108.08 Cone and Integral

Introduction

The process of deriving an equation by finding the value of a quantity in two different ways and then equating those values with each other has been there in mathematics since antiquity. In this Note, we employ the same technique to evaluate an integral by finding the volume of a right circular cone in two different ways.

Description of the cone

Let us consider a right circular cone that has a circular base of radius R in the horizontal xy-plane. The centre of the circle lies at the origin, O, of coordinates. The apex, D, of the cone lies on the vertical z-axis at the point with (x, y, z) coordinates $(0, 0, H)$, where $H > 0$ is the cone's height. Let A be the point of intersection of the cone with the positive x-axis, B be any point on the surface of the cone above the xy-plane and C be a point on the z-axis having the same z coordinate as B. In terms of spherical polar coordinates r, θ, ϕ, we take θ to be the angle measured in space from the z-axis and r to be the distance from O. The diagram is as follows:

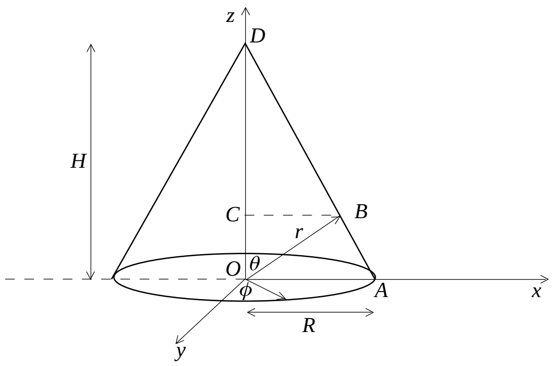

FIGURE 1

The surface of the cone is then $r = f(\theta)$, $OC = r\cos\theta$ and $CB = r\sin\theta$. Since $\Delta DCB \sim \Delta DOA$,

$$\frac{CD}{CB} = \frac{OD}{OA} \Rightarrow \frac{CD}{r\sin\theta} = \frac{H}{R} \Rightarrow CD = \frac{Hr\sin\theta}{R}.$$

Since, $OC + CD = OD$,

$$r\cos\theta + \frac{Hr\sin\theta}{R} = H$$

which after some simplification gives,

$$r = f(\theta) = \frac{RH}{R\cos\theta + H\sin\theta}. \tag{1}$$

Volume integral and its evaluation

From multivariable calculus it is known that, volume (V) of any three dimensional solid (U) is given by,

$$V = \iiint_U dx\, dy\, dz$$

since the above integral becomes complicated in Cartesian coordinates (as the limits involved are variables). It is therefore convenient to evaluate it in spherical polar coordinates (where 5 of the 6 limits are constant, [1]). Hence, transforming the coordinate system from Cartesian to spherical polars, we obtain

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{r(\theta)} r^2 \sin\theta \, dr\, d\theta\, d\phi = 2\pi \int_0^{\pi/2} \int_0^{r(\theta)} r^2 \sin\theta \, dr\, d\theta.$$

This can be simplified as

$$V = 2\pi \int_0^{\pi/2} \left[\sin\theta \frac{r^3}{3}\right]_0^{r(\theta)} d\theta = \frac{2\pi}{3} \int_0^{\pi/2} r^3(\theta) \sin\theta \, d\theta.$$

Substituting (1) in the above integral, we have,

$$V = \frac{2\pi}{3} \int_0^{\pi/2} \left(\frac{RH}{R\cos\theta + H\sin\theta}\right)^3 \sin\theta \, d\theta = \frac{2\pi R^3}{3} \int_0^{\pi/2} \frac{\sin\theta}{\left(\frac{R}{H}\cos\theta + \sin\theta\right)^3} d\theta$$

which can be found in [2].

We define the parameter $a = \frac{R}{H}$. We also re-express the integral by making the substitution $u = \cos\theta \Rightarrow du = -\sin\theta \, d\theta$, from which we obtain,

$$V = \frac{2\pi R^3}{3} \int_1^0 \frac{-du}{\left(au + \sqrt{1 - u^2}\right)^3}$$

since the volume of any right circular cone of height H and radius R is given by,

$$V = \frac{\pi R^2 H}{3}.$$

We therefore have,

$$\frac{\pi R^2 H}{3} = \frac{2\pi R^3}{3} \int_0^1 \frac{du}{\left(au + \sqrt{1 - u^2}\right)^3} \Rightarrow \int_0^1 \frac{du}{\left(au + \sqrt{1 - u^2}\right)^3} = \frac{1}{2a}.$$

Thus we have the following result:

Theorem: If the parameter $a > 0$, then

$$\int_0^1 \frac{du}{\left(au + \sqrt{1 - u^2}\right)^3} = \frac{1}{2a} \qquad (2)$$

where the condition $a > 0$ must be satisfied since $a \leqslant 0$ will give rise to a non-integrable singularity within the interval of integration.

Corollary

Substituting $a = \frac{1}{b}$ where $b > 0$ in (2) and dividing the main result by b^3, we have,

$$\int_0^1 \frac{du}{\left(u + b\sqrt{1 - u^2}\right)^3} = \frac{1}{2b^2}. \qquad (3)$$

If we differentiate (3) with respect to b repeatedly, n times, we obtain after some simplification,

$$\int_0^1 \frac{\left(1 - u^2\right)^{n/2}}{\left(u + b\sqrt{1 - u^2}\right)^3} = \frac{1}{b^{n+2}(n + 2)} \qquad (4)$$

where $n = -1, 0, 1, 2, 3, \ldots$. The detailed proof of (4) and why is it valid for $n = -1$ and not for any other negative integer, is left as an exercise for the reader.

Acknowledgement

I would like to thank the anonymous referee for carefully going through this Note and improving the quality of it.

References

1. James Stewart, *Multivariable Calculus* (7th edn.), Brooks/Cole (2012), p. 1057, 15.9.
2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (7th edn.), Academic Press (2007) p. 408, 3.669.

10.1017/mag.2024.25

ANGAD SINGH
Control and Computing,
Department of
Electrical Engineering,
Indian Institute of Technology, Bombay, India
e-mail: *angadsingh1729@gmail.com*