Amazing Proofs: Episode 6

Aim:

In this episode of "Amazing Proofs", we are going to derive Ramanujan's first congruence for the partition function using EPNT (proved in episode 4) and the special case identity proved in episode 5.

Statement:

If $n \in \mathbb{N}$, then

$$p(5n-1) \equiv 0 \pmod{5}$$

Materials required:

Basics of q-series, Series and product manipulation, Modular arithmetic, Partition function

Procedure:

We know from that if $|q| < 1$, then

$$(q)_\infty = (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^{-n} q^{n(3n+1)/2} \tag{1}$$

and

$$(q)_\infty^3 = (q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \tag{2}$$

multiplying (1) and (2), we obtain,

$$(q)_\infty^4 = (q;q)_\infty^4 = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2} (-1)^{-n} q^{n(3n+1)/2}$$

after simplification, we have,

$$q(q)_\infty^4 = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} (-1)^{m-n} (2m+1) q^k$$

where, $k = \frac{m(m+1)+n(3n+1)}{2} + 1 \implies 2(n+1)^2 + (2m+1)^2 = 8k - 10n^2 - 5$.
Observe that,

$$k \equiv 0 \pmod 5 \implies 2(n+1)^2 + (2m+1)^2 \equiv 0 \pmod 5$$

since $a^2 \equiv 0, 1, 4 \pmod 5$, we have,

$$k \equiv 0 \pmod 5 \implies 2(n+1)^2 \equiv 0 \pmod 5 \text{ and } (2m+1)^2 \equiv 0 \pmod 5$$

Now let us define a simple notation/operator as $(f(x))_n \pmod p$, which is simply the remainder when the coefficient of x^n in the series expansion of $f(x)$ is divided by p, where p is a prime.
Thus,

$$(q(q)^4_\infty)_{5n} \equiv 0 \pmod 5 \tag{3}$$

observe that,

$$(1-q^5) \equiv (1-q)^5 \pmod 5$$

hence

$$(1-q^5)(1-q^{10})(1-q^{15})... \equiv (1-q)^5(1-q^2)^5(1-q^3)^5... \pmod 5$$

therefore,

$$\left(\frac{(q^5)_\infty}{(q)^5_\infty}\right)_{n+1} \equiv 0 \pmod 5 \tag{4}$$

multiplying (3) and (4), we obtain,

$$\left(q\frac{(q^5)_\infty}{(q)_\infty}\right)_{5n} \equiv 0 \pmod 5$$

using EPNT, we obtain

$$((q^5)_\infty)_n \equiv -1, 0, 1 \pmod 5$$

and

$$\frac{q}{(q)_\infty} = q\left(\sum_{n=0}^{\infty} p(n)q^n\right) = q + \sum_{n=2}^{\infty} p(n-1)q^n$$

thus,

$$\left(\frac{q}{(q)_\infty}\right)_{5n} \equiv p(5n-1) \pmod 5$$

hence,

$$\left(q\frac{(q^5)_\infty}{(q)_\infty}\right)_{5n} \equiv p(5n-1) \;\equiv\; 0 \pmod 5$$

this completes the proof.

Corollary:

Using the same identity along with some minor changes, we will derive Ramanujan's second congruence in the next episode.

Conclusion:

This proof uses elementary properties of modular arithmetic and series/product manipulation. Ramanujan, in his 1919 paper, proved the following identity

$$\sum_{n=1}^{\infty} p(5n-1)q^n = 5q\frac{(q^5)^5_\infty}{(q)^6_\infty}$$

using Eisenstein series from which his first congruence follows immediately. There exist other complicated congruences (ones involving higher moduli) which can be proved using advanced mathematics (theory of modular forms).