A note on Euler-Mascheroni Constant

1 Introduction

The Euler-Mascheroni constant γ, is defined as,

$$\gamma = \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - ln(n)\right)$$

or

$$\gamma = \lim_{n\to\infty} H_n - ln(n)$$

where H_n is the n^{th} harmonic number and $ln(n)$ is the natural logarithm of n.
It has the numerical value,

$$\gamma = 0.57721566490153286060651209008240243104215933593992...$$

In this paper, I prove the existence of the above constant in a different way.

2 Definition

Let us define a function $G(x)$ in the following way,

$$G(x) = \int_0^1 ln(t)ln(1 - t^x)dt$$

where $x \in \mathbb{R}^+ - \{0\}$.
Now, using the known expansion of $ln(1 - x)$ we have,

$$-ln(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \tag{1}$$

Thus using (1), we get,

$$\int_0^1 ln(t)ln(1-t^x)dx = \int_0^1 ln(t) \sum_{k=1}^{\infty} \frac{-t^{xk}}{k} dt = -\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 t^{kx} ln(t)dt = \sum_{k=1}^{\infty} \frac{1}{k(kx + 1)^2}$$

Thus,

$$G(x) = \sum_{k=1}^{\infty} \frac{1}{k(kx + 1)^2}$$

Now let us prove that, if

$$L = \int_0^1 G(1/x)dx \tag{2}$$

then, L exists and $L \geq 0$.
Proof: Observe that,

$$G(x) \geq 0$$

for all $x \in \mathbb{R}^+ - \{0\}$.
Hence,

$$G(1/x) \geq 0$$

thus,

$$L = \int_0^1 G(1/x)dx \geq 0$$

therefore,

$$L \geq 0 \tag{3}$$

Now consider,

$$G(1/x) = \sum_{k=1}^{\infty} \frac{x^2}{k(k+x)^2} \leq \sum_{k=1}^{\infty} \frac{x^2}{k^3} = \zeta(3)x^2$$

thus,

$$L = \int_0^1 G(1/x)dx \leq \int_0^1 \zeta(3)x^2 dx = \frac{\zeta(3)}{3}$$

Hence,

$$L \leq \frac{\zeta(3)}{3} \tag{4}$$

Using (3) and (4),

$$0 \leq L \leq \frac{\zeta(3)}{3} \tag{5}$$

Hence, L exists and $L \geq 0$.
Now let us calculate L,

$$L = \int_0^1 G(1/x)dx = \int_0^1 \sum_{k=1}^{\infty} \frac{x^2}{k(k+x)^2}dx = \sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \frac{x^2}{(k+x)^2}dx$$

simplifying it further we have,

$$L = \sum_{k=1}^{\infty} \left(\frac{1}{k(k+1)} - 2ln\Big(\frac{k+1}{k}\Big) + \frac{2}{k+1} \right)$$

now,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1$$

thus,

$$L = 1 + 2\sum_{k=1}^{\infty} \left(\frac{1}{k+1} - ln\Big(\frac{k+1}{k}\Big) \right) = -1 + 2\lim_{n\to\infty} H_n - ln(n)$$

using (5),

$$0 \leq -1 + 2\lim_{n\to\infty} H_n - ln(n) \leq \frac{\zeta(3)}{3}$$

finally,

$$\frac{1}{2} \leq \gamma \leq \frac{1}{2} + \frac{\zeta(3)}{6}$$

3 References

[1] B. C. Berndt, Ramanujans Notebooks, Part I, SpringerVerlag, New York, 1985.

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