UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 3

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Exercise 3. 1 Fourier Transform

The Fourier Transform is defined as

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

i).

Plug in the definition of $f(x)$

$$f(x) = \Pi_{a,b}(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases} \quad , \quad a, b \in \mathbb{R}$$

The Fourier transform is then calculated as

$$\begin{aligned}
\mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{a} 0 \cdot e^{-i\omega t} dt + \int_{a}^{b} e^{-i\omega t} dt + \int_{b}^{\infty} 0 \cdot e^{-i\omega t} dt \\
&= \int_{a}^{b} e^{-i\omega t} dt \\
&= \left. \frac{e^{-i\omega t}}{-i\omega} \right|_{a}^{b} = \frac{e^{-i\omega b} - e^{-i\omega a}}{-i\omega}
\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{e^{-i\omega b} - e^{-i\omega a}}{-i\omega\sqrt{2\pi}}$$

ii).

$$f(x) = e^{-a|x|}$$

Plug it in Fourier transform, which yields to

$$\begin{aligned}
\mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{0} e^{ax} e^{-i\omega x} dx + \int_{0}^{+\infty} e^{-ax} e^{-i\omega x} dx \\
&= \left. \frac{e^{x(a-i\omega)}}{a - i\omega} \right|_{-\infty}^{0} + \left. \frac{e^{-x(a+i\omega)}}{-a - i\omega} \right|_{0}^{+\infty} \\
&= -\frac{1}{a - i\omega} + \frac{1}{-a - i\omega} \\
&= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} \\
&= \frac{2a}{a^2 + \omega^2}
\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{a\sqrt{\frac{2}{\pi}}}{a^2 + \omega^2}$$

iii).

Plug in $f(t) = e^{-ax^2}$

$$\begin{aligned}\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-at^2}e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\frac{b^2}{4a}-\left(\sqrt{a}t+\frac{b}{2\sqrt{a}}\right)^2}\mathrm{d}t\end{aligned}$$

Then we apply the substitution Rule, we define $u := (\sqrt{a}t + \frac{b}{2\sqrt{a}})$, then

$$\mathrm{d}u = \sqrt{a}\mathrm{d}t$$

The original equation then becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{e}^{\frac{b^2}{4a}-\left(\sqrt{a}t+\frac{b}{2\sqrt{a}}\right)^2}\mathrm{d}t \\ =&\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\frac{b^2}{4a}-u^2}}{\sqrt{a}}\mathrm{d}u \\ =&\frac{\mathrm{e}^{\frac{b^2}{4a}}}{\sqrt{a}}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-u^2}}{\sqrt{a}}du\end{aligned}$$

Note that for the third step, we define

$$b = i\omega$$

We recall the definition of Gauss Error function, which is of the similar form

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}dt$$

which yields to

$$\begin{aligned}\mathcal{F}(\omega)\cdot\sqrt{2\pi} =&\frac{\sqrt{\pi}\mathrm{e}^{\frac{b^2}{4a}}}{\sqrt{a}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-u^2}}{\sqrt{\pi}}\mathrm{d}u \\ =&\frac{\sqrt{\pi}\mathrm{e}^{\frac{b^2}{4a}}}{\sqrt{a}}\mathrm{erf}(x)\Big|_0^{+\infty} \\ =&\frac{\sqrt{\pi}\mathrm{e}^{\frac{-\omega^2}{4a}}}{\sqrt{a}}(1-0) \\ =&\frac{\sqrt{\pi}\mathrm{e}^{\frac{-\omega^2}{4a}}}{\sqrt{a}}\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{e^{-\frac{w^2}{4a}}}{\sqrt{2a}}$$

iv).

Plug in $f(x) = \text{cos}(x)e^{-x^2}$,

$$\begin{aligned}
\mathcal{F}(\omega)\cdot\sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} \text{cos}(t)e^{-t^2}e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} \frac{e^{it}+e^{-it}}{2}e^{-t^2}e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} \frac{e^{it}}{2}e^{-t^2}e^{-i\omega t}dt + \int_{-\infty}^{\infty} \frac{e^{-it}}{2}e^{-t^2}e^{-i\omega t}dt
\end{aligned}$$

Then the strategy is mostly alike 3.1.iii). We simplify the procedure, and the answer is given as

$$\begin{aligned}
\mathcal{F}(\omega)\cdot\sqrt{2\pi} &= \frac{\sqrt{\pi}\mathrm{e}^{-\frac{w^2}{4}+\frac{w}{2}-\frac{1}{4}}}{2} + \frac{\sqrt{\pi}\mathrm{e}^{-\frac{u^2}{4}-\frac{w}{2}-\frac{1}{4}}}{2} \\
&= \frac{\sqrt{\pi}\left(\mathrm{e}^{w}+1\right)\mathrm{e}^{-\frac{w^2}{4}-\frac{w}{2}-\frac{1}{4}}}{2}
\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{\left(\mathrm{e}^{w}+1\right)\mathrm{e}^{-\frac{w^2}{4}-\frac{w}{2}-\frac{1}{4}}}{2\sqrt{2}}$$

v).

We consider the property that Plug in $f(x) = \text{cos}(2x)/\left(4+x^2\right)$

$$\begin{aligned}
\mathcal{F}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\text{cos}(2t)}{4+t^2}e^{-i\omega t}dt \\
&= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{2it}+e^{-2it}}{4+t^2}e^{-i\omega t}dt \\
&= \frac{1}{2\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} \frac{e^{2it}}{4+t^2}e^{-i\omega t}dt + \int_{-\infty}^{\infty} \frac{e^{-2it}}{4+t^2}e^{-i\omega t}dt\right)
\end{aligned}$$

From the solution of 3.1.3,

$$\widehat{e^{-a|x|}} = \frac{a\sqrt{\frac{2}{\pi}}}{a^2+\omega^2}$$

Plug in $a = 2$, it yields to

$$\widehat{e^{-2|x|}} = \frac{2\sqrt{\frac{2}{\pi}}}{4+\omega^2}$$

From the definition of Fourier Transform

$$\widehat{\widehat{f(t)}} = f(-t)$$

We could conclude that

$$\frac{1}{2\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} \frac{e^{2it}}{4+t^2}e^{-i\omega t}dt\right) = \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-2|\omega-2|}$$

So the whole Fourier Transform then becomes

$$\mathcal{F}(\omega) = \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-2|\omega-2|} + \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-2|\omega+2|}$$

vi).

Consider the property

$$\widehat{\varphi * \psi} = (2\pi)^{n/2}\hat{\varphi} \cdot \hat{\psi}$$

Here $n = 1$. Then

$$\widehat{xe^{-x^2} * e^{-x^2}} = (2\pi)^{\frac{1}{2}} \widehat{xe^{-x^2}} \cdot \widehat{e^{-x^2}}$$

From the result before

$$\widehat{e^{-x^2}} = \frac{e^{-\frac{w^2}{4}}}{\sqrt{2}}$$

Then we apply Fourier Transform on xe^{-x^2}

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} te^{-t^2} e^{-i\omega t} dt = \frac{\mathrm{i}\omega \mathrm{e}^{-\frac{w^2}{4}}}{2\sqrt{2}}$$

So the total integral is given by

$$\begin{aligned} \widehat{xe^{-x^2} * e^{-x^2}} &= (2\pi)^{\frac{1}{2}} \cdot \frac{e^{-\frac{w^2}{4}}}{\sqrt{2}} \cdot \frac{\mathrm{i}\omega \mathrm{e}^{-\frac{w^2}{4}}}{2\sqrt{2}} \\ &= \frac{\sqrt{2\pi} e^{-\frac{w^2}{4}} i\omega e^{-\frac{w^2}{4}}}{4} \end{aligned}$$

Exercise 3. 2 Fourier on $\mathcal{S}'(\mathbb{R})$

i).

$$g(x) = \begin{cases} e^{-\varepsilon x} & x \geq 1 \\ 0 & x < 1 \end{cases} \qquad \varepsilon > 0$$

$$\begin{aligned} \hat{T}_g \varphi = T_g \hat{\varphi} &= \int_1^{\infty} e^{-\varepsilon \xi} \cdot \hat{\varphi}(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \int_{-\infty}^{\infty} e^{-\varepsilon \xi} e^{-ix\xi} \varphi(x) dx d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \int_{-\infty}^{\infty} e^{(-\varepsilon - ix)\xi} \varphi(x) dx d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_1^{\infty} e^{(-\varepsilon - ix)\xi} d\xi \right) \varphi(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\varepsilon - ix}}{\varepsilon + ix} \varphi(x) dx \end{aligned}$$

So

$$g(\hat{x}) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\varepsilon - ix}}{\varepsilon + ix}$$

ii).

$$\sin(3x-2) = \frac{e^{i(3x-2)} - e^{-i(3x-2)}}{2i} = \frac{e^{-2i}}{2i}e^{i3x} - \frac{e^{2i}}{2i}e^{-i3x}$$

$$\hat{T}_{\delta(\xi-\xi_0)}\varphi = \int_{-\infty}^{\infty} \delta\left(\xi - \xi_0\right) \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ix\xi}\varphi(x)dxd\xi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ix\xi_0}\varphi(x)dx$$

$$\hat{\delta}\left(\xi - \xi_0\right) = \frac{1}{\sqrt{2\pi}} e^{-i\xi_0 x}$$

,thus

$$e^{-i\xi_0 x} = \sqrt{2\pi}\hat{\delta}\left(\xi - \xi_0\right)$$

$$\begin{aligned}
\hat{T}_{\sin(3x-2)}\varphi &= T\frac{e^{-2i}}{2i}\delta(\xi+3)\hat{\hat{\varphi}} - T\frac{e^{2i}}{2i}\delta(\xi-3)\hat{\hat{\varphi}} \\
&= \frac{e^{-2i}}{2i}\int_{-\infty}^{\infty}\sqrt{2\pi}\delta(\xi+3)\varphi(-\xi)d\xi - \frac{e^{2i}}{2i}\int_{-\infty}^{\infty}\sqrt{2\pi}\delta(\xi-3)\varphi(-\xi)d\xi \\
&= \sqrt{2\pi}\left[\frac{e^{-2i}}{2i}\varphi(3) - \frac{e^{2i}}{2i}\varphi(-3)\right]
\end{aligned}$$

which gives the final answer

$$\widehat{\sin(3x-2)} = i\sqrt{\frac{\pi}{2}}\left[e^{2i}\delta(\xi+3) - e^{-2i}\delta(\xi-3)\right]$$

iii).

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

So

$$\begin{aligned}
\hat{T}_g\varphi = T_g\hat{\varphi} &= \int_{-\infty}^{\infty}\xi^2\cos(\xi)\hat{\varphi}(\xi)d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi^2\cos(\xi)e^{-ix\xi}\varphi(x)dxd\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(\xi)\left[\int_{-\infty}^{\infty}\xi^2 e^{-i\xi x}\varphi(x)dx\right]d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\xi}+e^{-i\xi}}{2}\left(\int_{-\infty}^{\infty}-e^{-i\xi x}\varphi''(x)dx\right)d\xi \\
&= -\sqrt{\frac{\pi}{2}}\varphi''(1) - \sqrt{\frac{\pi}{2}}\varphi''(-1)
\end{aligned}$$

which yields to the answer

$$\widehat{x^2\cos x} = -\sqrt{\frac{\pi}{2}}\delta''(\xi-1) - \sqrt{\frac{\pi}{2}}\delta''(\xi+1)$$

iv).

$f(x) = xH(x-2)$

$$F\left[xH\left(x-2\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xH\left(x-2\right)e^{-j\omega x}dx = \frac{je^{-2j\omega}(\pi e^{2j\omega}\omega^2\delta^{'}\left(\omega\right) - 2\omega + j)}{\omega^2\sqrt{2\pi}}$$

where the δ(x) means the Dirac Delta Function.

v).

$x^2\delta(x-1)$

$$F\left[x^2\delta\left(x-1\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\delta\left(x-1\right)e^{-j\omega x}dx = \frac{e^{-j\omega}}{\sqrt{2\pi}}$$

Exercise 3. 3

We know that

$$u_{tt} - u_{xx} = 0,\ so\ u_{tt} = u_{xx}$$

Taking Fourier Transform on both sides, we have

$$\hat{u_{tt}} = \hat{u_{xx}}$$

So

$$\int_{-\infty}^{\infty} u_{tt}e^{-jx\xi}dx = \left(-j\xi\right)^2\int_{-\infty}^{\infty} ue^{-jx\xi}dx$$

So

$$\frac{d^2\hat{u}}{dt^2} = -\xi^2\hat{u}$$

This is an ordinary second order homogeneous differential equation with constant coefficient.
So

$$\hat{u} = Acos(\xi t) + Bsin(\xi t)$$

We know that

$$\begin{cases} u\left(x,0\right) = f\left(x\right) \\ u_t\left(x,0\right) = g\left(x\right) \end{cases}$$

So

$$\begin{cases} \hat{u}\left(\xi,0\right) = \hat{f}\left(\xi\right) \\ \hat{u_t}\left(\xi,0\right) = \hat{g}\left(\xi\right) \end{cases}$$

So

$$\begin{cases} A = \hat{f}\left(\xi\right) \\ B = \frac{\hat{g}\left(\xi\right)}{\xi} \end{cases}$$

So we get

$$\hat{u}\left(\xi,t\right) = \hat{f}\left(\xi\right)cos\left(\xi t\right) + \frac{\hat{g}\left(\xi\right)}{\xi}sin(\xi t)$$

So

$$u\left(x,t\right)=F^{-1}[\hat{f}\left(\xi\right)cos\left(\xi t\right)]+F^{-1}[\frac{\hat{g}\left(\xi\right)}{\xi}sin(\xi t)]$$

By Shifting property, we have

$$\begin{cases} F[f\left(x-t\right)]=e^{jt\xi}\hat{f}(\xi) \\ F\left[f\left(x+t\right)\right]=e^{-jt\xi}\hat{f}(\xi) \end{cases}$$

So

$$F\left[f\left(x-t\right)+f\left(x+t\right)\right]=\left(e^{jt\xi}+e^{-jt\xi}\right)\hat{f}\left(\xi\right)=2cos(\xi t)\hat{f}(\xi)$$

So

$$F^{-1}\left[\cos\left(\xi t\right)\hat{f}\left(\xi\right)\right]=\frac{f\left(x-t\right)+f\left(x+t\right)}{2}$$

Now

$$F\left[f\left(x\right)\right]=\frac{2sin(\xi t)}{\xi},\ for\ \xi\neq 0$$

where

$$f(x)=\begin{cases} 1,\ |x|\ <t \\ 0,\ |x|\ >t \end{cases}$$

So

$$\frac{1}{2}f\left(x\right)=F^{-1}[\frac{sin(\xi t)}{\xi}]$$

By convolution theorem,

$$F^{-1}\left[\frac{\hat{g}\left(\xi\right)}{\xi}sin\left(\xi t\right)\right]=F^{-1}\left[\frac{\sin\left(\xi t\right)}{\xi}\hat{g}\left(\xi\right)\right]=\left[\frac{1}{2}f\left(x\right)\right]*g\left(x\right)=\int_{-\infty}^{\infty}\frac{1}{2}f\left(u\right)g\left(x-u\right)du=\frac{1}{2}\int_{-t}^{t}g\left(u\right)du$$

Since

$$f\left(x\right)=1\ when\ |x|\ <t$$

So

$$u\left(x,t\right)=\frac{f\left(x-t\right)+f\left(x+t\right)}{2}+\frac{1}{2}\int_{-t}^{t}g\left(x\right)dx$$