7.5 THE METHOD OF IMAGES

The *Green's function* K for any of the PDEs of Section 7.1 can be expressed in the form

$$K = K_F + K_G, \tag{7.5.1}$$

where K_F is the *free space Green's function*. We recall that the free space Green's function satisfies the same differential equation as the Green's function K. In addition, K_F satisfies (backward) causality conditions in the hyperbolic and parabolic cases, and appropriate conditions at infinity in the elliptic case. Consequently, K_G satisfies a homogeneous differential equation with homogeneous end conditions at $t = T$ if these are relevant. The boundary conditions for K_G are no longer homogeneous, however. For example, if K is required to vanish on ∂G, then $K_G = -K_F$ on ∂G. Although it may be possible to use *eigenfunction expansions* or *transform methods* to determine K_G, we do not use these approaches here since K itself can just as easily be determined in the same fashion, as we have seen in Sections 7.3 and 7.4. However, the results for K_G obtained by expansion or transform methods may be better suited for numerical evaluation than those found for K, since K_G is not singular within G.

In this section we construct Green's functions only for the equations of Section 7.1 that have constant coefficients with boundaries ∂G of a special form. We decompose the Green's function K as in (7.5.1) and use the *method of images* to determine K_G. We assume that the boundaries for the given problem are made up of (portions of) lines or planes, or (portions of) circles or spheres. For a prescribed singular point of the Dirac delta function in the equation for K, we consider all possible image points obtained by reflection through lines and planes and inversion through circles and spheres. (The inversion process is defined later.) If none of the resulting image points lies in the interior of the region in which the problem is specified and certain additional conditions are met, the Green's function K can be specified in a simple manner. We do not describe the most general regions and equations for which the method of images works. Instead, we consider a number of problems in the text and the exercises that exhibit the basic features of the method. Clearly, it is necessary to know the free space Green's functions K_F for the given equations in order to apply the method of images, and most of the relevant ones have already been determined.

Laplace's Equation in a Half-Space

We consider Laplace's equation in the half-space $z > 0$. The *Green's function* $K = K(x, y, z; \xi, \eta, \zeta)$ satisfies the equation

$$\nabla^2 K(x, y, z; \xi, \eta, \zeta) = -\delta(x - \xi)\delta(y - \eta)\delta(z - \zeta) \tag{7.5.2}$$

in the region G defined as the half-space $z > 0$. On the boundary ∂G [i.e., the (x, y)-plane $z = 0$ on which $\partial/\partial n = -\partial/\partial z$], we have

$$\alpha\, K(x, y, 0; \xi, \eta, \zeta) - \beta\, \frac{\partial K(x, y, 0; \xi, \eta, \zeta)}{\partial z} = 0. \tag{7.5.3}$$

We assume that α and β are constants and consider three cases. For the *Dirichlet problem* we set $\alpha = 1$ and $\beta = 0$; for the *Neumann problem* we put $\alpha = 0$ and $\beta = -1$; for the *Robin problem* we set $\alpha = h$ and $\beta = 1$ with $h > 0$. The Green's function $K(x, y, z; \xi, \eta, \zeta)$ is required to vanish at infinity.

We recall that the *free space Green's function* for Laplace's equation is

$$K_F(x,y,z;\xi,\eta,\zeta) = \frac{1}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{-1/2}, \tag{7.5.4}$$

as was shown in Example 6.13. It follows from our discussion in that example that $\nabla^2 K_F(x,y,z;\xi,\eta,\zeta) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$. The Green's function $K(x,y,z;\xi,\eta,\zeta)$ is then expressed as $K(x,y,z;\xi,\eta,\zeta) = K_F(x,y,z;\xi,\eta,\zeta) + K_G(x,y,z;\xi,\eta,\zeta)$. As a result, $\nabla^2 K_F(x,y,z;\xi,\eta,\zeta) + \nabla^2 K_G(x,y,z;\xi,\eta,\zeta) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$ implies that $\nabla^2 K_G(x,y,z;\xi,\eta,\zeta) = 0$, so that $K_G(x,y,z;\xi,\eta,\zeta)$ is a *harmonic function* (i.e., a solution of Laplace's equation). In view of (7.5.3), the boundary condition for $K_G(x,y,z;\xi,\eta,\zeta)$ is

$$\begin{aligned}&\alpha K_G(x,y,0;\xi,\eta,\zeta) - \beta\,\frac{\partial K_G(x,y,0;\xi,\eta,\zeta)}{\partial z}\\ &\qquad = -\alpha K_F(x,y,0;\xi,\eta,\zeta) + \beta\,\frac{\partial K_F(x,y,0;\xi,\eta,\zeta)}{\partial z}.\end{aligned} \tag{7.5.5}$$

The point (ξ,η,ζ) is the source (or singular) point for the Green's functions $K(x,y,z;\xi,\eta,\zeta)$ and $K_F(x,y,z;\xi,\eta,\zeta)$. Now $K_F(x,y,z;\xi,\eta,\zeta)$ is given in terms of the distance from the observation point (x,y,z) to the source point (ξ,η,ζ). As shown in Figure 7.4, if we introduce the image (source) point $(\xi,\eta,-\zeta)$—that is, the reflection of (ξ,η,ζ) in the plane $z = 0$—then as the observation point (x,y,z) tends to a boundary point $(x,y,0)$, its distance from (ξ,η,ζ) equals its distance from $(\xi,\eta,-\zeta)$.

Consequently, if we introduce the function

$$\hat{K}_G(x,y,z;\xi,\eta,\zeta) = \frac{1}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2\right]^{-1/2}, \tag{7.5.6}$$

we have in effect a free space Green's function that corresponds to the image source point $(\xi,\eta,-\zeta)$. At the boundary $z = 0$, $\hat{K}_G(x,y,0;\xi,\eta,\zeta) = \frac{1}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{-1/2} = K_F(x,y,0;\xi,\eta,\zeta)$, $\partial\hat{K}_G(x,y,0;\ \xi,\eta,\zeta)/\partial z = -\zeta/(4\pi[(x-\xi)^2 + (y-\eta)^2 + \zeta^2]^{3/2}) = -\partial K_F(x,y,0;\xi,\eta,\zeta)/\partial z$.

Furthermore, $\nabla^2\hat{K}_G(x,y,z;\xi,\eta,\zeta) = -\delta(x-\xi)\delta(y-\eta)\delta(z+\zeta)$, and the right-hand side vanishes in the half-space $z > 0$ since $\delta(z+\zeta) = 0$ there. Therefore, if we set $K_G(x,y,z;\xi,\eta,\zeta) = -\hat{K}_G(x,y,z;\xi,\eta,\zeta)$ for the *Dirichlet problem* and $K_G(x,y,z;\xi,\eta,\zeta) = \hat{K}_G(x,y,z;\xi,\eta,\zeta)$ for the *Neumann problem*, the Green's function $K(x,y,z;\xi,\eta,\zeta) = K_F(x,y,z;\xi,\eta,\zeta) + K_G(x,y,z;\xi,\eta,\zeta)$ satisfies (7.5.2) in the half-space $z > 0$ as well as the boundary conditions $K(x,y,0;\xi,\eta,\zeta) = 0$ and $\partial K(x,y,0;\xi,\eta,\zeta)/\partial z = 0$, which are appropriate for the Dirichlet and Neumann problems, respectively.

The Green's function for the *Robin problem* cannot be obtained solely in terms of an image source point at $(\xi,\eta,-\zeta)$. Instead, we must introduce an entire line of

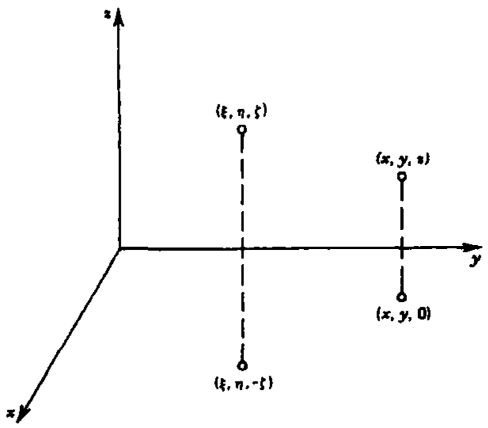

Figure 7.4 The source point (ξ, η, ζ) and its image point.

image sources on the line $x = \xi$ and $y = \eta$ with z extending from $z = -\zeta$ to $z = -\infty$ and a source density function to be determined. Let

$$K_G(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2\right]^{-1/2} + \frac{1}{4\pi}\int_{-\infty}^{-\zeta} \frac{\rho(s)}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-s)^2\right]^{1/2}}\, ds, \tag{7.5.7}$$

where $\rho(s)$ is the source density. When we put $h = 0$ in the boundary condition $\partial K(x, y, 0; \xi, \eta, \zeta)/\partial z - hK(x, y, 0; \xi, \eta, \zeta) = 0$, we expect (7.5.7) to reduce to the form appropriate for the Neumann problem, so we must have $\rho(s) = 0$ when $h = 0$. For this reason we have added the free space Green's function corresponding to an image source at $(\xi, \eta, -\zeta)$ to the integral term in (7.5.7).

It is assumed that $\rho(s)$ decays sufficiently rapidly at infinity that the integral in (7.5.7) converges and that differentiation under the integral sign is permitted. We then find that $\nabla^2 K_G(x, y, z; \xi, \eta, \zeta) = 0$ for $z > 0$, since all the singular points in (7.5.7) occur in the lower half-space $z < 0$. Applying the boundary condition at $z = 0$ gives, in view of (7.5.5),

$$\begin{aligned}
\left.\frac{\partial K_G}{\partial z} - hK_G\right|_{z=0} &= -\frac{\zeta}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{-3/2} \\
&\quad - \frac{1}{4\pi}\int_{-\infty}^{-\zeta} \rho(s)\frac{\partial}{\partial s}\left[(x-\xi)^2 + (y-\eta)^2 + s^2\right]^{-1/2} ds \\
&\quad - \frac{h}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{-1/2} - \frac{h}{4\pi}\int_{-\infty}^{-\zeta} \frac{\rho(s)\, ds}{\left[(x-\xi)^2 + (y-\eta)^2 + s^2\right]^{1/2}} \\
= \left.-\frac{\partial K_F}{\partial z} + hK_F\right|_{z=0} &= -\frac{\zeta}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{-3/2} \\
&\quad + \frac{h}{4\pi}\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{-1/2}.
\end{aligned} \tag{7.5.8}$$

The operator $\partial/\partial z$ has the same effect as $-\partial/\partial s$ at $z = 0$, and the use of $\partial/\partial s$ in the integral term enables us to integrate by parts. We have

$$\int_{-\infty}^{-\zeta} \rho(s)\frac{\partial}{\partial s}\left[(x-\xi)^2+(y-\eta)^2+s^2\right]^{-1/2} ds \tag{7.5.9}$$

$$= \rho(-\zeta)\left[(x-\xi)^2+(y-\eta)^2+\zeta^2\right]^{-1/2} - \int_{-\infty}^{-\zeta}\frac{\rho'(s)\,ds}{[(x-\xi)^2+(y-\eta)^2+s^2]^{1/2}}.$$

Combining results gives

$$\left[-\frac{h}{2\pi}-\frac{\rho(-\zeta)}{4\pi}\right]\left[(x-\xi)^2+(y-\eta)^2+\zeta^2\right]^{-1/2}$$

$$+\frac{1}{4\pi}\int_{-\infty}^{-\zeta}\frac{\rho'(s)-h\rho(s)}{[(x-\xi)^2+(y-\eta)^2+s^2]^{1/2}}\,ds = 0. \tag{7.5.10}$$

Therefore, the boundary condition is satisfied if we set

$$\rho'(s) - h\rho(s) = 0,\ s < -\zeta, \quad \rho(-\zeta) = -2h. \tag{7.5.11}$$

The solution of the initial value problem (7.5.11) is $\rho(s) = -2he^{h(s+\zeta)}$. We note that $\rho(s)$ vanishes for $h = 0$ and that it decays exponentially as $s \to -\infty$.

The Green's function for the *third boundary value problem* thus has the form

$$K(x,y,z;\xi,\eta,\zeta) = \frac{1}{4\pi}\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{-1/2}$$

$$+\frac{1}{4\pi}\left[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2\right]^{-1/2}$$

$$-\frac{h}{2\pi}\int_{-\infty}^{-\zeta}\frac{e^{h(s+\zeta)}}{[(x-\xi)^2+(y-\eta)^2+(z-s)^2]^{1/2}}\,ds. \tag{7.5.12}$$

With $h = 0$ this reduces to Green's function for the *Neumann problem.* The Green's function for the *Dirichlet problem* is

$$K(x,y,z;\xi,\eta,\zeta) = \frac{1}{4\pi}\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{-1/2}$$

$$-\frac{1}{4\pi}\left[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2\right]^{-1/2}. \tag{7.5.13}$$

On using the free space Green's function for Laplace's equation in two dimensions, it is easy to obtain the Green's function for the half-plane problem in two dimensions. Furthermore, on using the formulas given in Section 7.1, one can readily obtain the solutions of boundary value problems in the half-space or the half-plane for Laplace's equation, as shown in the exercises.

Hyperbolic Equations in a Semi-Infinite Interval

The free space Green's functions appropriate for the one-dimensional hyperbolic equation (7.1.45) with constant coefficients were obtained in Section 7.4. For the *Klein-Gordon equation*, the free space Green's function $K_F(x,t;\xi,\tau)$ satisfies (7.4.33) and is given as

$$K_F = \frac{1}{2\gamma} J_0\left[\frac{c}{\gamma}\sqrt{\gamma^2(t-\tau)^2-(x-\xi)^2}\right] H[x-\xi-\gamma(t-\tau)]H[\xi-x+\gamma(\tau-t)], \tag{7.5.14}$$

in view of (7.4.35). For the *modified telegrapher's equation* (7.4.36), we have

$$K_F = \frac{1}{2\gamma} I_0\left[\frac{\hat{c}}{\gamma}\sqrt{\gamma^2(t-\tau)^2-(x-\xi)^2}\right] H[x-\xi-\gamma(t-\tau)]H[\xi-x+\gamma(\tau-t)]. \tag{7.5.15}$$

The free space Green's function for the *wave equation* (7.4.27) is

$$K_F(x,t;\xi,\tau) = \frac{1}{2c} H[x-\xi-c(t-\tau)]H[\xi-x+c(\tau-t)]. \tag{7.5.16}$$

For the wave equation, $K_F(x,t;\xi,\tau)$ can also be expressed as

$$K_F(x,t;\xi,\tau) = \frac{1}{2c} H(\tau-t)H[c^2(t-\tau)^2-(x-\xi)^2]. \tag{7.5.17}$$

It is readily seen that (7.5.17) is consistent with (7.4.28) and that the product of the Heaviside functions given in (7.5.14)–(7.5.15), can also be expressed in the form (7.5.17) with appropriately modified constants. In connection with the application of the method of images, it is apparent from the form of $K_F(x,t;\xi,\tau)$ in (7.5.17) that an image source can be introduced at $x=-\xi$ if the Green's function in the interval $x>0$ is to be obtained. It can be shown by direct substitution that (7.5.17) is a solution of (7.4.27).

For each of the hyperbolic equations considered we now obtain *Green's functions* for the semi-infinite interval $x>0$. In the case of *Dirichlet boundary conditions* at $x=0$ [i.e., $K(0,t;\xi,\tau)=0$], the Green's function is given as

$$K(x,t;\xi,\tau) = K_F(x,t;\xi,\tau) - K_F(x,t;-\xi,\tau), \tag{7.5.18}$$

where $K_F(x,t;\xi,\tau)$ is the appropriate free space Green's function (7.5.14), (7.5.15), or (7.5.16). If a *Neumann boundary condition* is given at $x=0$ [i.e., $\partial K(0,t;\xi,\tau)/\partial x = 0$], the Green's function is

$$K(x,t;\xi,\tau) = K_F(x,t;\xi,\tau) + K_F(x,t;-\xi,\tau). \tag{7.5.19}$$

Finally, if a *Robin boundary condition* is assigned (i.e., $\partial K(0,t;\xi,\tau)/\partial x - hK(0,t;\xi,\tau) = 0$ with $h>0$), the Green's function is

$$K(x,t;\xi,\tau) = K_F(x,t;\xi,\tau) + K_F(x,t;-\xi,\tau) - 2h\int_{-\infty}^{-\zeta} e^{h(s+\xi)} K_F(x,t;s,\tau)\,ds. \tag{7.5.20}$$

By comparison with the methods used in the preceding example or by direct verification it can be determined that (7.5.18)–(7.5.20) give the required Green's functions for each of the boundary value problems considered.

Heat Equation in a Finite Interval

We construct the Green's function $K(x,t;\xi,\tau)$ for the equation of heat conduction in a finite interval $0 < x < l$. Thus $K(x,t;\xi,\tau)$ satisfies the equation

$$\frac{\partial K(x,t;\xi,\tau)}{\partial t} + c^2 \frac{\partial^2 K(x,t;\xi,\tau)}{\partial x^2} = -\delta(x-\xi)\delta(t-\tau),\ 0 < x,\xi < l,\ t,\tau < T. \tag{7.5.21}$$

It is assumed that $K(x,t;\xi,\tau)$ vanishes at the endpoints so that $K(0,t;\xi,\tau) = K(l,t;\xi,\tau) = 0$. In addition, we have the end condition $K(x,T;\xi,\tau) = 0$.

The free space Green's function $K_F(x,t;\xi,\tau)$ for the one-dimensional heat equation was found in Section 7.4 to be [see (7.4.6)]

$$K_F(x,t;\xi,\tau) = \frac{H(\tau - t)}{\sqrt{4\pi c^2(\tau - t)}} \exp\left[-\frac{(x-\xi)^2}{4c^2(\tau - t)}\right]. \tag{7.5.22}$$

We express $K(x,t;\xi,\tau)$ in the form (7.5.1) [i.e., $K(x,t;\xi,\tau) = K_F(x,t;\xi,\tau) + K_G(x,t;\xi,\tau)$] and use the method of images to specify $K_G(x,t;\xi,\tau)$. The source point $x = \xi$ must have an image with respect to $x = 0$ and $x = l$, and each of the image sources, in turn, must also have images with respect to $x = 0$ and $x = l$. Consequently, we are led to consider an infinite sequence of source points $\xi_n = \pm\xi \pm 2nl$, with $n = 0, 1, 2, 3, \ldots$. Some of these points are shown in Figure 7.5.

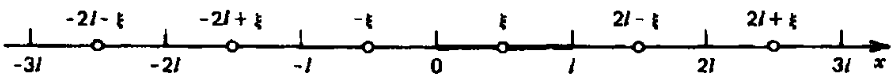

Figure 7.5 The source point and the image sources.

The Green's function $K = K(x,t;\xi,\tau)$ can then be written as an infinite series

$$K = \frac{H(\tau - t)}{\sqrt{4\pi c^2(\tau - t)}} \sum_{n=-\infty}^{\infty} \left[\exp\left[-\frac{(x-\xi-2nl)^2}{4c^2(\tau - t)}\right] - \exp\left[-\frac{(x+\xi-2nl)^2}{4c^2(\tau - t)}\right]\right]. \tag{7.5.23}$$

The term with $n = 0$ and a positive coefficient corresponds to $K_F(x,t;\xi,\tau)$. Clearly, $K(x,T;\xi,\tau) = 0$ since $\tau < T$. It can be shown that the series can be differentiated term by term. Inasmuch as each of the functions in the series except the term that corresponds to $K_F(x,t;\xi,\tau)$ has it source point ξ_n outside the interval $0 < x < l$, we see that (7.5.23) satisfies (7.5.2). Also, it is not difficult to see that at $x = 0$ and $x = l$ there corresponds to each term in the series with a positive coefficient an identical term with a negative coefficient. For example, Figure 7.5 shows that $x = \xi$ and

$x = -\xi$ are images with respect to $x = 0$ and that $x = \xi$ and $x = 2l - \xi$ are images with respect to $x = l$. The terms in the series (7.5.23) that correspond to the points $x = \xi$ and $x = -\xi$ are $\exp[-(x-\xi)^2/4c^2(\tau - t)] - \exp[-(x+\xi)^2/4c^2(\tau - t)]$, and this difference vanishes at $x = 0$. Also, for $x = \xi$ and $x = 2l - \xi$ we have $\exp[-(x-\xi)^2/4c^2(\tau - t)] - \exp[-(x+\xi-2l)^2/4c^2(\tau - t)]$, and this difference vanishes when $x = l$. In fact, if we replace n by $-n$ in the second series, it becomes identical to the first series at $x = 0$ and their difference equals zero. At $x = l$ we replace n by $-n+1$ in the second series and find that the two series are equal.

The solution of the initial and boundary value problem for the heat equation in a finite interval when the formula (7.1.35) of Section 7.1 is used has a form similar to that given in Example 5.12. As was shown there, this result is expected to be useful for small values of t, whereas that given by the finite Fourier transform method is more useful for t large.

Green's Function for Laplace's Equation in a Sphere

As shown above, the method of images can be applied to equations with constant coefficients of all three types if there are linear or planar boundaries. For Laplace's equation it is possible to extend the image method to problems that involve circular or spherical boundaries as demonstrated below.

We construct the Green's function for the *Dirichlet problem* for *Laplace's equation* in the interior of a sphere using inversion with respect to the sphere. The Green's function $K = K(x, y, z; \xi, \eta, \zeta)$ satisfies the equation

$$\nabla^2 K(x, y, z; \xi, \eta, \zeta) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \tag{7.5.24}$$

and the boundary condition $K(x, y, z; \xi, \eta, \zeta) = 0$ on the sphere of radius a with center at the origin. Let the observation point P be denoted by $P = (x, y, z)$. The source point is $P_0 = (\xi, \eta, \zeta)$ and the origin of coordinates (i.e., the center of the sphere) is $O = (0, 0, 0)$.

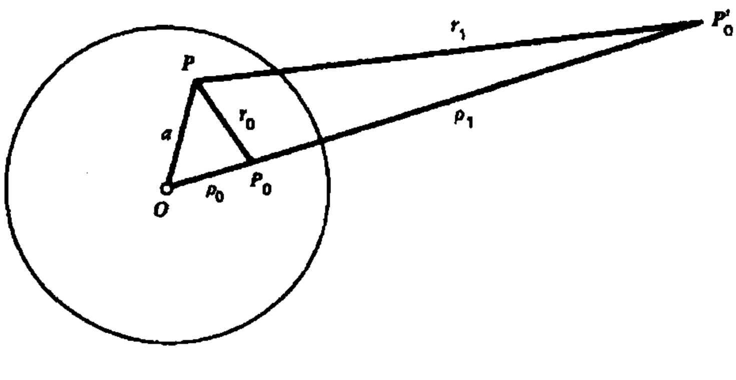

Figure 7.6 Inversion with respect to the sphere.

As shown in Figure 7.6, we introduce the image source point $P_0' = (\xi', \eta', \zeta')$. The point P_0' lies on the radial line extending from the origin O through the source point P_0. Its distance from the origin equals ρ_1, and the distance of P_0 from the origin is ρ_0. These distances are related by $\rho_0\rho_1 = a^2$, where a is the radius of the sphere. The foregoing process of obtaining P_0' from P_0 is called *inversion with respect to the sphere*. We note that if $a = 1$, then $\rho_1 = 1/\rho_0$, so that the distances ρ_0 and ρ_1 are inverse to one another. The observation point P is assumed to lie on the sphere in Figure 7.6.

The triangles ΔOPP_0 and $\Delta OPP_0'$ in the figure are similar, since they have a common angle $\angle POP_0$ and proportional sides $\overline{OP_0}/\overline{OP} = \overline{OP}/\overline{OP_0'}$. This follows since $\overline{OP} = a$ (the radius of the sphere), $\overline{OP_0} = \rho_0$, $\overline{OP_0'} = \rho_1$, and $\overline{OP_0}\,\overline{OP_0'} = \rho_0\rho_1 = \overline{OP}^{\,2} = a^2$ in view of the above. The similarity of the triangles implies that all three sides are proportional and we have

$$\frac{\rho_0}{a} = \frac{a}{\rho_1} = \frac{r_0}{r_1}, \tag{7.5.25}$$

where $r_0 = \overline{PP_0}$ and $r_1 = \overline{PP_0'}$.

To complete the solution of the problem we set $K = K_F + K_G$, where K_F is the free space Green's function (7.5.4) and K_G is a constant multiple of the free space Green's function with source point at $P_0' = (\xi', \eta', \zeta')$. That is,

$$K(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\left(\frac{1}{r_0} + \frac{c}{r_1}\right), \tag{7.5.26}$$

where $r_0^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ and $r_1^2 = (x-\xi')^2 + (y-\eta')^2 + (z-\zeta')^2$, with the constant c to be specified. Since P_0' lies outside the sphere, the second term in (7.5.26) is a solution of Laplace's equation within the sphere. Thus (7.5.26) is a solution of (7.5.24). On the sphere [i.e., when $P = (x, y, z)$ lies on the sphere] we have, in the notation of Figure 7.6,

$$K(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\left(\frac{1}{r_0} + \frac{c}{r_1}\right) = \frac{1}{4\pi r_0}\left(1 + \frac{\rho_0 c}{a}\right), \tag{7.5.27}$$

in view of (7.5.26). Thus if $c = -a/\rho_0$, the Green's function K vanishes on the sphere. Therefore, we obtain the Green's function

$$K(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\left(\frac{1}{r_0} - \frac{a}{\rho_0 r_1}\right). \tag{7.5.28}$$

Let G represent the interior of the sphere $x^2 + y^2 + z^2 = a^2$ with ∂G representing the sphere itself. The Dirichlet boundary value problem for *Poisson's equation*,

$$\nabla^2 u(x, y, z) = -F(x, y, z), \qquad (x, y, z) \in G, \tag{7.5.29}$$

with the boundary condition

$$u(x, y, z)|_{\partial G} = B(x, y, z), \tag{7.5.30}$$

has the solution, with $K = K(x,y,z;\xi,\eta,\zeta)$,

$$u(\xi,\eta,\zeta) = \iint_G KF(x,y,z)\,dv - \int_{\partial G} \frac{\partial K}{\partial n} B(x,y,z)\,ds, \tag{7.5.31}$$

where (ξ,η,ζ) as an arbitrary point within the sphere, as follows from (7.1.8).

To determine (7.5.31) more explicitly it is necessary to evaluate the exterior normal derivative $\partial K(x,y,z;\xi,\eta,\zeta)/\partial n$ on the surface of the sphere. Now

$$\frac{\partial K}{\partial n} = \frac{1}{4\pi}\left[\frac{\partial}{\partial r_0}\left(\frac{1}{r_0}\right)\frac{\partial r_0}{\partial n} - \frac{a}{\rho_0}\frac{\partial}{\partial r_1}\left(\frac{1}{r_1}\right)\frac{\partial r_1}{\partial n}\right] = \frac{1}{4\pi}\left[-\frac{1}{r_0^2}\frac{\partial r_0}{\partial n} + \frac{a}{\rho_0 r_1^2}\frac{\partial r_1}{\partial n}\right]. \tag{7.5.32}$$

We note that $r_0^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ and that $r_1^2 = (x-\xi')^2 + (y-\eta')^2 + (z-\zeta')^2$, with (x,y,z) given as a point on the sphere in the present discussion. On introducing coordinate systems with the origins at (ξ,η,ζ) and (ξ',η',ζ'), we immediately conclude that $\partial r_0/\partial n = \cos(\theta_0)$, $\partial r_1/\partial n = \cos(\theta_1)$. The angle θ_0 is the angle between the exterior unit normal vector to the sphere at the point $P = (x,y,z)$ and the vector extending from the source point $P_0 = (\xi,\eta,\zeta)$ to P. Similarly, θ_1 is the angle between the normal vector at P and the vector from $P_0' = (\xi',\eta',\zeta')$ to P. Referring to Figure 7.6 and using the law of cosines, we obtain $\cos(\theta_0) = (a^2 + r_0^2 - \rho_0^2)/2ar_0$, $\cos(\theta_1) = (a^2 + r_1^2 - \rho_1^2)/2ar_1$. Using (7.5.25) to replace ρ_1 and r_1 by ρ_0 and r_0 in $\cos(\theta_1)$ gives $\cos(\theta_1) = (\rho_0^2 + r_0^2 - a^2)/2\rho_0 r_0$. Combining these results we conclude that $\partial K/\partial n|_{\partial G} = (\rho_0^2 - a^2)/4\pi a r_0^3$. Thus the solution formula (7.5.31) reduces to

$$u(\xi,\eta,\zeta) = \frac{1}{4\pi}\iint_G \left(\frac{1}{r_0} - \frac{a}{\rho_0 r_1}\right) F\,dv + \frac{1}{4\pi a}\int_{\partial G} \frac{a^2 - \rho_0^2}{r_0^3} B\,ds, \tag{7.5.33}$$

with $F = F(x,y,z)$ and $B = B(x,y,z)$. By transforming to spherical coordinates with center at the origin, the second integral in (7.5.33) can be expressed as, with $\gamma = \angle POP_0$,

$$\frac{1}{4\pi a}\int_{\partial G} \frac{a^2 - \rho_0^2}{r_0^3} B\,ds = \frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(a^2 - \rho_0^2)B(a,\theta,\phi)\sin(\phi)}{[a^2 - 2a\rho_0\cos(\gamma) + \rho_0^2]^{3/2}}\,d\phi\,d\theta. \tag{7.5.34}$$

This expression is known as *Poisson's integral* for the sphere and can be obtained by using separation of variables for Laplace's equation in the sphere. It represents the solution of the Dirichlet problem for Laplace's equation, that is, (7.5.29)–(7.5.30) with $F(x,y,z) = 0$.

If we put $F(x,y,z) = 0$ in (7.5.33) and evaluate $u(\xi,\eta,\zeta)$ at the origin, we obtain

$$u(0,0,0) = \frac{1}{4\pi a^2}\int_{\partial G} B(x,y,z)\,ds, \tag{7.5.35}$$

since $\rho_0 = 0$ and $r_0 = a$ in that case. Now $4\pi a^2$ equals the surface area of the sphere and $B(x, y, z)$ is the value of $u(x, y, z)$ on the surface of the sphere. Thus $u(0, 0, 0)$ evaluated at the center of the sphere equals the average of its values on the surface of the sphere. This *mean value property* is valid for *harmonic functions* (i.e., solutions of Laplace's equation) in both two and three dimensions. It also arises in discrete formulations of Laplace's equation, as was seen in Section 1.3. This property can be derived in a more direct manner using Green's theorem as shown in the exercises.

The method of inversion that we have used to obtain Green's function for the Dirichlet problem does not work for the case of Neumann's problem or the third boundary value problem for the sphere. Similarly, it cannot be applied to other elliptic equations or, for that matter, to hyperbolic or parabolic equations. As an example, we consider the *reduced wave equation* in three dimensions,

$$\nabla^2 u(x, y, z) + k^2 u(x, y, z) = 0, \tag{7.5.36}$$

in a sphere and try to use inversion to find the Green's function for the Dirichlet problem. As shown in Example 6.13, the free space Green's function for this problem is

$$K_F(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi r}\, e^{ikr}, \tag{7.5.37}$$

with $r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$. This function satisfies the (three-dimensional) *radiation condition of Sommerfeld* $\lim_{r\to\infty} r\,[\partial K_F/\partial r - ikK_F] = 0$. The Green's function K is sought in the form

$$K = \frac{1}{4\pi}\left(\frac{1}{r_0}e^{ikr_0} + \frac{c}{r_1}e^{ikr_1}\right), \tag{7.5.38}$$

with r_0 and r_1 defined as above and the constant c to be determined. Clearly, c must have the same value given in the foregoing, (i.e., $c = -a/\rho_0$). However, on the sphere we have $r_1 = (a/\rho_0)r_0$, so that the exponentials have different arguments and no cancellation occurs. It is not possible to replace e^{ikr_1} by $e^{ik(\rho_0/a)r_1}$, in which case both exponentials would be equal on the sphere, since $(1/r_1)\exp[ik(\rho_0/a)r_1]$ is not a solution of the reduced wave equation.

There are other analytic techniques for obtaining *Green's functions* that we have not discussed. For example, *conformal mapping methods* are extremely useful in obtaining Green's functions for Laplace's equation in two dimensions. They require a knowledge of the theory of complex variables for their application. In addition, it is often possible to use known Green's functions to convert a differential equation to an *integral equation* and thereby construct other Green's functions. Again, it is rarely a simple matter to solve the integral equations that result, and most often this must be done approximately. Finally, there exist *perturbation* and *asymptotic methods* for finding certain Green's functions. An indication of how to use such methods is given in Chapters 9 and 10, which are devoted to approximation methods.

Exercises 7.5

7.5.1. Use the Green's functions obtained in the subsection on Laplace's equation in a half-space and appropriate forms of the solution formulas obtained in Section 7.1 to construct the solution of Laplace's equation $\nabla^2 u(x,y,z) = 0$ in the half-space $z > 0$ if the following boundary conditions are given: (a) $u(x,y,0) = f(x,y)$, $-\infty < x, y < \infty$; (b) $\partial u(x,y,0)/\partial z = f(x,y)$, $-\infty < x, y < \infty$; (c) $\partial u(x,y,0)/\partial z - hu(x,y,0) = f(x,y)$, $-\infty < x, y < \infty$.

7.5.2. Use the method of images to obtain the Green's function associated with the Dirichlet problem for $\nabla^2 u(x,y) = 0$ in the half-plane $y > 0$, in the form $K(x,y;\xi,\eta) = -(1/2\pi)\log(\rho/\hat{\rho})$, $0 < y, \eta < \infty$, $-\infty < x, \xi < \infty$, where $\rho^2 = (x-\xi)^2 + (y-\eta)^2$ and $\hat{\rho}^2 = (x-\xi)^2 + (y+\eta)^2$. Use this Green's function to obtain the solution (5.2.58) of the boundary value problem (5.2.50)–(5.2.51).

7.5.3. Apply the method of images to obtain the Green's function associated with the Neumann problem for $\nabla^2 u(x,y) = 0$, in the form $K(x,y;\xi,\eta) = -(1/2\pi)\log(\rho\hat{\rho})$, $0 < y, \eta < \infty$, $-\infty < x, \xi < \infty$, where ρ and $\hat{\rho}$ are defined as in Exercise 7.5.2. Use this Green's function to obtain the solution (5.2.66) of Laplace's equation with the boundary condition (5.2.60).

7.5.4. Obtain the Green's function that corresponds to (7.5.12) in the two-dimensional case.

7.5.5. Use the method of images to obtain the Green's function for Laplace's equation in a composite medium in three dimensions. That is, determine $K(x,y,z;\xi,\eta,\zeta)$ that satisfies $p(x,y,z)\nabla^2 K(x,y,z,\xi,\eta,\zeta) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$, $-\infty < x, \xi.y, \eta, z, \zeta < \infty$, with $z \neq 0$, the condition $K(x,y,z,\xi,\eta,\zeta) \to 0$ as $|z| \to \infty$, as well as the jump conditions at $z = 0$, $K(x,y,z,\xi,\eta,\zeta)$ and $p(x,y,z)$ $\partial K(x,y,z,\xi,\eta,\zeta)/\partial z$ continuous at $z = 0$, where $p(x,y,z) = p_1$ for $z < 0$, $p(x,y,z) = p_2$ for $z > 0$, and p_1 and p_2 are constants. *Hint*: Let $K(x,y,z,\xi,\eta,\zeta) = K_1(x,y,z,\xi,\eta,\zeta)$ for $z < 0$ and $K(x,y,z,\xi,\eta,\zeta) = K_2(x,y,z,\xi,\eta,\zeta)$ for $z > 0$. Since the source point lies in $z > 0$ use the free space Green's function and its image for $z > 0$ and the free space Green's function with a source at (ξ,η,ζ) in $z < 0$. Then apply the matching conditions.

7.5.6. Use the approach that led to (7.5.23) to construct a Green's function for Laplace's equation $\nabla^2 u(x,y,z) = 0$ in the region $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < \infty$ for the case of Dirichlet boundary conditions.

7.5.7. Apply the method of images to construct the Green's function for the Helmholtz equation $\nabla^2 u(x,y,z) + k^2 u(x,y,z) = 0$ in the half-space $z > 0$, in the case where $K(x,y,z;\xi,\eta,\zeta)$ satisfies one of the following boundary conditions: (a) $K(x,y,0;\xi,\eta,\zeta) = 0$; (b) $\partial K(x,y,0;\xi,\eta,\zeta)/\partial z = 0$; (c) $\partial K(x,y,0;\xi,\eta,\zeta)/\partial z - hK(x,y,0;\xi,\eta,\zeta) = 0$, with $-\infty < x, y, \xi, \eta < \infty$, and where $K(x,y,z;\xi,\eta,\zeta)$ satisfies the radiation condition at infinity in all cases. *Hint*: Use Example 6.13.

7.5.8. Construct the two-dimensional forms of the Green's functions obtained in Exercise 7.5.7 for the Helmholtz equation $\nabla^2 u(x,y) + k^2 u(x,y) = 0$ in the half-plane $y > 0$.