4. 題目 $\forall f, g \in 2^{\mathbb{N}}$: $d(f,g) := \sum_{n\in\mathbb{N}} \frac{|f(n)-g(n)|}{2^n}$

(1) Find all f s.t. $d(f,\tilde{0}) = 1$

(2) Given f, find all g s.t. $d(f,g) = \frac{1}{2}$

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(1) $\because$ $1 = \frac{1}{1-(\frac{1}{2})} - 1 = \sum_{n=0}^{\infty} (\frac{1}{2})^n - 1 = \sum_{n\in\mathbb{N}} (\frac{1}{2})^n$

$\therefore$ $\forall n\in\mathbb{N}$: $|f(n) - 0| = 1$ $\therefore$ $f(n) := 1$ $\forall n\in\mathbb{N}$

(2) $\because$

$$\sum_{n=1}^{\infty} \frac{|f(n)-g(n)|}{2^n} = \begin{cases} \frac{1}{2} + \sum_{n=2}^{\infty} \frac{|f(n)-g(n)|}{2^n} & \text{if } f(1) \neq g(1) \\ 0 + \sum_{n=2}^{\infty} \frac{|f(n)-g(n)|}{2^n} & \text{if } f(1) = g(1) \end{cases}$$

$$\sum_{n=2}^{\infty} \frac{|f(n)-g(n)|}{2^n} = \begin{cases} \frac{1}{2} = \sum_{n=0}^{\infty} (\frac{1}{2})^n - 1 - \frac{1}{2} & \text{if } f(n) \neq g(n) \\ 0 = \sum_{n=2}^{\infty} (\frac{0}{2^n}) & \text{if } f(n) = g(n) \\ \text{o/w} & \end{cases}$$

$$\therefore \begin{cases} g(1) = f(1) \wedge g(n) \neq f(n) \ \forall n \geq 2 \\ g(1) \neq f(1) \wedge g(n) = f(n) \ \forall n \geq 2 \end{cases}$$

I.p.1

1. (a) 略 (不要廢話，沒時間寫)

(b) $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ bounded [Show the Proof]

$$\Rightarrow \left.\begin{array}{l} b_{sup} := \sup(\operatorname{rng}((a_n)_{n\in\mathbb{N}})) \\ b_{inf} := \inf(\operatorname{rng}((a_n)_{n\in\mathbb{N}})) \end{array}\right\} \text{exists (by Thm.)}$$

case "increasing":

isn't an upper bound

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } a_N \in (b_{sup} - \varepsilon, b_{sup}]$$

$$\Rightarrow \forall n > N: |a_n - b_{sup}| = b_{sup} - a_n \le b_{sup} - a_N < \varepsilon$$

case "decreasing":

isn't a lower bound

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } a_N \in [b_{inf}, b_{inf} + \varepsilon)$$

$$\Rightarrow \forall n > N: |a_n - b_{inf}| = a_n - b_{inf} \le a_N - b_{inf} < \varepsilon$$

2. (a) 略

(b) Let $X = (\mathbb{R}, d_D)$, $Y = \mathbb{R} - \{0\}$

closed, bounded $\subseteq B_2(0)$ in d_D

open in d_D

Let $\{x \mid x \neq 0\}$ be open cover of Y

$\Rightarrow \forall S \subsetneq \{x \mid \_\}: S \not\supseteq Y \Rightarrow \nexists$ finite subcover

$\Rightarrow$ not compact.

I. p. 2

3. Let $X = (-\frac{\pi}{2}, \frac{\pi}{2})$, $Y = \mathbb{R}$ with d_E

$\Rightarrow$ tan is cont. $\wedge$ onto

but $[0, \frac{\pi}{2})$ bounded $\not\Rightarrow$ $[0, \infty)$ ~~bounded~~

$\downarrow$

by $B_3(0)$

4(a) false: as above $x_n \to \frac{\pi}{2}$ as $n \to \infty$ Cauchy

but $f(x_n) \to \infty$ as $n \to \infty$ not Cauchy $\leftarrow$ By Thm.

($\because$ not bounded)

4.(b) $\left| \frac{m^2 - n^2}{(n^2+1)(m^2+1)} \right| \leq \left| \frac{m^2}{(n^2+1)m^2} \right| +$

$= \frac{1}{n^2+1} \leq \frac{1}{n} < \frac{1}{N}$ Goal $\frac{c}{n^k} < \frac{c}{n}$

$< \frac{c}{N}$

4.(d) By complete: $a_n \to a$, $b_n \to b$, $c_n \to c$, $d_n \to d$

By Thm. $a_n b_n \to ab$, $c_n d_n \to cd$

$\Rightarrow (a_n b_n, c_n d_n)$ conv. $\underset{\text{(Thm.)}}{\Rightarrow}$ Cauchy

6. $\because \bigcup_{n=1}^{\infty} B_n(a) = X \Rightarrow \{B_n(a)\}_{n\in\mathbb{N}}$ open cover K

$\because K$ compact $\therefore \exists \{B_{i_k}(a)\}_{k=1}^{m}$ for some m

finite subcover

where $(i_k)_{k\in\mathbb{N}}$ increasing

lower

$0 \le$

$\therefore B_{i_m}(a) \supseteq K \therefore \forall x\in K: d(x,a) < \underline{i_m}$ upper

$\therefore \{d(x,a) \mid x\in K\} \subseteq B_{i_m}(0)$ bdd

4.(c).

$\forall \varepsilon>0\ \exists N_1\in\mathbb{N}$ s.t. $\forall n,m > N_1: d(p_m, p_n) < \frac{\varepsilon}{2}$ ①

$\forall \varepsilon>0\ \exists N_2\in\mathbb{N}$ s.t. $\forall k > N_2: d(p_{n_k}, p) < \frac{\varepsilon}{2}$ ②

let $N = \max\{N_1, N_2\}$

$$\forall m > N: d(p_m, p) \le \overset{①}{d(p_m, p_{n_{N+1}})} + \overset{②}{d(p_{n_{N+1}}, p)}$$

$\because n_{N+1} \ge \underline{N+1} > N_1, \underline{N_2}$

① ②

[Show the Proof]

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

II.p.1

1. (a) X (sequentially)

(b) Compact $\Rightarrow$ all Cauchy has convergent subsequence

(c) X $\Rightarrow$ converges $\Rightarrow$ complete

2 (a) $\mathbb{Q} \cap [0,1]$ is closed & bounded $\Leftrightarrow$ compact in $\mathbb{R}$

(X) (V) $\Leftrightarrow$ compact in $\mathbb{Q}$

$\because \mathbb{Q} \cap [0,1]$ dense in $[0,1]$

$\therefore \overline{\mathbb{Q} \cap [0,1]} = [0,1] \neq \mathbb{Q} \cap [0,1] \Rightarrow$ not closed

$\Rightarrow$ not compact $\square$

反例: $U_0 := (-1, \frac{1}{\pi})$ $\nearrow$ $\frac{1}{\pi} + \frac{1}{n+1} < \frac{1}{\pi} + \frac{1}{2} < \frac{1}{3} + \frac{2}{3} = 1$

$U_n := (\frac{1}{\pi} + \frac{1}{n+1}, 2) \quad \forall n \in \mathbb{N}$

$\Rightarrow \{U_i\}_{i=0}^{\infty}$ is an open cover of $[0,1] \cap \mathbb{Q}$

by: $\begin{cases} x \in [0, \frac{1}{\pi}) \cap \mathbb{Q} : x \in U_0 \\ x \in (\frac{1}{\pi}, 1] \cap \mathbb{Q} : \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n+1} < x - \frac{1}{\pi} \\ \qquad \Rightarrow x \in (\frac{1}{\pi} + \frac{1}{n+1}, 2) = U_n \end{cases}$

Assume $\exists$ finite subcover $\{U_{i_k}\}_{k=1}^{m}$ for some $m \in \mathbb{N}$

where $(i_k)_{k=1}^{m}$ increasing

$\Rightarrow \forall l \geq i_m + 1 : U_l \notin \{U_{i_k}\}_{k=1}^{m} \Rightarrow \frac{1}{i_m+1} \notin \bigcup_{k=1}^{m} (U_{i_k})$ $\divideontimes$

3 (a) $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n, m > N$:

$$\left| \frac{n}{n^2+1} - \frac{m}{m^2+1} \right| \overset{通分}{=} \left| \frac{n(m^2+1) - m(n^2+1)}{(n^2+1)(m^2+1)} \right|$$

$$\underset{\text{ineq.}}{\overset{\text{tri}}{\leq}} \left| \frac{n(m^2+1)}{(n^2+\cancel{1})(m^2+1)} \right| + \left| \frac{m(n^2+1)}{(n^2+1)(m^2+\cancel{1})} \right|$$

$$< \frac{n(m^2+1)}{n^2(m^2+1)} + \frac{m(n^2+1)}{m^2(n^2+1)} = \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \overbrace{\frac{1}{N} < \varepsilon}^{\text{let}}$$

②

3.(b). $\exists N \in \mathbb{N}$ s.t. $\forall n > N$: $d(p_n, p_{N+1}) < 1$ [State the Proof]

$\Rightarrow \exists b := \max\{1, d(p_n, p_{N+1}) \mid n = 1, \ldots, N\}$

$\Rightarrow rng((p_n)_{n\in\mathbb{N}}) \subseteq B_{b+1}(p_{N+1}) \Rightarrow$ bounded

4. A, B compact $\overset{\text{Thm}}{\Longrightarrow}$ A, B closed

$\Rightarrow A \cap B$ closed

$\because A \cap B$ closed $\subseteq A \overset{\text{Thm.}}{\Longrightarrow} A \cap B$ compact.

7. Let $X = \{1, 2, 3\}$, $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$

$\{1\}$ is open $\wedge$ $\{1\}$ is compact (all cover is finite)

HW: Assume $\exists \alpha, \beta > 0$ s.t. $\forall x, y \in X$:

$$\alpha\left(1 - \frac{1}{1+|x-y|}\right) \le d_E(x,y) \le \beta\left(1 - \frac{1}{1+|x-y|}\right) \le \beta$$

lim = 0

but $\forall x \in X$: $\exists y \in X$ s.t. $d_E(x,y) > \beta$ $\times$

or

$$\sup\{d_E(x,y) \mid x, y \in X\} = \infty$$

II - p.3

"somewhere dense" (6.)

Let $X = (\mathbb{R}, d_D)$, $Y = (\mathbb{R}, d_E)$ where d_D denotes discrete

$\{0\}$ is somewhere dense in X ? True

Claim $\exists$ open $U \subseteq X$ s.t. $Cl_X(U \cap \{0\}) \supseteq U$

$U = B_1(0) = \{0\}$ is open $\wedge$ $Cl_X(\{0\} \cap \{0\}) = \{0\} \supseteq U$

Let $f = id_{\mathbb{R}}$ $\because$ $\mathcal{T}_X$ of $X = \mathcal{P}(X)$

$\therefore$ $\forall$ open $A \subseteq Y$: $f^{-1}(A)$ open in $X \Rightarrow$ cont.

$\{0\} \xmapsto{f} \{0\}$ is not somewhere dense in Y

$\because$ $\forall$ open non-empty $V \subseteq Y$: $V \cap \{0\} = \emptyset$ or $\{0\}$

$\Rightarrow Cl_Y(V \cap \{0\}) = \emptyset$ or $\{0\} \supseteq V = \{0\}$ or $\emptyset$

$\times$