Foundations of Mathematics: On the Translations Between Arithmetic and Set Theory

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The main question: What are the appropriate axioms to prove the theorems of mathematics

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Aristote, Euclide, Descartes, Cauchy, Weierstraß, Dedekind, Peano, Frege, Russell, Cantor, Hilbert, Brouwer, Weyl, von Neumann, Skolem, Tarski, Heyting, Gödel, ...

The Set Theory of Zermelo and Fraënkel

Example

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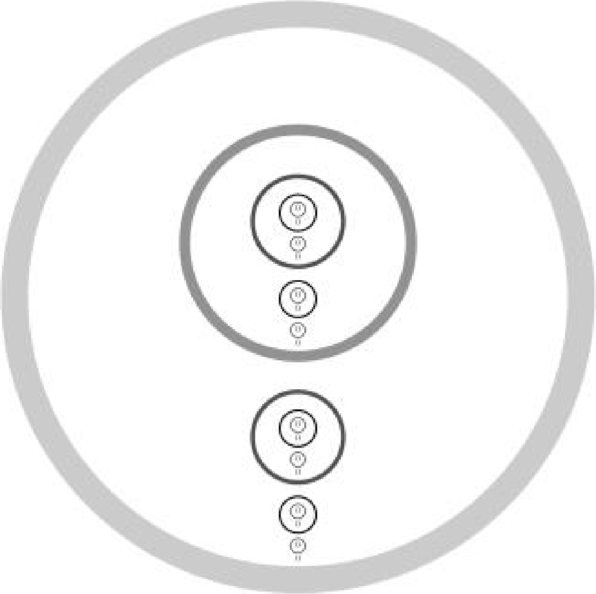
5. plus 5 others ...

The Von Neumann construction of $\mathbb{N}$

$$
\begin{aligned}
0 &= \{\ \} && = \emptyset, \\
1 &= \{0\} && = \{\emptyset\} \\
2 &= \{0, 1\} && = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} && = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\dots \\
n + 1 &= n \cup \{n\} && = \{0, 1, 2, \dots, n\} \\
&\dots
\end{aligned}
$$



The Realm of Second-order Arithmetic

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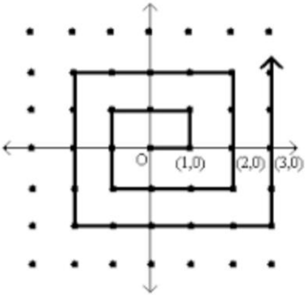
Example: $\mathbb{Z} \times \mathbb{Z}$ is countable



(Well-founded) Trees

Definition

A tree is a subset of finite sequences of natural numbers ($T \subseteq \mathbb{N}^{<\infty}$), such that leafs are the biggest sequences and root is the empty sequence. Moreover there is a branch in T from the root to any leaf.

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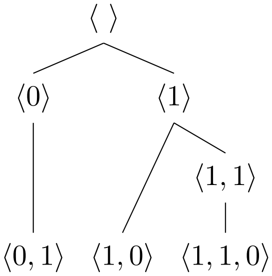
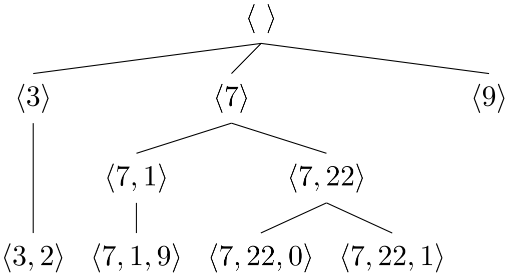


Another example of tree



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