Introduction to Interest Rates

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Readings: Hull Chapter 4

Time Value of Money

- A dollar today is worth more than a dollar tomorrow
- Let r be the interest rate per year: \$1 deposited for 1 year grows to \$ 1+r (**principal + interest**)
- **Compound interest:** earn interest on interest; \$1 deposited for t years grows to \$ $(1+r)^t$ (annual compounding)
- **Example (annual compounding):** *what amount does \$100 invested at 10% (or 7%) per year for 7 (or 10) years grow to?*

$$100(1+10\%)^7 = 195 \quad (\text{or } 100(1+7\%)^{10} = 197)$$

- Let r be the interest rate per year. \$1 deposited for 1 year with **monthly compounding** grows to

$$(1 + r/12)^{12}$$

- \$1 deposited for t years with interest rate r compounded m times per year grows to

$$(1 + r/m)^{mt}$$

- **Continuous compounding**: as $m \to \infty$

$$(1 + r/m)^{mt} \to e^{rt}$$

\$1 deposited with continuous compounding grows to e^{rt} at time t

- **Example:** *the value in 1 year of $1 deposited today with interest rate* $r = 10\%$ *and the following compounding frequencies*

annually	$m = 1$	$1 + r$	\$1.10000
semiannually	$m = 2$	$(1 + r/2)^2$	\$1.10250
quarterly	$m = 4$	$(1 + r/4)^4$	\$1.10381
monthly	$m = 12$	$(1 + r/12)^{12}$	\$1.10471
daily	$m = 365$	$(1 + r/365)^{365}$	\$1.10516
continuously	$m = \infty$	e^r	\$1.10517

  - When an interested rate is quoted, one should find out what compounding frequency is used

- In derivatives pricing, continuous compounding typically used

- Conversion between different compounding
  - r_m: interest rate per year compounded m times a year
  - r_c: interest rate per year with continuous compounding

$$(1 + r_m/m)^m = e^{r_c}$$

$$r_c = m \ln(1 + r_m/m), \qquad r_m = m(e^{\frac{r_c}{m}} - 1)$$

  - **Example:** *what is the corresponding annual compounding rate for continuous compounding interest rate 10% per year?*

$$(e^{0.1} - 1) = 10.517\%$$

Present and Future Values

- **Future value** of \$1 deposited today at the end of t^{th} year

  $(1+r)^t$ with interest rate r per year compounded annually

  e^{rt} with interest rate r per year compounded continuously

- **Present value** of \$1 at the end of t^{th} year

  $\dfrac{1}{(1+r)^t}$ with interest rate r compounded annually

  e^{-rt} with interest rate r compounded continuously

  These are **discount factors**

Mortgage

- Homeowner gets immediate cash from a bank to pay for a home, and makes monthly payments throughout its term to the bank.
- **Example (30-year fixed mortgage):** *a new home costs \$272,000. 10% is paid at the time of purchase. The remaining \$244,800 needs to be financed. What is the monthly payment of a 30-year fixed mortgage loan at rate 6.25%?*
  - Determine monthly payment so that present value of all 360 payments is \$244,800 at annual rate 6.25% with monthly compounding
  - Monthly payments usually start at the beginning of the second month

- Suppose the monthly payment is A. Denote $r = 6.25\%$. Then

$$244,800 = \frac{A}{1+r/12} + \frac{A}{(1+r/12)^2} + \cdots + \frac{A}{(1+r/12)^{360}},$$

Geometric sequence with ratio $d = 1/(1+r_m)$, $r_m = r/12$

$$244,800 = A\frac{d(1-d^{360})}{1-d} = \frac{A}{r_m}\left(1 - \frac{1}{(1+r_m)^{360}}\right)$$

$$A = 1507.28$$

- Q: for the first payment, how much is interest/principal?

Treasury bills

- **T-bills:** issued by U.S. government at discount to their face value
  Maturity: 4, 13, 26, 52 weeks
  Minimum face value: \$100
  Trading: auctioned off by the U.S. government, traded on the secondary market
  Example: *26-week T-bill with face value \$100 was auctioned off at \$99.026806*

Time	*0*	*182/365*
Cash flow	*-99.026806*	*+100*

Treasury notes/bonds

- **T-notes/bonds:** coupon payment every six months, the face value is paid at maturity
  Maturity: 2, 3, 5, 7, 10 years for T-notes; 30 years for T-bonds
  Minimum face value: \$100
  Example: *2-year T-note with face value \$100 and coupon rate 2.375% was auctioned off at \$99.990164. Coupon payment every 6 months:* $2.375\% \times 100/2 = 1.1875$

Time	0	1/2	1	3/2	2
Cash flow	-99.990164	+1.1875	+1.1875	+1.1875	+101.1875

Zero coupon bonds

- Sold at price lower than the face value; pay the face value at maturity; no coupons
- Return comes from the price difference
- **Zero rate:** continuously compounded interest rate implied by a zero-coupon bond
  - A zero coupon bond with face value F and maturity T sells for P; corresponding zero rate solves

$$P = e^{-rT}F, \qquad r = \frac{1}{T}\ln(F/P)$$

  - Zero rate is also called **spot rate:** the interest rate for borrowing/lending over a period of length T, starting **TODAY**

- **Example (T-bills):** *4-, 13-, 26- and 52-week T-bills with face value $100 were auctioned off at the following prices. What are the corresponding zero rates?*

$$r = \frac{1}{T} \ln(F/P),$$

Maturity	F	P	Zero rate
$T = 28/365$	100	99.867000	1.73%
$T = 91/365$	100	99.563000	1.76%
$T = 182/365$	100	99.026806	1.96%
$T = 364/365$	100	97.836222	2.19%

Term structure of interest rates

- **Term structure**: interest rate applied for borrowing/lending depends on the length of time
- **Zero curve**: plot of zero rates w.r.t. maturities
- Computing present value for a stream of cash flows: different discounting rate for each cash flow

Coupon bonds

- Pay coupons periodically and the face value at maturity (e.g., T-notes, T-bonds)
- Extend zero curve using coupon bonds: **Bootstrap method**
- **Example:** *Suppose the following bonds (face value \$100, coupons paid every half year) are traded:*

Maturity	*Coupon rate*	*Price*
0.5	0%	97.523
1.0	4.75%	99.792
1.5	4.50%	99.466

What are the zero rates for 0.5-, 1.0- and 1.5-year maturities?

- 0.5-year zero rate solves

$$97.523 = 100e^{-0.5r_{0.5}} \Rightarrow r_{0.5} = 5.02\%$$

- 1.0-year coupon bond pays \$2.375 at time 0.5, and \$102.375 at time 1. 1-year zero rate solves

$$99.792 = 2.375e^{-0.5r_{0.5}} + 102.375e^{-r_1} \qquad r_1 = 4.90\%$$

- 1.5-year coupon bond pays \$2.25 at time 0.5 and time 1, and 102.25 at time 1.5. 1.5-year zero rate solves

$$99.466 = 2.25e^{-0.5r_{0.5}} + 2.25e^{-r_1} + 102.25e^{-1.5r_{1.5}}, \quad r_{1.5} = 4.81\%$$

Bond yield

- **Bond yield** (yield to maturity): the constant "*interest rate*" implied by the bond so that the present value of all future payments is equal to the current bond price
- Suppose the bond pays c_i at time t_i, $1 \leq i \leq n$, and is selling at P, then the bond yield y solves

$$P = c_1 e^{-yt_1} + c_2 e^{-yt_2} + \cdots + c_n e^{-yt_n}.$$

- Other compounding may be used

- The yield of a zero-coupon bond is the zero rate
- **Example (Treasury note):** *A 2-year T-note with coupon rate 4.75% and face value \$100 was auctioned off at \$99.971. What is the yield of the T-note?*

  Each coupon payment is $4.75\% \times 100/2 = \$2.375$.

  $$2.375e^{-0.5y} + 2.375e^{-y} + 2.375e^{-1.5y} + 102.375e^{-2y} = 99.971$$

  The above equation can be solved in **Mathematica** using function **FindRoot** (or using Solver in Excel):

  $$y = 4.709\%$$

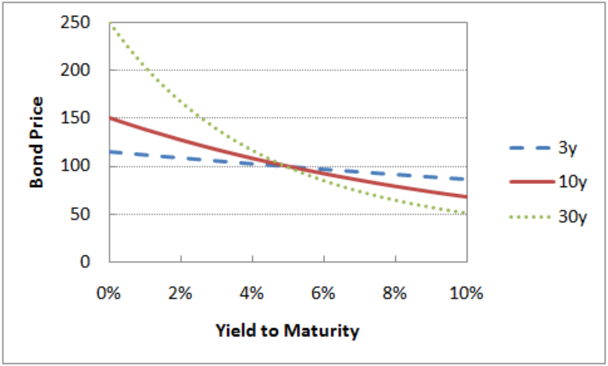
- **Yield curve:** the plot of bond yields w.r.t. maturities

Treasury yield curve

- US treasury yield curve **http://finance.yahoo.com/bonds**

Bond price sensitivity

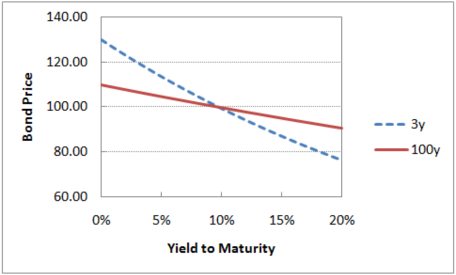
- Sensitivity of bond price w.r.t. yield
- Bond price is a decreasing function of yield
- Three coupon bonds with coupon rate 5%, face value \$100



- 30-year bond most sensitive to yield changes; 3-year bond least sensitive
- Shall we use maturity to measure bond price sensitivity?
- **Example:** *bond A with maturity 3 years, coupon rate 10%, face value \$100; bond B that pays 5 in 0.5 year, 105 in 1 year, and 0.01 in 100 years. Is bond B more sensitive to yield?*

$$
\begin{array}{lll}
A: & 5, 5, 5, 5, 5, 105 & \\
B: & 5, 105, & 0.01
\end{array}
$$

- Yield to maturity vs bond price



- Bond A more sensitive, although with smaller maturity;
- When using maturity to measure bond price sensitivity, should take individual payments into account

Duration

- **Duration**: weighted average of payment times. Suppose the bond pays c_i at time t_i. y is the bond yield. Duration is defined by

$$D = \sum_{i=1}^{n} t_i \left(\frac{c_i e^{-yt_i}}{P} \right), \qquad P = \sum_{i=1}^{n} c_i e^{-yt_i}$$

  Measures the slope of the price-yield curve

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i} = -DP, \qquad \Delta P \approx -DP\Delta y$$

- Fund managers, financial institutions match both present values and durations of their portfolio and future obligation

Convexity

- **Convexity** of a bond: weighted average of squared payment times

$$C = \sum_{i=1}^{n} t_i^2 \left(\frac{c_i e^{-y t_i}}{P} \right)$$

Measures the **curvature** of the price-yield curve: $\frac{\partial^2 P}{\partial y^2} = CP$

- Approximately we have

$$\Delta P = \frac{\partial P}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \Delta y^2 = -DP\Delta y + \frac{1}{2} CP \Delta y^2$$

- Matching both duration and convexity provides better protection from both small and large interest rate changes

Risk free interest rates

- Risk free interest rates are important for derivatives pricing
- **Treasury rates**
  - Treasury rates are risk free
  - Relatively lower due to federal regulations and taxation policies
- **LIBOR** (**L**ondon **I**nter**B**ank **O**ffer **R**ate): spot rates at which large banks in London are willing to lend money among themselves
  - Almost risk free
  - Overnight, 1-week, 2-week, 1 to 12-month rates available in major currencies
  - Used by derivative traders as risk free interest rates

- Selected LIBOR rates on a certain day (USD, **http://www.bba.org.uk**)

Maturity	LIBOR	Continuous compounding rate
1m (31d)	5.32000%	5.38157%
2m (59d)	5.34563%	5.39627%
3m (90d)	5.36000%	5.39836%

- \$1 million lend for 2 months (59 days) grows to

$$1 + 5.34563\% \frac{59}{360} = 1.0134 \text{ millions}$$

- **Day count convention:** LIBOR rates use actual/360 for USD
- Convert to continuously compounded rates (for the 2m rate)

$$1 + 5.34563\% \frac{59}{360} = e^{r \times 59/365} \Rightarrow r = 5.39627\%$$