CHAPTER 7

GREEN'S FUNCTIONS

The method of *Green's functions* is an important technique for solving boundary value, initial and boundary value, and Cauchy problems for partial differential equations. It is most commonly identified with the solution of boundary value problems for Laplace's equation and a Green's function has already been introduced in that context in Chapter 1. It was also seen in Section 1.3 and in our study of point source problems in Section 6.7 that the Green's function is often worthwhile determining in its own right rather than as a tool to be used only for solving another problem.

In this chapter we begin by constructing generalizations of *Green's second theorem* that are appropriate for the second order differential equations introduced in Chapter 4. These integral theorems [which are special cases of the general result (3.6.7) given in Section 3.6] are then used to show how boundary value, initial and boundary value, and Cauchy problems can be solved in terms of appropriately defined Green's functions for each of these problems. Even though the construction of Green's functions requires that a problem similar to the original (given) problem must be solved, it is often easier to solve the Green's function problem in a number of important cases, as we shall see. In this regard the *fundamental solutions* considered in Section 6.7, of which Green's functions are a special case, play an important role. Since the determination and use of Green's functions require the use of *generalized functions* such as the Dirac delta function, a brief discussion of the theory of generalized functions is given in this chapter. Most of the chapter, however, is devoted to the construction and use of Green's functions for problems involving equations of elliptic, hyperbolic, and parabolic types.

7.1 INTEGRAL THEOREMS AND GREEN'S FUNCTIONS

In this section we construct *integral theorems* appropriate for the *elliptic, hyperbolic,* and *parabolic equations* introduced in Section 4.1. Each of these theorems follows from an application of the divergence theorem and represents a generalization of Green's second theorem. These theorems form the basis for the construction of the Green's functions we consider in this chapter. Technically, the theorems are valid only if the functions occurring in the integrals are sufficiently smooth, and as we have seen in Section 6.7, this is generally not the case for Green's functions. Nevertheless, we shall assume that these theorems are formally valid in all cases and rely on the theory of generalized functions presented in Section 7.2 to form a basis for their validity, even though this is not demonstrated. We begin our discussion with problems in two or three space dimensions and present the one-dimensional results at the end of this section. Even though these integral theorems are special cases of the general result (3.6.7), we include some details of their derivation.

Integral Theorems and Green's Functions for Elliptic Equations

We start with the *elliptic equation*

$$Lu(\mathbf{x}) = -\nabla \cdot (p(\mathbf{x})\nabla u(\mathbf{x})) + q(\mathbf{x})u(\mathbf{x}) = \rho(\mathbf{x})F(\mathbf{x}) \tag{7.1.1}$$

in two or three dimensions given over a bounded region G with the boundary conditions

$$\alpha(\mathbf{x})\, u(\mathbf{x}) + \beta(\mathbf{x})\, \frac{\partial u(\mathbf{x})}{\partial n}\bigg|_{\partial G} = B(\mathbf{x}). \tag{7.1.2}$$

The conditions on the coefficients in (7.1.1) and (7.1.2) given in Section 4.1 are assumed to remain in effect. Introducing a function $w(\mathbf{x})$ whose properties are to be specified and proceeding as in Example 4.2, we obtain

$$\iint_G [wLu - uLw]\, dv = -\int_{\partial G} p[w\nabla u - u\nabla w]\cdot \mathbf{n}\, ds = \int_{\partial G} p\left[u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right] ds, \tag{7.1.3}$$

on applying the divergence theorem with $\mathbf{n}$ as the exterior unit normal on ∂G. Equation (7.1.3) is the basic *integral theorem* from which the *Green's function method* proceeds in the *elliptic case*.

The function $w(\mathbf{x})$ is now determined such that (7.1.3) expresses $u(\mathbf{x})$ at an arbitrary point ξ in the region G in terms of $w(\mathbf{x})$ and known functions in (7.1.1) and (7.1.2).

Let $w(\mathbf{x})$ be a solution of $Lw(\mathbf{x}) = \delta(\mathbf{x}-\boldsymbol{\xi})$, where $\delta(\mathbf{x}-\boldsymbol{\xi})$ is a two- or three-dimensional Dirac delta function. The substitution property of the delta function then yields

$$\iint_G u(\mathbf{x})Lw(\mathbf{x})\,dv = \iint_G u(\mathbf{x})\,\delta(\mathbf{x}-\boldsymbol{\xi})\,dv = u(\boldsymbol{\xi}). \tag{7.1.4}$$

In view of (7.1.1) we also have

$$\iint_G w(\mathbf{x})Lu(\mathbf{x})\,dv = \iint_G \rho(\mathbf{x})w(\mathbf{x})F(\mathbf{x})\,dv. \tag{7.1.5}$$

It now remains to choose boundary conditions for $w(\mathbf{x})$ on ∂G so that the boundary integral in (7.1.3) involves only $w(\mathbf{x})$ and known functions. This can be accomplished by requiring $w(\mathbf{x})$ to satisfy the homogeneous version of the boundary condition (7.1.2); that is, $\alpha(\mathbf{x})\,w(\mathbf{x}) + \beta(\mathbf{x})\,\partial w(\mathbf{x})/\partial n\big|_{\partial G} = 0$. If $\mathbf{x} \in S_1$ on ∂G, we have

$$u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n} = \frac{1}{\alpha}B\frac{\partial w}{\partial n}, \tag{7.1.6}$$

in view of (7.1.2). If $\mathbf{x} \in S_2 \cup S_3$ on ∂G, we have

$$u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n} = -\frac{1}{\beta}Bw. \tag{7.1.7}$$

The function $w(\mathbf{x})$ is called the *Green's function* for the boundary value problem (7.1.1)–(7.1.2). To indicate its dependence on the point $\boldsymbol{\xi}$, we denote the Green's function by $w(\mathbf{x}) = K(\mathbf{x};\boldsymbol{\xi})$, as in Section 1.3. In terms of the Green's function $K(\mathbf{x};\boldsymbol{\xi})$, the foregoing implies that (7.1.3) takes the form

$$u(\boldsymbol{\xi}) = \iint_G \rho K(\mathbf{x};\boldsymbol{\xi})F\,dv - \int_{S_1} \frac{pB}{\alpha}\frac{\partial K(\mathbf{x};\boldsymbol{\xi})}{\partial n}\,ds + \int_{S_2\cup S_3} \frac{pB}{\beta}K(\mathbf{x};\boldsymbol{\xi})\,ds. \tag{7.1.8}$$

The Green's function $K(\mathbf{x};\boldsymbol{\xi})$ thus satisfies the equation

$$-\nabla\cdot(p(\mathbf{x})\nabla K(\mathbf{x};\boldsymbol{\xi})) + q(\mathbf{x})K(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi}), \qquad \mathbf{x},\ \boldsymbol{\xi} \in G, \tag{7.1.9}$$

and the boundary condition

$$\alpha(\mathbf{x})K(\mathbf{x};\boldsymbol{\xi}) + \beta(\mathbf{x})\left.\frac{\partial K(\mathbf{x};\boldsymbol{\xi})}{\partial n}\right|_{\partial G} = 0, \tag{7.1.10}$$

with the derivatives taken in the $\mathbf{x}$-variables. It follows from (7.1.9) and Section 6.7 that the Green's function is a *fundamental solution* of (7.1.1). This fact will be exploited in the construction of certain Green's functions.

Not all Green's function problems (7.1.9)–(7.1.10) have solutions. In certain cases considered later, a generalized or modified Green's function must be constructed that satisfies an equation that differs from (7.1.9) or boundary conditions that differ from (7.1.10). However, once the Green's function has been determined, the formulas

(7.1.8) or slightly modified ones in the generalized case yield the solution $u(\mathbf{x})$ of the boundary value problem (7.1.1)–(7.1.2) at any point in G. By introducing appropriate assumptions on the behavior of the solutions at infinity, the Green's function technique can also be applied to problems over unbounded regions. We construct Green's functions for specific elliptic equations of the form (7.1.1) over bounded and unbounded regions.

Integral Theorems and Green's Functions for Hyperbolic Equations

We consider the initial and boundary value problem for the *hyperbolic equation*

$$\rho(\mathbf{x})u_{tt}(\mathbf{x},t) + Lu(\mathbf{x},t) = \rho(\mathbf{x})F(\mathbf{x},t), \qquad \mathbf{x} \in G,\ t > 0, \tag{7.1.11}$$

where the operator L is defined as in (7.1.1) and G is a bounded region in two or three-dimensional space. The initial conditions for $u(\mathbf{x},t)$ are

$$u(\mathbf{x},0) = f(\mathbf{x}), \qquad u_t(\mathbf{x},0) = g(\mathbf{x}), \qquad \mathbf{x} \in G. \tag{7.1.12}$$

The boundary conditions on ∂G are given as in Section 4.1 in the form

$$\alpha(\mathbf{x})u(\mathbf{x},t) + \beta(\mathbf{x})\left.\frac{\partial u(\mathbf{x},t)}{\partial n}\right|_{\partial G} = B(\mathbf{x},t), \qquad t > 0. \tag{7.1.13}$$

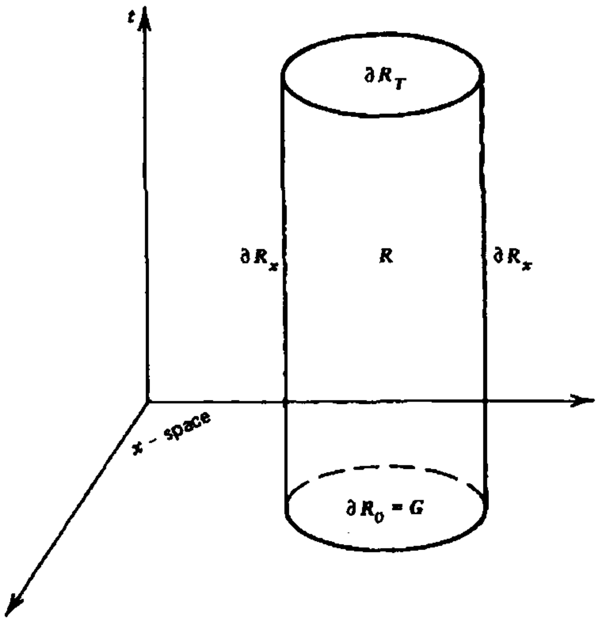

Figure 7.1 The region R.

The integral theorem appropriate for the problem is given over the bounded cylindrical region $R = G \times [0,\ T]$ in $(\mathbf{x},t)$-space , as shown in Figure 7.1 ($T > 0$ is an arbitrary number). The lateral boundary of R is denoted by $\partial R_{\mathbf{x}}$ and the two caps of the cylinder, which are portions of the planes $t = 0$ and $t = T$, are denoted by ∂R_0 and ∂R_T, respectively. ∂R_0 is identical to the region G and the initial conditions for $u(\mathbf{x}, t)$ are assigned on it. The boundary conditions for $u(\mathbf{x}, t)$ are assigned on $\partial R_{\mathbf{x}}$. The exterior unit normal $\mathbf{n}$ to ∂R has the form $\mathbf{n} = [\mathbf{n_x},\ 0]$ on $\partial R_{\mathbf{x}}$, where $\mathbf{n_x}$ is the exterior unit normal to ∂G. On ∂R_0, $\mathbf{n}$ has the form $\mathbf{n} = [\mathbf{0},\ -1]$, and on ∂R_T it has the form $\mathbf{n} = [\mathbf{0},\ 1]$.

It follows from Section 6.4 that for an arbitrary function $w(\mathbf{x}, t)$ we have

$$\iint_R [w(\rho u_{tt}+Lu)-u(\rho w_{tt}+Lw)]\,dv = \int_{\partial R} [-pw\nabla u+pu\nabla w, \rho wu_t-\rho uw_t]\cdot\mathbf{n}\,ds$$
$$=\int_{\partial R_{\mathbf{x}}} (-pw\nabla u+pu\nabla w)\cdot\mathbf{n_x}\,ds+\int_{\partial R_T} (\rho wu_t-\rho uw_t)\,d\mathbf{x}-\int_{\partial R_0} (\rho wu_t-\rho uw_t)\,d\mathbf{x}, \tag{7.1.14}$$

where $\tilde{\nabla} = [\nabla, \partial/\partial t]$, the gradient operator in space-time, the divergence theorem, as well as the foregoing results concerning the exterior unit normal to the boundary ∂R have been used. The *integral relation* (7.1.14) forms the basis for the *Green's function method* for solving the initial and boundary value problem (7.1.11)–(7.1.13).

We now show how $w(\mathbf{x}, t)$ is specified so that the solution $u(\mathbf{x}, t)$ of (7.1.11)–(7.1.13) can be determined at an arbitrary point $(\boldsymbol{\xi}, \tau)$ in the region R from (7.1.14). First we require that $w(\mathbf{x}, t)$ be a solution of

$$\rho(\mathbf{x})w_{tt}(\mathbf{x}, t) + Lw(\mathbf{x}, t) = \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau), \qquad \boldsymbol{\xi} \in G,\ 0 < \tau < T. \tag{7.1.15}$$

The product of the Dirac delta functions in (7.1.15) has the effect

$$\iint_R u(\rho w_{tt} + Lw)\,dv = \iint_R u\,\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau)\,dv = u(\boldsymbol{\xi}, \tau). \tag{7.1.16}$$

In addition, we obtain from (7.1.11),

$$\iint_R w(\rho u_{tt} + Lu)\,dv = \iint_R \rho wF\,dv, \tag{7.1.17}$$

so that this term is known once $w(\mathbf{x}, t)$ is specified.

Since

$$\int_{\partial R_{\mathbf{x}}} p(-w\nabla u + u\nabla w)\cdot\mathbf{n_x}\,ds = \int_{\partial R_{\mathbf{x}}} p\left(-w\frac{\partial u}{\partial n} + u\frac{\partial w}{\partial n}\right)\,ds, \tag{7.1.18}$$

we see that if we require, as in the elliptic case, that

$$\alpha w + \beta\,\frac{\partial w}{\partial n}\bigg|_{\partial R_{\mathbf{x}}} = 0, \tag{7.1.19}$$

we obtain

$$\int_{\partial R_{\mathbf{x}}} p\left(-w\frac{\partial u}{\partial n} + u\frac{\partial w}{\partial n}\right)\,ds = \int_{\hat{S}_1} \frac{p}{\alpha}\,B\,\frac{\partial w}{\partial n}\,ds - \int_{\hat{S}_2\cup\hat{S}_3} \frac{p}{\beta}\,Bw\,ds, \tag{7.1.20}$$

where $\hat{S}_1$, $\hat{S}_2$, and $\hat{S}_3$ are the portions of $\partial R_{\mathbf{x}}$ that correspond to S_1, S_2, and S_3 on ∂G, respectively.

To complete the determination of $w(\mathbf{x}, t)$, we expect that initial conditions must be assigned to it for some value of t. If $w(\mathbf{x}, t)$ and $w_t(\mathbf{x}, t)$ are specified at $t = 0$, the integral over ∂R_0 in (7.1.14) is determined completely since $u(\mathbf{x}, t)$ and $u_t(\mathbf{x}, t)$ are given at $t = 0$. However, $u(\mathbf{x}, t)$ and $u_t(\mathbf{x}, t)$ at $t = T$ (i.e., on ∂R_T) are not known. If we specify $w(\mathbf{x}, t)$ and $w_t(\mathbf{x}, t)$ at $t = T$, it must be done in such a way that the unknown values of $u(\mathbf{x}, t)$ and $u_t(\mathbf{x}, t)$ play no role in the integral over ∂R_T. The only possible choice is to set

$$w(\mathbf{x}, T) = 0, \qquad w_t(\mathbf{x}, T) = 0, \tag{7.1.21}$$

so that the entire integral over ∂R_T vanishes.

The equation (7.1.15) together with the boundary condition (7.1.19) and the conditions (7.1.21) at $t = T$ constitutes a *backward initial and boundary value problem* for the function $w(\mathbf{x}, t)$. It differs from the types of problems considered previously for hyperbolic equations (see, however, Section 1.2), where initial conditions were assigned at $t = 0$ and the problem was solved for $t > 0$. Here we assign end conditions at $t = T$ and solve the problem for $t < T$. The problem for $w(\mathbf{x}, t)$ is *well posed* because if t is replaced by $-t$ in $w_{tt}(\mathbf{x}, t)$ its sign is unchanged. We will refer to problems for which either initial conditions or end conditions are assigned as initial value problems. The function $w(\mathbf{x}, t)$ determined from (7.1.15), (7.1.19), and (7.1.21) is called the *Green's function* for the initial and boundary value problem (7.1.11)–(7.1.13) for $u(\mathbf{x}, t)$. It is denoted as $w(\mathbf{x}, t) = K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$.

Once the initial and boundary value problem for $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is solved, the values of $K(\mathbf{x}, 0; \boldsymbol{\xi}, \tau)$ and $K_t(\mathbf{x}, 0; \boldsymbol{\xi}, \tau)$ are known. Then the foregoing results yield the solution $u(\mathbf{x}, t)$ at an (arbitrary) point $(\boldsymbol{\xi}, \tau)$ as

$$\begin{aligned} u(\boldsymbol{\xi}, \tau) = & \iint_R \rho K F \, dv + \int_{\partial R_0} (\rho K g - \rho K_t f) \, d\mathbf{x} \\ & - \int_{\hat{S}_1} \frac{pB}{\alpha} \frac{\partial K}{\partial n} \, ds + \int_{\hat{S}_2 \cup \hat{S}_3} \frac{pBK}{\beta} \, ds. \end{aligned} \tag{7.1.22}$$

For completeness, we state the problem that the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ must satisfy. It is a solution of the equation

$$\rho(\mathbf{x}) K_{tt}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + LK(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \delta(\mathbf{x} - \boldsymbol{\xi}) \delta(t - \tau), \ \mathbf{x}, \boldsymbol{\xi} \in G, \ t, \tau < T, \ \tau > 0, \tag{7.1.23}$$

with the end conditions

$$K(\mathbf{x}, T; \boldsymbol{\xi}, \tau) = 0, \qquad K_t(\mathbf{x}, T; \boldsymbol{\xi}, \tau) = 0, \tag{7.1.24}$$

and the boundary condition

$$\alpha(\mathbf{x}) K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + \beta(\mathbf{x}) \left. \frac{\partial K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)}{\partial n} \right|_{\partial R_x} = 0, \qquad t < T. \tag{7.1.25}$$

It is shown in the exercises that $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = K(\boldsymbol{\xi}, -\tau; x, -t)$. Therefore, as a function of $\boldsymbol{\xi}$ and τ, $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ satisfies the same differential equation but with time running forward instead of backward. In these variables it represents the *causal fundamental solution* for the given hyperbolic operator and the boundary condition (7.1.25).

Integral Theorems and Green's Functions for Parabolic Equations

The *parabolic equation*

$$\rho(\mathbf{x})u_t(\mathbf{x},t) + Lu(\mathbf{x},t) = \rho(\mathbf{x})F(\mathbf{x},t), \qquad \mathbf{x} \in G,\ t > 0, \tag{7.1.26}$$

with the initial and boundary conditions

$$u(\mathbf{x},0) = f(\mathbf{x}),\ \mathbf{x} \in G,\ \alpha(\mathbf{x})u(\mathbf{x},t) + \beta(\mathbf{x})\left.\frac{\partial u(\mathbf{x},t)}{\partial n}\right|_{\partial G} = B(\mathbf{x},t),\ t > 0, \tag{7.1.27}$$

can be treated in the same way as the hyperbolic problem (7.1.11)–(7.1.13). The operator L, the regions R and G, and their boundaries are defined as in the foregoing hyperbolic problem.

We introduce the function $w(\mathbf{x},t)$ and consider the integral relation

$$\iint_R [w(\rho u_t + Lu) - u(-\rho w_t + Lw)]\, dv = \iint_R \tilde{\nabla} \cdot [-pw\nabla u + pu\nabla w,\ \rho wu]\, dv$$

$$= \int_{\partial R_{\mathbf{x}}} \left(-pw\frac{\partial u}{\partial n} + pu\frac{\partial w}{\partial n}\right) ds + \int_{\partial R_T} \rho wu\, d\mathbf{x} - \int_{\partial R_0} \rho wu\, d\mathbf{x}. \tag{7.1.28}$$

Again, $\tilde{\nabla} = [\nabla, \partial/\partial t]$ is the gradient operator in space-time and the region R and its boundaries are as shown in Figure 7.1. The result (7.1.28) is a consequence of the divergence theorem, but it differs from the preceding integral theorems for the elliptic and hyperbolic problems in the following respect. The operator $\rho(\partial/\partial t) + L$ in the parabolic equation (7.1.26) is not self-adjoint. Its *adjoint operator* is given as $-\rho(\partial/\partial t) + L$. With this choice for the adjoint operator we find that $w(\rho u_t + Lu) - u(-\rho w_t + Lw)$ is a divergence expression, as is shown in (7.1.28) (see Example 3.9).

We require $w(\mathbf{x},t)$ to be a solution of

$$-\rho(\mathbf{x})w_t(\mathbf{x},t) + Lw(\mathbf{x},t) = \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau), \qquad \boldsymbol{\xi} \in G,\ 0 < \tau < T, \tag{7.1.29}$$

with the end and boundary conditions

$$w(\mathbf{x},T) = 0, \quad \alpha(\mathbf{x})w(\mathbf{x},t) + \beta(\mathbf{x})\left.\frac{\partial w(\mathbf{x},t)}{\partial n}\right|_{\partial R_{\mathbf{x}}} = 0. \tag{7.1.30}$$

Then, $w(\mathbf{x},t) = K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ is the Green's function for the initial and boundary value problem (7.1.26)–(7.1.27). It follows from (7.1.28) that

$$u(\boldsymbol{\xi},\tau) = \iint_R \rho KF\, dv + \int_{\partial R_0} \rho Kf\, d\mathbf{x} - \int_{\hat{S}_1} \frac{p}{\alpha}\, B\, \frac{\partial K}{\partial n}\, ds + \int_{\hat{S}_2 \cup \hat{S}_3} \frac{p}{\beta}\, BK\, ds. \tag{7.1.31}$$

For completeness, we state the problem that the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ must satisfy. It is a solution of the equation

$$-\rho(\mathbf{x})K_t(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + LK(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau), \ \mathbf{x}, \boldsymbol{\xi} \in G, \ t, \tau < T, \ \tau > 0, \tag{7.1.32}$$

with the end and boundary conditions

$$K(\mathbf{x}, T; \boldsymbol{\xi}, \tau) = 0, \ \alpha(\mathbf{x})K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + \beta(\mathbf{x}) \left. \frac{\partial K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)}{\partial n} \right|_{\partial R_{\mathbf{x}}} = 0, \ t < T. \tag{7.1.33}$$

The equation (7.1.32) satisfied by the *Green's function* $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is a *backward parabolic equation* that results on reversing the direction of time in the (forward) parabolic equation (7.1.26). Since the problem for the Green's function is to be solved backward in time, the initial and boundary value problem (7.1.32)–(7.1.33) for K is *well posed*. [That is, we must determine $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ for $t < T$ with an end condition given at $t = T$.] Once $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ has been determined, all the terms on the right side of (7.1.31) are known and the solution $u(\mathbf{x}, t)$ of the initial and boundary value problem (7.1.26)–(7.1.27) is specified completely.

It is shown in the exercises that $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = K(\boldsymbol{\xi}, -\tau; x, -t)$. Therefore, as a function of $\boldsymbol{\xi}$ and τ, $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ satisfies a forward parabolic differential equation, but with time now running forwards instead of backwards. In these variables it represents the *causal fundamental solution* for the given parabolic operator and the boundary condition (7.1.33).

Causal Fundamental Solutions and Green's Functions for Cauchy Problems

The Green's functions $K(\mathbf{x}; \boldsymbol{\xi})$ and $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ defined above are *fundamental solutions* of the PDEs (7.1.9), (7.1.23) or (7.1.32) in the elliptic, hyperbolic, and parabolic cases, respectively. Each of these equations is the adjoint of the given equation for $u(\mathbf{x})$ or $u(\mathbf{x}, t)$. Since the elliptic and hyperbolic equations are self-adjoint, the Green's function is also a fundamental solution of the given equation. In the parabolic case, since the given equation is not self-adjoint, the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is only a fundamental solution of the adjoint equation. We have shown, however, that as a function of $\boldsymbol{\xi}$ and τ, $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is a fundamental solution of the given parabolic equation.

Let us consider an *instantaneous point source problem* for the hyperbolic and parabolic cases, with the source acting at the time $t = \tau$ and located at the point $\mathbf{x} = \boldsymbol{\xi}$. We require that the solution $u(\mathbf{x}, t)$ satisfy the homogeneous boundary condition $\alpha(\mathbf{x})u(\mathbf{x}, t) + \beta(\mathbf{x})\partial u(\mathbf{x}, t)/\partial n = 0$ on ∂G for $t > \tau$ and that $u(\mathbf{x}, t) = 0$ for $t < \tau$. Then the solution is called a *causal fundamental solution* for the initial and boundary value problem. (In Section 6.7 we found causal fundamental solutions over unbounded regions.) We have already indicated how to obtain these solutions in terms of the Green's functions $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$.

The Green's function method can also be used to solve *Cauchy problems* for hyperbolic and parabolic equations. In the hyperbolic case we assume that $u(\mathbf{x}, t)$ is a solution of (7.1.11) with initial data (7.1.12) and in the parabolic case $u(\mathbf{x}, t)$ is a solution of (7.1.26) that satisfies the initial condition (7.1.27). Both problems are given over the entire two- or three-dimensional space. The integral theorems (7.1.14) and (7.1.28) can be used for these problems if we assume that the (spatial) boundary $\partial R_{\mathbf{x}}$ tends to infinity and the solution $u(\mathbf{x}, t)$ and the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ are such that the contributions from these integrals vanish in the limit.

The Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ for the *hyperbolic case* is taken to be the solution of the backward Cauchy problem (7.1.23) and (7.1.24). Then the solution of the Cauchy problem (7.1.11)–(7.1.12) is given as

$$u(\boldsymbol{\xi}, \tau) = \int_0^T \int_{\mathbf{x}-space} \rho K F \, d\mathbf{x} \, dt + \int_{\mathbf{x}-space} [\rho K g - \rho K_t f]\big|_{t=0} \, d\mathbf{x}, \quad (7.1.34)$$

as is easily seen from the (modified) integral relation (7.1.14).

In the *parabolic case* the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ is chosen to satisfy the backward Cauchy problem (7.1.32)–(7.1.33), and it then follows from the (modified) integral theorem (7.1.28) that the solution of the Cauchy problem (7.1.26)–(7.1.27) takes the form

$$u(\boldsymbol{\xi}, \tau) = \int_0^T \int_{\mathbf{x}-space} \rho K F \, d\mathbf{x} \, dt + \int_{\mathbf{x}-space} [\rho K f]\big|_{t=0} \, d\mathbf{x}. \quad (7.1.35)$$

The foregoing results are easily modified to yield Green's functions and solution formulas for initial and boundary value problems for hyperbolic and parabolic equations given over semi-infinite spatial regions.

Green's Functions for Hyperbolic and Parabolic Equations: An Alternative Construction

There is an alternative approach to the construction of Green's functions that applies in the hyperbolic and parabolic cases. Instead of having $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ satisfy the inhomogeneous equations (7.1.23) and (7.1.32), we require that they be solutions of the homogeneous equations

$$\rho(\mathbf{x}) K_{tt}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + LK(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = 0, \qquad \mathbf{x}, \boldsymbol{\xi} \in G, \ t < \tau, \quad (7.1.36)$$

$$-\rho(\mathbf{x}) K_t(\mathbf{x}, t; \boldsymbol{\xi}, \tau) + LK(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = 0, \qquad \mathbf{x}, \boldsymbol{\xi} \in G, \ t < \tau, \quad (7.1.37)$$

in the hyperbolic and parabolic cases, respectively. The homogeneous initial conditions (7.1.24) and (7.1.33) are replaced by

$$K(\mathbf{x}, \tau; \boldsymbol{\xi}, \tau) = 0, \qquad K_t(\mathbf{x}, \tau; \boldsymbol{\xi}, \tau) = -\frac{\delta(\mathbf{x} - \boldsymbol{\xi})}{\rho(\mathbf{x})}, \qquad \boldsymbol{\xi} \in G, \quad (7.1.38)$$

$$K(\mathbf{x}, \tau; \boldsymbol{\xi}, \tau) = \frac{\delta(\mathbf{x} - \boldsymbol{\xi})}{\rho(\mathbf{x})}, \qquad \boldsymbol{\xi} \in G, \quad (7.1.39)$$

respectively. The boundary conditions (7.1.25) and (7.1.33) for $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ are retained.

In this formulation we obtain

$$\int_{\partial R_\tau} [\rho K u_t - \rho u K_t]\, d\mathbf{x} = \int_{\partial R_\tau} u\, \delta(\mathbf{x} - \boldsymbol{\xi})\, d\mathbf{x} = u(\boldsymbol{\xi}, \tau), \tag{7.1.40}$$

$$\int_{\partial R_\tau} \rho u K\, d\mathbf{x} = \int_{\partial R_\tau} u\, \delta(\mathbf{x} - \boldsymbol{\xi})\, d\mathbf{x} = u(\boldsymbol{\xi}, \tau), \tag{7.1.41}$$

for the hyperbolic and parabolic cases, respectively, when (7.1.38) and (7.1.39) are used. The solutions $u(\boldsymbol{\xi}, \tau)$ then have the form (7.1.22) and (7.1.31) in the hyperbolic and parabolic cases as is easily seen. The only difference is that the domain of integration in the original formulation of the Green's function problem extends from 0 to T whereas in the present formulation it extends from 0 to $\tau < T$. However, since the equation and the data for $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ are all homogeneous for $\tau < t < T$, the Green's function $K(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ vanishes identically in that interval. Consequently, the domains of integration are, in effect, identical for both formulations. The relation between these two approaches is connected with *Duhamel's principle* (see Section 4.5), which relates inhomogeneous equations with homogeneous initial conditions to homogeneous equations with inhomogeneous initial conditions.

Integral Theorems and Green's Functions in One Dimension

The preceding results are valid in two or three space dimensions. The case of one dimension for the *elliptic equation* (7.1.1) with the boundary condition (7.1.2), leads to the consideration of a boundary value problem for an ordinary differential equation for $u(x)$. It is given as, for $0 < x < l$,

$$-\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x) = \rho(x)F(x), \quad \begin{cases} \alpha_1 u(0) - \beta_1 u'(0) = B_1, \\ \alpha_2 u(l) + \beta_2 u'(l) = B_2, \end{cases} \tag{7.1.42}$$

where α_1, β_1, α_2, β_2 satisfy the conditions given in Chapter 4. The related Green's function $K(x; \xi)$ satisfies the equation

$$LK(x;\xi) = -\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial K(x;\xi)}{\partial x}\right) + q(x)K(x;\xi) = \delta(x-\xi), \qquad 0 < x,\ \xi < l, \tag{7.1.43}$$

with the boundary conditions

$$\alpha_1 K(0;\xi) - \beta_1 \partial K(0;\xi)/\partial x = 0, \quad \alpha_2 K(l;\xi) + \beta_2 \partial K(l;\xi)/\partial x = 0. \tag{7.1.44}$$

The solution of the boundary value problem (7.1.43)–(7.1.44) is expressed at a point ξ with $0 < \xi < l$ in terms of the solution formula (7.1.8) specialized to the one-dimensional case. The region G is the interval $0 < x < l$ and the S_i $(i = 1, 2, 3)$ correspond to boundary conditions of the first, second, or third kinds at $x = 0$ and

$x = l$. Thus the integral over G in (7.1.8) becomes an integral over $0 < x < l$. The integrals over the S_i reduce to the integrands evaluated at $x = 0$ or $x = l$. The normal derivative $\partial/\partial n$ equals $-\partial/\partial x$ at $x = 0$ and $\partial/\partial x$ at $x = l$. For example, in the case of mixed boundary conditions in (7.1.42) with $\alpha_1 \neq 0,\ \beta_1 = 0,\ \alpha_2 \neq 0,\ \beta_2 \neq 0$, we obtain the solution formula

$$u(\xi) = \int_0^l \rho(x)K(x;\xi)F(x)\,dx + \frac{p(0)}{\alpha_1}\,B_1\frac{\partial K(0;\xi)}{\partial x} + \frac{p(l)}{\beta_2}\,B_2K(l;\xi). \quad (7.1.45)$$

The one-dimensional versions of the *hyperbolic* and *parabolic equations* (7.1.11) and (7.1.26) lead to the consideration of the region R given as $[0,l] \times [0,T]$. The boundary ∂R is made up of the portion ∂R_x, which comprises the lines $x = 0$ and $x = l$ with $0 \le t \le T$, and ∂R_0 and ∂R_T, which represent the lines $t = 0$ and $t = T$, respectively, with $0 < x < l$. The region R is depicted in Figure 7.2.

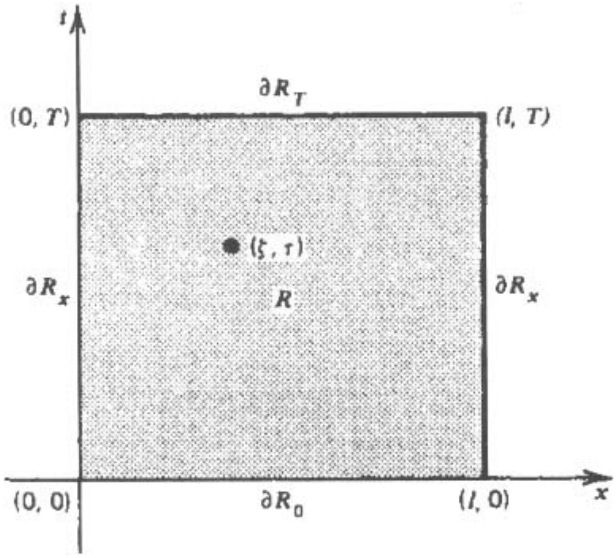

Figure 7.2 The region R.

For the *hyperbolic case* we have, with $0 < x < l,\ t > 0$,

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\,\frac{\partial u(x,t)}{\partial x}\right) + q(x)u(x,t) = \rho(x)F(x,t), \quad (7.1.46)$$

with the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 < x < l, \quad (7.1.47)$$

and the boundary conditions

$$\alpha_1 u(0,t) - \beta_1 u_x(0,t) = g_1(t),\ \alpha_2 u(l,t) + \beta_2 u_x(l,t) = g_2(t). \quad (7.1.48)$$

With the operator L defined as in (4.1.7), we have

$$\int_0^T \int_0^l [K(\rho u_{tt} + Lu) - u(\rho K_{tt} + LK)]\, dx\, dt$$

$$= -\int_0^l [\rho K u_t - \rho u K_t]\big|_{t=0}\, dx + \int_0^T [-pKu_x + puK_x]\big|_{x=l}\, dt$$

$$+ \int_0^l [\rho K u_t - \rho u K_t]\big|_{t=T}\, dx - \int_0^T [-pKu_x + puK_x]\big|_{x=0}\, dt, \quad (7.1.49)$$

where the space-time gradient operator $\tilde{\nabla} = [\partial/\partial x, \partial/\partial t]$ and Green's theorem in the plane were used.

For the one-dimensional *parabolic case* we consider the equation

$$\rho(x)\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\,\frac{\partial u(x,t)}{\partial x}\right) + q(x)u(x,t) = \rho(x)F(x,t), \quad (7.1.50)$$

with $0 < x < l,\ t > 0$, the initial condition

$$u(x,0) = f(x), \quad (7.1.51)$$

and the boundary conditions (7.1.48). Proceeding as in the higher-dimensional case, with $\tilde{\nabla} = [\partial/\partial x, \partial/\partial t]$ as the space-time gradient operator, we obtain

$$\int_0^T \int_0^l [K(\rho u_t + Lu) - u(-\rho K_t + LK)]\, dx\, dt$$

$$= -\int_0^l [\rho K u]\big|_{t=0}\, dx + \int_0^T [-pKu_x + puK_x]\big|_{x=l}\, dt$$

$$+ \int_0^l [\rho K u]\big|_{t=T}\, dx - \int_0^T [-pKu_x + puK_x]\big|_{x=0}\, dt. \quad (7.1.52)$$

With $(\xi,\ \tau)$ as an interior point in the region R, we choose $K(x,t;\xi,\tau)$ to be a solution of

$$\rho(x)K_{tt}(x,t;\xi,\tau) + LK(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau), \qquad t < T, \quad (7.1.53)$$

$$-\rho(x)K_t(x,t;\xi,\tau) + LK(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau), \qquad t < T, \quad (7.1.54)$$

with $0 < x,\ \xi < l$, and $0 < t,\ \tau < T$ in the hyperbolic and parabolic problems, respectively. In addition, we have the end conditions

$$K(x,T;\xi,\tau) = 0, \qquad K_t(x,T;\xi,\tau) = 0, \quad (7.1.55)$$

in the former case and

$$K(x,T;\xi,\tau) = 0 \quad (7.1.56)$$

in the latter case. In both cases $K(x,t;\xi,\tau)$ is required to satisfy the homogeneous version of (7.1.48). If we have $\beta_1 = \beta_2 = 0$ in the boundary conditions, we obtain

$$\int_0^T [-pKu_x + puK_x]\big|_{x=l}\, dt = \int_0^T \frac{p(l)}{\alpha_2} g_2(t) K_x(l,t;\xi,\tau)\, dt, \tag{7.1.57}$$

$$\int_0^T [-pKu_x + puK_x]\big|_{x=0}\, dt = \int_0^T \frac{p(0)}{\alpha_1} g_1(t) K_x(0,t;\xi,\tau)\, dt. \tag{7.1.58}$$

Using these results, we easily obtain the following expression for the solution u at the arbitrary point $(\xi,\ \tau)$ in the hyperbolic case

$$\begin{aligned} u(\xi,\tau) = & \int_0^T \int_0^l \rho(x) F(x,t) K(x,t;\xi,\tau)\, dx\, dt \\ & + \int_0^l \rho(x)[g(x)K(x,0;\xi,\tau) - f(x)K_t(x,0;\xi,\tau)]\, dx \\ & + \int_0^T \left\{ \frac{p(0)g_1(t)}{\alpha_1} K_x(0,t;\xi,\tau) - \frac{p(l)g_2(t)}{\alpha_2} K_x(l,t;\xi,\tau) \right\} dt \end{aligned} \tag{7.1.59}$$

when $\beta_1 = \beta_2 = 0$ in the boundary conditions. For the parabolic case we have

$$\begin{aligned} u(\xi,\tau) = & \int_0^T \int_0^l \rho(x) F(x,t) K(x,t;\xi,\tau)\, dx\, dt + \int_0^l \rho(x) f(x) K(x,0;\xi,\tau)\, dx \\ & + \int_0^T \left\{ \frac{p(0)g_1(t)}{\alpha_1} K_x(0,t;\xi,\tau) - \frac{p(l)g_2(t)}{\alpha_2} K_x(l,t;\xi,\tau) \right\} dt, \end{aligned} \tag{7.1.60}$$

with $\beta_1 = \beta_2 = 0$ in the boundary conditions.

If β_1 and β_2 are not zero or if there are mixed boundary conditions, a somewhat different expression for the solutions is readily obtained. Also, if the problem is given over a semi-infinite interval or we are dealing with the Cauchy problem over the infinite interval, appropriate expressions for the solution are easily found in a manner similar to that used previously in the higher-dimensional problems. Further, $K(x,t;\xi,\tau)$ can be characterized in an alternative manner as was done in (7.1.36)–(7.1.40) for higher dimensions.

Green's Functions for Nonself-Adjoint Elliptic Equations

The initial (or end) data for the Green's functions $K(\mathbf{x},\tau;\boldsymbol{\xi},\tau)$ in the hyperbolic and parabolic cases are assigned either at $t = T$ or at $t = \tau$. For the given problem for $u(\mathbf{x},t)$, however, the data are prescribed at $t = 0$. This results from the fact that the Green's function is the solution of an adjoint problem in which time runs backward rather than forward. The general form of the boundary conditions is the same for $K(\mathbf{x};\boldsymbol{\xi})$ and $u(\mathbf{x})$ in the elliptic problems, and for $K(\mathbf{x},\tau;\boldsymbol{\xi},\tau)$ and $u(\mathbf{x},t)$ in the hyperbolic and parabolic problems considered. This occurs because the elliptic operator L that occurs in all three equations is *self-adjoint*, and it determines the choice of the boundary condition for the (adjoint) Green's function problem in each case. To see what happens if the elliptic (spatial) operator is *not self-adjoint*, we now consider a boundary value problem for a nonself-adjoint elliptic equation and determine the corresponding Green's function problem.

Consider the *elliptic equation*

$$\tilde{L}u(\mathbf{x}) = Lu(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) = \rho(\mathbf{x})F(\mathbf{x}) \tag{7.1.61}$$

in a bounded region G, where L is defined in (7.1.1). With $\mathbf{b}(\mathbf{x}) \neq \mathbf{0}$, the operator $\tilde{L}$ is not self-adjoint. The boundary conditions for $u(\mathbf{x})$ are

$$\alpha(\mathbf{x})\, u(\mathbf{x}) + \beta(\mathbf{x})\, \frac{\partial u(\mathbf{x})}{\partial n}\bigg|_{\partial G} = B(\mathbf{x}), \tag{7.1.62}$$

as in the problem (7.1.1)–(7.1.2). The operator $\tilde{L}^*$, defined as $\tilde{L}^*u(\mathbf{x}) = Lu(\mathbf{x}) - \nabla \cdot (u(\mathbf{x})\mathbf{b}(\mathbf{x}))$, is the adjoint of L, as is easily checked, and we have

$$\iint_G [w\tilde{L}u - u\tilde{L}^*w]\, dv = \iint_G \nabla \cdot [pu\nabla w - pw\nabla u + wu\mathbf{b}]\, dv$$

$$= \int_{\partial G} \left[pu\frac{\partial w}{\partial n} - pw\frac{\partial u}{\partial n} + wu\mathbf{b}\cdot\mathbf{n}\right] ds \tag{7.1.63}$$

on using the divergence theorem.

Proceeding as in the problem (7.1.1)–(7.1.2), we put $w(\mathbf{x}) = K(\mathbf{x};\boldsymbol{\xi})$, the *Green's function*, and require that $K(\mathbf{x};\boldsymbol{\xi})$ be a solution of

$$\tilde{L}^*K(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi}), \qquad \mathbf{x},\ \boldsymbol{\xi} \in G. \tag{7.1.64}$$

Then $K(\mathbf{x};\boldsymbol{\xi})$ must be specified on ∂G so that $u(\boldsymbol{\xi})$ is determined completely in terms of the boundary values for $u(\mathbf{x})$ and $K(\mathbf{x};\boldsymbol{\xi})$. If $\mathbf{x} \in S_1$ on ∂G (see Section 4.1) we have

$$pu\frac{\partial K}{\partial n} - pK\frac{\partial u}{\partial n} + Ku\mathbf{b}\cdot\mathbf{n} = \frac{p}{\alpha}B\frac{\partial K}{\partial n} + K\left(u\mathbf{b}\cdot\mathbf{n} - p\frac{\partial u}{\partial n}\right) = \frac{p}{\alpha}B\frac{\partial K}{\partial n}, \tag{7.1.65}$$

if we set $K(\mathbf{x};\boldsymbol{\xi}) = 0$. If $\mathbf{x} \in S_2 \cup S_3$ on ∂G, we have

$$pu\frac{\partial K}{\partial n} - pK\frac{\partial u}{\partial n} + Ku\mathbf{b}\cdot\mathbf{n} = -\frac{p}{\beta}BK + u\left(p\frac{\partial K}{\partial n} + \frac{p\alpha}{\beta}K + K\mathbf{b}\cdot\mathbf{n}\right) = -\frac{p}{\beta}BK \tag{7.1.66}$$

if we set $p(\mathbf{x})\partial K(\mathbf{x};\boldsymbol{\xi})/\partial n + (p(\mathbf{x})\alpha(\mathbf{x})/\beta(\mathbf{x}))K(\mathbf{x};\boldsymbol{\xi}) + K(\mathbf{x};\boldsymbol{\xi})\mathbf{b}(\mathbf{x})\cdot\mathbf{n} = 0$. Thus, the (adjoint) boundary conditions for $K(\mathbf{x};\boldsymbol{\xi})$ are

$$K(\mathbf{x};\boldsymbol{\xi}) = 0, \quad \mathbf{x} \in S_1, \tag{7.1.67}$$

$$\frac{p(\mathbf{x})}{\beta(\mathbf{x})}\left(\alpha(\mathbf{x})K(\mathbf{x};\boldsymbol{\xi}) + \beta(\mathbf{x})\frac{\partial K(\mathbf{x};\boldsymbol{\xi})}{\partial n}\right) + K(\mathbf{x};\boldsymbol{\xi})\mathbf{b}(\mathbf{x})\cdot\mathbf{n}, \quad \mathbf{x} \in S_2 \cup S_3.$$

In terms of the Green's function $K(\mathbf{x};\boldsymbol{\xi})$ that satisfies the adjoint equation (7.1.64) and the adjoint boundary conditions (7.1.67), the solution $u(\mathbf{x})$ of the problem (7.1.61)–(7.1.62) is given by the formula (7.1.8). We observe that only in the case of Dirichlet boundary conditions for $u(\mathbf{x})$ does the Green's function $K(\mathbf{x};\boldsymbol{\xi})$ satisfy the (homogeneous) Dirichlet condition $K(\mathbf{x};\boldsymbol{\xi}) = 0$ on the boundary. For boundary conditions of the second and third kind, $K(\mathbf{x};\boldsymbol{\xi})$ satisfies modified boundary conditions as given in (7.1.67) [unless $\mathbf{b}(\mathbf{x})\cdot\mathbf{n} = 0$ on the boundary]. The adjoint boundary conditions in the one-dimensional case are considered in the exercises.

If we replace the operator L in (7.1.11) and (7.1.26) (the *hyperbolic* and *parabolic cases*) by the operator $\tilde{L}$, but leave the initial and boundary conditions for these equations unchanged, we find that the Green's functions $K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ for these problems are defined in terms of the adjoints of the modified equations. They have the same end conditions as before, but on the (lateral) boundary ∂R_x they satisfy (7.1.67).

Because of the greater difficulty involved in determining Green's functions for the foregoing nonself-adjoint problems, we restrict our discussion to problems with self-adjoint elliptic operators. However, before proceeding to construct Green's functions for various problems, we present the theory of generalized functions in the following section since these functions play an important role in Green's function theory.

Exercises 7.1

7.1.1. Show that for the elliptic case the Green's function $K(\mathbf{x};\boldsymbol{\xi})$ determined from (7.1.9)–(7.1.10) is symmetric [i.e., $K(\mathbf{x};\boldsymbol{\xi}) = K(\boldsymbol{\xi},\mathbf{x})$]. *Hint*: Let $u(\mathbf{x}) = K(\mathbf{x};\hat{\boldsymbol{\xi}})$ and $w(\mathbf{x}) = K(\mathbf{x};\boldsymbol{\xi})$ in (7.1.3).

7.1.2. Show that the Green's function for the hyperbolic problem (7.1.23)–(7.1.25) satisfies the equation $K(\mathbf{x},t;\boldsymbol{\xi},\tau) = K(\boldsymbol{\xi},-\tau;\mathbf{x},-t)$. *Hint*: Let $u(\mathbf{x},t) = K(\mathbf{x},-t;\hat{\boldsymbol{\xi}},-\hat{\tau})$ and $w(\mathbf{x},t) = K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ in (7.1.14).

7.1.3. Show that the Green's function for the parabolic problem (7.1.32)–(7.1.33) satisfies $K(\mathbf{x},t;\boldsymbol{\xi},\tau) = K(\boldsymbol{\xi},-\tau;\mathbf{x},-t)$. *Hint*: Let $u(\mathbf{x},t) = K(\mathbf{x},-t;\hat{\boldsymbol{\xi}},-\hat{\tau})$ and $w(\mathbf{x},t) = K(\mathbf{x},t;\boldsymbol{\xi},\tau)$ in (7.1.28).

7.1.4. Let $F = 0$ in (7.1.8) and let B have delta function behavior with the singular point at $\mathbf{x} = \hat{\mathbf{x}}$ on the boundary so that (7.1.8) reduces to $u(\boldsymbol{\xi}) = -p/\alpha\, \partial K(\hat{\mathbf{x}};\boldsymbol{\xi})/\partial n$, $\hat{\mathbf{x}} \in S_1$, $u(\boldsymbol{\xi}) = p/\beta\, K(\hat{\mathbf{x}};\boldsymbol{\xi})$, $\hat{\mathbf{x}} \in S_2 \cup S_3$. Use these results to show that not only can the solution of (7.1.1)–(7.1.2) with $B = 0$ and $F \neq 0$ be expressed as the superposition of the solutions of point source or singularity problems, but the same can be done for (7.1.1)–(7.1.2) if $F = 0$ and $B \neq 0$. In the latter case the point sources lie on the boundary.

7.1.5. Use the expression (7.1.22) to characterize the solution of the initial and boundary value problem for the hyperbolic equation (7.1.11) as a superposition of solutions of point source problems, as in Exercise 7.1.4.