UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 1

Group 22

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Exercise 1. 1

i).

For the intervals on

$$0 \le x \le \xi - \frac{1}{2n} \cup \xi + \frac{1}{2n} \le x \le 1$$

We have

$$u_n(x)'' = 0 \Rightarrow \begin{cases} u_{n1}(x) = x^2 + a \cdot x + b & 0 \le x \le \xi - \dfrac{1}{2n} \\ u_{n2}(x) = x^2 + c \cdot x + d & \xi + \dfrac{1}{2n} \le x \le 1 \end{cases}$$

For the intervals on

$$\xi - \frac{1}{2n} \le x \le \xi + \frac{1}{2n}$$

The function is given as

$$u_{n3}(x)'' = -n \Rightarrow u_{n3}(x) = -\frac{n \cdot x^2}{2} + \alpha \cdot x + \beta$$

For u_{n1}, u_{n2}, u_{n3}, they should satisfy

$$\begin{cases} u_{n1}(\xi - \dfrac{1}{2n}) = u_{n3}(\xi - \dfrac{1}{2n}) \\ u_{n2}(\xi + \dfrac{1}{2n}) = u_{n3}(\xi + \dfrac{1}{2n}) \\ \underset{x \to \xi - \frac{1}{2n}^-}{lim} u'_{n1}(x) = \underset{x \to \xi - \frac{1}{2n}^+}{lim} u'_{n3}(x) \\ \underset{x \to \xi + \frac{1}{2n}^-}{lim} u'_{n3}(x) = \underset{x \to \xi + \frac{1}{2n}^+}{lim} u'_{n2}(x) \end{cases}$$

This will give a linear equation set (four equations) with four variables. ($a = 0, d = 0$) And the final result will be then the solution provided:

$$u_n(x) = \begin{cases} (1 - \xi) \cdot x & 0 \le x \le \xi - \dfrac{1}{2n} \\ (1 - \xi) \cdot x - \dfrac{n}{2}(x - \xi + 1/(2n))^2 & \xi - \dfrac{1}{2n} < x < \xi + \dfrac{1}{2n} \\ \xi \cdot (1 - x) & \xi + \dfrac{1}{2n} \le x \le 1 \end{cases}$$

ii).

As $n \to \infty$,

$$\xi - \frac{1}{2n} = \xi = \xi + \frac{1}{2n}$$

And the term

$$(1 - \xi) \cdot x - \frac{n}{2}(x - \xi + 1/(2n))^2 x$$

is canceled as $n \to \infty$, then

$$\underset{n \to \infty}{lim} u_n(x) = \begin{cases} (1 - \xi)x & 0 \le x \le \xi \\ \xi(1 - x) & \xi \le x \le 1 \end{cases}$$

Then since $\underset{n \to \infty}{lim} u_n(x) = g(x; \xi)$, it's sufficient to say

$$\underset{n \to \infty}{lim} |u_n(x) - g(x, \xi)| = 0$$

□

iii).

No, it's not uniform convergence. The definition of uniform convergence is

$$\forall \varepsilon > 0, \ \ \exists N_0 \in N, \ \ \forall n > N_0, \ \ |f_n(x) - f(x)| < \varepsilon$$

Obviously, the convergence is uniform on $0 \leq x \leq \xi - \frac{1}{2n}$ and $\xi + \frac{1}{2n} \leq x \leq 1$. Since $u_n(x) = g(x;\xi)$ on these two intervals.

Next we will prove on $\xi - \frac{1}{2n} < x < \xi + \frac{1}{2n}$, it's not uniform convergence. We choose $\varepsilon = \varepsilon_0 < 1/4$, and we consider the interval on $[\xi, \xi + \frac{1}{2n})$.

$$\begin{aligned}
|u_n(x) - g(x;\xi)| &= |(1-\xi) \cdot x - \frac{n}{2}(x - \xi + 1/(2n))^2 - (1-x)\xi| \\
&= |t - \frac{n}{2}(t + \frac{1}{2n})^2| \quad , t \leftarrow (x - \xi)
\end{aligned}$$

Since $\forall x$ on the interval,

$$|u_n(x) - g(x;\xi)| < \varepsilon_0$$

However, we evaluate $x = \xi + \frac{1}{2n} - \varepsilon_1$

$$|u_n(x) - g(x;\xi)| = |\frac{1-n}{2n}| \to \frac{1}{2} > \varepsilon_0$$

which means it is not uniform convergence.

Exercise 1. 2

i).

We first test whether T is a linear functional.

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda\varphi_1(-10) + \lambda\varphi_2(-10) = \lambda T\varphi_1 + \mu T\varphi_2$$

It's linear. Then we test its continuity

$$\varphi_m \to 0, \ |T\varphi_m| = |\varphi_m(-10)| \leq |\sup \varphi_m(x)| = 0$$

So we can conclude **it's a distribution**.

ii).

We first test whether T is a linear functional.

$$\begin{aligned}
T(\lambda\varphi_1 + \mu\varphi_2) &= [\lambda\varphi_1(0) + \mu\varphi_2(0)]^2 \\
&= \lambda^2\varphi_1^2(0) + \mu^2\varphi_2^2(0) + 2\lambda\mu\varphi_1(0)\varphi_2(0) \\
&\neq \lambda\varphi_1^2(0) + \mu\varphi_2^2(0)
\end{aligned}$$

So

$$T(\lambda\varphi_1 + \mu\varphi_2) \neq \lambda T\varphi_1 + \mu T\varphi_2$$

which means **it's not a distribution**.

iii).

Obviously, T maps to $\mathbb{C}^n$ instead of $\mathbb{C}$, so it's not a linear functional, thus **it's not a distribution**.

iv).

We first test whether T is a linear functional.

$$\begin{aligned} T(\lambda\varphi_1 + \mu\varphi_2) &= \lambda\varphi_1(0) + \mu\varphi_2(0) + \lambda\varphi_1(1) + \mu\varphi_2(1) + \lambda\varphi_1(2) + \mu\varphi_2(2) + ... \\ &= \lambda(\varphi_1(0) + \varphi_1(1) + \varphi_1(2) + ...) + \mu(\varphi_2(0) + \varphi_2(1) + \varphi_2(2) + ...) \\ &= \lambda T\varphi_1 + \mu T\varphi_2 \end{aligned}$$

It's linear. Then we test its continuity

$$\varphi_m \to 0, \ \ |T\varphi_m| = |\phi_m(0) + \phi_m(1) + \phi_m(2) + ...| \leq 0 + 0 + 0 + ... + 0 = 0$$

So we can conclude **it's a distribution**.

v).

We first test whether T is a linear functional.

$$\begin{aligned} T(\lambda\varphi_1 + \mu\varphi_2) &= \int_{S^{n-1}} (\lambda\varphi_1 + \mu\varphi_2) \\ &= \int_{S^{n-1}} \lambda\varphi_1 + \int_{S^{n-1}} \mu\varphi_2 \\ &= \lambda T\varphi_1 + \mu T\varphi_2 \end{aligned}$$

It's linear. Then we test its continuity

$$\varphi_m \to 0, \ \ |T\varphi_m| = |\int_{S^{n-1}} \varphi_m| \leq \sup|\varphi_m| = 0$$

So we can conclude **it's a distribution**.

vi).

vi).a.

We first test whether T is a linear functional.

$$\begin{aligned} T(\lambda\varphi_1 + \mu\varphi_2) &= \int \frac{\lambda\varphi_1 + \mu\varphi_2}{x} \\ &= \int \frac{\lambda\varphi_1}{x} + \int \frac{\mu\varphi_2}{x} \\ &= \lambda T\varphi_1 + \mu T\varphi_2 \end{aligned}$$

It's linear. Then we test its continuity

$$\begin{aligned} \varphi_m \to 0, \ \ |T\varphi_m| &= |\int \frac{1}{x}\varphi_m dx| \\ &= |\ \ln(x)\varphi_m(x)|_{-\infty}^{+\infty} - \int \ln|x|\varphi'(x)dx| \\ &= \int \ln|x|\varphi'(x)dx \end{aligned}$$

As $m \to \infty$,

$$\varphi_m'(x) \to 0, \ \ \sup|D^\alpha|\varphi_m(x) \to 0$$

So we could conclude it's a distribution.

vi).b.

$$\begin{aligned}\varphi_m \to 0,\ \ |T\varphi_m| &= |\int \frac{1}{\sqrt{|x|}}\varphi_m(x)dx| \\ &\le (\int \frac{1}{\sqrt{|x|}}dx)\cdot sup|\varphi_m(x)| \\ &= 2\sqrt{|x|}\cdot sup|\varphi_m(x)|\end{aligned}$$

So we can conclude it's a distribution.

vi).c.

$$\begin{aligned}\varphi_m \to 0,\ \ |T\varphi_m| &= |\int \frac{1}{x^2}\varphi_m(x)dx| \\ &= -\frac{1}{x}\varphi(x)\Big|_{-\infty}^{\infty} - \int (-\frac{1}{x})\varphi'(x)dx \\ &= ln|x|\varphi'(x)|_{-\infty}^{\infty} - \int ln|x|\phi''(x)dx \\ &= \int ln|x|\varphi''(x)dx\end{aligned}$$

Sine

$$sup|D^\alpha|\varphi_m(x) \to 0$$

So we can conclude it's a distribution.