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交大密西根学院

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UM-SJTU Joint Institute  
VV557 Methods of Applied Math II

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Assignment 3

Group 22

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### Exercise 3.1 Fourier Transform

The Fourier Transform is defined as

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

i).

Plug in the definition of  $f(x)$

$$f(x) = \Pi_{a,b}(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}, \quad a, b \in \mathbb{R}$$

The Fourier transform is then calculated as

$$\begin{aligned} \mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^a 0 \cdot e^{-i\omega t} dt + \int_a^b e^{-i\omega t} dt + \int_b^{\infty} 0 \cdot e^{-i\omega t} dt \\ &= \int_a^b e^{-i\omega t} dt \\ &= \left. \frac{e^{-i\omega t}}{-i\omega} \right|_a^b = \frac{e^{-i\omega b} - e^{-i\omega a}}{-i\omega} \end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{e^{-i\omega b} - e^{-i\omega a}}{-i\omega\sqrt{2\pi}}$$

ii).

$$f(x) = e^{-a|x|}$$

Plug it in Fourier transform, which yields to

$$\begin{aligned} \mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^0 e^{ax} e^{-i\omega x} dx + \int_0^{+\infty} e^{-ax} e^{-i\omega x} dx \\ &= \left. \frac{e^{x(a-i\omega)}}{a-i\omega} \right|_{-\infty}^0 + \left. \frac{e^{-x(a+i\omega)}}{-a-i\omega} \right|_0^{+\infty} \\ &= -\frac{1}{a-i\omega} + \frac{1}{-a-i\omega} \\ &= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{a\sqrt{\frac{2}{\pi}}}{a^2 + \omega^2}$$

iii).

Plug in  $f(t) = e^{-ax^2}$

$$\begin{aligned}\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} e^{-at^2} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} e^{\frac{b^2}{4a} - \left(\sqrt{a}t + \frac{b}{2\sqrt{a}}\right)^2} dt\end{aligned}$$

Then we apply the substitution Rule, we define  $u := (\sqrt{a}t + \frac{b}{2\sqrt{a}})$ , then

$$du = \sqrt{a}dt$$

The original equation then becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{\frac{b^2}{4a} - \left(\sqrt{a}t + \frac{b}{2\sqrt{a}}\right)^2} dt \\ &= \int_{-\infty}^{\infty} \frac{e^{\frac{b^2}{4a} - u^2}}{\sqrt{a}} du \\ &= \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{a}} du\end{aligned}$$

Note that for the third step, we define

$$b = i\omega$$

We recall the definition of Gauss Error function, which is of the similar form

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^x e^{-t^2} dt$$

which yields to

$$\begin{aligned}\mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \frac{\sqrt{\pi} e^{\frac{b^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du \\ &= \frac{\sqrt{\pi} e^{\frac{b^2}{4a}}}{\sqrt{a}} \text{erf}(x)|_0^{+\infty} \\ &= \frac{\sqrt{\pi} e^{\frac{-\omega^2}{4a}}}{\sqrt{a}} (1 - 0) \\ &= \frac{\sqrt{\pi} e^{\frac{-\omega^2}{4a}}}{\sqrt{a}}\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2a}}$$

iv).

Plug in  $f(x) = \cos(x)e^{-x^2}$ ,

$$\begin{aligned}\mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \cos(t)e^{-t^2} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{e^{it} + e^{-it}}{2} e^{-t^2} e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{e^{it}}{2} e^{-t^2} e^{-i\omega t} dt + \int_{-\infty}^{\infty} \frac{e^{-it}}{2} e^{-t^2} e^{-i\omega t} dt\end{aligned}$$

Then the strategy is mostly alike 3.1.iii). We simplify the procedure, and the answer is given as

$$\begin{aligned}\mathcal{F}(\omega) \cdot \sqrt{2\pi} &= \frac{\sqrt{\pi}e^{-\frac{\omega^2}{4} + \frac{\omega}{2} - \frac{1}{4}}}{2} + \frac{\sqrt{\pi}e^{-\frac{\omega^2}{4} - \frac{\omega}{2} - \frac{1}{4}}}{2} \\ &= \frac{\sqrt{\pi}(e^{\omega} + 1)e^{-\frac{\omega^2}{4} - \frac{\omega}{2} - \frac{1}{4}}}{2}\end{aligned}$$

So

$$\mathcal{F}(\omega) = \frac{(e^{\omega} + 1)e^{-\frac{\omega^2}{4} - \frac{\omega}{2} - \frac{1}{4}}}{2\sqrt{2}}$$

v).

We consider the property that Plug in  $f(x) = \cos(2x)/(4 + x^2)$

$$\begin{aligned}\mathcal{F}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos(2t)}{4 + t^2} e^{-i\omega t} dt \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{2it} + e^{-2it}}{4 + t^2} e^{-i\omega t} dt \\ &= \frac{1}{2\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} \frac{e^{2it}}{4 + t^2} e^{-i\omega t} dt + \int_{-\infty}^{\infty} \frac{e^{-2it}}{4 + t^2} e^{-i\omega t} dt \right)\end{aligned}$$

From the solution of 3.1.3,

$$\widehat{e^{-a|x|}} = \frac{a\sqrt{\frac{2}{\pi}}}{a^2 + \omega^2}$$

Plug in  $a = 2$ , it yields to

$$\widehat{e^{-2|x|}} = \frac{2\sqrt{\frac{2}{\pi}}}{4 + \omega^2}$$

From the definition of Fourier Transform

$$\widehat{\widehat{f(t)}} = f(-t)$$

We could conclude that

$$\frac{1}{2\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} \frac{e^{2it}}{4 + t^2} e^{-i\omega t} dt \right) = \frac{1}{4} \sqrt{\frac{\pi}{2}} e^{-2|\omega-2|}$$

So the whole Fourier Transform then becomes

$$\mathcal{F}(\omega) = \frac{1}{4} \sqrt{\frac{\pi}{2}} e^{-2|\omega-2|} + \frac{1}{4} \sqrt{\frac{\pi}{2}} e^{-2|\omega+2|}$$

vi).

Consider the property

$$\widehat{\varphi * \psi} = (2\pi)^{n/2} \hat{\varphi} \cdot \hat{\psi}$$

Here  $n = 1$ . Then

$$xe^{-x^2} * e^{-x^2} = (2\pi)^{\frac{1}{2}} \widehat{xe^{-x^2}} \cdot \widehat{e^{-x^2}}$$

From the result before

$$\widehat{e^{-x^2}} = \frac{e^{-\frac{w^2}{4}}}{\sqrt{2}}$$

Then we apply Fourier Transform on  $xe^{-x^2}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} te^{-t^2} e^{-i\omega t} dt = \frac{i\omega e^{-\frac{w^2}{4}}}{2\sqrt{2}}$$

So the total integral is given by

$$\begin{aligned} xe^{-x^2} * e^{-x^2} &= (2\pi)^{\frac{1}{2}} \cdot \frac{e^{-\frac{w^2}{4}}}{\sqrt{2}} \cdot \frac{i\omega e^{-\frac{w^2}{4}}}{2\sqrt{2}} \\ &= \frac{\sqrt{2\pi} e^{-\frac{w^2}{4}} i\omega e^{-\frac{w^2}{4}}}{4} \end{aligned}$$

### Exercise 3. 2

i).

$$g(x) = \begin{cases} e^{-\varepsilon x} & x \geq 1 \\ 0 & x < 1 \end{cases} \quad \varepsilon > 0$$

$$\begin{aligned} \hat{T}_g \varphi &= T_g \hat{\varphi} = \int_1^{\infty} e^{-\varepsilon \xi} \cdot \hat{\varphi}(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \int_{-\infty}^{\infty} e^{-\varepsilon \xi} e^{-ix\xi} \varphi(x) dx d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \int_{-\infty}^{\infty} e^{(-\varepsilon - ix)\xi} \varphi(x) dx d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_1^{\infty} e^{(-\varepsilon - ix)\xi} d\xi \right) \varphi(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\varepsilon - ix}}{\varepsilon + ix} \varphi(x) dx \end{aligned}$$

So

$$g(\hat{x}) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\varepsilon - ix}}{\varepsilon + ix}$$

ii).

$$\sin(3x-2) = \frac{e^{i(3x-2)} - e^{-i(3x-2)}}{2i} = \frac{e^{-2i}}{2i} e^{i3x} - \frac{e^{2i}}{2i} e^{-i3x}$$

$$\hat{T}_{\delta(\xi-\xi_0)}\varphi = \int_{-\infty}^{\infty} \delta(\xi-\xi_0) \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ix\xi} \varphi(x) dx d\xi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ix\xi_0} \varphi(x) dx$$

$$\hat{\delta}(\xi-\xi_0) = \frac{1}{\sqrt{2\pi}} e^{-i\xi_0 x}$$

,thus

$$e^{-i\xi_0 x} = \sqrt{2\pi} \hat{\delta}(\xi-\xi_0)$$

$$\begin{aligned} \hat{T}_{\sin(3x-2)}\varphi &= T \frac{e^{-2i}}{2i} \delta(\xi+3) \hat{\varphi} - T \frac{e^{2i}}{2i} \delta(\xi-3) \hat{\varphi} \\ &= \frac{e^{-2i}}{2i} \int_{-\infty}^{\infty} \sqrt{2\pi} \delta(\xi+3) \varphi(-\xi) d\xi - \frac{e^{2i}}{2i} \int_{-\infty}^{\infty} \sqrt{2\pi} \delta(\xi-3) \varphi(-\xi) d\xi \\ &= \sqrt{2\pi} \left[ \frac{e^{-2i}}{2i} \varphi(3) - \frac{e^{2i}}{2i} \varphi(-3) \right] \end{aligned}$$

which gives the final answer

$$\widehat{\sin(3x-2)} = i\sqrt{\frac{\pi}{2}} [e^{2i}\delta(\xi+3) - e^{-2i}\delta(\xi-3)]$$

iii).

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

So

$$\begin{aligned} \hat{T}_g\varphi &= T_g\hat{\varphi} = \int_{-\infty}^{\infty} \xi^2 \cos(\xi) \hat{\varphi}(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^2 \cos(\xi) e^{-ix\xi} \varphi(x) dx d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\xi) \left[ \int_{-\infty}^{\infty} \xi^2 e^{-i\xi x} \varphi(x) dx \right] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi} + e^{-i\xi}}{2} \left( \int_{-\infty}^{\infty} -e^{-i\xi x} \varphi''(x) dx \right) d\xi \\ &= -\sqrt{\frac{\pi}{2}} \varphi''(1) - \sqrt{\frac{\pi}{2}} \varphi''(-1) \end{aligned}$$

which yields to the answer

$$\widehat{x^2 \cos x} = -\sqrt{\frac{\pi}{2}} \delta''(\xi-1) - \sqrt{\frac{\pi}{2}} \delta''(\xi+1)$$