UM–SJTU Joint Institute

VE477 Intro to Algorithms

Homework 2

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Question 1 Basic complexity

1. a)

We first prove that $n^3 - 3n^2 - n + 1 = \mathcal{O}(n^3)$. We choose $c = 2$ and $n = 4$, next we calculate

$$c \cdot g(n) - f(n) = 2n^3 - (n^3 - 3n^2 - n + 1) = n((n - \frac{3}{2})^2 - \frac{13}{4})$$

For $n > 4$, obviously the former equation yields to a result greater than 0. Since we have found the c and n to make the condition validate, which means $n^3 - 3n^2 - n + 1 = \mathcal{O}(n^3)$.

Next is $n^3 - 3n^2 - n + 1 = \Omega(n^3)$. We choose $c = \frac{1}{2}$ and $n = 7$.

$$f(n) - c \cdot g(n) = (n^3 - 3n^2 - n + 1) - \frac{1}{2}n^3 = \frac{1}{2}n[(n-3)^2 - 11] + 1$$

For $n \geq 7$, the former equation yields to a result greater than 0, which means $n^3 - 3n^2 - n + 1 = \Omega(n^3)$.

Since $n^3 - 3n^2 - n + 1 = \mathcal{O}(n^3)$ and $n^3 - 3n^2 - n + 1 = \Omega(n^3)$, we could conclude that

$$n^3 - 3n^2 - n + 1 = \Theta(n^3)$$

□

1. b)

We set $c = 1$ and $n = 2$. We will find that when $n = 2$, $2^n = n^2$, for easier comparison, we transform them into log basis. which is $2 \log n$ and $n \log 2$

then we use

$$f(x) = \int f'(x)$$

So next we need to compare $\frac{d}{dn} 2 \log n = \frac{2}{n}$ and $\log 2$.

Obviously, $\frac{2}{n} \leq 1$, $\forall n \geq 2$, so we have

$$\frac{d}{dn} 2 \log n \leq \frac{d}{dn} n \log 2$$

And then

$$f(n) = 2 \log 2 + \int_2^n f'(n)$$

and

$$g(n) = 2 \log 2 + \int_2^n g'n$$

So $\forall n \geq 2$, and $c = 1$,

$$f(n) \leq g(n)$$

namely

$$n^2 = \mathcal{O}(2^)$$

□

2. a)

$f(n) = \mathcal{O}(g(n))$. We choose $c = 1$ and $n = 9$. For the base case, namely $f(9)$ and $g(9)$, $f(n) \leq g(n)$. And we apply the same methods as 1.b), since $f'(n) < g'(n)$, $\forall n \geq 9$, we could conclude that

$$f(n) = \mathcal{O}(g(n))$$

3. a)

Not exist.

3. b)

$f(n) = n$, $g(n) = 10$

4

When n is approaching ∞,

$$f_4(n) > f_1(n) > f_3(n) > f_2(n)$$

It is easy to obtain the order of f_2 and f_3,

$$\frac{f_3}{f_2} = \frac{\sqrt{n}}{\sqrt{\log n}} > 1 \Rightarrow f_3 > f_2$$

Next we need to compare f_3 and f_1. After observing the form of f_3 and f_1, we divide them into pairs, namely $p_i = \sqrt{i} + \sqrt{n+1-i}$ for f_1 and $q = 2\sqrt{n}$ for f_3.

Note that $f_1 = \sum_{i=1}^{n/2} p_i$ and $f_3 = \sum_{i=1}^{n/2} q$. Then we calculate $p_1^2 - q^2$,

$$p_1^2 - q^2 = n + 2\sqrt{n} + 1 - 4\log n > n - 4\log n$$

when $n \geq 9$, we will have $f_1^2 - f_3^2 > 0$. Similarly, we can derive that for every pair, $p_i > q$. And this tells $f_1 > f_3$.

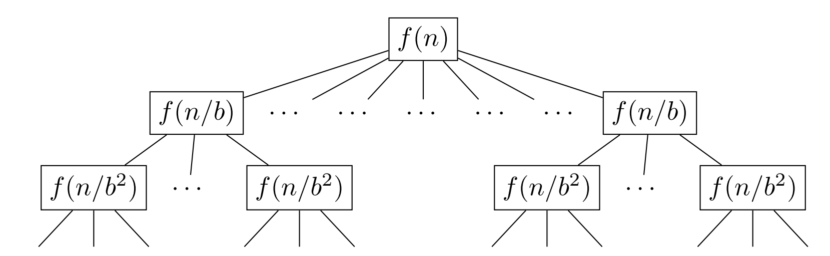
$f_4 > f_1$ is also obvious. That

$$f_4 > n\sqrt{n} > \underbrace{\sqrt{n} + \sqrt{n} + \cdots + \sqrt{n}}_{\text{totally n items}} > 1 + \sqrt{2} + \cdots + \sqrt{n} = f_1$$

$\square$

Question 2 Master Theorem

1 a)



where each node has b number of child nodes.

1 b)

i) The depth of the tree is $\log_b n$

ii) The leaves are $a^{depth} = a^{\log_b n}$

iii) In each level, denote as level j, the cost is $a^j \cdot f(n/b^j)$

iv) T(n) is the rest items plus the calculation on each level,

$$T(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j) + \Theta(n^{log_b a})$$

Please note that the layer is calculated to the $log_b n - 1$ recursive call of $f(x)$, And for the bottom, namely the leaves, it will run several operations, which is $\Theta(a^{log_b n}) = \Theta(n^{log_b a})$, which is derived from the property of logarithm.

2 a)

1. We prove from the definition, that $\exists c1, c2, n_0$ s.t. $\forall n \geq n_0$, $c_1 \cdot n^{\log_b a} \leq f(n) \leq c_2 \cdot n^{\log_b a}$.

   We first prove the upper bound, and the lower bound can also be derived symmetrically.

$$g(n) \leq \sum_{j=0}^{\log_b n-1} a^j (c_2 \frac{n}{b^j})^{\log_b a} = c_2^{\log_b a} \left(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \right)$$

   Similarly,

$$g(n) \geq c_1^{\log_b a} \left(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \right)$$

   So we find $c_2 = c_1^{\log_b a}$ and $c_3 = c_2^{\log_b a}$ s.t., $\forall n \geq n_0$,

$$c_1^{\log_b a} \left(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \right) \leq g(n) \leq c_2^{\log_b a} \left(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \right)$$

   which means

$$g(n) = \Theta(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a})$$

$\square$

2.

3. We use the result from *2.a.ii*, that

$$\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} = n^{\log_b a} \log_b n$$

   And we substitute this into the result above, that

$$g(n) = \Theta(n^{\log_b a} \log_b n)$$

2 b)

i) Not done yet.

ii) By observation, we can see that actually we need to show

$$\sum_{j=0}^{\log_b n-1} \frac{a^j}{(b^j)^{\log_b a-\varepsilon}} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1}$$

And we apply transformations to the left part inside the sum, which is

$$\sum_{j=0}^{\log_b n-1} \frac{a^j}{(b^{\log_b a-\varepsilon})^j} = \sum_{j=0}^{\log_b n-1} b^{\varepsilon j} = \frac{1-b^{\varepsilon\cdot\log_b n}}{1-b^\varepsilon} = \frac{1-(b^{\log_b n})^\varepsilon}{1-b^\varepsilon} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1}$$

$\square$

iii)

$$\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon} = \frac{n^\varepsilon - 1}{(b^\varepsilon - 1)\cdot n^\varepsilon} n^{log_b a}$$

We set $c_0 = 1$, and solve

$$\frac{n^\varepsilon - 1}{(b^\varepsilon - 1)\cdot n^\varepsilon} \leq 1$$

which is

$$n^\varepsilon \cdot (2 - b^\varepsilon) \leq 1$$

if $2 - b^\varepsilon \leq 0$, this equation obviously holds, and then we could conclude that $\forall n > n_0$, $\exists c_0 = 1$ s.t.

$$\frac{n^\varepsilon - 1}{b^\varepsilon - 1} n^{\log_b a-\varepsilon} \leq C \cdot n^{\log_b a}$$

which means

$$g(n) = O(n^{\log_b a})$$

2 c)

1. Simply set $c = 1$, and it is obvious that

$$\begin{aligned} g(n) &= a^0 f(n/b^0) + af(n/b) + a^2 f(n/b^2) \\ &> a^0 f(n/b^0) = f(n) \end{aligned}$$

which means

$$g(n) = \Omega(f(n))$$

$\square$

2. Let $t = n/b^{j-1}$,

$$a^j f(n/b^j) = a^j f(t/b) = a^{j-1} \cdot (a \cdot f(t/b)) \leq a^{j-1} c f(t) = c(a^{j-1} f(n/b^{j-1}))$$

Similarly, we apply the same method to $a^{j-1} f(n/b^{j-1})$, and so on and so forth,

$$a^j f(n/b^j \leq c \cdot a^{j-1} f(n/b^{j-1}) \leq c^2 \cdot a^{j-1} f(n/b^{j-2} \leq \cdots \leq c^j f(n)$$

$\square$

3. We recall the definition of $g(n)$ is that

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$$

So

$$\begin{aligned}
g(n) &= a^0 f(n/b^0) + af(n/b) + a^2 f(n/b^2) + \cdots + a^{\log_b n-1} f(n/b^{\log_b n-1}) \\
&\leq f(n) + cf(n) + c^2 f(n) + \cdots + c^{\log_b n-1} f(n) \\
&< \frac{1}{1-c} f(n) \ \text{(Derived from infinite geometric sequence sum)}
\end{aligned}$$

Let $c_0 = \frac{1}{1-c}$, this yields to

$$g(n) = \mathcal{O}(f(n))$$

4. Since we can find c_0, c_1 s.t.

$$c_0 f(n) \leq g(n) \leq c_1 f(n)$$

so

$$g(n) = \Theta(f(n))$$