MATH 480: Transience and Recurrence

Tiffany Yong

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Recall that $\mathscr{C}(a \leftrightarrow \infty) > 0 \iff$ the random walk is transient, since $C(a \leftrightarrow \infty) = P[a \to \infty]\pi(a)$, and $\mathscr{C}(a \leftrightarrow \infty) > 0 \Rightarrow P[a \to \infty] > 0$. Thus, if $\mathscr{C}(a \leftrightarrow \infty) = 0$, then the random walk is recurrent. To show recurrence, we can express a lower bound l on effective resistance

$$\mathscr{R}(a \leftrightarrow \infty) = \frac{1}{\mathscr{C}(a \leftrightarrow \infty)} \geq l,$$

and show that l is infinite.

We want to use the energy formulation of effective resistance defined before, where if i is a unit current from A to Z, then $\mathscr{E}(i) = \mathscr{R}(A \leftrightarrow Z)$.

Definition 1 (Cutset on finite networks)**.** Let A and Z be two disjoint sets of vertices. Then, a set Π of edges separates A and Z if every path with one endpoint in A and the other endpoint in Z must include an edge in Π. We call this set of edges Π a cutset.

Theorem 2 (Nash-Williams Inequality)**.** *If a and z are distinct vertices in a finite network that are separated by pairwise disjoint cutsets $\Pi_1, \ldots, \Pi_n$, then*

$$\mathscr{R}(a \leftrightarrow z) \geq \sum_{k=1}^{n} \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Proof. By the energy formulation of effective resistance, it suffices to show that the unit current flow i from a to z has energy at least the right-hand side.

We are given a finite cutset Π that separates a from z. Let Z be the set of endpoints of Π that are separated by Π from a, and let K denote the set of vertices that are not separated from a by Π. Let $H := G \restriction (K \cup Z)$ be the subnetwork of G induced by $K \cup Z$. Then, i induces a unit flow i_H from a to Z.

Applying flow conservation to H gives us that

$$1 = -\sum_{x \in Z} d^* i_H(x) = -\sum_{e^- \in Z, e \in H} i(e).$$

If both the head and tail of e lie in Z, then $i(e)$ and $i(-e)$ are both in the right-hand side summation, so they cancel out. All edges in H with only one endpoint in Z must lie in Π, by definition. This gives us that

$$1 = -\sum_{e^- \in Z, e \in H} i(e) = \sum_{e \in \Pi} |i(e)|.$$

Then, by the Cauchy-Schwarz inequality, we get:

$$\sum_{e \in \Pi} i(e)^2 r(e) \sum_{e \in \Pi} c(e) \geq \left(\sum_{e \in \Pi} |i(e)| \sqrt{r(e)} \sqrt{c(e)} \right)^2 = \left(\sum_{e \in \Pi} |i(e)| \right)^2 \geq 1.$$

So, we get that

$$\sum_{e \in \Pi} i(e)^2 r(e) \geq \left(\sum_{e \in \Pi} c(e) \right)^{-1}.$$

If we substitute Π for Π_k and take the summation over $k = 1, \ldots, n$, we finally get

$$\mathscr{R}(a \leftrightarrow z) = \mathscr{E}(i) = \sum_{k=1}^{n} \left(\sum_{e \in \Pi_k} i(e)^2 r(e) \right) \geq \sum_{k=1}^{n} \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

□

In order to apply this same principle to infinite networks, we define a cutset on infinite networks.

Definition 3 (Cutset on infinite networks)**.** We say a set of edges Π separates a and ∞ if every infinite simple path from a must include an edge in Π. We call Π a cutset as well.

Theorem 4 (Nash-Williams Criterion)**.** *If $\langle \Pi_n \rangle$ is a sequence of pairwise disjoint finite cutsets in a locally finite network G, each of which separates a from ∞, then*

$$\mathscr{R}(a \leftrightarrow \infty) \geq \sum_n \left(\sum_{e \in \Pi_n} c(e) \right)^{-1}.$$

In particular, if the right-hand side is infinite, then G is recurrent.

Proof. For each $n \geq 1$, we choose a finite subnetwork G_n which contains $\bigcup_{k=1}^n \Pi_k$, and identify $G \setminus G_n$ to a single vertex, z_n to form the finite network G_n^W. Then, as we have shown before,

$$\mathscr{R}(a \leftrightarrow \infty) = \lim_{n \to \infty} \mathscr{R}(a \leftrightarrow z_n),$$

so the Nash-Williams Inequality gives us the conclusion. □

Remark 5. If the cutsets can be ordered so that Π_1 separates a from Π_2 and for all $n > 1$, Π_n separates Π_{n-1} from Π_{n+1}, then the summation in the statement of the Nash-Williams Criterion has a natural interpretation.

Identify all vertices between Π_n and Π_{n+1} into a single vertex U_n. Identify all the vertices that Π_1 separates from ∞ into one vertex U_0. Then, only the parallel edges of Π_n join U_{n-1} to U_n. Finally, replace these edges by a single edge of resistance $\left(\sum_{e \in \Pi_n} c(e)\right)^{-1}$. Then, this new network is a series network with effective resistance from U_0 to ∞ equal to the right-hand side of the Nash-Williams Criterion.

Here, Rayleigh's monotonicity principle shows that the effective resistance from a to ∞ in G is at least the right-hand side of the Nash-Williams Criterion.

Theorem 6 (First part of Pólya's Theorem)**.** *Simple random walk on the nearest-neighbour graph of $\mathbb{Z}^d$ is recurrent for $d = 1, 2$.*

Proof. For $d = 1, 2$, we can use the Nash-Williams criterion with the cutsets

$$\Pi_n := \left\{ e : d(\mathbf{0}, e^-) = n - 1,\ d(\mathbf{0}, e^+) = n \right\},$$

where $\mathbf{0}$ is the origin and $d(\cdot, \cdot)$ is the graph distance.

When $d = 1$, we know that

$$\mathscr{R}(a \leftrightarrow \infty) \geq \sum_n |\Pi_n|^{-1} = \sum_n \frac{1}{2} = \infty.$$

When $d = 2$, we know that

$$\mathscr{R}(a \leftrightarrow \infty) \geq \sum_n |\Pi_n|^{-1} = \sum_n \frac{1}{8k + 4} = \infty,$$

so the random walk on the nearest-neighbour graph of $\mathbb{Z}^d$ is recurrent for $d = 1, 2$. □

We also want to show that a simple random walk on $\mathbb{Z}^d$ is transient for $d \geq 3$, but this requires a more involved technique.

Let $G = (V, E)$ be a denumerable network, and we use the definition of $\ell^2(V)$, $\ell^2_-(E, r)$ as well as the coboundary operator d and boundary operator d^* covered in last class. Here, E might be infinite, so we require that $\sum_{e^- = x} |\theta(e)| < \infty$ for our definition of the boundary operator to be valid.

Suppose that V is finite and $\sum_e |\theta(e)| < \infty$. Then, we know from before that the coboundary operator and boundary operator are adjoint. Then, both flow conservation and Lemma 6 ($(\theta, df) = \text{Strength}(\theta)(\alpha - \zeta)$) from the previous class continue to hold. Finally, the rest of the theorems covered in last class (Thomson's Principle, reciprocity law, Rayleigh's Monotonicity) also holds because of the following consequence of Cauchy-Schwarz:

$$\forall x \in V \qquad \sum_{e^- = x} |\theta(e)| \leq \sqrt{\sum_{e^- = x} \theta(e)^2 / c(e) \cdot \sum_{e^- = x} c(e)} \leq \sqrt{\mathscr{E}(\theta) \pi(x)}.$$

In particular, we know that if $\mathscr{E}(\theta) \leq \infty$, then $\sum_{e^- = x} |\theta(e)| < \infty$, and $d^*\theta$ is defined.

Definition 7 (Flow and unit flow)**.** We call an antisymmetric function θ on the edges E of a possibly infinite graph a flow if

$$\forall x \in V \qquad \sum_{e^- = x} |\theta(e)| < \infty.$$

If θ also satisfies $d^*(\theta)(x) = \mathbb{1}_{\{a\}}(x)$ then θ is a unit flow from $a \in V$ to ∞.

Theorem 8 (Energy and Transience)**.** *Let G be a denumerable connected network. Random walk on G is transient iff there is a unit flow on G of finite energy from some (every) vertex to ∞.*

Proof. ($\Rightarrow$) Let G_n be the sequence of finite induced subgraphs that exhaust G. Recall that G_n^W is the graph obtained from G by identifying $G \setminus G_n$ to a single vertex z_n, and removing loops (but keeping multiple edges). Fix any vertex $a \in G$ which, without loss of generality, belongs to each G_n. Then, by defnition, $\mathscr{R}(a \leftrightarrow \infty) = \lim_{n\to\infty} \mathscr{R}(a \leftrightarrow z_n)$. Let i_n be the unit current flow in G_n^W from a to z_n, and v_n be the corresponding voltage. Then, $\mathscr{E}(i_n) = \mathscr{R}(a \leftrightarrow z_n)$, so $\mathscr{R}(a \leftrightarrow \infty) < \infty \Leftrightarrow \lim_{n\to\infty} \mathscr{E}(i_n) < \infty$.

Notice that each edge of G_n^W comes from an edge in G, and can be identified with it, even if one endpoint might be different.

If θ is a unit flow on G from a to ∞ that has finite energy, then the restriction $\theta \restriction G_n^W$ of θ to G_n^W is a unit flow from a to z_n. Then, Thomson's principle gives us:

$$\mathscr{E}(i_n) \le \mathscr{E}(\theta \restriction G_n^W) \le \mathscr{E}(\theta) < \infty.$$

In particular, we get that $\lim_{n\to\infty} \mathscr{E}(i_n) < \infty$, so the random walk is transient.

($\Leftarrow$) Suppose G is transient. Then, there must be some $M < \infty$ such that $\mathscr{E}(i_n) \le M$ for all n. Begin a random walk at a. Let $Y_n(x)$ be the number of visits to x before hitting $G \setminus G_n$, and $Y(x)$ be the total number of visits to x. This is merely Green's function as defined before. Then, $Y_n(x)$ increases monotonically to $Y(x)$. By the monotone convergence theorem and the properties of Green's function as voltage, we get

$$E[Y(x)] = \lim_{n\to\infty} E[Y_n(x)] = \lim_{n\to\infty} \pi(x)v_n(x) =: \pi(x)v(x).$$

By transience, we know that $E[Y(x)] < \infty$, so $v(x) < \infty$ as well. So, $i := c \cdot dv = \lim_{n\to\infty} c \cdot dv_n = \lim_{n\to\infty} i_n$ exists and is a unit flow from a to ∞ of energy at most M. $\square$

So, we can use the rest of our electrical terminology on infinite networks.

Proposition 9. *Let G be a transient connected network and G_n be finite induced subnetworks that contain a vertex a and that exhaust G. Identify $G \setminus G_n$ to z_n, forming G_n^W. Let i_n be the unit current flow in G_n^W from a to z_n. Then $\langle i_n \rangle$ has a pointwise limit i on G, which is the unique unit flow on G from a to ∞ of minimum energy. Let v_n be the voltages on G_n^W corresponding to i_n and with $v_n(z_n) := 0$. Then, $v := \lim v_n$ exists on G and has the following properties:*

$$\begin{aligned} dv &= ir \\ v(a) &= \mathscr{E}(i) = \mathscr{R}(a \leftrightarrow \infty), \\ \forall x \quad v(x)/v(a) &= P_x[\tau_a < \infty] \end{aligned}$$

Start a random walk at a. For all vertices x, the expected number of visits to x is $\mathscr{G}(a, x) = \pi(x)v(x)$. For all edges, the expected signed number of crossings of e is $i(e)$.

Proof. We saw in the proof above that v and i exist, $dv = ir$ and $\mathscr{G}(a, x) = \pi(x)v(x)$. We have also proved before that for all edges e, the expected signed number of crossings of e is $i(e)$.

Since the events $[\tau_a < \tau_{G\setminus G_n}]$ are increasing in n with union $[\tau_a < \infty]$, we have

$$\frac{v(x)}{v(a)} = \lim_{n\to\infty} \frac{v_n(x)}{v_n(a)} = \lim_{n\to\infty} P_x^{G_n^W}[\tau_a < \tau_{z_n}] = \lim_{n\to\infty} P_x^G[\tau_a < \tau_{G\setminus G_n}] = P_x^G[\tau_a < \infty].$$

We have that $v(a) = \lim v_n(a) = \lim \mathscr{E}(i_n) = \lim \mathscr{R}(a \leftrightarrow z_n) = \mathscr{R}(a \leftrightarrow \infty)$. We know that $\mathscr{E}(i) \le \liminf \mathscr{E}(i_n)$. Since $\mathscr{E}(i_n) \le \mathscr{E}(i)$ as shown above, we have that $\mathscr{E}(i) = \lim \mathscr{E}(i_n) = v(a)$. Likewise, we know that $\mathscr{E}(i_n) \le \mathscr{E}(\theta)$ for every unit flow from a to ∞, so i has minimum energy.

Finally, we establish the uniqueness of a unit flow from a to ∞ with minimum energy. Note that for all flows θ, θ',

$$\frac{\mathscr{E}(\theta) + \mathscr{E}(\theta')}{2} = \mathscr{E}\left(\frac{\theta + \theta'}{2}\right) + \mathscr{E}\left(\frac{\theta - \theta'}{2}\right).$$

If both θ and θ' both have minimum energy, so does $(\theta + \theta')/2$, so $\mathscr{E}((\theta - \theta'/2) = 0$, which gives us that $\theta = \theta'$. $\square$

So we can call i the unit current flow and v the voltage on G. We can think of G as having 0 voltage at ∞.

By the theorem above and Rayleigh's monotonicity principle, the type of random walk does not change when the conductances are changed by bounded factors.

Now the question is how we determine if there is a flow from a to ∞ of finite energy. A useful technique involves flows created from random paths. Suppose P is a probability measure on paths $\langle e_n : n \geq 0 \rangle$ from a to z on a finite graph, or from a to ∞ on an infinite graph. An infinite path is said to go to ∞ when no vertex is visited infinitely many times. Define

$$\theta(e) := \sum_{n \geq 0} (P[e_n = e] - P[e_n = -e]),$$

given that $\sum_{n \geq 0}(P[e_n = e] + P[e_n = -e]) < \infty$. For example, this condition holds when paths are edge-simple, since the sum equals to the expected number of times that e is traversed in either direction. Each path $\langle e_n : n \geq 0 \rangle$ determines a unit flow ψ from a to z by sending 1 along each edge in the path:

$$\psi := \sum_{n \geq 0} \boldsymbol{\chi}^{e_n}.$$

If the condition holds for all e, then θ is defined everywhere. Now, θ is an expectation of a random unit flow, so θ is a unit flow itself. We saw in the proposition above that the walk $\langle X_n : n \geq 0 \rangle$ gives rise to the path $\langle e_n : n \geq 0 \rangle$ with $e_n := \langle X_n, X_{n+1} \rangle$ if we view current as edge crossings. However, there are other useful pairs of random paths and their expected flows as well.

Theorem 10 (Second part of Pólya's Theorem)**.** *Simple random walk on the nearest-neighbour graph of $\mathbb{Z}^d$ is transient for all $d \geq 3$.*

Proof. By Rayleigh's monotonicity principle, it suffices to show transience for $d = 3$, since conductance in the random walk will only decrease as dimension increases.

Let L be a random uniformly distributed ray from the origin $\mathbf{0}$ of $\mathbb{R}^3$ to ∞. Let $\mathcal{P}(L)$ be a simple path in $\mathbb{Z}^3$ from $\mathbf{0}$ to ∞ that stays within distance 4 of L, chosen to be the (almost surely unique) closest path to L in the Hausdorff metric.

Define the flow θ by

$$\theta(e) := \sum_{n \geq 0} (P[\mathcal{P}(L) = e] - P[\mathcal{P}(L) = -e]).$$

Then, θ is a unit flow from $\mathbf{0}$ to ∞. We claim it has finite energy. There is some constant A such that if e is an edge whose midpoint is at Euclidean distance R from $\mathbf{0}$, then $P[e \in \mathcal{P}(L)] \leq A/R^2$. Since all edge centers are separated from each other by Euclidean distance at least $1/\sqrt{2}$, there is also a constant B such that there are at most Bn^2 edge centers whose distance from the origin is between n and $n+1$. It follows that the energy of θ is at most $\sum_n \theta(e)^2 Bn^2 = \sum_n (A/n^2)^2 Bn^2 = \sum_n A^2B/n^2$, which is finite. So transience follows from Theorem 8 above. $\square$

Remark 11. The continuous case — Brownian motion in $\mathbb{R}^3$ — is easier to handle because of its spherical symmetry. Here, we are approximating this continuous case in our solution. We can in fact use the transience of the continuous case to deduce the transience of the discrete case.

Since the harmonic series which arises in the recurrence of $\mathbb{Z}^2$ just barely diverges, it seems that the change from recurrence to transience occurs "just after" dimension 2, rather than somewhere else in $[2, 3]$. To explore this, we care about the type of spaces which are between $\mathbb{Z}^2$ and $\mathbb{Z}^3$. For example, consider the wedge

$$W_f := \{(x, y, z) : |z| \leq f(|x|)\},$$

where $f : \mathbb{N} \to \mathbb{N}$ is an increasing function. The number of edges that leave the portion $W_f \cap \{(x, y, z) : |x| \vee |y| \leq n\}$ is of the order $n(f(n) + 1)$, so that according to the Nash-Williams criterion,

$$\sum_{n \geq 1} \frac{1}{n(f(n) + 1)} = \infty$$

is sufficient for recurrence.

Since a simple random walk on $\mathbb{Z}^2$ is recurrent, the effective resistance from the origin to distance n tends to infinity. We care about the speed of this divergence.

Proposition 12. *There are positive constants C_1, C_2 such that if one identifies to a single vertex z_n all vertices of $\mathbb{Z}^2$ that are at distance more than n from $\mathbf{0}$, then*

$$C_1 \log n \leq \mathscr{R}(\mathbf{0} \leftrightarrow z_n) \leq C_2 \log n.$$

Proof. The lower bound is a direct consequence of the Nash-Williams Inequality, where the pairwise disjoint cutsets Π_k are the ones we used in our proof of the first part of Pólya's theorem. The upper bound follows from the estimate of the energy of the unit flow analogous to that used for the transience of $\mathbb{Z}^3$. That is, let there be a uniform ray emanating from the origin, and define $\mathcal{P}(L)$ and $\theta(e)$ as above. Then, θ defines a unit flow from $\mathbf{0}$ to z_n, and its energy is bounded by $C_2 \log n$. $\square$

Proposition 13. *For $d \geq 2$, there is a positive constant C_d such that if G_n is the subnetwork of $\mathbb{Z}^d$ induced on the vertices in a box of side length n, then for any pair of vertices x, y in G_n at mutual distance k,*

$$\mathscr{R}(x \leftrightarrow y : G_n) \in \begin{cases}]C_d^{-1} \log k, C_d \log k[& \text{if } d = 2 \\ ]C_d^{-1}, C_d[& \text{if } d \geq 3. \end{cases}$$

Proof. The lower bounds follow from the Nash-Williams Inequality. For the upper bound, we prove the $d = 2$ case. Let M be the straight line joining x and y. There is a straight line segment L of length k inside the portion of $\mathbb{R}^2$ which corresponds to G_n, such that L meets the midpoint of M in a right angle.

Let Q be a uniformly random point on L, and write $L(Q)$ for the union of the two straight line segments from x to Q and Q to y. Let $\mathcal{P}(Q)$ be a path in G_n from x to y that is closest to $L(Q)$, and define the unit flow θ as before. Then, $\mathscr{E}(\theta) \leq C_2 \log k$ for some C_2, as before. $\square$