Determinacy in and out second order arithmetic

An introduction to the proof theoretic strength of the determinacy scale

Thibaut Kouptchinsky

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Program

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$\begin{array}{llllllllll} \text{I:} & a_0 & & a_2 & & & a_{2n} & & & \\ & & & & & \cdots & & & \cdots & (a_i)_{i<\omega} \overset{?}{\in} X \\ \text{II:} & & a_1 & & a_3 & & & a_{2n+1} & & \end{array}$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): “All these games are determined”.
(False in ZF + C.)

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

ZF + AD *proves that every sets of real numbers is Lebesgue measurable, has the Baire property, and has the perfect set property.*

Study these properties for projective Σ^1_n sets in ω^ω.

Are Σ^1_3 sets Lebesgue measurable?
Kechris and Martin: "Yes, provided $\mathsf{Det}(\Pi^1_2)$ $(+\mathsf{AC}_\omega(\omega^\omega))$".

Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.