

$$(\Omega, \mathcal{F}_\Omega, P) \xrightarrow{X} (\mathbb{R}, \mathcal{F}_\mathbb{R}, P_X) \quad \begin{cases} \mathcal{X} = X(\Omega) \\ P_X: \mathcal{F}_\mathbb{R} \rightarrow \mathbb{R} \end{cases} \quad \begin{cases} f_X: \mathbb{R} \rightarrow \mathbb{R} \\ F_X: \mathbb{R} \rightarrow \mathbb{R} \end{cases}$$

$$P_X(A \in \mathcal{F}_\mathbb{R}) := P(X^{-1}(A) \in \mathcal{F}_\Omega)$$

$$f_X(x \in \mathbb{R}) := \begin{cases} P_X(\{x\}) & \text{if } x \in \mathcal{X} \\ 0 & \text{o.w.} \end{cases} \quad \begin{cases} 1. f_X(x) \geq 0 \quad \forall x \in \mathcal{X} \\ 2. f_X(x) = 0 \quad \forall x \notin \mathcal{X} \end{cases}$$

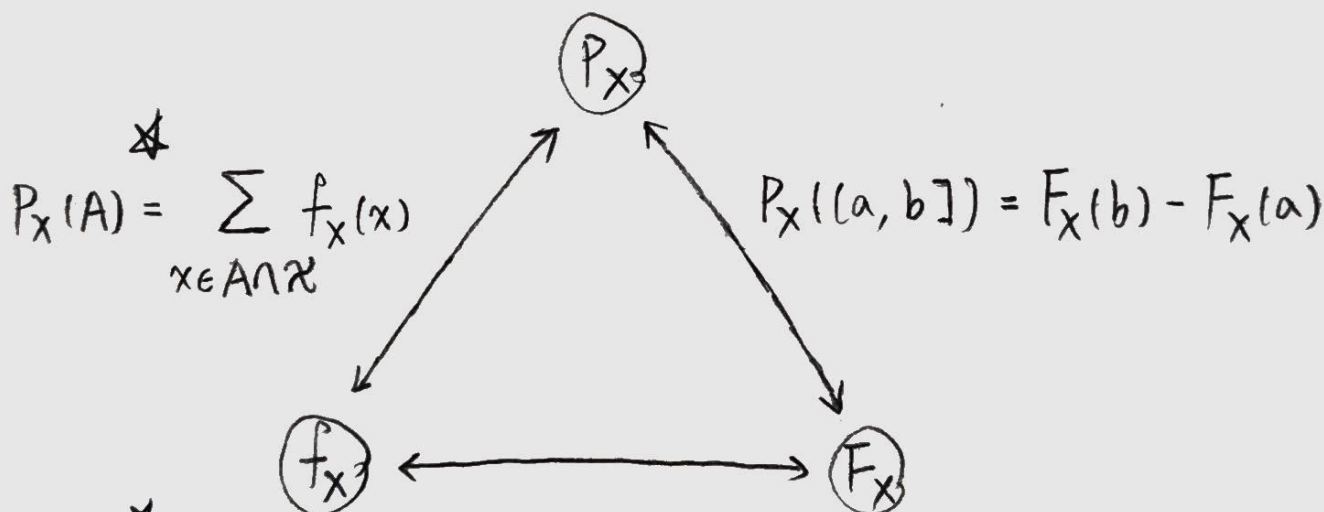
$$F_X(x \in \mathbb{R}) := P_X((-\infty, x])$$

$$3. \sum_{x \in \mathcal{X}} f_X(x) = 1 \quad \star$$

$$\left\{ \begin{array}{l} F_X \text{ non-decreasing} \end{array} \right.$$

$$\left\{ \begin{array}{l} F_X \text{ continuous from the right } \left( \lim_{t \rightarrow x^+} F_X(t) = F_X(x) \right) \end{array} \right.$$

$$\lim_{t \rightarrow -\infty} F_X(t) = 0 \quad \wedge \quad \lim_{t \rightarrow \infty} F_X(t) = 1 \quad \star \quad \Rightarrow \quad 0 \leq F_X(x) \leq 1 \quad \forall x \in \mathbb{R}$$



$\left\{ \begin{array}{l} \text{discrete r.v.: } \mathcal{X} \text{ countable} \\ \text{continuous r.v.: } \mathcal{X} \text{ uncountable} \end{array} \right.$

$$\left\{ \begin{array}{l} P_Y(A_Y) = P_X(g^{-1}(A_Y)) \quad \star \\ f_Y(y) = \sum_{x \in g^{-1}(y) \cap \mathcal{X}} f_X(x) \end{array} \right.$$

$$E(X) := \sum_{x \in \mathcal{X}} (f_X(x) \cdot x) \quad E(g(X)) = \sum_{x \in \mathcal{X}} (f_X(x) \cdot g(x))$$

$$\begin{cases} \mu_X := E(\text{id}(X)) \\ \sigma_X^2 := E((X - \mu_X)^2) \end{cases} \quad \begin{cases} E(aX + b) = a \cdot E(X) + b \\ \sigma_{(aX+b)}^2 = a^2 \cdot \sigma_X^2 \end{cases} \quad [\text{linear trans}]$$

$$= E(X^2) - \mu_X^2 \quad [\text{mean square} - \text{square mean}]$$

$$= E((X - c)^2) - (c - \mu_X)^2 \quad [\text{mean square error} - \text{square bias}]$$


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$$X_n \sim \text{binomial}(n, p)$$

$$\begin{cases} P_{X_n}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \\ \mu_{X_n} = np \\ \sigma_{X_n}^2 = np(1-p) \end{cases} \quad \begin{cases} n \in \{1, 2, 3, \dots\} = \# \text{ of trials} \\ \text{rng}(X_n) = \{0, \dots, n\} = \\ \# \text{ of } E_0 \text{ occurred} \end{cases}$$

$$Y_r \sim \text{neg-binomial}(r, p)$$

$$\begin{cases} P_{Y_r}(\{k\}) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \\ \mu_{Y_r} = \frac{r}{p} \\ \sigma_{Y_r}^2 = \frac{r}{p} \cdot \frac{(1-p)}{p} \end{cases} \quad \begin{cases} r \in \{1, 2, 3, \dots\} = \# \text{ of } E_0 \text{ occurred} \\ \text{rng}(Y_r) = \{r, r+1, r+2, \dots\} = \\ \# \text{ of trials needed } (\infty \text{ ok!}) \end{cases}$$

$$\cdot \text{convert} = P_{X_n}(\{x < r\}) = P_{Y_r}(\{y > n\})$$

$$\cdot \text{memoryless} = P_{Y_r}(\{y > s+t\} | \{y > s\}) = P_{Y_r}(\{y > t\})$$

$$X \sim \text{Poisson}(\lambda)$$

$$\begin{cases} P_X(\{k\}) = \frac{e^{-\lambda} \lambda^k}{k!} \\ \mu_X = \lambda \\ \sigma_X^2 = \lambda \end{cases} \begin{cases} \lambda > 0 : \text{expectation} \\ \text{rng}(X) = \{0, 1, 2, \dots\} : \\ \# \text{ of } E_0 \text{ occurred in } \infty \text{ trials} \end{cases}$$

$$N_t \sim \text{Poisson}(\lambda = \lambda_0 t)$$

$$N_t - N_s \sim \text{Poisson}(\lambda = \lambda_0(t-s)) \begin{cases} \lambda_0 > 0 : \text{expectation / unit-time} \\ \text{rng}(N_t) = \{0, 1, 2, \dots\} : \\ \# \text{ of } E_0 \text{ occurred in } [0, t) \end{cases}$$

$$X \sim \text{binomial}(n, p = p_n) \approx X_p \sim \text{Poisson}(\lambda = np_n) \rightarrow$$

$$P_X(\{k\}) \approx P_{X_p}(\{k\}) \text{ if } \textcircled{3} k \ll n \quad \text{if } \textcircled{1} n \text{ suff. large}$$

$$[\text{assume: } p_n \rightarrow 0, np_n \rightarrow \lambda \text{ as } n \rightarrow \infty] \quad \textcircled{2} p_n \approx 0$$

$$X \sim \text{hypergeo}(n, N = N_{\hat{i}}, R = R_{\hat{i}}) \approx X_B \sim \text{binomial}(n, p = \frac{R_{\hat{i}}}{N_{\hat{i}}})$$

$$\hookrightarrow \text{if } \textcircled{1} N_{\hat{i}}, R_{\hat{i}} \text{ suff. large } \textcircled{2} N_{\hat{i}} \gg n$$

$$[\text{assume: } N_{\hat{i}}, R_{\hat{i}} \rightarrow \infty, \frac{R_{\hat{i}}}{N_{\hat{i}}} \rightarrow p \text{ as } \hat{i} \rightarrow \infty, n \text{ fixed}]$$

$$X \sim \text{hypergeo}(n, N, R)$$

$$\begin{cases} P_X(\{k\}) = \frac{\binom{R}{k} \binom{N-R}{n-k}}{\binom{N}{n}} \\ \mu_X = n \left( \frac{R}{N} \right) \\ \sigma_X^2 = n \left( \frac{R}{N} \right) \left( 1 - \frac{R}{N} \right) \left( 1 - \frac{n-1}{N-1} \right) \end{cases} \quad \text{rng}(X) = \{0, \dots, n\}$$

$(n, R, N-R)$  in  $\{1, 2, 3, \dots\}$  with  $n, R \leq N = (\text{trials, red, white})$

Probability Space:  $(\Omega, \mathcal{F}_\Omega, P)$ ,  $\omega \in \Omega$ ,  $E \in \mathcal{F}_\Omega$  (measurable)

[classical approach]

$$P(E) = \frac{\#E}{\#\Omega} \text{ where } \begin{cases} 1. \Omega \text{ is finite} \\ 2. P \text{ is symmetric } (P(\{\omega\}) = \frac{1}{\#\Omega} \forall \omega) \end{cases}$$

[modern approach]

$$P: \mathcal{F}_\Omega \rightarrow \mathbb{R} \text{ where } \begin{cases} 1. P(E) \geq 0 \quad \forall E \\ 2. P(\Omega) = 1 \\ 3. \forall \text{ pairwise disjoint } (E_n)_{n \in \mathbb{N}} \text{ in } \mathcal{F}_\Omega: \end{cases}$$

$E_i \cap E_j = \emptyset \quad \forall i \neq j$

$$P\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} P(E_n)$$

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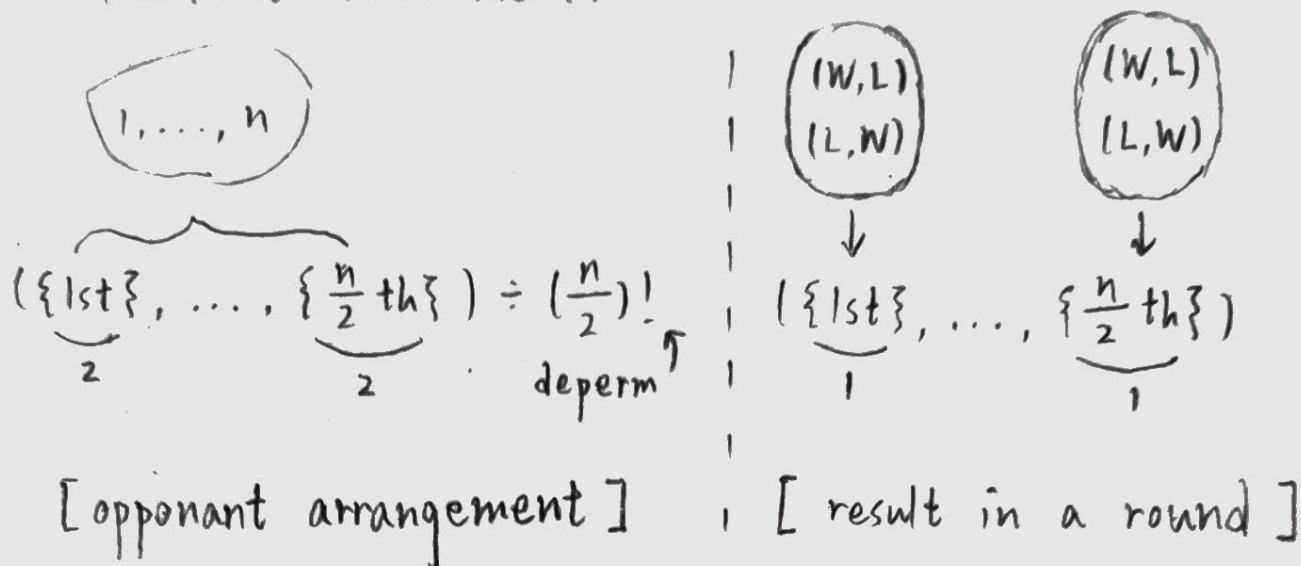
Thm.  $\forall A, B \in \mathcal{F}_\Omega: P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Thm.  $\forall (E_1, \dots, E_n) \text{ in } \mathcal{F}_\Omega = (\text{no disjoint ok!})$

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i) - \sum_{1 \leq i < j \leq n} P(E_i \cap E_j) + \dots$$



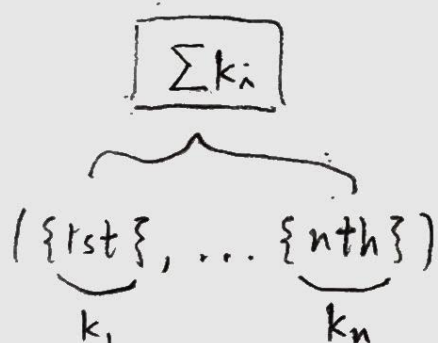
# Ex. Knockout Tournament.



Ans. # outcomes in 1st round =  $\left(\frac{n!}{(2!)^{\frac{n}{2}}}\right) \cdot 2^{\frac{n}{2}} \cdot \left(\frac{1}{(\frac{n}{2})!}\right) = \frac{n!}{(\frac{n}{2})!}$

# total outcomes =  $\frac{n!}{(\frac{n}{2})!} \cdot \dots \cdot \frac{4!}{2!} \cdot \frac{2!}{1!} = n!$

## Ex.



# =  $\frac{(\sum k_i)!}{k_1! \dots k_n!} = ( ) \cdot ( ) \cdot ( ) \cdot \dots$

[Partition]

## Ex. Number of Integer Solution

