# Foundations of Mathematics: On the Translations Between Arithmetic and Set Theory

Thibaut Kouptchinsky

COLABS Tohoku University

UCLouvain

February 2023

;

## **The main question**: What are the appropriate axioms to prove the theorems of mathematics

*Fondations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge*

---

*Stephen G. Simpson*
*SSOA (second edition)*

## **The main question**: What are the appropriate axioms to prove the theorems of mathematics

> *Fondations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge*
>
> ---
>
> *Stephen G. Simpson*
> *SSOA (second edition)*

Aristote, Euclide, Descartes, Cauchy, Weierstraß, Dedekind, Peano, Frege, Russell, Cantor, Hilbert, Brouwer, Weyl, von Neumann, Skolem, Tarski, Heyting, Gödel, ...

# The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots, 2^{82,589,933} - 1, \ldots\}$.

# The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots, 2^{82,589,933} - 1, \ldots\}$.
2. The graph of the function $\sin\colon \mathbb{R} \to \mathbb{R}$: $\{(x, \sin(x)) : x \in \mathbb{R}\}$.

# The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots, 2^{82,589,933} - 1, \ldots\}$.
2. The graph of the function $\sin\colon \mathbb{R} \to \mathbb{R}$: $\{(x, \sin(x)) : x \in \mathbb{R}\}$.
3. The set of the subsets of the complex number $\mathcal{P}(\mathbb{C}) = \{A : A \subseteq \mathbb{C}\}$.

# The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \dots, 2^{82,589,933} - 1, \dots\}$.
2. The graph of the function $\sin\colon \mathbb{R} \to \mathbb{R}$: $\{(x, \sin(x)) : x \in \mathbb{R}\}$.
3. The set of the subsets of the complex number $\mathcal{P}(\mathbb{C}) = \{A : A \subseteq \mathbb{C}\}$.

A counter-example, the Russel's Paradox:

$$S = \{X : X \notin X\}$$

## The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots, 2^{82,589,933} - 1, \ldots\}$.
2. The graph of the function $\sin\colon \mathbb{R} \to \mathbb{R}$: $\{(x, \sin(x)) : x \in \mathbb{R}\}$.
3. The set of the subsets of the complex number $\mathcal{P}(\mathbb{C}) = \{A : A \subseteq \mathbb{C}\}$.

A counter-example, the Russel's Paradox:

$$S = \{X : X \notin X\}$$

- If $S \in S$, then $S \notin S$;

# The Set Theory of Zermelo and Fraënkel

**Example**

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots, 2^{82,589,933} - 1, \ldots\}$.
2. The graph of the function $\sin\colon \mathbb{R} \to \mathbb{R}$: $\{(x, \sin(x)) : x \in \mathbb{R}\}$.
3. The set of the subsets of the complex number $\mathcal{P}(\mathbb{C}) = \{A : A \subseteq \mathbb{C}\}$.

A counter-example, the Russel's Paradox:

$$S = \{X : X \notin X\}$$

- If $S \in S$, then $S \notin S$;
- If $S \notin S$, then $S \in S$.

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.
2. EXTENSIONALITY Two sets are equal if they have the same elements.

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.
2. Extensionality Two sets are equal if they have the same elements.
3. Pair If $a$ and $b$ are sets, we can form $\{a, b\}$.

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.
2. Extensionality Two sets are equal if they have the same elements.
3. Pair If $a$ and $b$ are sets, we can form $\{a, b\}$.
4. Separation If $P$ is property and $X$ is a set, we can form

$$\{u \in X \mid P(u)\}.$$

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.
2. EXTENSIONALITY Two sets are equal if they have the same elements.
3. PAIR If $a$ and $b$ are sets, we can form $\{a, b\}$.
4. SEPARATION If $P$ is property and $X$ is a set, we can form

$$\{u \in X \mid P(u)\}.$$

5. INFINITY There exists an infinite set.

# The axioms of ZF

1. There exists a set and this set is empty; $\emptyset$.
2. EXTENSIONALITY Two sets are equal if they have the same elements.
3. PAIR If $a$ and $b$ are sets, we can form $\{a, b\}$.
4. SEPARATION If $P$ is property and $X$ is a set, we can form

$$\{u \in X \mid P(u)\}.$$

5. INFINITY There exists an infinite set.
6. POWER SET If $X$ is a set, we can form $\mathcal{P}(X)$

# The axioms of ZF

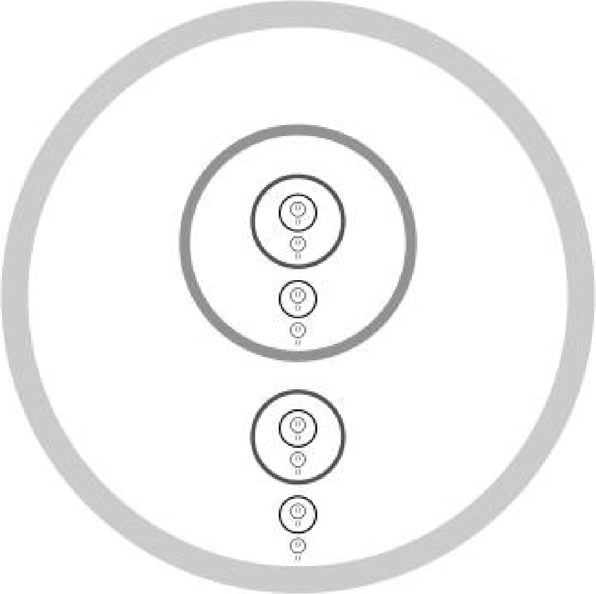
1. There exists a set and this set is empty; $\emptyset$.
2. EXTENSIONALITY Two sets are equal if they have the same elements.
3. PAIR If $a$ and $b$ are sets, we can form $\{a, b\}$.
4. SEPARATION If $P$ is property and $X$ is a set, we can form

$$\{u \in X \mid P(u)\}.$$

5. INFINITY There exists an infinite set.
6. POWER SET If $X$ is a set, we can form $\mathcal{P}(X)$
7. plus $3$ others ...

# The Von Neumann construction of $\mathbb{N}$

$$
\begin{aligned}
0 &= \{\ \} && = \emptyset, \\
1 &= \{0\} && = \{\emptyset\} \\
2 &= \{0, 1\} && = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} && = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\dots \\
n + 1 &= n \cup \{n\} && = \{0, 1, 2, \dots, n\} \\
&\dots
\end{aligned}
$$



# The Realm of Second-order Arithmetic

Friedman and Simpson: Encoding the ordinary mathematics with only

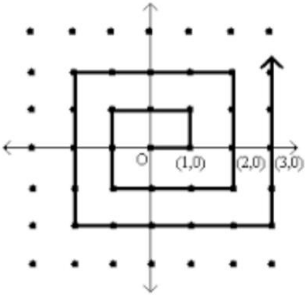
1. Natural numbers $n \in \mathbb{N}$;
2. Sets of natural numbers $S \subseteq \mathbb{N}$.

# The Realm of Second-order Arithmetic

Friedman and Simpson: Encoding the ordinary mathematics with only

1. Natural numbers $n \in \mathbb{N}$;
2. Sets of natural numbers $S \subseteq \mathbb{N}$.

Example: $\mathbb{Z} \times \mathbb{Z}$ is countable



# Trees

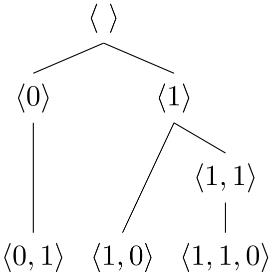
**Definition**

A tree is a subset of finite sequences of natural numbers ($T \subseteq \mathbb{N}^{<\infty}$) such that if $\sigma = \langle m_0, m_1, \ldots m_i, \ldots m_k\rangle$ is in $T$, then every initial segment of $\sigma$ (for example $\tau = \langle m_0, m_1, \ldots m_i\rangle$) is also in $T$.

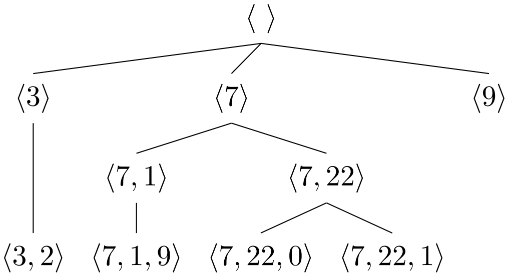
# Trees

### Definition

A tree is a subset of finite sequences of natural numbers ($T \subseteq \mathbb{N}^{<\infty}$) such that if $\sigma = \langle m_0, m_1, \ldots m_i, \ldots m_k \rangle$ is in $T$, then every initial segment of $\sigma$ (for example $\tau = \langle m_0, m_1, \ldots m_i \rangle$) is also in $T$.



# Another example of tree



# Set Theory in Second Order Arithmetic

We encode a set by a tree $T$ where

## Set Theory in Second Order Arithmetic

We encode a set by a tree $T$ where

1. If $\sigma \in T$ is an end node (a leaf), then $|\sigma|$ code the emptyset,

# Set Theory in Second Order Arithmetic

We encode a set by a tree $T$ where

1. If $\sigma \in T$ is an end node (a leaf), then $|\sigma|$ code the emptyset,
2. If $\tau$ is a direct child node of $\sigma$, then it code $|\tau| \in |\sigma|$.

# Set Theory in Second Order Arithmetic

We encode a set by a tree $T$ where

1. If $\sigma \in T$ is an end node (a leaf), then $|\sigma|$ code the emptyset,
2. If $\tau$ is a direct child node of $\sigma$, then it code $|\tau| \in |\sigma|$.