Amazing Proofs: Episode 3

Aim:

In this episode of "Amazing Proofs", we are going to derive the famous formula for Riemann zeta function at positive even values which was first discovered by Euler in 1739.

Statement:

For $n \in \mathbb{N}$, we have,

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

where B_n is the n^{th} Bernoulli number.

Materials required:

Infinite product of $\sin(x)$, Hyperbolic trigonometric functions, Bernoulli numbers, Method of comparing coefficients

Procedure:

For $z \in \mathbb{C}$, it is known that,

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

logarithmically differentiating both the sides such that $z \notin \mathbb{Z}$, we obtain,

$$\pi \cot(\pi z) = \frac{1}{z} - 2z \sum_{k=1}^{\infty} \frac{1}{k^2 - z^2}$$

observe that if $|z| < 1$, then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - z^2} = \sum_{k=1}^{\infty} k^{-2} \sum_{n=1}^{\infty} \left(\frac{z^2}{k^2}\right)^{n-1} = \frac{1}{z^2} \sum_{n=1}^{\infty} z^{2n} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{1}{z^2} \sum_{n=1}^{\infty} \zeta(2n) z^{2n}$$

hence

$$\pi\cot(\pi z) = \frac{1}{z} - \frac{2}{z}\sum_{n=1}^{\infty}\zeta(2n)z^{2n} \tag{1}$$

It is known that for $|z| < 2\pi$,

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty}\frac{B_n z^n}{n!}$$

using the above equation, we have,

$$\coth(z) = 1+\frac{1}{z}\left(\frac{2z}{e^{2z}-1}\right) = 1+\frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{B_n(2z)^n}{n!}\right) = 1+\frac{1}{z}\left(-z+\sum_{n=0}^{\infty}\frac{B_{2n}(2z)^{2n}}{(2n)!}\right)$$

hence,

$$\coth(z) = \frac{1}{z} + \frac{1}{z}\sum_{n=1}^{\infty}\frac{B_{2n}(2z)^{2n}}{(2n)!}$$

finally substituting $z = \pi i z$, we obtain,

$$\pi\cot(\pi z) = \frac{1}{z} + \frac{1}{z}\sum_{k=1}^{\infty}\frac{(-1)^n B_{2n}(2\pi z)^{2n}}{(2n)!} \tag{2}$$

Comparing the coefficient of z^{2n} in equation (1) and (2), we have,

$$-2\zeta(2n) = \frac{(-1)^n B_{2n}(2\pi)^{2n}}{(2n)!}$$

which completes the proof.

Corollary:

1. $$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \ldots$$

2. $$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}}$$

3. $$\zeta(2n) \text{ is transcedental } \forall n \in \mathbb{N}.$$

Conclusion:

Not much is known about the irratioanlity, transcendence and closed forms of $\zeta(2n+1)$ (except for the irrationality of $\zeta(3)$). But for all the readers interested in $\zeta(3)$, here is another beautiful series

$$\zeta(3) = \frac{\pi^2}{7}\left(1 - 4\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2^{2k}(2k+1)(2k+2)}\right)$$

which was given by Euler in 1772.