

# ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

## PROJECT N°2 REPORT

RANDOMIZED NYSTRÖM

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*Authors:*

Tara FJELLMAN

Amal SEDDAS

*Professor:*

Laura GRIGORI

The EPFL logo is rendered in a bold, red, sans-serif typeface. The letters are thick and blocky, with the 'E' and 'F' featuring a distinctive stepped or 'Z' shape on their right-hand side.

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# 1 Introduction

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## 2 Algorithms

### 2.1 Sketching Matrices

### 2.2 Randomized Nyström low rank approximation

The goal of this project is to study the randomized Nyström algorithm for computing a rank-  $k$  approximation of a matrix  $A \in \mathbb{R}^{n \times n}$  that is symmetric positive semidefinite. Given a sketching matrix  $\Omega \in \mathbb{R}^{n \times l}$ , where  $l$  is the sketch dimension,  $l > k$ , the randomized Nyström approximation relies on the following formula:

$$A_{Nyst} = (A\Omega) (\Omega^T A\Omega)^+ (\Omega^T A)$$

where  $(\Omega^T A\Omega)^+$  denotes the pseudoinverse of  $\Omega^T A\Omega$ . This formula provides an approximation of  $A$  of rank at most  $l$ . Several solutions are possible for computing a rank- $k$  approximation from  $A_{Nyst}$ , that we denote as  $[A_{Nyst}]_k$ . One consists of computing a rank- $k$  decomposition of the matrix  $B = \Omega^T A\Omega$ , while another one consists of computing a rank- $k$  truncation of the Nyström approximation  $A_{Nyst}$ . The second approach is considered for this project. Different solutions are proposed in the literature, see one solution in the slides from the lecture on Randomized algorithms for low rank matrix approximation, where the algorithm relies on the Cholesky decomposition or the eigenvalue decomposition of  $B$ . Other reference is [44]. Two different data sets should be used to validate your considered randomized Nyström low rank approximation, that are described later on. Hence this project should allow you to identify a randomized algorithm that is numerically stable for the considered data sets and scales reasonably well in parallel.

### 3 Data sets

We consider two different types of data sets in this project. The first is synthetic and is a part of those described in section 5 of [4]. More precisely, we consider polynomial and exponential decay diagonal matrices with different decay rates, which we call slow, medium and fast in the rest of the report.

The second data set we use described for example in [1] (section 4). It consists of a similarity matrix build from the MNIST dataset with help of the radial basis function  $e^{-\|x_i - x_j\|^2 / c^2}$ . With the first  $n$  rows of the MNIST dataset leads to a dense  $n \times n$  matrix  $A$ . The parameter  $c$  can be varied to tune how much to penalise dissimilarity. To make the exploration of this value simpler, we decided to first normalize the MNIST dataset by dividing the entries by the maximal intensity across the dataset.

[ADRESS THIS] The dimension  $n$  should be taken depending on what your code can support in terms of memory consumption.

### 4 Stability analysis

- bullet points about comments to include

### 5 Parallelisation

Describe how you parallelize randomized Nyström and provide pseudo-code of the parallel algorithm. For the parallelization, the matrix  $A$  should be distributed among processors by using a two-dimensional block distribution while the matrix  $\Omega$  should be distributed using a block row distribution. You can consider that the number of processors is a power of 2 such that you can easily distribute the matrices among processors. For example, when  $P = \sqrt{P} \times \sqrt{P} = 9$ , the matrices  $A$  and  $\Omega$  are distributed as:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}$$
$$B = \Omega^T A \Omega = \begin{pmatrix} \Omega_1^T & \Omega_2^T & \Omega_3^T \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}$$

Compute  $C = A\Omega, C \in \mathbb{R}^{m \times I}$  Compute  $B = \Omega^T C, B \in \mathbb{R}^{I \times I}$  Compute truncated rank- $k$  eigenvalue decomposition of  $B = U \Lambda U^T$   $B_k^+ = U(:, 1:k) \Lambda_k^+ (U(:, 1:k))^T$  rank- $k$  approximation of  $B$  Compute thin QR decomposition  $C = QR$  Compute eigendecomposition  $RB_k^+ R^T = U_k \Lambda_k U_k^T$  Compute  $\hat{U}_k = QU_k$  Output factorization  $A_{Nyst,k} = \hat{U}_k \Lambda_k \hat{U}_k^T$

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**Algorithm 1** Randomized Nystrom algorithm. The syntax was adapted from this Overleaf example.

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**Require:**  $n \geq 0$

**Ensure:**  $y = x^n$

$C \leftarrow A\Omega$

$B \leftarrow \Omega^T C$

$L, Failed \leftarrow Cholesky(B)$

**if** Failed **then**

$U, \Lambda \leftarrow EigDecomp(B)$

**else**

$Z \leftarrow CL^{-T}$   $\triangleright$  Substitution : No computation of  $(L^T)^{-1}$

$Q, R \leftarrow QR(Z)$

$U_k, \Sigma_k, V_k \leftarrow TruncSVD(R)$

$\hat{U}_k \leftarrow QU_k$

$[A_{Nyst}]_k \leftarrow \hat{U}_k \Sigma_k^2 \hat{U}_k^T$

**end if**

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## 6 Experimental procedure

- averaging to increase precision of results - when running in parallel, was limited by number of cores  $p = 2^{2s}$  with  $s \in \mathbb{N}$  (due to perfect square constraint of the parallelised matrix multiplication and the SHRT algorithm using the Hadamard transform), so we ran results for 1,4,16,64 - ran results on the Helvetios cluster -  $k$  was taken often around  $n/20$  as we deemed this value relevant for real applications -  $n$  was made vary as powers of 2 due to the Hadamard transformation while  $l$  was taken logarithmically spaced just to make the visuals more easy to interpret

## 7 Algorithm Performance

### 7.1 Sequential Performance

- in general (both plots) observe that  $k$  rank approximation represents minority of runtime (as expected as of order  $k^3 + nk^2$ ). Indeed for shown examples represents

at most 10%. This would be different if one were to take big values of  $k$ . Also does not really seem to depend on sketching method which makes sense given the computations are the same regardless of which sketching method was used to obtain the B and C matrices

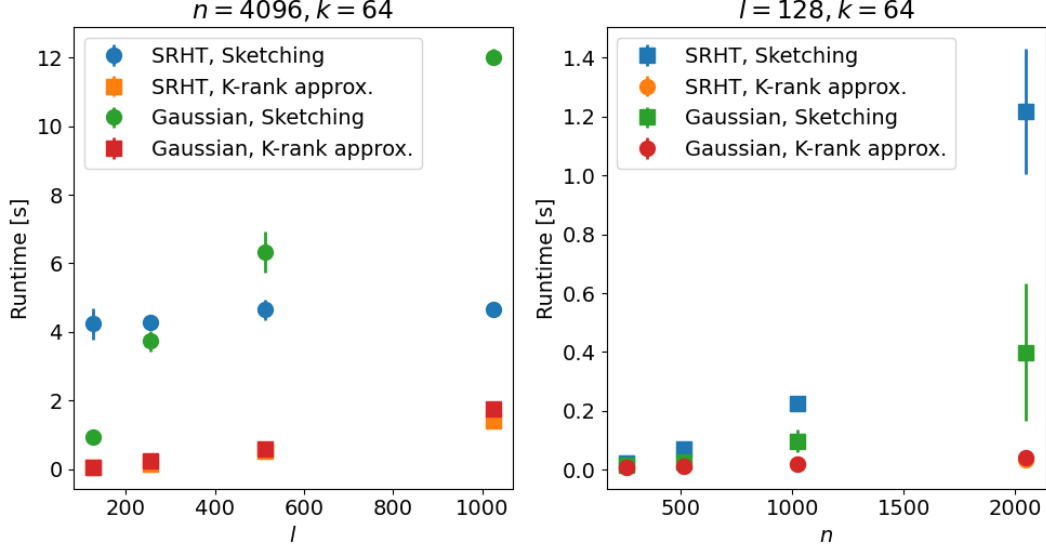


Figure 1: Runtimes associated to the tested sketching methods : as a function of  $l$  (left) and as a function of  $n$  (right). Runtimes are broken down in that associated to the computation of  $A\Omega, \Omega^T A\Omega$  and that associated to the computation of the  $k$  rank approximation.

- different behaviours as a function of  $l$  and  $n$  : as a function of  $l$  SRHT looks almost flat, having really slow increase, while gaussian increases steadily given its complexity is of order  $nl^2 + ln^2$ . This gives advantage for SRHT if we are embedding "dense spaces" as it scales much better than gaussian as function of  $l$ . For  $n$  : both algos show significant increase, but SRHT shows faster runtime growth. This makes sense due to extra  $\log_2(n)$  term in its complexity. This would give advantage to gaussian sketching if we are dealing with sparse information within really high dimensional spaces. -

## 7.2 Parallel Performance

- selected  $l = n/64$  given we are using TSQR for the QR factorisation and we need the matrix to be tall-skinny to actually observe a speed-up. To have a good speed-up for a bigger  $l$ , while keeping numerical stability for ill-conditioned matrices another algo should be used.

- for small example : some speed-up is observed as a function of number of

cores but communication costs are non neglectable and even dominate the total runtime for  $p=64$ . This is expected as the computation is rather small to be run in parallel. The runtimes for the  $k$ -rank are the most affected as these represented the smallest part of the computation to start with. for the specific values selected the gaussian sketching is faster (indeed  $l$  is taken only as a small fraction of  $n$ ).

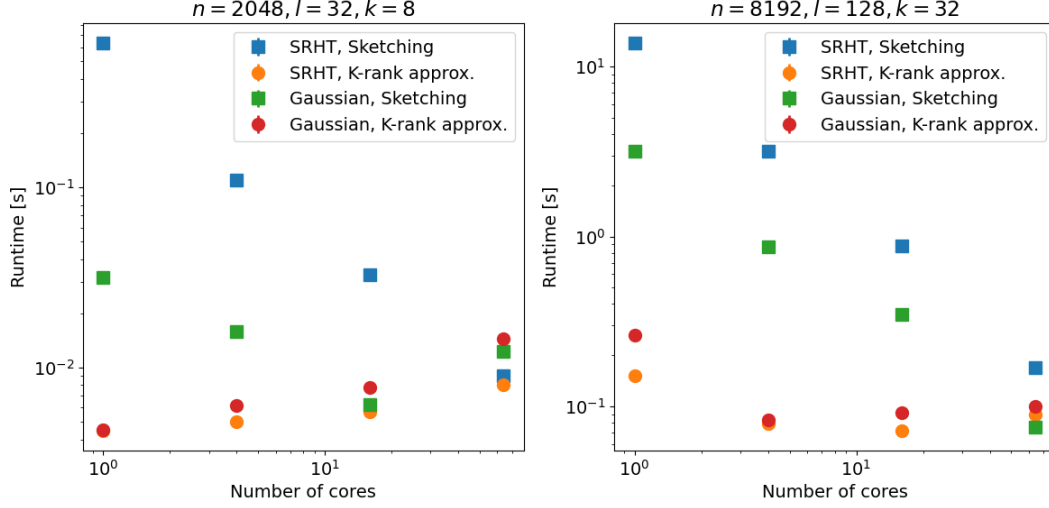


Figure 2: Runtimes associated to the tested sketching methods as a function of the number of cores : for a “small” example (left) and a “large” one (right). Runtimes are broken down in that associated to the computation of  $A\Omega, \Omega^T A\Omega$  and that associated to the computation of the  $k$  rank approximation.

- for the large : communication is not much of an issue, and big speed ups (quasi linear) are observed for the sketching computation. This means the parallelization is successful and scales well for big computations. We can see the speedup is not as good for  $k$ -rank partly because the computation was smaller to start with. The fact that the matrix is not that tall-skinny is also expected to have contributed to the low speed-up of that part. -

## 8 Conclusion

## Aknowledegments

## References