MATH 254: Honours Analysis 1

Fall 2020, Summary Notes

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Contents

1 Logic, sets, functions and other preliminaries

1.1 Mathematical Induction

Definition 1.1. (Axiom of Induction, Version 1) Let S be a subset of natural numbers ($S \subseteq \mathbb{N}$) such that the following holds:

(a) **Base case:** 1 is in S $(1 \in S)$

(b) **Induction step:** Assuming a natural number n is in S, then the natural number $n + 1$ (the successor for n) is also in S.

$\Rightarrow S = \mathbb{N}$

Definition 1.2. (Axiom of Induction, Version 2) Let ($S \subseteq \mathbb{N}$) such that the following holds:

(a) **Base case:** A natural number, m, is in S $(m \in S)$

(b) **Induction step:** Assuming a natural number n is in S, then the natural number $n + 1$ (the successor for n) is also in S.

$\Rightarrow \{m, m + 1, m + 2, \dots\} \subseteq S$

Definition 1.3. (Axiom of Induction, Version 3) Let ($S \subseteq \mathbb{N}$) such that the following holds:

(a) **Base case:** ($1 \in S$)

(b) **Induction step:** Assuming the natural numbers $1, 2, \ldots, n-1$ are in S, then the natural number n is also in S.

$\Rightarrow S = \mathbb{N}$

Definition 1.4. (Axiom of Induction, Version 4) Let ($S \subseteq \mathbb{N}$) such that the following holds:

(a) **Base case:** A natural number, m, is in S ($m \in S$)

(b) **Induction step:** Assuming $m, m+1, \ldots, m+n$ are in S, then $m+n+1$ is in S.

$\Rightarrow \{m, m+1, m+2, \ldots\} \subseteq S$

Example 1.1. (Prime Factorisation) Every natural number $n > 1$ is a product of (one or more) prime numbers.

Proof. Proof with Axiom of Induction version 4. Let S be the set of all natural numbers bigger than 1 that are products of prime numbers.
Base case: ($n = 2$)
By inspection, 2 is an element of S, since it is a prime number.
Inductive step: ($n \to n+1$)
Assume that $2, 3, \ldots, 2+n$ are in S.
If the $2+(n+1)$ number is a prime number, result is as follows. If not, it can be written as the product of two natural numbers, a and b that are bigger than 1. By assumption, $a, b \in S$. Thus, a and b can be written as the product of prime numbers. Thus, $2+(n+1)$ is also the product of prime numbers.
Hence, by the Axiom of Induction Version 4, $S = \{2, 3, \ldots\}$. □

Example 1.2. (Bernoulli's Inequality) $\forall x \geq -1,\ \forall n \in \mathbb{N} : (1+x)^n \geq 1 + n(x)$

Proof. Proof with Axiom of Induction Version 1.
Base case: ($n = 1$)

$$\begin{aligned} (1+x)^1 &= 1 + x = 1 + (1)x \\ \Rightarrow (1+x)^1 &\geq 1 + x \end{aligned}$$

Inductive step: ($n \to n+1$)
Assume that $(1+x)^n \geq 1 + nx$ for some $n \in \mathbb{N}$

$$\begin{aligned} (1+x)^n &\geq 1 + nx \\ \Rightarrow (1+x)(1+x)^n &\geq (1+x)(1+nx) \\ \Rightarrow (1+x)^{n+1} &\geq 1 + x + nx + nx^2 \\ \Rightarrow (1+x)^{n+1} &\geq 1 + (n+1)x + nx^2 \\ \Rightarrow (1+x)^{n+1} &\geq 1 + (n+1)x, \text{ since } nx^2 \geq 0 \end{aligned}$$

$$(1+x)^n \geq 1 + nx \ \forall x \geq -1 \ \forall n \in \mathbb{R}$$

□

Example 1.3. (The Well-Ordering Property of $\mathbb{N}$) Let $S \subseteq \mathbb{N}$ be any **non-empty** subset of $\mathbb{N}$. Then, S has least (or minimal) element (i.e. $\exists s \in S, \forall t \in S : s \leq t$)

Proof. **Case 1: S is finite.** Proof by Axiom of Induction Version 2.
Base case: ($n = 1$), where n is the cardinality of the set S.
$S = \{a_1\}$ for some $a_1 \in \mathbb{N}$. Since there is only one element of the set, it has to be the least element. Thus, a_1 is the least element of S.
Inductive step: ($n \to n + 1$)
Assume that every subset of $\mathbb{N}$ with n elements has a least element.
Let S have $n + 1$ elements, $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$. Let S' have n elements, $\{a_1, a_2, \ldots, a_n\}$.
Since S' has n elements, it has a least element. We can call this least element a_i.
Case 1: If $a_i < a_{n+1} \Rightarrow a_i$ is the least element of S.
Case 2: If $a_{n+1} < a_i \Rightarrow a_{n+1}$ is the least element of S. $\Rightarrow S$ has a least element, which implies that every finite subset of $\mathbb{N}$ has a least element.
Case 2: S is infinite, especially $S \neq \emptyset$.
Let $t_0 \in S$. Let $S' := S \cap \{1, 2, 3, \ldots, t_0\}$.
Let $t \in S \backslash S'$. $\Rightarrow t > t_0$. Since S' is finite, it has a least element, s.
$\Rightarrow s \leq t \ \forall t \in S'$. $\Rightarrow s \leq t_0 \leq t \ \forall t \in S \backslash S'$. $\Rightarrow s \leq t \ \forall t \in S$.
Thus, s is the least element of S. □

Example 1.4. Prove that $n^3 + 2n$ is divisible by 3 $\forall n \in \mathbb{N}$.

Proof. Let S be the set of all $n \in \mathbb{N}$ such that $n^3 + 2n$ is divisible by 3.
Base case: ($n = 1$)
$1 + 2 = 3$ which is divisible by 3, so $1 \in S$.
Inductive step: ($n \to n + 1$)
Assume that $n^3 + 2n$ is divisible by 3 for some $n \in \mathbb{N}$. Then,

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= (n^3 + 2n) + 3n^2 + 3n + 3 \\ &= (n^3 + 2n) + 3(n^2 + n + 1)\end{aligned}$$

Since we know that n^3+2n is divisible by 3, and the other terms are divisible by 3, so $n+1 \in S$.
$\Rightarrow S = \mathbb{N}$, and this completes the proof. □

Question 1.1. Conjecture a formula for the sum

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

and prove your conjecture using the Axiom of Induction.

Proof. The formula for the sum of $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$ is $\frac{n}{2n+1}$. We prove this with the Axiom of Induction. Let S be the set of all $n \in \mathbb{N}$ such that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.
Base case: ($n = 1$)
When $n = 1$, $\frac{1}{1\cdot 3} = \frac{1}{3} = \frac{1}{2(1)+1}$. $\Rightarrow 1 \in S$.
Inductive step: ($n \to n + 1$)

Assume that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for some $n\in\mathbb{N}$. Then,

$$\begin{aligned}\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)} &= \frac{n}{(2n+1)}+\frac{1}{(2n+1)(2n+3)}\\ &= \frac{n(2n+3)+1}{(2n+1)(2n+3)}\\ &= \frac{n(2n+1)+2n+1}{(2n+1)(2n+3)}\\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)}\\ &= \frac{n+1}{2(n+1)+1}\end{aligned}$$

$\Rightarrow n+1\in S\Rightarrow S=\mathbb{N}$. □

Question 1.2. Prove by induction that

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n\text{ nested square roots}}=2\cos\left(\frac{\pi}{2^{n+1}}\right) \tag{1}$$

for all $n\in\mathbb{N}$.

Proof. We prove this with the Axiom of Induction. Let S be the set of all $n\in\mathbb{N}$ such that $\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n\text{ nested square roots}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$.

Base case: $(n=1)$
When $n=1$, $2\cos(\frac{\pi}{2^2})=2\cos(\frac{\pi}{4})=\sqrt{2}\Rightarrow 1\in S$.
Inductive step: $(n\to n+1)$
Assume that $\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n\text{ nested square roots}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$ for some $n\in\mathbb{N}$. Then,

$$\begin{aligned}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n+1\text{ nested square roots}} &= \sqrt{2+2\cos\left(\frac{\pi}{2^{n+1}}\right)}\\ &= 2\sqrt{\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2^{n+1}}\right)}\\ &= 2\cos\left(\frac{\pi}{2^{n+2}}\right)\end{aligned}$$

$\Rightarrow n+1\in S\Rightarrow S=\mathbb{N}$. □

Question 1.3. Recall that the binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k}=\frac{n!}{k!(n-k)!}$. Prove by induction on n that $\sum_{k=0}^{n}\binom{n}{k}=2^n$ for all $n\in\mathbb{N}_0$. You may use, without proof, the well-known identity $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ for all $n\in\mathbb{N}_0$ and $1\le k\le n$.

Proof. We prove this with the Axiom of Induction. Let S be the set of all $n \in \mathbb{N}$ such that $\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$.
Base case: $(n = 0)$
When $n = 0$, $\sum_{k=0}^{n} \binom{n}{k} = \binom{0}{0} = 1 = 2^0 \Rightarrow 1 \in S$.
Inductive step: $(n \to n + 1)$
Assume that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for some $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \binom{n+1}{0} + \binom{n+1}{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} \\
&= 2 + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=1}^{n} \binom{n}{k-1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} + \binom{n}{n} + \sum_{k=0}^{n-1} \binom{n}{k} \\
&= 2 \sum_{k=0}^{n} \binom{n}{k} \\
&= 2(2^n) \\
&= 2^{n+1}
\end{aligned}$$

$\Rightarrow n + 1 \in S \Rightarrow S = \mathbb{N}_0$. $\square$

1.2 Construction of Real Numbers

From $\mathbb{N}$ to $\mathbb{Z}$, we seek to find a number system that is closed under subtraction.
From $\mathbb{Z}$ to $\mathbb{Q}$, we seek to find a number system that is closed under division (the inverse of multiplication).
Within $\mathbb{Q}$, we can do operations of addition, multiplication and comparison. These operations have the following properties:

1. Property of distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$
2. Property of associativity: $x + (y + z) = (x + y) + z$
3. Property of comparison: if $x < y$ and $y < z$ then $x < z$

The rationals ($\mathbb{Q}$), together with "+", "·", and relation "<" form a mathematical structure that is called an ordered field. "Ordered" gives the relation of comparison, and "field" is the properties of addition and multiplication.
From $\mathbb{Q}$ to $\mathbb{R}$, we seek to find a number system that fulfills the Axiom of Completeness, since there exists numbers that are not rational. This makes $\mathbb{Q}$ insufficient for analysis and geometry. Analysis is based on the notion of length, so rational numbers cannot be used for the development of analysis.

Example 1.5. (Taking square root of a non-perfect square) For any $k \in \mathbb{Z}$, if k is not a perfect square, then $\sqrt{k}$ is irrational.

Proof. This is proven by contradiction. Assume that $\sqrt{k}$ is rational, thus $\exists a, b \in \mathbb{N}$ s.t.

$\mathrm{GCD}(a, b) = 1$ and $\sqrt{k} = \frac{a}{b}$.

$$\begin{aligned}
\sqrt{k}b &= a \\
kb^2 &= a^2 \Rightarrow a^2 \text{ is divisible by k} \Rightarrow a \text{ is divisible by k} \\
&\Rightarrow \exists c \in \mathbb{N} : a = kc \\
&\Rightarrow kb^2 = a^2 = (kc)^2 = k^2c^2 \\
&\Rightarrow b^2 = kc^2 \Rightarrow b^2 \text{ is divisible by k} \Rightarrow b \text{ is divisible by k}
\end{aligned}$$

$\Rightarrow$ Both a and b are divisible by k, which is a contradiction, so $\sqrt{k}$ is irrational. □

1.3 Notation and Preliminaries

Definition 1.5. (Index Sets) Let J be some set. Then we may use J to index a collection of other sets, A_α for $\alpha \in J$.
For example, if $J = \{a, b, c\}$, our sets could be $A_a = \{1\}, A_b = \{1, 2, 3\}, A_c = \{1, 3, 6\}$.
We may write arbitrary unions and intersections over an index set J, denoted by $\bigcup_{\alpha \in J} A_\alpha$ and $\bigcap_{\alpha \in J} A_\alpha$ respectively. For the example above, $\bigcup_{\alpha \in J} A_\alpha = \{1, 2, 3, 6\}$ and $\bigcap_{\alpha \in J} A_\alpha = \{1\}$.
We can use $\mathbb{N}$ to index our sets. Using $J = \mathbb{N}$, we can set $A_n = \{1, 2, 3, \ldots, n\}$. Then, $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} A_n = \{1\}$. We can have $J = \mathbb{R}$. We might define $A_\alpha = \{x \in \mathbb{R} : x > \alpha\}$.

$$\begin{aligned}
\bigcap_{n=1}^{\infty} = \bigcap_{n=\mathbb{N}} &= \{x : x \in A_n, \forall n \in \mathbb{N}\} \\
\bigcup_{n=1}^{\infty} = \bigcup_{n=\mathbb{N}} &= \{x : \exists n \in \mathbb{N}, x \in A_n\}
\end{aligned}$$

Definition 1.6. If $A \subseteq X$, then the complement of A relative to X, denoted A^c, is:

$$A^c = \{x : x \in X, x \notin A\}$$

It is often implicitly understood that $x \in X$, due to how X is defined. A simpler definition is thus $A^c = \{x : x \notin A\}$. $(A^c)^c = A$, since $A^c = B^c \iff A = B$.

Proposition 1.1. (De Morgan's Laws)

$$\left(\bigcap_{x=1}^{\infty} A_n\right)^c = \bigcup_{x=1}^{\infty} {A_n}^c \tag{1}$$

$$\left(\bigcup_{x=1}^{\infty} A_n\right)^c = \bigcap_{x=1}^{\infty} {A_n}^c \tag{2}$$

Proof.

$$\begin{aligned}
\left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \{x : \exists n \in \mathbb{N}, x \notin A_n\} \\
&= \bigcup_{n=1}^{\infty} \{x \notin A_n\} \\
&= \bigcup_{n=1}^{\infty} {A_n}^c \\
\left(\bigcup_{n=1}^{\infty} A_n\right)^c &= \{x : x \notin A_n, \forall n \in \mathbb{N}\} \\
&= \bigcap_{n=1}^{\infty} \{x : x \notin A_n\} \\
&= \bigcap_{n=1}^{\infty} {A_n}^c
\end{aligned}$$

□

We can also prove (1) using (2).

Proof.

$$\begin{aligned}
\left(\bigcup_{n=1}^{\infty} {A_n}^c\right)^c &= \bigcap_{n=1}^{\infty} ({A_n}^c)^c && \text{by (2)} \\
&= \bigcap_{n=1}^{\infty} A_n && \text{since } ({A_n}^c)^c = A_n \\
\Rightarrow \left(\bigcap_{n=1}^{\infty} A_n\right)^c &= \left(\left(\bigcup_{n=1}^{\infty} {A_n}^c\right)^c\right)^c \\
&= \bigcup_{n=1}^{\infty} {A_n}^c
\end{aligned}$$

□

Exercise 1.1.

$$\overline{E} \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} E_m \quad \underline{E} \coloneqq \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} E_m$$

Prove that:
(1) $\overline{E} = \{x : x \in E_m \text{ for infinitely many } m\}$
(2) $\underline{E} = \{x : x \notin E_m \text{ for only finitely many } m\}$

Proof. (1) $X = \{x : x \in E_m \text{ for infinitely many } m \in \mathbb{N}\}$. First, prove that (i) $\overline{E} \subseteq X$.
Proof by contradiction. $x \in \overline{E}$. If x belongs to E_m for only finitely many m, let k be the greatest value of m where $x \in E_m$. Then, let $c = k + 1$.
If $\bigcap_{n=c}^{\infty} \bigcup_{m>n} E_m = \emptyset$, then $\bigcap_{n=1}^{\infty} \bigcup_{m>n} E_m = \emptyset$, which is a contradiction. Thus, $\overline{E} \subseteq X$.
Alternative proof: Let $x \in \overline{E}$. Then, $x \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} E_m \Rightarrow x \in \bigcup_{m \geq n} E_m \forall n \in \mathbb{N}$.
For $n = 1$: $x \in \bigcup_{m \geq 1} E_m \Rightarrow \exists E_{m_1}, m_1 \geq 1$ s.t. $x \in E_{m_1}$
For $n = m_1 + 1$: $x \in \bigcup_{m \geq M_1+1} E_m \Rightarrow \exists E_{m_2}, m_2 \geq m_1 + 1$ s.t. $x \in E_{m_2}$

Continuing in this way, we construct $E_{m_{k+1}}$ from E_{m_k} by noticing that $x \in \bigcup_{m \geq m_k+1}$ and $\exists E_{m_{k+1}}, m_{k+1} \geq m_k + 1$, s.t. $x \in E_{m_{k+1}}$.
Thus, we obtain a sequence $E_{m_1}, E_{m_2}, \ldots, E_{m_k}, E_{m_{k+1}}, \ldots$ each containing x. So, $x \in E_m$ for infinitely many $m \in \mathbb{N}$. $\Rightarrow x \in X$.
Next, we prove that (ii) $X \subseteq \overline{E}$.
Let $x \in X$. Then, x belongs in infinitely many E_m's. Let now $S = \{m \in \mathbb{N} : x \in E_m\}$. By assumption, S is infinite, so for any $n \in \mathbb{N}$, we cannot have $S \subseteq \{1, 2, \ldots, n\}$.
$\Rightarrow \exists k \in S$ s.t. $k \notin \{1, 2, \ldots, n\} \Rightarrow k > n \Rightarrow x \in E_k \subseteq \bigcup_{m \geq n} E_m$.
Since $n \in \mathbb{N}$ was arbitrary, we can infer that $x \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} E_m = \overline{E}$.
Since $\overline{E} \subseteq X$ and $X \subseteq \overline{E}$, $X = \overline{E}$, and this completes the proof of (1).
(2) $X = \{x : x \notin E_m$ for only finitely many $m \in \mathbb{N}\}$ First, we prove (i) $\underline{E} \in X$.
Let $x \in \underline{E}$. $\Rightarrow x \in \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} E_m \Rightarrow \exists k \in \mathbb{N}$ s.t. $\forall m \geq k, x \in E_m$.
Let $S = \{m \in \mathbb{N} : x \notin E_m$. The largest this set can be is $S = \{1, 2, \ldots, k\}$, which makes S finite. Thus, $\underline{E} \in X$.
Next, we prove (ii) $X \in \underline{E}$.
Let $x \in X$. $S = \{m \in \mathbb{N} : x \notin E_m\}$. Since S is finite, it has to have a greatest element, b. $\Rightarrow \forall m \geq b+1, x \in E_m \Rightarrow x \in \bigcap_{m \geq b+1} E_m \Rightarrow x \in \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} E_m \Rightarrow x \in \underline{E}$
Since $\underline{E} \subseteq X$ and $X \subseteq \underline{E}$, $\underline{E} = X$. This completes the proof of (2). ∎

Question 1.4. Let $E_n, n = 1, 2, \ldots$ be an infinite sequence of sets. Let

$$\overline{E} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m, \quad \underline{E} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m.$$

Prove that

$$\bigcap_{n=1}^{\infty} E_n \subseteq \underline{E} \subseteq \overline{E} \subseteq \bigcup_{n=1}^{\infty} E_n.$$

Proof. First, prove that $\bigcap_{n=1}^{\infty} E_n \subseteq \underline{E}$.

$$\begin{aligned} x \in \bigcap_{n=1}^{\infty} E_n &\Rightarrow \forall n \in \mathbb{N}, x \in E_n \\ &\Rightarrow \exists n \in \mathbb{N} \text{ s.t. } \forall m \geq n, x \in E_m \text{ (setting } n = 1) \\ &\Rightarrow x \in \underline{E} \end{aligned}$$

Next, prove that $\underline{E} \subseteq \overline{E}$

$$x \in \underline{E} \Rightarrow \exists n_1 \in \mathbb{N} \text{ s.t. } \forall m \geq n_1, x \in E_m$$

$\forall n \in \mathbb{N}$, if $n < n_1$, then set $k = n_1$, since $x \in E_{n_1}$. If $n \geq n_1$, then set $k = n$, since $x \in E_n$ when $n \geq n_1$.

$$\begin{aligned} &\Rightarrow \forall n \in \mathbb{N}, \exists k \geq n \text{ s.t. } x \in E_k \\ &\Rightarrow x \in \overline{E} \end{aligned}$$

Finally, prove that $\overline{E} \subseteq \bigcup_{n=1}^{\infty} E_n$.

$$\begin{aligned} x \in \overline{E} &\Rightarrow \forall n \in \mathbb{N}, \exists m \geq n \text{ s.t. } x \in E_m \\ &\Rightarrow \exists n \in \mathbb{N} \text{ s.t. } x \in E_n \\ &\Rightarrow x \in \bigcup_{n=1}^{\infty} E_n \end{aligned}$$

This proves the statement. ∎

1.4 Elementary Logic

Definition 1.7. (A statement) A statement is any expression that is either True or False.

Definition 1.8. (Ex Falso Quodlibet) From a false statement, anything can follow.

1.5 Functions

Definition 1.9. (Function) Given two sets A and B, a function f from A to B, denoted $f : A \mapsto B$, is a rule that associates **each element** x of A to a **unique** element $f(x)$ of the set B. In analysis, this is much more general than functions merely described by formulas

Example 1.6. (Dirichlet Function) $f : \mathbb{R} \mapsto \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$$

Definition 1.10. (Domain) If $f : A \mapsto B$, the set A is called the domain of the function f.

Definition 1.11. (Range) The set $f(A) = \{f(x) : x \in A\}$ is called range of f, and is the set of all images of elements of set A. $f(A) \subseteq B$, but often $f(A) \neq B$.

Definition 1.12. (Direct Image) If $C \subseteq A$, then the set $f(C) = \{f(x) : x \in C\}$ is called the direct image of C by f.

Proposition 1.2. Let $f : A \mapsto B$ and let $C_1, C_2 \subseteq A$. Then, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

Proof. We will show that

$$f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2) \tag{1}$$

$$f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2) \tag{2}$$

Together, they imply the result. First, we prove (1):

$$\begin{aligned} y \in f(C_1 \cup C_2) &\Rightarrow \exists x \in C_1 \cup C_2 \text{ s.t. } y = f(x). \\ &\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ or } \exists x \in C_2 \text{ s.t. } y = f(x) \\ &\Rightarrow y \in f(C_1) \text{ or } y \in f(C_2) \\ &\Rightarrow y \in f(C_1) \cup f(C_2) \end{aligned}$$

Next, we prove (2):

$$\begin{aligned} y \in f(C_1) \cup f(C_2) &\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ or } \exists x \in C_2 \text{ s.t. } y = f(x) \\ &\Rightarrow \exists x \in C_1 \cup C_2 \text{ s.t. } y = f(x) \\ &\Rightarrow y \in f(C_1 \cup C_2) \end{aligned}$$

This proves the statement. ∎

Example 1.7. Let $f : A \mapsto B$ and let $C_n, n = 1, 2, \ldots$ be a sequence of subsets in A. Then,

$$f\left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} f(C_n)$$

Proof. We prove both directions like the above.

$$\begin{aligned}
y \in f\left(\bigcup_{n=1}^{\infty} C_n\right) &\Rightarrow \exists x \in \bigcup_{n=1}^{\infty} C_n \text{ s.t. } y = f(x). \\
&\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ or } \exists x \in C_2 \text{ s.t. } y = f(x) \text{ or...} \\
&\Rightarrow y \in f(C_1) \text{ or } y \in f(C_2) \text{ or...} \\
&\Rightarrow y \in \bigcup_{n=1}^{\infty} f(C_n)
\end{aligned}$$

$$\begin{aligned}
y \in \bigcup_{n=1}^{\infty} f(C_n) &\Rightarrow y \in f(C_1) \text{ or } y \in f(C_2) \text{ or...} \\
&\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ or } \exists x \in C_2 \text{ s.t. } y = f(x) \text{ or...} \\
&\Rightarrow \exists x \in \bigcup_{n=1}^{\infty} C_n \text{ s.t. } y = f(x) \\
&\Rightarrow y \in f\left(\bigcup_{n=1}^{\infty} C_n\right)
\end{aligned}$$

This proves the statement. □

Proposition 1.3. Let $f : A \mapsto B$ and let $C_1, C_2 \subseteq A$. Then, $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

Proof.

$$\begin{aligned}
y \in f(C_1 \cap C_2) &\Rightarrow \exists x \in C_1 \cap C_2 \text{ s.t. } y = f(x) \\
&\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ and } x \in C_2 \text{ s.t. } y = f(x) \\
&\Rightarrow y \in f(C_1) \cap f(C_2)
\end{aligned}$$

This proves the statement. □

It is possible that $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$, and there are many examples of this.

Example 1.8. ($f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$) Take $A = \mathbb{R}, B = \mathbb{R}, C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$. $C_1 \cap C_2 = \{2\pi\}$. Take $f(x) = \sin(x)$.

$$\begin{gathered}
f(C_1 \cap C_2) = \{0\} \\
f(C_1) = [-1, 1]; f(C_2) = [-1, 1] \\
f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)
\end{gathered}$$

This is true for any function that visits the same range twice.

Example 1.9. Let $f : A \mapsto B$ and let $C_n, n = 1, 2, \ldots$ be a sequence of subsets in A. Then,

$$f\left(\bigcap_{n=1}^{\infty} C_n\right) \subset \bigcap_{n=1}^{\infty} f(C_n)$$

Proof.

$$\begin{aligned}
y \in f\left(\bigcap_{n=1}^{\infty} C_n\right) &\Rightarrow \exists x \in \bigcap_{n=1}^{\infty} C_n \text{ s.t. } y = f(x) \\
&\Rightarrow \exists x \in C_1 \text{ s.t. } y = f(x) \text{ and } \exists x \in C_2 \text{ s.t. } y = f(x) \text{ and...} \\
&\Rightarrow y \in f(C_1) \text{ and } y \in f(C_2) \text{ and...} \\
&\Rightarrow y \in \bigcap_{n=1}^{\infty} f(C_n)
\end{aligned}$$

This proves the statement. □

Definition 1.13. (Inverse Image) Let $f : A \mapsto B$ and let $D \subseteq B$. The inverse image of the set D under function f is denoted by $f^{-1}(D)$ and is defined by:

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

Proposition 1.4. Let $f : A \mapsto B$ and let $D_1, D_2 \subseteq B$. Then,

$$f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2) \tag{1}$$

$$f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2) \tag{2}$$

Proof.

$$\begin{aligned} x \in f^{-1}(D_1 \cup D_2) &\iff \exists y \in D_1 \cup D_2 \text{ s.t. } y = f(x) \\ &\iff \exists y \in D_1 \text{ s.t. } y = f(x) \text{ or } \exists y \in D_2 \text{ s.t. } y = f(x) \\ &\iff x \in f^{-1}(D_1) \text{ or } x \in f^{-1}(D_2) \\ &\iff x \in f^{-1}(D_1) \cup f^{-1}(D_2) \end{aligned}$$

This proves statement (1).

$$\begin{aligned} x \in f^{-1}(D_1 \cap D_2) &\iff f(x) \in D_1 \cap D_2 \\ &\iff f(x) \in D_1 \text{ and } f(x) \in D_2 \\ &\iff x \in f^{-1}(D_1) \text{ and } x \in f^{-1}(D_2) \\ &\iff x \in f^{-1}(D_1) \cap f^{-1}(D_2) \end{aligned}$$

This proves statement (2). ☐

Proposition 1.5. Let $f : A \mapsto B$ and let $D_n \subseteq B$ for $n = 1, 2, 3, \ldots$. Then:

$$f^{-1}\left(\bigcup_{n=1}^{\infty} D_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(D_n) \tag{1}$$

$$f^{-1}\left(\bigcap_{n=1}^{\infty} D_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(D_n) \tag{2}$$

Proof.

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{n=1}^{\infty} D_n\right) &\iff f(x) \in \bigcup_{n=1}^{\infty} D_n \\ &\iff \exists n \in \mathbb{N} \text{ s.t. } f(x) \in D_n \\ &\iff \exists n \in \mathbb{N} \text{ s.t. } x \in f{-}1(D_1) \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(D_n) \end{aligned}$$

This proves statement (1).

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{n=1}^{\infty} D_n\right) &\iff f(x) \in \bigcap_{n=1}^{\infty} D_n \\ &\iff f(x) \in D_n, \forall n \in \mathbb{N} \\ &\iff x \in f^{-1}(D_n), \forall n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(D_n) \end{aligned}$$

☐

In general, there is **no relation** between the direct image of a complement of a set and a complement of the direct image. However, this is not true for the inverse image.

Proposition 1.6. (Relation between inverse image of a complement of a set and a complement of the inverse image) Let $f : A \mapsto B$ and let $D \subseteq B$. Then,

$$f^{-1}(D^c) = (f^{-1}(D))^c$$

Proof.

$$\begin{aligned} f^{-1}(D^c) &= \{x \in A : f(x) \in D^c\} = \{x \in A : f(x) \notin D\} \\ (f^{-1}(D))^c &= \{x \in A : f(x) \in D\}^c = \{x \in A : f(x) \notin D\} \end{aligned}$$

This proves the statement. □

Definition 1.14. (Absolute value function) $|x| := \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

For $a, b \in \mathbb{R}$, it satisfies:

1. $|a||b| = |a \cdot b|$
2. $|a - b| = |b - a|$
3. $|a + b| \leq |a| + |b|$ (Triangle Inequality)

We can also introduce other real numbers:

$$|a - b| = |a - c + c - b| \leq |a - c| + |c - b|$$

$$|a - b| = |a - c + c - d + d - b| \leq |a - c| + |c - d| + |d - b|$$

Exercise 1.2. Prove that $\forall a, b \in \mathbb{R}, ||a| - |b|| \leq |a - b|$

Proof. We must prove that $|a| - |b| \leq |a - b|$, and $|b| - |a| \leq |a - b|$.

$$\begin{aligned} |a - 0| &= |a - b + b - 0| \leq |a - b| + |b - 0| = |a - b| + |b| \\ &\Rightarrow |a| \leq |a - b| + |b| \\ \Rightarrow |a| - |b| &\leq |a - b| \\ |b - 0| &= |b - a + a - 0| \leq |b - a| + |a - 0| = |b - a| + |a| \\ &\Rightarrow |b| \leq |b - a| + |a| \\ \Rightarrow |b| - |a| &\leq |a - b| \end{aligned}$$

□

Exercise 1.3. (Connect AI and Triangle Inequality) Prove that for every natural number n and $x \in \mathbb{R}$, we have that

$$|\sin(nx)| \leq n|\sin(x)| \tag{1}$$

Proof. Let S denote the set of natural numbers such that (1) holds.
Base case: $|\sin(x)| \leq |\sin(x)|$
Inductive step: Assume that $|\sin(nx)| \leq n|\sin(x)|$

$$\begin{aligned}
|\sin((n+1)x)| &= |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \\
&\leq |\sin(nx)\cos(x)| + |\cos(nx)\sin(x)| \\
&= |\sin(nx)||\cos(x)| + |\cos(nx)||\sin(x)| \\
&\leq |\sin(nx)| + |\sin(x)| \\
&\leq n|\sin(x)| + |\sin(x)| \\
&= (n+1)|\sin(x)| \\
&\Rightarrow (n+1) \in S
\end{aligned}$$

We can conclude by the Axiom of Induction that $S = \mathbb{N}$. □

Example 1.10. Let $A, B \subseteq \mathbb{R}$ and $f : A \mapsto B$ be an injective function. Let $C \subseteq A$ be any set. Then, $f^{-1}(f(C)) = C$.

Proof. Recall that $f(C) = \{f(c) : c \in C\}$, and $f^{-1}(D) = \{a \in A : f(a) \in D\}$. We must prove that $C \subseteq f^{-1}(f(C))$ and $f^{-1}(f(C)) \subseteq C$.
To prove that $C \subseteq f^{-1}(f(C))$: Take $x \in C \Rightarrow f(x) \in f(C) \iff x \in f^{-1}(f(C))$.
To prove that $f^{-1}(f(C)) \subseteq C$: Take $x \in f^{-1}(f(C)) \iff f(x) \in f(C) \Rightarrow f(x) \in \{f(c) : c \in C\} \Rightarrow \exists c \in C$ such that $f(x) = f(c)$. Then, using that f is injective, we see that $x = c$ and thus $x = c \in C$. This completes the proof. □

NOTE: Although the inclusion $f^{-1}(f(C)) \subseteq C$ is true for all functions f as above, the injectivity assumption was essential to guarantee equality. Consider for instance the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = |x|$. Then, $f^{-1}(f([0,1])) = f^{-1}([0,1]) = [-1,1] \neq [0,1]$.

Question 1.5. Let $f : D \mapsto E$ be a function and let $A \subseteq D$, $B \subseteq E$. Prove the following:

(a) $f(f^{-1}(B)) \subseteq B$.

Proof.

$$\begin{aligned}
y \in f(f^{-1}(B)) &\Rightarrow \exists x \in f^{-1}(B) \text{ s.t. } y = f(x) \\
&\Rightarrow y \in B
\end{aligned}$$

□

(b) $f^{-1}(f(A)) \supseteq A$.

Proof.

$$\begin{aligned}
x \in A &\Rightarrow \exists y \in f(A) \text{ s.t. } y = f(x) \\
&\Rightarrow x \in f^{-1}(f(A))
\end{aligned}$$

□

2 The Real Numbers

2.1 Axiom of Completeness

We are left with **"holes"** in the number line after the introduction of rational numbers. One such "hole" is a place where $\sqrt{2}$ should be.
Real numbers fill the "holes" left by rational numbers so we get a number system that becomes a solid line, which allows for the development of analysis based on the number system $\mathbb{R}$.
Other than being an **ordered field** with respect to addition, multiplication and comparison, real numbers have an **additional property** that is summarised in the **Axiom of Completeness**.

Definition 2.1. (Axiom of Completeness) Every non-empty set of real numbers that is bounded from above has a least upper bound.

Definition 2.2. (Upper bound) A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that

$$x \leq b, \forall x \in A.$$

Such b is called an upper bound for the set A.

Definition 2.3. (Lower bound) A set A is bounded from below if there exists a number $l \in \mathbb{R}$ such that

$$x \geq l, \forall x \in A.$$

Such l is called a lower bound for A.

Definition 2.4. (Least upper bound) A real number S is the least upper bound for a set $A \subseteq \mathbb{R}$ if the following holds:

(a) S is an upper bound for A.

(b) If b is any upper bound for A, then $S \leq b$.

If the least upper bound exists, it is unique.
The least upper bound of A is denoted as $S = \sup A$ (supremum of A).

Definition 2.5. (Greatest lower bound) A real number i is the greatest lower bound for a set $A \subseteq \mathbb{R}$ if the following holds:

(a) i is an lower bound for A.

(b) If l is any other lower bound for A, then $i \geq l$.

If the greatest lower bound exists, it is unique.
The greatest lower bound of A is denoted as $i = \inf A$ (infimum of A).

The following is an important characterisation of the supremum.

Proposition 2.1. Let S be an upper bound for set $A \subseteq \mathbb{R}$.
Then, $S = \sup A \iff \forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x$.

Proof. First, we prove one direction, that $S = \sup A \Rightarrow \forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x$.
Proof by contradiction. Assume $S = \sup A$. If $\exists \epsilon > 0$ s.t. $S - \epsilon \geq x, \forall x \in A$, then $S - \epsilon$ is an upper bound of A. Since $\epsilon > 0$, then $S - \epsilon < S$, which implies that $S - \epsilon$ is the least upper bound of the set A, which is a contradiction.
Hence, $S = \sup A \Rightarrow \forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x$.
Next, we prove the other direction, that $\forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x \Rightarrow S = \sup A$.

Proof by contradiction. Assume $\forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x$. If $S \neq \sup A$, then $\exists b \geq x, \forall x \in A$, where $b < S$. Let $\epsilon = S - b > 0$. From our assumption, we know that $S - \epsilon < x \Rightarrow S - S + b < x \Rightarrow b < x$, which is a contradiction.
Hence, $\forall \epsilon > 0, \exists x \in A$ s.t. $S - \epsilon < x \Rightarrow S = \sup A$.
The above proves the proposition. □

Proposition 2.2. Let $i \in \mathbb{R}$ be a lower bound for a set $A \subseteq \mathbb{R}$. Then, $i = \inf A \iff \forall \epsilon > 0, \exists x \in A$ s.t. $x < i + \epsilon$.

Proof. First, we prove one direction, that $i = \inf A \Rightarrow \forall \epsilon > 0, \exists x \in A$ s.t. $x < i + \epsilon$.
We have that $i = \inf A$, and $\epsilon > 0$. Since $i < i + \epsilon$, $i + \epsilon$ cannot be a lower bound of the set A, otherwise $i + \epsilon = \inf A$, which is a contradiction. By the definition of not being a lower bound, this means that $\exists x \in A$ s.t. $x < i + \epsilon$.
Next, we prove the other direction, that $\forall \epsilon > 0, \exists x \in A$ s.t. $x < i + \epsilon \Rightarrow i = \inf A$.
Proof by contradiction. Assume that $\forall \epsilon > 0, \exists x \in A$ s.t. $x < i + \epsilon$. If $i \neq \inf A, \exists l \leq x, \forall x \in A$ s.t. $l > i$. Let $\epsilon = l - i > 0$. From our assumption, we know that $x < i + \epsilon \Rightarrow x < i - i + l \Rightarrow x < l$, which is a contradiction.
Hence, $\forall \epsilon > 0, \exists x \in A$ s.t. $x < i + \epsilon \Rightarrow i = \inf A$.
The above proves the proposition. □

We define $-A = \{-x : x \in A\}$. If $z \in \mathbb{R}$ is an upper bound for A, then $-z$ is a lower bound for $-A$. If $z \in \mathbb{R}$ is a lower bound for A, then $-z$ is an upper bound for $-A$.

Exercise 2.1. Prove that if A is bounded from above, then $-\sup(A) = \inf(-A)$.

Proof. Since A is bounded from above, $\exists S = \sup(A)$, $S \geq x \ \forall x \in A \Rightarrow -S \leq -x \ \forall x \in A \Rightarrow -S \leq x \ \forall x \in -A \Rightarrow -\sup(A) \leq x \ \forall x \in -A$. Thus, we have that $-\sup(A)$ is a lower bound of $-A$, and by the Axiom of Completeness, $-A$ has a greatest lower bound.
Let l be any lower bound for $-A$. $l \leq -x \ \forall x \in A \Rightarrow x \leq -l \ \forall x \in A$. Thus, $-l$ is an upper bound for A. $\sup(A) \leq -l \Rightarrow l \leq -\sup(A) \Rightarrow -\sup(A)$ is the greatest lower bound of $-A$. □

Exercise 2.2. Prove that if A is bounded from below, then $-\inf(A) = \sup(-A)$.

Proof. Since A is bounded from below, $\exists i = \inf(A)$, $i \leq x \ \forall x \in A \Rightarrow -x \leq -i \ \forall x \in A \Rightarrow x \leq -i \ \forall x \in -A \Rightarrow x \leq -\inf(A) \ \forall x \in -A$. Thus, we have that $-\inf(A)$ is an upper bound of $-A$, and by the Axiom of Completeness, $-A$ has a least upper bound.
Let b be any upper bound for $-A$. $b \geq -x \ \forall x \in A \Rightarrow -b \leq x \ \forall x \in A$. Thus, $-b$ is a lower bound for A. $\inf(A) \geq -b \Rightarrow -\inf(A) \leq b \Rightarrow -\inf(A)$ is the least upper bound of $-A$. □

It follows from this exercise that the formulation of the Axiom of Completeness (AC) can be equivalently done in terms of the greatest lower bound.

Definition 2.6. (AC Version 2) Every non-empty set of real numbers that is bounded from below has a greatest lower bound (infimum).

Definition 2.7. (Maximum) Let $A \subseteq \mathbb{R}$. An element $M \in A$ is called **maximum** of A if $x \leq M \ \forall x \in A$. M is both an upper bound of A and an element of A. If A has a maximum M, the obviously $\sup A = M$.

Definition 2.8. (Minimum) Let $A \subseteq \mathbb{R}$. An element $m \in A$ is called **minimum** of A if $x \geq m \ \forall x \in A$. m is both a lower bound of A, and an element of A. If A has a minimum m, then obviously $\inf A = M$.

However, notions of minimum and maximum should not be confused with supremum and infimum. A set $A \subseteq \mathbb{R}$ that is bounded from above must have a supremum, by AC. However, this set may or may not have a maximum.

Exercise 2.3. $A = [0, 1[= \{x \in \mathbb{R} : 0 \leq x < 1\}$. Prove that $\sup A = 1$, but A has no maximum.

Proof. Let $\epsilon > 0$. $1 - \frac{\epsilon}{2} < 1 \Rightarrow 1 - \frac{\epsilon}{2} \in A$. $1 - \epsilon < 1 - \frac{\epsilon}{2} \Rightarrow \forall \epsilon > 0, \exists x = 1 - \frac{\epsilon}{2} \in A$ s.t. $1 - \epsilon < x$ $\Rightarrow \sup A = 1 \Rightarrow x < 1 \leq b\ \forall x \in A$, where b is all the upper bounds of the set $A \Rightarrow x < b$, and A has no maximum. □

Exercise 2.4. Prove that A has a maximum $\iff$ $\sup A \in A$.

Proof. First, prove that A has a maximum $\Rightarrow \sup A \in A$. If A has a maximum M, then $M = \sup A$, since $x \leq M\ \forall x \in A$ and $M \leq b$ where b is all the upper bounds of the set A, since $M \in A$.
Next, prove that $\sup A \in A \Rightarrow A$ has a maximum. If $\sup A \in A$, then $\sup A$ is an upper bound of A that is also an element of A, thus $\sup A$ is the maximum of A, and A has a maximum. □

A similar consideration applies to minimum and infimum.

Example 2.1. Let $A \subset \mathbb{R}$, non-empty, bounded from above. Let $c > 0$.
Define $cA = \{ca : a \in A\}$. Prove that $\sup(cA)$ exists, and equals $c \sup(A)$.

Proof. Since $A \subseteq \mathbb{R}$ is nonempty, cA is nonempty as well. Since $\sup(A)$ is an upper bound of A:

$$\begin{aligned} a &\leq \sup(A), \forall a \in A \\ \Rightarrow ca &\leq c \sup(A), \forall ca \in cA\ (c > 0) \end{aligned}$$

$\Rightarrow cA$ is bounded from above $\Rightarrow \sup(cA)$ exists, by the Axiom of Completeness.
Let b be all the upper bounds of cA.

$$\begin{aligned} ca &\leq b, \forall a \in A \\ a &\leq \frac{b}{c}, \forall a \in A\ (c > 0) \\ a &\leq \sup(A) \leq \frac{b}{c}, \text{by definition of supremum} \\ ca &\leq c \sup(A) \leq b \end{aligned}$$

$\Rightarrow \sup(cA) = c \sup(A)$. □

Example 2.2. Let $A, B \in \mathbb{R}$, both non-empty and $A \cap B = \emptyset$. Assume $A \cup B = \mathbb{R}$. If $a < b, \forall a \in A, \forall b \in B$, prove $\exists c \in \mathbb{R}$ s.t. $a \leq c, \forall a \in A$ and $b \geq c, \forall b \in B$.

Proof. Since $a < b, \forall a \in A, \forall b \in B$, A is bounded from above with b as its upper bounds, and B is bounded from below with a as its lower bounds. By the Axiom of Completeness, this implies that A has a least upper bound, and B has a greatest lower bound. Let $c = \sup(A) \Rightarrow a \leq c, \forall a \in A$.
Since $b \in B$ are all upper bounds of A, $c = \sup(A) \leq b, \forall b \in B$. This proves the statement. □

Example 2.3. $A, B \subseteq \mathbb{R}$, non-empty, bounded from above. $A + B = \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B)$ exists and equals $\sup(A) + \sup(B)$.

Proof. Let M_1 and M_2 be the upper bounds of A and B respectively. Then, $k \leq M_1 + M_2, \forall k \in A + B$. Thus, $A + B$ is bounded from above, and by the Axiom of Completeness, has a least

upper bound.
Let u be an arbitrary upper bound of $A + B$.

$$\begin{aligned} a + b &\leq u \\ a &\leq u - b \\ \sup(A) &\leq u - b \\ b &\leq u - \sup(A) \\ \sup(B) &\leq u - \sup(A) \\ \sup(A) + \sup(B) &\leq u \end{aligned}$$

$\Rightarrow \sup(A + B) = \sup(A) + \sup(B)$. This proves the statement. $\square$

Example 2.4. Let $A \subseteq \mathbb{R}$, non-empty, bounded from below. Consider $B = \{b \in \mathbb{R} : b \text{ is a lower bound of } A\}$. Then, $\sup B$ exists and is the greatest lower bound of A.

Proof. Since A is bounded from below, B must be non-empty. $b \leq a$, $\forall b \in B$ and $\forall a \in A$. Thus, b is bounded from above by $a \in A$, and $\sup B$ exists. Since all $a \in A$ are upper bounds of B, then $\sup B \leq a, \forall a \in A$. Thus, $\sup B$ is a lower bound of A. Let l be any lower bound of A. Then, $l \in B \Rightarrow l \leq \sup B$. Thus, $\sup B$ is the greatest lower bound of A. This completes the proof. $\square$

Question 2.1. Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ be two sets bounded from above. The sum of A and B is the set

$$A + B = \{a + b : a \in A, b \in N\}.$$

Prove that $A + B$ is bounded from above and that

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Since A is bounded from above, $\exists M_a \in \mathbb{R}$ such that $a \leq M_a, \forall a \in A$. Since B is bounded from above, $\exists M_b \in \mathbb{R}$ such that $b \leq M_b, \forall b \in B$. Then, $a + b \leq M_a + M_b, \forall a \in A, \forall b \in B$. $\Rightarrow A + B$ is bounded from above. By the Axiom of Completeness, it has a least upper bound. Since $a \leq \sup(A), \forall a \in A$ and $b \leq \sup(B), \forall b \in B$, then $a + b \leq \sup(A) + \sup(B), \forall a \in A, \forall b \in B$. Thus, $\sup(A) + \sup(B)$ is an upper bound for $A + B$.
Let u be any upper bound of $A + B$.

$$\begin{aligned} a + b &\leq u, \forall a \in A, \forall b \in B \\ \Rightarrow a &\leq u - b \\ \Rightarrow \sup(A) &\leq u - b \\ \Rightarrow b &\leq u - \sup(A) \\ \Rightarrow \sup(B) &\leq u - \sup(A) \\ \Rightarrow \sup(A) + \sup(B) &\leq u \end{aligned}$$

Thus, $\sup(A + B) = \sup(A) + \sup(B)$. $\square$

Question 2.2. Let $x, y, x \in \mathbb{R}$. Show that $|x - y| + |y - z| = |x - z|$ if and only if $x \leq y \leq z$ or $x \geq y \geq z$.

Proof. We analyse under what condition equality holds in the triangle inequality $|a+b \leq |a|+|b|$.

$$\begin{aligned} |a + b| = |a| + |b| &\iff |a + b|^2 = (|a| + |b|)^2 \\ &\iff (a + b)^2 = |a|^2 + 2|a||b| + |b|^2 \\ &\iff a^2 + 2ab + b^2 = a^2 + 2|a||b| + b^2 \\ &\iff ab = |ab| \\ &\iff ab \geq 0 \end{aligned}$$

$\Rightarrow |a+b| = |a| + |b| \iff ab \geq 0.$ □

Now we prove the problem: Let $a := x-y$ and $b := y-z$. Then $a+b = (x-y)+(y-z) = x-z$. Thus, as shown above:

$$\begin{aligned} |x-y| + |y-z| = |x-z| &\iff (x-y)(y-z) \geq 0 \\ &\iff (x-y \geq 0 \text{ and } y-z \geq 0) \text{ or } (x-y \leq 0 \text{ and } y-z \leq 0) \\ &\iff (x \geq y \text{ and } y \geq z) \text{ or } (x \leq y \text{ and } y \leq z) \\ &\iff (x \geq y \geq z) \text{ or } (x \leq y \leq z) \end{aligned}$$

Question 2.3. Let A and B be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if both A and B are bounded above. If it is the case, prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.

Proof. (1) A and B are bounded from above $\Rightarrow A \cup B$ is bounded from above.
If A is bounded from above, $\exists M_a \in \mathbb{R}$ such that $a \leq M_a, \forall a \in A$. If B is bounded from above, $\exists M_b \in \mathbb{R}$ such that $b \leq M_b, \forall b \in B$. Take $M = \max\{M_a, M_b\}$. Let $C = A \cup B$. Then, $c \leq M, \forall c \in C$. $\Rightarrow M$ is an upper bound for $A \cup B$.
(2) $A \cup B$ is bounded from above $\Rightarrow A$ and B are bounded from above.
If $A \cup B$ is bounded from above, $\exists M \in \mathbb{R}$ such that $a \leq M, \forall a \in A$ and $b \leq M, \forall b \in B$. But then, M is an upper bound for both A and B. Thus, A and B are bounded from above.
(3) Prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.
First, we prove that $\sup(\sup A, \sup B)$ is an upper bound of $A \cup B$. Let $C = A \cup B$.

$$\begin{aligned} a &\leq \sup A, \forall a \in A \\ b &\leq \sup B, \forall b \in B \\ \sup A &\leq \sup(\sup A, \sup B) \\ \sup B &\leq \sup(\sup A, \sup B) \\ &\Rightarrow c \leq \sup(\sup A, \sup B), \forall c \in C \end{aligned}$$

$\Rightarrow \sup(\sup A, \sup B)$ is an upper bound of $A \cup B$.
Let u be any upper bound of $A \cup B$. Then, $c \leq u, \forall c \in C$. If $c \in A$, $c \leq \sup A \leq u$. If $c \in B$, $c \leq \sup B \leq u$. Thus, we have $c \leq \sup(\sup A, \sup B) \leq u$. □

Question 2.4. If f is a function $f : D \mapsto \mathbb{R}$, one says that f is bounded above (resp. bounded below, bounded) if the image of D under f i.e. $f(D) = \{f(x) : x \in D\}$ is bounded above (resp. bounded below, bounded). If f is bounded above (resp. bounded below), then one denotes by $\sup f$ the supremum of $f(D)$ (resp. by $\inf f$ the infimum of $f(D)$).
Assume that two functions $f : D \mapsto \mathbb{R}$ and $g : D \mapsto \mathbb{R}$ are bounded above.

(a) Prove that $f(x) \leq g(x)$ for all $x \in D$ implies $\sup f \leq \sup g$.

Proof. If $f(x) \leq g(x), \forall x \in D$, then $g(x)$ is an upper bound of f for all $x \in D$. Take some $k \in D$. Then, $\sup f \leq g(k) \leq \sup g \Rightarrow \sup f \leq \sup g$. This proves the statement. □

(b) Show that the converse is not true by providing a counterexample.

Proof. Let $D := [-1, 1]$, $f(x) := 0$, $g(x) := x$ for all $x \in D$. Then, $\sup g = 1$ (g is increasing in D), and $\sup f = 0$, so $\sup f \leq \sup g$. But then, at $x = -1$, $g(x) = -1 \leq 0 = f(x)$. Thus, $\sup f \leq \sup g \not\Rightarrow f(x) \leq g(x)\ \forall x \in D$. □

(c) Prove that $f(x) \leq g(y)$ for all $x, y \in D$ if and only if $\sup f \leq \inf g$.

Proof. (1) $\sup f \leq \inf g \Rightarrow f(x) \leq g(y)$ for all $x, y \in D$.
If $\sup f \leq \inf g$, then knowing that $f(x) \leq \sup f, \forall x \in D$ and $\inf g \leq g(x), \forall x \in D$, we have that $f(x) \leq \sup f \leq \inf g \leq g(x), \forall x \in D \Rightarrow f(x) \leq g(x), \forall x \in D$. This proves (1).
(2) $f(x) \leq g(y)$ for all $x, y \in D \Rightarrow \sup f \leq \inf g$.
If $f(x) \leq g(y), \forall x, y \in D$, this implies that $g(y)$ are all upper bounds of f. Then, we have that $\sup f \leq g(y), \forall y \in D$. This implies that $\sup f$ is a lower bound of g. Thus, we have that $\sup f \leq \inf g$. This proves the statement. □

2.2 Consequences of Completeness

The first consequence is the **nested interval property**, which is a mathematically rigorous version of the intuitive statement that the real number system is a "solid" line.

Theorem 2.1. (Nested Interval Property) Let $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}$ where $n = 1, 2, \dots$ is an infinite sequence of closed intervals such that each I_n contains I_{n+1}. ($I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$)
Then, $\bigcap_{n=1}^{\infty} \neq \emptyset$.

Proof. We can see that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. The first implication of this relation is that the sequence a_n of the left end points of the intervals I_n is increasing,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \dots$$

and similarly, the sequence b_n of the right end points is decreasing.

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \leq \dots$$

We also have that for all natural numbers n and k,

$$a_n \leq b_k \tag{1}$$

Consider the case where $n \leq k$, then $a_n \leq a_k \leq b_k$. Consider the case where $k \leq n$, then $a_n \leq b_n \leq b_k$. This proves equation (1).
Let $A = \{a_n : n \in \mathbb{N}\}$. By (1), every right end-point b_n is an upper bound of A.
Set $x = \sup A$, which exists by the Axiom of Completeness. Then, for every $n \in \mathbb{N}$, $a_n \leq x$.
We also have that $x \leq b_n$ $\forall n \in \mathbb{N}$, since x is the least upper bound for A and b_n is an upper bound for A.
$\Rightarrow \forall n \in \mathbb{N}$, $a_n \leq x \leq b_n$. $\Rightarrow \forall n \in \mathbb{N}, x \in I_n$.
Hence, $x \in \bigcap_{n=1}^{\infty} I_n$ and $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 2.2. (Archimedean property of $\mathbb{R}$)

(a) For any $x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > x$.

(b) For any real number $y > 0, \exists n \in N$ satisfying $\frac{1}{n} < y$.

Part (a) states that the set of natural numbers (as a subset of $\mathbb{R}$), is not bounded from above. Part (b) follows from (a), by setting $x = \frac{1}{y}$.

Proof. Proof of (a), by contradiction. Suppose that (a) does not hold, and the set of natural numbers is bounded from above in $\mathbb{R}$. Then, by the Axiom of Completeness, $\alpha = \sup \mathbb{N}$ exists. Since α is the least upper bound for $\mathbb{N}$, $\alpha - 1$ is not an upper bound for $\mathbb{N}$. Hence, $\exists n \in \mathbb{N}$ s.t. $\alpha - 1 < n \Rightarrow \alpha < n + 1$. But this contradicts the fact that $\alpha = \sup \mathbb{N}$, namely that α is an upper bound for $\mathbb{N}$. Thus, (a) must hold. □

Theorem 2.3. (Density of $\mathbb{Q}$ in $\mathbb{R}$) For any two real numbers a and b with $a < b$, there exists a rational number x such that $a < x < b$.

Proof. By the Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, or equivalently, $na + 1 < nb$.
Let now $m \in \mathbb{Z}$ s.t. $m - 1 \leq na < m$. Such m exists, since intervals with integer end points make partition of the real line. $\Rightarrow a < \frac{m}{n}$, and $m \leq na + 1 < nb$, so $\frac{m}{n} < b$. It follows that the rational number $x = \frac{m}{n}$ satisfies $a < x < b$. This proves the theorem.
Alternative proof: If $na + 1$ is a natural number, then $na < na + 1 < nb \Rightarrow a < \frac{na+1}{n} < b$, where $\frac{na+1}{n} \in \mathbb{Q}$.
Otherwise, in between na and $na + 1$, there must be a natural number m. $na < m < na + 1 < nb \Rightarrow a < \frac{m}{n} < b$, where $\frac{m}{n} \in \mathbb{Q}$.
Thus, $\exists x \in \mathbb{Q}$ s.t. $a < x < b$. $\square$

Exercise 2.5. Given a real number z, $\exists m \in \mathbb{Z}$ s.t. $m - 1 \leq z < m$. Proof is based on the Well-Ordering Property of $\mathbb{N}$.

Proof. If $z > 0$, let $S = \{n \in \mathbb{N} : n > z\}$. Then, $\exists m \in S$ s.t. $m \leq n, \forall n \in S$. m is the least element that is greater than z, so $m - 1 \leq z \Rightarrow m - 1 \leq z < m$.
If $z < 0$, let $S = \{n \in \mathbb{N} : -n \leq -z + 1\}$. Then, $\exists k \in S$ s.t. $k \leq n, \forall n \in S$. $-k$ is the least element that is less than or equal to $-z + 1$, so $-z + 1 < -k + 1 \Rightarrow -k - 1 \leq z < -k$. Let $m = -k$, then $m - 1 \leq z < m$. This proves the statement. $\square$

Corollary 2.1. Given any 2 real numbers a and b satisfying $a < b$, there exists an irrational number y such that $a < y < b$. The irrational numbers $\mathbb{I}$ are defined as $\mathbb{I} = \{x : x \in \mathbb{R}, x \notin \mathbb{Q}\}$.

Proof. Let y_0 be an arbitrary irrational number. Then, if $a, b \in \mathbb{R}$ s.t. $a < b$, $a - y_0 < b - y_0$. By the theorem of density of the rationals in $\mathbb{R}$, $\exists x \in \mathbb{Q}$ s.t. $a - y_0 < x < b - y_0$. Then, $a < x + y_0 < b$, where $x + y_0 \in \mathbb{I}$. Thus, $\exists y = x + y_0$ such that $a < y < b$. $\square$

Theorem 2.4. (Existence of square roots) There exists $\alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof. Let $A = \{x \in \mathbb{R} : x^2 < 2\}$. A is non-empty and bounded from above and below; if $x \in A$, then $|x| \leq 2$. Set $\alpha = \sup A$. We will prove that $\alpha^2 = 2$ by showing that both $\alpha^2 < 2$ and $\alpha^2 > 2$ lead to a contradiction.
Case 1: Suppose $\alpha^2 < 2$.
Then, $\forall n \in \mathbb{N}$, we have:

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &\leq \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

Note that $\alpha = \sup A > 0$. Choose now $n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}$$

Now, we have that:

$$\left(\alpha + \frac{1}{n_0}\right)^2 \leq \alpha + \frac{2\alpha + 1}{n_0} < \alpha^2 + \frac{2\alpha + 1}{1} \cdot \frac{2 - \alpha^2}{2\alpha + 1} = \alpha^2 + 2 - \alpha^2 = 2$$

Since $\left(\alpha + \frac{1}{n_0}\right)^2 < 2$, then $\left(\alpha + \frac{1}{n_0}\right) \in A$, which means that α cannot be an upper bound for A. This contradicts the assumption that $\alpha = \sup A$, and thus it is not possible that $\alpha^2 < 2$.
Case 2: $\alpha^2 > 2$. Then, $\forall n \in \mathbb{N}$, we have:

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &\geq \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

Note that $\alpha = \sup A > 0$. Choose now $n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$$

Now we have that:

$$\left(\alpha - \frac{1}{n_0}\right)^2 \geq \alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 - \frac{2\alpha}{1} \cdot \frac{\alpha^2 - 2}{2\alpha} = \alpha^2 - \alpha^2 + 2 = 2$$

Since $2 < \left(\alpha - \frac{1}{n_0}\right)^2 < \alpha^2$, then α cannot be the least upper bound for A. This contradicts the assumption that $\alpha = \sup A$, and thus it is not possible that $\alpha^2 > 2$.
Since $\alpha^2 \geq 2$ and $\alpha^2 \leq 2$, it must be that $\alpha^2 = 2$. This completes the proof. □

Theorem 2.5. (Existence of square roots) There exists $\alpha \in \mathbb{R}$ s.t. $\alpha^2 = x$.

Proof. Similar argument as the above. □

Exercise 2.6. Using binomial formula for $(a+b)^m$, $m \in \mathbb{N}$ and following the lines of the above argument, show that for any $x \geq 0$, $\sqrt[m]{x}$ exists; namely that $\exists \alpha \in \mathbb{R}$ s.t. $\alpha^m = x$.

Proof. Let $A = \{y \in \mathbb{R} : y^m < x\}$. A is non-empty and bounded from above; if $y \in A$, then $y \leq x$. Set $\alpha = \sup A$. We will prove that $\alpha^m = x$ by showing that both $\alpha^m < x$ and $\alpha^m > x$ lead to a contradiction.
Case 1: Suppose $\alpha^m < x$.
Then, $\forall n \in \mathbb{N}$, we have:

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^m &= \alpha^m + \binom{m}{1} \cdot \frac{\alpha^{m-1}}{n^1} + \binom{m}{2} \cdot \frac{\alpha^{m-2}}{n^2} + \cdots + \binom{m}{m} \cdot \frac{1}{n^m} \\ &\leq \alpha^m + \binom{m}{1} \cdot \frac{\alpha^{m-1}}{n} + \binom{m}{2} \cdot \frac{\alpha^{m-2}}{n} + \cdots + \binom{m}{m} \cdot \frac{1}{n} \\ &= \alpha^m + \frac{\binom{m}{1} \cdot \alpha^{m-1} + \binom{m}{2} \cdot \alpha^{m-2} + \cdots + \binom{m}{m} \cdot 1}{n}\end{aligned}$$

Note that $\alpha = \sup A > 0$. Choose now $n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{x - \alpha^m}{\binom{m}{1} \cdot \alpha^{m-1} + \binom{m}{2} \cdot \alpha^{m-2} + \cdots + \binom{m}{m} \cdot 1}$$

Now, we have that:

$$\left(\alpha + \frac{1}{n_0}\right)^m \leq \alpha^m + \frac{\binom{m}{1} \cdot \alpha^{m-1} + \binom{m}{2} \cdot \alpha^{m-2} + \cdots + \binom{m}{m} \cdot 1}{n_0} < a^m + x - a^m = x$$

Since $\left(\alpha+\frac{1}{n_0}\right)^m < x$, then $\left(\alpha+\frac{1}{n_0}\right)^m \in A$, which means that α cannot be an upper bound for A. This contradicts our assumption that $\alpha = \sup A$, and thus it is not possible that $\alpha^m < x$.
Case 2: Suppose $\alpha^m > x$. Then, $\forall n \in \mathbb{N}$, we have:

$$\begin{aligned}\left(\alpha-\frac{1}{n}\right)^m &= \alpha^m - \sum_{k=1}^{m}\binom{m}{k}\frac{\alpha^{m-k}}{(-n)^k}\\ &\geq \alpha^m - \sum_{k=1}^{m/2}\binom{m}{2k+1}\frac{\alpha^{m-(2k-1)}}{(-n)^{2k-1}}\\ &\geq \alpha^m - \sum_{k=1}^{m/2}\binom{m}{2k+1}\frac{\alpha^{m-(2k-1)}}{n}\\ &= \alpha^m - \frac{\sum_{k=1}^{m/2}\binom{m}{2k+1}\alpha^{m-(2k-1)}}{n}\end{aligned}$$

Note that $\alpha = \sup A > 0$. Choose now $n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{a^m - x}{\sum_{k=1}^{m/2}\binom{m}{2k+1}\alpha^{m-(2k-1)}}$$

Now we have that:

$$\left(\alpha-\frac{1}{n_0}\right)^m \geq \alpha^m - \frac{\sum_{k=1}^{m/2}\binom{m}{2k+1}\alpha^{m-(2k-1)}}{n_0} > a^m - a^m + x = x$$

Since $\left(\alpha-\frac{1}{n_0}\right)^m > x$, then $\left(\alpha-\frac{1}{n_0}\right)^m$ is an upper bound, which means that α cannot be the least upper bound for A. This contradicts our assumption that $\alpha = \sup A$, and thus it is not possible that $\alpha^m > x$.
Since $\alpha^m \geq x$ and $\alpha^m \leq x$, it must be that $\alpha^m = x$. This completes the proof. $\square$

The existence of $\sqrt{2}$ and our previous proof that there exists $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$ shows that $\sqrt{2}$ is an irrational number.
The above proof shows that if the Axiom of Completeness would hold in $\mathbb{Q}$ and if $B = \{x \in \mathbb{Q} : x^2 < 2\}$, then $\alpha = \sup B$ would satisfy $\alpha^2 = 2$, so we would have a rational number α such that $\alpha^2 = 2$, which is impossible.
$\Rightarrow$ Axiom of Completeness does not hold for $\mathbb{Q}$.

Exercise 2.7. $A \subseteq \mathbb{R}$ is nonempty and bounded from above. Let $s \in \mathbb{R}$ have the property that $\forall n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Prove that $s = \sup(A)$.

Proof. s is an upper bound for A. We prove this by contradiction. If $\exists x \in A$ such that $x > s$, then let $\epsilon = x - s > 0$. By the Archimedean property, $\exists m \in N$ such that $m > \frac{1}{\epsilon} \Rightarrow \epsilon > \frac{1}{m}$. Then, $s + \epsilon = x > s + \frac{1}{m}$. But then $s + \frac{1}{m} < x$ for some $x \in A$ and this contradicts the fact that $s + \frac{1}{m}$ is an upper bound for A. Thus, $\forall x \in A, x \leq s$ and s is an upper bound for A.
$\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$, by the Archimedean property. $\Rightarrow \epsilon > \frac{1}{n} \Rightarrow s - \epsilon < s - \frac{1}{n}$. Since $s - \frac{1}{n}$ is not an upper bound, $\forall n \in \mathbb{N}$, $\exists x \in A$ such that $s - \frac{1}{n} < x$. Thus, we have $s - \epsilon < s - \frac{1}{n} < x$, $x \in A$. Since $\forall \epsilon > 0$, $\exists x \in A$ such that $s - \epsilon < x$, $s = \sup(A)$. $\square$

Exercise 2.8. Prove that $\bigcap_{n=1}^{\infty} \left]0, \frac{1}{n}\right[= \emptyset$.

Proof. Let $x \in \mathbb{R}$. We need to show that $x \notin]0, \frac{1}{n}[$ for some $n \in \mathbb{N}$.
Case 1: $x \leq 0$ or $x \geq 1$. $\Rightarrow$ Take $n = 1$, then $x \notin]0, 1[$.
Case 2: $0 < x < 1$. Then, by the Archimedean property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{x} \Rightarrow x > \frac{1}{n}$. Then, $x \notin]0, \frac{1}{n}[$.
$\Rightarrow$ Any $x > 0$ cannot be in $\bigcap_{n=1}^{\infty}]0, \frac{1}{n}[$, so $\bigcap_{n=1}^{\infty}]0, \frac{1}{n}[= \emptyset$. $\square$

Exercise 2.9. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Prove that $\sup(T) = b$.

Proof. Proof by contradiction. Since $a \leq t \leq b, \forall t \in T$, b must be an upper bound of T. This implies that T has a least upper bound. Suppose that this least upper bound $x \in \mathbb{R}$ is such that $x < b$. Then, by the Density of $\mathbb{Q}$ in $\mathbb{R}$, there must exist $p \in \mathbb{Q}$ such that $x < p < b$. But then, $p \in T$, and x cannot be an upper bound of T, which is a contradiction. Thus, we have that $\sup(T) = b$. $\square$

Exercise 2.10. Consider $A = [0, 1[$. Prove that $\sup(A) = 1$.

Proof. Since $0 \leq a < 1, \forall a \in A$, 1 must be an upper bound of A. This implies that A has a least upper bound. Let b be an upper bound for A. Proceed by contradiction. If $b < 1$, then we have $b < \frac{b+1}{2} \in [0, 1[$. Hence, b is not an upper bound for A, which is a contradiction. Thus, 1 is the least upper bound of A. $\square$

Exercise 2.11. Prove that given a real number z, there exists $m \in \mathbb{Z}$ such that $m \leq z < m+1$.

Proof. Let $A \coloneqq \{n \in \mathbb{Z} : n \leq z\}$. By the Archimedean property, $\exists N \in \mathbb{N}$ such that $N > z$. By the Archimedean property again, $\exists M \in \mathbb{N}$ such that $M > -z$. $\Rightarrow -M < z < N$, and the set A is non-empty and bounded from above. By the Axiom of Completeness, we have a supremum $s = \sup(A)$. Since s is a supremum, $\exists m \in A$ such that $s - 1 < m \leq s$. Since $m \in A$, trivially $m \leq z$. Then, $s - 1 < m \Rightarrow s < m + 1 \Rightarrow (m + 1) \notin A \Rightarrow m + 1 > z$. $\Rightarrow m \leq z < m + 1$. $\square$

Example 2.5. Prove that $\bigcup_{n=1}^{\infty}]\frac{1}{n}, n[=]0, \infty[$.

Proof. Clearly, $]\frac{1}{n}, n[\subseteq]0, \infty[\ \forall n \in \mathbb{N}$. $\Rightarrow \bigcup_{n=1}^{\infty}]\frac{1}{n}, n[\subseteq]0, \infty[$.
Conversely, let $x \in]0, \infty[$. By applying the Archimedean property, $\exists n_1 \in \mathbb{N}$ such that $n_1 > x$. On the other hand, as $x > 0$, we have that $\exists n_2 \in \mathbb{N}$ such that $\frac{1}{n_2} < x$. Let now $N \coloneqq \max\{n_1, n_2\}$ so that $N \geq n_1$ and $N \geq n_2$. $\Rightarrow \frac{1}{N} \leq \frac{1}{n_2} < x < n_1 \leq N$. $\Rightarrow x \in]\frac{1}{N}, N[\subseteq \bigcup_{n=1}^{\infty}]\frac{1}{n}, n[$. This concludes the proof. $\square$

Question 2.5. Let x be a real number. Show that, for any $\epsilon > 0$, there exist two rationals q and q' such that $q < x < q'$ and $|q - q'| < \epsilon$.

Proof. Let $\epsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number q' such that $x < q' < x + \frac{\epsilon}{2}$. We also have that there exists a rational number q such that $x - \frac{\epsilon}{2} < q < x$. Then, we have that $q < x < q'$, and $|q - q'| = q' - q < x + \frac{\epsilon}{2} - (x - \frac{\epsilon}{2}) = \epsilon \Rightarrow |q - q'| < \epsilon$. This proves the statement $\square$

Question 2.6. Let S be a nonempty and bounded subset of $\mathbb{R}$.

(a) Prove that $S \subseteq [\inf S, \sup S]$.

Proof. If $x \in S$, then $x \leq \sup S$, $x \geq \inf S$. $\Rightarrow x \in [\inf S, \sup S] \Rightarrow S \subseteq [\inf S, \sup S]$. $\square$

(b) Prove that if J is a closed interval containing S, then $[\inf S, \sup S] \subseteq J$.

Proof. We first assume that J is bounded, so $J \coloneqq [a, b]$, where $a, b \in \mathbb{R}$. If J is a closed interval containing S, then $a \leq x, \forall x \in S$, and $b \geq x, \forall x \in S$. We can see that a is a lower bound of S, and b is an upper bound of S. Then, $a \leq \inf S$ and $b \geq \sup S$. $\Rightarrow [\inf S, \sup S] \subseteq [a, b] = J$.
It remains to prove the statement in the case that J is unbounded. Since S is bounded, there exists a closed and bounded interval I with $S \subseteq I$. Let $\tilde{J} \coloneqq J \cap I$. Then, $\tilde{J}$ is closed, and also bounded since I is bounded. Furthermore, since $S \subseteq I$ and $S \subseteq J$, we have $S \subseteq \tilde{J}$. We can thus apply our result above to $\tilde{J}$ instead of J and obtain $[\inf S, \sup S] \subseteq \tilde{J}$, which implies $[\inf S, \sup S] \subseteq J$. This proves the statement. $\square$

Question 2.7. For any $n \in \mathbb{N}$, let $I_n =]0, \frac{1}{n}[$ and $J_n = [0, \frac{1}{n}]$. Show that $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$ and $\bigcap_{n\in\mathbb{N}} J_n = \{0\}$.

Proof. Let $x \in \mathbb{R}$. To prove that $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$, we need to show that $x \notin]0, \frac{1}{n}[$ for some $n \in \mathbb{N}$. If $x \leq 0$ or $x \geq 1$, then take $n = 1$. $x \notin]0, 1[$. When $0 < x < 1$, then by the Archimedean property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{x} \Rightarrow x > \frac{1}{n} \Rightarrow x \notin]0, \frac{1}{n}[$. Thus, $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$.
Let $y = 0$. Then, $y \in [0, \frac{1}{n}], \forall n \in \mathbb{N} \Rightarrow \{0\} \in \bigcap_{n\in\mathbb{N}} J_n$. Let $x \in \mathbb{R} \setminus \{0\}$. To prove that $\bigcap_{n\in\mathbb{N}} J_n = \{0\}$, we need to show that $x \notin [0, \frac{1}{n}]$. When $x < 0$ or $x > 1$, take $n = 1$. $x \notin [0, 1]$. When $0 < x \leq 1$, then by the Archimedean property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{x} \Rightarrow x > \frac{1}{n} \Rightarrow x \notin [0, \frac{1}{n}]$. Thus, $\bigcap_{n\in\mathbb{N}} J_n = \{0\}$. $\square$

3 Cardinality

The concept of cardinality refers to the size of sets.

Definition 3.1. (Injective) A function $f : A \mapsto B$ is called one-to-one (or 1-1, or **injective**), if $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. Each element of A has a unique image in B.

Definition 3.2. (Surjective) A function f is called onto (or **surjective**) if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. Every element of B is in the image of f.

Definition 3.3. (Bijective) The function f is called **bijective** if it is injective and surjective.

Definition 3.4. (Composition of functions) If $f : A \mapsto B$ and $g : B \mapsto C$, the composite function $h : A \mapsto C$ is defined as $h(x) = g(f(x)),\ x \in A$. This is represented as $h = g \circ f$.

Exercise 3.1. (a) Prove that if f and g are injective, then so is the composite function $h = g \circ f$.

(b) Prove that if f and g are bijections, then so is h.

(c) Show that for any $E \subseteq C, h^{-1}(E) = f^{-1}(g^{-1}(E))$.

Proof. (a) Since f is injective, then $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$. Since g is injective, then $f(a_1) \neq f(a_2) \Rightarrow g(f(a_1)) \neq g(f(a_2))$. Thus, $a_1 \neq a_2 \Rightarrow h(a_1) \neq h(a_2)$.

(b) From the above, if f and g are bijections, then h is injective. Since g is surjective, $\forall c \in C$, $\exists b \in B$ such that $g(b) = c$. Since f is surjective, $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. Thus, $\forall c \in C$, $\exists a \in A$ such that $h(c) = a$.

(c) Since $h : A \mapsto C$ and $E \subseteq C$, then $h^{-1}(E) \coloneqq \{a \in A : h(a) \in E\}$. We have that $g^{-1}(E) = \{b \in B : g(b) \in E\}$, and $f^{-1}(g^{-1}(E)) = \{a \in A : g(f(a)) \in E\}$. Thus, $h^{-1}(E) = f^{-1}(g^{-1}(E))$.

$\square$

The notion of inverse function is defined for bijective maps.
If $f : A \mapsto B$ is a bijection, the inverse map $f^{-1} : B \mapsto A$ is defined by $f^{-1}(y) = x$, where x is such that $f(x) = y$.

Example 3.1. Check that $f^{-1} : B \mapsto A$ is also a bijection.

Proof. The fact that f is injective means that f^{-1} is well-defined (no 1 input going to 2 outputs). The fact that f is surjective means that $\forall b \in B, \exists a \in A : f(a) = b$. Since $\forall a \in A$ there is a unique mapping to some $b \in B$, this means that f^{-1} is one-one as well. For any $a \in A$ that undergoes f and spans all of B, f^{-1} takes it back to A, so f^{-1} is surjective. $\square$

NOTE: Do not confuse inverse functions with inverse image of a set. Inverse image of a set is always defined for any function. The inverse of a function is a function and is defined only for bijective maps.
If $f : A \mapsto B$ is bijective and $y \in B$, then

$$f^{-1}(y) = f^{-1}(\{y\})$$

where the convenient function to find all elements of the set does not always exist, but the set $f^{-1}(\{y\})$ always exists.

Example 3.2. $f : \mathbb{R} \mapsto]0, \infty[\ f(x) = e^x$. Then the inverse map $f^{-1} :]0, \infty[\mapsto \mathbb{R}$ is $f(x) = \ln x$.

Example 3.3. $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\mapsto \mathbb{R}$ $f(x) = \tan x = \frac{\sin x}{\cos x}$. Then the inverse map $f^{-1}(x) = \arctan x$.

After these preliminaries, we turn to the concept of cardinality. If A is a finite set, this concept is elementary.

Definition 3.5. (Cardinal number - finite set) $|A|$ is the cardinal number of A, which is defined as the number of elements of A.

Definition 3.6. (Finite set) We say that A is a finite set with n elements if there exists bijection $f : \{1, 2, 3, \ldots, n\} \mapsto A$, and then $|A| = n$. The bijection f allows to enumerate elements of A. Setting $f(k) = x_k, k = 1, 2, \ldots, n \Rightarrow A = \{x_1, x_2, \ldots, x_n\}$.

The concept of cardinality becomes interesting in the case of sets which are not finite (infinite sets). There are two approaches to the general notion of cardinality.

1. Compare "sizes" by using notions of bijections and injections.

2. Using the notion of cardinal numbers.

Both notions are elementary in the case of finite sets. If A and B are finite, they have the same cardinality if there exists a bijection $f : A \mapsto B$. This is the same as saying that A and B have the same cardinal number ($|A| = |B|$).
For infinite sets, the above notion of cardinality extends; the notion of cardinal number for infinite sets is much more complex, and will not be discussed.

Definition 3.7. The set A has the same cardinality as the set B if there exists a bijection $f : A \mapsto B$. In this case, we write $A \sim B$.

Exercise 3.2. $\sim$ is a notion of equivalence. Prove that the following holds for any 3 sets, A, B, and C.

(a) $A \sim A$.

(b) If $A \sim B \Rightarrow B \sim A$.

(c) If $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

Proof. (a) Let $f : A \mapsto A$ be defined by $f(a) = a$. Then, f is a bijection, so $A \sim A$.

(b) If $A \sim B$, there exists a bijection $f : A \mapsto B$. Then, $f^{-1} : B \mapsto A$ is also a bijection as proven above. Thus, $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, there exists a bijection $f : A \mapsto B$ and a bijection $g : B \mapsto C$. Then, $h = g \circ f$ is a bijection as proven above, so $A \sim C$.

□

Example 3.4. Let E be the set of all even natural numbers $\Rightarrow E = \{2, 4, 6, 8, \ldots\}$. Let $f : \mathbb{N} \mapsto E$ be defined by $f(n) = 2n$. Then, f is a bijection, so $\mathbb{N} \sim E$.
$\Rightarrow \mathbb{N}$ and E have the same "size", even if obviously $\mathbb{N}$ has more elements than E.

Exercise 3.3. Prove that $\mathbb{N} \sim \mathbb{Z}$.

Proof.

$$f(n) = \begin{cases} \frac{n}{2}, & \text{for } n \text{ even.} \\ \frac{-n+1}{2}, & \text{for } n \text{ odd.} \end{cases}$$

f is a bijection where $f : \mathbb{N} \mapsto \mathbb{Z} \Rightarrow \mathbb{N} \sim \mathbb{Z}$. □

Two sets have the same cardinality if they have the same size, and there is no intuition for infinite sets.

Definition 3.8. (Countable sets) A set A is countable if $\mathbb{N} \sim A$.

Definition 3.9. (Uncountable sets) An infinite set that is not countable is called an uncountable set.

Theorem 3.1. (Basic Fact I) Suppose that $A \subseteq B$.

(a) If B is finite or countable, then A is also finite or countable.

(b) If A is uncountable, then B is also uncountable.

(b) follows from (a). The proof of (a) is complex, so it is omitted here.

Definition 3.10. (Cartesian product of two sets) If A and B are two sets, their cartesian product is defined by:

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Theorem 3.2. (Basic Fact II) The product set $\mathbb{N} \times \mathbb{N}$ is countable. The theorem asserts that there exists a bijection between $\mathbb{N} \times \mathbb{N} = \{(n, m) : n, m \in \mathbb{N}\}$ and $\mathbb{N}$. The proof exhibits explicit bijection $\Rightarrow f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$.

Proof. To describe f, recall that in the first class, we have proven by induction that $\forall n \in \mathbb{N}$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
$$\Psi(n) = \frac{n(n+1)}{2}$$

Define $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ by the formula $f(m, n) = \Psi(m + n - 2) + m, (m, n) \in \mathbb{N} \times \mathbb{N}$. Then, f is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.
There is a simple intuition why $\mathbb{N} \times \mathbb{N}$ is countable.

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \ldots
\end{array}$$

Using this diagonal argument, we can intuitively see that eventually we would exhaust the table.

Full Proof

The k-th diagonal of the table below has k points. Let $\Psi(k) :=$ total number of points in diagonals 1 through $k = 1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$. Ψ is strictly increasing, with

$$\Psi(k+1) = \Psi(k) + k + 1 \tag{1}$$

NOTE: $(m, n) \in \mathbb{N} \times \mathbb{N}$ lies on the kth diagonal $k = m + n - 1$.

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & \ldots
\end{array}$$

Thus, we count (m, n) by counting the points in the first $k-1(= m+n-2)$ diagonals and then add m. $h : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$, $[h(m,n) \coloneqq \Psi(m+n-2)+m]$, which is our counting function (CF). We want to prove that h is a bijection.
Injectivity: Note that if $(m,n) \neq (m',n')$, then either $m+n \neq m'+n'$ or $m+n = m'+n'$ but $m \neq m'$.

1. Assume $m+n \neq m'+n'$
   Without loss of generality, assume $m+n < m'+n'$. Then,

$$\begin{aligned} h(m,n) &= \Psi(m+n-2)+m \\ &\leq \Psi(m+n-2)+m+n-1 \quad (n \geq 1) \\ &= \Psi(m+n-1) \quad \text{[using (1)]} \\ &\leq \Psi(m'+n'-2) \quad (\Psi \text{ is strictly increasing}) \\ &< \Psi(m'+n'-2)+m' \\ &= h(m',n') \\ \Rightarrow h(m,n) &\neq h(m',n') \end{aligned}$$

2. Assume that $m+n = m'+n'$ but $m \neq m'$ and $n \neq n'$

$$\begin{aligned} h(m,n) - m &= \Psi(m+n-2) \\ &= \Psi(m'+n'-2) \\ &= h(m',n') - m' \end{aligned}$$

   Since $m \neq m'$, we conclude that $h(m,n) \neq h(m',n')$, so h is injective.

Surjectivity: Recall f is surjective ($f : A \mapsto B$) if $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$.
For $n = 1$, $h(1,1) = \Psi(2-2)+1 = \Psi(0)+1 = 1$.
Now let $p \in \mathbb{N}$ with $p \geq 2$. We want to find a pair $(m_p, n_p) \in \mathbb{N} \times \mathbb{N}$ with $h(m_p, n_p) = p$. Since $p < \Psi(p)$ by the definition of Ψ and $p \geq 2$, the set $E_p \coloneqq \{k \in \mathbb{N} : p \leq \Psi(k)\}$ is non-empty. By applying the Well-Ordering Property, we can let $k_p > 1$ be the least element in E_p (note that $k_p > 1$ since $\Psi(1) = 1$).

$$\begin{aligned} \Psi(k_p - 1) &< p \text{ (since } k_p \text{ is the least element)} \\ &\leq \Psi(k_p) \ (k_p \in E_p) \\ &= \Psi(kp-1) + k_p \end{aligned}$$

Hence we can let $m_p \coloneqq p - \Psi(k_p - 1)$ so that $1 \leq m_p \leq k_p$ and we can let $n_p \coloneqq k_p - m_p + 1$ such that $1 \leq n_p \leq k_p$. This makes $m_p + n_p - 1 = k_p$. Thus,

$$\begin{aligned} h(m_p, n_p) &= \Psi(m_p + n_p - 2) + m_p \\ &= \Psi(k_p - 1) + m_p \\ &= p \end{aligned}$$

$\Rightarrow h$ is surjective.
$\Rightarrow h$ is bijective $\Rightarrow \mathbb{N} \times \mathbb{N}$ is countably infinite. $\square$

Exercise 3.4. Using Basic Fact II, prove that if A and B are countable sets then their product $A \times B$ is also countable.

Proof. If A and B are countable, there exists bijection $f : A \mapsto \mathbb{N}$ and bijection $g : B \mapsto \mathbb{N}$. Thus, $A \sim \mathbb{N}$ and $B \sim \mathbb{N}$. From Basic Fact II, we know that there exists a bijection $h : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$, such that $\mathbb{N} \times \mathbb{N}$ is countable. We can define a bijective function $k : A \times B \mapsto \mathbb{N} \times \mathbb{N}$, where $k(a, b) = (f(a), g(b))$. Thus, since $\mathbb{N} \times \mathbb{N}$ is countable, $A \times B$ must be countable. □

Theorem 3.3. (Basic Fact III) The following statements are equivalent.

(a) A is either finite or countable set.

(b) There exists a surjection of $\mathbb{N}$ onto A.

(c) There exists an injection of A onto $\mathbb{N}$.

Proof. We will show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).
(a) $\Rightarrow$ (b)
If A is a finite set with size $|A| = N$, then there exists a bijection $h : \{1, 2, \ldots, N\} \mapsto A$. We define a function $f : \mathbb{N} \mapsto A$ which is a surjection, by the following:

$$f(n) = \begin{cases} h(n), & \text{if } n \leq N \\ h(N), & \text{if } n > N \end{cases}$$

If A is a countable set, it must have a bijection $f : \mathbb{N} \mapsto A$. Bijections are also surjections, so the relation follows.
(b) $\Rightarrow$ (c)
If there exists a surjection of $\mathbb{N}$ onto A, then there exists $f : \mathbb{N} \mapsto A$ such that $\forall a \in A, \exists n \in \mathbb{N}$ such that $f(n) = a$. We know that each element in A has a mapping from $\mathbb{N}$, but we want to create a unique mapping. We now define a function $g : A \mapsto \mathbb{N}$:

$$g(a) = \min\{n \in \mathbb{N} : f(n) = a\}$$

This function is an injection. If $g(a_1) = g(a_2)$, then $f(n) = a_1 = a_2$. Thus, if $a_1 \neq a_2$, then $g(a_1) \neq g(a_2)$.
(c) $\Rightarrow$ (a)
If there exists an injection of A onto $\mathbb{N}$, then there exists $f : A \mapsto \mathbb{N}$ such that if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Let $B = f(A)$. We know that $A \sim B$ with f as a bijection, and $B \subseteq \mathbb{N}$. By Basic Fact I, since $\mathbb{N}$ is countable, then B is finite or countable. Since $B \sim A$, then A must be finite or countable too.
This proves the statement that all three statements are equivalent. □

Theorem 3.4. Let $A_n,\ n = 1, 2, \ldots$ be an infinite sequence of sets such that every A_n is either finite or countable. Then,

$$A = \bigcup_{n=1}^{\infty} A_n \text{ is either finite or countable.}$$

Proof. We know that every A_n is either finite or countable. Thus, by Basic Fact III, we know that for each n there exists surjection $\varphi_n : \mathbb{N} \mapsto A_n$. If there is a surjection φ_n for each $n \in \mathbb{N}$, there exists a surjection $f : \mathbb{N} \times \mathbb{N} \mapsto \bigcup_{n=1}^{\infty} A_n$, where $f(m, n) = \varphi_n(m)$. By Basic Fact II, $g : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ is a bijection. Thus, the composition $h = g \circ f : \mathbb{N} \mapsto \bigcup_{n=1}^{\infty} A_n$ is a surjection. By Basic Fact III again, we have that $\bigcup_{n=1}^{\infty} A_n$ is either finite or countable. □

Corollary 3.1. Let $A_n,\ n = 1, 2, \ldots$ be an infinite sequence of countable sets. Then $A = \bigcup_{n=1}^{\infty} A_n$ is also countable. This is an immediate consequence of the previous theorem because A is infinite.

Exercise 3.5. State and prove the finite union variant of the previous theorem and corollary.

Proof. Let $A_n,\ n = 1, 2, \dots, M$ be a finite sequence of sets such that every A_n is either finite or countable. Then,

$$A = \bigcup_{n=1}^{M} A_n \text{ is either finite or countable.}$$

We know that every A_n is either finite or countable. Thus, by Basic Fact III, for each $n \in \mathbb{N}, n \leq M$, we have a surjection $\varphi_n : \mathbb{N} \mapsto A_n$. If there is a surjection for each n, we can define a map $f : \mathbb{N} \times \mathbb{N} \mapsto \bigcup_{n=1}^{N} A_n$ such that:

$$f(k, n) = \begin{cases} \varphi_n(k), & \text{when } n \leq M \\ \varphi_M(k), & \text{when } n > M \end{cases}$$

which is a surjection. By Basic Fact II, $g : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ is a bijection. Thus, the composition $h = g \circ f : \mathbb{N} \mapsto \bigcup_{n=1}^{M} A_n$ is a surjection. By Basic Fact III again, we have that $\bigcup_{n=1}^{M} A_n$ is either finite or countable.
The corollary is: let $A_n,\ n = 1, 2, \dots, M$ be an finite sequence of countable sets. Then $A = \bigcup_{n=1}^{M} A_n$ is also countable. This is an immediate consequence of the previous theorem because A is infinite. ☐

Theorem 3.5. The set $\mathbb{Q}$ of rational numbers is countable.

Proof. Write $\mathbb{Q} = A_1 \cup A_2 \cup A_3$, where $A_1 = \{0\}$, $A_2 = \{\frac{m}{n} : m, n \in \mathbb{N}\}$, $A_3 = \{-\frac{m}{n} : m, n \in \mathbb{N}\}$. We will prove that A_2 is countable. Consider a map $g : \mathbb{N} \times \mathbb{N} \mapsto A_2$, where $h(n, m) = \frac{m}{n}$. So h is obviously a surjection. By Basic Fact II, $f : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ is a bijection, so $g \circ f : \mathbb{N} \mapsto A_2$ is a surjection. By Basic Fact III, we have that A_2 is countable. If A_2 is countable, there is a bijection $h : A_2 \mapsto A_3$ where $h(a_2) = -a_3$, so A_3 must be countable. A_1 is finite and thus countable. The finite union of finite or countable sets must be countable by the theorem before. Thus, $\mathbb{Q}$ must be countable, and this proves the statement. ☐

Theorem 3.6. The set $\mathbb{R}$ of real numbers is uncountable.

Proof. Obviously, $\mathbb{R}$ is infinite. To prove the statement, we will argue by contradiction. Suppose that $\mathbb{R}$ is countable, namely that there exists a bijection $f : \mathbb{N} \mapsto \mathbb{R}$. Denoting $f(1) = x_1$, $f(2) = x_2$, $f(n) = x_n$, $\dots$, we have that $\mathbb{R} = \{x_1, x_2, \dots, x_n, \dots\}$.
Choose now a non-empty closed interval $I_1 = [a_1, b_1]$ such that $x_1 \notin I_1$. If $x_2 \notin I_1$, then we set that $I_2 = I_1$ If $x_2 \in I_1$, then we divide I_1 in four equal parts (closed subintervals):

So, if $x_2 \in I_1 \in [a_1, b_1]$, then either $x \in I_2'$ or $x \in I_2''$, since these two intervals are disjoint. If $x_2 \notin I_2'$, we set $I_2 = I_2'$, and if $x_2 \notin I_2''$, we set $I_2 = I_2''$. If $x_2 \notin I_2' \cup I_2''$, it does not matter which we set it to.
We now have two intervals I_1 and I_2 such that $I_1 \supseteq I_2$, $x_1 \notin I_1$, $x_2 \notin I_2$, and in particular, $x_1, x_2 \notin I_2$.
We now look at x_3. If $x_3 \notin I_2$, then we set $I_3 = I_2$. If $x_3 \in I_2$, we divide I_2 into four equal parts (closed subintervals) and proceed in exactly the same way as above to obtain a closed non-empty interval I_3 such that: $I_1 \supseteq I_2 \supseteq I_3$ and $x_1, x_2, x_3 \notin I_3$.
Continuing in this way, we obtain a sequence $I_n = [a_n, b_n]$ of non-empty closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n \supseteq \dots$ and such that $\forall n \in \mathbb{N}, x_n \notin I_n$.

Consider now the intersection $\bigcap_{n=1}^{\infty} I_n$. $\forall m \in \mathbb{N}$, $x_m \notin I_m$ and so $x_m \notin \bigcap_{n=1}^{\infty} I_n$.
Since $\mathbb{R} = \{x_1, x_2, \ldots, x_n, \ldots\}$ we have that $\mathbb{R} \cap (\bigcap_{n=1}^{\infty} I_n) = \emptyset$, or $\bigcap_{n=1}^{\infty} I_n = \emptyset$. On the other hand, by the nested interval property and our construction, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.
So, assuming $\mathbb{R}$ is countable, we have arrived at a contradiction: that there exists a sequence of $I_n,\ n = 1, 2, \ldots$ satisfying the conditions of nested interval property theorem but having an empty intersection. Thus, our initial assumption that $\mathbb{R}$ is not countable cannot be correct. This proves the statement. □

NOTE: The above proof uses crucially the Axiom of Completeness. This should not be surprising. Since $\mathbb{Q}$ is countable, the proof that $\mathbb{R}$ is uncountable has to use critically the axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Corollary 3.2. The set $\mathbb{I}$ of irrational numbers in $\mathbb{R}$ is uncountable.

Proof. Proof by contradiction. Suppose that $\mathbb{I}$ is countable. From before, we have that $\mathbb{Q}$ is countable. $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$, and the finite union of countable sets is countable. We arrive at $\mathbb{R}$ being countable, which is a contradiction. Thus, we must have that $\mathbb{I}$ is uncountable. □

Theorem 3.7. For any interval $]a, b[$, with $a < b$, $]a, b[\ \sim \mathbb{R}$.

Proof. There exists a bijection $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\ \mapsto \mathbb{R}$, where $f(x) = \tan x$. Thus, $]-\frac{\pi}{2}, \frac{\pi}{2}[\ \sim \mathbb{R}$. On the other hand, the linear map $g :]a, b[\ \mapsto\ ]-\frac{\pi}{2}, \frac{\pi}{2}[$, where $g(x) = \frac{x-a}{b-a}\pi - \frac{\pi}{2}$ is a bijection. $\Rightarrow\ ]a, b[\ \sim\ ]-\frac{\pi}{2}, \frac{\pi}{2}[\ \Rightarrow\ ]a, b[\ \sim \mathbb{R}$. □

REMARK: We have used basic properties of $\tan x$ established in calculus, and we will do such things in the course even if we have not defined $\sin x$ and $\cos x$ rigorously.

Exercise 3.6. The book uses a simpler function to start the argument, namely $f :]-1, 1[\ \mapsto \mathbb{R}$, $f(x) = \frac{x}{x^2-1}$ is a bijection. Show that the above is a bijection, and thus prove that $]a, b[\ \sim \mathbb{R}$.

Proof. $y = \frac{x}{x^2-1} \Rightarrow yx^2 - y - x = 0 \Rightarrow x = \frac{1}{2y}(1 \pm \sqrt{1 + 4y^2})$. By observation, we can see that $x^2 - 1 < 0\ \forall x \in\ ]-1, 1[$. Thus, x and y must have opposite signs, so we choose the solution $x = \frac{1}{2y}(1 - \sqrt{1 + 4y^2})$. Since there is no constraint on y, $y \in \mathbb{R}$. Since each x provides a unique solution, the function is also injective. Thus, $]-1, 1[\ \sim \mathbb{R}$.
On the other hand, the linear map $g :]a, b[\ \mapsto\ ]-1, 1[$, where $g(x) = \frac{x-a}{b-a}(2) - 1$ is a bijection. $\Rightarrow\ ]a, b[\ \sim\ ]-1, 1[\ \Rightarrow\ ]a, b[\ \sim \mathbb{R}$. □

Theorem 3.8. $[0, 1[\ \sim\ ]0, 1[$.

Proof. Let A be the set of all rational numbers in $[0, 1[$ and B the set of all rational numbers in $]0, 1[$. Both sets A and B are infinite, and $A, B \subseteq \mathbb{Q}$. Then by the result we have discussed before, $\mathbb{N} \sim A$ and $\mathbb{N} \sim B$. $\Rightarrow A \sim B$ and there exists a bijection $h : A \mapsto B$.
We now define a map $f : [0, 1[\ \mapsto\ ]0, 1[$ by setting:

$$f(x) = \begin{cases} h(x), x \in A \\ x, x \notin A \end{cases}$$

Then f is a bijection, since h is a bijection and mapping x to itself is a bijection. $\Rightarrow [0, 1[\ \sim\ ]0, 1[$. □

Exercise 3.7. Prove that $[0, 1] \sim\ ]0, 1[$.

Proof. Let A be the set of all rational numbers in $[0, 1]$ and B the set of all rational numbers in $]0, 1[$. Both sets A and B are infinite, and $A, B \subseteq \mathbb{Q}$. Then by the result we have discussed before, $\mathbb{N} \sim A$ and $\mathbb{N} \sim B$. $\Rightarrow A \sim B$ and there exists a bijection $h : A \mapsto B$.
We now define a map $f : [0, 1] \mapsto]0, 1[$ by setting:

$$f(x) = \begin{cases} h(x), x \in A \\ x, x \notin A \end{cases}$$

Then f is a bijection, and $[0, 1] \sim]0, 1[$. □

Definition 3.11. (Power set) Given a set A, the power set of A, denoted by $P(A)$, is the collection of all subsets of A.

Theorem 3.9. (Cantor's Power Set Theorem) If A is any set, then there is no surjection of A onto $\mathcal{P}(A)$.

Proof. We argue by contradiction. Suppose that there exists a surjection $f : A \mapsto \mathcal{P}(A)$. For any $a \in A$, the image $f(a)$ is a subset of A. So we have two possibilities. Either $a \in f(a)$, or $a \notin f(a)$. Let $D = \{a \in A : a \notin f(a)\}$. Since D is a subset of A, $D \in \mathcal{P}(A)$, and since we assumed that f is a surjection, $\exists a_0 \in A$ such that $f(a_0) = D$. Now we have two possibilities.

1. $a_0 \in D$. But then $a_0 \notin f(a_0) = D$ by definition of D, which leads to a contradiction.
2. $a_0 \notin D$. But then $a_0 \notin f(a_0)$, so $a_0 \in D$ by definition of D, which leads to a contradiction.

So the assumption that there is a surjection $f : A \mapsto \mathcal{P}(A)$ led to a conclusion that there exists a subset $D \subseteq A$ and $a_0 \in A$ such that neither $a_0 \in D$ nor $a_0 \notin D$. We have a contradiction and our assumption that there exists a surjection $f : A \mapsto \mathcal{P}(A)$ cannot be true. □

It follows from Cantor's Theorem that the set $\mathcal{P}(A)$ for any A, is "strictly larger" than A. Taking an infinite sequence of sets $A, \mathcal{P}(A), \mathcal{P}(\mathcal{P}(A)), \mathcal{P}(\mathcal{P}(\mathcal{P}(A))), \ldots,$ we will end with an infinite sequence of strictly bigger sets.
One consequence is that $\mathcal{P}(\mathbb{N})$ is not countable. In fact, one can show that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. Then, $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ is strictly bigger than $\mathbb{R}$.

4 Sequences

Definition 4.1. (Sequences and Real Sequences) Let A be a set. The map $X : \mathbb{N} \mapsto A$ is called an A-valued sequence indexed by natural numbers. Setting $X(n) = x_n$, one often writes sequences by listing its elements $(x_1, x_2, \dots, x_n, \dots)$ and that is also often abbreviated by $(x_n)_{n\in\mathbb{N}}$.
There are wide ranges of possibilities for set A. The sequence can also be defined more generally by replacing the index set $\mathbb{N}$ by some other set $(\mathbb{N} \times \mathbb{N}, \mathbb{Z})$.
In this course, we will primarily deal with the case where $A = \mathbb{R}$ and index set is $\mathbb{N}$. Such a sequence will be called a real sequence, and we will just drop the real and write (x_n) for the sequence.

Definition 4.2. (Convergence of sequences) A sequence (x_n) converges to a real number x if for every $\epsilon > 0$, there exists natural number $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|x_n - x| < \epsilon$.
One writes $\lim_{n\to\infty} x_n = x$ for the statement that (x_n) converges to x.

The idea behind the concept is that as ϵ gets small, only finitely many elements of the sequence are not in the ϵ neighbourhood of x.

Example 4.1. Let (x_n) be given by $x_n = \frac{1}{n}, n \geq 1$. Then, $\lim_{n\to\infty} = 0$.

Proof. Let $\epsilon > 0$. Set $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, by the Archimedean property. When $n \geq N, \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Thus, $\forall \epsilon > 0, \exists N = \frac{1}{\epsilon}$ such that $\forall n \geq N$, $|x_n - 0| = |\frac{1}{n} - 0| < \epsilon$. Thus, $lim_{n\to\infty} = 0$. $\square$

Example 4.2. Prove that $\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1$.

Proof. Let $\epsilon > 0$. Set $N \in \mathbb{N}$ such that $N > \frac{1}{\sqrt{\epsilon}}$, by the Archimedean property. When $n \geq N$, $|x_n - 1| = |\frac{n^2+1}{n^2} - 1| = |\frac{n^2-1-n^2}{n^2}| = \frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$, and $\lim_{n\to\infty} x_n = 1$. $\square$

Definition 4.3. (Divergence of sequences) If (x_n) is a sequence that does not converge to any real number x, then we say that (x_n) is a divergent sequence, or that (x_n) is divergent.

Example 4.3. Let $x_n = (-1)^n, n = 1, 2, \dots$. Then the sequence (x_n) is divergent.

Proof. Proof by contradiction. Suppose that there exists a real number x such that $\lim_{n\to\infty} x_n = x$. Take $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, we have $|x_n - x| < 1$. Then, for all $n \geq N$ we have: $|x_n - x_{n+1}| = |(-1)^n - (-1)^{n+1}| = 2$. On the other hand, $|x_n - x_{n+1}| = |x_n - x + x - x_{n+1}| \leq |x_n - x| + |x - x_{n+1}| < 1 + 1 = 2$. So the above give that $\forall n \geq N$, $|x_n - x_{n+1} = 2$ and $|x_n - x_{n+1}| < 2$, which is impossible. Thus, our initial assumption that the sequence x_n is convergent cannot be true. $\square$

Exercise 4.1. $\lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$.

Proof. Let $\epsilon > 0$. Set $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^3}$, by the Archimedean property. When $n \geq N$, $|x_n - 0| = \left|\frac{\sin n}{\sqrt[3]{n}} - 0\right| \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon$, and $\lim_{n\to\infty} x_n = 0$. $\square$

Exercise 4.2. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Proof. Let $\epsilon > 0$. Set $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, by the Archimedean property. When $n \geq N$, $|x_n - 0| = \left|\frac{n!}{n^n}\right| = \left|\frac{1}{n} \cdot \frac{2}{n} \cdot \dots \cdot \frac{n}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$, and $\lim_{n\to\infty} x_n = 0$. $\square$

Exercise 4.3. $\lim_{n\to\infty} \frac{(1+2+\dots+n)^2}{n^4} = \frac{1}{4}$.

Proof. We use crucially a lemma that we prove with the Axiom of Induction.
Lemma: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Base case: $n = 1$
Let S be the set of all $n \in \mathbb{N}$ for which Lemma 1 holds. $1 = \frac{1(2)}{2} \Rightarrow 1 \in S$.
Inductive step: $n \to n + 1$
Assume that Lemma 1 holds for n. Then:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \end{aligned}$$

$n \in S \Rightarrow n + 1 \in S$. Thus, $S = \mathbb{N}$.
Let $\epsilon > 0$. Set $N_1 \in \mathbb{N}$ such that $N_1 > \frac{1}{\epsilon}$, by the Archimedean property. Set $N_2 \in \mathbb{N}$ such that $N_2 > \frac{1}{\sqrt{2\epsilon}}$, by the Archimedean property. Let $N = \max\{N_1, N_2\}$. Then, when $n \geq N$, $|x_n - \frac{1}{4}| = \left|\frac{(1+2+\ldots+n)^2}{n^4} - \frac{1}{4}\right| = \left|\frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} - \frac{1}{4}\right| = \left|\frac{n^2(n^2+2n+1)}{4n^4} - \frac{1}{4}\right| = \left|\frac{n^4+2n^3+n^2}{4n^4} - \frac{1}{4}\right| = \left|\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} - \frac{1}{4}\right| = \left|\frac{1}{2n} + \frac{1}{4n^2}\right| \leq \frac{1}{2n} + \frac{1}{4n^2} < \frac{\epsilon}{2} + \frac{1}{4n^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. $\Rightarrow \lim_{n\to\infty} x_n = \frac{1}{4}$. □

4.1 Properties of the Limits

Theorem 4.1. The limit of a sequence, when it exists, it is unique.

Proof. Proof by contradiction. Suppose that there exists real numbers x_1 and x_2 such that $\lim_{n\to\infty} x_n = x_1 = x_2$, but $x_1 \neq x_2$. Then, $\forall \epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N$, we have $|x_n - x_1| < \epsilon$, and there exists $N_2 \in N$ such that $\forall n \geq N$, we have $|x_n - x_2| < \epsilon$. Take $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, $|x_n - x_1| < \epsilon$ and $|x_n - x_2| < \epsilon$. Let $\epsilon = \frac{|x_1 - x_2|}{2}$. Then, $|x_1 - x_2| = |x_1 - x_n + x_n - x_2| \leq |x_n - x_1| + |x_n - x_2| < \epsilon + \epsilon = |x_1 - x_2|$. So $|x_1 - x_2| < |x_1 - x_2|$, which is impossible. Thus, the limit of a sequence, when it exists, is unique. □

Theorem 4.2. Every convergent sequence is bounded. In other words, if (x_n) is a convergent sequence there exists a real number $M > 0$ such that $|x_n| \leq M, \forall n \geq 1$.

Proof. Let x be the limit of the sequence (x_n) such that $\lim_{n\to\infty} x_n = x$. Let $\epsilon = 1$. Then, $\forall N \in \mathbb{N}$ such that for every $n \geq N$, we have $|x_n - x| < \epsilon = 1$. This implies that $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$ for all $n \in \mathbb{N}$. Take $M = \max\{|x_1|, \ldots, |x_N|, 1 + |x|\}$. We claim that $|x_n| \leq M$ $\forall n \geq 1$. To prove this claim, we distinguish two cases:

(a) $1 \leq n \leq N$. Then, since $M = \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$, we certainly have $|x_n| \leq M$.

(b) $n > N$. Then, we have shown that the choice of N implies that $|x_n| < |x| + 1$, and so given our choice of M, we have that $|x_n| \leq M$ also for $n > N$.

This proves the statement. □

Theorem 4.3. (Algebraic Limit Theorems) Let (x_n) and (y_n) be two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} 6y_n = y$. Then,

1. For any real number $c \in \mathbb{R}$, $\lim_{n\to\infty} cx_n = cx$.

2. $\lim_{n\to\infty}(x_n + y_n) = x + y$.

3. $\lim_{n\to\infty}(x_n \cdot y_n = x \cdot y$.

4. Suppose that $y \neq 0$ and that $y_n \neq 0, \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

NOTE: The converse statements in the Algebraic Limit Theorem may not hold. For example, $\lim_{n\to\infty}(x_n + y_n)$ may exist even if individual sequences (x_n) and (y_n) are divergent. Take (x_n) to be any divergent sequence and define $y_n = -x_n$. Then, $x_n + y_n = 0$, so $\lim_{n\to\infty}(x_n + y_n) = 0$ exists, but both (x_n) and (y_n) are divergent. Similarly for (3), if $(x_n) = (-1)^n$ and $(y_n) = (-1)^{n+1}$, then $x_n \cdot y_n = -1, \forall n \in \mathbb{N}$, so $\lim_{n\to\infty} x_n \cdot y_n = -1$, but both (x_n) and (y_n) are divergent sequences.

Proof. (1)
(2)
(3)
(4)

□

5 Limits and Continuity

6 Differentiation