Integral Problem

Prove that,

$$erfc(1) < 1/e$$

Proof: Let us define a function $\psi(m, a)$ as,

$$\psi(m,a) = \int_0^\infty \frac{e^{-x^2} x^{m-1}}{(1+x^2)^a} dx$$

It is easy to observe that, if $m - 1 \geq 2a$ and $a \neq 0$, then,

$$\psi(m,a) < \frac{\Gamma(\frac{m}{2} - a)}{2}$$

since $(1+x^2)^a > x^{2a}$ $\forall x \in \mathbb{R}$. Hence,

$$\psi(m,a) < \int_0^\infty e^{-x^2} x^{m-2a-1} dx = \frac{1}{2} \int_0^\infty e^{-x} x^{\frac{m}{2}-a-1} dx = \frac{\Gamma(\frac{m}{2} - a)}{2}$$

Substituting $m = 2$ and $a = 1/2$, we obtain,

$$\psi(2, 1/2) < \frac{\sqrt{\pi}}{2} \tag{1}$$

We know that, the error function is defined as,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{2}$$

and the complimentary error function is defined as,

$$erfc(x) = 1 - erf(x)$$

Differentiating (2) w.r.t x, we get,

$$\frac{d}{dx} erf(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

Generalising the above result we have,

$$\frac{d}{dx} erf(f(x)) = \frac{2}{\sqrt{\pi}} e^{-(f(x))^2} f'(x)$$

Put $f(x) = (1+x^2)^{1/2}$ and integrating both the sides from 0 to ∞ w.r.t x,

$$erf(\infty) - erf(1) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(1+x^2)}x}{(1+x^2)^{1/2}} dx = \frac{2}{e\sqrt{\pi}} \psi(2, 1/2)$$

Since $erf(\infty) = 1$, thus,

$$\psi(2, 1/2) = \frac{e\sqrt{\pi} erfc(1)}{2}$$

Using (1), we have,

$$\frac{e\sqrt{\pi} erfc(1)}{2} < \frac{\sqrt{\pi}}{2}$$

Finally,

$$erfc(1) < 1/e$$

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