Using Correlation for Time Delay Estimation

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I. Example

Auto-correlation and cross-correlation functions can be used to measure the time delay of a certain signal. In this example, let us assume that the original signal is $x(t) = 5\cos(2\pi f_0 t)\, e^{-42(t-\Delta)^2}$ and the received signal $y(t)$ is a delayed version of $x(t)$. We assume that $y(t) = x(t-\tau) + n(t)$, where $\tau > 0$ is the time delay and $n(t)$ is noise.

Let f_s be the sampling rate, then $T = 1/f_s$ is the time duration between two consecutive samples. Moreover, time-related quantities can be understood/expressed as follows:

- We replace t by $i \times T$, where i implies the i-th sample.
- We rewrite $\tau = d \times T$, where the time delay τ is corresponding to the d-th sample.
- We rewrite $\Delta = L \times T$, where L implies the corresponding sample.

Now, we can express $x(t)$ and $y(t)$ as follows:

$$x[i] = 5\cos(2\pi f_0 iT)\, e^{-42(iT-LT)^2}, \tag{1}$$

$$y[i] = 5\cos\big(2\pi f_0 (i-d)T)\big) e^{-42[(i-d)T-LT]^2} + n[i]. \tag{2}$$

Let N is the number of samples. Given the sequence $\{x[i]\}_{i=0}^{N}$ and the sequence $\{y[i]\}_{i=0}^{N}$, we can estimate the value of d. Note that finding d is corresponding to finding τ. In this tutorial, we use the auto-correlation and cross-correlation functions to estimate d.

Denote $Corr(x)$ as the auto-correlation of $\{x[i]\}$. Denote $Corr(y,x)$ as the cross-correlation of $\{y[i]\}$ and $\{x[i]\}$. Please note that $Corr(y,x)$ is different from $Corr(x,y)$. Then, we find the optimal indices $i^\star_{xx}$ and $i^\star_{yx}$ as follows:

$$i^\star_{xx} = \arg\max_i Corr(x), \tag{3}$$

$$i^\star_{yx} = \arg\max_i Corr(y,x). \tag{4}$$

Finally, the delay d can be calculated as $d = i^\star_{yx} - i^\star_{xx}$.

II. Visualization with Matlab

```
fs = 100; % sampling rate
T = 1/fs;
f0 = 1500; % carrier frequency
L = 30; % the main signal energy is centered about sample number L

%
num_samples = 1024;
sample_idx = 0:(num_samples-1);
TimeInstants_for_Tx = sample_idx*T;
delay_idx = 180;
TimeInstants_for_Rx = (sample_idx-delay_idx)*T;

% Original signal
x = 5.*cos(2*pi*f0*TimeInstants_for_Tx).*exp(-42.*((TimeInstants_for_Tx-L*T).^2));
% Received signal without noise
y = 5.*cos(2*pi*f0*TimeInstants_for_Rx).*exp(-42.*((TimeInstants_for_Rx-L*T).^2));
% Noise and Received noisy signal
noise = rand(1,num_samples);
y = y + noise;

% Auto-Correlation
[corr_xx, lag_xx] = xcorr(x, x); % Auto-correlation of x
% Cross-Correlation . NOTE: xcorr(x,y) is DIFFERENT from xcorr(y,x)
[corr_yx, lag_yx] = xcorr(y, x); % Cross-correlation of y and x

% find the maximum of the auto-correlation function
[peak_xx, idx_peak_xx] = max(corr_xx);

% find the maximum of the cross-correlation function
[peak_yx, idx_peak_yx] = max(corr_yx);

% estimate the delay
delay_est = idx_peak_yx - idx_peak_xx


% plot signals
figure
plot(sample_idx,x)
hold on
plot(sample_idx, y)
```

```
xlabel('Sample index','FontSize',12)
ylabel('Signal','FontSize',12)
legend('Original signal','Received noisy signal','FontSize',10)

% plot correlation functions
figure
plot(lag_yx, corr_yx,'r','LineWidth', 1.4);
hold on
plot(lag_yx, corr_xx,'--b','LineWidth', 1.4);
xlabel('Lag','FontSize',12)
ylabel('Correlation','FontSize',12)
legend('Cross-correlation xcorr(y,x)','Auto-correlation xcorr(x)', ...
    'FontSize',10,'Location','best')
```

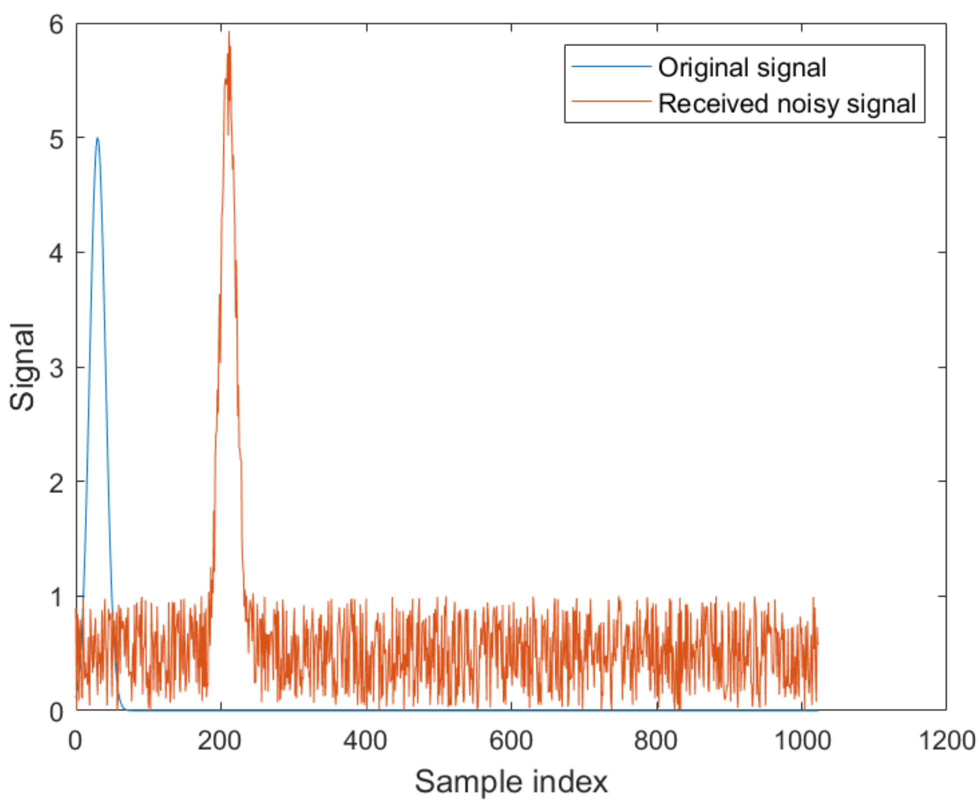

Fig. 1. The original signal and the received noisy signals are depicted in the time domain.

REFERENCES

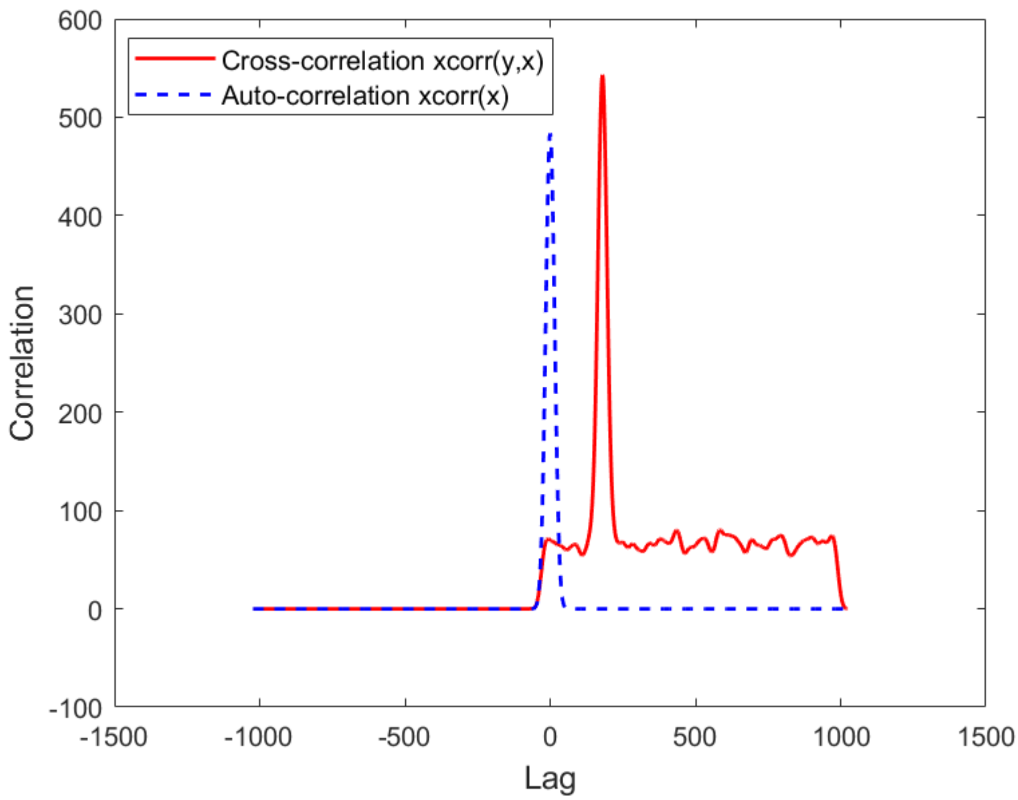

Fig. 2. Auto-correlation and cross-correlation functions. It shows $d = i^{\star}_{yx} - i^{\star}_{xx}$ = 180.