On the product $\prod_{k=1}^{m-1}\left(1+\frac{1}{2k}\right)^{k^3}$

1 Introduction

In this paper, I prove an inequality related to the mentioned product using the sum of the square roots of first n natural numbers.

2 Main Proof

Consider the sum,

$$\sum_{k=1}^{n}\sqrt{k} \tag{1}$$

Let us find a lower bound for this sum.
We know that, using the method of continued fractions,

$$\sqrt{n^2+d} \approx n+\frac{2nd}{4n^2+d} \tag{2}$$

Also, if d is positive then,

$$\sqrt{n^2+d} \geq n+\frac{2nd}{4n^2+d} \tag{3}$$

Let us divide (1) into three parts, that is, the sum over perfect squares, the sum over the numbers between two consecutive perfect squares and the sum over the numbers greater than the largest perfect square. Thus, we have,

$$\sum_{k=1}^{n}\sqrt{k}=\sum_{k=1}^{m}\sqrt{k^2}+\sum_{k=1}^{m-1}\sum_{i=k^2+1}^{(k+1)^2-1}\sqrt{i}+\sum_{k=m^2+1}^{n}\sqrt{k} \tag{4}$$

where $m=\lfloor\sqrt{n}\rfloor$.
Now simplifying (4) by substituting (2) and by noting (3), we obtain,

$$\sum_{k=1}^{n}\sqrt{k} \geq \frac{m(m+1)}{2}+\sum_{k=1}^{m-1}\sum_{i=k^2+1}^{(k+1)^2-1}\left(k+\frac{2k(i-k^2)}{4k^2+(i-k^2)}\right)+\sum_{k=m^2+1}^{n}\left(m+\frac{2m(k-m^2)}{3m^2+k}\right)$$

Simplifying the above sum further, we obtain,

$$\sum_{k=1}^{n}\sqrt{k} \geq f(m,n) - 8m^3\sum_{k=1}^{n-m^2}\Big(\frac{1}{4m^2+k}\Big) - 8\sum_{k=1}^{m-1}k^3\sum_{i=1}^{2k}\Big(\frac{1}{4k^2+i}\Big) \qquad (5)$$

where $f(m,n) = \frac{m(m+1)}{2} + m(m-1)(2m-1) + 3m(n-m^2)$.
It is known from elementary calculus that,

$$\sum_{k=1}^{n}\frac{1}{k+m} \approx ln\Big(1+\frac{n}{m}\Big) \qquad (6)$$

Thus using (6) in (5), we obtain,

$$\sum_{k=1}^{n}\sqrt{k} \geq f(m,n) - 8m^3 ln\Big(\frac{3}{4}+\frac{n}{4m^2}\Big) - 8ln(P(m))$$

where $P(m) = \prod_{k=1}^{m-1}\left(1+\frac{1}{2k}\right)^{k^3}$.
Put $n = m^2$, to get,

$$ln(P(m)) \geq \frac{1}{8}\Big(\frac{m(m+1)}{2} + m(m-1)(2m-1) - \sum_{k=1}^{m^2}\sqrt{k}\Big)$$

Using the same method, we can obtain a similar kind of inequality using the sum of the cube roots of first n^3 natural numbers.

Corollary: If $a(k) := a = \lfloor\frac{k^2(3k+2)}{2k+1}\rfloor$ and $b(k) := b = \lfloor\frac{3k^3+7k^2+5k+1)}{2k+1}\rfloor$, then,

$$\sum_{k=1}^{n^3}k^{1/3} < \frac{n(n+1)}{2} + \sum_{k=1}^{n-1}\Big(ka + (k+1)b + \frac{a(a+1)}{6k^2} - \frac{b(b+1)}{6(k+1)^2}\Big)$$

3 References

[1] On the sum of the square roots of the first n natural numbers, *Journal of the Indian Mathematical Society*, VII, 1915, 173 – 175
[2] The sum of the r^{th} roots of first n natural numbers and new formula

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