Supplementary Calculus Notes

Econ 3020 - Winter 2026

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Disclaimer

Important Note

This document is intended as a **supplement** to, not a substitute for, the official lectures

- This is **unofficial reference material**. Please always refer to the Professor's lectures and slides for definitive course content and notation.
- **Rule of Thumb:** In the event of any discrepancy between these notes and the Professor's materials, **the Professor's materials take precedence.**

1 Basic Derivatives

Consider the function where $x \in \mathbb{R}$:

$$f(x) = x^2 \tag{1}$$

First Derivative (f')

Using the Power Rule:

$$\frac{df}{dx} = 2x^{2-1}$$
$$\frac{df}{dx} = 2x^{1}$$
$$\frac{df}{dx} = 2x$$

Or simply: $f'(x) = 2x$.

Second Derivative (f'')

We differentiate the first derivative with respect to x:

$$\begin{aligned}\frac{d}{dx}\left(\frac{df}{dx}\right) &= \frac{d}{dx}(2x) \\ &= (2)(1)x^{1-1} \\ \frac{d^2 f}{dx^2} &= 2x^0 \\ \frac{d^2 f}{dx^2} &= 2\end{aligned}$$

Third Derivative (f''')

We differentiate the second derivative with respect to x:

$$\begin{aligned}\frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) &= \frac{d}{dx}(2) \\ \frac{d^3 f}{dx^3} &= 0\end{aligned}$$

(The derivative of a constant is always 0).

Example 2: Constant Multiples

Consider the function where $x \in \mathbb{R}$:

$$f(x) = 2x^2 \tag{2}$$

Derivatives:

$$\frac{df}{dx} = 4x$$
$$\frac{d^2 f}{dx^2} = 4$$
$$\frac{d^3 f}{dx^3} = 0$$

Example 3: Additive Rule

Consider the function where $x \in \mathbb{R}$:

$$f(x) = 2x^2 + x^3 \tag{3}$$

Apply the power rule to each term separately:

$$f'(x) = 2 \cdot (2x^{2-1}) + 3x^{3-1}$$
$$f'(x) = 4x^1 + 3x^2$$

Example 4: Constants

Consider the function below.
The derivative of a constant is 0 because it does not change.

$$f(x) = x^2 + 3 \tag{4}$$

$$\frac{df}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3)$$
$$\frac{df}{dx} = 2x^{2-1} + 0$$
$$\frac{df}{dx} = 2x$$

Example 5: Chain Rule

Consider the function:

$$f(x) = (x^2 + 3)^3 \tag{5}$$

The rule is: *differentiate the outside first, then the inside.*

$$
\begin{aligned}
f'(x) &= 3(x^2+3)^{3-1} \cdot \frac{d}{dx}(x^2+3) \\
f'(x) &= 3(x^2+3)^2 \cdot (2x) \\
f'(x) &= 6x(x^2+3)^2
\end{aligned}
$$

Addition Note

It is helpful to distinguish between the **result** and the **operation**:

- $\frac{df}{dx}$ **(The Result):** This symbol represents the derivative of the function f. It is a noun. It refers to the final answer.
- $\frac{d}{dx}$ **(The Operator):** This symbol is an instruction or command. It represents the **action** of taking the derivative with respect to x. It tells you to differentiate whatever comes immediately after it.

Example:

$$\frac{d}{dx}(x^2) = 2x \tag{6}$$

In this equation:

- $\frac{d}{dx}$ is the instruction: "Differentiate x^2".
- $2x$ is the result ($\frac{df}{dx}$).

Example 6: Product Rule

Let $x \in \mathbb{R}_{++}$.

The rule is **Derivative of the first times the second, PLUS derivative of the second times the first.**

$$(uv)' = u'v + uv'$$

First Derivative (f')

Using the Product Rule:

$$\begin{aligned}
f(x) &= x^2(x^2+3)^3 \\
f'(x) &= \frac{d}{dx}(x^2)\cdot(x^2+3)^3 + x^2\cdot\frac{d}{dx}\left((x^2+3)^3\right) \\
&= 2x(x^2+3)^3 + x^2\cdot 3(x^2+3)^2\cdot(2x) \\
&= 2x(x^2+3)^3 + 6x^3(x^2+3)^2 \\
&= 2x(x^2+3)^2\left[(x^2+3)+3x^2\right] \\
&= 2x(x^2+3)^2(4x^2+3)
\end{aligned}$$

Example 7: Negative Exponents

Let $x \in \mathbb{R}_{++}$.

- $\mathbb{R}_{++}$ means strictly positive numbers (excluding 0).
- $\mathbb{R}_{+}$ means positive numbers (including 0).

Consider:

$$f(x) = \frac{1}{x} = x^{-1} \tag{7}$$

Apply the power rule:

$$\begin{aligned}
f'(x) &= -1\cdot x^{-1-1} \\
f'(x) &= -1\cdot x^{-2} \\
f'(x) &= -\frac{1}{x^2}
\end{aligned}$$

Practice Question

Find $\frac{df}{dx}$ for:

$$f(x) = \frac{1}{(x^2+3)^3} = (x^2+3)^{-3} \tag{8}$$

Solution:

$$\begin{aligned}
f'(x) &= (-3)(x^2+3)^{-3-1}\cdot\frac{d}{dx}(x^2+3) \\
f'(x) &= (-3)(x^2+3)^{-4}\cdot(2x) \\
f'(x) &= -6x(x^2+3)^{-4}
\end{aligned}$$

To write this as a fraction, the negative exponent moves to the denominator and becomes positive:

$$f'(x) = \frac{-6x}{(x^2+3)^4} \tag{9}$$

Example 8: Quotient Rule

Let $x \in \mathbb{R}_{++}$.

$$f(x) = \frac{x^2}{2x+1} \tag{10}$$

The Quotient Rule is: "Low d-High minus High d-Low, over Low squared":

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Applying the rule:

$$\begin{aligned} f'(x) &= \frac{(2x)(2x+1) - (x^2)(2)}{(2x+1)^2} \\ &= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} \\ &= \frac{2x^2 + 2x}{(2x+1)^2} \end{aligned}$$

2 Two Variables and Partial Derivatives

Example 9: Cobb-Douglas Function

Consider a standard Cobb-Douglas function (Multiplicative):

$$f(x,y) = x^\alpha y^\beta \tag{11}$$

Find $\frac{\partial f}{\partial x}$: We treat y as a constant.

$$\frac{\partial f}{\partial x} = \alpha x^{\alpha-1} \cdot y^\beta$$

Example 10: Additive Function

Consider the function where $x, y \in \mathbb{R}$:

$$f(x, y) = x^2 + y^2 \tag{12}$$

Find $\frac{\partial f}{\partial x}$: We treat y as a constant. Since y is constant, y^2 is also a constant, so its derivative is 0.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2)$$
$$\frac{\partial f}{\partial x} = 2x + 0$$
$$\frac{\partial f}{\partial x} = 2x$$

The rest part is the same, so I will not cover.

Example 11: Exponents and Logs

Let $x \in \mathbb{R}_{++}$ (so that $\ln(x)$ is defined).

$$f(x) = e^x \tag{13}$$

find $f'(x)$

$$\frac{d}{dx}(a^x) = a^x \cdot \ln(a) \tag{14}$$

then

$$\frac{d}{dx}(e^x) = e^x \cdot \ln(e) \tag{15}$$

$$\frac{d}{dx}(e^x) = e^x \cdot 1 \tag{16}$$

for the e the difference thing we will use the chain rule

$$f(x) = e^{3x} \tag{17}$$

$$\frac{d}{dx}(e^{3x}) = e^{3x} \cdot \frac{d}{dx}3x \text{ (chain rule)} \tag{18}$$

$$\frac{d}{dx}(e^{3x}) = e^{3x} \cdot 3 \tag{19}$$

$$\frac{d}{dx}(e^{3x}) = 3e^{3x} \tag{20}$$

$$f(x) = \ln(x) \quad \Rightarrow \quad f'(x) = \frac{1}{x} \tag{21}$$

$$f'(x) = \frac{1}{x} \tag{22}$$

$$f(x) = \ln(x^2) \tag{23}$$

Method 1

$$\begin{aligned} f'(x) &= \frac{1}{x^2} \cdot 2x \\ &= f'(x) = \frac{2}{x} \end{aligned}$$

Method 2

$$\begin{aligned} f(x) &= \ln(x^2) \\ &= f(x) = 2 \cdot \ln(x) \\ &= f'(x) = 2 \cdot \frac{1}{x} \\ &= f'(x) = \frac{2}{x} \end{aligned}$$

Consider the function:

$$f(x) = \ln\left[(x^3 + 4)^2\right] \tag{24}$$

find $f'(x)$

Method 1

$$\begin{aligned} f'(x) &= \frac{1}{(x^3+4)^2} \cdot \frac{d}{dx}((x^3+4)^2) \\ &= \frac{1}{(x^3+4)^2} \cdot \left[2(x^3+4)^1 \cdot 3x^2\right] \\ &= \frac{6x^2(x^3+4)}{(x^3+4)^2} \\ &= \frac{6x^2}{x^3+4} \end{aligned}$$

Method 2

First, bring the exponent down using the rule $\ln(a^b) = b \ln a$:

$$f(x) = 2\ln(x^3+4) \tag{25}$$

$$\begin{aligned} f'(x) &= 2 \cdot \frac{d}{dx}(\ln(x^3+4)) \\ &= 2 \cdot \frac{1}{x^3+4} \cdot (3x^2) \\ &= \frac{6x^2}{x^3+4} \end{aligned}$$

Example 12: Total Differential

Mathematical Assumption

For the Total Differential to exist and for the linear approximation to work, we assume the function $f(x, y)$ is **"Smooth"**. Mathematically, this means:

1. **Continuity:** The function has no holes or jumps.
2. **Continuous Partial Derivatives:** The rate of change does not jump suddenly. The surface has no sharp corners or kinks.

Now, let's start with the Cobb-Douglas utility function:

$$U(x, y) = x^{1-a}y^a \tag{26}$$

where $a \in (0, 1)$.

The formula for the **Total Differential** is:

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \tag{27}$$

Step 1: Find the Marginal Utility of x ($\frac{\partial U}{\partial x}$) Treat y as a constant. Apply the Power Rule to x^{1-a}:

$$\begin{aligned} \frac{\partial U}{\partial x} &= (1-a)x^{(1-a)-1} \cdot y^a \\ &= (1-a)x^{-a}y^a \end{aligned}$$

This represents the rate at which utility increases as we add more of good x, holding good y constant.

Step 2: Find the Marginal Utility of y ($\frac{\partial U}{\partial y}$) Treat x as a constant. Apply the Power Rule to y^a:

$$\begin{aligned} \frac{\partial U}{\partial y} &= x^{1-a} \cdot ay^{a-1} \\ &= ax^{1-a}y^{a-1} \end{aligned}$$

Similarly, this is the rate of utility change from an increase in good y, holding good x constant.

Step 3: Combine into the Total Differential

$$dU = \left[(1-a)x^{-a}y^a\right] dx + \left[ax^{1-a}y^{a-1}\right] dy \tag{28}$$

This equation sums up the effects. It tells us the total change in utility when the quantities of both x and y change at the same time.

Indifference Curve and MRS

By the definition of an indifference curve, the consumer is **indifferent** between bundles along the curve. Mathematically, this implies utility is constant, so $dU = 0$.

$$0 = MU_x dx + MU_y dy \tag{29}$$

Rearranging terms to put dx and dy on opposite sides:

$$-MU_y dy = MU_x dx \tag{30}$$

To find the slope of the indifference curve ($\frac{dy}{dx}$), we divide both sides by dx and MU_y:

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y} \tag{31}$$

Assumption: Strict Monotonicity

Theoretically, the axiom of **Local Non-Satiation (LNS)** ensures there is always a direction to make the consumer better off. However, for simplicity, we assume **Strict Monotonicity** ("More is Better").

This ensures:

1. Marginal Utilities are strictly positive: $MU_x > 0$ and $MU_y > 0$.
2. The solution lies **on** the budget line.

Marginal Rate of Substitution (MRS)

The **Marginal Rate of Substitution** is defined as the absolute value of the slope of the indifference curve:

$$MRS_{xy} = \left|\frac{dy}{dx}\right| = \frac{MU_x}{MU_y} \tag{32}$$

Optimization will be covered in the file `addition_calculus_2`. (If you cannot find it, I probably haven't finished it yet).

Postscript

I have always wanted to write a set of supplementary notes, both to review the math content for myself and to provide extra help for students who need it. However, I could never seem to find the opportunity to sit down and start writing.

The first draft of these notes was finally finished on a train during my 2025 Winter Break. Thanks to the poor internet connection on board, I was able to disconnect and focus entirely on this work.