Amazing Proofs: Episode 10

Aim:

In this episode of "Amazing Proofs", we are going to derive Rogers-Ramanujan identities.

Statement:

If $|q| < 1$, then,

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

and

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

Materials Required:

Basics of q-series, Series and product manipulation, Jacobi Triple Product Identity (JTPI proved in episode 4)

Prerequisites:

Let us define a few functions first,

$$P_r = \prod_{s=1}^{r} \frac{1}{1-q^s} = \frac{1}{(q;q)_r}, \quad Q_r(a) = \prod_{s=r}^{\infty} \frac{1}{1-aq^s}, \quad \lambda(r) = \frac{r(5r+1)}{2}$$

Now let us define an operator η as,

$$\eta f(a) = f(aq)$$

Finally, let us define a simple sum,

$$H_m(a) = \sum_{r=0}^{\infty} (-1)^r a^{2r} q^{\lambda(r)-mr} (1 - a^m q^{2mr}) P_r Q_r$$

where, $P_0 = 1$, $|a| < 1$, $r \in \mathbb{N} \cup \{0\}$ and $m = 0, 1, 2$.

Procedure:

Let us start by first observing that,

$$H_m - H_{m-1} = \sum_{r=0}^{\infty} (-1)^r a^{2r} q^{\lambda(r)} P_r Q_r C_{mr} \tag{1}$$

where,

$$C_{mr} = \frac{1 - a^m q^{2mr}}{q^{mr}} - \frac{1 - a^{m-1} q^{2(m-1)r}}{q^{(m-1)r}}$$

which can be simplified to,

$$C_{mr} = a^{m-1} q^{r(m-1)} (1 - aq^r) + q^{-mr} (1 - q^r)$$

observe again that,

$$(1 - aq^r) Q_r(a) = Q_{r+1}(a) = Q_{r+1}$$

and

$$(1 - q^r) P_r = P_{r-1}$$

thus, (1) becomes,

$$H_m - H_{m-1} = \sum_{r=0}^{\infty} (-1)^r a^{2r} q^{\lambda(r)} (a^{m-1} q^{r(m-1)} P_r Q_{r+1} + q^{-mr} P_{r-1} Q_r)$$

since, $P_0 = 1$, we have by definition, $P_{-1} = 0$, therefore,

$$H_m - H_{m-1} = \sum_{r=0}^{\infty} (-1)^r a^{2r+m-1} q^{\lambda(r)+r(m-1)} P_r Q_{r+1} + \sum_{r=1}^{\infty} (-1)^r a^{2r} q^{\lambda(r)-mr} P_{r-1} Q_r$$

replacing r by $r + 1$ in the second summation, we have,

$$H_m - H_{m-1} = \sum_{r=0}^{\infty} (-1)^r P_r Q_{r+1} D_{mr} \tag{2}$$

where,

$$D_{mr} = a^{2r+m-1} q^{\lambda(r)+r(m-1)} - a^{2(r+1)} q^{\lambda(r+1)-m(r+1)}$$

since $\lambda(r + 1) - \lambda(r) = 5r + 3$, the above equation can be written as,

$$D_{mr} = a^{2r+m-1} q^{\lambda(r)+r(m-1)} (1 - a^{3-m} q^{(2r+1)(3-m)})$$

using the definition of η, we obtain,

$$D_{mr} = a^{m-1}\eta(a^{2r}q^{\lambda(r)-r(3-m)}(1-a^{3-m}q^{2r(3-m)}))$$

since $\eta Q_r = Q_{r+1}$, (2) becomes,

$$H_m - H_{m-1} = \sum_{r=0}^{\infty}(-1)^r P_r(\eta Q_r)a^{m-1}\eta(a^{2r}q^{\lambda(r)-r(3-m)}(1-a^{3-m}q^{2r(3-m)}))$$

which can be written as,

$$H_m - H_{m-1} = a^{m-1}\eta\left(\sum_{r=0}^{\infty}(-1)^r P_r Q_r a^{2r}q^{\lambda(r)-r(3-m)}(1-a^{3-m}q^{2r(3-m)})\right)$$

therefore,

$$H_m - H_{m-1} = a^{m-1}\eta H_{3-m}$$

Substituting $m = 1$ and $m = 2$ in the above equation, we have,

$$H_1 - H_0 = \eta H_2, \quad H_2 - H_1 = a\eta H_1$$

since $H_0 = 0$, adding both of them, we obtain,

$$H_2 - H_0 = \eta H_2 + a\eta H_1 \implies H_2 = \eta H_2 + a\eta^2 H_2$$

Now, let us now expand $H_2(a)$ in terms of the variable a, thus,

$$H_2(a) = \sum_{n=0}^{\infty} c_n a^n \tag{3}$$

where, $c_0 = 1$. Using the equation $H_2 = \eta H_2 + a\eta^2 H_2$, we have,

$$\sum_{n=0}^{\infty} c_n a^n = \sum_{n=0}^{\infty} c_n a^n q^n + \sum_{n=0}^{\infty} c_n a^{n+1} q^{2n}$$

comparing the coefficients of a^n, we obtain,

$$c_n = c_n q^n + c_{n-1}q^{2n-2} \implies c_n = \frac{q^{2n-2}}{1-q^n}c_{n-1}$$

solving this recurrence relation, we obtain,

$$c_n = \frac{q^{2n-2}}{1-q^n}\frac{q^{2n-4}}{1-q^{n-1}}\frac{q^{2n-6}}{1-q^{n-2}}\dots\frac{q^2}{1-q^2}\frac{q^0}{1-q}c_0 = \frac{q^{n(n-1)}}{(q;q)_n} = q^{n(n-1)}P_n$$

finally, (3) becomes,

$$H_2(a) = \sum_{n=0}^{\infty} q^{n(n-1)}P_n a^n$$

using the definition of $H_2(a)$ (from the "Prerequisites" section), we have,

$$H_2(a) = \sum_{r=0}^{\infty}(-1)^r a^{2r} q^{\lambda(r)-2r}(1-a^2q^{4r})P_rQ_r$$

substituting $a = q$ in the above equations, we obtain,

$$H_2(q) = 1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \sum_{r=0}^{\infty}(-1)^r q^{\lambda(r)}(1-q^{4r+2})P_rQ_r(q)$$

Let us simplify the second sum in the above equation by first observing that,

$$P_rQ_r(q) = \prod_{s=1}^{r}\frac{1}{1-q^s}\prod_{s=r}^{\infty}\frac{1}{1-q^{s+1}} = \prod_{s=1}^{\infty}\frac{1}{1-q^s} = P_\infty = \frac{1}{(q;q)_\infty}$$

the remaining part inside the summation can be written as,

$$\sum_{r=0}^{\infty}(-1)^r q^{\lambda(r)}(1-q^{4r+2}) = 1 + \sum_{r=1}^{\infty}(-1)^r q^{\lambda(r)} - \sum_{r=1}^{\infty}(-1)^r q^{\lambda(r-1)+4(r-1)+2}$$

which can be simplified further as,

$$\sum_{r=0}^{\infty}(-1)^r q^{\lambda(r)}(1-q^{4r+2}) = 1 + \sum_{r=1}^{\infty}(-1)^r\left(q^{\frac{r(5r+1)}{2}} + q^{\frac{r(5r-1)}{2}}\right)$$

thus,

$$H_2(q) = \frac{1}{(q;q)_\infty}\left(1 + \sum_{r=1}^{\infty}(-1)^r\left(q^{\frac{r(5r+1)}{2}} + q^{\frac{r(5r-1)}{2}}\right)\right) \tag{4}$$

It is known from JTPI that,

$$(q^2;q^2)_\infty(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}z^n$$

replacing q by $q^{5/2}$ and z by $-q^{1/2}$ in JTPI, we obtain,

$$(q^5;q^5)_\infty(q^3;q^5)_\infty(q^2;q^5)_\infty = \sum_{n=-\infty}^{\infty} q^{5n^2/2}(-q)^{n/2} = 1+\sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{n(5n+1)}{2}}+q^{\frac{n(5n-1)}{2}}\right)$$

therefore, (4) becomes,

$$H_2(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

this completes the proof of the first identity.
We will follow the same lines of proof to prove the second identity, observe that,

$$H_1 = \eta H_2 = \sum_{n=0}^{\infty} q^{n^2} P_n a^n$$

using the definition of $H_1(a)$ (from the "Prerequisites" section), we have,

$$H_1(a) = \sum_{r=0}^{\infty}(-1)^r a^{2r} q^{\lambda(r)-r}(1-aq^{2r})P_r Q_r$$

substituting $a = q$ in the above equations and replacing q by $q^{5/2}$ and z by $-q^{3/2}$ in JTPI, we obtain,

$$\sum_{n=0}^{\infty} q^{n^2+n} P_n = P_\infty \sum_{r=0}^{\infty}(-1)^r q^{\lambda(r)+r}(1-q^{2r+1}) = \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}$$

this completes the proof of the second identity.

Corollary:

In the next episode, we will derive Rogers-Ramanujan continued fraction.

Conclusion:

If we interpret the above identity combinatorially, we conclude that the number of partitions of n such that the adjacent parts differ by at least 2 is the same as the number of partitions of n such that each part is congruent to either 1 or 4 modulo 5. Furthermore these identities are used in forming the well known Rogers-Ramanujan continued fraction and various other beautiful identities related to modular forms which will be discussed in the later episodes.