Geometric Mechanics

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by

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Abstract

This report contains a brief understanding of the first few chapters of the book **Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions**. The initial section of the book delves into finite-dimensional conservative mechanical systems, with a focus on modern formulations employing the language of differential geometry and differential topology in Lagrangian and Hamiltonian mechanics. This approach offers several advantages, including applicability to systems on general manifolds with constraints, coordinate independence, and relevance to infinite-dimensional systems. The geometric perspective yields an elegant viewpoint, illustrating connections between rigid body motion and geodesic motion on the rotation group. Symmetries of mechanical systems are mathematically represented through Lie group actions, facilitating a reduction in system dimensions by grouping equivalent states and exploiting conserved quantities. The book also explores Lie group symmetries, Poisson reduction, and momentum maps in a general context, subsequently specialising in cases where the configuration space is a Lie group or a Lie group product with a vector space.

Contents

Chapter 1

Introduction

1.1 From Newtonian to Lagrangian Mechanics

While Newtonian mechanics is a powerful and intuitive framework for understanding classical mechanics, Lagrangian (as well as Hamiltonian) mechanics provides a more elegant and versatile approach in many situations because of the following reasons:

- **Coordinate Dependence**: Lagrangian mechanics uses generalised coordinates, whereas Newton's laws are formulated in terms of Cartesian coordinates, and the equations of motion are explicitly dependent on the choice of coordinates.
- **Invariance Under Coordinate Transformations**: The Lagrangian formulation is invariant under general coordinate transformations, whereas Newton's equations of motion are not.
- **Conservation Laws**: Conservation laws are more naturally derived from the symmetries of the Lagrangian. Noether's theorem provides a systematic way

to associate symmetries with conserved quantities, whereas the conservation laws are not immediately apparent from Newton's laws.[1]

The theorem mentioned below gives some insight about the transition from Newtonian to Lagrangian mechanics.

Theorem: The well known equations of motion (i.e., Newtonian potential system), given by,

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V}{\partial \mathbf{q}_i}$$

where $i = 1, ..., N$, is equivalent to the Euler - Lagrange equations (Lagrangian system), given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{1.1}$$

for the Lagrangian $L : \mathbb{R}^{2dN} \to \mathbb{R}$, defined by,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{N} \frac{1}{2} m_i ||\dot{\mathbf{q}}_i||^2 - V(\mathbf{q})$$

where, a Lagrangian system on a configuration space $\mathbb{R}^{dN}$ is the system of ODEs in (1.1), that is the Euler–Lagrange equations, for some function $L : \mathbb{R}^{2dN} \to \mathbb{R}$ called the Lagrangian.

Proof. Let L be defined as above then, we have,

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\sum_{i=1}^{N} \frac{1}{2} m_i ||\dot{\mathbf{q}}_i||^2 - V(\mathbf{q})\right) = \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial}{\partial \dot{\mathbf{q}}}\left(||\dot{\mathbf{q}}_i||^2\right) = m_i \dot{\mathbf{q}}_i$$

therefore,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \frac{d}{dt}\left(m_i \dot{\mathbf{q}}_i\right) = m_i \ddot{\mathbf{q}}_i$$

also,

$$\frac{\partial L}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}\left(\sum_{i=1}^{N}\frac{1}{2}m_i||\dot{\mathbf{q}}_i||^2 - V(\mathbf{q})\right) = -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}_i}$$

finally, we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = m_i\ddot{\mathbf{q}}_i + \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}_i} = 0$$

□

Note: The above Lagrangian(L) can vary as per the problem, that is, not all Lagrangian systems are Newtonian potential systems.

Example: The Lagrangian in the case of electromagnetic theory is given by,

$$L = \frac{1}{2}m||\dot{\mathbf{q}}||^2 + e\mathbf{A}(\mathbf{q})\cdot\dot{\mathbf{q}}$$

where,

m is the mass of the point charge,

e is the electric charge carried by a single electron,

$\mathbf{A}(\mathbf{q})$ is the vector potential of $\mathbf{B}(\mathbf{q})$ from which the Lorentz force law ($\mathbf{F} = e\dot{\mathbf{q}} \times \mathbf{B}$) can be derived.

Observe that the Lagrangian system is equivalent to the Hamilton's principle of stationary action, that is,

$$\delta S = 0$$

where, S, the action functional is given by,

$$S[\mathbf{q}(\cdot)] = \int_a^b L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt$$

Example *(The Catenary Problem)*:

Suppose we wish to find the curve that an ideal hanging chain or cable assumes under its own weight when supported only at its ends $x = -a$ and $x = a$ in a uniform gravitational field, then we will use Lagrangian mechanics or the action principle to find the equation of that curve.

Observe that the constraint on the hanging cable is:

$$I = \int_{-a}^{a} \sqrt{1 + (y')^2} dx$$

where,

$\sqrt{1 + (y')^2} dx$ is the infinitesimal length of the cable,

I is the length of the cable from $x = -a$ to $x = a$ and

y is the equation of the curve which we want to find

We want to find such y which minimises the potential energy of the hanging cable out of all curves that exist, that is, we have to find a y which minimises,

$$J = \int_{-a}^{a} \mu g y \sqrt{1 + (y')^2} dx$$

where,

J is the potential energy of the cable from $x = -a$ to $x = a$,

μ is the mass per unit length of the cable and

g is the acceleration due to gravity,

Hence, using Lagrange's method of undetermined coefficients we obtain the functional S which we wish to minimise (S includes the constraint too),

$$S = J + kI = \int_{-a}^{a} \left(\mu g y \sqrt{1 + (y')^2} + k \sqrt{1 + (y')^2} \right) dx$$

where, the Lagrangian L is given by,

$$L(x, y, \dot{y}) = \mu g y \sqrt{1 + (y')^2} + k \sqrt{1 + (y')^2}$$

since, $\frac{\partial L}{\partial \mathbf{x}} = 0$, we will employ a special case of Euler - Lagrange equation, namely the Beltrami identity,

$$L - \dot{\mathbf{y}}\left(\frac{\partial L}{\partial \dot{\mathbf{y}}}\right) = c_1$$

substituting the values and simplifying it, we have,

$$\mu g y\sqrt{1+(y')^2} + k\sqrt{1+(y')^2} - \frac{(y')^2(\mu g y + k)}{\sqrt{1+(y')^2}} = c_1 \implies \frac{(\mu g y + k)}{\sqrt{1+(y')^2}} = c_1$$

separating the variables,

$$\int \frac{dy}{\sqrt{\frac{(\mu g y + k)^2}{{c_1}^2} - 1}} = \int dx \implies \frac{c_1}{\mu g}\cosh^{-1}\left(\frac{\mu g y + k}{c_1}\right) = x + c_2$$

Since the curve y passes through the points $(-a, 0)$ and $(a, 0)$, we therefore have,

$$\frac{c_1}{\mu g}\cosh^{-1}\left(\frac{k}{c_1}\right) = a + c_2$$

$$\frac{c_1}{\mu g}\cosh^{-1}\left(\frac{k}{c_1}\right) = -a + c_2$$

which implies, $c_2 = 0$, from which we obtain,

$$k = c_1\cosh\left(\frac{\mu g a}{c_1}\right) \implies y = \frac{c_1}{\mu g}\left(\cosh\left(\frac{\mu g x}{c_1}\right) - \cosh\left(\frac{\mu g a}{c_1}\right)\right)$$

where, c_1 can be obtained using the constraint put on the length of the cable.

The above techniques from Lagrangian mechanics can also be used to derive the Euler equations of motion for rigid bodies.

1.2 Introduction to Manifolds

We define a (real) n-dimensional **manifold** as a **topological space** M for which every point $x \in M$ has a neighbourhood **homeomorphic** to Euclidean space $\mathbb{R}^n$.

To understand the above definition in a clear manner, let us first understand what a **topological space** is:

A collection T of subsets of M is a topology on M if,

1. $\phi, M \in T$

2. The union of an arbitrary collection of sets in T is in T

3. The intersection of finite number of sets in T is in T

The pair (M, T) is called as a topological space, where, generally M is referred to as an topological space if T is clear from the context.

Example: A metric space is a topological space, which is an ordered pair (M, d) where M is a set and d is a metric on M, that is, a function $d : M \times M \to \mathbb{R}$ satisfying the following axioms for all points $x, y, z \in M$:

1. The distance from a point to itself is zero: $d(x, x) = 0$

2. The distance between two distinct points is always positive: if $x \neq y$, then $d(x, y) > 0$

3. The distance from x to y is always the same as the distance from y to x: $d(x, y) = d(y, x)$

4. The triangle inequality holds: $d(x, z) \leq d(x, y) + d(y, z)$

Now let us understand what a **homeomorphism** is:

A homeomorphism is a bijective and continuous function between topological spaces that has a continuous inverse function.

Example: For all $a < b$, the set (a, b) is homeomorphic to $\mathbb{R}$. Since, the set (a, b) is homeomorphic to $(0, 1)$ via the map $x \to \frac{x-a}{b-a}$ (this map is a continuous bijection and its inverse, $x \to (b-a)x + a$, is also continuous). Similarly, $(0, 1)$ is homeomorphic to $\mathbb{R}$ via the map $x \to \tan(\pi x - \pi/2)$ (this map is a continuous bijection too and its inverse, $x \to \frac{1}{\pi}(\tan^{-1}(x) + \pi/2)$, is also continuous.

Some examples of **manifolds** are as follows:

1. One-dimensional manifolds include lines and circles, but not self-crossing curves such as a figure 8 (because that will violate the property that a manifold locally resembles an Euclidean space).
2. Two-dimensional manifolds are also called surfaces which include the plane, the sphere, and the torus
3. Real $n \times n$ matrices $M_n(\mathbb{R})$
4. General linear group: $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$
5. Special linear group: $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$
6. Orthogonal group: $O_n(R) = \{A \in GL_n(\mathbb{R}) \mid A^T = A^{-1}\}$
7. Special orthogonal group: $SO_n(\mathbb{R}) = O_n(\mathbb{R}) \cap SL_n(\mathbb{R})$

Tangent vector: A tangent vector can be thought of as an element in a tangent plane, or submanifold tangent space. More formally, if M is a submanifold of $\mathbb{R}^n$. A tangent vector to M is $g'(0)$ for some smooth path $g : \mathbb{R} \to M$ such that $g(0) = x$.

Example: Consider a particle moving on a submanifold M of $\mathbb{R}^n$. If $g(t)$ is the position of a particle at time t, then the velocity vector at time t is the derivative $g'(t)$. The vector $g'(t)$ is called a tangent vector to M, based at the point $g(t) \in M$.

Tangent space: The tangent space to M at x is the set of all tangent vectors based at x, denoted by T_xM.

Example: A plane passing through the point x on the surface of a 2 dimensional sphere represents the set of all tangent vectors to S^2 at x.

Tangent bundle: The tangent bundle of a submanifold M of $\mathbb{R}^n$, denoted by TM, is the union of all of the tangent spaces to M:

$$TM = \bigcup_{x \in M} T_x M$$

1.3 Elementary Differential Geometry

The concept of differential geometry starts with defining what a **vector field** is. A vector field on a manifold M is a map X from M to TM such that $X(z) \in T_z M$ for all $z \in M$, where TM is the tangent bundle of M.
Example: The vector field X on $\mathbb{R}^2$ defined by $X(x, y) = (-y, x)$ corresponds to the system of differential equations, $x' = -y$ and $y' = x$.
It has solution curves of the form $(x(t), y(t)) = (r\cos t + \omega, r\sin t + \omega)$.

Tensors: A covariant k-tensor on V is a multilinear map $V^k \to \mathbb{R}$, where $V^k = V \times \cdots \times V$, with k copies of V, and 'multilinear' means that the map is linear in each of its variables separately, holding the others fixed.
A contravariant k-tensor on V is a multilinear map $(V^*)^k \to \mathbb{R}$.
Example: First of all observe that, a rank 0 tensor is a scalar (a real number, invariant under rotation), a rank 1 tensor is a vector (a row of real numbers, the components of which, will change under rotations, although the vector will remain the same, in the sense that its magnitude and direction will be left unchanged), finally a rank w tensor is a tensor (a square array of real numbers, can be written

in the form of a matrix).

Covariant tensor (rank 2): A covariant tensor is a tensor, the components (A_{kl}) of which, transforms to (A'_{ij}) in the following manner:

$$A'_{ij} = \sum_{kl} \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_j} A_{kl}$$

Contravariant tensor (rank 2): A contravariant tensor is a tensor, the components (B^{kl}) of which, transforms to (B'^{ij}) in the following manner:

$$B'^{ij} = \sum_{kl} \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} B^{kl}$$

Riemannian manifold: A Riemannian metric on a manifold Q is a smooth symmetric positive-definite covariant 2-tensor field g on Q. The pair (Q, g) is called a Riemannian manifold.

Example: The well known Pythagoras theorem, when extended to n-dimensions, gives,

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2$$

Here, $Q = \mathbb{R}^n$, which makes it an example of a Riemannian manifold when the Riemannian metric becomes equal to the Euclidean metric for the case of Cartesian coordinates, that is,

$$g_{ij} = \delta_{ij}$$

where, δ_{ij} is the usual Kronecker delta (since, $ds^2 = g_{ij} dx^i dx^j$).

Similar expression can be obtained in the case of polar coordinates, by performing the transformation $(x, y) = (r\cos(\theta), r\sin(\theta))$, which gives,

$$ds^2 = (dr)^2 + r^2(d\theta)^2$$

Symplectic manifold: A symplectic form is a closed non-degenerate 2-form. If ω is a symplectic form on a manifold M, then the pair (M, ω) is called a symplectic manifold.

Example: The cross product of two vectors (that is finding the area of the parallelogram determined by the two vectors) in $\mathbb{R}^2$ is similar to the canonical symplectic bilinear form, that is,

$$\omega((x_1, y_1), (x_2, y_2)) := x1 \cdot y2 - x2 \cdot y1$$

Its matrix, with respect to the standard basis, is $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

1.4 Lie Groups and Lie Algebra

Matrix Lie Groups: A matrix group is a subset of $M(n, \mathbb{R})$, or $M(n, \mathbb{C})$, that is a group, with the group operation being matrix multiplication. A matrix Lie group is a matrix group that is also a submanifold of $M(n, \mathbb{R})$ or $M(n, \mathbb{C})$.

Example:

1. General linear group: $GL_n(\mathbb{R})$ (dimension $= n^2$)
2. Special linear group: $SL_n(\mathbb{R})$ (dimension $= n^2 - 1$)
3. Orthogonal group: $O_n(R)$ (dimension $= (n^2 - n)/2$)
4. Special orthogonal group: $SO_n(\mathbb{R})$ (dimension $= (n^2 - n)/2$)
5. Symplectic group: $\mathrm{Sp}(2n, \mathbb{R})$ (dimension $= 2n^2 + n$)
6. Special Euclidean group: $SE(3)$

where, $\mathrm{Sp}(2n, \mathbb{R}) = \{U \in GL_{2n}(\mathbb{R}) : U^T J U = J\}$ and $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ and $SE(3)$

is the Lie group of 4×4 matrices of the form $\begin{bmatrix} \mathbf{R} & \mathbf{v} \\ 0 & 1 \end{bmatrix}$ where, $\mathbf{R} \in SO(3)$ and $\mathbf{v} \in \mathbb{R}^3$.

Lie Groups: A Lie group is a smooth manifold that is also a group, with the property that the operations of group multiplication, $(g, h) \to gh$ and inversion, $g \to g^{-1}$, are smooth.

Example:

1. For dimension one, we have the real line $\mathbb{R}$ (with the group operation being addition) and the circle group S^1 of complex numbers with absolute value one (with the group operation being multiplication).

2. For two dimensions, we have a simply connected Lie group such as $\mathbb{R}^2$ (with the group operation being vector addition).

3. Every matrix Lie group is a Lie group. Indeed, it is a manifold by definition, and the operations of matrix multiplication and matrix inversion are smooth because they are rational operations on the matrix entries.

Lie Algebra: A (real) Lie algebra is a (real) vector space A together with a bilinear operation $(v, w) \in V \to [v, w] \in V$, called the bracket, such that

1. $[v, w] = -[w, v]$ for all $v,\ w \in V$ (skew-symmetry),

2. $[[v, w], u] + [[u, v], w] + [[w, u], v] = 0$ for all $v,\ w,\ u \in V$ (Jacobi identity).

Example:

1. The vector space $\mathbb{R}^3$ is a Lie algebra when endowed with a bracket given by the usual vector cross-product $[x, y] := x \times y$.

2. Every matrix Lie algebra, with bracket given by commutator $[A, B] := AB - BA$, is a Lie algebra.

3. The space of linear maps on a vector space V. The bracket is $[f, g] = f \circ g - g \circ f$.

1.5 Practical Implications

Geometric mechanics has profound applications in robotics and automation, autonomous vehicles in marine, aerospace, and other environments, flight control, problems in nuclear magnetic resonance, fluid mechanics, etc. The book under review specifically mentions how the theory of geometric mechanics has applications in soliton theory, image analysis and fluid mechanics.
Some of them are listed below:

- **Symplectic Geometry and Hamiltonian Formulation**: Soliton equations often have a Hamiltonian structure. The study of solitons within the symplectic geometry framework allows for a deeper understanding of the underlying Hamiltonian structure of the associated PDEs. The soliton equations can be reformulated in terms of Hamiltonian systems, and the associated symplectic structure can be used to study the conservation laws and integrability properties of solitons.

- **Diffeomorphic Image Registration:** Geometric mechanics can contribute to the development of diffeomorphic image registration techniques. Diffeomorphic transformations, which are smoothly varying and invertible, can be modeled using the geometric structures provided by symplectic manifolds.

- **Hamiltonian Formulation of Fluid Dynamics**: Geometric mechanics provides a natural framework for the Hamiltonian formulation of fluid dynamics. The fluid equations, such as the Euler and Navier-Stokes equations, can be recast in Hamiltonian form, revealing the underlying symplectic structure of the flow.

Chapter 2

Conclusions

2.1 Study Summary

The first few chapters of the book under review aim to present a unified perspective on Lagrangian and Hamiltonian mechanics, adopting a coordinate-free language rooted in differential geometry, influenced by the Marsden-Ratiu school. While sharing content similarities with Marsden and Ratiu's formal treatise, it maintains a less formal tone.

The book primarily focuses on applying Lagrangian and Hamiltonian mechanics to a diverse range of systems, including N-particle systems, rigid bodies, and continua like fluids and electromagnetic systems. This approach, grounded in differential geometry, allows for a seamless transition between finite-dimensional conservative systems (covered in the first part) and the infinite-dimensional scenarios explored in the second part. The central theme throughout the book is the role of symmetry in enabling a reduction in system dimensions and leveraging conserved quantities, such as momentum maps.

The first part systematically introduces concepts in a clear and progressive manner. Starting with a recapitulation of Newtonian, Lagrangian, and Hamiltonian mechanics in Euclidean space, the groundwork for extending these mechanics to systems on manifolds is laid out in Chapters 2 and 3. Chapter 4 introduces symmetry, reduction, and conservation laws, setting the stage for the subsequent exploration of Lie groups and algebras in Chapters 5 and 6. These chapters are crucial for understanding mechanical systems with configuration spaces represented as Lie groups.

The book's clarity is evident through systematic illustrations, well-chosen examples, and exercises (with selected solutions) to aid understanding. Prerequisites include a solid background in linear algebra, multivariate calculus, standard methods for solving ODEs and PDEs, and possibly variational principles.

In conclusion, the book offers a comprehensive presentation of geometric mechanics, suitable for graduate students and specialists seeking a modern and unified formulation of Lagrangian and Hamiltonian mechanics. Its emphasis on applications across various domains, coupled with clear explanations and illustrative examples, makes it a valuable resource for those delving into the intricacies of advanced classical mechanics.

2.2 Scope for Future Work

The future scope of geometric mechanics and symmetry in research in various branches of engineering holds tremendous potential for advancing our understanding of complex physical systems and developing innovative applications across various scientific and engineering domains. Here are some of the potential avenues for future research:

- **Incorporation in Robotics and Control Systems**: Applying geometric mechanics and symmetry principles to the design and control of robotic systems. Investigating and understanding the underlying geometry of robotic motion can lead to more efficient control algorithms, improved stability, and better overall performance in robotic applications.

- **Machine Learning and Data Analysis:** The field of machine learning and data analysis can be explore deeply using insights from geometric mechanics. Developing algorithms that leverage the inherent symmetries in data to improve learning efficiency, enhance pattern recognition, and provide more robust solutions in various domains, including computer vision and natural language processing.

- **Distributed and Multi-agent Systems**: Applying techniques from the theory of geometric control we can address various challenges in distributed and multi-agent systems. Investigating how symmetry principles can be utilized to design decentralized controllers that ensure stability, coordination, and optimal performance in large-scale networks of interconnected systems. Exploring applications in fields such as swarm robotics, autonomous vehicles, and networked control systems.

- **Adaptive Control and Learning Systems**: Incorporating symmetry considerations into adaptive control algorithms could help us in developing adaptive controllers that exploit the inherent symmetries in the system dynamics, leading to improved performance, robustness, and faster convergence in learning-based control systems. Investigating the interplay between geometric structures, symmetry, and machine learning techniques we can enhance the adaptability of control systems in uncertain environments.

- **Geometric Formulation of Fluid Equations**: Geometric mechanics provides a powerful framework for expressing fluid dynamics equations in a coordinate - free manner. This approach can lead to more elegant and insightful formulations of the governing equations, making it easier to uncover underlying geometric structures inherent in fluid flow.

In summary, the future of geometric mechanics and symmetry holds promise for groundbreaking discoveries and applications across a wide range of scientific and technological domains. Embracing interdisciplinary approaches and leveraging advanced mathematical tools will likely lead to transformative insights and innovations in our understanding of the natural world. Especially, the future scope of geometric mechanics in control system research involves pushing the boundaries of traditional control theory to address the challenges posed by increasingly complex and interconnected systems. The integration of geometric methods and symmetry principles is expected to open new avenues for designing more efficient, adaptive, and reliable control systems across diverse domains.

References

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