We have a sequential game represented in normal form. This sequential game has 2 players, Alice and Bob. Each player takes turns making actions. We assume that the game is general, without repeated outcomes for strategy profiles. We assume that no player plays twice in a row. Finally, we assume crucially that there is no node where only one action is possible. If there exists a node where only one action is possible, this action is forced to be part of every strategy profile. We would not be able to meaningfully differentiate where this action is in the game.

Proof. Let the game have n nodes, such that strategy profiles consist of n actions. We refer to the l-th node as N_l, and we call the set of all nodes $N = \{N_1 \dots N_n\}$. Let f_l be the number of actions available at node N_l. Then, we name the i-th action available at node l as $x_{l,i}$, where the first number refers to the node number and the second number refers to the action number. We call an end of the sequential game an end of the graph where an payoff is reached.

Let's say the normal form representation of the game has the dimensions $a \times b$. Since the payoff at each end of the sequential game is unique, if the payoff of a strategy profile matches the payoff of another strategy profile, both profiles go to the same end of the graph. We follow the following procedure:

Let K be the set of all the payoffs in the normal form game. We choose an arbitrary payoff $k \in K$ from the normal form game. Let S_k be the set of all the strategy profiles that give us this payoff, where $s_\mu \in S_k$, a strategy profile, is a set of actions that make up that strategy profile.

Let U_k be the set of all nodes that are not essential to reaching the payoff k. For a node to be non-essential to reach the payoff, this means that to reach payoff k, we do not need to traverse this node, so whichever action we take at this node is trivial.

Let $x_{j,\alpha}$ and $x_{j,\beta}$ be two arbitrary actions possible at arbitrary node N_j. If $\exists s_1, s_2 \in S_k$ such that $x_{j,\alpha} \in s_1$, and $x_{j,\beta} \in s_2$, then this shows that payoff k can be reached regardless of the action taken at N_j. Thus, $N_j \in U_k$.

We know that we can get a subset of U_k using the following criteria, but we want to prove that this condition is true for all $N_j \in U_k$. For this, we use crucially the idea that each node has at least 2 actions from our assumptions above. Then, at node N_j, there will be at least 2 actions $x_{j,1}$ and $x_{j,2}$. If this node N_j is not traversed to reach the payoff k, then there must exist strategy profile $s_1 \in S_k$ such that $x_{j,1} \in s_1$, and another strategy profile $s_2 \in S_k$ such that $x_{j,2} \in s_2$. Thus, all non-essential nodes are collected.

Now that we have the set of all the non-essential nodes U_k, let E_k be all the other nodes that we have now found essential, such that $E_k = N \setminus U_k$. All these nodes in E_k have a very specific feature. $\forall N_\epsilon \in E_k$, we have that $\exists r \in [1, f_\epsilon]$ such that $x_{\epsilon,r} \in s, \forall s \in S_k$, where r is a specific and unique integer. This action is part of the path to reach payoff k. For each node in E_k, we can find an action which fulfils this condition, so we let A_k be the set of all these actions that consist of the path (shortest walk) to k. We call A_k the action path to k.

We repeat this procedure for all of the payoffs that appear in the normal form game, and we let $A = \{A_k : \forall k \in K\}$, which is the set of action paths to all payoffs. We can already find the deepest tree of the game. The action path with the deepest tree is $\{A_{k_0} : |A_{k_0}| \geq |A_k|, \forall k \in K\}$. The size of this set A_{k_0} is the longest path in the game.

Not only that, but we can also reconstruct the whole sequential game. To do this, note that A_k consists of the path to the payoff k. Iterating through $x \in A_k \in A$, we can easily find a structure. If an action x_2 never appears without an action x_1 for all $A_k \in A$, but $\exists A_k \in A$ such that action $x_1 \in A_k$ but $x_2 \notin A_k$, then action x_1 is taken at a parent node N_p of the child node N_c where action x_2 is taken. Following through this process, using again the assumption that no node only has one action available, we can recreate the entire structure. This is the case because if the action x_1 is taken at N_p and action x_2 is taken at N_c, there must exist some other action x_3 at node N_c based on our assumption. Then, $\{x_1, x_3\}$ is in some action path in A, as it leads to a different payoff, due to the fact that no two payoffs are the same.
We also have that this creates a unique graph, because as each parent-child relation is uncovered, there will only be one way to write the graph such that all of them hold. We create this unique graph like so:

1. We start with the root of the graph, which is the highest node. We take the power set of A, $\mathcal{P}(A)$, which is the set of all the possible subsets of A. $\forall B \in \mathcal{P}(A)$, we take the intersection of $\bigcap_{A_k \in B} A_k$. If the $|\bigcap_{A_k \in B} A_k| = 1$, then the shared action in the action profiles $A_k \in B$ must be connected to the root of the graph. In this way, we find all the actions connected to the root of the graph, since to reach a child node, we must pass through a parent node.

2. We create nodes at the end of all actions connected to the root of the graph. Then, repeat step 1 at each of the nodes. We iterate until we retrieve the full graph.

The simple process without proof is as follows:

1. Choose a payoff k in the normal form representation of the game.

2. Write down all the strategy profiles that get you that payoff.

3. If an action is shared in all the strategy profiles that get you that payoff, add it to a set A_k which we call the action path to k.

4. Iterate through steps 1-3 for all payoffs in the normal form representation of the game. Then, the deepest path of the game is the length of the longest A_k.

5. Then, start with the root of the graph, which is the highest node. We take the power set of A, $\mathcal{P}(A)$. $\forall B \in \mathcal{P}(A)$, we take the intersection of $\bigcap_{A_k \in B} A_k$. If the $|\bigcap_{A_k \in B} A_k| = 1$, then the shared action in the action profiles $A_k \in B$ is connected to the root of the graph. In this way, we find all the actions connected to the root of the graph.

6. We create nodes at the end of all actions connected to the root of the graph. Then, repeat step 1 at each of the nodes. We iterate until we retrieve the full graph.

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