Forward and Futures Contracts

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Readings: Hull Chapters 2, 3, 5

Forward contracts

- **Forward** contract: agreement to buy/sell an asset at a future time T (maturity or expiration date) for a certain price K (delivery price)
- **Long** position: agrees to buy; **short** position: agrees to sell
- **Settlement**: at maturity, short position delivers asset, long position delivers cash of amount K
- May be settled in cash without physical delivery of the asset
- Long position **payoff** $S_T - K$, short position payoff $K - S_T$ (S_T : asset price at T)

Forward price

- **Forward price**: specific delivery price K so that the forward contract has no initial cost
- The value of the forward contract changes over time
- *How to compute forward price? How to value an existing forward contract?*
- Notations
  - $t = 0$ and T: forward inception and maturity
  - S_0, S_T : underlying asset prices at $t = 0$ and T
  - K : delivery price
  - F_0 : forward price
  - r : risk free interest rate per year for the time period $[0, T]$, with continuous compounding

Assumptions

- No transaction costs
- No trading restrictions (such as short selling) unless mentioned
- No tax issues
- Market participants can borrow and lend at the same risk free interest rate
- **No arbitrage**

Investment assets with no income

- Consider forwards on investment assets (not commodities)
- **Investment asset**: held mainly for investment purposes: stocks, bonds
    - **Consumption asset**: held primarily for consumption: oil, wheat
- Assume that the asset doesn't pay income such as coupons, dividends during $[0, T]$

- **No arbitrage** implies that $F_0 = S_0 e^{rT}$
  - Buy an asset now for S_0, and take a short forward position to sell the asset at T for F_0. This is risk free and should earn risk free interest rate:

$$S_0 = F_0 e^{-rT}$$

- Create a **synthetic forward** contract

Position	Time 0	Time T
Long forward	no cost	$S_T - F_0$
Synthetic long forward	borrow S_0, buy asset	$S_T - S_0 e^{rT}$

- Suppose that $F_0 > S_0 e^{rT}$, consider the following trading strategy:

  At time 0:

  - Buy the asset for S_0
  - Short forward (to sell the asset for F_0 at maturity)
  - Borrow S_0 at interest rate r

  Zero cost (no capital is needed from the investor)

  At time T:

  - Forward contract: deliver the asset and receive F_0
  - Repay the loan: $S_0 e^{rT}$

  A costless, riskless income $F_0 - S_0 e^{rT} > 0$, arbitrage!

- Suppose that $F_0 < S_0 e^{rT}$, consider the following trading strategy:

  At time 0:

  - Short sell the asset for S_0 and deposit S_0 at interest rate r
  - Long forward (to buy the asset at maturity)

  At time T:

  - Deposit account: receive $S_0 e^{rT}$
  - Forward contract: buy the underlying asset for F_0
  - Close the short selling account: return the asset

  A costless, riskless income $S_0 e^{rT} - F_0 > 0$, arbitrage!

- With restriction on short selling
  - Arbitrage opportunities exist for investors who hold the asset for long turn investment
  - Sell the asset now, and buy back at T and enjoy a riskfree profit $S_0 e^{rT} - F_0$

Example (forward price of a non-dividend paying stock)

Current stock price of Amazon is \$80/share. 3-month LIBOR rate is 5% with continuous compounding. Are there arbitrage opportunities if the 3-month forward price quoted by a bank is 82?

Forward price:

$$F_0 = S_0 e^{rT} = 80 \times e^{5\%/4} = 81.01$$

Arbitrage (**buy low sell high**):

- **Now**: borrow 80 at 5%, buy one Amazon share, short forward
- **At time T**: sell the share for 82, repay $80 \times e^{5\%/4} = 81.01$, earn 0.99

Investment assets with known income

- Suppose the asset pays $c_1, c_2, \cdots, c_n$ at times $t_1, t_2, \cdots, t_n \in [0, T]$
- **Cost** of the asset: $S_0 - I_0$, where I_0 is the present value of all payments. Suppose the risk free interest rates are $r_1, r_2, \cdots, r_n$ for time periods $[0, t_1], [0, t_2], \cdots, [0, t_n]$:

$$I_0 = c_1 e^{-r_1 t_1} + \cdots + c_n e^{-r_n t_n}$$

- **No arbitrage** implies that $F_0 = (S_0 - I_0)e^{rT}$

- Suppose $F_0 < (S_0 - I_0)e^{rT}$, consider the following trading strategy:

  At time 0

  - Short sell the asset for S_0, long forward
  - Deposit $c_k e^{-r_k t_k}$ at rate r_k for time period $[0, t_k]$, deposit $S_0 - I_0$ at rate r for time period $[0, T]$

  At time t_k

  - Get c_k, return it to the asset lender

  At time T

  - Get $(S_0 - I_0)e^{rT}$, buy the asset for F_0, return the asset to the lender

  Costless, riskless income $(S_0 - I_0)e^{rT} - F_0 > 0$**, arbitrage!**

- Short sell an asset with income, need to return those income

- Suppose $F_0 > (S_0 - I_0)e^{rT}$, consider the following trading strategy:

  At time 0
  - Buy the asset for S_0, short forward
  - Borrow $c_k e^{-r_k t_k}$ at rate r_k for the time period $[0, t_k]$, borrow $S_0 - I_0$ at rate r for the time period $[0, T]$

  At time t_k
  - Receive c_k and repay the loan that is due at t_k

  At time T
  - Sell the asset for F_0, repay the loan $(S_0 - I_0)e^{rT}$

  A costless, riskless income $F_0 - (S_0 - I_0)e^{rT} > 0$, arbitrage!

Example (forward price of a stock paying dividends)

GM stock price is \$30/share today. It will pay dividends of \$0.25/share in 2 months and in 5 months. What should be the forward price of a 9-month contract? What arbitrage opportunities exist if a bank quotes 30.5? 2-, 5-, and 9-month LIBOR rates are 5.2%, 5.3% and 5.4%, respectively (continuous compounding).

The forward price should be

$$\begin{aligned} F_0 &= (S_0 - I_0)e^{rT} \\ &= (30 - 0.25e^{-5.2\%\times 2/12} - 0.25e^{-5.3\%\times 5/12})e^{5.4\%\times 9/12} \\ &= 30.73 \end{aligned}$$

If the bank quotes 30.5, an investor planning to hold GM shares for more than 9 months can (**buy low sell high**)

- sell shares for 30 each, deposit $0.25e^{-5.2\%\times 2/12}$ for 2 months, deposit $0.25e^{-5.3\%\times 5/12}$ for 5 months, deposit the remaining at rate 5.4% for 9 months, take a long forward position
- In 2 months, receive 0.25, in 5 months, receive 0.25
- At maturity, get $(30 - 0.25e^{-5.2\%\times 2/12} - 0.25e^{-5.3\%\times 5/12})e^{5.4\%\times 9/12} = 30.73$, buy back shares for 30.5, earn \$0.23/share

Investment assets with known yield

- Suppose the asset provides a known yield at rate q per year (continuous compounding): one unit of the asset at time 0 grows to e^{qT} units at T
- E.g., when the asset is GBP
  - The interest rate for GBP is q with continuous compounding
  - One pound today is worth e^{qT} pounds at time T
  - One pound at time T is worth e^{-qT} pounds today
- Cost (at time 0) of one unit of the asset (at time T): $S_0 e^{-qT}$
- Forward price for delivery of one unit of the asset at T:

$$F_0 = S_0 e^{-qT} e^{rT} = S_0 e^{(r-q)T}$$

- Create a synthetic forward contract

Position	Time 0	Time T
Short forward	no cost	$F_0 - S_T$
Synthetic short forward	Short e^{-qT} assets, deposit $S_0 e^{-qT}$	$S_0 e^{(r-q)T} - S_T$

- Suppose $F_0 > S_0 e^{(r-q)T}$, consider the following trading strategy

  At time 0

  - Buy e^{-qT} units of the asset
  - Short forward (to sell one unit of the asset)
  - Borrow $e^{-qT} S_0$ at rate r

  At time T, e^{-qT} units of the asset grow into one unit of the asset

  - Forward contract: sell the asset and receive F_0
  - Repay the loan: $e^{(r-q)T} S_0$

  A costless, riskless income $F_0 - e^{(r-q)T} S_0 > 0$, arbitrage!

- Suppose $F_0 < S_0 e^{(r-q)T}$, consider the following trading strategy

At time 0

- Long forward (to buy one unit of the asset)
- **Short sell e^{-qT} units** of the asset and receive $S_0 e^{-qT}$
- Deposit $S_0 e^{-qT}$ at rate r

At time T

- Deposit account: get $e^{(r-q)T} S_0$
- Forward contract: buy the asset for F_0
- Short selling account: **return one unit** of the asset

A costless, riskless income $e^{(r-q)T} S_0 - F_0 > 0$, arbitrage!

Example (forward price of a foreign currency)

USD/GBP exchange rate is 2 today. 6-month LIBOR rates for USD and GBP are 5.2% and 6.2%, respectively. Are there arbitrage opportunities if the 6-month forward price is 1.95?

Forward price:

$$F_0 = S_0 e^{(r-q)T} = 2e^{(5.2\%-6.2\%)/2} = 1.99$$

Arbitrage (**buy low sell high**):

- Long forward to buy £1 at T, borrow £e^{-qT}, sell for \$$S_0 e^{-qT}$, deposit at rate r
- At maturity: get $S_0 e^{(r-q)T} = 1.99$, buy £1 for 1.95 and repay the loan

Valuing existing forward contracts

- Forward contracts have zero cost at time 0 by letting $K = F_0$
- At an arbitrary time $t \in (0, T]$, the value of the forward is not necessary 0
- Banks evaluate forward contracts frequently (**marking to market**)
- Notations
  - $t \in [0, T]$: valuation time
  - S_t : underlying asset price at time t
  - F_t : forward price determined at time t with maturity T
  - V_t : value at time t of the long forward contract with delivery price K

- Decompose the payoff of a (long) forward contract
  - Payoff of the long forward contract at maturity

$$S_T - K = (S_T - F_t) + (F_t - K)$$

  - A forward contract with forward price F_t as the delivery price
  - A contract with fixed payoff $F_t - K$ at time T
- Suppose that the riskfree interest rate for the time period $[t, T]$ is r, then

$$V_t = e^{-r(T-t)}(F_t - K)$$

- Pricing forward contracts: **discount the payoff as if the forward price were realized**

- Summary of forward contracts for investment assets ($t \in [0, T]$)
  - **Investment assets with no income**

$$F_t = e^{r(T-t)} S_t$$

  - **Investment assets with known income**

$$F_t = e^{r(T-t)} (S_t - I_t)$$

    I_t : present value at time t of income received during $[t, T]$
  - **Investment assets with known continuous yield**

$$F_t = e^{(r-q)(T-t)} S_t$$

  - **Value of the (long) forward contract** (with delivery price K)

$$V_t = e^{-r(T-t)} (F_t - K)$$

Example (valuing forward contracts)

The delivery price of a 3-month forward contract to buy one Amazon share was determined to be \$81.01. The forward contract expires in 1 month. The current share price is \$85/share. 1-month LIBOR rate is 5.2%. What is the value of the long forward contract?

The value of the long forward contract is

$$V_t = e^{-r(T-t)}(F_t - K) = e^{-5.2\%/12}(F_t - 81.01) = 4.34$$

The current forward price F_t is

$$F_t = e^{r(T-t)}S_t = 85e^{5.2\%/12} = 85.37$$

Forward rate agreement (FRA)

Example (forward rate agreement)

GM needs to borrow \$1 million in 2 months. The money will be borrowed for 8 months. Currently, interest rates are relatively low. GM would like to lock the rate today to reduce interest rate risk. GM enters an FRA with a bank.

- **FRA**: agreement to borrow/lend L (principal) at a certain rate r_K for a certain future time period $[t_1, t_2]$
- An important interest rate derivative: total notional amount of outstanding FRAs was \$51,749 billions by End-Dec-2009
- No cost to enter an FRA by selecting appropriate interest rate: **forward rate**; *How to determine the forward rate?*

Determining forward rate

- Notations (assume continuous compounding)
  - r_1: interest rate for period $[0, t_1]$
  - r_2: interest rate for period $[0, t_2]$
  - r_0^f: forward rate to be applied for period $[t_1, t_2]$
- **No arbitrage** implies that $r_0^f = (r_2 t_2 - r_1 t_1)/(t_2 - t_1)$.
  - Deposit \$1 at rate r_1 for the period $[0, t_1]$, deposit $e^{r_1 t_1}$ at rate r_0^f for the period $[t_1, t_2]$
  - Deposit \$1 at rate r_2 for the period $[0, t_2]$

$$e^{r_2 t_2} = e^{r_1 t_1} e^{r_0^f (t_2 - t_1)} \Rightarrow r_0^f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$$

- If the interest rate agreed on is $r > r_0^f$, consider the following:

At time 0

- Deposit 1 for the period $[0, t_1]$ at rate r_1
- Enter an FRA to deposit $e^{r_1 t_1}$ at rate r for time period $[t_1, t_2]$
- Borrow 1 for the period $[0, t_2]$ at rate r_2

At time t_1

- Receive $e^{r_1 t_1}$, deposit for the period $[t_1, t_2]$ at rate r

At time t_2

- Receive $e^{r_1 t_1} e^{r(t_2 - t_1)}$, repay the loan $e^{r_2 t_2}$

A costless, riskless income of $e^{r_1 t_1 + r(t_2 - t_1)} - e^{r_2 t_2} > 0$, arbitrage!

- Suppose $r < r_0^f$, consider the following strategy:

  At time 0

  - Borrow 1 for the period $[0, t_1]$ at rate r_1
  - Enter an FRA to borrow $e^{r_1 t_1}$ at rate r for the period $[t_1, t_2]$
  - Deposit 1 for the period $[0, t_2]$ at rate r_2

  At time t_1

  - Borrow $e^{r_1 t_1}$ at rate r for period $[t_1, t_2]$ and repay the first loan

  At time t_2

  - Receive $e^{r_2 t_2}$, and repay the second loan $e^{r_1 t_1} e^{r(t_2 - t_1)}$

  A costless, riskless income of $e^{r_2 t_2} - e^{r_1 t_1 + r(t_2 - t_1)} > 0$, arbitrage!

Speculating on interest rate

- Zero rates and forward rates

Year	Zero rate	Forward rate
1	3.0	
2	4.0	5.0
3	4.6	5.8

- Forward rate seems higher than 1-year zero rate
- A speculator may bet that 1-year zero rate will be consistently lower than 1-year forward rate resulted from the zero curve: borrow for 1-year and roll over, deposit for longer periods
- Used by the Treasurer at **Orange County**; Led to \$1.5 billion loss when interest rate rose, and Orange County bankrupted

Valuing FRAs

- FRA has no cost when entered at time 0 by letting $r_K = r_0^f$
- Value of FRA no more zero after the inception of the contract
- Notation
  - $t \in [0, t_1]$: valuation time
  - r_1^*: interest rate for period $[t, t_1]$
  - r_2^*: interest rate for period $[t, t_2]$
  - r_t^f: time t forward rate for period $[t_1, t_2]$

$$r_t^f = \frac{r_2^*(t_2 - t) - r_1^*(t_1 - t)}{t_2 - t_1}$$

  - V_t: time t value of FRA to the lender (lend L at rate r_K for the period $[t_1, t_2]$)

- The lender will deposit L at rate r_K at time t_1, and receive $Le^{r_K(t_2-t_1)}$
- Decompose the FRA into two contracts
  - Deposit L at rate r_t^f at time t_1, and receive $Le^{r_t^f(t_2-t_1)}$ at t_2
  - Receive $Le^{r_K(t_2-t_1)} - Le^{r_t^f(t_2-t_1)}$ at t_2
- Value at time t of the FRA to the lender

$$V_t = Le^{-r_2^*(t_2-t)}(e^{r_K(t_2-t_1)} - e^{r_t^f(t_2-t_1)})$$

- In particular, at $t = t_1$, $r_t^f = r_2^*$,

$$V_{t_1} = L(e^{(r_K-r_2^*)(t_2-t_1)} - 1)$$

- FRA often settled at t_1. Borrower pays V_{t_1} to the lender

Example (forward rate agreement)

GM needs to borrow \$1 million in 2 months. The money will be borrowed for 8 months. 2- and 10-month LIBOR rates are 5.24% and 5.35%, respectively. GM enters an FRA with a bank. Suppose in 2 months, the 8-month LIBOR rate turns out to be 5.6%. How should the FRA be settled?

Forward rate

$$r_0^f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} = \frac{5.35\% \times 10/12 - 5.24\% \times 2/12}{8/12} = 5.38\%$$

Value of the FRA to GM two months later:

$$1\text{m} \cdot (1 - e^{(r_K - r_2^*)(t_2 - t_1)}) = 1\text{m} \cdot (1 - e^{(5.38\% - 5.6\%) \times 8/12}) = \$1465.59$$

Forward contracts on commodities

- **Commodities for investment:** gold
- Commodities are subject to **storage costs**
- Let U be the present value of storage costs for each unit of the asset
- Adjusted cost of the asset: $S_0 + U$
- **Forward price** for a commodity as an investment asset

$$F_0 = (S_0 + U)e^{rT}$$

- If the asset receives income, U is the PV of storage costs, net of income

- **Consumption commodities:** wheat
- No arbitrage implies that $F_0 \leq (S_0 + U)e^{rT}$ (otherwise, buy now and short a forward contract)
- To show $F_0 \geq (S_0 + U)e^{rT}$ by contradiction, need to short sell: often not possible (assets held for consumption)
- **Forward price** for a consumption asset: $F_0 \leq (S_0 + U)e^{rT}$
- **Convenience yield** y is such that $F_0 e^{yT} = (S_0 + U)e^{rT}$

Futures contracts

- **Futures contract**: agreement to buy/sell an asset at a future time for a certain price (**delivery price**)
- Specific delivery price so that the futures contract has no initial cost: **futures price**
- Forwards are traded in the over-the-counter market
- Futures are **exchange-traded, highly standardized**
  - The exchange specifies: **what to deliver, when to deliver, where to deliver**
  - Futures are settled daily

Terms specified by the exchange

- **The asset:** quality of a commodity, maturity of treasuries
- **Contract size:** amount of the asset to be delivered per contract; e.g., a corn futures contract on CBOT is for 5000 bushels of corn
- **Delivery location:** if alternatives are specified, seller selects the location
- **Maturity:** when maturity is a whole month, seller picks the date
- **Price movement limits, position limits:** prevent speculators from manipulating the market

CBOT Corn futures

Corn Futures

Contract Size	5,000 bushels (~ 127 Metric Tons)	
Deliverable Grade	#2 Yellow at contract Price, #1 Yellow at a 1.5 cent/bushel premium #3 Yellow at a 1.5 cent/bushel discount	
Pricing Unit	Cents per bushel	
Tick Size (minimum fluctuation)	1/4 of one cent per bushel ($12.50 per contract)	
Contract Months/Symbols	March (H), May (K), July (N), September (U) & December (Z)	
Trading Hours	CME Globex (Electronic Platform)	6:00 pm - 7:15 am and 9:30 am - 1:15 pm central time, Sunday - Friday Central Time
	Open Outcry (Trading Floor)	9:30 am - 1:15 pm Monday - Friday Central Time

Daily settlement and margining

Example (daily settlement)

Consider a long futures contract for 100 ounces of gold traded on NYMEX (**http://www.nymex.com**). The current futures price is \$880 per ounce

- The investor needs to open a **margin account**
- **Initial margin:** the amount required to be deposited at the contract inception, e.g., \$4000;
- **Marking to market:** the margin account is adjusted at the end of each trading day to reflect the investor's gain/loss

Date	Current futures price		Initial margin
day 1	$F_0 = 880$		4000
Date	End of day futures price	Gain	Margin account balance
day 1	$F_1 = 874$	-600	3400
day 2	$F_2 = 876$	200	3600
day 3	$F_3 = 860$	-1600	$2000 \to 4000$ **(-2000)**
day 4	$F_4 = 865$	500	4500

- At the end of day 1, \$600 is transferred to the margin account of a short position through the exchange **clearinghouse**
- In effect, a futures contract is closed out every day and rewritten at a new futures price

- **Maintenance margin:** minimum amount required in the margin account, e.g., $3000
- **Margin call:** if the balance falls below the maintenance margin, receives a margin call and needs to top up the margin account to the initial margin
- May withdraw any balance in excess of the initial margin
- The exchange specifies the minimum initial and maintenance margins. Depend on the variability of the underlying asset price
- Credit risk is minimized through **the margining system**

- Suppose day 4 is the maturity: futures price = spot price = 865
- **Futures price converges to spot price as time to maturity decreases**
- Payoff of the long futures position:

$$
\begin{aligned}
-1500 &= -600 + 200 - 1600 + 500 \\
&= 100(F_1 - F_0) + 100(F_2 - F_1) + 100(F_3 - F_2) + 100(F_4 - F_3) \\
&= 100(F_4 - F_0) \;=\; 100(S_T - F_0)
\end{aligned}
$$

  equal to the payoff of a long forward with delivery price F_0
- Profit is realized at maturity for a forward contract; It is realized over time for a futures contract.

Futures price

- **Deterministic interest rates**: futures price = forward price
- **Stochastic interest rates:**
  - Asset price and interest rate **positively correlated:** futures price > forward price typically. For long futures,
    - Asset price increases ⇒ immediate profit + invested at higher rate
    - Asset price decreases ⇒ immediate loss + financed at lower rate
  - Asset price and interest rate **negatively correlated:** futures price < forward price typically
- Difference not significant for short maturities

Currency futures

- Currency forwards and futures widely used to hedge exchange rate risks
- Popular currency futures on CME

Currency	Contract size	Minimum fluctuation
Euro	125,000 Euro	\$0.0001/Euro = \$12.5/contract
GBP	£62,500	\$0.0001/GBP = \$6.25/contract
Japanese yen	¥12,500,000	\$0.000001/yen = \$12.5/contract
CAD	100,000 CAD	\$0.0001/CAD = \$10/contract
Swiss francs	125,000 CHF	\$0.0001/ = \$12.5/contract

Index futures

- Index futures: underlying financial variables are major market indices
- Used by investors to manage risk of stock market fluctuation (hedging, changing portfolio beta, etc)
- Popular index futures on CME:

Index	Contract size	Minimum fluctuation
E-mini S&P 500	50	\$0.25/index = \$12.5/contract
E-mini NASDAQ-100	20	\$0.25/index = \$5/contract
S&P 500	250	\$0.10/index = \$25/contract
NASDAQ-100	100	\$0.25/index = \$25/contract
E-mini Russell 2000	100	\$0.1/index = \$10/contract

Hedging using futures

- Hedgers use futures to reduce risks due to: asset price fluctuation, exchange rate movement, interest rate uncertainty
- Long and short hedges

Current position	*Risk*	*Hedge*	*Example*
will sell an asset	*asset price may fall*	**short**	*farmer who will harvest corn*
will buy an asset	*asset price may rise*	**long**	*company that will buy corn*

- Compared to forward contracts: futures are traded on exchanges (more liquid), almost no credit risk

Short hedge example

Example (short hedge)

A heating oil producer enters a contract to sell 420,000 gallons of heating oil at the market price on December 15. How can the producer hedge its position? Current December futures price is \$2.08/gal

- Short hedge: fixes the price at \$2.08/gal (potential loss due to price drop canceled by gain in short futures)
- Perfect hedge possible since
  - heating oil futures are traded (NYMEX)
  - each heating oil futures contract is on 42,000 gallons
  - December futures contracts are available

Basis risk

- Perfect hedging often not possible because
  - futures with desired maturity not available
  - futures with desired underlying (asset or size) not available
- Suppose the sale/purchase will occur at T, futures with maturity T not available:
  - Choose the first available maturity after T
  - The sale/purchase price cannot be fixed at the futures price: **basis risk**
  - **Basis** $=$ spot price at time T - futures price of the contract at time T
  - If maturity of the futures $= T$, no basis risk: basis $= 0$

Basis risk example

Example (basis risk)

A company will receive and sell ¥50 million in 5 months. The next available delivery month for Yen futures on CME is in 7 months. Current futures price $F_0 = 0.7800$ cents/yen. In 5 months, spot price $S_1 = 0.7200$, futures price with 2 months to maturity $F_1 = 0.7250$.

- **Short hedge**: desired maturity not available; yen futures with 7 months to maturity used
- Short 4 futures (1 contract is for ¥12.5 million). In 5 months,
  - **Gain on the futures:** $F_0 - F_1 = 0.7800 - 0.7250 = 0.0550$
  - Sell yen: **Effective price:** $F_0 + \underbrace{S_1 - F_1}_{\textbf{basis risk}} = 0.7750$ cents/yen

Cross hedging

- Futures with desired underlying not available
  - Choose contracts with best matching underlying asset (**cross hedging**)
  - American Airlines need to buy jet fuel in 3 months (**long hedge**). Jet fuel futures not available. Use heating oil futures
- Cross hedging
  - **Hedge ratio:** amount of asset underlying the futures contracts for each unit of the asset to be hedged
  - American Airlines: *the amount of heating oil underlying (long) futures contracts to hedge the purchase of 1 gallon of jet fuel*
  - **Minimize the variance** of the outcome of the hedged position

- Suppose maturity of hedge is T (buy/sell at this time)
- Futures with maturity $\geq T$ used, with underlying asset highly correlated with the asset to be hedge. h: hedge ratio

0	T	Long hedge	Short hedge
S_0	S_T	buy at S_T	sell at S_T
F_0	F_T	gain $h(F_T - F_0) = h\Delta F$	gain $h(F_0 - F_T) = -h\Delta F$
		effective price $S_T - h\Delta F$	effective price $S_T - h\Delta F$

- With perfect hedging (same maturity, futures with the same underlying), $h = 1$, $S_T = F_T$, effective price $= F_0$
- In general, effective price $F_0 + (S_T - h\Delta F - F_0) \neq F_0$

Hedge ratio

- Notations
  - ΔS : change in spot price during the life of the hedge $[0, T]$
  - ΔF : change in futures price during the life of the hedge
  - σ_S, σ_F : standard deviation of ΔS and ΔF
  - ρ : correlation coefficient between ΔS and ΔF
  - h^* : minimum variance hedge ratio
- Minimize the variance of $\Delta S - h\Delta F$

$$V(\Delta S - h\Delta F) = \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F, \quad \frac{\partial V}{\partial h} = 0$$

  minimum variance achieved when $h = h^* = \rho\sigma_S/\sigma_F$
- ρ, σ_S, σ_F estimated from historical data

- Linear regression interpretation

$$\Delta S = h\Delta F + \text{const} + \text{random error}$$

least square estimation of the coefficient h

$$h^* = \frac{\hat{\text{cov}}(\Delta S, \Delta F)}{\hat{\text{var}}(\Delta F)}$$

- Hedging with the minimum variance hedge ratio h^*,

$$V(\Delta S - h^* \Delta F) = (1 - \rho^2)\sigma_S^2$$

in contrast to (without hedging) $V(\Delta S) = \sigma_S^2$

- **Effectiveness of cross hedging**
  - hedging reduces the variance by a factor of $1-\rho^2$
  - for the underlying asset of the futures contract, choose one that is highly correlated with the original asset
- **Number of futures contracts needed:**

  N_A : amount of the asset to be hedged

  Q_F : amount of the asset underlying each futures contracts

  Number of futures contracts needed: $h^* \times N_A/Q_F$

Cross hedging example

Example (cross hedging)

American Airlines need to buy 2 million gallons of jet fuel in 1 month and will use heating oil futures for hedging. Each heating oil futures contract is on 42000 gallons of heating oil. The standard deviations of the spot price change and futures price change in 1 month are estimated to be 0.0263 and 0.0313, respectively. The correlation coefficient is estimated to be 0.928. How many futures contracts should be used?

$$h^* = \rho \frac{\sigma_S}{\sigma_F} = 0.928 \times 0.0263/0.0313 = 0.78$$

Number of contracts needed:

$$h^* \times N_A/Q_F = 0.78 \times 2000,000/42000 = 37.14 \approx 37$$

Hedging using stock index futures

- Index futures are used to manage risks due to stock market fluctuation
- An investor holding a well diversified equity portfolio faces the risk of a market crash
- **Cross hedge:** short a stock index futures (a stock index is highly correlated to a well diversified equity portfolio)
- Compute the optimal number of futures contracts

- **Notations:**
  S, I, F : current portfolio value, index value, index futures price
  ΔS : the value change of the portfolio
  ΔF : the futures price change
  ΔI : the value change of the index
  σ_S, σ_F : standard deviation of ΔS and ΔF
  ρ : correlation coefficient between ΔS and ΔF
  Q_F : the amount of underlying assets for each futures contract (e.g., for CME S&P 500 index futures, $Q_F = 250$)
- Optimal hedge ratio

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

- **Beta** of a portfolio measures the variability of the portfolio relative to the overall market (represented by a stock index)

$$\beta = \frac{\text{cov}(\Delta S/S, \Delta I/I)}{\text{var}(\Delta I/I)} = \frac{I}{S}\frac{\text{cov}(\Delta S, \Delta I)}{\text{var}(\Delta I)}$$

- Assume maturities of the hedge and the futures are the same

$$\Delta F = \Delta I + const, \beta = \frac{I}{S}\frac{\text{cov}(\Delta S, \Delta F)}{\text{var}(\Delta F)} = \frac{I}{S}\frac{\rho\sigma_S}{\sigma_F} = \frac{I}{S}h^* \Rightarrow h^* = \frac{\beta S}{I}$$

- The **number of futures contracts needed**

$$h^* \times Q_A/Q_F = h^*/Q_F = \beta S/(IQ_F)$$

Example (hedging using index futures)

A fund manager holding a portfolio worries that there will be a big change in the value of the portfolio in a month (due to expected release of news). S&P 500 index futures will be used for the short hedge. The current values of the portfolio and the S&P 500 index are 5000,000 and 1000, respectively. Each futures contract is on 250 indices. The beta of the portfolio is 1.5. How many futures contracts should be used?

$$\frac{\beta S}{IQ_F} = \frac{1.5 \times 5000,000}{1000 \times 250} = 30$$

Manage portfolio beta

- Beta of the hedged portfolio in the previous example
  - Current value of the hedged portfolio: $P = S$
  - Value change of the hedged portfolio

$$\Delta P = \Delta S + 30 \cdot 250 \cdot (F - I_T) = \Delta S + 30 \cdot 250 \cdot (F - I - \Delta I)$$

  - Beta of the hedged portfolio

$$\beta^* = \frac{\mathrm{cov}(\frac{\Delta P}{P}, \frac{\Delta I}{I})}{\mathrm{var}(\frac{\Delta I}{I})} = \frac{\mathrm{cov}(\frac{\Delta S - 30 \cdot 250 \cdot \Delta I}{S}, \frac{\Delta I}{I})}{\mathrm{var}(\frac{\Delta I}{I})} = \beta - 30 \cdot \frac{250 I}{S} = 0$$

- If desired beta is 0.75: short 15 futures

$$\beta^* = \beta - 15 \times \frac{250I}{S} = 1.5 - \frac{1}{2} \times 1.5 = 0.75$$

- If desired beta is 2: long 10 futures

$$\beta^* = \beta + 10 \times \frac{250I}{S} = 1.5 + \frac{1}{3} \times 1.5 = 2$$

Hedging going wrong

- **Metallgesellschaft** (MG) enter 5- and 10-year supply contracts with its customers to sell heating oil
- Long futures to hedge its positions: all available maturities not long enough
- Rolling the hedge forward: enter another short maturity long futures when the previous expires
- Oil price fell, margin requirements put huge short term cash flow pressure on MG, causing a loss of $1.33 billion

Speculating using futures

- GBP/USD futures on CME is on 62,500 British pounds. Initial margin is $2430 (maintenance margin $1800, **http://www.cmegroup.com**). Current exchange rate is 1.5541 USD/GBP. Current 3-month futures price is 1.5524 USD/GBP
- An investor who believes that GBP will strengthen relative to USD and with $24,300 available can
  1. Buy 15,636.06 GBPs with $24,300
  2. Enter 10 long futures contacts on 625,000 GBPs

- Profit when the exchange rate moves

	spot rate/futures price	
	1.5735/1.5718	1.5347/1.5330
strategy 1	15,636.06(1.5735-1.5541)=\$303.34	\$-303.34
strategy 2	625,000(1.5718-1.5524)=\$12,111.67	\$-12,111.67

- Leverage effect