Vv557 Methods of Applied Mathematics II
Green Functions for Partial Differential Equations

Assignment 7

Date Due: 1:00 PM, Thursday, the 18$^{\text{th}}$ of April 2018

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This assignment has a total of **(13 Marks)**.

Exercise 7.1
Suppose that $Bu = u|_{\partial\Omega}$ (Dirichlet boundary condition) and let

$$M = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \colon Bu = 0\}.$$

Show that if v satisfies

$$\int_{\partial\Omega} J(u, v)\, d\vec{\sigma} = 0 \qquad\qquad \text{for all } u \in M$$

then

$$v \in M.$$

This proves $M^* \subset M$ for the case of Dirichlet boundary conditions.
(2 Marks)

Exercise 7.2
The goal of this exercise is to obtain the d'Alembert solution formula for the Cauchy problem for the wave equation

$$u_{tt} - u_{xx} = F(x, t), \qquad x \in \mathbb{R},\ t > 0, \qquad\qquad u(x, 0) = f(x), \qquad\qquad u_t(x, 0) = h(x),$$

from the general solution formula.

i) Let $I = (-L, L) \subset \mathbb{R}$, $L > 0$, be an interval, let $T > 0$ be fixed and set $\Omega = I \times (0, T)$. Consider the wave equation problem

$$Lu = u_{tt} - u_{xx} = F \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}\Big|_{\partial I} = 0, \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = h(x). \qquad (*)$$

Suppose that Green's function $g(x; \xi)$ is known. Write down as explicitly as possible the solution formula for the problem:

- Find L^* and adjoint boundary conditions.
- Find the conjunct of L
- Use Green's formula to find the solution formula. **You should go into detail in how the various integrals are evaluated!**

(6 Marks)

ii) Keeping $T > 0$ fixed, verify that if L is large enough, the fundamental solution $E(x, t; \xi, \tau) = \frac{1}{2}H(t - \tau - |x - \xi|)$ satisfies the boundary conditions of $(*)$.
(2 Marks)

iii) Using E and the solution formula obtained in (iii), let $L \to \infty$ to obtain d'Alembert's formula for the solution of the Cauchy problem,

$$u(x, t) = \frac{f(x + t) + f(x - t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy + \frac{1}{2}\iint_{\triangle(x,t)} F(y, s)\, dy\, ds$$

where

$$\triangle(x, t) = \{(y, s) \in \mathbb{R}^2 \colon 0 \le s \le t - |x - y|\}.$$

(3 Marks)