1.

let $g: \mathbb{N} \to X$ bijective, then $f \circ g : \mathbb{N} \to Y$ is onto.

$\Rightarrow$ let $h(y) :=$ smallest number in $B_y = \{ b \in \mathbb{N} \mid f \circ g (b) = y \}$

$\Rightarrow$ $h : Y \to \mathbb{N}$ is 1-1 $\Rightarrow$ $|Y| \le |\mathbb{N}|$ $\Rightarrow$ Y is countable

2.

let $f : A \to B$, $g : C \to D$ bijective. then,

define $f \times g : A \times C \to B \times D$, $f \times g\,(a, c) = (f(a), g(c))$

if $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$ then,

$f(a_1) = f(a_2)$, $g(c_1) = g(c_2) \Rightarrow a_1 = a_2$, $c_1 = c_2$, since f, g 1-1

$\forall (c, d) \in C \times D$, $\exists$ $a \in A$, $b \in B$ s.t. $f(a) = c$, $g(b) = d$, f, g onto.

$\Rightarrow f \times g (a, b) = (c, d)$

Hence, $f \times g$ bijective $\Rightarrow$ $|A \times B| = |C \times D|$

3,

$\mathbb{Q}$ countable $\Rightarrow$ let $f : \mathbb{Q} \to \mathbb{N}$ bijective

$\Rightarrow$ let $g : \mathbb{N} \to \mathbb{Q}$, $g(n) = \frac{1}{n}$ then, $g \circ f (\mathbb{Q}) \subseteq \mathbb{Q} \cap [0,1]$

and $g \circ f$ 1-1 $p : \mathbb{Q} \cap [0,1] \to \mathbb{Q}$, $p(x) = x$ clear 1-1

$\Rightarrow$ $|\mathbb{Q}| = |\mathbb{Q} \cap [0,1]|$ by S-B thm.

4.

(a) ex.

(b) $f: (0,1) \to \mathbb{R}$, $f^2 : (0,1)^2 \to \mathbb{R}^2$, $f^2(x,y) = (f(x), f(y))$. bijective.

$$\Rightarrow \mathbb{R} \xrightarrow{f} (0,1) \xrightarrow{\text{by a}} (0,1)^2 \xrightarrow{f^2} \mathbb{R}^2$$

(c)

by induction.

5. $f : S^1 \to D^2$ 1-1

$D^2 \to \mathbb{R}^2 \to \mathbb{R} \to (0,1) \to S^1$ 1-1 □

7. $|GL(n,\mathbb{R})| \leq |\mathbb{R}^{n^2}| = |\mathbb{R}|$

8.

let $S_k = \{ r \in \mathbb{R} \mid p(r) = 0 \text{ for some } \deg p = k \}$.

then. $|S_k| \leq k \times |\mathbb{N}^{k+1}|$ countable

$\Rightarrow A \subseteq \bigcup S_k \Rightarrow A$ countable.