UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 5

Group 22

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Exercise 5. 1

L^* is the same as L since $a_1 = a_0 = 0$.

$$L^* = \frac{d^2}{dx^2}$$

Green's formula thus becomes

$$\int_0^1 (vLu - uL^*v) = \int_0^1 (vu'' - uv'') = v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0)$$

The set M consists of all functions u s.t.

$$u(0) = 0$$

Apply these constraints, the right hand side simplifies to

$$v(1)u'(1) - u(1)v'(1) - v(0)u'(0)$$

where $u'(1), u(1), u(0)$ are arbitrary. The adjoint boundary functionals can then be expressed as

$$\begin{cases} B_1^* v = v(1) = 0 \\ B_2^* v = v'(1) = 0 \\ B_3^* v = v(0) = 0 \end{cases}$$

Exercise 5. 2

i).

$g(x;\xi)$ should satisfy

$$\begin{cases} Lg(x;\xi) = \delta(x-\xi) \\ g(0) = g'''(0) = g(1) = g''(1) = 0 \end{cases}$$

The solution is in the form of

$$g(x;\xi) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + ax^3 + bx^2 + cx + d$$

where a, b, c, d are real numbers. Plug in the conditions, it will yield to

$$\begin{cases} u(0) = 0 & \Rightarrow \quad d = 0 \\ u'''(0) = 0 & \Rightarrow \quad a = 0 \\ u(1) = 0 & \Rightarrow \quad \dfrac{(1-\xi)^3}{6} + b + c = 0 \\ u''(1) = 0 & \Rightarrow \quad 1 - \xi + 2b = 0 \end{cases}$$

So

$$g(x;\xi) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + \frac{\xi - 1}{2}x^2 + \frac{\xi^3 - 3\xi^2 + 2}{6}x$$

ii).

Through *Integral by parts*

$$\begin{aligned}\int u^{(4)}v &= u^{(3)}v - \int u^{(3)}v' \\ &= u^{(3)}v - u^{(2)}v' + \int u^{(2)}v^{(2)} \\ &= u^{(3)}v - u^{(2)}v' + u'v^{(2)} - \int u'v^{(3)} \\ &= u^{(3)}v - u^{(2)}v' + u'v^{(2)} - uv^{(3)} + \int uv^{(4)}\end{aligned}$$

Since we have

$$L^* = L = \frac{d^4}{dx^4}$$

So Greens' formula is

$$\int vLu - uL^*v = u^{(3)}v - u^{(2)}v' + u'v^{(2)} - uv^{(3)}$$

Plug in the boundaries 0 and 1,

$$\begin{aligned}\int_0^1 vLu - uL^*v = \quad & u^{(3)}(0)v(0) - u^{(2)}(0)v'(0) + u'(0)v^{(2)}(0) - u(0)v^{(3)}(0) \\ & - \Big(u^{(3)}(1)v(1) - u^{(2)}(1)v'(1) + u'(1)v^{(2)}(1) - u(1)v^{(3)}(1)\Big)\end{aligned}$$

With boundary conditions

$$B_1u = u(0), \quad B_2u = u'''(0), \quad B_3 = u(1), \quad B_4 = u''(1)$$

The RHS of green's formula then becomes

$$-u^{(2)}(0)v'(0) + u'(0)v^{(2)}(0) - u^{(3)}(1)v(1) - u'(1)v^{(2)}(1)$$

which is independent of u. So the boundary conditions are

$$\begin{cases} B_1^*v = v'(0) = 0 \\ B_2^*v = v^{(2)}(0) = 0 \\ B_3^*v = v(1) = 0 \\ B_4^*v = v^{(2)}(1) = 0 \end{cases}$$

With the same strategy, we calculate $v(x) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + ax^3 + bx^2 + cx + d$

$$\begin{cases} v'(0) = 0 & \Rightarrow \quad c = 0 \\ v''(0) = 0 & \Rightarrow \quad b = 0 \\ v(1) = 0 & \Rightarrow \quad \dfrac{(1-\xi)^3}{6} + a + d = 0 \\ v''(1) = 0 & \Rightarrow \quad 1 - \xi + 6a = 0 \end{cases}$$

So the solution is given as

$$g^*(x;\xi) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + \frac{\xi - 1}{6}x^3 + \frac{\xi^3 - 3\xi^2 + 2\xi}{6}$$

iii).

It is always true for adjoint Green function,

$$g^*(x,\xi) = g(\xi, x)$$

If we want

$$g(x,\xi) = g(\xi, x)$$

This means $g = g^*$. However, from our previous calculation, $g \neq g^*$. which proves it's impossible for

$$g(x,\xi) = g(\xi, x)$$

$\square$