Fundamental Groups of Hamming Graphs

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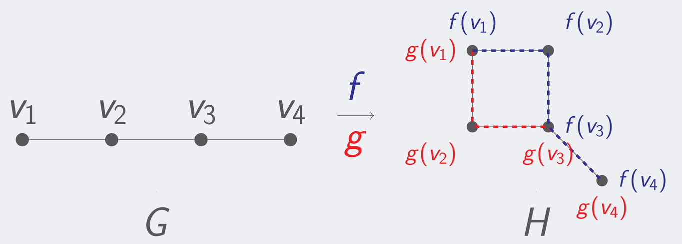
Abstract

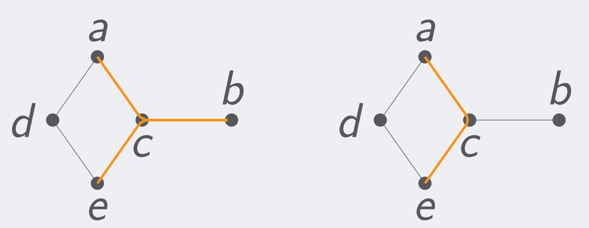
Recently there has been growing interest in discrete homotopies and homotopies of graphs beyond treating graphs as 1-dimensional simplicial spaces. One such type of homotopy is $\times$-homotopy. Recent work by Chih-Scull has developed a homotopy category, a fundamental group for graphs under this homotopy, and a way of computing covers of graphs that lift homotopy via this fundamental group. In this research, we compute the fundamental groups of all Hamming graphs, show that they are direct products of cyclic groups, and use this result to describe some $\times$-homotopy covers of Hamming graphs.

Homotopy Definitions

We say $f, g : G \to H$ are a spider pair if there exists a single vertex v such that $f(u) = g(u)$ for all $u \neq v$ and $f(v) \neq g(v)$. When we replace f with g we refer to it as a spider move.



We define a prune of f to be given by a walk f' obtained by deleting the vertices v_i and v_{i+1} from the walk when $v_i = v_{i+2}$.



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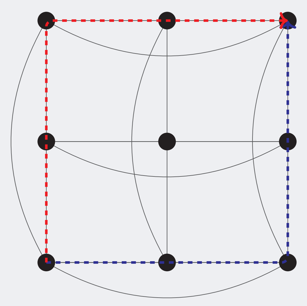
Two walks f, g are equivalent (also denoted $f \simeq g$) if we can transform f to g via a series of prunes, anti-prunes and spider moves.



We define a fundamental group to be the group generated by equivalence classes of closed walks with concatenation as the operation.

References

- JA Bondy and USR Murty, *Graph theory. 2008*, Grad. Texts Math, Springer, 2008
- T Chih and L Scull, *A Homotopy Category for Graphs* , JACO **53** (2021)
- T Chih and L Scull, *Fundamental Groupoids for Graphs* , CGAS **16** (2023)
- T Chih and L Scull, *Homotopy Covers for Graphs* , (In Progress)

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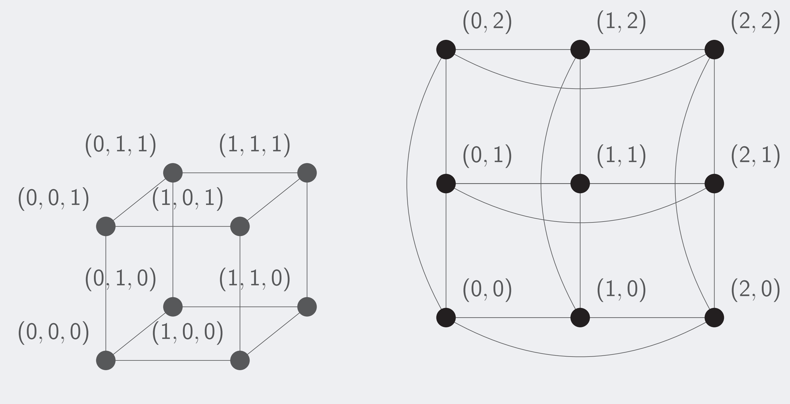
Hamming Graph

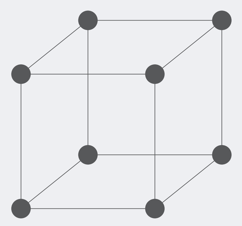
We define a Hamming graph $H(d, q)$ to be a graph where the vertices are the d-tuples with entries in a q element set, and two vertices are adjacent if they differ in exactly one coordinate.



On the left $H(3, 2)$ and on the right $H(2, 3)$.

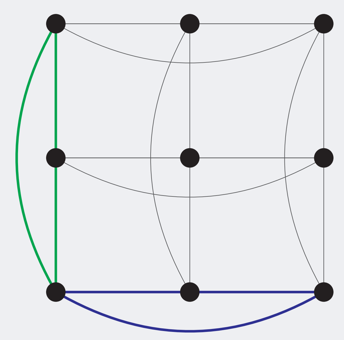
$\Pi(H(d, 2))$ is trivial

We first showed that $\Pi(H(d, 2)) \cong \{e\}$ because each closed walk in $H(d, 2)$ is trivial.



Ground Walks Generate

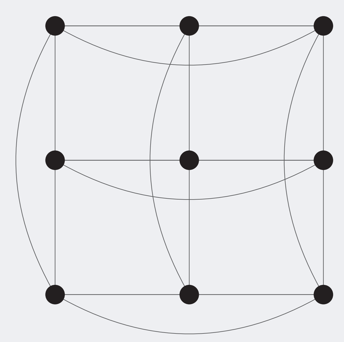
Let U_i be a walk in $H(d, q)$ of length three where the only nonzero coordinate is i. Thus, $U_i = (\mathbf{0}(0, -, a_i, -, 0)(0, -, b_i, -, 0)\mathbf{0})$. We call this a ground walk in the ith coordinate.



We found that for Hamming graphs, every closed walk W is equivalent to a product of these ground walks, and thus the ground walks are the generators of W.

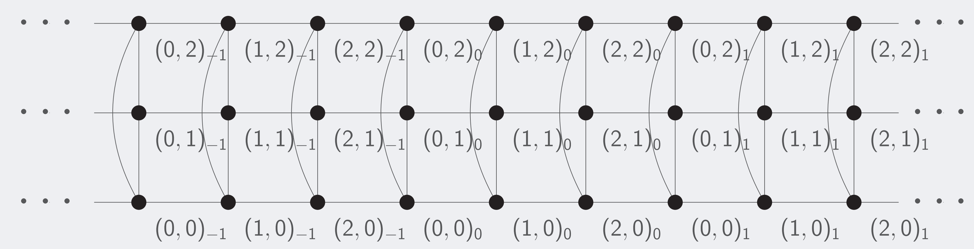
Ground Walks Commute

We also found that our ground walks U_i and U_j commute by performing a series of spider moves and prunes.

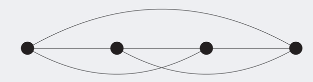


Order

If $q = 3$, the order of a ground walk is infinity. Consider the following cover of $H(2, 3)$:



If $q \neq 3$, the order of a ground walk is 2. Consider the following cover of K_4:



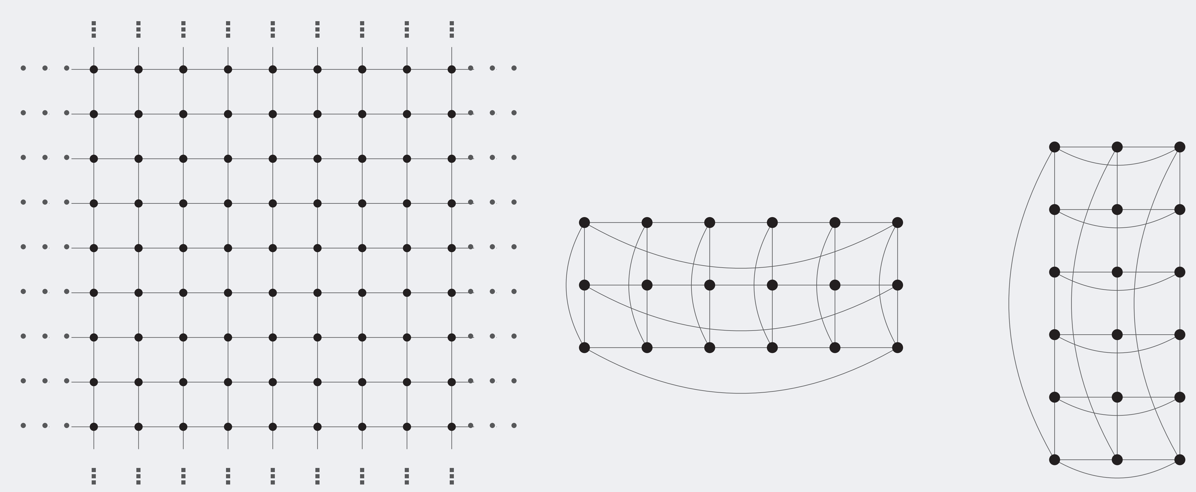
Minimal Generating Set

We also showed that for a walk U_i, U_i can not be a product of all the U_j where $i \neq j$ because spider moves preserve the number of changes in each coordinate and prunes preserve the parity of the changes in each coordinate. Therefore, $\{U_i\}$ is a minimal generating set.

Classifying the fundamental groups for Hamming Graphs

Since the ground walks are a minimal generating set and they commute, $\Pi(H(d, q))$ is a direct product of the groups generated by the ground walks. That is $\Pi(H(d, q)) \cong \Pi(K_q)^d$, and thus we showed that $\Pi(H(d, 3)) \cong \mathbb{Z}^d$, and $\Pi(H(d, q)) \cong \mathbb{Z}_2^d$ when $q \neq 3$.

Covers of $H(2, 3)$



Covers from left to right: Universal cover, $U/\langle U_1^2, U_2\rangle$, and $U/\langle U_1, U_2^2\rangle$.

Conclusion

We found that the fundamental groups of Hamming graphs $H(q, d)$ are $\mathbb{Z}^d$ when $q = 3$ and $\mathbb{Z}_2^d$ when $q > 3$. We also showed that all covers of Hamming graphs are generated by ground walks, and these generators commute and are unique. Thus, Hamming graphs are direct products of cyclic groups.