Determinacy on the edge of second order arithmetic

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Janvier 12, 2023

Program

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$\begin{array}{lccccccccc} \text{I:} & a_0 & & a_2 & & & a_{2n} & & & \\ & & & & \cdots & & & \cdots & & (a_i)_{i<\omega} \overset{?}{\in} X \\ \text{II:} & & a_1 & & a_3 & & & a_{2n+1} & & \end{array}$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): “All these games are determined”.
(False in ZF + C.)

The Borel and projective hierarchy

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

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- Study these properties for projective Σ^1_n sets in ω^ω (Blackwell, 1967).
- Are Σ^1_2, Σ^1_3, etc sets Lebesgue measurable?
- Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.

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Theorem (Martin, ZFC)

All Borel games are determined.

The theorem of Wolfe

Theorem ($\mathsf{ZC}^- + \Sigma_1$-Replacement)

All Σ^0_2 games are determined.

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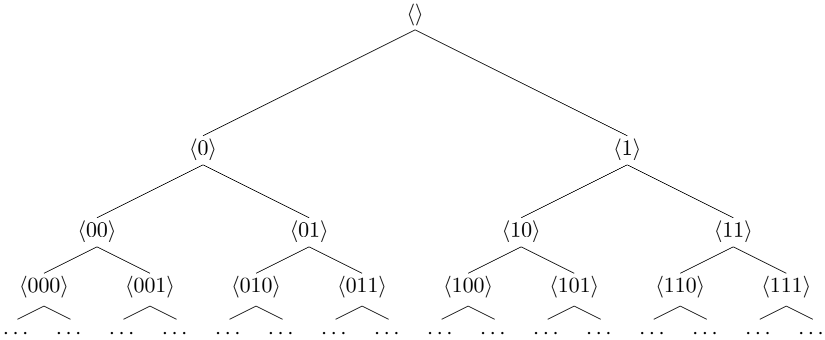
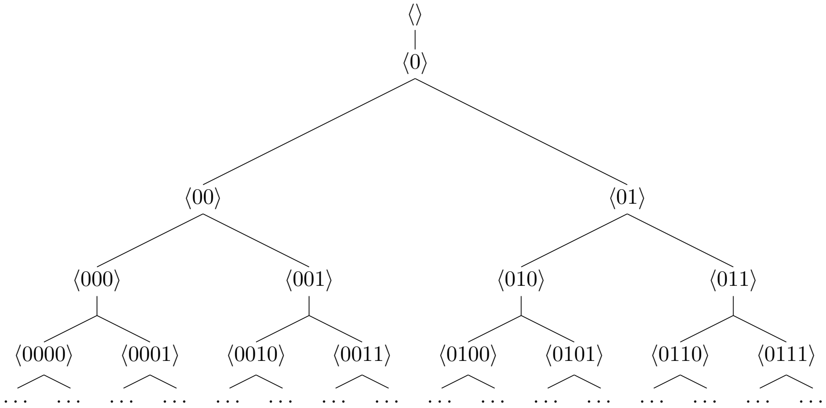
In a quasistrategy the player's response has not to be unique.

Example: A binary game



Strategy and quasistrategy



A technical lemma

Lemma

Let $B \subseteq A \subseteq [T]$ with B being closed. If player I has no winning strategy in the game $G(T, A)$, then there is a strategy τ for II such that every $x \in [\tau]$ has a finite initial segment p verifying

$[T_p] \cap B = \emptyset$ *and* *I has still no winning strategy in $G(T_p, A)$*

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- Our τ avoid all the A_i and hence is winning for II.

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$\mathsf{Det}(n\text{-}\Pi^0_3)$ is the Π^1_3 sentence

$$\forall X \ \exists Y \ \forall Z \ (X \in n\text{-}\Pi^0_3) \to \begin{cases} Y \in S_I \wedge Z \in S_{II} \to Y \bigoplus Z \in X; \\ Y \in S_{II} \wedge Z \in S_I \to Z \bigoplus Y \in X. \end{cases}$$

Consider $\emptyset = A_m \subseteq \cdots \subseteq A_1 \subseteq A_0$, Π^3_0 sets.

$$A_i = \bigcap_{k<\omega} A_{i,k} \qquad \text{and} \qquad A_{i,k} = \bigcup_{j<\omega} A_{i,k,j}.$$

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Strong Σ^1_{m+2}-DC_0 is Π^1_4 conservative over Π^1_{m+2}-CA_0.

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A quasistrategy U witnesses $P^s(S)$ if U is as required in the appropriate clause, the latter being a $\Pi^1_{|s|+1}$ sentence.

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A quasistrategy U for I locally witnesses $P^s(S)$ if $|s| = n + 1$ and l is even if: $\exists D \subseteq S \; \forall d \in D$, there is a quasistrategy R^d for II in S_d such that:

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We observe that "U locally witnesses $P^s(S)$" is a $\Sigma^1_{|s|+2}$ sentence.

Lemma (1)

If U locally witnesses $P^s(S)$, then U witnesses $P^s(S)$.

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- By induction hypothesis we have to build for $n > 1$, $\hat{U}$, $\hat{D}$ and $\{\hat{R}^d : d \in \hat{D}\}$ locally witnessing $P^{s[n-1]}(\hat{S})$.

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- We use $\Sigma^1_{|s|}$-AC$_0$ to pick up witnesses for $P^{s[n-1]}(\hat{R}^d)$ which will be our $\hat{U}_d \cap \hat{R}^d$.

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Lemma (2)

If $P^s(S)$ fails, then there is a quasistrategy W for I if l is odd in S such that $P^s(W)$ fails everywhere.

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- We now consider the game $G(S, B)$ where

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- Using preceding lemma, $G(S, B)$ is not a win for I.
- Again we define W as non-losing II's quasistrategy (and use preceding lemma to show it is as required).

Definition

For $|s| = n + 1$, W strongly witnesses $P^s(S)$ if, for all $p \in W$, W_p witnesses $P^s(S_p)$, that is, W witnesses $P^s(S)$ and $P^{s[n]}(W)$ fails everywhere. This is a $\Pi^1_{|s+1|}$ sentence.

Lemma (3)

If $P^s(S)$, then there is a W that strongly witnesses it.

Proof of lemma 3

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Apply preceding lemma to get a W such $P^{s[n]}(W)$ fails everywhere.

Lemma (4)

If $|s| = n + 1$, *then at least one of* $P^s(S)$ *and* $P^{s[n]}(S)$ *holds.*

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- Using strong Σ^1_{m+2}-DC$_0$, we define by induction a quasistrategy U for II in S along with $D \subseteq S$ and R^d for $d \in D$ showing that

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  2. We make sure $x \notin A_{m-n-1,s(n),j}$, $\forall j < \omega$ such that $x \notin A_{m-n-1}$.

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  1. We define U such that $x \in \bar{A} \cup A_{m-n-2,j}$, $\forall j < \omega$ such that $x \notin \bar{A} \cup A_{m-n-2,\cdot}$.
  2. We make sure $x \notin A_{m-n-1,s(n),j}$, $\forall j < \omega$ such that $x \notin A_{m-n-1}$.
  3. We then use $\bar{A} \cup A_{m-n-2} \setminus A_{m-n-1} \subseteq \bar{A}$.

Proof of the theorem

- Suppose m is odd and $G(A,T)$ is not a win for II; $P^{\emptyset}(T)$ fails.

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- We define a quasistrategy U for I in $W^{\emptyset}$ by induction on $|p|$ for $p \in U$. In the same time, we use strong Σ^1_3-DC_0 to define for $|p| = j + 1$ a quasistrategy W^p for I such that

$$W^p \text{ strongly witnesses } P^{\langle j \rangle}(W_p^{p[j]}).$$

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- By preceding lemma $P^{\langle j \rangle}(W^p_q)$ and we choose a W^q that strongly witnesses it.

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- Then $\forall j \; x \in A \cup A_{m-1,j}$,
- But $\bigcap_{j<\omega} A_{m-1,j} = A_{m-1} \subseteq A$.

Thank you for your attention!