Determinacy in and out second order arithmetic

An introduction to the proof theoretic strength of the determinacy scale

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Program

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$\begin{array}{llllllllll} \text{I:} & a_0 & & a_2 & & & a_{2n} & & & \\ & & & & & \cdots & & & \cdots & (a_i)_{i<\omega} \overset{?}{\in} X \\ \text{II:} & & a_1 & & a_3 & & & a_{2n+1} & & \end{array}$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): "All these games are determined".
(False in ZF + C.)

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

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- Study these properties for projective Σ^1_n sets in ω^ω (Blackwell, 1967).
- Are Σ^1_2, Σ^1_3, etc sets Lebesgue measurable?
- Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.

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Theorem (Martin, ZFC)

All Borel games are determined.

Theorem ($ZC^- + \Sigma_1$-Replacement)

All Σ^0_2 games are determined.

Lemma

Let $B \subseteq A \subseteq [T]$ with B being closed. If player I has no winning strategy in the game $G(T, A)$, then there is a strategy τ for II such that every $x \in [\tau]$ has a finite initial segment p verifying

$[T_p] \cap B = \emptyset$ *and* *I has still no winning strategy in* $G(T_p, A)$

Proof of the theorem of Wolfe

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Theorem (MedSalem and Tanaka)

Borel determinacy does not imply Δ^1_2-CA_0.

Definition

We define $\Sigma^1_{|s|+2}$ relations $P^s(S)$ by induction on $|s| \leq m$:

- When $|s| = 0$, $P^\emptyset(S)$ iff

$$\text{I (II) has a winning strategy in } G(A,S) \text{ if } l \text{ is even (odd).} \tag{1}$$

- For $|s| = n+1$ and l is even, $P^s(S)$ iff there is a quasistrategy U for I in S such that

$$[U] \subseteq A \cup A_{l,s(n)} \quad \text{and} \quad P^{s[n]}(U) \text{ fails.} \tag{2}$$

- For $|s| = n+1$ and l is odd, $P^s(S)$ iff there is a quasistrategy U for II in S such that

$$[U] \subseteq \bar{A} \cup A_{l,s(n)} \quad \text{and} \quad P^{s[n]}(U) \text{ fails.} \tag{3}$$

Definition

A quasistrategy U locally witnesses $P^s(S)$ if $|s| = n+1$ and U is a quasistrategy for I (II) if l is even (odd) and there is $D \subseteq S$ such that, for every $d \in D$, there is a quasistrategy R^d for II (I) if l is even (odd) in S_d such that the following conditions are satisfied:

1. $\forall d \in D \cap U,\ \ U_d \cap R^d$ witnesses $P^s(R^d)$.
2. $[U] \setminus \bigcup_{d \in D}[R^d] \subseteq A$ (resp. $\bar{A}$).
3. $\forall p \in S\ \exists^{\leq 1} d \in D,\ d \subseteq p \wedge p \in R^d$.

We observe that "U locally witnesses $P^s(S)$" is a $\Sigma^1_{|s|+2}$ sentence.

Lemma (1)

If U locally witnesses $P^s(S)$, then U witnesses $P^s(S)$.

Proof of lemma 1

Definition

We say that $P^s(S)$ fails everywhere if $P^s(S_p)$ fails for every $p \in S$. This is a $\Pi^1_{|s|+2}$ sentence.

Lemma (2)

If $P^s(S)$ fails, then there is a quasistrategy W for I (II) if l is odd (even) in S such that $P^s(W)$ fails everywhere.

Proof of lemma 2

Definition

For $|s| = n+1$, W strongly witnesses $P^s(S)$ if, for all $p \in W$, W_p witnesses $P^s(S_p)$, that is, W witnesses $P^s(S)$ and $P^{s[n]}(W)$ fails everywhere. This is a $\Pi_{|s+1|^1}$ sentence.

Lemma (3)

If $P^s(S)$, then there is a W that strongly witnesses it.

Proof of lemma 3

Lemma (4)

If $|s| = n + 1$*, then at least one of* $P^s(S)$ *and* $P^{s[n]}(S)$ *holds.*

Proof of lemma 4

Proof of the theorem

Thank you for your attention!