Determinacy in and out second order arithmetic

An introduction to the proof theoretic strength of the determinacy scale

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December 22, 2022

Program

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5 Perspectives and Material

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$\begin{array}{llllllllll} \text{I:} & a_0 & & a_2 & & & a_{2n} & & & \\ & & & & \cdots & & & & \cdots & (a_i)_{i<\omega} \overset{?}{\in} X \\ \text{II:} & & a_1 & & a_3 & & & a_{2n+1} & & \end{array}$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): "All these games are determined".
(False in ZF + C.)

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

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- Are Σ^1_2, Σ^1_3, etc sets Lebesgue measurable?
- Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.

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Theorem (Martin, ZFC)

All Borel games are determined.

Sketch of the proof

A conservation result

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ZFC^- *is a* Π^1_4 *conservative extension of* Z_2.

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- We can construct L(X) in Z_2;
- We can show in that Z_2 that L(X) $\models$ ZFC$^-$;
- We use Shoenfield Absoluteness theorem ($\Pi^1_1 - CA_0$).

Some right axioms systems

Theorem (Steel, Simpson)

Over RCA_0, $\mathsf{Det}(\Sigma^0_1)$ *is equivalent to* ATR_0.

Theorem (Tanaka)

$\mathsf{Det}(\Sigma^0_2)$ *is equivalent to* $\Sigma^1_1 - \mathsf{MI}$.

How much determinacy can we prove in Z_2?

Theorem (Montalbán and Shore)

Π^1_{n+2}-$\mathsf{CA}_0 \vdash \mathsf{Det}(n\text{-}\Pi^0_3)$ *but* Δ^1_{n+2}-$\mathsf{CA}_0 \nvdash \mathsf{Det}(n\text{-}\Pi^0_3)$.

However, Π^1_{n+2}-CA_0 is not the right set of axioms for $\mathsf{Det}(n\text{-}\Pi^0_3)$.

Theorem (MedSalem and Tanaka)

Borel determinacy does not imply Δ^1_2-CA_0.

Reversals

Theorem (Hachtman)

$\mathsf{Det}(\Sigma^0_3)$ *is equivalent to the existence of a* $\beta-$*model satisfying* $\Pi^2_1 - \mathsf{MI}$.

Theorem (Aguilera and Welch)

Over Π^1_1-CA_0*, for each* $m \in \mathbb{N}$ *we have an equivalence between*

1. $\mathsf{Det}(m\text{-}\Pi^0_3)$,
2. *Every real belongs to a* β*-model of* Π^1_{m+1}-MI.

Game encoding models

Theorem (Friedman $<$ Martin $<$ M. and S.)

We cannot prove $\mathsf{Det}(\Sigma^5_0) < \mathsf{Det}(\Sigma^4_0) < \mathsf{Det}(\omega\text{-}\Pi^3_0)$ *in* ZCF^-

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- Same technique as for the limitative result of Friedman for Borel determinacy.
- The games are deemed to encode fragments of set theory using Gödel numbering.

Sketch of the proof for $\mathrm{Det}(\Sigma^4_0)$

Measurability properties

Theorem (Kechris, Martin)

In $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$, $\mathsf{Det}(\Pi^1_n)$ *proves that every* Σ^1_{n+1} *sets of reals satisfies M1, M2 and M3.*

Theorem (Shelah-Woodin)

Given $n \in \omega$, *if there are* n *Woodin cardinals with a measurable cardinal above them, then every* Σ^1_{n+2} *sets of reals satisfies M1, M2 and M3.*

Determinacy and high cardinal hypotheses

Theorem

Given $n \in \omega$, if there are n Woodin cardinals with a measurable cardinal above them, then $\mathrm{Det}(\Pi^1_{n+1})$.

Remark

This is a corollary from a theorem of Martin-Steel, which is out of the scope of the present talk.