Amazing Proofs: Episode 4

Aim:

In this episode of "Amazing Proofs", we are going to derive the Jacobi triple product identity (JTPI) and the Euler pentagonal number theorem (EPNT).

Statement:

JTPI states that, for $z, q \in \mathbb{C}$ where $|q| < 1$ and $z \neq 0$

$$(q^2;q^2)_\infty(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n$$

and EPNT, which is itself a special case of the JTPI, states that,

$$(q)_\infty := (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

Materials required:

Basics of q-series and partitions, Series and product manipulation

Prerequisite:

If $z, q \in \mathbb{C}$ where $|q| < 1$ and $z \neq 0$, then define $(z;q)_n$ as

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k)$$

then

$$(-z;q)_\infty = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n \tag{1}$$

and

$$\frac{1}{(z;q)_\infty} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q;q)_n} \tag{2}$$

Proof: Observe that,

$$(-z;q)_\infty = (1+z)\prod_{n=1}^{\infty}(1+zq^n) = (1+z)\prod_{n=0}^{\infty}(1+(zq)q^n) = (1+z)(-zq;q)_\infty$$

now let the series expansion of $(-z;q)_\infty$ be of the form given below,

$$(-z;q)_\infty = \sum_{n=0}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} a_n z^n$$

where a_n is a function of q and n and $a_0 = 1$, then using the above functional equation we obtain,

$$(-z;q)_\infty = \sum_{n=0}^{\infty} a_n z^n = (1+z)\sum_{n=0}^{\infty} a_n (zq)^n$$

comparing the coefficients of z^n we have,

$$a_n = a_n q^n + a_{n-1} q^{n-1}$$

solving it we obtain,

$$a_n = \frac{q^{n(n-1)/2}}{(q;q)_n}$$

which completes the proof of (1), similarly we can prove (2).

Procedure:

Replacing q by q^2 and z by zq in (1), we obtain,

$$(-zq;q^2)_\infty = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} z^n$$

observe that,

$$\frac{1}{(q^2;q^2)_n} = \frac{(q^{2n+2};q^2)_\infty}{(q^2;q^2)_\infty}$$

hence,

$$(-zq;q^2)_\infty = 1 + \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}(q^{2n+2};q^2)_\infty q^{n^2} z^n$$

observe that, if n is a negative integer then,

$$(q^{2n+2};q^2)_\infty = 0$$

thus,

$$(q^2;q^2)_\infty(-zq;q^2)_\infty = \sum_{n=-\infty}^{\infty}(q^{2n+2};q^2)_\infty q^{n^2}z^n$$

replacing q by q^2 and z by $-q^{2n+2}$ in (1), we obtain,

$$(q^{2n+2};q^2)_\infty = 1 + \sum_{m=1}^{\infty}\frac{(-1)^m q^{m^2+2mn+m}}{(q^2;q^2)_m}$$

hence,

$$(q^2;q^2)_\infty(-zq;q^2)_\infty = \sum_{n=-\infty}^{\infty}\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m q^{m^2+2mn+m}}{(q^2;q^2)_m}\right)q^{n^2}z^n$$

simplifying it further we obtain,

$$(q^2;q^2)_\infty(-zq;q^2)_\infty = \sum_{n=-\infty}^{\infty}q^{n^2}z^n + \sum_{m=1}^{\infty}\frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m}\sum_{n=-\infty}^{\infty}q^{(m+n)^2}z^{m+n}$$

again observe that,

$$\sum_{n=-\infty}^{\infty}q^{(m+n)^2}z^{m+n} = \sum_{n=-\infty}^{\infty}q^{n^2}z^n$$

therefore

$$(q^2;q^2)_\infty(-zq;q^2)_\infty = \sum_{n=-\infty}^{\infty}q^{n^2}z^n\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m}\right)$$

again replacing q by q^2 and z by $-qz^{-1}$ in (2), we obtain,

$$\frac{1}{(-qz^{-1};q^2)_\infty} = 1 + \sum_{m=1}^{\infty}\frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m}$$

finally

$$(q^2;q^2)_\infty(-zq;q^2)_\infty = \sum_{n=-\infty}^{\infty}\frac{q^{n^2}z^n}{(-qz^{-1};q^2)_\infty}$$

which after some manipulation gives

$$(q^2;q^2)_\infty(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \tag{3}$$

which proves the JTPI.
Replacing q by $q^{3/2}$ and z by $-q^{-1/2}$ in (3), we obtain,

$$(q^3;q^3)_\infty(q;q^3)_\infty(q^2;q^3)_\infty = (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}$$

which proves the EPNT.

Corollary:

It is known that if $p(n)$ is the partition function, then

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n$$

hence

$$(q)_\infty \sum_{n=0}^{\infty} p(n) q^n = 1 \implies \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}\right)\left(\sum_{n=0}^{\infty} p(n) q^n\right) = 1$$

using the above equation we can obtain a simple recurrence relation for the partition function,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + ...$$

Conclusion:

JTPI, EPNT and other similar infinite products like the Dirichlet eta function can all be expressed using Ramanujan's theta function. The recurrence relation is a nice way to calculate $p(n)$, but is too cumbersome, hence for the interested readers, here is an asymptotic formula

$$p(n) \sim \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{\frac{2\pi}{\sqrt{6}}\sqrt{n-1/24}}}{\sqrt{n-1/24}}\right) + \frac{(-1)^n}{2\pi} \frac{d}{dn}\left(\frac{e^{\frac{\pi}{\sqrt{6}}\sqrt{n-1/24}}}{\sqrt{n-1/24}}\right)$$