104.30 An integral relating π and $\zeta(3)$

Introduction

Bernhard Riemann in his 1859 article "On the Number of Primes Less Than a Given Magnitude" defined the zeta function of a complex variable $s = \sigma + it$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $Re(s) > 1$ for absolute convergence.

There are various known integral representations of Riemann's Zeta Function for $Re(s) > 1$, such as the well known

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

given in [1] and [2]. Some slightly complicated looking integrals, like

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^{\infty} \frac{\sin(s \arctan x)}{(1+x^2)^{s/2}\left(e^{2\pi x} - 1\right)}\, dx$$

$$\zeta(s) = \frac{\pi^{s/2}}{s(s-1)\Gamma(s/2)} + \frac{\pi^{s/2}}{\Gamma(s/2)} \int_1^{\infty} \left(x^{s/2} + x^{(1-s)/2}\right) \frac{\omega(x)}{x}\, dx$$

where

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

and so on.

In this Note, I prove an interesting integral representation of $\zeta(3)$. It is interesting because the proof uses elementary calculus techniques and infinite series manipulations, thus making it simple and easy to understand.

Preliminaries

Theorem 1: For every $a, b \in \mathbb{R}^+$ we have,

$$\int_0^{\infty} \frac{1}{\left(x^2 + a^2\right)\left(x^2 + b^2\right)}\, dx = \frac{\pi}{2ab(a+b)}. \tag{1}$$

This result can be proved easily by solving the integral using partial fractions.

Theorem 2:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)} = 2\zeta(3). \tag{2}$$

Proof: Let us write this double summation as follows,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn(m+n)} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 \frac{x^{m+n-1}}{mn}\,dx \\
&= \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{m+n-1}}{mn}\,dx \\
&= \int_0^1 \sum_{m=1}^{\infty}\frac{x^m}{m}\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\,dx \\
&= \int_0^1 \frac{\log^2(1-x)}{x}\,dx \\
&= \int_0^1 \frac{\log^2 x}{1-x}\,dx \\
&= \int_0^1 \log^2 x \sum_{i=0}^{\infty} x^i\,dx \\
&= \sum_{i=0}^{\infty} x^i \log^2 x\,dx \\
&= \sum_{i=0}^{\infty}\frac{(-1)^2(2!)}{(i+1)^{2+1}} = \sum_{i=1}^{\infty}\frac{2}{i^3} = 2\zeta(3)
\end{aligned}$$

This proves Theorem 2.

Theorem 3: For any real value of x, we have,

$$\frac{\pi\coth(\pi x)}{2x} - \frac{1}{2x^2} = \sum_{k=1}^{\infty}\frac{1}{k^2+x^2}. \tag{3}$$

It is a very well known result and can be proved by logarithmically differentiating the infinite product of $\sin x$ and then by replacing x by $i\pi x$.

Care should be taken at $x = 0$, as the right hand side of the equation becomes the well known Basel Problem, that is the sum of the reciprocal of the squares of natural numbers.

While the left hand side is determined by taking the limit as x tends to zero,

$$\lim_{x\to 0}\frac{\pi\coth(\pi x)}{2x} - \frac{1}{2x^2}.$$

Now substituting $x = \pi x$ in the series expansion of $\coth x$ and using it in the above limit, we have,

$$\lim_{x\to 0}\frac{\pi}{2x}\left(\frac{1}{\pi x} + \frac{\pi x}{3} - \frac{\pi^3 x^3}{45}\dots\right) - \frac{1}{2x^2} = \frac{\pi^2}{6}.$$

Thus we have established that at $x = 0$, (3) becomes, as given in [3]

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Final Result

Consider integral (1),

$$\int_0^\infty \frac{1}{(x^2 + a^2)(x^2 + b^2)}\,dx = \frac{\pi}{2ab(a + b)}.$$

Summing both the sides from $a = 1$ to $a = \infty$, we obtain,

$$\sum_{a=1}^{\infty} \int_0^\infty \frac{1}{(x^2 + a^2)(x^2 + b^2)}\,dx = \sum_{a=1}^{\infty} \frac{\pi}{2ab(a + b)}.$$

Interchanging the integral and summation sign and then using (3), we have,

$$\int_0^\infty \left(\frac{\pi \coth(\pi x)}{2x} - \frac{1}{2x^2}\right)\frac{1}{(x^2 + b^2)}\,dx = \sum_{a=1}^{\infty} \frac{\pi}{2ab(a + b)}.$$

Summing both the sides from $b = 1$ to $b = \infty$, we obtain,

$$\sum_{b=1}^{\infty} \int_0^\infty \left(\frac{\pi \coth(\pi x)}{2x} - \frac{1}{2x^2}\right)\frac{1}{(x^2 + b^2)}\,dx = \sum_{b=1}^{\infty} \sum_{a=1}^{\infty} \frac{\pi}{2ab(a + b)}.$$

Interchanging the integral and summation sign and using (3) on the left-hand side and employing (2) on the right-hand side, we arrive at,

$$\int_0^\infty \left(\frac{\pi \coth(\pi x)}{2x} - \frac{1}{2x^2}\right)^2 dx = \pi\zeta(3).$$

Hence, we obtain the desired result.

Reference

1. B. C. Berndt, An unpublished manuscript of Ramanujan on infinite series identities, *J. Ramanujan Math. Soc.* **19** (2004) pp. 57-74.
2. S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto, On rapidly convergent series for the Riemann zeta-values via the modular relation, Abh. Math. Sem. Univ. Hamburg **72** (2002) pp. 187-206.
3. On some integrals involving the Hurwitz zeta function. II, *Ramanujan J.* **6**(4) (2002) pp. 449-468.

10.1017/mag.2020.108

ANGAD SINGH

Department of Electronics and Telecommunications, Pune Institute of Computer Technology, Pune, India

e-mail: *angadsingh1729@gmail.com*