Introduction to Quantum Computing

Module 2 — Part I

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ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

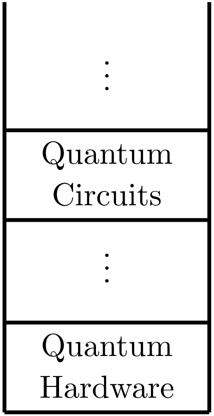
Inria informatiques mathématiques

Academic Year 2024/2025

Part I

This Module: Contents and Motivations

What?



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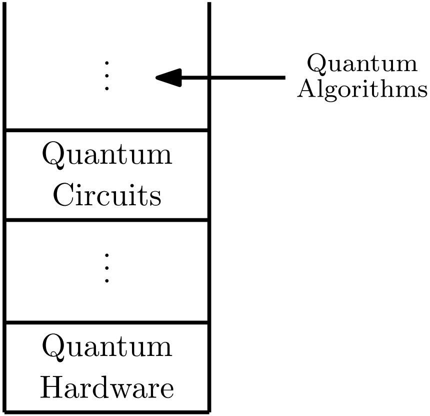

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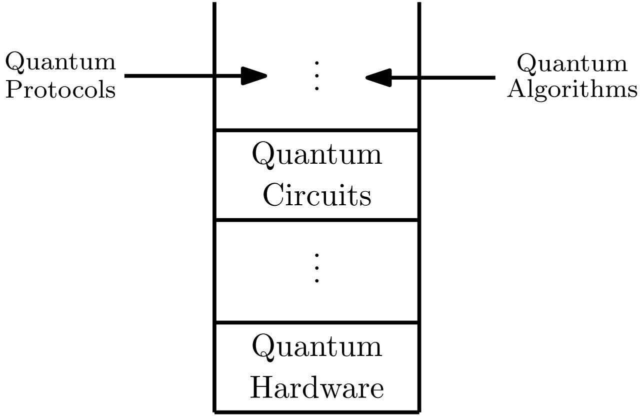
Quantum Protocols
Quantum Algorithms
Quantum Circuits
Quantum Hardware

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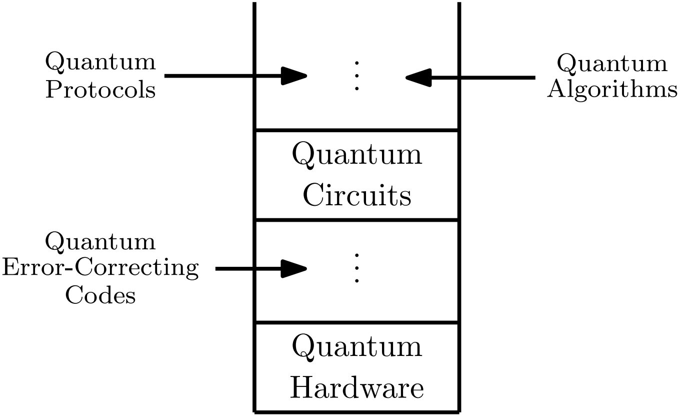
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Quantum Error-Correcting Codes
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- Please feel free to ask any question during lectures, or to write me at `ugo.dallago@unibo.it`

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- It's important to understand **where** this potential comes from.
  - You know what a quantum circuit **is**.
  - But perhaps you do not know in which sense quantum circuits can be **helpful**.
- But **beware**: most of what we are going to say is about the *quantum circuit model*, and nobody currently knows whether this model is realistic when the number of qubits grows.
  - That is why we will also talk about quantum error correcting codes.

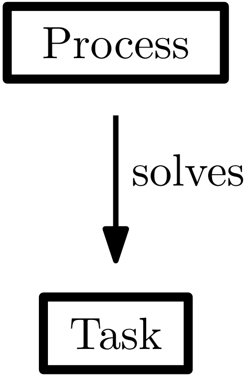
This Slideset

- A glimpse of **circuit complexity**.
  - Classical circuit complexity.
  - Quantum circuit complexity.
  - How all this relates to Turing machines.
- Some basic **quantum algorithmics**.
  - A Recap on Deutsch and Bernstein-Vazirani Algorithms.
  - Simon Algorithm.
  - Shor Algorithm.
    - We will try to give most details, but we will perhaps skip some of them
  - Grover Algorithm

Part II

Basic Circuit Complexity

Tasks and Processes



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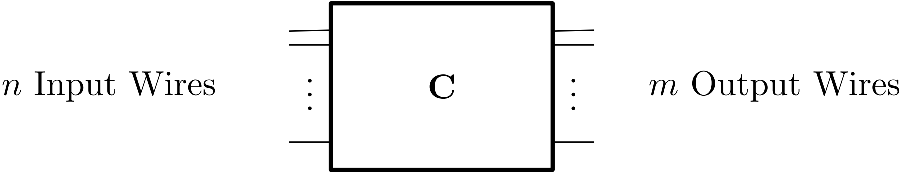
- A particularly interesting alphabet is $\Sigma = \{0, 1\}$.
- We are interested in tasks formulated as mathematical functions from $\{0, 1\}^n$ to $\{0, 1\}^m$. The set of all such **boolean functions** is indicated as $\mathcal{F}^n_m$.
- Sometimes, we are also interested in **parametric boolean functions** namely in functionals which turn a function in $\mathcal{F}^p_q$ to a function in $\mathcal{F}^n_m$.
  - An example when $p = q = 2$, $n = 0$ and $m = 1$ is the task of checking whether the input function $f \in \mathcal{F}^p_q$ returns 00 for all possible inputs.

Classical Processes

- A *classical process* will be a **boolean circuit** :



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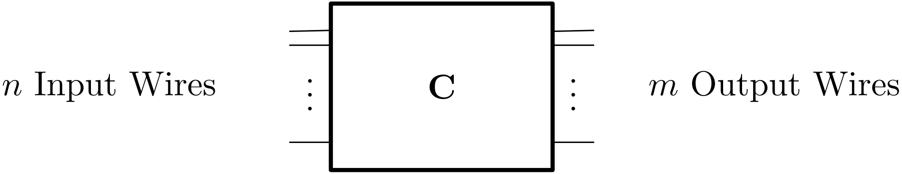

- One such circuit $\mathbf{C}$ is said to **compute** the boolean function $f_{\mathbf{C}}$ in $\mathcal{F}^n_m$ such that $f_{\mathbf{C}}(x) = y$ precisely when $\mathbf{C}$ when executed on x on the n input wires produces y on its m output wires.

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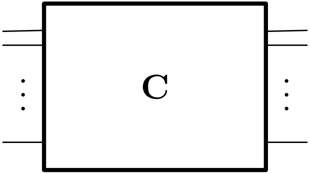

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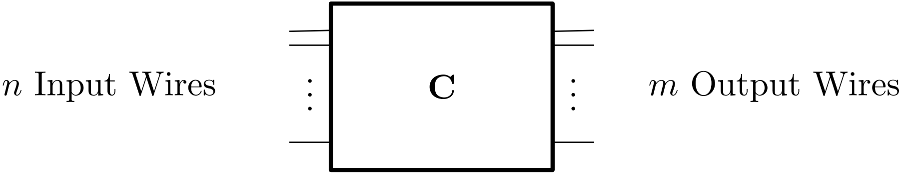

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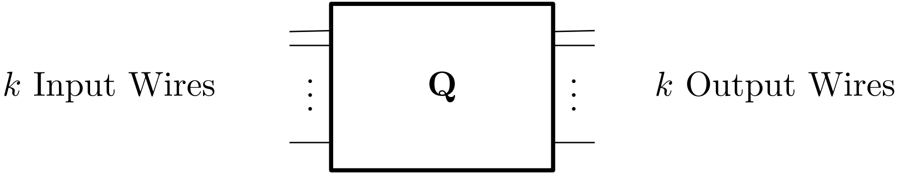
- The *size* of $\mathbf{C}$ is a measure of how many resources $\mathbf{C}$ would consume when executed.
  - The size of $\mathbf{C}$, that we write as $|\mathbf{C}|$, can be measured in different ways, e.g., as the *number of gates*, as the *width*, or as the *depth*.
  - An interesting size parameter of any parametric boolean circuit $\mathbf{C}$ is the number of calls to the special gate computing the parameter function, which we indicate as $|\mathbf{C}|_{\mathsf{P}}$.

Quantum Processes

- Similarly, a *quantum process* is just a **quantum circuit** :



- We can generalize all what we have said about boolean circuits, keeping in mind that
  - Quantum circuits, due to measurements, can have a probabilistic behaviour. We then say that one such circuit *computes* a function $f_{\mathbf{Q}}$ in $\mathcal{F}_m^n$ if it does so up to a *small* probability of error.
  - The circuit $\mathbf{Q}$, if we do not consider measurements, must be reversible. It is thus convenient to assume that for such a circuit to compute $f \in \mathcal{F}_m^n$, it must be that $\mathbf{Q}$ has $n+m+r$ inputs and $n+m+r$ outputs an that, with high probability, on input $|x\rangle \otimes |y\rangle \otimes \left|0^{\otimes r}\right\rangle$ it produces (with high probability) the value $|x\rangle \otimes |f(x) \oplus y\rangle \otimes |\psi\rangle$ in output.
    - The r qubits are auxiliary, and their output value is not relevant.

The Fundamental Question

- The question now is the following: Is there any (parametric) function f such that there is a quantum circuit $\mathbf{Q}$ computing f such that all (known) boolean circuits $\mathbf{C}$ computing f are such that $|\mathbf{Q}| < |\mathbf{C}|$?

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- We will give some examples of such functions:
  - Starting from **toy** functions, for which you have already seen the corresponding algorithms.
  - And ending in **relevant** functions.
  - Many of them will be **parametric** functions.

A Bridge to "Uniform" Complexity Theory

- In classical computation, a *circuit family* is a family $\{\mathbf{C}_n\}_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, the circuit $\mathbf{C}_n$ computes a function in $\mathcal{F}_1^n$.
  - Such a circuit family $\{\mathbf{C}_n\}_{n\in\mathbf{N}}$ can thus be seen as computing a function $f_{\{\mathbf{C}_n\}} : \{0,1\}^* \to \{0,1\}$ where $\{0,1\}^* = \cup_{n=0}^{\infty}\{0,1\}^n$.

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- A very similar result holds for polysize *quantum* circuit families generated in *classical* polynomial time and a class called $\mathtt{BQP}$

Part III

Basic Quantum Algorithmics

The Deutsch Problem

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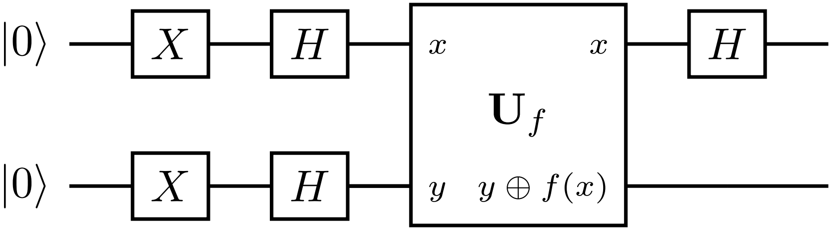
- Classically, we do not have any hope of getting the job done without invoking the f *twice*.
- In quantum computing, one can do strictly better, and invoke f just *once*!
- In doing so, we will assume that there is a gate $\mathbf{U}_f$ computing f without using any auxiliary qubit:

$$\mathbf{U}_f(|x\rangle\,|y\rangle) = |x\rangle\,|y \oplus f(x)\rangle$$

The Deutsch Circuit $\mathbf{C}_{\mathsf{DEUTSCH}}$

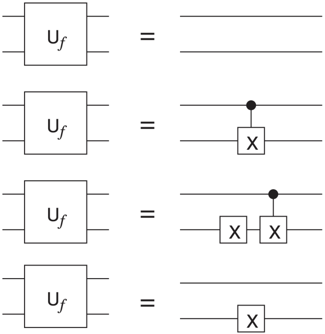


Correctness

$$(\mathbf{H} \otimes \mathbf{1})\mathbf{U}_f(\mathbf{H} \otimes \mathbf{H})(\mathbf{X} \otimes \mathbf{X})(|0\rangle\,|0\rangle)$$

$$= \begin{cases} |1\rangle \frac{1}{\sqrt{2}}\Big(|f(0)\rangle - |\tilde{f}(0)\rangle\Big), & f(0) = f(1), \\ |0\rangle \frac{1}{\sqrt{2}}\Big(|f(0)\rangle - |\tilde{f}(0)\rangle\Big), & f(0) \neq f(1). \end{cases}$$

Equational Interpretation



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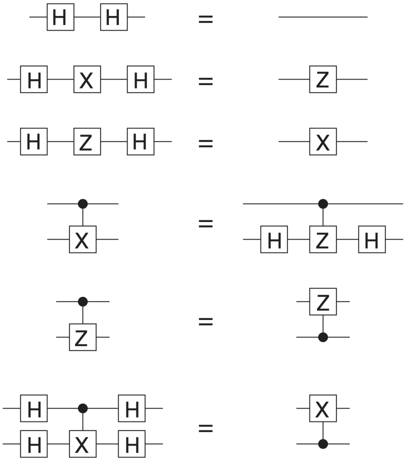

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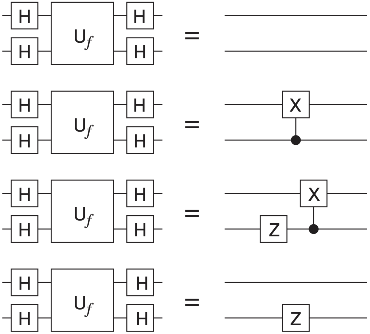

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The Bernstein-Vazirani Problem

- Given $a, x \in \{0,1\}^n$. We write $a \cdot x$ for the bitwise inner-product of a and x:

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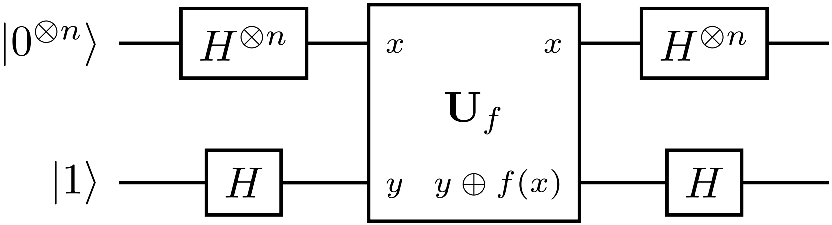
- Simply, we want to determine a invoking f_a, but trying to minimize the amount of times we invoke the function.
- Classically, we cannot do that by invoking f_a less than n times.
- There is a quantum circuit which invokes f_a **just once**.

The Bernstein-Vazirani Circuit $\mathbf{C}_{\mathsf{BV}}$



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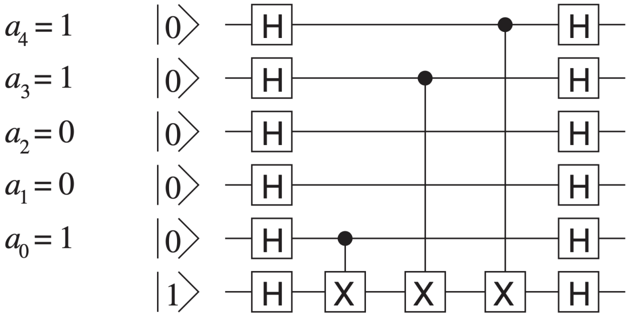

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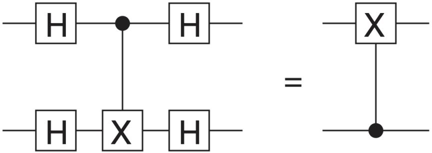

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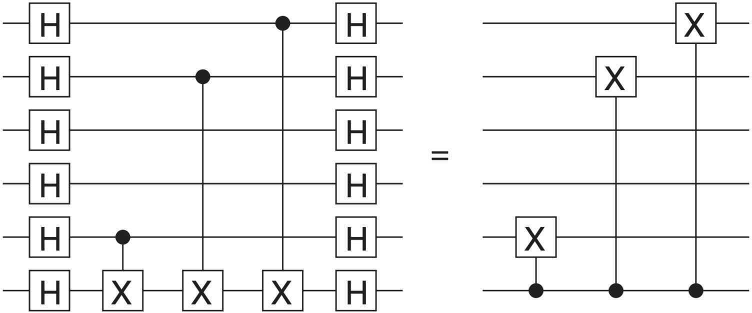

Part IV

The Simon Problem

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- Classically, one cannot succeed (with high probability) unless one invokes f around 2^n times, so **exponentially many** times.
- There are, instead, quantum circuits which succeed in determining a by invoking f an amount of time which is **linear** in n.

Classical Strategy

- Any classical algorithm querying a function $f \in \mathcal{F}_n^n$ which is periodic modulo a *accumulates knowledge* as follows:
  - If the queries done so far, namely

    $$y_1 = f(x_1), \ldots, y_k = f(x_k)$$

    are such that $y_i \neq y_j$ for every distinct $i, j \in \{1, \ldots, k\}$, then the only thing the algorithm knows is that for every distinct $i, j \in \{1, \ldots, k\}$, it holds that $a \neq x_i \oplus x_j$.
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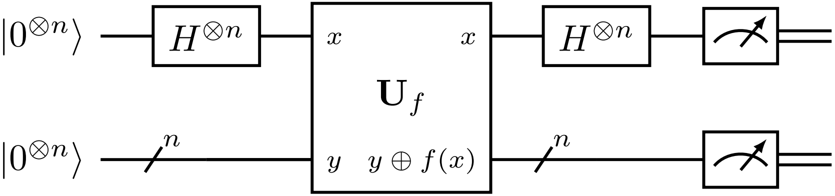
  - If, instead, $y_i = y_j$, then a can be easily computed as $x_i \oplus x_j$.
- Since there are only $\frac{k(k-1)}{2}$ pairs of distinct indices in $\{1, \ldots, k\}$, in the worst case the number of queries is has to be exponential.

The Simon Circuit $\mathbf{C}_{\mathsf{SIMON}}$



The Simon Algorithm

- The actual Simon's Algorithm has the following structure:
  1. $i \leftarrow 1$.
  2. Apply $\mathbf{C}_{\mathsf{SIMON}}$ and obtain in the first n qubits the value $|x_i\rangle$
  3. If $i = n + 3$, then go to step 4, otherwise, increment i by 1 and go back to 2.
  4. Solve the linear system of equations

$$\begin{cases} a \cdot x_1 = 0 \\ \vdots \\ a \cdot x_{n+3} = 0 \end{cases}$$

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- The system of equation above has a unique solution with probability at least $\frac{2}{3}$.

Part V

Shor's Algorithm

A Groundbreaking Result

- Shor's Algorithm is considered as **one of the major results** in the theory of computation since the inception of the discipline, in the sixties.

A Groundbreaking Result

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- Ultimately, however, Shor's Algorithm is an algorithm about **period finding**.

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  - If N is the product of two prime numbers p and q, then $\Phi(N) = (p - 1)(q - 1)$.
  - Euler's theorem tells us that every $a \in \mathbb{G}_N$, it holds that

$$a^{\Phi(N)} \equiv 1 \mod N$$

Number-Theoretic Preliminaries — II

- The **order** of $n \in \mathbb{G}_N$ is the smallest number r such that

$$n^r \equiv 1 \mod N$$

  - The order of n always divides $\Phi(N)$.
- A **subgroup** of $\mathbb{G}_N$ generated by $n \in \mathbb{G}_N$ is the subset $\langle n \rangle$ of $\mathbb{G}_N$ of those elements of $\mathbb{G}_N$ which can be written as $n^k \mod N$, where $k \in \mathbb{Z}$.
- If $\langle n \rangle = \langle m \rangle$ then n and m have the same order.

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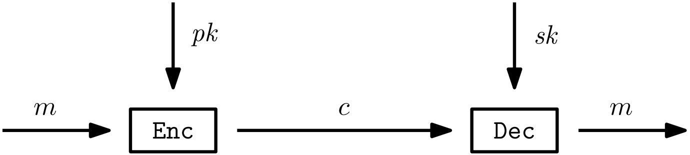

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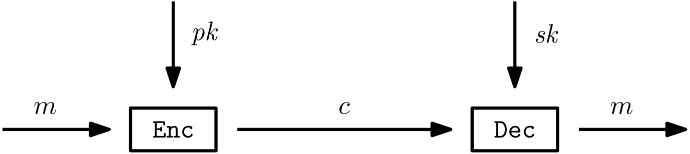

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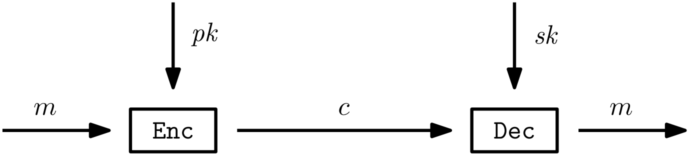

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- Messages are simply elements of $\mathbb{G}_N$.
- Given a message $m \in \mathbb{G}_N$ and a public key (N, e), its encryption is $\mathtt{Enc}(m, (N, e)) = m^e \mod N$.
- Given a ciphertext $c \in \mathbb{G}_N$ and a private key (N, d), its decryption is $\mathtt{Dec}(c, (N, d)) = c^d \mod N$.

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$$\begin{tikzcd} & {} \arrow[d, "pk"] & & {} \arrow[d, "sk"] & \\ {} \arrow[r, "m"] & \boxed{\texttt{Enc}} \arrow[rr, "c"] & & \boxed{\texttt{Dec}} \arrow[r, "m"] & {} \end{tikzcd}$$

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- Given a ciphertext $c \in \mathbb{G}_N$ and a private key (N, d), its decryption is $\texttt{Dec}(c, (N, d)) = c^d \mod N$.
- Noticeably, $\texttt{Dec}(\texttt{Enc}(m, (N, e)), (N, d)) = m$.

Breaking RSA

- Clearly, **one way** of breaking RSA consists in, given $N = pq$ and $e \in \mathbb{G}_{(p-1)(q-1)}$, determine d such that
$$ed \equiv 1 \mod (p-1)(q-1)$$
  - This way one can determine the private key from the public key.
  - We do not even know $(p-1)(q-1)$, however.
- If we can find, given a and N, the smallest number $r \geq 1$ such that
$$c^r \equiv 1 \mod N$$
(where c is the ciphertext) then we can retrieve the message from the cryptogram in a natural way.
  - Such an r is nothing more and nothing less than the period of the function $x \mapsto c^x \mod N$.

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$$|\Psi\rangle_n = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle \, .$$

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  where m is the smallest integer such that $mr + x_0 \geq 2^n$
- There are two problems, however, namely:
  - The presence of x_0
  - The fact that f is not necessarily easy to be computed.

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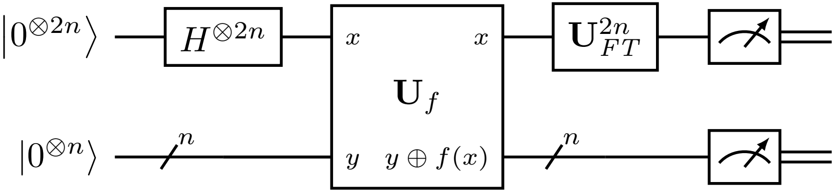
- Remarkably, $\mathbf{U}_{FT}^{n}$ can be implemented with a circuit of quadratic size consisting of unary gates, only.
- If one applies $\mathbf{U}_{FT}^{n}$ to $\left|\Psi\right\rangle_n$, after that measuring *all bits*, then the probability of observing y when measuring the input qubits is

$$p(y) = \frac{1}{2^n m} \left| \sum_{k=0}^{m-1} e^{2\pi i k r y/2^n} \right|$$

Shor's Circuit $\mathbf{C}_{\mathsf{SHOR}}$



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- Out of them, we can finally find r.

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- Just iterating the multiplication in the obvious way does *not* work, because the number of iterations will be in the worst case equal to N.
- We need a more clever algorithm, called **fast exponentiation**, working on the input register x, the output register y and a work register w.
  - Initially, $y = 1$ and $w = b$.
  - We then update y and w in an iterative way, squaring w and multiplying y by w only if the corresponding bit of x is 1.

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- However, the same algorithm can be used to solve other problems of interest:
  - **Integer Factorization**.
    - After checking if the number N in input is prime, generate $1 < a < N$ at random, and if a is prime with N, compute the period r of $x \mapsto a^x \mod N$. If r is even, then any factor of N is either a factor of $a^{\frac{r}{2}} - 1$ or a factor of $a^{\frac{r}{2}} + 1$.
  - **Discrete Logarithm**.

Part VI

Grover Algorithm

Grover's Algorithm

- Grover's algorithm solves the following computational problem:
  - *Input*: a boolean circuit (whose structure is not accessible) computing an unknown function $f : \{0,1\}^n \to \{0,1\}$ such that there exists $a \in \{0,1\}^n$ with $f(x) = 1$ iff $x = a$.
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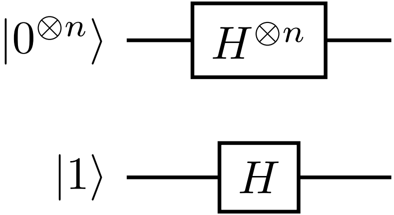
- Grover's Algorithm, by way of a technique called **amplitude amplification**, manages to solve the same problem in time at most $O(\sqrt{2^n})$.

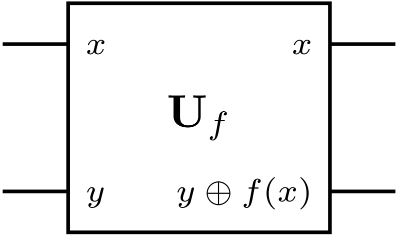
First Component



The resulting state is

$$|\phi\rangle \otimes H(|1\rangle) = \left(\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \right) \otimes H(|1\rangle)$$

Second Component

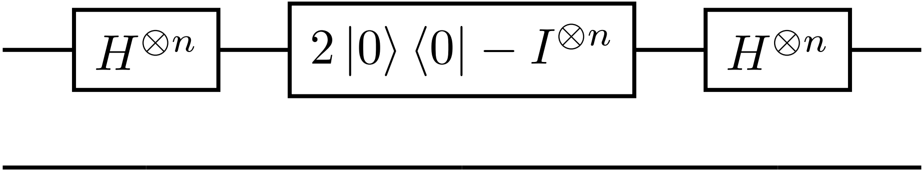


On an input $|x\rangle \otimes H(|1\rangle)$, the output is

$$(-1)^{f(x)}\,|x\rangle \otimes H(|1\rangle)$$

and so there must be a unitary transformation $\mathbf{V}$ on n qubits capturing its behaviour.

Third Component



The action of the component on the first n qubits can be seen as that of a unitary transformation, call it $\mathbf{W}$

A Geometric Analysis

- Let us take a look at how $\mathbf{V}$ and $\mathbf{W}$ behave on the states $|a\rangle$ and $|\phi\rangle$:

$$\mathbf{V}\,|a\rangle = -\,|a\rangle\,; \qquad \mathbf{V}\,|\phi\rangle = |\phi\rangle - \frac{2}{2^{n/2}}\,|a\rangle\,;$$

$$\mathbf{W}\,|a\rangle = \frac{2}{2^{n/2}}\,|\phi\rangle - |a\rangle \qquad \mathbf{W}\,|\phi\rangle = |\phi\rangle$$

- Applying $\mathbf{V}$ and $\mathbf{W}$, then keep the following invariant true: the state is in the form $|\psi\rangle \otimes H(|1\rangle)$, where $|\psi\rangle$ lives in the two-dimensional space of all linear combinations $\alpha\,|\phi\rangle + \beta\,|a\rangle$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$.
- The states $|a\rangle$ and $|\phi\rangle$ are almost orthogonal, since $\theta = \langle a|\phi\rangle = 2^{-n/2}$. There is in particular a vector forming a small angle with $|\phi\rangle$ and being orthogonal with $|a\rangle$, call it $\left|a^{\perp}\right\rangle$.
- Both $\mathbf{V}$ and $\mathbf{W}$ are reflections, and their product $\mathbf{WV}$ is thus a rotation.

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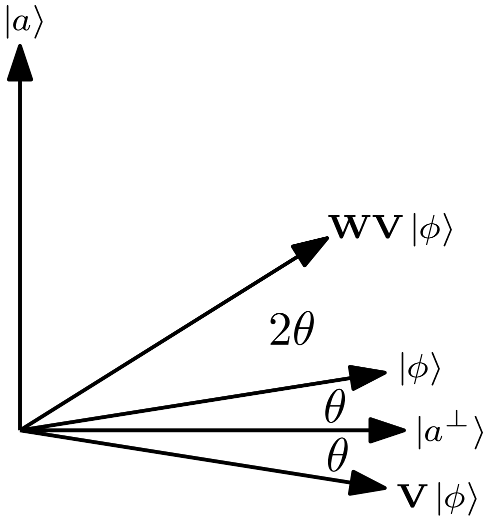

Grover's Circuit $\mathbf{C}_{\mathsf{GROVER}}$