Determinacy in and out second order arithmetic

An introduction to the proof theoretic strength of the determinacy scale

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Program

Academic me

- Master student at Catholic University of Louvain-la-Neuve (Belgium).

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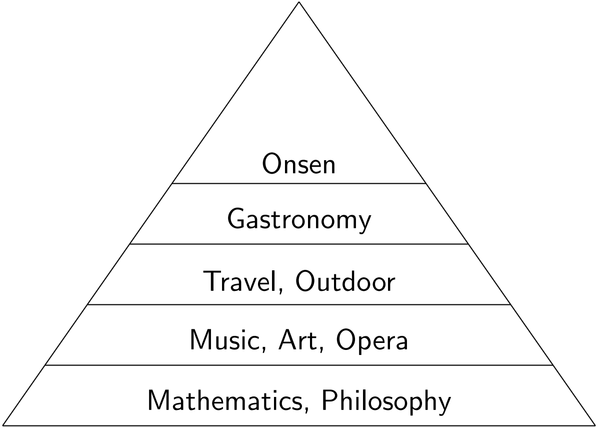
- Master Thesis with Profs. J.P. Aguilera (TU Wien) and T. Van der Linden (UCLouvain).
- Foundations of mathematics, reverse mathematics, β-models, determinacy, model theory, set theory, surreal numbers, topos theory etc.

Hobbies


Onsen
Gastronomy
Travel, Outdoor
Music, Art, Opera
Mathematics, Philosophy

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$\begin{array}{lcccccccccc} \text{I:} & a_0 & & a_2 & & & a_{2n} & & & & \\ & & & & \cdots & & & & \cdots & & (a_i)_{i<\omega} \overset{?}{\in} X \\ \text{II:} & & a_1 & & a_3 & & & a_{2n+1} & & & \end{array}$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): "All these games are determined".
(False in ZF + C.)

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

ZF + AD *proves that every set of real numbers is Lebesgue measurable (M1), has the Baire property (M2), and has the perfect set property (M3).*

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- Study these properties for projective Σ^1_n sets in ω^ω (Blackwell, 1967).
- Are Σ^1_2, Σ^1_3, etc sets Lebesgue measurable?
- Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.

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Theorem (Martin, ZFC)

All Borel games are determined.

A conservation result

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- We use Shoenfield Absoluteness theorem ($\Pi^1_1 - \mathsf{CA}_0$).

Some right axioms systems

Theorem (Steel, Simpson)

Over RCA_0, $\mathrm{Det}(\Sigma^0_1)$ *is equivalent to* ATR_0.

Theorem (Tanaka)

$\mathrm{Det}(\Sigma^0_2)$ *is equivalent to* $\Sigma^1_1 - \mathrm{MI}$.

How much determinacy can we prove in Z_2?

Theorem (Montalbán and Shore)

$\forall n \in \omega$, Π^1_{n+2}-$\mathsf{CA}_0 \vdash \mathsf{Det}(n\text{-}\Pi^0_3)$ *but* Δ^1_{n+2}-$\mathsf{CA}_0 \nvdash \mathsf{Det}(n\text{-}\Pi^0_3)$.

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Theorem (MedSalem and Tanaka)

Borel determinacy does not imply Δ^1_2-CA_0.

Reversals

Theorem (Hachtman)

Over Π^1_1-CA_0, $\mathrm{Det}(\Sigma^0_3)$ *(lightface) is equivalent to the existence of a* $\beta-$*model satisfying* $\Pi^1_2 - \mathrm{MI}$.

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Theorem (Aguilera and Welch)

Over Π^1_1-CA_0, *for each* $m \in \mathbb{N}$ *we have an equivalence between*

1. $\mathrm{Det}(m\text{-}\Pi^0_3)$,
2. *Every real belongs to a* β*-model of* Π^1_{m+1}-MI.

Game encoding models

Theorem (Friedman < Martin < M. and S.)

We cannot prove $\mathrm{Det}(\Sigma^0_5) < \mathrm{Det}(\Sigma^0_4) < \mathrm{Det}(\omega\text{-}\Pi^0_3)$ *in* ZFC^-

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- Same technique as for the limitative result of Friedman for Borel determinacy.
- The games are deemed to encode fragments of set theory using Gödel numbering.

Ingretients of the proof for $\mathrm{Det}(\Sigma^4_0)$

- I and II are playing S_I and S_{II} with

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- I wins if she plays L_{β_0} but looses if he doesn't byt II does;
- The models are countable and characterised by the subsets of ω they contain;
- Gödel-Tarski undefinability of truth.

Measurability properties

Theorem (Kechris, Martin)

In $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$, $\mathsf{Det}(\Pi^1_n)$ *proves that every* Σ^1_{n+1} *sets of reals satisfies M1, M2 and M3.*

Theorem (Shelah-Woodin)

Given $n \in \omega$, *if there are* n *Woodin cardinals with a measurable cardinal above them, then every* Σ^1_{n+2} *sets of reals satisfies M1, M2 and M3.*

Determinacy and high cardinal hypotheses

Theorem

Given $n \in \omega$, if there are n Woodin cardinals with a measurable cardinal above them, then $\mathrm{Det}(\Pi^1_{n+1})$.

Remark

This is a corollary from a theorem of Martin-Steel, which is out of the scope of the present talk.

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- SoSOA (Simpson);
- The limits of determinacy in second-order arithmetic (MS).

Thank you for your attention!