On Ramanujan's 133^{rd} Birth Anniversary: Famous constants related to Srinivasa Ramanujan

1 Almost Integer (Ramanujan's Constant)

Ramanujan's constant is the transcendental number $e^{\pi\sqrt{163}}$, which is an almost integer, in that it is very close to an integer:

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925... \approx 640320^3 + 744$$

This number was discovered in 1859 by the mathematician Charles Hermite. In a 1975 April Fool article in Scientific American magazine, "Mathematical Games" columnist Martin Gardner made the hoax claim that the number was in fact an integer, and that the Indian mathematical genius Srinivasa Ramanujan had predicted it—hence its name. This coincidence is explained by complex multiplication and the q-expansion of the j-invariant.

2 Landau–Ramanujan constant

In mathematics and the field of number theory, the Landau–Ramanujan constant is the positive real number K that occurs in a theorem proved by Edmund Landau in 1908, stating that for large x, the number of positive integers below x that are the sum of two square numbers behaves asymptotically as

$$\frac{Kx}{\sqrt{\ln(x)}}$$

This constant K was rediscovered in 1913 by Srinivasa Ramanujan, in the first letter he wrote to G.H. Hardy.
Landau's theorem states that if $N(x)$ is the number of positive integers less than x that are the sum of two squares, then

$$\lim_{x\to\infty} \frac{N(x)}{x/\sqrt{\ln(x)}} = K = 0.76422365358922066299069873125009232811679054 1...$$

where K is the Landau–Ramanujan constant.
This constant was stated by Landau in the limit form above; Ramanujan

instead approximated $N(x)$ as an integral, with the same constant of proportionality, and with a slowly growing error term.

3 Ramanujan–Soldner constant

In mathematics, the Ramanujan–Soldner constant (also called the Soldner constant) is a mathematical constant defined as the unique positive zero of the logarithmic integral function. It is named after Srinivasa Ramanujan and Johann Georg von Soldner.
Its value is approximately $\mu \approx 1.45136923488338105028396848589202744949303228...$
Since the logarithmic integral is defined by

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln(t)}$$

we have

$$\mathrm{li}(x) = \mathrm{li}(x) - \mathrm{li}(\mu)$$

thus

$$\mathrm{li}(x) = \int_\mu^x \frac{dt}{\ln(t)}$$

where $\mathrm{li}(\mu) = 0$.

4 Reciprocal of π

Ramanujan is famous for his 17 series on $1/\pi$ such as:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{(k!)^4} \frac{26390k + 1103}{396^{4k}}$$

To see more about his work on $1/\pi$ and his paper "Modular equations and approximations to π", click on the below link:
http://ramanujan.sirinudi.org/Volumes/published/ram06.pdf

5 Sum of all natural numbers

Srinivasa Ramanujan presented two derivations of $1+2+3+4+... = -1/12$ in chapter 8 of his first notebook.

First derivation:

$$c = 1 + 2 + 3 + 4 + 5 + 6 + ...$$

$$4c = 4 + 8 + 12$$

$$c - 4c = 1 - 2 + 3 - 4 + 5 - 6 + ...$$

The second key insight is that the alternating series $1 - 2 + 3 - 4 + ...$ is the formal power series expansion of the function $1/(1+x)^2$ but with x defined as 1. Accordingly, Ramanujan writes:

$$-3c = 1 - 2 + 3 - 4 + = \frac{1}{(1+1)^2} = \frac{1}{4}$$

Dividing both sides by 3, one gets $c = -1/12$.

Second derivation (using Ramanujan Summation):
Ramanujan summation is a technique invented by the mathematician Srinivasa Ramanujan for assigning a value to divergent infinite series. Although the Ramanujan summation of a divergent series is not a sum in the traditional sense, it has properties which make it mathematically useful in the study of divergent infinite series, for which conventional summation is undefined.
For a function f, the classical Ramanujan sum of the series $\sum_{k=1}^{\infty} f(k)$ is defined as:

$$\sum_{k=1}^{\infty} f(k) = -\frac{1}{2} f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{2k-1}(0)$$

where $f^{(2k-1)}$ is the $(2k-1)$-th derivative of f and B_{2k} is the $2k$-th Bernoulli number: $B_2 = 1/6, B_4 = -1/30$, and so on. Setting $f(x) = x$, the first derivative of f is 1, and every other term vanishes, so:

$$\sum_{k=1}^{\infty} k = -\frac{1}{6} \times \frac{1}{2!} = -\frac{1}{12}$$

6 Taxicab Number

The n-th taxicab number, typically denoted Ta(n) or Taxicab(n), also called the n-th Hardy–Ramanujan number, is defined as the smallest integer that can be expressed as a sum of two positive integer cubes in n distinct ways. The most famous taxicab number is $1729 = \mathrm{Ta}(2) = 1^3 + 12^3 = 9^3 + 10^3$.

The name is derived from a conversation in about 1919 involving mathematicians G. H. Hardy and Srinivasa Ramanujan. As told by Hardy:

I remember once going to see him [Ramanujan] when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to be rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways.

So far, the following 6 taxicab numbers are known:

$$\mathrm{Ta}(1) = 2 = 1^3 + 1^3$$

$$\begin{aligned}\mathrm{Ta}(2) = 1729 &= 1^3 + 12^3\\ &= 9^3 + 10^3\end{aligned}$$

$$\begin{aligned}\mathrm{Ta}(3) = 87539319 &= 167^3 + 436^3\\ &= 228^3 + 423^3\\ &= 255^3 + 414^3\end{aligned}$$

$$\begin{aligned}\mathrm{Ta}(4) = 6963472309248 &= 2421^3 + 19083^3\\ &= 5436^3 + 18948^3\\ &= 10200^3 + 18072^3\\ &= 13322^3 + 16630^3\end{aligned}$$

$$\begin{aligned}\mathrm{Ta}(5) = 48988659276962496 &= 38787^3 + 365757^3\\ &= 107839^3 + 362753^3\\ &= 205292^3 + 342952^3\\ &= 221424^3 + 336588^3\\ &= 231518^3 + 331954^3\end{aligned}$$

$$\begin{aligned}\mathrm{Ta}(6) = 24153319581254312065344 &= 582162^3 + 28906206^3\\ &= 3064173^3 + 28894803^3\\ &= 8519281^3 + 28657487^3\\ &= 16218068^3 + 27093208^3\\ &= 17492496^3 + 26590452^3\\ &= 18289922^3 + 26224366^3\end{aligned}$$

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