UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 5

Group 22

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Exercise 5. 1

L^* is the same as L since $a_1 = a_0 = 0$.

$$L^* = \frac{d^2}{dx^2}$$

Green's formula thus becomes

$$\int_0^1 (vLu - uL^*v) = \int_0^1 (vu'' - uv'') = v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0)$$

The set M consists of all functions u s.t.

$$u(0) = 0$$

Apply these constraints, the right hand side simplifies to

$$v(1)u'(1) - u(1)v'(1) - v(0)u'(0)$$

where $u'(1), u(1), u(0)$ are arbitrary. The adjoint boundary functionals can then be expressed as

$$\begin{cases} B_1^*v = v(1) = 0 \\ B_2^*v = v'(1) = 0 \\ B_3^*v = v(0) = 0 \end{cases}$$

Exercise 5. 2

i).

$g(x;\xi)$ should satisfy

$$\begin{cases} Lg(x;\xi) = \delta(x-\xi) \\ g(0) = g'''(0) = g(1) = g''(1) = 0 \end{cases}$$

The solution is in the form of

$$g(x;\xi) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + ax^3 + bx^2 + cx + d$$

where a, b, c, d are real numbers. Plug in the conditions, it will yield to

$$\begin{cases} u(0) = 0 & \Rightarrow \quad d = 0 \\ u'''(0) = 0 & \Rightarrow \quad a = 0 \\ u(1) = 0 & \Rightarrow \quad \dfrac{(1-\xi)^3}{6} + b + c = 0 \\ u''(1) = 0 & \Rightarrow \quad 1 - \xi + 2b = 0 \end{cases}$$

So

$$g(x;\xi) = H(x-\xi) \cdot \frac{(x-\xi)^3}{6} + \frac{\xi - 1}{2}x^2 + \frac{\xi^3 - 3\xi^2 + 2}{6}x$$

ii).

Through *Integral by parts* (Note that here we denote n^{th} order derivative of u as $u^{(n)}$)

$$\begin{aligned}\int u^{(4)}v &= u^{(3)}v - \int u^{(3)}v' \\ &= u^{(3)}v - u^{(2)}v' + \int u^{(2)}v^{(2)} \\ &= u^{(3)}v - u^{(2)}v' + u'v^{(2)} - \int u'v^{(3)} \\ &= u^{(3)}v - u^{(2)}v' + u'v^{(2)} - uv^{(3)} + \int uv^{(4)}\end{aligned}$$

From the calculation above, now we have

$$L^* = L = \frac{d^4}{dx^4}$$

So Greens' formula is

$$\int vLu - uL^*v = u^{(3)}v - u^{(2)}v' + u'v^{(2)} - uv^{(3)}$$

Plug in the boundaries 0 and 1,

$$\begin{aligned}\int_0^1 vLu - uL^*v = \quad & u^{(3)}(0)v(0) - u^{(2)}(0)v'(0) + u'(0)v^{(2)}(0) - u(0)v^{(3)}(0) \\ & - \Big(u^{(3)}(1)v(1) - u^{(2)}(1)v'(1) + u'(1)v^{(2)}(1) - u(1)v^{(3)}(1)\Big)\end{aligned}$$

With boundary conditions

$$B_1u = u(0), \quad B_2u = u'''(0), \quad B_3 = u(1), \quad B_4 = u''(1)$$

The RHS of green's formula then becomes

$$-u^{(2)}(0)v'(0) + u'(0)v^{(2)}(0) - u^{(3)}(1)v(1) - u'(1)v^{(2)}(1)$$

which is independent of u. So the boundary conditions are

$$\begin{cases} B_1^*v = v'(0) = 0 \\ B_2^*v = v^{(2)}(0) = 0 \\ B_3^*v = v(1) = 0 \\ B_4^*v = v^{(2)}(1) = 0 \end{cases}$$

With the same strategy, we calculate $v(x) = H(x-\xi)\cdot\frac{(x-\xi)^3}{6} + ax^3 + bx^2 + cx + d$

$$\begin{cases} v'(0) = 0 & \Rightarrow \quad c = 0 \\ v''(0) = 0 & \Rightarrow \quad b = 0 \\ v(1) = 0 & \Rightarrow \quad \dfrac{(1-\xi)^3}{6} + a + d = 0 \\ v''(1) = 0 & \Rightarrow \quad 1 - \xi + 6a = 0 \end{cases}$$

So the solution is given as

$$g^*(x;\xi) = H(x-\xi)\cdot\frac{(x-\xi)^3}{6} + \frac{\xi-1}{6}x^3 + \frac{\xi^3 - 3\xi^2 + 2\xi}{6}$$

iii).

It is always true for adjoint Green function,

$$g^*(x, \xi) = g(\xi, x)$$

If we want

$$g(x, \xi) = g(\xi, x)$$

This means $g = g^*$. However, from our previous calculation, it's impossible for $g(x, \xi) = g(\xi, x)$, which proves

$$g \neq g^*$$

□

Exercise 5. 3

The fully homogeneous adjoint problem is

$$\begin{cases} -v'' - v = 0 \quad -\pi < x < \pi \\ v(\pi) - v(-\pi) = 0 \\ v'(\pi) - v'(-\pi) = 0 \end{cases}$$

which has a non-trivial solution $v(x) = c$ or $v(x) = c \cdot \sin(x)$ or $v(x) = c \cdot \cos(x)$. Now that we have

$$\begin{aligned} J(u, v)|_{-\pi}^{\pi} &= -u'v + uv'|_{-\pi}^{\pi} \\ &= -u'(\pi)v(\pi) + u(\pi)v'(\pi) + u'(-\pi)v(-\pi) - u(-\pi)v'(-\pi) \\ &= [u(\pi) - u(-\pi)]v'(\pi) - [u'(\pi) - u'(-\pi)]v(\pi) + u(-\pi)v'(\pi) - u'(-\pi)v(\pi) + u'(-\pi)v(-\pi) - u(-\pi)v'(-\pi) \\ &= [u(\pi) - u(-\pi)]v'(\pi) - [u'(\pi) - u'(-\pi)]v(\pi) + [v'(\pi) - v'(-\pi)]u(-\pi) + [v(-\pi) - v(\pi)]u'(-\pi) \end{aligned}$$

Plug in boundary conditions determined by u and v,

$$\begin{aligned} J(u, v)|_{-\pi}^{\pi} &= [u(\pi) - u(-\pi)]v'(\pi) - [u'(\pi) - u'(-\pi)]v(\pi) \\ &= \gamma_1 v'(\pi) - \gamma_2 v(\pi) \\ &= B_1 u B_2^* v - B_2 u B_1^* v \end{aligned}$$

For solution $v = c \cdot \sin(x)$,

$$\int_{-\pi}^{\pi} f(x) \sin(x) \mathrm{d}x = \gamma_1 \sin'(\pi) - \gamma_2 \sin(\pi) = -\gamma_1$$

For solution $v = c \cdot \cos(x)$,

$$\int_{-\pi}^{\pi} f(x) \cos(x) \mathrm{d}x = \gamma_1 \cos'(\pi) - \gamma_2 \cos(\pi) = \gamma_2$$

So the conditions are

$$\int_{-\pi}^{\pi} f(x) \sin(x) \mathrm{d}x = -\gamma_1$$

$$\int_{-\pi}^{\pi} f(x) \cos(x) \mathrm{d}x = \gamma_2$$

The type of forcing function that can give a periodic solution, i.e.

$$\int_{-\pi}^{\pi} f(x) \sin(x) \mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} f(x) \cos(x) \mathrm{d}x = 0$$

Exercise 5. 4

We first find the adjoint problem of the original one.

$$\begin{aligned}\int_0^1 (u'' + \pi^2 u)v &= \int_0^1 u''v + \int_0^1 \pi^2 uv \\ &= u'v - uv' + \int_0^1 uv'' + \int_0^1 \pi^2 uv \\ &= u'v - uv' + \int_0^1 (v'' + \pi^2 v)u\end{aligned}$$

So $L^* = \frac{d^2}{dx^2} + \pi^2$. Analyzing the same equation given above,

$$\begin{aligned}\int_0^1 vLu - uL^*v dx = (u'v - uv')|_0^1 \\ &= u'(1)v(1) - u(1)v'(1) - u'(0)v(0) + u(0)v'(0) \\ &= [u'(0) + u'(1)]v(1) - u'(0)v(1) - [u(0) + u(1)]v'(1) + u(0)v'(1) - u'(0)v(0) + u(0)v'(0) \\ &= \underbrace{[u'(0) + u'(1)]}_{B_2 u} v(1) - \underbrace{[u(0) + u(1)]}_{B_1 u} v'(1) - \underbrace{[v(0) + v(1)]}_{B_1^* v} u'(0) + \underbrace{[v'(0) + v'(1)]}_{B_2^* v} u(0)\end{aligned}$$

So the adjoint problem M^* is given as

$$\begin{aligned}L^* &= \frac{d^2}{dx^2} + \pi^2 \\ B_1^* &= v(0) + v(1) \\ B_2^* &= v'(0) + v'(1)\end{aligned}$$

Solving this equation, it will lead to two orthonormal non-trivial solutions:

$$\begin{aligned}v_1 &= \sqrt{2} \cdot \cos(\pi x) \\ v_2 &= \sqrt{2} \cdot \sin(\pi x)\end{aligned}$$

The fundamental solution $E(x;\xi)$ is calculated at $x = \xi$ with $u(\xi) = 0$ and $u'(\xi) = 1$, so it is

$$E(x;\xi) = H(x-\xi) \cdot \frac{\sin(\pi(x-\xi))}{\pi}$$

Next we find w_1 and w_2 s.t.

$$\begin{aligned}w_1'' + \pi^2 w_1 = Lw_1 = v_1 = \sqrt{2} \cdot \cos(\pi x) \\ w_2'' + \pi^2 w_2 = Lw_2 = v_2 = \sqrt{2} \cdot \sin(\pi x)\end{aligned}$$

Solving by Mathematica with code

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DSolve[y''[x] + \[Pi]^2 y[x] == c*Sin[\[Pi] x], y[x], x]
DSolve[y''[x] + \[Pi]^2 y[x] == c*Cos[\[Pi] x], y[x], x]
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It gives two solutions

$$\begin{aligned}
w_1 &= \frac{2\pi\left(cx + 2\pi c_2\right)\sin(\pi x) + \left(c + 4\pi^2 c_1\right)\cos(\pi x)}{4\pi^2} \\
&= \frac{-2\sqrt{2}\pi x\cos(\pi x) + \sqrt{2}\sin\pi x}{4\pi^2} + c_1\cos(\pi x) + c_2\sin(\pi x) \\
&= -\frac{\sqrt{2}x\cos(\pi x)}{2\pi} + c_1\cos(\pi x) + c_2' sin(\pi x) \\
w_2 &= \frac{\sqrt{2}\cos(\pi x) + 2\pi\sqrt{2}x\sin(\pi x)}{4\pi^2} + c_3\cos(\pi x) + c_4\sin(\pi x) \\
&= +\frac{\sqrt{2}x\sin(\pi x)}{2\pi} + c_3'\cos(\pi x) + c_4 sin(\pi x)
\end{aligned}$$

Here we can set c_1 and c_2 and c_3 and c_4 to 0 since we are focusing on non-zero parts, and $\cos(\pi x)$ and $\sin(\pi x)$ will be added to the equation later as a term.

$$\begin{aligned}
g_M(x;\xi) &= H(x-\xi)\cdot\frac{\sin\left(\pi(x-\xi)\right)}{\pi} - v_1(\xi)w_1(x) - v_2(\xi)w_2(x) + c_1\cos(\pi x) + c_2\sin(\pi x) \\
&= H(x-\xi)\cdot\frac{\sin\left(\pi(x-\xi)\right)}{\pi} - \frac{x\sin(\pi x - \pi\xi)}{\pi} + c_1\cos(\pi x) + c_2\sin(\pi x)
\end{aligned}$$

We need the above equation to satisfy

$$B_1 g = B_2 g = 0$$

, which means

$$g(0) + g(1) = 0 \qquad g'(0) + g'(1) = 0$$

We plug in the conditions, and we find it automatically satisfies the boundary equations. So we can conclude

$$g_M(x;\xi) = H(x-\xi)\cdot\frac{\sin\left(\pi(x-\xi)\right)}{\pi} - \frac{x\sin(\pi x - \pi\xi)}{\pi}$$

Exercise 5. 5

i).

The nontrivial solutions: $u^{(1)} = 1,\ u^{(2)} = x$

ii).

We want to show that the problem is self-adjoint, we need to show that $L = L^*$:

$$J(u,v)\,|_0^1 = u^{(3)}(1)\,v(1) - u^{''}(1)\,v^{'}(1) + u^{'}(1)\,v^{''}(1) - u(1)\,v^{(3)}(1) - u^{(3)}(0)\,v(0) + u^{''}(0)\,v^{'}(0) - u^{'}(0)\,v^{''}(0) + u(0)\,v^{(3)}(0)$$

Then it yields to

$$\begin{cases} B_1^* = v^{''}(0) \\ B_2^* = v^{(3)}(0) \\ B_3^* = v^{''}(1) \\ B_4^* = v^{(3)}(1) \end{cases}$$

and

$$L = L^*$$

So $M = M^*$. So, the problem is self-adjoint.

iii).

iii) Constructing the Modified Green Function:

$$v^{(1)} = 1,\ v^{(2)} = 2\sqrt{3}(\xi - \frac{1}{2})$$

are the k non-trivial, orthonormalized solutions of the adjoint problem.
Find a fundamental solution $E(x,\xi)$such that $LE = \delta(x-\xi)$.

$$u_\xi(x) = \frac{1}{6}(x-\xi)^3$$

, so $E(x,\xi) = \frac{1}{6}H(x-\xi)\,(x-\xi)^3$
Find 2 solutions of the inhomogeneous equations Lw=v: $w^{(1)} = \frac{x^4}{24},\ w^{(2)} = \frac{\sqrt{3}x^5}{60} - \frac{\sqrt{3}x^4}{24}$
Find p independent solutions of the homogeneous equation Lu = 0 and add them to $E(x,\xi) - \left(v^{(1)}(\xi)\,w^{(1)}(x) + v^{(2)}(\xi)\,w^{(2)}(x)\right)$ in order to satisfy the boundary conditions $B_1 g = ... = B_p g = 0$.

$$g_m = \frac{1}{6}H(x-\xi)\,(x-\xi)^3 - \frac{x^4}{24} - 2\sqrt{3}\left(\xi - \frac{1}{2}\right)\left(\frac{\sqrt{3}x^5}{60} - \frac{\sqrt{3}x^4}{24}\right) + ax^3 + bx^2 + cx + d$$

$$\begin{cases} B_1 g_m = 0 \\ B_2 g_m = 0 \\ B_3 g_m = 0 \\ B_4 g_m = 0 \end{cases}$$

$$=> \begin{cases} a = 0 \\ b = 0 \\ c\ is\ arbitrary \\ d\ is\ arbitrary \end{cases}$$

So, $g_m = \frac{1}{6}H\left(x-\xi\right)\left(x-\xi\right)^3 - \frac{x^4}{24} - \left(2\xi-1\right)\left(\frac{x^5}{20} - \frac{x^4}{8}\right)$.

iv).

We know that $u = \int_0^1 g_m\left(x,\xi\right) f\left(\xi\right) d\xi$

$$u^{(4)} = f$$

, where f satisfies the solvability conditions:

$$\begin{cases} \int_0^1 f\left(\xi\right) v^{(1)} d\xi = 0 \\ \int_0^1 f\left(\xi\right) v^{(2)} d\xi = 0 \end{cases}$$

=> $\begin{cases} \int_0^1 f\left(\xi\right) d\xi = 0 \\ \int_0^1 f\left(\xi\right) 2\sqrt{3}(\xi - \frac{1}{2}) d\xi = 0 \end{cases}$

So, $u = \int_0^1 (\frac{1}{6}H\left(x-\xi\right)\left(x-\xi\right)^3 - \frac{x^4}{24} - \left(2\xi-1\right)\left(\frac{x^5}{20} - \frac{x^4}{8}\right)) f\left(\xi\right) d\xi = \int_0^x \frac{1}{6}(x-\xi)^3 f\left(\xi\right) d\xi$