Amazing Proofs: Episode 9

Aim:

In this episode of "Amazing Proofs", we are going to derive the functional equation of the Riemann's zeta function.

Statement:

If $\Re(s) > 0$ and $\xi(s)$ is defined as,

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

then

$$\xi(s) = \xi(1-s)$$

where $s \neq 1$, $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann's zeta function.

Materials Required:

Functional equation of $\theta(t)$:

We proved in the previous episode for all real $\Re(t) > 0$, that

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\Big(\frac{1}{t}\Big)$$

where $\theta(t)$ is defined as,

$$\theta(t) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t} = 1 + 2\sum_{k=1}^{\infty} e^{-\pi k^2 t}$$

Procedure:

It is known that from the definition of gamma function for $\Re(s) > 0$ that,

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx \implies \Gamma(s/2) = \int_0^\infty x^{s/2-1}e^{-x}dx$$

substituting $x = k^2 t$ where $k \in N$, we obtain,

$$\Gamma(s/2) = k^s \int_0^\infty t^{s/2-1} e^{-k^2 t} dt \implies \frac{\Gamma(s/2)}{k^s} = \int_0^\infty t^{s/2-1} e^{-k^2 t} dt$$

summing up both the sides from $k = 1$ to $k = \infty$, we have,

$$\Gamma(s/2)\zeta(s) = \int_0^\infty t^{s/2-1} \left(\sum_{k=1}^\infty e^{-k^2 t} \right) dt = \int_0^\infty t^{s/2-1} \left(\frac{\theta(t/\pi) - 1}{2} \right) dt$$

thus, we have,

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty t^{s/2-1} \left(\frac{\theta(t) - 1}{2} \right) dt$$

Now observe that,

$$2\xi(s) = \int_0^1 t^{s/2-1}(\theta(t) - 1)dt + \int_1^\infty t^{s/2-1}(\theta(t) - 1)dt$$

which can be written as,

$$2\xi(s) = -\frac{2}{s} + \int_0^1 t^{s/2-1}\theta(t)dt + \int_1^\infty t^{s/2-1}(\theta(t) - 1)dt$$

substituting $t = 1/y$ in the first integral and replacing y by t, we have,

$$2\xi(s) = -\frac{2}{s} + \int_1^\infty t^{-s/2-1}\theta(1/t)dt + \int_1^\infty t^{s/2-1}(\theta(t) - 1)dt$$

using the functional equation of the theta function, the first integral can be written as,

$$\int_1^\infty t^{-s/2-1}\theta(1/t)dt = \int_1^\infty t^{-s/2-1/2}\theta(t)dt = -\frac{2}{1-s} + \int_1^\infty t^{-s/2-1/2}(\theta(t) - 1)dt$$

therefore,

$$2\xi(s) = -\frac{2}{s} - \frac{2}{1-s} + \int_1^\infty t^{-s/2-1/2}(\theta(t) - 1)dt + \int_1^\infty t^{s/2-1}(\theta(t) - 1)dt$$

finally,

$$\xi(s) = -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2}\int_1^\infty (t^{(1-s)/2} + t^{s/2})(\theta(t) - 1)\frac{dt}{t}$$

replacing s by $1 - s$, it can be shown that, $\xi(s) = \xi(1 - s)$, which completes the proof.

Corollary:

From $\xi(s) = \xi(1-s)$, we have,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)$$

or

$$\zeta(1-s) = \sqrt{\pi}\pi^{-s}\frac{\Gamma(s/2)}{\Gamma\Big(\frac{1-s}{2}\Big)}\zeta(s)$$

it is known from Legendre's duplication formula that,

$$\Gamma(2s) = \pi^{-1/2}2^{2s-1}\Gamma(s)\Gamma\Big(s+\frac{1}{2}\Big) \implies \Gamma(s) = \pi^{-1/2}2^{s-1}\Gamma(s/2)\Gamma\Big(\frac{1+s}{2}\Big)$$

or

$$\Gamma(s) = \pi^{-1/2}2^{s-1}\frac{\Gamma(s/2)}{\Gamma\Big(\frac{1-s}{2}\Big)}\Gamma\Big(\frac{1+s}{2}\Big)\Gamma\Big(\frac{1-s}{2}\Big)$$

using Euler's reflection formula (proved in episode 1), we have,

$$\Gamma\Big(\frac{1+s}{2}\Big)\Gamma\Big(\frac{1-s}{2}\Big) = \frac{\pi}{\cos(\pi s/2)}$$

thus,

$$\Gamma(s) = \pi^{-1/2}2^{s-1}\frac{\Gamma(s/2)}{\Gamma\Big(\frac{1-s}{2}\Big)}\frac{\pi}{\cos(\pi s/2)} \implies \frac{\Gamma(s/2)}{\Gamma\Big(\frac{1-s}{2}\Big)} = \pi^{-1/2}2^{1-s}\cos(\pi s/2)\Gamma(s)$$

finally,

$$\zeta(1-s) = \sqrt{\pi}\pi^{-s}\Big(\pi^{-1/2}2^{1-s}\cos(\pi s/2)\Gamma(s)\Big)\zeta(s) = 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2)\zeta(s)$$

Conclusion:

Using the above functional equation, it can be shown that, $\zeta(s)$ has all the trivial zeroes in the region $\Re(s) < 0$, specifically at $s = -2, -4, -6, ...$ and no zeroes in the region $\Re(s) > 1$. This implies that $\zeta(s)$ has all the non-trivial zeroes in the region $0 < \Re(s) < 1$ (Riemann's hypothesis states that all the non trivial zeroes of $\zeta(s)$ lies at $s = 1/2$).