Metric Space & Topological Space

(1) $d : X^2 \to \mathbb{R}$ where
1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \Leftrightarrow x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) + d(z,y) \geq d(x,y)$

inherited d_A on $A \subseteq X$

$\Updownarrow$

$d_A : A^2 \to \mathbb{R}$

where $d_A(x,y) = d(x,y)$

$\Updownarrow$

subspace

(2) $\mathcal{T} \subseteq \mathcal{P}(X)$ where
1. $\emptyset, X \in \mathcal{T}$
2. $\forall A_i \in \mathcal{T} : \bigcap_{i \in I,\ \text{finite}} A_i \in \mathcal{T}$ (交)
3. $\forall A_i \in \mathcal{T} : \bigcup_{i \in I,\ \text{any}} A_i \in \mathcal{T}$

(3) induced $\mathcal{T} := \{ A \subseteq X \mid \forall p \in A \ \exists r > 0 \text{ s.t. } B_r(p) \subseteq A \}$

$p \in \text{int}_X(A) \Leftrightarrow \exists r > 0 \text{ s.t. } B_r(p) \subseteq A$

$\text{int}_X(A) = A \Leftrightarrow A \in \mathcal{T}$ i.e. A is open in X

互斥

(4) $p \in \text{Cl}_X(A) \Leftrightarrow \forall r > 0 : B_r(p) \cap A \neq \emptyset$

$\Leftrightarrow \exists (p_n)_{n \in \mathbb{N}}$ in A s.t. $\lim_{n \to \infty} p_n = p$

$\text{Cl}_X(A) = A \Leftrightarrow A$ is closed i.e. $X - A$ is open in X

(5) $\forall A_i$ is closed in X : $\bigcup_{i \in I,\ \text{finite}} A_i$, $\bigcap_{i \in I,\ \text{any}} A_i$ closed in X

(6) $(p_n)_{n\in\mathbb{N}}$ in A : $p_n := f(n)$ where $f: \mathbb{N} \to A$

(7) $\lim\limits_{n\to\infty} p_n = p \Leftrightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N: d(p_n, p) < \varepsilon$

$\lim\limits_{x\to\infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \ \exists N > 0$ s.t. $\forall x > N: |f(x) - L| < \varepsilon$

(8). $\mathrm{int}_X(A) \subseteq A$, $\mathrm{int}_X \circ \mathrm{int}_X(A) = \mathrm{int}_X(A)$ i.e. $\mathrm{int}_X(A) = \bigcup\limits_{\text{open}, S \subseteq A} S$

$Cl_X(A) \supseteq A$, $Cl_X \circ Cl_X(A) = Cl_X(A)$ i.e. $Cl_X(A) = \bigcap\limits_{\text{closed}; S \supseteq A} S$

(9) A is dense in X : $Cl_X(A) = X$

somewhere dense : $\exists Y$ open in X s.t $Cl_Y(Y \cap A) = Y$

nowhere dense : $(\neg)$

$\Updownarrow$

$Cl_X(Y \cap A) \supseteq Y$

(10) $\forall A \subseteq \mathbb{Z}$: A is closed in $\mathbb{R}$

(Cor.) $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$

(11) $\mathbb{Q}$ is dense in $\mathbb{R}$

(12) $B_r(p) := \{ x \in X \mid d(x, p) < r \}$, $B_r(p)$ is open (always)

$C_r(p) := \{ x \in X \mid d(x, p) = r \}$

(13) $\forall p \in X$: $\{p\}$ is closed

$\forall A \subseteq X$ where A is finite : $A = \bigcup\limits_{i \in I, \text{ finite}} \{p_i\}$ is closed

(14) For d as discrete metric : $\mathcal{T} = P(X)$ i.e. all are open.

(15) absolute difference: $d(x,y) := |x-y|$

taxicab distance: $d(x,y) := \sum_{n=1}^{N} |x_n - y_n|$

euclidean metric: $d(x,y) := \|x-y\|$

discrete metric: $d(x,y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$\|x\| := \sqrt{\langle x, x \rangle}$

$\langle x, y \rangle := \sum_{n=1}^{N} x_n y_n$

(16) 柯西不等式

1. $\|x\| \cdot \|y\| \geq |\langle x, y \rangle|$

2. $\left(\sum_{n=1}^{N} x_n^2\right) \cdot \left(\sum_{n=1}^{N} y_n^2\right) \geq \left(\sum_{n=1}^{N} x_n y_n\right)^2$

1-1: $\forall x, y \in \mathrm{Dom}(f): f(x) = f(y) \Rightarrow x = y$

onto: $\forall y \in \mathrm{Cod}(f): \exists x \in \mathrm{Dom}(f)$ s.t. $f(x) = y$

(1) SB Thm.: $|A| \leq |B| \wedge |A| \geq |B| \Leftrightarrow |A| = |B|$

(2) finite: $\exists$ bijective $f: \{1, \ldots, N \in \mathbb{N}\} \to X$

denumerable: $\exists$ bijective $f: \mathbb{N} \to X$

(3) $|A| \leq |B| \Leftrightarrow \exists$ 1-1 $f: A \to B$, $|A| = |B| \Leftrightarrow \exists$ bijective f

(4) X is countable $\wedge \exists$ 1-1 $f: A \to X \Rightarrow A$ is countable

(5) $\forall X_i$ countable: $\bigcup_{i \in I,\ \text{countable}} X_i$ is countable

(6) A, B countable: $A \times B$ countable

A countable: A^n countable $\forall n \in \mathbb{N}$

(7) $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

(8) $|[0,1)| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |(0,1)|$, $|\mathcal{F}(\mathbb{N}, \{0,1\})| = |\mathcal{P}(\mathbb{N})|$

(9) $|(0,1) \times (0,1)|$ uncountable

$|\mathbb{R}^n|$ $\forall n \in \mathbb{N}$ uncountable

(10) Well-ordering principle

$\forall A \subseteq \mathbb{N}$ with $A \neq \emptyset$: $\exists$ smallest element of A