An Interesting Zeta Series

Introduction

It is known that, if $s \in \mathbb{C}$ and $\Re(s) > 1$, then Riemann's zeta function can be expressed as,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

There exist many interesting infinite sums related to $\zeta(s)$ in which $\zeta(s)$ appears in various forms but rarely do we see $\zeta(s)$ in the exponent.

In this paper, we prove the convergence of such an infinite series in which $\zeta(s)$ appears in the exponent using elementary methods.

1 Preliminaries

In order to prove the final result, we state and prove three lemmas.

Lemma 1: If $k \in \mathbb{N}$, $k \geq 3$ and $f(k) = 2^{k-1} - k - 1$, then

$$f(k) \geq 0$$

Proof: Observe that, if $x \in \mathbb{R}^+$ and $F(x) = 2^{x-1} - x - 1$ then $F'(x) = 2^{x-1}\ln(2) - 1$ which means $F'(x) = 0 \implies x = 1 - \frac{\ln(\ln(2))}{\ln(2)} \approx 1.5287...$ Since, $F'(1) < 0$ and $F'(2) > 0$, therefore $F(x)$ is increasing for all $x > 1.5287...$. Since $F(3) = 0$, therefore $F(x) \geq 0$ for all $x \geq 3$ (where equality holds at $x = 3$). Since f replicates F whenever x is a positive integer (k), this completes the proof of Lemma 1.

Lemma 2: If $k \in \mathbb{N}$ and $k \geq 3$,then

$$\zeta(k) < 1 + 1/k$$

Proof: Observe that,

$$\frac{1}{2^k} + \frac{1}{3^k} < \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}<\underbrace{\left(\frac{1}{4^k}+...\right)}_{\text{4 times}}=\frac{1}{2^{2k-2}}$$

$$\frac{1}{8^k}+\frac{1}{9^k}+\frac{1}{10^k}+\frac{1}{11^k}+\frac{1}{12^k}+\frac{1}{13^k}+\frac{1}{14^k}+\frac{1}{15^k}<\underbrace{\left(\frac{1}{8^k}+...\right)}_{\text{8 times}}=\frac{1}{2^{3k-3}}$$

and so on. Adding up the above inequalities, we obtain,

$$\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\frac{1}{5^k}+...+<\frac{1}{2^{k-1}}+\frac{1}{2^{2k-2}}+\frac{1}{2^{3k-3}}+...+$$

which becomes,

$$\zeta(k)-1<\frac{1}{2^{k-1}-1}$$

using lemma 1, we have $f(k)\geq 0 \implies 2^{k-1}-1>k \implies \frac{1}{2^{k-1}-1}<\frac{1}{k}$ for all $k\geq 3$, therefore,

$$\zeta(k)-1<\frac{1}{2^{k-1}-1}<\frac{1}{k}$$

for all $k\in\mathbb{N}$ and $k\geq 3$. This completes the proof of the second lemma.

Lemma 3: If $x\in\mathbb{R}^+$ and $x>1$, then,

$$\frac{1}{x}-\frac{1}{x^{1+1/x}}<\frac{\ln(x)}{x^2}$$

Proof: Let

$$g(x)=\frac{\ln(x)}{x}+\frac{1}{x^{1/x}}-1$$

then,

$$g'(x)=\left(\frac{1-\ln(x)}{x^2}\right)\left(1-\frac{1}{x^{1/x}}\right)$$

Case(i): $x\in(1,e)$
Observe that $g(1)=g'(1)=0$, therefore $g(x)$ just touches the x-axis at $x=1$ and since $g'(x)>0$ in $(1,e)$, thus $g(x)$ is increasing in $(1,e)$ which implies $g(x)>0$.
Case(ii): $x=e$

Observe that $g(e) > 0$, $g'(e) = 0$ and $g(x)$ attains a maxima at $x = e$.
Case(iii): $x \in (e, \infty)$
Here, $g'(x) < 0$, thus $g(x)$ is decreasing. Since $g(e) > 0$ and $g(x), g'(x) \to 0$ as $x \to \infty$, we have $g(x) > 0$ in (e, ∞).
Thus $g(x) > 0$ in $(1, \infty)$ which implies $g(x)/x > 0$ in $(1, \infty)$ too. This completes the proof of lemma 3.

2 Final result

Theorem: The infinite series

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k^{\zeta(k)}} \right)$$

converges.

Proof: Using lemma 2, we have,

$$\zeta(k) < 1 + 1/k \implies \frac{1}{k} - \frac{1}{k^{\zeta(k)}} < \frac{1}{k} - \frac{1}{k^{1+1/k}}$$

using lemma 3, we obtain,

$$\frac{1}{k} - \frac{1}{k^{\zeta(k)}} < \frac{1}{k} - \frac{1}{k^{1+1/k}} < \frac{\ln(k)}{k^2}$$

summing up both the sides from $k = 3$ to $k = \infty$, we have,

$$\sum_{k=3}^{\infty} \left(\frac{1}{k} - \frac{1}{k^{\zeta(k)}} \right) < \sum_{k=3}^{\infty} \frac{\ln(k)}{k^2} < \sum_{k=3}^{\infty} \frac{\sqrt{k}}{k^2} = \zeta(3/2) - 1 - \frac{1}{2\sqrt{2}}$$

since $3/2 > 1 \implies \zeta(3/2)$ is finite. Thus,

$$\sum_{k=3}^{\infty} \left(\frac{1}{k} - \frac{1}{k^{\zeta(k)}} \right)$$

converges. Using the known limit $\lim_{k \to 1} (k-1)\zeta(k) = 1$, we can prove that,

$$\lim_{k \to 1} \left(\frac{1}{k} - \frac{1}{k^{\zeta(k)}} \right) = 1 - \frac{1}{e}$$

finally adding $\frac{1}{2} - \frac{1}{2^{\zeta(2)}}$ to the above series we complete the proof.

References

[1] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers* (Oxford University Press, 2008).

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ANGAD SINGH
email-id: angadsingh1729@gmail.com