Amazing Proofs: Episode 5

Aim:

In this episode of "Amazing Proofs", we are going to derive a special case of Jacobi's triple product identity.

Statement:

If $|q| < 1$, then

$$(q)_\infty^3 = (q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}$$

Materials required:

Basics of q-series, Series and product manipulation, Jacobi's triple product identity, Partition function

Procedure:

JTPI states that if $|q| < 1$ and $z \neq 0$, then,

$$(q^2;q^2)_\infty (-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n$$

replacing z by zq and q by $q^{1/2}$ in the above equation, we obtain,

$$(q;q)_\infty (-zq;q)_\infty (-z^{-1};q)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n$$

which after simplification becomes,

$$(1+z^{-1})(q;q)_\infty (-zq;q)_\infty (-z^{-1}q;q)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n$$

observe that,

$$\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n = \sum_{n=0}^{\infty} q^{n(n+1)/2}(z^n + z^{-n-1}) = \sum_{n=0}^{\infty} q^{n(n+1)/2} z^{-n}(z^{2n} + z^{-1})$$

thus,

$$(q;q)_\infty(-zq;q)_\infty(-z^{-1}q;q)_\infty = \sum_{n=0}^{\infty} \frac{(z^{2n}+z^{-1})}{(1+z^{-1})} q^{n(n+1)/2} z^{-n} \tag{1}$$

since,

$$\lim_{z\to -1} \frac{(z^{2n}+z^{-1})}{(1+z^{-1})} = 2n+1$$

letting $z \to -1$ in (1) completes the proof.

Corollary:

In the next episode, we will derive Ramanujan's first congruence using this special case identity as a corollary.

Conclusion:

This identity can be used to find many beautiful identities related to q-series and arithmetic/modular properties of the partition function $p(n)$.