

Vv557 Methods of Applied Mathematics II

Green Functions for Partial Differential Equations



JOINT INSTITUTE
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Assignment 1

Date Due: 1:00 PM, Thursday, the 7th of March 2018

This assignment has a total of **(16 Marks)**.

Recall that a sequence (f_n) of functions $f_n: I \rightarrow \mathbb{C}$, where $I \subset \mathbb{R}$, converges *pointwise* to a function $f: I \rightarrow \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |f_n(x) - f(x)| = 0 \quad \text{for all } x \in I.$$

The convergence is *uniform* if

$$\lim_{n \rightarrow \infty} \sup_{x \in I} |f_n(x) - f(x)| = 0.$$

Exercise 1.1

Let $\xi \in (0, 1) \subset \mathbb{R}$ be fixed. Solve the problem

$$-u'' = f_n(x; \xi), \quad 0 < x < 1, \quad u(0) = u(1) = 0 \quad (1)$$

for

$$f_n(x; \xi) = \begin{cases} n, & |x - \xi| < 1/2n, \\ 0 & \text{otherwise,} \end{cases}$$

with $1/n$ smaller than $\min\{\xi, 1 - \xi\}$.

i) For n as above, find the solution u_n of (1). *Solution:*

$$u_n(x) = \begin{cases} (1 - \xi) \cdot x & 0 \leq x \leq \xi - \frac{1}{2n}, \\ (1 - \xi) \cdot x - \frac{n}{2} \left(x - \xi + 1/(2n) \right)^2 & \xi - \frac{1}{2n} < x < \xi + \frac{1}{2n}, \\ \xi \cdot (1 - x) & \xi + \frac{1}{2n} \leq x \leq 1. \end{cases}$$

(3 Marks)

ii) Verify that the sequence of solutions $u_n(x; \xi)$ converges pointwise on $[0, 1]$ as $n \rightarrow \infty$ to the Green function $g(x, \xi)$ derived in the lecture.

(3 Marks)

iii) Is the convergence uniform on $[0, 1]$? Prove your assertion!

(2 Marks)

Exercise 1.2

Which of the following are distributions? Justify your response!

- i) $T: \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}, \varphi \mapsto \varphi(-10),$
- ii) $T: \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}, \varphi \mapsto \varphi(0)^2,$
- iii) $T: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{C}^n, \varphi \mapsto \text{grad } \varphi(0),$
- iv) $T: \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}, \varphi \mapsto \varphi(0) + \varphi(1) + \varphi(2) + \varphi(3) + \dots,$
- v) $T: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{C}, \varphi \mapsto \int_{S^{n-1}} \varphi, \text{ where } S^{n-1} = \{x \in \mathbb{R}^n: |x| = 1\}.$
- vi) $T_f: \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{R}, \varphi \mapsto \int_{\mathbb{R}} f(x) \varphi(x) dx, \text{ with}$

 - (a) $f(x) = 1/x,$
 - (b) $f(x) = 1/\sqrt{|x|},$
 - (c) $f(x) = 1/x^2.$

(8 Marks)