UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Lecture Notes Hints

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Notes 1 PP 330

To calculate a surface integral in $\mathbb{R}^n$ of $S \subset \mathbb{R}^n$ is a hypersurface.

i). Step 1: Find parametrization γ

$$\gamma : [a_1, b_1] \times ... \times [a_{n+1}, b_{n+1}] \to S \subset \mathbb{R}^n$$

$$\gamma : (s_1, ..., s_{n+1}) \mapsto \begin{pmatrix} \gamma_1(s_1, ...s_{n+1}) \\ \gamma_2(s_1, ...s_{n+1}) \\ \gamma_3(s_1, ...s_{n+1}) \\ \vdots \\ \gamma_{n+1}(s_1, ...s_{n+1}) \end{pmatrix}$$

A simple example is given as

$$(\phi, \theta) = \begin{pmatrix} \cos \pi \sin \theta \\ \sin \pi \sin \theta \\ \cos \theta \end{pmatrix}$$

ii). Step 2: Find a normal vector $\vec{n} \perp S$

iii).

$$\int_S f ds = \int \int \int ... \int f(\gamma(s_1, ..., s_{n+1})) \det(\frac{\partial \gamma}{\partial s_1}, ..., \vec{a})$$

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$$\int_\varphi f ds = \int_a^b f \circ \gamma(t) \cdot |\gamma'(t)| dt$$

where φ indicates curve/line integral on $\mathbb{R}^n$. by

$$\gamma(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$t \in R$, $\gamma'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\gamma'(t)| = 1$. So

$$
\begin{aligned}
&\int_{\partial \mathbb{H}} h(\cdot) \frac{\partial g(x;\cdot)}{\partial n} ds \\
&= \int_{-\infty}^{\infty} h(\gamma(t)) \cdot \frac{-\partial g(x;\cdot)}{\partial \xi_2} |_{\gamma(t)} dt \\
&= -\int_{-\infty}^{\infty} h(\xi_2) \cdot \frac{-\partial g(x;\xi_1,\xi_2)}{\partial \xi_1} |_{\xi_2=0} dt
\end{aligned}
$$

Notes 3 Method of Images

This part is to raise an intuitive understanding of **Method of Images** chapter.

Electrostatics by Maxwell

$$-\Delta \underbrace{V}_{\text{potential}} = 4\pi \underbrace{\rho}_{\text{charge density}}$$

For a unit point charge located at ξ, it has charge density $\delta(x-\xi)$. The potential for a point charge is

$$V(x;\xi) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{|x-\xi|}$$

This solves $-\Delta V = 4\pi\delta(x-\xi)$

E.g. a charge of 1 Coulomb at ξ and 2 Coulomb at ξ^*,

$$-\Delta V = 4\pi(\delta(x-\xi) + 2\delta(x-\xi^*))$$

which is also taken as

$$V = V_1 + V_2$$

where according to the superposition principle

$$
\begin{aligned}
-\Delta V_1 &= -4\pi\delta(x-\xi) && (1) \\
-\Delta V_2 &= -4\pi\delta(x-\xi^*) && (2)
\end{aligned}
$$

If V_1 solves equation (1), V_2 also solves equation (2).

$$V_2(x;\xi^*) = 2V_1(x;\xi^*)$$

The effect of having a **ground plate** at x plane is equal to have a ξ^* charge.

Notes 4 Neumann Problem

$$\frac{\partial g}{\partial n} = 0 \quad g = \text{physical potential} \tag{3}$$

which means g satisfy $\Delta g = \delta$ and boundary condition.

$$\langle \nabla g, \vec{n} \rangle = 0 \tag{4}$$

This equation means the force perpendicular to the boundary vanishes. This equation (4) means at boundary the force is 0. Please note that for Neumann problems, the potential at boundary points is arbitrary as long as (3) is satisfied.

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Consider the reflection of ξ according to two boundaries.

We have

$$\Delta g = \delta(x - \xi), g|_{\partial\Omega} = 0 \tag{5}$$

So

$$E(x, \xi) = \frac{1}{2\pi} \ln |x - \xi| \tag{6}$$

and the sequence of images are

$$\xi_n^{\pm} = (\xi_1, \pm\xi_2 \pm 2n\pi)$$

Then we have

$$\begin{aligned} g(x; \xi) &= \sum_{n\in\mathbb{Z}} E(x, \xi_n^+) - \sum_{n\in\mathbb{Z}} E(x, \xi_n^-) &(7)\\ &= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \ln |x - \xi_n^+| - \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \ln |x - \xi_n^-| &(8) \end{aligned}$$

Two comments:

1. We ignore convergence issue
2. Introduce complex numbers:

$$\left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| = |x_1 + ix_2| = \sqrt{x_1^2 + x_2^2} \tag{9}$$

Then we could write

$$\begin{aligned} |x - \xi_n^+| = \left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \xi_2 + 2n\pi \end{pmatrix} \right| &= |(x_1 - \xi_1) + (x_2 - \xi_2 + 2n\pi)i| \\ &= |x - \xi + 2n\pi i| \quad x = x_1 + x_2 i \end{aligned}$$

Similarly,

$$|x - \xi_n^-| = |x - \bar{\xi} + 2n\pi i||$$

SO we have

$$\sum_{n\in\mathbb{Z}} \ln |x - \xi_n^+| = \sum_{n\in\mathbb{Z}} \ln |x - \xi + 2n\pi i| = \ln \left(\prod_{n\in\mathbb{Z}} |x - \xi + 2n\pi i| \right) \tag{10}$$

Now

$$g(x,\xi) = \frac{1}{2\pi} \ln \left(\prod_{n\in\mathbb{Z}} \frac{|x-\xi+2n\pi i|}{|x-\bar{\xi}+2n\pi i|} \right) \tag{11}$$

$$= \frac{1}{2\pi} \ln \left| \left(\prod_{n\in\mathbb{Z}} \frac{\frac{x-\xi}{2in\pi}-1}{\frac{x-\bar{\xi}}{2n\pi i}-1} \right) \cdot \frac{x-\xi}{x-\bar{\xi}} \right| \tag{12}$$