The Mathematical Gazette, July 2021

105.19 An inverted cone and Fermat's last theorem

Introduction

Pierre de Fermat conjectured around 1637 that there are no three positive integers a, b and c that satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles, and formally published in 1995 using elliptic curves and the modularity theorem. Here we provide a simple application of Fermat's theorem.

Main proof

Theorem: Consider a closed right circular cone of height h and radius r whose vertex is at O and whose base is horizontal. It is partially filled with water up to a height h_1 (measured from its base). The upper part of the cone is filled with air. The cone is then inverted and the height of the water level becomes h_2 (measured from O). Then, at least one of h, h_1 or h_2 is irrational.

Proof: Let V_1 be the volume of air and V_2 the volume of water before inverting the cone. As water is incompressible, these volumes are unchanged when the cone is inverted. Let the volume of the containing cone be V. Before inversion the air fills a cone of height $h - h_1$. After inversion the water fills a cone of height h_2. Both these cones are similar to the containing cone, so $V_1 : V_2 : V = (h - h_1)^3 : h_2^3 : h^3$. But $V_1 + V_2 = V$, so

$$(h - h_1)^3 + h_2^3 = h^3. \qquad (1)$$

It is known from Fermat's last theorem that there are no solutions to (1) in integers, as a result there are no solutions in rationals either. Hence, at least one of h, h_1 or h_2 is irrational.

10.1017/mag.2021.62

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