (9)$$

Then (9) has a particular solution in the form

$$n^m \left(\sum_{i=0}^{t} q_i n^i \right) s^n$$

Inhomogeneous Linear Recurrence Relations

Theorem (Continued)

where $m = 0$ if s is not a root of the characteristic polynomial of the homogeneous recurrence relation associated with (9), and if s is a root of the characteristic polynomial of the homogeneous recurrence relation associated with (9), then m is the multiplicity of that root.

Examples for Recurrence Relations

- Homogeneous Linear Recurrence Relation:
  1. Distinct Solutions for Characteristic Polynomial
  $$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$
  2. Solutions with Multiplicities for Characteristic Polynomial
  $$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$
- Inhomogeneous Linear Recurrence Relation:
  1. $f(x)$ where x is not the solution for characteristic polynomial
  $$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$
  2. $f(x)$ where x is the solution for characteristic polynomial
  $$a_n = 6a_{n-1} - 9a_{n-2} + 3^n$$

Examples for Recurrence Relations

e.g.
Let (a_n) be the sequence such that $a_0 = 0, a_1 = 1,$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3^n$$

Determine a_n as function of n $(n \in \mathbb{N})$.

Examples for Recurrence Relations

Solution:
Suppose $a_n = b_n + c_n$ where

$$b_n = 5b_{n-1} - 6b_{n-2} + 2^n \qquad c_n = 5c_{n-1} - 6c_{n-2} + 3^n$$

We obtain

$$b_n = p_1 2^n + p_2 3^n + p_3 n2^n \qquad c_n = q_1 2^n + q_2 3^n + q_3 n3^n$$

Thus

$$a_n = Q_1 2^n + Q_2 3^n + Q_3 n2^n + Q_4 n3^n$$

with $a_0 = 0, a_1 = 1, a_2 = 18, a_3 = 119$ Solving all the coefficient we have

$$a_n = 4 \cdot 2^n - 4 \cdot 3^n - 2n2^n + 3n3^n$$

Divide-and-Conquer Algorithm

Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c, \qquad a \geq 1, b \in \mathbb{N} \setminus \{0,1\}, c > 0$$

whenever n is divisible by b, Then

$$f(n) = \begin{cases} O(n^{log_b a}) & \textit{if } a > 1 \\ O(logn) & \textit{if } a = 1 \end{cases}$$

Furthermore, whenever $a > 1$ *and* $n = b^k$ *for some* $k \in \mathbb{Z}^+$

$$f(n) = C_1 n^{log_b a} + C_2$$

with

$$C_1 = f(1) + \frac{c}{a-1}, \qquad C_2 = -\frac{c}{a-1}$$

The Master Theorem

Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d, \qquad a \geq 1, b \in \mathbb{N} \setminus \{0, 1\}, c > 0, d \geq 0$$

whenever n is b^k, Then

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$