Determinacy in and out second order arithmetic

An introduction to the proof theoretic strength of the determinacy scale

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Program

Academic me

- Master student at Catholic University of Louvain-la-Neuve (Belgium).

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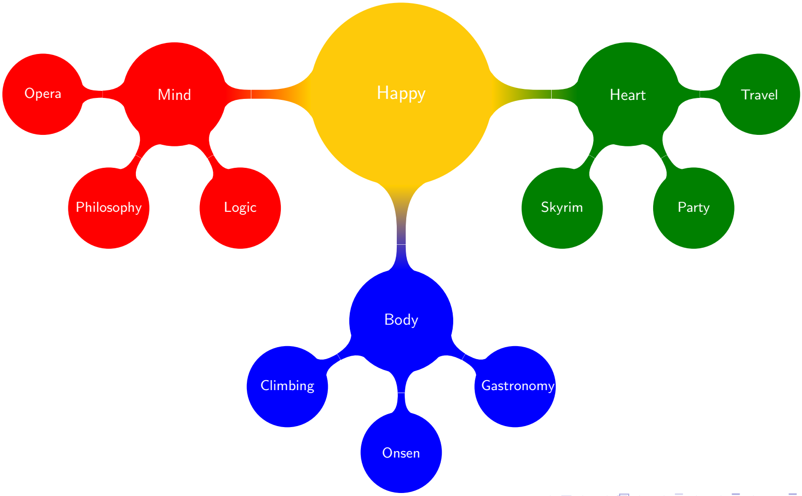
- (I like logic).

Hobbies


Happy
Mind
Opera
Philosophy
Logic
Heart
Travel
Skyrim
Party
Body
Climbing
Onsen
Gastronomy

What is determinacy?

Consider a set A and a payoff set $X \subseteq A^\omega$.

$$
\begin{array}{lccccccc}
\text{I:} & a_0 & & a_2 & & a_{2n} & & \\
 & & & & \cdots & & \cdots & (a_i)_{i<\omega} \overset{?}{\in} X \\
\text{II:} & & a_1 & & a_3 & & a_{2n+1} &
\end{array}
$$

Player I wins if yes. Otherwise player II wins.

Axiom of determinacy (AD): “All these games are determined”.
(False in ZF + C.)

Motivations and applications

Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis)

ZF + AD *proves that every sets of real numbers is Lebesgue measurable (M1), has the Baire property (M2), and has the perfect set property (M3).*

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- Study these properties for projective Σ^1_n sets in ω^ω (Blackwell, 1967).
- Are Σ^1_2, Σ^1_3, etc sets Lebesgue measurable?
- Applications in measure theory, descriptive set theory, harmonic analysis, ergodic theory, dynamical systems etc.

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Theorem (Martin, ZFC)

All Borel games are determined.

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ZFC^- is a Π^1_4 conservative extension of Z_2.

- For any $X \subset \mathbb{N}$, we can construct $\mathsf{L}(\mathsf{X})$ in Z_2;
- We can show in Z_2 that $\mathsf{L}(\mathsf{X}) \models \mathsf{ZFC}^-$ (Simpson);

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Theorem

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- For any $X \subset \mathbb{N}$, we can construct $\mathsf{L}(\mathsf{X})$ in Z_2;
- We can show in Z_2 that $\mathsf{L}(\mathsf{X}) \models$ ZFC$^-$ (Simpson);
- We use Shoenfield Absoluteness theorem ($\Pi^1_1 - CA_0$).

Some right axioms systems

Theorem (Steel, Simpson)

Over RCA_0, $\mathrm{Det}(\Sigma^0_1)$ *is equivalent to* ATR_0.

Theorem (Tanaka)

$\mathrm{Det}(\Sigma^0_2)$ *is equivalent to* $\Sigma^1_1 - \mathrm{MI}$.

How much determinacy can we prove in Z_2?

Theorem (Montalbán and Shore)

Π^1_{n+2}-CA$_0 \vdash$ Det$(n$-$\Pi^0_3)$ *but* Δ^1_{n+2}-CA$_0 \nvdash$ Det$(n$-$\Pi^0_3)$.

However, Π^1_{n+2}-CA$_0$ is not the right set of axioms for Det$(n$-$\Pi^0_3)$.

Theorem (MedSalem and Tanaka)

Borel determinacy does not imply Δ^1_2-CA$_0$.

Reversals

Theorem (Hachtman)

$\mathrm{Det}(\Sigma^0_3)$ (lightface) is equivalent to the existence of a $\beta-$model satisfying $\Pi^2_1 - \mathrm{MI}$.

Theorem (Aguilera and Welch)

Over Π^1_1-CA_0, for each $m \in \mathbb{N}$ we have an equivalence between

1. $\mathrm{Det}(m\text{-}\Pi^0_3)$,
2. *Every real belongs to a β-model of Π^1_{m+1}-MI.*

Game encoding models

Theorem (Friedman < Martin < M. and S.)

We cannot prove $\mathsf{Det}(\Sigma^5_0) < \mathsf{Det}(\Sigma^4_0) < \mathsf{Det}(\omega\text{-}\Pi^3_0)$ *in* ZFC^-

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- Same technique as for the limitative result of Friedman for Borel determinacy.
- The games are deemed to encode fragments of set theory using Gödel numbering.

Ingretients of the proof for $\mathrm{Det}(\Sigma^4_0)$

Measurability properties

Theorem (Kechris, Martin)

In $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$, $\mathsf{Det}(\Pi^1_n)$ *proves that every* Σ^1_{n+1} *sets of reals satisfies M1, M2 and M3.*

Theorem (Shelah-Woodin)

Given $n \in \omega$, *if there are* n *Woodin cardinals with a measurable cardinal above them, then every* Σ^1_{n+2} *sets of reals satisfies M1, M2 and M3.*

Determinacy and high cardinal hypotheses

Theorem

Given $n \in \omega$, if there are n Woodin cardinals with a measurable cardinal above them, then $\mathrm{Det}(\Pi^1_{n+1})$.

Remark

This is a corollary from a theorem of Martin-Steel, which is out of the scope of the present talk.

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- Set Theory (Jech), Descriptive Set Theory (Kechris, Moschovakis);
- Determinacy of Infinitely Long Games (Martin), The Higher Infinite (Martin);
- SoSOA (Simpson);
- The limits of determinacy in second-order arithmetic (MS).

Thank you for your attention!