Discrete Mathematics Recitation Class

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Contents

Cardinality (P96)

Lemma

If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection.

Proof.

For all $z \in C$, there exists a unique y that $(y, z) \in g$. Similarly, for all $y \in B$, there exists a unique X that $(x, y) \in f$. This means for all $z \in C$, there exists a unique x that $(x, z) \in g \circ f$, thus $g \circ f$ is a bijection. □

Definitions

1. *equal cardinality*: bijection
2. *small or equal cardinality*: injection

If $|A| \leqslant |B|$, then $|A| = |C|$ for some $C \subseteq B$.

Examples for Cardinality (P97-P98)

e.g.

1. $|\mathbb{N}| = |2\mathbb{N}|$ $(f : \mathbb{N} \to \mathbb{N}, f(n) = 2n)$
2. $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$ since

$$f : \mathbb{N} \to \mathbb{N}, f(n) = \begin{cases} 0 & n = 0 \\ n+1 & n > 0 \end{cases}$$

3. $|\mathbb{Z}| = |\mathbb{N} \setminus \{1\}|$ since

$$f : \mathbb{N} \to \mathbb{N}, f((-1)^k n) = \begin{cases} 0 & n = 0 \\ 2n+k & n > 0 \end{cases}$$

Theorem

$|\mathbb{Z}| = |\mathbb{N}|$ *(according to e.g.2 and e.g.3)*

Countable Sets & Infinite Sets (P99-P100)

Definitions
For a set A

1. *infinite*: $f : A \to A$ is injective but not surjective.
2. *countable*: $|A| \leqslant |N|$
3. *countably infinite*: both countable and infinite

Lemma

If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is an injective function.

Proof.

$f' : A \to ran f, f' = f$ is a bijection. $g' : ran f \to ran(g \circ f), g' = g$ is a bijection. Thus $g' \circ f' : A \to ran(g \circ f)$ is a bijection. Thus $g \circ f : A \to C$ is an injection. □

Countable Sets & Infinite Sets (P101)

Lemma

If B is a countable set and $A \subseteq B$ then A is countable.

Proof.

$|A| \leqslant |B| \leqslant |\mathbb{N}|$ (we can construct an injective function $f : A \to B, f(x) = x$ for $x \in A$) $\square$

Cantor's Pairing Function (P102)

Theorem

$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

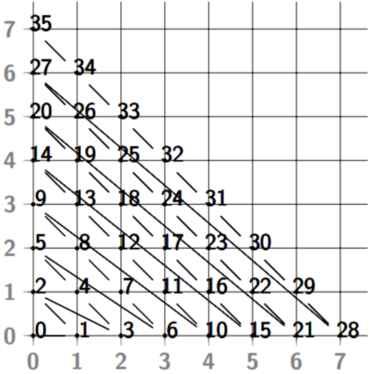
Proof.

Cantor Pairing Function

$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$\pi(x, y) = \frac{1}{2}(x+y)(x+y+1)+y$$

$\square$



Theorem

Cantor's Pairing Function

$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *is a bijection (according to the figure on the right).*

Cantor's Theorem (P104)

Definition

$|A| < |B|$: exists injective functions, exists no bijective functions

Theorem

If A is a set, then there is no injection $f : \mathcal{P}(A) \to A$.

Proof.

This is a proof by contradiction. Let A be a set. Suppose that $f : \mathcal{P}(A) \to A$ is an injection. Since f is injective, $f^{-1} : ranf \to \mathcal{P}(A)$ is a bijection. Let $Z = \{x \in ranf | x \notin f^{-1}(x)\}$. Note that $Z \subseteq A$, and let z=$f(Z)$. Now if $z \in f^{-1}(z) = Z$, then $z \notin f^{-1}(z)$, which is a contradiction. And if $z \notin f^{-1}(z)$, then $z \in Z = f^{-1}(z)$, which is a contradiction (recall Russell's Paradox). $\square$

Cantor's Theorem (P104)

Corollary (Cantor's Theorem)

If A is a set, then $|A| < |\mathcal{P}(A)|$.

Proof.

The function $f = \{(x, \{x\}) \in A \times \mathcal{P}(A) | x \in A\}$ is an injection. $\square$

Uncountable Sets (P105)

Definition

For a set A:

uncountable: not countable ($|A| > |\mathbb{N}|$, recall the definition of countable)

Cantor's Paradox in Naive Set Theory:

If V is the set of all sets, then $\mathcal{P}(V) \subseteq V$, which leads to a contradiction.

Morphisms & Isomorphisms

- isomorphism: $(x, y) \in R$ iff $(f(x), f(y)) \in S$
- homomorphism: *if* $(x, y) \in R$, then $(f(x), f(y)) \in S$

Isomorphisms are definitely homomorphisms.

Order-Preserving Functions (P108)

Definition

Compare with monotone function in Calculus.

e.g.

- Let $a \in \mathbb{N}$ with $a \neq 0$. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = ax$ is order-preserving from $(\mathbb{N}, |)$ to $(\mathbb{N}, |)$.
- The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n - 1$ is order-preserving from $(\mathbb{Z}, \leqslant)$ to $(\mathbb{Z}, \leqslant)$, but $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = -n$ is not.

Fixed Points (P109)

Definition

$f(x) = x$

e.g.

The function $f : \mathcal{P}(\mathbb{N} \to \mathcal{P}(\mathbb{N})$ defined by $f(X) = X \setminus \{0\}$ has the property that if $A \subseteq \mathbb{N}$ is such that $0 \notin A$, then A is a fixed point of f.

Tarski-Knaster Theorem (P110-P111)

Theorem

Let $(L, \preceq)$ be a complete lattice. If $f : (L, \preceq) \to (L, \preceq)$ is an order-preserving function, then f has a (least) fixed point.

Proof.

Let $f : (L, \preceq) \to (L, \preceq)$ be order preserving. Consider $X = \{x \in L | f(x) \preceq x\}$ and $a \in \bigwedge X$

Claim I: If $x \in X$, then $f(x) \in X$. To see this, let $x \in X$. Therefore $f(x) \preceq x$. Since f is order preserving, $f(f(x)) \preceq f(x)$. This shows that $f(x) \in X$.

Claim II: $f(a)$ is a lower bound on X. Since f is order preserving. $f(a) \preceq f(x)$. Since $f(x) \preceq x$, it follows that $f(a) \preceq x$.

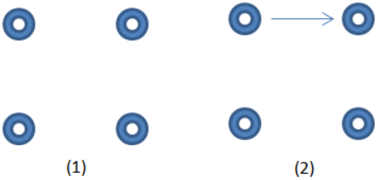
It follows from Claim II that $f(a) \preceq a$, because a is the g.l.b. of X. Therefore $a \in X$. So, by Claim I, $f(a) \in X$. Therefore $a \preceq f(a)$ and $a = f(a)$. So a is a fixed point of f. □

Problems in Assignment 2

Q1. Draw diagrams representing every possible partial order (up to isomorphism) with exactly 4 elements. Note that it is probably helpful for you to use a notation that allows you to omit relationships that are forced by the axioms of reflexivity and transitivity. Which of these posets are chain complete? Which of these posets are linear orders? Which of these posets are well-orders? Which of these posets are lattices? Which of these posets are complete lattices?

Problems in Assignment 2

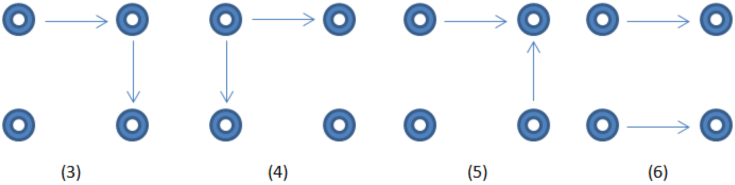
Solution.
Find all 16 graphs in the order of numbers of arrows.



(1) No arrow
(2) One arrow

Problems in Assignment 2

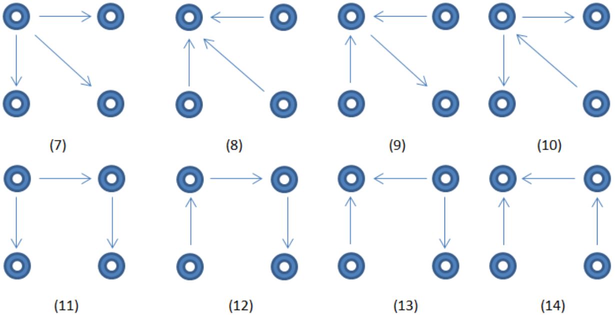
Solution(continued).



(3) One in&out
(4) One two-out
(5) One two-in
(6) Parallel

Problems in Assignment 2

Solution(continued).



(7) One three-out

(8) One three-in

(9) One two-in-one-out

(10) One two-out-one-in

(11) two-out + in&out

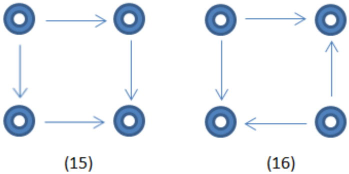
(12) two in&out

(13) two-out + two-in

(14) two-in + in&out

Problems in Assignment 2

Solution(continued).



(15) two-in + two in&out + two-out
(16) two two-in + two two-out

Solution(continued).

	chain complete	linear order	well-order	lattice	complete lattice
no arrow	N	N	N	N	N
one arrow	N	N	N	N	N
one two-out	N	N	N	N	N
one two-in	N	N	N	N	N
one in&out	N	N	N	N	N
parallel	N	N	N	N	N
one three-out	Y	N	N	N	N
one three-in	N	N	N	N	N
two-in + two-out	N	N	N	N	N
two-in + in&out	N	N	N	N	N
two-out + in&out	Y	N	N	N	N
two in&out	Y	Y	Y	Y	Y
one two-in-one-out	Y	N	N	N	N
one two-out-one-in	N	N	N	N	N
two-out + two in&out + two-in	Y	N	N	Y	Y
two two-out + two two-in	N	N	N	N	N

Problems in Assignment 2

Q2.
(i) Prove that every complete lattice has a unique maximal element.
(ii) Give an example of an infinite chain complete poset with no unique maximal element.
(iii) Prove that any closed interval on $\mathbb{R}$ ($[a, b]$) with the usual order ($\leqslant$) is a complete lattice (you may assume the properties of $\mathbb{R}$ that you assume in Calculus class).
(iv) Say that a poset is *almost chain complete* if every nonempty chain has an l.u.b. Give an example of an almost chain complete poset with no minimal element.

Problems in Assignment 2

Solution.

(i)

- Existence: $L \subseteq L$
- Uniqueness: Proof of Contradiction

(ii) We define (A,R) is a poset and all its subset $X \subseteq A$ is a chain. We define

$$A^* = A \cup \{u.b._1, u.b._2\}$$

$$R^* = R \cup \{(x, u.b._1), (x, u.b._2), (u.b._1, u.b._1), (u.b._2, u.b._2)\}$$

for all $x \in A$, thus we have (A^*, R^*) satisfy the requirement.

Problems in Assignment 2

Solution(continued).

(iii) Suppose $X \subseteq \mathbb{R}([a, b])$ is a closed interval $[c, d]$ where $a \leqslant c \leqslant d \leqslant b$, $Y \subseteq X$. Consider $(Y, \leqslant)$:

- Y is finite.
- Y is infinite. We can consider a sequence S_y whose elements are in increasing order. Since the sequence is monotonously increasing and it has lower bound c and upper bound d, this sequence must converge on $\mathbb{R}$. Thus its l.u.b. is either the upper limit or the largest element. Its g.l.b. is either the lower limit or the smallest element. Thus all $X's$ subsets have g.l.b. and l.u.b. X is a complete lattice.

(iv) $(\mathbb{R}(0, 1], \leqslant)$.