UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 6

Group 22

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Exercise 6. 1

Given that v satisfies

$$\int_{\partial\Omega} J(u,v)d\vec{\sigma} = \int_{\partial\Omega} p(u\frac{\partial v}{\partial n} - v\frac{\partial u}{dn}) = 0 \tag{1}$$

Since $u \in M$,

$$u|_{\partial\Omega} = 0$$

Plug in eqn. (1), it yields to

$$\int_{\partial\Omega} J(u,v)d\vec{\sigma} = \int_{\partial\Omega} p(0 \cdot \frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})d\vec{\sigma} = -\int_{\partial\Omega} pv\frac{\partial u}{\partial n}d\vec{\sigma} \tag{2}$$

From condition of u, we know nothing about the term $\frac{\partial u}{\partial n}$, which means it could be arbitrary. Also, by definition, $p > 0$. To let equation (2) evaluate to 0, it must satisfy $v|_{\partial\Omega} = 0$, which means

$$v \in M$$

□

Exercise 6. 2

i).

(a)

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}, \quad L^* = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = L$$

$$\int_\Omega (vLu - uL^*v)\, d(x,t) = \int_\Omega (vLu - uLv)d(x,t) = \int_{\partial\Omega} J(u,v)d\sigma$$

Then it could be expressed as

$$\begin{aligned}
\int_{\partial\Omega} J(u,v)d\sigma &= \int_{\partial\Omega} \begin{pmatrix} v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t} \\ u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} \end{pmatrix} d\sigma \\
&= \int_0^T (uv_x - vu_x)|_{-L}^{L}\, dt + \int_{-L}^{L} (vu_t - uv_t)|_0^T\, dx \\
&= \int_0^T u\; v_x|_{-L}^{L}\, dt + \int_{-L}^{L} [v(x,T)u_t(x,T) - u(x,T)v_t(x,T)]\, dx
\end{aligned}$$

Then the adjoint boundary conditions are

$$\left.\frac{\partial v}{\partial n}\right|_{\partial I} = 0, v(x,T) = 0, v_t(x,T) = 0$$

(b)

$$J(u,v) = \begin{pmatrix} v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t} \\ u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} \end{pmatrix}$$

(c) The Green function satisfies

$$Lg(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau)$$

Since $Lu = F, and\ \frac{\partial v}{\partial n}\big|_{\partial I} = 0, u(x,0) = f(x), \quad u_t(x,0) = h(x)$ We have,

$$u(\xi,\tau) = \int_\Omega gFd(x,t) - \int_{\partial\Omega} J\left(u,g\right) d\xi d\tau$$

$$u(x,t) = \int_\Omega g(\xi,\tau;x,t)F(\xi,\tau)d(\xi,\tau) + \int_{-L}^{L} \left[h(\xi)g(\xi,0;x,t) + f(\xi)g_t(\xi,0;x,t)\right] d\xi$$

ii).

$$E(x,t;\xi,\tau) = \frac{1}{2}H(t-\tau-|x-\xi|)$$

When $\quad t = 0, \tau > 0, |x-\xi| \geq 0$

$$E(x,0;\xi,\tau) = \frac{1}{2}H(-\tau-|x-\xi|) = 0$$

Since

$$T_{\frac{\partial E}{\partial t}}\varphi = -\int_R \frac{1}{2}H(t-\tau-|x-\xi|)\frac{\varphi(t)}{\partial t}dt = T_{\frac{1}{2}\delta(t-\tau-|x-\xi|)}\varphi$$

,

$$E_t(x,t;\xi,\tau) = \delta(t-\tau-|x-\xi|)$$

If L is Large enough,

$$\frac{\partial E}{\partial x}\bigg|_{x=-L} = -\frac{1}{2}\,\mathrm{sgn}(x-\xi)\cdot\delta(t-\tau-|x-\xi|)\bigg|_{x=-L} = -\frac{1}{2}\delta\left(t-\tau-L-\xi\right) = 0$$

$$\frac{\partial E}{\partial x}\bigg|_{x=L} = -\frac{1}{2}\,\mathrm{sgn}(x-\xi)\cdot\delta(t-\tau-|x-\xi|)\bigg|_{x=L} = \frac{1}{2}\delta\left(t-\tau-L+\xi\right) = 0$$

$$\frac{\partial E}{\partial x}\bigg|_{x=-L} = \frac{\partial E}{\partial x}\bigg|_{x=L} = 0, \quad \frac{\partial E}{\partial n}\bigg|_{\partial I} = 0$$

Therefore, if L is large enough, the fundamental solution$E(x,t;\xi,\tau) = \frac{1}{2}H(t-\tau-|x-\xi|)$ satisfies the boundary conditions.

iii).

Since E is the Green function and T is fixed, we have

$$u\left(x,t\right) = \int_\Omega g(x,t;\xi,\tau)F(\xi,\tau)d\xi d\tau + \int_{-L}^{L} h\left(\xi\right)g\left(x,t;\xi,0\right) - f(\xi)g_t\left(x,t;\xi,0\right)d\xi$$

From the results in i) and ii), and the additional boundary conditions (*), we have

$$\int_{-L}^{L} h\left(\xi\right)g\left(x,t;\xi,0\right) - f\left(\xi\right)g_t\left(x,t;\xi,0\right)d\xi$$

$$= \int_{t=T}\left(-\frac{1}{2}H\left(t-\tau-|x-\xi|\right)\right)h\left(\xi,\tau\right) + \frac{1}{2}f\left(\xi,\tau\right)\delta\left(t-\tau-|x-\xi|\right)d\xi$$

$$= \frac{f\left(T-\tau+x,\tau\right) + f\left(x-T+\tau,\tau\right)}{2} - \int_{x-t+\tau}^{t-\tau+x} h\left(\sigma,\tau\right)d\sigma$$

$$= \frac{f\left(x+T\right)+f\left(x-T\right)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h\left(\sigma\right)d\sigma$$

While

$$\int_{\Omega} g(x,t;\xi,\tau)F(\xi,\tau)d\xi d\tau = \int_{\Omega} \frac{1}{2}H(t-|x-\xi|)F(\xi,\tau)d\xi d\tau = \frac{1}{2}\iint_{\Delta(x,t)} F(\xi,\tau)d\xi d\tau$$

where $\Delta\left(x,t\right) = \{(\xi,\tau) \in R^2 : 0 \leq \tau \leq t-|x-\xi|\}$
When τ=0,

$$u\left(x,t\right) = \frac{f\left(x+t\right)+f\left(x-t\right)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h\left(y\right)dy + \frac{1}{2}\iint_{\Delta(x,t)} F(y,s)dyds$$

where $\Delta\left(x,t\right) = \{(y,s) \in R^2 : 0 \leq s \leq t-|x-y|\}$.