Sum of Divisors Function Inequality

Introduction

In this article, we will derive some inequalities related to $\sigma(n)$ using q-series.

Prerequisites

Definition 1

For all $n \in \mathbb{N}$, we define $\sigma(n)$ as the sum of all the divisors of n, that is,

$$\sigma(n) = \sum_{d|n} d$$

Definition 2

If $m, n \in \mathbb{N}$ and $1 \leq m \leq n$, then,

$$G(m, n) \coloneqq G(m, n; q) = (e^{\pi i m/n} q)_\infty$$

Lemma 1

If $|q| < 1$, then we have,

$$-\ln((q)_\infty) = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k} q^k$$

Proof: We know that if $|q| < 1$, then,

$$(q)_\infty = \prod_{k=1}^{\infty} (1 - q^k)$$

therefore,

$$\ln((q)_\infty) = \ln\left(\prod_{k=1}^{\infty}(1 - q^k)\right) = \sum_{k=1}^{\infty} \ln(1 - q^k) = -\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{mk}}{m}$$

the above equation can be written as,

$$\ln((q)_\infty) = -\sum_{k=1}^{\infty}\left(\sum_{d|k}\frac{1}{d}\right)q^k = -\sum_{k=1}^{\infty}\left(\sum_{d|k}\frac{d}{k}\right)q^k = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{d|k}d\right)q^k$$

using definition 1, we finally have,

$$\ln((q)_\infty) = -\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k$$

which completes the proof of lemma 1.

Lemma 2

If $z \in \mathbb{C}$, then we have,

$$|\ln(z)| \geq \ln(|z|)$$

Proof: Let $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$, then,

$$|\ln(z)| = |\ln(r)+i\theta| = \sqrt{\ln^2(r) + \theta^2} \geq \sqrt{\ln^2(r)} = |\ln(r)| = |\ln(|z|)| \geq \ln(|z|)$$

which completes the proof of lemma 2.

Lemma 3

If $|\alpha| = 1$ and $|q| < 1$, then we have,

$$2\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k \geq \ln(|(\alpha q)_\infty|^2)$$

Proof: If $|q| < 1$, then by using lemma 1, we have,

$$-\ln((q)_\infty) = \sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k$$

replacing q by αq, where $|\alpha| = 1$ we have,

$$-\ln((\alpha q)_\infty) = \sum_{k=1}^{\infty}\frac{\sigma(k)}{k}(\alpha q)^k$$

observe that,

$$|\ln((\alpha q)_\infty)| = \left|\sum_{k=1}^{\infty} \frac{\sigma(k)}{k}(\alpha q)^k\right| \le \sum_{k=1}^{\infty}\left|\frac{\sigma(k)}{k}(\alpha q)^k\right| = \sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k$$

using lemma 2, we obtain,

$$|\ln((\alpha q)_\infty)| \ge \ln(|(\alpha q)_\infty|)$$

therefore,

$$\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k \ge |\ln((\alpha q)_\infty)| \ge \ln(|(\alpha q)_\infty|)$$

which completes the proof of lemma 3.

Inequality

If $m, n \in \mathbb{N}$ and $1 \le m \le n$, then we have the following inequality,

$$2\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k \ge \ln(|G(m,n)|^2)$$

Proof: From [1], definition 2 and lemma 3, we obtain,

$$\ln(|G(m,n)|^2) = \ln(|(e^{i\pi m/n}q)_\infty|^2) \le 2\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k$$

substituting $m = 1$ and $n = 2$, we have,

$$2\sum_{k=1}^{\infty}\frac{\sigma(k)}{k}q^k \ge \ln\left(\frac{(q^4)_\infty^8}{(q^2)_\infty^3(q^8)_\infty^3}\right)$$

comparing the coefficient of q^{8k} in both the sides, we have,

$$2\frac{\sigma(8k)}{8k} \ge -8\frac{\sigma(2k)}{2k} + 3\frac{\sigma(4k)}{4k} + 3\frac{\sigma(k)}{k}$$

thus,

$$16\sigma(2k) + \sigma(8k) \ge 12\sigma(k) + 3\sigma(4k)$$

similarly, for $m = 2$ and $n = 3$, we have,

$$12\sigma(3k) + \sigma(9k) \ge 9\sigma(k)$$

and for $m = 1$ and $n = 3$, we have,

$$36\sigma(k)+9\sigma(4k)+72\sigma(6k)+4\sigma(9k)+3\sigma(36k) \ge 54\sigma(2k)+48\sigma(3k)+12\sigma(12k)+6\sigma(18k)$$

References

[1] Angad Singh *Special function and q-series* (Romanian Mathematical Magazine, 2021).

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