Vv557 Methods of Applied Mathematics II

Green Functions for Partial Differential Equations

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Office Hours, Email, Canvas

- Please read the Course Profile, which has been uploaded to the Resources section on the Canvas course site.
- My office is Room 441c in in the UM-SJTU JI Building.
- My email is horst@sjtu.edu.cn and I'll try to answer email queries within 24 hours.
- Office hours will be announced on Canvas.
- Please also make use of the Discussion tab on Canvas for asking questions, making comments or giving feedback on the course.

Coursework

- There will be weekly coursework throughout the term.
- You will be randomly assigned into assignment groups of three students; you are expected to collaborate within each group and hand in a single, common solution paper to each coursework.
- Each student must achieve 60% of the total coursework points by the end of the term in order to obtain a passing grade for the course. However, the assignment points have no effect on the course grade.
- Each member of an assignment group will receive the same number of points for each submission. However, there will be an opportunity for team members to anonymously evaluate each others' contributions to the assignments. In cases where one or more group members consistently do not contribute a commensurate share of the work, individual group members may lose some or all of their marks.

LaTeX Policy

As engineers, you should be familiar yourselves with a mathematical typesetting program called LaTeX. This is open-source software, and there are various implementations available. I suggest that you use Baidu or Google to find a suitable implementation for your computer and OS.

Written coursework will be required to be submitted as a typed LaTeX manuscript.

Use of External Sources; Honor Code Policy

- The correct way of using outside sources is to understand the contents of your source and then to write in your own words and without referring back to the source the solution of the problem. Your solution should differ in style significantly from the published solution. If you are not sure whether you are incorporating too much material from your source in your solutions, then you must cite the source that you used.
- You may and are required to collaborate freely with other students in your assignment group. However, you may not communicate at all about concrete coursework with students from other groups. However, discussing general questions regarding the lecture contents with any other student is of course fine and encouraged.

  Do not show or explain your solutions to any student outside your assignment group.

Use of External Sources; Honor Code Policy

In this course, the following actions are examples of violations of the Honor Code ("another student" means a student outside your assignment group):

- Showing another student your written solution to a problem.
- Sending a screenshot of your solution via QQ, email or other means to another student.
- Showing another student the written solution of a third student; distributing some student's solution to other students.
- Viewing another student's written solution.
- Copying your solution in electronic form (LaTeX source, PDF, JPG image etc.) to the computer hardware (flash drive, hard disk etc.) of another student. Having another student's solution in electronic form on your computer hardware.

If you have any questions regarding the application of the Honor Code, please contact me or any of the TAs.

Grading Policy

The course grade will be composed as follows:

- Midterm exam: 50%
- Final exam: 50%

Class Attendance and Absence for Medical Reasons

If you are unable to attend an exam or a class, you should notify me. The following rules apply:

- Absence for illness should be supported by a hospital/doctors certificate. A note that a student visited a medical facility is not sufficient excuse for missing a Tuesday class or an exam. The note must specifically indicate that the student was incapable of attending a class or taking the exam due to medical problems.
- **Late** medical excuses must satisfy the following criteria to be valid:
  - (i) The problem must be confirmed by the doctor to be so severe that the student could not participate in the exam.
  - (ii) The problem must have occurred so suddenly that it was impractical to contact me in advance.
  - (iii) The student must be in contact with me immediately after the exam or Tuesday class with the required documentation.

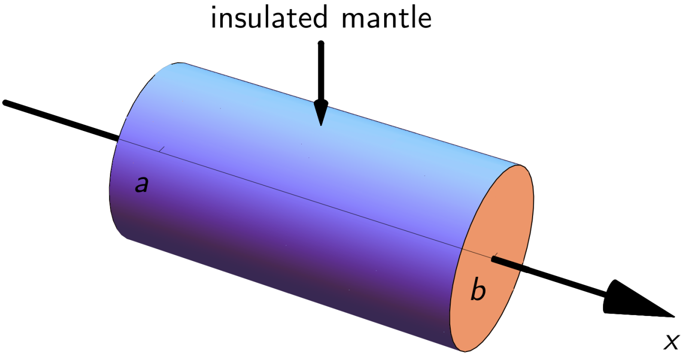
Literature

We will use various textbook sources for the course. These include

- Y. Pinchover and J. Rubinstein, An Introduction to Partial Differential Equations, Cambridge University Press 2005;
- I. Stakgold and M. Holst, Greens Functions and Boundary Value Problems, 3rd Ed., Wiley 2011;
- E. Zauderer, Partial Differential Equations of Applied Mathematics, 2nd Ed., Wiley 1989.

The Heat Equation for a Cylinder

Consider a cylinder with an insulated mantle, as shown below:



We are interested in the temperature θ within the cylinder.

The Heat Equation for a Cylinder

Assumption: The temperature is a function of x and t only,

$$\theta = \theta(x, t), \qquad x \in [a, b], \quad t \geq 0.$$

Heat density H related to temperature via

$$H(x, t) = \varrho \cdot c \cdot \theta(x, t).$$

material density specific heat capacity

$$\text{Total heat} = \int_a^b H(x, t)\, dx.$$

Change of Heat in the Cylinder

Change in heat is due to two physical processes:

(i) Heat is generated from heat sources.

$$\text{Heat created at time } t = Q(t).$$

(ii) Heat enters or leaves through the faces of the cylinder.

Heat flux in the positive x-direction:

$$B(x,t) = -k\frac{\partial\theta(x,t)}{\partial x}$$

heat conduction coefficient

(Fourier's law of heat conduction)

Change of Heat in the Cylinder

Total heat change between time t and time $t + \Delta t$:

$$\int_a^b H(x, t + \Delta t)\, dx - \int_a^b H(x, t)\, dx = \int_t^{t+\Delta t} \big(B(a, \tau) - B(b, \tau) + Q(\tau)\big)\, d\tau$$

We divide the equation by Δt and let $\Delta t \to 0$ to obtain the instantaneous change in heat:

$$\int_a^b \frac{\partial H}{\partial t}\, dx = B(a, t) - B(b, t) + Q(t)$$

Change of Heat in the Cylinder

The heat-temperature relation and Fourier's law give

$$\begin{aligned}\varrho c \int_a^b \frac{\partial \theta}{\partial t}\,dx &= k\,\frac{\partial \theta}{\partial x}\bigg|_{x=b} - k\,\frac{\partial \theta}{\partial x}\bigg|_{x=a} + Q(t)\\ &= k\int_a^b \frac{\partial^2 \theta}{\partial x^2}\,dx + Q(t)\end{aligned}$$

(Fundamental equation for the temperature)

Assumption: The heat sources can be expressed in terms of an integrable heat source density q, i.e.,

$$Q(t) = \int_a^b q(x,t)\,dx.$$

The Classical Heat Equation

Then

$$\varrho c \int_a^b \frac{\partial \theta}{\partial t}\,dx = \int_a^b \left(k \frac{\partial^2 \theta}{\partial x^2} + q(x,t) \right)\,dx.$$

Letting $a = x$, $b = x + \varepsilon$, divide by ε and let $\varepsilon \to 0$,

$$\boxed{\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial x^2}} + \frac{1}{\varrho c} q(x,t),$$

with thermal diffusivity

$$\alpha^2 = \frac{k}{\varrho c} > 0.$$

(Classical heat equation)

The Stationary Heat Equation

Heat equation: $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial x^2} + q(x,t), \qquad 0 < x < 1,\ t > 0.$

Suppose $q(x,t) = f(x)$ and search for an equilibrium solution

$$\theta(x,t) = u(x).$$

We obtain the Stationary Heat Equation

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1. \tag{I.1a}$$

We impose boundary conditions

$$u(0) = \alpha, \qquad u(1) = \beta, \qquad \alpha, \beta \in \mathbb{R}. \tag{I.1b}$$

Data for the Equilibrium Heat Equation

The triple (f, α, β) is called the data for the equilibrium heat equation.

Superposition Principle: If

- u_1 satisfies (I.1) with data (f_1, α_1, β_1) and
- u_2 satisfies (I.1) with data (f_2, α_2, β_2)

then

- $u_1 + u_2$ satisfies (I.1) with data $(f_1 + f_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$.

Application:

Solution for $(f, 0, 0)$ + Solution for $(0, \alpha, \beta)$ = Solution for (f, α, β)

Classical Solutions

Consider the ODE

$$a_p(x)u^{(p)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = f(x)$$

on the interval $(a, b) \subset \mathbb{R}$ with

- $a_0, \ldots, a_p$ continuous on $[a, b]$
- f piecewise continuous on $[a, b]$

A classical solution is a function u such that

- u is continuous on $[a, b]$
- u is $p - 1$ times continuously differentiable on (a, b)
- u is p times differentiable at all points in (a, b) where f is continuous. At these points, u solves the ODE.

Classical Solutions

Thus, a classical solution to the boundary value problem

$$-\frac{d^2u}{dx^2} = 0, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0$$

can not be something ridiculous like

$$u(x) = \begin{cases} 1 & 0 < x < 1. \\ 0 & x = 0 \text{ or } x = 1. \end{cases}$$

However: A classical solution does not have to be p-times differentiable at points where f has jumps!

Discontinuous Inhomogeneities

Example: Fix $0 < \xi < 1$ and consider the problem

$$-\frac{d^2u}{dx^2} = H_\xi(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

with the Heaviside function

$$H_\xi(x) = \begin{cases} 0 & x < \xi \\ 1 & x > \xi \end{cases}$$

and we can define $H_\xi(\xi)$ any way we like.

Denote a solution of the boundary value problem by $u(x;\xi)$.

Discontinuous Inhomogeneities

Solve separately in the intervals $(0, \xi)$ and $(\xi, 1)$:

$$u(x;\xi) = \begin{cases} Ax & 0 < x < \xi, \\ -(x-1)^2/2 + B(1-x) & \xi < x < 1, \end{cases}$$

with integration constants $A, B \in \mathbb{R}$.

Continuity of u implies

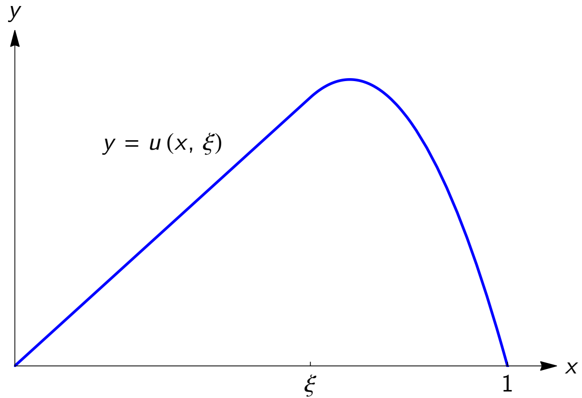
$$\lim_{x \nearrow \xi} u(x;\xi) = \lim_{x \searrow \xi} u(x;\xi) \quad \Rightarrow \quad A\xi = -(\xi-1)^2/2 + B(1-\xi).$$

Continuous differentiability implies

$$\lim_{x \nearrow \xi} u'(x;\xi) = \lim_{x \searrow \xi} u'(x;\xi) \qquad \Rightarrow \qquad A = -(\xi - 1) - B.$$

Discontinuous Inhomogeneities

$$u(x;\xi) = \begin{cases} \frac{(\xi-1)^2}{2}x & 0 < x < \xi, \\ -(x-1)^2/2 + \frac{1-\xi^2}{2}(1-x) & \xi < x < 1. \end{cases}$$



A Point Source

Fundamental equation for the temperature ($\alpha^2 = 1$):

$$\int_0^1 \frac{\partial \theta}{\partial t}\,dx = \int_0^1 \frac{\partial^2 \theta}{\partial x^2}\,dx + Q(t)$$

Assumption: Point heat source located at $0 < \xi < 1$ such that

$$Q(t) = 1$$

There exists no density q such that

$$Q(t) = \int_0^1 q(x, t)\,dx$$

Differential Equation for a Point Source

Equilibrium equation, $\theta(x, t) = u(x)$:

$$-\frac{d^2u}{dx^2} = 0, \qquad\qquad 0 < x < 1,\ x \neq \xi.$$

(Differential Equation is not defined for $x = \xi$!)

Denote by $g(x, \xi)$ the solution with

$$g(0, \xi) = g(1, \xi) = 0 \qquad \text{and} \qquad Q(t) = 1.$$

(Green's Function)

The Heat Balance Equation

The differential equation implies

$$g(x,\xi) = \begin{cases} Ax & 0 < x < \xi, \\ B(1-x) & \xi < x < 1, \end{cases} \qquad A, B \in \mathbb{R}. \tag{I.2}$$

Problem: g classical $\Rightarrow A = B = 0$

The equation does not take the point source into account.

Consider instead the heat balance equation

$$\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx + Q(t) = 0$$

which holds for any $a, b \in [0,1]$.

The Jump Condition

In particular, for $\varepsilon > 0$,

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \frac{\partial^2 g}{\partial x^2}\, dx = -Q = -1$$

so

$$g'(x,\xi)|_{x=\xi+\varepsilon} - g'(x,\xi)|_{x=\xi-\varepsilon} = -1.$$

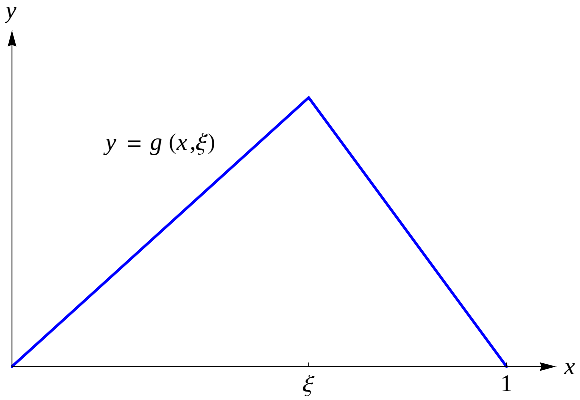
or

$$\lim_{x \nearrow \xi} g'(x,\xi) - \lim_{x \searrow \xi} g'(x,\xi) = -1$$

(Jump Condition)

The Solution for a Point Source

$$g(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi, \\ (1-x)\xi & \xi \le x \le 1. \end{cases}$$



Several Point Sources

Generalization: Two point sources

- located at ξ_1 and ξ_2
- generating heat q_1 and q_2

The problem is linear, so the solution (temperature distribution) is

$$u(x) = q_1 \cdot g(x, \xi_1) + q_2 \cdot g(x, \xi_2).$$

Check:

(i) u satisfies the boundary conditions $u(0) = u(1) = 0$.

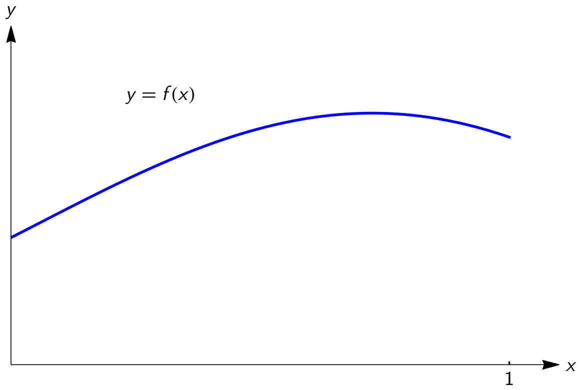
(ii) u satisfies $u''(x) = 0$ for any $x \neq \xi_1, \xi_2$.

(iii) u satisfies the heat balance on any subinterval of $[0, 1]$.

The General Inhomogeneous Equation

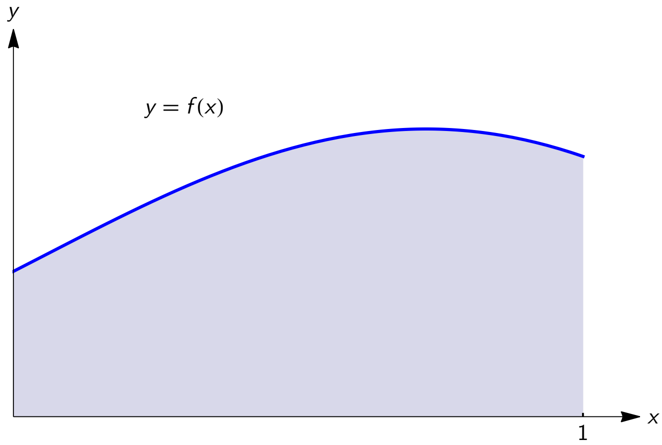
$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

where f is an integrable function.



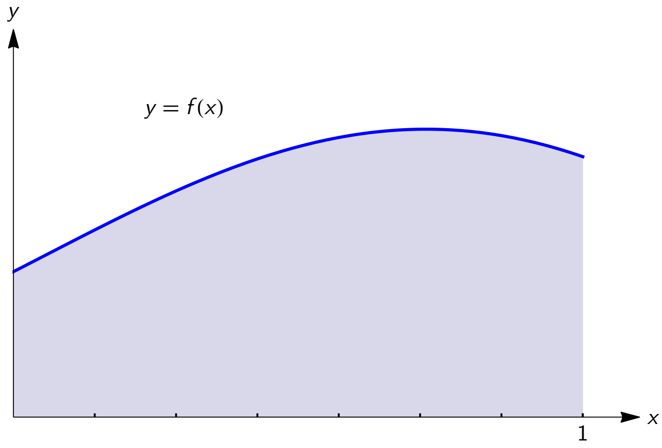
Approximating the Inhomogeneity

The integral of f gives the generated heat.



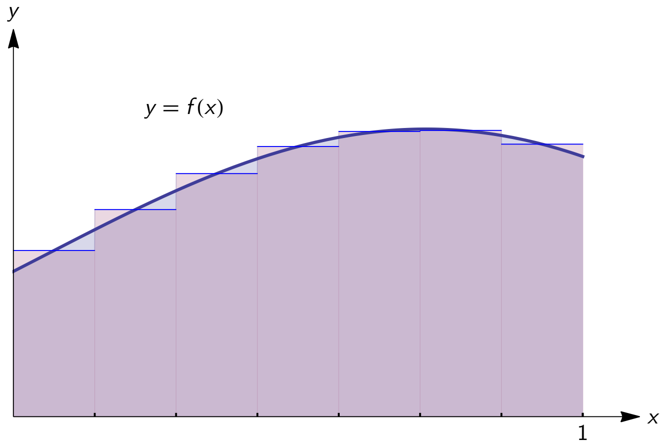
Approximating the Inhomogeneity

Divide the interval into n equal parts of width $\Delta\xi = 1/n$:



Approximating the Inhomogeneity

Use the midpoint value of f to approximate the generated heat:



Approximating the Inhomogeneity

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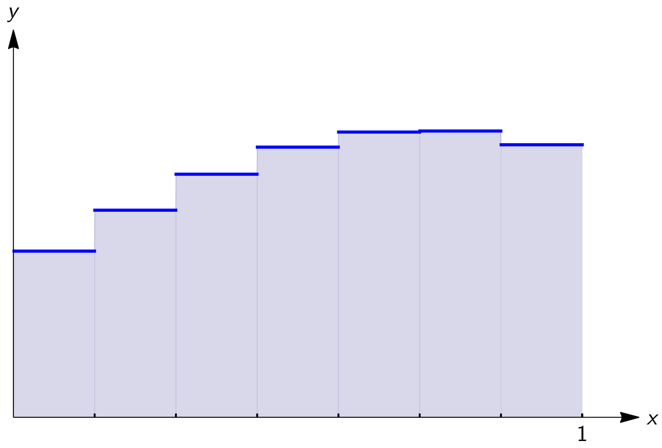
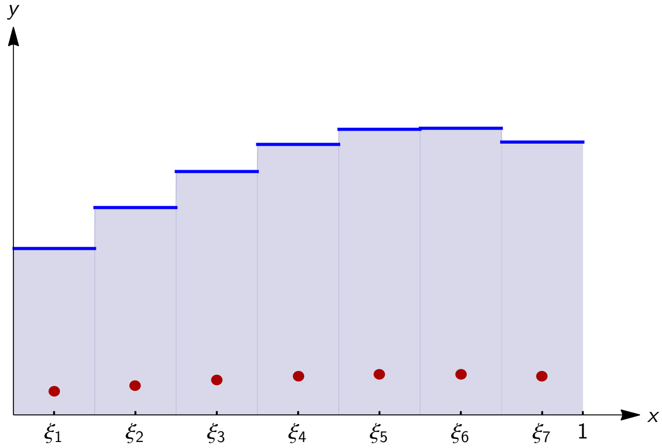


Approximating the Inhomogeneity

Place point sources at the midpoints $\xi_1, \dots, \xi_n$ of equal strength to the generated heat, $q_k = f(\xi_k) \cdot \Delta\xi = f(\xi_k)/n$:



Point Source Approximation

Use these n point sources to find an approximate solution u_n of $-u'' = f$:

$$u_n(x) = \sum_{k=1}^{n} f(\xi_k) \Delta\xi \cdot g(x, \xi_k).$$

Letting $n \to \infty$, the right-hand side converges to

$$u(x) := \int_0^1 f(\xi) g(x, \xi)\, d\xi.$$

Intuitively, this should be a solution of

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

The General Solution

The solution of

$$-\frac{d^2u}{dx^2} = 0, \qquad 0 < x < 1, \qquad u(0) = \alpha, \qquad u(1) = \beta,$$

is

$$u(x) = \alpha(1-x) + \beta x.$$

Hence, the general solution for data (f, α, β) is

$$u(x; \alpha, \beta) = \int_0^1 f(\xi) g(x, \xi)\, d\xi + \alpha(1-x) + \beta x.$$

(by the Superposition Principle)

The Equilibrium Heat Equation

Theorem.
Let $\alpha, \beta \in \mathbb{R}$ and $f \in C([0,1])$ be given. Then the unique solution of

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = \alpha, \qquad u(1) = \beta$$

is given by

$$u(x; \alpha, \beta) = \int_0^1 f(\xi) g(x, \xi)\, d\xi + \alpha(1 - x) + \beta x.$$

Existence of a Solution

Existence. We write

$$g(x,\xi) = \begin{cases} (1-\xi)x & 0 \le x < \xi \\ (1-x)\xi & \xi \le x \le 1 \end{cases}$$

$$= \begin{cases} l(x,\xi) & 0 \le x < \xi, \\ r(x,\xi) & \xi \le x \le 1. \end{cases}$$

Since

$$g(0,\xi) = g(1,\xi) = 0$$

it is sufficient to consider $\alpha = \beta = 0$.

Existence of a Solution

The derivative of $u(x;0,0)$ is given by

$$\frac{du}{dx} = \frac{d}{dx}\int_0^x f(\xi)r(x,\xi)\,d\xi + \frac{d}{dx}\int_x^1 f(\xi)l(x,\xi)\,d\xi$$

By the chain rule

$$\begin{aligned}\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} h(x,y)\,dy = \int_{\alpha(x)}^{\beta(x)} \frac{dh}{dx}(x,y)\,dy \\ + \beta'(x)h(x,\beta(x)_-) - \alpha'(x)h(x,\alpha(x)_+).\end{aligned}$$

where

$$f(x_\pm) := \lim_{\varepsilon\to 0} f(x \pm \varepsilon)$$

for any function f.

Existence of a Solution

We hence obtain

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\, d\xi + \int_x^1 f(\xi) l_x(x,\xi)\, d\xi + r(x,x_-)f(x_-) - l(x,x_+)f(x_+)$$

Since f and g are continuous,

$$f(x_-) = f(x_+) \qquad \text{and} \qquad r(x,x_-) = l(x,x_+),$$

so

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\, d\xi + \int_x^1 f(\xi) l_x(x,\xi)\, d\xi$$

Existence of a Solution

Differentiating

$$\frac{du}{dx} = \int_0^x f(\xi) r_x(x,\xi)\,d\xi + \int_x^1 f(\xi) l_x(x,\xi)\,d\xi$$

once more:

$$\begin{aligned}\frac{d^2u}{d^2x} &= \int_0^x f(\xi) r_{xx}(x,\xi)\,d\xi + \int_x^1 f(\xi) l_{xx}(x,\xi)\,d\xi \\ &\quad + f(x)\big(r_x(x,x_-) - l_x(x,x_+)\big) \\ &= -f(x).\end{aligned}$$

This proves that a solution exists and is given by

$$u(x;\alpha,\beta) = \int_0^1 f(\xi) g(x,\xi)\,d\xi + \alpha(1-x) + \beta x.$$

Uniqueness of the Solution

Uniqueness. Suppose that u_1 and u_2 are two classical solutions of

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = \alpha, \qquad u(1) = \beta$$

Then

$$v = u_1 - u_2$$

is twice continuously differentiable and satisfies

$$-v''(x) = 0, \qquad 0 < x < 1, \qquad v(0) = 0, \qquad v(1) = 0$$

Uniqueness of the Solution

In particular, $v''(x) = 0$ for all $x \in (0, 1)$ and

$$v(x) = Ax + B \qquad \text{for } 0 < x < 1 \text{ and some } A, B \in \mathbb{R}.$$

Since $v(0) = v(1) = 0$ and v is continuous on $[0, 1]$,

$$v(x) = 0$$

for all $x \in [0, 1]$.

Hence,

$$u_1 = u_2.$$

This proves that there exists only one solution.

Eigenfunction Expansion of the Green Function

Different approach to the Green function for the problem

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Consider the associated eigenvalue problem

$$-u''(x) = \lambda u, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

One finds

- eigenvalues $\lambda_n = n^2\pi^2$, $n \in \mathbb{N}$
- eigenfunctions $u_n(x) = \sin(n\pi x)$

Eigenfunction Expansion of Green's Function

Multiply $-u'' = f$ with u_n and integrate:

$$-\int_0^1 u_n(x)u''(x)\,dx = \int_0^1 f(x)u_n(x)\,dx.$$

Integrate the LHS by parts twice, use $-u_n'' = \lambda_n u_n$:

$$\int_0^1 u_n(x)u(x)\,dx = \frac{1}{\lambda_n}\int_0^1 f(x)u_n(x)\,dx.$$

Now expand u into a Fourier-sine series:

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} 2\int_0^1 u_n(\xi)u(\xi)\,d\xi\ \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\int_0^1 f(\xi)u_n(\xi)\,d\xi\ \sin(n\pi x). \end{aligned}$$

Eigenfunction Expansion of the Green Function

Interchange the infinite series with the integral:

$$u(x) = \int_0^1 g(x, \xi) f(\xi)\, d\xi$$

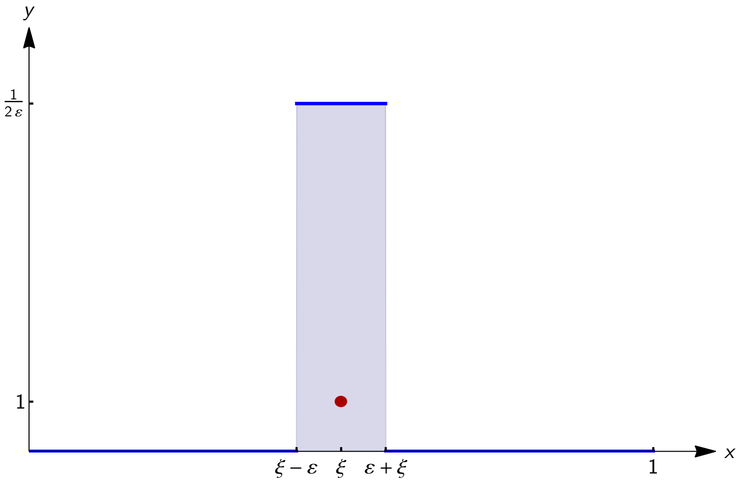
where

$$g(x, \xi) = \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x) \sin(n\pi \xi)}{n^2 \pi^2}.$$

(Eigenfunction Expansion of the Green function)

Physical Model of a Point Source

Heat $Q(t) = 1$ is generated uniformly in the interval $[\xi - \varepsilon, \xi + \varepsilon]$



Physical Approach to the Green Function

Then $Q = \int_0^1 q_\varepsilon(x;\xi)\,dx$ where

$$q_\varepsilon(x;\xi) = \begin{cases} 0 & |x-\xi| > \varepsilon, \\ \frac{1}{2\varepsilon} & |x-\xi| \le \varepsilon. \end{cases}$$

Of course, $\lim_{\varepsilon\to 0} q_\varepsilon(x;\xi)$ does not exist!

But: For any $\varepsilon > 0$ there exists a classical solution $u_\varepsilon(x,\xi)$ of

$$-u'' = q_\varepsilon(x;\xi), \qquad\qquad u(0) = u(1) = 0.$$

It can be shown that

$$g(x,\xi) = \lim_{\varepsilon\to 0} u_\varepsilon(x,\xi).$$

The Dirac Delta "Function" δ_ξ

Symbolically, we want to write

$$-g''(x,\xi) = \delta_\xi(x),$$

where

$$\delta_\xi(x) \overset{?}{:=} \lim_{\varepsilon\to 0} q_\varepsilon(x;\xi).$$

Necessary Properties of δ_ξ

(i) $\delta_\xi(x) = 0$ if $x \neq \xi$ (there is no heat source at $x \neq \xi$)

(ii) $\int_0^1 \delta_\xi(x)\,dx = 1$ (the total heat generated is equal to one)

But such a function does not exist!

Goals for this Course

- Define generalized functions, that include objects like $\delta_\xi(x)$
- Extend calculus to generalized functions.
  Upshot: (nearly) all functions can be differentiated.
- Define non-classical solutions to differential equations.
- Formalize the concept of Green functions and their use in solving differential equations.
- Investigate methods for finding Green functions.
- Introduce a numerical method that uses Green functions.

Part I

Basic Theory of Distributions

Distributions

Operations on Distributions

Families of Distributions

The Classical Fourier Transform

Tempered Distributions and the Fourier Transform

Distributions

Operations on Distributions

Families of Distributions

The Classical Fourier Transform

Tempered Distributions and the Fourier Transform

Smooth Functions

Definitions:

- A domain is an open and simply connected set in $\mathbb{R}^n$. We reserve the symbol Ω for domains.
- The set of k times continuously differentiable functions on a domain Ω:

$$C^k(\Omega) := \{\varphi\colon \Omega \to \mathbb{C} : \text{all partial derivatives of } \varphi \text{ of order } k \text{ exist and are continuous}\}.$$

- The set of smooth functions on Ω:

$$C^\infty(\Omega) := \{\varphi\colon \Omega \to \mathbb{C} : \text{all partial derivatives of } \varphi \text{ of any order exist}\}.$$

Compactly Supported Functions

- The support of a function $\varphi\colon \Omega \to \mathbb{C}$:

$$\operatorname{supp}\varphi := \overline{\{x \in \Omega\colon \varphi(x) \neq 0\}}.$$

($\overline{A}$ denotes the closure of a set $A \subset \mathbb{R}^n$.)

- The set of smooth functions with compact support:

$$C_0^\infty(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n)\colon \operatorname{supp}\varphi \text{ is a bounded set}\}$$

$$C_0^\infty(\Omega) := \{\varphi \in C_0^\infty(\mathbb{R}^n)\colon \operatorname{supp}\varphi \subset \Omega\}$$

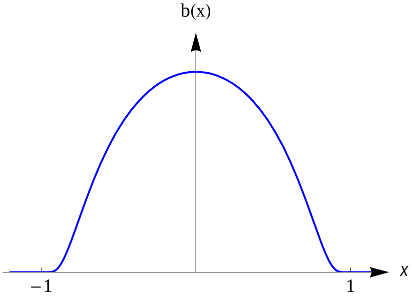
Problem: Do there even exist functions in $C_0^\infty(\mathbb{R}^n)$?

The Bump Function

Define the bump function

$$b\colon \mathbb{R} \to \mathbb{R}, \qquad b(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & -1 \le x \le 1, \\ 0 & |x| > 1. \end{cases}$$

(For $\mathbb{R}^n$, consider simply $b(|x|)$.)

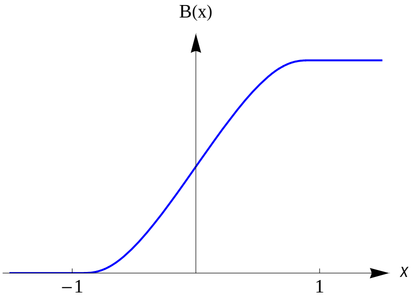


The Smooth Step

The integral of the bump function,

$$B(x) := \int_{-\infty}^{x} b(y)\,dy,$$

creates a "smooth step":



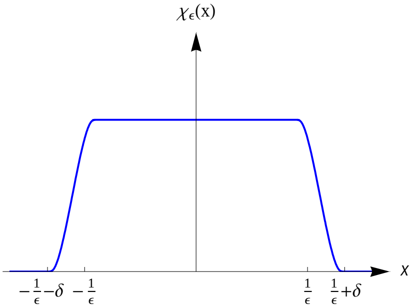
Cut-Off Functions

Shift and scale B to create a cut-off function χ_ε with the properties

(i) $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$,

(ii) $\chi_\varepsilon(x) = 1$ if $|x| < 1/\varepsilon$,

(iii) $\chi_\varepsilon(x) = 0$ if $|x| > 1/\varepsilon + \delta$ (here $\delta > 0$ may depend on ε)



Constructing Smooth, Compactly Supported Functions

If $f \in C^\infty(\mathbb{R})$, the function f_ε defined by

$$f_\varepsilon(x) := \chi_\varepsilon(x) \cdot f(x)$$

satisfies

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{for } |x| < 1/\varepsilon, \\ 0 & \text{for } |x| > 1/\varepsilon + \varepsilon. \end{cases}$$

Furthermore, $f_\varepsilon \in C^\infty(\mathbb{R})$ since both χ_ε and f are smooth functions. Hence,

$$f_\varepsilon \in C_0^\infty(\mathbb{R}).$$

Conclusion: There are many functions in $C_0^\infty(\mathbb{R})$!

Multi-Index Notation for Smooth Functions

A multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of natural numbers. We define

- Degree of α:
$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$
- Derivatives:
$$D^\alpha u := \frac{\partial^\alpha u}{\partial x^\alpha} := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$
- Monomials:
$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$
- Factorials:
$$\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$$

Convergence in $C_0^\infty(\mathbb{R})$

Important for following discussion:
Sequences of smooth, compactly supported functions.

How to define convergence in $C_0^\infty(\mathbb{R})$?

Let (φ_m) be a sequence in $C_0^\infty(\mathbb{R})$. Then, given some $\varphi \in C_0^\infty(\mathbb{R})$,

$$\varphi_m \to \varphi \qquad \text{if and only if} \qquad \underbrace{\varphi_m - \varphi}_{=:\psi_m} \to 0.$$

It is sufficient to define what it means that a sequence (ψ_m) converges to zero. Such a sequence will be called a null sequence.

Null Sequences

Definition (φ_m) is a null sequence in $C_0^\infty(\mathbb{R}^n)$ if

(i) There exists some $R > 0$ such that for all $m \in \mathbb{N}$,

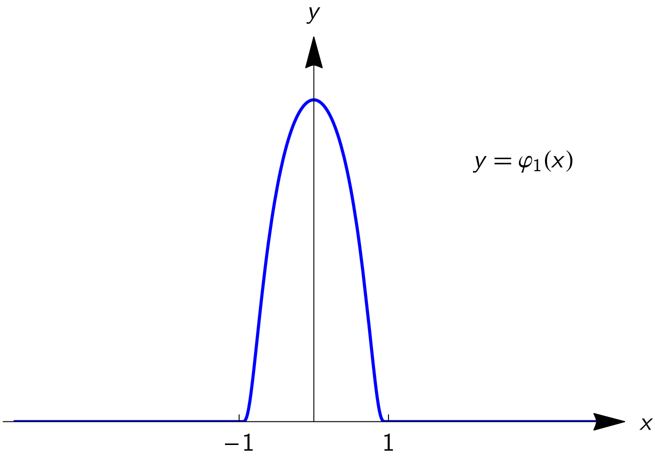
$$\operatorname{supp} \varphi_m \subset B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}.$$

(ii) For every multi-index $\alpha \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |D^\alpha \varphi_m(x)| \xrightarrow{m \to \infty} 0.$$

Example of a Null Sequence

Let $\varphi \in C_0^\infty(\mathbb{R})$ and define $\varphi_m(x) := \varphi(x)/m$.



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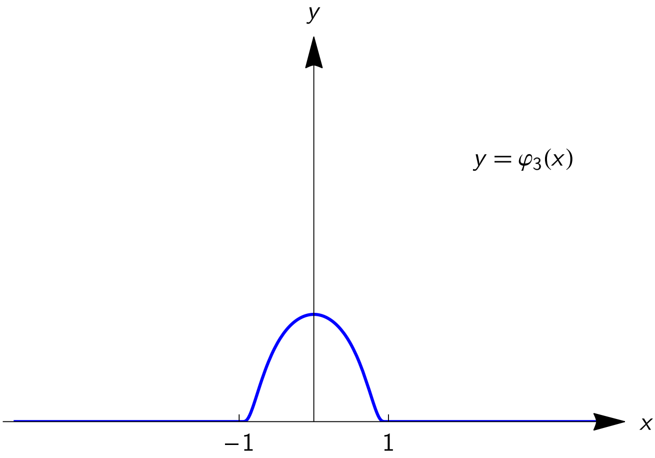

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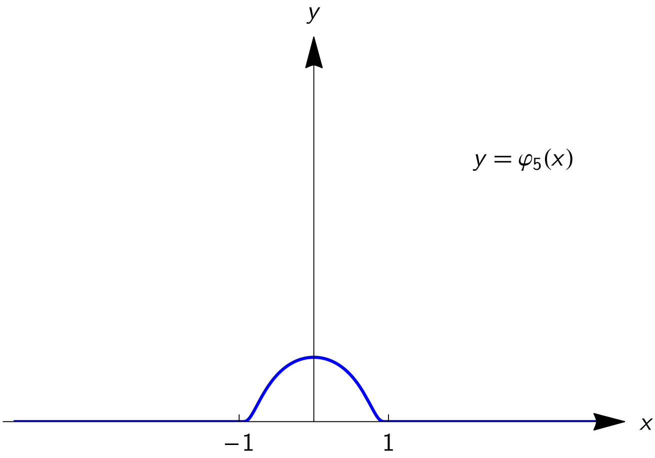

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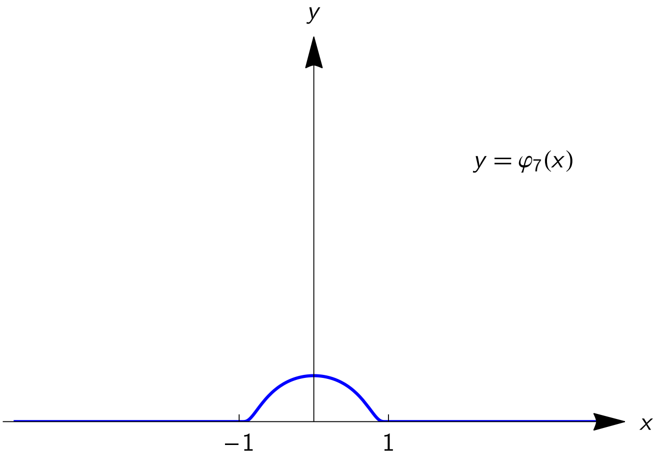
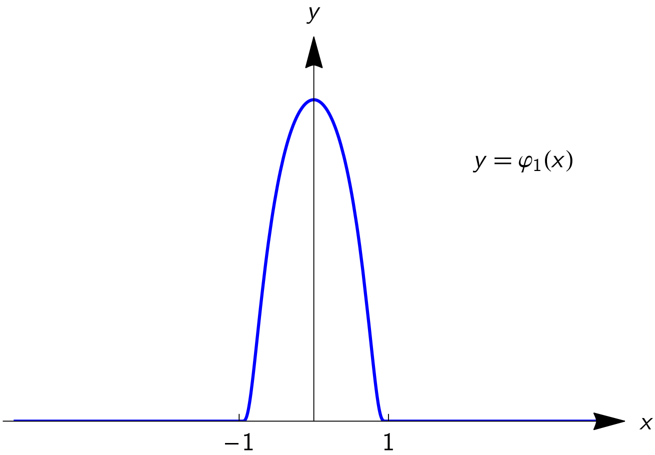


Not a Null Sequence

Let $\varphi \in C_0^\infty(\mathbb{R})$ and define $\varphi_m(x) := \varphi(x/m)/m$.



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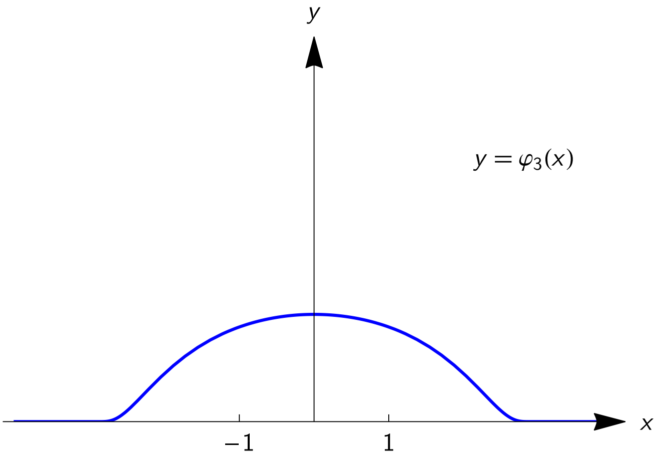

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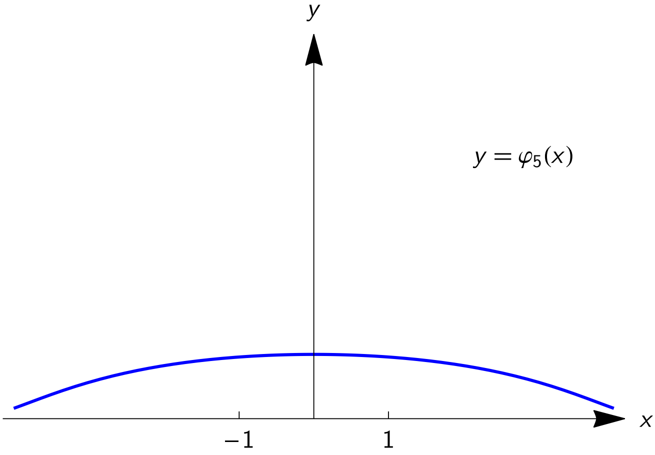

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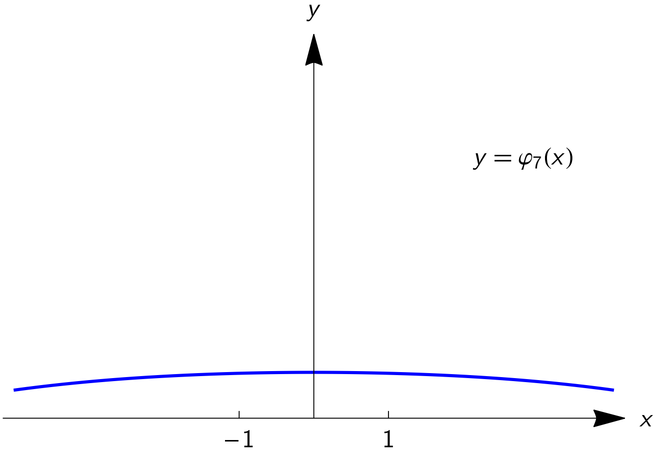

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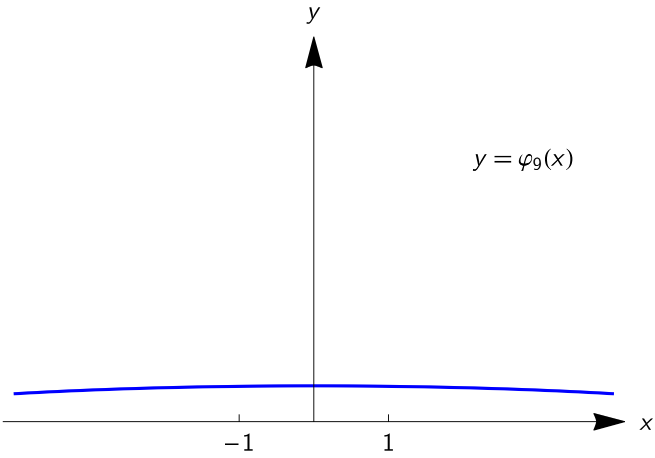
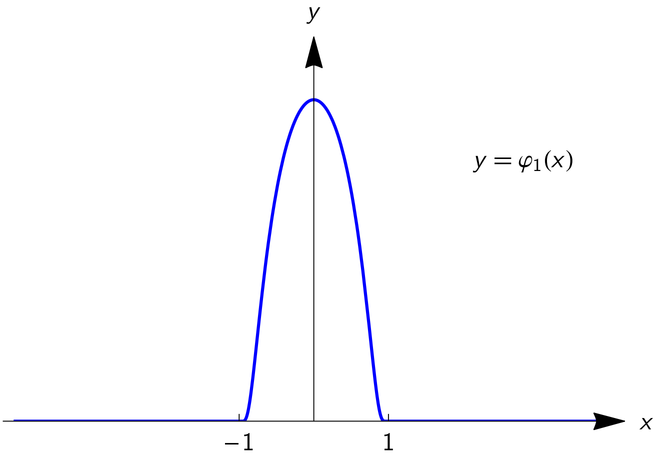


Not a Null Sequence

Let $\varphi \in C_0^\infty(\mathbb{R})$ and define $\varphi_m(x) := \varphi(m \cdot x)/m$.



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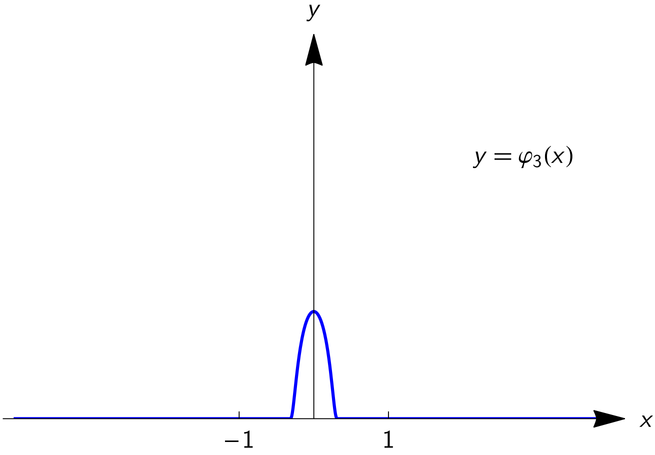

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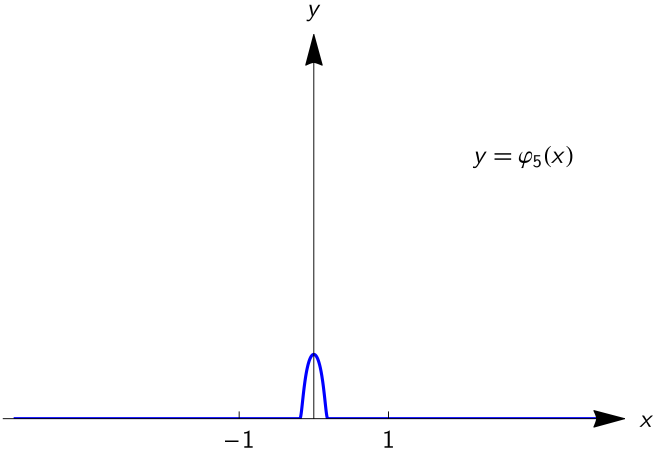

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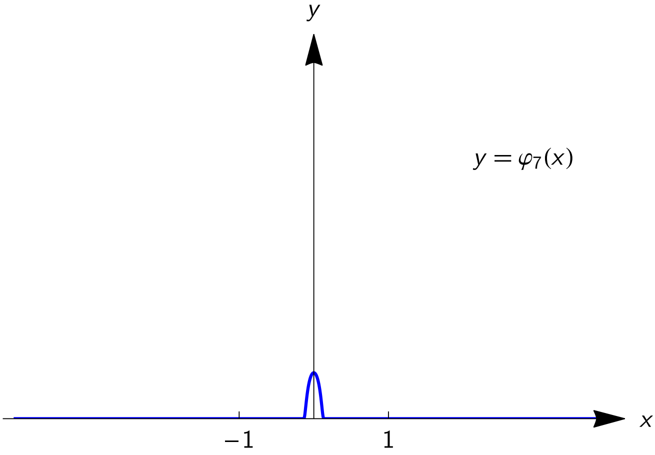

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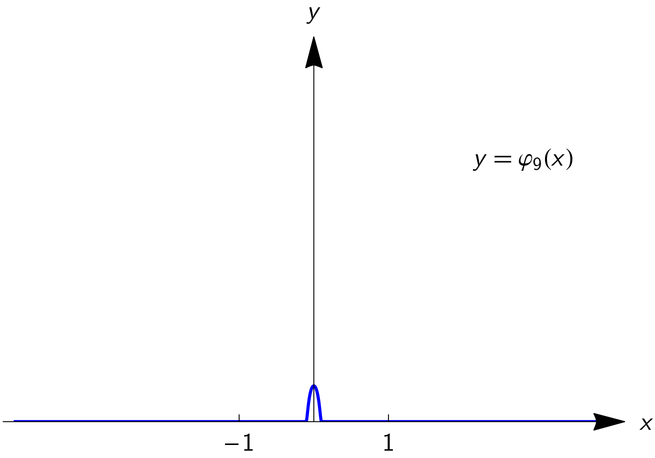

Test Functions and Linear Functionals

$\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$

(Space of Test Functions in $\mathbb{R}^n$)

$\mathcal{D}(\mathbb{R}^n)$ is a (complex) vector space.

Definition. A linear functional on $\mathcal{D}(\mathbb{R}^n)$ is a map

$$T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$$

such that

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda T\varphi_1 + \mu T\varphi_2$$

for $\varphi_1, \varphi_2 \in \mathcal{D}$, $\lambda, \mu \in \mathbb{C}$.

Examples of Linear Functionals

Linear functionals on $\mathcal{D}(\mathbb{R})$:

(i) $T\varphi := \int_0^\infty \varphi(x)\,dx$

(ii) $T\varphi := \varphi(0)$

(iii) $T\varphi := \varphi'(1)$

Linear functionals on $\mathcal{D}(\mathbb{R}^n)$:

(i) $T\varphi := \int_{\mathbb{R}^n} \varphi(x)\,dx$

(ii) $T\varphi := \int_S \varphi\,d\sigma$, where S is a surface in $\mathbb{R}^n$

(iii) $T\varphi := \int_S \operatorname{grad}\varphi\,d\vec{\sigma}$

Continuous Linear Functionals

Definition. A linear functional T is said to be continuous if

$$\underbrace{\varphi_m}_{\text{null sequence in } \mathcal{D}(\mathbb{R}^n)} \to 0 \qquad \Rightarrow \qquad T\underbrace{\varphi_m}_{\text{sequence in } \mathbb{C}} \to 0$$

A continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$ is called a distribution.

The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

$\mathcal{D}'(\mathbb{R}^n)$ is a vector space.

Examples. All previous examples of linear functionals are distributions.

Delta is not a function.

Locally Integrable Functions

Definition. A function $g\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\int_\Omega |g(x)|\,dx < \infty \qquad \text{for any bounded set } \Omega \subset \mathbb{R}^n$$

is said to be locally integrable.

The space of locally integrable functions is denoted by $L^1_{\text{loc}}(\mathbb{R}^n)$.

Example. The following functions $f\colon \mathbb{R} \to \mathbb{R}$ are locally integrable:

(i) $f(x) = x^2$

(ii) $f(x) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ (Heaviside function)

(iii) $f(x) = \begin{cases} \ln(x) & x > 0 \\ 0 & x \leq 0 \end{cases}$

Regular and Singular Distributions

If $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ then

$$T_g : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}, \qquad \varphi \mapsto \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

defines a distribution.

Definition. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ so that

$$T\varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

for some $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be a regular distribution.

A distribution that is not regular is said to be singular.

Regular and Singular Distributions

Example. The distribution $T \in \mathcal{D}'(\mathbb{R})$ given by

$$T\varphi = \int_0^\infty \varphi(x)\,dx = \int_{-\infty}^\infty H(x)\varphi(x)\,dx$$

is regular.

The Dirac delta distribution $T_\delta \in \mathcal{D}'(\mathbb{R})$ given by

$$T_\delta \varphi := \varphi(0)$$

is singular.

Proof that T_δ is Singular

Suppose that there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$T_\delta \varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx = \varphi(0).$$

For $a > 0$ define $\psi_a \in \mathcal{D}(\mathbb{R}^n)$,

$$\psi_a(x) = \begin{cases} e^{-a^2/(|x|^2+a^2)} & |x| < a, \\ 0 & \text{otherwise} \end{cases}$$

and note

$$|\psi_a(x)| \le \frac{1}{e}.$$

Proof that T_δ is Singular

Then

$$\begin{aligned} |T_\delta \psi_a| &= \left| \int_{\mathbb{R}^n} g(x)\psi_a(x)\,dx \right| \\ &\le \frac{1}{e} \int_{|x|<a} |g(x)|\,dx \\ &\xrightarrow{a \to 0} 0. \end{aligned}$$

But

$$T_\delta \psi_a = \psi_a(0) = \frac{1}{e} \not\to 0 \qquad \text{as } a \to 0.$$

Contradiction!

Outlook

Purely formally / symbolically:

$$T_\delta \varphi = \int_{\mathbb{R}^n} \delta(x)\varphi(x)\,dx = \varphi(0)$$

Dirac delta "function"

To Do:

- Prove that T_δ / $\delta(x)$ represents a "point source".
- Consider also "point dipoles" and similar objects.

Outlook

Natural identification

$$g \in L^1_{\text{loc}}(\mathbb{R}^n) \qquad \leftrightarrow \qquad T_g \in \{T \in \mathcal{D}' : T \text{ regular}\}$$

leads to

$$L^1_{\text{loc}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

To Do:

- Extend operations of calculus (differentiation, multiplication of functions, etc.) to $\mathcal{D}'(\mathbb{R}^n)$
- Define convergence of sequences of distributions
- Discuss the Fourier transform
- How to solve equations in distributions?

Distributions

Operations on Distributions

Families of Distributions

The Classical Fourier Transform

Tempered Distributions and the Fourier Transform

Extension by Duality

Basic Idea:

- Any operation in calculus can be performed on test functions $\varphi \in \mathcal{D} = C_0^\infty$.
- Use the "dual pairing"

$$T_g\varphi = \int_{\mathbb{R}^n} g(x)\varphi(x)\,dx$$

  to see what equivalent operation can be performed on "sufficiently nice" $g \in L^1_{\text{loc}}$.
- Define the operation on distributions in terms of an equivalent operation on test functions.

Dilation

Dilation operator: For $\alpha > 0$, define

$$D_\alpha \colon \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (D_\alpha \varphi)(x) = \alpha^{n/2} \cdot \varphi(\alpha x).$$

Then for a regular distribution T_g,

$$\begin{aligned} T_g(D_\alpha \varphi) &= \alpha^{n/2} \int_{\mathbb{R}^n} g(x)\varphi(\alpha x)\, dx \\ &= \frac{1}{\alpha^{n/2}} \int_{\mathbb{R}^n} g\left(\frac{x}{\alpha}\right) \varphi(x)\, dx \\ &= T_{D_{1/\alpha} g} \varphi \\ &=: (D_{1/\alpha} T_g)\varphi. \end{aligned}$$

Dilation

Definition. The dilation operator

$$D_\alpha \colon \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n)$$

is defined by

$$(D_\alpha T)(\varphi) := T(D_{1/\alpha}\varphi).$$

This definition ensures that

$$T_{D_\alpha g} = D_\alpha T_g$$

and extends the definition of the dilation from $L^1_{\text{loc}}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

Translation

Translation operator: For $y \in \mathbb{R}^n$, define

$$\tau_y : \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (\tau_y \varphi)(x) = \varphi(x - y).$$

Then for a regular distribution T_g,

$$T_g(\tau_y \varphi) = \int_{\mathbb{R}^n} g(x)\varphi(x-y)\, dx = \int_{\mathbb{R}^n} g(x+y)\varphi(x)\, dx = T_{\tau_{-y} g}\varphi.$$

Definition. We define the translation operator

$$\tau_y : \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n) \qquad (\tau_y T)(\varphi) := T(\tau_{-y}\varphi)$$

The Translation of Distributions

Example. Let $T_\delta \varphi = \varphi(0)$. Then

$$(\tau_\xi T_\delta)(\varphi) = T_\delta(\tau_{-\xi}\varphi) = \varphi(x+\xi)\big|_{x=0} = \varphi(\xi).$$

Sum and Scalar Multiplication

$\mathcal{D}'(\mathbb{R}^n)$ is a vector space:

- $T_1, T_2 \in \mathcal{D}'$ implies $T_1 + T_2 \in \mathcal{D}'$ with
$$(T_1 + T_2)(\varphi) = T_1\varphi + T_2\varphi$$
- $T \in \mathcal{D}'$, $\lambda \in \mathbb{C}$ implies $\lambda T \in \mathcal{D}'$ with
$$(\lambda T)(\varphi) = \lambda \cdot T\varphi$$

The pointwise sum and scalar multiple of functions generalize automatically to distributions:

$$T_{f+g} = T_f + T_g, \qquad\qquad T_{\lambda f} = \lambda T_f.$$

Multiplication by Smooth Functions

Multiplication operator: For $h \in C^\infty(\mathbb{R}^n)$, define

$$M_h \colon \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}(\mathbb{R}^n), \qquad (M_h\varphi)(x) = h(x)\varphi(x).$$

Then for a regular distribution T_g,

$$T_g(M_h\varphi) = \int_{\mathbb{R}^n} g(x)h(x)\varphi(x)\,dx = T_{M_h g}\varphi.$$

Definition. We define the mutliplication operator

$$M_h \colon \mathcal{D}'(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n) \qquad (M_h T)(\varphi) := T(M_h\varphi)$$

Warning: We can not multiply a distribution with a non-smooth function or another distribution!

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The Weak Derivative

Suppose $f \in L^1_{\text{loc}}(\mathbb{R})$ is differentiable and $f' \in L^1_{\text{loc}}(\mathbb{R})$. Then

$$T_{f'}\varphi = \int_{\mathbb{R}} f'(x)\varphi(x)\,dx = -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx$$

for $\varphi \in \mathcal{D}(\mathbb{R})$.

Note:

- $\varphi \in \mathcal{D}(\mathbb{R}) \quad \Rightarrow \quad \varphi' \in \mathcal{D}(\mathbb{R})$
- $\varphi \mapsto -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx$ is continuous linear functional
- RHS defines a distribution even if f is not differentiable.

The Weak Derivative

Definition. For $T \in \mathcal{D}'(\mathbb{R}^n)$, $\alpha \in \mathbb{N}^n$ a multi-index, $D^\alpha T \in \mathcal{D}'(\mathbb{R}^n)$ defined by

$$(D^\alpha T)\varphi := (-1)^{|\alpha|} T(D^\alpha \varphi)$$

is said to be the weak derivative or distributional derivative of T.

Note:

- Every distribution is differentiable in the weak sense
- If T is a regular distribution $T = T_f$ and if f is differentiable with $D^\alpha f \in L^1_{\text{loc}}$, then

$$D^\alpha T_f = T_{D^\alpha f}.$$

Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$T_f \varphi = \int_{\mathbb{R}} x^2 \varphi(x)\, dx.$$

The weak derivative is

$$\begin{aligned} T_f' \varphi &= -\int_{-\infty}^{\infty} x^2 \varphi'(x)\, dx \\ &= \underbrace{-x^2 \varphi(x)\Big|_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} 2x\varphi(x)\, dx \\ &= T_{f'}\varphi. \end{aligned}$$

Example: $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

$$T_f\varphi = \int_{\mathbb{R}} |x|\varphi(x)\,dx.$$

The weak derivative is

$$\begin{aligned} T_f'\varphi &= -\int_{-\infty}^{\infty} |x|\varphi'(x)\,dx = \int_{-\infty}^{0} x\varphi'(x)\,dx - \int_{0}^{\infty} x\varphi'(x)\,dx \\ &= \underbrace{x\varphi(x)\big|_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} \varphi(x)\,dx + \underbrace{x\varphi(x)\big|_{0}^{\infty}}_{=0} + \int_{0}^{\infty} \varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} \operatorname{sgn}(x)\varphi(x)\,dx, \end{aligned}$$

where $\operatorname{sgn}(x) = x/|x|$ is the sign function.

Example: Heaviside function and Delta Distribution

$$T_H\varphi = \int_{\mathbb{R}} H(x)\varphi(x)\,dx = \int_0^\infty \varphi(x)\,dx$$

The weak derivative is

$$\begin{aligned} T_H'\varphi &= -\int_0^\infty \varphi'(x)\,dx \\ &= -\varphi(x)\big|_0^\infty \\ &= \varphi(0) = T_\delta\varphi. \end{aligned}$$

The derivative of the Heaviside function is the Dirac distribution δ.

Furthermore,

$$T_\delta'\varphi = -T_\delta\varphi' = -\varphi'(0).$$

$f(x) = 1/x$ as a Distribution

Problem:

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad f(x) = \frac{1}{x}$$

is not locally integrable.

But we would like to have a distribution analogous to this function!

Approach:

$$g : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad g(x) = \ln(|x|)$$

is locally integrable. Furthermore,

$$g'(x) = f(x) \qquad \text{and} \qquad T_g' \quad \text{exists.}$$

Distributional Derivative of the Logarithm

$$\begin{aligned}
T_g'\varphi &= -\int_{-\infty}^{\infty} \varphi'(x)\ln(|x|)\,dx \\
&= -\int_{-\infty}^{0} \varphi'(x)\ln(-x)\,dx - \int_{0}^{\infty} \varphi'(x)\ln(x)\,dx \\
&= -\int_{0}^{\infty} \varphi'(-x)\ln(x)\,dx - \int_{0}^{\infty} \varphi'(x)\ln(x)\,dx \\
&= \lim_{\varepsilon\to 0}\biggl(-\int_{\varepsilon}^{\infty} (\varphi'(x)+\varphi'(-x))\ln(x)\,dx\biggr) \\
&= \lim_{\varepsilon\to 0}\biggl(-(\varphi(x)-\varphi(-x))\ln(x)\Big|_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx\biggr) \\
&= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx
\end{aligned}$$

The Principle Value of $1/x$

Definition. The distribution $\mathcal{P}(1/x) \in \mathcal{D}'(\mathbb{R})$ is given by

$$\mathcal{P}\left(\frac{1}{x}\right)(\varphi) := \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx$$

The right-hand side is called

- the Cauchy principal part integral of φ or
- the Cauchy principal value of $\displaystyle\int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$.

The Cauchy principal part integral converges for any $\varphi \in \mathcal{D}(\mathbb{R})$.

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

$$f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad\qquad f(x) = \frac{1}{|x|}$$

is locally integrable (even about the origin - use polar coordinates!)

The Laplacian in $\mathbb{R}^n$ is the linear differential operator

$$\Delta := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

Then

$$\begin{aligned}(\Delta T_f)(\varphi) = T_f(\Delta\varphi) &= \int_{\mathbb{R}^3} \frac{\Delta\varphi(x)}{|x|}\,dx \\ &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx\end{aligned}$$

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

Green's second identity:

$$\int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx = \int_{|x|>\varepsilon} \varphi(x)\,\Delta\left(\frac{1}{|x|}\right)\,dx + \int_{|x|=\varepsilon} \left(\frac{1}{|x|}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)\right)\,d\sigma$$

normal derivative (inward pointing)

Spherical coordinates (r, ϕ, θ):

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$$

and

$$\Delta\left(\frac{1}{|x|}\right) = \Delta\frac{1}{r} = 0 \qquad \text{for } r \neq 0$$

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The Laplacian of $1/|x|$ in $\mathbb{R}^3$

Hence

$$\int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\,dx = -\int_{r=\varepsilon}\left(\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r^2}\right)\,d\sigma.$$

$\varphi \in \mathcal{D}(\mathbb{R}^3)$ implies $\frac{\partial\varphi}{\partial r}$ bounded, so

$$\left|\int_{r=\varepsilon} \frac{1}{r}\frac{\partial\varphi}{\partial r}\,d\sigma\right| \le \frac{\text{constant}}{\varepsilon}\cdot 4\pi\varepsilon^2 \xrightarrow{\varepsilon\to 0} 0$$

Using spherical coordinates,

$$\int_{r=\varepsilon} \frac{\varphi}{r^2}\,d\sigma \xrightarrow{\varepsilon\to 0} 4\pi\varphi(0).$$

The Laplacian of $1/|x|$ in $\mathbb{R}^3$

In summary:

$$(\Delta T_f)(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\Delta\varphi(x)}{|x|}\, dx = 4\pi\varphi(0),$$

so that

$$\Delta \frac{1}{|x|} = -4\pi\delta(x)$$

in the sense of distributions.

Distributions

Operations on Distributions

Families of Distributions

The Classical Fourier Transform

Tempered Distributions and the Fourier Transform

Families of Functions

Example. The functions $f_n\colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} n & |x| < 1/(2n), \\ 0 & \text{otherwise}, \end{cases} \qquad n \in \mathbb{N},$$

define a sequence $(f_n)_{n \in \mathbb{N} \setminus \{0\}}$.

More generally, for any $\varepsilon > 0$ the functions $f_\varepsilon\colon \mathbb{R} \to \mathbb{R}$

$$f_\varepsilon(x) = \begin{cases} 1/\varepsilon & |x| < \varepsilon/2, \\ 0 & \text{otherwise}, \end{cases}$$

define a family of functions $(f_\varepsilon)_{\varepsilon > 0}$.

Weak Convergence of Distributions

Definition. Suppose

- $I \subset \mathbb{R}$ index set
- $\{T_\alpha\}_{\alpha \in I}$ family of distributions, $T_\alpha \in \mathcal{D}'(\mathbb{R}^n)$
- $\alpha_0 \in \bar{I}$ in the index set or boundary point of index set.
- $T \in \mathcal{D}'(\mathbb{R}^n)$ given.

Then

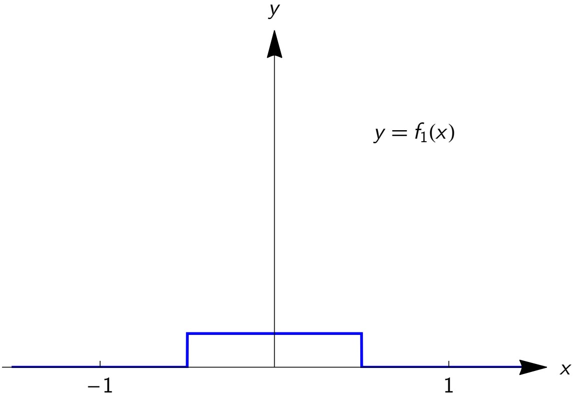
$$\lim_{\alpha \to \alpha_0} T_\alpha = T \qquad :\Leftrightarrow \qquad \lim_{\alpha \to \alpha_0} T_\alpha \varphi = T\varphi$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(Weak Convergence or Distributional Convergence)

Example: Convergence to a Point Source

$$f_n \colon \mathbb{R} \to \mathbb{R} \qquad f_n(x) = \begin{cases} n & |x| < 1/(2n), \\ 0 & \text{otherwise.} \end{cases}$$



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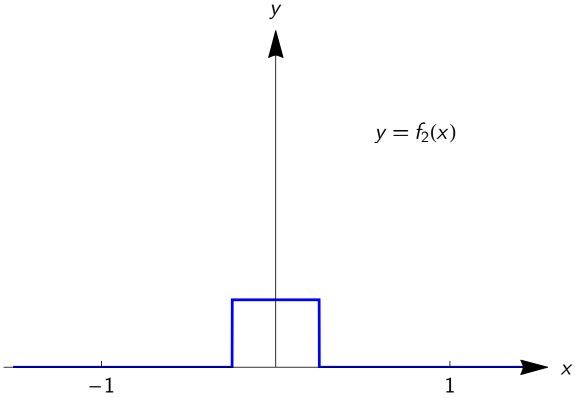
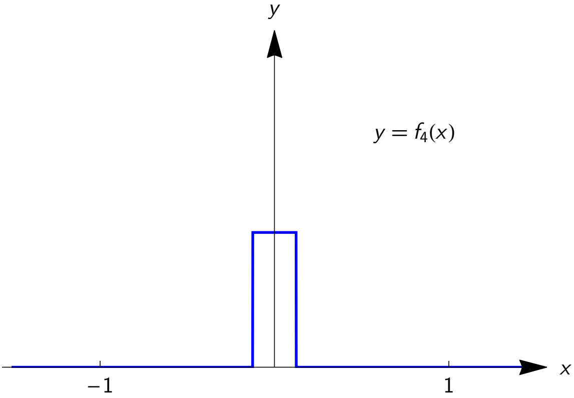


Example: Convergence to a Point Source

$f_n \colon \mathbb{R} \to \mathbb{R}$ $$f_n(x) = \begin{cases} n & |x| < 1/(2n), \\ 0 & \text{otherwise.} \end{cases}$$



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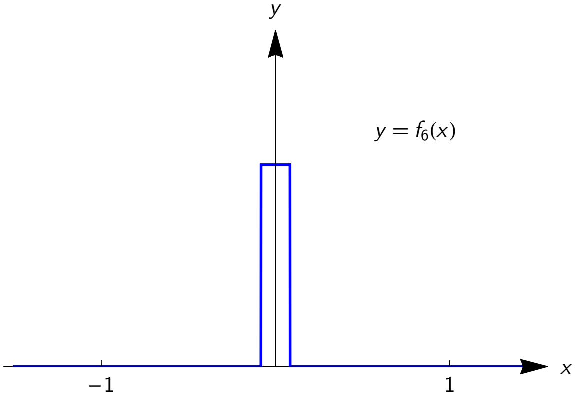

Example: Convergence to a Point Source

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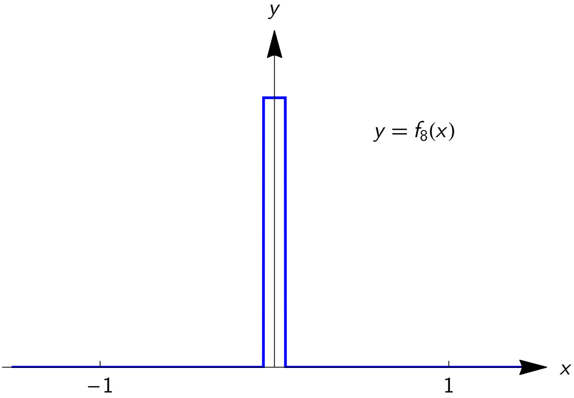


Weak Convergence of (f_n)

For any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} f_n(x)\varphi(x)\,dx &= n\int_{-1/(2n)}^{1/(2n)} \varphi(x)\,dx \\ &= \varphi(0) + n\int_{-1/(2n)}^{1/(2n)} (\varphi(x)-\varphi(0))\,dx\end{aligned}$$

Hence,

$$\begin{aligned}|T_{f_n}\varphi - \varphi(0)| &\le n\int_{-1/(2n)}^{1/(2n)} |\varphi(x)-\varphi(0)|\,dx \\ &\le n\cdot\frac{1}{n}\sup_{|x|\le 1/(2n)} |\varphi(x)-\varphi(0)| \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

Weak Convergence of (f_n)

Hence

$$T_{f_n}\varphi \xrightarrow{n\to\infty} \varphi(0) = T_\delta\varphi$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Therefore,

$$T_{f_n} \xrightarrow{n\to\infty} T_\delta.$$

Formally,

$$f_n \xrightarrow{n\to\infty} \delta$$

in the sense of distributions.

Since (f_n) is a physical model for a point source, this shows:

A physical point source is represented by the Dirac distribution T_δ.

Criterion for Weak Convergence

Lemma. Suppose

- $f_n \in L^1_{\text{loc}}(\mathbb{R}^n)$
- For any $R > 0$, $\sup_{|x|<R} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Then

$$f_n \to f \qquad\qquad \text{distributionally.}$$

Criterion for Weak Convergence

Proof.
Suppose $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}\varphi \subset \{x\colon |x| < R\}$ for some $R > 0$

$$\begin{aligned}|T_{f_n}\varphi - T_f\varphi| &\leq \int_{\mathbb{R}^n} |f_n(x) - f(x)| \cdot |\varphi(x)|\, dx \\ &\leq \underbrace{\sup_{|x|<R} |f_n(x) - f(x)|}_{\to 0} \cdot \underbrace{\int_{|x|<R} |\varphi(x)|\, dx}_{=:C}\end{aligned}$$

$\square$

Note:
Pointwise convergence is not necessary and not sufficient for distributional comnvergence.

Delta Families and Delta Sequences

Definition. Suppose

- $I \subset \mathbb{R}$ (index set),
- $f_\alpha \in L^1_{\text{loc}}(\mathbb{R}^n)$ for all $\alpha \in I$

Then $\{f_\alpha\}_{\alpha \in I}$ is a delta family (as $\alpha \to \alpha_0$) if

$$\lim_{\alpha \to \alpha_0} f_\alpha = \delta.$$

If $I = \mathbb{N}$ and $\alpha_0 = \infty$ then $\{f_\alpha\}_{\alpha \in I}$ is a delta sequence.

Constructing Delta Families

Theorem. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

- $f(x) \geq 0$ for all $x \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} f(x)\,dx = 1$

Then

$$f_\alpha(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right) \qquad \text{for } \alpha > 0$$

defines a delta family $\{f_\alpha\}_{\alpha>0}$ as $\alpha \to 0$. In particular,

$$\lim_{\alpha \searrow 0} \int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\,dx = \varphi(0)$$

for any φ that is bounded and continuous at $x = 0$.

Constructing Delta Families

Proof.

Suppose $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is bounded and continuous at $x = 0$.

$$T_{f_\alpha}\varphi = \int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\,dx = \varphi(0) + \int_{\mathbb{R}^n} f_\alpha(x)\underbrace{(\varphi(x) - \varphi(0))}_{=:\psi(x)}\,dx$$

Prove:

$$\lim_{\alpha\to 0}\int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx = 0$$

Show that for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that

$$\alpha < \gamma \quad \Rightarrow \quad \left|\int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx\right| < \varepsilon.$$

Constructing Delta Families

(i) For any $\alpha > 0$,

$$\int_{\mathbb{R}^n} f_\alpha(x)\,dx = 1$$

(ii) For any $R > 0$,

$$\lim_{\alpha\to 0} \int_{|x|>R} f_\alpha(x)\,dx = 0$$

(iii) For any $R > 0$,

$$\lim_{\alpha\to 0} \int_{|x|<R} f_\alpha(x)\,dx = 1$$

Constructing Delta Families

Since $f \geq 0$, for any $R > 0$,

$$\left| \int_{|x|<R} f_\alpha(x)\psi(x)\,dx \right| \leq \underbrace{\max_{|x|\leq R} |\psi(x)|}_{=:c(R)} \cdot \underbrace{\int_{|x|<R} f_\alpha(x)\,dx}_{\leq 1}.$$

Furthermore,

$$\left| \int_{|x|>R} f_\alpha(x)\psi(x)\,dx \right| \leq \underbrace{\sup_{x\in\mathbb{R}^n} |\psi(x)|}_{=:M} \cdot \int_{|x|>R} f_\alpha(x)\,dx.$$

Then

$$\left| \int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\,dx \right| \leq c(R) + M \int_{|x|>R} f_\alpha(x)\,dx$$

Constructing Delta Families

Fix $\varepsilon > 0$.

- Choose $R > 0$ small enough so that $c(R) < \varepsilon/2$.
- Choose $\gamma > 0$ small enough so that

$$\left| \int_{|x|>R} f_\alpha(x)\, dx \right| < \frac{\varepsilon}{M} \qquad \text{for } \alpha < \gamma$$

Then

$$\left| \int_{\mathbb{R}^n} f_\alpha(x)\psi(x)\, dx \right| < \varepsilon$$

for all $\alpha < \gamma$.

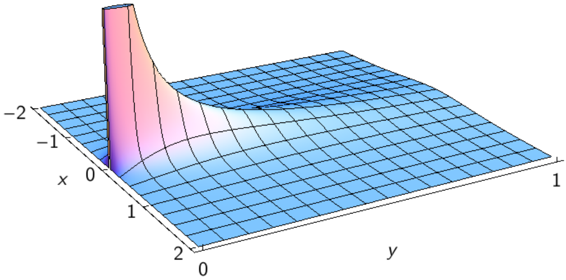
This completes the proof.

Example: $f(x) = \frac{1}{\pi(x^2+1)}$

$$f_y(x) = \frac{1}{y} f\left(\frac{x}{y}\right) = \frac{y}{\pi(x^2+y^2)}, \qquad\qquad y > 0,$$

with

$$f_y(x) \to \delta(x) \qquad\qquad \text{as } y \searrow 0$$

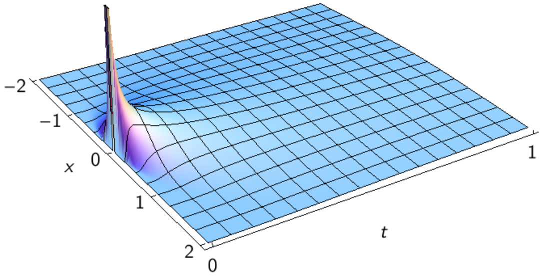


Example: $f(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$

$$f_t(x) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}, \qquad t > 0,$$

with

$$f_t(x) \to \delta(x) \qquad \text{as } t \searrow 0$$

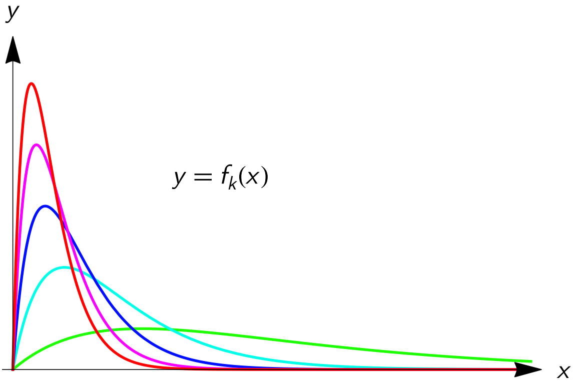


Example: $f(x) = H(x)xe^{-x}$

$$f_k(x) = k^2 H(x) x e^{-kx}, \qquad k \in \mathbb{N},$$

with

$$f_k(x) \to \delta(x) \qquad \text{as } k \to \infty$$

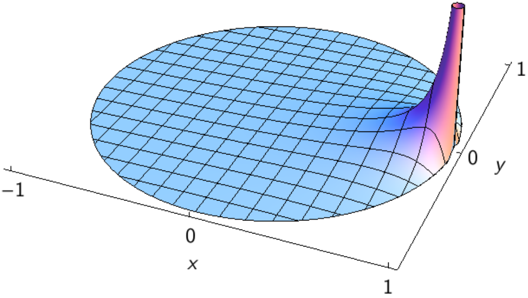


The Poisson Kernel

$$f_r(\theta) = \begin{cases} \frac{1}{2\pi} \cdot \frac{1-r^2}{1+r^2-2r\cos\theta} & |\theta| \le \pi, \\ 0 & |\theta| > \pi, \end{cases} \qquad 0 \le r < 1,$$

where

$$f_r(\theta) \to \delta(\theta) \qquad \text{as } r \nearrow 1$$

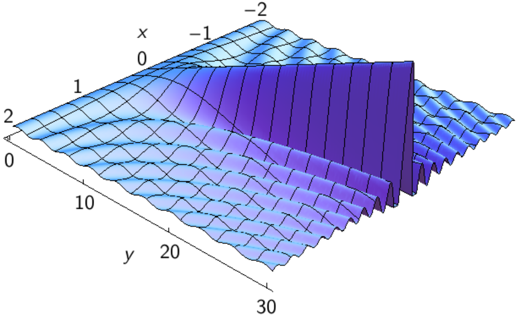


The Dirichlet Kernel

$$f_R(x) = \frac{1}{2\pi}\int_{-R}^{R} e^{i\omega x}\,d\omega = \frac{\sin(Rx)}{\pi x}, \qquad R > 0, \qquad (1.3.1)$$

where

$$f_R(x) \to \delta(x) \qquad \text{as } R \to \infty$$



Distributions

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Functions of Rapid Decrease

Definition.

$$\mathcal{S}(\mathbb{R}^n) := \left\{\varphi \in C^\infty(\mathbb{R}^n) \colon \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^n\right\}$$

is called the space of

- Schwartz functions or
- functions of rapid decrease.

Examples.

(i) $e^{-x^2} \in \mathcal{S}(\mathbb{R})$

(ii) $\frac{1}{1+x^2} \notin \mathcal{S}(\mathbb{R})$ (decay too slow)

(iii) $e^{-|x|} \notin \mathcal{S}(\mathbb{R})$ (not smooth at $x = 0$)

Properties of Functions of Rapid Decrease

- $\mathcal{S}(\mathbb{R}^n)$ is a vector space
- If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $x^\alpha D^\beta \varphi \in \mathcal{S}(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}^n$
- If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$|\varphi(x)| \leq \frac{C}{1 + |x|^{2n}}$$

for some $C > 0$. Therefore,

$$\int_{\mathbb{R}^n} |\varphi(x)|\, dx \leq C \int_{\mathbb{R}^n} \frac{dx}{1 + |x|^{2n}} < \infty$$

The Fourier Transform

Definition. The Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is the function $\mathcal{F}\varphi$ given by

$$(\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx, \qquad \xi \in \mathbb{R}^n.$$

We also write $\hat{\varphi}$ for $\mathcal{F}\varphi$. Here

$$\langle x, \xi\rangle = \sum_{k=1}^{n} x_k \xi_k$$

(Euclidean scalar product)

The integral on the right exists because

$$\left| \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx \right| \le \int_{\mathbb{R}^n} \underbrace{\left| e^{-i\langle x,\xi\rangle} \right|}_{=1} \cdot |\varphi(x)|\, dx < \infty.$$

Basic Property of the Fourier Transform

Proposition. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\mathcal{F}\big[D^\alpha\big((-ix)^\beta\varphi(x)\big)\big](\xi) = (i\xi)^\alpha D^\beta(\mathcal{F}\varphi)(\xi)$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^n$.

Proof for $n = 1$.

$$\begin{aligned}\widehat{\varphi'}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi'(x)e^{-i\xi x}\,dx \\ &= \underbrace{\frac{1}{\sqrt{2\pi}}(-i\xi)e^{-i\xi x}\varphi(x)\Big|_{-\infty}^{\infty}}_{=0} - \frac{(-i\xi)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)e^{-i\xi x}\,dx \\ &= (i\xi)\hat{\varphi}(\xi).\end{aligned}$$

Basic Property of the Fourier Transform

$$\begin{aligned}
\widehat{(-ix\varphi)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-ix)\varphi(x)e^{-i\xi x}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\frac{d}{d\xi}e^{-i\xi x}\,dx \\
&= \hat{\varphi}'(\xi).
\end{aligned}$$

The proof for general n is completely analogous.

Example: Fourier Transform of $f(x) = e^{-x^2/2}$

$$f : \mathbb{R} \to \mathbb{R}, \qquad\qquad f(x) = e^{-x^2/2}.$$

Instead of calculating

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{-x^2/2}\, dx.$$

(e.g., by contour integration in the complex plane) we use the properties of the Fourier transform.

Example: Fourier Transform of $f(x) = e^{-x^2/2}$

Consider

$$g(x) = -xe^{-x^2/2} = -i(-ix)f(x) = \frac{d}{dx}f(x).$$

Then

$$\hat{g}(\xi) = \widehat{f'}(\xi) = i\xi\hat{f}(\xi), \qquad \hat{g}(\xi) = -i \cdot \widehat{(-ix)f}(\xi) = -i\hat{f}'(\xi).$$

so

$$\hat{f}'(\xi) = -\xi\hat{f}(\xi)$$

Furthermore,

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\cdot 0}e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}\,dx = 1$$

Example: Fourier Transform of $f(x) = e^{-x^2/2}$

Initial value problem:

$$\hat{f}'(\xi) = -\xi \hat{f}(\xi), \qquad \hat{f}(0) = 1$$

Unique solution:

$$\hat{f}(\xi) = e^{-\xi^2/2}.$$

Thus,

$$\hat{f} = f$$

(f is a fixed point of the Fourier transform)

Convergence and Continuity in $\mathcal{S}(\mathbb{R}^n)$

Definition. Let (φ_m) be a sequence with $\varphi_m \in \mathcal{S}(\mathbb{R}^n)$, $m \in \mathbb{N}$.

(i) (φ_m) is a null sequence in $\mathcal{S}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta \varphi_m(x)| \xrightarrow{m \to \infty} 0.$$

(ii) A linear map $L \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is said to be continuous if

$$L\varphi_m \xrightarrow{m \to \infty} 0$$

for all null sequences (φ_m) in $\mathcal{S}(\mathbb{R}^n)$.

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

Theorem. The Fourier transform is a continuous, linear map

$$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Proof for $n = 1$.

1) $\varphi \in \mathcal{S}(\mathbb{R}) \;\Rightarrow\; \hat{\varphi} \in \mathcal{S}(\mathbb{R})$

We need to show that $\hat{\varphi} \in C^\infty(\mathbb{R})$ and

$$\sup_{\xi \in \mathbb{R}} \left| \xi^j \frac{d^k \hat{\varphi}(\xi)}{d\xi^k} \right| < \infty$$

for all $j, k \in \mathbb{N}$.

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

$\hat{\varphi} \in C^\infty(\mathbb{R})$:

$$\frac{d^k}{d\xi^k}\hat{\varphi}(\xi) = [\widehat{(-ix)^k\varphi}](\xi)$$

The right-hand side exists since $(-ix)^k\varphi \in \mathcal{S}(\mathbb{R})$ for any $k \in \mathbb{N}$.

$\sup\limits_{\xi\in\mathbb{R}} \left|\xi^j \frac{d^k\hat{\varphi}(\xi)}{d\xi^k}\right| < \infty$:

$$\begin{aligned}\left|(-i\xi)^j \frac{d^k\hat{\varphi}(\xi)}{d\xi^k}\right| &\le \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|\frac{d^j}{dx^j}\big((ix)^k\varphi(x)\big)\right| \cdot |e^{-ix\xi}|\,dx \\ &\le \frac{1}{\sqrt{2\pi}}\sup_{x\in\mathbb{R}}\left|(1+x^2)\underbrace{\frac{d^j}{dx^j}\big((ix)^k\varphi(x)\big)}_{\in\mathcal{S}(\mathbb{R})}\right|\underbrace{\int_{\mathbb{R}}\frac{1}{1+x^2}\,dx}_{<\infty} \\ &< \infty\end{aligned}$$

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

2) $\mathcal{F}$ is linear

For $\lambda, \mu \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\mathcal{F}[\lambda\varphi + \mu\psi] &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} [\lambda\varphi(x) + \mu\psi(x)] e^{-ix\xi}\, d\xi \\
&= \lambda \cdot (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, d\xi \\
&\qquad + \mu \cdot (2\pi)^{-n/2} \int_{\mathbb{R}^n} \psi(x) e^{-ix\xi}\, d\xi \\
&= \lambda\mathcal{F}\varphi + \mu\mathcal{F}\psi
\end{aligned}$$

$\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is Linear and Continuous

3) $\mathcal{F}$ is continuous

We have seen that for some $C > 0$

$$\sup_{\xi\in\mathbb{R}} \left|\xi^j \frac{d^k \hat{\varphi}(\xi)}{d\xi^k}\right| \le C \cdot \sup_{x\in\mathbb{R}} \left|(1+x^2)\frac{d^j}{dx^j}\big((ix)^k\varphi(x)\big)\right|$$

If (φ_m) is a null sequence, the right-hand side converges to zero. Therefore, the left-hand side converges to zero and $(\widehat{\varphi_m})$ is also a null sequence.

This completes the proof.

The Fourier Inversion Formula

Theorem. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) e^{i\langle x, \xi \rangle} \, d\xi$$

Proof for $n = 1$.

Suppose first that $\varphi \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{\varphi}(\xi) e^{ix\xi} \, d\xi &= \frac{1}{2\pi} \int_{-R}^{R} e^{ix\xi} \int_{-\infty}^{\infty} \varphi(\omega) e^{-i\omega\xi} \, d\omega \, d\xi \\ &= \int_{-\infty}^{\infty} \varphi(\omega) \frac{1}{2\pi} \int_{-R}^{R} e^{i(x-\omega)\xi} \, d\xi \, d\omega \\ &= \int_{-\infty}^{\infty} \varphi(x-y) \frac{1}{2\pi} \int_{-R}^{R} e^{iy\xi} \, d\xi \, dy \end{aligned}$$

The Fourier Inversion Formula

Recall that the Dirichlet kernel

$$\frac{1}{2\pi}\int_{-R}^{R} e^{iy\xi}\,d\xi$$

is a delta family that converges to $\delta(y)$ as $R \to \infty$.

Therefore,

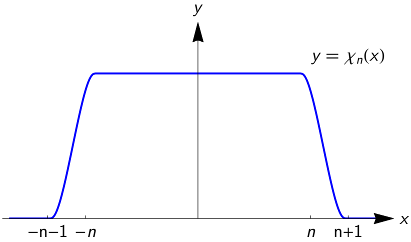
$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(\xi)e^{ix\xi}\,d\xi &= \lim_{R\to\infty}\int_{-\infty}^{\infty}\varphi(x-y)\frac{1}{2\pi}\int_{-R}^{R} e^{iy\xi}\,d\xi\,dy \\ &= \varphi(x-0) \\ &= \varphi(x)\end{aligned}$$

This proves the statement for $\varphi \in \mathcal{D}(\mathbb{R})$.

The Fourier Inversion Formula

Let $\chi_n \in \mathcal{D}(\mathbb{R})$, $n \in \mathbb{N}$, be cut-off functions with

$$\chi_n(x) = \begin{cases} 1 & |x| < n \\ 0 & |x| > n+1 \end{cases}$$



Now suppose $\varphi \in \mathcal{S}(\mathbb{R})$. Then $\chi_n\varphi \in \mathcal{D}(\mathbb{R})$ and

$$\chi_n\varphi \xrightarrow{n\to\infty} \varphi$$

in $\mathcal{S}(\mathbb{R})$.

The Fourier Inversion Formula

The Fourier inversion formula states simply that

$$\hat{\hat{\varphi}}(-x) = \varphi(x) \qquad\qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

We have proven the inversion formula for all test functions, so

$$\widehat{\widehat{\chi_n\varphi}}(-x) = \chi_n\varphi(x) \qquad\qquad \text{for all } n \in \mathbb{N}.$$

Since the Fourier transform and the reflection $\varphi(x) \mapsto \varphi(-x)$ are continuous, we can let $n \to \infty$ on both sides, yielding

$$\hat{\hat{\varphi}}(-x) = \varphi(x).$$

This completes the proof.

Properties of the Fourier Transform

Suppose $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

(i) (Dilation) For $\alpha \in \mathbb{R}_+$ we define $D_\alpha \varphi(x) = \alpha^{n/2}\varphi(\alpha x)$. Then

$$\mathcal{F}(D_\alpha \varphi) = D_{1/\alpha}\mathcal{F}\varphi.$$

(ii) (Translation) For $y \in \mathbb{R}^n$ we define $\tau_y \varphi(x) = \varphi(x - y)$. Then

$$(\mathcal{F}\tau_y\varphi)(\xi) = e^{-i\langle y,\xi\rangle}\mathcal{F}\varphi(\xi).$$

(iii) (Unitarity) Let $\langle \varphi, \psi \rangle_{L^2} := \int_{\mathbb{R}^n} \overline{\varphi(x)}\psi(x)\,dx$. Then

$$\langle \hat{\varphi}, \hat{\psi} \rangle_{L^2} = \langle \varphi, \psi \rangle_{L^2}.$$

The Convolution

Definition. The convolution of $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$(\varphi * \psi)(y) := \int_{\mathbb{R}^n} \varphi(y-x)\psi(x)\,dx.$$

Properties. For $\varphi, \psi, \chi \in \mathcal{S}(\mathbb{R}^n)$,

i) $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$

ii) $\varphi * \psi = \psi * \varphi$

iii) $\varphi * (\psi * \chi) = (\varphi * \psi) * \chi$

iv) $(2\pi)^{n/2}\widehat{\varphi \cdot \psi} = \hat{\varphi} * \hat{\psi}$

v) $\widehat{\varphi * \psi} = (2\pi)^{n/2}\hat{\varphi} \cdot \hat{\psi}$

Distributions

Operations on Distributions

Families of Distributions

The Classical Fourier Transform

Tempered Distributions and the Fourier Transform

Tempered Distributions

A linear functional on $\mathcal{S}(\mathbb{R}^n)$ is a map $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ such that

$$T(\lambda\varphi_1 + \mu\varphi_2) = \lambda T\varphi_1 + \mu T\varphi_2$$

for $\varphi_1, \varphi_2 \in \mathcal{D}$, $\lambda, \mu \in \mathbb{C}$.

T is said to be continuous if

$$\varphi_m \to 0 \qquad \Rightarrow \qquad T\varphi_m \to 0$$

A continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ is called a tempered distribution on $\mathbb{R}^n$.

The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

$\mathcal{S}'(\mathbb{R}^n)$ is a vector space.

Tempered Distributions

Since

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$$

it is easy to see that

$$\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

so every tempered distribution is also a distribution.

If $T_g \in \mathcal{S}'(\mathbb{R}^n)$ is given by

$$T_g\varphi := \int_{\mathbb{R}^n} g(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

for some function g we simply write $g \in \mathcal{S}'(\mathbb{R}^n)$.

Examples of Tempered Distributions

- $T_\delta \colon \varphi \mapsto \varphi(0)$ is a tempered distribution on $\mathbb{R}^n$
- $g(x) = x^2$ is a tempered distribution on $\mathbb{R}$.
- $g(x) = e^{x^2}$ is not a tempered distribution on $\mathbb{R}$, since

$$\int_{-\infty}^{\infty} e^{x^2} \varphi(x)\, dx$$

  does not exist for all Schwartz functions φ, e.g., not for $\varphi(x) = e^{-x^2}$.

The term "tempered" refers to the growth of g at infinity, which can not be too rapid.

The Fourier Transform for Tempered Distributions

Definition. The Fourier transform of $T \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\hat{T}\varphi := T\hat{\varphi}.$$

where $\hat{\varphi}$ is the Fourier transform of the Schwartz function φ.

We also write

$$\mathcal{F}T \quad \text{for} \quad \hat{T}.$$

Remarks.

- Since $\hat{\varphi} \in \mathcal{S}$ if $\varphi \in \mathcal{S}$, the right-hand side is well-defined.
- Since the Fourier transform and T are continuous and linear, $\hat{T}$ will be continuous and linear. Therefore, $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$.

The Fourier Transform for Tempered Distributions

Since

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

we check that our definition is compatible with the previous one for Schwartz functions: If $g \in \mathcal{S}(\mathbb{R}^n)$, then $T_g \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\begin{aligned}
(\hat{T}_g)\varphi = T_g(\hat{\varphi}) &= \int_{\mathbb{R}^n} g(\xi)\hat{\varphi}(\xi)\,d\xi \\
&= \int_{\mathbb{R}^n} g(\xi)\cdot(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)\,dx\,d\xi \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\xi)e^{-i\langle x,\xi\rangle}\,d\xi\,\varphi(x)\,dx \\
&= \int_{\mathbb{R}^n} \hat{g}(x)\varphi(x)\,dx \\
&= T_{\hat{g}}\varphi.
\end{aligned}$$

The Fourier Transform for Tempered Distributions

We have extended the Fourier transform to a continuous, bijective map

$$\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

The inverse is given by

$$(\mathcal{F}^{-1}T)(\varphi) = T(\mathcal{F}^{-1}\varphi).$$

Example.

$$\begin{aligned}\hat{T}_\delta\varphi = T_\delta\hat{\varphi} = \hat{\varphi}(0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{e^{-ix\cdot 0}}_{=1}\varphi(x)\,dx \\ &= T_{1/\sqrt{2\pi}}\varphi\end{aligned}$$

so

$$\hat{\delta} = 1/\sqrt{2\pi}.$$

Example: $g \in \mathcal{S}'(\mathbb{R})$, $g(x) = 1$

Since $T_g = \sqrt{2\pi}\,\hat{T}_\delta$,

$$\hat{T}_g\varphi = T_g\hat{\varphi} = \sqrt{2\pi}\,\hat{T}_\delta\hat{\varphi} = \sqrt{2\pi}\,T_\delta\hat{\hat{\varphi}}$$

Since $\hat{\hat{\varphi}}(x) = \varphi(-x)$, we have

$$\hat{T}_g\varphi = \sqrt{2\pi}\varphi(-0) = \sqrt{2\pi}\,T_\delta\varphi$$

so

$$\hat{1} = \sqrt{2\pi}\delta$$

Formally,

$$\hat{1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\,d\xi = \sqrt{2\pi}\delta(x)$$

which coincides with the Dirichlet kernel as a delta family.

Example: $g \in \mathcal{S}'(\mathbb{R})$, $g(x) = x$

$$
\begin{aligned}
\hat{T}_g \varphi = T_g \hat{\varphi} &= \int_{-\infty}^{\infty} \xi \cdot \hat{\varphi}(\xi)\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi e^{-ix\xi} \varphi(x)\, dx\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i \frac{d}{dx}\big(e^{-ix\xi}\big) \varphi(x)\, dx\, d\xi \\
&= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\xi} \varphi'(x)\, dx\, d\xi \\
&= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{-ix\xi}\, d\xi}_{=2\pi\delta(x)}\, \varphi'(x)\, dx \\
&= -i\sqrt{2\pi}\varphi'(0).
\end{aligned}
$$

Example: The Heaviside function $H \in \mathcal{S}'(\mathbb{R})$

$$\hat{T}_H\varphi = T_H\hat{\varphi} = \int_0^\infty \hat{\varphi}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_{-\infty}^\infty e^{-ix\xi}\varphi(x)\,dx\,d\xi$$

$$= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(x) \int_0^R e^{-ix\xi}\,d\xi\,dx$$

$$= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(x) \frac{e^{-iRx}-1}{-ix}\,dx$$

$$= \lim_{R\to\infty} \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(x) \frac{\cos(Rx)-1}{x}\,dx$$

$$+ \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(x) \frac{\sin(Rx)}{x}\,dx$$

Example: The Heaviside function $H \in \mathcal{S}'(\mathbb{R})$

It can be shown that

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \varphi(x) \frac{1-\cos(Rx)}{x}\, dx = \mathcal{P}\left(\frac{1}{x}\right)\varphi.$$

Furthermore,

$$\frac{\sin(Rx)}{\pi x}$$

is a delta family as $R \to \infty$ (the Dirichlet kernel again) so

$$\hat{H}(\xi) = \frac{-i}{\sqrt{2\pi}}\mathcal{P}\left(\frac{1}{\xi}\right) + \sqrt{\frac{\pi}{2}}\delta(\xi).$$

The Convolution for Tempered Distributions

For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ the convolution is defined by

$$(\varphi * \psi)(y) := \int_{\mathbb{R}^n} \varphi(y-x)\psi(x)\,dx.$$

Does not work for two distributions!

Definition. Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then $T * \psi \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$(T * \psi)(\varphi) := T(\widetilde{\psi} * \varphi),$$

where $\widetilde{\psi}(x) = \psi(-x)$.

If $T = T_g$ for some $g \in \mathcal{S}(\mathbb{R}^n)$,

$$T_g * \psi = T_{g*\psi}.$$

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The Convolution for Tempered Distributions

Example. The convolution of the Dirac distribution with a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\begin{aligned}(T_\delta * \psi)(\varphi) &= T_\delta\biggl(\int_{\mathbb{R}} \psi(-x)\varphi((\,\cdot\,) - x)\,dx\biggr)\\ &= \int_{\mathbb{R}} \psi(-x)\varphi(0-x)\,dx\\ &= \int_{\mathbb{R}} \psi(x)\varphi(x)\,dx = T_\psi\varphi,\end{aligned}$$

so

$$\delta * \psi = \psi$$

Properties of the Convolution

For $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi, \chi \in \mathcal{S}(\mathbb{R}^n)$

(i) $D^\beta(T * \psi) = (D^\beta T) * \psi = T * D^\beta \psi,$

(ii) $(T * \psi) * \chi = T * (\psi * \chi)$

(iii) $\widehat{T * \psi} = (2\pi)^{n/2} \hat{\psi} \hat{T}$ where

$$\hat{\psi} \hat{T}(\varphi) = \hat{T}(\hat{\psi} \varphi).$$

Very useful for solving partial differential equations!

Example: The Heat Equation

Heat equation on $\mathbb{R}^n$:

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Initial condition:

$$u(x,0) = f(x), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

Assumption:

$$u(\,\cdot\,,t) \in \mathcal{S}'(\mathbb{R}^n) \qquad \text{for all } t \geq 0$$

Example: The Heat Equation

Treat $t \geq 0$ as a parameter and apply Fourier transform "with respect to the x-variable". Then

$$\frac{\partial \hat{u}}{\partial t} + |\xi|^2 \hat{u} = 0, \qquad\qquad (\xi, t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

with initial condition

$$\hat{u}(\xi, t) = \hat{f}(\xi)$$

Unique solution:

$$\hat{u}(\xi, t) = e^{-t|\xi|^2} \hat{f}(\xi)$$

Set

$$\hat{\psi}(\xi, t) := e^{-t|\xi|^2}$$

Example: The Heat Equation

Then

$$\hat{u}(\xi, t) = \hat{\psi}(\xi, t)\hat{f}(\xi)$$

By convolution properties, for $t > 0$,

$$u(x, t) = (2\pi)^{-n/2} f * \psi(\,\cdot\,, t)$$

From $\hat{\psi}(\xi, t) := e^{-t|\xi|^2}$,

$$\psi(x, t) = (2t)^{-n/2} e^{-|x|^2/(4t)}$$

Then

$$u(x, t) = f * p(\,\cdot\,, t)$$

where

$$p(x, t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

(Heat kernel)

Example: The Heat Equation

Theorem. The heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{1.5.1}$$

with initial condition

$$u(x,0) = f(x), \qquad f \in \mathcal{S}'(\mathbb{R}^n).$$

has the unique solution $u(\,\cdot\,,t) \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$u(x,t) = f * p(x,t), \qquad t > 0,$$

where

$$p(x,t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

Example: The Heat Equation

If $f \in \mathcal{S}(\mathbb{R}^n)$, then $u(\,\cdot\,, t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t > 0$.

Furthermore,

$$u(x, t) = f * p(x, t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/(4t)}\, dy$$

Since $p(\,\cdot\,, t)$ is a delta family as $t \searrow 0$, we see that

$$\lim_{t \searrow 0} u(x, t) = f(x),$$

as expected.

These formulas hold also if f is only continuous and bounded.

The uniqueness of the solution requires $u(\,\cdot\,, t) \in \mathcal{S}'(\mathbb{R}^n)$. There exist other solutions of the heat equation with initial condition that “blow up” at infinity.

Part II

Boundary Value Problems for Ordinary Differential Equations

Differential Operators and Types of Solutions

Causal Fundamental Solutions and Initial Value Problems

Second-Order Boundary Value Problems

Adjoint BVPs and Higher-Order Equations

Solvability Conditions and Modified Green's Functions

Differential Operators and Types of Solutions

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Linear Ordinary Differential Operators

In this section, we consider linear, ordinary differential operators of order p:

$$L = \sum_{k=0}^{p} a_k(x) \frac{d^k}{dx^k},$$

with coefficient functions $a_k \in C^\infty((a, b), \mathbb{R})$, $[a, b] \subset \mathbb{R}$.

The Formal Adjoint

Recall that for $T \in \mathcal{D}'(\mathbb{R})$

$$T'\varphi := -T(\varphi') \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

More generally:

Definition. The operator L^* such that

$$(LT)(\varphi) = T(L^*\varphi)$$

for any $T \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$ is called the formal adjoint of L.

If $L = L^*$, we say that L is formally self-adjoint.

Example: Second-Order Operator

$$L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

For $T \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}(LT)(\varphi) &= a_2 \cdot T''(\varphi) + a_1 \cdot T'(\varphi) + a_0 \cdot T(\varphi)\\ &= T''(a_2 \cdot \varphi) + T'(a_1 \cdot \varphi) + T(a_0 \cdot \varphi)\\ &= T\left(a_2''\varphi + 2a_2'\varphi' + a_2\varphi'' - a_1'\varphi - a_1\varphi' + a_0\varphi\right)\end{aligned}$$

so the formal adjoint is

$$L^* = a_2\frac{d^2}{dx^2} + (2a_2' - a_1)\frac{d}{dx} + (a_2'' - a_1' + a_0)$$

Green's Formula and the Conjunct

On $C([a,b],\mathbb{R})$ we can define an inner product by

$$\langle v, u\rangle_{L^2([a,b])} := \int_a^b v(x)u(x)\,dx.$$

Definition. The relation

$$\langle v, Lu\rangle_{L^2([a,b])} - \langle L^* v, u\rangle_{L^2([a,b])} = J(u,v)\big|_a^b$$

obtained by integration by parts is called Green's formula for L.

The bilinear form J is the conjunct of L.

Example: Second-Order Operator

Suppose $u, v \in C^2((a,b),\mathbb{R})$. Then

$$\begin{aligned}\langle v, Lu\rangle_{L^2([a,b])} &:= \int_a^b v(x)(Lu)(x)\,dx \\ &= \int_a^b v(x)\big(a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x)\big)\,dx \\ &= \langle L^* v, u\rangle_{L^2([a,b])} + J(u,v)\big|_a^b\end{aligned}$$

where the conjunct is

$$J(u,v) = a_2\big(vu' - uv'\big) + \big(a_1 - a_2'\big)uv$$

General Ordinary Differential Operators

If

$$Lu(x) = \sum_{k=0}^{p} a_k(x) \frac{d^k}{dx^k} u(x),$$

then

$$L^* v(x) = \sum_{k=0}^{p} (-1)^k \frac{d^k}{dx^k} \big(a_k(x) v(x) \big).$$

The conjunct is

$$J(u, v) = \sum_{k=1}^{p} \sum_{i+j=k-1} (-1)^i D^i(a_k v) D^j u$$

J contains only derivatives up to order $p-1$.

Lagrange's identity

Fix $a \in \mathbb{R}$ and consider $b = x$ variable. Differentiating Green's formula

$$\int_a^x v(y)(Lu)(y)\,dy - \int_a^x L^*v(y)u(y)\,dy = J(u,v)\big|_a^x$$

yields

$$vLu - uL^*v = \frac{d}{dx}J(u,v).$$

Lagrange's identity for L.

Classical Solutions

Consider the differential equation

$$Lu = f \qquad \text{on } \Omega$$

where

- L is an ordinary or partial differential operator
- Ω is a domain in $\mathbb{R}^n$
- f is a continuous function on Ω.

A classical solution is a function $u \in C^p(\Omega)$ such that

$$Lu = f \qquad \text{on } \Omega$$

in the usual sense.

Weak Solutions

Now let

$$Lu = f \qquad \text{on } \Omega$$

where $f \in L^1_{\text{loc}}(\Omega)$, i.e.,

$$\int_B |f(x)|\,dx < \infty \qquad \text{for any bounded set } B \subset \Omega$$

A weak solution is a function $u \in L^1_{\text{loc}}(\Omega)$ such that

$$(LT_u)(\varphi) = T_f\varphi$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \Omega$.

Example: $xu'(x) = 0$ on $\mathbb{R}$

All classical solutions have the form

$$u(x) = c, \qquad c \in \mathbb{R}$$

We show that the Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0 \end{cases}$$

is a weak solution.

Example: $xu'(x) = 0$ on $\mathbb{R}$

For any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$
\begin{aligned}
xT_H'(\varphi) &= T_H'(x\varphi) \\
&= -T_H((x\varphi)') \\
&= -\int_{\mathbb{R}} H(x)\big(x\varphi(x)\big)'\,dx \\
&= -\int_0^\infty \big(\varphi(x) + x\varphi'(x)\big)\,dx \\
&= -\int_0^\infty \varphi(x)\,dx - x\varphi(x)\big|_0^\infty + \int_0^\infty \varphi(x)\,dx \\
&= 0
\end{aligned}
$$

Example: $\dfrac{\partial u}{\partial x_1}(x_1, x_2) = 0$ on $\mathbb{R}^2$

Any locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ that does not depend on x_1 is a weak solution, since

$$\begin{aligned}
\left(\frac{\partial}{\partial x_1} T_f\right)\varphi &= T_f\left(-\frac{\partial \varphi}{\partial x_1}\right) \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_2)\varphi_{x_1}(x_1, x_2)\,dx_1\,dx_2 \\
&= -\int_{-\infty}^{\infty} f(x_2)\underbrace{\int_{-\infty}^{\infty}\varphi_{x_1}(x_1, x_2)\,dx_1}_{=0}\,dx_2 \\
&= 0.
\end{aligned}$$

Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

d'Alembert's classical solution to the wave equation:

$$u(x,t) = f(x-t) + g(x+t)$$

for any $f, g \in C^2(\mathbb{R})$.

We show that

$$u(x,t) = H(x-t)$$

is a weak solution.

Note

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}, \qquad L^* = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = L$$

Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$(LT_{H(x-t)})(\varphi) = T_{H(x-t)}(L\varphi) = \int_{\mathbb{R}^2} H(x-t)L\varphi(x,t)\,dx\,dt = 0.$$

We perform a change of variables in the integral, setting

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \begin{pmatrix} x-t \\ x+t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ t \end{pmatrix}.$$

We note that $\det A = 2$ and define $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^2)$ by

$$\widetilde{\varphi}(\xi,\tau)\big|_{(\xi,\tau)=(x-t,x+t)} = \widetilde{\varphi}(x-t, x+t) := \varphi(x,t).$$

Example: $u_{xx} - u_{tt} = 0$ on $\mathbb{R}^2$

Then

$$\varphi_{xx}(x,t) - \varphi_{tt}(x,t) = \widetilde{\varphi}_{\xi\tau}(\xi,\tau).$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} H(x-t) L\varphi(x,t)\,dx\,dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\xi)\widetilde{\varphi}_{\xi\tau}(\xi,\tau)\,d\xi\,d\tau \\ &= \int_0^{\infty}\int_{-\infty}^{\infty} \widetilde{\varphi}_{\xi\tau}(\xi,\tau)\,d\tau\,d\xi \\ &= \int_0^{\infty} \underbrace{\widetilde{\varphi}_{\xi}(\xi,\tau)\Big|_{-\infty}^{\infty}}_{=0}\,d\xi \\ &= 0\end{aligned}$$

which verifies the assertion.

Classical and Weak Solutions

Lemma. Let $f \in C(\Omega)$. Then

(i) a classical solution of $Lu = f$ is also a weak solution.

(ii) a weak solution u such that $u \in C^p(\Omega)$ is also a classical solution.

Proof.

(i) Let u be a classical solution of $Lu = f$. Then for any $\varphi \in \mathcal{D}(\Omega)$,

$$(LT_u)(\varphi) = T_u(L^*\varphi) = \int_\Omega uL^*\varphi = \underbrace{-J(u,\varphi)|_{\partial\Omega}}_{\substack{= 0 \text{ since} \\ \operatorname{supp}\varphi \subset \Omega}} + \int_\Omega \varphi Lu$$

$$= \int_\Omega f\varphi = T_f\varphi.$$

Classical and Weak Solutions

(ii) Let $u \in C^p(\Omega)$ be a weak solution of $Lu = f$. Then for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega f\varphi = T_u(L^*\varphi) = \int_\Omega uL^*\varphi = \underbrace{-J(u,\varphi)|_{\partial\Omega}}_{=0} + \int_\Omega \varphi Lu,$$

so

$$\int_\Omega (Lu - f)\varphi = 0 \qquad\qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We will show that this implies

$$Lu(x) = f(x) \qquad\qquad \text{for all } x \in \Omega$$

so u is a classical solution.

Classical and Weak Solutions

Suppose that

$$Lu(x_0) - f(x_0) > 0 \qquad \text{for some } x_0 \in \Omega$$

Since $Lu - f$ is continuous, there exists some neighborhood $B_\varepsilon(x_0)$ such that $Lu - f > 0$ on $B_\varepsilon(x_0)$.

We can find a cut-off function $\varphi \in C_0^\infty(B_\varepsilon(x_0))$ such that $\varphi \geq 0$ on $B_\varepsilon(x_0)$.

But then

$$\int_\Omega (Lu - f)\varphi > 0$$

which is a contradiction.

Thus, $Lu = f$ on Ω. This completes the proof.

Differential Operators and Types of Solutions

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Distributional Solutions

The most general problem for a differential operator L on a domain $\Omega \subset \mathbb{R}^n$ is

$$LT = S \qquad\qquad \text{on } \Omega$$

with given $S \in \mathcal{D}'(\mathbb{R}^n)$.

$T \in \mathcal{D}'(\mathbb{R}^n)$ is said to be a distributional solution if

$$(LT)(\varphi) = S\varphi$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \in \Omega$.

Note: If S is a regular distribution, then a regular distributional solution T is also a weak solution.

Fundamental Solutions

Definition. Let $\xi \in \mathbb{R}^n$ be fixed. A solution $E(\,\cdot\,;\xi) \in \mathcal{D}'(\mathbb{R}^n)$ of

$$LE(x;\xi) = \delta_\xi(x) = \delta(x-\xi)$$

is said to be a a fundamental solution for L with pole at ξ.

Note:

(i) E is a distributional solution of $LE = \delta(x-\xi)$. Often, but always, E is a locally integrable function..

(ii) Fundamental solutions are not unique; they may differ by addition of a solution of $Lu = 0$.

(iii) If the operator L has constant coefficients, then

$$E(x;\xi) = E(x-\xi;0).$$

Fundamental Solutions

Example. We have seen that

$$\Delta\left(\frac{1}{4\pi}\frac{1}{|x|}\right)=\delta(x) \qquad\qquad \text{in } \mathbb{R}^3.$$

Hence,

$$E(x;\xi)=\frac{1}{4\pi}\frac{1}{|x-\xi|}$$

is a fundamental solution with pole at ξ.

The same is true for

$$E(x;\xi)+u(x)$$

where u is any solution of $\Delta u=0$, e.g.,

$$u(x_1,x_2,x_3)=x_1x_2.$$

Causal Fundamental Solutions

Definition. Let

$$L = a_p(t)\frac{d^p}{dt^p} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)$$

where $a_0, a_1, \dots, a_p$ are continuous functions defined on $\mathbb{R}$.

A fundamental solution $E(\,\cdot\,;\xi)\colon \mathbb{R} \to \mathbb{C}$ with pole at ξ is said to be causal if

$$E(x,\xi) = 0 \qquad\qquad \text{for } x < \xi.$$

Heuristic Construction

Suppose $E(t;\tau)$ is a causal fundamental solution with pole at τ, i.e.,

$$LE = a_p(t)\frac{d^pE}{dt^p} + \cdots + a_1(t)\frac{dE}{dt} + a_0(t)E = \delta(t-\tau)$$

and $E(t;\tau) = 0$ for $t < \tau$.

Assumption. $a_p(\tau) \neq 0$.

Here $E \in \mathcal{D}'(\mathbb{R})$. Let $E_{\text{Prim}} \in \mathcal{D}'(\mathbb{R})$ be a primitive of E, i.e., a distribution such that

$$E'_{\text{Prim}} = E.$$

(It can be shown that for any E such a distribution exists.)

Heuristic Construction

Suppose that E_{Prim} satisfies

$$a_p(t)\frac{d^p E_{\text{Prim}}}{dt^p} + \cdots + a_0(t)E_{\text{Prim}} = H(t-\tau).$$

Then $LE = \delta(t-\tau)$ and E is a fundamental solution.

The right-hand side is a locally integrable function which is discontinuous only at $t=\tau$.

We expect a classical solution E_{Prim}, i.e.,

$$E_{\text{Prim}} \in C^{(p-1)}(\mathbb{R}) \cap C^p(\mathbb{R}\setminus\{\tau\})$$

We also suppose

$$E_{\text{Prim}}(t,\tau) = 0 \qquad \text{for } t<\tau.$$

Heuristic Construction

Then for any $t < \tau$

$$E_{\text{Prim}}(t;\tau) = E'_{\text{Prim}}(t;\tau) = \cdots = E^{(p-1)}_{\text{Prim}}(t;\tau) = 0.$$

Since E_{Prim} is a classical solution,

$$E_{\text{Prim}}(\tau;\tau) = E'_{\text{Prim}}(\tau;\tau) = \cdots = E^{(p-1)}_{\text{Prim}}(\tau;\tau) = 0.$$

This implies

$$E(\tau;\tau) = E'(\tau;\tau) = \cdots = E^{(p-2)}(\tau;\tau) = 0.$$

We need one more initial condition.

Heuristic Construction

We divide

$$LE = a_p(t)\frac{d^pE}{dt^p} + \cdots + a_1(t)\frac{dE}{dt} + a_0(t)E = \delta(t-\tau)$$

by a_p, integrate and taking the limit:

On the right-hand side

$$\lim_{\varepsilon\to 0}\int_{\tau-\varepsilon}^{\tau+\varepsilon} \frac{1}{a_p(t)}\delta(t-\tau)\,dt = \frac{1}{a_p(\tau)}$$

On the left, by continuity,

$$\lim_{\varepsilon\to 0}\int_{\tau-\varepsilon}^{\tau+\varepsilon} \left(\frac{d^pE}{dt^p} + \cdots + \frac{a_1(t)}{a_p(t)}\frac{dE}{dt} + \frac{a_0(t)}{a_p(t)}E\right)dt = E^{(p-1)}(\tau;\tau)$$

Candidate for the Causal Fundamental Solution

We expect that, at least for $t > \tau$, $E(t; \tau)$ coincides with the solution $u_\tau(t)$ of

$$\frac{d^p u_\tau}{dt^p} + \cdots + a_1(t)\frac{du_\tau}{dt} + a_0(t)u_\tau = 0$$

with initial conditions

$$u_\tau(\tau) = u'_\tau(\tau) = \cdots = u_\tau^{(p-2)}(\tau) = 0, \qquad u_\tau^{(p-1)}(\tau) = \frac{1}{a_p(\tau)}.$$

We hence define the candidate

$$E(t; \tau) := H(t - \tau)u_\tau(t)$$

for the causal fundamental solution. We need to verify that

$$LE(t; \tau) = \delta(t - \tau)$$

Verification of the Causal Fundamental Solution

By Green's formula

$$
\begin{aligned}
LT_E\varphi = T_E(L^*\varphi) &= \int_{-\infty}^{\infty} H(t-\tau)u_\tau(t)L^*\varphi\,d\varphi \\
&= \int_{\tau}^{\infty} u_\tau(t)L^*\varphi\,d\varphi \\
&= \int_{\tau}^{\infty} \underbrace{(Lu_\tau)(t)}_{=0}\varphi(t)\,dt + J(u_\tau,\varphi)\Big|_{t=\tau}^{t=\infty}.
\end{aligned}
$$

Recall that

$$
J(u_\tau,\varphi) = \sum_{k=1}^{p} \sum_{i+j=k-1} (-1)^i D^i(a_k\varphi)D^j u_\tau.
$$

Since $\varphi \in C_0^\infty(\mathbb{R})$, J vanishes at infinity.

Verification of the Causal Fundamental Solution

All derivatives of u_τ of order less that $p-1$ vanish at $t=\tau$, so

$$J(u_\tau,\varphi)|_{t=\tau} = a_p(\tau)\varphi(\tau)\underbrace{D^{p-1}u_\tau(\tau)}_{=1/a_p(\tau)} = \varphi(\tau),$$

and hence

$$LT_E\varphi = \varphi(\tau),$$

as desired.

We have hereby established a method for finding causal fundamental solutions for ordinary differential operators.

Ordinary Differential Equations

Consider

$$Lu = f \qquad\qquad \text{on an open interval } I \subset \mathbb{R}$$

where

$$L = a_p(x)\frac{d^p}{dx^p} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

and

- f is piecewise continuous on the closure $\bar{I}$ of I,
- $a_0, a_1, \dots, a_p \in C(\bar{I})$,
- $a_p(x) \neq 0$ for all $x \in I$.

Initial Value Problems

Definition. An initial value problem (IVP) for L on I consists of the equation

$$Lu = f \qquad \text{on } I$$

and initial conditions at a point $x_0 \in \overline{I}$ given by

$$u(x_0) = \gamma_1, \qquad u'(x_0) = \gamma_2, \qquad \dots, \qquad u^{(p-1)}(x_0) = \gamma_p.$$

for some numbers $\gamma_1, \dots, \gamma_p \in \mathbb{R}$.

The data for the IVP is summarized by writing

$$\{f; \gamma_1, \gamma_2, \dots, \gamma_p\}_{x_0}.$$

Classical Solutions

Recall that a classical solution of the ODE

- is continuous on $\bar{I}$,
- is $p-1$ times continuously differentiable on I,
- is p times differentiable for all $x \in I$ where f is continuous,
- satisfies $Lu = f$ at all points in I where where f is continuous.

Theorem. The initial value problem

$$\begin{aligned} Lu &= f \quad \text{on } I, \\ u(x_0) &= \gamma_1, \\ &\vdots \\ u^{(p-1)}(x_0) &= \gamma_p, \end{aligned}$$

has a unique classical solution on $\bar{I}$.

Existence and Uniqueness of Solutions

The condition $a_p(x) \neq 0$ on $\bar{I}$ is essential.

Examples.

- The initial value problem

$$xu' - 2u = 0, \quad x \in \mathbb{R}, \qquad\qquad u(0) = 0$$

  has more than one solution.

- The initial value problem

$$xu' + u = 0, \quad x \in \mathbb{R}, \qquad\qquad u(0) = 0$$

  has no solution.

Linear Independence

Definition. A family $\{f_k\}_{k=1}^n$ of functions $f_1, \dots, f_n \colon I \to \mathbb{C}$ is said to be (linearly) independent if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad \text{for all } x \in I$$

with $c_1, \dots, c_n \in \mathbb{C}$ implies

$$c_1 = c_2 = \cdots = c_n = 0.$$

If $\{f_k\}_{k=1}^n$ is not independent, we say that the family is (linearly) dependent.

The Wronskian

Definition. For $f_1, \dots, f_n \in C^{(p-1)}(I)$

$$W(f_1, \dots, f_n; x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}$$

is called the Wronskian of $\{f_k\}_{k=1}^n$.

Note:

If $\{f_k\}_{k=1}^n$ is dependent, then $W(f_1, \dots, f_n; x) = 0$.

The converse is in general false!

The Wronskian

Example. Suppose $f_1, f_2 \colon (-1,1) \to \mathbb{R}$ are given by

$$f_1(x) = x^2, \qquad\qquad f_2(x) = |x| \cdot x.$$

Then f_1 and f_2 are independent, but

$$W(f_1, f_2; x) = 0 \qquad\qquad \text{for all } x \in (-1,1).$$

Abel's Formula for the Wronskian

Suppose that $u_1, \dots, u_p$ are p solutions of

$$Lu = 0 \qquad\qquad \text{on } I \subset \mathbb{R}.$$

where L is given as in the previous section.

Then Abel's formula for the Wronskian is

$$W(u_1, \dots, u_p; x) = C \cdot e^{-m(x)} \qquad \text{for all } x \in I$$

where $C \in \mathbb{R}$ is some constant and m is a particular solution of

$$m'(x) = \frac{a_{p-1}(x)}{a_p(x)}.$$

Consequence of Abel's Formula

If $u_1, \dots, u_p$ are solutions of $Lu = 0$, then

$$W(u_1, \dots, u_p; x) = 0 \quad \text{for all } x \in I$$

if and only if

$$W(u_1, \dots, u_p; x_0) = 0 \quad \text{for a single } x_0 \in I.$$

Independence of Solutions

Theorem. Let $u_1, \dots, u_p$ be solutions of $Lu = 0$. Then

- $u_1, \dots, u_p$ are dependent

if and only if

- $W(u_1, \dots, u_p; x_0) = 0$ for some $x_0 \in I$.

Proof.
The Wronskian vanishes at a single point if and only if it vanishes everywhere on I.

If the solutions are dependent, then the Wronskian vanishes.

However, the converse is not obvious.

Independence of Solutions

Suppose that $W(u_1, \dots, u_p; x_0) = 0$.

Consider the system of equations

$$\begin{aligned}
u_1(x_0)y_1 &+ u_2(x_0)y_2 + \dots + u_p(x_0)y_p = 0,\\
&\vdots\\
u_1^{(p-1)}(x_0)y_1 &+ u_2^{(p-1)}(x_0)y_2 + \dots + u_p^{(p-1)}(x_0)y_p = 0,
\end{aligned}$$

for the p unknowns $y_1, \dots, y_p$.

Since $W(u_1, \dots, u_p; x_0) = 0$, this system has a non-trivial solution

$$(y_1, \dots, y_p) \in \mathbb{C}^p.$$

Independence of Solutions

Define

$$U(x) := y_1 u_1(x) + \cdots + y_p u_p(x).$$

Then $U(x)$ solves $Lu = 0$ with

$$\begin{aligned} U(x_0) &= 0 \\ U'(x_0) &= 0 \\ &\vdots \\ U^{(p-1)}(x_0) &= 0. \end{aligned}$$

Independence of Solutions

Since the solution of an initial value problem is unique,

$$U(x) = y_1 u_1(x) + \cdots + y_p u_p(x) = 0$$

for all $x \in I$, even though not all of the $y_k \in \mathbb{C}$ vanish.

Hence, the functions $(u_1, \dots, u_p)$ are dependent.

Basis of Solutions

2.2.1. Theorem. Let $u_1, \ldots, u_p$ be solutions to the initial value problem for L on I with data

- $\{0; 1, 0, \ldots, 0\}_{x_0}$ in the case of u_1,
- $\{0; 0, 1, 0, \ldots, 0\}_{x_0}$ in the case of u_2,

$\vdots$

- $\{0; 0, \ldots, 0, 1\}_{x_0}$ in the case of u_n.

Then $\{u_1, \ldots, u_p\}$ is an independent set.

Any solution of $Lu = 0$ on I can be written in the form

$$u(x) = c_1 u_1(x) + \cdots + c_p u_p(x)$$

for some $c_1, \ldots, c_p \in \mathbb{C}$.

Basis of Solutions

The set is independent because $W(u_1, \dots, u_p; x_0) = 1 \neq 0$.

Any solution u_0 of $Lu = 0$ is completely determined by its initial values at some $x_0 \in \bar{I}$.

Since

$$u(x) := \underbrace{u_0(x_0)}_{=:c_1} u_1(x) + \underbrace{u_0'(x_0)}_{=:c_2} u_2(x) + \cdots + \underbrace{u_0^{(p-1)}(x_0)}_{=:c_p} u_p(x).$$

has just these initial values and solves $Lu = 0$,

$$u(x) = u_0(x).$$

This gives the desired representation.

Ordinary Differential Equations

We now discuss the inhomogeneous equation

$$Lu = f \qquad\qquad \text{on an open interval } I \subset \mathbb{R}$$

where

$$L = a_p(x)\frac{d^p}{dx^p} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

and

- f is piecewise continuous on the closure $\bar{I}$ of I,
- $a_0, a_1, \dots, a_p \in C(\bar{I})$,
- $a_p(x) \neq 0$ for all $x \in I$.

The Function u_ξ

Recall how to construct a causal fundamental solution for L:

Take $I = \mathbb{R}$ and fix $\xi \in \mathbb{R}$. Define u_ξ to satisfy

$$Lu_\xi = 0 \quad \text{on } \mathbb{R}$$

with initial values

$$u_\xi(\xi) = 0, \qquad \dots, \qquad u_\xi^{(p-2)}(\xi) = 0, \qquad u_\xi^{(p-1)}(\xi) = \frac{1}{a_p(\xi)}.$$

u_ξ is the solution of the IVP for L on $I = \mathbb{R}$ with data

$$\left\{0; 0, \dots, 0, \frac{1}{a_p(\xi)}\right\}_\xi .$$

Interpretation of u_ξ

We set

$$E(x,\xi) := H(x-\xi)u_\xi(x), \tag{2.2.1}$$

where H is the Heaviside function.

If the coefficient functions are smooth, $a_1 \ldots, a_p \in C^\infty(\mathbb{R})$, we can interpret L as acting on the distribution E and find

$$LE = \delta(x-\xi).$$

Solution Formula for the Inhomogeneous Equation

We would therefore expect that the solution of

$$Lu = f \quad \text{on } \mathbb{R}, \qquad u(x_0) = 0, \qquad \dots, \qquad u^{(p-1)}(x_0) = 0,$$

is given by

$$u(x) = \int_{x_0}^{\infty} E(x,\xi) f(\xi)\, d\xi = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi.$$

Note: By the chain rule,

$$u'(x) = u_x(x) f(x) + \int_{x_0}^{x} u_\xi'(x) f(\xi)\, d\xi$$

where of course

$$u_x(x) = u_\xi(x)\big|_{\xi=x}.$$

Verification of the Solution Formula

Since $u_\xi(\xi) = 0$ for any $\xi \in \mathbb{R}$, we have

$$\begin{aligned} u(x) &= \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi, \\ u'(x) &= \underbrace{u_x(x)}_{=0} f(x) + \int_{x_0}^{x} u'_\xi(x) f(\xi)\, d\xi \\ &= \int_{x_0}^{x} u'_\xi(x) f(\xi)\, d\xi \end{aligned}$$

and

$$u(x_0) = u'(x_0) = 0.$$

Verification of the Solution Formula

We continue to differentiate, yielding

$$u^{(p-1)}(x) = \underbrace{u_x^{(p-2)}(x)}_{=0} f(x) + \int_{x_0}^{x} u_\xi^{(p-1)}(x) f(\xi)\, d\xi,$$

so that u satisfies the initial conditions

$$\begin{aligned} u(x_0) &= 0, \\ u'(x_0) &= 0, \\ &\vdots \\ u^{(p-1)}(x_0) &= 0. \end{aligned}$$

Verification of the Solution Formula

Finally, at all points $x \in I$ where f is continuous,

$$\begin{aligned} u^{(p)}(x) &= \underbrace{u_x^{(p-1)}(x)}_{=1/a_p(x)} f(x) + \int_{x_0}^{x} u_\xi^{(p)}(x) f(\xi)\, d\xi \\ &= \frac{f(x)}{a_p(x)} + \int_{x_0}^{x} u_\xi^{(p)}(x) f(\xi)\, d\xi. \end{aligned}$$

This implies that

$$\begin{aligned} Lu &= a_p(x) u^{(p)}(x) + \cdots + a_0(x) u(x) \\ &= f(x) + \int_{x_0}^{x} \underbrace{(Lu_\xi)(x)}_{=0} f(\xi)\, d\xi \\ &= f(x). \end{aligned}$$

The Solution Formula

Theorem. The unique classical solution of the initial value problem with data

$$\{f; 0, \dots, 0\}_{x_0}$$

is given by

$$u(x) = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi$$

where u_ξ is the solution of the initial value problem with data

$$\{0; 0, \dots, 0, 1/a_p(\xi)\}_\xi$$

If the coefficients $a_1, \dots, a_p$ of L are constants, then

$$u_\xi(x) = u_0(x - \xi)$$

The Inhomogeneous Equation with General Data

The solution of the initial value problem with data

$$\{f; \gamma_1, \dots, \gamma_p\}_{x_0}$$

is given by

$$u(x) = \int_{x_0}^{x} u_\xi(x) f(\xi)\, d\xi + \gamma_1 u_1(x) + \dots + \gamma_p u_p(x),$$

where u_k, $k = 1, \dots, p$, is a solution of the equation with data

$$\{0; 0, \dots, 0, \underset{\substack{\nwarrow \\ k}}{1}, 0, \dots, 0\}_{x_0}$$

Differential Operators and Types of Solutions

Causal Fundamental Solutions and Initial Value Problems

Second-Order Boundary Value Problems

Adjoint BVPs and Higher-Order Equations

Solvability Conditions and Modified Green's Functions

The Second-Order Equation

We consider

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = f \qquad \text{on } (a, b) \subset \mathbb{R}$$

where

- f is piecewise continuous on $[a, b]$,
- $a_0, a_1, a_2 \in C([a, b])$,
- $a_p(x) \neq 0$ for all $x \in [a, b]$.

As usual, a classical solution

- is continuous on $[a, b]$,
- is continuously differentiable on (a, b),
- is twice differentiable and satisfies $Lu = f$ at all points in (a, b) where where f is continuous.

Boundary Conditions

We impose the boundary conditions

$$\begin{aligned} B_1 u &:= \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = \gamma_1, \\ B_2 u &:= \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2, \end{aligned}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_j \in \mathbb{R}$, $i, j = 1, 2$, and the row vectors

$$(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12}) \qquad \text{and} \qquad (\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$$

are assumed to be independent.

B_1 and B_2 are called boundary functionals.

We say that $\{f; \gamma_1, \gamma_2\}$ is the data for the boundary value problem (L, B_1, B_2).

Types of Boundary Conditions

- Homogeneous boundary conditions: $\gamma_1 = \gamma_2 = 0$
- Fully homogeneous boundary value problem: data $\{0; 0, 0\}$
- Unmixed or separated boundary conditions:

$$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a) = \gamma_1,$$
$$B_2 u = \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2,$$

- Initial conditions:

$$B_1 u = u(a) = \gamma_1,$$
$$B_2 u = u'(a) = \gamma_2.$$

Superposition principle

Suppose

- u solves (L, B_1, B_2) with data $\{f; \gamma_1, \gamma_2\}$
- $\widetilde{u}$ solves (L, B_1, B_2) with data $\{\widetilde{f}; \widetilde{\gamma}_1, \widetilde{\gamma}_2\}$

Then

- $c_1 u + c_2 \widetilde{u}$ solves (L, B_1, B_2) with data

$$\{c_1 f + c_2 \widetilde{f}; c_1\gamma_1 + c_2\widetilde{\gamma}_1, c_1\gamma_1 + c_2\widetilde{\gamma}_2\}$$

Existence and Uniqueness

- If the problem with data $\{0; 0, 0\}$ has only the trivial solution $u \equiv 0$, then the problem $\{f; \gamma_1, \gamma_2\}$ will have at most one classical solution.
- If the problem with data $\{0; 0, 0\}$ has a non-trivial solution, then the problem $\{f; \gamma_1, \gamma_2\}$ will have either no classical solution or an infinite number of classical solutions.

Major Assumption: Unless otherwise stated, we will always suppose that the problem with data $\{0; 0, 0\}$ has only the trivial solution.

Example of Non-Uniqueness / Non-Existence

$$-u'' = f(x), \quad 0 < x < 1, \qquad u'(0) = \gamma_1, \qquad u'(1) = \gamma_2.$$

The problem with data $\{0; 0, 0\}$ has the non-trivial solution $u(x) = 1$.

There can only be a solution of this problem if

$$\int_0^1 f(x)\,dx = \gamma_1 - \gamma_2.$$

- For data $\{1; 0, 0\}$ there are no classical solutions.
- For data $\{\sin(2\pi x); 0, 0\}$ there is an infinite number of solutions:

$$u(x) = C - \frac{x}{2\pi} + \frac{1}{4\pi^2}\sin(2\pi x), \qquad C \in \mathbb{R}.$$

Fundamental Solution

A fundamental solution $E(x,\xi)$ for L with pole at $\xi \in [a,b]$ satisfies

$$LE = \delta(x-\xi), \qquad\qquad x,\xi \in (a,b)$$

in the distributional sense.

We can construct a fundamental solution by imposing

- $LE = 0$ for $a < x < \xi$ and $\xi < x < b$
- E continuous on $[a,b]$, including at $x = \xi$
- Jump condition: $\displaystyle \lim_{\varepsilon \searrow 0} \left(\frac{dE}{dx}\bigg|_{x=\xi+\varepsilon} - \frac{dE}{dx}\bigg|_{x=\xi-\varepsilon} \right) = \frac{1}{a_2(\xi)}$

Green's Function

Green's function $g(x,\xi)$ for (L, B_1, B_2) is defined by the following properties

- $g(\,\cdot\,,\xi)$ is a fundamental solution with pole at $\xi \in (a,b)$
- $B_1 g = B_2 g = 0$

Since the difference of any two such functions has a continuous first derivative at $x = \xi$ and satisfies the problem with data $\{0; 0, 0\}$ (which has only the trivial solution), Green's function is uniquely defined, if it exists at all.

We write

$$Lg = \delta(x-\xi), \qquad x, \xi \in (a,b), \qquad B_1 g = 0, \qquad B_2 g = 0.$$

Green's Function for Unmixed Boundary Conditions

We consider (L, B_1, B_2) with

$$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a)$$
$$B_2 u = \beta_{21} u(b) + \beta_{22} u'(b)$$

Major Assumption: We suppose that the fully homogeneous problem has only the trivial solution.

Our goal is to find Green's function satisfying

$$\begin{aligned} Lg &= \delta(x - \xi), & x, \xi &\in (a, b), \\ B_1 g &= 0, \\ B_2 g &= 0. \end{aligned}$$

Two Basic Functions

Let u_1 satisfy the initial value problem

$$Lu_1 = 0, \qquad u_1(a) = \alpha_{12}, \qquad u_1'(a) = -\alpha_{11}.$$

Then u_1 satisfies

$$Lu_1 = 0, \qquad B_1 u_1 = 0.$$

Similarly, we can find u_2 such that

$$Lu_2 = 0, \qquad B_2 u_2 = 0.$$

From the Major Assumption, it follows that u_1 and u_2 must be independent.

Construction of Green's Function

Green's function has the form

$$g(x,\xi) = \begin{cases} c_1 \cdot u_1(x) & x < \xi, \\ c_2 \cdot u_2(x) & x > \xi \end{cases}$$

for some $c_1, c_2 \in \mathbb{R}$.

The continuity of g and the jump condition at $x = \xi$ give

$$\begin{aligned} c_1 \cdot u_1(\xi) - c_2 \cdot u_2(\xi) &= 0, \\ -c_1 \cdot u_1'(\xi) + c_2 \cdot u_2'(\xi) &= \frac{1}{a_2(\xi)}. \end{aligned}$$

Construction of Green's Function

Since u_1 and u_2 are independent, the Wronskian satisfies

$$W(u_1, u_2; \xi) \neq 0$$

Hence, by Cramer's rule,

$$c_1 = \frac{u_2(\xi)}{a_2(\xi)W(u_1, u_2; \xi)}, \qquad c_2 = \frac{u_1(\xi)}{a_2(\xi)W(u_1, u_2; \xi)}.$$

In summary, we see that

$$g(x, \xi) = \begin{cases} \dfrac{u_1(x)u_2(\xi)}{a_2(\xi)W(u_1, u_2; \xi)} & x < \xi, \\ \dfrac{u_1(\xi)u_2(x)}{a_2(\xi)W(u_1, u_2; \xi)} & x > \xi. \end{cases}$$

Construction of Green's Function

For short, we write

$$x_< := \min\{x, \xi\}, \qquad x_> := \max\{x, \xi\}$$

so

$$g(x, \xi) = \frac{u_1(x_<)u_2(x_>)}{a_2(\xi)W(u_1, u_2; \xi)}.$$

If $L = L^*$ there exists a constant $c \in \mathbb{C}$ such that

$$g(x, \xi) = c \cdot u_1(x_<)u_2(x_>)$$

Non-Homogeneous Boundary Conditions

Given that u_1 and u_2 satisfy

$$Lu_1 = 0, \qquad Lu_2 = 0, \qquad B_1 u_1 = 0, \qquad B_2 u_2 = 0,$$

we also see that

$$v(x) = \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x)$$

satisfies

$$Lv = 0, \qquad B_1 v = \gamma_1, \qquad B_2 v = \gamma_2.$$

Mixed Boundary Conditions

In the general case, we have

$$\begin{aligned} B_1 g &:= \alpha_{11} g(a) + \alpha_{12} g'(a) + \beta_{11} g(b) + \beta_{12} g'(b) = 0, \\ B_2 g &:= \alpha_{21} g(a) + \alpha_{22} g'(a) + \beta_{21} g(b) + \beta_{22} g'(b) = 0, \end{aligned}$$

It is possible to find a non-trivial function u_1 satisfying

$$Lu_1 = 0, \qquad\qquad B_1 u_1 = 0.$$

by solving $Lu_1 = 0$ with the separated boundary conditions

$$\begin{aligned} \alpha_{11} u_1(a) + \alpha_{12} u_1'(a) &= 1, \\ \beta_{11} u_1(b) + \beta_{12} u_1'(b) &= -1. \end{aligned}$$

Similarly, there exists a non-trivial u_2 such that

$$Lu_2 = 0, \qquad\qquad B_2 u_2 = 0.$$

Green's Function for Mixed Boundary Conditions

We construct Green's function from the sum of the causal fundamental solution

$$E(x,\xi) = H(x-\xi)u_\xi(x)$$

and u_1 and u_2:

$$g(x,\xi) = H(x-\xi)u_\xi(x) + c_1 \cdot u_1(x) + c_2 \cdot u_2(x)$$

where $c_1, c_2 \in \mathbb{C}$ may depend on ξ.

The constants are determined through

$$\begin{aligned} B_1 g &= \beta_{11} u_\xi(b) + \beta_{12} u_\xi'(b) + c_2 \cdot B_1 u_2 &= 0, \\ B_2 g &= \beta_{21} u_\xi(b) + \beta_{22} u_\xi'(b) + c_1 \cdot B_2 u_1 &= 0. \end{aligned}$$

Example for Mixed Boundary Conditions

$$\begin{aligned} Lu &= u'' && \text{on } (0,1) \subset \mathbb{R}, \\ B_1 u &= u(0) + u(1) \\ B_2 u &= u'(0) + u'(1) \end{aligned}$$

We first find a causal fundamental solution by solving

$$u_\xi'' = 0, \qquad u_\xi(\xi) = 0, \qquad u_\xi'(\xi) = 1.$$

This gives

$$u_\xi(x) = x - \xi$$

so the casual fundamental solution is

$$E(x, \xi) = H(x - \xi) \cdot (x - \xi).$$

Example for Mixed Boundary Conditions

We find a non-trivial function u_1 such that

$$u_1'' = 0, \qquad B_1 u_1 = u_1(0) + u_1(1) = 0.$$

We take

$$u_1(x) = 1 - 2x.$$

Next we choose a function u_2 such that

$$u_2'' = 0, \qquad B_2 u_2 = u_2'(0) + u_2'(1) = 0.$$

and we can take

$$u_2(x) = 1.$$

Example for Mixed Boundary Conditions

Then Green's function is

$$g(x, \xi) = H(x - \xi) \cdot (x - \xi) + c_1(1 - 2x) + c_2, \qquad 0 < \xi < 1,$$

and the parameters $c_1, c_2 \in \mathbb{R}$ are determined through

$$\begin{aligned}
B_1 g &= g(0, \xi) + g(1, \xi) \\
&= c_1 + c_2 + 1 - \xi - c_1 + c_2 \\
&= 0, \\
B_2 g &= g'(0, \xi) + g'(1, \xi) \\
&= -2c_1 + 1 - 2c_1 \\
&= 0
\end{aligned}$$

which gives

$$c_1 = \frac{1}{4}, \qquad\qquad c_2 = \frac{\xi - 1}{2}.$$

Example for Mixed Boundary Conditions

We finally have

$$g(x,\xi) = H(x-\xi)\cdot(x-\xi) - \frac{x-\xi}{2} - \frac{1}{4}$$

$$= \begin{cases} \dfrac{\xi - x}{2} - \dfrac{1}{4} & x < \xi, \\ \dfrac{x-\xi}{2} - \dfrac{1}{4} & x > \xi. \end{cases}$$

Note: The construction worked because the completely homogeneous problem has only the trivial solution, as can be easily checked.

Solution Formula for the General Problem

Theorem. If the completely homogeneous problem (L, B_1, B_2) has only the trivial solution, the problem with data $\{f; \gamma_1, \gamma_2\}$ has the unique solution

$$u(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi + \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x).$$

Proof. We have seen in the study of initial value problems that the integral satisfies the inhomogeneous differential equation while u_1 and u_2 solve the homogeneous equation. Thus, the sum solves $Lu = f$.

From

$$B_1 g = B_2 g = 0, \qquad B_1 u_1 = 0, \qquad B_2 u_2 = 0$$

we see that u satisfies $B_u = \gamma_1$ and $B_2 u = \gamma_2$.

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The Formal Adjoint and Green's Formula

The formal adjoint of

$$L = a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0$$

is

$$L^* = a_2 \frac{d^2}{dx^2} + (2a_2' - a_1) \frac{d}{dx} + (a_2'' - a_1' + a_0).$$

Green's identity is

$$\int_a^b \big(vLu - uL^*v\big) = J(u, v)\big|_a^b$$

with the conjunct

$$J(u, v) = a_2 \big(vu' - uv'\big) + \big(a_1 - a_2'\big) uv.$$

Adjoint Boundary Value Problems

We want to solve the problem (L, B_1, B_2) on $(a, b) \subset \mathbb{R}$ with

$$\begin{aligned} B_1 u &= \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b), \\ B_2 u &= \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b), \end{aligned}$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i, j = 1, 2$.

Suppose that u satisfies

$$B_1 u = B_2 u = 0.$$

Question: For which functions v is $J(u, v)\big|_a^b = 0$?

Adjoint Boundary Functionals

Definition.

$$M := \{u \in C^2(a,b) \colon B_1 u = B_2 u = 0\},$$
$$M^* := \{v \in C^2(a,b) \colon J(u,v)\big|_a^b = 0 \text{ for all } u \in M\}.$$

There exist so-called adjoint boundary functionals B_1^* and B_2^* such that

$$M^* = \{v \in C^2(a,b) \colon B_1^* v = B_2^* v = 0\}$$

The adjoint boundary functionals have the form

$$B_1^* u = \alpha_{11}^* u(a) + \alpha_{12}^* u'(a) + \beta_{11}^* u(b) + \beta_{12}^* u'(b),$$
$$B_2^* u = \alpha_{21}^* u(a) + \alpha_{22}^* u'(a) + \beta_{21}^* u(b) + \beta_{22}^* u'(b),$$

where $\alpha_{ij}^*, \beta_{ij}^* \in \mathbb{R}$, $i, j = 1, 2$.

Adjoint Boundary Functionals

The existence of B_1^* and B_2^* follows from

$$J(u,v)\Big|_a^b = a_2\big(vu' - uv'\big)\Big|_a^b + \big(a_1 - a_2'\big)uv\Big|_a^b$$

and then “factoring out” B_1u and B_2u in the equation

$$J(u,v)\Big|_a^b = 0.$$

While M^* is completely determined by M, B_1^*, B_2^* are not unique.

For example, we can replace B_1^*, B_2^* by

$$\widetilde{B}_1^* = B_1^* + B_2^*, \qquad \widetilde{B}_2^* = B_1^* - B_2^*$$

without affecting M^*.

Example of Adjoint Boundary Value Functionals

$$L = \frac{d^2}{dx^2} \qquad \text{on } (0,1) \subset \mathbb{R}$$

with

$$B_1 u = u'(0) - u(1), \qquad\qquad B_2 u = u'(1).$$

The conjunct is

$$\begin{aligned} J(u,v)\big|_0^1 &= vu' - uv'\big|_0^1 \\ &= v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0). \end{aligned}$$

Now if $u \in M = \{u \in C^2([0,1]) \colon B_1 u = B_2 u = 0\}$, then

$$J(u,v)\big|_0^1 = -u'(0)[v'(1) + v(0)] + u(0)v'(0).$$

Example of Adjoint Boundary Value Functionals

Hence,

$$\begin{aligned} M^* &= \{v \in C^2([0,1]) \colon J(u,v)\big|_a^b = 0 \text{ for all } u \in M\} \\ &= \{v \in C^2([0,1]) \colon v'(1) + v(0) = 0 \text{ and } v'(0) = 0\} \end{aligned}$$

A possible choice of adjoint boundary functionals is

$$B_1^* v = v'(1) + v(0), \qquad\qquad B_2^* v = v'(0).$$

Adjoint Boundary Value Problems

Definition. The boundary value problem

$$(L^*, B_1^*, B_2^*)$$

is said to be the adjoint of

$$(L, B_1, B_2)$$

.

(L, B_1, B_2) is called self-adjoint if

$$L = L^* \qquad \text{and} \qquad M = M^*.$$

The Adjoint Green Function

Definition. We call the solution $g(\,\cdot\,,\xi)$ of

$$Lg(x,\xi) = \delta(x-\xi), \qquad x \in (a,b), \qquad g \in M,$$

the direct Green function.

The solution $g^*(\,\cdot\,,\xi)$ of

$$L^*g^*(x,\xi) = \delta(x-\xi), \qquad x \in (a,b), \qquad g^* \in M^*$$

will be called the adjoint Green function.

The Adjoint Green Function

Lemma. The adjoint Green function satisfies

$$g^*(x,\xi) = g(\xi,x).$$

Proof. From the formal properties of the delta-distribution,

$$\begin{aligned} g^*(\xi,\eta) - g(\eta,\xi) &= \int_a^b \Big(g^*(x,\eta)\underbrace{Lg(x,\xi)}_{=\delta(x-\xi)} - g(x,\xi)\underbrace{L^*g^*(x,\eta)}_{=\delta(x-\eta)}\Big)\,dx \\ &= J(g,g^*)\big|_a^b = 0. \end{aligned}$$

Reciprocity principle. If (L, B_1, B_2) is self-adjoint, $g = g^*$ and hence

$$g(x,\xi) = g(\xi,x).$$

A New Perspective on the Solution Formula

Suppose u satisfies

$$Lu = f, \qquad x \in (a, b), \qquad B_1 u = 0, \qquad B_2 u = 0.$$

Then Green's formula yields

$$\int_a^b \big(g^*(x,\xi) \underbrace{Lu(x)}_{=f(x)} - u(x) \underbrace{L^* g^*(x,\xi)}_{=\delta(x-\xi)}\big)\, dx = \int_a^b g^*(x,\xi) f(x)\, dx - u(\xi).$$

On the other hand, since $u \in M$ and $g^* \in M^*$,

$$\int_a^b \big(g^*(x,\xi) Lu(x) - u(x) L^* g^*(x,\xi)\big)\, dx = J(u, g^*)\big|_a^b = 0.$$

A New Perspective on the Solution Formula

This yields

$$u(\xi) = \int_a^b g^*(x, \xi) f(x)\, dx,$$

or, relabeling the variables,

$$u(x) = \int_a^b g^*(\xi, x) f(\xi)\, d\xi = \int_a^b g(x, \xi) f(\xi)\, d\xi$$

This is the familiar formula for data $\{f, 0, 0\}$.

A New Perspective: Data $\{0; \gamma_1, \gamma_2\}$

Suppose u satisfies

$$Lu = 0, \qquad x \in (a, b), \qquad B_1 u = \gamma_1, \qquad B_2 u = \gamma_2.$$

As before, Green's formula yields

$$-u(\xi) = \int_a^b \big(g^*(x,\xi)Lu(x) - L^* g^*(x,\xi)u(x)\big)\,dx = J(u, g(\xi,\,\cdot\,))\big|_a^b.$$

Relabeling the variables,

$$u(x) = -J(u, g(x,\,\cdot\,))\big|_a^b.$$

This is a new formula!

The Solution Formula

The inhomogeneous problem

$$Lu = f, \qquad x \in (a, b), \qquad B_1 u = \gamma_1, \qquad B_2 u = \gamma_2,$$

is hence solved by

$$u(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi - J(u, g(x, \cdot))\big|_a^b.$$

This formula can be generalized to partial differential equations, while the earlier formula

$$u(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi + \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x),$$

does not generalize.

Example

$$L = \frac{d^2}{dx^2} \qquad \text{on } (0,1) \subset \mathbb{R}$$

with

$$B_1 u = u'(0) - u(1), \qquad\qquad B_2 u = u'(1).$$

Green's function is found in the usual manner:

$$g(x,\xi) = (x-\xi)H(x-\xi) - x + \xi - 1.$$

We have seen that for $L = \frac{d^2}{dx^2}$,

$$J(u,v) = uv' - u'v,$$

Example

Using g and setting

$$B_1 u = u'(0) - u(1) = \gamma_1, \qquad B_2 u = u'(1) = \gamma_2,$$

we obtain

$$\begin{aligned}
-J(u, g(x,\,\cdot\,))\big|_0^1 &= -\Big(g(x,\xi)u'(\xi) - u(\xi)\frac{dg(x,\xi)}{d\xi}\Big)\Big|_{\xi=0}^{\xi=1} \\
&= -g(x,1)u'(1) + u(1)\left.\frac{dg(x,\xi)}{d\xi}\right|_{\xi=1} \\
&\qquad + g(x,0)u'(0) - u(0)\left.\frac{dg(x,\xi)}{d\xi}\right|_{\xi=0} \\
&= x\gamma_2 - \gamma_1.
\end{aligned}$$

Example

The general solution for data $\{f, \gamma_1, \gamma_2\}$ is then

$$u(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi + x\gamma_2 - \gamma_1.$$

Boundary Value Problems of Order p

Consider the problem $(L, B_1, \dots, B_p)$ on $[a, b] \subset \mathbb{R}$, where

$$L = a_p \frac{d^p}{dx^p} + a_{p-1} \frac{d^{p-1}}{dx^{p-1}} + \dots + a_1 \frac{d}{dx} + a_0.$$

with $a_0, \dots, a_p \in C([a, b])$ and $a_p(x) \neq 0$ for all $x \in [a, b]$.

We have boundary functionals

$$B_1 u := \sum_{k=1}^{p} \alpha_{1k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{1k} u^{(k-1)}(b),$$

$$\vdots$$

$$B_p u := \sum_{k=1}^{p} \alpha_{pk} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{pk} u^{(k-1)}(b).$$

Boundary Value Problems of Order p

Assumptions:

(i) The row vectors

$$(\alpha_{i1}, \dots, \alpha_{ip}, \beta_{i1}, \dots, \beta_{ip})$$

are independent.

(ii) The completely homogeneous problem has only the trivial solution.

We seek to solve the problem $(L, B_1, \dots, B_p)$ for data

$$\{f; \gamma_1, \dots, \gamma_p\}.$$

Boundary Value Problems of Order p

We define

$$M := \{u \in C^p(a,b) \colon B_1 u = \cdots = B_p u = 0\},$$
$$M^* := \{v \in C^p(a,b) \colon J(u,v)|_a^b = 0 \text{ for all } u \in M\}.$$

The boundary value problem $(L, B_1, \dots, B_p)$ is said to be self-adjoint if

$$L = L^* \qquad \text{and} \qquad M = M^*.$$

Goal: characterize M^* through adjoint boundary functionals

$$B_1^*, \dots, B_p^*$$

The Conjunct

Recall that

$$J(u,v) = \sum_{k=1}^{p} \sum_{i+j=k-1} (-1)^i D^i(a_m v) D^j u.$$

We express $J(u,v)\big|_a^b$ in the form

$$J(u,v)\big|_a^b = \sum_{k=1}^{p} (A_{2p+1-k}v)u^{(k-1)}(a) + \sum_{k=1}^{p} (A_{p+1-k}v)u^{(k-1)}(b)$$

with boundary functionals A_k, $k = 1, \dots, 2p$.

The right-hand side is a linear combination of the $2p$ terms

$$u(a), \dots, u^{(p-1)}(a), \qquad u(b), \dots, u^{(p-1)}(b).$$

Additional Boundary Functionals

We now define p additional boundary functionals as follows:

$$B_{p+1}u := \sum_{k=1}^{p} \alpha_{(p+1)k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{(p+1)k} u^{(k-1)}(b)$$

$$\vdots$$

$$B_{2p}u := \sum_{k=1}^{p} \alpha_{(2p)k} u^{(k-1)}(a) + \sum_{k=1}^{p} \beta_{(2p)k} u^{(k-1)}(b)$$

such that all $2p$ row vectors

$$(\alpha_{i1}, \dots, \alpha_{ip}, \beta_{i1}, \dots, \beta_{ip}), \qquad\qquad i = 1, \dots, 2p$$

are independent.

Adjoint Boundary Functionals

We can then write

$$\begin{aligned} J(u,v)\Big|_a^b &= \sum_{k=1}^{2p} (B^*_{2p+1-k}v)\cdot B_k u \\ &= (B^*_{2p}v)B_1 u + \cdots + (B^*_{p+1}v)B_p u \\ &\qquad + (B^*_p v)B_{p+1} u + \cdots + (B^*_1 v)B_{2p} u. \end{aligned}$$

with certain boundary functionals B^*_{2p+1-k}, $k = 1, \dots, 2p$.

If $u \in M$, $J(u,v)\big|_a^b$ vanishes if v satisfies

$$B^*_1 v = \cdots = B^*_p v = 0,$$

so these are just the adjoint boundary functionals.

Example

$$L = \frac{d^2}{dx^2} + x^2 \frac{d}{dx} + 1 \qquad \text{on } (0, 1) \subset \mathbb{R}$$

with

$$B_1 u = u(0) + u(1), \qquad\qquad B_2 u = u'(1).$$

The boundary functionals correspond to row vectors

$$(1, 0, 1, 0) \qquad \text{and} \qquad (0, 0, 0, 1).$$

We add two functionals, $B_3 u = u(1)$ and $B_4 u = u'(0)$, which correspond to row vectors

$$(0, 0, 1, 0) \qquad \text{and} \qquad (0, 1, 0, 0).$$

Example

The conjunct is

$$J(u,v) = (u'v - uv') + 2xuv.$$

and

$$J(u,v)\big|_0^1 = v'(0)u(0) - v(0)u'(0) + \big(2v(1) - v'(1)\big)u(1) + v(1)u'(1)$$

Since $u(0) = B_1 u - B_3 u$, we have

$$J(u,v)\big|_0^1 = \underbrace{v'(0)}_{=: B_4^* v} B_1 u + \underbrace{v(1)}_{=: B_3^* v} \cdot B_2 u + \underbrace{\big(2v(1) - v'(1) - v'(0)\big)}_{=: B_2^* v} B_3 u$$
$$+ \underbrace{v(0)}_{=: B_1^* v} B_4 u$$

Example

We hence obtain the adjoint boundary functionals

$$B_1^* v = v(0), \qquad B_2^* v = v'(0) + 2v(1) - v'(1)$$

Solution Formula

As in the previous section, we define the direct and adjoint Green functions to satisfy

$$Lg(x,\xi) = \delta(x-\xi), \qquad B_1 g = \cdots = B_p g = 0,$$

$$L^* g^*(x,\xi) = \delta(x-\xi), \qquad B_1^* g^* = \cdots = B_p^* g^* = 0,$$

and we can again show that

$$g^*(x,\xi) = g(\xi,x).$$

Then the solution of $(L, B_1, \dots, B_p)$ with data $\{f; \gamma_1, \dots, \gamma_p\}$ is

$$u(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi - J(u, g(x,\,\cdot\,))\big|_a^b.$$

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Existence and Uniqueness

If the boundary value problem

$$(L, B_1, \dots, B_p)$$

with data

$$(0; 0, \dots, 0)$$

has only the trivial solution,

- Green's function can be constructed and
- there exists a solution formula for any data $\{f; \gamma_1, \dots, \gamma_p\}$.

(Existence and uniqueness of the solution for the general problem)

The Fredholm Alternative

Fredholm Alternative.

- Either the completely homogeneous problem has a non-trivial solution,
- Or the solution to the problem with data $\{f; 0, \dots, 0\}$ exists and is unique.

Relationship to the Adjoint Problem

The completely homogeneous direct problem is

$$Lu = 0, \qquad x \in (a, b), \qquad B_1 u = \cdots = B_p u = 0. \qquad (*)$$

There is a relationship to the adjoint problem

$$L^* v = 0, \qquad x \in (a, b), \qquad B_1^* v = \cdots = B_p^* v = 0 \qquad (**)$$

as follows:

- If $(*)$ has only the trivial solution, then $(**)$ also has only the trivial solution.
- If there are k independent, non-trivial solutions $u^{(1)}, \dots, u^{(k)}$ of $(*)$, then $(**)$ also has k independent, non-trivial solutions $v^{(1)}, \dots, v^{(k)}$.

Solvability via the Adjoint Problem

Consider now the problem

$$Lu = f, \qquad x \in (a, b), \qquad B_1 u = \cdots = B_p u = 0.$$

Suppose there exists a solution u and let v be any non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \dots, B_p^*)$. Then

$$\begin{aligned}\int_a^b f(x)v(x)\,dx &= \int_a^b \big(v(x)Lu(x) - u(x)L^*v(x)\big)\,dx \\ &= J(u,v)\big|_a^b \\ &= 0\end{aligned}$$

Solvability via the Adjoint Problem

Hence, if $v^{(1)}, \dots, v^{(k)}$ are k independent, non-trivial solutions of the completely homogeneously adjoint problem $(L^*, B_1^*, \dots, B_p^*)$, a necessary condition for the solvability of

$$(L, B_1, \dots, B_p) \qquad \text{with data} \qquad (f; 0, \dots, 0)$$

is

$$\int_a^b f(x)v^{(1)}(x)\,dx = \cdots = \int_a^b f(x)v^{(k)}(x)\,dx = 0.$$

It can be shown that this condition is also sufficient, i.e., a solution exists if and only if f satisfies these k equations.

Example

$$\begin{aligned} -u'' + u' &= f, && 0 < x < 1, \\ u(1) - u(0) &= 0, \\ u'(1) - u'(0) &= 0. \end{aligned}$$

The fully homogeneous adjoint problem is

$$\begin{aligned} -v'' - v' &= 0, && 0 < x < 1, \\ v(1) - v(0) &= 0, \\ v'(1) - v'(0) &= 0. \end{aligned}$$

which has non-trivial solution $v(x) = c$, $c \in \mathbb{R}$.

Hence, a solution will exist if and only if

$$\int_0^1 f(x)\, dx = 0.$$

Example

$$u' + u = f, \qquad 0 < x < 1, \qquad u(0) - e \cdot u(1) = 0.$$

The adjoint homogeneous problem is

$$-v' + v = 0, \qquad 0 < x < 1, \qquad -e \cdot v(0) + v(1) = 0.$$

which has non-trivial solution

$$v(x) = c \cdot e^x, \qquad c \in \mathbb{R}.$$

The (necessary and sufficient) solvability condition is

$$\int_0^1 f(x)e^x \, dx = 0.$$

Solvability of the General Inhomogeneous Problem

Consider now the problem

$$Lu = f, \qquad x \in (a,b), \qquad B_1 u = \gamma_1, \qquad \dots, \qquad B_p u = \gamma_p.$$

Suppose u is a solution and v any non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \dots, B_p^*)$. Then

$$\begin{aligned}\int_a^b f(x)v(x)\,dx &= \int_a^b \big(v(x)Lu(x) - u(x)L^*v(x)\big)\,dx \\ &= J(u,v)\big|_a^b \\ &= \gamma_1 B_{2p}^* v + \dots + \gamma_p B_{p+1}^* v\end{aligned}$$

where $B_{p+1}^*, \dots, B_{2p}^*$ are the additional adjoint boundary functionals introduced previously.

Solvability of the General Inhomogeneous Problem

If $v^{(1)}, \dots, v^{(k)}$ are k non-trivial solution of the completely homogeneous adjoint problem $(L^*, B_1^*, \dots, B_p^*)$, the solvability conditions are

$$\int_a^b f(x) v^{(1)}(x)\, dx = \gamma_1 B_{2p}^* v^{(1)} + \cdots + \gamma_p B_{p+1}^* v^{(1)},$$

$$\int_a^b f(x) v^{(2)}(x)\, dx = \gamma_1 B_{2p}^* v^{(2)} + \cdots + \gamma_p B_{p+1}^* v^{(2)},$$

$$\vdots$$

$$\int_a^b f(x) v^{(k)}(x)\, dx = \gamma_1 B_{2p}^* v^{(k)} + \cdots + \gamma_p B_{p+1}^* v^{(k)}.$$

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Example

$$u' + u = f, \qquad 0 < x < 1, \qquad u(0) - e \cdot u(1) = \gamma_1.$$

We have $L^* v = -v' + v$ and

$$\begin{aligned} J(u,v)\big|_0^1 &= u(1)v(1) - u(0)v(0) \\ &= \underbrace{\big(u(0) - e \cdot u(1)\big)}_{=B_1 u} \underbrace{-v(0)}_{=B_2^* v} + \underbrace{u(1)}_{=B_2 u} \underbrace{\big(v(1) - ev(0)\big)}_{=B_1^* v} \end{aligned}$$

We have already seen that

$$v(x) = c \cdot e^x$$

solves the fully homogeneous adjoint problem (L^*, B_1^*).

Example

Then

$$J(u,v)\Big|_0^1 = \gamma_1 B_2^* v = -\gamma_1 \cdot c.$$

A solution to (L, B_1) with data (f, γ_1) exists if and only if

$$\int_0^1 c \cdot e^x f(x)\,dx = -c\gamma_1$$

or

$$\int_0^1 e^x f(x)\,dx = -\gamma_1.$$

Existence of Green's function

Solvability conditions for

$$Lu(x,\xi) = f, \quad a < x < b, \qquad B_1 u = \cdots = B_p u = 0$$

are

$$\int_a^b f(x) v^{(j)}(x)\, dx = 0 \qquad j = 1, \dots, k. \tag{2.5.1}$$

where $v^{(1)}, \dots, v^{(k)}$ are independent solutions of the completely homogeneous adjoint problem.

If $f(x) = \delta(x-\xi)$ these are not satisfied for all $\xi \in (a,b)$, so Green's function doesn't exist.

Example: Formally Self-Adjoint Problem

$$\begin{aligned} -u'' + u' &= \delta(x-\xi), \qquad && 0 < x < 1, \\ u(1) - u(0) &= 0, \\ u'(1) - u'(0) &= 0 \end{aligned}$$

$v(x) = 1$ is a solution of the completely homogeneous adjoint problem.

Solvability condition for Green's function:

$$\int_0^1 \delta(x-\xi)\,dx = 0.$$

Not satisfied.

Approach: Derive a "modified" Green function instead.

Orthonormalization

Non-trivial solutions of the completely homogeneous adjoint problem:

$$v^{(1)}, \dots, v^{(k)}$$

Orthonormalize with respect to

$$\langle f, g \rangle_{L^2([a,b])} := \int_a^b f(x)g(x)\,dx.$$

so that

$$\int_a^b v^{(i)}(x)v^{(j)}(x)\,dx = \delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Modified Equation for Green's Function

Instead of $Lu = \delta(x-\xi)$ solve

$$\begin{aligned} Lu &= \delta(x-\xi) - v^{(1)}(\xi)v^{(1)}(x) - \dots - v^{(k)}(\xi)v^{(k)}(x) \\ &=: f(x) \end{aligned}$$

Solvability conditions for $j = 1, \dots, k$ are satisfied:

$$\begin{aligned} \int_a^b f(x)v^{(j)}(x)\,dx &= \int_a^b \Big(\delta(x-\xi) - \sum_{i=1}^k v^{(i)}(\xi)v^{(i)}(x)\Big)v^{(j)}(x)\,dx \\ &= v^{(j)}(\xi) - \sum_{i=1}^k v^{(i)}(\xi) \underbrace{\int_a^b v^{(i)}(x)v^{(j)}(x)\,dx}_{=\delta_{ij}} \\ &= 0. \end{aligned}$$

Modified Green function

Definition. The modified (direct) Green function is defined by

$$
\begin{aligned}
Lg_M(x,\xi) &= \delta(x-\xi) - \sum_{i=1}^{k} v^{(i)}(\xi)v^{(i)}(x), \\
B_1 g_M &= 0 \\
&\vdots \\
B_p g_M &= 0.
\end{aligned}
$$

where $v^{(1)}, \dots, v^{(k)}$ are the k orthonormalized non-trivial solutions of the completely homogeneous adjoint problem.

Constructing the Modified Green Function

Suppose that $v^{(1)}, \dots, v^{(k)}$ are the k non-trivial, orthonormalized solutions of the adjoint problem.

(i) Find a fundamental solution $E(x, \xi)$ such that

$$LE = \delta(x - \xi)$$

(ii) Find k solutions

$$w^{(1)}, \dots, w^{(k)}$$

of the inhomogeneous equations

$$Lw^{(i)} = v^{(i)}, \qquad i = 1, \dots, k,$$

(without regard to boundary conditions). Then

$$L\Big(E(x, \xi) - \sum_{i=1}^{k} v^{(i)}(\xi) w^{(i)}(x)\Big) = \delta(x - \xi) - \sum_{i=1}^{k} v^{(i)}(\xi) v^{(i)}(x)$$

Constructing the Modified Green Function

(iii) Find p independent solutions of the homogeneous equation $Lu = 0$ and add them to

$$E(x,\xi) - \sum_{i=1}^{k} v^{(i)}(\xi) w^{(i)}(x)$$

in order to satisfy the boundary conditions

$$B_1 g = \cdots = B_p g = 0.$$

Example

$$\begin{aligned} Lu &= u'', & 0 < x < 1, \\ B_1 u &= u(0) + u(1), \\ B_2 u &= u'(0) - u'(1) \end{aligned}$$

The completely homogeneous problem has a non-trivial solution,

$$u^{(1)}(x) = 1 - 2x.$$

Green's formula is

$$\int_0^1 (vu'' - uv'')\, dx = vu' - uv'|_0^1.$$

We set

$$B_3 u = u(0), \qquad B_4 u = u'(0).$$

Example

$$
\begin{aligned}
J(u,v)\big|_0^1 &= v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0) \\
&= -v(1)B_2u + v(1)B_4u - v'(1)B_1u + v'(1)B_3u \\
&\quad - v(0)B_4u + v'(0)B_3u \\
&= -v'(1)B_1u - v(1)B_2u + (v'(1) + v'(0))B_3u \\
&\quad + (v(1) - v(0))B_4u
\end{aligned}
$$

The adjoint boundary conditions are

$$
\begin{aligned}
B_1^*v &= v(1) - v(0), & B_2^*v &= v'(1) + v'(0), \\
B_3^*v &= -v(1), & B_4^*v &= -v'(1).
\end{aligned}
$$

Example

The completely homogeneous adjoint problem

$$\begin{aligned} v'' &= 0, && 0 < x < 1, \\ v(0) &= v(1), \\ v'(0) &= -v'(1) \end{aligned}$$

has a non-trivial solution

$$v^{(1)}(x) = 1.$$

Hence the problem

$$Lu = f, \qquad 0 < x < 1, \qquad B_1 u = 0, \qquad B_2 u = 0$$

is solvable if and only if

$$\int_0^1 f(x)\, dx = 0.$$

Example

Green's function doesn't exist.

Construct a modified Green function:

(i) Causal fundamental solution for L:

$$E(x,\xi) = H(x-\xi)\cdot(x-\xi)$$

(ii) Find a solution of

$$Lw = v^{(1)}(x) = 1.$$

Choose

$$w(x) = \frac{x^2}{2}.$$

Example

(iii) Add solutions of the homogeneous equation $Lu = 0$,

$$g_M(x,\xi) = H(x-\xi)(x-\xi) - \frac{x^2}{2} + a + bx, \qquad a, b \in \mathbb{R},$$

to satisfy

$$B_1 g_M = B_2 g_M = 0.$$

In particular,

$$0 = g_M(1,\xi) + g_M(0,\xi) = (1-\xi) - \frac{1}{2} + a + b + a$$

so $2a + b = 1/2 - \xi$ and

$$0 = g_M'(1,\xi) - g_M'(0,\xi) = 1 - 1 + b - b$$

so b is arbitrary.

Construction of the Modified Green's Function

Set $b = 0$ and obtain

$$g_M(x, \xi) = H(x-\xi)(x-\xi) - \frac{x^2 - \xi}{2} - \frac{1}{4}.$$

Modified Adjoint Green function

The modified adjoint Green g_M^* satisfies

$$
\begin{aligned}
L^* g_M^*(x,\xi) &= \delta(x-\xi) - \sum_{i=1}^{k} u^{(i)}(\xi) u^{(i)}(x), \\
B_1^* g_M^* &= 0 \\
&\vdots \\
B_p^* g_M^* &= 0.
\end{aligned}
$$

where $u^{(1)}, \dots, u^{(k)}$ are the k orthonormalized non-trivial solutions of the completely homogeneous direct problem.

A Solution Formula

Suppose that u is a solution of

$$Lu = f, \qquad\qquad B_1 u = \cdots = B_p u = 0.$$

Then

$$\begin{aligned}
0 &= J(u, g_M^*)\Big|_a^b \\
&= \int_a^b \big(g_M^*(x,\xi) Lu(x) - u(x) L^* g_M^*(x,\xi)\big)\,dx \\
&= \int_a^b g_M^*(x,\xi) f(x) - u(x)\delta(x-\xi) + u(x)\sum_{i=1}^k u^{(i)}(\xi) u^{(i)}(x)\,dx \\
&= -u(\xi) + \int_a^b g_M^*(x,\xi) f(x)\,dx + \sum_{i=1}^k \langle u, u^{(i)}\rangle u^{(i)}(\xi).
\end{aligned}$$

A Solution Formula

This implies

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x)$$

Since we can always add solutions to the completely homogeneous problem to u, we can take

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi$$

Express the solution formula in terms of the modified direct Green function.

The Modified Direct and Adjoint Green's functions

By Green's formula,

$$\begin{aligned}
0 &= J(g_M, g_M^*)\Big|_a^b \\
&= \int_a^b g_M^*(x,\eta) L g_M(x,\xi) - g_M(x,\xi) L^* g_M^*(x,\eta)\, dx \\
&= g_M^*(\xi,\eta) - g_M(\eta,\xi) - \sum_{i=1}^k v^{(i)}(\xi) \underbrace{\int_a^b v^{(i)}(x) g_M^*(x,\eta)\, dx}_{=\langle v^{(i)}, g_M^*(\,\cdot\,,\eta)\rangle} \\
&\qquad + \sum_{i=1}^k u^{(i)}(\eta) \underbrace{\int_a^b u^{(i)}(x) g_M(x,\xi)\, dx}_{=\langle u^{(i)}, g_M(\,\cdot\,,\xi)\rangle}
\end{aligned}$$

The Modified Direct and Adjoint Green's functions

$$\begin{aligned} g_M^*(\xi, x) &= g_M(x, \xi) \\ &\qquad + \sum_{i=1}^{k} \big(v^{(i)}(\xi) \langle v^{(i)}, g_M^*(\,\cdot\,, x) \rangle - u^{(i)}(x) \langle u^{(i)}, g_M(\,\cdot\,, \xi) \rangle \big) \end{aligned}$$

Then

$$u(x) = \int_a^b g_M^*(\xi, x) f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x)$$

becomes

$$\begin{aligned} u(x) &= \int_a^b g_M(x, \xi) f(\xi)\, d\xi \\ &\qquad - \sum_{i=1}^{k} u^{(i)}(x) \int_a^b \langle u^{(i)}, g_M(\,\cdot\,, \xi) \rangle f(\xi)\, d\xi + \sum_{i=1}^{k} \langle u, u^{(i)} \rangle u^{(i)}(x) \end{aligned}$$

A Solution Formula

Write the last equation as

$$u(x)-\int_a^b g_M(x,\xi)f(\xi)\,d\xi = \sum_{i=1}^k \left\langle u - \int_a^b g_M(\,\cdot\,,\xi)f(\xi)\,d\xi, u^{(i)} \right\rangle u^{(i)}(x)$$

Geometrically,

$$u - \int_a^b g_M(\,\cdot\,,\xi)f(\xi)\,d\xi \ \in\ \operatorname{span}\{u^{(1)},\dots,u^{(k)}\},$$

i.e.,

$$u(x) = \int_a^b g_M(x,\xi)f(\xi)\,d\xi + \sum_{i=1}^k c_i u^{(i)}(x)$$

where $c_1,\dots,c_k$ are arbitrary constants.

Part III

Boundary Value Problems for Partial Differential Equations

Boundary Value Problems for PDEs

Eigenfunction Expansions

The Method of Images

The Boundary Element Method

Boundary Value Problems for PDEs

Eigenfunction Expansions

The Method of Images

The Boundary Element Method

Basic Quantities

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded, connected, open set.

The boundary of Ω is denoted $\partial\Omega$.

Define

$$Lu := -\operatorname{div}\big(p(x)\operatorname{grad} u\big) + q(x)u, \qquad\qquad x \in \Omega,$$

where

- $p, q\colon \Omega \to \mathbb{R}$ are sufficiently smooth
- $p(x) > 0$ for all $x \in \Omega$
- $q(x) \geq 0$ for all $x \in \Omega$

Basic Quantities

Let $I \subset \mathbb{R}$ be an open interval and

$$F\colon \Omega \to \mathbb{R} \qquad \text{or} \qquad F\colon \underset{\text{position } x}{\Omega} \times \underset{\text{time } t}{I} \to \mathbb{R}$$

be a forcing function.

Let

$$\varrho\colon \Omega \to [0, \infty) \subset \mathbb{R}$$

Depending on context,

$$u\colon \Omega \to \mathbb{C} \qquad \text{or} \qquad u\colon \Omega \times I \to \mathbb{C}$$

Second-Order Equations

- Elliptic equation

$$Lu = \varrho(x)F(x), \qquad x \in \Omega,$$

- Parabolic equation

$$\varrho(x)\frac{\partial u}{\partial t} + Lu = \varrho(x)F(x,t), \qquad (x,t) \in \Omega \times I,$$

- Hyperbolic equation

$$\varrho(x)\frac{\partial^2 u}{\partial t^2} + Lu = \varrho(x)F(x,t) \qquad (x,t) \in \Omega \times I.$$

Boundary Conditions

Boundary operator

$$Bu := \alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$$

where $\alpha, \beta\colon \partial\Omega \to \mathbb{R}$ with

$$\alpha(x) \geq 0, \qquad \beta(x) \geq 0, \qquad \alpha(x) + \beta(x) > 0 \qquad \text{on } \partial\Omega.$$

Special cases: boundary conditions

- of the first kind (Dirichlet): $\beta(x) = 0$ for all x
- of the second kind (Neumann): $\alpha(x) = 0$ for all x
- of the third kind (Robin): $\alpha(x), \beta(x) \neq 0$ for all x

Boundary Conditions

We write

$$\partial\Omega = S_1 \cup S_2 \cup S_3$$

where

- S_1, S_2, S_3 are pairwise disjoint
- boundary conditions of the kth kind are imposed on S_k.

If any two of these sets are non-empty, we say that we have

mixed boundary conditions.

The Boundary Value Problem

The equations

$$Lu = \varrho F, \qquad\qquad Bu = \gamma$$

where

$$\gamma\colon \partial\Omega \to \mathbb{R} \qquad \text{or} \qquad \gamma\colon \partial\Omega \times I \to \mathbb{R}$$

constitute the boundary value problem (L, B) with data $\{\varrho F, \gamma\}$

The Formal Adjoint

Definition. Let L be a partial differential operator on $\mathbb{R}^n$. The operator L^* such that

$$(LT)(\varphi) = T(L^*\varphi)$$

for any $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is called the formal adjoint of L.

If $L = L^*$, we say that L is formally self-adjoint.

Example.

$$L = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(x)\frac{\partial}{\partial x_i}\right) + q(x)$$

is formally self-adjoint.

Lagrange's Identity

For $u, v \in C^2(\mathbb{R}^n)$,

$$\begin{aligned} vLu &= v \operatorname{div}(p \operatorname{grad} u) \\ &= \operatorname{div}(pv \operatorname{grad} u) - p\langle \operatorname{grad} v, \operatorname{grad} u \rangle, \\ uLv &= \operatorname{div}(pu \operatorname{grad} v) - p\langle \operatorname{grad} u, \operatorname{grad} v \rangle \end{aligned}$$

so that

$$vLu - uLv = \operatorname{div}(pu \operatorname{grad} v) - \operatorname{div}(pv \operatorname{grad} u)$$

(Lagrange's Identity)

Green's Formula and the Conjunct

Using the divergence theorem,

$$\begin{aligned}\int_\Omega (vLu - uLv)\,dx &= -\int_\Omega \operatorname{div}(pv\,\operatorname{grad} u - pu\,\operatorname{grad} v)\,dx \\ &= -\int_{\partial\Omega} p(v\,\operatorname{grad} u - u\,\operatorname{grad} v)\,d\vec{\sigma} \\ &= \int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\,d\sigma\end{aligned}$$

(Green's Formula)

The conjunct of L is

$$J(u,v) = -p(v\,\operatorname{grad} u - u\,\operatorname{grad} v)$$

Adjoint Boundary Value Problems

Definition. Let (L, B) be a boundary value problem. Set

$$M := \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \colon Bu = 0\},$$

$$M^* := \Big\{v \in C^2(\Omega) \cap C(\overline{\Omega}) \colon \int_{\partial\Omega} J(u, v)\, d\vec{\sigma} = 0 \quad \text{for all } u \in M.\Big\}$$

A boundary operator B^* such that

$$M^* = \{v \in C^2(\Omega) \colon B^* v = 0\}$$

is said to be the adjoint operator to B.

(L^*, B^*) is the adjoint boundary value problem to (L, B).

If $L = L^*$ and $M = M^*$, (L, B) is said to be self-adjoint.

Self-Adjointness

Example. We show that $M = M^*$ for

$$L = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(x)\frac{\partial}{\partial x_i}\right) + q(x).$$

Suppose that $u, v \in M$. Then

$$\alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad \alpha(x)v + \beta(x)\frac{\partial v}{\partial n}\bigg|_{\partial\Omega} = 0.$$

Fix $x \in \partial\Omega$ and regard $\alpha(x), \beta(x)$ as solutions of the system

$$\begin{pmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Self-Adjointness

This implies

$$0 = \det \begin{pmatrix} u & \frac{\partial u}{\partial n} \\ v & \frac{\partial v}{\partial n} \end{pmatrix} = u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

on $\partial\Omega$.

Hence, if $u, v \in M$, then

$$\int_{\partial\Omega} J(u, v)\, d\vec{\sigma} = \int_{\partial\Omega} p \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma = 0.$$

This shows

$$v \in M \quad \Rightarrow \quad v \in M^*$$

The converse can be shown by considering S_1, S_2 and S_3 separately.

Direct and Adjoint Green Functions

Definition. The (direct) Green function $g(x, \xi)$ for (L, B) satisfies

$$Lg = \delta(x - \xi), \qquad Bg = 0.$$

while the adjoint Green function g^* satisfies

$$L^* g^* = \delta(x - \xi), \qquad B^* g^* = 0.$$

Solution Formula for the Elliptic Problem

Suppose u solves (L, B) with data $(\varrho F, \gamma)$ and $g^* = g$ is the Green function for $(L^*, B^*) = (L, B)$.

Then Green's formula gives

$$
\begin{aligned}
u(\xi) &= \int_\Omega g(x,\xi)\varrho(x)F(x)\,dx - \int_{\partial\Omega} p\left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right)\,d\sigma \\
&= \int_\Omega g(x,\xi)\varrho(x)F(x)\,dx \\
&\qquad - \int_{S_1} \frac{p}{\alpha}\gamma\frac{\partial g(\,\cdot\,,\xi)}{\partial n}\,d\sigma + \int_{S_2\cup S_3} \frac{p}{\beta}\gamma g(\,\cdot\,,\xi)\,d\sigma
\end{aligned}
$$

The Parabolic Boundary Value Problem

Recall

$$L = -\operatorname{div}(p(x)\operatorname{grad}) + q(x), \qquad x \in \Omega \subset \mathbb{R}^n,$$

and

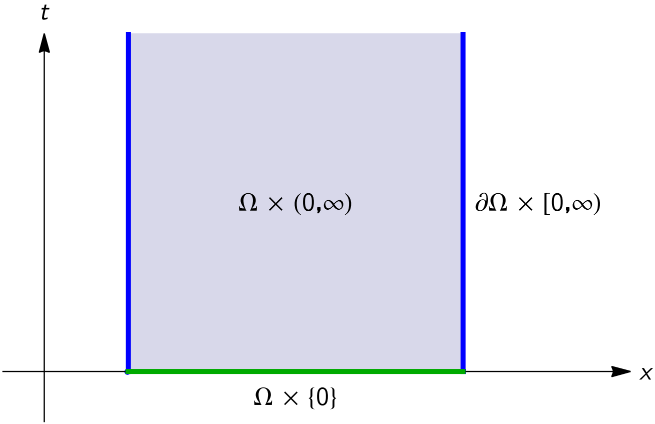
$$\widetilde{L} = \varrho(x)\frac{\partial}{\partial t} + L, \qquad (x,t) \in \Omega \times (0,\infty)$$

$\widetilde{L}$ is not self-adjoint:

$$\widetilde{L}^* = -\varrho(x)\frac{\partial}{\partial t} + L$$

Domain and Boundary of the Parabolic Problem

$$\partial(\Omega \times (0,\infty)) = \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times [0,\infty)\big)$$



Boundary Conditions

Let $\alpha, \beta \colon \partial\Omega \to \mathbb{C}$.

$$Bu := \alpha \cdot u\big|_{\partial\Omega\times[0,\infty)} + \beta \cdot \frac{\partial u}{\partial n}\Big|_{\partial\Omega\times[0,\infty)},$$
$$\widetilde{B}_1 u := u\big|_{\Omega\times\{0\}}.$$

We impose

$$Bu = \gamma(x,t), \qquad (x,t) \in \partial\Omega \times (0,\infty),$$

(Boundary Condition)

$$\widetilde{B}_1 u = u(x,0) = f(x) \qquad x \in \Omega$$

(Initial Condition)

Green's Formula and the Conjunct

Recall Lagrange's identity for L:

$$vLu - uLv = \operatorname{div}_x(pu \operatorname{grad}_x v - pv \operatorname{grad}_x u)$$

where div_x and grad_x emphasize the variables of differentiation.

Let $V \subset \mathbb{R}^{n+1}$ be a bounded domain. Then

$$\begin{aligned}
&\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) \\
&= \int_V \Big(\operatorname{div}_x(pu \operatorname{grad}_x v - pv \operatorname{grad}_x u) + \frac{d}{dt}(\varrho uv)\Big)\, d(x,t) \\
&= \int_V \operatorname{div}_{(x,t)} \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho uv \end{pmatrix} d(x,t)
\end{aligned}$$

Green's Formula and the Conjunct

Using the divergence theorem in $\mathbb{R}^{n+1}$,

$$\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) = \int_{\partial V} \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho u v \end{pmatrix} d\vec{\sigma}$$

so

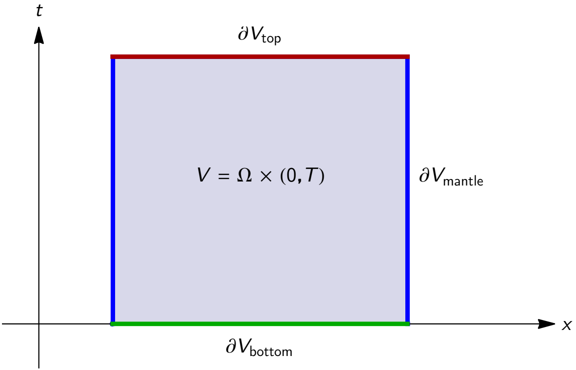
$$J(u,v) = \begin{pmatrix} p(u \operatorname{grad}_x v - v \operatorname{grad}_x u) \\ \varrho u v \end{pmatrix}.$$

is the conjunct for $\widetilde{L}$.

Restriction to a Bounded cylinder

Fix $T > 0$ and restrict the PDE to the cylinder

$$V = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$$



Boundary Conditions on the Bounded Cylinder

Then

$$\begin{aligned}\partial V &= \underbrace{\Omega \times \{0\}}_{\text{“bottom”}} \cup \underbrace{\partial\Omega \times [0, T]}_{\text{“mantle”}} \cup \underbrace{\Omega \times \{T\}}_{\text{“top”}} \\ &= \partial V_{\text{bottom}} \cup \partial V_{\text{mantle}} \cup \partial V_{\text{top}}\end{aligned}$$

We have

- boundary conditions on $\partial V_{\text{mantle}}$
- initial conditions on $\partial V_{\text{bottom}}$
- no conditions on ∂V_{top}

Green's Formula for the Bounded Cylinder

From

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial V_{\text{top}}} J(u,v)\,d\vec{\sigma} + \int_{\partial V_{\text{bottom}}} J(u,v)\,d\vec{\sigma} + \int_{\partial V_{\text{mantle}}} J(u,v)\,d\vec{\sigma}$$

we find

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial\Omega}\int_0^T p(u\,\mathrm{grad}\,v - v\,\mathrm{grad}\,u)\,dt\,d\vec{\sigma} + \int_{\Omega} \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\,dx$$

Green's Formula for the Bounded Cylinder

$$
\begin{aligned}
&\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) \\
&= \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\, dt\, d\vec{\sigma} \\
&\qquad + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\, dx
\end{aligned}
$$

Adjoint Boundary Conditions

As usual,

$$M = \big\{u \in C^2(V) \colon Bu = \widetilde{B}_1 u = 0\big\},$$
$$M^* = \Big\{v \in C^2(V) \colon \int_{\partial V} J(u,v)\,d\vec{\sigma} = 0 \text{ for all } u \in M\Big\}.$$

Since

$$\int_{\partial V} J(u,v)\,d\vec{\sigma} = \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\,dt\,d\vec{\sigma} + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\,dx$$

Adjoint Boundary Conditions

As usual,

$$M = \left\{u \in C^2(V) \colon Bu = \widetilde{B}_1 u = 0\right\},$$
$$M^* = \left\{v \in C^2(V) \colon \int_{\partial V} J(u,v)\, d\vec{\sigma} = 0 \text{ for all } u \in M\right\}.$$

Since

$$\int_{\partial V} J(u,v)\, d\vec{\sigma} = \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\, dt\, d\vec{\sigma}$$
$$+ \int_{\Omega} \varrho(x)\big(u(x,T)\underbrace{v(x,T)}_{\widetilde{B}_1^* v} - \underbrace{u(x,0)}_{\widetilde{B}_1 u} v(x,0)\big)\, dx$$

Adjoint Boundary Conditions

As usual,

$$M = \left\{u \in C^2(V) \colon Bu = \widetilde{B}_1 u = 0\right\},$$
$$M^* = \left\{v \in C^2(V) \colon \int_{\partial V} J(u,v)\, d\vec{\sigma} = 0 \text{ for all } u \in M\right\}.$$

We see that

$$M^* = \left\{v \in C^2(V) \colon B^* v = \widetilde{B}_1^* v = 0\right\}$$

where

$$B^* v = Bv, \qquad \widetilde{B}_1^* v = v(x,T) = v|_{\partial V_{\text{top}}}$$

(Adjoint Boundary Conditions)

Adjoint Green Function

The adjoint Green function satisfies

$$\widetilde{L}^* g^*(x, t; \xi, \tau) = \delta\big((x, t) - (\xi, \tau)\big),$$

for

$$x, \xi \in \Omega, \qquad\qquad t, \tau \in (0, T),$$

with boundary conditions

$$\begin{aligned} Bg^* &= 0, \\ \widetilde{B}_1^* g^* &= g^*(x, T; \xi, \tau) = 0. \end{aligned}$$

Solution Formula

Start from Green's formula,

$$\int_V (v\widetilde{L}u - u\widetilde{L}^* v)\, d(x,t) = \int_{\partial\Omega}\int_0^T p(u \operatorname{grad} v - v \operatorname{grad} u)\, dt\, d\vec{\sigma} + \int_\Omega \varrho(x)\big(u(x,T)v(x,T) - u(x,0)v(x,0)\big)\, dx$$

Suppose

$$\widetilde{L}u = \varrho F(x,t), \qquad u(x,0) = f(x), \qquad Bu = \gamma(x,t)$$

and $v = g^*$ satisfies

$$\widetilde{L}^* g^* = \delta\big((x,t) - (\xi,\tau)\big), \qquad g^*(x,T;\xi,\tau) = 0, \qquad Bg^* = 0$$

Solution Formula

Then

$$
\begin{aligned}
u(\xi,\tau) = \int_V & \varrho(x) F(x,t) g^*(x,t;\xi,\tau)\, d(x,t) \\
& + \int_\Omega \varrho(x) g^*(x,0;\xi,\tau) f(x)\, dx \\
& - \int_{\widetilde{S}_1} \frac{p}{\alpha}\gamma \frac{\partial g^*(\,\cdot\,;\xi,\tau)}{\partial n_x}\, d\sigma + \int_{\widetilde{S}_2 \cup \widetilde{S}_3} \frac{p}{\beta}\gamma g^*(\,\cdot\,;\xi,\tau)\, d\sigma
\end{aligned}
$$

where

$$
\widetilde{S}_k = S_k \times [0,T] \subset \partial V_{\text{mantle}}, \qquad k = 1,2,3.
$$

Causal Fundamental Solutions

A fundamental solution $E(x, t; \xi, \tau)$ for a time-dependent PDE satisfies

$$\widetilde{L}E = \delta\big((x,t) - (\xi,\tau)\big), \qquad x, \xi \in \Omega, \quad t, \tau \in \mathbb{R}$$

E is said to be causal if

$$E(x, t; \xi, \tau) = 0 \qquad \text{whenever } t < \tau.$$

Causal Fundamental Solution

The direct Green function $g(x, t; \xi, \tau)$ for the parabolic problem in a bounded cylinder satisfies

$$\begin{aligned}
\widetilde{L}g &= \delta\big((x, t) - (\xi, \tau)\big), \qquad && x, \xi \in \Omega, \quad t, \tau \in (0, T). \\
Bg &= 0, \\
\widetilde{B}_1 g &= 0.
\end{aligned}$$

A causal fundamental solution E already satisfies the first and the third equation!

Just as for ODEs, causal fundamental solutions may be constructed by solving PDEs with no time singularity.

Causal Fundamental Solution for an ODE

A causal fundamental solution for a first-order initial value problem:

$$\begin{aligned} a_1(t)E'(t;\tau) + a_0(t)E(t;\tau) &= \delta(t-\tau), & t,\tau &\in \mathbb{R}, \\ E(t;\tau) &= 0 & t &< \tau \end{aligned}$$

can be found by solving

$$\begin{aligned} a_1(t)u'(t;\tau) + a_0(t)u(t;\tau) &= 0, & t,\tau &\in \mathbb{R}, \\ u(\tau;\tau) &= \frac{1}{a_1(\tau)} \end{aligned}$$

and setting

$$E(t;\tau) = H(t-\tau)u(t;\tau)$$

Analogy of the Parabolic Problem

A causal fundamental solution for the parabolic problem:

$$\begin{aligned} \varrho(x)\frac{\partial}{\partial t}E(x,t;\xi,\tau) + LE(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), && t,\tau\in\mathbb{R}, \\ E(x,t;\xi,\tau) &= 0 && t<\tau \end{aligned}$$

can be found by solving

$$\begin{aligned} \varrho(x)\frac{\partial}{\partial t}u(x,t;\xi,\tau) + Lu(x,t;\xi,\tau) &= 0, && t,\tau\in\mathbb{R}, \\ u(x,\tau;\xi,\tau) &= \frac{1}{\varrho(x)}\delta(x-\xi) \end{aligned}$$

and setting

$$E(x,t;\xi,\tau) = H(t-\tau)u(x,t;\xi,\tau)$$

Example: Heat Equation

$$p(x,t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

solves

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \qquad (3.1.1)$$

with initial condition

$$u(x,0) = \delta(x)$$

Hence,

$$E(x,t;\xi,\tau) = H(t-\tau)p(x-\xi,t-\tau)$$

Boundary Value Problems for PDEs

Eigenfunction Expansions

The Method of Images

The Boundary Element Method

The Eigenvalue Problem for the Elliptic Operator

$$Lu := -\operatorname{div}\big(p(x)\operatorname{grad} u\big) + q(x)u, \qquad x \in \Omega \subset \mathbb{R}^n,$$

with boundary values

$$Bu := \alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = \gamma(x,t).$$

We set

$$M := \big\{u \in C^2(\Omega) \colon Bu = 0\big\}.$$

The Eigenvalue Problem for the Elliptic Operator

The eigenvalue problem for L is

$$Lu = \lambda u, \qquad\qquad u \in M,$$

complex eigenvalues

Scalar product:

$$\langle u, v\rangle_{L^2} := \int_\Omega \overline{u(x)} v(x)\,dx, \qquad\qquad u, v \in C^2(\Omega)$$

complex conjugate of $u(x)$

(L, B) is a self-adjoint boundary value problem, so

$$\langle v, Lu\rangle_{L^2} = \langle Lv, u\rangle_{L^2}, \qquad\qquad \text{for } u, v \in M.$$

Eigenvalues and Eigenfunctions

This implies:

- eigenvalues λ are real numbers
- eigenfunctions u to different eigenvalues are orthogonal

A high-powered theorem (from theory of compact operators) gives:

- Eigenvalues exist.
- The eigenvalues form an infinite sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ with

$$\lambda_1 \le \lambda_2 \le \dots \qquad \text{and} \qquad \lambda_n \xrightarrow{n\to\infty} \infty.$$

- The eigenfunctions $\{u_n\}$ give an orthonormal basis of $C^2(\Omega)$.

Positivity of Eigenvalues

We will prove that the eigenvalues can not be strictly negative:

$$
\begin{aligned}
\lambda\langle u, u\rangle_{L^2} = \langle u, Lu\rangle_{L^2} &= \int_\Omega \overline{u(x)}(Lu)(x)\,dx \\
&= \int_\Omega \overline{u(x)}\big[-\operatorname{div}\big(p(x)\operatorname{grad}u(x)\big) + q(x)u(x)\big]\,dx \\
&= -\int_\Omega \big[\operatorname{div}\big(\overline{u(x)}p(x)\operatorname{grad}u(x)\big) - p(x)|\operatorname{grad}u(x)|^2\big]\,dx \\
&\quad + \int_\Omega q(x)|u(x)|^2\,dx \\
&= \int_\Omega \left(p(x)|\operatorname{grad}u(x)|^2 + q(x)|u(x)|^2\right)\,dx - \int_{\partial\Omega} p\bar{u}\frac{\partial u}{\partial n}\,d\sigma
\end{aligned}
$$

Positivity of Eigenvalues

Setting $\partial\Omega = S_1 \cup S_2 \cup S_3$,

$$\begin{aligned} u\Big|_{S_1} &= 0, \\ \frac{\partial u}{\partial n}\Big|_{S_2} &= 0, \\ \frac{\partial u}{\partial n}\Big|_{S_3} &= -\frac{\alpha}{\beta}u \qquad\qquad \text{for } u \in M, \end{aligned}$$

Therefore,

$$\langle u, Lu\rangle_{L^2} = \int_\Omega \left(p(x)|\operatorname{grad} u(x)|^2 + q(x)|u(x)|^2\right)\,dx + \int_{S_3} p\frac{\alpha}{\beta}|u|^2\,d\sigma$$

so

$$\langle u, Lu\rangle_{L^2} \geq 0 \quad \text{and} \quad \langle u, Lu\rangle_{L^2} = 0 \quad \Leftrightarrow \quad q(x) \equiv \alpha(x) \equiv 0$$

Finding the Green function for the Elliptic Operator

Let $\{u_n\}$ be a basis of orthonormal eigenfunctions for (L, B).

Let g be the (unknown) Green function.

$$\int_\Omega \overline{u_n(x)} Lg(x;\xi)\, dx = \overline{u_n(\xi)}$$

Since (L, B) is a self-adjoint boundary value problem,

$$\int_\Omega \overline{u_n(x)} Lg(x;\xi)\, dx = \int_\Omega g(x;\xi)\overline{Lu_n(x)}\, dx = \lambda_n \int_\Omega g(x;\xi)\overline{u_n(x)}\, dx.$$

It then follows that

$$\int_\Omega g(x;\xi)\overline{u_n(x)}\, dx = \frac{\overline{u_n(\xi)}}{\lambda_n}.$$

Full Eigenfunction Expansion of Green's Function

The eigenfunctions $\{u_n\}$ are an orthonormal basis of $C^2(\Omega)$.

Green's function can be expanded in a series

$$\begin{aligned} g(x;\xi) &= \sum_n \langle u_n, g(\,\cdot\,;\xi)\rangle_{L^2} u_n(x) \\ &= \sum_n \frac{u_n(x)\overline{u_n(\xi)}}{\lambda_n}. \end{aligned}$$

(Full eigenfunction expansion of g)

Example: Dirichlet Problem on a Rectangle

Let

$$L = -\Delta, \qquad \Omega = [0, a] \times [0, b] \subset \mathbb{R}^2, \qquad Bu = u|_{\partial\Omega}$$

Orthonormalized eigenfunctions are

$$u_{m,n}(x_1, x_2) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right), \quad m, n = 1, 2, 3, \dots$$

Hence,

$$g(x;\xi) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab} \frac{\sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right) \sin\left(\frac{m\pi \xi_1}{a}\right) \sin\left(\frac{n\pi \xi_2}{b}\right)}{m^2\pi^2/a^2 + n^2\pi^2/b^2}$$

Example: Dirichlet Problem on a Disk

Let

$$L = -\Delta, \qquad \Omega = \{x \in \mathbb{R}^2 \colon |x| \le 1\}, \qquad Bu = u|_{\partial\Omega}$$

Orthonormalized eigenfunctions are

$$u_{m,n}(r,\varphi) = \frac{e^{in\varphi}}{\sqrt{\pi} J_n'(\alpha_{n,m})} J_n(\alpha_{n,m} r), \quad m = 1, 2, 3, \dots, \quad n \in \mathbb{Z}, \dots$$

where $\alpha_{n,m}$ is the mth positive zero of the nth Bessel function of the first kind, J_n. Hence,

$$g(r,\varphi;\varrho,\theta) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{in(\varphi-\theta)} J_n(\alpha_{n,m} r) J_n(\alpha_{n,m}\varrho)}{J_n'(\alpha_{n,m})^2 \alpha_{n,m}^2}$$

Partial Eigenfunction Expansions

Full eigenfunction expansion: involves multiple series,

- unwieldy and difficult to evaluate approximately

Partial eigenfunction expansion: involves fewer series.

Strategy ($\Omega \subset \mathbb{R}^2$):

- Separate variables and solve eigenvalue problem for each variable x_1 and x_2
- Expand Green function in terms of x_1 or x_2 eigenfunctions
- Find coefficients by solving a Green's function problem for an ODE

Example: Dirichlet Problem on a Rectangle

Wanted: Green function g satisfying

$$-\Delta g(x;\xi) = \delta(x-\xi), \qquad\qquad x \in \Omega = (0,a)\times(0,b),$$

with Dirichlet conditions

$$\begin{aligned} g(x;\xi)\big|_{x_1=0} &= g(x;\xi)\big|_{x_1=a} = 0, \qquad & x_2 \in [0,b], \\ g(x;\xi)\big|_{x_2=0} &= g(x;\xi)\big|_{x_2=b} = 0, \qquad & x_1 \in [0,a], \end{aligned}$$

for fixed $\xi \in \Omega$.

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Step 1: Separation of Variables

Formally solve

$$-\Delta u = 0 \qquad \text{on } \Omega, \qquad\qquad u\big|_{\partial\Omega} = 0$$

by setting

$$u(x_1, x_2) = X_1(x_1)X_2(x_2).$$

Eigenvalue problems:

$$\begin{aligned} X_1'' &= -\lambda X_1, && 0 < x_1 < a, && X_1(0) = X_1(a) = 0, \\ X_2'' &= \lambda X_2, && 0 < x_2 < b, && X_2(0) = X_2(b) = 0. \end{aligned}$$

Orthonormal eigenfunctions and eigenvalues

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi x_1}{a}\right) \right\}_{m=1}^{\infty}, \qquad\qquad \lambda_m = \left(\frac{m\pi}{a}\right)^2$$

Step 2: Expand the Green function

Fix (for example) $x_2 \in [0, b]$.

Expand g in terms of the x_1-eigenfunctions:

$$g(x_1, x_2; \xi) = \sum_{m=1}^{\infty} g_m(x_2; \xi) \sin\left(\frac{m\pi x_1}{a}\right).$$

Here

$$g_m(x_2; \xi) = \frac{2}{a} \int_0^a g(x; \xi) \sin\left(\frac{m\pi x_1}{a}\right)\, dx_1.$$

Step 3: Determine the Coefficients

$$\int_0^a \frac{2}{a}\sin\left(\frac{m\pi x_1}{a}\right)(-\Delta)g(x;\xi)\,dx_1 = \frac{2}{a}\sin\left(\frac{m\pi \xi_1}{a}\right)\delta(x_2-\xi_2)$$

Writing out the left-hand side,

$$\begin{aligned}
&-\int_0^a \frac{2}{a}\sin\left(\frac{m\pi x_1}{a}\right)\Delta g(x;\xi)\,dx_1 \\
&= -\int_0^a \frac{2}{a}\sin\left(\frac{m\pi x_1}{a}\right)\frac{\partial^2 g}{\partial x_1^2}\,dx_1 - \int_0^a \frac{2}{a}\sin\left(\frac{m\pi x_1}{a}\right)\frac{\partial^2 g}{\partial x_2^2}\,dx_1 \\
&= \frac{m^2\pi^2}{a^2}g_m(x_2;\xi) - \frac{\partial^2 g_m}{\partial x_2^2}
\end{aligned}$$

Step 3: Determine the Coefficients

Using the boundary conditions on Ω, g_m satisfies

$$\frac{\partial^2 g_m}{\partial x_2^2} - \frac{m^2\pi^2}{a^2} g_m(x_2;\xi) = -\frac{2}{a}\sin\left(\frac{m\pi\xi_1}{a}\right)\delta(x_2-\xi_2)$$

for $0 < x_2 < b$, with

$$g_m(0;\xi) = g_m(b;\xi) = 0.$$

This is a Green function problem for an ODE!

We obtain

$$g_m(x_2;\xi) = \begin{cases} \frac{2}{m\pi}\frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)}\sinh\left(\frac{m\pi x_2}{a}\right)\sinh\left(\frac{m\pi(b-\xi_2)}{a}\right), & x_2 < \xi_2, \\ \frac{2}{m\pi}\frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)}\sinh\left(\frac{m\pi\xi_2}{a}\right)\sinh\left(\frac{m\pi(b-x_2)}{a}\right), & x_2 > \xi_2. \end{cases}$$

Partial Eigenfunction Expansion for the Rectangle

We set

$$y_< := \min\{x_2, \xi_2\}, \qquad y_> := \max\{x_2, \xi_2\}$$

and write

$$g_m(x_2; \xi) = \frac{2}{m\pi} \frac{\sin(m\pi\xi_1/a)}{\sinh(m\pi b/a)} \sinh\left(\frac{m\pi(b - y_>)}{a}\right) \sinh\left(\frac{m\pi y_<}{a}\right)$$

Finally,

$$g(x; \xi) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \frac{\sin\left(\frac{m\pi\xi_1}{a}\right) \sin\left(\frac{m\pi x_1}{a}\right) \sinh\left(\frac{m\pi(b-y_>)}{a}\right) \sinh\left(\frac{m\pi y_<}{a}\right)}{\sinh(m\pi b/a)}.$$

(Partial eigenfunction expansion)

Partial Eigenfunction Expansion for the Rectangle

We could also have expanded Green's function in terms of the x_2 eigenfunctions; this would have yielded

$$g(x;\xi) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\sin\left(\frac{n\pi\xi_2}{b}\right)\sin\left(\frac{n\pi x_2}{b}\right)\sinh\left(\frac{n\pi(a-z_>)}{b}\right)\sinh\left(\frac{n\pi z_<}{a}\right)}{\sinh(n\pi a/b)}.$$

where

$$z_< := \min\{x_1,\xi_1\}, \qquad z_> := \max\{x_1,\xi_1\}.$$

Both partial expansions give the same Green's function, as does the full eigenfunction expansion.

Which of the partial expansions is more suitable in a given situation depends on the problem.

Boundary Value Problems for PDEs

Eigenfunction Expansions

The Method of Images

The Boundary Element Method

The Method of Images

Wanted: Green function satisfying

$$Lg(x;\xi) = \delta(x-\xi), \qquad x,\xi \in \Omega \subset \mathbb{R}^n, \qquad Bg = 0.$$

Standard Approach: Let $E(x;\xi)$ be a fundamental solution "with pole at ξ " and set

$$g(x;\xi) = E(x;\xi) + v(x)$$

where v satisfies

$$Lv = 0, \qquad x \in \Omega \subset \mathbb{R}^n, \qquad Bv = -BE(\,\cdot\,,\xi).$$

The Method of Images

Method of Images: Use the fundamental solution E to construct v, exploiting the symmetry of Ω.

Basic idea:

$$LE(x;\xi^*) = 0, \qquad x \in \Omega, \qquad \xi^* \notin \Omega$$

Choose ξ^* to satisfy the boundary conditions for the original problem.

Green's Function for $L = -\Delta$ on the Half-Plane

Half-Plane: $\mathbb{H} := \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$

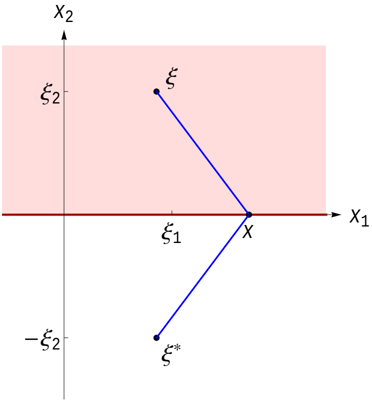
Boundary: $\partial\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = 0\}$

Dirichlet Conditions: $g|_{\partial\mathbb{H}} = 0$

Fundamental Solution: $E(x; \xi) = \frac{1}{2\pi} \ln|x - \xi|$

$$-\Delta E = \delta(x - \xi)$$

Green's Function for the Half-Plane



For $\xi = (\xi_1, \xi_2) \in \mathbb{H}$ set

$$\xi^* := (\xi_1, -\xi_2) \notin \mathbb{H}.$$

Then

$$|x - \xi| = |x - \xi^*| \quad \text{for } x \in \partial\mathbb{H}$$

and

$$g(x;\xi) = E(x;\xi) - E(x;\xi^*)$$

will vanish for $x \in \partial\mathbb{H}$.

The Dirichlet Problem on the Half-Plane

For example, the solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, && x \in \mathbb{H}, \\ u(x_1, 0) &= h(x_1), && x_1 \in \mathbb{R}, \quad h \in L^1_{\mathrm{loc}}(\mathbb{R}) \end{aligned}$$

is given by

$$u(x) = \int_{\partial\mathbb{H}} h \cdot \frac{\partial g}{\partial n}\, ds = -\int_{\mathbb{R}} h(\xi_1) \cdot \frac{\partial g(x; \xi_1, \xi_2)}{\partial \xi_2}\bigg|_{\xi_2 = 0} d\xi_1.$$

An easy calculation yields

$$\frac{\partial g(x; \xi_1, \xi_2)}{\partial \xi_2}\bigg|_{\xi_2 = 0} = -\frac{1}{\pi} \frac{x_2}{x_2^2 + (x_1 - \xi_1)^2}$$

The Dirichlet Problem on the Half-Plane

Solution formula:

$$u(x_1, x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(y)\frac{x_2}{x_2^2 + (x_1 - y)^2}\,dy.$$

- Check directly that u satisfies the equation
- Check that u satisfies boundary conditions:

$$\begin{aligned} u(x_1, x_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty} h(y)\frac{x_2}{x_2^2 + (x_1 - y)^2}\,dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} h(y + x_1)\frac{x_2}{x_2^2 + y^2}\,dy \quad \xrightarrow{x_2 \to 0} \quad h(x_1) \end{aligned}$$

since $\dfrac{1}{\pi}\dfrac{x_2}{x_2^2 + y^2}$ is a delta family as $x_2 \to 0$.

The Dirichlet Problem on the Unit Disk

Method of images exploits symmetry of a domain.

Discuss examples in $\mathbb{R}^2$ for $L = -\Delta$ using

$$E(x;\xi) = \frac{1}{2\pi}\ln|x-\xi|$$

Consider the unit disk

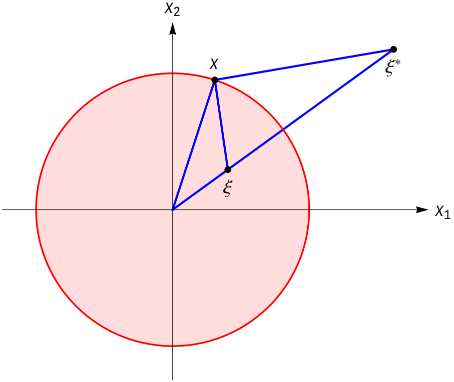
$$D = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq 1\}.$$

For $\xi \in D$ set

$$\xi^* = \frac{\xi}{|\xi|^2}.$$

The Dirichlet Problem on the Unit Disk

For $\xi \in D$ set $\xi^* = \dfrac{\xi}{|\xi|^2}$.



$\triangle(0, x, \xi)$ is similar to $\triangle(0, x, \xi^*)$

Therefore,

$$\frac{|x - \xi|}{|\xi|} = \frac{|x - \xi^*|}{|x|}.$$

Since $|x| = 1$,

$$|x - \xi| = |\xi| \cdot |x - \xi^*|.$$

The Dirichlet Problem on the Unit Disk

Algebraically:

$$\begin{aligned}|x-\xi|^2 &= 1+|\xi|^2-2\langle x,\xi\rangle\\ &= \left\langle |\xi|x-\frac{\xi}{|\xi|}, |\xi|x-\frac{\xi}{|\xi|}\right\rangle\\ &= |\xi|^2\cdot|x-\xi^*|^2\end{aligned}$$

It follows that we can take Green's function to be

$$\begin{aligned}g(x;\xi) &= E(x;\xi)-\frac{1}{2\pi}\ln\big(|\xi|\cdot|x-\xi^*|\big)\\ &= E(x;\xi)-E(x;\xi^*)-\frac{1}{2\pi}\ln(|\xi|).\end{aligned}$$

The Dirichlet Problem on the Unit Disk

The Dirichlet problem

$$\Delta u = 0, \qquad x \in D, \qquad u\big|_{\partial D} = h$$

then has the solution

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)K(r,\theta;a,\varphi)\big|_{a=1}\,d\varphi$$

(Poisson's integral formula)

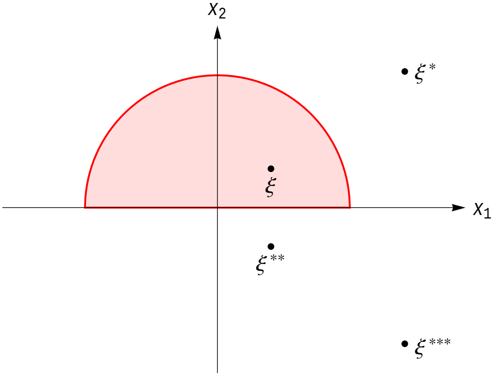
$$K(r,\theta;a,\varphi) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}.$$

(Poisson kernel)

The Dirichlet Problem on the Semi-Disk

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < 1,\ x_2 > 0\}$$

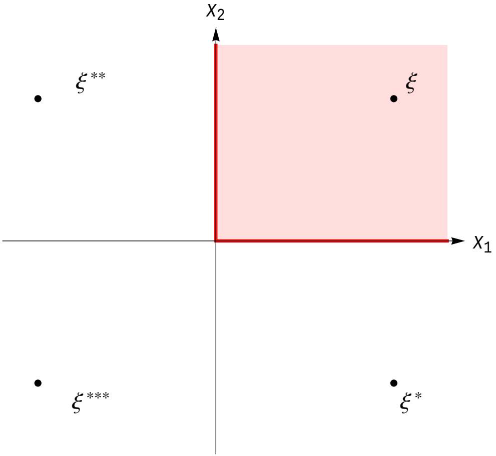
Place image charges as follows:



The Dirichlet Problem on the First Quadrant

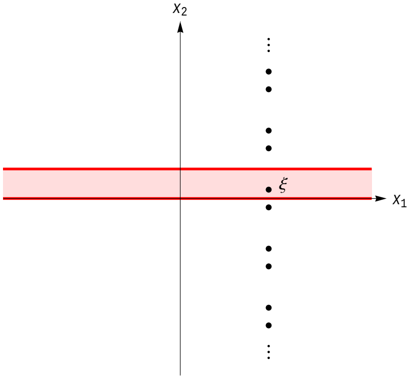
$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$$

Place image charges as follows:



The Dirichlet Problem on the Infinite Strip

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < 1\}$$



Needed: an infinite number of image charges.

For $\xi = (\xi_1, \xi_2)$:

$$\xi_{2n}^{+} := (\xi_1, 2n + \xi_2)$$

and

$$\xi_{2n}^{-} := (\xi_1, 2n - \xi_2)$$

with $n \in \mathbb{Z}$.

An Infinite Series of Point Charges

Green's function:

$$\begin{aligned} g(x;\xi) &= \sum_{n\in\mathbb{Z}} \big(E(x;\xi_{2n}^{+}) - E(x;\xi_{2n}^{-})\big) \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \big(\ln\big(|x-\xi_{2n}^{+}|\big) - \ln\big(|x-\xi_{2n}^{-}|\big)\big) \end{aligned}$$

Does this series converge?

We note that

$$\begin{aligned} |x-\xi_{2n}^{\pm}|^2 &= (x_1-\xi_1)^2 + (x_2 \mp \xi_2 - 2n)^2 \\ &= (x_1-\xi_1)^2 + (x_2 \mp \xi_2)^2 - 4n(x_2 \mp \xi_2) + 4n^2 \end{aligned}$$

An Infinite Series of Point Charges

Then

$$\begin{aligned} |x - \xi_{2n}^{\pm}| &= \sqrt{(x_1 - \xi_1)^2 + (x_2 \mp \xi_2)^2 - 4n(x_2 \mp \xi_2) + 4n^2} \\ &= 2|n|\sqrt{1 + \frac{(x_1 - \xi_1)^2 + (x_2 \mp \xi_2)^2}{4n^2} - \frac{(x_2 \mp \xi_2)}{n}} \end{aligned}$$

and

$$\ln\left(|x - \xi_{2n}^{\pm}|\right) = \ln(2|n|) - \frac{x_2 \mp \xi_2}{2n} + O\left(\frac{1}{n^2}\right) \qquad \text{as } n \to \infty.$$

- the $\ln(2|n|)$ summand disappears in the difference of the series
- the $O(1/n)$ terms do not cancel, so the series for g does not converge

Modified Green's Function

Set

$$E_{\text{mod}}(x;\xi_{2n}^{\pm}) := E(x;\xi_{2n}^{\pm}) - \underbrace{\frac{1}{2\pi}\ln(2|n|) + \frac{x_2 \mp \xi_2}{4\pi n}}_{\text{harmonic}}.$$

Then the modified Green's function is given by

$$\begin{aligned} g_{\text{mod}}(x;\xi) &= \sum_{n\in\mathbb{Z}} \big(E_{\text{mod}}(x;\xi_{2n}^{+}) - E_{\text{mod}}(x;\xi_{2n}^{-})\big) \\ &= \sum_{n\in\mathbb{Z}} E_{\text{mod}}(x;\xi_{2n}^{+}) - \sum_{n\in\mathbb{Z}} E_{\text{mod}}(x;\xi_{2n}^{-}) \end{aligned}$$

and both series converge separately.

Note that in the difference of the infinite series the added harmonic terms all cancel on the boundary, so that the boundary conditions remain satisfied.

Green Functions for the Upper Half-Space

We consider the half-space

$$\mathbb{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

Fundamental solution in $\mathbb{R}^3$:

$$E(x; \xi) = -\frac{1}{4\pi} \frac{1}{|x - \xi|}, \qquad x, \xi \in \mathbb{R}^3.$$

Goal: find Green function for

$$L = -\Delta$$

with Dirichlet, Neumann and Robin conditions on

$$\partial\mathbb{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}.$$

Dirichlet Problem for the Upper Half-Space

For $\xi = (\xi_1, \xi_2, \xi_3)$, set $\xi^* = (\xi_1, \xi_2, -\xi_3)$.

Then

$$E(x;\xi) = -\frac{1}{4\pi}\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2}},$$
$$E(x;\xi^*) = -\frac{1}{4\pi}\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3+\xi_3)^2}}$$

so

$$E(x,\xi^*) = E(x,\xi) \qquad\qquad \text{when } x \in \partial\mathbb{H}.$$

$$g(x;\xi) = E(x,\xi) - E(x;\xi^*)$$

is the Green function for the Dirichlet problem on $\mathbb{H}$.

Neumann Problem for the Upper Half-Space

$$\begin{aligned}
\frac{\partial E}{\partial n}\bigg|_{\partial\mathbb{H}} &= -\frac{\partial E}{\partial x_3}\bigg|_{x_3=0} \\
&= \frac{1}{4\pi}\frac{\xi_3 - x_3}{((x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2)^{3/2}}\bigg|_{x_3=0} \\
&= \frac{1}{4\pi}\frac{\xi_3}{((x_1-\xi_1)^2+(x_2-\xi_2)^2+\xi_3^2)^{3/2}}.
\end{aligned}$$

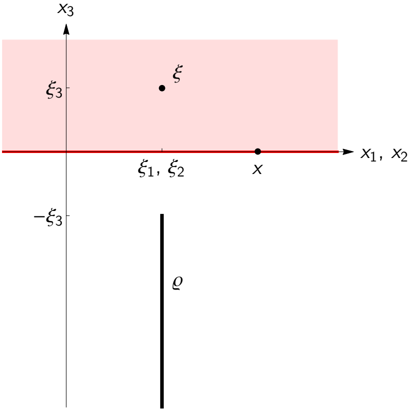
Hence,

$$\frac{\partial E(x;\xi)}{\partial n}\bigg|_{\partial\mathbb{H}} = -\frac{\partial E(x;\xi^*)}{\partial n}\bigg|_{\partial\mathbb{H}}$$

$$g_N(x;\xi) = E(x,\xi) + E(x;\xi^*)$$

is the Green function for the Neumann problem on $\mathbb{H}$

Robin Problem for the Upper Half-Space



We require

$$\frac{\partial g_R(\,\cdot\,;\xi)}{\partial n} + \alpha g_R(\,\cdot\,;\xi) = 0$$

on $\partial\mathbb{H}$ for a fixed $\alpha \geq 0$.

Use line charge with charge density ϱ, require

$$\lim_{s\to-\infty} \varrho(s) = 0.$$

$$\begin{aligned} g_R(x;\xi) &= E(x;\xi) + E(x;\xi^*) \\ &\quad + \frac{1}{4\pi}\int_{-\infty}^{-\xi_3} \frac{\varrho(s)}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-s)^2}}\,ds \end{aligned}$$

Robin Problem for the Upper Half-Space

Robin boundary condition yields

$$\varrho'(s) - \alpha\varrho(s) = 0, \qquad s < -\xi_3,$$

with initial condition

$$\varrho(-\xi_3) = -2\alpha.$$

Solution:

$$\varrho(s) = -2\alpha e^{\alpha(s+\xi_3)}.$$

Boundary Value Problems for PDEs

Eigenfunction Expansions

The Method of Images

The Boundary Element Method

A Boundary Value Problem in Two Dimensions

Consider for simplicity $\Omega \subset \mathbb{R}^2$,

$$\partial\Omega = S_1 \cup S_2, \qquad\qquad L = -\Delta$$

with

- Dirichlet boundary conditions on S_1
- Neumann boundary conditions on S_2

Fundamental solution

$$E(x;\xi) = \frac{1}{2\pi} \ln|x - \xi|$$

The Boundary Integral Solution

Green's formula for $\varphi, \psi \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$\int_\Omega (\psi \Delta \varphi - \varphi \Delta \psi)\, dx = \int_{\partial\Omega} \Big(\psi \cdot \frac{\partial \varphi}{\partial n} - \varphi \cdot \frac{\partial \psi}{\partial n} \Big)\, ds,$$

Suppose that $-\Delta u = 0$ on Ω. Then

$$\int_{\partial\Omega} \Big(u \cdot \frac{\partial E(\,\cdot\,;\xi)}{\partial n} - E(\,\cdot\,;\xi) \cdot \frac{\partial u}{\partial n} \Big)\, ds = \begin{cases} u(\xi) & \xi \in \Omega, \\ 0 & \xi \notin \overline{\Omega} \end{cases}$$

The case $\xi \in \partial\Omega$ is treated by integrating an ε-semicircle around ξ and letting $\varepsilon \to 0$.

The Boundary Integral Solution

$$\int_{\partial\Omega}\Big(u\cdot\frac{\partial E(\,\cdot\,;\xi)}{\partial n}-E(\,\cdot\,;\xi)\cdot\frac{\partial u}{\partial n}\Big)\,ds=\begin{cases}0 & \xi\notin\overline{\Omega},\\ u(\xi) & \xi\in\Omega,\\ \frac{1}{2}u(\xi) & \xi\in\partial\Omega.\end{cases}$$

Writing

$$\lambda_\Omega(\xi):=\begin{cases}0 & \xi\notin\overline{\Omega},\\ 1 & \xi\in\Omega,\\ 1/2 & \xi\in\partial\Omega,\end{cases}$$

we have

$$\lambda_\Omega(\xi)u(\xi)=\int_{\partial\Omega}\Big(u\cdot\frac{\partial E(\,\cdot\,;\xi)}{\partial n}-E(\,\cdot\,;\xi)\cdot\frac{\partial u}{\partial n}\Big)\,ds$$

(Boundary integral solution)

The Boundary Element Method (BEM)

Step 1: Approximate Ω by a polygon

- Choose $x^{(k)} \in \partial\Omega$, $k = 1, \dots, N$, and join by straight line segments (Boundary elements).

The line element joining $x^{(k)}$ to $x^{(k+1)}$ is denoted $\mathcal{C}^{(k)}$ ($x^{(N+1)} := x^{(1)}$).

Step 2: Approximate boundary data

- Find midpoint $\overline{x}^{(k)}$ of $\mathcal{C}^{(k)}$
- Take

$$u\Big|_{\mathcal{C}^{(k)}} \approx \overline{u}^{(k)} = u(x^{(k)}) \qquad \text{on } S_1$$

$$\frac{\partial u}{\partial n}\Big|_{\mathcal{C}^{(k)}} \approx \overline{p}^{(k)} = \frac{\partial u}{\partial n}\Big|_{x^{(k)}} \qquad \text{on } S_2$$

The Boundary Element Method (BEM)

Then

$$\lambda_\Omega(\xi)u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\xi) - \overline{p}^{(k)} I_1^{(k)}(\xi).$$

with

$$I_1^{(k)}(\xi) := \int_{\mathcal{C}^{(k)}} E(\,\cdot\,;\xi)\,ds, \qquad I_2^{(k)}(\xi) := \int_{\mathcal{C}^{(k)}} \frac{\partial E(\,\cdot\,;\xi)}{\partial n}\,ds.$$

Easily calculated!

The Boundary Element Method (BEM)

Choose $\xi = \overline{x}^{(k)}$:

$$\frac{1}{2}\underbrace{u(\overline{x}^{(k)})}_{=\overline{u}^{(k)}} = \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\overline{x}^{(k)}) - \overline{p}^{(k)} I_1^{(k)}(\overline{x}^{(k)})$$

- Linear system of N algebraic equations
- $2N$ unknowns $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$, $k = 1, \dots, N$
- N unknowns given by boundary data

$\Rightarrow$ Find all coefficients $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$.

The Boundary Element Method (BEM)

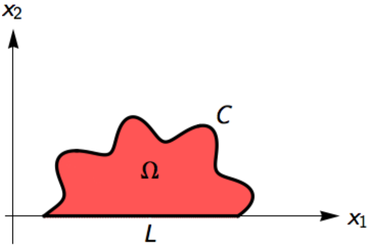
Then

$$u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot I_2^{(k)}(\xi) - \overline{p}^{(k)} I_1^{(k)}(\xi)$$

for $\xi \in \Omega$.

Green Functions and the BEM

Suppose $\partial\Omega = L \cup \mathcal{C}$, $\mathcal{C} = S_1 \cup S_2$.



Require

$$u\big|_L = 0$$

and

$$u\big|_{S_1} = f, \qquad \frac{du}{dn}\bigg|_{S_2} = g,$$

Use Green's function for upper half-plane,

$$g(x;\xi) = E(x,\xi) - E(x;\xi^*)$$

with $\xi^* = (\xi_1, -\xi_2)$

Green Functions and the BEM

We find

$$\lambda_{\Omega'}(\xi)u(\xi) = \int_{\mathcal{C}} \Big(u \cdot \frac{\partial g(\,\cdot\,;\xi)}{\partial n} - g(\,\cdot\,;\xi) \cdot \frac{\partial u}{\partial n} \Big) \, ds$$

where

$$\lambda_{\Omega'}(\xi) = \begin{cases} 0 & \xi \in L \cup \overline{\Omega}^c, \\ 1 & \xi \in \Omega, \\ 1/2 & \xi \in \mathcal{C}. \end{cases}$$

Note: We do not need to integrate over L, as

$$u\big|_L = g\big|_L = 0$$

Green Functions and the BEM

Discretize $\mathcal{C}$: $x^{(1)}$, $x^{(N+1)}$ are the endpoints of $\mathcal{C}$.

Then

$$\lambda_\Omega(\xi)u(\xi) \approx \sum_{k=1}^{N} \overline{u}^{(k)} \cdot \int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds - \overline{p}^{(k)} \int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds.$$

where

$$\begin{aligned}\int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds &= I_1^{(k)}(\xi) - I_1^{(k)}(\xi^*),\\ \int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds &= I_2^{(k)}(\xi) - I_2^{(k)}(\xi^*).\end{aligned}$$

As before, find the N unknown parameters of $\overline{u}^{(k)}$ and $\overline{p}^{(k)}$.

Green Functions and the BEM

Advantage of Green functions in BEM:

- Smaller part of $\partial\Omega$ to discretize, fewer equations / unknowns.

Disdvantage of Green functions in BEM:

- The integrals

$$\int_{\mathcal{C}^{(k)}} g(\,\cdot\,;\xi)\,ds \qquad \text{and} \qquad \int_{\mathcal{C}^{(k)}} \frac{\partial g}{\partial n}(\,\cdot\,;\xi)\,ds$$

may be harder to evaluate