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Maxmin and Minmax

Prove that max min $\leq$ min max for any finite game.

Proof. First, we prove that $\min_{s_2} u(s_1, s_2) \leq \max_{s_1} u(s_1, s_2)$, for all finite games. By the definition of $\max_{s_1} u(s_1, s_2)$, for any arbitrary $k_2 \in S_2$,

$$u(s_1, k_2) \leq \max_{s_1} u(s_1, k_2), \forall s_1 \in S_1 \tag{1}$$

By the definition of $\min_{s_2} u(s_1, s_2)$, for any arbitrary $k_1 \in S_1$,

$$\min_{s_2} u(k_1, s_2) \leq u(k_1, s_2), \forall s_2 \in S_2 \tag{2}$$

Setting $s_2 = k_2$ in equation (2) and $s_1 = k_1$ in equation (1), we have

$$\begin{aligned} \min_{s_2} u(k_1, s_2) &\leq u(k_1, k_2) \\ u(k_1, k_2) &\leq \max_{s_1} u(s_1, k_2) \\ \min_{s_2} u(k_1, s_2) &\leq \max_{s_1} u(s_1, k_2) \end{aligned}$$

Since this inequality is true for any arbitrary k_1 and k_2, we have that

$$\min_{s_2} u(s_1, s_2) \leq \max_{s_1} u(s_1, s_2), \forall s_1 \in S_1, \forall s_2 \in S_2$$

Let m be the set of all $\{\min_{s_2} u(s_1, s_2) : s_1 \in S_1\}$ and M be the set of all $\{\max_{s_1} u(s_1, s_2) : s_2 \in S_2\}$. Then,

$$\begin{aligned} \max_{s_1} \min_{s_2} u(s_1, s_2) &\in m \\ \min_{s_2} \max_{s_1} u(s_1, s_2) &\in M \\ m_k &\leq M_k, \forall m_k \in m, \forall M_k \in M \\ \max_{s_1} \min_{s_2} u(s_1, s_2) &\leq M_k, \forall M_k \in M \\ \Rightarrow \max_{s_1} \min_{s_2} u(s_1, s_2) &\leq \min_{s_2} \max_{s_1} u(s_1, s_2) \end{aligned}$$

□

If the game is a zero-sum game, can you say more?

Proof. If the game is a zero-sum game,

$$\max_{s_1} \min_{s_2} u(s_1, s_2) = \min_{s_2} \max_{s_1} u(s_1, s_2) \tag{3}$$

This is because when the game is zero-sum, $\forall s \in S, u_1(s) + u_2(s) = 0$. Thus, when $\min_{s_2} u(s_1, s_2)$ is achieved, $\max_{s_1} u(s_1, s_2)$ is achieved simultaneously. Then, when $\max_{s_1} \min_{s_2} u(s_1, s_2)$ is achieved, $\min_{s_2} \max_{s_1} u(s_1, s_2)$ is achieved simultaneously. □