Black-Scholes-Merton Model

Liming Feng

Dept. of Industrial & Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Readings: Hull Chapters 12, 13

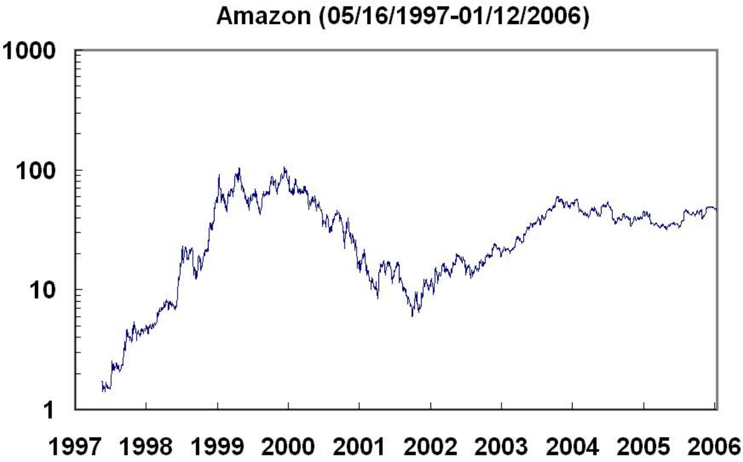
©Liming Feng. Do not distribute without permission of the author

Binomial model review

- From the binomial model
  - **No arbitrage pricing**: derivative price = cost of constructing a **replicating portfolio**
  - **Delta hedging**: short a derivative and hold Δ_t units of the underlying asset to cancel the risk
  - **Risk neutral pricing**: derivative price = risk neutral expectation of the payoff discounted at the risk free rate; underlying asset earns risk free rate in risk neutral world
- Similar treatments for options pricing in the **Black-Scholes-Merton model**

Brownian motion: brief history

- Brownian motion: observed in 1827 by **Robert Brown**; highly irregular movement of pollen particles in water
- **Albert Einstein**'s 1905 paper popularized BM
- **Louis Bachelier**'s 1900 doctoral dissertation: *The theory of speculation*
- Motivation: irregularity of stock price process (see Figure)
- Bachelier's work rediscovered, improved, **Black-Scholes** formula for European options invented (1973)


Amazon (05/16/1997-01/12/2006)
1000
100
10
1
1997
1998
1999
2000
2001
2002
2003
2004
2005
2006

Log return process of an asset

- **Log return** of an asset over the period $[0, t]$

$$X_t = \ln(S_t/S_0) \quad \text{so that} \quad S_t = S_0 e^{X_t}$$

Current price of a non dividend paying stock is \$100. Stock price in 6 months will be \$110. Then

$$X_t = \ln(110/100) = 9.53\%$$

X_t is continuously compounded rate of return of the stock over $[0, t]$ (**not annualized**). Generally, X_t is random

- In the Black-Scholes-Merton model, we assume
  - Returns over non overlapping time periods are independent
  - Its distribution only depends on length of the time period
  - It is normally distributed
- Suppose returns over periods of length 1 are $N(0,1)$
- **Return over period** $[0,2]$

$$X_2 = \ln(S_2/S_0) = \ln(S_2/S_1) + \ln(S_1/S_0) \sim N(0,2)$$

- **Return over period** $[0, 1/2] \sim N(0, 1/2)$

$$\ln(S_{1/2}/S_0) + \ln(S_1/S_{1/2}) = \ln(S_1/S_0) \sim N(0, 1)$$

Generally, return over period $[0, t]$: $X_t \sim N(0, t)$.

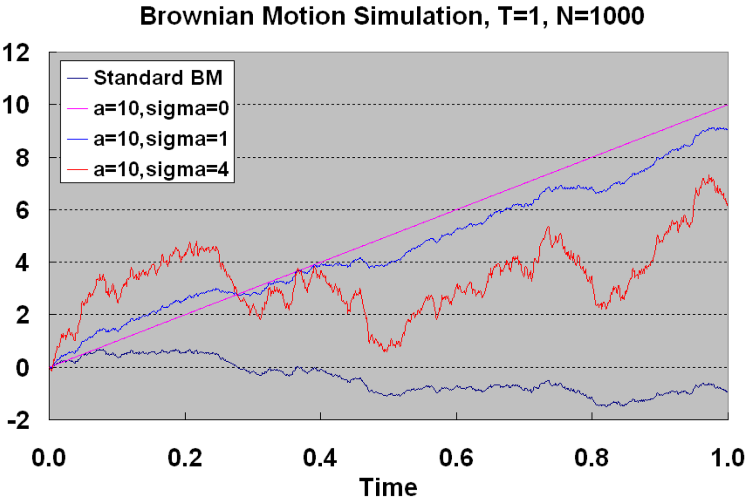
- **Return over period** $[t_1, t_2]$:

$$\ln(S_{t_2}/S_{t_1}) = \ln(S_{t_2}/S_0) - \ln(S_{t_1}/S_0) = X_{t_2} - X_{t_1} \sim N(0, t_2 - t_1)$$

- Properties of the previous return process $X_t = \ln(S_t/S_0)$
  - On non-overlapping time periods, returns are independent
  - The return process starts at 0
  - Return over a given time period is normal: mean 0, variance $=$ length of time period
- $\{X_t\}$ is a **standard Brownian motion**

- $\{B_t, t \geq 0\}$ is a **standard Brownian motion** if
  - it starts at zero: $B_0 = 0$
  - over non-overlapping periods $[s, t]$, $[u, v]$, $B_t - B_s$ and $B_v - B_u$ are independent (**independent increment**)
  - over period $[s, t]$, $B_t - B_s \sim N(0, t - s)$ (**stationary increment**)
- Also known as **Wiener process** (named after Norbert Wiener)

- If the stock return process is modeled by a standard Brownian motion
$$X_t = \ln(S_t/S_0) = B_t$$
  – $\mathbb{E}[X_t] = 0$ for $\forall t \geq 0$
  – $\text{var}(X_t) = t$, increases as t increases
- To allow deterministic trend in return and more flexibility in variance, let
$$X_t = at + \sigma B_t$$
then $X_t \sim N(at, \sigma^2 t)$; X_t is a **Brownian motion with drift a and volatility σ**


Brownian Motion Simulation, T=1, N=1000
Standard BM
a=10,sigma=0
a=10,sigma=1
a=10,sigma=4
12
10
8
6
4
2
0
-2
0.0
0.2
0.4
0.6
0.8
1.0
Time

Simulate a Brownian motion

- Simulate a BM with drift a and volatility σ over period $[0, T]$

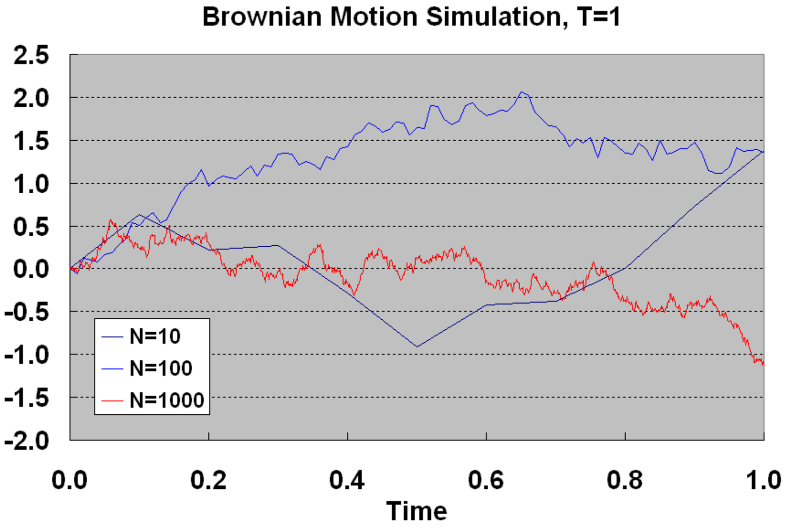
$$X_t = at + \sigma B_t$$

  - Divide $[0, T]$ into N subintervals: $\Delta t = T/N$
  - Start from $X_0 = 0$, simulate $X_{\Delta t}, X_{2\Delta t}, \cdots$ and connect
  - It suffices to simulate $B_{\Delta t}, B_{2\Delta t}, \cdots$

$$B_{\Delta t} = \sqrt{\Delta t}\ \epsilon_1,$$

$$B_{2\Delta t} - B_{\Delta t} = \sqrt{\Delta t}\ \epsilon_2, \quad \cdots$$

    where ϵ_i are independent standard normal r.v.'s
  - Select large N to get better approximation to a true BM


Brownian Motion Simulation, T=1
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
N=10
N=100
N=1000
0.0
0.2
0.4
0.6
0.8
1.0
Time

Stochastic differential equation

- Increment of X_t over time period $[t, t + \Delta t]$

$$X_{t+\Delta t} - X_t = a\Delta t + \sigma(B_{t+\Delta t} - B_t)$$

Denote $\Delta X_t = X_{t+\Delta t} - X_t, \Delta B_t = B_{t+\Delta t} - B_t$,

$$\Delta X_t = a\Delta t + \sigma \Delta B_t$$

- **Stochastic differential equation** for X_t (increment of X_t over $[t, t + dt]$)

$$dX_t = adt + \sigma dB_t$$

describes how the return process evolves over time

- X_t is not differentiable, **cannot write**

$$\frac{dX_t}{dt} = a + \sigma \frac{dB_t}{dt}$$

Note that $\Delta B_t \sim N(0, \Delta t) = \sqrt{\Delta t} N(0,1) >> \Delta t$

- In contrast, for a differentiable function, e.g., $f(t) = t^2$, we write $df(t) = 2t\,dt$ or $\frac{df(t)}{dt} = 2t$
- When handling stochastic differential equations, **ordinary calculus** that handles differentiable functions **cannot be applied**
- **Stochastic calculus** should be applied

- For options pricing, need to find out how option price $f(t, X_t)$ change over time
- **Chain rule in ordinary calculus**, if $g(t)$ is differentiable, $f(t, x)$ is differentiable

$$df(t, g(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dg(t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} g'(t) dt$$

- **Chain rule in stochastic calculus**?

- Suppose $dX_t = adt + \sigma dB_t$. What is the SDE for $f(t, X_t)$ (f differentiable, but X_t not differentiable)

$$df(t, X_t) = ?$$

- Taylor series of $f(t, x)$

$$f(t+\Delta t, x+\Delta x) - f(t, x) = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2 + \frac{\partial^2 f}{\partial t \partial x}\Delta t \Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 + \cdots$$

- Apply Taylor series on $f(t, X_t)$, replace Δt by dt, ΔX_t by dX_t

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial t^2} dt^2 + \frac{\partial^2 f}{\partial t \partial x} dt \cdot dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

- Omit high order terms in dt above

$$\frac{1}{2}\frac{\partial^2 f}{\partial t^2} dt^2$$

$$dt \cdot dX_t = dt(adt + \sigma dB_t) \sim dt(adt + \sigma\sqrt{dt}N(0,1))$$

Recall that $\Delta B_t \sim N(0, \Delta t) = \sqrt{\Delta t}N(0,1)$

- $(dX_t)^2 = (adt + \sigma dB_t)^2 = \sigma^2 (dB_t)^2$
- A useful rule: $(dB_t)^2 = dt$: note that $\Delta B_t \sim N(0, \Delta t)$

$$\mathbb{E}[(\Delta B_t)^2] = \Delta t, \quad \text{var}\left((\Delta B_t)^2\right) = 2\Delta t^2$$

  In limit, we have $(dB_t)^2 = dt$
- **Itô formula**: if $dX_t = adt + \sigma dB_t$, then

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 \\ &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x}(adt + \sigma dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt \\ &= (\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}) dt + \sigma \frac{\partial f}{\partial x} dB_t \end{aligned}$$

Black-Scholes-Merton model

- In the Black-Scholes-Merton model, asset price follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

  - LHS: instantaneous rate of return of the asset over $[t, t + dt]$
  - RHS: a deterministic trend + a stochastic term
  - μ is the **expected instantaneous rate of return**
  - σ is the **volatility**
- What is the distribution of S_t?

- Risk free investment at rate r: 1 invested at time 0 grows to $L_t = e^{rt}$ at time t

$$\frac{dL_t}{L_t} = \frac{re^{rt}}{e^{rt}}dt = rdt$$

- If $\sigma = 0$ in the Black-Scholes-Merton model

$$\frac{dS_t}{S_t} = \mu dt \Rightarrow S_t = S_0 e^{\mu t} \Rightarrow X_t = \ln(S_t/S_0) = \mu t$$

- Consider the return process $X_t = \ln(S_t/S_0)$
- By Itô formula, $f(t, S_t) = \ln(S_t/S_0)$,

$$\frac{\partial f}{\partial t}(t, S_t) = 0, \quad \frac{\partial f}{\partial S}(t, S_t) = \frac{1}{S_t}, \quad \frac{\partial^2 f}{\partial S^2}(t, S_t) = -\frac{1}{S_t^2}$$

$$dX_t = \frac{1}{S_t} \cdot dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right) \cdot (dS_t)^2$$

Plug in $dS_t = \mu S_t dt + \sigma S_t dB_t$,

$$dX_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$$

- In the Black-Scholes-Merton model,

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dB_t \\ dX_t &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t \\ X_t &= (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \\ S_t = S_0 e^{X_t} &= S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right) \end{aligned}$$

- S_t follows a **geometric Brownian motion**

Lognormal distribution

- S_t has a lognormal distribution

$$\ln(S_t) = \ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$$

  normally distributed: mean $\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t$, variance $\sigma^2 t$

- If $\ln(X) \sim N(m, s^2)$, then

$$\mathbb{E}[X] = e^{m+s^2/2}$$

- For the Black-Scholes-Merton model,

$$\mathbb{E}[S_t] = S_0 e^{\mu t}$$

Example (Black-Scholes-Merton model)

Suppose stock price follows geometric Brownian motion with expected instantaneous rate of return 10% and volatility 20%. Current stock price is \$100. What is the expected stock price in 6 months? What is the probability the stock price in 6 months is greater than 120?

Expected stock price in 6 months: $S_0 e^{\mu T} = 100e^{10\%/2} = 105.13$

$$\mathbb{P}(S_T > 120) = \mathbb{P}((\mu - \frac{1}{2}\sigma^2)T + \sigma B_T > \ln(\frac{120}{S_0}))$$

$$= \mathbb{P}(Z > \frac{\ln(\frac{120}{S_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) = 1 - N(1.0064) = 15.71\%$$

Estimating volatility

- Two main parameters in the model: μ, σ

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t$$

- For pricing derivatives, μ doesn't matter, work in the risk neutral world
- To estimate σ
  - **Use historical data**
  - Volatility implied by observed option prices in the market

- Given daily stock prices

$$S_0, S_1, \cdots, S_n$$

observed at times $0, \Delta t, \cdots, n\Delta t$

- Over the period $[0, \Delta t]$,

$$S_1 = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma B_{\Delta t}\right)$$

- Daily log return

$$u_1 = \ln(S_1/S_0) \sim N\left((\mu - \frac{1}{2}\sigma^2)\Delta t, \sigma^2 \Delta t\right)$$

similarly, define daily log returns $u_2, \cdots, u_n$

- Want to estimate $\sigma^2 \Delta t$, use sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2$$

- Volatility estimation using historical data

$$\hat{\sigma} = \frac{s}{\sqrt{\Delta t}}$$

- 252 trading days in 1 year; when daily prices used

$$\Delta t = 1/252$$

Pricing European options

- Pricing a European option in the BSM model when underlying asset does not pay income and follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

- In binomial model: one can create a risk free (hedged) position by shorting an option and hold Δ_t units of assets
- Will derive an equation for European option price based on the same idea

- Consider a European option with maturity T, option price at time t: $f(t, S_t)$
- Dynamics of asset price S_t

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- Dynamics of option price $f(t, S_t)$

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt + \frac{\partial f}{\partial S}\sigma S_t dB_t \end{aligned}$$

Delta hedging

- Short an option, long Δ_t assets at any time $0 \le t \le T$; select Δ_t so that the hedged position is risk free
- Value of the portfolio at time $t \in [0, T]$

$$\Pi_t = -f(t, S_t) + \Delta_t S_t$$

- Dynamics of the portfolio value

$$\begin{aligned} d\Pi_t &= -df + \Delta_t dS_t \\ &= -\left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt - \frac{\partial f}{\partial S}\sigma S_t dB_t + \Delta_t(\mu S_t dt + \sigma S_t dB_t) \\ &= -\left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} - \Delta_t \mu S_t\right) dt - (\frac{\partial f}{\partial S} - \Delta_t)\sigma S_t dB_t \end{aligned}$$

- To make the hegded position risk free, select $\Delta_t = \frac{\partial f}{\partial S}(t, S_t)$

$$\begin{aligned} d\Pi_t &= -\left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} - \frac{\partial f}{\partial S}\mu S_t\right) dt \\ &= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) dt \end{aligned}$$

- Risk free investment earns risk free rate: $d\Pi_t = r\Pi_t dt$

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}\right) = r\Pi_t = r(-f + \frac{\partial f}{\partial S} S_t)$$

Intuition behind delta hedging

- Δ_t is the **rate of change** of the option price w.r.t. the underlying asset price

$$\Delta_t = \frac{\partial f}{\partial S} \approx \frac{\Delta f}{\Delta S}, \quad \Delta f \approx \Delta_t \times \Delta S$$

- Suppose for a European call, $\Delta_t = 0.4$

$$\Delta f \approx 0.4 \Delta S$$

  Hedged position: short a call, long 0.4 shares: if stock price increases by 1 cent: $\Delta S = 1$ cent; call price increases by 0.4 cents;

- Loss due to the increase in the call value canceled by gain due to the increase of the value of the shares

Black-Scholes-Merton equation

- European option price solves

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$

  This is known as the **Black-Scholes-Merton equation**
- **Terminal condition**: e.g., for a European call option

$$f(T, S) = (S - K)^+$$

- Solve the PDE to obtain $f(t, S)$: option price at t when underlying asset price is S (*solutions for simple cases can be derived; numerical methods used otherwise*)

Example (Black-Scholes-Merton equation)

Verify that the value of a long forward contract with maturity T and delivery price K on an asset with no income satisfies the Black-Scholes-Merton equaiotn.

The value of a long forward contract at time t is

$$V(t,S) = e^{-r(T-t)}(F_t - K) = e^{-r(T-t)}(Se^{r(T-t)} - K) = S - Ke^{-r(T-t)}$$

$$\frac{\partial V}{\partial t} = -rKe^{-r(T-t)}, \quad \frac{\partial V}{\partial S} = 1, \quad \frac{\partial^2 V}{\partial S^2} = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -rKe^{-r(T-t)} + rS - r(S - Ke^{-r(T-t)}) = 0$$

Risk neutral valuation

- Risk neutral pricing of derivatives in binomial model
  - Stock earns risk free rate in the risk neutral world
  - Compute risk neutral expectation of the payoff
  - Discount at the risk free rate

- Black-Scholes-Merton model
  - Stock earns risk free rate in the risk neutral world

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

so that

$$\mathbb{E}^*[S_t] = S_0 e^{rt}$$

  - European derivative price at time 0

$$f = e^{-rT}\mathbb{E}^*[\text{payoff at time } T]$$

Example (risk neutral valuation)

Verify the formula for the value of a long forward contract at time zero with maturity T and delivery price K on an asset with no income

Using the risk neutral pricing formula,

$$V(0,S) = e^{-rT}\mathbb{E}^*[S_T - K] = e^{-rT}(S_0 e^{rT} - K) = S_0 - Ke^{-rT}$$

Black-Scholes formula

- European call/put with strike K

$$c = e^{-rT}\mathbb{E}^*[(S_T - K)^+]$$

$$p = e^{-rT}\mathbb{E}^*[(K - S_T)^+]$$

where

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma B_T\right)$$

- S_T log-normally distributed, the above can be computed

- **Black-Scholes formula** for European vanilla options on assets with no income

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $N(x)$ is the cdf of $N(0,1)$,

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Example (put-call parity in the BSM model)

Verify the put-call parity for European puts and calls with maturity T and strike K on an asset with no income in the Black-Scholes-Merton model.

Put-call parity should hold in any reasonable model

$$
\begin{aligned}
c + Ke^{-rT} &= S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} \\
&= Ke^{-rT} N(-d_2) + S_0(1 - N(-d_1)) \;=\; p + S_0
\end{aligned}
$$

Options on assets with no income

- In the risk neutral world, price of an asset with no income follows
$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$
$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$
- Risk neutral pricing of derivatives
$$\text{derivative price} = e^{-rT}\mathbb{E}^*[\text{derivative payoff } (S_T)]$$
where $\mathbb{E}^*$ is risk neutral expectation

Options on assets with continuous yield

- Asset with continuous yield q: $S_0 \Rightarrow e^{qt} S_t$
- In the risk neutral world, it earns risk free rate

$$\mathbb{E}^*[e^{qt} S_t] = S_0 e^{rt}, \quad \text{or} \quad \mathbb{E}^*[S_t] = S_0 e^{(r-q)t}$$

- Corresponding geometric Brownian motion and SDE

$$S_t = S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)t + \sigma B_t \right)$$

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dB_t$$

- Risk neutral pricing: compute $e^{-rT}\mathbb{E}^*[\text{derivative payoff } (S_T)]$ using the above S_T

- **Black-Scholes formula** for European vanilla options on assets with continuous yield q

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where $N(x)$ is the cdf of $N(0,1)$,

$$d_1 = \frac{\ln(S_0 e^{-qT}/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Options on assets with known discrete income

- **Stocks with known dividends** (time and amount) before the option maturity T
- Stock price = present value of future dividends (deterministic) + risky component (geometric Brownian motion)
- For European options, in the Black-Scholes formula, replace the initial stock price S_0 by

$$S_0 - D$$

S_0 initial stock price, D present value of dividends in $[0, T]$

- American calls/puts on stocks with dividend: early exercise may be optimal
- American calls: **early exercise can only be optimal right before the dividend is paid** (stock price reduced by the amount of dividend payment)
- Option priced numerically using binomial method (construct a binomial tree for the risky component; add back the deterministic component to obtain the tree for the stock price; check whether it is optimal to early exercise on dividend payment dates)