GFDM - lineare Elastizität

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Kapitel 1

Elastostatik

1.1 Grundgleichungen

Gleichungen für das Verschiebungsfeld

Für das Verschiebungsfeld

$$\boldsymbol{u} = \begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{pmatrix}$$

gilt:

Auf Ω:

$$\begin{aligned} u_{,xx}\,(\lambda + 2\mu) + u_{,yy}\,\mu + v_{,xy}\,(\lambda + \mu) &= 0 \\ v_{,yy}\,(\lambda + 2\mu) + v_{,xx}\,\mu + u_{,xy}\,(\lambda + \mu) &= 0 \end{aligned}$$

Auf $\partial\Omega_d$:

$$\begin{aligned} u &= \bar{u} \\ v &= \bar{v} \end{aligned}$$

Auf $\partial\Omega_n$:

$$\begin{aligned} (\lambda + 2\mu)\, n_x\, u_{,x} + \mu\, n_y\, u_{,y} + \mu\, n_y\, v_{,x} + \lambda\, n_x\, v_{,y} &= p_x \\ (\lambda + 2\mu)\, n_y\, v_{,y} + \mu\, n_x\, v_{,x} + \mu\, n_x\, u_{,y} + \lambda\, n_y\, u_{,x} &= p_y \end{aligned}$$

Dehnungen

$$\begin{aligned}
\varepsilon_{xx} &= u_{,x} \\
\varepsilon_{xy} &= \frac{1}{2}\left(u_{,y} + v_{,x}\right) \\
\varepsilon_{yy} &= v_{,y}
\end{aligned}$$

Spannungen

$$\begin{aligned}
\sigma_{xx} &= \varepsilon_{xx}\,(\lambda + 2\mu) + \varepsilon_{yy}\,\lambda \\
\sigma_{xy} &= \varepsilon_{xy}\,2\mu \\
\sigma_{yy} &= \varepsilon_{yy}\,(\lambda + 2\mu) + \varepsilon_{xx}\,\lambda
\end{aligned}$$

1.2 Moving-Least-Square Approximation des Verschiebungsfeldes

Die Komponenten des Verschiebungsfeldes u und v werden an der Stelle $\boldsymbol{x}_i$ durch Taylorpolynome $\tilde{u}_i$ und $\tilde{v}_i$ approximiert. An jeder Stelle x_j mit $j \in S_i$ lautet ergibt sich der Wert aus der Reihenentwicklung:

$$\begin{aligned}
\tilde{u}_i(\boldsymbol{x}_j) = {} & u(\boldsymbol{x}_i) + u_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + u_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
& \frac{1}{2}\,u_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + u_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,u_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2 \\
\tilde{v}_i(\boldsymbol{x}_j) = {} & v(\boldsymbol{x}_i) + v_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + v_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
& \frac{1}{2}\,v_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + v_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,v_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2
\end{aligned}$$

Damit ergibt sich für alle x_j mit $j \in S_i$ und $|S| = n$ der numerische Fehler

$$\begin{aligned}
e^u_{ij} &= \tilde{u}_i(\boldsymbol{x}_j) - u(\boldsymbol{x}_j) \\
e^v_{ij} &= \tilde{u}_i(\boldsymbol{x}_j) - u(\boldsymbol{x}_j)
\end{aligned}$$

Mit den Definitionen:

$$\boldsymbol{e}_i^u = \begin{pmatrix} e_{i1}^u \\ \vdots \\ e_{in}^u \end{pmatrix} \in \mathbb{R}^n, \quad \boldsymbol{e}_i^v = \begin{pmatrix} e_{i1}^v \\ \vdots \\ e_{in}^v \end{pmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{a}_i^u = \begin{pmatrix} u_i \\ u_{i,x} \\ u_{i,y} \\ u_{i,xx} \\ u_{i,xy} \\ u_{i,yy} \end{pmatrix} \in \mathbb{R}^6, \quad \boldsymbol{a}_i^v = \begin{pmatrix} v_i \\ v_{i,x} \\ v_{i,y} \\ v_{i,xx} \\ v_{i,xy} \\ v_{i,yy} \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{b}_i^u = \begin{pmatrix} u_{j1} \\ \vdots \\ u_{jn} \end{pmatrix} \in \mathbb{R}^n, \quad \boldsymbol{b}_i^v = \begin{pmatrix} v_{j1} \\ \vdots \\ v_{jn} \end{pmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{D}_i = \begin{pmatrix} 1 & \Delta x_{i1} & \Delta y_{i1} & \frac{1}{2}\,\Delta x_{i1}^2 & \frac{1}{2}\,\Delta x_{i1}\,\Delta y_{i1} & \frac{1}{2}\,\Delta y_{i1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta x_{in} & \Delta y_{in} & \frac{1}{2}\,\Delta x_{in}^2 & \frac{1}{2}\,\Delta x_{in}\,\Delta y_{in} & \frac{1}{2}\,\Delta y_{in}^2 \end{pmatrix} \in \mathbb{R}^{n\times 6}$$

ergibt sich:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^v \end{pmatrix}}_{\in\mathbb{R}^{2n}} = \underbrace{\begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{pmatrix}}_{\in\mathbb{R}^{2n\times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^v \end{pmatrix}}_{\boldsymbol{a}_i\in\mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^v \end{pmatrix}}_{\in\mathbb{R}^{2n}}$$

1.3 Approximation auf Ω

Für einen Punkt $\boldsymbol{x} \in \Omega$ gilt:

$$\begin{aligned} u_{i,xx} + u_{i,yy}\,a_0 + v_{i,xy}\,a_1 &= e_{i\Omega}^u \\ v_{i,yy} + v_{i,xx}\,a_0 + u_{i,xy}\,a_1 &= e_{i\Omega}^v \end{aligned}$$

mit

$$a_0 = \frac{\mu}{\lambda + 2\mu}, \quad a_1 = \frac{\lambda+\mu}{\lambda+2\mu} \tag{1.1}$$

Dies führt zu folgenden Minimierungsproblem:

$$\min J_i = \sum_{j \in S_i} W_{ij}^2 \,(e_{ij}^u)^2 + W_{ij}^2 \,(e_{ij}^v)^2 + W_\Omega^2 \,(e_\Omega^u)^2 + W_\Omega^2 \,(e_\Omega^v)^2$$

Mit

$$\boldsymbol{e}_{i\Omega} = \begin{pmatrix} e_{i\Omega}^u \\ e_{i\Omega}^v \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{i\Omega} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$$

$$\boldsymbol{G}_\Omega = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & a_0 & 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 12}$$

folgt:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_{i\Omega} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+2}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+2 \times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^v \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_{i\Omega} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+2}}$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$\boldsymbol{W} = \mathrm{diag}\,(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega) \in \mathbb{R}^{2n+2 \times 2n+2} \tag{1.2}$$

lautet:

$$\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i \right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12 \times 2n+2}} \cdot \boldsymbol{b}_i \tag{1.3}$$

Daraus resultiert das Gleichungssystem

$$\begin{aligned} u_i &= \sum_{j \in S_i} C_i\,[0, 0:n]\; u_j + \sum_{j \in S_i} C_i\,[0, n:2n]\; v_j \\ v_i &= \sum_{j \in S_i} C_i\,[6, 0:n]\; u_j + \sum_{j \in S_i} C_i\,[6, n:2n]\; v_j \end{aligned}$$

oder

$$
\begin{aligned}
u_i\left(1 - C_i\left[0,0\right]\right) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\left[0,1:n\right]\, u_j - \sum_{j\in S_i} C_i\left[0,n:2n\right]\, v_j = 0 \\
v_i\left(1 - C_i\left[6,n\right]\right) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\left[6,n+1:2n\right]\, v_j - \sum_{j\in S_i} C_i\left[6,0:n\right]\, u_j = 0
\end{aligned}
$$

Für die partiellen Ableitungen von u und v gilt:

$$
\begin{aligned}
u_{i,x} &= \sum_{j\in S_i} C_i\left[1,0:n\right]\, u_j + \sum_{j\in S_i} C_i\left[1,n:2n\right]\, v_j \\
u_{i,y} &= \sum_{j\in S_i} C_i\left[2,0:n\right]\, u_j + \sum_{j\in S_i} C_i\left[2,n:2n\right]\, v_j \\
v_{i,x} &= \sum_{j\in S_i} C_i\left[7,0:n\right]\, u_j + \sum_{j\in S_i} C_i\left[7,n:2n\right]\, v_j \\
v_{i,y} &= \sum_{j\in S_i} C_i\left[8,0:n\right]\, u_j + \sum_{j\in S_i} C_i\left[8,n:2n\right]\, v_j
\end{aligned}
$$

1.4 Approximation auf $\partial\Omega_d$

Für einen Punkt $\boldsymbol{x} \in \partial\Omega_d$ gilt:

$$
\begin{aligned}
u_{i,xx} + u_{i,yy}\, a_0 + v_{i,xy}\, a_1 &= e^u_{i\Omega} \\
v_{i,yy} + v_{i,xx}\, a_0 + u_{i,xy}\, a_1 &= e^v_{i\Omega} \\
u - \bar{u} &= e^u_{id} \\
v - \bar{v} &= e^v_{id}
\end{aligned}
$$

Dies führt zu folgenden Minimierungsproblem:

$$
\min J_i = \sum_{j\in S_i} W_{ij}^2\left(e^u_{ij}\right)^2 + W_{ij}^2\left(e^v_{ij}\right)^2 + W_\Omega^2\left(e^u_\Omega\right)^2 + W_\Omega^2\left(e^v_\Omega\right)^2 + W_d^2\left(e^u_{id}\right)^2 + W_d^2\left(e^v_{id}\right)^2
$$

Mit

$$\boldsymbol{e}_{id} = \begin{pmatrix} e^u_{id} \\ e^v_{id} \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{id} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in \mathbb{R}^2$$

$$\boldsymbol{G}_{id} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 12}$$

folgt:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}^u_i \\ \boldsymbol{e}^v_i \\ \boldsymbol{e}_{i\Omega} \\ \boldsymbol{e}_{id} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+4}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \\ \multicolumn{2}{c}{\boldsymbol{G}_{id}} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+4\times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}^u_i \\ \boldsymbol{a}^v_i \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}^u_i \\ \boldsymbol{b}^v_i \\ \boldsymbol{b}_{i\Omega} \\ \boldsymbol{b}_{id} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+4}}$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$\boldsymbol{W} = \operatorname{diag}\left(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega, W_d, W_d\right) \in \mathbb{R}^{2n+2\times 2n+4} \tag{1.4}$$

lautet:

$$\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i\right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2\right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12\times 2n+4}} \cdot \boldsymbol{b}_i \tag{1.5}$$

Daraus resultiert das Gleichungssystem

$$\begin{aligned} u_i &= \sum_{j\in S_i} C_i\left[0, 0:n\right] u_j + \sum_{j\in S_i} C_i\left[0, n:2n\right] v_j + C_i\left[0, n+2\right] \bar{u}_i + C_i\left[0, n+3\right] \bar{v}_i \\ v_i &= \sum_{j\in S_i} C_i\left[6, 0:n\right] u_j + \sum_{j\in S_i} C_i\left[6, n:2n\right] v_j + C_i\left[6, n+2\right] \bar{u}_i + C_i\left[6, n+3\right] \bar{v}_i \end{aligned}$$

oder

$$u_i\,(1 - C_i\,[0,0]) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\,[0,1:n]\;u_j - \sum_{j\in S_i} C_i\,[0,n:2n]\;v_j =$$
$$C_i\,[0,n+2]\,\bar{u}_i + C_i\,[0,n+3]\,\bar{v}_i$$

$$v_i\,(1 - C_i\,[6,n]) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\,[6,n+1:2n]\;v_j - \sum_{j\in S_i} C_i\,[6,0\ldots n]\;u_j =$$
$$C_i\,[6,n+2]\,\bar{u}_i + C_i\,[6,n+3]\,\bar{v}_i$$

Für die partiellen Ableitungen gilt:

$$\begin{aligned}
u_{i,x} &= \sum_{j\in S_i} C_i\,[1,0:n]\;u_j + \sum_{j\in S_i} C_i\,[1,n:2n]\;v_j + C_i\,[1,n+2]\,\bar{u}_i + C_i\,[1,n+3]\,\bar{v}_i\\
u_{i,y} &= \sum_{j\in S_i} C_i\,[2,0:n]\;u_j + \sum_{j\in S_i} C_i\,[2,n:2n]\;v_j + C_i\,[2,n+2]\,\bar{u}_i + C_i\,[2,n+3]\,\bar{v}_i\\
v_{i,x} &= \sum_{j\in S_i} C_i\,[7,0:n]\;u_j + \sum_{j\in S_i} C_i\,[7,n:2n]\;v_j + C_i\,[7,n+2]\,\bar{u}_i + C_i\,[7,n+3]\,\bar{v}_i\\
v_{i,y} &= \sum_{j\in S_i} C_i\,[8,0:n]\;u_j + \sum_{j\in S_i} C_i\,[8,n:2n]\;v_j + C_i\,[8,n+2]\,\bar{u}_i + C_i\,[8,n+3]\,\bar{v}_i
\end{aligned}$$

1.5 Approximation auf $\partial\Omega_n$

Für einen Punkt $\boldsymbol{x} \in \partial\Omega_n$ gilt:

$$\begin{aligned}
u_{i,xx} + u_{i,yy}\,a_0 + v_{i,xy}\,a_1 &= e^u_{i\Omega}\\
v_{i,yy} + v_{i,xx}\,a_0 + u_{i,xy}\,a_1 &= e^v_{i\Omega}\\
n_{ix}\,u_{i,x} + a_0\,n_{iy}\,u_{i,y} + a_0\,n_{iy}\,v_{i,x} + a_2\,n_{ix}\,v_{i,y} - a_3\,p_{ix} &= e^u_{in}\\
n_{iy}\,v_{i,y} + a_0\,n_{ix}\,v_{i,x} + a_0\,n_{ix}\,u_{i,y} + a_2\,n_{iy}\,u_{i,x} - a_3\,p_{iy} &= e^v_{in}
\end{aligned}$$

mit

$$a_2 = \frac{\lambda}{\lambda + 2\mu}, \quad a_3 = \frac{1}{\lambda + 2\mu} \tag{1.6}$$

Dies führt zu folgenden Minimierungsproblem:

$$\min J_i = \sum_{j \in S_i} W_{ij}^2 \left(e_{ij}^u\right)^2 + W_{ij}^2 \left(e_{ij}^v\right)^2 + W_\Omega^2 \left(e_\Omega^u\right)^2 + W_\Omega^2 \left(e_\Omega^v\right)^2 + W_n^2 \left(e_{in}^u\right)^2 + W_n^2 \left(e_{in}^v\right)^2$$

Mit

$$\boldsymbol{e}_{id} = \begin{pmatrix} e_{in}^u \\ e_{in}^v \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{in} = \begin{pmatrix} a_3\, p_{ix} \\ a_3\, p_{iy} \end{pmatrix} \in \mathbb{R}^2$$

$$\boldsymbol{G}_{in} = \begin{pmatrix} 0 & n_x & a_0\, n_y & 0 & 0 & 0 & 0 & a_0\, n_y & a_2\, n_x & 0 & 0 & 0 \\ 0 & a_2\, n_y & a_0\, n_x & 0 & 0 & 0 & 0 & a_0\, n_x & n_y & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 12}$$

folgt:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_{i\Omega} \\ \boldsymbol{e}_{in} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+4}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \\ \multicolumn{2}{c}{\boldsymbol{G}_{in}} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+4\times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^v \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_{i\Omega} \\ \boldsymbol{b}_{in} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+4}}$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$\boldsymbol{W} = \operatorname{diag}\left(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega, W_n, W_n\right) \in \mathbb{R}^{2n+2\times 2n+4} \tag{1.7}$$

lautet:

$$\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i\right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2\right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12\times 2n+4}} \cdot \boldsymbol{b}_i \tag{1.8}$$

Daraus resultiert das Gleichungssystem

$$u_i = \sum_{j \in S_i} C_i\,[0, 0:n]\; u_j + \sum_{j \in S_i} C_i\,[0, n:2n]\; v_j + C_i\,[0, n+2]\; a_3\, p_{ix} + C_i\,[0, n+3]\; a_3\, p_{iy}$$
$$v_i = \sum_{j \in S_i} C_i\,[6, 0:n]\; u_j + \sum_{j \in S_i} C_i\,[6, n:2n]\; v_j + C_i\,[6, n+2]\; a_3\, p_{ix} + C_i\,[6, n+3]\; a_3\, p_{iy}$$

oder

$$u_i\,(1 - C_i\,[0,0]) - \sum_{\substack{j \in S_i \\ j \neq i}} C_i\,[0,1:n]\;u_j - \sum_{j \in S_i} C_i\,[0,n:2n]\;v_j =$$
$$C_i\,[0,n+2]\;a_3\,p_{ix} + C_i\,[0,n+3]\;a_3\,p_{iy}$$

$$v_i\,(1 - C_i\,[6,n]) - \sum_{\substack{j \in S_i \\ j \neq i}} C_i\,[6,n+1:2n]\;v_j - \sum_{j \in S_i} C_i\,[6,0\ldots n]\;u_j =$$
$$C_i\,[6,n+2]\;a_3\,p_{ix} + C_i\,[6,n+3]\;a_3\,p_{iy}$$

Für die partiellen Ableitungen gilt:

$$u_{i,x} = \sum_{j \in S_i} C_i\,[1,0:n]\;u_j + \sum_{j \in S_i} C_i\,[1,n:2n]\;v_j + C_i\,[1,n+2]\;a_3\,p_{ix} + C_i\,[1,n+3]\;a_3\,p_{iy}$$
$$u_{i,y} = \sum_{j \in S_i} C_i\,[2,0:n]\;u_j + \sum_{j \in S_i} C_i\,[2,n:2n]\;v_j + C_i\,[2,n+2]\;a_3\,p_{ix} + C_i\,[2,n+3]\;a_3\,p_{iy}$$
$$v_{i,x} = \sum_{j \in S_i} C_i\,[7,0:n]\;u_j + \sum_{j \in S_i} C_i\,[7,n:2n]\;v_j + C_i\,[7,n+2]\;a_3\,p_{ix} + C_i\,[7,n+3]\;a_3\,p_{iy}$$
$$v_{i,y} = \sum_{j \in S_i} C_i\,[8,0:n]\;u_j + \sum_{j \in S_i} C_i\,[8,n:2n]\;v_j + C_i\,[8,n+2]\;a_3\,p_{ix} + C_i\,[8,n+3]\;a_3\,p_{iy}$$

Kapitel 2

Elastodynamik

2.1 Grundgleichungen

Gleichungen für das Verschiebungsfeld

Für das Verschiebungsfeld

$$\boldsymbol{u} = \begin{pmatrix} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{pmatrix}$$

gilt:

Auf Ω:

$$\begin{aligned} u_{,xx}\,(\lambda + 2\mu) + u_{,yy}\,\mu + v_{,xy}\,(\lambda + \mu) &= \rho\,\ddot{u} \\ v_{,yy}\,(\lambda + 2\mu) + v_{,xx}\,\mu + u_{,xy}\,(\lambda + \mu) &= \rho\,\ddot{v} \end{aligned}$$

Auf $\partial\Omega_d$:

$$\begin{aligned} u &= \bar{u} \\ v &= \bar{v} \end{aligned}$$

Auf $\partial\Omega_n$:

$$\begin{aligned} (\lambda + 2\mu)\,n_x\,u_{,x} + \mu\,n_y\,u_{,y} + \mu\,n_y\,v_{,x} + \lambda\,n_x\,v_{,y} &= p_x \\ (\lambda + 2\mu)\,n_y\,v_{,y} + \mu\,n_x\,v_{,x} + \mu\,n_x\,u_{,y} + \lambda\,n_y\,u_{,x} &= p_y \end{aligned}$$

Dehnungen

$$
\begin{aligned}
\varepsilon_{xx} &= u_{,x} \\
\varepsilon_{xy} &= \frac{1}{2}\left(u_{,y} + v_{,x}\right) \\
\varepsilon_{yy} &= v_{,y}
\end{aligned}
$$

Spannungen

$$
\begin{aligned}
\sigma_{xx} &= \varepsilon_{xx}\,(\lambda + 2\mu) + \varepsilon_{yy}\,\lambda \\
\sigma_{xy} &= \varepsilon_{xy}\,2\mu \\
\sigma_{yy} &= \varepsilon_{yy}\,(\lambda + 2\mu) + \varepsilon_{xx}\,\lambda
\end{aligned}
$$

2.2 Approximation auf Ω

Approximation der Zeitableitungen:

$$
\begin{aligned}
\ddot{u}_i &= f^2\,\left(u_i - 2\,u_i^{k-1} + u_i^{k-2}\right) \\
\ddot{v}_i &= f^2\,\left(v_i - 2\,v_i^{k-1} + v_i^{k-2}\right)
\end{aligned}
$$

Für einen Punkt $\boldsymbol{x} \in \Omega$ gilt:

$$
\begin{aligned}
a_0\,u_i - u_{i,xx} - a_1\,u_{i,yy} - a_2\,v_{i,xy} - a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) &= e_{i\Omega}^u \\
a_0\,v_i - v_{i,yy} - a_1\,v_{i,xx} - a_2\,u_{i,xy} - a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) &= e_{i\Omega}^v
\end{aligned}
$$

mit

$$
a_0 = \frac{\rho\,f^2}{\lambda + 2\mu}, \quad a_1 = \frac{\mu}{\lambda + 2\mu}, \quad a_2 = \frac{\lambda + \mu}{\lambda + 2\mu}
$$

Dies führt zu folgenden Minimierungsproblem:

$$
\min J_i = \sum_{j \in S_i} W_{ij}^2\,(e_{ij}^u)^2 + W_{ij}^2\,(e_{ij}^v)^2 + W_{\Omega}^2\,(e_{\Omega}^u)^2 + W_{\Omega}^2\,(e_{\Omega}^v)^2
$$

Mit

$$\boldsymbol{e}_{i\Omega} = \begin{pmatrix} e^u_{i\Omega} \\ e^v_{i\Omega} \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{i\Omega} = \begin{pmatrix} a_0 \left(2\,u_i^{k-1} - u_i^{k-2}\right) \\ a_0 \left(2\,v_i^{k-1} - v_i^{k-2}\right) \end{pmatrix} \in \mathbb{R}^2$$

$$\boldsymbol{G}_\Omega = \begin{pmatrix} a_0 & 0 & 0 & 1 & 0 & a_1 & 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & a_0 & 0 & 0 & a_1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 12}$$

folgt:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}^u_i \\ \boldsymbol{e}^v_i \\ \boldsymbol{e}_{i\Omega} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+2}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+2\times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}^u_i \\ \boldsymbol{a}^v_i \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}^u_i \\ \boldsymbol{b}^v_i \\ \boldsymbol{b}_{i\Omega} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+2}}$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$\boldsymbol{W} = \operatorname{diag}\left(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega\right) \in \mathbb{R}^{2n+2\times 2n+2} \tag{2.1}$$

lautet:

$$\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i\right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2\right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12\times 2n+2}} \cdot \boldsymbol{b}_i \tag{2.2}$$

Daraus resultiert das Gleichungssystem

$$\begin{aligned} u_i = & \sum_{j\in S_i} C_i\,[0, 0:n]\; u_j + \sum_{j\in S_i} C_i\,[0, n:2n]\; v_j + \\ & C_i\,[0, n]\; a_0\;\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[0, n+1]\; a_0\;\left(2\,v_i^{k-1} - v_i^{k-2}\right) \end{aligned}$$

$$\begin{aligned} v_i = & \sum_{j\in S_i} C_i\,[6, 0:n]\; u_j + \sum_{j\in S_i} C_i\,[6, n:2n]\; v_j + \\ & C_i\,[6, n]\; a_0\;\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[6, n+1]\; a_0\;\left(2\,v_i^{k-1} - v_i^{k-2}\right) \end{aligned}$$

oder

$$u_i\left(1-C_i\left[0,0\right]\right)-\sum_{\substack{j\in S_i\\ j\neq i}}C_i\left[0,1:n\right]\,u_j-\sum_{j\in S_i}C_i\left[0,n:2n\right]\,v_j=$$
$$C_i\left[0,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[0,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)$$

$$v_i\left(1-C_i\left[6,n\right]\right)-\sum_{\substack{j\in S_i\\ j\neq i}}C_i\left[6,n+1:2n\right]\,v_j-\sum_{j\in S_i}C_i\left[6,0:n\right]\,u_j=$$
$$C_i\left[6,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[6,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)$$

Für die partiellen Ableitungen von u und v gilt:

$$\begin{aligned}u_{i,x}=&\sum_{j\in S_i}C_i\left[1,0:n\right]\,u_j+\sum_{j\in S_i}C_i\left[1,n:2n\right]\,v_j+\\&C_i\left[1,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[1,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)\end{aligned}$$

$$\begin{aligned}u_{i,y}=&\sum_{j\in S_i}C_i\left[2,0:n\right]\,u_j+\sum_{j\in S_i}C_i\left[2,n:2n\right]\,v_j+\\&C_i\left[2,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[2,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)\end{aligned}$$

$$\begin{aligned}v_{i,x}=&\sum_{j\in S_i}C_i\left[7,0:n\right]\,u_j+\sum_{j\in S_i}C_i\left[7,n:2n\right]\,v_j+\\&C_i\left[7,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[7,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)\end{aligned}$$

$$\begin{aligned}v_{i,y}=&\sum_{j\in S_i}C_i\left[8,0:n\right]\,u_j+\sum_{j\in S_i}C_i\left[8,n:2n\right]\,v_j+\\&C_i\left[8,n\right]\,a_0\,\left(2\,u_i^{k-1}-u_i^{k-2}\right)+C_i\left[8,n+1\right]\,a_0\,\left(2\,v_i^{k-1}-v_i^{k-2}\right)\end{aligned}$$

2.3 Approximation auf $\partial\Omega_d$

Für einen Punkt $\boldsymbol{x}\in\partial\Omega_d$ gilt:

$$
\begin{aligned}
a_0\, u_i - u_{i,xx} - a_1\, u_{i,yy} - a_2\, v_{i,xy} - a_0\, \left(2\, u_i^{k-1} - u_i^{k-2}\right) &= e_{i\Omega}^u \\
a_0\, v_i - v_{i,yy} - a_1\, v_{i,xx} - a_2\, u_{i,xy} - a_0\, \left(2\, v_i^{k-1} - v_i^{k-2}\right) &= e_{i\Omega}^v \\
u - \bar{u} &= e_{id}^u \\
v - \bar{v} &= e_{id}^v
\end{aligned}
$$

Dies führt zu folgenden Minimierungsproblem:

$$
\min J_i = \sum_{j \in S_i} W_{ij}^2\, (e_{ij}^u)^2 + W_{ij}^2\, (e_{ij}^v)^2 + W_\Omega^2\, (e_\Omega^u)^2 + W_\Omega^2\, (e_\Omega^v)^2 + W_d^2\, (e_{id}^u)^2 + W_d^2\, (e_{id}^v)^2
$$

Mit

$$
\boldsymbol{e}_{id} = \begin{pmatrix} e_{id}^u \\ e_{id}^v \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{id} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in \mathbb{R}^2
$$

$$
\boldsymbol{G}_{id} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 12}
$$

folgt:

$$
\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_{i\Omega} \\ \boldsymbol{e}_{id} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+4}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \\ \multicolumn{2}{c}{\boldsymbol{G}_{id}} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+4 \times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^v \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_{i\Omega} \\ \boldsymbol{b}_{id} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+4}}
$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$
\boldsymbol{W} = \operatorname{diag}\left(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega, W_d, W_d\right) \in \mathbb{R}^{2n+2 \times 2n+4} \tag{2.3}
$$

lautet:

$$
\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i\right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2\right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12 \times 2n+4}} \cdot \boldsymbol{b}_i \tag{2.4}
$$

Daraus resultiert das Gleichungssystem

$$\begin{aligned} u_i = &\sum_{j\in S_i} C_i\,[0,0:n]\;u_j + \sum_{j\in S_i} C_i\,[0,n:2n]\;v_j + \\ &C_i\,[0,n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[0,n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\ &C_i\,[0,n+2]\,\bar{u}_i + C_i\,[0,n+3]\,\bar{v}_i \end{aligned}$$

$$\begin{aligned} v_i = &\sum_{j\in S_i} C_i\,[6,0:n]\;u_j + \sum_{j\in S_i} C_i\,[6,n:2n]\;v_j + \\ &C_i\,[6,n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[6,n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\ &C_i\,[6,n+2]\,\bar{u}_i + C_i\,[6,n+3]\,\bar{v}_i \end{aligned}$$

oder

$$\begin{aligned} &u_i\,(1 - C_i\,[0,0]) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\,[0,1:n]\;u_j - \sum_{j\in S_i} C_i\,[0,n:2n]\;v_j = \\ &C_i\,[0,n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[0,n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\ &C_i\,[0,n+2]\,\bar{u}_i + C_i\,[0,n+3]\,\bar{v}_i \end{aligned}$$

$$\begin{aligned} &v_i\,(1 - C_i\,[6,n]) - \sum_{\substack{j\in S_i\\ j\neq i}} C_i\,[6,n+1:2n]\;v_j - \sum_{j\in S_i} C_i\,[6,0:n]\;u_j = \\ &C_i\,[6,n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[6,n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\ &C_i\,[6,n+2]\,\bar{u}_i + C_i\,[6,n+3]\,\bar{v}_i \end{aligned}$$

Für die partiellen Ableitungen von u und v gilt:

$$
\begin{aligned}
u_{i,x} = & \sum_{j\in S_i} C_i\,[1,0:n]\; u_j + \sum_{j\in S_i} C_i\,[1,n:2n]\; v_j + \\
& C_i\,[1,n]\; a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[1,n+1]\; a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
& C_i\,[1,n+2]\,\bar{u}_i + C_i\,[1,n+3]\,\bar{v}_i
\end{aligned}
$$

$$
\begin{aligned}
u_{i,y} = & \sum_{j\in S_i} C_i\,[2,0:n]\; u_j + \sum_{j\in S_i} C_i\,[2,n:2n]\; v_j + \\
& C_i\,[2,n]\; a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[2,n+1]\; a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
& C_i\,[2,n+2]\,\bar{u}_i + C_i\,[2,n+3]\,\bar{v}_i
\end{aligned}
$$

$$
\begin{aligned}
v_{i,x} = & \sum_{j\in S_i} C_i\,[7,0:n]\; u_j + \sum_{j\in S_i} C_i\,[7,n:2n]\; v_j + \\
& C_i\,[7,n]\; a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[7,n+1]\; a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
& C_i\,[7,n+2]\,\bar{u}_i + C_i\,[7,n+3]\,\bar{v}_i
\end{aligned}
$$

$$
\begin{aligned}
v_{i,y} = & \sum_{j\in S_i} C_i\,[8,0:n]\; u_j + \sum_{j\in S_i} C_i\,[8,n:2n]\; v_j + \\
& C_i\,[8,n]\; a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[8,n+1]\; a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
& C_i\,[8,n+2]\,\bar{u}_i + C_i\,[8,n+3]\,\bar{v}_i
\end{aligned}
$$

2.4 Approximation auf $\partial\Omega_n$

Für einen Punkt $\boldsymbol{x} \in \partial\Omega_n$ gilt:

$$
\begin{aligned}
a_0\,u_i - u_{i,xx} - a_1\,u_{i,yy} - a_2\,v_{i,xy} - a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) &= e_{i\Omega}^u \\
a_0\,v_i - v_{i,yy} - a_1\,v_{i,xx} - a_2\,u_{i,xy} - a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) &= e_{i\Omega}^v \\
n_{ix}\,u_{i,x} + a_1\,n_{iy}\,u_{i,y} + a_1\,n_{iy}\,v_{i,x} + a_3\,n_{ix}\,v_{i,y} - a_4\,p_{ix} &= e_{in}^u \\
n_{iy}\,v_{i,y} + a_1\,n_{ix}\,v_{i,x} + a_1\,n_{ix}\,u_{i,y} + a_3\,n_{iy}\,u_{i,x} - a_4\,p_{iy} &= e_{in}^v
\end{aligned}
$$

mit

$$
a_3 = \frac{\lambda}{\lambda + 2\mu}, \quad a_4 = \frac{1}{\lambda + 2\mu} \tag{2.5}
$$

Dies führt zu folgenden Minimierungsproblem:

$$\min J_i = \sum_{j \in S_i} W_{ij}^2 \, (e_{ij}^u)^2 + W_{ij}^2 \, (e_{ij}^v)^2 + W_\Omega^2 \, (e_\Omega^u)^2 + W_\Omega^2 \, (e_\Omega^v)^2 + W_n^2 \, (e_{in}^u)^2 + W_n^2 \, (e_{in}^v)^2$$

Mit

$$\boldsymbol{e}_{id} = \begin{pmatrix} e_{in}^u \\ e_{in}^v \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{b}_{in} = \begin{pmatrix} a_4 \, p_{ix} \\ a_4 \, p_{iy} \end{pmatrix} \in \mathbb{R}^2$$

$$\boldsymbol{G}_{in} = \begin{pmatrix} 0 & n_x & a_1 \, n_y & 0 & 0 & 0 & 0 & a_1 \, n_y & a_3 \, n_x & 0 & 0 & 0 \\ 0 & a_3 \, n_y & a_1 \, n_x & 0 & 0 & 0 & 0 & a_1 \, n_x & n_y & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 12}$$

folgt:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_{i\Omega} \\ \boldsymbol{e}_{in} \end{pmatrix}}_{\boldsymbol{e}_i \in \mathbb{R}^{2n+4}} = \underbrace{\begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_i \\ \multicolumn{2}{c}{\boldsymbol{G}_\Omega} \\ \multicolumn{2}{c}{\boldsymbol{G}_{in}} \end{pmatrix}}_{\boldsymbol{M} \in \mathbb{R}^{2n+4 \times 12}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^v \end{pmatrix}}_{\boldsymbol{a}_i \in \mathbb{R}^{12}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_{i\Omega} \\ \boldsymbol{b}_{in} \end{pmatrix}}_{\boldsymbol{b}_i \in \mathbb{R}^{2n+4}}$$

Die Lösung des Minimierungsproblems mit der Wichtungsmatrix

$$\boldsymbol{W} = \operatorname{diag}\left(W_{i1}, \cdots, W_{in}, W_{i1}, \cdots, W_{in}, W_\Omega, W_\Omega, W_n, W_n\right) \in \mathbb{R}^{2n+2 \times 2n+4} \tag{2.6}$$

lautet:

$$\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i \right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \right]}_{\boldsymbol{C}_i \in \mathbb{R}^{12 \times 2n+4}} \cdot \boldsymbol{b}_i \tag{2.7}$$

Daraus resultiert das Gleichungssystem

$$
\begin{aligned}
u_i = &\sum_{j \in S_i} C_i\,[0, 0:n]\;u_j + \sum_{j \in S_i} C_i\,[0, n:2n]\;v_j + \\
&C_i\,[0, n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[0, n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[0, n+2]\;a_3\,p_{ix} + C_i\,[0, n+3]\;a_3\,p_{iy}
\end{aligned}
$$

$$
\begin{aligned}
v_i = &\sum_{j \in S_i} C_i\,[6, 0:n]\;u_j + \sum_{j \in S_i} C_i\,[6, n:2n]\;v_j + \\
&C_i\,[6, n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[6, n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[6, n+2]\;a_3\,p_{ix} + C_i\,[6, n+3]\;a_3\,p_{iy}
\end{aligned}
$$

oder

$$
\begin{aligned}
&u_i\,(1 - C_i\,[0, 0]) - \sum_{\substack{j \in S_i \\ j \neq i}} C_i\,[0, 1:n]\;u_j - \sum_{j \in S_i} C_i\,[0, n:2n]\;v_j = \\
&C_i\,[0, n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[0, n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[0, n+2]\;a_3\,p_{ix} + C_i\,[0, n+3]\;a_3\,p_{iy}
\end{aligned}
$$

$$
\begin{aligned}
&v_i\,(1 - C_i\,[6, n]) - \sum_{\substack{j \in S_i \\ j \neq i}} C_i\,[6, n+1:2n]\;v_j - \sum_{j \in S_i} C_i\,[6, 0:n]\;u_j = \\
&C_i\,[6, n]\;a_0\,\left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[6, n+1]\;a_0\,\left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[6, n+2]\;a_3\,p_{ix} + C_i\,[6, n+3]\;a_3\,p_{iy}
\end{aligned}
$$

Für die partiellen Ableitungen von u und v gilt:

$$
\begin{aligned}
u_{i,x} = &\sum_{j \in S_i} C_i\,[1, 0:n]\ u_j + \sum_{j \in S_i} C_i\,[1, n:2n]\ v_j + \\
&C_i\,[1, n]\ a_0\ \left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[1, n+1]\ a_0\ \left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[1, n+2]\ a_3\,p_{ix} + C_i\,[1, n+3]\ a_3\,p_{iy}
\end{aligned}
$$

$$
\begin{aligned}
u_{i,y} = &\sum_{j \in S_i} C_i\,[2, 0:n]\ u_j + \sum_{j \in S_i} C_i\,[2, n:2n]\ v_j + \\
&C_i\,[2, n]\ a_0\ \left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[2, n+1]\ a_0\ \left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[2, n+2]\ a_3\,p_{ix} + C_i\,[2, n+3]\ a_3\,p_{iy}
\end{aligned}
$$

$$
\begin{aligned}
v_{i,x} = &\sum_{j \in S_i} C_i\,[7, 0:n]\ u_j + \sum_{j \in S_i} C_i\,[7, n:2n]\ v_j + \\
&C_i\,[7, n]\ a_0\ \left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[7, n+1]\ a_0\ \left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[7, n+2]\ a_3\,p_{ix} + C_i\,[7, n+3]\ a_3\,p_{iy}
\end{aligned}
$$

$$
\begin{aligned}
v_{i,y} = &\sum_{j \in S_i} C_i\,[8, 0:n]\ u_j + \sum_{j \in S_i} C_i\,[8, n:2n]\ v_j + \\
&C_i\,[8, n]\ a_0\ \left(2\,u_i^{k-1} - u_i^{k-2}\right) + C_i\,[8, n+1]\ a_0\ \left(2\,v_i^{k-1} - v_i^{k-2}\right) + \\
&C_i\,[8, n+2]\ a_3\,p_{ix} + C_i\,[8, n+3]\ a_3\,p_{iy}
\end{aligned}
$$

Kapitel 3

Elastodynamik im Zustandsraum

3.1 Grundgleichungen

Der Zustand des Systems wird durch den Vektor

$$\boldsymbol{u} = \begin{pmatrix} u(\boldsymbol{x}, t) \\ \dot{u}(\boldsymbol{x}, t) \\ v(\boldsymbol{x}, t) \\ \dot{v}(\boldsymbol{x}, t) \end{pmatrix}$$

beschrieben. Damit lauten die Grundgleichungen:

Auf Ω:

$$\begin{aligned} u_{,xx}\,(\lambda + 2\mu) + u_{,yy}\,\mu + v_{,xy}\,(\lambda + \mu) &= \rho\,\frac{d}{dt}\dot{u} \\ v_{,yy}\,(\lambda + 2\mu) + v_{,xx}\,\mu + u_{,xy}\,(\lambda + \mu) &= \rho\,\frac{d}{dt}\dot{v} \end{aligned}$$

Auf $\partial\Omega_d$:

$$\begin{aligned} u &= \bar{u} \\ v &= \bar{v} \end{aligned}$$

Auf $\partial\Omega_n$:

$$\begin{aligned} (\lambda + 2\mu)\,n_x\,u_{,x} + \mu\,n_y\,u_{,y} + \mu\,n_y\,v_{,x} + \lambda\,n_x\,v_{,y} &= p_x \\ (\lambda + 2\mu)\,n_y\,v_{,y} + \mu\,n_x\,v_{,x} + \mu\,n_x\,u_{,y} + \lambda\,n_y\,u_{,x} &= p_y \end{aligned}$$

Zuslich muss in jedem Punkt gelten:

$$\frac{d}{dt}u = \dot{u}$$
$$\frac{d}{dt}v = \dot{v}$$

3.2 Moving-Least-Square Approximation des Verschiebungsfeldes

Die Komponenten des Zustandsvektors u,$\dot{u}$,v,$\dot{v}$ werden an der Stelle $\boldsymbol{x}_i$ durch Taylorpolynome $\tilde{u}_i$,$\tilde{\dot{u}}_i$,$\tilde{v}_i$,$\tilde{\dot{v}}_i$ approximiert. An jeder Stelle x_j mit $j \in S_i$ ergibt sich der Wert aus der Reihenentwicklung:

$$\begin{aligned}
\tilde{u}_i(\boldsymbol{x}_j) = u(\boldsymbol{x}_i) + u_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + u_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
\frac{1}{2}\,u_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + u_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,u_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2 \\
\tilde{\dot{u}}_i(\boldsymbol{x}_j) = \dot{u}(\boldsymbol{x}_i) + \dot{u}_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + \dot{u}_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
\frac{1}{2}\,\dot{u}_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + \dot{u}_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,\dot{u}_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2 \\
\tilde{v}_i(\boldsymbol{x}_j) = v(\boldsymbol{x}_i) + v_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + v_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
\frac{1}{2}\,v_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + v_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,v_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2 \\
\tilde{\dot{v}}_i(\boldsymbol{x}_j) = \dot{v}(\boldsymbol{x}_i) + \dot{v}_{,x}(\boldsymbol{x}_i)\,(x_j - x_i) + \dot{v}_{,y}(\boldsymbol{x}_i)\,(y_j - y_i) + \\
\frac{1}{2}\,\dot{v}_{,xx}(\boldsymbol{x}_i)\,(x_j - x_i)^2 + \dot{v}_{,xy}(\boldsymbol{x}_i)\,(x_j - x_i)\,(y_j - y_i) + \frac{1}{2}\,\dot{v}_{,yy}(\boldsymbol{x}_i)\,(y_j - y_i)^2
\end{aligned}$$

Damit ergibt sich fr alle x_j mit $j \in S_i$ und $|S| = n$ der numerische Fehler

$$\begin{aligned}
e_{ij}^{u} &= \tilde{u}_i(\boldsymbol{x}_j) - u(\boldsymbol{x}_j) \\
e_{ij}^{\dot{u}} &= \tilde{\dot{u}}_i(\boldsymbol{x}_j) - \dot{u}(\boldsymbol{x}_j) \\
e_{ij}^{v} &= \tilde{u}_i(\boldsymbol{x}_j) - u(\boldsymbol{x}_j) \\
e_{ij}^{\dot{v}} &= \tilde{\dot{v}}_i(\boldsymbol{x}_j) - \dot{v}(\boldsymbol{x}_j)
\end{aligned}$$

Mit den Definitionen:

$$\boldsymbol{e}_i^u = \begin{pmatrix} e_{i1}^u \\ \vdots \\ e_{in}^u \end{pmatrix}, \quad \boldsymbol{e}_i^{\dot{u}} = \begin{pmatrix} e_{i1}^{\dot{u}} \\ \vdots \\ e_{in}^{\dot{u}} \end{pmatrix}, \quad \boldsymbol{e}_i^v = \begin{pmatrix} e_{i1}^v \\ \vdots \\ e_{in}^v \end{pmatrix}, \quad \boldsymbol{e}_i^{\dot{v}} = \begin{pmatrix} e_{i1}^{\dot{v}} \\ \vdots \\ e_{in}^{\dot{v}} \end{pmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{a}_i^u = \begin{pmatrix} u_i \\ u_{i,x} \\ u_{i,y} \\ u_{i,xx} \\ u_{i,xy} \\ u_{i,yy} \end{pmatrix}, \quad \boldsymbol{a}_i^{\dot{u}} = \begin{pmatrix} \dot{u}_i \\ \dot{u}_{i,x} \\ \dot{u}_{i,y} \\ \dot{u}_{i,xx} \\ \dot{u}_{i,xy} \\ \dot{u}_{i,yy} \end{pmatrix}, \quad \boldsymbol{a}_i^v = \begin{pmatrix} v_i \\ v_{i,x} \\ v_{i,y} \\ v_{i,xx} \\ v_{i,xy} \\ v_{i,yy} \end{pmatrix}, \quad \boldsymbol{a}_i^{\dot{v}} = \begin{pmatrix} \dot{v}_i \\ \dot{v}_{i,x} \\ \dot{v}_{i,y} \\ \dot{v}_{i,xx} \\ \dot{v}_{i,xy} \\ \dot{v}_{i,yy} \end{pmatrix} \in \mathbb{R}^6$$

$$\boldsymbol{b}_i^u = \begin{pmatrix} u_{j1} \\ \vdots \\ u_{jn} \end{pmatrix}, \quad \boldsymbol{b}_i^{\dot{u}} = \begin{pmatrix} \dot{u}_{j1} \\ \vdots \\ \dot{u}_{jn} \end{pmatrix}, \quad \boldsymbol{b}_i^v = \begin{pmatrix} v_{j1} \\ \vdots \\ v_{jn} \end{pmatrix}, \quad \boldsymbol{b}_i^{\dot{v}} = \begin{pmatrix} \dot{v}_{j1} \\ \vdots \\ \dot{v}_{jn} \end{pmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{D}_i = \begin{pmatrix} 1 & \Delta x_{i1} & \Delta y_{i1} & \frac{1}{2}\Delta x_{i1}^2 & \frac{1}{2}\Delta x_{i1}\Delta y_{i1} & \frac{1}{2}\Delta y_{i1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta x_{in} & \Delta y_{in} & \frac{1}{2}\Delta x_{in}^2 & \frac{1}{2}\Delta x_{in}\Delta y_{in} & \frac{1}{2}\Delta y_{in}^2 \end{pmatrix} \in \mathbb{R}^{n\times 6}$$

ergibt sicht:

$$\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^{\dot{u}} \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_i^{\dot{v}} \end{pmatrix}}_{\in\mathbb{R}^{4n}} = \underbrace{\begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} \end{pmatrix}}_{\in\mathbb{R}^{4n\times 24}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^{\dot{u}} \\ \boldsymbol{a}_i^v \\ \boldsymbol{a}_i^{\dot{v}} \end{pmatrix}}_{\boldsymbol{a}_i\in\mathbb{R}^{24}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^{\dot{u}} \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_i^{\dot{v}} \end{pmatrix}}_{\in\mathbb{R}^{4n}}$$

3.3 Approximation auf Ω

Approximation der Zeitableitungen:

$$
\begin{aligned}
\frac{d}{dt}u_i &= f\,\left(u_i - u_i^{k-1}\right)\\
\frac{d}{dt}\dot{u}_i &= f\,\left(\dot{u}_i - \dot{u}_i^{k-1}\right)\\
\frac{d}{dt}v_i &= f\,\left(v_i - v_i^{k-1}\right)\\
\frac{d}{dt}\dot{v}_i &= f\,\left(\dot{v}_i - \dot{v}_i^{k-1}\right)
\end{aligned}
$$

Fr einen Punkt $\boldsymbol{x} \in \Omega$ gilt:

$$
\begin{aligned}
a_0\,\dot{u}_i - u_{i,xx} - a_1\,u_{i,yy} - a_2\,v_{i,xy} - a_0\,\dot{u}_i^{k-1} &= e_{i\Omega}^{u}\\
f\,u_i - \dot{u}_i - f\,u_i^{k-1} &= e_{i\Omega}^{\dot{u}}\\
a_0\,\dot{v}_i - v_{i,xx} - a_1\,v_{i,yy} - a_2\,u_{i,xy} - a_0\,\dot{v}_i^{k-1} &= e_{i\Omega}^{v}\\
f\,v_i - \dot{v}_i - f\,v_i^{k-1} &= e_{i\Omega}^{\dot{v}}
\end{aligned}
$$

mit

$$
a_0 = \frac{\rho\,f}{\lambda + 2\mu}, \quad a_1 = \frac{\mu}{\lambda + 2\mu}, \quad a_2 = \frac{\lambda + \mu}{\lambda + 2\mu}
$$

Dies fhrt zu folgenden Minimierungsproblem:

$$
\begin{aligned}
\min J_i = \sum_{j \in S_i} & W_{ij}^2\,(e_{ij}^{u})^2 + W_{ij}^2\,(e_{ij}^{\dot{u}})^2 + (e_{ij}^{v})^2 + W_{ij}^2\,(e_{ij}^{\dot{v}})^2 +\\
& W_{\Omega}^2\,(e_{\Omega}^{u})^2 + W_{\Omega}^2\,(e_{\Omega}^{\dot{u}})^2 + W_{\Omega}^2\,(e_{\Omega}^{v})^2 + W_{\Omega}^2\,(e_{\Omega}^{\dot{v}})^2
\end{aligned}
$$

Mit

$$
\boldsymbol{e}_{i\Omega} = \begin{pmatrix} e_{i\Omega}^{u}\\ e_{i\Omega}^{\dot{u}}\\ e_{i\Omega}^{v}\\ e_{i\Omega}^{\dot{v}} \end{pmatrix} \in \mathbb{R}^4, \quad \boldsymbol{b}_{i\Omega} = \begin{pmatrix} a_0\,\dot{u}_i^{k-1}\\ f\,u_i^{k-1}\\ a_0\,\dot{v}_i^{k-1}\\ f\,v_i^{k-1} \end{pmatrix} \in \mathbb{R}^4
$$

Die Matrix $\boldsymbol{G}_{i\Omega} \in \mathbb{R}^{4\times 24}$ hat die folgenden Eintr:

$$
\begin{aligned}
G_{i\Omega}\,[0,3] &= -\,1\\
G_{i\Omega}\,[0,4] &= -\,a_2\\
G_{i\Omega}\,[0,5] &= -\,a_1\\
G_{i\Omega}\,[0,6] &= a_0\\
G_{i\Omega}\,[1,0] &= f\\
G_{i\Omega}\,[1,6] &= -\,1\\
G_{i\Omega}\,[2,15] &= -\,1\\
G_{i\Omega}\,[2,16] &= -\,a_2\\
G_{i\Omega}\,[2,17] &= -\,a_1\\
G_{i\Omega}\,[2,18] &= a_0\\
G_{i\Omega}\,[3,12] &= f\\
G_{i\Omega}\,[3,18] &= -\,1
\end{aligned}
$$

Damit setzt sich das Gesamtgleichungssystem folgenderman zusammen:

$$
\underbrace{\begin{pmatrix} \boldsymbol{e}_i^u \\ \boldsymbol{e}_i^{\dot{u}} \\ \boldsymbol{e}_i^v \\ \boldsymbol{e}_i^{\dot{v}} \\ \boldsymbol{e}_{i\Omega} \end{pmatrix}}_{\in\mathbb{R}^{4n+4}} = \underbrace{\begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D} \\ & \boldsymbol{G}_{i\Omega} & & \end{pmatrix}}_{\in\mathbb{R}^{(4n+4)\times 24}} \cdot \underbrace{\begin{pmatrix} \boldsymbol{a}_i^u \\ \boldsymbol{a}_i^{\dot{u}} \\ \boldsymbol{a}_i^v \\ \boldsymbol{a}_i^{\dot{v}} \end{pmatrix}}_{\boldsymbol{a}_i\in\mathbb{R}^{24}} - \underbrace{\begin{pmatrix} \boldsymbol{b}_i^u \\ \boldsymbol{b}_i^{\dot{u}} \\ \boldsymbol{b}_i^v \\ \boldsymbol{b}_i^{\dot{v}} \\ \boldsymbol{b}_{i\Omega} \end{pmatrix}}_{\in\mathbb{R}^{4n+4}}
$$

Die Lsung des Minimierungsproblems mit der Wichtungsmatrix

$$
\begin{aligned}
\boldsymbol{W} = \operatorname{diag}(&W_{i1},\cdots,W_{in},\\
&W_{i1},\cdots,W_{in},\\
&W_{i1},\cdots,W_{in},\\
&W_{i1},\cdots,W_{in},\\
&W_{\Omega},W_{\Omega},W_{\Omega},W_{\Omega}) \in \mathbb{R}^{(4n+4)\times(4n+4)}
\end{aligned}
$$

lautet:

$$
\boldsymbol{a}_i = \underbrace{\left[\left(\boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2 \cdot \boldsymbol{M}_i\right)^{-1} \cdot \boldsymbol{M}_i^T \cdot \boldsymbol{W}_i^2\right]}_{\boldsymbol{C}_i\in\mathbb{R}^{24\times(4n+4)}} \cdot \boldsymbol{b}_i \tag{3.1}
$$

Daraus resultiert das Gleichungssystem

Daraus resultiert das Gleichungssystem

$$
\begin{aligned}
u_i = &\sum_{j \in S_i} C_i\,[0, 0:n]\ u_j + \\
&\sum_{j \in S_i} C_i\,[0, n:2n]\ \dot{u}_j + \\
&\sum_{j \in S_i} C_i\,[0, 2n:3n]\ v_j + \\
&\sum_{j \in S_i} C_i\,[0, 3n:4n]\ \dot{v}_j +
\end{aligned}
$$