UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Lecture Notes Hints

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Note 1 Distribution

$$\mathcal{D}(\mathbb{R}) := C_0^\infty(\mathbb{R}^n)$$

Everyone will be confused when they just first see these notations. This is the definition of Test function space. Together test function space, we have introduced distribution, linear functional, etc... Here we will give an intuitive understanding that helps you to understand these terrifying concepts.

i). A naive understanding of Distribution

A distribution can be understood in analogy to a function. Recall the definition of a function:

> A function takes a number as input and output a number.

Similarly,

> A distribution takes a function as a input and generates a number ($\mathbb{C}$)

Note 2 Boundary Value Problems of Higher order p

An example problem:

$$L = D^2(b_2D^2) + D(b_1D) + b_0 \tag{2.1}$$

where b_2, b_1, b_0 are functions of x.

We first calculate conjunction and L^*. Apply integral chain rule,

$$\int vLu = \int v[(b_2u'')'' + (b_1u')' + b_0]$$

$$\begin{aligned}\int v(b_2u'')'' &= v(b_2u'')' - \int v'(b_2u'')' \\ &= v(b_2u'')' - v'(b_2u'') + \int v''b_2u'' \\ &= v(b_2u'')' - v'(b_2u'') + u'v''b_2 - u(v''b)' + \int (v''b_2)''y \end{aligned}\tag{2.2}$$

$$\begin{aligned}\int v(b_1u')' &= v(b_1u') - \int v'(b_1u') \\ &= v(b_1u') - ub_1v' + \int u(b_1v')'' \end{aligned}\tag{2.3}$$

We synthesis (2.2) and (2.3), which yields to

$$L^* = L \tag{2.4}$$

and

$$J(u,v) = v(b_2u'')' - v'(b_2u'') + u'v''b_2 - u(v''b)' + v(b_1u') - ub_1v'|_a^b \tag{2.5}$$

Two cases to be considered as self adjoint.

- $u(a), u''(a), u(b), u''(b) = 0$
- $u(a), u'(a), u(b), u'(b) = 0$

However, if three conditions for a and one condition for b, not self-adjoint. *This could be easily verified as plug in the conditions and see what need to happen to make conjunction to be 0 for v.*

Note 3 Solvability Conditions

Solve

$$\begin{aligned} Lu &= \delta(x-\xi) - v^{(1)}(\xi)v^{(1)}(x) - \cdots - v^{(k)}(\xi)v^{(k)}(x) && (3.1) \\ &=: f(x) && (3.2) \end{aligned}$$

instead of originally $Lu = \delta(x-\xi)$

Note 4 Partial Boundary Value Problems General Notes

i). Basic Quantities

$$Lu := -\nabla \cdot (p(x)\nabla u) + q(x)u \tag{4.1}$$

ii). Derivation of Green's Formula

Green's Formula is given as

$$\int_{\Omega} (vLu - uLv)dx = \int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma \tag{4.2}$$

The steps are: According to divergence delta rule

$$\nabla \cdot (f\vec{v}) = f(\nabla \cdot \vec{v}) + \vec{v} \cdot (\nabla f) \tag{4.3}$$

Rearrange the items, it will lead to

$$f(\nabla \cdot \vec{v}) = \nabla \cdot (f\vec{v}) - \vec{v} \cdot (\nabla f) \tag{4.4}$$

where as $f = p$ and $\vec{v} = p\nabla u$

Note 5 PP 330

To calculate a surface integral in $\mathbb{R}^n$ of $S \subset \mathbb{R}^n$ is a hypersurface.

i). Step 1: Find parametrization γ

$$\gamma : [a_1, b_1] \times ... \times [a_{n+1}, b_{n+1}] \to S \subset \mathbb{R}^n$$

$$\gamma : (s_1, ..., s_{n+1}) \mapsto \begin{pmatrix} \gamma_1(s_1, ...s_{n+1}) \\ \gamma_2(s_1, ...s_{n+1}) \\ \gamma_3(s_1, ...s_{n+1}) \\ \vdots \\ \gamma_{n+1}(s_1, ...s_{n+1}) \end{pmatrix}$$

A simple example is given as

$$(\phi, \theta) = \begin{pmatrix} \cos\pi\sin\theta \\ \sin\pi\sin\theta \\ \cos\theta \end{pmatrix}$$

ii). Step 2: Find a normal vector $\vec{n} \perp S$

iii).

$$\int_S f ds = \int\int\int ... \int f(\gamma(s_1, ..., s_{n+1})) \det(\frac{\partial\gamma}{\partial s_1}, ..., \vec{a})$$

Note 6 PP 366

$$\int_\varphi f ds = \int_a^b f \circ \gamma(t) \cdot |\gamma'(t)| dt$$

where φ indicates curve/line integral on $\mathbb{R}^n$. by

$$\gamma(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$t \in R$, $\gamma'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\gamma'(t)| = 1$. So

$$\begin{aligned} &\int_{\partial\mathbb{H}} h(\cdot) \frac{\partial g(x; \cdot)}{\partial n} ds \\ &= \int_{-\infty}^{\infty} h(\gamma(t)) \cdot \frac{-\partial g(x; \cdot)}{\partial \xi_2} |_{\gamma(t)} dt \\ &= -\int_{-\infty}^{\infty} h(\xi_2) \cdot \frac{-\partial g(x; \xi_1, \xi_2)}{\partial \xi_1} |_{\xi_2 = 0} dt \end{aligned}$$

Note 7 Method of Images

This part is to raise an intuitive understanding of **Method of Images** chapter.

Electrostatics by Maxwell

$$-\Delta \underbrace{V}_{\text{potential}} = 4\pi \underbrace{\rho}_{\text{charge density}}$$

For a unit point charge located at ξ, it has charge density $\delta(x-\xi)$. The potential for a point charge is

$$V(x;\xi) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{|x-\xi|}$$

This solves $-\Delta V = 4\pi\delta(x-\xi)$

E.g. a charge of 1 Coulomb at ξ and 2 Coulomb at ξ^*,

$$-\Delta V = 4\pi(\delta(x-\xi) + 2\delta(x-\xi^*))$$

which is also taken as

$$V = V_1 + V_2$$

where according to the superposition principle

$$-\Delta V_1 = -4\pi\delta(x-\xi) \tag{7.1}$$
$$-\Delta V_2 = -4\pi\delta(x-\xi^*) \tag{7.2}$$

If V_1 solves equation (7.1), V_2 also solves equation (7.2).

$$V_2(x;\xi^*) = 2V_1(x;\xi^*)$$

The effect of having a **ground plate** at x plane is equal to have a ξ^* charge.

Note 8 Neumann Problem

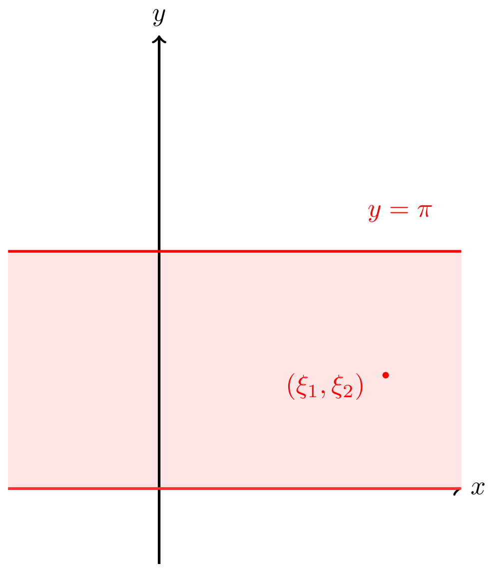
$$\frac{\partial g}{\partial n} = 0 \quad g = \text{physical potential} \tag{8.1}$$

which means g satisfy $\Delta g = \delta$ and boundary condition.

$$\langle \nabla g, \vec{n} \rangle = 0 \tag{8.2}$$

This equation means the force perpendicular to the boundary vanishes. This equation (8.2) means at boundary the force is 0. Please note that for Neumann problems, the potential at boundary points is arbitrary as long as (8.1) is satisfied.

Note 9 PP 374



Consider the reflection of ξ according to two boundaries.

We have

$$\Delta g = \delta(x - \xi), g|_{\partial\Omega} = 0 \tag{9.1}$$

So

$$E(x, \xi) = \frac{1}{2\pi} \ln |x - \xi| \tag{9.2}$$

and the sequence of images are

$$\xi_n^{\pm} = (\xi_1, \pm\xi_2 \pm 2n\pi)$$

Then we have

$$\begin{aligned} g(x;\xi) &= \sum_{n\in\mathbb{Z}} E(x, \xi_n^+) - \sum_{n\in\mathbb{Z}} E(x, \xi_n^-) \\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \ln |x - \xi_n^+| - \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \ln |x - \xi_n^-| \end{aligned} \tag{9.3}$$

Two comments:

1. We ignore convergence issue
2. Introduce complex numbers:

$$\left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| = |x_1 + ix_2| = \sqrt{x_1^2 + x_2^2} \tag{9.4}$$

Then we could write

$$|x-\xi_n^+| = \left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \xi_2 + 2n\pi \end{pmatrix} \right| = |(x_1-\xi_1)+(x_2-\xi_2+2n\pi)i|$$
$$= |x-\xi+2n\pi i| \quad x = x_1 + x_2 i$$

Similarly,

$$|x-\xi_n^-| = |x-\bar{\xi}+2n\pi i||$$

So we have

$$\sum_{n\in\mathbb{Z}} \ln|x-\xi_n^+| = \sum_{n\in\mathbb{Z}} \ln|x-\xi+2n\pi i| = \ln\left(\prod_{n\in\mathbb{Z}} |x-\xi+2n\pi i|\right) \tag{9.5}$$

Now

$$\begin{aligned} g(x,\xi) &= \frac{1}{2\pi}\ln\left(\prod_{n\in\mathbb{Z}} \frac{|x-\xi+2n\pi i|}{|x-\bar{\xi}+2n\pi i|}\right) \\ &= \frac{1}{2\pi}\ln\left|\left(\prod_{n\in\mathbb{Z}} \frac{\frac{x-\xi}{2in\pi}-1}{\frac{x-\bar{\xi}}{2n\pi i}-1}\right)\cdot\frac{x-\xi}{x-\bar{\xi}}\right| \end{aligned} \tag{9.6}$$

According to Weierstraß factorization theorem,

$$\begin{aligned} &\sinh(z) = \frac{e^z-e^{-z}}{2} \\ &\sinh(ix) = -i\sin(x) \\ &\sinh(z) = 2\cdot\prod_{n\in Z\backslash 0}(z-in\pi) = 2\prod_{n\in\mathbb{Z}\backslash 0}(1-\frac{z}{in\pi}) \end{aligned} \tag{9.7}$$

Here 9.7 is to evaluate hyperbolic function. And we have

$$\begin{aligned} g(x,\xi) &= \frac{1}{2\pi}\ln\left|\frac{\sinh((x-\xi)/2)}{\sinh((x-\bar{\xi})/2)}\right| \\ &= \frac{1}{4\pi}\ln\left|\frac{\sinh^2((x-\xi)/2)}{\sinh^2((x-\bar{\xi})/2)}\right| \end{aligned} \tag{9.8}$$

Also, we have an equation

$$\sinh^2(a+bi) = \cosh^2(a)-\cos^2(b) \tag{9.9}$$

Plug (9.9) into (9.8) it yields to

$$g(x,\xi) = \frac{1}{4\pi}\ln\left(\frac{\cosh^2((x_1-\xi_1)/2)+\cos^2((x_2-\xi_2)/2)}{\cosh^2((x_1-\xi_1)/2)+\cos^2((x_2+\xi_2)/2)}\right) \tag{9.10}$$

Then in this case

$$u(x) = \int_{\delta\Omega} \frac{\partial g(x,\cdot)}{\partial n}|_{\partial\Omega}\cdot f \tag{9.11}$$

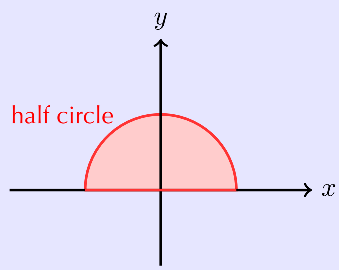
$$\begin{aligned} \frac{\partial g}{\partial n} &= \langle \nabla g, \hat{n}\rangle \\ &= \langle \begin{pmatrix} \frac{\partial g}{\partial \xi_1} \\ \frac{\partial g}{\partial \xi_2} \end{pmatrix}, \pm\begin{pmatrix} 0 \\ 1 \end{pmatrix}\rangle \\ &= \pm\frac{\partial g}{\partial \xi_2} \end{aligned} \tag{9.12}$$

Plug in $\xi_2 = \pi$ into (9.11)

$$\frac{\partial g}{\partial \xi_2}|_{\xi_2=\pi} = \frac{1}{2\pi}\frac{sin(x_2)}{cos(x_1-\xi_1)+cos(x_2)} \tag{9.13}$$

So the final answer

$$u(x) = \frac{\sin(x_2)}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{f_1(\xi_1)}{\cosh(x_1-\xi_1)+\cos(x_2)} + \frac{f_2(\xi_1)}{\cosh(x_1-\xi_1)-\cos(x_2)}\right)d\xi_1$$



Partial Eigenfunction Expansion (Draw a semi circle with boundary to be red here) $-\Delta g = \delta, \quad g|_{\partial\Omega} = 0$

$$\Delta_{(r,\theta)} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{9.14}$$

1) First find partial eigenfunctions $-\Delta u = 0$, $u = u(r,\theta) = R(r)\cdot\Theta(\theta)$

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2}R\Theta + \frac{1}{r}\cdot\frac{\partial}{\partial r}R\Theta + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}R\Theta = 0\\
\Leftrightarrow &R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0\\
\Leftrightarrow &\frac{r^2}{R}R'' + \frac{r}{R}R' = -\frac{\Theta''}{\Theta} = \lambda \in \mathbb{R} \qquad (9.15)
\end{aligned}$$

Then we transform into an eigenvalue problem

$$\Theta'' + \lambda\Theta = 0 \qquad 0 < \Theta < r \tag{9.16}$$
$$r^2R'' + rR' - \lambda R = 0 \qquad 0 < r < 1, R(1) = 0 \tag{9.17}$$

2) Choose and find a set of eigenfunctions

$$\Theta_m(\theta) = \underbrace{\sqrt{\frac{2}{\pi}}}_{\text{Normalize coordinate}}\cdot\sin(m\theta), \qquad m \in \mathbb{N}^* \tag{9.18}$$

3) Expand the unknown green function in terms of eigenfunction

$$g(r,\theta;\rho,\varphi) = \sum_{n=1}^{+\infty} g_m(r,\rho,\varphi)\cdot\Theta_m(\theta) \tag{9.19}$$
$$g_m(r,\rho,\varphi) = \sqrt{\frac{2}{\pi}}\int_0^{\pi} g(r,\theta;\rho,\varphi)\sin(m\theta)d\theta \tag{9.20}$$

4) Determine the ODE Green's function problem

$$-\Delta_{(r,\theta)} g(r,\theta;\rho,\varphi) = \frac{\delta(r-\rho)\delta(\theta-\varphi)}{r} \tag{9.21}$$

Multiply (9.21) with $\frac{2}{\pi}\sin(m\theta)$, then integrate

$$\begin{aligned}
&-\int \frac{2}{\pi}\sin(m\theta)\left(\Delta_{(r,\theta)} g\right) = \frac{2}{\pi r}\sin(m\varphi)\delta(r-\rho) \\
&\Leftrightarrow -\left(\frac{\partial^2}{\partial v^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) g_m(r,\rho,\varphi) - \frac{1}{r^2}\int_0^\pi \frac{2}{\pi}\sin(m\theta)\cdot\frac{\partial^2 g}{\partial\theta^2} d\theta = \frac{2}{\pi r}\sin(m\varphi)\delta(r-\rho) \\
&\Leftrightarrow -\frac{\partial^2}{\partial r^2} g_m + \frac{1}{r}\frac{\partial}{\partial r} g_m + \frac{m^2}{r^2} g_m = \frac{2}{\pi r}\sin(m\varphi)\delta(r-\rho)
\end{aligned} \tag{9.22}$$

Consider $g_m = g_m(r)$, then we can write this as

$$-r^2 g_m'' + r g_m + m^2 g_m = \frac{2}{\pi} r \sin(m\varphi)\cdot\delta(r-\rho) \tag{9.23}$$

Note that here

$$r\cdot\delta(r-\rho) = \rho\cdot\delta(r-\rho)$$

which is due to

$$r T_{\delta(r-\rho)}\varphi = T_{\delta(r-\rho)}(r\cdot\varphi(r)) = \rho\cdot\varphi(\rho) \tag{9.24}$$

Now we choose g_m to satisfy

$$g_m|r = 0 = g_m|r = 1 = 0$$

Here for $r = 0$, we cannot choose g_m arbitrarily. At the origin, θ is not properly defined,

5) Solve it !

$$\begin{aligned}
&-r^2 g_m'' - r g_m' + m^2 g_m = 0 \\
&\Rightarrow g_m(r) = r^\lambda \Rightarrow \lambda = \pm m
\end{aligned} \tag{9.25}$$

Then find g_m.

$$g_m(r,\rho,\varphi) = \begin{cases} \dfrac{1}{m}\dfrac{r^m(\rho^{-m}-\rho^m)}{\pi}\sin(m\varphi) & r < \rho \\ \dfrac{1}{m}\dfrac{\rho^m(r^{-m}-r^m)}{\pi}\sin(m\varphi) & r > \rho \end{cases} \tag{9.26}$$

6) Put it together

$$g(r,\theta;\rho,\varphi) = \sum_{m=1}^{\infty} g_M(r,\rho,\varphi)\sin(m\theta) \tag{9.27}$$

Note 10 HW5 EX3

The Fourier expansion can be

$$f(x) = c_0\langle f, 1\rangle + \sum_{n=1}^{\infty} c_n\langle f, \cos(nx)\rangle\cos(nx) \tag{10.1}$$

$$+\sum_{n=1}^{\infty} d_n\langle f, \sin(nx)\rangle\sin(nx) \tag{10.2}$$

Note 11 HW5 EX5

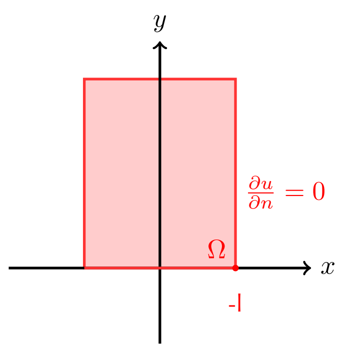
iv.

$$u(x) = \int_0^1 g_M(x,\xi) f(\xi) d\xi + c_0 + c_1 x \tag{11.1}$$

Note 12 HW6 EX2

$$u_{tt} - u_{xx} = F(x,t) \qquad u(x,0) = f(x) \quad u_t(x,0) = h(x) \tag{12.1}$$

The greens formula work for bounded regions.



$$B_1^* u = u(x,T)$$
$$B_2^* u = u_t(x,T)$$

Now we have

$$\begin{aligned}
\langle u, Lv\rangle - \langle v, Lu\rangle &= \iint u(x,t)[v_{tt} - v_{xx}] - v[u_{tt} - u_{xx}] dxdt \\
&= \int_{-L}^{L}\int_0^T (uv_{tt} - vu_{tt}) dtdx - \int_0^T \int_{-L}^{L} (uv_{xx} - vu_{xx}) dxdt \\
&= \int_{-L}^{L} [uv_t|_0^T - \cancel{\int_0^T u_t v_t dt} - vu_t|_0^T + \cancel{\int_0^T u_t v_t dt}] - \ldots \\
&= \int_{-L}^{L} (uv_t - vu_t)|_0^T dx - \int_0^T (u_x v_x - vu_x)|_{-L}^{L} dt \\
&= \int_{-L}^{L}\int_0^T \frac{d}{dt}(uv_t - vu_t) dtdx - \int_0^T \int_{-L}^{L} \frac{d}{dx}(uv_x - vu_x) dxdt \\
&= \iint_\Omega div_{(x,t)} \underbrace{\begin{pmatrix} vu_x - uv_x \\ uv_t - vu_t \end{pmatrix}}_{J(u,v)} d(x,t)
\end{aligned} \tag{12.2}$$

Recall that green's function

$$\langle u, Lv\rangle - \langle v, Lu\rangle = \int Jd\vec{\sigma}\langle a\rangle \tag{12.3}$$

Recall the Causal F.S.:

$$E(x,t;\xi,\tau) = \frac{1}{2}H(t-\tau-|x-\xi|) \tag{12.4}$$

Adjoint boundary conditions

$$M = \{u \in C^2(\Omega) \cap C(\bar{\Omega}) u(x,0) = 0, u_t(x,0) = 0, u_x(-L,t) = u_x(L,t) = 0\} \tag{12.5}$$

Solution formula:

- u satisfies

$$u_{xx} - u_{tt} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = h(x), u_x(\pm L, t) = 0 \tag{12.6}$$

- v satisfies

$$v_{xx} - v_{tt} = \delta, \quad v(x,T) = 0, \quad v_t(x,T) = 0, \quad v_x(\pm L, t) = 0 \tag{12.7}$$

  Also we have $v = g^*$

So here we have

$$\begin{aligned} \langle u, Lv\rangle - \langle v, Lu\rangle &= u(\xi) - \int_\Omega g^* F \\ &= \int_{-L}^{L} u(x,0)g_t^*(x,0) - u_t(x,0)g^*(x,0)dx \end{aligned} \tag{12.8}$$

Then

$$u(\xi,\tau) = \int_\Omega g^*(x,t;\xi,\tau)F(x,t)d(x,t) + \cdots \tag{12.9}$$

Fix ξ and τ, For L large enough, at the boundary, the conditions for $\partial u/\partial n = 0$ automatically satisfies.

$$E(\pm L, t; \xi, \tau) = \frac{1}{2}H(t-\tau-|\pm L - \xi|) = 0 \qquad \text{if } L \text{ large} \tag{12.10}$$

$$E(x,0;\xi,\tau) = \frac{1}{2}H(T-\tau-|x-\xi|) = 0 \tag{12.11}$$

Equation (12.11) is always 0 because $\tau > 0$. From (12.10) and (12.11), we can conclude

$$E \in M$$

We take

$$E^*(x,t;\xi,\tau) = E(\xi,\tau;x,t) \tag{12.12}$$

as g^*. So that (12.9) can be then expressed as

$$\begin{aligned} u(\xi,\tau) &= \int_\Omega g^*(x,t;\xi,\tau)F(x,t)d(x,t) + \cdots \\ &= \iint_\Omega \frac{1}{2}H(\tau - t - |x-\xi|)F(x,t)d(x,t) + \cdots \qquad (12.13) \\ &= \frac{1}{2}\int_0^\tau \int_{\xi-(\tau-t)}^{\xi+(\tau-t)} F(x,t)dxdt \qquad (12.14) \end{aligned}$$

From (12.13) to (12.14), it is due to the equation evaluates to 1 only if

$$|x-\xi| < \tau - t$$

Note 13 Adjoint and direct boundary conditions for PDE

Direct Boundary conditions

$$M = \{u \in C^2(V) : Bu = \tilde{B}_1 u = 0\} \tag{13.1}$$

Here

$$Bu = \gamma(x,t) \quad (x,t) \in \partial\Omega \times (0,\infty) \tag{13.2}$$

$$\tilde{B}_1 u = u(x,0) = f(x) \tag{13.3}$$

Adjoint Boundary condition

$$M^* = v \in C^2(V), B^* v = \tilde{B}_1^* v = 0 \tag{13.4}$$

where

$$B^* v = Bv \tag{13.5}$$

$$\tilde{B}_1^* v = v(x,T) \tag{13.6}$$

Modified greens' function Non-separapble ODE **Method of Images, Eigenvalue**

Note 14 How to solve these problems in Chpter 4

i). Solvability

1. Find adjoint problem: For adjoint problem, it has the same **trivial or non-trivial** solution as the original problem.
2. check $\int fv = 0$.
3. If $\int fv = 0$, it means that the function is solvable. Otherwise, it's not solvable.
4. Then we need to use modified green function.

$$\begin{aligned} \mathsf{L}\mathsf{g}_M(x,\xi) &= \delta(x-\xi) - \sum_{i=1}^{k} v^{(i)}(\xi) v^{(i)}(x) \\ B_1 g_M &= 0 \\ &\vdots \\ B_p g_M &= 0 \end{aligned} \tag{14.1}$$

   Note here v is orthonormal solution.

5. The modified adjoint green function is

$$\begin{aligned} L^* g_M^*(x,\xi) &= \delta(x-\xi) - \sum_{i=1}^{k} u^{(i)}(\xi) u^{(i)}(x) \\ B_1^* g_M^* &= 0 \\ &\vdots \\ B_p^* g_M^* &= 0 \end{aligned} \tag{14.2}$$

6. Then the final solution is given as

$$u(x) = \int_a^b g_M(x,\xi) f(\xi) d\xi + \sum_{i=1}^{k} c_i u^{(i)}(x) \tag{14.3}$$