106.01 The number of divisors of the LCM of the first *n* natural numbers

We begin by defining three functions L, d and π.

$L(n)$ gives the least number divisible by all natural numbers from 1 to n. For example, if $n = 5$, then $L(5) = 60$.

$d(n)$ gives the number of divisors of a natural number n. For $n = 10$, $d(10) = 4$, the factors of 10 being 1, 2, 5 and 10.

$\pi(n)$ defines the number of primes less than or equal to n. $\pi(20)$, for example, is 8 since the primes are 2, 3, 5, 7, 11, 13, 17 and 19.

For a natural number n and prime p, we have

$$L(n) = \prod_{p \leqslant n} p^{\lfloor \log_p(n) \rfloor} \tag{1}$$

for if m is the highest power of p such that $p^m \leqslant n$, then $m \leqslant \log_p(n)$ or $m = \lfloor \log_p(n) \rfloor$.

We now show that

$$d(L(n)) = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{\pi(\sqrt[k]{n})}$$

for $n \in \mathbb{N}$.

We can represent $L(n)$, given in (1), as

$$L(n) = \prod_{\sqrt{n} < p \leqslant n} p^{\lfloor \log_p(n) \rfloor} \prod_{\sqrt[3]{n} < p \leqslant \sqrt{n}} p^{\lfloor \log_p(n) \rfloor} \prod_{\sqrt[4]{n} < p \leqslant \sqrt[3]{n}} p^{\lfloor \log_p(n) \rfloor} \dots . \tag{2}$$

Here we consider $\lfloor \log_p(n) \rfloor$ in the intervals as shown.

To find $\lfloor \log_p(n) \rfloor$ in $(\sqrt{n}, n]$, observe that any prime p lying in the interval $(\sqrt{n}, n]$ occurs only once in the prime factorisation of $L(n)$ because $p > \sqrt{n} \Rightarrow p^2 > n$, hence p^2 lies outside the interval $(\sqrt{n}, n]$. Therefore, $\lfloor \log_p(n) \rfloor = 1$ for the interval $(\sqrt{n}, n]$.

Similarly, any prime p in the interval $(\sqrt[3]{n}, \sqrt{n}]$ occurs exactly twice in the prime factorisation of $L(n)$ because $p > \sqrt[3]{n} \Rightarrow p^3 > n$. Hence p^3 lies outside the interval $(\sqrt[3]{n}, \sqrt{n}]$. Thus, $\lfloor \log_p(n) \rfloor = 2$ for the interval $(\sqrt[3]{n}, \sqrt{n}]$ and so on. Hence, (2) becomes

$$L(n) = \prod_{\sqrt{n} < p \leqslant n} p \prod_{\sqrt[3]{n} < p \leqslant \sqrt{n}} p^2 \prod_{\sqrt[4]{n} < p \leqslant \sqrt[3]{n}} p^3 \dots . \tag{3}$$

For example, let $n = 30$. Then primes in the interval $(\sqrt{30} < p \leqslant 30]$ are 7, 11, 13, 17, 19, 23 and 29, the prime in the interval $(\sqrt[3]{30} < p \leqslant \sqrt{30}]$ is 5, the prime in the interval $(\sqrt[4]{30} < p \leqslant \sqrt[3]{30}]$ is 3 and the prime in the interval $(\sqrt[5]{30} < p \leqslant \sqrt[4]{30}]$ is 2. Thus,

$$\begin{aligned} L(30) &= (7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29)(5^2)(3^3)(2^4) \\ &= 2329089562800. \end{aligned}$$

Now, if $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$, then

$$d(n) = (1 + a_1)(1 + a_2)(1 + a_3)\dots(1 + a_k).$$

Using this relation in (3), we obtain

$$d(L(n)) = (1 + 1)^{\pi(n) - \pi(\sqrt{n})}(1 + 2)^{\pi(\sqrt{n}) - \pi(\sqrt[3]{n})}(1 + 3)^{\pi(\sqrt[3]{n}) - \pi(\sqrt[4]{n})}\dots .$$

Simplifying further we obtain

$$d(L(n)) = \left(2^{\pi(n) - \pi(\sqrt{n})}\right)\left(3^{\pi(\sqrt{n}) - \pi(\sqrt[3]{n})}\right)\left(4^{\pi(\sqrt[3]{n}) - \pi(\sqrt[4]{n})}\right)\dots .$$

Finally this becomes

$$d(L(n)) = \prod_{k=1}^{\infty} (k + 1)^{\pi(\sqrt[k]{n}) - \pi(\sqrt[k+1]{n})}$$

which, after some algebra, gives us

$$d(L(n)) = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{\pi(\sqrt[k]{n})}$$

as required.

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