7.2.22. Determine the finite part of the integral $\int_0^c (1+x)/x^{3/2}\,dx$.

7.2.23. Given a (piecewise smooth) surface S, consider the generalized function $F(x,y,z)\delta(S)$ that vanishes at all points P not on the surface S and has the property that $(F(x,y,z)\delta(S), \phi(x,y,z)) = \int_S F(x,y,z)\phi(x,y,z)\;ds$. The function $F(x,y,z)\delta(S)$ is called a *single-layer distribution* and represents a generalization of the one-dimensional Dirac delta function. Show that if $f(x,y,z)$ is a piecewise smooth function with a jump discontinuity across the surface S, then $\partial f(x,y,z)/\partial x = \partial f(x,y,z)/\partial x\big|_{P\notin S} + [f(x,y,z)]_S\,\mathbf{n}\cdot\mathbf{i}\,\delta(S)$, where $\mathbf{n}$ is a unit normal to the surface S and $[f(x,y,z)]_S$ is the jump in $f(x,y,z)$ across S, given as the difference between the value of $f(x,y,z)$ on the side of S that $\mathbf{n}$ points to and the side it points away from. Also, obtain the corresponding formula for the other two partial derivatives of $f(x,y,z)$. *Hint*: Use the definition of the generalized (partial) derivative of $f(x,y,z)$ and the divergence theorem.

7.2.24. With the single-layer distribution $F(x,y,z)\delta(S)$ given as in Exercise 7.2.23, the *double-layer distribution* is given as $-\partial/\partial n(F(x,y,z)\delta(S))$, where $\partial/\partial n$ is the normal derivative on S and $(-\partial(F(x,y,z)\delta(S))/\partial n), \phi(x,y,z)) = \int_S F(x,y,z)\partial\phi$ $(x,y,z)/\partial n\;ds$. This corresponds to a generalization of $-\delta'(x)$. Demonstrate that $\nabla^2 f(x,y,z) = \nabla^2 f(x,y,z)\big|_{P\notin S} + \partial([f(x,y,z)]_S\delta(S))/\partial n + [\partial f(x,y,z)/\partial n]_S$ $\delta(S)$. Single- and double-layer distributions can be used to describe some of the results in Chapter 6 relating to concentrated source terms in the language of the theory of generalized functions.

7.2.25. Show that (a) $(1/2\pi)\int_{-\infty}^{\infty}\exp(i\lambda x)\exp(-i\hat{\lambda}x)\,dx = \delta(\lambda-\hat{\lambda})$; (b) $(2/\pi)\int_0^{\infty}$ $\cos(\lambda x)\cos(\hat{\lambda}x)\;dx = \delta(\lambda-\hat{\lambda})$, $\lambda,\ \hat{\lambda}\geq 0$; (c) $(2/\pi)\int_0^{\infty}\sin(\lambda x)\sin(\hat{\lambda}x)dx = \delta(\lambda-\hat{\lambda})$, $\lambda,\ \hat{\lambda}\geq 0$. Demonstrate that these results can be interpreted as orthogonality conditions for the functions arising in the Fourier transform, the Fourier cosine transform, and the Fourier sine transform. In part (a) the Hermitian inner product must be used.

7.3 GREEN'S FUNCTIONS FOR BOUNDED REGIONS

A general procedure for determining Green's functions for problems given over bounded (spatial) regions is the method of finite Fourier transforms presented in Section 4.6. For the equations (7.1.9), (7.1.23), and (7.1.32) given in Section 7.1, the Green's function is expanded in a series of eigenfunctions of the elliptic operator L defined in (7.1.1). The coefficients in the series of eigenfunctions are specified in the manner shown in Section 4.6. Although this procedure is identical to that given in Section 4.6, the solutions of the given problems expressed in terms of the Green's functions have a somewhat different form than that given earlier. However, the uniqueness theorems guarantee that there can be only one solution for each of these problems. We do not demonstrate in the general case the equivalence of the various solution forms. Our main emphasis in this section lies in the construction of Green's functions.

Green's Functions for Elliptic PDEs

We begin by constructing the Green's function $K(\mathbf{x};\boldsymbol{\xi})$ for the elliptic problem. As shown in Section 7.1, the function $K(\mathbf{x};\boldsymbol{\xi})$ satisfies the equation

$$LK(\mathbf{x};\boldsymbol{\xi}) \equiv -\nabla\cdot(p(\mathbf{x})\nabla K(\mathbf{x};\boldsymbol{\xi}))+q(\mathbf{x})K(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi}), \quad \mathbf{x},\ \boldsymbol{\xi}\in G \tag{7.3.1}$$

and the boundary condition

$$\alpha(\mathbf{x})K(\mathbf{x};\boldsymbol{\xi}) + \beta(\mathbf{x})\frac{\partial K(\mathbf{x};\boldsymbol{\xi})}{\partial n}\bigg|_{\partial G} = 0, \tag{7.3.2}$$

where derivatives are taken with respect to the variable $\mathbf{x}$. Let $K(\mathbf{x};\boldsymbol{\xi})$ be the orthonormalized set of eigenfunctions of the operator L; that is,

$$LM_k(\mathbf{x}) = \lambda_k\rho(\mathbf{x})M_k(\mathbf{x}), \qquad k=1,2,\dots, \tag{7.3.3}$$

where the $M_k(\mathbf{x})$ are the eigenvalues of L and $\rho(\mathbf{x})$ is a given weight function. The boundary condition for the $M_k(\mathbf{x})$ is (7.3.2) with $K(\mathbf{x};\boldsymbol{\xi})$ replaced by $M_k(\mathbf{x})$.

We express $K(\mathbf{x};\boldsymbol{\xi})$ as a series of eigenfunctions

$$K(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty} N_k(\boldsymbol{\xi})M_k(\mathbf{x}), \tag{7.3.4}$$

as in (4.6.3) with the (Fourier) coefficients $N_k(\boldsymbol{\xi})$ to be determined. Proceeding as in Section 4.6, we multiply (7.3.1) by $M_k(\mathbf{x})$, $(k=1,2,\dots)$ and integrate over the region G. [Note that the operator K introduced in Section 4.6 is unrelated to the Green's function $K(\mathbf{x};\boldsymbol{\xi})$ of the present chapter.] Using the results (4.6.5)–(4.6.8) and noting that both $K(\mathbf{x};\boldsymbol{\xi})$ and $M_k(\mathbf{x})$ satisfy homogeneous boundary conditions of the form (7.3.2), we obtain

$$\lambda_k N_k(\boldsymbol{\xi}) = \iint_G \delta(\mathbf{x}-\boldsymbol{\xi})M_k(\mathbf{x})\,dx = M_k(\boldsymbol{\xi}), \qquad k=1,2,\dots, \tag{7.3.5}$$

since $\boldsymbol{\xi}\in G$. Then if all the $\lambda_k>0$, we have $N_k(\boldsymbol{\xi}) = M_k(\boldsymbol{\xi})/\lambda_k$, and this yields the *bilinear expansion* of the Green's function $K(\mathbf{x};\boldsymbol{\xi})$,

$$K(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty}\frac{M_k(\mathbf{x})M_k(\boldsymbol{\xi})}{\lambda_k}. \tag{7.3.6}$$

We notice that $K(\mathbf{x};\boldsymbol{\xi}) = K(\boldsymbol{\xi};\mathbf{x})$, so that the Green's function for the elliptic problem is *symmetric*. This result can be proven directly for the Green's function without the use of the bilinear expansion (see Exercise 7.1.1). The symmetry is a consequence of the fact that the operator $(1/\rho)L$ taken together with the boundary conditions is *self-adjoint*. It implies that the interchange of the *source point* $\boldsymbol{\xi}$ and the *observation point* $\mathbf{x}$ does not alter the solution.

It was shown in Section 4.2 that the eigenvalues λ_k for the eigenvalue problem (7.3.3) with the boundary condition (7.3.2) [with $K(\mathbf{x};\boldsymbol{\xi})$ replaced by $M_k(\mathbf{x})$] are nonnegative. It has also been demonstrated in Exercise 4.2.5 that $\lambda_0 = 0$ is an eigenvalue for this problem if and only if $q(\mathbf{x}) = 0$ in the operator L and $\alpha(\mathbf{x}) = 0$ in the boundary condition. The corresponding eigenfunction $M_0(\mathbf{x})$ is clearly a constant in that case. Consequently, unless zero is an eigenvalue, the bilinear expansion (7.3.6) yields a formal solution of the boundary value problem (7.3.1)–(7.3.2) for the Green's function $K(\mathbf{x};\boldsymbol{\xi})$. It may be verified directly that (7.3.6) is a (generalized) solution of (7.3.1)–(7.3.2).

In the one-dimensional case, (7.3.1) is an ODE and the bilinear expansion (7.3.6) represents an expansion of the Green's function $K(x;\xi)$ in terms of the eigenvalues and eigenfunctions of a Sturm-Liouville problem. In that case, it is often preferable to determine the Green's function by solving the differential equation directly. Additionally, for certain higher-dimensional problems it is possible to construct the Green's function in terms of eigenfunctions for lower-dimensional problems. Since the solution of higher-dimensional eigenvalue problems is generally not a simple task, this can lead to a substantial simplification of the Green's function problem, especially if the resulting eigenvalue problems are one-dimensional.

Proceeding as in Section 4.2, we consider the function $u = u(\mathbf{x}, y)$ where $\mathbf{x}$ is a point in the (bounded) region G and y is a scalar variable defined over the interval $0 < y < \hat{l}$. In place of (7.1.1) we now consider the equation

$$\rho(\mathbf{x})u_{yy}(\mathbf{x}, y) - Lu(\mathbf{x}, y) = -\rho(\mathbf{x})F(\mathbf{x}, y), \qquad \mathbf{x} \in G,\ 0 < y < \hat{l}, \tag{7.3.7}$$

and the boundary conditions (7.1.2) and (4.2.8) if G is a two-dimensional region. If G is one-dimensional and is given as $0 < x < l$, $u(x, y)$ satisfies the boundary conditions (7.1.47), where t is replaced by y.

The Green's function for the problem satisfies the equation

$$\rho(\mathbf{x})\frac{\partial^2 K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)}{\partial y^2} - LK(\mathbf{x}, y; \boldsymbol{\xi}, \eta) = -\delta(\mathbf{x}-\boldsymbol{\xi})\delta(y-\eta),\ \mathbf{x}, \boldsymbol{\xi} \in G,\ 0 < y, \eta < \hat{l}, \tag{7.3.8}$$

and the boundary condition (7.3.2)—if G is two-dimensional—as well as a homogeneous form of the boundary condition (4.2.8). The differential operator in (7.3.7) is self-adjoint. The Green's function is expressed in the form $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ and the derivatives in (7.3.8) are taken with respect to the variables $\mathbf{x}$ and y. If the region G is one-dimensional, $K(x, y; \xi, \eta)$ satisfies the boundary conditions (7.1.47) (where t is replaced by y) with $u(x, y)$ replaced by $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ and $g_1(y) = g_2(y) = 0$.

To determine $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ we use the finite Fourier transform procedure of Section 4.6. With the set $\{M_k(\mathbf{x})\}$ $(k = 1, 2, \ldots)$ as the eigenfunctions of the operator L in (7.3.8), we construct the eigenfunction expansion

$$K(\mathbf{x}, y; \boldsymbol{\xi}, \eta) = \sum_{k=1}^{\infty} N_k(y)M_k(\mathbf{x}). \tag{7.3.9}$$

The $N_k(y)$ are the Fourier coefficients of $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ given as

$$N_k(y) = (K(\mathbf{x}, y; \boldsymbol{\xi}, \eta), M_k(\mathbf{x})) = \iint_G \rho(\mathbf{x}) K(\mathbf{x}, y; \boldsymbol{\xi}, \eta) M_k(\mathbf{x})\, d\mathbf{x}, \ k \geq 1, \tag{7.3.10}$$

assuming that G is two-dimensional. We multiply (7.3.8) by $M_k(\mathbf{x})$ and integrate over the region G. Again using the results (4.6.5)–(4.6.8), we obtain

$$N_k''(y) - \lambda_k N_k(y) = -M_k(\boldsymbol{\xi}) \delta(y - \eta), \ 0 < y, \ \eta < \hat{l}, \ k = 1, 2, \ldots. \tag{7.3.11}$$

Since $K(\mathbf{x}, 0; \boldsymbol{\xi}, \eta) = K(\mathbf{x}, \hat{l}; \boldsymbol{\xi}, \eta) = 0$, we must have

$$N_k(0) = 0, \qquad N_k(\hat{l}) = 0, \tag{7.3.12}$$

as the boundary conditions for $N_k(y)$. To determine the $N_k(y)$, we must construct a *Green's function*, given here as $N_k(y)/M_k(\boldsymbol{\xi})$, for an ODE in a finite interval. Although $N_k(y)$ may also be found using eigenfunction expansions, we use a more direct and concrete approach in the following example.

Example 7.5. Green's Function for an Ordinary Differential Equation. Using the notation given in Example 7.1, we consider the Green's function $K(x; \xi)$ that satisfies the equation

$$\frac{\partial^2 K(x; \xi)}{\partial x^2} - c^2 K(x; \xi) = -\delta(x - \xi), \qquad 0 < x, \ \xi < l, \tag{7.3.13}$$

with the homogeneous (Dirichlet) boundary conditions

$$K(0; \xi) = 0, \qquad K(l; \xi) = 0. \tag{7.3.14}$$

From our discussion in Example 7.2 we find that $K(x; \xi)$ satisfies the homogeneous equation

$$\frac{\partial^2 K(x; \xi)}{\partial x^2} - c^2 K(x; \xi) = 0, \qquad x \neq \xi, \tag{7.3.15}$$

and the continuity and jump conditions at $x = \xi$:

$$K(x; \xi) \text{ continuous at } x = \xi, \quad \left[\frac{\partial K(x; \xi)}{\partial x} \right]_{x=\xi} = -1, \tag{7.3.16}$$

where the brackets represent the jump in the first derivative of $K(x; \xi)$ across $x = \xi$.

To solve this problem we denote $K(x; \xi)$ by $K_1(x; \xi)$ for $x < \xi$ and $K_2(x; \xi)$ for $x > \xi$. Both $K_1(x; \xi)$ and $K_2(x; \xi)$ satisfy the homogeneous equation (7.3.15). $K_1(x; \xi)$ vanishes at $x = 0$ while $K_2(x; \xi)$ vanishes at $x = l$. We easily conclude that

$$K_1(x; \xi) = a_1 \sinh(cx), \quad K_2(x; \xi) = a_2 \sinh[c(l - x)], \tag{7.3.17}$$

where a_1 and a_2 are as yet unspecified constants (note that c is assumed to be a constant). The continuity and jump conditions (7.3.16) imply that $a_1 \sinh(c\xi) = a_2 \sinh[c(l-\xi)]$, and $-a_2 c\cosh[c(l-\xi)] - a_1 c\cosh(c\xi) = -1$. The solution of the system for a_1 and a_2 is given as $a_1 = \sinh[c(l-\xi)]/(c\sinh(cl))$, $a_2 = \sinh[c\xi]/(c\sinh(cl))$. Inserting these expressions into (7.3.17), we obtain the Green's function $K(x;\xi)$ as

$$K(x;\xi) = \frac{1}{c\sinh(cl)} \begin{cases} \sinh[c(l-\xi)]\sinh(cx), & 0 < x < \xi, \\ \sinh(c\xi)\sinh[c(l-x)], & \xi < x < l. \end{cases} \tag{7.3.18}$$

We observe that $K(x;\xi) = K(\xi;x)$, so that the Green's function is symmetric.

In the limit as $c \to 0$, we obtain the Green's function $K(x;\xi)$ that satisfies the equation

$$\frac{\partial^2 K(x;\xi)}{\partial x^2} = -\delta(x-\xi), \qquad 0 < x,\ \xi < l, \tag{7.3.19}$$

and (7.3.14). Since $\sinh(x) \approx x$ as $x \to 0$, it is given

$$K(x;\xi) = \frac{1}{l} \begin{cases} x(l-\xi), & 0 < x < \xi, \\ \xi(l-x), & \xi < x < l. \end{cases} \tag{7.3.20}$$

We continue our discussion of (7.3.8) and assume that the eigenvalues λ_k of (7.3.3) are all positive. Then on using the results of the preceding example, we find that the solution of (7.3.11)–(7.3.12) is given as

$$N_k(y) = \frac{M_k(\boldsymbol{\xi})}{\sqrt{\lambda_k}\sinh(\sqrt{\lambda_k}\,\hat{l})} \begin{cases} \sinh[\sqrt{\lambda_k}(\hat{l}-\eta)]\sinh(\sqrt{\lambda_k}\,y), & 0 < y < \eta, \\ \sinh(\sqrt{\lambda_k}\,\eta)\sinh[\sqrt{\lambda_k}(\hat{l}-y)], & \eta < y < \hat{l}. \end{cases} \tag{7.3.21}$$

Inserting (7.3.21) into (7.3.9) completes the formal solution of the problem (7.3.8) for the Green's function $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ for the given special region.

The preceding technique can be generalized to deal with elliptic equations

$$\rho(\mathbf{x})\hat{L}u(\mathbf{x}, y) - Lu(\mathbf{x}, y) = -\rho(\mathbf{x})F(\mathbf{x}, y), \qquad \mathbf{x} \in G,\ 0 < y < \hat{l}, \tag{7.3.22}$$

where

$$\hat{L}u(\mathbf{x}, y) = \frac{\partial}{\partial y}\left[a(y)\frac{\partial u(\mathbf{x}, y)}{\partial y}\right] + b(y)u(\mathbf{x}, y), \tag{7.3.23}$$

so that $\hat{L}$ is a self-adjoint operator. The $(\mathbf{x}, y)$-region may be defined as in the foregoing, and the boundary conditions in the y-variable may be of the more general form (7.1.47), appropriately modified.

The Green's function $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ satisfies the equation

$$\rho(\mathbf{x})\hat{L}K(\mathbf{x}, y; \boldsymbol{\xi}, \eta) - LK(\mathbf{x}, y; \boldsymbol{\xi}, \eta) = -\delta(\mathbf{x}-\boldsymbol{\xi})\delta(y-\eta), \tag{7.3.24}$$

where $\mathbf{x}, \boldsymbol{\xi} \in G$, $0 < y, \eta < \hat{l}$, and $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ satisfies a homogeneous version of the boundary conditions for $u(\mathbf{x}, y)$. Expanding $K(\mathbf{x}, y; \boldsymbol{\xi}, \eta)$ as in (7.3.9), we conclude that $N_k(y)$ satisfies

$$\hat{L}N_k(y) - \lambda_k N_k(y) = -M_k(\boldsymbol{\xi})\delta(y-\eta),\ 0 < y,\ \eta < \hat{l}, \quad k = 1, 2, \ldots \tag{7.3.25}$$

and $N_k(y)$ satisfies appropriate boundary conditions at $y = 0$ and $y = \hat{l}$. Thus we again obtain a Green's function problem for a self-adjoint ordinary differential equation. A simple case of the Green's function problem was considered in Example 7.5, and further cases are studied in the exercises. We have further occasion to consider the technique of reducing the Green's function problem to a lower dimensional one later in this chapter.

Example 7.6. Laplace's Equation: Green's Function for a Rectangle. We consider the rectangular region G, given as $0 < x < l$ and $0 < y < \hat{l}$, and construct the Green's function $K(x, y; \xi, \eta)$ that satisfies the equation

$$\frac{\partial^2 K(x,y;\xi,\eta)}{\partial x^2} + \frac{\partial^2 K(x,y;\xi,\eta)}{\partial y^2} = -\,\delta(x-\xi)\delta(y-\eta), \quad (x,y) \in G, \tag{7.3.26}$$

with $0 < \xi < l$ and $0 < \eta < \hat{l}$ and the Dirichlet boundary condition

$$K(x,y;\xi,\eta) = 0, \qquad (x,y) \in \partial G. \tag{7.3.27}$$

We begin by applying the first of the two methods presented above. Thus, we must solve the following Dirichlet *eigenvalue problem* in the region G:

$$-\nabla^2 M(x,y) = \lambda M(x,y),\ (x,y) \in G, \quad M(x,y) = 0,\ (x,y) \in \partial G. \tag{7.3.28}$$

The eigenvalues and eigenfunctions are determined using the *method of separation of variables*.

Let $M(x, y) = F(x)G(y)$ and insert this expression into (7.3.28). We have $M_{xx}(x,y) + M_{yy}(x,y) + \lambda M(x,y) = F''(x)G(y) + F(x)G''(y) + \lambda F(x)G(y) = 0$. Dividing by $F(x)G(y)$ and separating variables gives $F''(x)/F(x) + \lambda = -G''(y)/G(y) = k^2$, where k^2 is the separation constant. The equations for $F(x)$ and $G(y)$ are

$$F''(x) + (\lambda - k^2)F(x) = 0, \quad G''(y) + k^2 G(y) = 0. \tag{7.3.29}$$

The boundary condition in (7.3.28) implies that

$$F(0) = F(l) = 0, \quad G(0) = G(\hat{l}) = 0. \tag{7.3.30}$$

Consequently, we are led to consider one-dimensional eigenvalue problems for $F(x)$ and $G(y)$ of a type studied in Section 4.3.

For the eigenvalue problem (7.3.29)–(7.3.30) for $G(x)$ we obtain as the eigenvalues and the eigenfunctions

$$k_m^2 = \left(\frac{\pi m}{\hat{l}}\right)^2, \quad G_m(y) = \sin\left(\frac{\pi m y}{\hat{l}}\right), \quad m = 1, 2, \ldots. \tag{7.3.31}$$

For each of these eigenvalues, we have an eigenvalue problem (7.3.29)–(7.3.30) for $F(x)$. As in Section 4.3, we find the eigenvalues

$$\lambda_{nm} - k_m^2 = \left(\frac{\pi n}{l}\right)^2, \qquad m = 1, 2, \ldots, \ n = 1, 2, \ldots, \tag{7.3.32}$$

and the eigenfunctions

$$F_n(x) = \sin\left(\frac{\pi n x}{l}\right), \qquad n = 1, 2, \ldots. \tag{7.3.33}$$

Combining the results obtained, we conclude that the *eigenvalues* and the *eigenfunctions* for the problem (7.3.28) are given as

$$\lambda_{nm} = \left(\frac{\pi n}{l}\right)^2 + \left(\frac{\pi m}{\hat{l}}\right)^2, \quad \hat{M}_{nm}(x, y) = \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi m y}{\hat{l}}\right), \tag{7.3.34}$$

with $n, m = 1, 2, \ldots$. The unnormalized eigenfunctions are denoted as $\hat{M}_{nm}(x, y) = F_n(x)G_m(y)$. It may be noted that the eigenvalues λ_{nm} are positive, infinite in number, and tend to infinity as n and m tend to infinity.

The appropriate inner product for the rectangular region G is given as

$$(f(x, y), g(x, y)) = \int_0^{\hat{l}}\int_0^l f(x, y)g(x, y)\, dx\, dy. \tag{7.3.35}$$

If we consider the two pairs (n, m) and (j, k) with $(n, m) \neq (j, k)$, we have

$$(\hat{M}_{nm}, \hat{M}_{jk}) = \int_0^{\hat{l}} \sin\left(\frac{\pi m y}{\hat{l}}\right)\sin\left(\frac{\pi k y}{\hat{l}}\right)dy \int_0^l \sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi j x}{l}\right)dx = 0, \tag{7.3.36}$$

since $\{\sin(\pi n x/l)\}$ and $\{\sin(\pi m y/\hat{l})\}$ are orthogonal sets and $m \neq k$ and/or $n \neq j$. Thus eigenfunctions $\{\hat{M}_{nm}(x, y)\}$ form an *orthogonal set* that we now orthonormalize. The square of the norm of $M_{nm}(x, y)$ is given as

$$||\hat{M}_{nm}||^2 = \int_0^{\hat{l}} \sin^2\left(\frac{\pi m y}{\hat{l}}\right)dy \int_0^l \sin^2\left(\frac{\pi n x}{l}\right)dx = \frac{\hat{l}}{2}\frac{l}{2}, \tag{7.3.37}$$

on using Section 4.3. Thus an orthonormal set of eigenfunctions is

$$M_{nm}(x, y) = \frac{2}{\sqrt{l\hat{l}}}\sin\left(\frac{\pi n x}{l}\right)\sin\left(\frac{\pi m y}{\hat{l}}\right), \qquad n, m = 1, 2, \ldots. \tag{7.3.38}$$

Different sets of values of (n, m) do not necessarily yield distinct eigenvalues λ_{nm}. For example, if $l = \hat{l}$, we see that $\lambda_{12} = (\pi/l)^2 + (2\pi/l)^2 = \lambda_{21}$. Nevertheless, the set of eigenvalues λ_{nm} can be arranged in a sequence that corresponds to the positive integers $k = 1, 2, 3, \ldots$ with an equivalent arrangement for the eigenfunctions $M_{nm}(x, y)$. It is then possible to speak of the set of eigenvalues λ_k and the eigenfunctions $M_k(x, y)$, with $k = 1, 2, 3, \ldots$, and even if λ_k is a multiple eigenvalue, the corresponding eigenfunctions are orthogonal, as we have shown. The single-subscript notation was used in our earlier discussions of multidimensional eigenvalue problems.

Given a function $f(x, y)$, defined in the rectangular region $0 < x < l$ and $0 < y < \hat{l}$, that satisfies certain smoothness conditions, we have the expansion

$$f(x,y) = \frac{2}{\sqrt{l\hat{l}}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi m y}{\hat{l}}\right), \tag{7.3.39}$$

with the *Fourier coefficients* c_{nm} given as

$$c_{nm} = \frac{2}{\sqrt{l\hat{l}}} \int_0^{\hat{l}} \int_0^{l} f(x,y) \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi m y}{\hat{l}}\right) dx\, dy. \tag{7.3.40}$$

The expansion (7.3.39) is known as a *double Fourier sine series.*

Having determined the eigenfunctions and eigenvalues for the problem (7.3.28) we are now in a position to construct the Green's function $K(x, y; \xi, \eta)$. In view of (7.3.6), we obtain

$$K(x,y;\xi,\eta) = \frac{4}{l\hat{l}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\pi n x/l)\sin(\pi n \xi/l)\sin\left(\pi m y/\hat{l}\right)\sin\left(\pi m \eta/\hat{l}\right)}{(\pi n/l)^2 + (\pi m/\hat{l})^2} \tag{7.3.41}$$

as the *eigenfunction expansion* of the Green's function for the region G.

The alternative method presented above for the construction of the Green's function for the rectangular region G has better convergence properties than those of the series (7.3.41), as we now demonstrate. The equation (7.3.26) for the Green's function $K(x, y; \xi, \eta)$ is written as

$$\frac{\partial^2 K(x,y;\xi,\eta)}{\partial y^2} - LK(x,y;\xi,\eta) = -\,\delta(x-\xi)\delta(y-\eta), \tag{7.3.42}$$

with the operator L given as $L = -(\partial^2/\partial x^2)$. The appropriate eigenvalue problem is

$$LM(x) = -M''(x) = \lambda M(x),\ 0 < x < l, \quad M(0) = M(l) = 0. \tag{7.3.43}$$

This represents a *Sturm-Liouville problem* which was solved in Example 4.4. The eigenvalues and orthonormalized eigenfunctions are

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2, \ M_k(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi k x}{l}\right), \quad k = 1, 2, \ldots. \tag{7.3.44}$$

With $N_k(y)$ defined as in (7.3.21), the Green's function is given by the *eigenfunction expansion*

$$K(x, y; \xi, \eta) = \sum_{k=1}^{\infty} N_k(y) M_k(x). \tag{7.3.45}$$

For y different from η, the hyperbolic functions that occur in $N_k(y)$ can be approximated by exponentials as was done in Example 4.11, and the series can be shown to converge fairly rapidly if $|y - \eta|$ is not small.

Both (7.3.41) and (7.3.45) can be used to solving Dirichlet's problem for Laplace's equation in a rectangle. It has been shown in Example 4.11 that a direct separation of variables approach yields quite satisfactory results.

Modified Green's Functions for Elliptic PDEs

The eigenfunction expansion method for constructing Green's functions for the elliptic equation (7.1.1) with the boundary condition (7.1.2) fails if $\lambda_0 = 0$ is an eigenvalue of the associated eigenvalue problem. As indicated, this occurs for the Green's function problem (7.3.1)–(7.3.2) if and only if $q(\mathbf{x}) = 0$ in (7.3.1) and $\alpha(\mathbf{x}) = 0$ in the boundary condition (7.3.2).

Let $\lambda_0 = 0$ be an eigenvalue of the operator L [see (7.3.3)] and $M_0(\mathbf{x})$ be the eigenfunction corresponding to λ_0. Then (7.3.5) implies that $\lambda_0 N_0 = 0 = M_0(\boldsymbol{\xi})$, and this is not possible since we cannot have $M_0 \equiv 0$. Clearly then, the Green's function cannot be constructed in the given manner if $\lambda_0 = 0$ is an eigenvalue. There are two methods whereby a *modified Green's function* can be constructed in the special case of a zero eigenvalue. For each of the two modified Green's functions it is possible to solve the corresponding boundary value problem for $u(\mathbf{x})$ in a manner similar to that presented for the (ordinary) Green's function $K(\mathbf{x}; \boldsymbol{\xi})$ in Section 7.1, and this will be demonstrated below.

We put $q(\mathbf{x}) = 0$ in (7.1.1) and consider the *elliptic equation*

$$Lu(\mathbf{x}) = -\nabla \cdot (p(\mathbf{x})\nabla u(\mathbf{x})) = \rho(\mathbf{x})F(\mathbf{x}), \qquad \mathbf{x} \in G, \tag{7.3.46}$$

with the *Neumann boundary condition*

$$\beta(\mathbf{x}) \left. \frac{\partial u(\mathbf{x})}{\partial n} \right|_{\partial G} = B(\mathbf{x}). \tag{7.3.47}$$

The eigenvalue problem that corresponds to (7.3.46)–(7.3.47) is given as

$$LM(\mathbf{x}) = \lambda \rho(\mathbf{x}) M(\mathbf{x}), \quad \mathbf{x} \in G, \qquad \left. \frac{\partial M(\mathbf{x})}{\partial n} \right|_{\partial G} = 0, \tag{7.3.48}$$

and $\lambda_0 = 0$ is an eigenvalue. It follows from the results of Section 4.6 that

$$\lambda_0 N_0 = 0 = \iint_G \rho(\mathbf{x}) F(\mathbf{x}) M_0(\mathbf{x})\, d\mathbf{x} + \int_{\partial G} \frac{p(\mathbf{x})}{\beta(\mathbf{x})} M_0(\mathbf{x}) B(\mathbf{x})\, ds. \tag{7.3.49}$$

Since $M_0(\mathbf{x})$ must be a constant (see Exercise 4.2.5), we conclude that

$$\iint_G \rho(\mathbf{x})F(\mathbf{x})\,d\mathbf{x} + \int_{\partial G} \frac{p(\mathbf{x})}{\beta(\mathbf{x})}B(\mathbf{x})\,ds = 0. \tag{7.3.50}$$

Unless $F(\mathbf{x})$ and $B(\mathbf{x})$ are such that the *compatibility condition* (7.3.50) is satisfied, the boundary value problem (7.3.46)–(7.3.47) has no solution. We assume that (7.3.50) is satisfied and construct the modified Green's function.

In the first method, the *modified Green's function,* which we denote by $\hat{K}(\mathbf{x};\boldsymbol{\xi})$, is expanded in a series of eigenfunctions of the operator L,

$$\hat{K}(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty} N_k(\boldsymbol{\xi})M_k(\mathbf{x}), \tag{7.3.51}$$

where the $M_k(\mathbf{x})$, $k = 1,2,\ldots$ correspond to the positive eigenvalues of L. This expansion differs from that given in (7.3.4) since the eigenfunction $M_0(\mathbf{x})$ is absent from the series, even though $\lambda_0 = 0$ is an eigenvalue for this problem. We have removed the term $N_0(\boldsymbol{\xi})M_0(\mathbf{x})$ from the expansion so as to avoid obtaining the contradictory result $\lambda_0 N_0(\boldsymbol{\xi}) = 0 = M_0(\boldsymbol{\xi})$, which was derived in the foregoing.

Since the complete set of eigenfunctions is given as $\{M_k(\mathbf{x})\}$, $k = 0,1,2,\ldots$, the eigenfunction expansion of $\delta(\mathbf{x}-\boldsymbol{\xi})$ is

$$\delta(\mathbf{x}-\boldsymbol{\xi}) = \sum_{k=0}^{\infty}(\delta(\mathbf{x}-\boldsymbol{\xi}), M_k(\mathbf{x}))M_k(\mathbf{x}) = \rho(\boldsymbol{\xi})\sum_{k=0}^{\infty} M_k(\boldsymbol{\xi})M_k(\mathbf{x}). \tag{7.3.52}$$

This series can be expressed as $\delta(\mathbf{x}-\boldsymbol{\xi}) = \rho(\mathbf{x})\sum_{k=0}^{\infty} M_k(\boldsymbol{\xi})M_k(\mathbf{x})$. Further,

$$L\hat{K}(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty} N_k(\boldsymbol{\xi})LM_k(\mathbf{x}) = \rho(\mathbf{x})\sum_{k=1}^{\infty}\lambda_k N_k(\boldsymbol{\xi})M_k(\mathbf{x}). \tag{7.3.53}$$

Comparing (7.3.52) with (7.3.53) shows that if we set $N_k(\boldsymbol{\xi}) = M_k(\boldsymbol{\xi})/\lambda_k$, $k = 1,2,\ldots$, the modified Green's function satisfies the equation

$$L\hat{K}(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty}\rho(\mathbf{x})M_k(\boldsymbol{\xi})M_k(\mathbf{x}) = \delta(\mathbf{x}-\boldsymbol{\xi}) - \rho(\mathbf{x})M_0(\boldsymbol{\xi})M_0(\mathbf{x}). \tag{7.3.54}$$

Noting the foregoing discussion, we define the *modified Green's function* $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ to be a solution of the equation

$$L\hat{K}(\mathbf{x};\boldsymbol{\xi}) \equiv -\nabla\cdot(p(\mathbf{x})\nabla\hat{K}(\mathbf{x};\boldsymbol{\xi})) = \delta(\mathbf{x}-\boldsymbol{\xi}) - \rho(\mathbf{x})M_0(\boldsymbol{\xi})M_0(\mathbf{x}),\ \mathbf{x},\boldsymbol{\xi}\in G \tag{7.3.55}$$

with the Neumann boundary condition

$$\left.\frac{\partial\hat{K}(\mathbf{x};\boldsymbol{\xi})}{\partial n}\right|_{\partial G} = 0. \tag{7.3.56}$$

The bilinear series of eigenfunctions for $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ has the form

$$\hat{K}(\mathbf{x};\boldsymbol{\xi}) = \sum_{k=1}^{\infty} \frac{M_k(\mathbf{x})M_k(\boldsymbol{\xi})}{\lambda_k}. \tag{7.3.57}$$

Given the boundary value problem (7.3.46)–(7.3.47), we now construct a *solution formula* for $u(\mathbf{x})$ in terms of the modified Green's function $\hat{K}(\mathbf{x};\boldsymbol{\xi})$. Following the procedure given in Section 7.1, we set $w = \hat{K}$ in (7.3) and note that (7.1.4) gives

$$\iint_G u(\mathbf{x})L\hat{K}(\mathbf{x};\boldsymbol{\xi})\, dv = u(\boldsymbol{\xi}) - M_0(\boldsymbol{\xi}) \iint_G \rho(\mathbf{x})u(\mathbf{x})M_0(\mathbf{x})\, dv. \tag{7.3.58}$$

The last integral in (7.3.58) is the Fourier coefficient of $u(\mathbf{x})$ with respect to the eigenfunction $M_0(\mathbf{x})$, and it can be expressed as $(u(\mathbf{x}), M_0(\mathbf{x}))$. Then we obtain from (7.1.8) the solution formula

$$u(\boldsymbol{\xi}) = \iint_G \rho F\hat{K}(\mathbf{x};\boldsymbol{\xi})\, dv + \int_{\partial G} \frac{pB}{\beta}\hat{K}(\mathbf{x};\boldsymbol{\xi})\, ds + (u(\mathbf{x}), M_0(\mathbf{x}))M_0(\boldsymbol{\xi}), \tag{7.3.59}$$

where ρ, p, F, B, β are all functions of $\mathbf{x}$. The solution exists only if the compatibility condition (7.3.50) is satisfied, and even then it is determined only up to an arbitrary constant multiple of the constant eigenfunction $M_0(\mathbf{x})$ as seen from the last term in (7.3.59). Consequently, as has been noted in Section 6.8, the solution to the boundary value problem (7.3.46)–(7.3.47) is not unique. Any constant can be added to it.

Similarly, the modified Green's function $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ is determined only up to an arbitrary constant. By expressing $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ in the form (7.3.51) and (7.3.57), we have, in effect, equated the arbitrary constant to zero. As a result, the modified Green's function $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ is symmetric. The modified Green's function $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ differs from the ordinary Green's function $K(\mathbf{x};\boldsymbol{\xi})$ in that $L\hat{K}$ does not equal $\delta(\mathbf{x}-\boldsymbol{\xi})$, but is given as in (7.3.55). However, $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ does satisfy the homogeneous boundary condition (7.3.56).

There is an alternative construction of a *modified Green's function*, which we denote by $\tilde{K}(\mathbf{x};\boldsymbol{\xi})$, associated with the boundary value problem (7.3.46)–(7.3.47), for which we have

$$L\tilde{K}(\mathbf{x};\boldsymbol{\xi}) \equiv -\nabla\cdot(p(\mathbf{x})\nabla\tilde{K}(\mathbf{x};\boldsymbol{\xi})) = \delta(\mathbf{x}-\boldsymbol{\xi}), \qquad \mathbf{x},\ \boldsymbol{\xi} \in G. \tag{7.3.60}$$

An application of the divergence theorem shows that

$$-\iint_G \nabla\cdot(p\,\nabla\tilde{K})\, dv = -\int_{\partial G} p\frac{\partial\tilde{K}}{\partial n}\, ds = \iint_G \delta(\mathbf{x}-\boldsymbol{\xi})\, dv = 1. \tag{7.3.61}$$

Consequently, if we set $L\tilde{K}(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi})$, we cannot have $\partial\tilde{K}(\mathbf{x};\boldsymbol{\xi})/\partial n = 0$ on ∂G, as was the case for $\hat{K}(\mathbf{x};\boldsymbol{\xi})$. Instead, we must set $\partial\tilde{K}(\mathbf{x};\boldsymbol{\xi})/\partial n\big|_{\partial G} = b(\mathbf{x})$, where $b(\mathbf{x})$ is any function for which $\int_{\partial G} p(\mathbf{x})b(\mathbf{x})\, ds = -1$, in view of (7.3.61). [Note that $b(\mathbf{x})$ may be taken to be a constant.]

The *solution formula* for $u(\mathbf{x})$ that satisfies (7.3.46)–(7.3.47) is then obtained from the results given at the beginning of Section 7.1. With $w(\mathbf{x}) = \tilde{K}(\mathbf{x};\boldsymbol{\xi})$ we obtain in place of (7.1.7),

$$u(\mathbf{x})\frac{\partial \tilde{K}(\mathbf{x};\boldsymbol{\xi})}{\partial n} - \tilde{K}(\mathbf{x};\boldsymbol{\xi})\frac{\partial u(\mathbf{x})}{\partial n} = u(\mathbf{x})b(\mathbf{x}) - \frac{1}{\beta(\mathbf{x})}B(\mathbf{x})\tilde{K}(\mathbf{x};\boldsymbol{\xi}) \qquad (7.3.62)$$

and the solution is obtained in the form

$$u(\boldsymbol{\xi}) = \iint_G \rho F\tilde{K}(\mathbf{x};\boldsymbol{\xi})\,dv + \int_{\partial G}\frac{pB}{\beta}\tilde{K}(\mathbf{x};\boldsymbol{\xi})\,ds - \int_{\partial G} pbu(\mathbf{x})\,ds, \qquad (7.3.63)$$

where ρ, p, F, B, β, b are all functions of $\mathbf{x}$, on using (7.1.8). The last integral in (7.3.63) is an arbitrary constant since $b(\mathbf{x})$ is arbitrary.

The expansion of $\tilde{K}(\mathbf{x};\boldsymbol{\xi})$ in a series of eigenfunctions is not as straightforward as that for $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ since $\tilde{K}(\mathbf{x};\boldsymbol{\xi})$ does not satisfy a homogeneous boundary condition. Nevertheless, it can be carried out using the finite Fourier transform techniques of Section 4.6. A relation between $\hat{K}(\mathbf{x};\boldsymbol{\xi})$ and $\tilde{K}(\mathbf{x};\boldsymbol{\xi})$ can be established by way of the procedure developed in Chapter 4, whereby inhomogeneous boundary conditions can be transformed to homogeneous conditions. This is considered in the exercises.

Example 7.7. Laplace's Equation: The Modified Green's Function in a Rectangle. Given the rectangular region G defined as $0 < x < l$ and $0 < y < \hat{l}$, we construct the modified Green's function $\hat{K}(x, y; \xi, \eta)$ that satisfies the equation [(see (7.3.55)]

$$\frac{\partial^2 \hat{K}(x,y;\xi,\eta)}{\partial x^2} + \frac{\partial^2 \hat{K}(x,y;\xi,\eta)}{\partial y^2} = -\,\delta(x-\xi)\delta(y-\eta) + M_0(\boldsymbol{\xi})M_0(\mathbf{x}), \quad (7.3.64)$$

with $(x, y) \in G$ and $(\xi, \eta) \in G$, and the *Neumann condition* $\partial\hat{K}(x, y; \xi, \eta)/\partial n\big|_{\partial G} = 0$, where $\partial\hat{K}(x, y; \xi, \eta)/\partial n$ is an exterior normal derivative.

To solve for $\hat{K}(x, y; \xi, \eta)$, we must determine the eigenvalues and eigenfunctions of the problem

$$-\nabla^2 M(x,y) = \lambda M(x,y), \; (x,y)\in G, \quad \left.\frac{\partial M(x,y)}{\partial n}\right|_{\partial G} = 0. \qquad (7.3.65)$$

Using separation of variables, we set $M(x, y) = F(x)G(y)$ in (7.3.65) and obtain $M_{xx}(x, y) + M_{yy}(x, y) + \lambda M = F''(x)G(y) + F(x)G''(y) + \lambda F(x)G(y) = 0$. Dividing by $F(x)G(y)$ and separating variables gives $F''(x)/F(x) + \lambda = -G''(y)/G(y) = k^2$, where k^2 is the separation constant. Then $F(x)$ satisfies the boundary value problem

$$F''(x) + (\lambda - k^2)F(x) = 0, \; 0 < x < l, \quad F'(0) = F'(l) = 0. \qquad (7.3.66)$$

Also, $G(y)$ satisfies

$$G''(y) + k^2 G(y) = 0, \; 0 < y < \hat{l}, \quad G'(0) = G'(\hat{l}) = 0. \qquad (7.3.67)$$

The eigenvalue problems for $F(x)$ and $G(y)$ were studied in Section 4.3.

The eigenvalues and eigenfunctions for (7.3.67) are

$$k_m^2 = \left(\frac{\pi m}{\hat{l}}\right)^2, \; G_m(y) = \cos\left(\frac{\pi m y}{\hat{l}}\right), \quad m = 0, 1, 2, \ldots. \tag{7.3.68}$$

For $F(x)$ we find the eigenvalues

$$\lambda_{nm} - k_m^2 = \left(\frac{\pi n}{l}\right)^2, \qquad n = 0, 1, 2, \ldots, \; m = 0, 1, 2, \ldots, \tag{7.3.69}$$

and the eigenfunctions

$$F_n(x) = \cos\left(\frac{\pi n x}{l}\right), \qquad n = 0, 1, 2, \ldots. \tag{7.3.70}$$

Then the *eigenvalues* and the corresponding *eigenfunctions* for (7.3.65) are

$$\lambda_{nm} = \left(\frac{\pi n}{l}\right)^2 + \left(\frac{\pi m}{\hat{l}}\right)^2, \; \hat{M}_{nm}(x, y) = \cos\left(\frac{\pi n x}{l}\right)\cos\left(\frac{\pi m y}{\hat{l}}\right), \tag{7.3.71}$$

with $n, m = 0, 1, 2, \ldots$. The unnormalized eigenfunctions are denoted as $\hat{M}_{nm}(x, y)$. It may be noted that the eigenvalues λ_{nm} are nonnegative, infinite in number, and tend to infinity as n and m tend to infinity.

The inner product for the region G is given as in (7.3.35), and it is easily shown that $\{\hat{M}_{nm}(x, y)\}$ is an orthogonal set using Section 4.3. Also, the square of the norm of $\hat{M}_{nm}(x, y)$ is $||\hat{M}_{nm}(x, y)||^2 = (\hat{M}_{nm}(x, y), \hat{M}_{nm}(x, y)) = l\hat{l}/4$, $n, m = 1, 2, \ldots$. For $\hat{M}_{00}(x, y)$, $||\hat{M}_{00}(x, y)||^2 = l\hat{l}$, and for $\hat{M}_{n0}(x, y)$ and $\hat{M}_{0m}(x, y)$, $||\hat{M}_{n0}(x, y)||^2 = l\hat{l}/2 = ||\hat{M}_{0m}(x, y)||^2$, with $n, m = 1, 2, \ldots$.

Consequently, the orthonormal set of eigenfunctions is given as

$$M_{nm}(x, y) = \begin{cases} \sqrt{1/l\hat{l}}, & m = n = 0, \\ \sqrt{2/l\hat{l}}\cos(\pi n x/l), & m = 0, \; n = 1, 2, \ldots, \\ \sqrt{2/l\hat{l}}\cos\left(\pi m y/\hat{l}\right), & n = 0, \; m = 1, 2, \ldots, \\ \sqrt{4/l\hat{l}}\cos(\pi n x/l)\cos\left(\pi m y/\hat{l}\right), & n, m = 1, 2, \ldots. \end{cases} \tag{7.3.72}$$

We see that $\lambda_{00} = 0$ is an eigenvalue whose eigenfunction $M_{00}(x, y)$ is constant.

Having determined the eigenfunctions $M_{nm}(x, y)$, we expand the modified Green's function in a series of these functions as in (7.3.57). The eigenfunction $M_{00}(x, y)$ must be excluded from the series. We have

$$\hat{K}(x,y;\xi,\eta) = \frac{2}{l\hat{l}}\sum_{m=1}^{\infty}\frac{\cos\left(\pi m y/\hat{l}\right)\cos\left(\pi m \eta/\hat{l}\right)}{(\pi m/\hat{l})^2} + \frac{2}{l\hat{l}}\sum_{n=1}^{\infty}\frac{\cos(\pi n x/l)\cos(\pi n \xi/l)}{(\pi n/l)^2}$$

$$+\frac{4}{l\hat{l}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos(\pi n x/l)\cos(\pi n\xi/l)\cos\left(\pi m y/\hat{l}\right)\cos\left(\pi m\eta/\hat{l}\right)}{(\pi n/l)^2+(\pi m/\hat{l})^2}. \qquad (7.3.73)$$

The alternative procedure presented above for the construction of Green's functions in terms of eigenfunctions for lower-dimensional problems can also be used to determine modified Green's functions. Its application to the problem of Example 7.7 is considered in the exercises. We do not construct Green's functions for nonselfadjoint problems because of difficulties that arise in connection with eigenfunction expansions for nonselfadjoint eigenvalue problems.

Green's Functions for Hyperbolic PDEs

The Green's function for the *hyperbolic problem* considered in Section 7.1 is expressed as $K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ and satisfies the equation

$$\rho(\mathbf{x})\frac{\partial^2 K(\mathbf{x},t;\boldsymbol{\xi},\tau)}{\partial t^2} + LK(\mathbf{x},t;\boldsymbol{\xi},\tau) = \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau),\ \mathbf{x},\boldsymbol{\xi}\in G,\ t,\tau<T, \qquad (7.3.74)$$

with G and T defined as before. In addition, $K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ satisfies the initial conditions $K(\mathbf{x},T;\boldsymbol{\xi},\tau) = \partial K(\mathbf{x},T;\boldsymbol{\xi},\tau)/\partial t = 0$ and the boundary condition

$$\alpha(\mathbf{x})K(\mathbf{x},t;\boldsymbol{\xi},\tau) + \beta(\mathbf{x})\frac{\partial K(\mathbf{x},t;\boldsymbol{\xi},\tau)}{\partial n}\bigg|_{\partial G} = 0, \qquad t<T. \qquad (7.3.75)$$

If the region G is one-dimensional (i.e., $\mathbf{x} = x$ and $0 < x < l$), the boundary condition (7.3.75) is replaced by a one-dimensional form as in (7.1.44).

Proceeding as in Section 4.6, we expand $K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ in a series of eigenfunctions, $K(\mathbf{x},t;\boldsymbol{\xi},\tau) = \sum_{k=1}^{\infty} N_k(t)M_k(\mathbf{x})$, where $M_k(\mathbf{x})$ are the eigenfunctions of the operator L, that satisfy a boundary condition of the form (7.3.75). To determine the $N_k(t)$, (7.3.74) is multiplied by $M_k(\mathbf{x})$ and integrated over G. This yields the equations

$$N_k''(t) + \lambda_k N_k(t) = M_k(\boldsymbol{\xi})\delta(t-\tau), \qquad t,\tau<T,\ k=1,2,\ldots. \qquad (7.3.76)$$

The initial conditions for $N_k(t)$ are $N_k(T) = N_k'(T) = 0$.

On the basis of Example 7.2 we conclude that $N_k(t)$ is continuous at $t=\tau$ and that $N_k'(t)$ has a jump discontinuity $[N_k'(t)]_{t=\tau} = M_k(\boldsymbol{\xi})$. The conditions at $t=T$ imply that $N_k(t) = 0,\ \tau < t \le T$. Additionally, the continuity of $N_k(t)$ and the jump condition on $N_k'(t)$ at $t=\tau$ imply that $N_k(\tau) = 0,\ N_k'(\tau) = -M_k(\boldsymbol{\xi})$. These serve as initial conditions for $N_k(t)$ in the interval $t < \tau$. Since $\delta(t-\tau) = 0$ for $t < \tau$, $N_k(t)$ satisfies the equation

$$N_k''(t) + \lambda_k N_k(t) = 0, \qquad t < \tau. \tag{7.3.77}$$

The solution of the problem for $N_k(t)$ is easily found and can be given as

$$N_k(t) = \frac{1}{\sqrt{\lambda_k}} \sin\left[\sqrt{\lambda_k}(\tau - t)\right] M_k(\boldsymbol{\xi}) H(\tau - t), \tag{7.3.78}$$

where $H(x)$ is the Heaviside function (7.2.18). The *Green's function* $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ thus has the form

$$K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \left[\sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \sin\left[\sqrt{\lambda_k}(\tau - t)\right] M_k(\boldsymbol{\xi}) M_k(\mathbf{x})\right] H(\tau - t). \tag{7.3.79}$$

We observe that $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is symmetric in $\mathbf{x}$ and $\boldsymbol{\xi}$ but not in t and τ. As noted in the discussion following (7.1.25), the function $S(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = K(\mathbf{x}, -t; \boldsymbol{\xi}, -\tau)$ is the *causal fundamental solution* for the given problem and is given as

$$S(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \left[\sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \sin\left[\sqrt{\lambda_k}(t - \tau)\right] M_k(\boldsymbol{\xi}) M_k(\mathbf{x})\right] H(t - \tau). \tag{7.3.80}$$

It may be verified directly that (7.3.79) is a generalized solution of (7.3.74). It has been assumed that $\lambda_0 = 0$ is not an eigenvalue. If $\lambda_0 = 0$ is an eigenvalue, an additional term $(\tau - t)M_0(\boldsymbol{\xi})M_0(\mathbf{x})$ must be added to the series in (7.3.79), as shown in the exercises. Some specific examples of eigenfunction expansions of Green's functions for hyperbolic problems are also considered in the exercises.

We note that the Green's function (7.3.79) is a solution of the homogeneous version of (7.3.74) for $t < \tau$. At $t = \tau$ the Green's function (7.3.79) vanishes, and its time derivative is $-\delta(\mathbf{x} - \boldsymbol{\xi})/\rho(\boldsymbol{\xi})$. Also, the boundary condition (7.3.75) is satisfied by the Green's function for $t < \tau$. These results follow easily from our discussion. Thus, for $t < \tau$ the Green's functions determined by each of the methods given in Section 7.1 are identical.

Green's Functions for Parabolic PDEs

The Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ for the *parabolic problem* of Section 7.1 satisfies the equation

$$-\rho(\mathbf{x}) \frac{\partial K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)}{\partial t} + LK(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau), \; \mathbf{x}, \boldsymbol{\xi} \in G, \; t, \tau < T, \tag{7.3.81}$$

and the initial and boundary conditions $K(\mathbf{x}, T; \boldsymbol{\xi}, \tau) = 0$, $\alpha(\mathbf{x})K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + \beta(\mathbf{x})\partial K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)/\partial n\big|_{\partial G} = 0$, $t < T$, respectively, if G is a two- or three-dimensional region.

As was done for the hyperbolic problem, we expand $K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ in a series of eigenfunctions $K(\mathbf{x},t;\boldsymbol{\xi},\tau) = \sum_{k=1}^{\infty} N_k(t)M_k(\mathbf{x})$. Multiplying (7.3.81) by $M_k(\mathbf{x})$ and integrating over the region G, we obtain

$$-N_k'(t) + \lambda_k N_k(t) = M_k(\boldsymbol{\xi})\delta(t-\tau), \qquad t,\tau < T,\ k = 1,2,\ldots. \tag{7.3.82}$$

Since $\delta(t-\tau) = \delta(\tau - t)$, we can write (7.3.82) as

$$\frac{d}{dt}[e^{-\lambda_k t}N_k(t)] = -M_k(\boldsymbol{\xi})e^{-\lambda_k t}\delta(\tau - t) = -M_k(\boldsymbol{\xi})e^{-\lambda_k \tau}\delta(\tau - t), \tag{7.3.83}$$

where (7.2.14) was used. Integrating (7.3.83) and using the initial condition $N_k(T) = 0$, we obtain

$$N_k(t) = e^{\lambda_k(t-\tau)}M_k(\boldsymbol{\xi})H(\tau - t). \tag{7.3.84}$$

[We recall that $dH(x)/dx = \delta(x)$.] Thus the *Green's function* is

$$K(\mathbf{x},t;\boldsymbol{\xi},\tau) = \left[\sum_{k=1}^{\infty} e^{\lambda_k(t-\tau)}M_k(\boldsymbol{\xi})M_k(\mathbf{x})\right]H(\tau - t). \tag{7.3.85}$$

As noted in the discussion following (7.1.33), $S(\mathbf{x},t;\boldsymbol{\xi},\tau) = K(\mathbf{x},-t;\boldsymbol{\xi},-\tau)$ is the *causal fundamental solution* for the given problem, and has the form

$$S(\mathbf{x},t;\boldsymbol{\xi},\tau) = \left[\sum_{k=1}^{\infty} e^{\lambda_k(\tau-t)}M_k(\boldsymbol{\xi})M_k(\mathbf{x})\right]H(t - \tau). \tag{7.3.86}$$

It is not difficult to show directly (see the exercises) that the series (7.3.85) is a generalized solution of (7.3.81). If $\lambda_0 = 0$ is an eigenvalue, the series (7.3.85) must be modified as shown in the exercises. Some specific examples of eigenfunction expansions of Green's functions for parabolic problems are also considered in the exercises.

We conclude with the observation that the Green's function (7.3.85) is a solution of the homogeneous version of (7.3.81) for $t < \tau$. It takes the value $\delta(\mathbf{x} - \boldsymbol{\xi})/\rho(\boldsymbol{\xi})$ at $t = \tau$. Also, the boundary condition for the problem is satisfied by the Green's function for $t < \tau$. This follows easily from our discussion. Consequently, we find that for $t < \tau$ the Green's functions determined by each of the methods given in Section 7.1 for the parabolic case are identical.

Exercises 7.3

7.3.1. Obtain the Green's function (7.3.20) for the problem (7.3.19) and (7.3.14) directly—that is, not as a limit of the function (7.3.18). Use this Green's function to solve the problem $u''(x) = f(x),\ 0 < x < l,\ u(0) = u(l) = 0$.

7.3.2. Solve the boundary value problem: $u''(x) - c^2u(x) = -f(x),\ 0 < x < l,\ u(0) = u(l) = 0$ using the Green's function (7.3.18).