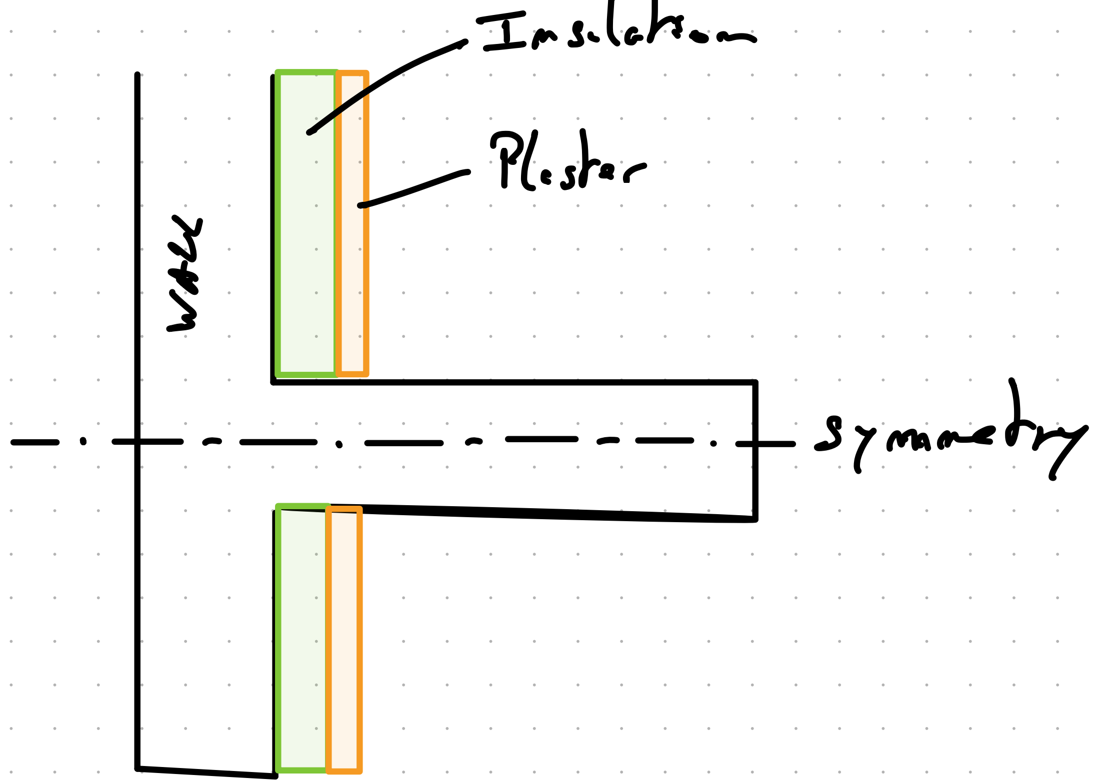
Numerical Methods

2024 - 2025

Calcul de pont thermique



Node at interface

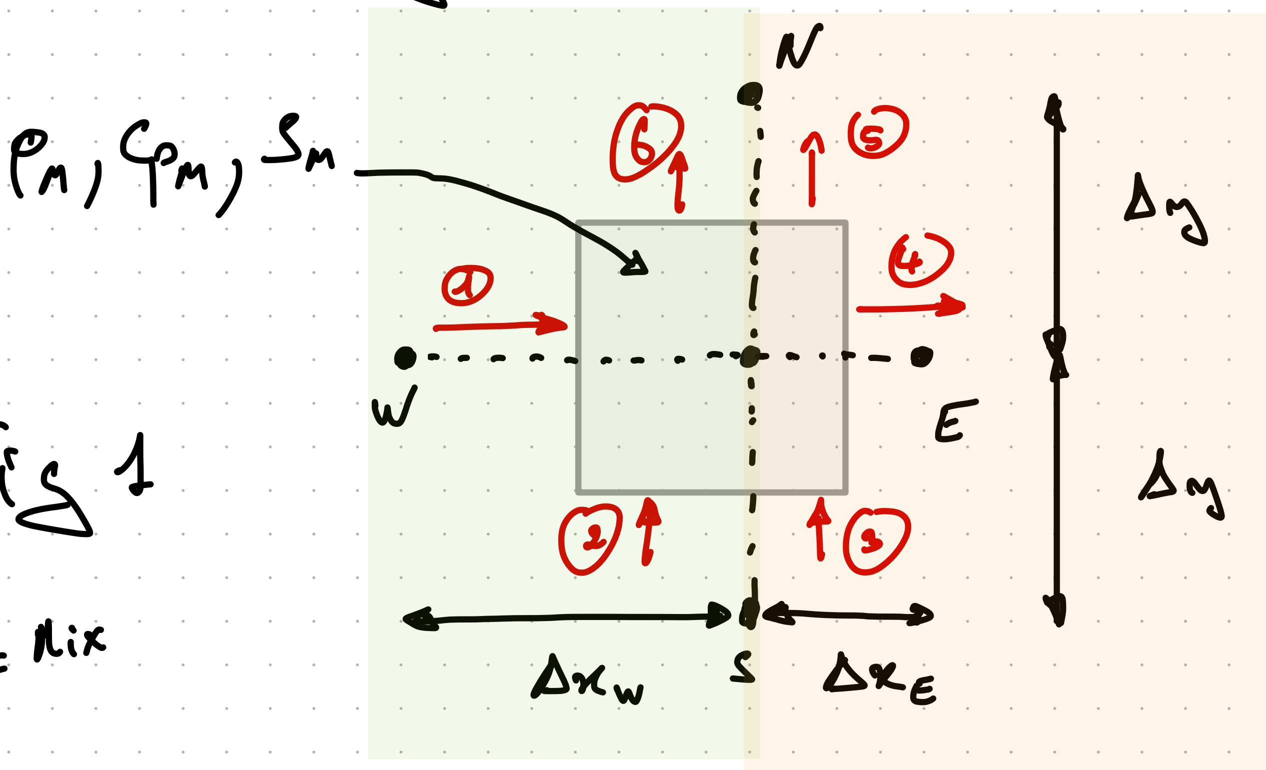


Fig 1

M = Mix

Def

$$\rho_M C_{pM} S_M = \rho_W C_{pW} \frac{\Delta x_W \Delta y}{2} + \rho_E C_{pE} \frac{\Delta x_E \Delta y}{2}$$

$$\boxed{S_M = \Delta y \left(\frac{\Delta x_W + \Delta x_E}{2}\right)}$$

so $$C_{p_M} = \frac{\rho_W}{\rho_M} C_{p_W} \frac{\Delta x_W}{\Delta x_E + \Delta x_E} + \dots$$

$$\dots \frac{\rho_E}{\rho_M} C_{p_E} \frac{\Delta x_E}{\Delta x_W + \Delta x_E}$$

and $\rho_M = \frac{m_M}{V_M}$ and $m_E + m_W = m_M$

$\rho_E = \frac{m_E}{V_E}$, $\rho_W = \frac{m_W}{V_W}$

$$\rho_M = \left(\rho_E V_E + \rho_W V_W\right) \times \frac{1}{V_M}$$

$$\rho_M = \left(\rho_E S_E + \rho_W S_W\right) \frac{1}{S_M}$$

$$\rho_M = \rho_E \frac{S_E}{S_M} + \rho_W \frac{S_W}{S_M}$$

$$\rho_M = r_E \rho_E + r_W \rho_W$$

$\boxed{r_E = \frac{\Delta x_E}{\Delta x_E + \Delta x_W}}$ and $\boxed{r_W = \frac{\Delta x_W}{\Delta x_E + \Delta x_W}}$

so
$$\boxed{\begin{aligned} \rho_M &= r_E \rho_E + r_W \rho_W \\ C_{p_M} &= r_W \frac{\rho_W}{\rho_M} C_{p_W} + r_E \frac{\rho_E}{\rho_M} C_{p_E} \end{aligned}}$$
and

From Fig 1 $\phi_1 + \phi_2 + \phi_3 - \{\phi_4 + \phi_5 + \phi_6\} = \rho_M C_{p_M} S_n \dfrac{T^{p+1} - T^p}{\Delta t}$ (eq 1)

$\boxed{⚠\ \phi \text{ in } W.m^{-1}}$

remove e the thickness

check: $\rho_n C_{p_n} S_n \dfrac{T^{p+1} - T^p}{\Delta t} = \dfrac{\cancel{kg}}{m^{\cancel{3}}} \dfrac{J}{\cancel{kg}\,\cancel{K}} \dfrac{\cancel{m^2}\,\cancel{K}}{s} = \dfrac{J}{m.s} = \dfrac{W}{m}$!! ok

$$\phi_1 = -k_W \Delta y \frac{T^{p+1} - T_W^{p+1}}{\Delta x_W} \; ; \; \phi_2 = -k_W \frac{\Delta x_W}{2} \frac{T^{p+1} - T_S^{p+1}}{\Delta y}$$

$$\phi_3 = -k_E \frac{\Delta x_E}{2} \frac{T^{p+1} - T_S^{p+1}}{\Delta y}$$

$$\phi_4 = -k_E \Delta y \frac{T_E^{p+1} - T^{p+1}}{\Delta x_E} \; ; \; \phi_5 = -k_E \frac{\Delta x_E}{2} \frac{T_N^{p+1} - T^{p+1}}{\Delta y}$$

$$\phi_6 = -k_W \frac{\Delta x_W}{2} \frac{T_N^{p+1} - T^{p+1}}{\Delta y}$$

From eq 1 coeff from ϕ_1 i.e $\times \dfrac{\Delta t}{\rho_M C_{p_M} S_M}$

on ϕ_1: $$-\frac{k_W \Delta y \Delta t}{\rho_M C_{p_M} S_M \Delta x_W} = \frac{-k_W \cancel{\Delta y} \Delta t}{\rho_M C_{p_M} \cancel{\Delta y} \left(\dfrac{\Delta x_E + \Delta x_W}{2}\right) \Delta x_W}$$

$$\frac{-k_W \Delta t}{\rho_M C_{p_M} \Delta x_W \dfrac{\Delta x_W}{2 r_W}} = \frac{-k_W \Delta t}{\dfrac{\rho_M C_{p_M} \Delta x_W^2}{2 r_W}} = -2 r_W Fo_{Mx}^{W}$$

on ϕ_2 : $-k_W \dfrac{\Delta x_W \; \Delta t}{\cancel{2}\Delta y \; \rho_M C_{p_M} \Delta y \left(\dfrac{\Delta x_E + \Delta x_W}{\cancel{2}}\right)}$

$$: -\frac{k_W \Delta t}{\rho_M C_{p_M} \dfrac{\Delta y^2}{r_W}} = -r_W \frac{k_W \Delta t}{\rho_M C_{p_M} \Delta y^2} = -r_W Fo^W_{My}$$

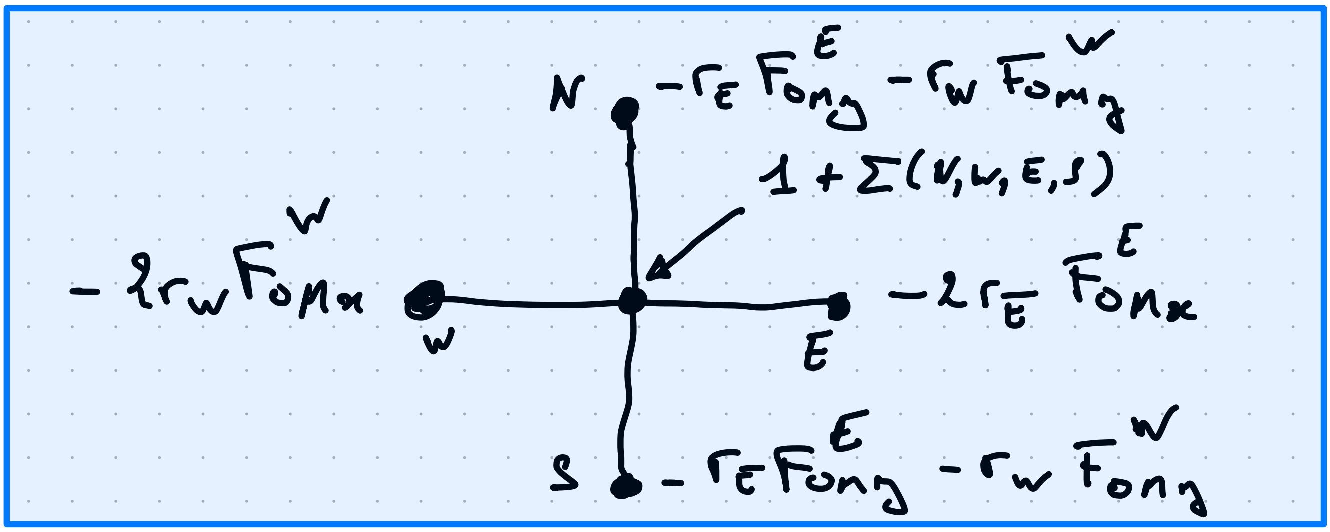
on ϕ_3 : $-k_E \dfrac{\Delta x_E \; \Delta t}{\cancel{2} \; \Delta y \, \rho_M C_{p_M} \Delta y \left(\dfrac{\Delta x_E + \Delta x_W}{\cancel{2}}\right)}$

$$-r_E \frac{k_E \Delta t}{\rho_M C_{p_M} \Delta y^2} = -r_E Fo^E_{My}$$

so for ϕ_4 : $-2 r_E Fo^E_{Mx} = -2 r_E \dfrac{k_E \Delta t}{\rho_M C_{p_M} \Delta x_E^2}$

ϕ_5 : $-r_E Fo^E_{My}$ cf ϕ_2

ϕ_6 : $-r_W Fo^W_{My}$ cf ϕ_3



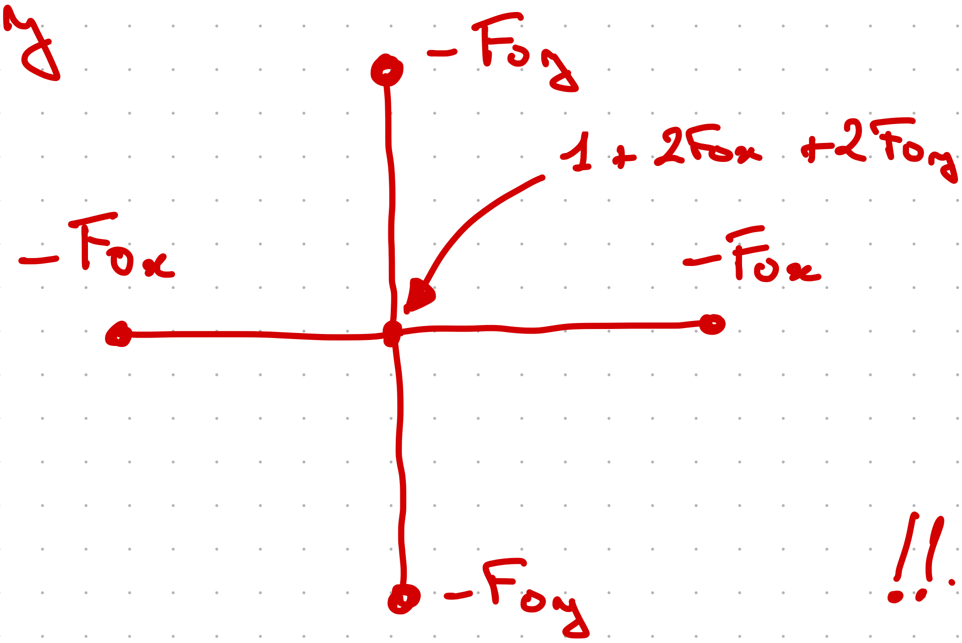
Check if $\underbrace{r_E = r_W}$ and Material W = Material E

$r_E = r_W = \frac{1}{2}$

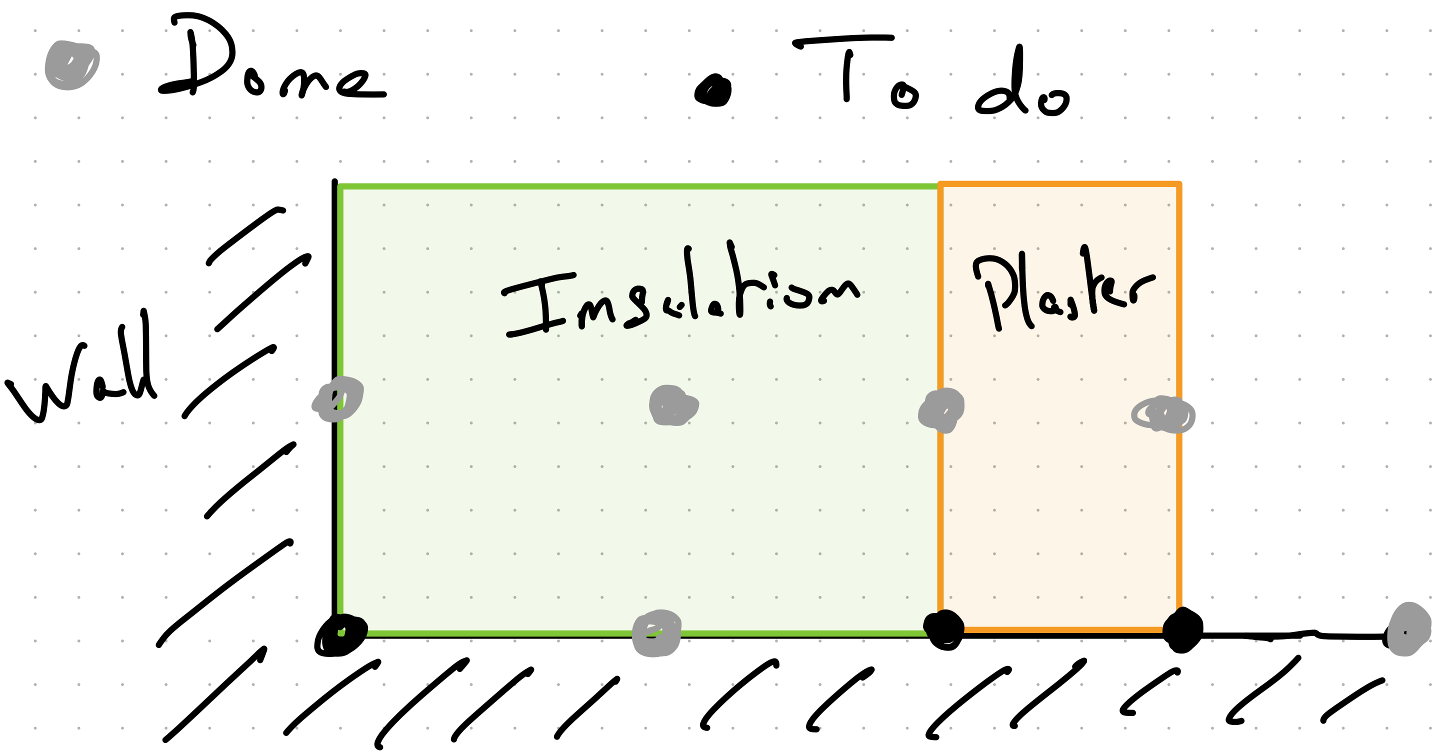
$k_W, \rho_W, c_{pW} = k_E, \rho_E, c_{pE}$

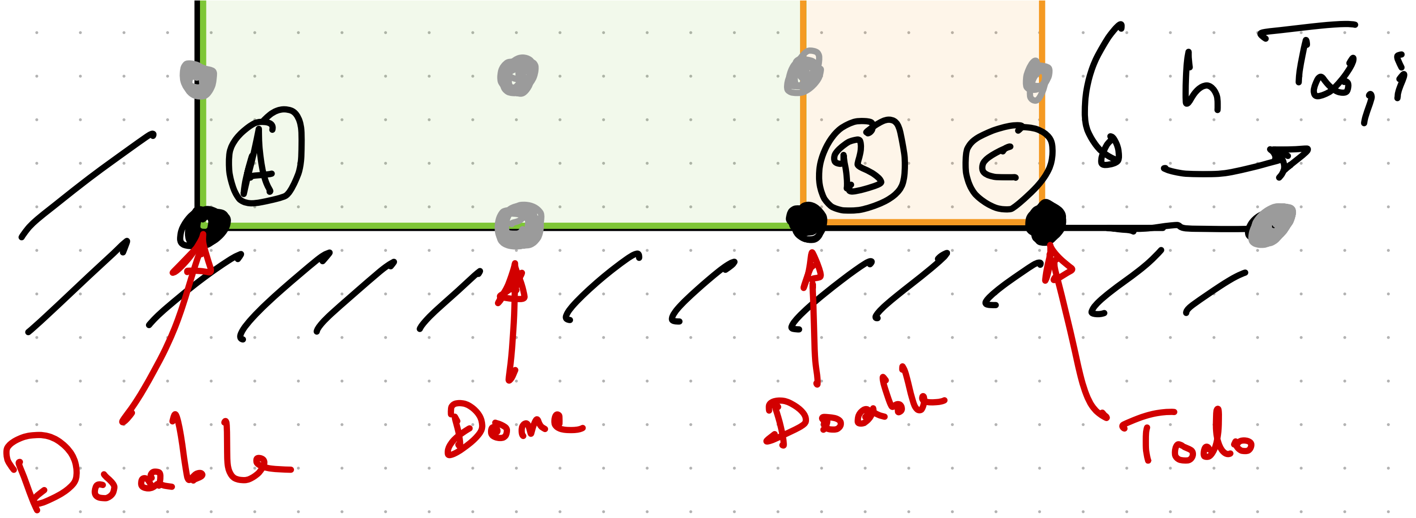
$= k_n, \rho_n, c_{pn}$

$\Delta x \neq \Delta y$

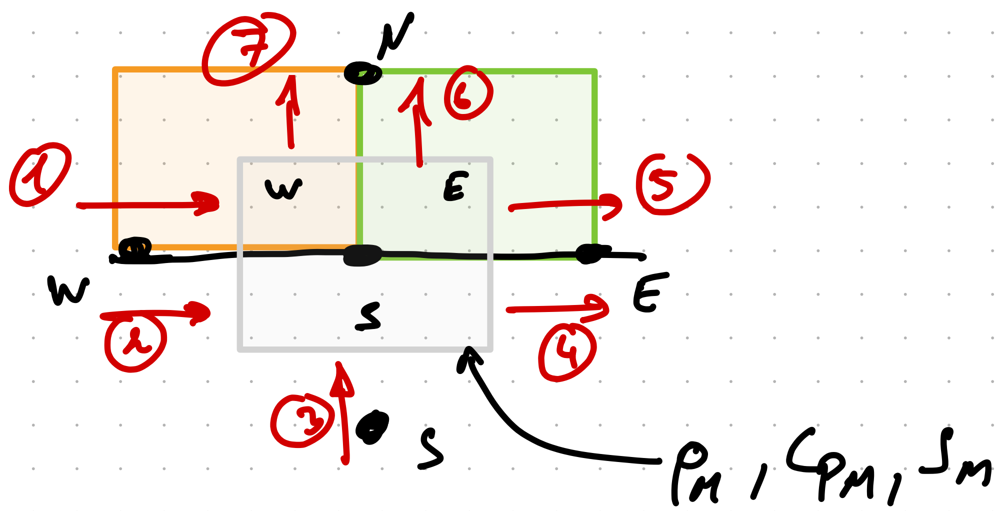


!!. OK !!





B



$$\rho_M = \frac{(m_E + m_W + m_S)}{V_M}$$

ρ_M en fct ρ_E, ρ_W, ρ_S $\quad$ r_E, r_W

$$\rho_M = \frac{(\rho_E V_E + \rho_W V_W + \rho_S V_S)}{V_M}$$

$$\rho_M = \rho_E \frac{S_E}{S_M} + \rho_W \frac{S_W}{S_M} + \rho_S \underbrace{\frac{S_S}{S_M}}_{1/2} = \rho_E \frac{S_E}{S_M} + \rho_W \frac{S_W}{S_M} + \frac{1}{2} \rho_S$$

$$S_E = \frac{\Delta y}{2} \times \frac{\Delta x_E}{2} \qquad S_W = \frac{\Delta y}{2} \times \frac{\Delta x_W}{2}$$

$$\boxed{S_M = \Delta y \times \left(\frac{\Delta x_E + \Delta x_W}{2}\right)}$$

$$r_E = \frac{\Delta x_E}{\Delta x_E + \Delta x_W}$$

$$\frac{S_E}{S_M} = \frac{r_E}{2} \quad / \quad \frac{S_W}{S_M} = \frac{r_W}{2} \qquad r_W = \frac{\Delta x_W}{\Delta x_E + \Delta x_W}$$

$$\boxed{\rho_M = \frac{r_E}{2}\rho_E + \frac{r_W}{2}\rho_W + \frac{\rho_S}{2}}$$

$$m_M C_{p_M} = m_E C_{p_E} + m_W C_{p_W} + m_S C_{p_S}$$

$$\rho_M S_M C_{p_M} = \rho_E S_E C_{p_E} + \rho_W S_W C_{p_W} + \rho_S S_S C_{p_S}$$

$$C_{p_M} = \frac{\rho_E S_E}{\rho_M S_M} C_{p_E} + \frac{\rho_W S_W}{\rho_M S_M} C_{p_W} + \frac{\rho_S S_S}{\rho_M S_M} C_{p_S}$$

$$\boxed{C_{p_M} = \frac{r_E}{2}\frac{\rho_E}{\rho_M} C_{p_E} + \frac{r_W}{2}\frac{\rho_W}{\rho_M} C_{p_W} + \frac{\rho_S}{2\rho_M} C_{p_S}}$$

$$\phi_1 : \quad - \frac{k_W \,\Delta y\, \Delta t}{2\,\Delta x_W\, \rho_M C_{p_M} S_M} = - \frac{k_W\, \Delta t}{\cancel{2}\rho_M C_{p_M} \Delta x_W \left(\frac{\Delta x_E + \Delta x_W}{\cancel{2}}\right)}$$

$$: - r_W \frac{k_W\, \Delta t}{\rho_M C_{p_M} \Delta x_W^2} = - r_W\, Fo_{M x}^{W}$$

$$\phi_2 : \quad -\frac{k_s \, \Delta y \, \Delta t}{2 \Delta x_w \, \rho_M c_{p_M} s_M} = -r_w \, Fo^{s}_{Mx}$$

$$\phi_3 : \quad -\frac{k_s \left(\frac{\Delta x_E + \Delta x_w}{2} \right) \Delta t}{\Delta y \, \rho_M \, c_{p_M} \, s_M}$$

$$: \quad -\frac{k_s \, \Delta t}{\rho_M c_{p_M} \Delta y^2} = - Fo^{s}_{My}$$

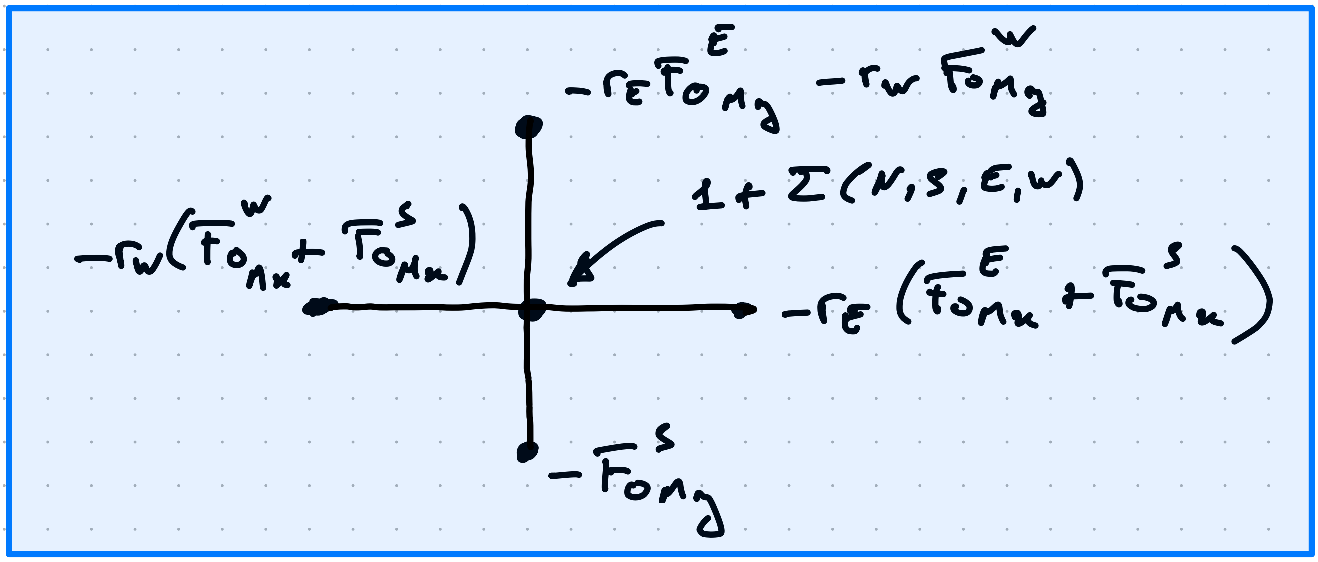
$\phi_4 : \quad -r_E \, Fo^{s}_{Mx}$ (analog ϕ_2) Δx_E

$\phi_5 : \quad -r_E \, Fo^{E}_{Mx}$ (analog ϕ_1) Δx_E

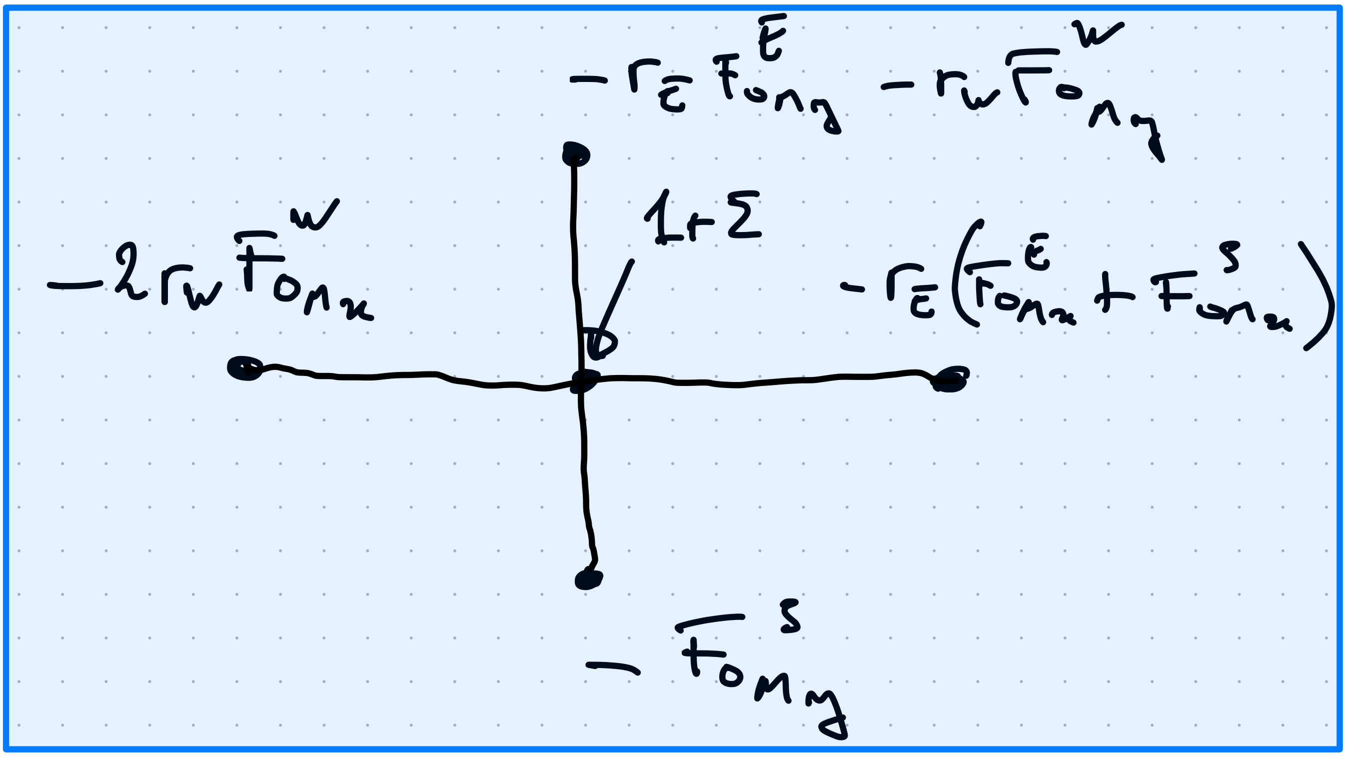
$\phi_6 : \quad -r_E \, Fo^{E}_{My}$ (analog ϕ_5 previous case)

$\phi_7 : \quad -r_w \, Fo^{W}_{My}$ (analog ϕ_6 previous case)

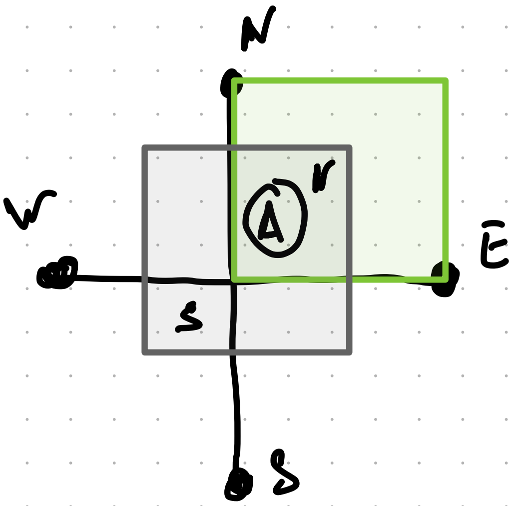
B



A



with



$$S_M = \Delta y \frac{(\Delta x_E + \Delta x_W)}{2}$$

$$\rho_M = \frac{m_S + m_N}{V_M}$$

$$\rho_M = \rho_S \frac{V_S}{V_M} + \rho_N \frac{V_N}{V_M} = \rho_S \frac{S_S}{S_M} + \rho_N \frac{S_N}{S_M}$$

$$\frac{S_S}{S_M} = \frac{\cancel{\Delta y}\,\Delta x_E/\cancel{2} + \cancel{\Delta y}/_2\,\Delta x_W/\cancel{2}}{\cancel{\Delta y}(\Delta x_E + \Delta x_W)/\cancel{2}} = \frac{2\Delta x_E + \Delta x_W}{2(\Delta x_E + \Delta x_W)}$$

$$\frac{S_S}{S_M} = \frac{\Delta x_W\left(1 + 2\frac{r_E}{r_W}\right)}{2(\Delta x_E + \Delta x_W)} = \frac{1 + \frac{2 r_E}{r_W}}{2/r_W} = \frac{r_W + 2 r_E}{2} = \frac{1 + r_E}{2}$$

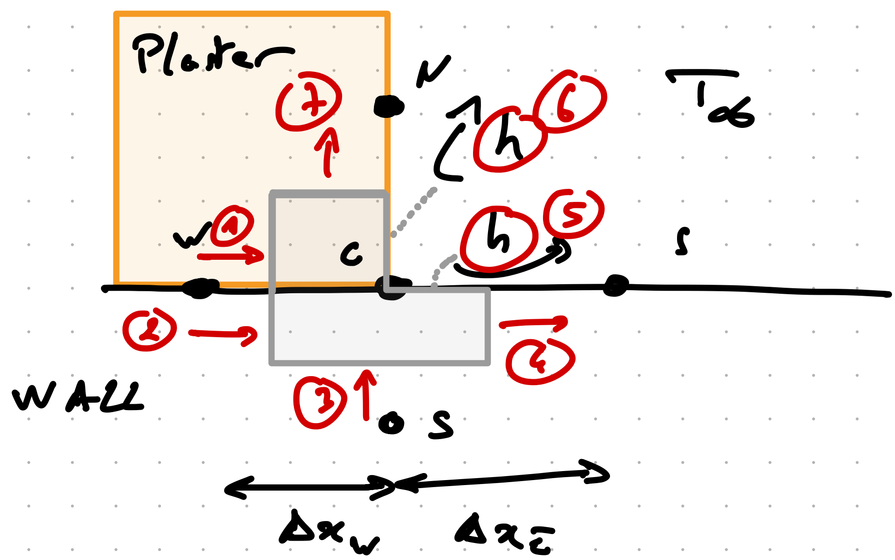
$$\frac{S_N}{S_M} = \frac{\cancel{\Delta y}/2 \, \Delta x_W \, \cancel{1/2}}{\cancel{\Delta y}\,(\Delta x_E + \Delta x_W)\,\cancel{/2}} = \frac{\Delta x_W}{2(\Delta x_E + \Delta x_W)} = \frac{r_W}{2}$$

$$\boxed{\rho_M = \left(\frac{1 + r_E}{2}\right) \rho_S + \frac{r_W}{2} \rho_N}$$

$$\rho_M S_M c_{pM} = \rho_S S_S c_{pS} + \rho_N S_N c_{pN}$$

$$\boxed{c_{pM} = \left(\frac{1 + r_E}{2}\right) \frac{\rho_S}{\rho_M} c_{pS} + \frac{r_W}{2} \frac{\rho_N}{\rho_M} c_{pN}}$$

(C)



$\phi_1, \phi_2, \phi_3, \phi_4$ and ϕ_7 already done for point (B)

ϕ_5 : Convection $\quad h \frac{\Delta x_E}{2} (T - T_\infty)$

ϕ_6 : " $\quad h \frac{\Delta y}{2} (T - T_\infty)$

$$\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6 - \phi_7 = \dots$$

$$\dots \rho_M \, C_{p_M} \, S_M \, \frac{T^{p+1} - T}{\Delta t}$$

$$\boxed{S_M = \frac{\Delta y}{2}\left(\frac{\Delta x_E + \Delta x_W}{2}\right) + \frac{\Delta y}{2}\left(\frac{\Delta x_W}{2}\right)} = \frac{\Delta y}{4}\left(\Delta x_E + \Delta x_W + \Delta x_W\right)$$

$$\rho_M = \rho_W \frac{S_W}{S_M} + \frac{\rho_S S_S}{S_M} \quad ; \quad S_W = \frac{\Delta x_W}{2} \times \frac{\Delta y}{2}$$

$$S_S = \frac{\Delta y}{2} \times \left(\frac{\Delta x_W + \Delta x_E}{2}\right)$$

$$\rho_M = \rho_W \frac{\Delta x_W \cancel{\Delta y}}{\cancel{\Delta y}(\Delta x_E + \Delta x_W + \Delta x_W)} +$$

$$\rho_S \frac{\cancel{\Delta y}(\Delta x_W + \Delta x_E)}{\cancel{\Delta y}(\Delta x_E + \Delta x_W + \Delta x_W)}$$

$$\rho_M = \rho_W \frac{1}{1 + \frac{1}{r_W}} + \rho_S \frac{1}{1 + r_W}$$

$$\boxed{\rho_M = \frac{1}{1 + r_W}\left(r_W \rho_W + \rho_S\right)}$$

$$\rho_M S_M C_{p_M} = \rho_W S_W C_{p_W} + \rho_S S_S C_{p_S}$$

$$C_{p_M} = \frac{\rho_W S_W}{\rho_M S_M} C_{p_W} + \frac{\rho_S S_S}{\rho_M S_M} C_{p_S}$$

$$\frac{S_W}{S_M} = \frac{\Delta x_W \cancel{\Delta y}}{\cancel{\Delta y}(\Delta x_W + \Delta x_E + \Delta x_W)} = \frac{1}{1 + \frac{1}{r_W}} \quad \text{and} \quad \frac{S_S}{S_M} = \frac{1}{1 + r_W}$$

$$\boxed{C_{p_M} = \frac{1}{1 + r_W}\left(r_W \frac{\rho_W}{\rho_M} C_{p_W} + \frac{\rho_S}{\rho_M} C_{p_S}\right)}$$

$$\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6 - \phi_7 = \dots$$

$$\dots \rho_M C_{pM} S_M \frac{T^{pM} - T}{\Delta t}$$

ϕ_1 is <u>NOT</u> similar to previous case because $\underline{S_M}$ is different

$$\phi_1 : -\frac{k_w \Delta y/2 \; \Delta t}{\rho_M C_{pM} \; S_M \quad \Delta x_w}$$

or $\boxed{\dfrac{\Delta y}{2 S_M}} = \dfrac{\cancel{\Delta y}}{\frac{\cancel{\Delta y}}{2}(\Delta x_E + \Delta x_w + \Delta x_w)} = \dfrac{2}{\Delta x_w (1 + \frac{1}{r_w})} = \dfrac{2 r_w}{(1+r_w)\Delta x_w}$

$$\phi_1 : -\frac{2 r_w}{1+r_w} \frac{k_w \Delta t}{\rho_M C_{pM} (\Delta x_w)^2}$$

$$\phi_1 : -\frac{2 r_w}{1+r_w} F_{o_{nx}}^{w}$$

Cf above

$$\phi_2 : -\frac{k_s \quad \Delta y/2 \quad \Delta t}{\rho_M \; C_{pM} \; S_M \; \Delta x_w}$$

$$\phi_2 : -\frac{2 r_w}{1+r_w} \frac{k_s \Delta t}{\rho_M C_{pM} \Delta x_w^2} = -\frac{2 r_w}{1+r_w} F_{o_{nx}}^{s}$$

$$\phi_3 : \frac{-k_s (\Delta x_E + \Delta x_w)/2 \ \Delta t}{\rho_m C_{pm} \ S_m \ \Delta y}$$

$$\frac{\Delta x_E + \Delta x_w}{2 S_m} = \frac{\Delta x_E + \Delta x_w}{\frac{\Delta y}{2}(\Delta x_E + \Delta x_w + \Delta x_w)}$$

$$= \frac{2}{\Delta y (1 + r_w)}$$

$$\phi_3 : -\frac{2}{1+r_w} \frac{k_s \Delta t}{\rho_m C_{pm} \Delta y^2} = -\frac{2}{1+r_w} \overline{Fo}^{s}_{my}$$

$$\phi_4 : \frac{-k_s \ \Delta y/2 \ \Delta t}{\rho_m C_{pm} \ S_m \ \Delta x_E}$$

$$\frac{\Delta y}{2 S_m} = \frac{\cancel{\Delta y}}{\frac{\cancel{\Delta y}}{2}(\Delta x_E + \Delta x_w + \Delta x_w)} = \frac{2}{\Delta x_E \left(\frac{1}{r_E} + \frac{r_w}{r_E} \right)}$$

$$= \frac{2}{\Delta x_E \left(\frac{1 + r_w}{r_E} \right)} = \frac{2 r_E}{\Delta x_E (1 + r_w)} \quad \left(\begin{array}{l} \text{note} \\ r_E + r_w = 1 \end{array} \right)$$

$$\phi_4 : -\frac{2 r_E}{1+r_w} \frac{k_s \Delta t}{\rho_m C_{pm} (\Delta x_E)^2} = -\frac{2 r_E}{1+r_w} \overline{Fo}^{s}_{mx}$$

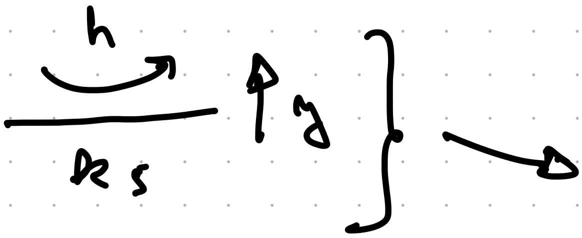
ok

$$\phi_5 : \frac{h \, \frac{\Delta x_E}{2} \, \Delta t}{\rho_M \, c_{pM} \, S_M}$$

or $$\frac{\Delta x_E}{2 S_M} = \frac{\Delta x_E}{\frac{\Delta y}{2}\left(\Delta x_E + \Delta x_w + \Delta x_w\right)}$$

$$\frac{\Delta x_E}{2 S_M} = \frac{2}{\Delta y\left(\frac{1}{r_E} + \frac{\Delta x_w}{\Delta x_E}\right)} = \frac{2}{\Delta y\left(\frac{1}{r_E} + \frac{r_w}{r_E}\right)} = \frac{2 r_E}{\Delta y (1 + r_w)}$$

$$\phi_5 : \frac{2 r_E}{1 + r_w} \frac{h \, \Delta t}{\rho_M C_{pM} \Delta y} = \frac{2 r_E}{1 + r_w} B_{iy}^{s} \, Fo_{My}^{s}$$

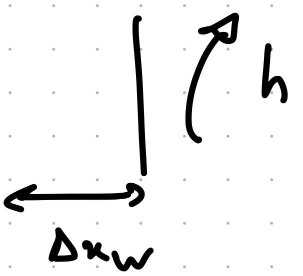


$$\boxed{\frac{h \, \Delta y}{k_s}} \qquad \boxed{\frac{k_s \, \Delta t}{\rho_M C_{pM} \Delta y^2}}$$

$$\phi_6 : \frac{h \, \frac{\Delta y}{2} \, \Delta t}{\rho_M C_{pM} S_M} \longrightarrow \frac{\Delta y}{2 S_M} = \frac{2 r_E}{\Delta x_E} = \frac{2 r_w}{(1 + r_w) \Delta x_w}$$

$$\phi_6 : \frac{2 r_w}{(1 + r_w)} \frac{h \, \Delta t}{\rho_M C_{pM} \Delta x_w} = \frac{2 r_w}{1 + r_w} B_{ix}^{w} \, Fo_{Mx}^{w}$$

$$\boxed{\frac{h \, \Delta x_w}{k_w}} \qquad \boxed{\frac{k_w}{\rho_M C_{pM} (\Delta x_w)^2}}$$



$$\phi_7 : \frac{-k_N \, \Delta x_w/2 \, \Delta t}{\rho_M c_{p_M} S_M \, \Delta y}$$

$$\frac{\Delta x_w}{2 S_M} = \frac{\Delta x_w}{\frac{\Delta y}{2}(\Delta x_E + \Delta x_w + \Delta x_w)} = \frac{2}{\Delta y\left(\frac{1}{r_w} + 1\right)}$$

$$= \frac{2 r_w}{\Delta y (1 + r_w)}$$

$$\phi_7 = -\frac{2 r_w}{1 + r_w} \frac{k^N \Delta t}{\rho_M c_{p_M} \Delta y^2} = -\frac{2 r_w}{1 + r_w} F_{o_{M y}}^{N}$$

$$\phi_1 + \phi_2 + \phi_3 - \left\{ \phi_4 + \phi_5 + \phi_6 + \phi_7 \right\} = \ldots$$

$$\ldots \rho_M c_{p_M} S_M \frac{T^{p+1} - T}{\Delta t}$$

$$-\frac{2 r_w}{1 + r_w} F_{o_{M x}}^{W} \left(T^{p+1} - T_W^{p+1}\right) - \frac{2 r_w}{1 + r_w} F_{o_{M x}}^{S} \left(T^{p+1} - T_W^{p+1}\right)$$

$$-\frac{2}{1 + r_w} F_{o_{M y}}^{S} \left(T^{p+1} - T_S^{p+1}\right) - \left\{ -\frac{2 r_E}{1 + r_w} F_{o_{M x}}^{S} \left(T_E^{p+1} - T^{p+1}\right)\right.$$

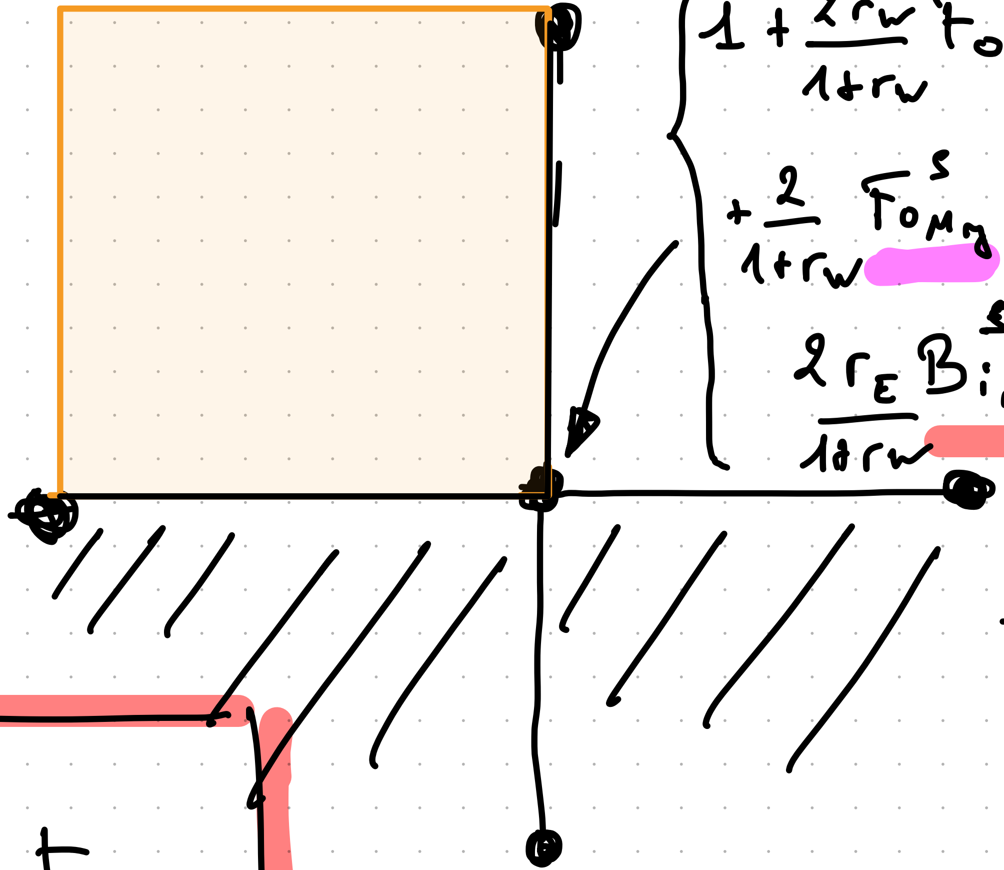
$$-\frac{2 r_E}{1 + r_w} B_{i_y}^{s} F_{o_{M y}}^{s} \left(T^{p+1} - T_\infty\right) - \frac{2 r_w}{1 + r_w} B_{i_x}^{W} F_{o_{M x}}^{W} \left(T^{p+1} - T_\infty\right)$$

$$\left. -\frac{2 r_w}{1 + r_w} F_{o_{M y}}^{N} \right\} = T^{p+1} - T$$

P_t ©

(c)

$$-\frac{2r_W}{1+r_W} Fo_{my}^N$$



$$\left\{ 1 + \frac{2r_W}{1+r_W} Fo_{my}^N + \frac{2r_W}{1+r_W}\left(Fo_{mx}^W + Fo_{mx}^S\right) + \frac{2}{1+r_W} Fo_{my}^S + \frac{2r_E}{1+r_W} Fo_{mx}^S + \frac{2r_E B_{iy}^S}{1+r_W} Fo_{my}^S + \frac{2r_W B_{ix}^W}{1+r_W} Fo_{mx}^W \right.$$

$$-\frac{2r_W}{1+r_W}\left(Fo_{mx}^W + Fo_{mx}^S\right)$$

$$-\frac{2r_E}{1+r_W} Fo_{mx}^S$$

$$-\frac{2}{1+r_W} Fo_{my}^S$$

$$\left(\frac{2r_E B_{iy}^S}{1+r_W} Fo_{my}^S + \frac{2r_W}{1+r_W} B_{ix}^W Fo_{mx}^W\right) T_\infty$$

⚠ in RHS

Check   Same Material and $\Delta x = \Delta y$

so $r_E : r_W = \frac{1}{2}$

$-\frac{2}{3} F_o$ $-\frac{2}{3} F_o$

$$\left\{ \overbrace{1 + \frac{2}{3} F_o + \frac{4 F_o}{3} + \frac{4}{3} F_o + \frac{2}{3} F_o}^{4 F_o} + \frac{2}{3} Bi F_o + \frac{2}{3} Bi F_o \right.$$

$-\frac{4}{3} F_o$ $-\frac{4}{3} F_o$ $-\frac{2}{3} F_o$ $-\frac{2}{3} F_o$

$1 + 4 F_o + \frac{4 Bi F_o}{3}$

$-4/_3 F_o$

$-4/_3 F_o$

From Eq 5.38

$$\frac{r_W}{1+r_W} = \frac{1/2}{3/2} = 1/_3$$

$$\frac{1}{1+r_W} = \frac{2}{3}$$

!! OK