$$(\Omega, \mathcal{F}_\Omega, P) \xrightarrow{X} (\mathbb{R}, \mathcal{F}_\mathbb{R}, P_X) \quad \begin{cases} \mathcal{X} = X(\Omega) \\ P_X : \mathcal{F}_\mathbb{R} \to \mathbb{R} \quad \begin{cases} f_X : \mathbb{R} \to \mathbb{R} \\ F_X : \mathbb{R} \to \mathbb{R} \end{cases} \end{cases}$$

$$P_X(A \in \mathcal{F}_\mathbb{R}) := P(X^{\leftarrow}(A) \in \mathcal{F}_\Omega)$$

$$f_X(x \in \mathbb{R}) := \begin{cases} P_X(\{x\}) & \text{if } x \in \mathcal{X} \\ 0 & \text{o.w.} \end{cases} \qquad \begin{cases} 1.\ f_X(x) \geq 0 \quad \forall x \in \mathcal{X} \\ 2.\ f_X(x) = 0 \quad \forall x \notin \mathcal{X} \\ 3.\ \sum_{x \in \mathcal{X}} f(x) = 1 \quad \star \end{cases}$$

$$F_X(x \in \mathbb{R}) := P_X((-\infty, x])$$

$$\begin{cases} F_X \text{ non-decreasing} \\ F_X \text{ continuous from the right } \left(\lim_{t \to x^+} F_X(t) = F(x) \right) \\ \lim_{t \to -\infty} F_X(t) = 0 \ \wedge \ \lim_{t \to \infty} F_X(t) = 1 \quad \star \quad \Rightarrow \quad 0 \leq F_X(x) \leq 1 \quad \forall x \in \mathbb{R} \end{cases}$$

$$\begin{tikzcd}
 & P_X \arrow[dl, leftrightarrow, "{\star\ P_X(A) = \sum_{x \in A \cap \mathcal{X}} f_X(x)}"'] \arrow[dr, leftrightarrow, "{P_X((a,b]) = F_X(b) - F_X(a)}"] & \\
f_X \arrow[rr, leftrightarrow] & & F_X
\end{tikzcd}$$

$$\star \quad f_X(x) = F_X(x) - \lim_{t \to x^-} F_X(t) \quad \Big| \quad F_X(x) = \sum_{t \in (-\infty, x] \cap \mathcal{X}} f_X(t)$$

$$\begin{cases} \text{discrete r.v.: } \mathcal{X} \text{ countable} \\ \text{continuous r.v.: } \mathcal{X} \text{ uncountable} \end{cases} \qquad \begin{cases} P_Y(A_Y) = P_X(g^{\leftarrow}(A_Y)) \quad \star \\ f_Y(y) = \sum_{x \in g^{\leftarrow}(y) \cap \mathcal{X}} f_X(x) \end{cases}$$

$E(X) := \sum_{x \in \mathcal{X}} (f_X(x) \cdot x)$ $\quad E(g(X)) = \sum_{x \in \mathcal{X}} (f_X(x) \cdot g(x))$

$$\begin{cases} \mu_X := E(id(X)) \\ \sigma_X^2 := E((X-\mu_X)^2) \end{cases} \qquad \begin{cases} E(aX+b) = a \cdot E(X) + b \\ \sigma_{(aX+b)}^2 = a^2 \cdot \sigma_X^2 \end{cases}$$ [linear trans]

$= E(X^2) - \mu_X^2$ [mean square - square mean]

$= E((X-c)^2) - (c-\mu_X)^2$ [mean square error - square bias]

$X_n \sim binomial(n, p)$

$$\begin{cases} P_{X_n}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \\ \mu_{X_n} = np \\ \sigma_{X_n}^2 = np(1-p) \end{cases}$$

$\begin{cases} n \in \{1, 2, 3, \dots\} : \text{\# of trials} \\ rng(X_n) = \{0, \dots, n\} : \end{cases}$

of E_0 occured

$Y_r \sim neg\text{-}binomial(r, p)$

$$\begin{cases} P_{Y_r}(\{k\}) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \\ \mu_{Y_r} = \frac{r}{p} \\ \sigma_{Y_r}^2 = \frac{r}{p} \cdot \frac{(1-p)}{p} \end{cases}$$

$\begin{cases} r \in \{1, 2, 3, \dots\} : \text{\# of } E_0 \text{ occured} \\ rng(Y_r) = \{r, r+1, r+2, \dots\} : \end{cases}$

# of trials needed (∞ ok !)

- convert : $P_{X_n}(\{x < r\}) = P_{Y_r}(\{y > n\})$
- memoryless : $P_{Y_r}(\{y > s+t\} | \{y > s\}) = P_{Y_r}(\{y > t\})$

$X \sim \text{Poisson}(\lambda)$

$$\begin{cases} P_X(\{k\}) = \dfrac{e^{-\lambda}\lambda^k}{k!} \\ \mu_X = \lambda \\ \sigma_X^2 = \lambda \end{cases} \quad \begin{cases} \lambda > 0 : \text{ expectation} \\ \text{rng}(X) = \{0, 1, 2, \dots\} : \\ \#\text{ of } E_0 \text{ occured in } \infty \text{ trials} \end{cases}$$

$N_t \sim \text{Poisson}(\lambda = \lambda_0 t)$

$N_t - N_s \sim \text{Poisson}(\lambda = \lambda_0 (t-s))$

$$\begin{cases} \lambda_0 > 0 : \text{ expectation / unit-time} \\ \text{rng}(N_t) = \{0, 1, 2, \dots\} : \\ \#\text{ of } E_0 \text{ occured in } [0, t) \end{cases}$$

$X \sim \text{binomial}(n, p = p_n) \approx X_p \sim \text{Poisson}(\lambda = n p_n)$

$P_X(\{k\}) \approx P_{X_p}(\{k\})$ if ③ $k \ll n$ if ① n suff. large

[assume: $p_n \to 0$, $np_n \to \lambda$ as $n \to \infty$] ② $p_n \approx 0$

$X \sim \text{hypergeo}(n, N = N_i, R = R_i) \approx X_B \sim \text{binomial}\left(n, p = \dfrac{R_i}{N_i}\right)$

↳ if ① N_i, R_i suff. large ② $N_i \gg n$

[assume: $N_i, R_i \to \infty$, $\dfrac{R_i}{N_i} \to p$ as $i \to \infty$, n fixed]

$X \sim \text{hypergeo}(n, N, R)$

$\text{rng}(X) = \{0, \dots, n\}$

$$\begin{cases} P_X(\{k\}) = \dfrac{\binom{R}{k}\binom{N-R}{n-k}}{\binom{N}{n}} \\ \\ (n, R, N-R) \text{ in } \{1, 2, 3, \dots\} \text{ with } n, R \le N : (\text{trials, red, white}) \end{cases} \quad \begin{cases} \mu_X = n\left(\dfrac{R}{N}\right) \\ \sigma_X^2 = n\left(\dfrac{R}{N}\right)\left(1 - \dfrac{R}{N}\right)\left(1 - \dfrac{n-1}{N-1}\right) \end{cases}$$

Probability Space : $(\Omega, \mathcal{F}_\Omega, P)$, $\omega \in \Omega$, $E \in \mathcal{F}_\Omega$ (measurable)

[classical approach]

$P(E) = \dfrac{\#E}{\#\Omega}$ where $\begin{cases} 1.\ \Omega \text{ is finite} \\ 2.\ P \text{ is symmetric } \left(P(\{\omega\}) = \dfrac{1}{\#\Omega}\ \forall \omega\right) \end{cases}$

[modern approach]

$P : \mathcal{F}_\Omega \to \mathbb{R}$ where $\begin{cases} 1.\ P(E) \geq 0 \quad \forall E \\ 2.\ P(\Omega) = 1 \\ 3.\ \forall \text{ pairwise disjoint } (E_n)_{n\in\mathbb{N}} \text{ in } \mathcal{F}_\Omega : \end{cases}$

(measure)

$E_i \cap E_j = \emptyset \quad \forall i \neq j$

$$P\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} P(E_n)$$

Thm. $\forall A, B \in \mathcal{F}_\Omega$: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Thm. $\forall (E_1, \ldots, E_n)$ in $\mathcal{F}_\Omega$: (no disjoint ok!)

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \leq i < j \leq n} P(E_i \cap E_j) + \ldots$$

Ex. Knockout Tournament.

$$\overbrace{(\underbrace{\{1st\}}_{2}, \ldots, \underbrace{\{\tfrac{n}{2}th\}}_{2})}^{1, \ldots, n} \div \underbrace{(\tfrac{n}{2})!}_{\text{deperm}}$$

[opponant arrangement]

$$(\underbrace{\{1st\}}_{1}, \ldots, \underbrace{\{\tfrac{n}{2}th\}}_{1})$$

(W, L) (L, W) ↓ ... (W, L) (L, W) ↓

[result in a round]

Ans. # outcomes in 1st round $= \left(\frac{n!}{(2!)^{\frac{n}{2}}}\right) \cdot 2^{\frac{n}{2}} \cdot \left(\frac{1}{(\frac{n}{2})!}\right) = \frac{n!}{(\frac{n}{2})!}$

total outcomes $= \frac{n!}{\cancel{(\frac{n}{2})!}} \cdot \ldots \cdot \frac{\cancel{4!}}{\cancel{2!}} \cdot \frac{\cancel{2!}}{1!} = n!$

Ex.

$$\overbrace{(\underbrace{\{1st\}}_{k_1}, \ldots \underbrace{\{nth\}}_{k_n})}^{\boxed{\sum k_i}}$$

$\# = \frac{(\sum k_i)!}{k_1! \ldots k_n!} = (\quad) \cdot (\quad) \cdot (\quad) \cdot \ldots$

[Partition]

Ex. Number of Integer Solution

$\boxed{n = \sum n_i}$ (視為相同)(分堆)

$$\overbrace{(\underbrace{\{1st\}}_{\substack{n_1 \\ (??)}}, \ldots, \underbrace{\{rth\}}_{\substack{n_r \\ (??)}})}$$

$\Downarrow$

$\boxed{1st, \ldots, (n+r-1)th}$ (挑位子)

$$\overbrace{(\underbrace{\{\bigcirc\}}_{n}, \underbrace{\{\square\}}_{r-1})}$$

$\# = \binom{n+r-1}{n}$