Introduction to Quantum Computing

Module 2 — Part II

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Inria informatiques mathématiques

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Part I

Quantum Protocols

Circuits vs. Protocols

- **Quantum Circuits**
  - These are sequences of (unitary) operations applied to qubits in a prescribed manner.
  - Meant to be executed by quantum hardware or simulated through classical hardware.
  - They are seen as ways to compute functions from $\{0,1\}^n$ to $\{0,1\}^m$.

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  - Meant to be executed by quantum hardware or simulated through classical hardware.
  - They are seen as ways to compute functions from $\{0,1\}^n$ to $\{0,1\}^m$.
- **Quantum Protocols**
  - Communication protocols, i.e., sequence of computation and communication steps, perormed by a set of (possibly distributed) agents.
  - Besides using and exchanging classical data, the agents can use and exchange quantum data in the form of qubits.
  - The underlying task to be solved can be different from the mere computation of a function, e.g.,
    - Transmission or distribution of some information.
    - Consensus.
    - Commitment.
    - $\ldots$

What *is* a Quantum Protocol?

- One can formally define what a quantum protocol actually is, and there are *many* languages and models of quantum protocols.
- We will stay very **informal**, and describe quantum protocols by:
  - Either giving a circuit, describing who owns each of the qubits, and/or when they are exchanges between the parties.
  - Or/and describing what each of the agents is supposed to do with its data.

Part II

Quantum Teleportation

Teleporting a Qubit?

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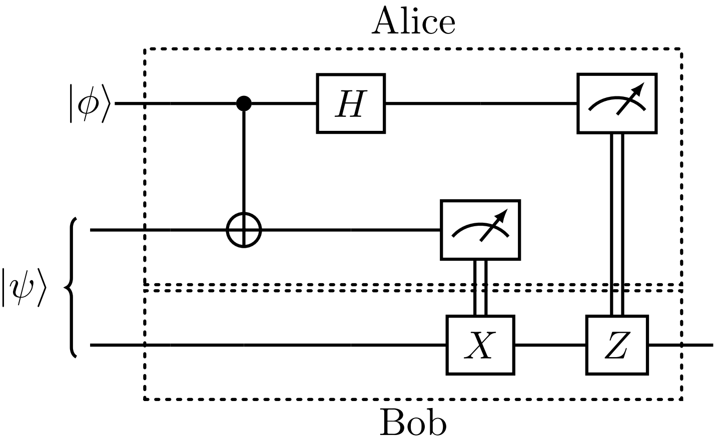
- In doing so, se is allowed to actually send classical information, but that she cannot send quantum information.
- Surprisingly, this can be achieved, provided Alice and Bob share a pair on entrangled qubits $|\psi\rangle$, which can be setup *prior* to the creation of Alice's qubit.
- The byproduct of the communication is that $|\psi\rangle$ is destroyed.

Quantum Teleportation as a Circuit



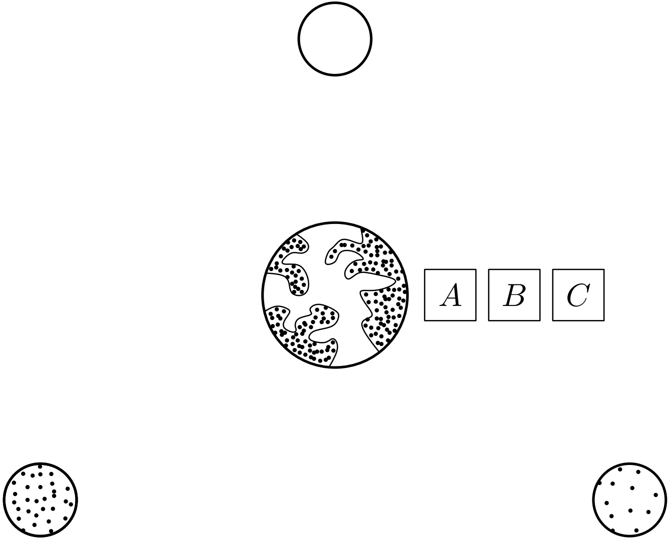
Part III

Quantum Pseudotelepathy

Quantum Pseudotelepahy

- This term refers to protocols which uses entanglement for the sake of proving that (part of the) communication between the parties can be avoided.
  - This is often provably impossible in classical computing.
- We will introduce only a very simple example of quantum pseudotelepathy, through a game.

A Game



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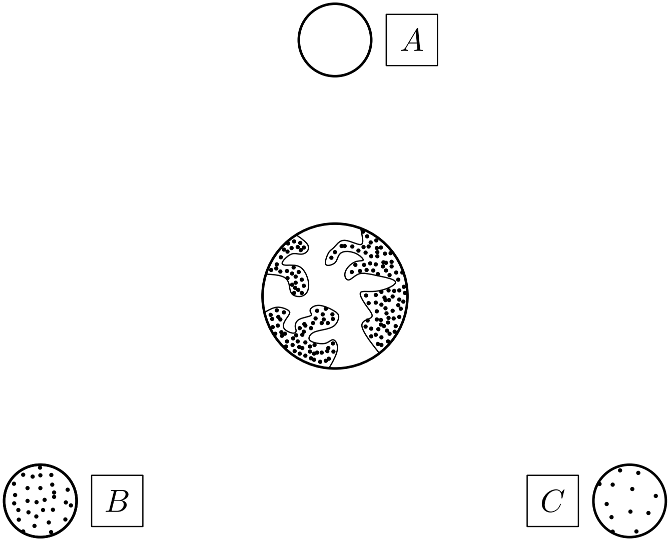

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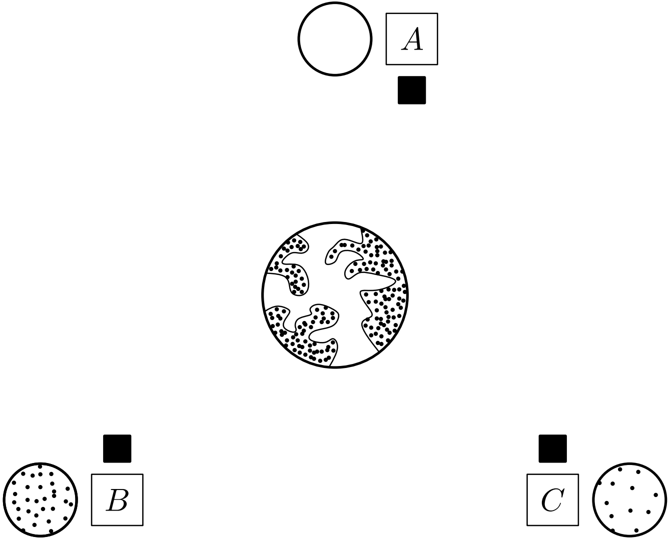

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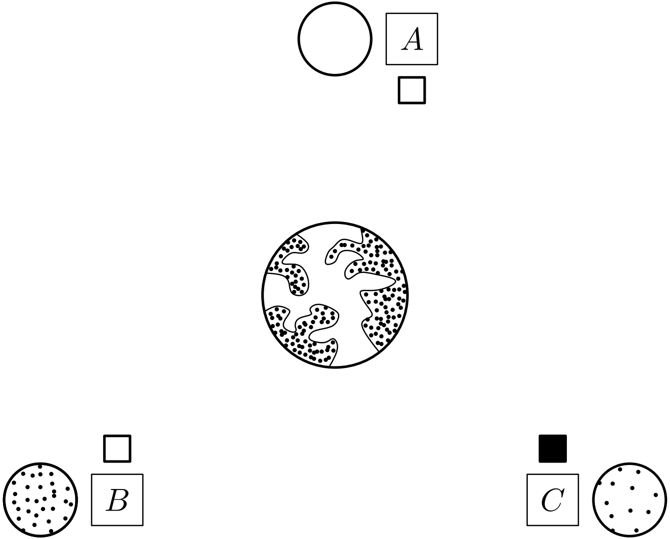

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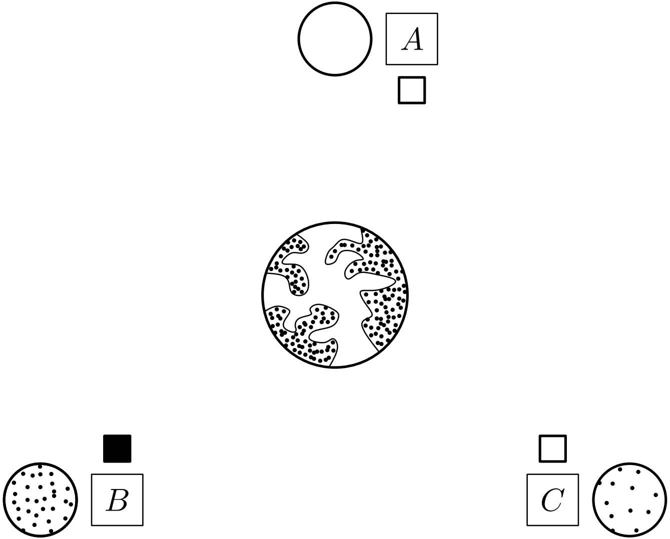

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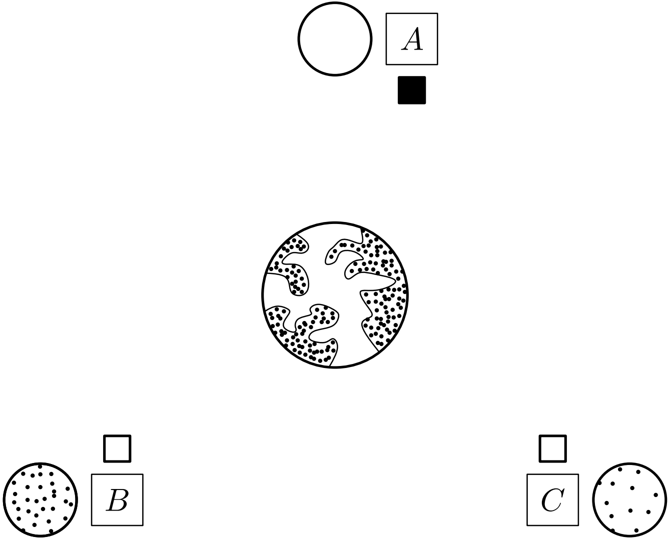

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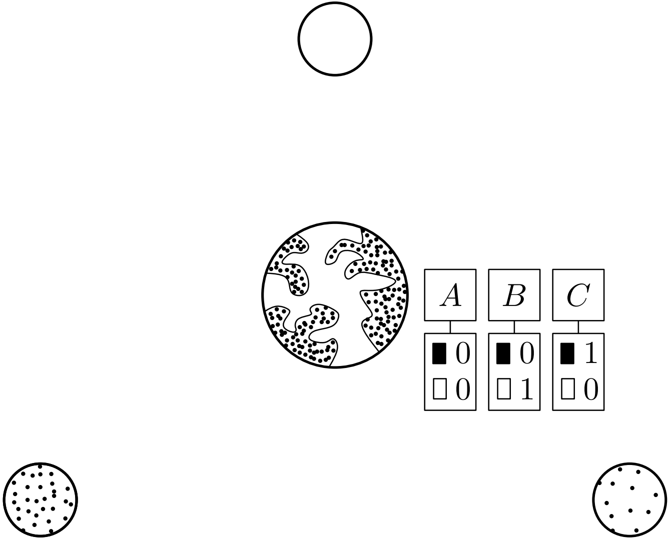

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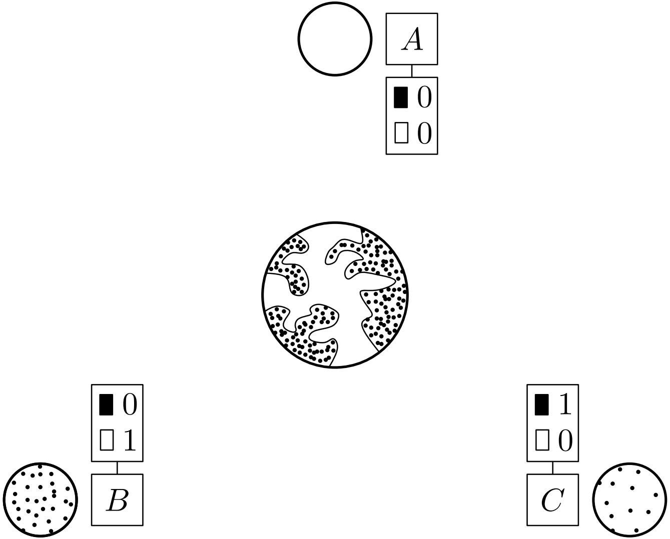

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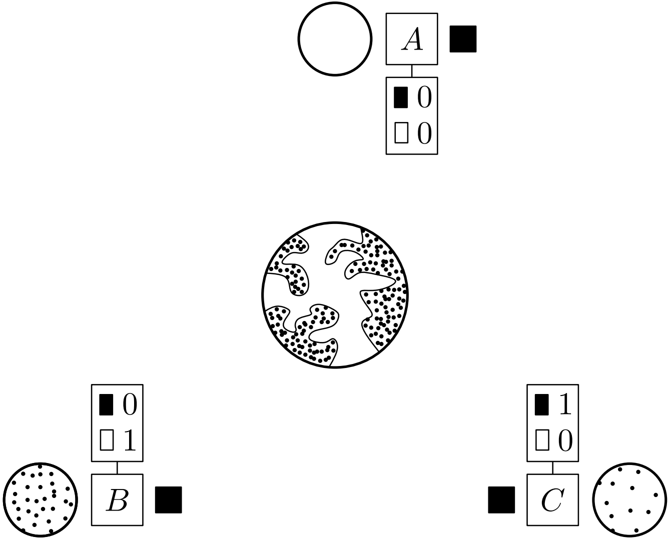

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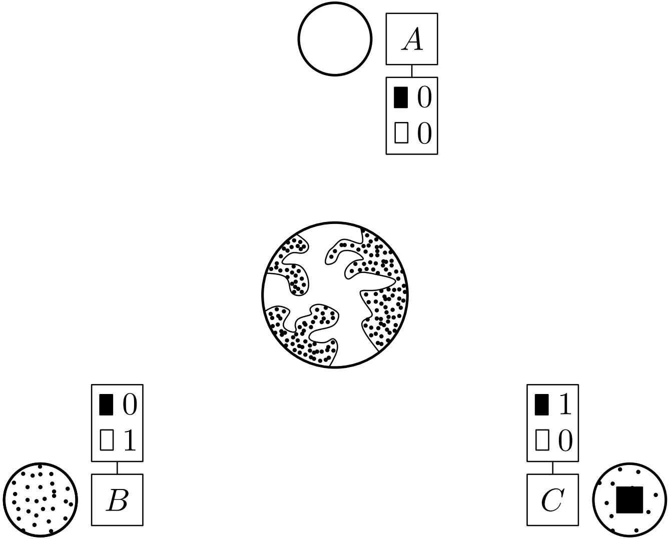

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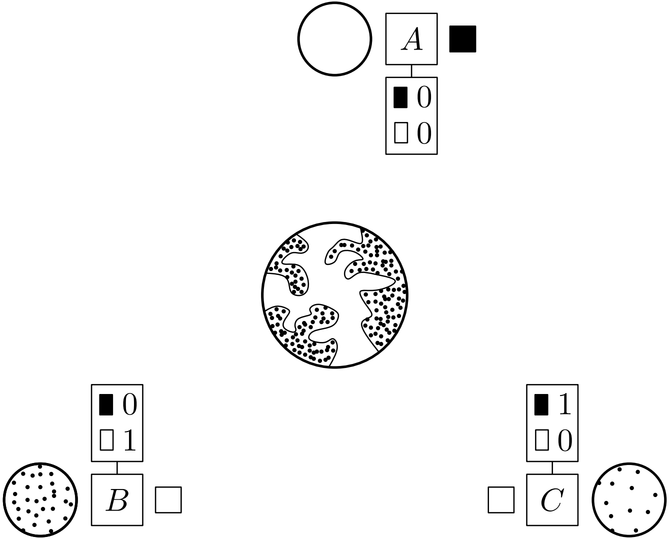

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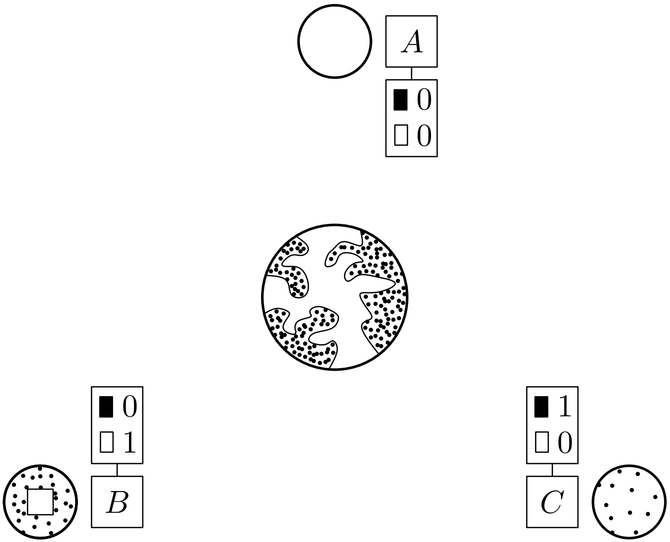
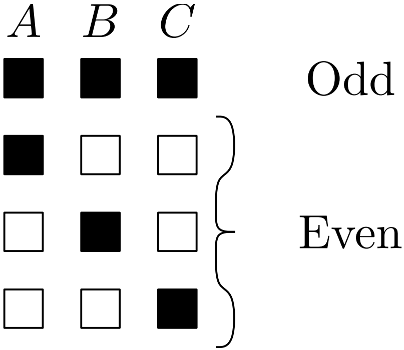


When do A, B and C Win?



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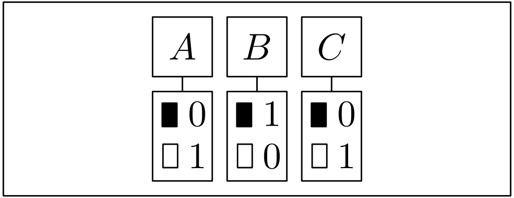

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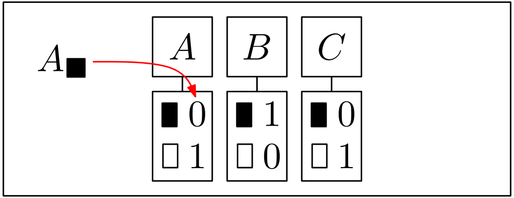

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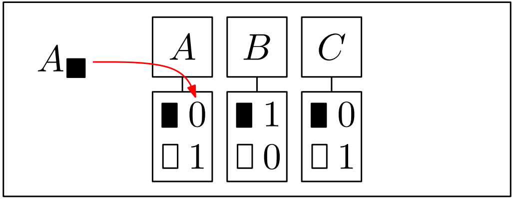

$$A_{\blacksquare} + B_{\blacksquare} + C_{\blacksquare} = \text{Odd}$$

$$A_{\blacksquare} + B_{\square} + C_{\square} = \text{Even}$$

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A, B e C Cannot Win!

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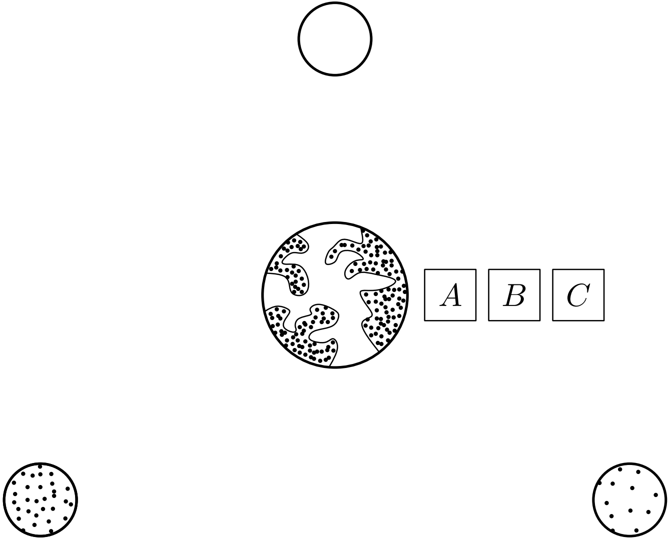

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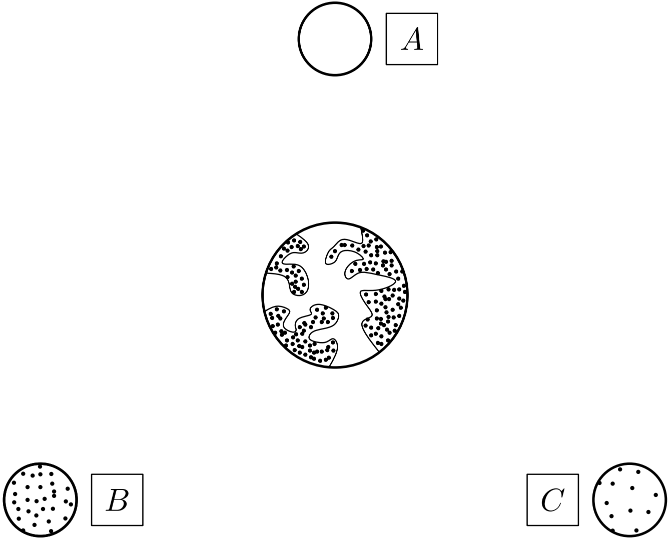

Quantum Strategies

$$\boxed{A}$$

$\square$

$$|1yz\rangle \rightarrow \sqrt{\tfrac{1}{2}}|1yz\rangle + \sqrt{\tfrac{1}{2}}|0yz\rangle$$

$$|0yz\rangle \rightarrow \sqrt{\tfrac{1}{2}}|1yz\rangle - \sqrt{-\tfrac{1}{2}}|0yz\rangle$$

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If ■ arrives…

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If $\blacksquare$ arrives...

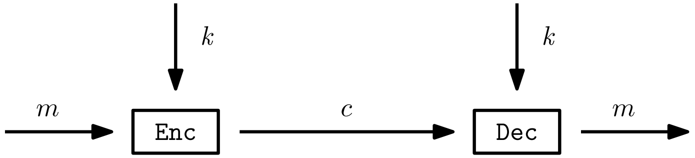
$$\sqrt{\frac{1}{2}}\,|111\rangle + \sqrt{\frac{1}{2}}\,|000\rangle \to \frac{1}{2}\,|111\rangle + \frac{1}{2}\,|011\rangle + \frac{1}{2}\,|100\rangle - \frac{1}{2}\,|000\rangle$$

Part IV

Quantum Key Distribution

Private-Key Cryptography

- Sometimes, encryption is done **symmetrically** rather than with distinct public and private keys:



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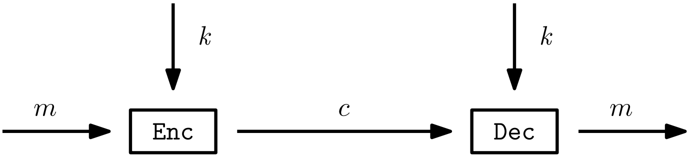

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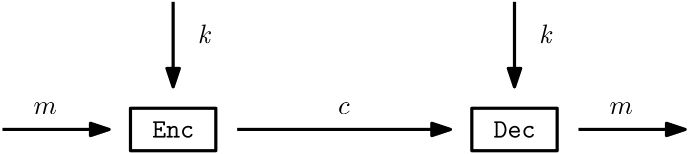

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- Symmetric encryption schemes, however, pose a challenge: how could we let the sender and receiver **share** a key k in the first place?
- Many solutions possible:
  - Using a **secure** channel.
  - Using **public-key** encryption to share the key, then switching to the private-key setting.
  - Relying on **quantum computing**

Exploiting Quantum Channels

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  - Observing some piece of data becomes a **quantum measurement**, and duplicating it is forbidden, due to the **No-Cloning Theorem**.
- Can this be exploited? Is there a way to *take advantage of that*?

Working with Two Orthonormal Bases

$$|\rightarrow\rangle = |0\rangle \qquad |\nwarrow\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

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- If the state of your system is in $\mathbf{P}$, and you measure it with respect to $\mathbf{T}$ you will end up in either $|\nwarrow\rangle$ or $|\nearrow\rangle$ each with probability $\frac{1}{2}$.

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The BB84 Protocol

Quantum
Insecure Channel

$$\text{Alice} \xrightarrow{\hspace{6em}} \text{Bob}$$

0	1	0	0	1	0	$\cdots$
P	**P**	**T**	**P**	**T**	**P**	$\cdots$
$\lvert\rightarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nwarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nearrow\rangle$	$\lvert\rightarrow\rangle$	$\cdots$

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$$\text{Alice} \xrightarrow[\text{Insecure Channel}]{\text{Quantum}} \text{Bob}$$

0	1	0	0	1	0	$\cdots$	0	1	0	0	1	0	$\cdots$
P	**P**	**T**	**P**	**T**	**P**	$\cdots$	**T**	**P**	**T**	**T**	**P**	**P**	$\cdots$
$\lvert\rightarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nwarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nearrow\rangle$	$\lvert\rightarrow\rangle$	$\cdots$	?	$\lvert\uparrow\rangle$	$\lvert\nwarrow\rangle$	?	?	$\lvert\rightarrow\rangle$	$\cdots$

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$\lvert\rightarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nwarrow\rangle$	$\lvert\uparrow\rangle$	$\lvert\nearrow\rangle$	$\lvert\rightarrow\rangle$	$\cdots$	?	$\lvert\uparrow\rangle$	$\lvert\nwarrow\rangle$	?	?	$\lvert\rightarrow\rangle$	$\cdots$

- Alice and Bob, **after** having transmitted sequence of $\{0, 1\}$, can just compare their sequences of $\{\mathbf{T}, \mathbf{P}\}$, and they can do it on an insecure channel.
- They can also **realize** somebody has seen or altered the sequence, by comparing *some* (around *half*) of the exchanged values.

Part V

Quantum Commitment

Commitment Protocols

- A **commitment protocol** allows two parties Alice and Bob to interact so as to allow Alice to commit to a chosen value, guaranteeing that:
  - The value chosen by Alice remains *hidden* until Alice decides to reveal it.
  - Neither Alice nor Bob can *change* the value chosen by Alice, i.e., the protocol is **binding**.

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- In (classical) cryptography, various forms of commitment protocols have been introduced. All of them rely on cyrptographic assumptions, like the existence of one-way functions.
- Quantum computing, given the fact that observations can possibly alter the value of data, seems like a promising approach.

Quantum Bit Commitment

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- Suppose Alice and Bob wants to commit on the value of a single bit, chosen by Alice.
- Alice chooses a value b, and prepares n qubits $|x_1\rangle, \ldots, |x_n\rangle$ as follows, after having drawn n bits at random, obtaining values $v_1, \ldots, v_n \in \{0,1\}$:
  - If $b = 1$, then $|x_i\rangle$ will be set to $|v_i\rangle$.
  - If $b = 0$, then $|x_i\rangle$ will be set to $H(|v_i\rangle)$.

  She then noted the values $v_1 \ldots, v_n$, and send $|x_1\rangle, \ldots, |x_n\rangle$ to Bob, who keeps them in a safe place.
- When the time comes for Alice to reveal the value of b, she tells Bob not only b, but also $v_1, \ldots, v_n$. Bob can then verify whether Alice is cheating by just measuring $|x_1\rangle, \ldots, |x_n\rangle$.

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- The protocol, however, **cannot be considered secure**:
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  - The qubits Bob receives do not have states of their own, being entangled with the qubits Alice keeps for herself.
  - Before revealing her choice, Alice makes a direct measurement on each of the qubits she has kept and correctly informs Bob. Crucially, however, **she can change the value of** b by applying the H gate to its qubit.

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- There is more: as proved by Mayers, unconditionally secure quantum bit commitment is impossible.