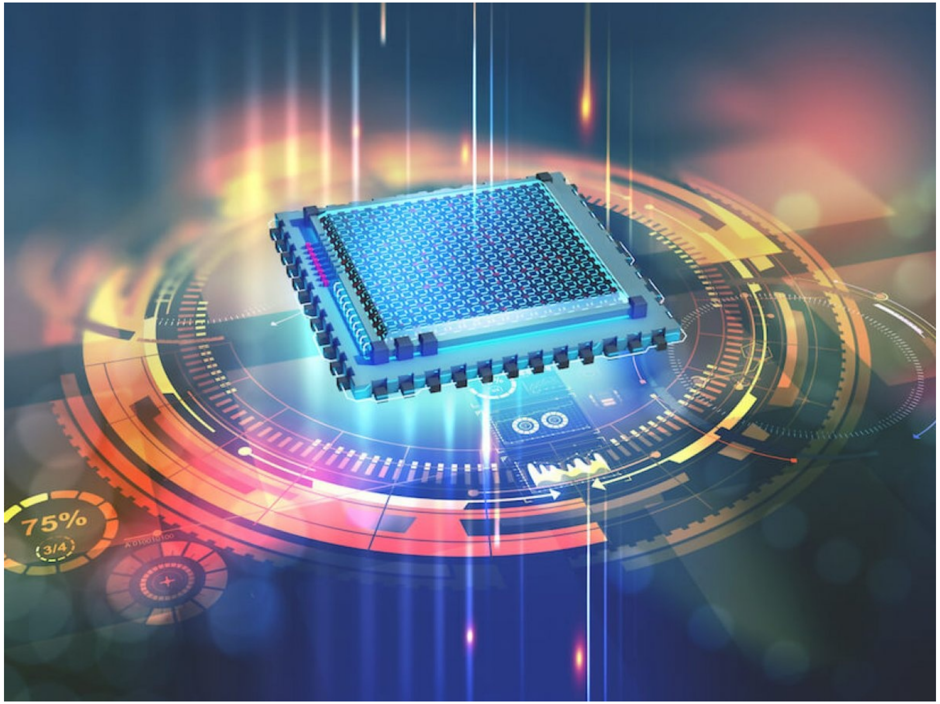
Universality



Last class

- Bloch sphere

Today

- Universal sets of gates

Universality

- We have seen a few sample circuits, as well as a few sample gates
- In order to build circuits it is useful to have universal sets of gates
- What does it mean for a set of gates to be universal?

Universality

- Which behaviors are all the possible behaviors?
  - Quantum computations are described by unitary matrices
- Quantum states have continuous sets of values
  - It makes no sense to require to build exactly a given unitary matrix
  - But we can require to reduce the error as much as desired

Measuring errors

- We need to define a distance between matrices

$$E(U, V) = max_{|\psi\rangle} \left\| \left(U|\psi\rangle - V|\psi\rangle \right) \right\|$$

where

$$\left\| |\psi\rangle \right\| = \langle \psi | \psi \rangle$$

- Matrices are close if they send each vector to close by vectors

Universality: formal definition

- A set of gates is *universal* if, for any integer $n \geq 1$, given a unitary matrix U with size 2^n and an error ε we can build using only gates in the set a circuit whose unitary matrix V is such that $E(U,V) < \varepsilon$
- Above, n is the number of qubits

What we already know

- From quantum state preparation we know that any set of 1-qubit gates is not universal
- E.g., H is not universal
- Rationale: any set of 1-qubit gates cannot transform a non-entangled state into an entangled one
- There are other criteria to show that some sets of gates are NOT universal

Need for superposition

- Any set of classical gates is not universal
- They send classical states to classical states
- E.g., you can not create Hadamard using classical gates
- E.g., CNOT is not universal

Need for complex coefficients

- Composing matrices with real coefficients always produces matrices with real coefficients
- It will never be possible to approximate matrices with coefficients with non-zero imaginary part
- E.g., the set {CNOT, H} is not universal

Anything else?

- Actually yes, but it starts getting more complex
- The set {CNOT,H,P} satisfies all the conditions we have seen (hence can show many relevant quantum effects) but it is not universal
- Circuits obtained from them are called stabilizer circuits
- You will find them again when discussing error correction

Entangling gates

- A 2-qubit gate is an entangling gate if there is an input product state such that the output is not a product state
- That is, the gate can create entanglement
- E.g., CNOT is an entangling gate

A first universal set

- Any set composed of all 1-qubit gates and an entangling gate is universal
- E.g., all 1-qubit gates and a CNOT
- We will not give a proof of this
- We will however refine this result

A first result: limitations

- Any set composed of all 1-qubit gates and an entangling gate is universal
- This provides us a universal set of gates which is infinite
- Not really satisfactory

A first universal set, refined

- Any set composed of all 1-qubit gates and an entangling gate is universal
- However, any 1-qubit gate is a rotation in the Bloch sphere
- Any rotation can be decomposed into rotations around any two non-parallel axes
- Any set composed of all 1-qubit gates corresponding to rotations around two non-parallel axes and an entangling gate is universal

Approximating rotations

- Fix an axis and consider a rotation around it of φ radians
- Applying n gates in a row gives rotations of $n\varphi$
- However, rotations of 2π are the identity
- If the ratio between φ and 2π is irrational then we will approximate all the rotations around the considered axis

Approximating rotations

- If the ratio between φ and 2π is irrational then we will approximate all the rotations around the considered axis
- Fix an error ε and divide the circle in portions of size ε
- Put in the circle all values $n\varphi$
- Since there are at most $2\pi/\varepsilon$ portions, for $n>2\pi/\varepsilon$ we have at least two points, corresponding to tries n_1 and n_2, closer than ε (pigeonhole principle)
- Hence applying our rotation n2-n1 times allows us to move less than ε (and more than 0 due to irrationality)
- Hence we can cover the whole circumference with steps less than ε

A finite universal set

- Any set composed of two 1-qubit gates corresponding to rotations around two non-parallel axes whose step has an irrational ratio with 2π and an entangling gate is universal
- Consider $R_{\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} = T$
- One can show that HTHT and THTH satisfy the conditions above

A finite universal set

- Any set composed of two 1-qubit gates corresponding to rotations around two non-parallel axes whose step has an irrational ratio with 2π and an entangling gate is universal
- One can show that HTHT and THTH satisfy the conditions above
- Hence {H,T,CNOT} is a universal set of gates