UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 2

Group 22

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Exercise 2. 1

Since $g \in \mathcal{D}'\left(\mathbb{R}^2\right), g(x) = -\frac{1}{2\pi}\log|x|$, g(x) is locally integrable, we have

$$\left(\Delta T_g\right)(\varphi) = T_g(\Delta\varphi) = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} g(x)\varphi(x)dx$$

In polar coordinates, $r = |x|, g(r) = -\frac{1}{2\pi}\log(r), r \geq 0$
According to Green's second identity:

$$\int_{r>\varepsilon} g(r)\Delta\varphi(r)dr = \int_{r>\varepsilon}\varphi(r)\Delta g(r)dr + \int_{r=\varepsilon}\left(g\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial g}{\partial n}\right)d\sigma$$

since $\Delta g = \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial^2 r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial\theta} = 0, \int_{r>\varepsilon}\varphi(r)\Delta g(r)dr = 0,$

$$\int_{r=\varepsilon}\left(g\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial g}{\partial n}\right)d\sigma = \int_{r=\varepsilon}\left(\varphi\frac{\partial g}{\partial r} - g\frac{\partial\varphi}{\partial r}\right)d\sigma$$

$$= \int_{r=\varepsilon}\left[\left(-\frac{1}{2\pi}\right)\cdot\frac{1}{r}\varphi - \left(-\frac{1}{2\pi}\right)\cdot\log(r)\frac{\partial\varphi}{\partial r}\right]d\theta$$

$\varphi \in \mathcal{D}'\left(\mathbb{R}^2\right)$ implies $\frac{\partial\varphi}{\partial r}$ bounded, so

$$\left|\int_{r=\varepsilon}\frac{1}{2\pi}\cdot log(r)\frac{\partial\varphi}{\partial r}d\sigma\right| \leq constant\cdot\frac{1}{2\pi}\cdot\log\varepsilon\cdot 2\pi\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0$$

$$\int_{r=\varepsilon}\left(-\frac{1}{2\pi}\right)\frac{1}{r}\cdot\varphi d\sigma \overset{\varepsilon\to 0}{\longrightarrow} -\frac{1}{2\pi}\cdot\frac{1}{\varepsilon}\cdot\varphi(x)\cdot 2\pi\varepsilon = -\varphi(0)$$

Therefore, $\Delta T_g\varphi(x) = -\varphi(0), \Delta g = -\delta(x)$

Exercise 2. 2

u: $R^2 \to R$ is given by u(x,t)=$\begin{cases} \frac{1}{2} & t - |x| > 0, \\ 0 & otherwise \end{cases}$

Now, u(x,t)=$\begin{cases} \frac{1}{2} & t > |x|, \\ 0 & t \leq |x| \end{cases}$ –(0)

So, in any case u(x,t) is a constant function, neither it depends on the values of t nor x.

Now, differentiate (0) partially with respect to t

$$u_t = \begin{cases} 0 & t > |x|, \\ 0 & t \leq |x| \end{cases} \tag{1}$$

Again differentiate (1) partially with respect to t

=> u_{tt}=0 for any t

Now, differentiate (0) partially with respect to x

$$u_x = \begin{cases} 0 & t > |x|, \\ 0 & t \leq |x| \end{cases} \tag{2}$$

Again differentiate (2) partially with respect to x

=> u_{xx}=0 for any x

So $u_{tt} - u_{xx} = 0 - 0 = 0$

Exercise 2. 3

The distribution of $P(\frac{1}{x})$ is given as

$$\mathcal{P}\left(\frac{1}{x}\right)(\varphi) := \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

Observing the distribution of $P(\frac{1}{x^2})$ is given as

$$\mathcal{P}\left(\frac{1}{x^2}\right)(\varphi) := \lim_{\varepsilon \searrow 0} \int_{|x| > \varepsilon} \frac{1}{x^2}(\varphi(x) - \varphi(0)) dx$$

According to the definition of weak derivative

$$\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi) = -\mathcal{P}\left(\frac{1}{x}\right)(\frac{d}{dx}\varphi)$$

We can thus express it into

$$\begin{aligned}\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi) &= -\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi'(x)}{x} dx \\ &= -\lim_{\varepsilon \to 0} (\int_{\varepsilon}^{\infty} \frac{\varphi'(x)}{x} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi'(x)}{x} dx)\end{aligned}$$

Then we investigate on $-\lim\limits_{\varepsilon \to 0} \displaystyle\int_{-\infty}^{-\varepsilon} \frac{\varphi'(x)}{x} dx$. By applying the basic integral rule $\int u'v = uv + \int uv'$,

$$\begin{aligned}-\lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} \frac{\varphi'(x)}{x} dx &= -\lim_{\varepsilon \to 0} \left(\frac{1}{x} \cdot \varphi(x) \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x'} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left(\frac{1}{x} \cdot \varphi(x) \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left(\frac{1}{x} \cdot \varphi(x) \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{(\varphi(x) - \varphi(0)) + \varphi(0)}{x^2} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left(\frac{1}{x} \cdot \varphi(x) \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{(\varphi(x) - \varphi(0))}{x^2} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(0)}{x^2} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left((\frac{\varphi(-\varepsilon)}{-\varepsilon} - 0) + \int_{-\infty}^{-\varepsilon} \frac{(\varphi(x) - \varphi(0))}{x^2} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(0)}{x^2} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left(\frac{\varphi(-\varepsilon)}{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{(\varphi(x) - \varphi(0))}{x^2} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(0)}{x^2} dx \right)\end{aligned}$$

Similarly, we can derive the expression of $-\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\varphi'(x)}{x} dx$. Then

$$\begin{aligned}\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi) &= -\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi'(x)}{x} dx \\ &= -\lim_{\varepsilon \to 0} (\int_{\varepsilon}^{\infty} \frac{\varphi'(x)}{x} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi'(x)}{x} dx)\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi) &= -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi'(x)}{x}dx \\
&= -\lim_{\varepsilon\to 0}(\int_{\varepsilon}^{\infty}\frac{\varphi'(x)}{x}dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi'(x)}{x}dx) \\
&= -\lim_{\varepsilon\to 0}\left(\frac{\varphi(-\varepsilon)}{-\varepsilon} + \frac{\varphi(\varepsilon)}{\varepsilon} + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(0)}{x^2}dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi(0)}{x^2}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(0)}{x^2}dx\right)
\end{aligned}
$$

Sine $\varphi(x) \in C_0^\infty$, it is continuous at $x = 0$, so

$$\varphi(0^-) = \varphi(0^+)$$

which means

$$\lim_{\varepsilon\to 0}\varphi(-\varepsilon) = \lim_{\varepsilon\to 0}\varphi(\varepsilon)$$

Divide by same denominator ε, it yields to

$$\lim_{\varepsilon\to 0}\frac{\varphi(-\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\varphi(\varepsilon)}{\varepsilon}$$

Which means

$$\lim_{\varepsilon\to 0}\left(\frac{\varphi(-\varepsilon)}{-\varepsilon} + \frac{\varphi(\varepsilon)}{\varepsilon}\right) = 0$$

Also, $\int \frac{1}{x^2} = -\frac{1}{x}$, which is an odd function, and then integral given by

$$\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi(0)}{x^2}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(0)}{x^2}dx\right)$$

is defined on a symmetric interval, which yields to the total integral value to be 0. Then $\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi)$ becomes

$$\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right)(\varphi) = -\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)-\varphi(0)}{x^2}dx\right) = -\lim_{\varepsilon\searrow 0}\int_{|x|>\varepsilon}\frac{1}{x^2}(\varphi(x)-\varphi(0))dx$$

which means

$$\frac{d}{dx}\mathcal{P}\left(\frac{1}{x}\right) = -\mathcal{P}\left(\frac{1}{x^2}\right)$$

$\square$