Amazing Proofs: Episode 1

Aim:

In this episode of "Amazing Proofs", we are going to prove Weierstrass product for gamma function and Euler's reflection formula as a corollary.

Statement:

For every $z \in \mathbb{C}$, we have

$$\frac{1}{\Gamma(z)} = ze^{z\gamma}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}$$

and for $z \notin \mathbb{Z}$, we have

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Materials required:

Entire function, Harmonic numbers, Euler-Mascheroni constant, Gamma function, Analytic continuation, Infinite product of $\sin(x)$

Procedure:

Consider the following product for $n, m \in \mathbb{N}$,

$$n! = \prod_{k=1}^{n} k = \left(\prod_{j=1}^{m}\frac{n+j}{n+j}\right)\left(\prod_{k=1}^{n} k\right) = \left(\frac{(n+m)!}{m^n m!}\right)m^n m!\prod_{k=1}^{m}\frac{1}{n+k}$$

now,

$$\frac{(n+m)!}{m^n m!} = \prod_{k=1}^{n}\frac{m+n-k+1}{m}$$

thus,

$$\lim_{m\to\infty}\frac{(n+m)!}{m^n m!} = \lim_{m\to\infty}\prod_{k=1}^{n}\left(1+\frac{n-k+1}{m}\right) = 1$$

hence,

$$n! = \lim_{m\to\infty} m^n m! \prod_{k=1}^{m} \frac{1}{n+k} = \lim_{m\to\infty} m^n \prod_{k=1}^{m} \left(1+\frac{n}{k}\right)^{-1}$$

observe that,

$$m^n = e^{\ln(m^n)} = e^{n\ln(m)} = e^{n(H_m-(H_m-\ln(m)))}$$

therefore,

$$n! = \lim_{m\to\infty} e^{n(H_m-(H_m-\ln(m)))} \prod_{k=1}^{m} \left(1+\frac{n}{k}\right)^{-1}$$

simplifying it further we obtain,

$$n! = e^{-n\gamma} \prod_{k=1}^{\infty} \left(1+\frac{n}{k}\right)^{-1} e^{n/k}$$

extending the above definiton to complex numbers using analytic continuation ($z \in \mathbb{C}$) we have,

$$\Gamma(1+z) = e^{-z\gamma} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^{-1} e^{z/k}$$

finally

$$\frac{1}{\Gamma(z)} = ze^{z\gamma} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}$$

this completes the proof.

Corollary:

Replacing z by $-z$ and multiplying both the products, we obtain,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = \left[ze^{z\gamma} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}\right] \left[-ze^{-z\gamma} \prod_{k=1}^{\infty} \left(1-\frac{z}{k}\right) e^{z/k}\right]$$

thus,

$$\frac{1}{-z^2\Gamma(z)\Gamma(-z)} = \prod_{k=1}^{\infty} \left(1-\frac{z^2}{k^2}\right)$$

It is known that,

$$\sin(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right)$$

hence,

$$\frac{1}{-z^2 \Gamma(z)\Gamma(-z)} = \frac{\sin(\pi z)}{\pi z}$$

since $\Gamma(1-z) = -z\Gamma(-z)$, we have,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

this proves the reflection formula.

Conclusion:

We conclude that, $\Gamma(z)$ is a blessing to mankind beacause of its diverse nature.