Amazing Proofs: Episode 8

Aim:

In this episode of "Amazing Proofs", we are going to derive Poisson's summation formula and a famous corollary.

Statement:

If $f : \mathbb{R} \to \mathbb{C}$ be a Schwarz function and let $\hat{f}$ be its Fourier transform, then,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Materials Required:

Fourier Transform:

Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function, then the Fourier transform of f is defined by,

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx$$

where, $t \in \mathbb{R}$.

Fourier Coefficients:

Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function (a periodic function having period 1), then f can be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

where the Fourier coefficients c_n are given by,

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} dx$$

where, $n \in \mathbb{Z}$.

Schwarz Function:

A function f is Schwarz iff (if and only if) for all $c \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have,

$$|f^{(n)}(x)| = o(|x^c|)$$

where, $f^{(n)}(x)$ is the n^{th} derivative of f, o is the little-o notation and $\mathbb{N}_0 = \mathbb{N} \cup 0$.

Procedure:

Let us define a function $F(x)$ as,

$$F(x) = \sum_{k=-\infty}^{\infty} f(x+k)$$

clearly, $F(x) = F(x+1)$, which implies that $F(x)$ is 1-periodic.
Thus we have,

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

where,

$$c_n = \int_0^1 F(x) e^{-2\pi i n x} dx = \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k) e^{-2\pi i n x} dx$$

interchanging the integral and summation sign (since f is Schwarz), we have,

$$c_n = \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x} dx$$

replacing x by $x-k$, we obtain,

$$c_n = \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x-k+k) e^{-2\pi i n x} e^{2\pi i n k} d(x-k) = \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x) e^{-2\pi i n x} dx$$

observe that,

$$\sum_{k=-\infty}^{\infty} \int_k^{k+1} dx := \int_{-\infty}^{\infty} dx$$

finally,

$$c_n = \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx = \hat{f}(n)$$

Therefore,

$$F(x) = \sum_{k=-\infty}^{\infty} f(x+k) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

substituting $x = 0$, we get,

$$F(0) = \sum_{k=-\infty}^{\infty} f(k) = \sum_{n=-\infty}^{\infty} c_n = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

which completes the proof.

Corollary:

Substituting $f(x) = e^{-\pi x^2 t}$, we obtain (after some simplification),

$$\sum_{k=-\infty}^{\infty} e^{-\pi k^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}$$

It is known from the theory of theta functions that,

$$\theta(t) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t}$$

hence, the functional equation of theta function becomes,

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)$$

which will be used to prove the functional equation of Riemann's zeta function in the next episode.

Conclusion:

Poisson summation formula is a powerful tool/method to evaluate exponential sums, find functional equations of famous functions containing the term $e^{-\pi x^2}$ (using the fact that the Fourier transform of the function $e^{-\pi x^2}$ is the function itself) and their explicit values at certain points. It can also be used to prove some of the problems/results found in Ramanujan's notebooks.