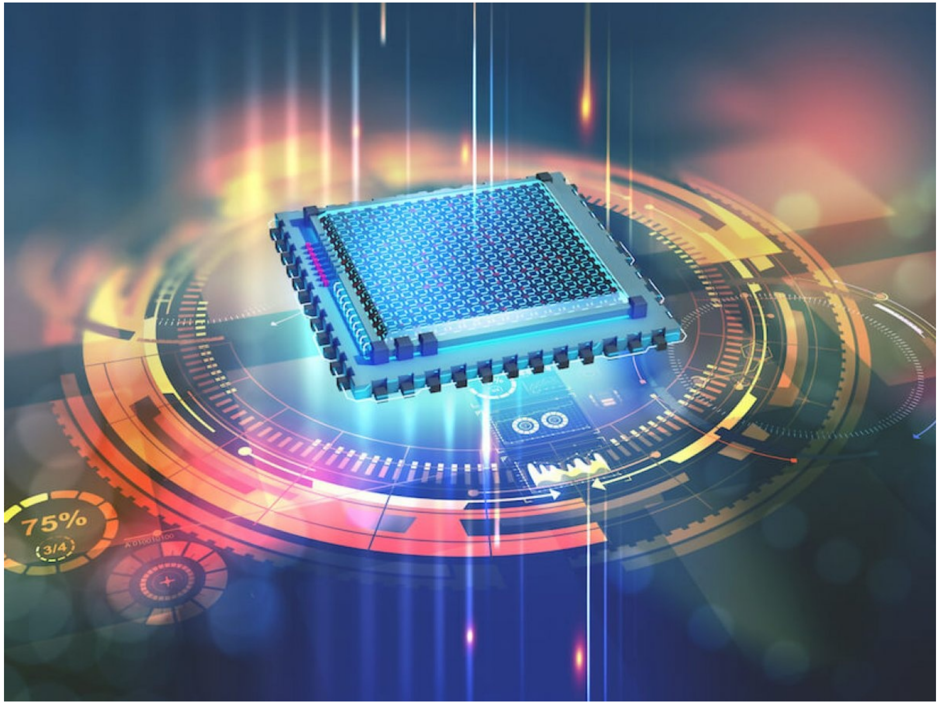
The Bloch sphere



Last class

- Bernastein-Vazirani problem
- Linear speed-up w.r.t. classical setting
- You will see in the second module examples of exponential speed-up as well

Today

- Bloch sphere
- A different representation for qubits
- Will enable us easier reasoning for next lesson
- Generalization of measurements

Polar form of complex numbers

- Remember that a qubit is represented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- We can equivalently represent coefficients in polar form

$$\alpha = a + ib$$

$$\alpha = r(\cos\phi + i\sin\phi) \quad \text{where} \quad r = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\phi = \frac{b}{a}$$

- This captures the distance from the origin and rotation from the x axis

Exponential form

- $\alpha = a + ib$

  $\alpha = r\left(\cos\phi + i\sin\phi\right) \quad \text{where} \quad r = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\phi = \frac{b}{a}$
- By using Euler formula we get

  $e^{i\phi} = \cos\phi + i\sin\phi$

  $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle = r_1 e^{i\phi_1}\left|0\right\rangle + r_2 e^{i\phi_2}\left|1\right\rangle = e^{i\phi_1}\left(r_1\left|0\right\rangle + r_2 e^{i(\phi_2 - \phi_1)}\left|1\right\rangle\right)$
- Here $e^{i\phi_1}$ is called a global phase, and has no physical relevance

Global phase

- $|\psi\rangle = e^{i\phi_1}(r_1|0\rangle + r_2 e^{i(\phi_2-\phi_1)}|1\rangle)$
- Let us compute the probability of getting $|0\rangle$
- $Pr_{|\psi\rangle}(0) = |e^{i\phi_1} r_1|^2 = |(\cos\phi_1 + i\sin\phi_1) r_1|^2 = r_1^2(\cos^2\phi_1 + \sin^2\phi_1) = r_1^2$
- The probability does not depend on ϕ_1
- If we apply a unitary to vectors differing only for the global phase then we get vectors differing only for the global phase, thanks to linearity

Simpler representation

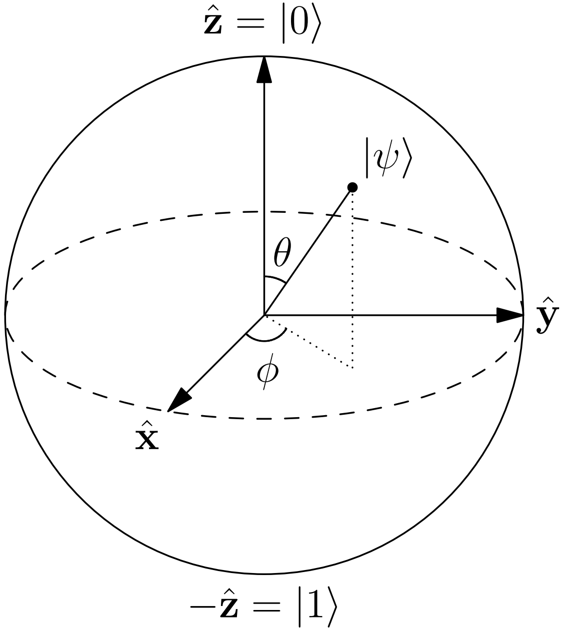
- $|\psi\rangle = e^{i\phi_1}\left(r_1|0\rangle + r_2 e^{i(\phi_2-\phi_1)}|1\rangle\right)$
- Since the global phase has no physical meaning we can set it to 0
- We also know that $r_1^2 + r_2^2 = 1$, hence we are down to only two parameters
- Parameters r_1 and r_2 are on a circle, hence we can derive r_1 and r_2

Simpler representation

- $\left|\psi\right\rangle = r_1\left|0\right\rangle + r_2 e^{i(\phi_2-\phi_1)}\left|1\right\rangle$
- Parameters r_1 and r_2 are on a circle, hence we can derive r_1 and r_2
- Set $r_1 = \cos \theta/2$, $r_2 = \sin \theta/2$ and $\phi=\phi_2-\phi_1$
- $\left|\psi\right\rangle = \cos\frac{\theta}{2}\left|0\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|1\right\rangle$

Bloch sphere

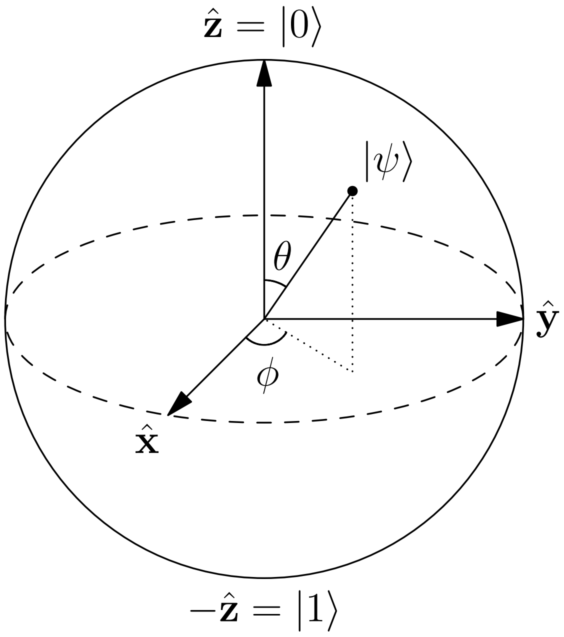
- $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
- θ and ϕ are real numbers
- $0 \leq \theta < \pi \quad 0 \leq \phi < 2\pi$
- x = sin θ cos ϕ
  y = sin θ sin ϕ
  z = cos θ



- We have a 3-dimensional representation of the states of a qubit

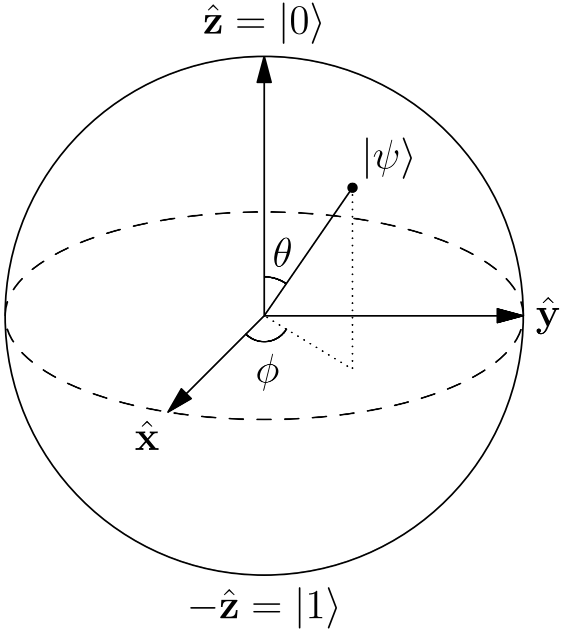
Bloch sphere

- $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
- The z axis is the probability of getting 0 or 1 upon measurement
- The x axis represents the real part of the state vector
- The y axis the imaginary part
- Axis x endpoints are |+⟩ and |-⟩
- Axis y endpoints are |i⟩=1/√2(|0⟩+i|1⟩) and |-i⟩=1/√2(|0⟩-i|1⟩)



Properties of the Bloch sphere

- $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
- Unitary transformations correspond to rotations
- Two orthogonal vectors are mapped to opposite points on the sphere
- Let us prove the latter

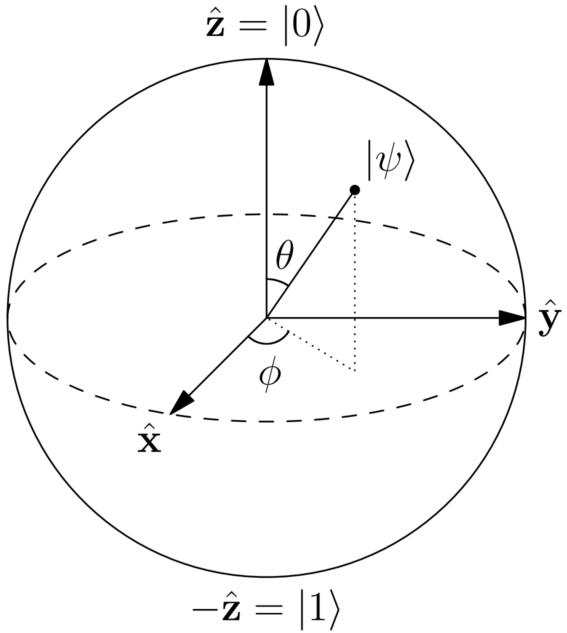


Orthogonal vectors

- $\left|\psi_1\right\rangle = \cos\frac{\theta_1}{2}\left|0\right\rangle + e^{i\phi_1}\sin\frac{\theta_1}{2}\left|1\right\rangle$

  $\left|\psi_2\right\rangle = \cos\frac{\theta_2}{2}\left|0\right\rangle + e^{i\phi_2}\sin\frac{\theta_2}{2}\left|1\right\rangle$
- Orthogonal: $\left\langle\psi_1\middle|\psi_2\right\rangle = 0$
- $\left\langle\psi_1\middle|\psi_2\right\rangle = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$

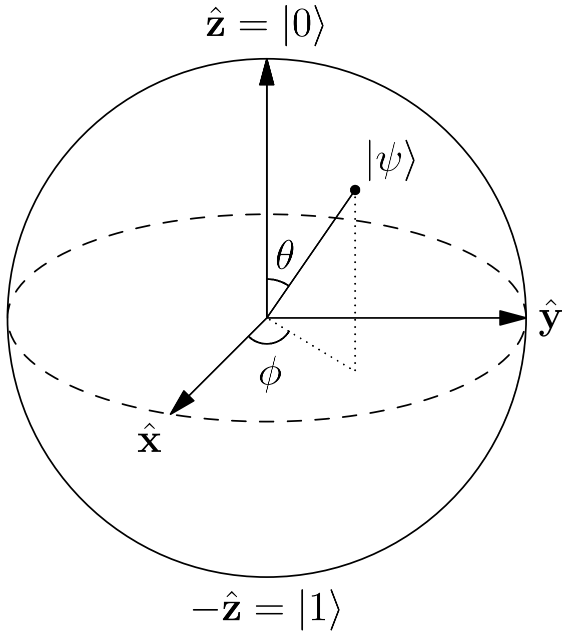
  $\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = -e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$



Orthogonal vectors (2)

- $\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = -e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$
- $e^{i(\phi_2-\phi_1)} = \cos(\phi_2-\phi_1) + i\sin(\phi_2-\phi_1)$ must be real
- Hence $\sin(\phi_2-\phi_1)=0$, thus $\phi_2-\phi_1 = n\pi$
- Looking at the absolute values:
  $\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right| = \left|\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|$

Orthogonal vectors (3)



- $$\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right| = \left|\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|$$

  $$\sin\frac{\theta_1}{2} = \cos\left(\frac{\pi-\theta_1}{2}\right)$$

  $$\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right| = \left|\cos\frac{\pi-\theta_1}{2}\cos\frac{\pi-\theta_2}{2}\right|$$

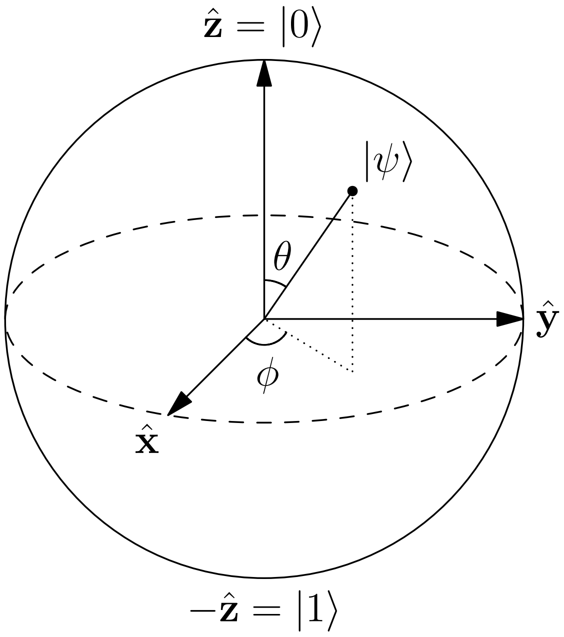
- We have $\theta_1 = \theta_2 - \pi$ using cos π-a = -cos a and cos -a = cos a
- The solution can be checked in the Bloch sphere or by turning to Cartesian coordinates

The Phase (P) gate

- The Phase maps $\alpha \mid 0\rangle + \beta \mid 1\rangle$ to $\alpha \mid 0\rangle + i\beta \mid 1\rangle$
- Exercise: compute the corresponding matrix
- Exercise: show that $P = \sqrt{Z}$
- Exercise: show that applying P does not change the probability of getting 0 or 1 after a measurement
- Remember that $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Phase shift

- $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
- Phase shifts just change the phase
- They do not change the probability of getting 0 or 1
- $R_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$
- This is also denoted as $R_z(\phi)$ since it is a rotation around axis z

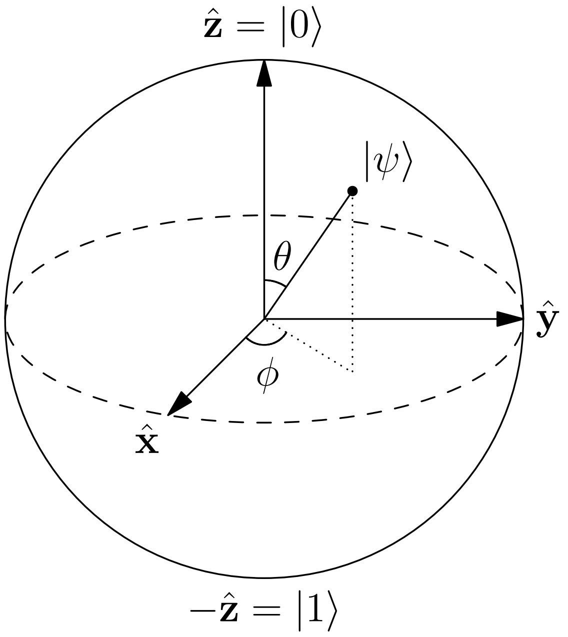


Relevant phase shifts

$$R_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$$R_\pi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

$$R_{\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = P$$



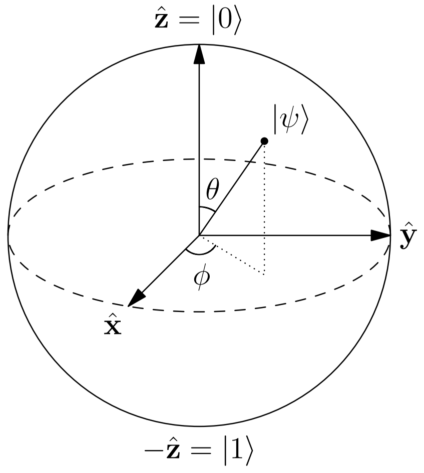
Rotations, in general

- Rotations are related to Pauli matrices

$$R_x(\phi) = \cos\frac{\phi}{2} I - i \sin\frac{\phi}{2} X \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R_y(\phi) = \cos\frac{\phi}{2} I - i \sin\frac{\phi}{2} Y \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$R_z(\phi) = \cos\frac{\phi}{2} I - i \sin\frac{\phi}{2} Z \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Projective measurements

- We have discussed measurements w.r.t. the computational basis
- In general, one can measure w.r.t. other basis, and measure multiple qubits at once
- This can be formulated in terms of observables
- (A more general class of measurements exists, but projective measurements+unitary transformations are equivalent to them)

Observables

- An observable is represented by a self-adjoint matrix (also called Hermitian) M, namely $M^{\dagger}=M$
- M can be expressed as $M=\Sigma_m \, mP_m$ where m are eigenvalues of M
- P_m are called projectors
- Projectors have the form $\{|v_i\rangle\langle v_i|\}$ where $\{v_i\}$ form an orthonormal (orthogonal vectors of length 1) basis

Eigenvalues and eigenvectors

- Whenever $Av=\lambda v$ we say that v is an eigenvalue of A and v the corresponding eigenvector
- Eigenvalues of self-adjoint matrices are real numbers

Projective measurements

- Given an observable $M=\Sigma_m mP_m$ we have that there are m possible outcomes of the measurement
- The probability p(m) is given by $\langle \psi | P_m | \psi \rangle$
- The state after the measurement is $\dfrac{P_m|\psi\rangle}{\sqrt{p(m)}}$

Computational base recovered

- Let us take M = I, and $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- $|\psi\rangle$ denotes the conjugate transpose of $\langle\psi|$
- $M = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I$
- Then the probability of getting the first outcome is

$$\langle\psi|P_m|\psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = |\alpha|^2$$

- The state collapses to

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}}{\sqrt{|\alpha|^2}} = \frac{1}{\sqrt{|\alpha|^2}} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \frac{\alpha}{\sqrt{|\alpha|^2}}|0\rangle$$

Global phase, physically irrelevant

Further example

- Let us take M = I, and $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- $M = |+\rangle\langle+| + |-\rangle\langle-| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Then the probability of getting the first outcome is

$$\langle\psi|P_m|\psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \frac{1}{2}\begin{bmatrix} \alpha+\beta \\ \alpha+\beta \end{bmatrix} =$$

$$= \frac{1}{2}\left(\alpha^*\alpha + \alpha^*\beta + \beta^*\alpha + \beta^*\beta\right)$$

- For $\alpha=0$ or $\beta=0$ you get 0.5 as expected

Further example: Bell basis

- The following vectors form an orthonormal basis
- $\frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}}, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}}$
- This cannot be expressed as successive measurements of single qubits
- Exercise: measure the state $|00\rangle$
- Exercise: measure the state $|0+\rangle$

Final notes on measurements

- We can perform multiple measurements at once
- By using tensor product we can measure qubits independently
- We can also measure qubits entanglment