MATH 480: Energy

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In this section, we will consider only finite graphs, where V and E are both finite. We define $\ell^2(V)$ to be the real Hilbert space of functions on $V(G)$ with the inner product

$$(f, g) \coloneqq \sum_{x \in V} f(x)g(x).$$

Since we are interested in flows — which can occur in both directions — we define each edge occurring with both orientations, so a positive flow from $x \in V$ to $y \in V$ is read as a negative flow from y to x. Thus, we are only interested in antisymmetric functions θ on E, where

$$\theta(-e) = -\theta(e),\ \forall e \in E.$$

We define $\ell^2_-(E)$ as the space of anti-symmetric functions θ on E, with the inner product

$$(\theta, \theta') \coloneqq \frac{1}{2} \sum_{e \in E} \theta(e)\theta'(e) = \sum_{e \in E_{1/2}} \theta(e)\theta'(e),$$

where $E_{1/2} \subset E$ is the set of edges containing exactly one direction of each edge in the graph.

We want to define maps between $\ell^2(V)$ and $\ell^2_-(E)$, which leads us to our definition of the coboundary operator and the boundary operator.

Definition 1 (Coboundary operator)**.** We define the coboundary operator $d : \ell^2(V) \to \ell^2_-(E)$ to be

$$(df)(e) \coloneqq f(e^-) - f(e^+).$$

Since current flows from greater to lesser voltage, the more useful definition is to take the difference between voltage at the tail and the head, rather than the other way round.

Definition 2 (Boundary operator)**.** We define the boundary operator $d^* : \ell^2_-(E) \to \ell^2(V)$ to be

$$(d^*\theta)(x) \coloneqq \sum_{e^- = x} \theta(e)$$

Both the coboundary and boundary operators are linear, and they are adjoint.

Lemma 3. *The coboundary and boundary operator are adjoints of each other, so $\forall f \in \ell^2(V)$ and $\forall \theta \in \ell^2_-(E)$,*

$$(\theta, df) = (d^*\theta, f).$$

Proof.

$$\begin{aligned}
(d^*\theta, f) &= \sum_{x \in V} d^*\theta(x) f(x) \\
&= \sum_{x \in V} \left(\sum_{e^- = x} \theta(e) \right) f(x) = \sum_{x \in V} \left(\sum_{e^- = x} \theta(e) f(e^-) \right) \\
&= \sum_{e \in E_{1/2}} \theta(e)(f(e^-) - f(e^+)) = \sum_{e \in E_{1/2}} \theta(e) df(e) \\
&= (\theta, df)
\end{aligned}$$

□

This allows us to rewrite expressions with these operators:

- **Ohm's Law:** $dv(e) = i(e)r(e)$, $\forall e \in E$.
- **Kirchhoff's Node Law:** $d^*i(x) = 0$, $\forall x \notin A \cup Z$.

This also allows us to define some flow terminology, abstracting from current. Let $\theta \in \ell^2_-(E)$ be a flow function. Then, the flow into a network at a vertex a is $d^*\theta(a)$. So we call $\theta \in \ell^2_-(E)$ a **flow between** A to Z if $d^*\theta$ is 0 off of $A \cup Z$ — if it is nonnegative on A and nonpositive on Z, we say θ is a flow **from** A to Z.

Definition 4 (Strength)**.** We define the strength of the flow as the total amount flowing into the network

$$\text{Strength}(\theta) \coloneqq \sum_{a \in A} d^*\theta(a).$$

We want to prove a crucial property, flow conservation, in order to show that the strength of the network is also the total amount flowing out of the network.

Lemma 5 (Flow Conservation)**.** *Let G be a finite graph and A and Z be two disjoint subsets of its vertices. If θ is a flow between A and Z, then*

$$\sum_{a \in A} d^*\theta(a) = -\sum_{z \in Z} d^*\theta(z).$$

Proof.

$$\begin{aligned}
\sum_{a \in A} d^*\theta(a) + \sum_{z \in Z} d^*\theta(z) &= \sum_{x \in A \cup Z} d^*\theta(x) = \sum_{x \in V} d^*\theta(x) \mathbb{1}_{[x \in A \cup Z]} \\
&= (d^*\theta, \mathbb{1}_{[x \in A \cup Z]}) = (\theta, d\mathbb{1}_{[x \in A \cup Z]}) \\
&= (\theta, \mathbb{1}_{[\text{one of } e^-, e^+ \in A \cup Z]}) \\
&= 0
\end{aligned}$$

□

Lemma 6. *Let G be a finite graph and A and Z be two disjoint subsets of its vertices. If θ is a flow from A to Z and $f \restriction A$, $f \restriction Z$ are constants α and ζ respectively, then*

$$(\theta, df) = \textit{Strength}(\theta)(\alpha - \zeta).$$

Proof.

$$\begin{aligned}
(\theta, df) = (d^*\theta, f) &= \sum_{x \in V} d^*\theta(x) f(x) = \sum_{a \in A} d^*\theta(a) f(a) + \sum_{z \in Z} d^*\theta(z) f(z) \\
&= \sum_{a \in A} d^*\theta(a)\alpha + \sum_{z \in Z} d^*\theta(z)\zeta = \sum_{a \in A} d^*\theta(a)\alpha - \sum_{a \in A} d^*\theta(a)\zeta \\
&= \text{Strength}(\theta)(\alpha - \zeta)
\end{aligned}$$

□

We define the following notation:

$$(f, g)_h \coloneqq (fh, g) = (f, gh)$$

$$\|f\|_h \coloneqq \sqrt{(f, f)_h}.$$

Definition 7 (Energy)**.** For an antisymmetric function θ, we define its energy to be

$$\mathscr{E}(\theta) \coloneqq \|\theta\|_r^2$$

where r is the collection of resistances.

This gives us that $\mathscr{E}(i) = (i, i)_r = (i, dv)$. If i is a unit current from A to Z with voltages v_A and v_Z constant on A and Z respectively, then by the lemma above,

$$\mathscr{E}(i) = v_A - v_Z = \mathscr{R}(A \leftrightarrow Z).$$

We can also use the inner product to express Kirchhoff's laws. Let $\boldsymbol{\chi}^e := \mathbb{1}_{\{e\}} - \mathbb{1}_{\{-e\}}$ denote the unit flow along e represented as an antisymmetric function in $\ell^2_-(E)$. Note that for every antisymmetric function θ and every e, we get

$$(\boldsymbol{\chi}^e, \theta)_r = \sum_{e \in E_{1/2}} (\mathbb{1}_{\{e\}} - \mathbb{1}_{\{-e\}})\theta(e)r(e) = \theta(e)r(e),$$

which gives us that

$$\left(\sum_{e^- = x} c(e)\boldsymbol{\chi}^e, \theta \right)_r = \sum_{e^- = x} \theta(e)r(e)c(e) = d^*\theta(x).$$

Let i be any current. Then:

- **Kirchhoff's Node Law:** For every $x \notin A \cup Z$, we have that

$$\left(\sum_{e^- = x} c(e)\boldsymbol{\chi}^e, i \right)_r = d^*i(x) = 0$$

- **Kirchhoff's Cycle Law:** If $e_1, e_2, \ldots, e_n$ is an oriented cycle in G, then

$$\left(\sum_{k=1}^{n} \boldsymbol{\chi}^{e_k}, i \right)_r = 0$$

Finally, we define $\sum_{e^- = x} c(e)\boldsymbol{\chi}^e$ to be the star at x, and we let $\bigstar$ denote the subspace in $\ell^2_-(E)$ spanned by all the stars, which we call the star space of G. We let $\diamondsuit$ denote the subspace spanned by all the cycles $\sum_{k=1}^{n} \boldsymbol{\chi}^{e_k}$, where $e_1, e_2, \ldots, e_n$ forms an oriented cycle, which we call the cycle space of G.

We can note that these subspaces are orthogonal to each other, and the sum of $\bigstar$ and $\diamondsuit$ is all of $\ell^2_-(E)$. To see this, suppose a flow $\theta \in \ell^2_-(E, r)$ was orthogonal to both $\bigstar$ and $\diamondsuit$. Since θ is orthogonal to $\diamondsuit$, it satisfies Kirchhoff's cycle law, and there is a function F such that $\theta = c\, dF$. Since θ is orthogonal to $\bigstar$, F must be harmonic. Since G is assumed to be finite in this chapter, the uniqueness principle implies that F is constant on each component of G, so $\theta = 0$. Thus we have proven that only the zero vector is orthogonal to both $\bigstar$ and $\diamondsuit$.

By Kirchhoff's Cycle Law above, we know that i is orthogonal to $\diamondsuit$, so it is in $\bigstar$. We also know that any $i \in \bigstar$ is a current, if we set $W = \{x : d^*i(x) = 0\}$. If θ is any antisymmetric function such that $d^*\theta = d^*i$, then $\theta - i$ is a sourceless flow, so it is orthogonal to $\bigstar$ and is an element of $\diamondsuit$. Thus,

$$\theta = i + (\theta + i)$$

is an orthogonal decomposition of θ relative to $\ell^2_-(E, r) = \bigstar \oplus \diamondsuit$. Crucially, let $P_\bigstar : \ell^2_-(E, r) \to \bigstar$ be the orthogonal projection onto the star space. Then,

$$i = P_\bigstar \theta$$

and

$$\|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2.$$

Theorem 8 (Thomson's Principle)**.** *Let G be a finite network and A and Z be two disjoint subsets of its vertices. Let θ be a flow from A to Z and i be the current flow from A to Z with $d^*i = d^*\theta$. Then $\mathscr{E}(\theta) > \mathscr{E}(i)$ unless $\theta = i$.*

Proof. This follows directly from the decomposition of $\|\theta\|_r^2$ above. □

We can construct an orthogonal basis $\{\boldsymbol{\chi}^e : e \in E_{1/2}\}$ of $\ell^2_-(E, r)$, but these vectors are not necessarily unit vectors, since $\|\boldsymbol{\chi}^e\|_r^2 = r(e)$. When we represent $P_\bigstar$ in the this basis, we get:

$$(P_\bigstar \boldsymbol{\chi}^e, \boldsymbol{\chi}^{e'})_r = (i^e, \boldsymbol{\chi}^{e'})_r = i^e(e')r(e')$$

where i^e is the unit current from e^- to e^+. Thus, we get that the matrix coefficient at (e', e) is

$$\frac{(P_\bigstar \boldsymbol{\chi}^e, \boldsymbol{\chi}^{e'})_r}{\|\boldsymbol{\chi}^{e'}\|_r^2} = i^e(e') =: Y(e, e')$$

which is the current that flows across e' when a unit current is imposed between endpoints of e. We call this matrix the transfer current matrix.

Corollary 9 (Reciprocity law)**.** *Since $P_\bigstar$ is an orthogonal projection, it is self-adjoint, so $(P_\bigstar \boldsymbol{\chi}^e, \boldsymbol{\chi}^{e'})_r = (\boldsymbol{\chi}^e, P_\bigstar \boldsymbol{\chi}^{e'})_r$. This gives us that*

$$Y(e, e')r(e') = Y(e', e)r(e)$$

We are interested in using these new definitions to analyse the escape probabilities $P[a \to Z]$ to determine if a network is transient or recurrent, especially when an edge is removed or added, or the conductance of an edge changes. Since we know from before that

$$P[a \to Z] = \frac{\mathscr{C}(a \leftrightarrow Z)}{\pi(a)},$$

then if no edge incident to a is affected, we only need to study the change in effective conductance.

Theorem 10 (Rayleigh's Monotonicity Principle)**.** *Let G be a connected graph with two assignments, c and c', of conductances on G with $c \le c'$ (everywhere).*

(i) If G is finite and A and Z two disjoint subsets of its vertices, then $\mathscr{C}_c(A \leftrightarrow Z) \le \mathscr{C}_{c'}(A \leftrightarrow Z)$.

(ii) If G is infinite and a is one of its vertices, then $\mathscr{C}_c(a \leftrightarrow \infty) \le \mathscr{C}_{c'}(a \leftrightarrow \infty)$. In particular, if (G, c) is transient, then so is (G, c').

Proof. If i is a unit current flow from A to Z, then

$$\mathscr{C}(A \leftrightarrow Z) = \frac{1}{\mathscr{R}(A \leftrightarrow Z)} = \frac{1}{\mathscr{E}(i)}.$$

We know that

$$\mathscr{E}_c(i_c) \ge \mathscr{E}_{c'}(i_c) \ge \mathscr{E}_{c'}(i_{c'}).$$

The first inequality comes from the definition of energy, since when conductance increases and current remains constant, the $\mathscr{E} = (i, i)_r$ identity shows that energy decreases. The second inequality comes from Thomson's principle above. Then, taking the reciprocals gives the result. □

Removing an edge decreases effective conductance, so if the edge is not incident to a, its removal decreases $P[a \leftrightarrow Z]$. Contracting an edge increases the effective conductance between any set of vertices, so it increases $P[a \leftrightarrow Z]$.