

# Introduction to Algorithms

## Chapter 1: Basics on algorithms

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Fall 2017

# Outline

- ① A brief introduction to algorithms
- ② Common data structures
- ③ Basic algorithm paradigms

# Algorithm

**Algorithm:** Recipe telling the computer how to solve a problem.

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Example.

I am the “computer”, detail an algorithm such that I can prepare a jam sandwich.

*Actions:* cut, listen, spread, sleep, read, take, eat, dip

*Things:* knife, guitar, bread, honey, jam jar, sword

# Algorithm

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I am the “computer”, detail an algorithm such that I can prepare a jam sandwich.

*Actions:* cut, listen, spread, sleep, read, take, eat, dip

*Things:* knife, guitar, bread, honey, jam jar, sword

Algorithm. (*Sandwich making*)

---

**Input** : 1 bread, 1 jamjar, 1 knife

**Output:** 1 jam sandwich

- 1 take the knife and cut 2 slices of bread;
  - 2 dip the knife into the jamjar;
  - 3 spread the jam on the bread, **using the knife**;
  - 4 assemble the 2 slices together, **jam on the inside**;
-

## A more formal view

An algorithm systematically solves a *well-defined* problem:

- The *Input* is clearly expressed
- The *Output* solves the problem
- The *Algorithm* provides a precise step-by-step procedure starting from the Input and leading to the Output

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Algorithms can be described using one of the three following ways:

- English
- Pseudocode
- Programming language

## A first example

Algorithm. (*Insertion Sort*)

---

**Input** :  $a_1, \dots, a_n$ ,  $n$  unsorted elements

**Output**: the  $a_i, 1 \leq i \leq n$ , in increasing order

```
1 for  $j \leftarrow 2$  to  $n$  do
2    $i \leftarrow 1$ ;
3   while  $a_j > a_i$  do  $i \leftarrow i + 1$ ;
4    $m \leftarrow a_j$ ;
5   for  $k \leftarrow 0$  to  $j - i - 1$  do  $a_{j-k} \leftarrow a_{j-k-1}$ ;
6    $a_i \leftarrow m$ 
7 end for
8 return  $(a_1, \dots, a_n)$ 
```

---



## A first problem

**Setup:** a robot arm solders chips on a board in  $n$  contact points

**Goal:** minimize the time to attach a chip to the board

**Assumptions:**

- The arm moves at constant speed
- Once a chip has been attached another one is soldered

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Defining the Input and Output:

- Input: a set  $S$  of  $n$  points in the plane
- Output: the shortest path visiting all the points in  $S$

## A first solution

Algorithm. (*Nearest neighbor*)

---

**Input** : a set  $S = \{s_0, \dots, s_{n-1}\}$  of  $n$  points in the plane

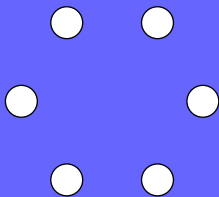
**Output**: the shortest cycle visiting all the points in  $S$

```
1  $p_0 \leftarrow s_0$ ;  
2 for  $i \leftarrow 1$  to  $n - 1$  do  
3    $p_i \leftarrow$  closest unvisited neighbor to  $p_{i-1}$ ;  
4   Visit  $p_i$ ;  
5 end for  
6 return  $\langle p_0, \dots, p_{n-1} \rangle$ 
```

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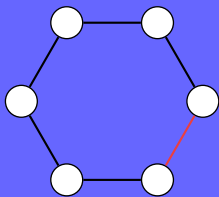
## A first failure

How does the nearest neighbor algorithm (1.7) perform in the following three cases?



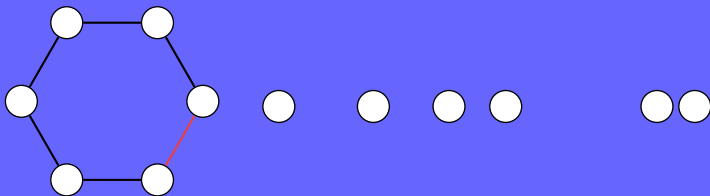
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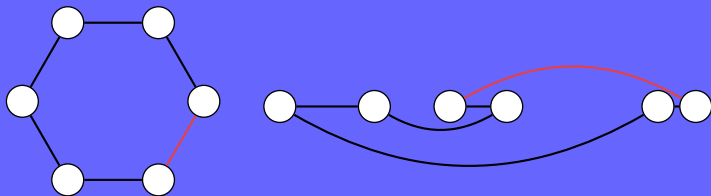
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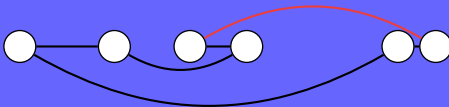
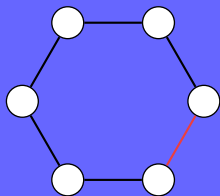
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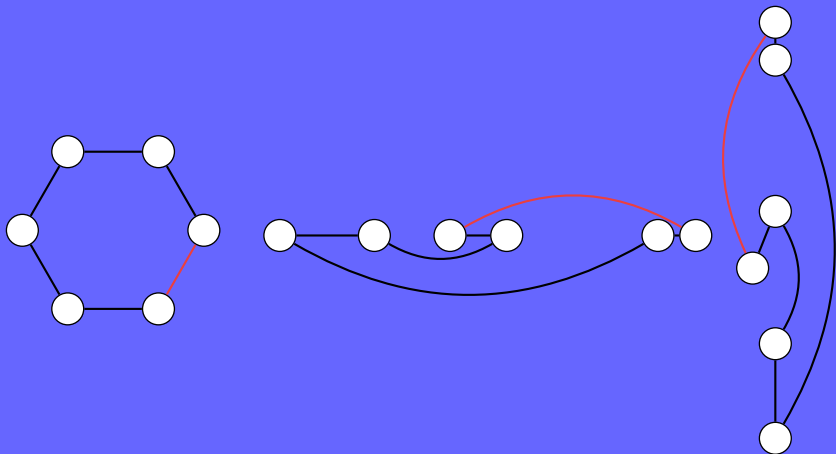
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## A first failure

How does the nearest neighbor algorithm (1.7) perform in the following three cases?



## A second solution

Algorithm. (*Closest pair*)

---

**Input** : a set  $S$  of  $n$  points in the plane

**Output**: the shortest cycle visiting all the points in  $S$

```
1 for  $i \leftarrow 1$  to  $n - 1$  do
2    $d \leftarrow \infty$ ;
3   foreach pair of end points  $\langle s, t \rangle$  from distinct vertex chains
4     do
5       if  $\text{dist}(s, t) \leq d$  then
6          $s_m \leftarrow s; t_m \leftarrow t; d \leftarrow \text{dist}(s, t)$ ;
7       end if
8   end foreach
9   Connect  $s_m$  and  $t_m$  by an edge;
10 end for
11 return all the points starting from one of the two end points
```

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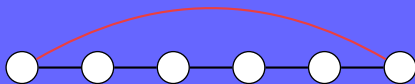
## A second failure

Applying the closest pair algorithm (1.9) on the following vertices arrangement yields the two graphs:



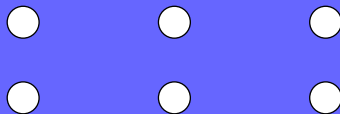
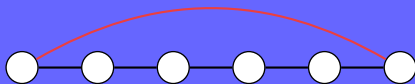
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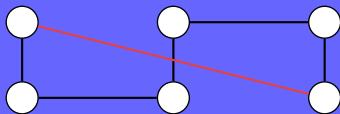
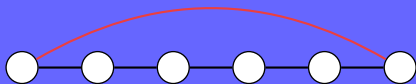
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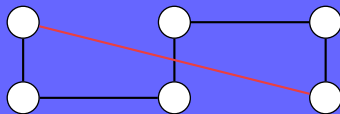
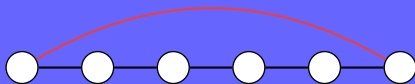
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Possible strategy to ensure the most optimal path:

- Enumerate all the possible paths
- Select the one that minimizes the total length

**Drawback:** for only 20 vertices  $20! = 2432902008176640000$  paths have to be explored...

# The first few lessons

A difference:

- Algorithm: always output a correct result
- Heuristic: idea serving as a guide to solve a problem with no guarantee of always providing a good solution

Correctness and efficiency:

- An algorithm working on a set of input does not imply it will work on all instances
- Efficient algorithm totally solving a problem might not exist



# Solving problems using algorithms

Common traps when defining the Input and Output:

- Are they precise enough?
- Can all the Input be easily and efficiently generated?
- Could there be any confusion on the expected Output?

Example.

For an Output, what does it mean to “find the best route”?

# Solving problems using algorithms

Common traps when defining the Input and Output:

- Are they precise enough?
- Can all the Input be easily and efficiently generated?
- Could there be any confusion on the expected Output?

Example.

For an Output, what does it mean to “find the best route”?

The shortest in distance, the fastest in time, or the one minimizing the number of turns?

**Conclusion:** where to start (Input) and where to go (Output) must be expressed in simple, precise, and clear terms.

# Incorrectness

Finding good counter-examples:

- Seek simplicity: make it clear why the algorithm fails
- Think small: algorithms failing for large Input often fail for smaller one
- Test the extremes: study special cases, e.g. inputs equal, tiny, huge...
- Think exhaustively: test whether all the possible cases are covered by the algorithm
- Track weaknesses: check if the underlying idea behind the algorithm has any “unexpected” impact on the output

# Outline

- 1 A brief introduction to algorithms
- 2 Common data structures
- 3 Basic algorithm paradigms

# Continuous vs. linked

Data structures can be split into two main categories:

- **Continuous:** a single piece of memory  
e.g. array, matrices, hash tables. . .
- **Linked data structures:** distinct chunks of memory connected together  
e.g. linked list, trees, graph adjacency lists. . .

Each type has some relative advantages. Choosing the proper data type is of a major importance when designing algorithms.

# Arrays

Each element can be efficiently located using its index:

- **Constant access time:** each index maps to a memory address
- **Space efficiency:** no space wasted with links or information on the data
- **Memory locality:** data is contiguous so *cache* can be used to speed up successive data accesses

**Downside:** the size cannot be easily adjusted during the program's execution

# Linked structures

A linked structure is composed of nodes. Each one contains:

- One or more fields on data
- A pointer to at least another node

The most common operations are:

- Search: find an item in the list
- Insert: add an item to the list
- Delete: remove an item from the list

Search can be implemented either iteratively or recursively

# Comparison

## Linked structure

- Overflow only occurs when memory is full
- Insertion/deletion are simple and fast
- Moving pointers is faster than moving the actual data

## Array

- No extra space wasted for the pointer field
- Efficient random access is possible
- Better memory locality and cache performance



# Containers

Common data structures allowing the storage and retrieval of data independently of the content:

- **Stack:**
  - LIFO order
  - Simple to implement and very efficient
- **Queue:**
  - FIFO order
  - Minimize the maximum waiting time
  - Trickier to implement than stacks

Both can be implemented using either linked lists or arrays, depending if the size of the container is known in advance

# Dictionaries

Data type allowing access by content. Primary operations:


- **Search:** search a value in a given dictionary
- **Insert:** add an element to the dictionary
- **Delete:** remove an element from the dictionary

Most common operations:

- **Max/Min:** retrieve the largest/smallest element from the dictionary
- **Predecessor/Successor:** retrieve the element just before/after a given element; before/after are defined with respect to a sorted order

# Dictionary using arrays

Let  $n$  be the number of elements in the array

Operation	Unsorted array	Sorted array
search( $D, k$ )	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$
insert( $D, k$ )	$\mathcal{O}(1)$	$\mathcal{O}(n)$ 
delete( $D, k$ )	$\mathcal{O}(1)^*$	$\mathcal{O}(n)$
predecessor( $D, k$ )	$\mathcal{O}(n)$	$\mathcal{O}(1)$
successor( $D, k$ )	$\mathcal{O}(n)$	$\mathcal{O}(1)$
minimum( $D$ )	$\mathcal{O}(n)$	$\mathcal{O}(1)$
maximum( $D$ )	$\mathcal{O}(n)$	$\mathcal{O}(1)$

\* Assuming a pointer to the key  $k$  is given how to get  $\mathcal{O}(1)$ ?

Just delete the entry of  $k$ , the empty address can be used later with no consequence.

# Dictionary using linked structures

Let  $n$  be the number of elements in the list

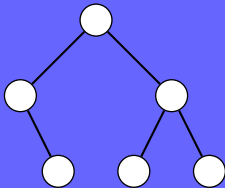
Operation	Unsorted		Sorted	
	Single	Double	Single	Double
<code>search(D,k)</code>	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
<code>insert(D,k)</code>	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
<code>delete(D,k)</code>	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(n)^*$	$\mathcal{O}(1)$
<code>predecessor(D,k)</code>	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)^*$	$\mathcal{O}(1)$
<code>successor(D,k)</code>	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
<code>minimum(D)</code>	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
<code>maximum(D)</code>	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)^\dagger$	$\mathcal{O}(1)$

\* Why are singly linked lists slower? **Need to be shifted after removing one element.**

† How to achieve  $\mathcal{O}(1)$  for singly sorted lists?

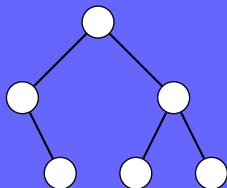
Use a pointer to record the minimum element each time when we insert elements.

# Binary search trees



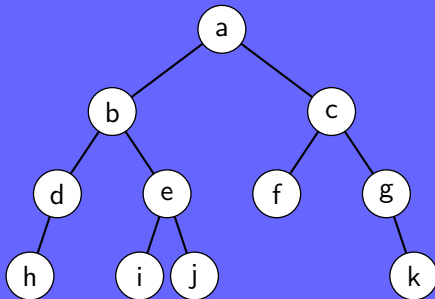
- Constructed from doubly linked list
- First object is the root of the tree
- Second object is a left child if it precedes the root and a right child if it succeeds it
- Third and further object are sorted along the tree following a similar pattern

# Binary search trees



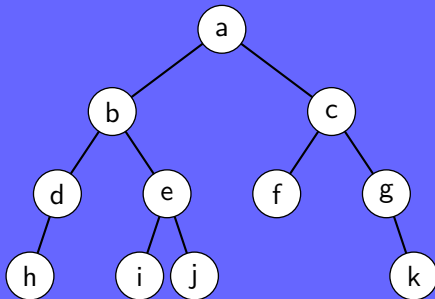
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- 
- The three primary dictionary operations take  $\mathcal{O}(h)$ , with  $h$  the height of the tree
  - Binary search trees balance the search time and flexible update
  - How to handle deletion?  
**Need further rearranging the whole tree**

# Binary search tree traversal



- Preorder traversal:

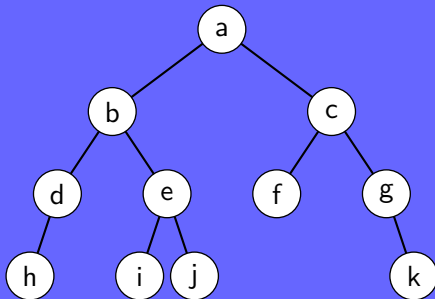
# Binary search tree traversal



- Preorder traversal:  
a, b, d, h, e, i, j, c, f, g, k
- Inorder traversal:

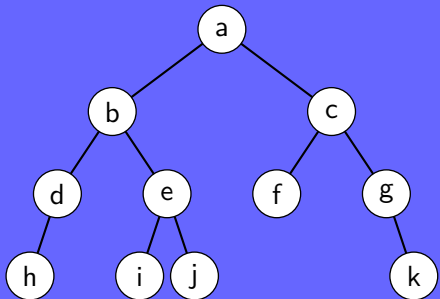


# Binary search tree traversal



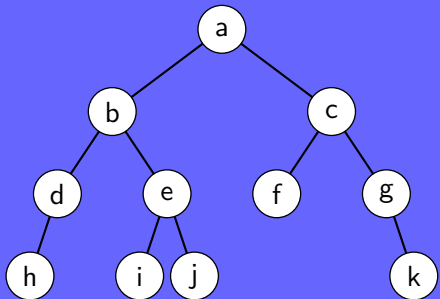
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h, d, b, i, e, j, a, f, c, g, k
- Postorder traversal:

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# Binary search tree traversal



- Preorder traversal:  
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h, d, i, j, e, b, f, k, g, c, a

How to implement inorder tree traversal?

**Visit the left/right subtree in-order recursively**

# Priority queue

Primary operations for priority queues:

- **Insert:** add an element to the queue
- **Find minimum/maximum:** return the last/first element in the queue
- **Delete minimum/maximum:** remove the last/first element in the queue

Operation	Array		Balanced tree
	Unsorted	Sorted	
<code>insert(Q,x)</code>	$\mathcal{O}(1)$	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$
<code>find_min(Q)</code>	$\mathcal{O}(1)^*$	$\mathcal{O}(1)$	$\mathcal{O}(1)^*$
<code>delete_min(Q)</code>	$\mathcal{O}(n)$	$\mathcal{O}(1)^\dagger$	$\mathcal{O}(\log n)$

Just exchange the min and the max, then delete and remove the min slot

Add a pointer to record the minimum each time insert.

\* How to reach  $\mathcal{O}(1)$  for an unsorted array and a balanced tree?

† How to reach  $\mathcal{O}(1)$  when deleting the min in a sorted array?

# Hash tables

Practical way to maintain a dictionary where:

- The data is stored in an array
- Each key is hashed and stored at index “the hash of the key”
- Keys with a similar hash are store in a linked list

**Good hash function:** all indices occur with equiprobability

# Hash tables

Practical way to maintain a dictionary where:

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Example.

A common choice is  $H(k) = k \bmod m$ , with  $m$  a prime not too close from a power of 2.

For  $n = 2000$ , 701 would be a good choice if one desires to have about three keys stored at each index.

## Other common data structures

- **Strings:** array of characters; use suffix trees/arrays for pattern matching
- **Geometric element:** define regions as polygons using segments and points in an array or a tree
- **Graphs:** consider the adjacency matrix or an adjacency list; graph algorithms vary depending on the structure
- **Sets:** bit vector where the element in the set is the index and the value store is 1 or 0 depending whether the element is in the set; dictionaries can be used for fast membership queries

# Summary

A few points to remember when selecting a data structure:

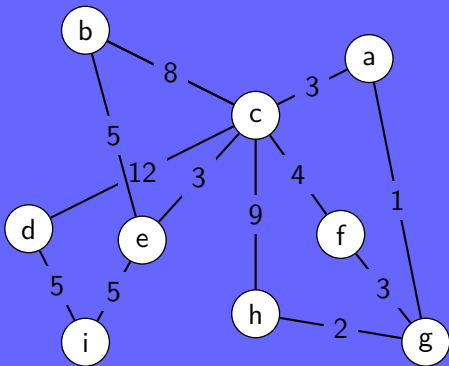
- Data can be represented in many ways
- No data structure is fast in all aspects
- Choosing the wrong data structure can be disastrous in terms of performance
- Several choices are often possible
- Identifying the best data structure is often not critical
- Always aim for clear, simple, and efficient data structures



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# A first graph problem



Simple problem:

- We have  $n$  computers connected by wires
- Using different wires implies different costs
- We want:
  - All the computers to be connected to the network
  - Minimize the cost

# Minimum spanning tree

## Problem (Minimum Spanning Tree (MST))

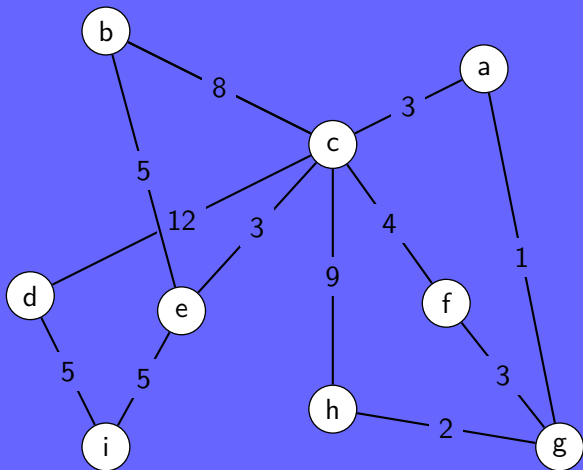
Given a weighted graph  $G$ , find a subgraph  $T$  such that:

- ① All the vertices on  $G$  are connected on  $T$ ,
- ② The total weight, defined as the sum of the weight of all the edges in  $T$ , is minimized.

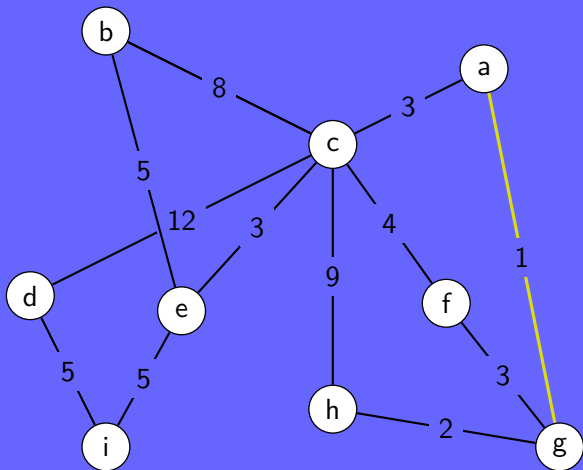
The graph  $T$  is a *minimum spanning tree* for  $G$ .

Note that  $T$  is a tree: if it contained a cycle, at least one edge could be removed, allowing a lower weight while preserving the connected property of  $T$ .

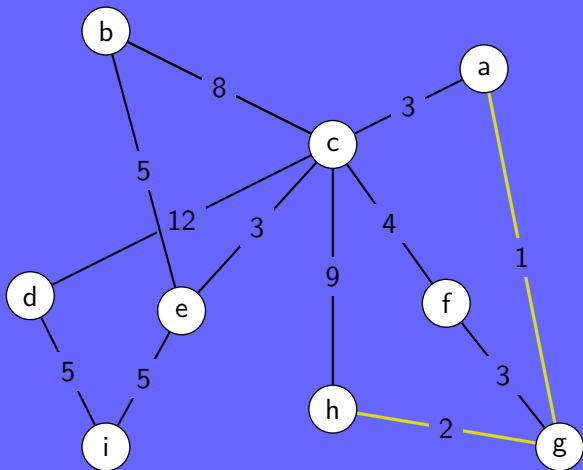
## A greedy approach



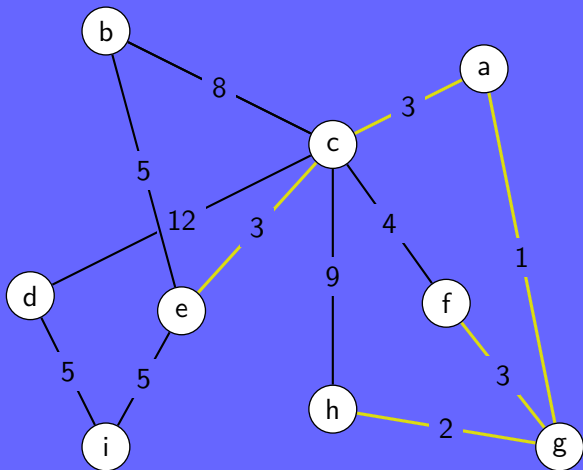
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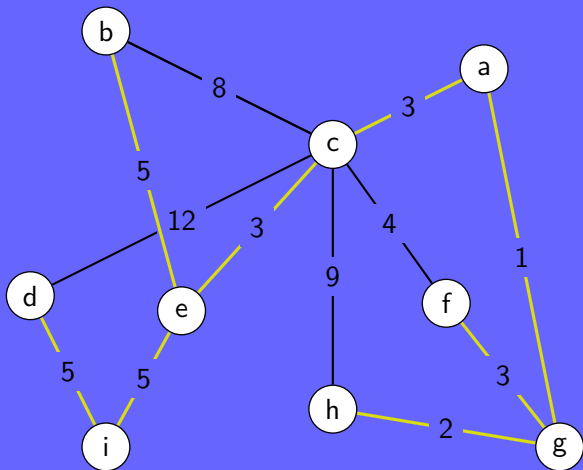
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## A greedy approach





# Kruskal's algorithm

Algorithm. (*Kruskal*)

---

**Input** : A graph  $G = \langle V, E \rangle$

**Output:** A minimum spanning tree  $T$  for  $G$

```
1 Sort the edges  $G.E$  by weight;  
2  $T \leftarrow \emptyset$ ;  
3 for edges  $(u, v)$  in  $G.E$ , in non-decreasing order do  
4   | if adding  $(u, v)$  does not create a cycle then  
5   |   | add edge  $(u, v)$  to  $T$   
6   | end if  
7 end for  
8 return  $T$ 
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---

What needs to be specified?

# Correctness of Kruskal's algorithm

## Theorem

Assuming the previous notations, Kruskal's algorithm produces a minimum spanning tree for  $G$ .

Proof.

Let  $G = \langle V, E \rangle$  be a graph and let  $v$  and  $w$  be two vertices connected by an edge. If  $S$  is the set of all the vertices with a path to  $v$  before  $e$  is added, then  $w \notin S$ , otherwise this would define a cycle. Moreover if there was an edge with smaller weight than  $e$ , connecting  $S$  and  $V - S$ , then it would have already been added. Therefore  $e$  is the cheapest edge connecting  $V - S$  to  $S$ , and as such belongs to a minimum spanning tree of  $G$ .

Clearly by design the algorithm will not generate any cycle.

Moreover as  $G$  is connected and all the edges are explored  $V - S$  and  $S$  will be linked at some stage. Hence  $T$  is connected.  $\square$

## Back to the algorithm

**Issue:** how to represent the data such that whether or not adding an edge creates a cycle can be efficiently tested?

For each edge joining two vertices  $v$  and  $w$ :

- Identify all the connected components of  $v$  and  $w$
- If the edge is to be included, merge the two components

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**Issue:** how to represent the data such that whether or not adding an edge creates a cycle can be efficiently tested?

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- Identify all the connected components of  $v$  and  $w$
- If the edge is to be included, merge the two components

Extra notes:

- No edge needs to be removed
- No component needs to be split
- Everything must be done efficiently

# Toward a new data structure

Representing data using:

- **An array:** testing can be done in constant time; merging requires linear time
- **A graph:** merging is only adding an edge; testing requires a full graph traversal

# Toward a new data structure

Representing data using:

- **An array:** testing can be done in constant time; merging requires linear time
- **A graph:** merging is only adding an edge; testing requires a full graph traversal

Implement a new data structure containing:

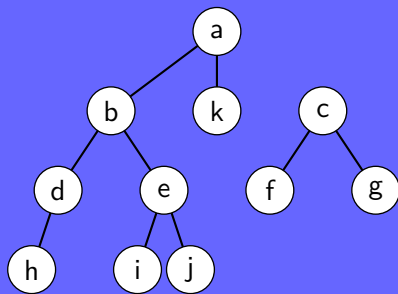
- A pointer to the parent
- The *rank*, or depth, of the sub-tree

# The union-find data structure

The two operations are:

- $\text{Find}(v)$ : find the root of the tree for vertex  $v$
- $\text{Union}(v,w)$ : link the root of the tree containing  $v$  to the root of the tree containing  $w$  (or the other way around)

- We have two sub-trees

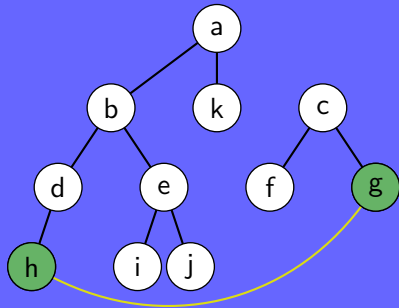




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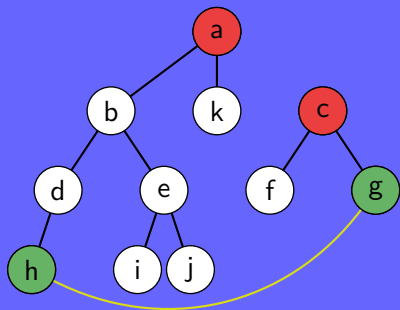


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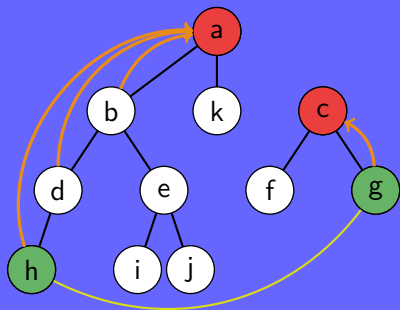


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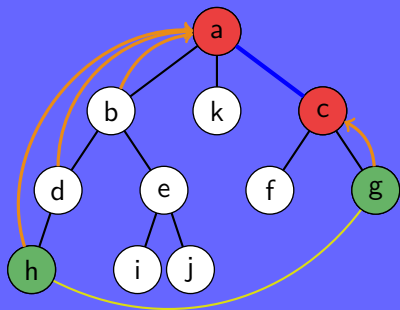


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- Update parents for  $h, d, b$  and  $g$
- $\text{Union}$  connects  $c$  to  $a$

# Handling a union-find structure

Algorithm.

---

```
1 Function GenSet(x):  
2   |   x.parent  $\leftarrow$  x; x.rank  $\leftarrow$  0;  
3 end  
4 Function Find(x):  
5   |   if x.parent  $\neq$  x then x.parent  $\leftarrow$  Find(x.parent) ;  
6   |   return x.parent  
7 end  
8 Function Union(x,y):  
9   |   X  $\leftarrow$  Find(x); Y  $\leftarrow$  Find(y);  
10  |   if X.rank > Y.rank then Y.parent  $\leftarrow$  X;  
11  |   else X.parent  $\leftarrow$  Y;  
12  |   if X.rank = Y.rank then Y.rank++;  
13 end
```

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# Revisiting Kruskal's algorithm

Algorithm. (*Kruskal with find-union*)

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**Input** : A graph  $G = \langle V, E \rangle$

**Output**: A minimum spanning tree  $T$

```
1 Sort the edges  $G.E$  by weight;
2  $T \leftarrow \emptyset$ ;
3 for edges  $(u, v)$  in  $G.E$ , in non-decreasing order do
4   | if Find( $u$ )  $\neq$  Find( $v$ ) then
5   |   | add edge  $(u, v)$  to  $T$ ;
6   |   | Union( $u, v$ )
7   | end if
8 end for
9 return  $T$ 
```

---

# Counting inversions

## Problem (Counting inversions)

Given a list of  $n$  elements  $a_0, \dots, a_{n-1}$ , determine how many pairs  $(a_i, a_j)_{\substack{0 \leq i, j \leq n \\ i \neq j}}$  are not in ascending order. Such pairs are called *inversions*.

Remark.

This problem has numerous applications such as

- Voting theory
- Analysis of search engines ranking
- Collaborative filtering

## Example

Given 6 movies compare the ranking of two users:

Movie	A	B	C	D	E	F
First user	1	2	3	4	5	6
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Inversions:  $(3, 2)$ ,  $(5, 2)$ ,  $(5, 4)$

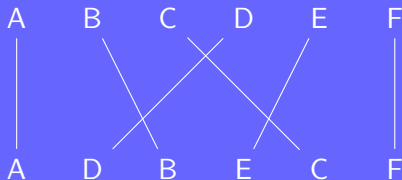
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A simple geometrical view: **How many crosses and how many inversions**



## Divide and conquer approach

Strategy for solving the counting inversions problem (1.40):

- ① **Divide:** split the list  $L$  into two halves  $L_1$  and  $L_2$
- ② **Conquer:** recursively count inversions in each list
- ③ **Combine:** count inversions for the pairs  $(l_i, l_j)$  with  $l_i$  and  $l_j$  belonging to  $L_1$  and  $L_2$  respectively

The sum of the three counts is the total number of inversion in  $L$

Example.

1 5 4 8 10 2 6 9 3 7

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(4,2),(4,3),(5,2),(5,3),(8,2),(8,3),(8,6),(8,7),(10,2),(10,3),(10,6),(10,7),(10,9)

# Merge and count

Algorithm. (*Merge and count*)

---

**Input** : Two sorted lists:  $L_1 = (l_{1,1}, \dots, l_{1,n_1})$  and  
 $L_2 = (l_{2,1}, \dots, l_{2,n_2})$

**Output:** The number of inversions *count*, and  $L_1$  and  $L_2$  merged into  $L$

```
1 Function MergeCount( $L_1, L_2$ ):
2    $count \leftarrow 0$ ;  $L \leftarrow \emptyset$ ;  $i \leftarrow 1$ ;  $j \leftarrow 1$ ;
3   while  $i \leq n_1$  and  $j \leq n_2$  do
4     if  $l_{1,i} \leq l_{2,j}$  then
5       | append  $l_{1,i}$  to  $L$ ;  $i++$ ;
6     else
7       | append  $l_{2,j}$  to  $L$ ;  $count \leftarrow count + n_1 - i + 1$ ;  $j++$ ;
8       | end if
9   end while
10  if  $i > n_1$  then append  $l_{2,j}, \dots, l_{2,n_2}$  to  $L$ ;
11  else append  $l_{1,i}, \dots, l_{1,n_1}$  to  $L$ ;
12  return  $count$  and  $L$ 
```

*all  $> l_{2,j}$ , add to count  
( $l_{1,i}, l_{1,i+1}, \dots, l_{1,n_1}$ )*

## Sort and count

Algorithm. (*Sort and count*)

---

**Input** : A list  $L = (l_1, \dots, l_n)$

**Output:** The number of inversions *count* and  $L$  sorted

```
1 Function SortCount( $L$ ):
2   if  $n=1$  then return 0 and  $L$ ;
3   else
4     Split  $L$  into  $L_1 = (l_1, \dots, l_{\lceil n/2 \rceil})$  and  $L_2 =$ 
        $(l_{\lceil n/2 \rceil+1}, \dots, l_n)$ ;
5      $count_1, L_1 \leftarrow \text{SortCount}(L_1)$ ;
6      $count_2, L_2 \leftarrow \text{SortCount}(L_2)$ ;
7      $count, L \leftarrow \text{MergeCount}(L_1, L_2)$ ;
8   end if
9    $count \leftarrow count_1 + count_2 + count$ ;
10  return  $count$  and  $L$ 
11 end
```

---

## Key points

- How to present pseudocode?
- What are the two main categories of data structure?
- What is a greedy algorithm?
- Describe the divide and conquer strategy
- How is the Union-Find data structure working?



Thank you!