A Flawed Proof

Introduction

In this article, we will demonstrate how some common mistakes in complex analysis lead to the formula for the sum of the first n natural numbers, thus making it a faulty/flawed/misleading proof.

Prerequisites

If z_k is a complex number such that,

$$z_k = e^{2\pi ik/n}$$

where $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, then the following holds,

$$z_0 z_1 z_2 ... z_{n-1} = (-1)^{n-1} \tag{1}$$

We can prove (1) by using the fact that the product of the roots of the equation

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... a_0 = 0 \tag{2}$$

is given by,

$$\alpha_1 \alpha_2 \alpha_3 ... \alpha_n = (-1)^n \frac{a_0}{a_n}$$

where, $a_0, a_1, a_2 ... a_n \in \mathbb{R}$, $a_n \neq 0$ and $\alpha_1, \alpha_2, \alpha_3 ... \alpha_n$ are the roots of (2).

Faulty Proof

Let us take the complex logarithm of both the sides in (1),

$$\ln(z_0) + \ln(z_1) + \ln(z_2) + ... + \ln(z_{n-1}) = (n-1)\ln(-1)$$

thus,

$$\sum_{k=0}^{n-1} \ln(z_k) = (n-1)i\pi$$

since, $\ln(z_k) = 2\pi ik/n$, we obtain,

$$\sum_{k=0}^{n-1} \frac{2\pi ik}{n} = (n-1)i\pi \implies \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2} \implies \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Conclusion

Though we got the correct result, some of the steps/formulae we used in the above section were incorrect. One should have a sound understanding of complex roots, single/multivalued functions, branch points, branch cut, principal branch, principal values, Riemann surfaces, etc to avoid these kind of mistakes.
The first mistake/ambiguity arises in (1), where we could have taken $(-1)^{n+1}$ instead of $(-1)^{n-1}$, leading to an incorrect result.

Secondly, while taking the logarithm, one should take care that $\ln(x^y) = y\ln(x)$ is valid only when x is a positive real number and y is a real number. This mistake can be seen from the equation we used earlier, that is,

$$\ln((-1)^{n-1}) = (n-1)\ln(-1)$$

which is not true for some n.

Lastly, one should also take care of the fact that $\ln(z)$ is a multivalued function and has a branch point at $z = 0$, as a result, adding any integral multiple of 2π in $arg(z)$ gives another possible solution, thus one should use the formula

$$\ln(a+ib) = \ln(\sqrt{a^2+b^2}) + i\arg(a+ib) + 2\pi im$$

where m is an integer and then define the principal branch of the complex logarithm function where $\arg(z)$ (hence $\ln(z)$) becomes single-valued and thus takes principal values only. This case appears while taking the complex logarithm of -1 and z_k.

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Web: http://www.ssmrmh.ro

ANGAD SINGH
email-id: angadsingh1729@gmail.com