UM–SJTU Joint Institute

VV557 Methods of Applied Math II

Assignment 4

Group 22

Sui, Zijian 515370910038

Wang, Tianze 515370910202

Xu, Yisu 118370910021

Exercise 3. 1

i).

We know that $\frac{d^4 g}{dx^4} = \delta\left(x-\xi\right)$.

We hence define the candidate $E\left(x,\xi\right) = H(x-\xi)g_\xi(x)$for the casual fundamental solution, with initial conditions $g_\xi\left(\xi\right) = g_\xi^{'}\left(\xi\right) = g_\xi^{''}\left(\xi\right) = 0,\ g_\xi^{(3)}\left(\xi\right) = 1$.

Assume that $g_\xi\left(x\right) = ax^3 + bx^2 + cx + d$, where a, b, c, d are real numbers.

With initial conditions $\begin{cases} a\xi^3 + b\xi^2 + c\xi + d = 0 \\ 3a\xi^2 + 2b\xi + c = 0 \\ 6a\xi + 2b = 0 \\ 6a = 1 \end{cases}$, we get $\begin{cases} a = \frac{1}{6} \\ b = -\frac{\xi}{2} \\ c = \frac{\xi^2}{2} \\ d = -\frac{\xi^3}{6} \end{cases}$.

So, $E\left(x,\xi\right) = H(x-\xi)(\frac{1}{6}x^3 - \frac{\xi}{2}x^2 + \frac{\xi^2}{2}x - \frac{\xi^3}{6})$.

ii).

The general solution is $g_\xi\left(x,\xi\right) = E\left(x,\xi\right) + u(x)$, where $u\left(x\right) = ax^3 + bx^2 + cx + d$.

So, $g_\xi\left(x,\xi\right) = \begin{cases} ax^3 + bx^2 + cx + d\ , x < \xi \\ \left(a + \frac{1}{6}\right)x^3 + \left(b - \frac{\xi}{2}\right)x^2 + \left(c + \frac{\xi^2}{2}\right)x + \left(d - \frac{\xi^3}{6}\right)\ , x > \xi \end{cases}$

The boundary conditions $g\left(0,\xi\right) = g^{''}\left(0,\xi\right) = g\left(1,\xi\right) = g^{''}\left(1,\xi\right) = 0$.

Then we get $\begin{cases} d = 0 \\ 2b = 0 \\ \left(a + \frac{1}{6}\right) + \left(b - \frac{\xi}{2}\right) + \left(c + \frac{\xi^2}{2}\right) + \left(d - \frac{\xi^3}{6}\right) = 0 \\ (6a+1) + (2b - \xi) = 0 \end{cases}$.

So, $\begin{cases} a = \frac{\xi - 1}{6} \\ b = 0 \\ c = \frac{\xi^3}{6} - \frac{\xi^2}{2} + \frac{\xi}{3} \\ d = 0 \end{cases}$

So, $g_\xi\left(x,\xi\right) = \begin{cases} \frac{\xi-1}{6}x^3 + (\frac{\xi^3}{6} - \frac{\xi^2}{2} + \frac{\xi}{3})x\ , x < \xi \\ \frac{\xi}{6}x^3 - \frac{\xi}{2}x^2 + \left(\frac{\xi^3}{6} + \frac{\xi}{3}\right)x - \frac{\xi^3}{6}, x > \xi \end{cases}$

Exercise 3. 2

i).

We set:

$$E(x,\xi) = H(x-\xi)u_\xi(x),$$

where H is the Heaviside function.
Solve $\quad Lu = 0$:

$$Ly = -y'' - k^2 y = 0$$
$$-\lambda^2 - k^2 = 0$$

Solve the equation, we obtain

$$\lambda = \pm ik$$

Thus $y = e^{0x}\left(C_1 \cos kx + C_2 \sin kx\right) = C_1 \cos kx + C_2 \sin kx$

$$Lu_\xi = 0, \quad u_\xi(\xi) = 0, \quad u'_\xi(\xi) = -1$$

We have,

$$\begin{cases} C_1 \cos k\xi + C_2 \sin k\xi = 0 \\ -C_1 k \sin k\xi + C_2 k \cos k\xi = -1 \end{cases}$$

Solve the equation set, we have

$$C_1 = \frac{\sin k\xi}{k}, \quad C_2 = -\frac{\cos k\xi}{k}$$
$$u_\xi(x) = \frac{\sin k\xi}{k} \cos k\xi - \frac{\cos k\xi}{k} \sin kx$$

Therefore,

$$\begin{aligned} E(x,\xi) &= H(x-\xi)u_\xi(x) \\ &= H(x-\xi)\left(\frac{\sin k\xi}{k} \cos kx - \frac{\cos k\xi}{k} \sin kx\right) \end{aligned}$$

ii).

The general solution of the homogeneous equation $-\frac{d^2u}{dx^2} - k^2 u = 0$ is

$$u(x) = C_3 \cos kx + C_4 \sin kx$$

$$g(x,\xi) = E(x,\xi) + u(x) = \begin{cases} C_3 \cos kx + C_4 \sin kx \quad 0 < x < \xi \\ \left(\frac{\sin k\xi}{k} + C_3\right) \cos kx + \left(-\frac{\cos k\xi}{k} + C_4\right) \sin kx \quad \xi < x < 1 \end{cases}$$

We impose the boundary conditions,

$$\begin{cases} g(0,\xi) = C_3 = 0 \\ g(1,\xi) = \left(\frac{\sin k\xi}{k} + C_3\right) \cos k + \left(-\frac{\cos k\xi}{k} + C_4\right) \sin k = 0 \end{cases}$$

Then we obtain,

$$C_3 = 0, C_4 = \frac{\sin(k - k\xi)}{k \sin k}$$

Therefore,

$$g(x,\xi) = \begin{cases} \frac{\sin(k-k\xi)}{k \sin k} \sin kx, 0 < x < \xi \\ \frac{\sin k\xi}{k} \cos kx - \frac{\cos k \sin k\xi}{k \sin k} \sin kx, \xi < x < 1 \end{cases}$$

iii).

We apply Fourier transform on both sides

$$-\widehat{\frac{d^2E}{dx^2}} - k^2\widehat{E} = \delta\widehat{(x-\xi)}$$

According to Fourier Property

$$\delta\widehat{(x-\xi)}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\omega x}\delta(x-\xi)dx = \frac{e^{-i\omega\xi}}{\sqrt{2\pi}}$$

Also, we have

$$\widehat{\frac{d^2E}{dx^2}} = \widehat{D^2E} = (i\xi)^2\hat{E} = -\xi^2\hat{E}$$

So the equation then becomes

$$\xi^2\widehat{E} - k^2\widehat{E} = \frac{e^{-i\omega\xi}}{\sqrt{2\pi}}$$

Which means

$$\widehat{E} = \frac{e^{-i\omega\xi}}{\sqrt{2\pi}(\omega^2-k^2)} = \frac{e^{-i\omega\xi}}{\sqrt{2\pi}} \cdot \frac{1}{2k}(\frac{1}{\omega-k} - \frac{1}{\omega+k})$$

A useful Fourier transformation is (According to the equation $\mathcal{F}D^\alpha(-ix)^\beta\varphi(x) = (i\xi)^\alpha D^\beta\widehat{\varphi(x)}$)

$$\mathcal{F}(sgn(x))(\omega) = \frac{1}{\sqrt{2\pi}}\frac{2}{i\omega}$$

So

$$\mathcal{F}^{-1}\left(\frac{1}{w}\right) = -i\sqrt{\frac{\pi}{2}}\,\mathsf{sgn}(x)$$

So the inverse Fourier of $(\frac{1}{\omega-k} - \frac{1}{\omega+k})$ is given as

$$\mathcal{F}^{-1}\left(\frac{1}{\omega-k} - \frac{1}{\omega+k}\right) = (e^{-ikx} - e^{ikx}) \cdot (-i\sqrt{\frac{\pi}{2}}sgn(x)) = -2i\sin(kx)sgn(x) \cdot (-i\sqrt{\frac{\pi}{2}}sgn(x))$$

So the total inverse Fourier transform is

$$E = \mathcal{F}^{-1}(\frac{e^{-i\omega\xi}}{\sqrt{2\pi}\,(\omega^2-k^2)}) = -\frac{\sin(k|x-\xi|)}{2k}$$

iv).

We have $g = u + E$. Since $u(x)$ satisfies

$$-\frac{d^2u}{dx^2} - k^2u = 0$$

We can set

$$u(x) = \alpha\cos(kx) + \beta\sin(kx)$$

According to two boundary conditions,

$$\begin{cases} g(0,\xi) = -\dfrac{\sin(k\xi)}{2k} + \alpha = 0 \\ g(1,\xi) = -\dfrac{\sin(k(1-\xi))}{2k} + \alpha\cos k + \beta\sin k = 0 \end{cases}$$

Synthesis two equations, the answer is given as

$$\begin{cases} \alpha = \dfrac{\sin(k\xi)}{2k} \\ \beta = \dfrac{\frac{\sin(k(1-\xi))}{2k} - \frac{\sin(k\xi)\cos k}{2k}}{\sin k} \end{cases}$$

So the generation solution is given as

$$g(x) = -\frac{\sin(k|x-\xi|)}{2k} + \frac{\sin(k\xi)}{2k}\cos(kx) + \left(\frac{\frac{\sin(k(1-\xi))}{2k} - \frac{\sin(k\xi)\cos k}{2k}}{\sin k}\right)\sin(kx)$$