Geometric Mechanics

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What's geometric mechanics?

- Originated from 17th century classical mechanics
- Incorporates Lagrangian and Hamiltonian formalism
- Goals:
  - to apply particular geometric methods to many areas of mechanics
  - applies to systems for which the configuration space shows group structure
  - apply reduction techniques to systems having symmetries (conserved quantities)
- Most work done by Vladimir Arnold (1966), Stephen Smale (1970) and Jean-Marie Souriau (1970), Ralph Abraham and Jerrold Eldon Marsden (1967)

Newtonian Mechanics

- Based on Newton's Laws of motion (specifically the second one)
- Assumes that the concepts of distance, time, and mass, are absolute
- Motion is in an inertial frame
- Governed by the equation

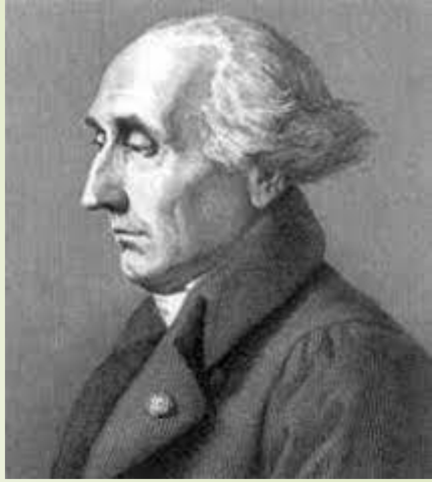
$$\boldsymbol{F_i} = m\boldsymbol{a} \rightarrow \boldsymbol{F_i} = m_i \ddot{\boldsymbol{q}}_i$$

- Newtonian potential system

$$m_i \ddot{\boldsymbol{q}}_i = -\frac{\partial V}{\partial \boldsymbol{q}_i}$$

- Total energy, linear and angular momentum is conserved (imposing conditions on V)
- Conserved quantities → Symmetries (Noether's theorem)

Lagrangian Mechanics
(same but novel)



- Newtonian approach → Manipulation of vector quantities
- Lagrangian approach → Involves kinetic and potential energies which involve only scalar functions
- **Advantages:** Coordinate-independent and equations of motion are easily handled in constrained problems.
- Equations of motion are obtained by → Euler - Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = 0$$

where L is the Lagrangian (a scalar function), given by,

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T - V = \sum_{i=1}^{N} \frac{1}{2} m_i \|\dot{\boldsymbol{q}}_i\|^2 - V(\boldsymbol{q})$$

Simple Pendulum Example

$$x = \ell \sin\theta \quad y = \ell - \ell\cos\theta$$

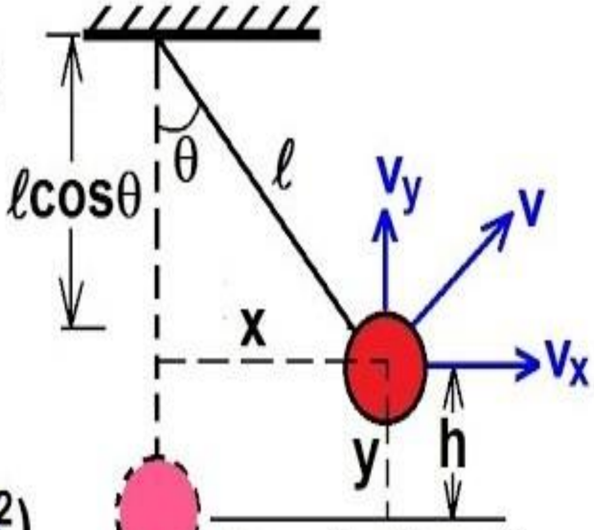
$$v_x = \frac{dx}{dt} = \dot{x} = \ell\cos\theta\,\dot{\theta}$$

$$v_y = \frac{dx}{dt} = \dot{y} = \ell\,\sin\theta\,\dot{\theta}$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2)$$

$$KE = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$L = KE - PE$$



$$h = \ell - \ell\cos\theta$$

$$KE = \frac{1}{2}mv^2$$

$$PE = mgh$$

$$L = KE - PE = T - V$$

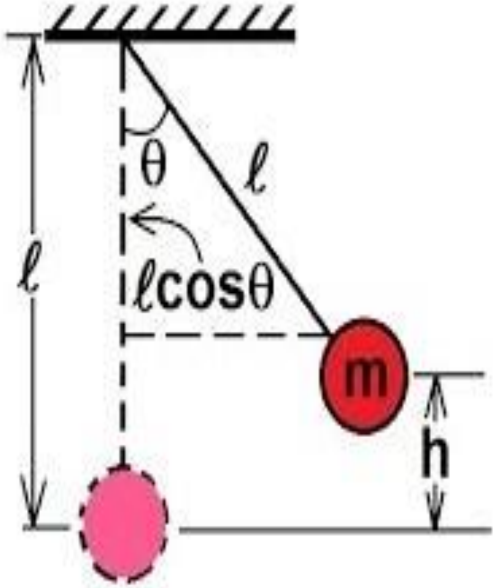
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

$$m\ell^2\ddot{\theta} - (-mg\,\ell\,\sin\theta) = 0$$

$$\ell\ddot{\theta} + g\sin\theta = 0$$

$$\ell\ddot{\theta} + g = 0 \qquad \theta = A\sin(\omega t)$$

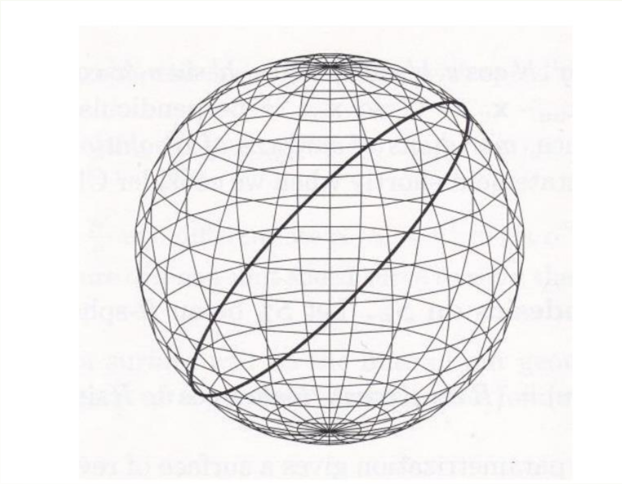


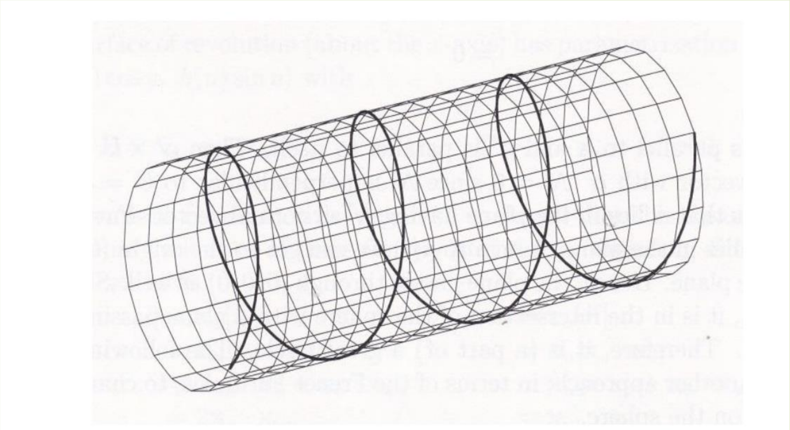
Standard Model Lagrangian

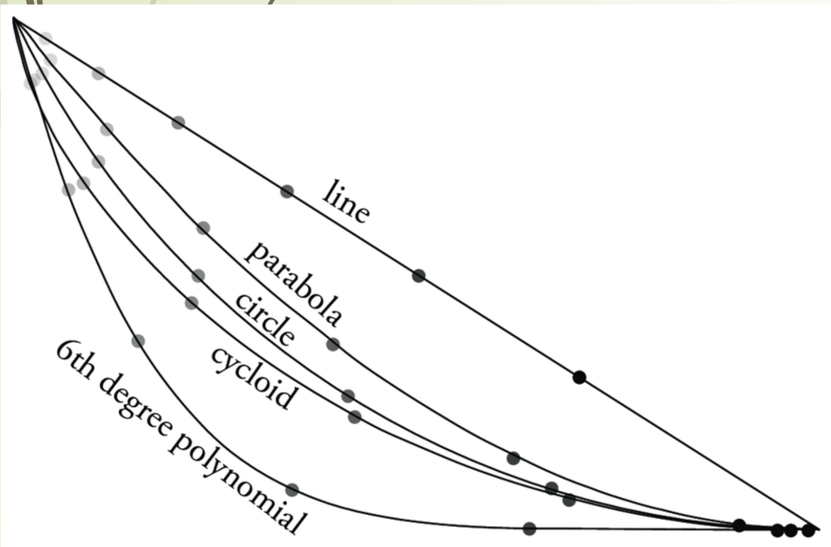
$$
\begin{aligned}
\mathcal{L}_{SM} = \quad & \underbrace{\frac{1}{4} W_{\mu\nu} \cdot W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G^{\alpha}_{\mu\nu} G^{\mu\nu}_{\alpha}}_{\text{kinetic energies and self-interactions of the gauge bosons}} \\
+ \quad & \underbrace{\overline{L} \gamma^{\mu} \left(i\partial_{\mu} - \frac{1}{2} g \tau \cdot W_{\mu} - \frac{1}{2} g' Y B_{\mu} \right) L + \overline{R} \gamma^{\mu} \left(i\partial_{\mu} - \frac{1}{2} g' Y B_{\mu} \right) R}_{\text{kinetic energies and electroweak interactions of fermions}} \\
+ \quad & \underbrace{\frac{1}{2} \left| \left(i\partial_{\mu} - \frac{1}{2} g \tau \cdot W_{\mu} - \frac{1}{2} g' Y B_{\mu} \right) \phi \right|^{2} - V(\phi)}_{W^{\pm}, Z, \gamma \text{ and Higgs masses and couplings}} \\
+ \quad & \underbrace{g'' \left(\overline{q} \gamma^{\mu} T_{a} q \right) G^{\alpha}_{\mu}}_{\text{interactions between quarks and gluons}} + \underbrace{\left(G_{1} \overline{L} \phi R + G_{2} \overline{L} \phi_{c} R + h.c. \right)}_{\text{fermion masses and couplings to Higgs}}
\end{aligned}
$$

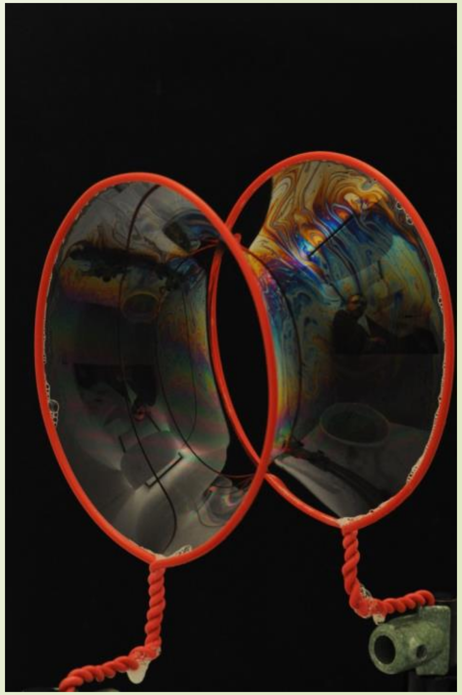
Application of Lagrangian Mechanics

- Problems in calculus of variations
  - Brachistochrone curve problem
  - Hanging chain (Catenary Problem)
  - Finding geodesics
  - Minimal surfaces

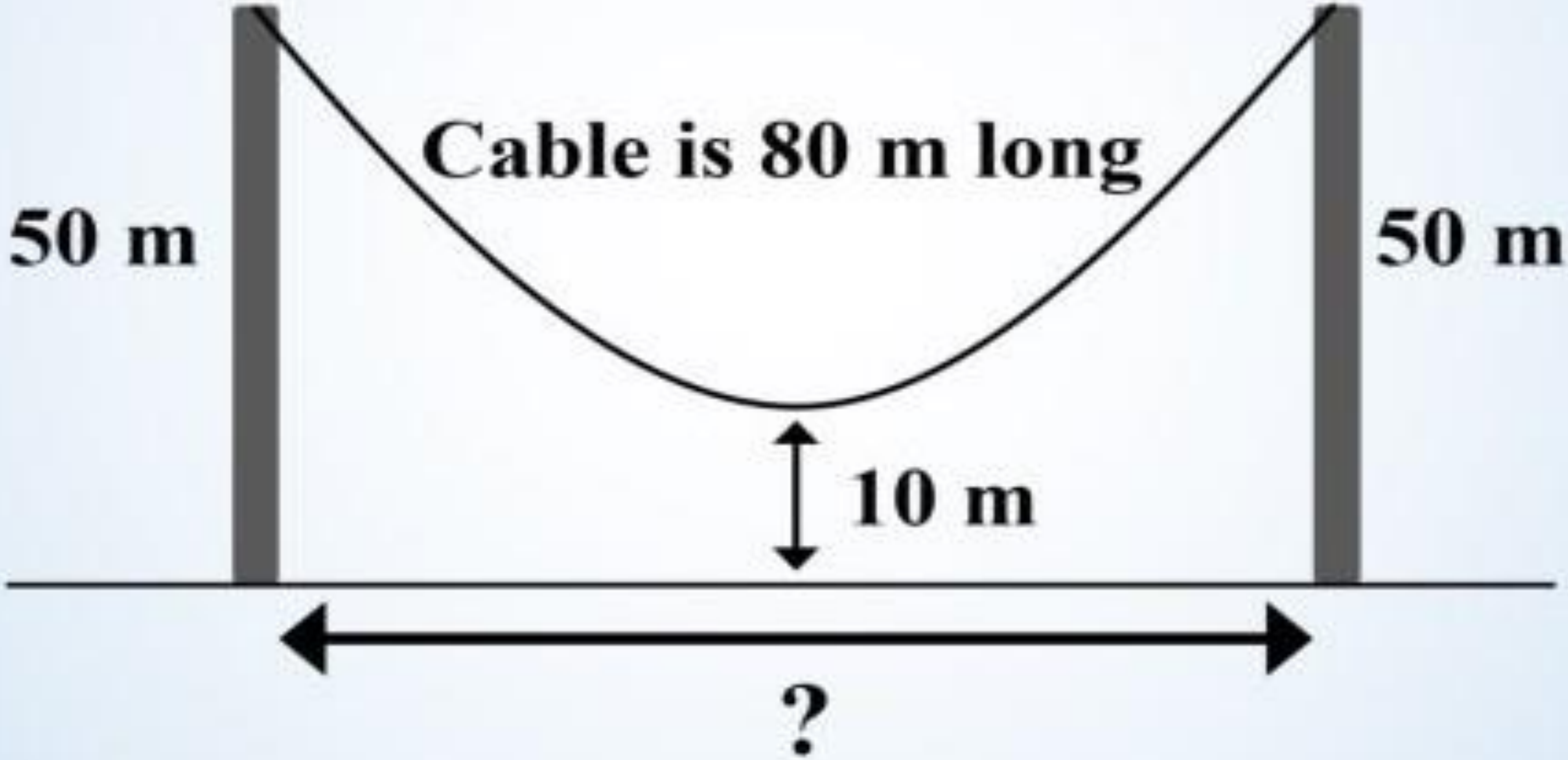








How Far Apart Are The Poles?

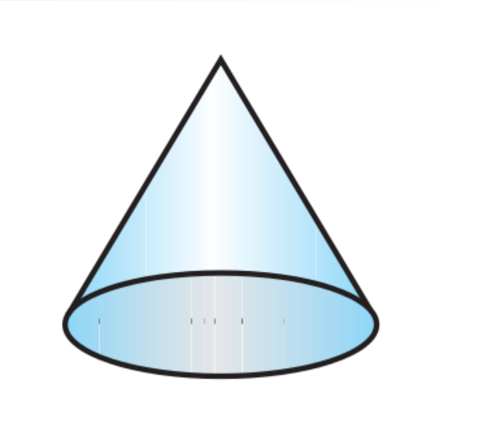


Manifolds: A Brief Overview

- A n-dimensional manifold is a **topological space** M for which every point $x \in M$ has a neighbourhood **homeomorphic** to Euclidean space $\mathbb{R}^n$
- In simple words, it is a space that locally resembles a Euclidean space, but may have a more complicated global structure

Manifolds

Not manifolds



Topological Space

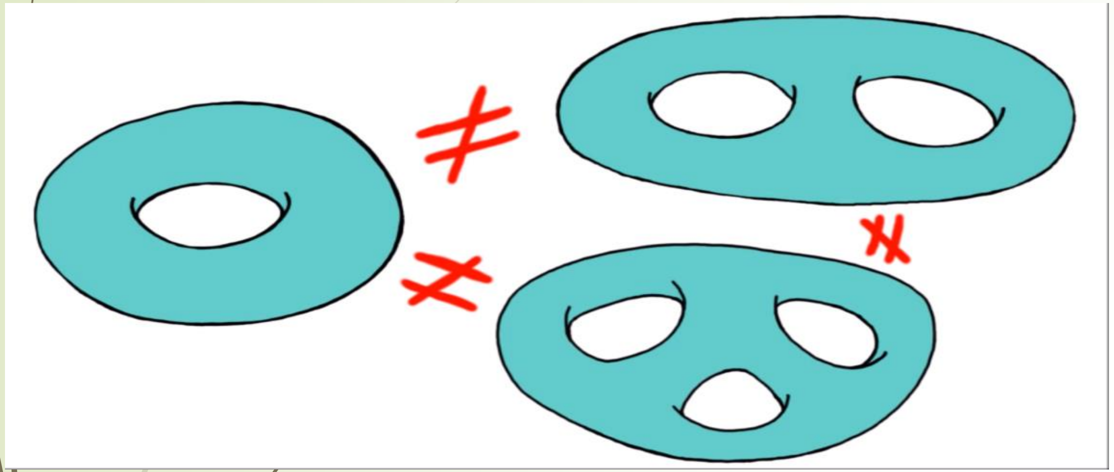
- A collection T of subsets of M is a topology on M if,
  - $\emptyset, M \in T$
  - The union of an arbitrary collection of sets in T is in T
  - The intersection of finite number of sets in T is in T
- $(M, T) \rightarrow$ topological space
- Generally, M is referred to as an topological space if T is clear from the context
- Examples:
  - Real Line ($\mathbb{R}$)
  - Euclidean Space ($\mathbb{R}^n$)
  - Sphere (S^n)
  - Closed Interval [0,1]:

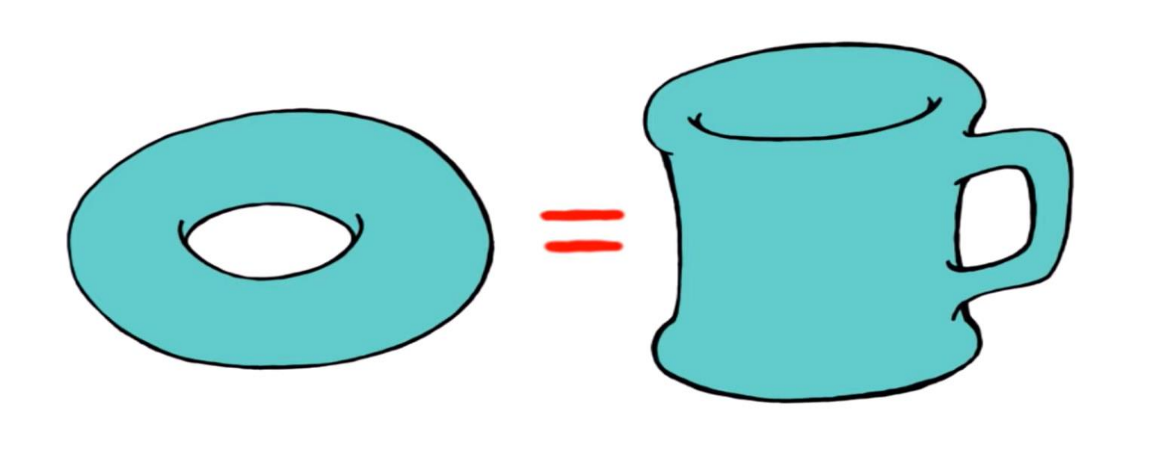
Homeomorphism

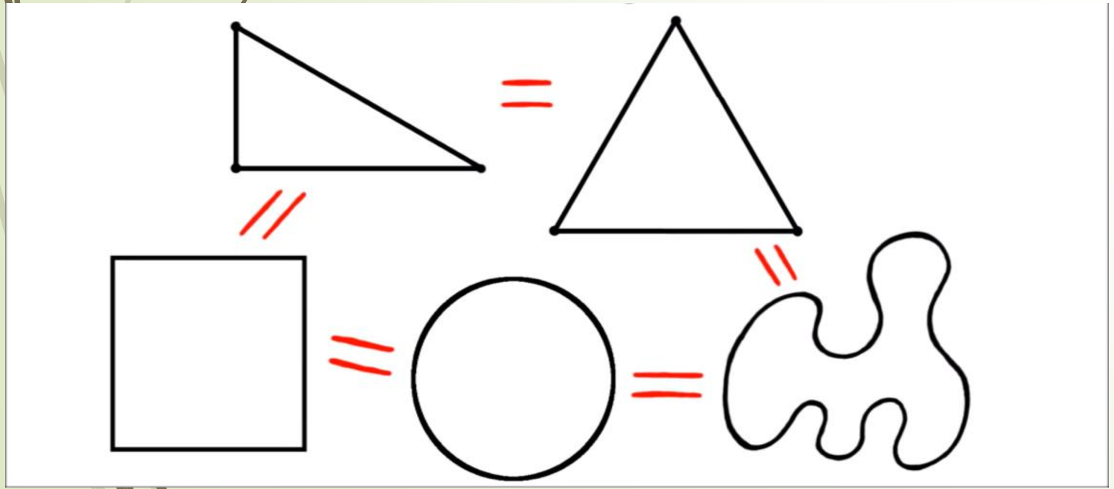
- Bijection and its inverse is continuous → Topological spaces are homeomorphic to each other
- Homeomorphism ↔ Topological equivalence
- In simple words, two objects are topologically equivalent to each other if one can be molded into other by:
  - Twisting
  - Stretching

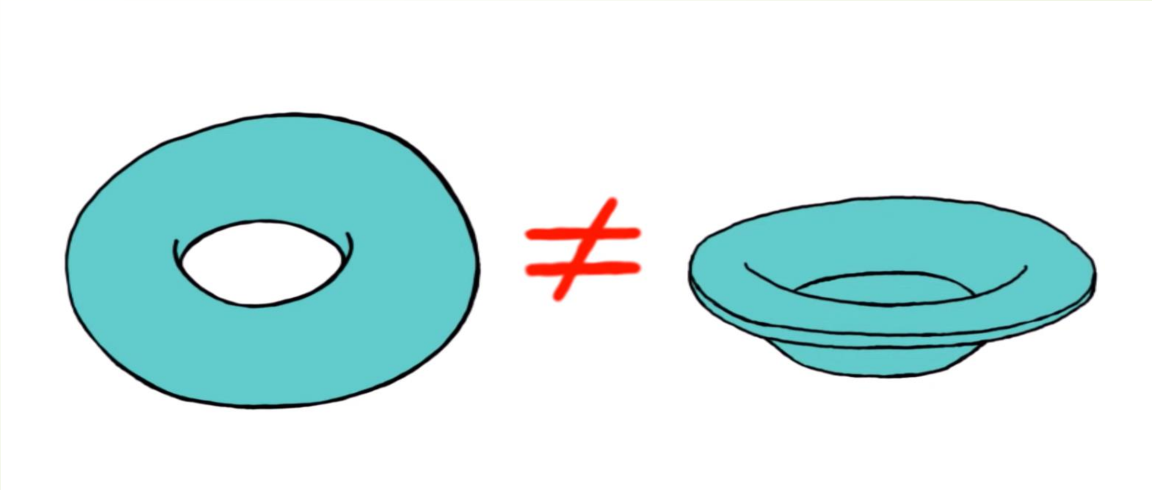
  but not by
  - Tearing
  - Cutting
  - Gluing
- Metrics to check equivalence: Loops and genus

Examples

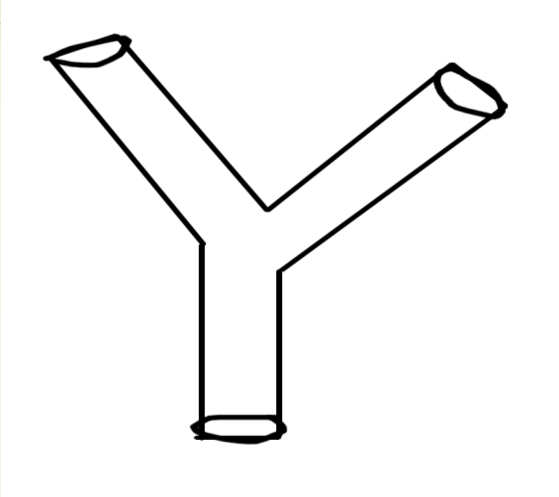


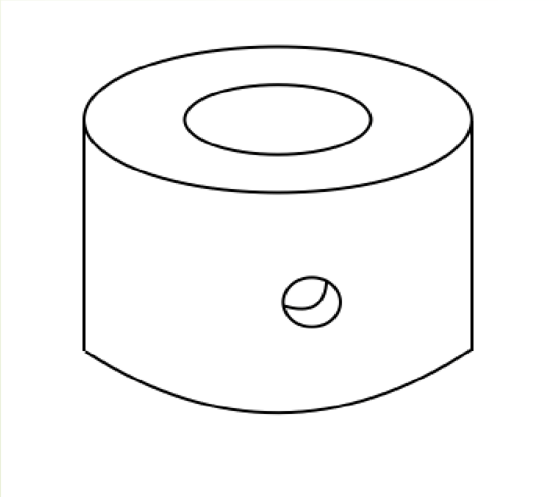


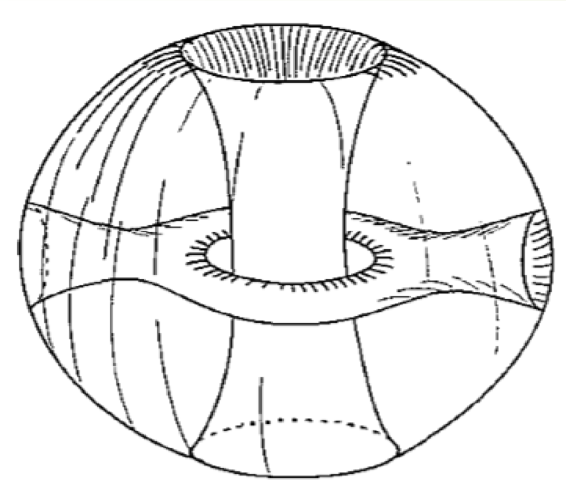




Find the number of holes in the below figures

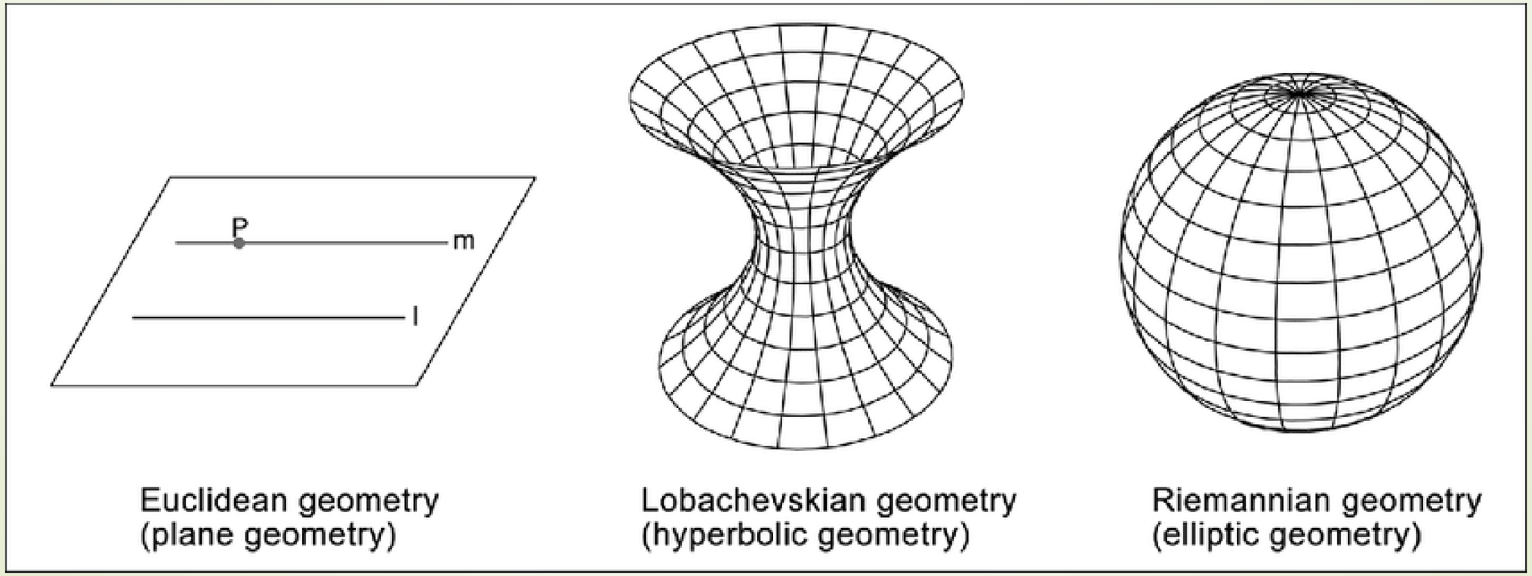






Relation with Differential Geometry
(geometry using advanced calculus)

- Mechanics and geometry go hand in hand
- Objects like fields are required to perform calculus on general manifolds
- Mostly surfaces are curved → Requires generalised notion of length, area, volume and so on…
- Broadly divided into: Riemannian and Symplectic geometry



Lie Groups

- A group which is also a smooth manifold
- Group: Set satisfying the below properties:
  - $\forall\, a, b \in G$ we have $a \cdot b \in G$
  - $\forall\, a \in G\ \exists\, e \in G \mid a \cdot e = e \cdot a = a$
  - $\forall\, a \in G\ \exists\, a^{-1} \in G \mid a \cdot a^{-1} = a^{-1} \cdot a = e$
  - $\forall\, a, b, c \in G$ we have $\left((a \cdot b) \cdot c\right) = (a \cdot (b \cdot c))$
- Smooth manifold: Manifold on which functions can be differentiated arbitrarily many times

Examples

- General linear group: $GL_n(\mathbb{R})$
- Special linear group: $SL_n(\mathbb{R})$
- Orthogonal group: $OL_n(\mathbb{R})$
- Special orthogonal group: $SOL_n(\mathbb{R})$
- Symplectic group: $Sp_{2n}(\mathbb{R})$
- Special Euclidean group: $SE(3)$

Lie Algebra

- A Lie algebra is a vector space V over some field F together with a binary operation $[\cdot,\cdot]$ called the Lie bracket satisfying the following axioms:
  - $\forall\, x, y, z \in V$ and $\forall\, a, b \in F$ we have $[ax + by, z] = [ax + by, z] + [ax + by, z]$
  - $\forall\, x \in V$ we have $[x, x] = 0$
  - $\forall\, x, y, z \in V$ we have $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$
- Above three implies: $[x, y] = -[y, x]$

Examples

- $V = \mathbb{R}^3$, $F = \mathbb{R}$ and $[x, y] = x \times y$ (the usual cross product of two vectors)
- $V = \mathbb{R}^{n \times n}$, $F = \mathbb{R}$ and $[x, y] = AB - BA$ (difference of matrix multiplications)

Applications of Lie Theory

- Analysis (Harmonic analysis and the Peter-Weyl theorem),
- Algebraic topology (Principal bundles and characteristic classes),
- Algebraic geometry (Algebraic groups and flag varieties),
- Combinatorics (Root systems and Coxeter groups),
- Differential geometry (Connections and Chern-Weil theory),
- Number theory (Automorphic forms and the Langlands program),
- Low-dimensional topology (Quantum groups and Chern-Simons theory),
- Riemannian geometry (Holonomy and symmetric spaces),
- Finite group theory (The finite simple groups of Lie type)

and so on…

Practical Applications of Geometric Mechanics

- Robotics
- Celestial Mechanics
- Fluid Dynamics
- Optimal Control
- Cosmology
- Quantum Mechanics
- Machine Learning Integration

and so on …

Conclusion

- Newtonian mechanics ↔ Lagrangian mechanics but much better
- Lagrangian on manifolds and curved surfaces
- Geometric mechanics → Study systems for which the configuration space is a Lie group
- Symmetry → Noether's theorem → Conserved quantities → Reduction

Scope in Future Work

- Incorporation in robotics and control systems
- Machine learning and data analysis
- Distributed and multi-agent systems
- Adaptive control and learning systems
- Geometric formulation of fluid equations

Interdisciplinary scope → collaborations across physics, engineering, mathematics, computer science, and other domains

References

- Holm, D.D. and Schmah, T. and Stoica, C. (2009). *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*, Oxford University Press, Oxford

Any questions…?