Binomial Model

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Readings: Hull Chapter 11

Introduction

- Derivatives discussed: forwards, futures, options
- **Arbitrage arguments** (model independent)
  - forward price/rate, valuation of forward contracts
  - arbitrage relationships for options
- Valuation and risk management of options
- Need to model the dynamics of the underlying asset
  - **Binomial model**
  - **Black-Scholes-Merton model**
- First consider *European contracts on assets with no income*

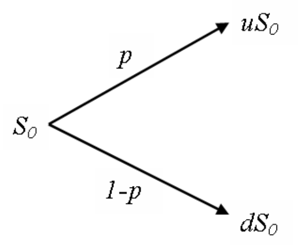
One-step binomial model

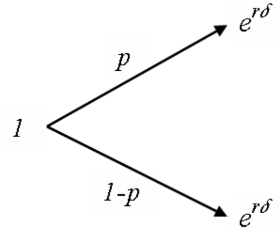
- **Risky asset:** e.g., a stock
  - Stock price process over the period $[0, \delta]$: $S = (S_0, S_\delta)$

$$S_\delta = \begin{cases} uS_0, & \text{with prob } p \\ dS_0, & \text{with prob } 1-p \end{cases}$$

  where $p \in (0,1)$, $0 < d < u$ (a Bernoulli trial)
- **Risk free investment:**
  - r : risk free interest rate per year (continuous compounding)
  - \$1 today worths $e^{r\delta}$ at time δ

- One-step binomial model over time period $[0, \delta]$





- To avoid arbitrage, $d < e^{r\delta} < u$ (**no arbitrage condition**)
  - If $d < u \leq e^{r\delta}$, short stock and deposit at rate r
  - If $u > d \geq e^{r\delta}$, borrow at rate r and buy stock
- Simple model, deep results!
  - Relation b/w **no arbitrage pricing** and **risk neutral pricing**
  - Tractable approximation to **continuous time models**

No arbitrage pricing

- Pricing a **long forward** contract with delivery price K and maturity δ
- **Method used before**: decompose the payoff of the long forward contract

$$S_\delta - K = S_\delta - F_0 + F_0 - K$$

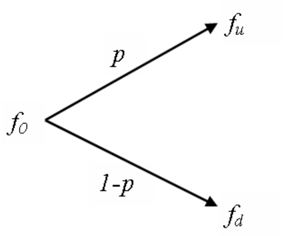
- Value of the long forward contract at time 0

$$V_0 = e^{-r\delta}(F_0 - K) = e^{-r\delta}(S_0 e^{r\delta} - K) = S_0 - Ke^{-r\delta}$$

Replicating portfolio

- **Alternatively**: **replicate** the long forward contract by trading the underlying stock and risk free investing
  - *Buy 1 share, borrow* $Ke^{-r\delta}$ *at rate* r (cost $S_0 - Ke^{-r\delta}$)
  - Payoff is also $S_\delta - K$
  - Same cost at time 0: $V_0 = S_0 - Ke^{-r\delta}$
- **No arbitrage pricing**
  - Find a **replicating portfolio** for the contract (how many shares to buy, how much to borrow)
  - No arbitrage $\Rightarrow$ value of the replicating portfolio = value of the contract
  - Extension to calls with payoff $(S_\delta - K)^+$, puts with payoff $(K - S_\delta)^+$

- Consider a derivative with payoff $f(S_\delta)$: $f_u = f(uS_0)$, $f_d = f(dS_0)$



- For calls, $f_u = (uS_0 - K)^+$, $f_d = (dS_0 - K)^+$

- **Replicating portfolio:** buy Δ shares and borrow Ψ
  - Value at time δ: $\Delta S_\delta - \Psi e^{r\delta}$
  - Δ and Ψ solve

$$\Delta \cdot uS_0 - \Psi e^{r\delta} = f_u,$$
$$\Delta \cdot dS_0 - \Psi e^{r\delta} = f_d$$

  - Unique solution:

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}, \quad \Psi = \frac{df_u - uf_d}{e^{r\delta}(u-d)}$$

- No arbitrage $\Rightarrow$ **value of the derivative at time** 0 should be

$$f_0 = \Delta S_0 - \Psi = \frac{f_u - f_d}{u-d} - \frac{df_u - uf_d}{e^{r\delta}(u-d)}$$

Example (pricing a call)

Suppose the current stock price is 20. $p = 0.6$, $u = 2$, $d = 0.5$, $\delta = 1$. The risk free interest rate is $r = \ln(1.25)$ with continuous compounding. What is the price of a call with strike price 25?

- Call payoff: $f_u = 15, f_d = 0$
- **Replicating portfolio**: buy Δ shares, borrow Ψ

$$40\Delta - 1.25\Psi = 15$$

$$10\Delta - 1.25\Psi = 0$$

$$\Delta = 1/2, \Psi = 4$$

- Call price $= \Delta S_0 - \Psi = 6$

Risk neutral pricing

- No arbitrage pricing leads to

$$
\begin{aligned}
f_0 &= \frac{f_u - f_d}{u-d} - \frac{df_u - uf_d}{e^{r\delta}(u-d)} \\
&= e^{-r\delta}\left(\frac{e^{r\delta}-d}{u-d} f_u + \frac{u-e^{r\delta}}{u-d} f_d\right) \\
&= e^{-r\delta}(p^* f_u + (1-p^*) f_d)
\end{aligned}
$$

where

$$
p^* = \frac{e^{r\delta}-d}{u-d} = \frac{1.25-0.5}{2-0.5} = 0.5 \text{ in the example}
$$

- By the no arbitrage condition $d < e^{r\delta} < u$, p^* is a well defined **probability**

$$0 < p^* < 1$$

- p^* is called the **risk neutral probability**

$$S_0 = e^{-r\delta}\mathbb{E}^*[S_\delta] = e^{-r\delta}(p^* \cdot uS_0 + (1 - p^*) \cdot dS_0)$$

$$f_0 = e^{-r\delta}\mathbb{E}^*[f(S_\delta)] = e^{-r\delta}(p^* f_u + (1 - p^*) f_d)$$

  In the **“risk neutral world”**, risky investments earn risk free interest rate

- **Risk neutral pricing**: **Derivative price = risk neutral expectation of the payoff discounted at the risk free rate**

- In the **"physical world"** where the actual probability p is used, stock earns more than the risk free interest rate,

$$\mathbb{E}[S_\delta] = puS_0 + (1-p)dS_0 = 0.6\times 40 + 0.4\times 10 = 28 = S_0 e^{\mu\delta} \to \mu = \ln(1.4) > \ln(1.25)$$

call earns more than the risk free interest rate

$$\mathbb{E}[f(S_\delta)] = pf_u + (1-p)f_d = 0.6\times 15 + 0.4\times 0 = 9 = f_0 e^{\mu\delta} \to \mu = \ln(1.5) > \ln(1.25)$$

- For derivatives pricing, one **should not discount the expected payoff in the physical world at the risk free rate**
- From CAPM, different **risk adjusted discount rates** should be used for different risky investments

- **Actual probability** p doesn't matter in pricing derivatives: the info has been contained in S_0 already
- Summary

Physical world	Risk neutral world
Where we live	Where we price derivatives
Stock price goes up with prob p	Stock price goes up with prob p^*
Stock earns risk adjusted rate	Stock earns risk free rate

- Risk neutral pricing procedure: (1). find risk neutral probability; (2). compute risk neutral expected payoff; (3). discount at the risk free rate

Example (risk neutral pricing)

Suppose the current stock price is 20. $p = 0.6$, $u = 2$, $d = 0.5$, $\delta = 1$. The risk free interest rate is $r = \ln(1.25)$ with continuous compounding. What is the price of a call option with strike price 25? What is the price of a put option with strike price 25?

- Compute the risk neutral probability:

$$p^* = \frac{e^{r\delta} - d}{u - d} = \frac{1.25 - 0.5}{2 - 0.5} = 0.5$$

By **risk neutral pricing**, call price at time 0:

$$\begin{aligned} f_0 &= e^{-r\delta}(p^* f_u + (1 - p^*) f_d) \\ &= (0.5 \times 15 + 0.5 \times 0)/1.25 = 6 \end{aligned}$$

- Put payoff: $f_u = 0, f_d = 15$
  Replicating portfolio: buy Δ shares, borrow Ψ

$$40\Delta - 1.25\Psi = 0$$

$$10\Delta - 1.25\Psi = 15$$

  with solution $\Delta = -0.5, \Psi = -16$ (short sell 0.5 share, deposit \$16). Therefore, put price $= \Delta S_0 - \Psi = 6$
- **Risk neutral pricing**:

$$f_0 = e^{-r\delta}(p^* f_u + (1 - p^*) f_d) = (0.5 \times 0 + 0.5 \times 15)/1.25 = 6$$

- **European put call parity**: call price $+ Ke^{-r\delta} =$ put price $+ S_0$

Delta hedging

- Risk management is important to derivative traders
  - How to hedge a **short** position in a derivative contract
  - Derivative contracts can be replicated by trading the underlying asset and risk free investment
  - The replicating portfolio contains Δ shares

$$\Delta = \frac{f_u - f_d}{S_0(u-d)}$$

  - Sell a derivative and buy Δ shares to cancel the risk

- Consider a **hedged position**: short a derivative, long Δ shares
- Value of the hedged position at time δ
  - *Stock price goes up*

$$-f_u + \Delta \cdot uS_0 = \frac{df_u - uf_d}{u - d}$$

  - *Stock price goes down*

$$-f_d + \Delta \cdot dS_0 = \frac{df_u - uf_d}{u - d}$$

- **The hedged position is risk free**
- **Delta hedging**: sell a derivative, long Δ shares

Example (delta hedging)

The current stock price is 20. $u = 2$, $d = 0.5$, $\delta = 1$. The risk free interest rate is $r = \ln(1.25)$ with continuous compounding. Consider a call option with strike price 25. Compare writing a covered call with the Delta hedging.

- Write the call and earn \$6
- **Delta hedging**: buy $\Delta = 1/2$ share, initial investment \$4
  Up: buy extra 1/2 share, sell 1 share at strike, receive \$5
  Down: sell 1/2 share at market price, receive \$5
- **Rate of return** (annualized, continuous compounding)

$$5 = 4e^{R\delta} \quad \Rightarrow \quad R = \ln(1.25) = 22.3\% = r$$

- **Covered call**: buy one share, initial investment \$14

  Up: sell 1 share at strike, receive \$25

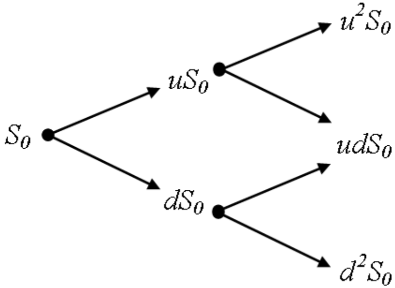
$$25 = 14e^{R\delta} \quad \Rightarrow \quad R = \ln(25/14) = 58.0\%$$

  Down: sell 1 share at market price, receive \$10

$$10 = 14e^{R\delta} \quad \Rightarrow \quad R = \ln(10/14) = -33.6\%$$

Two-step binomial model

- Two-step binomial model over time period $[0, 2\delta]$



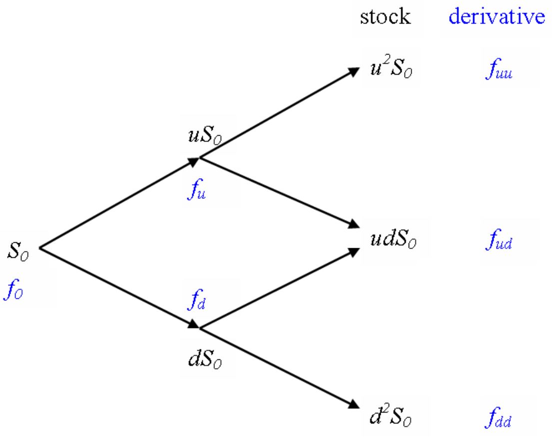
- Stock price process: $S = (S_0, S_\delta, S_{2\delta})$
- Risk free investment: $1 \Rightarrow e^{r\delta} \Rightarrow e^{2r\delta}$

- No arbitrage pricing of a European derivative: construct a **replicating portfolio**
  - To replicate a call option, buy stocks
  - Determine # shares to buy, amount to borrow at time 0
  - At time δ, need to adjust # of shares

    Stock price goes down: reduce # of shares
    Stock price goes up: increase # of shares

  - Need to determine how many shares to hold at time δ

- European derivative with payoff $f(S_{2\delta})$ at time 2δ: $f_{uu} = f(u^2 S_0), f_{ud} = f(udS_0), f_{dd} = f(d^2 S_0)$



- **Price the derivative**
  1. No arbitrage pricing: construct a *replicating portfolio*
  2. Risk neutral pricing: simplify calculations
- **Replicating portfolio**
  - Time 0: long Δ_0 shares, borrow Ψ_0 (cost f_0)
  - $S_\delta = uS_0$ at time δ: long Δ_u, borrow Ψ_u
  - $S_\delta = dS_0$ at time δ: long Δ_d, borrow Ψ_d
- **Select Δ's, Ψ's**

$$\text{Time } 2\delta: \text{ value of the replicating portfolio} = \text{derivative payoff}$$

- **Derivative price** $= f_0$ to avoid arbitrage

Replicating portfolio

- *Stock price goes up at time δ*:

$$\begin{aligned} \text{derivative payoff} &= \text{portfolio payoff} \\ f_{uu} &= \Delta_u \cdot u^2 S_0 - \Psi_u e^{r\delta} \\ f_{ud} &= \Delta_u \cdot udS_0 - \Psi_u e^{r\delta} \end{aligned}$$

$$\Delta_u = \frac{f_{uu} - f_{ud}}{uS_0(u-d)}, \quad \Psi_u = \frac{df_{uu} - uf_{ud}}{e^{r\delta}(u-d)}$$

Portfolio value at δ (when stock price goes up at δ, amount needed to replicate derivative payoff)

$$f_u = \Delta_u \cdot uS_0 - \Psi_u = e^{-r\delta}\left[p^* f_{uu} + (1-p^*) f_{ud}\right]$$

$$p^* = \frac{e^{r\delta} - d}{u - d}$$

- *Stock price goes down at time δ*:

$$\begin{aligned} \text{derivative payoff} &= \text{portfolio payoff} \\ f_{ud} &= \Delta_d \cdot udS_0 - \Psi_d e^{r\delta} \\ f_{dd} &= \Delta_d \cdot d^2 S_0 - \Psi_d e^{r\delta} \end{aligned}$$

$$\Delta_d = \frac{f_{ud} - f_{dd}}{dS_0(u-d)}, \quad \Psi_d = \frac{df_{ud} - uf_{dd}}{e^{r\delta}(u-d)}$$

Portfolio value at δ (when stock price goes down at δ, amount needed to replicate derivative payoff)

$$f_d = \Delta_d \cdot dS_0 - \Psi_d = e^{-r\delta}\left[p^* f_{ud} + (1-p^*) f_{dd}\right]$$

- *At time 0*:

$$\begin{aligned} \text{amount needed at } \delta &= \text{portfolio payoff} \\ f_u &= \Delta_0 \cdot uS_0 - \Psi_0 e^{r\delta} \\ f_d &= \Delta_0 \cdot dS_0 - \Psi_0 e^{r\delta} \end{aligned}$$

$$\Delta_0 = \frac{f_u - f_d}{S_0(u-d)}, \quad \Psi_0 = \frac{df_u - uf_d}{e^{r\delta}(u-d)}$$

Portfolio value at 0 (amount needed at time 0 to replicate required amounts at δ)

$$f_0 = \Delta_0 S_0 - \Psi_0 = e^{-r\delta}\left[p^* f_u + (1-p^*) f_d\right]$$

Example (price and hedge a call in 2-step binomial model)

Suppose current stock price is 20. $u = 2$, $d = 0.5$, $\delta = 1$. The risk free interest rate is $r = \ln(1.25)$ with continuous compounding. Price a European call with strike 15 and maturity 2 years

Possible call payoffs

$$f_{uu} = (80 - 15)^+ = 65, f_{ud} = (20 - 15)^+ = 5, f_{dd} = (5 - 15)^+ = 0$$

If stock price goes to 40 at δ,

$$\Delta_u = 1, \Psi_u = 12, \text{ cost } 40 - 12 = 28 = f_u$$

If stock price goes to 10 at δ,

$$\Delta_d = 1/3, \Psi_d = 4/3, \text{ cost } 10/3 - 4/3 = 2 = f_d$$

To replicate amount needed at δ (either 28 or 2),

$$\Delta_0 = 13/15, \Psi_0 = 16/3, \text{ cost } 13 \cdot 20/15 - 16/3 = 12 = f_0 = \text{call price}$$

Backward induction

- Starting from derivative payoff

$$f_{uu}, f_{ud}, f_{dd} \quad \Rightarrow \quad f_u, f_d \quad \Rightarrow \quad f_0$$

where, with $p^* = \frac{e^{r\delta}-d}{u-d}$,

$$f_u = e^{-r\delta}(p^* f_{uu} + (1-p^*)f_{ud})$$

$$f_d = e^{-r\delta}(p^* f_{ud} + (1-p^*)f_{dd})$$

$$f_0 = e^{-r\delta}(p^* f_u + (1-p^*)f_d)$$

- In the example, $p^* = 0.5$, $f_u = (0.5 \cdot 65 + 0.5 \cdot 5)/1.25 = 28$, $f_d = (0.5 \cdot 5 + 0.5 \cdot 0)/1.25 = 2$, $f_0 = (0.5 \cdot 28 + 0.5 \cdot 2)/1.25 = 12$

Risk neutral pricing

- p^* is the **risk neutral probability**. In the **risk neutral world**, stock earns risk free interest rate

$$(p^*)^2 = \text{prob(stock price goes to } u^2 S_0) = 1/4$$

$$2p^*(1-p^*) = \text{prob(stock price goes to } udS_0) = 1/2$$

$$(1-p^*)^2 = \text{prob(stock price goes to } d^2 S_0) = 1/4$$

$$\mathbb{E}^*[S_{2\delta}] = (p^*)^2 u^2 S_0 + 2p^*(1-p^*)udS_0 + (1-p^*)^2 d^2 S_0 = e^{2r\delta} S_0 = 31.25$$

- **Derivative price = risk neutral expected payoff discounted at the risk free rate**

$$\begin{aligned} f_0 &= e^{-2r\delta}\left((p^*)^2 f_{uu} + 2p^*(1-p^*)f_{ud} + (1-p^*)^2 f_{dd}\right) \\ &= e^{-2r\delta}\mathbb{E}^*[f(S_{2\delta})] = 12 \end{aligned}$$

Dynamic delta hedging

- Hedge a short position in a derivative contract
- Sell a derivative, hold Δ_t shares, $t = 0, \delta$
  - At time 0,
  $$\Delta_0 = \frac{f_u - f_d}{S_0(u-d)} = 13/15$$
  - Stock price goes up at time δ
  $$\Delta_u = \frac{f_{uu} - f_{ud}}{uS_0(u-d)} = 1$$
  - Stock price goes down at time δ
  $$\Delta_d = \frac{f_{ud} - f_{dd}}{dS_0(u-d)} = 1/3$$
- The hedged position is risk free

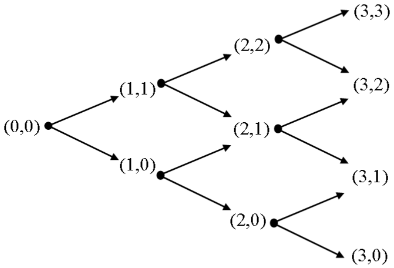
Example (price and hedge a call in 2-step binomial model)

$\Delta_0 = 13/15, \Delta_u = 1, \Delta_d = 1/3$

- Write a call to get 12, long $\frac{13}{15}$ shares, cost $\frac{13}{15} \cdot 20 - 12 = \frac{16}{3}$
- $S_\delta = 40$: value of the portfolio $\frac{13}{15} \cdot 40 - 28 = \frac{20}{3}$; borrow $\frac{2}{15} \cdot 40$ and buy extra $\frac{2}{15}$ shares; at maturity, sell 1 share for 15 and repay the loan, get $15 - \frac{80}{15} \cdot 1.25 = \frac{25}{3}$
- $S_\delta = 10$: value of the portfolio $\frac{13}{15} \cdot 10 - 2 = \frac{20}{3}$; sell $\frac{8}{15}$ shares and deposit $\frac{80}{15}$; at maturity, **either** buy $\frac{2}{3}$ shares at 20 and sell 1 share at strike 15 and get $\frac{80}{15} \cdot 1.25 - \frac{2}{3} \cdot 20 + 15 = \frac{25}{3}$, **or** sell $\frac{1}{3}$ shares and get $\frac{80}{15} \cdot 1.25 + \frac{5}{3} = \frac{25}{3}$
- In any case, end up with $\frac{25}{3}$; earn risk free rate: $\frac{16}{3} = \frac{25}{3} e^{-2r\delta}$

Multi-step binomial model

- Multi-step binomial model over time period $[0, N\delta = T]$



- Node (n, j): time $n\delta$, j is the number of up moves in the stock price
- Stock price at node (n, j): $S_{n,j} = u^j d^{n-j} S_0$

- Price a derivative with payoff $f(S_T)$ at time $T = N\delta$

$$f_{N,j} = f(S_{N,j}), \quad 0 \leq j \leq N$$

- **Risk neutral pricing formula**

$$f_0 = e^{-rT}\mathbb{E}^*[f(S_T)] = e^{-rT}\sum_{j=0}^{N}\begin{pmatrix} N \\ j \end{pmatrix}(p^*)^j(1-p^*)^{N-j}f_{N,j}$$

where p^* is the risk neutral probability

$$p^* = \frac{e^{r\delta} - d}{u - d}$$

and the number of paths leading to node (N, j) is

$$\begin{pmatrix} N \\ j \end{pmatrix} = \frac{N!}{j!(N-j)!}$$

- **Backward induction**
  - Start with

$$f_{N,j}, \quad j = 0, 1, \cdots, N$$

  - For $n = N-1, N-2, \cdots, 0$

$$f_{n,j} = e^{-r\delta}(p^* f_{n+1,j+1} + (1-p^*) f_{n+1,j}), \quad j = 0, 1, \cdots, n$$

- **Hedge a short position**:

$$\Delta_{n,j} = \frac{f_{n+1,j+1} - f_{n+1,j}}{S_{n,j}(u-d)}, \quad S_{n,j} = u^j d^{n-j} S_0$$

sell a derivative and hold $\Delta_{n,j}$ shares if stock price arrives at node (n, j)

Assets with continuous yield

- Derivatives on assets with **continuous yield**: currencies, stock indices
- 1 unit of the asset grows to $e^{q\delta}$ units over $[0, \delta]$
- Derivative payoff: $f_u = f(uS_0)$, $f_d = f(dS_0)$
- **Replicating portfolio**: buy Δ units, borrow Ψ

$$e^{q\delta}\Delta uS_0 - \Psi e^{r\delta} = f_u$$

$$e^{q\delta}\Delta dS_0 - \Psi e^{r\delta} = f_d$$

$$\Delta = \frac{f_u - f_d}{e^{q\delta}S_0(u-d)}, \Psi = \frac{df_u - uf_d}{e^{r\delta}(u-d)}$$

- **Risk neutral pricing formula**:

$$\begin{aligned} f_0 &= \Delta S_0 - \Psi \\ &= e^{-r\delta}(p^* f_u + (1-p^*) f_d) \end{aligned}$$

where the **risk neutral probability**

$$p^* = \frac{e^{(r-q)\delta} - d}{u - d}$$

- In the **risk neutral world**,

$$\mathbb{E}^* \left[e^{q\delta} S_\delta \right] = e^{q\delta} \left(p^* u S_0 + (1-p^*) d S_0 \right) = e^{r\delta} S_0$$

- Generalized to multi-step binomial models similarly

Example (currency options in binomial models)

The current exchange rate is 1.5 USD/EUR. $u = 1.2$, $d = 0.8$, $\delta = 1$. Risk free interest rates for USD and EUR are $r = 5\%$ and $q = 4\%$, respectively (continuous compounding, assume flat term structures). Price a 2-year ATM European call.

- Possible payoffs: $f_{uu} = (2.16 - 1.5)^+ = 0.66$, $f_{ud} = (1.44 - 1.5)^+ = 0$, $f_{dd} = (0.96 - 1.5)^+ = 0$
- Risk neutral probability

$$p^* = \frac{e^{(r-q)\delta} - d}{u - d} = \frac{e^{0.01} - 0.8}{1.2 - 0.8} = 0.5251$$

- Backward induction

$$f_u = e^{-0.05} \cdot 0.66 \cdot p^* = 0.3297, \ \ f_d = 0, \ \ f_0 = e^{-0.05} \cdot 0.3297 \cdot p^* = 0.1647$$

CRR binomial model

- Given option maturity T. Divide $[0, T]$ into N equal intervals: $\delta = T/N$
- **Variability** of the stock price
  - Binomial model: u and d
  - **Black-Scholes-Merton model**: volatility σ
- Select u and d in the binomial model as follows (**Cox-Ross-Rubinstein binomial model**)

$$u = e^{\sigma\sqrt{\delta}}, \quad d = e^{-\sigma\sqrt{\delta}}$$

- The CRR model converges to the Black-Scholes-Merton model as N gets large

- **Black-Scholes formula**
  - European call price

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

  - European put price

$$p = -S_0 e^{-qT} N(-d_1) + K e^{-rT} N(-d_2)$$

where $N(x)$ is the cdf of $N(0,1)$,

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Example (Black-Scholes formula)

The current exchange rate is 1.5 USD/EUR. Risk free interest rates for USD and EUR are $r = 5\%$ and $q = 4\%$, respectively (continuous compounding). Price a 1-year ATM European put in the Black-Scholes-Merton model when $\sigma = 20\%$.

- $S_0 = K = 1.5, r = 0.05, q = 0.04, T = 1, \sigma = 0.2$

$$d_1 = 0.15, d_2 = -0.05, N(-d_1) = 0.4404, N(-d_2) = 0.5199$$

$$p = -S_0 e^{-qT} N(-d_1) + K e^{-rT} N(-d_2) = 0.1072$$

American options

- Consider an **American option**, payoff when exercised at $n\delta$

$$\text{put: } (K - S_{n,j})^+, \quad \text{call: } (S_{n,j} - K)^+$$

- Backward induction for American put options pricing:
  - start with

$$f_{N,j} = (K - S_{N,j})^+, \quad j = 0, 1, \cdots, N$$

  - for $n = N - 1, \cdots, 0$,

$$f_{n,j} = \max\left((K - S_{n,j})^+, e^{-r\delta}(p^* f_{n+1,j+1} + (1 - p^*) f_{n+1,j})\right)$$

Example (pricing an American put)

Consider a two-step binomial model with $S_0 = 4$, $u = 2$, $d = 1/2$, $\delta = 1$, $r = \ln(1.25)$. Price a 2-year American put with strike 5.

- *Payoff at maturity*: $f_{uu} = 0, f_{ud} = 1, f_{dd} = 4$; **risk neutral probability** $p^* = (e^{r\delta} - d)/(u - d) = 0.5$
- *At time δ*,

$$f_u = \max(0, e^{-r\delta}(p^* f_{uu} + (1 - p^*) f_{ud}) = 0.4$$

$$f_d = \max(3, e^{-r\delta}(p^* f_{ud} + (1 - p^*) f_{dd})) = \max(3, 2) = 3,$$ **early exercise!**

- *At time 0*,

$$f_0 = \max(1, e^{-r\delta}(p^* f_u + (1 - p^*) f_d)) = \max(1, 1.36) = 1.36$$

Path dependent derivatives

- Derivative payoff depends on the whole path of the asset price process

$$f(S_0, S_\delta, \cdots, S_{n\delta})$$

  - **Lookback options**: payoff depends on maximum/minimum asset price
  - **Asian options**: payoff depends on average asset price
- No arbitrage pricing still works: construct replicating portfolio correspondingly; backward induction
- Risk neutral pricing still works, but need to differentiate different paths

$$f_0 = \mathbb{E}^*[e^{-rT} f(S_0, \cdots, S_{n\delta})] \neq e^{-rT} \sum_{j=0}^{N} \binom{N}{j} (p^*)^j (1-p^*)^{N-j} f_{N,j}$$

Implementing binomial models

- C/C++: always use **double** (instead of **float**)
- Start with payoff at maturity, use backward induction
- **Not efficient** to keep the whole tree
- Enough to use a $N+1$ vector
- **Project**: pricing and analyzing European/American options in the CRR model, note the due dates for the draft and the final report, **no extension possible**