Markowitz Portfolio Theory and CAPM

Liming Feng

Dept. of Industrial & Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Reference: Investment Science by David Luenberger, chapters 6, 7

Outline

- Understanding risk and return
- Markowitz mean-variance portfolio theory
- Capital asset pricing model (CAPM) and beta
- **Harry Markowitz** and **William Sharpe**, Nobel prize laureates in Economics (1990)

Rate of return

- An investment with initial payment X_0 and final receipt X_1
- **Rate of return**

$$r = \frac{X_1 - X_0}{X_0} \qquad (i.e., X_1 = (1 + r)X_0)$$

Example: *buy a stock for 100 and sell later for 110*

$$r = \frac{110 - 100}{100} = 10\%$$

Similarly, deposit 100 and get 105 later, $r = 5\%$

- Receive X_0 initially and pay X_1 later

$$r = \frac{(-X_1) - (-X_0)}{-X_0} = \frac{X_1 - X_0}{X_0}$$

  Example: *borrow 100 and return 105 later,* $r = 5\%$

- **Short selling**: selling an asset that is not owned and needs to be borrowed
  - Borrow an asset, sell and receive X_0
  - At a future time, buy back the asset for X_1 and return it
  - Gain if asset price falls. Loss if asset price rises

Short selling in practice

- Limited gain, unlimited potential loss
- **Margin account** required for the protection of the lender
    - All proceeds of the short sale will be put in the margin account
    - **Initial margin**: additional $x\%$ required in the margin account
    - **Maintenance margin**: $y\%$ in addition to the asset value required all the time
    - **Margin call**: margin account balance $<$ maintenance margin requirement

- **Example (short selling)**: *an investor instructs a broker to short sell 1000 IBM shares at $100/share. The initial margin is 50% and the maintenance margin is 30%. How far can the share price rise before the investor gets a margin call?*

share price	total value	initial margin	total margin	
100	100k	50%×100k=50k	150k	
share price	total value	maintenance margin	total margin	margin call
110	110k	30%×110k=33k	143k	N/A
120	120k	30%×120k=36k	156k	6k
$x = 115.38$	$1000x$	$30\% \times 1000x = 300x$	$1300x < 150$k	N/A
90	90k	50%×90k=45k	135k	can withdraw 15k

Portfolio rate of return

- We assume idealized short selling (no commission/margining)
- Portfolio of n assets and initial investment X_0
- w_i - fraction of the initial investment invested in asset i (amount invested in asset i: $w_i X_0$)

$$\sum_{i=1}^{n} w_i = 1$$

- Rate of return for the portfolio

$$r_p = \frac{\sum_{i=1}^{n}(1 + r_i) w_i X_0 - X_0}{X_0} = \sum_{i=1}^{n} w_i r_i$$

- Negative w_i refers to short selling
- **Example:** *Suppose current IBM and Microsoft share prices are* 100 *and* 20*, respectively. An investor buys 50 IBM shares and shorts 50 Microsoft shares. What is the portfolio rate of return if IBM share price rises to* 120 *and Microsoft share price drops to* 12*?*

$$X_0 = 4000, X_1 = 6000 - 600 = 5400$$

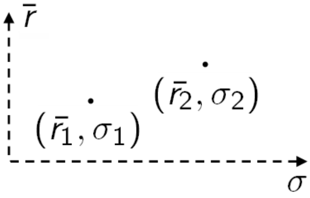
$$r_p = 1400/4000 = 35\%$$

$$w_1 = 125\%, w_2 = -25\%, r_1 = 20\%, r_2 = -40\%$$

$$r_p = w_1 r_1 + w_2 r_2 = 35\%$$

Uncertainty

- Rate of return r is a random variable
  - Expectation: $\bar{r} = \mathbb{E}[r]$
  - Variance measures risk: $\sigma^2 = \mathrm{var}(r)$
  - Covariance: $\sigma_{ij} = \mathrm{cov}(r_i, r_j)$ (in particular, $\sigma_{ii} = \sigma_i^2$)
- Mean standard deviation diagram



- Portfolio rate of return r_p:

$$r_p = w_1 r_1 + \cdots + w_n r_n$$

$$\mathbb{E}[r_p] = w_1 \bar{r}_1 + \cdots + w_n \bar{r}_n$$

$$\text{var}(r_p) = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

- Prefer higher rate of return and lower variance
- **Diversification**: including more assets in the portfolio

Diversification

- **Example**: *returns of three independent stocks have the same variance σ^2. What is the variance of the rate of return of a portfolio with equal weights in these stocks $(w_1 = w_2 = w_3 = 1/3)$?*
  - Variance of the portfolio rate of return:

$$\mathrm{var}(r_p) = \mathrm{var}\left(\frac{1}{3}r_1 + \frac{1}{3}r_2 + \frac{1}{3}r_3\right) = \frac{1}{3}\sigma^2$$

  - Extension to n assets: $\mathrm{var}(r_p) = \frac{1}{n}\sigma^2$

- **Example**: *returns of three correlated stocks have the same variance* σ^2 *and pairwise covariance* $0.3\sigma^2$. *What is the variance of the rate of return of a portfolio with equal weights in these stocks?*
  - Variance

$$\text{var}(r_p) = \text{var}\left(\frac{1}{3}r_1 + \frac{1}{3}r_2 + \frac{1}{3}r_3\right) = \frac{0.7}{3}\sigma^2 + 0.3\sigma^2$$

  - Extension to n assets: $\text{var}(r_p) = \frac{0.7}{n}\sigma^2 + 0.3\sigma^2$
- In general, diversification reduces variance
- It may reduce mean return while reducing variance
- Find a balance using **Markowitz portfolio theory**

Portfolio mean standard deviation diagram

- Given n risky assets, determine **feasible set**: pairs $(\sigma_p, \bar{r}_p)$ for all possible portfolios
- Portfolios of two assets

$$\text{Mean: } \bar{r}_p = w\bar{r}_1 + (1-w)\bar{r}_2$$
$$\text{Variance: } \sigma_p^2 = w^2\sigma_1^2 + 2w(1-w)\sigma_{12} + (1-w)^2\sigma_2^2$$

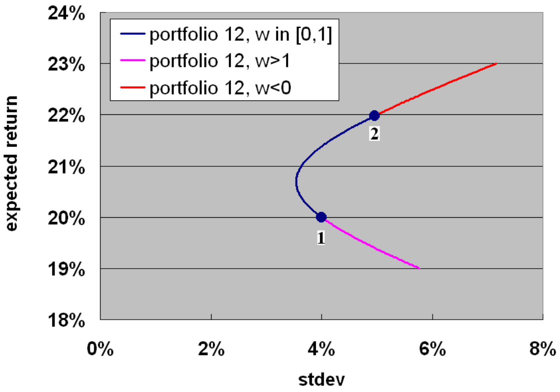
- **Example:** *consider the following two risky assets. What is the feasible set?*

	asset 1	*asset 2*
mean return	$\bar{r}_1 = 20\%$	$\bar{r}_2 = 22\%$
stdev	$\sigma_1 = 4\%$	$\sigma_2 = 5\%$
correlation	$\rho_{12} = 0.3$	

In Excel, let w varies from -0.5 to 1.5, compute $\bar{r}_p$ and σ_p, and draw $(\bar{r}_p, \sigma_p)$

	asset 1	asset 2	asset 3			
mean	20%	22%	24%			
stdev	4%	5%	4%			
corr 12, 23, 13	0.3	0.3	-0.1			
w	mean12	stdev12	mean 23	stdev23	mean13	stdev13
-0.50	0.2300	0.0716	0.2500	0.0577	0.2600	0.0651
-0.49	0.2298	0.0711	0.2498	0.0572	0.2596	0.0646
-0.48	0.2296	0.0707	0.2496	0.0568	0.2592	0.0640
-0.47	0.2294	0.0702	0.2494	0.0564	0.2588	0.0635

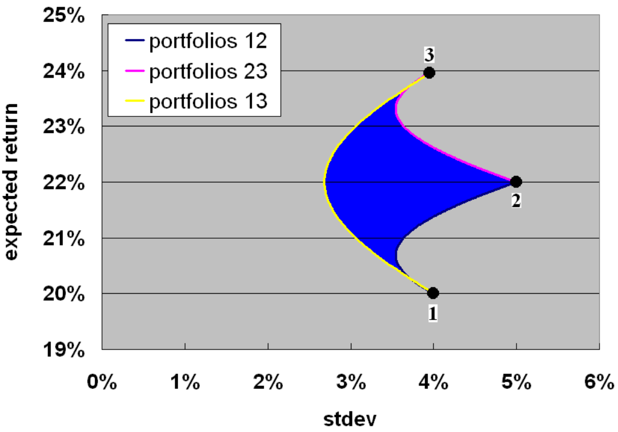
- Feasible set in the previous example



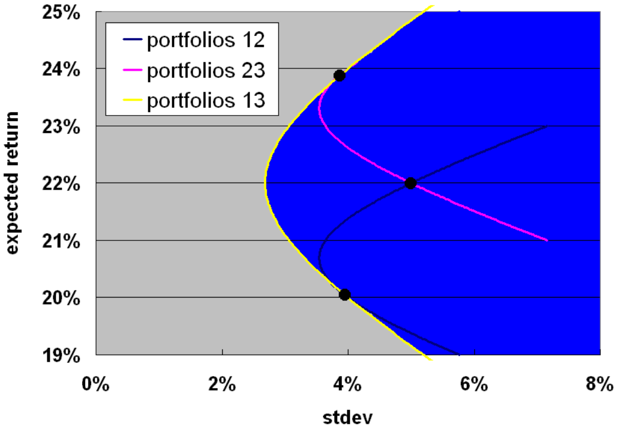
- Portfolios of three assets
- **Example:** *consider the following three risky assets. What is the feasible set?*

	asset 1	*asset 2*	*asset 3*
mean return	$\bar{r}_1 = 20\%$	$\bar{r}_2 = 22\%$	$\bar{r}_3 = 24\%$
stdev	$\sigma_1 = 4\%$	$\sigma_2 = 5\%$	$\sigma_3 = 4\%$
correlation	$\rho_{12} = 0.3$	$\rho_{23} = 0.3$	$\rho_{13} = -0.1$

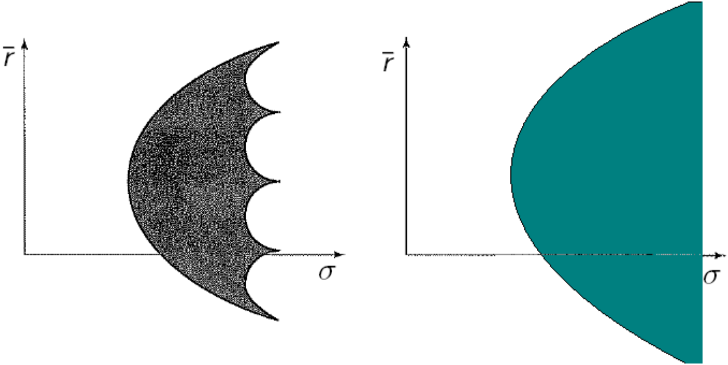
- Feasible set for the previous example with 3 assets (no short selling)



- With short selling



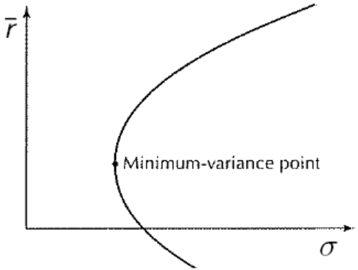
- Feasible set for portfolios of n assets



LHS: no short selling; RHS: with short selling

- **Minimum variance set**: left boundary of a feasible set
  - For given mean rate of return, the portfolio with smallest variance is on the minimum variance set
  - A **risk averse** investor prefers portfolios on the minimum variance set
- **Efficient frontier**: upper portion of the minimum variance set
  - For given standard deviation, the portfolio with largest return is on the efficient frontier
  - An investor wants more money prefers portfolios on the efficient frontier

- Minimum variance set and efficient frontier



LHS: Minimum variance set; RHS: efficient frontier

Markowitz model

- Finding the minimum variance portfolio for given expected rate of return $\bar{r}$

$$\min \quad \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

subject to

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}, \quad \sum_{i=1}^{n} w_i = 1$$

- The optimization problem can be solved numerically, e.g., in Excel

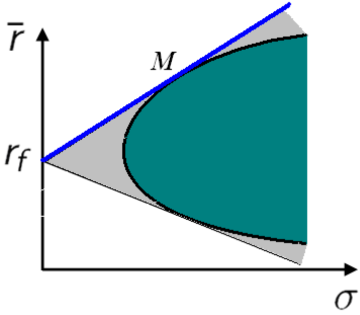
With a risk free asset

- Suppose there is a risk free asset (with deterministic return r_f and zero variance)
- Deposit \$1 at rate r_f, will receive $\$(1 + r_f)$; borrow \$1 at rate r_f, need to repay $\$(1 + r_f)$
- Consider portfolios of risky asset and risk free asset
    - How to balance risk and return
    - Determine the feasible set and efficient frontier

- Portfolio of n risky assets and a risk free asset: a fraction of w invested in the risk free asset, $1-w$ invested in a portfolio of risky assets (with mean return $\bar{r}_p$, variance σ_p^2)

$$\text{mean: } wr_f + (1-w)\bar{r}_p, \text{ variance: } (1-w)^2\sigma_p^2$$

- Feasible set



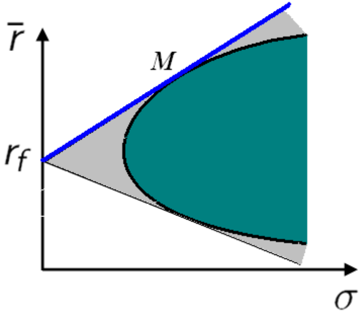
One fund theorem

- Efficient frontier is a straight line
- Tangent to the feasible set of risky assets
- **One fund theorem:** there is a portfolio M of risky assets such that any efficient portfolio (with both risky and risk free assets) is a linear combination of M and the risk free asset

Markowitz portfolio theory

- **One fund theorem**: one portfolio M of risky assets plus risk free investment
- If all investors prefer less risk for given rate of return, all of them will hold M
- M should be the **market portfolio** (of all risky assets on the market)

- **Capital market line**



- Relation between risk and return (of an efficient portfolio): as risk increases, expected rate of return increases

- For an arbitrary **efficient portfolio**

$$\bar{r}_p = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma_p$$

  - $\bar{r}_M$: expected rate of return of market portfolio
  - r_f: risk free rate of return;
  - σ_M: stdev of market portfolio rate of return
  - σ_p: stdev of the rate of return of the efficient portfolio
- **Price of risk** $(\bar{r}_M - r_f)/\sigma_M$: expected rate of return required for each unit of the stdev in addition to r_f

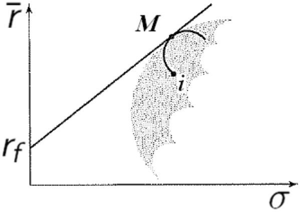
- Relation between risk and return of an individual asset?
- **Capital asset pricing model (CAPM)**: the expected return $\bar{r}_i$ of any asset i satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f), \qquad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

where σ_{iM}: covariance of returns of asset i and market portfolio

- Portfolio with w invested in asset i and $1-w$ invested in M
- On mean standard deviation diagram:

$$\bar{r} = w\bar{r}_i + (1-w)\bar{r}_M, \quad \sigma = (w^2\sigma_i^2 + 2w(1-w)\sigma_{iM} + (1-w)^2\sigma_M^2)^{1/2}$$



- Tangent to the capital market line at M (i.e., at $w = 0$)

- Expected **excess return** of asset i is proportional to expected excess return of market portfolio
- β_i – the **beta** of asset i
- In terms of risk and return: β_i measures riskiness of asset i relative to the market portfolio
- Beta of a portfolio of n assets with weights $w_1, \cdots, w_n$

$$\beta_p = \sum_{i=1}^{n} w_i \beta_i$$

- Amazon's beta on Sept 8, 2008 (**http://finance.aol.com**)

AMAZON COM INC

AMZN

81.16 1.97 2.49%

as of 04:01 PM EDT on 09/08/2008 in USD (NASDAQ Delay: 15 mins.)

Get Quote Details for: AMZN Go

Real-Time Quote
Only available during pre-market and regular market hours. about

Quote Details
- Option Chains
- Historical Prices
- Dividends & Splits
- Average Monthly Returns
- **Classic Chart**
- **Advanced Chart** BETA
- **Real-Time Headlines**
- News & Analysis
- Trade Publications
- Regional

Day Low	78.85	52-Wk Low	61.20
Day High	83.75	52-Wk High	101.09
Volume	9.49 M	Prev. Close	79.19
30-Day Avg. Vol.	7.21 M	Today's Open	82.25
Market Cap.	34.56 B	Dividend (TTM)	0.00
Shares Out.	425.92 M	Div. Yield (TTM)	0.00%
Revenue (LFY)	14.83 B	Beta	2.4596
Earnings (TTM)	476 M	P/E Ratio (TTM)	59.24
Next Earnings	10/22/08	P/E Ratio (Fwd.)	46.00
EPS (TTM)	1.37		

Systematic vs specific risk

- By CAPM, $\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$. Consider

$$\epsilon_i = r_i - r_f - \beta_i(r_M - r_f)$$

  - $\text{cov}(\epsilon_i, r_M) = \sigma_{iM} - \beta_i \sigma_M^2 = 0$
  - Decomposition of risk: $\sigma_i^2 = \text{var}(\epsilon_i) + \beta_i^2 \sigma_M^2$
  - **Systematic risk**: $\beta_i^2 \sigma_M^2$, risk associated with the market
  - **Specific risk**: $\text{var}(\epsilon_i)$, uncorrelated with the market
  - Q: what is the specific risk for a portfolio on the capital market line?

CAPM in practice

- By CAPM, all efficient portfolios consist of risk free asset and market portfolio
- Market portfolio often represented by major stock market indices: e.g., S&P 500
- Fund managers design portfolios that match stock market indices closely: **index funds**

Portfolio performance

- For an efficient portfolio (capital market line)

$$\bar{r}_p = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma_p$$

- Consider a portfolio with given historical data
  - estimate $\bar{r}_p$, σ_p
  - **Sharpe ratio**

$$S := \frac{\bar{r}_p - r_f}{\sigma_p}$$

  - not efficient if

$$S < \frac{\bar{r}_M - r_f}{\sigma_M}$$

- As a pricing model: for an asset with current price P and future price Q

$$\frac{\bar{Q} - P}{P} - r_f = \bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$$

Pricing formula

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}$$

Discounted expectation of future value, with **risk adjusted discount rate**

- Can be used for firms to select projects