VE477

Introduction to Algorithms

Assignment 1

Manuel — UM-JI (Fall 2018)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Hash tables*

In this exercise we want to estimate the maximum number of keys per slot we can expect when inserting n keys into n slots of a hash table.

Given a hash table with n slots, n keys are equiprobably hashed to each slot. Let M denote the maximum number of keys in a slot once they have all been inserted.

1. For any positive integer k, show that the probability P_k that exactly k keys hash to a same slot is

$$\left(\frac{1}{n}\right)^k \left(1-\frac{1}{n}\right)^{n-k} \binom{n}{k}.$$

2. Prove that the probability P'_k, for the slot with the most keys to have exactly k keys, is less or equal to nP_k.
3. Prove that $P_k < e^k/k^k$.

* 4. Show that for any positive integer $k \geq c \log n/ \log\log n$, for some constant $c > 1$, $P'_k < 1/n^2$.

5. Denoting the expected value of M by $E(M)$, observe that

$$E(M) \leq \Pr\left(M > \frac{c \log n}{\log\log n}\right) n + \Pr\left(M \leq \frac{c\log n}{\log\log n}\right) \frac{c\log n}{\log\log n},$$

and conclude that $E(M) = \mathcal{O}\left(\frac{\log n}{\log\log n}\right)$.

Hint: for question 3 apply Stirling formula.

Ex. 2 — *Minimum spanning tree*

Let G be a graph and T be a minimum spanning tree for G. Write the pseudocode of an algorithm which determines the minimum spanning tree of the graph G when the weight of an edge not in T is decreased.

Ex. 3 — *Simple algorithms*

* 1. Given two n-bits integers stored in two arrays, explain how to compute their sum in an $n+1$-bits array. Write the corresponding pseudocode.

2. One decides to multiply two integers x and y by writing a function `mult(x,y)` returning 0 if one of them is 0 and otherwise returning the sum of a recursive call on `mult`, with parameters $2x$ and $\lfloor y/2 \rfloor$, and $x \cdot (y \bmod 2)$.
   a) Express this algorithm as pseudo-code.
   b) Prove the correctness of this algorithm.

Ex. 4 — *Problem*

Given twenty five horses determine the three fastest ones, in the right order, knowing that no more than five can race at a time. What is the minimum number of races necessary? Detail a general algorithm which solves the problem.

Ex. 5 — *Critical thinking*

1. The *Knapsack problem* is defined as follows. Given a set S and a number n find a subset of S whose elements add up exactly to n. Which of the following algorithms solve the Knapsack problem?
   - Fit the knapsack with the smallest items first.
   - Fit the knapsack with the largest items first.
2. (*) In the course (Example 1.26) it is mentioned that m should be "a prime not too close from a power of 2" in order for the hash function $H(k) = k \bmod m$ to be a good choice. Explain.
3. Provide an example of a greedy algorithm which is locally optimal while not being globally optimal. Provide all the necessary details to support your claim.