Econ 5713 Assignment One

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Question one

0.1 Q1-a

We have a simple CAPM regression model as following

$$Y_t = \alpha_0 + \beta_0 X_t + \epsilon_t, \tag{1}$$

base on the figure one:

$$Y_t = \text{IBM Risk-Adjusted Return}, \quad X_t = \text{Market Risk-Adjusted Return}.$$

- **Dependent Variable:** The variable we are trying to explain, it is IBM_RISK_ADJ_RETURN
- **Method (Least Squares):** the method we using which is ordernery least squared
- **Sample / Included Observations:** The sample size which is 132 month here.
- **HAC Standard Errors & Covariance:** The method that use to adjust for possible autocorrelation and heteroskedasticity in the residuals. Due the poossible OLS cannot processing time series data propoly.
- **Coefficients (C(1) and C(2)):**
  - $\hat{\alpha}_0$ (labeled as C(1)) intercept.
  - $\hat{\beta}_0$ (labeled as C(2)) is the estimated slope of market risk-adjusted return.
- **Standard Error:** The standard error which measure the statistical uncertainty.
- **t-statistic:** The ratio of each coefficient to its standard error, where $\hat{\theta}$ is the estimated parameter (either $\hat{\alpha}_0$ or $\hat{\beta}_0$).
- **p-value:** describing the likelihood of obtaining the observed data under the null hypothesis of a statistical test.
- **R-squared:** How good the data fit the model
- **Adjusted R-squared:** improved version of R^2 adjusted for the number of regressors relative to the sample size.
- **Standard Error of Regression (SER):** The standard deviation of residuals.
- **Sum of Squared Residuals (SSR):** The sum of squared residuals.
- **Log Likelihood:** The value of the log-likelihood function for the fitted model.
- **Akaike Information Criterion (AIC) and Schwarz Criterion (BIC):** how good the variable is in the model, it will punish the variable that is not important.
- **F-statistic:** Tests of significance of the regression.
- **Prob(F-statistic):** The p-value of F-test.
- **Durbin-Watson Stat:** A test for autocorrelation in the residuals.

Q1-b

Recall the ols model:

$$y_t = \alpha_0 + \beta_0 x_t + \epsilon_t, \quad t = 1, 2, \ldots, n, \tag{2}$$

With the assumption

$$\epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2). \tag{3}$$

Otherwise you cannot do confdience interval
Step one: Drive the expression for the α_0 and β_0

$$\text{RSS} = \sum_{t=1}^{n} (y_t - \alpha_0 - \beta_0 x_t)^2 . \tag{4}$$

$$\hat{\beta}_0 = \frac{\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \tag{5}$$

We can rewrite it as following:

$$\frac{S_{xy}}{S_{xx}} \tag{6}$$

$$S_{xy} = \sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y}) \tag{7}$$

$$S_{xx} = \sum_{t=1}^{n}(x_t - \bar{x})^2. \tag{8}$$

For the intercept

$$\hat{\alpha}_0 = \bar{y} - \hat{\beta}_0 \bar{x}. \tag{9}$$

step two: apply the law or the large number (LLN) and Central Limit Theorem (CLT)
because we assume the model is stationary and $\epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$, we will get the finite variance.
by law the large number (LLN)
The sample mean $\bar{x}$ converges to $E(x_t)$. scaled sum of squares converges to the variance of the x_t

$$\frac{S_{xx}}{n} = \frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2 \xrightarrow{p} \text{Var}(x_t). \tag{10}$$

Therefore we can say we have a consistency of the estimatiors.
simplifiy:

$$S_{xy} = \sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y}) \tag{11}$$

$$y_t = \alpha_0 + \beta_0 x_t + \epsilon_t \tag{12}$$

$$\bar{y} = \alpha_0 + \beta_0 \bar{x}. \tag{13}$$

$$y_t - \bar{y} = (\alpha_0 + \beta_0 x_t + \epsilon_t) - (\alpha_0 + \beta_0 \bar{x}). \tag{14}$$

$$y_t - \bar{y} = \beta_0 (x_t - \bar{x}) + \epsilon_t. \tag{15}$$

$$S_{xy} = \sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t. \tag{16}$$

Recall the CLT

$$\sqrt{T}(\hat{\beta}_0 - \beta_0) \approx \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{S_{xx}/n} \xrightarrow{d} N\left(0, \frac{\sigma^2}{\mathrm{Var}(x_t)}\right). \tag{17}$$

$$\frac{S_{xx}}{n} = \frac{\sum_{t=1}^{n}(x_t - \bar{x})^2}{n} \tag{18}$$

$$\mathbb{E}[X^2] = \frac{\sum_{t=1}^{n}(x_t - \bar{x})^2}{n} \tag{19}$$

$$\sqrt{T}(\hat{\beta}_0 - \beta_0) \approx \frac{1}{\sqrt{T}} \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]} \xrightarrow{d} N\left(0, \frac{\sigma^2}{\mathrm{Var}(x_t)}\right). \tag{20}$$

Then the distribution for the β_0 is

$$\frac{\sqrt{T}(\hat{\beta}_0 - \beta_0)}{\frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]}} \tag{21}$$

The 95 confidence interval is:

$$-1.96 \leqslant \frac{\sqrt{T}(\hat{\beta}_0 - \beta_0)}{\frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]}} \leqslant 1.96 \tag{22}$$

$$\frac{-1.96}{\sqrt{T}} \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]} \leqslant (\hat{\beta}_0 - \beta_0) \leqslant \frac{1.96}{\sqrt{T}} \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]} \tag{23}$$

$$\hat{\beta}_0 - \frac{1.96}{\sqrt{T}} \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]} \leqslant \beta_0 \leqslant \hat{\beta}_0 + \frac{1.96}{\sqrt{T}} \frac{\sum_{t=1}^{n}(x_t - \bar{x})\epsilon_t}{\mathbb{E}[X^2]} \tag{24}$$

0.2 Q1-c

$$\hat{\alpha}_0 = 0.005851, \quad \text{Std. Error}(\hat{\alpha}_0) = 0.006054,$$
$$\hat{\beta}_0 = 1.188280, \quad \text{Std. Error}(\hat{\beta}_0) = 0.146546.$$

The sample size is $n = 132$. For a 95% confidence interval, we use:

$$t_{0.025,\,130} \approx 1.978$$

Confidence Interval for α_0

$$\begin{aligned}
\hat{\alpha}_0 \pm t_{0.025,130} \times \text{Std. Error}(\hat{\alpha}_0) &= 0.005851 \pm 1.978 \times 0.006054 \\
&= (0.005851 - 0.01197,\ 0.005851 + 0.01197) \\
&= (-0.0061,\ 0.0178).
\end{aligned}$$

Confidence Interval for β_0

$$\begin{aligned}
\hat{\beta}_0 \pm t_{0.025,130} \times \text{Std. Error}(\hat{\beta}_0) &= 1.188280 \pm 1.978 \times 0.146546 \\
&= (1.188280 - 0.2899,\ 1.188280 + 0.2899) \\
&= (0.8984,\ 1.4782).
\end{aligned}$$

Question Two

0.3 Q2-a

Recall the CLT Condition:

$$\bar{Z} = \frac{1}{T}\sum_{t=1}^{T} Z_t. \tag{25}$$

$$\sqrt{T}\bar{Z} \xrightarrow{d} \mathcal{N}\left(0, \mathrm{Var}(\sqrt{T}\bar{Z})\right), \quad \text{as } T \to \infty. \tag{26}$$

Therefore, for CLT to hold, the long-run variance:

$$\mathrm{Var}\left(\sqrt{T}\bar{Z}\right) = \mathbb{E}\left[\left(\sqrt{T}\bar{Z}\right)^2\right] \tag{27}$$

must be finite.
Then, we expanding the variance

$$\begin{aligned}
\mathrm{Var}\left(\sqrt{T}\bar{Z}\right) &= \mathbb{E}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t\right)^2\right] && (28)\\
&= \frac{1}{T}\mathbb{E}\left[\left(\sum_{t=1}^{T} Z_t\right)^2\right] && (29)\\
&= \frac{1}{T}\left[T\mathbb{E}[Z_0^2] + 2\sum_{m=1}^{T-1}(T-m)\mathbb{E}[Z_0 Z_m]\right] && (30)\\
&= \mathbb{E}[Z_0^2] + 2\sum_{m=1}^{T-1}\left(1 - \frac{m}{T}\right)\mathbb{E}[Z_0 Z_m]. && (31)
\end{aligned}$$

The second summation term,

$$\sum_{m=1}^{T-1}\left(1 - \frac{m}{T}\right)\mathbb{E}[Z_0 Z_m], \tag{32}$$

is Bartlett's kernel.
Recall, the AR(1) equation in the quesiton

$$Y_t = \alpha_0 + \beta Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d.}(0, \sigma^2). \tag{33}$$

assume β is a stationary and ergioic process
For this process, the autocovariance function is:

$$\mathbb{E}[Z_0 Z_m] = \gamma_m = \frac{\sigma^2}{1-\beta^2}\beta^m. \tag{34}$$

Substituting this into Bartlett's kernel:

$$\sum_{m=1}^{T-1}\left(1 - \frac{m}{T}\right)\frac{\sigma^2}{1-\beta^2}\beta^m. \tag{35}$$

Therefore β need to $|\beta| < 1$ to be stationary that the variance will not explore.
$|\beta| < 1$ (Stationary)

- The term β^m decays exponentially.

- The infinite sum $\sum_{m=1}^{\infty} \beta^m$ converges to $\frac{\beta}{1-\beta}$.
- Bartlett's kernel sum **remains finite**, meaning $\text{Var}(\sqrt{T}\bar{Z})$ is well-defined.
- Hence, CLT holds.

0.4 Q2-b

Question Three

0.5 Q3-a

Recall the equation for the AR(2) are:

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \epsilon_t \tag{36}$$

The first Yule-walker equation

$$E[X_t X_{t-1}] = a_1 E[X_{t-1}^2] + a_2 E[X_{t-1} X_{t-2}] + E[X_{t-1}\epsilon_t] \tag{37}$$

Since $E[X_{t-1}\epsilon_t] = 0$, we get:

$$\gamma(1) = a_1 \text{Var}(X) + a_2 \gamma(1) \tag{38}$$

$$\gamma(1) = \frac{a_1}{1 - a_2} \text{Var}(X) \tag{39}$$

The second Yule-walker equation

$$E[X_t X_{t-2}] = a_1 E[X_{t-1} X_{t-2}] + a_2 E[X_{t-2}^2] + E[X_{t-2}\epsilon_t] \tag{40}$$

Since $E[X_{t-2}\epsilon_t] = 0$, we get:

$$\gamma(2) = a_1 \gamma(1) + a_2 \text{Var}(X) \tag{41}$$

The third Yule-walker equation

$$E[X_t X_{t-3}] = a_1 E[X_{t-1} X_{t-3}] + a_2 E[X_{t-2} X_{t-3}] + E[X_{t-3}\epsilon_t] \tag{42}$$

Since $E[X_{t-3}\epsilon_t] = 0$, we get:

$$\gamma(3) = a_1 \gamma(2) + a_2 \gamma(1) \tag{43}$$

0.6 Q3-b

Recall the equation for the AR(5) are:

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + a_3 X_{t-3} + a_4 X_{t-4} + a_5 X_{t-5} + \epsilon_t \tag{44}$$

The first Yule-Walker equation

$$E[X_t X_{t-1}] = a_1 E[X_{t-1}^2] + a_2 E[X_{t-1} X_{t-2}] + a_3 E[X_{t-1} X_{t-3}] + a_4 E[X_{t-1} X_{t-4}] + a_5 E[X_{t-1} X_{t-5}] + E[X_{t-1}\epsilon_t] \tag{45}$$

Since $E[X_{t-1}\epsilon_t] = 0$, we obtain:

$$\gamma(1) = a_1 \text{Var}(X) + a_2 \gamma(1) + a_3 \gamma(2) + a_4 \gamma(3) + a_5 \gamma(4) \tag{46}$$

The Second Yule-Walker equation

$$E[X_tX_{t-2}] = a_1E[X_{t-1}X_{t-2}] + a_2E[X_{t-2}^2] + a_3E[X_{t-2}X_{t-3}] + a_4E[X_{t-2}X_{t-4}] + a_5E[X_{t-2}X_{t-5}] + E[X_{t-2}\epsilon_t] \tag{47}$$

Since $E[X_{t-2}\epsilon_t] = 0$, we obtain:

$$\gamma(2) = a_1\gamma(1) + a_2\mathrm{Var}(X) + a_3\gamma(1) + a_4\gamma(2) + a_5\gamma(3) \tag{48}$$

The Third Yule-Walker equation

$$E[X_tX_{t-3}] = a_1E[X_{t-1}X_{t-3}] + a_2E[X_{t-2}X_{t-3}] + a_3E[X_{t-3}^2] + a_4E[X_{t-3}X_{t-4}] + a_5E[X_{t-3}X_{t-5}] + E[X_{t-3}\epsilon_t] \tag{49}$$

Since $E[X_{t-3}\epsilon_t] = 0$, we obtain:

$$\gamma(3) = a_1\gamma(2) + a_2\gamma(1) + a_3\mathrm{Var}(X) + a_4\gamma(1) + a_5\gamma(2) \tag{50}$$

The Fourth Yule-Walker equation

$$E[X_tX_{t-4}] = a_1E[X_{t-1}X_{t-4}] + a_2E[X_{t-2}X_{t-4}] + a_3E[X_{t-3}X_{t-4}] + a_4E[X_{t-4}^2] + a_5E[X_{t-4}X_{t-5}] + E[X_{t-4}\epsilon_t] \tag{51}$$

Since $E[X_{t-4}\epsilon_t] = 0$, we obtain:

$$\gamma(4) = a_1\gamma(3) + a_2\gamma(2) + a_3\gamma(1) + a_4\mathrm{Var}(X) + a_5\gamma(1) \tag{52}$$

The Fifth Yule-Walker equation

$$E[X_tX_{t-5}] = a_1E[X_{t-1}X_{t-5}] + a_2E[X_{t-2}X_{t-5}] + a_3E[X_{t-3}X_{t-5}] + a_4E[X_{t-4}X_{t-5}] + a_5E[X_{t-5}^2] + E[X_{t-5}\epsilon_t] \tag{53}$$

Since $E[X_{t-5}\epsilon_t] = 0$, we obtain:

$$\gamma(5) = a_1\gamma(4) + a_2\gamma(3) + a_3\gamma(2) + a_4\gamma(1) + a_5\mathrm{Var}(X) \tag{54}$$

Question Four

0.7 Q4-a

Question Five

0.8 Q5-a

recall the FCLT

$$\frac{S_{\lfloor T\tau \rfloor}}{\sqrt{T}} \xrightarrow{d} \sigma_\epsilon B(\tau), \tag{55}$$

For large T, we approximate:

$$S_{1,t} \approx \sigma_\epsilon \sqrt{T} B\left(\frac{t}{T}\right). \tag{56}$$

Substituting the approximation into the sum

$$\sum_{t=1}^{T} S_{1,t}^2 \approx \sigma_\epsilon^2 T \sum_{t=1}^{T} B^2\left(\frac{t}{T}\right). \tag{57}$$

As $T \to \infty$, the Riemann sum $\sum_{t=1}^{T} B^2\left(\frac{t}{T}\right)$ converges to the integral $\int_0^1 B^2(\tau)\,d\tau$.

$$\int_0^1 B^2(\tau)\,d\tau \sim \frac{1}{3}\chi^2(1), \tag{58}$$

where $\chi^2(1)$ denotes a chi-squared distribution with 1 degree of freedom.

$$\sum_{t=1}^{T} S_{1,t}^2 \approx \sigma_\epsilon^2 T \cdot \frac{1}{3}\chi^2(1). \tag{59}$$

The sampling distribution of $\sum_{t=1}^{T} S_{1,t}^2$ is:

$$\sum_{t=1}^{T} S_{1,t}^2 \sim \frac{\sigma_\epsilon^2 T}{3}\chi^2(1). \tag{60}$$

0.9 Q5-b

Recall the FCLT

$$\frac{S_{\lfloor T\tau \rfloor}}{\sqrt{T}} \xrightarrow{d} \sigma_\epsilon B(\tau), \tag{61}$$

For large T, approximate:

$$S_{1,t} \approx \sigma_\epsilon \sqrt{T}\, B\left(\frac{t}{T}\right). \tag{62}$$

Substitute the approximation of $S_{1,t}$:

$$\sum_{t=1}^{T} S_{1,t}\eta_t \approx \sigma_\epsilon \sqrt{T} \sum_{t=1}^{T} B\left(\frac{t}{T}\right)\eta_t. \tag{63}$$

Since $B\left(\frac{t}{T}\right)$ varies slowly over t, approximate it as constant over small intervals:

$$\sum_{t=1}^{T} B\left(\frac{t}{T}\right)\eta_t \approx \sigma_\eta \int_0^1 B(\tau)\,dW(\tau), \tag{64}$$

where $W(\tau)$ is another Brownian motion representing the cumulative effect of η_t.

It is known that

$$\int_0^1 B(\tau)\,dW(\tau) \sim \mathcal{N}\left(0, \int_0^1 \mathbb{E}[B(\tau)^2]\,d\tau\right). \tag{65}$$

Since $\mathbb{E}[B(\tau)^2] = \tau$:

$$\int_0^1 \tau\,d\tau = \frac{1}{2}. \tag{66}$$

Thus,

$$\int_0^1 B(\tau)\,dW(\tau) \sim \mathcal{N}\left(0, \frac{1}{2}\right). \tag{67}$$

Combine the results:

$$\sum_{t=1}^{T} S_{1,t}\eta_t \approx \sigma_\epsilon \sqrt{T} \cdot \sigma_\eta \mathcal{N}\left(0, \frac{1}{2}\right). \tag{68}$$

Therefore, we conclude:

$$\sum_{t=1}^{T} S_{1,t}\eta_t \sim \mathcal{N}\left(0, \frac{\sigma_\epsilon^2 \sigma_\eta^2 T}{2}\right). \tag{69}$$