On Hardy's 144^{th} Birth Anniversary: Some results named after Godfrey Harold Hardy

1 Hardy–Littlewood conjecture for twin primes

Let $\pi_2(x)$ denote the number of primes $p \leq x$ such that $p+2$ is also prime. Define the twin prime constant C_2 as

$$C_2 = \prod_{\substack{p\ prime \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.66016181584686957392781211001455577...$$

then,

$$\pi_2(x) \sim 2C_2 \frac{x}{\ln^2(x)}$$

2 Hardy–Littlewood circle method

In mathematics, the Hardy–Littlewood circle method is a technique of analytic number theory. It is named after G. H. Hardy and J. E. Littlewood, who developed it in a series of papers on Waring's problem. The initial idea is usually attributed to the work of Hardy with Srinivasa Ramanujan a few years earlier, in 1916 and 1917, on the asymptotics of the partition function. Let us define $F(z)$ as,

$$F(z) = \sum_{k=0}^{\infty} a_k z^k$$

and let the radius of convergence of $F(z)$ be larger than 1, then using Hardy–Littlewood circle method, we have,

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}} dz$$

where C is the unit circle oriented counter-clockwise.

3 Hardy-Littlewood tauberian theorem

In mathematical analysis, the Hardy–Littlewood tauberian theorem is a tauberian theorem relating the asymptotics of the partial sums of a series with the asymptotics of its Abel summation. In this form, the theorem asserts that if, as $y \to 0$, the non-negative sequence a_n is such that there is an asymptotic equivalence

$$\sum_{n=0}^{\infty} a_n e^{-ny} \sim \frac{1}{y}$$

then there is also an asymptotic equivalence

$$\sum_{k=0}^{n} a_k \sim n$$

as $n \to \infty$.
The theorem was proved in 1914 by G. H. Hardy and J. E. Littlewood.

4 Hardy–Ramanujan asymptotic formula

In number theory, the partition function $p(n)$ represents the number of possible partitions of a non-negative integer n. For instance, $p(4) = 5$ because the integer 4 has the five partitions $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3$, $2 + 2$, and 4.
An asymptotic expression for $p(n)$ is given by,

$$p(n) \sim \frac{1}{4n\sqrt{3}} exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

This asymptotic formula was first obtained by G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920.

5 Hardy–Ramanujan theorem

In mathematics, the Hardy–Ramanujan theorem, proved by G. H. Hardy and Srinivasa Ramanujan (1917), states that the normal order of the number $\omega(n)$ of distinct prime factors of a number n is $\ln(\ln(n))$. Roughly speaking, this means that most numbers have about this number of distinct prime factors. It can also be written as, if $\psi(x)$ tends steadily to infinity with x, then

$$\ln(\ln(x)) - \psi(x)\sqrt{\ln(\ln(x))} < \omega(n) < \ln(\ln(x)) + \psi(x)\sqrt{\ln(\ln(x))}$$

for almost all numbers $n < x$.

6 Hardy about Ramanujan

Hardy's personal ratings of mathematicians on the basis of pure talent on a scale from 0 to 100 were as follows, Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100.

In an interview by Paul Erdős, when Hardy was asked what his greatest contribution to mathematics was, Hardy unhesitatingly replied that it was the discovery of Ramanujan. In a lecture on Ramanujan, Hardy said "my association with him is the one romantic incident in my life".

Hardy also quoted, "The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world ... It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. On the other hand there were things of which it was impossible that he would remain in ignorance ... so I had to try to teach him, and in a measure I succeeded, though I obviously learnt from him much more than he learnt from me".

ANGAD SINGH
Department of Electronics and Telecommunications, Pune Institute of Computer Technology, Pune, India
email-id: angadsingh1729@gmail.com