Vv557 Methods of Applied Mathematics II
Green Functions for Partial Differential Equations

Assignment 8

Date Due: 1:00 PM, Monday, the 29th of April 2018

This assignment has a total of **(8 Marks)**.

Exercise 8.1

Consider the boundary value problem for the heat equation on a finite interval $(0, L) \subset \mathbb{R}$:

$$\begin{aligned} u_t - c^2 u_{xx} &= F(x,t), && 0 < x < L, \\ u(0,t) &= \gamma_1, && 0 < t < T, \\ u(L,t) &= \gamma_2, && 0 < t < T, \\ u(x,0) &= f(x), && 0 < x < L. \end{aligned} \tag{$*$}$$

where $T > 0$ is some fixed time, $\gamma_1, \gamma_2 \in \mathbb{R}$, and $f\colon [0,L] \to \mathbb{R}$, $F\colon [0,L] \times \mathbb{R} \to \mathbb{R}$ suitably smooth functions.

i) A causal fundamental solution for the heat equation on $\mathbb{R}$ is given by

$$E(x,t;\xi,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi c^2 (t-\tau)}} e^{-\frac{(x-\xi)^2}{4c^2(t-\tau)}}.$$

Use the method of images to find an infinite series representation of $g(x,t;\xi,\tau)$ using suitable image charges. Draw a sketch!
(3 Marks)

ii) By using a separation-of-variables approach, find a partial eigenfunction expansion of $g(x,t;\xi,\tau)$ in terms of the x-eigenfunctions.
(3 Marks)

iii) Compare the series representations obtained in i) and ii). If one uses only a few terms of the series to approximate g, which of the two expansions is suitable for small times $t \ll 1$ and which is suitable for large times $t \gg 1$?
(2 Marks)