Foundations of Mathematics: On the Translations Between Arithmetic and Set Theory

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The main question: What are the appropriate axioms to prove the theorems of mathematics

Fondations of mathematics is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge

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Aristote, Euclide, Descartes, Cauchy, Weierstraß, Dedekind, Peano, Frege, Russell, Cantor, Hilbert, Brouwer, Weyl, von Neumann, Skolem, Tarski, Heyting, Gödel, ...

The Set Theory of Zermelo and Fraënkel

Example

1. The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \dots, 2^{82{,}589{,}933} - 1, \dots\}$.

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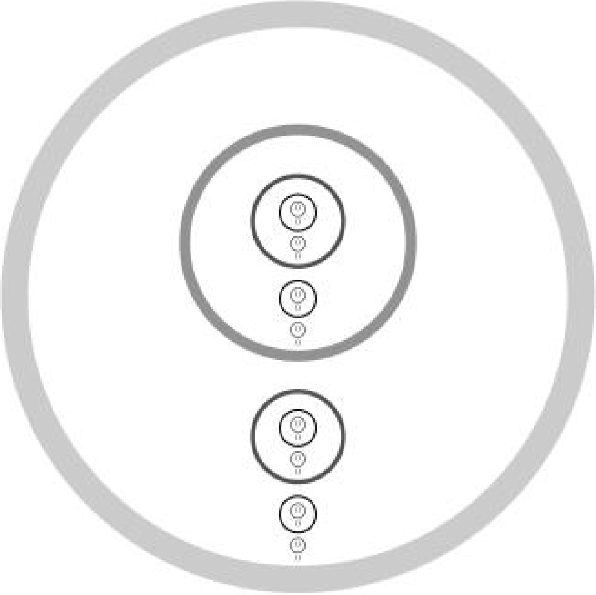
5. plus 5 others ...

The Von Neumann construction of $\mathbb{N}$

$$
\begin{aligned}
0 &= \{\ \} && = \emptyset, \\
1 &= \{0\} && = \{\emptyset\} \\
2 &= \{0, 1\} && = \{\emptyset, \{\emptyset\}\} \\
3 &= \{0, 1, 2\} && = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\dots \\
n + 1 &= n \cup \{n\} && = \{0, 1, 2, \dots, n\} \\
&\dots
\end{aligned}
$$



The Realm of Second-order Arithmetic

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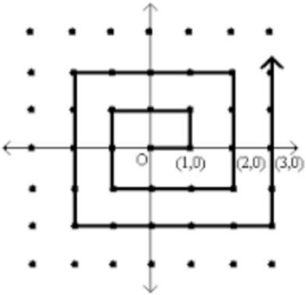
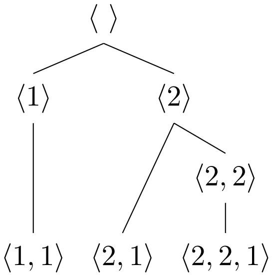
Example: $\mathbb{Z} \times \mathbb{Z}$ is countable



Trees


$\langle\ \rangle$
$\langle 1\rangle$
$\langle 2\rangle$
$\langle 2,2\rangle$
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