# Amazing Proofs: Episode 2

## Aim:

In this episode of "Amazing Proofs", we are going to solve the Diophantine equation $x^4 + y^4 = u^4 + v^4$ which was first solved by Euler.

## Statement:

If $x^4 + y^4 = u^4 + v^4$, then the simplest parametric solution is given by

$$x = a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6$$

$$y = a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7$$

$$u = a^7 + a^5b^2 - 2a^3b^4 - 3a^2b^5 + ab^6$$

$$v = a^7 + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7$$

## Materials required:

Basic knowledge of algebra and analysis

## Procedure:

Substituting

$$x = aw + c, y = bw - d, u = aw + d, v = bw + c \tag{1}$$

in $x^4 + y^4 = u^4 + v^4$ where $w \neq 0$, we obtain

$$(aw + c)^4 + (bw - d)^4 = (aw + d)^4 + (bw + c)^4$$

simplifying it we obtain,

$$4(a^3c - b^3d - a^3d - b^3c)w^3 + 6(a^2c^2 + b^2d^2 - a^2d^2 - b^2c^2)w^2 + 4(ac^3 - bd^3 - ad^3 - bc^3)w = 0$$

substituting $c = a^3 + b^3$ and $d = a^3 - b^3$, we have,

$$6(a^2c^2 + b^2d^2 - a^2d^2 - b^2c^2)w^2 + 4(ac^3 - bd^3 - ad^3 - bc^3)w = 0$$

hence,

$$w = \frac{2(ad^3 - ac^3 + bc^3 + bd^3)}{3(a^2 - b^2)(c^2 - d^2)}$$

substituting the values of $c, d$ and $w$ in (1), we complete the proof.

## Corollary:

When $a = 1$ and $b = 2$, we obtain the smallest solution,

$$133^4 + 134^4 = 158^4 + 59^4$$

## Conclusion:

The parametric solution given above is not complete, so finding a general solution is still an open problem. But as Pierre-Simon Laplace said, I would suggest,"Read Euler, read Euler, he is the master of us all."