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CHAPTER I

LARGE ELASTIC DEFORMATIONS

BY

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§ 1. Introduction

The rapid development of the theory of large elastic deformations during the past decade has been stimulated by the existence and widespread use of vulcanized rubber, a substance which, to a fair degree of approximation, may be regarded as an ideal, highly elastic material which is isotropic and incompressible. Although much of the basic mathematical theory was developed during the nineteenth century by Cauchy, Green and others, attention at that time was mainly concentrated on the classical infinitesimal theory for which there were obvious applications arising from the use of metals, concrete and other slightly elastic materials in engineering. Contributions were, however, made to the finite theory notably by FINGER [1894], E. and F. COSSERAT [1896], BRILLOUIN [1925] and by MURNAGHAN [1937] who has also [1951] derived approximate solutions to some simple problems. Nevertheless, owing to the non-linearity of the differential equations which need to be handled, progress was slow until the publication of a series of papers by RIVLIN [1948a,b,c,d, 1949a,b,c,] which demonstrated that the existence of an internal geometrical constraint, such as incompressibility, permitted the complete solution by inverse methods of problems which were otherwise intractable. Since then, the theory has been extended to deal with aeolotropic bodies and materials reinforced with inextensible cords or containing general systems of internal geometrical constraints. Systematic methods have also been developed for the derivation of approximate solutions.

Reviews of progress in the subject have been published by TRUESDELL [1952, 1953], who gives an historical treatment and an extensive bibliography, and by DOYLE and ERICKSEN [1956]. A systematic development of the theory as existing at the time of publication is given by GREEN and ZERNA [1954]. These works all use tensor methods, which are particularly suitable for the concise presentation of the theory, and consider in some detail the kinematics of deformation and the derivation of the fundamental equations of finite elasticity. A more extensive treatment of recent developments has been prepared by GREEN and ADKINS [1960]. Accounts of the subject which avoid the use of tensor notation have been given by RIVLIN [1956, 1958]. The former of these is of particular interest to readers mainly concerned with the physical aspects of the subject. TRELOAR [1958] gives an account of the theory of large elastic deformations resulting from the structural picture of rubber as a cross-linked network of long chain molecules. He includes in his work a description, from the physicist's point of view, of some of the theoretical and experimental results obtained by Rivlin and his co-workers.

The theory described in the present review assumes that the mechanical properties of an ideally elastic material can be described by means of a strain energy function, and the earlier sections are devoted mainly to a discussion of the influence upon the form of this function of such phenomenological properties as isotropy, aeolotropy, incompressibility and other internal geometrical constraints, and the variation in directional properties of the material from point to point (curvilinear aeolotropy). A description is then given of some of the more important exact solutions of the elastic equations which have so far been derived and the parametric representation of the stress deformation relations for these problems. Some of the results may be regarded as generalizations of those published in earlier work. A discussion of some of the experimental results obtained for rubber is also included. The remaining sections deal with the theory of highly elastic membranes and some of the general methods of deriving approximate solutions.

Since adequate accounts have been given in the works previously mentioned of the kinematics of deformation and the derivation of the stress-strain relations by variational procedures, only the main results required for a connected account of the subject are given here. To make the material accessible to the widest possible range of readers, tensor notation is kept to a minimum and such terms as are introduced are fully defined. For conciseness, the convention of summation over repeated suffixes is employed unless otherwise indicated.

Although many of the more important recent publications are included in the bibliography, the list is by no means complete. Further references may be found in the reviews and other major works cited.

I. SOME FUNDAMENTAL RESULTS

§ 2. Description of the Deformation

To describe the change in configuration of an elastic body, as a result of deformation, corresponding points of the unstrained and strained material may be referred to two coordinate frames which may or may not coincide. The deformation is then specified by means of functional relationships between the two sets of coordinates. Alternatively we may imagine a coordinate system which deforms with the body, the deformation being measured by the change in this system. This approach, which has been used by GREEN and ZERNA [1950, 1954] and others, is essentially that employed in the present section where curvilinear coordinates are required. For a fuller account of the measures of deformation which have been used in the development of the subject and the relation between them reference may be made to the works by TRUESDELL [1952, 1953].

We suppose that points of the undeformed body are referred to a fixed rectangular cartesian coordinate system x_i and that during deformation a point of the material initially at x_i is displaced to y_i in the same system. The displacement components u_i are therefore given by

$$u_i = y_i - x_i \,. \tag{2.1}$$

If elements of length in the undeformed body and in the deformed body are denoted by ds_0 and ds respectively, we have

$$\begin{aligned} ds_0^2 &= dx_i dx_i \,, \\ ds^2 &= dy_i dy_i = \frac{\partial y_i}{\partial x_r}\frac{\partial y_i}{\partial x_s}\, dx_r\, dx_s \,, \end{aligned} \tag{2.2}$$

summation being carried out over repeated suffixes. From these expressions we obtain

$$\tfrac{1}{2}(ds^2 - ds_0^2) = e_{ij}\, dx_i dx_j \,, \tag{2.3}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial y_r}{\partial x_i}\frac{\partial y_r}{\partial x_j} - \delta_{ij}\right) \tag{2.4}$$

are the usual Cauchy components of strain, which for classically small deformations become

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.5}$$

and δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = 1 \quad (i = j), \qquad \delta_{ij} = 0 \quad (i \neq j) . \tag{2.6}$$

The finite strain components (2.4) satisfy compatibility conditions which reduce to those of classical elasticity for infinitesimal deformations (see, for example, GREEN and ZERNA [1954]).

These relations are readily modified to give a measure of the deformation referred to curvilinear coordinates θ_i, for then, from (2.2) we may write

$$ds_0^2 = g_{ij} d\theta_i d\theta_j , \qquad ds^2 = G_{ij} d\theta_i d\theta_j , \tag{2.7}$$

where

$$g_{ij} = \frac{\partial x_r}{\partial \theta_i} \frac{\partial x_r}{\partial \theta_j} , \qquad G_{ij} = \frac{\partial y_r}{\partial \theta_i} \frac{\partial y_r}{\partial \theta_j} \tag{2.8}$$

are the covariant metric tensors associated with points in the undeformed body and in the deformed body respectively. The corresponding contravariant tensors are defined by

$$g^{ij} = \frac{\partial \theta_i}{\partial x_r} \frac{\partial \theta_j}{\partial x_r} , \qquad G^{ij} = \frac{\partial \theta_i}{\partial y_r} \frac{\partial \theta_j}{\partial y_r} . \tag{2.9}$$

The replacement of (2.3) by

$$\tfrac{1}{2}(ds^2 - ds_0^2) = \gamma_{ij} d\theta_i d\theta_j , \tag{2.10}$$

leads to the definition

$$\begin{aligned} \gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) &= \frac{1}{2} \left(\frac{\partial y_r}{\partial \theta_i} \frac{\partial y_r}{\partial \theta_j} - \frac{\partial x_r}{\partial \theta_i} \frac{\partial x_r}{\partial \theta_j} \right) \\ &= \frac{\partial x_r}{\partial \theta_i} \frac{\partial x_s}{\partial \theta_j} e_{rs} \end{aligned} \tag{2.11}$$

for the covariant strain tensor γ_{ij} associated with the curvilinear system θ_i. The components γ_{ij} unlike e_{ij} are not, in general, dimensionless quantities. Physical, dimensionless components of strain corresponding to the quantities e_{ij} may, however, readily be constructed when the coordinate frame θ_i is such that it coincides with an orthogonal system in the undeformed body. In this case $g_{ij} = 0$ when $i \neq j$ and from the first of (2.7) an element of length ds_i in the θ_i-direction is given by

$$ds_i = (\sqrt{g_{ii}})\, d\theta_i \quad (i \text{ not summed}) . \tag{2.12}$$

The formula for ds_0^2 may thus be written

$$ds_0^2 = dx_i' dx_i' \qquad (dx_i' = ds_i) ,$$

where x'_i is a local rectangular cartesian frame of reference with the x'_i-axes tangential to the θ_i curves at P. In terms of this system equation (2.3) becomes

$$\tfrac{1}{2}(\mathrm{d}s^2 - \mathrm{d}s_0^2) = e'_{ij}\mathrm{d}x'_i\mathrm{d}x'_j\,, \tag{2.13}$$

where

$$e'_{ij} = \frac{\gamma_{ij}}{\sqrt{(g_{ii}g_{jj})}} \quad (i, j \text{ not summed}) \tag{2.14}$$

are physical components of strain analogous to the quantities e_{ij} defined by (2.3). This specification of physical strain components relative to a local coordinate system x'_i is convenient for defining the elastic properties of a body where these vary in direction from point to point (curvilinear aeolotropy).

The curvilinear system θ_i may be chosen in any manner suitable for the problem under consideration. For convenience in handling the boundary conditions this choice is often made so that the system θ_i coincides with a reference frame related to points in the deformed material. The quantities G_{ij}, G^{ij} are then calculated from (2.8) and (2.9) and the relations $\theta_i = \theta_i(y_k)$ defining the curvilinear coordinate frame; the relations $y_i = y_i(x_k)$ which specify the deformation permit the calculation of g_{ij}, g^{ij} from (2.8) and (2.9). A corresponding procedure may be employed when the reference frame θ_i is associated with points in the undeformed body.

§ 3. The Strain Energy

We shall restrict consideration to elastic materials whose mechanical properties are specified by means of a strain energy function W. This function depends solely upon the state of deformation; it is independent of previous configurations of the material or the rate at which the given deformation has been produced or is varying at the instant under consideration. Static or dynamic hysteresis, stress relaxation and creep effects are thereby excluded. Moreover, the existence of a strain energy implies that the stresses form a conservative system. Materials not subject to this latter condition have been considered by NOLL [1955] and by SMITH and RIVLIN [1957]. It can be shown from thermodynamical considerations that a strain energy does, in fact, exist in at least two important physical cases. When the deformation is produced slowly, so that changes are effectively reversible and isothermal these changes may be identified with changes in the Helmholtz free energy of the body; for sufficiently rapid reversible changes, which are effectively adiabatic in character, the strain energy corresponds to the internal energy of the body. The corresponding functions W in these two cases are, in general, different. Thermodynamic aspects have been examined by CHU [1957], TRELOAR [1958] and others and are discussed in some detail by GREEN and ADKINS [1960] (p. 3).

When the strain energy depends purely on configurational changes, as represented by a functional relationship between the initial and final coordinates x_i, y_i of a typical point P of the material, it is natural to suppose that W is a function of the strain components e_{ij} or some other suitably defined functions of the deformation gradients $\partial y_i/\partial x_j$. This is the approach of the more usual theories of classical and finite elasticity (see, for example, GREEN and ZERNA [1954]). Somewhat more generally we may write

$$W = W\left(x_i, y_i, \frac{\partial y_i}{\partial x_j}, \frac{\partial^2 y_k}{\partial x_i \partial x_j}, \ldots\right). \tag{3.1}$$

The arguments y_i can be excluded from (3.1) from the consideration that the properties of the material are independent of its position in space; W must therefore be unaffected by rigid body translations of the deformed body. If the material is elastically homogeneous W must also be independent of the position of P in the undeformed body and the argument x_i cannot then appear explicitly in (3.1).

Further restrictions upon the form of W may be obtained from the requirement that the strain energy must be unaffected by arbitrary rigid body rotations of the deformed body. In such a motion, the point P at y_i is displaced to $\bar{y}_i$ where

$$\bar{y}_i = a_{ij}\, y_j\,, \tag{3.2}$$

the constants a_{ij} being connected by the orthogonality conditions

$$a_{ij}a_{ik} = a_{ji}a_{ki} = \delta_{jk}\,, \quad |a_{ij}| = 1\,. \tag{3.3}$$

From (3.1) we then have

$$\begin{aligned} & W\left(\frac{\partial y_p}{\partial x_i}, \quad \frac{\partial^2 y_q}{\partial x_i \partial x_j}, \ldots\right) \\ &= W\left(\frac{\partial \bar{y}_p}{\partial x_i}, \quad \frac{\partial^2 \bar{y}_q}{\partial x_i \partial x_j}, \ldots\right) \\ &= W\left(a_{pr}\frac{\partial y_r}{\partial x_i}, \quad a_{qs}\frac{\partial^2 y_s}{\partial x_i \partial x_j}, \ldots\right). \end{aligned} \tag{3.4}$$

When the strain energy is a general function of the first derivatives $\partial y_i/\partial x_j$ alone it has been shown by MURNAGHAN [1937] and by TRUESDELL [1952] that W is reducible to a function of the 6 quantities

$$\frac{\partial y_r}{\partial x_i}\frac{\partial y_r}{\partial x_j} = \delta_{ij} + 2e_{ij}\,, \tag{3.5}$$

where e_{ij} is defined by (2.4). An alternative reduction making use of classical invariant theory has been achieved when W is a polynomial in $\partial y_i/\partial x_j$ by SMITH and RIVLIN [1958]. This reduction depends upon the theorem (WEYL [1946]) that if Ψ is a polynomial in the components of n 3-dimensional vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_n$ which is invariant under the full group of proper orthogonal transformations defined by (3.2), (3.3), then Ψ is expressible as a polynomial in the scalar products

$$\boldsymbol{u}_i \cdot \boldsymbol{u}_j \quad (i, j = 1, 2, \ldots n)\,, \tag{3.6}$$

and the scalar triple products

$$[\boldsymbol{u}_i\boldsymbol{u}_j\boldsymbol{u}_k] \quad (i, j, k = 1, 2, \ldots n; \quad i \neq j \neq k \neq i)\,. \tag{3.7}$$

Since the squares of the triple products (3.7) can be expressed as polynomials in the quantities (3.6) it follows that Ψ can be expressed to any desired degree of approximation as a polynomial in the scalar products (3.6). Evidently the components of each of the triads

$$\left(\frac{\partial y_1}{\partial x_i}, \quad \frac{\partial y_2}{\partial x_i}, \quad \frac{\partial y_3}{\partial x_i}\right) \quad (i = 1, 2, 3) \tag{3.8}$$

behave as the components of a vector under the rotations of the y_i axes defined by (3.2), (3.3). The strain energy function can therefore be approximated by a polynomial in scalar products of the type (3.6); that is, by a polynomial in the quantities (3.5)†.

A similar reduction is possible if W contains the higher order derivatives $\partial^2 y_r/\partial x_i\partial x_j$, $\partial^3 y_r/\partial x_i\partial x_j\partial x_k, \ldots$. For example, when the first and second derivatives $\partial y_r/\partial x_i$, $\partial^2 y_r/\partial x_i\partial x_j$ are included in the polynomial for the strain energy, invariance under rigid body rotations leads to an expression for W as a function of the forty-five quantities

$$\frac{\partial y_r}{\partial x_i}\frac{\partial y_r}{\partial x_j}, \quad \frac{\partial y_r}{\partial x_i}\frac{\partial^2 y_r}{\partial x_j\partial x_k}, \quad \frac{\partial^2 y_r}{\partial x_i\partial x_j}\frac{\partial^2 y_r}{\partial x_k\partial x_l}. \tag{3.9}$$

These higher order derivatives are not therefore excluded from the

† The components e_{ij} are also invariant under the full orthogonal group of proper and improper transformations for which $|a_{ij}| = \pm 1$.

theory by invariance considerations alone. As indicated in § 11, however, their inclusion would lead to difficulties in defining the elastic problem uniquely with the usual number of boundary conditions, and for this reason they are not considered further. Remembering (3.5) we therefore suppose W for a homogeneous material to be expressible as a polynomial in the six strain components e_{ij} and write

$$W = W(e_{ij}) \,. \tag{3.10}$$

The invariance properties considered in this section have been described by NOLL [1955] as 'space isotropy'.

§ 4. Restrictions upon the Strain Energy

The reduction of the strain energy to the form (3.10) depends solely upon the fact that the elastic properties of the material are independent of its position or orientation in space. Further restrictions upon the form of W may be imposed by the elastic properties of the material itself. These further limitations may be due to

(i) symmetry properties
(ii) the existence of internal geometrical constraints
(iii) the variation in direction of the elastic properties throughout the material (curvilinear aeolotropy).

When the functional form of W has been determined from the foregoing considerations, still further limitations may be imposed by taking into account the expected behaviour of real materials. For example, one expects the strain energy to be a positive definite function of its arguments and usually this function is such that the material is unstressed in its initial undeformed state. In some instances the general pattern of the expected stress distribution may give further information. Thus for an isotropic body it is reasonable to suppose that the directions of the greatest principal stress and greatest principal extension will coincide; by a simple application of this idea BAKER and ERICKSEN [1954] have obtained inequalities restricting the form of W. Again we might expect St. Venant's principle to apply in several different situations. An obvious instance occurs where a system of forces applied over a limited region S of the boundary surface of a body B produces a local finite deformation. At distances sufficiently large compared with the dimensions of S the strain may be finite or infinitesimal, but under suitable conditions we might expect the resulting deformation pattern to be insensitive to the actual distribution of forces over S. As another example we may imagine the situation in which a large deformation is defined completely throughout B and a small superposed deformation is produced by a system of forces applied to S. In this case we should expect the superposed deformation pattern at sufficiently large distances from S to be relatively insensitive to the actual distribution of forces over S. For the classical infinitesimal theory STERNBERG [1954] has investigated such effects and his analysis depends upon the linear stress strain relations which then apply. From a corresponding investigation for the finite theory it may well emerge that if the effects described are to occur, the strain energy function must be restricted by further inequalities. Finally the work of HILL [1957] suggests that additional limitations may follow from uniqueness and stability considerations.

In this review we shall confine attention to the functional restrictions arising from the properties (i), (ii) and (iii) above.

§ 5. Symmetry Properties

In many crystalline bodies there is elastic symmetry at each point about given planes or axes fixed in direction throughout the material. This implies that the form of W, when expressed as a function of the components e_{ij}, must be unchanged by all transformations of the x_i coordinates which characterise the given properties. For example, if the material is symmetrical at each point about the planes x_1 = constant, then W must be unchanged in form by the transformation

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (-x_1, x_2, x_3)\,, \tag{5.1}$$

and since e_{12}, e_{13} are changed in sign by the reflection (5.1) we readily deduce that

$$W = W(e_{11}, e_{22}, e_{33}, e_{12}^2, e_{13}^2, e_{12}, e_{13}, e_{23})\,.$$

In general, the effect of symmetry properties is to limit the form of W so that it can be expressed in terms of functions of the strain components which are invariant under the required groups of transformations. We may then write

$$W = W(\Psi_r)\,, \tag{5.2}$$

where

$$\Psi_r = \Psi_r(e_{ij}) \quad (r = 1, 2, \ldots n) \tag{5.3}$$

are invariants for the group of transformations concerned whose number and form depend upon the symmetry properties of the body.

A reduction of the strain energy function W for each of the thirty-two crystal classes has been given by SMITH and RIVLIN [1958], assuming W to be a polynomial form in the components e_{ij} and making use of results in classical invariant theory. The forms of the strain energy for many of the crystal classes are not functionally distinct, but when the additional results for isotropic and transversely isotropic materials are taken into account the analysis shows that there are thirteen possible different polynomial forms for W. This number is reduced to ten for classically small deformations. For a description of the physical significance of the transformations employed reference may be made to the books on mineralogy by DANA and FORD [1932] and DANA and HURLBUT [1952]. Here we shall confine attention to isotropic, transversely isotropic and orthotropic (i.e. orthorhombic) materials.

§ 6. Geometrical Constraints

When the material is subject to internal geometrical constraints, so that it cannot be deformed in a completely arbitrary manner, there must exist functional relationships between the quantities employed to describe the deformation. For example, in the situation which could be simulated by the introduction into the undeformed elastic body of a system of thin, flexible, inelastic cords, each lying in the x_1-direction, any deformation to which the material is subjected must be limited by the condition that elements of length following the direction of the cords shall remain unchanged throughout. From (2.3) we see that this implies the condition $e_{11} = 0$. Similarly, the condition $\gamma_{11} = 0$ would be imposed by a system of cords following the θ_1-curves of the curvilinear system θ_i. A more general system of constraints related to the system θ_i would be represented by a functional relationship

$$f(\gamma_{ij}) = 0 \tag{6.1}$$

between the components γ_{ij}. Each independent constraint thus reduces by one the number of independent quantities available to describe the deformation. The possibility of six independent constraints must therefore be excluded since these would evidently yield six relations of the type (6.1) which could, in principle, be solved to give a set of discrete values for the components γ_{ij}. This would imply a material which is incapable of a continuously varying deformation.

Since each constraint is represented by a functional relationship between the strain components, it is possible to postulate symmetry properties analogous to those considered for the strain energy function W. Thus it may be possible to represent a given constraint by a functional relationship

$$f(\Psi_r) = 0 \tag{6.2}$$

between invariants of the type (5.3) or the corresponding quantities formed from the components γ_{ij}. An example of an isotropic constraint is provided by incompressible materials in which elements of volume remain unchanged throughout the deformation. This condition may be written either as

$$I_3 \equiv |\delta_{ij} + 2e_{ij}| = 1 \,, \tag{6.3}$$

or as

$$\sqrt{I_3} \equiv \left|\frac{\partial y_i}{\partial x_j}\right| = 1 \,, \tag{6.4}$$

I_3 being the third strain invariant employed in § 8.

When the system of constraints arises as an intrinsic property of the material the functional form of W may be regarded to some extent as arbitrary. For example, if

$$W = W(e_{ij})$$

represents the strain energy for an incompressible body then

$$W + f(I_3)$$

is also a possible form provided that f is a function which vanishes when $I_3 = 1$. Thus although W is completely defined for physically realisable strains, for others it may assume any values. When geometrical constraints arise from external mechanisms such as systems of inextensible cords, the elastic material and the external mechanisms (e.g. cords) may be regarded as separate systems. There is then no restriction or degree of arbitrariness in the form of W; the external constraints merely limit the types of deformation which can be allowed.

Materials subject to constraints have been studied by ERICKSEN and RIVLIN [1954] and by ADKINS [1956c, 1958b,c]. The case where constraints are introduced by systems of cords is discussed in greater detail in § 18.

§ 7. Curvilinear Aeolotropy

The third limitation upon the form of W for aeolotropic bodies arises if the elastic properties vary in direction from point to point. To obtain a clearer picture of the mathematical restriction which is thus imposed, we may consider again the case where the material is elastically homogeneous and the properties are uniform in direction; W is then a function, apart from physical constants which do not depend upon the deformation, only of the quantities e_{ij}. In any other system of orthogonal curvilinear coordinates θ_i, the derivatives $\partial\theta_r/\partial x_s$ appear explicitly in the form of W and we may write

$$W = W\left(\gamma_{ij}, \frac{\partial\theta_r}{\partial x_s}\right),$$

or

$$W = W\left(e'_{ij}, \frac{\partial\theta_r}{\partial x_s}\right),$$

by virtue of the relations (2.11) and (2.14). If, however, W is a function in which the quantities describing the deformation enter *only* as the components e'_{ij}, the derivatives $\partial\theta_r/\partial x_s$ being absent, the elastic properties at each point are again identical provided that the axes of reference x'_i are everywhere tangential to the coordinate curves θ_i.

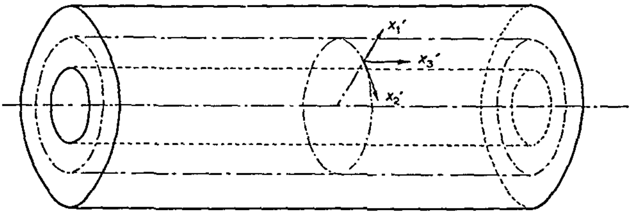

Fig. 1. System of local axes for definition of cylindrical aeolotropy.

In this case we may speak of curvilinear aeolotropy associated with the system θ_i, and the body may still be regarded as elastically homogeneous in contrast to one in which W is a function which involves not only the components e'_{ij} but also the coordinates θ_i explicitly. As an example of this type of homogeneous aeolotropy, we may imagine a circular cylindrical tube in which the elastic properties, although directional in character, are symmetrical about its axis. In this case the x'_i coordinate axes at each point of the body would be directed along the radial, transverse and axial directions (Fig. 1), and the tube could then be described as cylindrically aeolotropic. In terms of the reference frame x_i, the function W would contain, not only the components e_{ij} but also the derivatives $\partial x_r/\partial\theta_s$.

For these materials the form of W may again be restricted by the existence of local symmetry properties, defined at each point by transformations of the x'_i-axes, and also by the presence of internal geometrical constraints. Such materials have been studied by ADKINS [1955b,c, 1956c].

§ 8. Isotropic Bodies

An elastic material is defined as isotropic if the form of its strain energy function, when expressed in terms of e_{ij}, is independent of the choice of the x_i-axes in the undeformed body. This implies that W remains unchanged in form by all orthogonal transformations of the x_i coordinates defined by†

$$\bar{x}_i = a_{ij}x_j \,, \tag{8.1}$$

where

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij} \,. \tag{8.2}$$

From (2.4) and (8.1) the strain components $\bar{e}_{ij}$ referred to the system $\bar{x}_i$ are given by

$$\bar{e}_{ij} = a_{ir}a_{js}e_{rs} \,, \tag{8.3}$$

and the condition for isotropy becomes

$$W(e_{ij}) = W(\bar{e}_{ij}) = W(a_{ir}a_{js}e_{rs}) \,. \tag{8.4}$$

If it is assumed that W is a *polynomial* in the arguments e_{ij} the reduction to isotropic form is obtained as a natural consequence of the Cayley Hamilton theorem. For if we allow $\boldsymbol{e}$ to denote the symmetric 3×3 matrix with elements e_{ij}, then any polynomial in these quantities which is a scalar invariant under the group of orthogonal transformations (8.1) can be expressed in terms of the traces of powers of $\boldsymbol{e}$. Moreover, from the Cayley Hamilton theorem,

$$\boldsymbol{e}^3 = \varphi_1\boldsymbol{e}^2 + \varphi_2\boldsymbol{e} + \varphi_3\boldsymbol{I} \,, \tag{8.5}$$

† For *proper* orthogonal transformations the constants satisfy the condition $|a_{ij}| = 1$; in the case considered, when we allow $|a_{ij}| = \pm 1$ there is a centre of symmetry (or inversion) at each point. Both assumptions lead to the same functional form for W.

where

$$\varphi_1 = \operatorname{tr} e,$$
$$\varphi_2 = \tfrac{1}{2}[\operatorname{tr} e^2 - (\operatorname{tr} e)^2],$$
$$\varphi_3 = \tfrac{1}{6}[(\operatorname{tr} e)^3 - 3 \operatorname{tr} e \operatorname{tr} e^2 + 2 \operatorname{tr} e^3],$$

are scalar invariants and I is the unit matrix, it follows that

$$\operatorname{tr} e^n = \varphi_1 \operatorname{tr} e^{n-1} + \varphi_2 \operatorname{tr} e^{n-2} + \varphi_3 \operatorname{tr} e^{n-3} \quad (n = 4, 5, \ldots).$$

This expresses $\operatorname{tr} e^n$ for $n > 3$ as a polynomial in the traces of lower powers of e, and by successive applications of this result we may infer that W may be expressed as a polynomial in

$$\operatorname{tr} e = e_{ii}, \quad \operatorname{tr} e^2 = e_{ij}e_{ji}, \quad \operatorname{tr} e^3 = e_{ij}e_{jk}e_{ki}\,. \tag{8.6}$$

These quantities may be replaced by the more usual invariants of strain defined by

$$\begin{aligned} I_1 &= 3 + 2e_{rr} = \frac{1}{8}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\partial^2 I_3}{\partial e_{rr}\partial e_{ss}}\,, \\ I_2 &= 3 + 4e_{rr} + 2(e_{rr}e_{ss} - e_{rs}e_{sr}) = \frac{1}{2}\sum_{r=1}^{3}\frac{\partial I_3}{\partial e_{rr}}\,, \\ I_3 &= |\delta_{rs} + 2e_{rs}|\,, \end{aligned} \tag{8.7}$$

since

$$\begin{aligned} 2 \operatorname{tr} e &= I_1 - 3\,, \\ 4 \operatorname{tr} e^2 &= I_1^2 - 2I_2 - 2I_1 + 3, \\ 8 \operatorname{tr} e^3 &= 3I_3 - 3I_1I_2 + I_1^3 + 6I_2 - 3I_1^2 + 3I_1 - 3\,. \end{aligned} \tag{8.8}$$

For an isotropic body we may therefore write

$$W = W(I_1, I_2, I_3)\,. \tag{8.9}$$

An alternative set of invariants is provided by the system

$$\begin{aligned} J_1 &= I_1 - 3, \\ J_2 &= I_2 - 2I_1 + 3, \\ J_3 &= I_3 - I_2 + I_1 - 1\,. \end{aligned} \tag{8.10}$$

These quantities are of value in the development of approximate theories for compressible bodies, for when the displacement gradients are small and of order ε, then J_1, J_2, J_3 are of orders ε, ε^2 and ε^3 respectively.

When the material is incompressible, $I_3 = 1$ and (8.9) is replaced by

$$W = W(I_1, I_2)\,. \tag{8.11}$$

Particular forms of this function have been proposed for vulcanized rubber, which is effectively isotropic and incompressible, by various workers. A statistical theory of rubberlike elasticity, based on the molecular picture of rubber as a cross-linked network (see, for example, TRELOAR [1958]) yields the simple form

$$W = C(I_1 - 3)\,, \tag{8.12}$$

C being a constant. This has been described by RIVLIN [1948a] as a 'Neo-hookean' material. MOONEY [1940] on the basis of experiments in simple shear has suggested the two-constant formula

$$W = C_1(I_1 - 3) + C_2(I_2 - 3)\,, \tag{8.13}$$

while GENT and THOMAS [1958] have proposed the logarithmic form

$$W = C_1'(I_1 - 3) + C_2' \log_e(\tfrac{1}{3}I_2)\,. \tag{8.14}$$

With a suitable choice of values for the constants C_1', C_2' the form (8.14) provides an adequate description of the properties of rubber for much greater ranges of deformation than either of the forms (8.12), (8.13).

An alternative form of W for an isotropic body may be derived by returning again to the starting point for the discussion of § 3, that is by regarding the strain energy as a polynomial in the displacement gradients $\partial y_i/\partial x_j$. If now we impose the condition that W is unchanged by all orthogonal transformations of the coordinates x_i in the *undeformed* body, it follows by an argument analogous to that employed in deriving (3.10) that W may be represented by a polynomial form

$$W = W(C_{ij})\,, \tag{8.15}$$

where

$$C_{ij} = \frac{\partial y_i}{\partial x_r}\,\frac{\partial y_j}{\partial x_r}\,. \tag{8.16}$$

By means of the Cayley-Hamilton theorem W may now be reduced to a function of the invariants

$$C_{ii}, \quad C_{ik}C_{ki}, \quad C_{ij}C_{jk}C_{ki}\,, \tag{8.17}$$

analogous to (8.6). The formulations in terms of C_{ij} and e_{ij} may be shown to be equivalent.

§ 9. Orthotropic and Transversely Isotropic Materials

The effect of elastic symmetry upon the form of the strain energy function may be further illustrated by the common cases of orthotropy, in which there is symmetry about three orthogonal planes at each point, and transverse isotropy, in which there is rotational symmetry about an axis, these properties being related to the undeformed body in each case.

In the former instance, if the planes of symmetry coincide with the x_i coordinate planes, the form of W, when expressed in terms of e_{ij}, must remain unchanged by all transformations of the form

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\pm x_1, \pm x_2, \pm x_3) \,. \tag{9.1}$$

This implies that the form of W remains unchanged when the sign of any two of the components e_{12}, e_{13}, e_{23} is reversed and it follows from classical invariant theory (SMITH and RIVLIN [1958]) or by a direct evaluation of the functions which can occur (ADKINS [1958c]) that W must reduce to a function of the seven quantities

$$e_{11}, \quad e_{22}, \quad e_{33}, \quad e_{12}^2, \quad e_{13}^2, \quad e_{23}^2, \quad e_{12}e_{23}e_{31} \,. \tag{9.2}$$

By virtue of (8.7) the last of the invariants (9.2) may be replaced by I_3. If W is a polynomial in the components e_{ij} then it may also be expressed as a polynomial in the quantities (9.2). Since, however,

$$(e_{12}\, e_{23}\, e_{31})^2 = e_{12}^2\, e_{23}^2\, e_{13}^2 \,,$$

this expression is not unique unless we specify that W is linear in $e_{12}e_{23}e_{31}$ or is of some other particular functional form in this invariant.

In the classical theory of elasticity a material which is transversely isotropic relative to a given axis may be defined as one in which there is complete rotational symmetry about that axis. If, for example, we choose the x_i-axes in the undeformed body so that the x_3-axis coincides with the axis of symmetry, the strain energy function W when expressed in terms of the strain components e_{ij} must remain unchanged in form under all transformations

$$\bar{x}_\alpha = a_{\alpha\beta}\, x_\beta, \quad \bar{x}_3 = x_3 \,, \tag{9.3}$$

$$a_{\alpha\rho}\, a_{\beta\rho} = a_{\rho\alpha} a_{\rho\beta} = \delta_{\alpha\beta}, \quad |a_{\alpha\beta}| = +1 \,,$$

greek suffixes being restricted to the values 1, 2. The relations (9.3) describe rotations of the x_i-axes about the x_3-direction; these form a subgroup of the full orthogonal group defined by (8.1), (8.2). For infinitesimal deformations the resulting form of W is identical with that obtained for some of the crystal classes of the hexagonal system in which the x_3-axis is an axis of three-fold or six-fold symmetry. For finite deformations, however, when powers and products of higher order than the second in e_{ij} need to be considered, different forms of W are obtained for the two types of material. Furthermore if W is restricted to be a *polynomial* in the components e_{ij} different forms are obtained for materials which merely possess rotational symmetry as defined by the form invariance of W under the transformations (9.3) and those for which in addition W is invariant in form under reflections in planes containing the x_3-axis. If W is a polynomial in the quantities e_{ij} which is form invariant under the transformations (9.3), then it may be expressed as a polynomial in the invariants

$$e_{11} + e_{22}, \quad e_{33}, \quad e_{11}e_{22} - e_{12}^2, \quad e_{13}^2 + e_{23}^2, \quad |e_{ij}| \,, \tag{9.4}$$

and

$$(e_{11} - e_{22})e_{13}e_{23} - e_{12}(e_{13}^2 - e_{23}^2) \,. \tag{9.5}$$

If in addition W is unchanged in form by the transformation

$$\bar{x}_1 = -x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3 \,,$$

it may be expressed in terms of the quantities (9.4) alone. Such materials have been characterised as transversely isotropic by ERICKSEN and RIVLIN [1954]. An alternative definition of transverse isotropy adopted by the writer (ADKINS [1955b]) requires W to be form invariant under the class of transformations

$$\bar{x}_\alpha = \bar{x}_\alpha\,(x_1, x_2), \quad \bar{x}_3 = \pm x_3 \quad (\alpha = 1, 2) \,. \tag{9.6}$$

This again leads to an expression for W in terms of the invariants (9.4). In place of the quantities (9.4), we may by virtue of the definitions (8.7) employ the invariants

$$I_1, \quad I_2, \quad I_3, \quad e_{33}, \quad e_{13}^2 + e_{23}^2 \tag{9.7}$$

to specify the transversely isotropic form of W. Further possible invariant systems for these materials have been given by ADKINS [1960].

§ 10. Stress-Strain Relations

A discussion of the various forms which the stress strain relations of finite elasticity can assume has been given by TRUESDELL [1952]. Usually these relations are derived from a variational principle and different treatments of the problem have been given by MURNAGHAN [1937], RIVLIN [1948a], DOYLE and ERICKSEN [1956] and others. The analysis of GREEN and ZERNA [1954] leads to the convenient formula

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right), \tag{10.1}$$

for materials which are not incompressible or subject to other internal constraints. In this equation τ^{ij} is the contravariant stress tensor referred to curvilinear coordinates θ_i in the deformed body and the components γ_{ij} are defined by (2.11). When the curvilinear system θ_i forms an orthogonal reference frame in the *deformed* body, the physical (i.e. true) components of stress σ_{ij} are given by

$$\sigma_{ij} = \tau^{ij}\sqrt{(G_{ii}G_{jj})} \quad (i, j, \text{ not summed}). \tag{10.2}$$

These components are measured per unit area of deformed body and are directed along the θ_i-curves in the deformed material (see for example, GREEN and ZERNA [1954]).

When the elastic body is subject to internal geometrical constraints variations of the deformation are subject to restrictions of the form (6.1) and this has the effect of introducing additional parameters into the formulae (10.1). For example, if there are n constraints specified by relations

$$f_m(\bar{\gamma}_{ij}) = 0 \quad (m = 1, 2, \ldots n; n < 6) \tag{10.3}$$

between strain components $\bar{\gamma}_{ij}$ these components being defined for a curvilinear coordinate system θ_i by relations of the type (2.11), then the stress strain relations (10.1) are replaced by

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right) + \frac{1}{2}\sum_{m=1}^{n} p_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta_i}{\partial \bar{\theta}_r}\frac{\partial \theta_j}{\partial \bar{\theta}_s} + \frac{\partial \theta_i}{\partial \bar{\theta}_s}\frac{\partial \theta_j}{\partial \bar{\theta}_r}\right). \tag{10.4}$$

The Lagrangian multipliers p_m in (10.4) are functions of the coordinates θ_i (or $\bar{\theta}_i$) which are determined by the conditions of the problem. For incompressible materials, where there is the single constraint $I_3 = 1$, equations (10.4) yield

$$\tau^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right) + pG^{ij}. \tag{10.5}$$

These formulae for the stress tensor are readily modified to take account of restrictions upon the form of W arising from symmetries in the elastic material. If W is a function of t strain invariants Ψ_k defined by (5.3), then by making use of (2.11) we may rewrite (10.1) as

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}} \sum_{k=1}^{t} \left[\sum_{r=1}^{3} \sum_{s=1}^{3} \frac{\partial \theta_i}{\partial x_r} \frac{\partial \theta_j}{\partial x_s} \left(\frac{\partial \Psi_k}{\partial e_{rs}} + \frac{\partial \Psi_k}{\partial e_{sr}} \right) \right] \frac{\partial W}{\partial \Psi_k}, \tag{10.6}$$

with corresponding forms derived from (10.4) and (10.5). For compressible isotropic bodies we have three invariants Ψ_k which may be defined by (8.6), (8.7), (8.10) or (8.17); in the orthotropic case we have the seven invariants (9.2) while for transversely isotropic bodies we may choose for Ψ_k the five functions (9.4) or (9.7). When the material is isotropic and W is a function of the invariants (8.7), (10.1) and (10.5) yield for compressible and incompressible materials respectively the formulae

$$\tau^{ij} = \frac{2}{\sqrt{I_3}} \left(g^{ij} \frac{\partial W}{\partial I_1} + B^{ij} \frac{\partial W}{\partial I_2} + G^{i} I_3 \frac{\partial W}{\partial I_3} \right), \tag{10.7}$$

$$\tau^{ij} = 2\left(g^{ij} \frac{\partial W}{\partial I_1} + B^{ij} \frac{\partial W}{\partial I_2} \right) + pG^{ij}, \tag{10.8}$$

where

$$B^{ij} = g^{ij} I_1 - g^{ir} g^{js} G_{rs}, \tag{10.9}$$

and g_{ij}, g^{ij}, G_{ij} and G^{ij} are defined by (2.8) and (2.9).

§ 11. Equations of Equilibrium

With the equations of the preceding sections must be coupled the stress equations of equilibrium. In general coordinates θ_i these may be written, in the absence of body forces, as

$$\tau^{ik}_{,i} + \Gamma^{k}_{ir} \tau^{ir} + \Gamma^{r}_{ir} \tau^{ik} = 0, \tag{11.1}$$

where

$$\Gamma^{i}_{jk} = \tfrac{1}{2} G^{ir} (G_{jr,k} + G_{rk,j} - G_{jk,r}) \tag{11.2}$$

are the Christoffel symbols formed from the metric tensors G_{ij}, G^{ij} and the comma denotes partial differentiation with respect to the coordinates θ_i. When $\theta_i = y_i$, equations (11.1) reduce to

$$\frac{\partial \sigma_{ik}}{\partial y_k} = 0, \tag{11.3}$$

σ_{ik} being the physical components of stress referred to the rectangular cartesian coordinates y_i in the deformed body. When the reference frame θ_i coincides with a cylindrical polar coordinate system $(\varrho, \vartheta, y_3)$ in the deformed body so that

$$
\begin{gathered}
(\theta_1, \theta_2, \theta_3) = (\varrho, \vartheta, y_3)\,, \\
y_1 = \varrho \cos\vartheta, \quad y_2 = \varrho \sin\vartheta\,,
\end{gathered}
\tag{11.4}
$$

the relations (11.1) yield

$$
\begin{aligned}
\frac{\partial\sigma_{11}}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial\sigma_{12}}{\partial\vartheta} + \frac{\partial\sigma_{13}}{\partial y_3} + \frac{\sigma_{11} - \sigma_{22}}{\varrho} &= 0\,, \\
\frac{\partial\sigma_{12}}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial\sigma_{22}}{\partial\vartheta} + \frac{\partial\sigma_{23}}{\partial y_3} + 2\frac{\sigma_{12}}{\varrho} &= 0\,, \\
\frac{\partial\sigma_{13}}{\partial\varrho} + \frac{1}{\varrho}\frac{\partial\sigma_{23}}{\partial\vartheta} + \frac{\partial\sigma_{33}}{\partial y_3} + \frac{\sigma_{13}}{\varrho} &= 0\,.
\end{aligned}
\tag{11.5}
$$

In these equations σ_{ij} are the physical components of stress referred to the radial, transverse and longitudinal directions respectively and are given in terms of the components τ^{ij} by

$$
\begin{aligned}
&\sigma_{11} = \tau^{11}, \quad \sigma_{22} = \varrho^2\tau^{22}, \quad \sigma_{33} = \tau^{33}, \\
&\sigma_{12} = \varrho\tau^{12}, \quad \sigma_{13} = \tau^{13}, \quad \sigma_{23} = \varrho\tau^{23}\,.
\end{aligned}
\tag{11.6}
$$

When the expressions of § 10 for the stresses are substituted into (11.1) or into either of the particular forms (11.3) or (11.5), we obtain, in general three non-linear partial differential equations of the second order for the determination of the coordinates of points in the deformed body in terms of the initial coordinates of each particle. These equations must be solved subject to the appropriate stress or displacement boundary conditions. For example, if S is a boundary surface of the deformed body, l_i are the direction cosines of the normal at a point P of S, F_i are the components of force at P, measured per unit area of deformed surface relative to the y_i-directions and σ_{ij} are the physical stress components in the same coordinate system then we have at P

$$F_j = l_i\sigma_{ij}\,.$$

Evidently by the inclusion of the higher order displacement gradients $\partial^2 y_r/\partial x_i\partial x_j$, $\partial^3 y_r/\partial x_i\partial x_j\partial x_k, \ldots$ in the expression for W or in the stress strain relations it would be possible to obtain, from the equations of equilibrium, partial differential equations of higher order than the second for the determination of the coordinates y_i as functions of x_i. Under these circumstances more than the usual number of boundary conditions would be required to determine a state of strain uniquely, and this suggests that in any physically applicable theory of elastic materials such higher order derivatives must be excluded.

II. EXACT SOLUTIONS

§ 12. The Solution of Problems

As indicated in the preceding section the combination of the stress strain relations with the equations of equilibrium yields, in general, three non-linear partial differential equations for the determination of the quantities defining the deformation. In rectangular cartesian coordinates, for example, we obtain three equations for the determination of the final coordinates y_i of each particle in terms of its initial coordinates x_i or conversely. In the classical theory of elasticity these relations are linearized by retaining only the terms of the first order in the displacement components u_i and their derivatives; a variety of techniques is then available for dealing with the resulting equations. For large deformations, however, products and powers higher than the first in the displacement gradients and their derivatives are no longer negligible and this linearization procedure cannot be applied. Moreover, the differential equations to be solved are not completely defined until some specific form is assumed for the strain energy function W. In classical elasticity this difficulty does not arise since the neglect of the higher order terms in the displacement gradients automatically restricts W to be a quadratic form in the strain components e_{ij}.

When the elastic material is subject to one or more internal geometrical constraints it is possible to make further progress with certain types of problem. The constraint conditions provide one or more completely defined first order partial differential equations for the coordinates of points in the deformed body. When these have been solved, the Lagrangian multipliers p_m and p occurring in (10.4), (10.5) and (10.8) are determined by the equations of equilibrium and the boundary conditions. This procedure forms the basis of a number of solutions obtained for incompressible materials by RIVLIN

[1948d, 1949a,b,c], GREEN and SHIELD [1950], ERICKSEN and RIVLIN [1954], ADKINS [1955b,c] and others. In the simplest of these examples the deformation is uniform and the stresses have constant values which automatically satisfy the equations of equilibrium. The incompressibility condition then implies an algebraic relationship between the constants defining the deformation. In other problems possessing a suitable degree of symmetry the equations governing the deformation can, by a suitable choice of initial and final coordinate systems, be reduced to ordinary differential equations and the incompressibility condition can be integrated without difficulty.

§ 13. Uniform Deformation

When an elastic body is subject to the uniform deformation

$$y_i = c_{ij} x_j \, , \tag{13.1}$$

the displacement gradients $\partial y_i/\partial x_j = c_{ij}$ are constants. The strain components e_{ij} and the invariants I_r are also constant and this implies constant values throughout the material for W and for the terms involving W in the stress components σ_{ij} referred to the cartesian reference frame y_i. In the case of compressible materials the equations of equilibrium (11.3) are satisfied identically by the constant values of σ_{ij}. When the material is incompressible the equations of equilibrium reduce to $\partial p/\partial y_i = 0$ and the parameter p is then also a constant. Under these circumstances the boundary conditions yield a set of algebraic relations for the constants c_{ij} and in the incompressible case for p also.

The mathematical procedure may be illustrated by the example of pure homogeneous strain. We shall restrict attention to isotropic materials for which the stress strain relations (10.7), (10.8) apply. By a suitable choice of axes the problem may be derived as a special case of (13.1) by writing $c_{ii} = \lambda_i$, $c_{ij} = 0$ $(i \neq j)$ so that

$$y_1 = \lambda_1 x_1 \, , \quad y_2 = \lambda_2 x_2 \, , \quad y_3 = \lambda_3 x_3 \, , \tag{13.2}$$

λ_i being the principal extension ratios for this deformation. The invariants (8.7) become

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \, , \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \, , \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \, , \end{aligned} \tag{13.3}$$

and from (10.7) the physical components of stress σ_{ij} ($= \tau^{ij}$) are

$$\begin{aligned} \sigma_{ii} &= \frac{2\lambda_i^2}{\sqrt{I_3}} \left[\frac{\partial W}{\partial I_1} + (I_1 - \lambda_i^2)\frac{\partial W}{\partial I_2}\right] + 2\sqrt{I_3}\frac{\partial W}{\partial I_3}, \\ \sigma_{ij} &= 0 \quad (i \neq j)\,. \end{aligned} \tag{13.4}$$

Suppose now that the undeformed body is a unit cube with three adjacent edges coincident with the x_i-axis; the deformed body then becomes a cuboid with edges of lengths λ_1, λ_2, λ_3 (Fig. 2). The force f_1

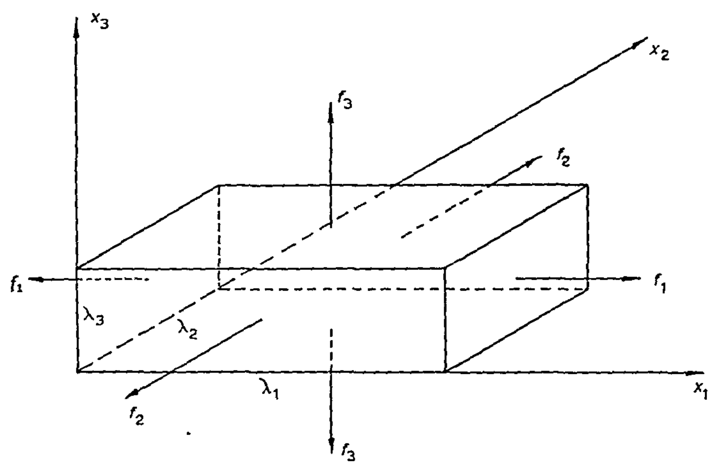

Fig. 2. Pure homogeneous deformation of unit cube.

on each face normal to the x_1-axis acts in the x_1-direction and is of magnitude

$$f_1 = \lambda_2\lambda_3\sigma_{11} = 2\lambda_1\left[\frac{\partial W}{\partial I_1} + (\lambda_2^2 + \lambda_3^2)\frac{\partial W}{\partial I_2} + \lambda_2^2\lambda_3^2\frac{\partial W}{\partial I_3}\right]. \tag{13.5}$$

Corresponding forces f_2, f_3 act on the other two pairs of faces. To calculate the forces f_i from the extension ratios λ_i, or conversely to determine the deformation when the force system is given, it is necessary to know the functional form of W.

In principle the relations between f_i and λ_i make it possible to evolve a procedure for the experimental determination of the form of W. If the deformation (13.2) can be produced, and the quantities λ_i, f_i measured satisfactorily, then I_1, I_2, I_3 may be calculated from (13.3), and (13.5) may then be regarded as a linear relationship between the three derivatives $\partial W/\partial I_r$ at known values of I_1, I_2, I_3. The expressions for f_2, f_3 furnish two similar equations, these three equations being linearly independent if the determinant

$$8\lambda_1\lambda_2\lambda_3(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)$$

of the coefficients of $\partial W/\partial I_r$ is non-zero. Provided therefore that no two extension ratios are equal it is possible to calculate the values of $\partial W/\partial I_r$ at known values of I_1, I_2, I_3. If now we assume that W can be approximated by a polynomial form in the invariants with a finite number of terms, so that

$$W = \sum_{r=1}^{m}\sum_{s=1}^{n}\sum_{t=1}^{p} C_{rst}\,(I_1-3)^r\,(I_2-3)^s\,(I_3-1)^t \quad (C_{000}=0)\,, \tag{13.6}$$

then we have

$$\frac{\partial W}{\partial I_1} = \sum_{r=1}^{m}\sum_{s=1}^{n}\sum_{t=1}^{p} r\,C_{rst}\,(I_1-3)^{r-1}\,(I_2-3)^s\,(I_3-1)^t\,, \tag{13.7}$$

with corresponding expressions for $\partial W/\partial I_2$, $\partial W/\partial I_3$. Of the quantities occurring in (13.7) we have seen that $\partial W/\partial I_1$ and the invariants I_r can be calculated from the measured values of f_r, λ_r. This equation may therefore be regarded as a linear relation between the constants C_{rst}. Two further relations are provided by the formulae for $\partial W/\partial I_2$, $\partial W/\partial I_3$. From each independent measurement we thus obtain three linear relations between the constants C_{rst}; a suitable number of measurements would provide a sufficient number of equations to determine the constants C_{rst} and hence the form of W.

When the material is incompressible, equations (13.3) and (13.4) are replaced by

$$\left.\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+\lambda_3^2,\\ I_2 &= \frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}, \end{aligned}\right\} \quad \lambda_1\lambda_2\lambda_3 = 1\,, \tag{13.8}$$

$$\sigma_{ii} = 2\left(\lambda_i^2\frac{\partial W}{\partial I_1} - \frac{1}{\lambda_i^2}\frac{\partial W}{\partial I_2}\right) + p', \quad \sigma_{ij} = 0 \quad (i \neq j)\,, \tag{13.9}$$

p' being an arbitrary isotropic pressure related to the parameter p of (10.8) by $p' = p + 2I_2(\partial W/\partial I_2)$. If $\sigma_{33} = 0$

$$\left.\begin{aligned} \sigma_{11} &= 2(\lambda_1^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_2^2\frac{\partial W}{\partial I_2}\right),\\ \sigma_{22} &= 2(\lambda_2^2-\lambda_3^2)\left(\frac{\partial W}{\partial I_1}+\lambda_1^2\frac{\partial W}{\partial I_2}\right). \end{aligned}\right. \tag{13.10}$$

This is the situation which arises when a uniform plane sheet, with its major surfaces parallel to the x_1, x_2-plane, is uniformly stretched by forces acting in its plane. Measurements of σ_{11}, σ_{22}, λ_1 and λ_2 for $\lambda_1 \neq \lambda_2$, are now sufficient to determine $\partial W/\partial I_1$, $\partial W/\partial I_2$ for given values of I_1, I_2 and to find, as before, a polynomial approximation for W. Essentially this formed the basis of the method used by RIVLIN and SAUNDERS [1951] to determine the form of the strain energy function for vulcanized rubber. Their procedure was in detail a little different, however, inasmuch as they chose the values of λ_1, λ_2 so that only one of the invariants varied during a series of readings the other remaining unchanged. They found for rubber the simple form

$$W = C(I_1 - 3) + f(I_2 - 3)\,, \tag{13.11}$$

in which C is a constant while the function f varies so that $(\partial f/\partial I_2)/C$ decreases from approximately 0.25 at low values of I_2 (< 5.0) to 0.03 at values of I_2 greater than 30.

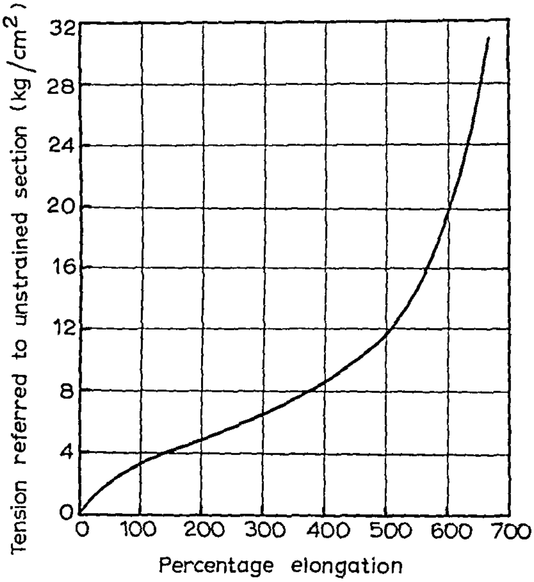

Fig. 3. Typical force-extension curve for vulcanized rubber.

Simple extension provides another example of the deformation (13.2). For a simple extension of ratio λ_1 in the x_1-direction, $\lambda_2 = \lambda_3$, and (13.4) yields

$$\begin{aligned} \sigma_{11} &= \frac{2(\lambda_1^2 - \lambda_2^2)}{\lambda_1\lambda_2^2}\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right), \\ \sigma_{22} = \sigma_{33} &= \frac{2}{\lambda_1}\left[\frac{\partial W}{\partial I_1} + (\lambda_1^2 + \lambda_2^2)\frac{\partial W}{\partial I_2} + \lambda_1^2\lambda_2^2\frac{\partial W}{\partial I_3}\right] = 0\,. \end{aligned} \tag{13.12}$$

The second of these relations serves to determine λ_2 in terms of λ_1 so that for compressible materials the amount of lateral contraction λ_2 due to a given extension λ_1 cannot be determined until the form of W is known. In the incompressible case this difficulty does not arise, since from the relations $\sigma_{22} = \sigma_{33} = 0$ and the incompressibility condition we obtain

$$\lambda_2^2 = \lambda_3^2 = \frac{1}{\lambda_1} = \frac{1}{\lambda} \quad \text{(say)}\,. \tag{13.13}$$

From (13.12) the tensile force N is given by

$$N = A_0\lambda_2\lambda_3\sigma_{11} = 2A_0\left(\lambda - \frac{1}{\lambda^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda}\,\frac{\partial W}{\partial I_2}\right), \tag{13.14}$$

where A_0 is the area of the cross-section of the undeformed test piece. A typical load extension curve† for vulcanized rubber is given in Fig. 3.

The uniform deformation (13.1) may be further illustrated by the simple shear of an incompressible isotropic cuboid. If the cuboid is bounded initially by the planes $x_i = 0$, $x_i = a_i$ and subjected to a simple shear in which planes parallel to the plane $x_2 = 0$ move parallel to the x_1-axis by amounts proportional to their x_2-coordinate (Fig.4) we have

$$y_1 = x_1 + Kx_2\,, \quad y_2 = x_2\,, \quad y_3 = x_3\,, \tag{13.15}$$

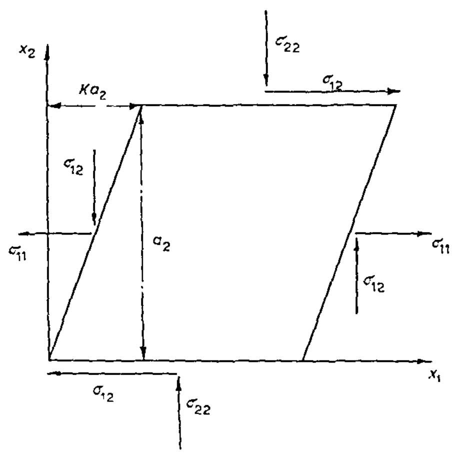

Fig. 4. Simple shear.

† The author is indebted to the Clarendon Press and to Dr. L. R. G. Treloar for permission to reproduce this curve from his book.

where K is a constant. The invariants I_1, I_2 are given by

$$I_1 = I_2 = 3 + K^2 . \tag{13.16}$$

The stress components τ^{ij} $(= \sigma_{ij})$ may be determined from (10.8) and if we choose p so that $\sigma_{33} = 0$ these become

$$\begin{aligned} \sigma_{11} &= 2K^2 \frac{\partial W}{\partial I_1}, & \sigma_{22} &= -2K^2 \frac{\partial W}{\partial I_2}, \\ \sigma_{12} &= 2K \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right), & \sigma_{13} &= \sigma_{23} = \sigma_{33} = 0 . \end{aligned} \tag{13.17}$$

To produce a finite shear therefore, it is necessary to apply a normal stress σ_{22} to the surfaces $x_2 = \pm a_2$ and a stress σ_{11} in the x_1-direction on the surfaces initially at $x_1 = \pm a_1$ in addition to the shearing stress σ_{12}. These additional stresses become second order effects for classically small deformations. We observe that when W is a linear function of the invariants I_1, I_2 the shear stress σ_{12} is proportional to the shear displacement K. It was in fact the observation of a linear load-deformation relation for rubber blocks in shear which led MOONEY [1940] to postulate the form (8.13) for rubber. The shear experiment is, however, a little deceptive, since even for quite large amounts of shear the invariants (13.16) do not become very large and if higher order derivatives of W are fairly small compared with $\partial W/\partial I_1$, $\partial W/\partial I_2$, as is actually the case for rubber, an approximately linear load-deformation relation would be expected.

§ 14. Cylindrical Symmetry

A class of problems which has been treated with some success is that involving cylindrical symmetry. In these problems, corresponding points of the undeformed body and of the deformed body are referred to cylindrical polar coordinates (r, θ, x_3), $(\varrho, \vartheta, y_3)$ respectively, where

$$x_1 + ix_2 = re^{i\theta} , \quad y_1 + iy_2 = \varrho e^{i\vartheta}, \tag{14.1}$$

and the deformation is defined by relations of the form

$$\begin{aligned} \varrho &= \varrho(r) \\ \vartheta &= \varphi(r) + a\theta + bx_3 , \\ y_3 &= w(r) + c\theta + dx_3 . \end{aligned} \tag{14.2}$$

In these equations ϱ, φ and w are functions of r only and a, b, c and d are constants. For simplicity we may suppose the reference frames (r, θ, x_3), $(\varrho, \vartheta, y_3)$ to coincide but this is not essential for the subsequent analysis; non-coincident reference frames would merely imply that the deformed body is subjected to an additional rigid-body displacement.

A physical picture of the problem thus defined is obtained if we think of the relations (14.2) as describing a composite deformation. Let the undeformed body be part of a cylindrical tube bounded in the coordinate system (r, θ, x_3) by the cylindrical surfaces $r = r_1$, $r = r_2$ $(r_1 > r_2)$, and the planes $\theta = \pm\theta_0$, $x_3 = \pm l$ (Fig. 5). The deformation

$$\varrho = \varrho(r)\,, \quad \vartheta = \theta\,, \quad y_3 = x_3\,, \tag{14.3}$$

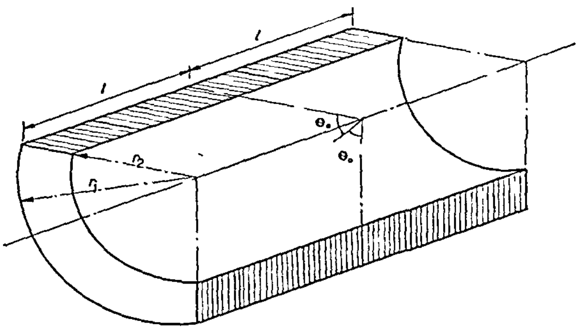

Fig. 5. Cylindrically symmetrical deformation: portion of undeformed tube.

represents a uniform inflation in which particles of the body move radially outwards from the axis $r = 0$. The relations

$$\varrho = r\,, \quad \vartheta = \theta + \varphi(r)\,, \quad y_3 = x_3\,, \tag{14.4}$$

$$\varrho = r\,, \quad \vartheta = \theta\,, \quad y_3 = x_3 + w(r)\,, \tag{14.5}$$

represent shearing deformations in which the concentric cylindrical surfaces r = constant are rotated about the x_3-axis and displaced parallel to this axis respectively. The constant d in (14.2) introduces a uniform extension in the x_3-direction while if $a \neq 1$ there is a change in the angular distance between any pair of radial planes. This latter type of deformation has been examined for various types of body by ADKINS *et al.* [1953] and by ADKINS [1955b,c]. With the appropriate form for the function $\varrho(r)$ and suitable assumptions about the elastic symmetry of the material we may obtain a flexure about the axis of the tube which may be sustained by flexural couples on the radial planes $\theta = \pm\theta_0$, the curved surfaces $r = r_1$, $r = r_2$ being free from applied traction. Evidently if the undeformed body is a complete tube ($\theta_0 = \pi$), deformations for which $a \neq 1$ must be excluded. The relations

$$\varrho = r, \quad \vartheta = \theta + bx_3, \quad y_3 = x_3, \tag{14.6}$$

describe a torsion in which planes perpendicular to the axis of the tube are rotated about this axis through an angle proportional to their distance from the origin. Finally, from the formulae

$$\varrho = r, \quad \vartheta = \theta, \quad y_3 = c\theta + x_3, \tag{14.7}$$

we may obtain a dislocation solution of the type discussed by VOLTERRA [1907] in which radial planes are sheared relative to each other. If the undeformed body is a circular tube, so that $\theta_0 = \pi$ we may imagine this tube to be cut down the radial plane $\theta = 0$ and rejoined after the deformation.

When the undeformed body possesses elastic symmetry about its axis partial differential equations may be avoided in the analysis of the deformation (14.2) and the special cases described by (14.3) to (14.7). The necessary degree of symmetry is exhibited by isotropic bodies, by materials which are transversely isotropic with respect to the x_3-direction, and by materials with cylindrical aeolotropy as discussed in § 7. The latter may be regarded as elastically homogeneous provided the axes of reference at each point of the undeformed body are chosen to coincide with the radial, transverse and longitudinal directions and W is a function only of e'_{ij}. In each of these cases the strain energy function W may be expressed as a function of r, ϱ, the derivatives ϱ_r, φ_r, w_r and the constants a, b, c, d, and may therefore be regarded as a function of the single variable r (or ϱ). In the case of an isotropic body, for example, if we choose the coordinate system θ_i so that

$$(\theta_1, \theta_2, \theta_3) = (\varrho, \vartheta, y_3), \tag{14.8}$$

and make use of (8.7) and the relations of §§ 2 and 8 we find that the invariants I_1, I_2, I_3 may be written

$$\begin{aligned} I_1 &= \varrho_r^2 + \varrho^2\varphi_r^2 + w_r^2 + \chi_1, \\ I_2 &= \frac{\lambda^2\varrho^2}{r^2} + \varrho_r^2\chi_1 + \chi_2, \\ I_3 &= \frac{\lambda^2\varrho^2\varrho_r^2}{r^2}, \end{aligned} \tag{14.9}$$

where

$$\chi_1 = \frac{1}{r^2}\left[\varrho^2(a^2 + b^2r^2) + c^2 + d^2r^2\right],$$
$$\chi_2 = \frac{\varrho^2}{r^2}\left[(c\varphi_r - aw_r)^2 + r^2(d\varphi_r - bw_r)^2\right], \tag{14.10}$$

are functions of r and $\lambda = ad - bc$.

From the stress strain relations given in § 10 it follows that the stress components τ^{ij}, and hence the physical components (11.6), referred to the reference frame θ_i (or ϱ, ϑ, y_3) are all functions of r, and the equations of equilibrium (11.5) in polar coordinates may be reduced to

$$\sigma_{22} = \frac{\mathrm{d}}{\mathrm{d}\varrho}(\varrho\sigma_{11}),$$
$$\frac{\mathrm{d}}{\mathrm{d}\varrho}(\varrho^2\sigma_{12}) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\varrho}(\varrho\sigma_{13}) = 0. \tag{14.11}$$

The second and third of these equations may evidently be integrated to yield

$$\varrho^2\sigma_{12} = A, \quad \varrho\sigma_{13} = B, \tag{14.12}$$

where A and B are constants. In the case of compressible bodies, the first of (14.11) with (14.12) yield three ordinary differential equations for the determination of ϱ, φ and w as functions of r. It is then necessary to know the form of W before further progress can be made. When the material is incompressible however, so that $I_3 = 1$, ϱ may be determined from the last of (14.9) in the form

$$\lambda\varrho^2 = r^2 + K, \tag{14.13}$$

K being a constant of integration. Since, from the relations of § 10 the stress components contain only first order derivatives of the displacement components, there follow from (14.12) a pair of first order differential equations for φ and w. If the material of the tube possesses suitable symmetry properties, σ_{12} and σ_{13} both vanish when $\varphi = w = 0$ and the second and third equations of equilibrium are then satisfied identically. In this case the first of (14.11) becomes a differential equation for ϱ in terms of r if the material is compressible, or for the isotropic pressure p when the body is incompressible.

The stress components for the isotropic incompressible case may be derived from (10.8) and (10.9) after evaluating the metric tensors (2.8), (2.9) with the help of (14.1), (14.2) and (14.8). We obtain

$$\sigma_{22} - \sigma_{11} = 2\Big\{\Big[\varrho^2\Big(\varphi_r^2 + \frac{a^2}{r^2} + b^2\Big) - \frac{r_2}{\lambda^2\varrho^2}\Big]\frac{\partial W}{\partial I_1}$$
$$+ \Big[\frac{1}{\lambda^2}(a^2 + b^2 r^2) + \frac{\lambda^2\varrho^2}{r^2} + \chi_2 - \frac{r^2}{\lambda^2\varrho^2}\chi_1\Big]\frac{\partial W}{\partial I_2}\Big\},$$

$$\sigma_{33} - \sigma_{11} = 2\Big\{\Big[w_r^2 + \frac{c^2}{r^2} + d^2 - \frac{r^2}{\lambda^2\varrho^2}\Big]\frac{\partial W}{\partial I_1}$$
$$+ \Big[\frac{1}{\lambda^2\varrho^2}(c^2 + d^2 r^2) + \frac{\lambda^2\varrho^2}{r^2} + \chi_2 - \frac{r^2}{\lambda^2\varrho^2}\chi_1\Big]\frac{\partial W}{\partial I_2}\Big\},$$
$$(14.14)$$

$$\sigma_{12} = \frac{2r}{\lambda}\Big\{\varphi_r\frac{\partial W}{\partial I_1} + \frac{1}{r^2}[(c^2 + d^2r^2)\varphi_r - (ac + bdr^2)w_r]\frac{\partial W}{\partial I_2}\Big\},$$

$$\sigma_{13} = \frac{2r}{\lambda\varrho}\Big\{w_r\frac{\partial W}{\partial I_1} + \frac{\varrho^2}{r^2}[(a^2 + b^2r^2)w_r - (ac + bdr^2)\varphi_r]\frac{\partial W}{\partial I_2}\Big\},$$

$$\sigma_{23} = 2\varrho\Big\{\varphi_r w_r\frac{\partial W}{\partial I_1} + \frac{1}{r^2}(ac + bdr^2)\Big(\frac{\partial W}{\partial I_1} + \frac{r^2}{\lambda^2\varrho^2}\frac{\partial W}{\partial I_2}\Big)\Big\}.$$

The first of these formulae, with the relation

$$\sigma_{11} = \int_{\varrho_0}^{\varrho}\frac{1}{\varrho}(\sigma_{22} - \sigma_{11})\,\mathrm{d}\varrho + [\sigma_{11}]_{\varrho=\varrho_0}, \qquad (14.15$$

derived from (14.11) gives an expression for σ_{11} which may be substituted back into (14.14), thus eliminating p.

When the material has the Mooney form of strain energy (8.13), $\partial W/\partial I_1$, $\partial W/\partial I_2$ take the constant values C_1, C_2. From (14.12), (14.14) we then derive a pair of differential equations which are linear in the unknown derivatives φ_r, w_r and may be integrated without difficulty. When $C_2 = 0$ there follow the simple results

$$\varphi(r) = \frac{\lambda^2 A}{2C_1K}\log\Big(\frac{Lr}{\sqrt{(r^2+K)}}\Big),$$
$$w(r) = \frac{\lambda B}{2C_1}\log(Mr), \qquad (14.16)$$

L and M being further constants of integration. By means of these relations and (14.13) the stresses (14.14) may be expressed as functions of r. The values of the constants a, b, c, d, A, B, K, L, M are determined by the boundary conditions.

§ 15. Some Generalizations of the Flexure Problem

Problems analogous to that discussed in the preceding section may be derived by replacing one or other or both of the cylindrical polar coordinate frames (r, θ, x_3), $(\varrho, \vartheta, y_3)$ by rectangular Cartesian systems. Corresponding to (14.2) we obtain the deformations

$$\varrho = \varrho(x_1)\,, \quad \vartheta = \varphi(x_1) + ax_2 + bx_3\,, \quad y_3 = w(x_1) + cx_2 + dx_3\,, \tag{15.1}$$

$$y_1 = y_1(r)\,, \quad y_2 = \varphi(r) + a\theta + bx_3\,, \quad y_3 = w(r) + c\theta + dx_3\,, \tag{15.2}$$

$$y_1 = y_1(x_1)\,, \quad y_2 = \varphi(x_1) + ax_2 + bx_3\,, \quad y_3 = w(x_1) + cx_2 + dx_3\,. \tag{15.3}$$

To examine the implications of (15.1) we may suppose the undeformed body to be a cuboid bounded by the planes $x_1 = A_1$, $x_1 = A_2$, $(A_1 > A_2 > 0)$, $x_2 = \pm B$, $x_3 = \pm C$. This cuboid is first subjected to the flexure

$$\varrho = \varrho(x_1)\,, \quad \vartheta = ax_2\,, \quad y_3 = x_3\,, \tag{15.4}$$

in which the planes x_1 = constant are deformed into the concentric cylindrical surfaces ϱ = constant and the planes x_2 = constant become radial planes through the axis $\varrho = 0$. Since displacements of all points of the body take place in the planes x_3 = constant this initial deformation may be regarded as an example of plane strain and has been treated from this point of view by ADKINS *et al.* [1953], GREEN and WILKES [1954] and ADKINS [1958a]. Upon this initial flexure there is superposed a uniform extension in the x_3-direction and further inflation, shearing and torsional deformations analogous to those described by (14.3) to (14.7) to give the final state of strain described by (15.1). This composite problem may be treated by the method of the preceding section. When the material is isotropic, transversely isotropic relative to the x_3-direction or uniformly aeolotropic relative to the x_i-axes the strain energy function W and the stress components τ^{ij} (or σ_{ij}) in the polar coordinate system $(\varrho, \vartheta, y_3)$ may be reduced to functions of the single variable x_1. The equations of equilibrium (14.11) and the integrated forms (14.12) continue to apply but for incompressible bodies the condition (14.13) is replaced by

$$\lambda \varrho^2 = 2x_1 + K\,. \tag{15.5}$$

Alternatively, the deformation (15.1) may be treated as a limiting case of (14.2). This approach has been adopted by ADKINS, GREEN and SHIELD [1953] and by ADKINS [1955b,c] and by this method it may be shown that results for the problem (15.1) may be derived from the corresponding formulae for the cylindrically symmetrical problem (14.2) by replacing $\mathrm{d}/\mathrm{d}r$ by $\mathrm{d}/\mathrm{d}x_1$ and a/r, c/r, λ/r by a, c, λ respectively wherever they occur. In addition, the quantities e'_{ij} are replaced by e_{ij} for aeolotropic materials.

A corresponding analysis may be performed for the inverse problem resulting from (15.2). In this case the undeformed body is the part of a cylindrical tube depicted in Fig. 5, in which there is again elastic symmetry about the axis. This is transformed by the inverse flexure

$$y_1 = y_1(r)\,, \quad y_2 = a\theta\,, \quad y_3 = x_3\,, \tag{15.6}$$

into a cuboid with its edges parallel to the y_i-axes and upon this deformation is superposed a system of shears and extensions analogous to those discussed in § 14 to give the more general state of strain described by (15.2). The strain energy function W may now be expressed in terms of the single variable r (or y_1) and a procedure similar to that of § 14 shows that for an isotropic incompressible material the stress deformation relations may be deduced from (14.14) by replacing ϱ_r by $\mathrm{d}y_1/\mathrm{d}r$ and then writing $\varrho = 1$ elsewhere and allowing σ_{ij} to represent the physical components of stress referred to the cartesian coordinates y_i in the deformed body. The equations of equilibrium (14.11) are therefore replaced by

$$\frac{\mathrm{d}\sigma_{11}}{\mathrm{d}y_1} = \frac{\mathrm{d}\sigma_{12}}{\mathrm{d}y_1} = \frac{\mathrm{d}\sigma_{13}}{\mathrm{d}y_1} = 0\,, \tag{15.7}$$

and from these we infer that

$$\sigma_{11} = A', \quad \sigma_{12} = B', \quad \sigma_{13} = C', \tag{15.8}$$

where A', B' and C' are constants determined by the boundary conditions. With (15.8) we must couple the stress strain relations corresponding to (14.14). For compressible bodies, this leads to three first-order differential equations for the determination of y_1, φ and w as functions of r. When the material is incompressible, the condition $I_3 = 1$ yields

$$\lambda\frac{\mathrm{d}y_1}{\mathrm{d}r} = r \quad \text{or} \quad 2\lambda y_1 = r^2 + K\,, \tag{15.9}$$

K being a constant. The second and third of (15.8) then serve to determine φ and w, while the formula for σ_{11} gives an immediate expression for the parameter p.

A similar discussion applies for the deformation (15.3). The stress strain relations for this problem may be derived from those for the deformation (15.1) by writing $\mathrm{d}y_1/\mathrm{d}r$ for ϱ_r and $\varrho = 1$ elsewhere and again allowing σ_{11} to represent the cartesian components of stress. The relationship between these two deformations is, in fact, analogous to that existing between (14.2) and (15.2). Alternatively we may derive the stress strain relations for (15.3) from those for the deformation (15.2) by replacing $\mathrm{d}/\mathrm{d}r$ by $\mathrm{d}/\mathrm{d}x_1$ and a/r, c/r, λ/r by a, c, λ respectively wherever they occur, the correspondence between these two problems being similar to that existing between (14.2) and (15.1). It follows that when the stress strain relations for the deformation (14.2) have been written down, those for the associated problems defined by (15.1) to (15.3) may be deduced by simple substitutions. Furthermore, it may be shown that this result applies not only for isotropic bodies, but for corresponding types of aeolotropic materials. The correct type of aeolotropy must then be assumed to ensure that W can be regarded as a function of a single variable (that is ϱ, r, y_1 or x_1 according to the problem being examined). If the undeformed body is defined by cylindrical polar coordinates (r, θ, x_3) as in (14.2) and (15.2), we must assume cylindrical aeolotropy; W is then a function

$$W = W(e'_{ij}) \,, \tag{15.10}$$

of the components e_{ij} evaluated at each point (as in § 7) with respect to a local cartesian reference frame with axes of x'_1, x'_2 and x'_3 directed along the radial, transverse and longitudinal directions respectively. If the undeformed body is defined in the cartesian system x_i, as in (15.1) and (15.3), the cartesian components e_{ij} must be used and

$$W = W(e_{ij}) \,, \tag{15.11}$$

the functional forms of (15.10) and (15.11) being the same for corresponding problems. These connexions between the cylindrically symmetrical and associated problems may be indicated by the scheme:

$$\begin{tikzcd}
 & (15.1) \arrow[dr, "\mathrm{II}"] & \\
(14.2) \arrow[ur, "\mathrm{I}"] \arrow[dr, "\mathrm{II}"'] & & (15.3)\,, \\
 & (15.2) \arrow[ur, "\mathrm{I}"'] &
\end{tikzcd}$$

where I and II represent the transitions

$$\mathrm{I}: \quad \left(\frac{a}{r}, \frac{c}{r}, \frac{\lambda}{r}\right) \to (a, c, \lambda) \,, \qquad \frac{\mathrm{d}}{\mathrm{d}r} \to \frac{\mathrm{d}}{\mathrm{d}x_1} \,, \qquad e'_{ij} \to e_{ij} \,,$$

$$\mathrm{II}: \quad \varrho_r \to \frac{\mathrm{d}y_1}{\mathrm{d}r} \,, \qquad \varrho \to 1 \,, \qquad [\sigma_{ij}]_{\varrho, \theta, y_3} \to [\sigma_{ij}]_{y_1, y_2, y_3} \,.$$

If we exclude spherically symmetrical deformations, all of the problems of finite elasticity which have so far been solved exactly, without restriction upon the form of W, are contained in the system of equations (14.2) and (15.1) to (15.3). Even for this limited class of problems, if the governing equations are to be integrated and the solution presented explicitly, we must either assume a specific form for W, or there must exist a sufficient number of cylindrically symmetrical geometrical constraints to provide the required number of integrable differential equations for the unknown functions of r. When the material is incompressible, for example, this condition is sufficient to determine ϱ or y_1, as a function of r or x_1, and the problems in which φ and w are identically zero can be solved completely.

In place of (15.2) we may examine the somewhat more general deformation

$$\begin{aligned} y_1 &= f(r) \;\; + h\theta + kx_3\,, \\ y_2 &= \varphi(r) \;\; + a\theta + bx_3\,, \\ y_3 &= w(r) + c\theta + dx_3\,, \end{aligned} \tag{15.12}$$

where h and k are additional constants. This represents the result of imposing a general system of shears and extensions upon a portion of a cylindrical tube which has first been deformed into a rectangular cuboid by the inverse flexure (15.6). Partial differential equations may again be excluded from the analysis provided the material is isotropic or cylindrically aeolotropic relative to the reference frame $(r,\ \theta,\ x_3)$. The stress strain relations yield expressions for the cartesian components σ_{ij} which contain, apart from constants, only the variables r and ϱ and the derivatives $f_r,\ \varphi_r,\ w_r$. These components may therefore be regarded as functions of the single variable r, and when the independent variables are changed from $y_1,\ y_2,\ y_3$ to $r,\ \theta,\ x_3$ in the equations of equilibrium

$$\frac{\partial \sigma_{ij}}{\partial y_j} = 0\,,$$

we obtain, for compressible bodies, three ordinary differential equations for f, φ and w. For the incompressible case, the condition $I_3 = 1$ yields

$$(ad - bc)f_r + (ck - hd)\varphi_r + (hb - ak)w_r = r\,,$$

or by integration

$$(ad - bc)f(r) + (ck - hd)\varphi(r) + (hb - ak)w(r) = \tfrac{1}{2}r^2 + K\,.$$

To determine f, φ and w it is again necessary to assume a particular form for W and to make use of the equations of equilibrium.

§ 16. The Torsion and Flexure Problems

An illustration of the cylindrically symmetrical deformation (14.2) is provided by the problem examined by RIVLIN [1949c] and others in which a cylindrical tube is subjected to a combined inflation, extension and torsion. This deformation may be obtained as a special case of (14.2) by writing

$$a = 1, \quad b = \lambda\psi, \quad c = 0, \quad d = \lambda, \quad \varphi(r) \equiv 0, \quad w(r) \equiv 0, \tag{16.1}$$

and since there is no longitudinal gap in the tube we have $\theta_0 = \pi$. We thus have the situation in which a cylindrical tube of length $2l$ and external and internal radii r_1, r_2 is subjected to a uniform extension of ratio λ in the x_3-direction, a uniform twist per unit length of stretched tube of magnitude ψ and an inflation in which the radial dimensions r_1, r_2 change to ϱ_1, ϱ_2 respectively.

If we suppose this deformation to be produced by means of a uniform inflating pressure P acting on the surface $\varrho = \varrho_2$, together with a longitudinal force N and a twisting couple M on the surfaces $y_3 = \pm\lambda l$, these forces may be calculated by introducing the conditions (16.1) into (14.14) and combining the resulting formulae derived from (14.14), (14.15) with the relations

$$\begin{gathered} P = -[\sigma_{11}]_{\varrho=\varrho_2} = 0\,, \qquad [\sigma_{11}]_{\varrho=\varrho_1} = 0\,, \\ N = 2\pi\int_{\varrho_2}^{\varrho_1} \varrho\sigma_{33}\,\mathrm{d}\varrho\,, \quad M = 2\pi\int_{\varrho_2}^{\varrho_1} \varrho^2\sigma_{23}\,\mathrm{d}\varrho\,. \end{gathered} \tag{16.2}$$

For an isotropic incompressible body, in which the condition (14.13) applies, this calculation yields the formulae

$$P = \frac{2}{\lambda}\int_{r_2}^{r_1} \left(1 - \frac{1}{\lambda^2\mu^4}\right)\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right)\frac{\mathrm{d}r}{r} + 2\psi^2\lambda\int_{r_2}^{r_1} \frac{\partial W}{\partial I_1} r\,\mathrm{d}r\,, \tag{16.3}$$

$$\begin{aligned} N = \frac{2\pi}{\lambda}\int_{r_2}^{r_1} \Big[2\Big(\lambda^2 - \frac{1}{\lambda^2\mu^2}\Big)\Big(\frac{\partial W}{\partial I_1} + \mu^2\frac{\partial W}{\partial I_2}\Big) \\ -\frac{1}{\lambda}\Big(1 - \frac{r_2^2}{r^2}\Big)\Big(1 - \frac{1}{\lambda^2\mu^4}\Big)\Big(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\Big)\Big] r\,\mathrm{d}r \\ - 2\pi\psi^2\int_{r_2}^{r_1} \Big[(r^2 - r_2^2)\frac{\partial W}{\partial I_1} + \frac{2r^2}{\lambda}\frac{\partial W}{\partial I_2}\Big] r\,\mathrm{d}r\,, \end{aligned} \tag{16.4}$$

$$M = 4\pi\psi\lambda\int_{r_2}^{r_1} \left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_2}\right)\mu^2 r^3\,\mathrm{d}r\,, \tag{16.5}$$

where $\mu = \varrho/r$ and I_1 and I_2 are given by

$$
\begin{aligned}
I_1 &= \lambda^2 + \mu^2 + \frac{1}{\lambda^2\mu^2} + \psi^2\lambda^2\mu^2 r^2\,, \\
I_2 &= \frac{1}{\lambda^2} + \frac{1}{\mu^2} + \lambda^2\mu^2 + \psi^2 r^2\,.
\end{aligned}
\tag{16.6}
$$

We observe that by virtue of the incompressibility condition

$$\lambda\varrho^2 = r^2 + K\,, \tag{16.7}$$

the deforming forces P, N and M are uniquely determined, for a given form of W, by (16.3) to (16.5) when the constants λ, ψ and K defining the deformation are known. Conversely if the applied loads P, N and M are prescribed, the relations (16.3) to (16.5) furnish three equations for the determination of λ, ψ and K. These relations are, however, non-linear in the unknown constants and further investigation is required to decide under what conditions the deformation is uniquely determined. The problem of uniqueness for finite deformations has been discussed for the problem of pure homogeneous strain by RIVLIN [1948b] and in more general terms by HILL [1957]. When $\psi = 0$ then $M = 0$ and (16.5) vanishes identically. The formulae for P and N then provide two relations for the determination of λ and K.

A particular example of the deformation (15.1) is provided by the simple flexure

$$\varrho = f(x_1)\,, \quad \vartheta = ax_2\,, \quad y_3 = dx_3\,. \tag{16.8}$$

If the undeformed body is the cuboid bounded by the plane faces $x_1 = A_1$, $x_1 = A_2$, $x_2 = \pm B$, $x_3 = \pm C$ the deformed body becomes a portion of a cylinder bounded by the curved surfaces $\varrho_1 = f(A_1)$, $\varrho_2 = f(A_2)$ and the planes $\vartheta = \pm aB$, $y_3 = \pm dC$. When the elastic material possesses a suitable degree of symmetry, the deformation may be sustained by means of normal tractions acting over the surfaces of the deformed body, the tractions over the curved surfaces being uniformly distributed. If both of the curved surfaces are free from applied stress, then the values of W at these two surfaces are equal and the forces on the planes $\vartheta = \pm aB$ reduce to a flexural couple M given by

$$M = \frac{2C}{a}\left[(A_1 - A_2)W_0 - \int_{A_2}^{A_1} W(x_1)\mathrm{d}x_1\right], \tag{16.9}$$

where

$$W_0 = W(A_1) = W(A_2)\,, \tag{16.10}$$

is the value of W at either of the curved surfaces. This result has been proved for incompressible isotropic bodies by RIVLIN [1949c] and for compressible or incompressible bodies which are orthotropic with the x_i-planes as planes of symmetry by GREEN and WILKES [1954] and by ADKINS [1955b]. When the elastic material is reinforced with one or two sets of thin, flexible inextensible cords, the cords of each set lying in parallel straight paths in the plane $x_1 = x_0$ $(A_1 \geqslant x_0 \geqslant A_2)$ it has been shown by ADKINS [1956a] that the formula (16.9) may be replaced by

$$M = \frac{2C}{a}\left[(A_1 - x_0)W(A_1) - (A_2 - x_0)W(A_2) - \int_{A_2}^{A_1} W(x_1)\mathrm{d}x_1\right], \quad (16.11)$$

with suitable assumptions regarding the properties of the cords. In this case, in general, $W(A_1) \neq W(A_2)$.

§ 17. Parametric Forms

The formulae for the stresses and the resultant forces on an elastic body may, for the problems of §§ 14 to 16, be written in elegant forms by regarding the strain components or the invariants appropriate to the material under consideration as functions of the parameters defining the deformation. An example of this type of approach (ADKINS [1955c]) is the cylindrically symmetrical deformation (14.2). In this case we regard

$$r,\ a,\ b,\ c,\ d,\ \varphi_r \ \text{and}\ w_r \qquad (17.1)$$

as independent variables and when the material is isotropic the invariants I_1, I_2, I_3 and the strain energy function W are all treated as functions of these quantities. If we assume the stresses σ_{ij} to be known functions of the parameters (17.1), then by virtue of (14.14), the quantities φ_r, w_r may, in principle, be eliminated from the first of the equations of equilibrium (14.11) by making use of (14.12). It follows that ϱ may be regarded as a function of r, a, b, c, d alone. A similar conclusion follows for incompressible bodies from (14.13). The constant K and additional arbitrary constants obtained from the integration of (14.11) may be regarded as independent of the parameters (17.1) and do not enter into the present discussion.

If, therefore, for an isotropic body we write

$$\begin{aligned} I_s &= I_s(r, a, b, c, d, \varphi_r, w_r) \quad (s = 1, 2, 3)\ , \\ W &= W(r, a, b, c, d, \varphi_r, w_r)\ , \\ \varrho &= \varrho(r, a, b, c, d)\ , \end{aligned} \qquad (17.2)$$

and allow ξ to denote any one of the parameters (17.1) we may evaluate the derivatives $\partial W/\partial\xi$ by combining (14.9) with the relations

$$\frac{\partial W}{\partial \xi} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \xi} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \xi} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \xi}.$$

By forming suitable combinations of the resulting expressions and applying the first of the equations of equilibrium (14.11) we obtain relations involving the stresses which, for incompressible materials, yield the comparatively simple formulae

$$\begin{aligned}
\sigma_{11} &= 2\int_{\varrho_0}^{\varrho}\left(a\frac{\partial W}{\partial a} + b\frac{\partial W}{\partial b} + \varphi_r\frac{\partial W}{\partial \varphi_r}\right)\frac{\mathrm{d}\varrho}{\varrho} + [\sigma_{11}]_{\varrho=\varrho_0} \\
&= 2\lambda\int_{\varrho_0}^{\varrho}\left(a\frac{\partial W}{\partial a} + c\frac{\partial W}{\partial c} + r\frac{\partial W}{\partial r}\right)\frac{\varrho\mathrm{d}\varrho}{r^2 - K} + [\sigma_{11}]_{\varrho=\varrho_0}, \\
\sigma_{22} &= \sigma_{11} + 2\left(a\frac{\partial W}{\partial a} + b\frac{\partial W}{\partial b} + \varphi_r\frac{\partial W}{\partial \varphi_r}\right), \\
\sigma_{33} &= \sigma_{11} + a\frac{\partial W}{\partial a} + b\frac{\partial W}{\partial b} + c\frac{\partial W}{\partial c} + d\frac{\partial W}{\partial d} + \varphi_r\frac{\partial W}{\partial \varphi_r} + w_r\frac{\partial W}{\partial w_r}, \\
\sigma_{23} &= \varrho\left(a\frac{\partial W}{\partial c} + b\frac{\partial W}{\partial d} + \varphi_r\frac{\partial W}{\partial w_r}\right) \\
&= \frac{1}{\varrho}\left(c\frac{\partial W}{\partial a} + d\frac{\partial W}{\partial b} + w_r\frac{\partial W}{\partial \varphi_r}\right), \\
\sigma_{12} &= \frac{r}{\lambda\varrho^2}\frac{\partial W}{\partial \varphi_r}, \\
\sigma_{13} &= \frac{r}{\lambda\varrho}\frac{\partial W}{\partial w_r},
\end{aligned} \tag{17.3}$$

K being defined as in (14.13).

When $\varphi(r) \equiv 0$, $w(r) \equiv 0$, $b = 0$, we may derive in the same way the additional relation

$$r^2\frac{\partial W}{\partial r} = -K\frac{\partial \sigma_{11}}{\partial r} - \frac{cr}{a\varrho}\sigma_{23}, \tag{17.4}$$

while when $\varphi(r) \equiv 0$, $w(r) \equiv 0$, $c = 0$ we have

$$r^2\frac{\partial W}{\partial r} = -K\frac{\partial \sigma_{11}}{\partial r} + \frac{br\varrho}{d}\sigma_{23}. \tag{17.5}$$

When $b = c = 0$ these relations both yield the expression

$$\sigma_{11} = [\sigma_{11}]_{r=r_0} - \frac{1}{K}\int_{r_0}^{r} r^2\frac{\partial W}{\partial r}\mathrm{d}r \tag{17.6}$$

for the radial stress σ_{11}. For the tube considered in § 14 we may deduce that, if both of the curved surfaces $r = r_1$, $r = r_2$ are free from applied traction, then

$$\int_{r_2}^{r_1} r^2 \frac{\partial W}{\partial r} \, \mathrm{d}r = 0 . \tag{17.7}$$

This condition applies if the tube is simply extended with no tractions applied to its curved surfaces; it also applies if it is turned inside out and allowed to rest in a deformed state with no traction applied to either of the curved surfaces and no resultant force on its plane ends. In this case (17.7) represents an equation for the determination of the extension ratio λ $(=d)$. For the inverted tube, $\varrho_2 > \varrho_1$ and from (14.13) λ is negative. A further example of the condition (17.7) is provided by the case in which the curved cuboid of § 14 is subjected to the simple flexure

$$\varrho = \varrho(r) , \quad \vartheta = a\theta , \quad y_3 = dx_3 ,$$

under the influence of tractions on its radial planes $\theta = \pm\theta_0$ and its plane ends $x_3 = \pm l$ only the surfaces $r = r_1$, $r = r_2$ remaining force free.

The choice of parameters (17.1) is evidently not unique. We may, for example, define a cylindrically symmetrical deformation of the type (14.2) by the relations

$$r = r(\varrho) , \quad \theta = \varphi_1(\varrho) + a_1\vartheta + b_1 y_3 , \quad x_3 = w_1(\varrho) + c_1\vartheta + d_1 y_3 ,$$

the coordinates in the deformed body being treated as independent variables, and allow the parameters

$$\varrho , \quad a_1, b_1, c_1, d_1 , \quad \mathrm{d}\varphi_1/\mathrm{d}\varrho \quad \text{and} \quad \mathrm{d}w_1/\mathrm{d}\varrho ,$$

to replace the set (17.1). Forms corresponding to (17.3) may then be derived. Again, for the deformation (15.12) we may define a set of nine parameters A_{ij} by the matrix

$$A_{ij} = \begin{bmatrix} f_r & h & k \\ \varphi_r & a & b \\ w_r & c & d \end{bmatrix} ,$$

and there then follow without difficulty for compressible bodies the formulae

$$\sigma_{ij}\sqrt{I_3} = A_{ik}\frac{\partial W}{\partial A_{jk}} = A_{jk}\frac{\partial W}{\partial A_{ik}} \qquad (\theta_i = y_i , \quad \sigma_{ij} = \tau^{ij}) ,$$

for the stress components σ_{ij}, summation being carried out over repeated suffixes. For the torsion problem defined by (16.1) and (14.2) formulae using the parameters r, ψ and λ have been given by GREEN [1955].

The stress deformation relations (17.3) to (17.7) apply not only for isotropic materials but also for materials which are transversely isotropic with the x_3-direction as the axis of anisotropy and for cylindrically aeolotropic bodies of the type described in § 7 in which the elastic properties are referred to the radial, transverse and axial directions at each point. The strain energy is then a function of the strain components e'_{ij} defined by (2.14). The functional form of W may be completely general, or it may be restricted by symmetry considerations, in which case we may write

$$W = W(\Psi_r)\,, \quad \Psi_r = \Psi_r(e'_{ij}) \quad (r = 1, 2, \ldots n)\,,$$

where Ψ_r are the invariants for the crystal class concerned. The components e'_{ij} and the invariants Ψ_r may be expressed in terms of the quantities ϱ, φ, w, a, b, c, d defining the deformation (14.2) by means of the formulae of § 2. When the parametric forms (17.3) have been derived, the stresses may be expressed in forms which exhibit the form of W for any given crystal class merely by making the substitutions

$$\frac{\partial W}{\partial \xi} = \sum_{r=1}^{n} \frac{\partial W}{\partial \Psi_r} \frac{\partial \Psi_r}{\partial \xi}\,,$$

where ξ represents any of the parameters (17.1).

§ 18. Reinforcement by Inextensible Cords

In some applications of rubberlike materials additional strength is given by the introduction of systems of relatively inextensible cords. This is done in tyre casings and in reinforced hosepipe. To render problems involving such materials mathematically tractable, ideal properties are attributed not only to the elastic material but also to the individual cords and to the composite body. It is assumed, for example, that the cords are each ideally thin, perfectly flexible and inextensible and that they are continuously distributed in families of non-intersecting smooth curves in smooth surfaces in the undeformed body. It is further postulated that there is no relative movement between each cord and the adjacent elastic material during deformation. The assumptions made are such, in fact, that each system of cords merely provides a means of introducing a geometrical constraint.

The analysis of such problems may proceed by several well defined stages as follows:

(i) the derivation of the geometrical conditions due to the cords,
(ii) the calculation of the stress strain relations for each system of cords,
(iii) the solution of the stress deformation problem for each region of the elastic material lying between adjacent layers of cords,
(iv) the determination of the interaction between each system of cords and the surrounding medium, and
(v) the calculation of the resultant force system for the composite body.

A number of symmetrical problems have been treated in this way by ADKINS and RIVLIN [1955] and by ADKINS [1956a]. Each system of cords is assumed to possess a suitable degree of symmetry so that partial differential equations need not be introduced into the analysis.

For illustration consider the cylindrically symmetrical deformation

$$\varrho = \varrho(r)\,, \quad \vartheta = a\theta + bx_3\,, \quad y_3 = c\theta + dx_3\,, \tag{18.1}$$

derived from (14.2) by putting $\varphi(r) \equiv 0$, $w(r) \equiv 0$. The symmetry is not disturbed by introducing into the cylindrical surface $r = r_0$ a layer of cords in which each cord follows a helical path

$$x_3 = r_0\theta \cot\alpha + \text{constant}\,, \tag{18.2}$$

α being the constant angle made by each cord with the x_3-direction. If we assume that the surface $r = r_0$ is displaced to $\varrho = \varrho_0$ by the deformation (18.1) and that the effect of the cords is to make the material inextensible along the paths (18.2), there follows the relationship

$$\varrho_0^2\,(a\sin\alpha + br_0\cos\alpha)^2 + (c\sin\alpha + dr_0\cos\alpha)^2 = r_0^2\,. \tag{18.3}$$

When the material is incompressible ϱ_0 can be eliminated from (18.3) by using the relation

$$\lambda\varrho_0^2 = r_0^2 + K \quad (\lambda = ad - bc > 0)\,,$$

which follows from (14.13). The layer of cords thus imposes a condition upon the constants a, b, c, d and K defining the deformation. Five such sets of cords would yield five relationships of this kind connecting the constants a, b, c, d, K and if these relationships were functionally independent a continuously varying deformation of the type (18.1) would be impossible. For this reason in the problems examined not more than four independent sets of cords have been allowed. Each layer of cords may be regarded as a thin flexible sheet and the stress resultants in this sheet can be expressed in terms of the tensions in the cords and the spacing between them by a direct calculation. The elastic problem for each tube of elastic material is of the kind considered in § 14. The forces of interaction between a layer of cords and the elastic medium on one side of it are everywhere radial and uniform and provide the boundary conditions for stage (iii). Each layer of cords on the other hand is in equilibrium under the action of the radial forces exerted by the elastic material and the tensions in the cords, the latter arising from the forces on the ends of the tube.

The general theory of thin reinforced sheets has been developed by ADKINS [1956c] and the allied problems involving networks of cords by RIVLIN [1955, 1959]. The writer [1958b] has also considered a more general three-dimensional problem of elastic bodies containing internal geometrical constraints.

§ 19. Solutions for a Mooney Material

Further progress is sometimes possible in deriving solutions of the equations of finite elasticity if a particular form can be assumed for W. We have seen, for example, that the equations for the cylindrically symmetrical deformation (14.2) can be solved completely if we restrict attention to an isotropic incompressible material having the linear Mooney form of strain energy function. In this case, the derivatives $\partial W/\partial I_1$, $\partial W/\partial I_2$ reduce to the constants C_1, C_2 respectively and this feature often simplifies the calculations considerably. Furthermore, this form gives a useful approximate description of the elastic properties of vulcanized rubber.

One class of problem which has been examined by ADKINS [1954] for this type of material is defined by the relations

$$[y_1, y_2, y_3] = [\lambda x_1, \lambda x_2, x_3/\lambda^2 + f(\lambda x_1, \lambda x_2)]\,, \tag{19.1}$$

or

$$[x_1, x_2, x_3] = [y_1/\lambda, y_2/\lambda, \lambda^2\{y_3 - f(y_1, y_2)\}]\,. \tag{19.1'}$$

When $\lambda = 1$ these equations describe a shearing deformation in which each point of the elastic body is displaced parallel to the x_3-axis through a distance which is dependent only upon its position in the

x_1, x_2-plane. When $\lambda \neq 1$ we obtain the slightly more general problem in which there is superposed a uniform finite extension of ratio $1/\lambda^2$ parallel to the x_3-axis.

We observe that the incompressibility condition $|\partial y_i/\partial x_j| = 1$ is automatically satisfied by the deformation (19.1) and by means of (10.8) (with $\theta_i = y_i$, $\tau^{ij} = \sigma_{ij}$) the stress components σ_{ij} may be expressed in terms of λ and the derivatives $\partial f/\partial y_1$, $\partial f/\partial y_2$. The third of the equations of equilibrium then yields the Poisson equation

$$\frac{\partial^2 f}{\partial y_1^2} + \frac{\partial^2 f}{\partial y_2^2} = A\,, \tag{19.2}$$

where A is an arbitrary constant, for the determination of f. The first and second of the equations of equilibrium give by integration an expression for the arbitrary isotropic pressure p.

When $\lambda = 1$, $A = 0$ we may obtain an approximation to a problem of practical importance. This problem arises from shearing deformations of rubber bush mountings. In these mountings the rubber is in the shape of a cylindrical annulus or tube with its inner and outer surfaces everywhere bounded to two, effectively rigid, long metal cylinders. When the generators of these cylinders are parallel to the x_3-axis the displacement of one cylinder through a distance K in the x_3-direction relative to the other produces a deformation of the type (19.1) with $\lambda = 1$, provided end effects can be neglected. If the surfaces of the cylinders adjacent to the rubber cut the x_1,x_2-plane in the curves S_1, S_2 the problem is reduced to that of finding the solution of the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = 0\,, \tag{19.3}$$

subject to the boundary conditions

$$f = 0 \quad \text{on} \quad S_1\,, \quad f = K \quad \text{on} \quad S_2\,. \tag{19.4}$$

This may be achieved in a number of ways depending upon the shapes of S_1, S_2; solutions using conformal transformations have been given by the writer. Special cases of this problem are provided by the cylindrically symmetrical deformation (14.5) and by the simple shear (13.15).

§ 20. Plane Strain

A further type of deformation which has been treated with some success is that of plane strain. This may be represented by the relations

$$y_\alpha = y_\alpha(x_1, x_2)\,, \quad y_3 = x_3 \quad (\alpha = 1, 2)\,, \tag{20.1}$$

the x_1, x_2-planes then being the planes of movement. As in the case of classical elasticity, the stress components σ_{ij} become functions of the two variables x_1, x_2 only. The equations of equilibrium may again be satisfied by introducing an Airy stress function Φ, but this is now related to points in the *deformed* body. The stresses in the x_1, x_2-plane are therefore given by

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial y_2^2}\,, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial y_1^2}\,, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial y_1 \partial y_2}\,. \tag{20.2}$$

When these stresses are expressed in terms of the deformation gradients $\partial y_i/\partial x_j$ by means of the stress-strain relations we obtain three differential equations for the determination of Φ, y_1, y_2 as functions of x_1, x_2. When the material is incompressible the forms derived from (20.2) contain the additional unknown p, but the incompressibility condition then completes the system of four equations required to determine Φ, y_1, y_2 and p. The general theory using this method of approach has been developed in papers by ADKINS *et al.* [1953, 1954] and GREEN and WILKES [1954].

When the material has the Mooney form of strain energy function, Φ and p may be eliminated from the equations (20.2) to yield the symmetrical relation

$$\frac{\partial(x_1, \nabla_1^2 x_1)}{\partial(y_1, y_2)} + \frac{\partial(x_2, \nabla_1^2 x_2)}{\partial(y_1, y_2)} = 0 \quad \left(\nabla_1^2 \equiv \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right), \tag{20.3}$$

which, by a change of independent variable may be expressed in the inverse form

$$\frac{\partial(y_1, \nabla_2^2 y_1)}{\partial(x_1, x_2)} + \frac{\partial(y_2, \nabla_2^2 y_2)}{\partial(x_1, x_2)} = 0 \quad \left(\nabla_2^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right). \tag{20.4}$$

With (20.3) and (20.4) we may couple the incompressibility condition in the forms

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = 1\,, \tag{20.5}$$

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = 1\,, \tag{20.6}$$

respectively. From the forms of these equations it may be deduced that if the relations

$$y_\alpha = f_\alpha(x_1, x_2) \qquad (\alpha = 1, 2), \tag{20.7}$$

represent a solution, in the absence of body forces, of the finite plane strain equations, then the inverse relations

$$x_\alpha = f_\alpha(y_1, y_2), \tag{20.8}$$

yield another solution, also with body forces absent. This reciprocal property has been extended to isotropic and transversely isotropic incompressible materials with a general form of strain energy function by ADKINS [1958a] who has also [1955a] obtained solutions of the Mooney equations of the form

$$x_1 = x_1(y_1), \quad x_2 = y_2 g(y_1) + f(y_1), \quad \left[\frac{\mathrm{d}x_1}{\mathrm{d}y_1} g(y_1) = 1\right]. \tag{20.9}$$

For these deformations the incompressibility condition is automatically satisfied and the equation of equilibrium (20.3) yields a pair of ordinary differential equations for the functions f and g, these equations arising as a result of equating separately to zero the coefficient of y_2 and the remaining terms derived from (20.3).

III. APPROXIMATION PROCEDURES

§ 21. Elastic Membranes

Much of the simplification that occurs in the theory of plane strain arises from the reduction of the number of independent variables from three to two in the equations governing the deformation. A corresponding simplification may be achieved when we consider the deformation of a thin plate or shell. With suitable restrictions upon the thickness and radii of curvature of the shell and upon the continuity of the deformation, we may, as in classical elasticity, replace the stress components acting at each point of the material by stress resultants obtained by integrating the stress-strain relations throughout the thickness of the deformed sheet. In practice, a membrane theory is found to be adequate for most purposes since the principal extension ratios in the middle surface of the sheet are usually much greater than unity. Stress couples and shearing stress resultants normal to the deformed middle surface are thus neglected, leaving a system of stress resultants acting at each point in the tangent plane to the deformed middle surface. These resultants are, in general, functions of position in this surface. A theory of plane deformation for isotropic sheets has been developed by ADKINS, GREEN and NICHOLAS [1954] and a more general theory for curved sheets is given by GREEN and ADKINS [1960] p. 3.

An interesting class of problems is that in which there is axial symmetry. In these a sheet possessing geometrical and elastic symmetry about an axis is subjected to a system of deforming forces which is symmetrical about the same axis. The middle surface of the sheet thus forms a surface of revolution both before and after deformation. It is further assumed that the deforming forces acting on the major surfaces of the deformed sheet are everywhere normal to these surfaces and form a continuous system so that there are no singularities on the deformed middle surface. The problem thus defined may be examined by considering the situation at a typical point B in the middle surface of the sheet. We suppose B to be situated initially a distance s along the meridian through this point from a given point A and a distance r from the axis of symmetry, these distances

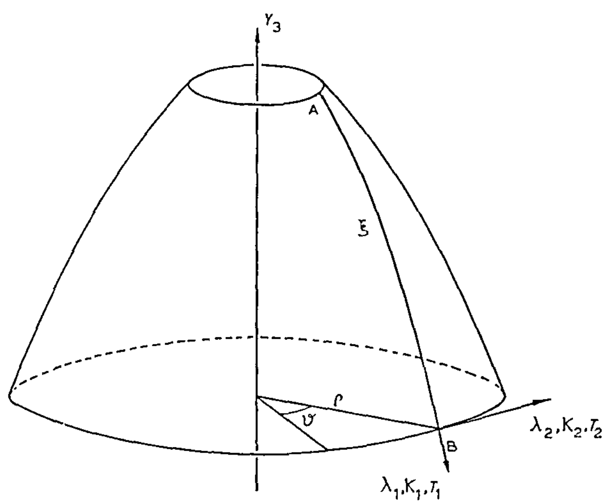

Fig. 6. Axially symmetrical deformation: portion of deformed surface of revolution.

being changed to ξ, ϱ respectively by the deformation. When the material is isotropic, or possesses suitable symmetry properties, the principal directions of stress and strain coincide with lines of longitude and latitude respectively in the deformed middle surface. If the principal extension ratios in these directions are denoted by λ_1, λ_2, the tensions by T_1, T_2 and the curvatures by $\varkappa_1$, $\varkappa_2$ respectively we obtain the equations

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \lambda_1 , \qquad \frac{\varrho}{r} = \lambda_2 , \tag{21.1}$$

$$T_1 = f_1(\lambda_1, \lambda_2, \lambda_3) , \qquad T_2 = f_2(\lambda_1, \lambda_2, \lambda_3) , \tag{21.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(T_1\varrho) = T_2 \frac{\mathrm{d}\varrho}{\mathrm{d}r} , \qquad \varkappa_1 T_1 + \varkappa_2 T_2 = P , \tag{21.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(\varkappa_2\varrho) = \varkappa_1 \frac{\mathrm{d}\varrho}{\mathrm{d}r} , \qquad \varkappa_2 = \frac{1}{\varrho}\left[1 - \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\xi}\right)^2\right]^{\frac{1}{2}} , \tag{21.4}$$

P being the resultant applied force acting normally to the deformed middle surface at B, and λ_3 the principal extension ratio in this direction. The situation in the deformed sheet is illustrated by Fig. 6. Eqs. (21.2) represent the expressions for the stress resultants obtained from the stress-strain relations, (21.3) are the equations of equilibrium, and (21.1) and (21.4) follow from geometrical considerations. For compressible bodies a relation between λ_1, λ_2, λ_3 is obtained from the stress-strain relations and the boundary conditions at the major surfaces. In the incompressible case we have

$$\lambda_3 = 1/(\lambda_1 \lambda_2) , \tag{21.5}$$

and when the material is isotropic the stress resultants take the form

$$\begin{aligned} T_1 &= 2\lambda_3 h\,(\lambda_1^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2 \frac{\partial W}{\partial I_2}\right) , \\ T_2 &= 2\lambda_3 h\,(\lambda_2^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2 \frac{\partial W}{\partial I_2}\right) , \end{aligned} \tag{21.6}$$

where h is the thickness of the undeformed sheet. With these, we must couple the expressions

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 , \qquad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} , \tag{21.7}$$

for the invariants. If the shape of the undeformed sheet is specified by a relation

$$s = f(r)$$

between s and r, so that

$$\frac{\mathrm{d}\varrho}{\mathrm{d}\xi} = \frac{\mathrm{d}\varrho}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\xi} = \frac{1}{\lambda_1 f'}\frac{\mathrm{d}\varrho}{\mathrm{d}r} , \tag{21.8}$$

the second of (21.1) and (21.3) to (21.6) provide eight equations for the determination of the eight unknowns ϱ, λ_1, λ_2, λ_3, $\varkappa_1$, $\varkappa_2$, T_1 and T_2 as functions of r.

When $s = r$ and P is a constant the equations describe a situation that can be reproduced experimentally by clamping a plane uniform circular rubber sheet around its edge and applying air pressure to one face. This problem has been solved numerically for a Mooney form of strain energy by ADKINS and RIVLIN [1952] and the theoretical results compared with experimental data published by TRELOAR [1944]. For small amounts of inflation good agreement was obtained. At high degrees of inflation the strain distribution in the deformed sheet proved to be very sensitive to the form of W. The Mooney form, with $C_2/C_1 = 0.1$ gave a much better agreement between theory and experiment than the Neohookean form ($C_2 = 0$) and such discrepancies as remained were consistent with the differences between the Mooney form (8.13) and the strain energy function (13.11) determined experimentally for vulcanized rubber.

The system of equations (21.1) to (21.6) also yields an exact solution for the inflation of a spherical rubber balloon. Here conditions are symmetrical at each point of the middle surface of the sheet so that we may write

$$\lambda_1 = \lambda_2 = \lambda\,, \qquad \varkappa_1 = \varkappa_2 = \varkappa, \qquad T_1 = T_2 = T\,. \tag{21.9}$$

If a is the initial radius of the balloon, the relation between P and λ becomes

$$P = \frac{4h}{\lambda a}\left(1 - \frac{1}{\lambda^6}\right)\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right). \tag{21.10}$$

When W has the form (13.11) determined experimentally for vulcanized rubber by RIVLIN and SAUNDERS [1951], calculations based on (21.10) show that the pressure rises fairly rapidly during the early stages of inflation to a maximum value around $\lambda = 1.5$ and that the subsequent fall in pressure as inflation proceeds is followed by a further steady rise at high degrees of inflation. This conclusion is readily verified experimentally. By considering the behaviour of $\mathrm{d}P/\mathrm{d}\lambda$ it may be shown theoretically that the same general characteristics would be exhibited by a material with the Mooney form of W (8.13) provided $0 < C_2/C_1 < 0.21$. If $C_2 = 0$ the initial rise in pressure is followed by a steady fall with no subsequent rise; if $C_2/C_1 > 0.21$, P is a monotonically increasing function of λ.

shell. Further aspects of the latter problem have been considered by ERICKSEN [1955] and by the writer [1955b] and the theory for the thick shell has been applied by GENT and LINDLEY [1958] to explain the breakdown of certain types of rubber mountings.

§ 22. Successive Approximations

When exact solutions of the equations of finite elasticity cannot be obtained it is natural to consider the possibility of deriving approximate forms based on the solutions of the classical infinitesimal theory of elasticity. One method is to assume that the components of displacement and stress can be expanded as power series in a real parameter ε. The justification for this procedure is provided by the work of STOPPELLI [1954, 1955] who has given a proof of existence and uniqueness of solution of the general elastic equations and has shown that the displacement components can be expanded as absolutely convergent power series in ε with non-zero radius of convergence provided sufficiently smooth solutions of the classical linear equations of elasticity exist.

We therefore assume expansions

$$\begin{aligned} u_i = y_i - x_i &= \sum_{n=1}^{\infty} \varepsilon^n ({}^n u_i) \,, \\ \sigma_{ij} &= \sum_{n=1}^{\infty} \varepsilon^n ({}^n \sigma_{ij}) \,, \end{aligned} \tag{22.1}$$

rectangular cartesian coordinates x_i being chosen for simplicity. In (22.1) ${}^n u_i$, ${}^n \sigma_{ij}$ are functions of the coordinates x_i (or y_i). The parameter ε must be chosen in a manner appropriate to the problem under consideration. For example, if the undeformed body is an infinite plane sheet containing a number of holes or inclusions and this is deformed by a uniform tension T in a given direction at infinity, we may conveniently choose $\varepsilon = T/E$ where E is Young's modulus for classically small deformations. As the tension T is increased the parameter ε also increases and higher order terms in the series (22.1) assume increasing importance. In other cases where displacement boundary conditions are prescribed ε may conveniently be related to these. When the first of the expansions (22.1) is substituted into the formula (2.4) for the strain components e_{ij} we obtain

$$\begin{aligned} e_{ij} = \frac{1}{2} \Big\{ \varepsilon \Big[\frac{\partial ({}^1 u_i)}{\partial x_j} + \frac{\partial ({}^1 u_j)}{\partial x_i} \Big] \\ + \varepsilon^2 \Big[\frac{\partial ({}^2 u_i)}{\partial x_j} + \frac{\partial ({}^2 u_j)}{\partial x_i} + \frac{\partial ({}^1 u_r)}{\partial x_i} \frac{\partial ({}^1 u_r)}{\partial x_j} \Big] + \ldots \Big\} \,. \end{aligned} \tag{22.2}$$

The coefficients of ε are thus similar in form to the familiar strain components of the infinitesimal theory; the coefficients of ε^2 yield corresponding linear forms in the components ${}^2u_i$ but with additional terms involving the first order quantities ${}^1u_i$. Thus, writing

$$e_{ij} = \sum_{n=1}^{\infty} \varepsilon^n({}^n e_{ij}) \,, \tag{22.3}$$

in (22.2) and equating to zero the coefficients of successive powers of ε we obtain a series of relations

$$\begin{aligned}
{}^1e_{ij} &= \frac{1}{2}\left[\frac{\partial({}^1u_i)}{\partial x_j} + \frac{\partial({}^1u_j)}{\partial x_i}\right], \\
{}^2e_{ij} &= \frac{1}{2}\left[\frac{\partial({}^2u_i)}{\partial x_j} + \frac{\partial({}^2u_j)}{\partial x_i}\right] + \frac{1}{2}\,\frac{\partial({}^1u_r)}{\partial x_i}\,\frac{\partial({}^1u_r)}{\partial x_j}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
{}^ne_{ij} &= \frac{1}{2}\left[\frac{\partial({}^nu_i)}{\partial x_j} + \frac{\partial({}^nu_j)}{\partial x_i}\right] + f_n\left(\frac{\partial({}^{n-1}u_k)}{\partial x_l}, \frac{\partial({}^{n-2}u_p)}{\partial x_q}, \ldots \frac{\partial({}^1u_r)}{\partial x_s}\right),
\end{aligned} \tag{22.4}$$

where f_n are known functions of the derivatives indicated. The strain invariants, strain energy function, stress strain relations, equations of equilibrium, boundary conditions and constraint conditions, where they exist, may be expanded similarly in powers of ε. From the coefficients of ε in these relations we deduce for the first order quantities the usual linear equations of the classical theory of elasticity. From the coefficients of ε^2 corresponding equations may be derived, linear in the second order quantities ${}^2u_i$, ${}^2\sigma_{ij}, \ldots$ but containing known additional functions of the first order quantities ${}^1u_i$, ${}^1\sigma_{ij}, \ldots$. If the classical first order problem can be solved, these additional functions can be evaluated explicitly as functions of the coordinates x_i. The linear second order equations can then often be solved by a procedure similar to that employed in deriving the first order solution. In principle the procedure could be continued to obtain solutions of the third and higher orders. In practice the complexity of the expressions obtained increases very rapidly with each successive degree of approximation and the process has so far been terminated with the second order terms.

This procedure is essentially that used by MURNAGHAN [1951] to derive approximate solutions to a number of simple problems. General equations for the derivation of second order solutions have been developed by GREEN and SPRATT [1954] who have used the theory to examine cylindrically symmetrical problems. The second-order torsion problem has been examined by GREEN and SHIELD [1951] and by RIVLIN [1953], who has also developed the general second-order theory using a somewhat different method of approach. For two-dimensional problems the linearity of the second-order equations makes possible complex variable techniques similar to those developed by MUSKHELISHVILI [1953] for the infinitesimal theory. The procedure has been worked out in detail by ADKINS *et al.* [1953, 1954, 1957] who have shown that by a suitable formulation a large number of problems may be solved simultaneously. For example it is possible to derive a solution which is inclusive of those for compressible or incompressible materials in plane stress or plane strain in which corresponding conditions are prescribed either on the stress or the displacement components over boundaries which are defined either in the deformed body or in the undeformed body, the appropriate results being selected by a suitable choice of parameters in this general solution. The corresponding results for the classical theory have been given by ADKINS [1956b].

For isotropic compressible materials five elastic constants are required to specify the mechanical properties of the material to a degree of approximation sufficient for the second order theory as compared with the two constants of classical elasticity. For incompressible bodies, two constants are required in place of the single constant of the classical theory. The Mooney constants C_1, C_2 are adequate for this purpose. In the case of aeolotropic bodies a large number of constants would be required for the second order theory this being particularly so when the form of the strain energy function is not greatly restricted by symmetry properties. Owing to this increase in complexity the second-order theory for aeolotropic bodies has received little attention although RIVLIN and TOPAKOGLU [1954] have indicated a general method of approach and BLACKBURN [1958] has solved some problems for transversely isotropic materials.

§ 23. Small Deformations Superposed upon Finite Deformations

The success of the approximation procedure outlined in the preceding section depends essentially upon the linearization of the relations governing the deformation so that one or other of the techniques available for the solution of linear differential equations can be applied. A second problem of this type occurs where an infinitesimal deformation is superposed upon a large deformation. Such problems may be treated in two stages. The first stage involves the solution if the elastic problem for the finite deformation alone; the second involves the determination of the quantities defining the additional infinitesimal strain. Since the equations for this latter part of the problem may be linearized, the usual techniques for dealing with linear partial differential equations are again often available.

To make this procedure clear we may suppose that a point initially at x_i in the rectangular coordinate system x_i is displaced to y_i by the finite deformation. Upon this deformation is superposed a small additional deformation specified by displacement components εv_i where ε is a small constant parameter. The final coordinates y_i of the point initially at x_i are therefore given by

$$y_i' = y_i + \varepsilon v_i = x_i + u_i + \varepsilon v_i \,, \tag{23.1}$$

u_i being the displacement components for the finite deformation. When the strain components are evaluated by means of (2.4) and squares of ε discarded we find that the quantities e_{ij} are replaced by $e_{ij} + \varepsilon e'_{ij}$ where

$$e'_{ij} = \frac{1}{2}\left(\frac{\partial y_r}{\partial x_i}\frac{\partial v_r}{\partial x_j} + \frac{\partial y_r}{\partial x_j}\frac{\partial v_r}{\partial x_i}\right), \tag{23.2}$$

may be regarded as the strain components for the superposed infinitesimal deformation. We observe that these latter quantities are linear in the derivatives $\partial v_i/\partial x_j$. For isotropic bodies the invariants I_r may be expanded in a similar manner. When squares and higher powers of ε are discarded, the invariants take the forms $I_r + \varepsilon I'_r$ where the additional quantities I'_r are again linear in $\partial v_i/\partial x_j$. A similar procedure may be followed in expanding the stress-strain relations, equations of equilibrium, boundary conditions and any relevant constraint conditions. In the stress-strain relations for the superposed deformation second-order derivatives of W with respect to the strains or invariants appear explicitly. If W is a function of I_r, for example, the derivatives $\partial W/\partial I_r$ are replaced in the perturbed state by

$$\frac{\partial W}{\partial I_r} + \varepsilon \sum_{s=1}^{3} I'_s \frac{\partial^2 W}{\partial I_r \partial I_s} \,,$$

the derivatives in this expression being given values appropriate to the finite deformation alone.

This procedure is closely related to that followed in investigations of stability. The equations of equilibrium for the superposed deformation are in fact identical with the equations of neutral equilibrium for the stability problem. The stability of a circular tube under end thrust has been examined by WILKES [1955] using this approach.

The general theory of small deformations superposed upon large deformations has been developed for isotropic materials by GREEN, RIVLIN and SHIELD [1951]. In applications of the theory attention has so far been confined to cases where the small deformation has been superposed on the pure homogeneous strain defined by (13.2). The deformation gradients are then constants, and the differential equations for the small deformation are linear in the derivatives $\partial v_i/\partial x_j$ with constant coefficients. As might be expected, the relationships obtained for the small deformation resemble those of classical elasticity for an orthotropic body when the principal extension ratios λ_i for the large strain are all different and those of a transversely isotropic material when two of the ratios are equal. In the latter case, the axis of anisotropy coincides with the direction of the unequal extension ratio. When the large deformation is inhomogeneous, the derivatives $\partial y_i/\partial x_j$ are no longer constant and the equations for the displacements v_i, although linear in $\partial v_i/\partial x_j$, then have coefficients which are functions of x_i. When the large deformation is defined by the general homogeneous strain (13.1) this difficulty does not appear but the equations for the small deformation are less simple in form than when the finite deformation is restricted to the form (13.2).

Problems examined by Green, Rivlin and Shield include the small indentation by a punch of a uniformly stretched material occupying the half space $x_3 \geqslant 0$, and a uniformly stretched plate subjected to additional small bending or generalized plane stress. GREEN and SHIELD [1951] have examined the problem of a rod of uniform arbitrary cross-section which is first subjected to a simple extension of ratio λ and then to a small torsion of amount ψ per unit length of extended rod, it being assumed that the cylinder is twisted about the line joining the centroids of the cross-sections. If λ_1 is the extension ratio uniform in all directions in planes perpendicular to the generators of the cylinder, the torsional couple M is given by

$$M = \frac{2\psi\lambda_1^2}{\lambda}\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right)[(\lambda^2 - \lambda_1^2)I_0 + \lambda_1^2 S_0]\,, \tag{23.3}$$

where I_0 is the moment of area of the unstrained cross-section about the line of centroids and S_0 is the torsional rigidity of the unextended cylinder. The tensile force for the simple extension is given by

$$N = \frac{2A_0}{\lambda}(\lambda^2 - \lambda_1^2)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2 \frac{\partial W}{\partial I_2}\right), \tag{23.4}$$

A_0 being the area of the unstrained cross-section. From (23.3) and (23.4) it follows that

$$\frac{N}{[M/\psi]_{\psi=0}} = \frac{(\lambda^2 - \lambda_1^2)A_0}{\lambda_1^2[(\lambda^2 - \lambda_1^2)I_0 + \lambda_1^2 S_0]}. \tag{23.5}$$

When the material is compressible, λ_1 is determined by relations of the form (13.12) and depends upon the form of W. When the rod is incompressible, however, $\lambda_1 = \lambda^{-\frac{1}{2}}$ and the ratio (23.5) reduces to

$$\frac{N}{[M/\psi]_{\psi=0}} = \frac{\lambda(\lambda^3 - 1)A_0}{(\lambda^3 - 1)I_0 + S_0}, \tag{23.6}$$

independent of the form of W. In these formulae $S_0 \leqslant I_0$, the equality sign holding only when the cross-section is a circular region or is a ring bounded by concentric circles. In this case we obtain for incompressible bodies the simple formula

$$\frac{Na^2}{[M/\psi]_{\psi=0}} = 2\left(\lambda - \frac{1}{\lambda^2}\right), \tag{23.7}$$

derived for a circular rod of radius a by RIVLIN [1949c] and verified experimentally by RIVLIN and SAUNDERS [1951].

§ 24. Other Methods of Approximation

Owing to the non-linearity of the governing equations for large elastic deformations, problems for which exact solutions have so far been obtained are limited either to those of the type described in §§ 13 to 18 in which the deformation possesses a considerable degree of symmetry or to those of the types considered in §§ 19 and 20 in which the deformation is less restricted but in which the strain energy function assumes a particularly simple form. These problems are solved by an inverse method: that is, a given type of deformation is assumed and the stress distribution required to sustain this deformation is calculated.

The range of investigation is extended somewhat by the approximation procedures described in §§ 22 and 23. These are characterised by the perturbation of a known deformation, the latter being finite in § 23 but of restricted magnitude in § 22. The equations for the displacement components describing the small perturbing deformation are linear in form and can often be solved by orthodox methods of approach. A further extension of the range in which the approximate solutions are valid might be obtained by proceeding to the third order in § 22 to find the components ${}^3u_i$ or by superposing a second order deformation upon the finite deformation in § 23, thus combining the procedures of §§ 22 and 23. No difficulty of principle is involved in either of these extensions but the amount of work involved in solving all but the simplest problems is likely to prove prohibitive.

In some instances further progress may be made by varying one or more of the other conditions which specify a given problem. Obvious examples are provided by the perturbation of the strain energy function, of the constraint conditions and of the shape of the boundaries of the undeformed material. The first of these types of problem has been examined by SPENCER [1959] who has restricted attention to isotropic bodies. In this case, it is assumed that the solution of a given problem is known for a particular material with strain energy

$$W = W(I_1, I_2, I_3)\,. \tag{24.1}$$

This solution may be represented by a functional relationship

$$y_i = y_i(x_k) = x_i + u_i \qquad [u_i = u_i(x_k)]\,, \tag{24.2}$$

between the initial and final coordinates x_i, y_i of each point of the material. For a body with a slightly different strain energy

$$W + \varepsilon W'\,, \tag{24.3}$$

we may expect to find the perturbed solution

$$y_i = x_i + u_i + \varepsilon v_i\,, \tag{24.4}$$

the boundary conditions being unchanged. As before, v_i is the additional displacement function, ε a small parameter and for simplicity we employ cartesian coordinates. The solution (24.4) leads to perturbed values $I_r + \varepsilon I'_r$ for the invariants I_r and to the first order in ε the new strain energy becomes

$$W(I_1 + \varepsilon I'_1, I_2 + \varepsilon I'_2, I_3 + \varepsilon I'_3) + \varepsilon W'(I_1 + \varepsilon I'_1, I_2 + \varepsilon I'_2, I_3 + \varepsilon I'_3)$$
$$= W(I_1, I_2, I_3) + \varepsilon \left[\sum_{r=1}^{3} I'_r \frac{\partial W(I_1, I_2, I_3)}{\partial I_r} + W'(I_1, I_2, I_3) \right].$$

The stress strain relations, constraint conditions and equations of equilibrium may be considered similarly and the analysis follows the lines of the problems examined in § 23, the main difference arising from the existence of the perturbing function W'. In principle the procedure can be applied to aeolotropic bodies and also conceivably to materials in which there is some perturbation of the coordinate system defining a particular type of curvilinear aeolotropy. The method is particularly valuable, however, in extending the solutions for an isotropic incompressible Mooney material to bodies with a slightly more general form of W and Spencer has examined the cylindrically symmetrical deformation

$$\varrho = \varrho(r), \quad \vartheta = \theta + \varphi(r), \quad y_3 = \lambda x_3 + w(r)$$

in this way.

References

ADKINS, J. E., 1954, Proc. Camb. Phil. Soc. **50** 334.
ADKINS, J. E., 1955a, Proc. Camb. Phil. Soc. **51** 363.
ADKINS, J. E., 1955b, Proc. Roy. Soc. A **229** 119.
ADKINS, J. E., 1955c, Proc. Roy. Soc. A **231** 75.
ADKINS, J. E., 1956a, J. Rat. Mech. Anal. **5** 189.
ADKINS, J. E., 1956b, J. Mech. Phys. Solids **4** 199.
ADKINS, J. E., 1956c, Phil. Trans. Roy. Soc. A **249** 125.
ADKINS, J. E., 1958a, J. Mech. Phys. Solids **6** 267.
ADKINS, J. E., 1958b, Quart. Appl. Math. Mech. **11** 88.
ADKINS, J. E., 1958c, Phil. Trans. Roy. Soc. A **250** 519.
ADKINS, J. E., 1960, Archive Rat. Mech. Anal. **4** 193.
ADKINS, J. E. and A. E. GREEN, 1957, Proc. Roy. Soc. A **239** 557.
ADKINS, J. E., A. E. GREEN and G. C. NICHOLAS, 1954, Phil. Trans. Roy. Soc. A **247** 279.
ADKINS, J. E., A. E. GREEN and R. T. SHIELD, 1953, Phil. Trans. Roy. Soc. A **246** 181.
ADKINS, J. E. and R. S. RIVLIN, 1952, Phil. Trans. Roy. Soc. A **244** 505.
ADKINS, J. E. and R. S. RIVLIN, 1955, Phil. Trans. Roy. Soc. A **248** 201.
BAKER, M. and J. L. ERICKSEN, 1954, J. Wash. Acad. Sci. **44** 33.
BLACKBURN, W. S., 1958, Quart. Appl. Math. Mech. **11** 142.
BRILLOUIN, L., 1925, Ann. de Phys. Paris (10) **3** 251.
CHU, BOA-TEH., 1957, Brown Univ. Rep. No. Nonr. 562 (20)/1.
COSSERAT, E. and F. COSSERAT, 1896, Ann. Fac. Sci. Toulouse **10**, I, 1-116.
DANA, E. S. and W. E. FORD, 1932, Dana's Manual of Mineralogy, 4th Edn. (John Wiley and Sons, New York, 1932).
DANA, J. D. and C. S. HURLBUT, 1952, Dana's Textbook of Mineralogy (John Wiley and Sons, New York, 1952).
DOYLE, T. C. and J. L. ERICKSEN, 1956, Advances in Applied Mechanics **4** 53.
ERICKSEN, J. L., 1955, Z. Angew. Math. Mech. **35** 382.
ERICKSEN, J. L. and R. S. RIVLIN, 1954, J. Rat. Mech. Anal. **3** 281.
FINGER, J., 1894, Akad. Wiss. Wien (IIa) **103** 163, 231, 1073.
GENT, A. N. and P. B. LINDLEY, 1958, Proc. Roy. Soc. A **249** 195.
GENT, A. N. and A. G. THOMAS, 1958, J. Polymer Sci. **28** 625.

GREEN, A. E., 1955, Proc. Roy. Soc. A **227** 271.
GREEN, A. E. and J. E. ADKINS, 1960, Large elastic deformations and non-linear continuum mechanics (Clarendon Press, Oxford, 1960).
GREEN, A. E., R. S. RIVLIN and R. T. SHIELD, 1951, Proc. Roy. Soc. A **211** 128.
GREEN, A. E. and R. T. SHIELD, 1950, Proc. Roy. Soc. A **202** 407.
GREEN, A. E. and R. T. SHIELD, 1951, Phil. Trans. Roy. Soc. A **244** 47.
GREEN, A. E. and E. B. SPRATT, 1954, Proc. Roy. Soc. A **224** 347.
GREEN, A. E. and E. W. WILKES, 1954, J. Rat. Mech. Anal. **3** 713.
GREEN, A. E. and W. ZERNA, 1950, Phil. Mag. **41** 313.
GREEN, A. E. and W. ZERNA, 1954, Theoretical elasticity (Clarendon Press, Oxford, 1954).
HILL, R., 1957, J. Mech. Phys. Solids **5** 229.
MOONEY, M., 1940, J. Appl. Phys. **11** 582.
MURNAGHAN, F. D., 1937, Amer. J. Math. **59** 235.
MURNAGHAN, F. D., 1951, Finite deformation of an elastic solid (John Wiley and Sons, New York, 1951).
MUSKHELISHVILI, N. I., 1953, Some basic problems of the mathematical theory of elasticity (translated from Russian by J. R. M. Radok) (P. Noordhoff Ltd., Groningen-Holland, 1953).
NOLL, W., 1955, J. Rat. Mech. Anal. **4** 3.
RIVLIN, R. S., 1948a, Phil. Trans. Roy. Soc. A **240** 459.
RIVLIN, R. S., 1948b, Phil. Trans. Roy. Soc. A **240** 491.
RIVLIN, R. S., 1948c, Phil. Trans. Roy. Soc. A **240** 509.
RIVLIN, R. S., 1948d, Phil. Trans. Roy. Soc. A **241** 379.
RIVLIN, R. S., 1949a, Proc. Camb. Phil. Soc. **45** 485.
RIVLIN, R. S., 1949b, Proc. Roy. Soc. A **195** 463.
RIVLIN, R. S., 1949c, Phil. Trans. Roy. Soc. A **242** 173.
RIVLIN, R. S., 1953, J. Rat. Mech. Anal. **2** 53.
RIVLIN, R. S., 1955, J. Rat. Mech. Anal. **4** 951.
RIVLIN, R. S., 1956, Rheology, Theory and Applications, vol. 1, edit. F. R. Eirich (Academic Press Inc., New York, 1956).
RIVLIN, R. S., 1958, Brown Univ. Rep. No. C11-43.
RIVLIN, R. S., 1959, Archive Rat. Mech. Anal. **2** 447.
RIVLIN, R. S. and D. W. SAUNDERS, 1951, Phil. Trans. Roy. Soc. A **243** 251.
RIVLIN, R. S. and C. TOPAKOGLU, 1954, J. Rat. Mech. Anal. **3** 581.
SMITH, G. F. and R. S. RIVLIN, 1957, Archive Rat. Mech. Anal. **2** 107.
SMITH, G. F. and R. S. RIVLIN, 1958, Trans. Amer. Math. Soc. **88** 175.
SPENCER, A. J. M., 1959, Quart. App. Math. Mech. **12** 129.
STERNBERG, E., 1954, Quart. App. Math. **11** 393.
STOPPELLI, F., 1954, Ric. di Mat. **3** 247.
STOPPELLI, F., 1955, Ric. di Mat. **4** 58.
TRELOAR, L. R. G., 1944, Trans. Inst. Rubber Ind. **19** 201.
TRELOAR, L. R. G., 1958, The Physics of Rubber Elasticity, 2nd Edn. (Oxford University Press, 1958).
TRUESDELL, C., 1952, J. Rat. Mech. Anal. **1** 125.
TRUESDELL, C., 1953, J. Rat. Mech. Anal. **2** 593.
VOLTERRA, V., 1907, Ann. École Norm. (3) **24** 401.
WEYL, H., 1946, The classical groups, their invariants and representations (Princeton University Press, 1946).
WILKES, E. W., 1955, Quart. Appl. Math. Mech. **8** 88.

CHAPTER II

ELASTIC WAVES IN ANISOTROPIC MEDIA

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CONTENTS

§ 1. Introduction

In recent years, the propagation of elastic waves in anisotropic media has been the subject of study from several points of view. These have included modern seismological interests and the understanding of the applications of ultrasonic techniques as well as the inherent appeal of the subject as a part of theoretical mechanics.

The subject has a lengthy history since it was for many years the means whereby optical phenomena in crystals were explained. In the theory of the elastic solid aether, however, there was no place for displacements involving dilatation and accordingly the theory of mechanical waves in an elastic solid was developed in an incomplete form, although as early as [1840] BLANCHET considered the propagation of elastic waves in an aeolotropic body and was aware of the existence of a wave surface having three sheets.

CHRISTOFFEL [1877] wrote a long memoir on the subject and it is clear that he, and also KELVIN [1904], had a fully developed and consistent picture of the behaviour of mechanical waves in crystalline media. After the turn of the century RUDSKI [1911] made some important contributions from a mathematical and seismological standpoint; much of his work has been reviewed in a recent general discussion by HELBIG [1958].

This article is intended to be a survey of the subject as a whole with particular reference to the developments of significance in the last decade.

GENERAL EXPOSITION

§ 2. Generalized Relation between Stress and Strain

If x_l represent the rectangular cartesian coordinates of a point in the medium and u_k its displacement, then the strain

$$\varepsilon_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}) = \varepsilon_{lk}\,, \tag{1}$$

where

$$u_{k,l} = \frac{\partial u_k}{\partial x_l} \quad (k, l = 1, 2, 3)\,.$$

A direct proportionality between stress σ_{ij} and strain ε_{kl} may then be expressed as

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}\,, \tag{2}$$

where summation is made over repeated suffixes.

If the elastic constants c_{ijkl} are real and satisfy the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}\,, \tag{3}$$

then (2) represents the generalized Hooke's Law and there may exist as many as 21 independent constants. Also there exists a strain energy function

$$W = \tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \geqslant 0\,, \tag{4}$$

so that

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij} \tag{5}$$

on account of (1) and (3), and the function specifies a unique state of strain for a given applied stress. The symmetry conditions (3) and the positive definite character of the strain energy function (4) impose some limits upon the values of the c_{ijkl}.

In recent years it has been suggested by LAVAL [1951], LE CORRE [1953] and RAMAN and VISWANATHAN [1955] that in dynamical problems notably those of wave propagation, the relation (2) between stress and strain is inadequate. They urge that the stress must also be related to the rotations $\omega_{kl} = \frac{1}{2}(u_{k,l} - u_{l,k})$, which are not homogeneous throughout the medium and they propose a set of nine equations:

$$\sigma_{ij} = d_{ijkl}u_{k,l}\,, \tag{6}$$

in which the d-coefficients obey symmetry conditions

$$d_{ijkl} = d_{jilk}\,. \tag{7}$$

Thus the matrix of stiffness constants is symmetrical but the stress and strain tensors are not, in general, symmetrical. There is, however, a unique strain energy function associated with any state of stress, strain and rotation.

Discussion of this proposition is current (JOEL and WOOSTER [1957, 1958a,b], KRISHNAN *et al.* [1958], MCCLINTOCK *et al.* [1958]), and HEARMON [1956] has shrewdly appraised the experimental evidence offered by Le Corre and others which is certainly not wholly convincing. HUNTINGTON [1958] has written a cogent discussion of the position from a theoretical standpoint. He makes the important point that rotations as such will not cause any change in the potential energy of a true lattice; change can occur only as a result of the rotations interacting with some polarizing field which may be internal to the specimen or externally imposed. Until such interactions have been shown with general satisfaction to have perceptible consequences, a rejection of Hooke's Law seems unwarranted and it would appear that the assumptions of classical elasticity will remain valid for many classes of solids in any case.

It is also possible to conceive that symmetry conditions (3) are limited to

$$c_{ijkl} = c_{jikl} = c_{ijlk}\,, \tag{3a}$$

when the stiffness matrix becomes non-symmetrical. In this case no unique strain energy function exists for a given composite state of stress and strain and it becomes possible to extract energy from the body by subjecting it to a suitable cycle of loading. Such a process violates the Second Law of Thermodynamics so it is clear that no material in an unstrained state may possess such a non-symmetric stiffness matrix. However, such a matrix is conceivable for a material in a state of internal stress or strain and, although it is clear that the matrix must become symmetrical when such strain is relieved, a finite amount of strain energy may become available from such a material during the process. An adequate analysis of this process would require consideration of the transition between symmetric and non-symmetric arrays. MAKINSON [1955] has made a preliminary investigation of the effects of a non-symmetric stiffness matrix on the behaviour of mechanical waves and has suggested that they may be observable in keratinous materials. Her ideas are interesting but no experimental evidence is adduced.

Accordingly, we shall develop the analysis in this article in terms of classical concepts but attempt to indicate the principal deviations which may occur when mechanical waves are propagated in non-Hookean media. In the general analysis, we shall follow closely the notation and methods of SYNGE [1956, 1957]; in dealing with particular symmetries, tensor calculus is rather less satisfactory and we shall use notation established in previous papers (MUSGRAVE [1954]).

§ 3. The Equations of Motion and Plane Waves

The equations of motion in an aeolotropic elastic continuum may be written

$$\sigma_{ij,j} = c_{ijkl}u_{k,lj} = \varrho \ddot{u}_i , \tag{8}$$

where u_i is a displacement and ϱ is the density.

For plane waves, we seek solutions of the form

$$\begin{aligned} u_k = u_k^* + \mathrm{i}u_k^\dagger &= Ap_k \exp \mathrm{i}\omega \left(\frac{n_j x_j}{v} - t\right) \\ &= Ap_k \exp \mathrm{i}\omega \,(s_j x_j - t) , \end{aligned} \tag{9}$$

where A is a scalar amplitude, t the time, ω the angular frequency, p_k a unit displacement vector, n_k a unit wave normal, x_k a space vector, v the magnitude of the phase velocity, and s_k the slowness vector. When all these quantities are real, u_k^* or $u_k^\dagger$ represents a plane progressive wave of zero attenuation. If p_k and s_k take complex values, u_k^* and $u_k^\dagger$ represent elliptically polarized waves whose amplitude is attenuated with distance (e.g. surface waves, waves in absorptive media).

Provided the c_{ijkl} are Hookean stiffnesses, the flux of energy associated with a wave at any instant is $-\sigma_{ij}^* \dot{u}_j^*$ which includes some fluctuating time-dependent terms. However, the energy actually carried forward by the wave in unit time may be represented by the vector

$$F_i^* = \lim_{t_2 - t_1 \to \infty} \frac{1}{t_2 - t_1} \int_{t_2}^{t_1} (-\sigma_{ij}^* \dot{u}_j^*) \mathrm{d}t . \tag{10}$$

For the displacement u_k^* in a medium of stiffnesses c_{ijkl},

$$F_i^* = \tfrac{1}{4} A^2 \omega^2 c_{ijkl} \,[p_j \bar{s}_k \bar{p}_l + \bar{p}_j s_k p_l] , \tag{10a}$$

where the bar indicates the complex conjugate.

Let us first investigate the case in which all these quantities are real, when either the real or imaginary part of (9) will represent an infinite plane progressive wave of zero attenuation. Substituting (9) in (8), we obtain equations for the components p_i:

$$[\varrho v^2 \delta_{ik} - \Gamma_{ik}]p_i = 0\,, \tag{11a}$$

where $\Gamma_{ik} = n_j n_l c_{ijkl}$ are the Christoffel stiffnesses and δ_{ik} the Kronecker symbol. Alternatively we may write

$$[\varrho \delta_{ik} - \alpha_{ik}]p_i = D_{ik} p_i = 0\,, \tag{11b}$$

where

$$\alpha_{ik} = s_j s_l c_{ijkl}\,.$$

§ 4. Velocity and Slowness Surfaces

The condition for non-zero p_i then yields the velocity equation

$$\det |\varrho v^2 \delta_{ik} - \Gamma_{ik}| = 0\,, \tag{12a}$$

which is of the twelfth degree in the components of velocity and represents the velocity surface of three sheets; alternatively we may obtain

$$\det |\varrho \delta_{ik} - \alpha_{ik}| = |D_{ik}| = 0\,, \tag{12b}$$

which is of the sixth degree in the components of slowness and represents the slowness surface of three sheets. In general, there are three real, positive and discrete $s_i^{(M)}$ $(M = 1, 2, 3)$ for a given wave normal n_i; associated with each slowness is a unique $p_i^{(M)}$ which is neither truly longitudinal (L) nor truly transverse (T) in character. The symmetry of eqs. (12) implies the mutual orthogonality of displacement vectors

$$p_i^{(M)} p_i^{(N)} = \delta_{MN}\,, \tag{13}$$

and the deviation from true L or T character may be expressed in the form

$$p_i^{(M)} n_i = \cos \delta^{(M)}\,, \tag{14}$$

where $\delta^{(M)}$ is the angle of deviation.

If we write $\alpha_{12}\alpha_{13}/\alpha_{23} = \beta_1^2$ and similarly, then we may express the displacement vector in a form analogous to that of KELVIN [1904]:

$$p_i = \frac{f\beta_i(\beta_j p_j)}{\varrho - \alpha_{ii} + (\beta_i)^2}\,, \tag{15}$$

where f is a normalizing factor dependent upon the value of the associated s_i. Multiplying by β_i and summing on the suffix i, we obtain another form for the slowness equation

$$\beta_i \left[\frac{\beta_i}{\varrho - \alpha_{ii} + (\beta_i)^2}\right] = 1 \,. \tag{12c}$$

§ 5. Wave Surface

Denoting (12b) and (12c) by

$$\varphi(s_i) = 0 \,, \tag{12d}$$

the corresponding wave surface

$$\psi(\xi_i) = 0 \,, \tag{16}$$

is the polar reciprocal of (12d) and thus the envelope of planes

$$\xi_i s_i = 1 \,, \tag{17}$$

where s_i is determined by (12d).

Differentiating (12d) and (17) and using a multiplier yields

$$\xi_i = \frac{\partial\varphi/\partial s_i}{[s_j \partial\varphi/\partial s_j]} \,, \tag{18}$$

as the coordinates of the point P on the wave surface which corresponds with the points S on the slowness surface.

In Fig. 1, if O is an origin of elastic disturbance, the plane wave element of slowness $\overrightarrow{OS}$ has an associated ray or direction of energy flux $\overrightarrow{OP}$ and, in general, for any wave normal there exist three extraordinary rays associated respectively with the slowness vectors, $s_i^{(M)}$. If we denote the unit ray direction by

$$r_i = \frac{\xi_i}{(\xi_j \xi_j)^{\frac{1}{2}}} \,, \tag{19}$$

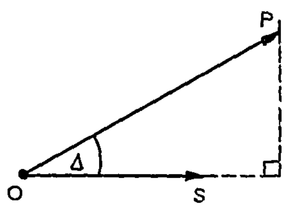


Fig. 1. Slowness vector $\overrightarrow{OS}$ and associated ray $\overrightarrow{OP}$.

then for any individual wave of slowness $s_i^{(M)}$

$$n_i r_i^{(M)} = \cos \Delta^{(M)} , \tag{20}$$

where $\Delta^{(M)}$ is the deviation of a ray from its associated wave normal.

The eq. (16) is not easily written in a more revealing way since, as polar reciprocal to a surface of the sixth degree, although it is necessarily of class 6 it may be of degree 150.

SYNGE [1957] has demonstrated that the direction of the energy flux F_i^* is the same as that of the ray r_i by considering

$$\frac{\partial \varphi}{\partial s_i} = C_{jk} \frac{\partial D_{jk}}{\partial s_i} \tag{21a}$$

where
$$C_{jk} D_{jl} = \delta_{kl} \varphi = 0 . \tag{22}$$

However, by (11b),

$$D_{jl} p_j = 0$$

and so

$$C_{jk} = g p_j p_k ,$$

where $g = g(s_i)$. Hence

$$\frac{\partial \varphi}{\partial s_i} = -2g c_{ijkl} p_j s_k p_l \tag{21b}$$

which is of the same form as F_i^* when the displacement and slowness vectors have real components.

§ 6. Correspondences between the Slowness and Wave Surfaces

The slowness and wave surfaces are polar reciprocal to each other and Table 1 lists some projective correspondences which are useful if the latter surface is to be inferred from the former.

TABLE 1

Projective correspondences between the slowness and wave surfaces

Slowness surface	*Wave surface*
1) Tangent plane P: normal r_i	Point p: co-ordinates ξ_i
2) Point Q: radius vector s_i	Tangent plane q: normal n_i
3) Tangent plane $\mathrm{P}^{(N)}$ with N contacts at points $\mathrm{Q}_{(N)}$	Point $\mathrm{p}^{(N)}$ with N tangent planes $\mathrm{q}_{(N)}$
4) Point $\mathrm{Q}^{(N)}$ with N tangent planes $\mathrm{P}_{(N)}$	Tangent plane $\mathrm{q}^{(N)}$ with N contacts at points $\mathrm{p}_{(N)}$
5) Tangent plane $\mathfrak{P}$ at a parabolic point $\mathfrak{Q}$	Cusp point $\mathfrak{p}$ (usually on a cuspidal edge)

If a correspondence of type (3) and (4) occurs, in which a tangent plane makes contact with a surface over a curve, a corresponding point having a cone of tangency exists on the polar reciprocal. Such correspondence implies that the phenomenon of conical refraction will be exhibited by the medium. Correspondences (3) and (4) where $N = \infty$ imply the existence of *external* and *internal* conical refraction respectively (see later).

Correspondences of type (5) are quite common but there are some general points worth mentioning. For all physically real media so far studied, the sheet of the slowness surface associated with the smallest values of slowness and with quasi-longitudinal displacement vectors (L) is completely enclosed by the other two sheets, associated with quasi-transverse displacement vectors (T), which usually have some points in common. DUFF [1960] has pointed out that under these circumstances, since the surface is of sixth degree, the L-sheet will always be convex and that parabolic points may occur only on the T-sheets. Hence the L-sheet of the wave surface will be convex and only the slower moving T-sheets will possess cuspidal edges.

MUSGRAVE [1957b] has shown that some simple inequalities between the elastic constants are satisfied when cusp points occur in planes of symmetry in orthorhombic, tetragonal, cubic and hexagonal media.

§ 7. Decay Function

The above analysis offers a means of calculating all the characteristics of interest for plane waves and most of those for waves from a point source. However, no decay function of the amplitude as the wave sheets propagate outward has been obtained. THOMAS [1957] has presented some general equations pertinent to the problem but in application he inserted the conditions for isotropy. More recently, BUCHWALD [1959] has developed from a method of LIGHTHILL [1960] another approach to the investigation of the wave surface by means of which he has found the form of the decay function.

Buchwald expresses the displacement at any point in space due to a point disturbance as a Fourier integral. He then estimates the values of this integral at points far from the source and by applying the method of stationary phase, shows that the amplitude of displacement at ξ_i on the wave surface is predominantly due to waves having slowness vectors s_i such that $\partial\varphi/\partial s_i$ is parallel to ξ_i (cf. (18)). His results indicate that, due to a single such wave,

$$|u_j| \sim |\xi_i|^{-1} \left|\frac{\partial \varphi}{\partial s_i}\right|^{-1} |K(s_i)|^{-\frac{1}{2}} |f_j(s_i)| , \tag{23a}$$

where $K(s_i)$ is the Gaussian curvature at s_i on $\varphi = 0$ and f_j is a function related to the source of disturbance. The total magnitude of the amplitude related to a given sheet of the slowness surface is then

$$|u_j| \sim |\xi_i|^{-1} \sum_{r=1}^{N} \left|\frac{\partial \varphi}{\partial s_i^{(r)}}\right|^{-1} |K(s_i^{(r)})|^{-\frac{1}{2}} |f_j(s_i^{(r)})| , \tag{23b}$$

where the $s_i^{(r)}$ are the appropriate slowness vectors.

The dependence (24) clearly breaks down if either or both of the principal curvatures at s_i is zero, i.e. at parabolic points. By considering third order terms in the expression for the form of $\varphi = 0$ in the neighbourhood of a point s_i, Buchwald shows that, for cusp points on $\psi = 0$ (see Table 1 correspondence (5)),

$$|u_j| \sim |\xi_i|^{-\frac{5}{6}} \left|\frac{\partial \varphi}{\partial s_i}\right|^{-1} |k_1|^{-\frac{1}{2}} |h_2|^{-\frac{1}{3}} |f_j(s_i)| , \tag{23c}$$

where k_1 is the non-zero principal curvature at s_i and h_2 is the third order coefficient involved. In a similar manner he has considered the case of correspondence (3) where $N \to \infty$, which gives rise to a conical point on the wave surface (see later). For such a point he finds

$$|u_j| \sim |\xi_i|^{-\frac{1}{2}} \int_{\mathrm{C}} \left|\frac{\partial \varphi}{\partial s_i^{(c)}}\right|^{-1} |k_1(s_i^{(c)})|^{-\frac{1}{2}} |f_j(s_i^{(c)})| \, \mathrm{d}c , \tag{23d}$$

where the integral is taken along the curve of contact C.

Thus Buchwald's analysis offers not only a solution for the slowness and wave surfaces but also the relative amplitude and decay function associated with any point on the latter surface.

We may summarize the findings so far by the following statement: for any wave normal n_i in an aeolotropic elastic medium, in general there exist three plane waves having three discrete phase velocities $n_i v^{(N)}$ or slownesses $s_i^{(N)}$ and associated unique displacement vectors, $p_i^{(N)}$. The $p_i^{(N)}$ form a mutually orthogonal triad oblique to the wave normal. The rays or paths of energy flux having unit vectors $r_i^{(N)}$ are oblique to the wave normal. The amplitude of disturbance for an ordinary point on the wave surface decays as $(|\xi_i|)^{-1}$; singular points may have weaker decay functions.

General solutions to the problems of progressive and standing waves excited in infinite layers of finite thickness have been given by SYNGE [1957].

Using similar notation, we shall now examine the reflection and refraction of plane waves at a plane boundary.

§ 8. Reflection and Refraction of Plane Waves at a Plane Boundary

Suppose the plane $x_3 = 0$ is the boundary between two aeolotropic media: let the elastic constants referred to the coordinate system x_i be c_{ijkl} for $x_3 > 0$ and c'_{ijkl} for $x_3 < 0$.

Let a given body wave of unit amplitude

$$u_k^{(\mathrm{I})} = p_k \exp i\omega (s_j^{(\mathrm{I})} x_j - t)$$

be incident upon $x_3 = 0$. We require to find the resultant reflected and refracted waves.

The boundary conditions at the interface require the continuity of displacement and of stress across $x_3 = 0$. These conditions may only be fulfilled for all times if the phase along the boundary is the same for all the waves, i.e.

$$s_1^{(\mathrm{I})} x_1 + s_2^{(\mathrm{I})} x_2 = s_1^{(M)} x_1 + s_2^{(M)} x_2 = s_1'^{(N)} x_1 + s_2'^{(N)} x_2\,, \qquad (24)$$

where (M) and (N) designate the possible slownesses. These may be found by representing the trace of the incident slowness on the interface by a vector and then drawing a line normal to the interface through its extremity. The points determined by the intersections of the line and the sheets of the slowness surface are the extremities of possible slowness vectors. They will have coordinates of the form $(s_1^{(\mathrm{I})}, s_2^{(\mathrm{I})}, s_3^{(M)})$ and $(s_1^{(\mathrm{I})}, s_2^{(\mathrm{I})}, s_3'^{(N)})$ the $s_3^{(M)}$ and $s_3'^{(N)}$ are roots of the slowness equations $|D_{ik}| = 0$, and $|D'_{ik}| = 0$ for given $s_j^{(M)} = s_j^{(\mathrm{I})} = s_j'^{(N)}$ $(j = 1, 2)$. Since slowness surfaces are of the sixth degree, we expect six roots $s_3^{(M)}$ or $s_3'^{(N)}$, $(M, N = 1, 2, \ldots 6)$, real or complex, from each equation. However, in the case of real roots, corresponding with body waves, three slownesses $s_3^{(M)}$, $(M = 1, 2, 3)$ have associated rays which travel from the boundary *into* the appropriate medium and the remaining $s_3^{(M)}$, $(M = 4, 5, 6)$ travel from the boundary *out* of the appropriate medium; in the case of complex roots, corresponding with surface waves, the slownesses $s_3^{(M)}$ will occur in conjugate pairs say 1 and 4, 2 and 5, 3 and 6 and from each pair the root which provides attenuation of amplitude with distance from the boundary in the appropriate medium is retained. SYNGE [1956] has shown that the energy flux of such waves is parallel to the interface. Consequently, three slownesses in each medium represent physically realizable waves and there are thus three reflected and three refracted waves whose amplitudes have yet to be found.

These amplitudes are now determined by the boundary conditions in displacement

$$p_k^{(\mathrm{I})} + \sum_{M=1}^{3} A^{(M)} p_k^{(M)} = \sum_{N=1}^{3} A'^{(N)} p_k'^{(N)} , \tag{25}$$

and in stress

$$\sigma_{3j} + \sum_{M=1}^{3} \sigma_{3j}^{(M)} = \sum_{N=1}^{3} \sigma_{3j}'^{(N)} , \tag{26}$$

where

$$\sigma_{3j}^{(M)} = c_{3jkl}\, \varepsilon_{kl}^{(M)} = \mathrm{i}\omega\, A^{(M)} c_{3jkl}\, p_k^{(M)} s_l^{(M)} \exp \mathrm{i}\omega\, (s_q^{(M)} x_q - t) , \tag{27}$$

and

$$\sigma_{3j}'^{(N)} = c_{3jkl}'\, \varepsilon_{kl}^{(N)} = \mathrm{i}\omega\, A'^{(N)} c_{3jkl}'\, p_k'^{(N)} s_l'^{(N)} \exp \mathrm{i}\omega\, (s_q'^{(N)} x_q - t) . \tag{28}$$

There are thus six equations, in general, complex, which determine uniquely the six amplitudes $A^{(M)}$, $A'^{(N)}$ $(M, N = 1, 2, 3)$.

A check on any numerical solution for such amplitudes is afforded by the balance of energy flux which must occur at the boundary and which requires that

$$F_3^{*(\mathrm{I})} + \sum_{M=1}^{3} F_3^{*(M)} = \sum_{N=1}^{3} F_3'^{*(N)} . \tag{29}$$

§ 9. Surface Waves

As mentioned already, the form of displacement (9) may represent a surface wave if p_k and s_k are complex. In the preceding paragraphs we have seen how such waves arise in a reflection-refraction problem when the trace of the slowness of the incident wave exceeds a critical magnitude. These elliptically polarized plane waves have, in general, a wave normal oblique to the interface while the associated energy flux is parallel to the interface, though not necessarily parallel to the trace of the slowness.

Analogously to Rayleigh waves in isotropic media, there may exist waves of compound elliptic displacement that travel in a medium and leave a plane surface free of stress. SYNGE [1957] has given a short general treatment from which our formulation is derived.

Consider a single semi-infinite medium ($x_3 \leqslant 0$) having constants

c_{ijkl}. The general form of displacement for a surface wave whose amplitude is attenuated with depth ($x_3 \to -\infty$) is

$$u_k = Ap_k \exp(+\omega \overset{\dagger}{s_3} x_3) \exp[i\omega(s_1x_1 + s_2x_2 + \overset{*}{s_3}x_3 - t)]\,. \tag{30}$$

There exist three such waves having discrete slownesses (s_1, s_2, $s_3^{(M)}$) ($M = 1, 2, 3$) where (s_1, s_2) is the common trace in the surface of their slownesses. Hence, if the displacements can be compounded in such a way as to leave $x_3 = 0$ free of stress, a generalized Rayleigh wave is possible. The boundary condition requires

$$\sum_{N=1}^{3} c_{3jkl}A^{(N)}p_k^{(N)}s_l^{(N)} = 0\,, \tag{31}$$

which represents three equations having a non-zero solution if

$$\det |K_{jN}| = 0\,, \tag{32}$$

where

$$K_{jN} = c_{3jkl}p_k^{(N)}s_l^{(N)}\,. \tag{33}$$

The determinant in (32) is third order with complex elements and equivalent to two real eqs. (32a) and (32b). When (s_1, s_2) can be found to satisfy (32a and b) simultaneously, it implies the existence of a generalized Rayleigh wave. In contrast to the degenerate case of isotropic media, these waves may propagate only in certain directions and may have energy fluxes parallel to planes of certain orientations with respect to the crystal symmetry. A further difference is that the variation of the composite amplitude with distance normal to the free surface is more complicated. The amplitude of the generalized Rayleigh wave is compounded of three complex displacement vectors, each of which is subject to both exponential and sinusoidal variation with depth. The resultant displacement is thus elliptically polarized with space-dependent eccentricity and lies in a plane whose orientation may also vary with distance from the free surface.

§ 10. Non-Hookean Quasi-Elastic Media

i) In media whose elasticity is governed by the eqs. (6) the characteristics of wave propagation are substantially unaltered. However, since under dynamic conditions some of the d_{ijkl} occur in pairs, while under static conditions only the Hookean members of the stiffness matrix are involved, it is necessary to obtain observations under both static and dynamic conditions in order to derive values for the individual d_{ijkl}.

ii) In the case of media having a non-symmetric stiffness matrix (eq. 3a), we have already pointed out the thermodynamic difficulties. However, to gain a first insight into the behaviour of waves in such a medium, MAKINSON [1955] examined the types of wave which could travel in a quartz-like structure without quantitatively considering the source or disposal of energy generated or dissipated in the medium. She found that waves, circularly polarized in right handed and left handed senses, were possible modes for a given wave normal; the senses of polarization being associated with exponentially increasing and decreasing amplitude respectively. No great emphasis can be placed on these findings until a more thorough consideration of the problems of energy involved provides some confirmation but they are challenging and interesting.

§ 11. Absorptive Media

Real solids, when submitted to experimental tests, invariably prove to be imperfectly elastic, albeit the degree of imperfection may be very slight. In wave propagation the deviation from the ideal Hookean elastic law is apparent chiefly as the absorption of energy during the transit of a wave. A wide variety of models of non-Hookean isotropic solids, involving the time-dependence of stress and strain have been propounded and discussed in some detail (see, for instance, REINER [1956]); some of these ideas have been carried over to the study of anisotropic materials (ZENER [1948]) but, for the most part, measurements of absorption as it varies with orientation over the three sheets of the velocity or slowness surface have yet to be made.

However, a large number of the non-Hookean models of solids have deformation laws which are linear in stress and strain and their time derivatives. As a result, the stress-strain relation for any oscillatory disturbance of constant frequency may be written

$$\sigma_{ij} = \mathscr{C}_{ijkl} u_{k,l} \, , \tag{34}$$

where the quantities are complex and for models which involve only first order time derivatives

$$\mathscr{C}_{ijkl} = c_{ijkl}(\omega) + \mathrm{i}\gamma_{ijkl}(\omega) \, , \tag{35}$$

where $c_{ijkl}(\omega)$ are stiffnesses akin to Hookean constants for a given frequency.

When relation (34) is combined with eqs. (8) and (9) in which p_k and s_k are complex, we obtain similar equations to (11a and b) for velocity, slowness and displacement:

$$[\varrho v^2 \delta_{ik} - \Gamma'_{ik}]p_i = 0\,, \tag{36a}$$

where $\Gamma'_{ik} = n_j n_l \mathscr{C}_{ijkl}$ may be termed complex Christoffel stiffnesses; alternatively,

$$[\varrho \delta_{ik} - \alpha'_{ik}]p_i = \mathscr{D}_{ik} p_i = 0\,. \tag{36b}$$

The slowness equation thus becomes

$$\det|\varrho \delta_{ik} - \alpha'_{ik}| = |\mathscr{D}_{ik}| = 0\,, \tag{37}$$

where each element has the form

$$\mathscr{D}_{ik} = \overset{*}{\mathscr{D}}_{ik} + \mathrm{i}\overset{\dagger}{\mathscr{D}}_{ik}\,, \tag{38}$$

$$\overset{*}{\mathscr{D}}_{ik} = \varrho\delta_{ik} - \{c_{ijkl}(\overset{*}{s}_j\overset{*}{s}_l - \overset{\dagger}{s}_j\overset{\dagger}{s}_l) - \gamma_{ijkl}(\overset{*}{s}_j\overset{\dagger}{s}_l + \overset{\dagger}{s}_j\overset{*}{s}_l)\}\,, \tag{39a}$$

$$\overset{\dagger}{\mathscr{D}}_{ik} = \gamma_{ijkl}(\overset{*}{s}_j\overset{*}{s}_l - \overset{\dagger}{s}_j\overset{\dagger}{s}_l) - c_{ijkl}(\overset{*}{s}_j\overset{\dagger}{s}_l + \overset{\dagger}{s}_j\overset{*}{s}_l)\,. \tag{39b}$$

The left side of (37) may thus be expanded into the sum of eight determinants, four real and four imaginary. It is clear that solutions for s^*_j and $s^\dagger_j$ will in general be very complicated.

However, if we choose to investigate a range or frequency such that $\gamma_{ijkl}(\omega) \ll c_{ijkl}(\omega)$, then we may reasonably expect $|s^\dagger_j|$ to be small. Under these circumstances, to the first order of small quantities,

$$\overset{*}{\mathscr{D}}_{ik} = \varrho\delta_{ik} - c_{ijkl}\overset{*}{s}_j\overset{*}{s}_l \tag{40a}$$

$$\overset{\dagger}{\mathscr{D}}_{ik} = \gamma_{ijkl}\overset{*}{s}_j\overset{*}{s}_l + c_{ijkl}\,(\overset{*}{s}_j\overset{\dagger}{s}_l + \overset{\dagger}{s}_j\overset{*}{s}_l) \tag{40b}$$

and the real part of (37) reduces to (12b). Thus the magnitudes of the slownesses of wave in slightly absorptive media remain unaltered to the first order of small quantities. For given $\mathscr{C}_{ijkl}$ we may now consider the imaginary part of (37) as an equation for $s^\dagger$ and making the reasonable assumption that absorption is constant over the plane of the wave, we may write

$$\mu \overset{*}{s}_j = \overset{\dagger}{s}_j\,, \tag{41}$$

where $\mu \ll 1$, and solve for μ as a function of the wave normal n_j. The equation is linear in μ for each of the three values of s^* and hence there is an absorption factor for each wave.

The displacement vector associated with each slowness will have the form

$$p_i^* + \mathrm{i} p_i^\dagger$$

where

$$p_i^* \sim \overset{\circ}{p}_i + \mathrm{O}(\mu^2), \quad \overset{\circ}{p}_i \text{ given by (15)} \tag{42a}$$

and

$$p_i^\dagger \sim \mathrm{O}(\mu) \,. \tag{42b}$$

Hence the displacement is in general elliptic, where the directions of the major axes form the same mutually orthogonal triad as specified by (15).

The flux of energy is now represented by

$$\tfrac{1}{4} A^2 \omega^2 \, |\mathscr{C}_{ijkl}| \, [p_j \bar{s}_k \bar{p}_l + \bar{p}_j s_k p_l] \cos \varphi_{ijkl} \,, \tag{43}$$

while the energy absorbed along the ray in unit time is given by

$$\tfrac{1}{4} A^2 \omega^2 \, |\mathscr{C}_{ijkl}| \, [p_j \bar{s}_k \bar{p}_l - \bar{p}_j s_k p_l] \sin \varphi_{ijkl} \,, \tag{44}$$

where

$$\tan \varphi_{ijkl} = \frac{\gamma_{ijkl}}{c_{ijkl}} \quad \text{(no summation)} \,.$$

Hence, as might be expected from the result for the slowness surface, the direction of the energy flux or ray is unaltered to the first order of small quantities.

Our general survey of the features of elastic wave propagation in aeolotropic media is now concluded and we turn our attention to some results appropriate to particular symmetries and media.

§ 12. Hexagonal Media

A method for obtaining the wave surface for any symmetry has been given by MUSGRAVE [1954 I] and it has been applied to media of hexagonal and cubic symmetry (MUSGRAVE [1954 II], MILLER and MUSGRAVE [1956]). It depends essentially on expressing the ray vector as the vector sum of two line segments which may be calculated and then performing the necessary geometrical construction. Thus the coordinates of the extremity of the ray vector may be written

$$\xi_i^{(M)} = \frac{k}{\varrho} [s_i^{(M)} + \Lambda_i^{(M)}] \quad (M = 1, 2, 3) \tag{45}$$

where k has the dimensions of an elastic constant and Λ_i has those of slowness. The vectors $k s_i/\varrho$ and $k \Lambda_i/\varrho$ then represent the line segments $\overrightarrow{\mathrm{OI}}$ and $\overrightarrow{\mathrm{IP}}$ in a velocity space illustrated in Fig. 2. Convenient choice of the constant k usually involves the case $v > k/\varrho v$ for quasi-longitudinal waves ($M = 1$, or suffix L) and $v < k/\varrho v$ for quasi-

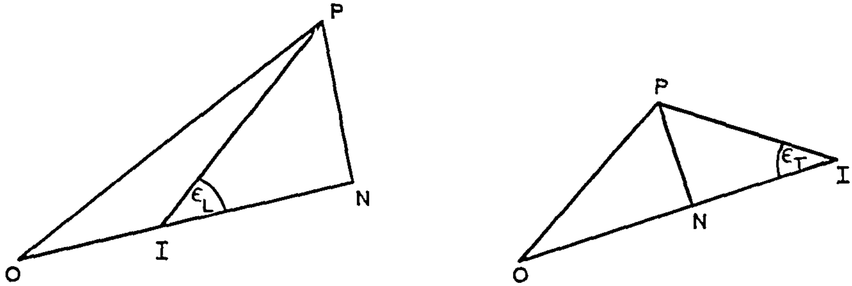


Fig. 2. Line segments $\vec{OI} = ks_i^{(M)}/\varrho$ and $\vec{IP} = k\Delta_i^{(M)}/\varrho$

for a) $M = 1$; or suffix L, $(v > k/\varrho v)$;
b) $M = 2, 3$; or suffix T_1, T_2, $(v < k/\varrho v)$.

transverse waves ($M = 2, 3$; or suffix T_1, T_2). The angle ε is given by

$$\cos \varepsilon^{(M)} = \frac{n_i \Delta_i^{(M)}}{|\Delta_j^{(M)}|} . \tag{46}$$

Fig. 3 shows the sections of the velocity, inverse and wave surfaces in a zonal plane for zinc. Rotation about the zonal axis yields the circularly symmetric surfaces.

The values of $\delta^{(M)}$ $(\equiv \delta_i)$, $M = 1$; or $i = \mathrm{L}$, and $\Delta^{(M)}$ $(\equiv \Delta_i)$, $M = 1$, 2, 3; or $i = \mathrm{L}$, T_1, T_2 are shown in Figs. 4 and 5.

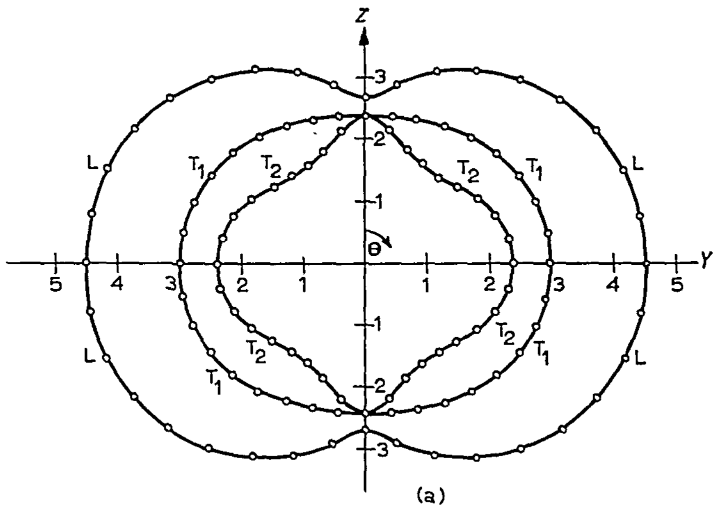
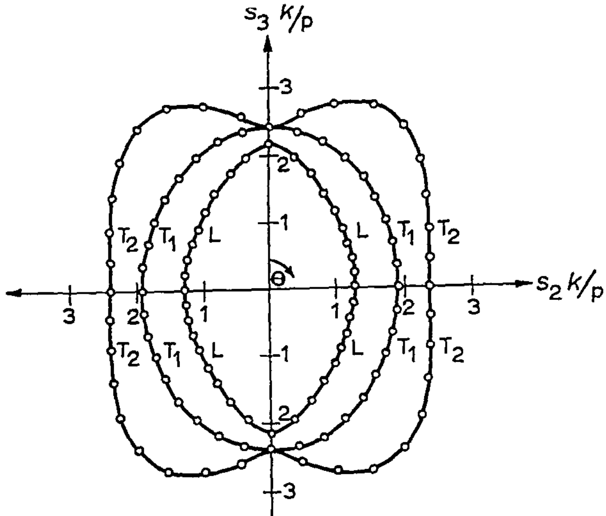
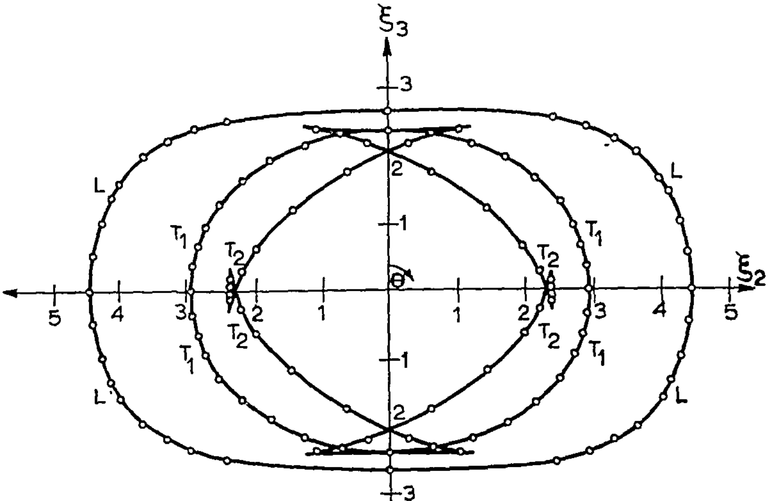


Fig. 3. Section in the zonal plane of
a) velocity surface, $v \propto \theta$,



(b)

b) inverse surface, $ks/\varrho \propto \theta$,



(c)

c) wave surface, $\xi \propto \theta$, for zinc.

For $M = 2$, $i = \mathrm{T}_1$, the displacement vector is truly transverse and parallel to the basal plane of the crystal.

It should be noted that the locus of the point I (called the inverse surface) has all the geometrical properties of the slowness surface while $\overrightarrow{\mathrm{OI}}$ has, in fact, dimensions of velocity. Consequently we may notice the cusp points about the ξ_2 and ξ_3 axes corresponding to the points of inflexion in the inverse section, MUSGRAVE [1957b] has also

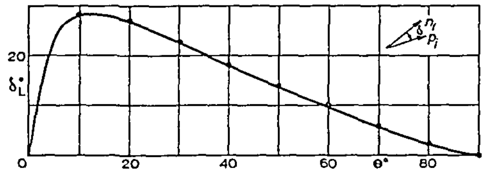


Fig. 4. Deviation $\delta^{(1)}$ ($\equiv \delta_L$) of quasi-longitudinal vector from wave normal.

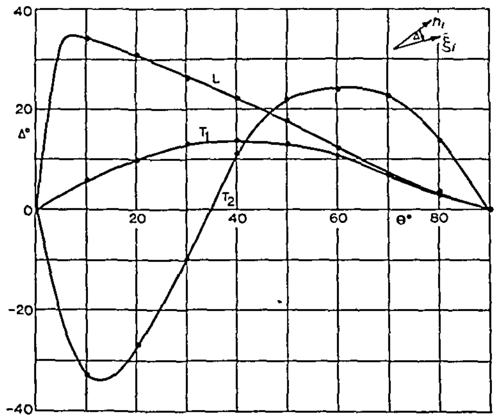


Fig. 5. Deviation $\Delta^{(M)}$ ($\equiv \Delta_i$), $M = 1, 2, 3$; $i = \mathrm{L}, \mathrm{T}_1, \mathrm{T}_2$; of rays from wave normal.

given inequalities between the elastic constants which must be satisfied if parabolic points on the inverse or slowness surface, and hence cusp points on the wave surface, are to exist about the axes of symmetry. These are derived by expressing the condition that the normal to a sheet of the slowness surface shall be parallel to an axis of symmetry. If this condition is satisfied by points of the slowness sheet not on the axis, it implies the existence of a parabolic point on the sheet. The points may be determined by the real intersections of a conic and the relevant slowness sheet. Thus, using the customary contraction of suffixes

$$\begin{array}{ll} 11 \rightarrow 1, & 23 \rightarrow 4 \\ 22 \rightarrow 2, & 31 \rightarrow 5 \\ 33 \rightarrow 3, & 12 \rightarrow 6 \end{array}$$

the slowness surface for a hexagonal medium having five independent elastic constants, c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, may be written

$$\varphi(r,z)\Phi(r,z) \equiv [\varrho - \tfrac{1}{2}(c_{11} - c_{12})r^2 + c_{44}z^2][\{\varrho - c_{44}(r^2 + z^2)\}^2 - \{\varrho - c_{44}(r^2 + z^2)\}\{ar^2 + hz^2\} + \{ah - d^2\}r^2z^2] = 0 \quad (47)$$

where $a = c_{11} - c_{44}$, $h = c_{33} - c_{44}$, $d = c_{13} + c_{44}$.

It is immediately clear that there exists a separable sheet, the spheroid

$$\varphi(r, z) \equiv \frac{1}{2\varrho}(c_{11} - c_{12})r^2 + \frac{c_{44}}{\varrho} z^2 - 1 = 0\,, \tag{48}$$

and this sheet can have no parabolic points. This is the sheet with the truly transverse displacement vector parallel to the basal plane. The remaining sheets have normals at (r,z) making an angle θ with the z-axis given by

$$\tan\theta = \frac{r[(c_{11} + c_{44})\{\varrho - c_{44}(r^2 + z^2)\} - c_{44}(ar^2 + hz^2) - (ah - d^2)z^2]}{z[(c_{33} + c_{44})\{\varrho - c_{44}(r^2 + z^2)\} - c_{44}(ar^2 + hz^2) - (ah - d^2)r^2]} \tag{49}$$

and clearly $\theta = 0$ for $r = 0$,

$$\text{or when } \frac{r^2}{A^2} + \frac{z^2}{B^2} = 1\,, \tag{50}$$

$$\text{where } \frac{A^2}{\varrho} = \frac{c_{11} + c_{44}}{2c_{11}c_{44}} \text{ and } \frac{B^2}{\varrho} = \frac{c_{11} + c_{44}}{c_{44}(c_{11} + c_{44}) + c_{11}h - d^2}\,.$$

Thus if the spheroid (50) makes real intersections with the L- and T_2-sheets of (47) ($\Phi(r,z) = 0$), the normals at these points will be parallel to $r = 0$. Real intersection can occur with the L-sheet if

$$\frac{B^2}{\varrho} < \frac{1}{c_{33}} \quad \text{i.e.} \quad c_{44}(c_{33} - c_{44}) < -(c_{33} + c_{44})^2\,,$$

which is impossible for real media, since for stability $c_{33} > c_{44} > 0$; and with the T_2-sheet if

$$\frac{B^2}{\varrho} > \frac{1}{c_{44}} \quad \text{i.e.} \quad c_{11}(c_{33} - c_{44}) < (c_{13} + c_{44})^2\,.$$

The elastic constants of zinc satisfy this latter relation and so there exists a plane of tangency P^∞ having a circular curve of contacts $\mathfrak{Q}_\infty$ on the T_2-sheet. There exists also a circular locus of parabolic points $\mathfrak{Q}$. Fig. 6 shows a section of the slowness and wave surfaces with correspondences between P^∞ and conical point p^∞, and the parabolic points $\mathfrak{Q}$ and the cuspidal edge $\mathfrak{p}$. We may recall that BUCHWALD [1959] has shown that for p^∞ the decay function of amplitude varies as $(|\xi_i|)^{-\frac{1}{2}}$ while for $\mathfrak{p}$ it varies as $(|\xi_i|)^{-\frac{5}{6}}$. The existence of the points $\mathfrak{Q}_{\mathrm{II}}$ may be established in a similar way.

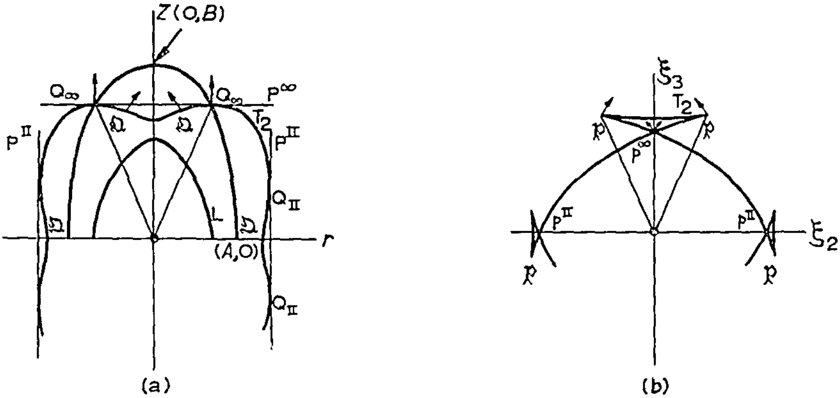


Fig. 6. Section of a) slowness sheets, b) wave sheet for zinc showing projective correspondences.

§ 13. Conical Refraction

Physically the point p^{∞} implies that all plane waves of normal OQ^{∞} have associated rays parallel to the z-axis. If these coincident rays were to meet a boundary normal to the zonal axis and pass into an isotropic medium, there would be sets of L- and T-waves satisfying the phase condition at the boundary and there would emerge two cones of rays in the isotropic medium (see Fig. 7). For this reason, a conical point on the wave surface might be said to give rise to double *external* conical refraction. When a conical point occurs on the slowness surface, there exists a degeneracy in the uniqueness of the displacement vectors for the T-waves. Any displacement vector normal to the L-vector can be propagated, but for each displacement there exists a different ray.

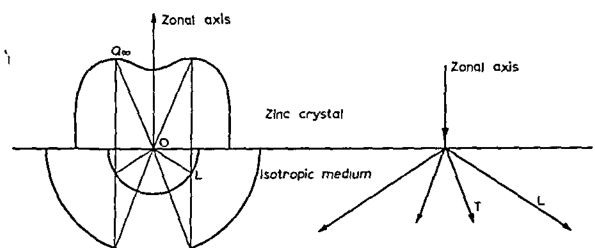


Fig. 7(a). Section of slowness sheets in a zinc crystal and an isotropic solid showing how two cones of external refraction, having longitudinal and transverse displacement respectively, may occur in the latter medium. $Q_{\infty}O$ represents a single incident slowness and $\overrightarrow{OL}$ and $\overrightarrow{OT}$ the possible refracted slownesses.
Fig. 7(b). Section of the multiple incident ray, common to many waves of different polarization, and the cones of externally refracted rays.

Hence, the rays form a cone about the single wave normal in the crystal and this is called *internal* conical refraction. An elliptic cone of internal refraction may occur in tetragonal crystals if the elastic constants satisfy the inequality

$$\frac{fh - d^2}{cf} > 0$$

where $c = c_{11} - c_{12} - 2c_{44}$; $f = c_{12} + c_{66}$ and d, h are as above. MUSGRAVE [1957a] has shown that the ellipse becomes a circle in the case of cubic symmetry. Furthermore, a conical point on the slowness surface along the (1, 1, 1) axis is an essential singularity in the geometry of the surface for all cubic media. The semi-angle Δ_c of this cone is given by MILLER and MUSGRAVE [1956] as

$$\tan \Delta_c = \left| \frac{c}{\sqrt{2}(c + 3c_{44})} \right|$$

and since $c = 0$ is the condition for isotropy, we see that the cone has zero angle in isotropic media. The magnitude of this angle may be as much as 40° in highly anisotropic cubic media (e.g. β-brass).

§ 14. Surface Waves - Particular Cases

STONELEY [1949, 1955], SYNGE [1957] and DERESIEWICZ and MINDLIN [1957] have variously contributed to the study of surface waves in hexagonal, cubic, orthorhombic and monoclinic media. STONELEY [1949] showed that true Rayleigh and Love waves having only exponential decrease in amplitude with depth may be propagated over the basal plane of a hexagonal medium. SYNGE [1957] has confirmed this finding and has shown that true Rayleigh waves may be propagated over any plane containing the zonal axis. STONELEY [1955] has also examined the case of cubic media and has discussed in detail the general type of surface wave which may occur in crystals. He shows that true Rayleigh wave be propagated over a (0, 0, 1) plane along the (1, 0, 0) direction and that their velocity is thus given by

$$c_{11}(\varrho v^2 - c_{44})\left[\varrho v^2 - \left(c_{11} - \frac{c_{12}^2}{c_{11}}\right)\right]^2 = c_{44}(\varrho v^2 - c_{11})(\varrho v^2)^2 .$$

This equation reduces to the usual Rayleigh wave equation for isotropic media if $c = 0$, i.e. $c_{12} = c_{11} - 2c_{44}$. DERESIEWICZ and MINDLIN [1957] have treated the case of surface waves which may travel along the digonal axis of a monoclinic crystal. The analysis is appropriate to an AT-cut quartz crystal and they find that a typical Rayleigh wave is possible. They have also dealt with the possibility of a surface wave propagated over a (0, 1, 0) plane in the (1, 0, 0) direction of an orthorhombic crystal. Again they find a true Rayleigh wave is possible.

No quantitative work on the reflection-refraction problem at a plane interface between two crystals has yet, to my knowledge, been published†, nor are observations of absorption as a function of orientation sufficiently comprehensive to merit discussion. Both types of problem are facets of the subject which seem worth examining.

In this article an attempt has been made to discuss the subject from an applied mathematical, rather than physical, point of view. Readers who wish for more detailed physical information are asked to refer to another article MUSGRAVE [1959] or the original references.

ACKNOWLEDGMENTS

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References

BLANCHET, P. H., 1840, Journal de Mathématiques (Liouville) **5** 1; **7** 13.
BUCHWALD, V. T., 1959, Proc. Roy. Soc. A **253** 563.
CHRISTOFFEL, E. B., 1877, Ann. di Mat. **8** 193.
DERESIEWICZ, H. and R. D. MINDLIN, 1957, Journal of App. Phys. **28** 669.
DUFF, G. F. D., 1960, Phil. Trans. Roy. Soc. A **252** 249.
HEARMON, R. F. S., 1956, Phil. Mag. Suppl. **5** 323.
HELBIG, K., 1958, Gerlands Beiträge zur Geophysik **67** Heft 3, 177.
HUNTINGTON, H. B., 1958, Solid State Physics **7** 213.
JOEL, N. and W. A. WOOSTER, 1957, Nature, Lond. **180** 430.
JOEL, N. and W. A. WOOSTER, 1958, Nature, Lond. **182** 1078, 1149.
KELVIN, LORD, 1904, Baltimore Lectures (Cambridge U.P.).
KRISHNAN, R. S. *et al.*, 1958, Nature, Lond. **182** 518.
LAVAL, J., 1951, Institut Internationale Physique Solvay Conseil, Brussels.
LE CORRE, Y., 1953, C. R. Acad. Sci. Paris **236** 1903.
LIGHTHILL, M. J., 1960, Phil. Trans. Roy. Soc. A **252** 397.
MAKINSON, K. R., 1955, Proc. International Wool Textile Research Conference Australia (Melbourne: C.S.I.R.O.) Pt. D, p. 54.
MCCLINTOCK, F. A. *et al.*, 1958, Nature, Lond. **182** 652.
MILLER, G. F. and M. J. P. MUSGRAVE, 1956, Proc. Roy. Soc. **236** 352.
MUSGRAVE, M. J. P., 1954, Proc. Roy. Soc. **226** I, 339; II, 356.

† Some quantitative results for reflection at the free boundary of a hexagonal crystal have recently appeared (MUSGRAVE [1960]).

MUSGRAVE, M. J. P., 1957a, Acta Cryst. **10** 316.
MUSGRAVE, M. J. P., 1957b, Camb. Phil. Soc. **53** 897.
MUSGRAVE, M. J. P., 1959, Reports on Progress in Physics (Phys. Soc. London).
MUSGRAVE, M. J. P., 1960, Geophysical Journal Roy. Astron. Soc. **3** 406.
RAMAN, SIR C. V. and K. S. VISWANATHAN, 1955, Proc. Indian Acad. Sci. A **42** 1, 51.
REINER, M. (editor), 1956, Rheology Vol. I (Academic Press).
RUDSKI, M. P., 1911, Anz. Akad. Wiss. Krakov, Bull. **8a** 503.
STONELEY, R., 1949, Mon. Not. Roy. Astr. Soc. (Geophysical Suppt) **5** 343.
STONELEY, R., 1955, Proc. Roy. Soc. **232** 447.
SYNGE, J. L., 1956, Proc. Roy. Irish Acad. **58** 13.
SYNGE, J. L., 1957, J. Math. Phys. **35** 323.
THOMAS, T. Y., 1957, J. Math. and Mech. **6** 759.
ZENER, C., 1948, Elasticity and Anelasticity of Metals (University of Chicago Press).

CHAPTER III

ELASTIC INCLUSIONS AND INHOMOGENEITIES

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CONTENTS

§ 1. Introduction

This review is concerned with the two following problems in the infinitesimal theory of elasticity, and with their inter-relation and generalization.

(i) The transformation problem.

A region (the 'inclusion') in a homogeneous elastic medium undergoes a permanent change of form which, in the absence of the constraint imposed by its surroundings (the 'matrix'), would be a prescribed uniform strain. To find the elastic field in matrix and inclusion.

(ii) The inhomogeneity problem.

A region in an otherwise homogeneous elastic medium has elastic constants differing from those of the remainder. To find how an applied stress, uniform at large distances, is disturbed by the inhomogeneity.

We shall not consider two-dimensional problems, where complex variable methods can be used, and the number of special cases which may be solved is unlimited. The three-dimensional inhomogeneity problem has been discussed extensively, particularly the special case of a cavity, i.e. an 'inhomogeneity' with vanishing elastic constants. The inclusion problem has received less attention, but is encountered in the discussion of various phenomena in solid-state physics, for example martensitic transformations and the formation of precipitates. STERNBERG [1958] has given an excellent annotated bibliography of the three-dimensional inhomogeneity (and inclusion) problems which have been solved.

We shall be largely concerned with the special case where the inclusion or inhomogeneity takes the form of the general ellipsoid with three unequal axes. There are two reasons for this. First, it appears to be the most general case whose solution can be given in a manageable form. Secondly, in this particular case there is a close connexion between the transformation and inhomogeneity problems. It stems from the fact that, as we shall see, the stress is constant throughout an ellipsoidal inclusion which has undergone a uniform transformation. As an illustration of this connexion suppose that it is required to solve the inhomogeneity problem for an ellipsoidal cavity. On the stress-field due to the inclusion superimpose everywhere a uniform stress equal and opposite to the uniform stress in the inclusion. The inclusion is then free of stress and may be removed. We are left with a stress-field which becomes uniform at infinity and which gives zero traction on the surface of the ellipsoid, as required. The general problem of an ellipsoidal inhomogeneity can be handled by an extension of this argument.

Among closed surfaces the ellipsoid alone has this convenient property. It shares it with other second-degree surfaces, and the analysis of §§ 3,4 can in fact be applied with trivial modifications to hyperboloids and paraboloids. However, the properties of such infinite 'inclusions' are not of much interest, and we shall not consider them. The corresponding in homogeneity problem is essentially the problem of stress-concentration by hyperboloidal notches. The cases which are of practical use have been discussed by NEUBER [1958].

We begin (§ 2) by finding a solution to problem (i) for an inclusion of arbitrary shape. The argument used is somewhat intuitive, but it is verified that the solution which is obtained does in fact satisfy the conditions of the problem. In § 3 the special case of an ellipsoidal inclusion is worked out to a point where numerical calculation is possible. Section 4 begins with a discussion of the inhomogeneity problem for an inhomogeneous region of arbitrary shape (though not much progress can be made) and the solution for the ellipsoidal inclusion is obtained from the results of § 2 in the way already indicated. By taking advantage of the connexion between inclusions and Somigliana dislocations (§ 5) it is possible to solve the problem of an ellipsoidal inhomogeneity perturbing a non-uniform stress-field. In § 6 we present some selected physical applications of the theory.

We shall use the familiar suffix notation. Repeated suffixes are to be summed over the values 1, 2, 3 and suffixes following a comma will denote differentiation with respect to the Cartesian coordinates x_1, x_2, x_3, e.g. $u_{i,j} = \partial u_i/\partial x_j$, $\psi_{,ijk} = \partial^3\psi/\partial x_i \partial x_j \partial x_k$.

Displacement u_i and strain e_{ij} are related by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \,. \tag{1.1}$$

Stress p_{ij} and strain are related by

$$p_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{1.2}$$

in an isotropic medium with Lamé constants λ,μ. If a second pair of quantities p'_{ij}, e_{ij} satisfy

$$p'_{ij} = \lambda e'_{kk}\delta_{ij} + 2\mu e'_{ij}$$

then

$$p_{ij}e'_{ij} = p'_{ij}e_{ij}\,. \tag{1.3}$$

A set of quantities bearing a common affix, e.g. $\overset{\mathrm{C}}{u_i}$, $\overset{\mathrm{C}}{e_{ij}}$, $\overset{\mathrm{C}}{p_{ij}}$ will be supposed to be related by (1.1) and (1.2) unless otherwise stated. It will sometimes be convenient to use the notation

$$f = f_{kk}\,, \qquad 'f_{ij} = f_{ij} - \tfrac{1}{3}f_{kk}\delta_{ij} \tag{1.4}$$

to denote the scalar and deviatoric parts of a second-order tensor.

We shall often make use of the formula

$$\nabla^2(pq) = p\nabla^2 q + q\nabla^2 p + 2p_{,k}q_{,k} \tag{1.5}$$

for calculating the Laplacian of a product, and of Gauss's theorem in the form

$$\int_S A_{\dots}\mathrm{d}S_k = \int_V A_{\dots,k}\mathrm{d}v \tag{1.6}$$

where S is a closed surface enclosing the volume V. Here and elsewhere $\mathrm{d}S_k$ is an abbreviation for $n_k\mathrm{d}S$, where dS is an element of surface and n_k is its normal.

§ 2. The General Transformed Inclusion

2.1. The elastic field

In this section we give a formal solution of the following problem:

A region bounded by a closed surface S in a homogeneous isotropic elastic medium undergoes a change of form which but for the constraint imposed by the surrounding material would be an arbitrary homogeneous strain $\overset{\mathrm{T}}{e_{ij}}$. To find the resulting elastic field inside and outside S.

It will be convenient to refer to the material inside S as the 'inclusion' and to the material outside S as the 'matrix'. The strain will be called the 'transformation strain', or, following Robinson [1951], the 'stress-free strain'. We assume that adjacent points immediately inside and outside S suffer no relative displacement; in other words the inclusion is 'bonded' to the matrix before, during and after the transformation. The problem can also be formulated as follows. To find a state of self-stress in an infinite body, with the following property: on making a cut over a prescribed closed surface S we are left with a stress-free cavity and a stress-free solid bounded by surfaces S_1, S_2 such that S_2 is transformed into S_1 by the homogeneous strain $-e_{ij}^{T}$.

TIMOSHENKO and GOODIER [1951] have given a method for finding the elastic field in a material in which each volume element alters its unconstrained shape. Our basic inclusion problem is merely the special case in which the change of shape is identical for all the volume elements inside a certain surface S and is zero for all elements outside S. The method of calculation we shall use (ESHELBY [1957]) is essentially equivalent to theirs.

It is first necessary to decide how to define the displacement. Let us fix our attention on some marked point $\boldsymbol{r}$ in the material, with coordinates $x_i(\boldsymbol{r})$ and suppose that, as we watch, the transformation takes place gradually, by some physical mechanism unspecified. Every point of the medium moves, and when the transformation is over the marked point will have different coordinates, $x_i(\boldsymbol{r}) + u_i^{C}(\boldsymbol{r})$ say. We take u_i^{C} as our displacement function. The state of zero displacement is the state of the material before the transformation has occurred.

We may contrast this definition of the displacement with another, perhaps equally natural, one. Suppose that after the transformation has occurred we make a cut over the surface S. Points on either side of the cut will move relatively as the stresses relax. For simplicity suppose that the two faces of the cut shrink away from each other everywhere, leaving a gap. During the relaxation every point of the matrix or inclusion suffers a certain displacement, $-u_i^{D}$ say. In place of u_i^{C} we might take u_i^{D} as our displacement function. In the matrix both u_i^{C} and u_i^{D} are measured from a state in which the matrix is free of stress. Consequently in the matrix $u_i^{C} = u_i^{D}$. In the inclusion u_i^{C} is measured from a state in which the inclusion is untransformed and unstressed, whereas u_i^{D} is measured from a state in which the inclusion has transformed but is free of stress because the constraint due to the matrix has been removed. Because of the gap which appears when matrix and inclusion are cut apart u_i^{D} is discontinuous across S, whereas u_i^{C} is continuous. It is generally more convenient to use u_i^{C}, but we shall refer to u_i^{D} again in § 5.

The displacement u_i^{C} will be calculated with the help of a sequence of imaginary cutting, straining and welding operations. This approach is somewhat alien to the usual methods of applied mathematics; the argument can, of course, be considered to be a purely heuristic one which points to a result (eq. (2.8)) whose validity has to be tested.

We shall suppose, to begin with, that the matrix extends to infinity. In the unstrained medium mark out the boundary S of the proposed transformed inclusion. Make a cut over S and remove the inclusion. Allow the transformation to occur. After it has suffered the uniform transformation $\overset{\mathrm{T}}{e}_{ij}$ the inclusion can no longer be fitted without strain into the cavity from which it was taken. Apply surface tractions $-\overset{\mathrm{T}}{p}_{ij}n_j$ to the surface of the inclusion, where

$$\overset{\mathrm{T}}{p}_{ij} = \lambda \overset{\mathrm{T}}{e}_{mm}\delta_{ij} + 2\mu \overset{\mathrm{T}}{e}_{ij} \,. \tag{2.1}$$

This produces a strain $-\overset{\mathrm{T}}{e}_{ij}$ in the inclusion and restores it to the form it had before transformation. Put the inclusion back in the cavity, still maintaining the surface tractions. Weld the material together across S. The surface tractions thus become an embedded layer of body force of amount

$$\mathrm{d}F_i = -\overset{\mathrm{T}}{p}_{ij}n_j\mathrm{dS} \tag{2.2}$$

on each element dS of S. The matrix is now unstressed and there is a uniform stress $-\overset{\mathrm{T}}{p}_{ij}$ in the inclusion. Further, every point in matrix or inclusion has the same position as it had before the transformation. That is, as regards displacement, the material is in the initial state from which we have agreed to measure $\overset{\mathrm{C}}{u}_i$. This state only differs from the required final state by the presence of the layer of body force (2.2). To get rid of this unwanted layer we apply an equal and opposite layer of body force

$$\mathrm{d}F_i = +\overset{\mathrm{T}}{p}_{ij}n_j\mathrm{dS} = \overset{\mathrm{T}}{p}_{ij}\,\mathrm{dS}_j \tag{2.3}$$

over S. The displacement induced by this operation is the displacement which we are trying to calculate.

A point force F_i at $\boldsymbol{r}'$ produces a displacement

$$u_i(\boldsymbol{r}) = U_{ij}F_j$$

at $\boldsymbol{r}$, where† (LOVE [1954])

$$U_{ij} = \frac{1}{4\pi\mu}\,\frac{\delta_{ij}}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{16\pi\mu(1-\sigma)}\,\frac{\partial^2}{\partial x_i \partial x_j}|\boldsymbol{r}-\boldsymbol{r}'|$$

$$= \frac{1}{4\pi\mu}\left(\tfrac{1}{2}\nabla^2 - \frac{1}{4(1-\sigma)}\,\frac{\partial^2}{\partial x_i \partial x_j}\right)|\boldsymbol{r}-\boldsymbol{r}'| \tag{2.4}$$

since

$$|\boldsymbol{r}-\boldsymbol{r}'|^{-1} = \tfrac{1}{2}\nabla^2|\boldsymbol{r}-\boldsymbol{r}'| \,. \tag{2.5}$$

† σ is Poisson's ratio.

Hence

$$u_i^{\mathrm{C}}(r) = \int_{\mathrm{S}} \mathrm{dS}_k p_{jk}^{\mathrm{T}} U_{ij}(|r - r'|) . \tag{2.6}$$

But by Gauss's theorem

$$\int |r - r'| \,\mathrm{dS}_k = \int \frac{\partial}{\partial x'_k} |r - r'| \,\mathrm{d}v = -\frac{\partial}{\partial x_k} \int |r - r'| \mathrm{d}v$$

and so

$$u_i^{\mathrm{C}} = \frac{1}{16\pi\mu(1-\sigma)} p_{jk}^{\mathrm{T}} \psi_{,ijk} - \frac{1}{4\pi\mu} p_{ik}^{\mathrm{T}} \varphi_{,k} \tag{2.7}$$

or

$$u_i^{\mathrm{C}} = \frac{1}{8\pi(1-\sigma)} e_{jk}^{\mathrm{T}} \psi_{,ijk} - \frac{1}{2\pi} e_{ik}^{\mathrm{T}} \varphi_{,k} - \frac{\sigma}{4\pi(1-\sigma)} e^{\mathrm{T}} \varphi_{,i} \tag{2.8}$$

where

$$\varphi(r) = \int_V \frac{\mathrm{d}v}{|r - r'|} \quad \text{and} \quad \psi(r) = \int_V |r - r'| \,\mathrm{d}v . \tag{2.9}$$

φ is the ordinary (harmonic or Newtonian) potential of attracting matter of unit density filling the volume V bounded by S. ψ is the corresponding biharmonic potential. Geometrically, ψ/V is the mean distance of the point r from all the points inside S.

From (2.5)

$$\nabla^2\psi = 2\varphi . \tag{2.10}$$

The following results follow from the theory of attraction (Poincaré [1899], MacMillan [1958]).

$$\nabla^4\psi = 2\nabla^2\varphi = \begin{cases} -8\pi \text{ outside S} \\ 0 \text{ inside S} \end{cases} \tag{2.11}$$

$$\varphi,\ \varphi_{,i} \text{ are continuous across S} \tag{2.12}$$

$$\varphi_{,ij}\,(\text{out}) - \varphi_{,ij}\,(\text{in}) = 4\pi n_i n_j . \tag{2.13}$$

The last equation gives the difference in the second derivative at two adjacent points immediately inside and outside S at a point where the normal to S is n_i. We shall use a similar notation for other quantities which are discontinuous across S. Eq. (2.13) is a re-statement of the result that the discontinuity in attraction across a double layer is 4π times its moment. More generally, a function satisfying

$$\nabla^2 U = -4\pi\varrho \tag{2.14}$$

suffers a discontinuity in its second derivatives

$$U_{,kl}\,(\text{out}) - U_{,kl}\,(\text{in}) = -4\pi\{\varrho(\text{out}) - \varrho(\text{in})\} n_k n_l \tag{2.15}$$

on crossing a surface across which ϱ is discontinuous. But $\psi_{,ij}$ satisfies (2.14) with $\varrho = -2\varphi_{,ij}/4\pi$ and so from (2.15) we obtain the relation

$$\psi_{,ijkl}\,(\text{out}) - \psi_{,ijkl}\,(\text{in}) = 8\pi n_i n_j n_k n_l\,. \tag{2.16}$$

By similar arguments one finds that

$$\psi\,,\ \psi_{,i}\,,\ \psi_{,ijk} \text{ are continuous across S.} \tag{2.17}$$

The stress in the matrix is

$$\overset{\mathrm{C}}{p}_{ij} = \lambda \overset{\mathrm{C}}{u}_{m,m}\,\delta_{ij} + \mu(\overset{\mathrm{C}}{u}_{i,j} + \overset{\mathrm{C}}{u}_{j,i})\,. \tag{2.18}$$

Since the inclusion was already subject to a uniform stress $-\overset{\mathrm{T}}{p}_{ij}$ before the body force (2.3) was applied, the stress in the inclusion is

$$\overset{\mathrm{I}}{p}_{ij} = \overset{\mathrm{C}}{p}_{ij} - \overset{\mathrm{T}}{p}_{ij} \tag{2.19}$$

with $\overset{\mathrm{C}}{p}_{ij}$ derived from $\overset{\mathrm{C}}{u}_{ij}$ as in (2.18).

We outline a method of verifying formally that the proposed solution (2.8), (2.18), (2.19) does in fact solve the inclusion problem. From (2.7) it follows that $\overset{\mathrm{C}}{u}_i$ satisfies the equilibrium equation

$$\mu\nabla^2 u_i + (\lambda + \mu) u_{m,mi} = 0$$

and from (2.12), (2.17) that it is continuous across S. If the stress is defined by (2.18) and (2.19) the relations (2.13) and (2.16) show that $p_{ij}n_j$ is continuous across S. Let an additional displacement $-\overset{\mathrm{C}}{u}_i(\boldsymbol{r}')$ be imposed on all points $\boldsymbol{r}'$ or the inner boundary of the matrix. By the uniqueness theorem of the theory of elasticity (LOVE [1954]) it will produce an additional displacement $-\overset{\mathrm{C}}{u}_i(\boldsymbol{r})$ throughout the matrix, and so leave it stress-free. Likewise an additional displacement $-\overset{\mathrm{C}}{u}_i(\boldsymbol{r}')$ imposed on all points $\boldsymbol{r}'$ of the surface of the inclusion induces a displacement $-\overset{\mathrm{C}}{u}_i(\boldsymbol{r})$ throughout its interior and so by (2.19) leaves it in a state of uniform stress $-\overset{\mathrm{T}}{p}_{ij}$. At this point the inner surface of the matrix and the outer surface of the inclusion still fit perfectly, because of the continuity of $\overset{\mathrm{C}}{u}_i$. If the tractions $-\overset{\mathrm{T}}{p}_{ij}n_j$ on the inclusion are reduced to zero the inclusion suffers a uniform strain $\overset{\mathrm{T}}{e}_{ij}$ and we are left with an unstressed matrix and an unstressed inclusion between whose surfaces there is the required misfit.

Equation (2.7) can be written in the Boussinesq-Papkovich-Neuber form

$$\overset{\mathrm{C}}{u}_i = B_i - \frac{1}{4\pi(1-\sigma)}\,(x_m B_m + \beta)_{,i} \tag{2.20}$$

with harmonic B_i and β:

$$4\pi\mu B_i = -p_{ik}^{\mathrm{T}}\varphi_{,k}, \qquad 4\pi\mu\beta = p_{jk}^{\mathrm{T}} f_{jk} \tag{2.21}$$

where

$$f_{ij} = x_i\varphi_{,j} - \psi_{,ij}\,. \tag{2.22}$$

That f_{ij} (and hence β) is harmonic inside and outside the inclusion follows from (1.5) and (2.10). Further f_{ij} behaves like r^{-1} for large r, while its normal derivative

$$\partial f_{ij}/\partial n = \varphi_{,j}n_i + x_i\varphi_{,jk}n_k - \psi_{,ijk}n_k$$

suffers a discontinuity $4\pi x_i n_j n_k n_k = 4\pi x_i n_j$ on passing through S, by (2.13) and (2.16). Hence f_{ij} is the harmonic potential of a layer of attracting matter distributed over S with surface density $x_i n_j$. In this way the biharmonic potential ψ can be replaced by the harmonic potential β of a certain surface layer.

It is interesting to see how much information can be obtained when our knowledge of φ and ψ is incomplete. We know in any case that φ and ψ behave as V/r and Vr for large r, and hence, from (2.7) that the field at large distance from the inclusion is given by

$$u_i^{\mathrm{C}}(r) = \frac{V e_{jk}^{\mathrm{T}} g_{ijk}}{8\pi(1-\sigma)r^2} \tag{2.23}$$

where

$$g_{ijk} = (1-2\sigma)(\delta_{ij}l_k + \delta_{ik}l_j - \delta_{jk}l_i) + 3l_il_jl_k$$

and l_i is a unit vector joining the origin to the point of observation $\boldsymbol{r}$. If only φ is known we can find the dilatation and rotation:

$$e^{\mathrm{C}} = -\frac{1-2\sigma}{8\pi\mu(1-\sigma)}\, p_{ik}^{\mathrm{T}}\varphi_{,ik} \tag{2.24}$$

$$4\pi\omega_{il}^{\mathrm{C}} = 2\pi(u_{i,l}^{\mathrm{C}} - u_{l,i}^{\mathrm{C}}) = e_{lk}^{\mathrm{T}}\varphi_{,ki} - e_{ik}^{\mathrm{T}}\varphi_{,kl}\,. \tag{2.25}$$

If e_{ij}^{T} happens to be a pure dilatation we can find the complete field in terms of φ:

$$u_i^{\mathrm{C}} = -\frac{1+\sigma}{12\pi(1-\sigma)}\, e^{\mathrm{T}}\varphi_{,i} \qquad (e_{ij}^{\mathrm{T}} = \tfrac{1}{3}e^{\mathrm{T}}\delta_{ij}) \tag{2.26}$$

(GOODIER [1937]). In this special case the dilatation has the constant value $e^{\mathrm{T}}(1+\sigma)/3(1-\sigma)$ inside the inclusion. In the matrix the dilatation is zero. Consequently the value of the bulk modulus $\varkappa^*$ of the matrix is irrelevant and (2.26) applies also to the case of an inclusion of elastic constants $\mu,\varkappa$ in a matrix with constants $\mu,\varkappa^*$ if we give to σ the value appropriate to the inclusion (CRUM, quoted by NABARRO [1940]; ROBINSON [1951]).

Again, it may happen that it is easier to calculate φ and ψ for points within the inclusion than for points outside it. (We shall see that this is so for an ellipsoidal inclusion.) Since the strains involve $\varphi_{,ij}$ and $\psi_{,ijkl}$ we can use (2.13) and (2.16) to find the strain $\overset{\mathrm{C}}{e}_{ij}(\text{out})$ (and hence the stress) at a point immediately outside the inclusion from the values $\overset{\mathrm{C}}{e}_{ij}(\text{in})$ at the adjacent point immediately inside the inclusion. Expressed in terms of the dilatational and deviatoric parts of $\overset{\mathrm{C}}{e}_{ij}$ the result is

$$e^{\mathrm{C}}(\text{out}) = e^{\mathrm{C}}(\text{in}) - \frac{1}{3}\,\frac{1+\sigma}{1-\sigma}\,e^{\mathrm{T}} - \frac{1-2\sigma}{1-\sigma}\,{}'\overset{\mathrm{T}}{e}_{ij}\,n_i n_j$$

$$\begin{aligned} {}'\overset{\mathrm{C}}{e}_{il}\,(\text{out}) = {}'\overset{\mathrm{C}}{e}_{il}\,(\text{in}) &+ \frac{1}{1-\sigma}\,{}'\overset{\mathrm{T}}{e}_{jk}\,n_j n_k n_i n_l - {}'\overset{\mathrm{T}}{e}_{ik}\,n_k n_l - {}'\overset{\mathrm{T}}{e}_{lk}\,n_k n_i \qquad (2.27)\\ &+ \frac{1-2\sigma}{3(1-\sigma)}\,{}'\overset{\mathrm{T}}{e}_{jk}\,n_j n_k\,\delta_{il} - \frac{1}{3}\,\frac{1+\sigma}{1-\sigma}\,e^{\mathrm{T}}(n_i n_l - \tfrac{1}{3}\delta_{il})\,. \end{aligned}$$

The foregoing results all refer to a transformed inclusion in an infinite matrix. If the matrix has a finite boundary the displacements will be of the form

$$\overset{\mathrm{F}}{u}_i = \overset{\mathrm{C}}{u}_i + \overset{\mathrm{im}}{u}_i$$

and the stress will be

$$\overset{\mathrm{F}}{p}_{ij} = \overset{\mathrm{C}}{p}_{ij} + \overset{\mathrm{im}}{p}_{ij}$$

in the matrix and

$$\overset{\mathrm{F}}{p}_{ij} - \overset{\mathrm{T}}{p}_{ij}$$

in the inclusion. The 'image field' $\overset{\mathrm{im}}{u}_i$, $\overset{\mathrm{im}}{p}_{ij}$ is free of singularities in the medium and is determined by the requirement that the sum of the C-field and the image-field shall satisfy whatever boundary conditions are imposed on the outer surface of the matrix. If the outer boundary S_0 is stress-free we must have

$$\overset{\mathrm{im}}{p}_{ij}\,n_j = -\overset{\mathrm{C}}{p}_{ij}\,n_j \quad \text{on} \quad S_0$$

that is, the image-field is the field produced by surface tractions $-\overset{\mathrm{C}}{p}_{ij}n_j$ acting on the outer surface. If the outer surface is held immovable we must have

$$\overset{\mathrm{im}}{u}_i = -\overset{\mathrm{C}}{u}_i \quad \text{on} \quad S_0\,. \qquad (2.28)$$

Thus when the C-quantities are known the determination of the image quantities reduces to a standard boundary-value problem.

It is sometimes convenient to have a formal expression analogous to (2.6) exhibiting u_i^{F} as the displacement induced by the layer of body force (2.3). When the boundary condition is (2.28) (rigidly held boundary) we may evidently write

$$u_i^{\mathrm{F}}(\boldsymbol{r}) = \int_{\mathrm{S}} \mathrm{dS}_k\, p_{jk}^{\mathrm{T}}\, U_{ij}(\boldsymbol{r}, \boldsymbol{r}') \tag{2.29}$$

where $F_j U_{ij}(\boldsymbol{r},\boldsymbol{r}')$ is the displacement at $\boldsymbol{r}$ due to a point-force $\boldsymbol{F} = (F_1, F_2, F_3)$ acting at $\boldsymbol{r}'$ in a body at whose surface the displacement is required to be zero.

With a suitable alternative definition of $U_{ij}(\boldsymbol{r}, \boldsymbol{r}')$ (2.29) also applies to the case where the outer boundary is stress-free. However, we cannot simply say that $F_j U_{ij}$ is the displacement at $\boldsymbol{r}$ due to a point-force at $\boldsymbol{r}'$ in a body with a stress-free surface. No such solution of the elastic equations exists, since the integral of the surface traction over S_0 must be equal to $-\boldsymbol{F}$. Instead we define $U_{ij}(\boldsymbol{r}, \boldsymbol{r}')$ as the displacement in the body with stress-free surfaces due to a point-force $\boldsymbol{F}$ at $\boldsymbol{r}'$, a point-force $-\boldsymbol{F}$ at the origin and a double force (LOVE [1954]) at the origin of moment $-\boldsymbol{F} \times \boldsymbol{r}'$. Since the resultant and moment of this set of forces are both zero the condition of zero traction over S_0 can be satisfied. It can easily be shown that when the elementary forces (2.3) are summed over S the corresponding auxiliary forces and moments at the origin add up to zero. Consequently, with this definition of U_{ij} the expression (2.29) actually gives the displacement due to the layer of body force (2.3) alone.

2.2. ENERGY RELATIONS

The elastic energy associated with the inclusion (i.e. the energy in the inclusion plus the energy in the matrix) can be calculated very simply by following the energy changes which occur during the imaginary operations leading to (2.6). Suppose first that the matrix is infinite. When the inclusion has been welded back into the matrix but is still held in its untransformed shape by the layer of body force (2.2) the energy in the matrix is zero and the energy in the inclusion is

$$\tfrac{1}{2} \int_V p_{ij}^{\mathrm{T}}\, e_{ij}^{\mathrm{T}}\, \mathrm{d}v\,. \tag{2.30}$$

When the layer of body force is relaxed each element of S moves through a distance $\overset{\mathrm{C}}{u_i}$ as the force on it falls from $\mathrm{d}F_i$ to zero. The amount of energy extracted from the elastic solid during the relaxation is thus

$$-\tfrac{1}{2}\int \overset{\mathrm{C}}{u_i}\,\mathrm{d}F_i = \tfrac{1}{2}\int_S \overset{\mathrm{T}}{p_{ij}}\,\overset{\mathrm{C}}{u_i}\,\mathrm{dS}_j = \tfrac{1}{2}\int \overset{\mathrm{T}}{p_{ij}}\,\overset{\mathrm{C}}{e_{ij}}\,\mathrm{d}v\,. \qquad (2.31)$$

The energy remaining in the medium is found by subtracting (2.31) from (2.30):

$$E_\infty = -\tfrac{1}{2}\int_V \overset{\mathrm{T}}{p_{ij}}\,(\overset{\mathrm{C}}{e_{ij}} - \overset{\mathrm{T}}{e_{ij}})\,\mathrm{d}v = -\tfrac{1}{2}\int_V \overset{\mathrm{I}}{p_{ij}}\,\overset{\mathrm{T}}{e_{ij}}\,\mathrm{d}v\,. \qquad (2.32)$$

The suffix ∞ emphasizes that this is the energy for an inclusion in an infinite matrix. Evidently we only need to know the elastic field inside the inclusion. If the stress-free strain is a pure dilatation $\overset{\mathrm{T}}{e_{ij}} = \frac{1}{3}e^{\mathrm{T}}\delta_{ij}$ we have by (2.26)

$$E_\infty = \frac{2}{9}\mu V \frac{1+\sigma}{1-\sigma}(e^{\mathrm{T}})^2 \qquad (2.33)$$

for an inclusion of any shape. According to the argument following (2.26) the expression is still correct if the bulk moduli of matrix and inclusion differ (NABARRO [1940], ROBINSON [1951]).

Suppose next that the inclusion is situated in a finite matrix bounded by the surface S_0. No difference is made to (2.30) if we cut away the part of the unstrained matrix exterior to S_0. The energy removed by relaxing the layer of body force is given by (2.31) with $\overset{\mathrm{C}}{u_i}$ replaced by $\overset{\mathrm{C}}{u_i} + \overset{\mathrm{im}}{u_i}$. The elastic energy due to the stress-field of the inclusion is thus

$$E_{\mathrm{inc}} = E_\infty + E_{\mathrm{im}}$$

where the 'image term'

$$E_{\mathrm{im}} = -\tfrac{1}{2}\int_V \overset{\mathrm{T}}{p_{ij}}\,\overset{\mathrm{im}}{e_{ij}}\,\mathrm{d}v = -\tfrac{1}{2}\int_V \overset{\mathrm{im}}{p_{ij}}\,\overset{\mathrm{T}}{e_{ij}}\,\mathrm{d}v$$

varies with the position of the inclusion. As an example we consider the case of a sphere of volume $V(1+e^{\mathrm{T}})$ forced into a spherical cavity of volume V in a semi-infinite solid. In the infinite solid $\overset{\mathrm{C}}{u_i}$ is given by (2.26) with $\varphi = V/r$. E_∞ is given by (2.33). To find E_{im} we have to calculate $\overset{\mathrm{im}}{e_{kk}}$. It is the dilatation produced by surface tractions $-\overset{\mathrm{C}}{p_{ij}}n_j$ acting on the free surface of the semi-infinite solid, and can be found by well-known methods (LOVE [1954]). As the dilatation is harmonic we need only find its value at the centre of the inclusion, since the mean value of harmonic function of a sphere is equal to its value at the centre. The result is

$$E_{\mathrm{inc}} = E_\infty\left\{1 - \frac{1}{4}(1+\sigma)\frac{a^3}{h^3}\right\} \qquad (2.34)$$

where a is the radius of the sphere and h is the distance of its centre from the free surface. If $\sigma = \frac{1}{3}$ the elastic energy is reduced to $\frac{2}{3}E_\infty$ if the inclusion just reaches the surface ($h = a$).

It is worth recalling at this point that, in thermodynamic terms, 'elastic energy' represents internal energy under adiabatic conditions and Helmholtz free energy under isothermal conditions (SOKOLNIKOFF [1946]). A calculation such as the foregoing covers both cases; the distinction only appears when we decide to insert either the adiabatic or isothermal values of the elastic constants.

In some applications it is necessary to consider the changes of energy which occur when an inclusion is formed in a body which is already stressed by externally applied loads. For simplicity we shall suppose that the body is stressed by surface tractions which do not vary when the outer surface S_0 of the body is slightly deformed by the introduction of the inclusion; in engineering language this is the case of 'dead loading'. The stress due to the inclusion must satisfy $\overset{\mathrm{F}}{p}_{ij}n_j = 0$ on S_0.

Let the external loads produce stress and strain $\overset{\mathrm{A}}{p}_{ij}$, $\overset{\mathrm{A}}{e}_{ij}$, not necessarily uniform. Before the inclusion has transformed the elastic energy in the material is

$$E_{\mathrm{A}} = \tfrac{1}{2} \int \overset{\mathrm{A}}{p}_{ij}\, \overset{\mathrm{A}}{e}_{ij}\, \mathrm{d}v \tag{2.35}$$

with the integral extending over the whole volume of the material. Suppose next that the body is subject to the combined action of the stresses due to the load and the internal stresses due to the inclusion. We might expect that the elastic energy would be the sum of (2.35), (2.34) and a cross term, representing an 'interaction energy'. But in fact the cross term is zero. To see this, suppose that the transformation occurs first, in the absence of external loads. The elastic energy is E_{inc}. Now let the load be applied. Within the limits of the usual infinitesimal theory of elasticity the body responds to external forces just as it would if it were not self-stressed by the transformed inclusion. The work done on the body by the load is thus

$$\tfrac{1}{2} \int_{S_0} \overset{\mathrm{A}}{p}_{ij}\, \overset{\mathrm{A}}{u}_i\, \mathrm{d}S_j = \tfrac{1}{2} \int \overset{\mathrm{A}}{p}_{ij}\, \overset{\mathrm{A}}{e}_{ij}\, \mathrm{d}v = E_{\mathrm{A}}$$

and so the total elastic energy is simply $E_{\mathrm{inc}} + E_{\mathrm{A}}$. The same conclusion can be reached analytically. If the volume integral of the energy density is converted into a surface integral over S and the outer surface of the matrix it will be found that the cross term between the A- and F-terms vanishes because $\overset{\mathrm{F}}{p}_{ij}n_j = 0$ on S_0.

Despite the lack of a cross term in the elastic energy it is possible to define a physically meaningful interaction energy. (For a general discussion in the context of solid state theory cf. PEACH [1951], ESHELBY [1951, 1956].) To introduce the concept of interaction energy we begin by enquiring whether in the presence of the stresses arising from the external load it is energetically possible for the inclusion to form spontaneously. At first sight it appears as if the answer is no, since the elastic energy increases by the necessarily positive quantity E_{inc} when the inclusion is introduced. However, we have to consider not simply the elastic energy of the material, but rather the total energy, E_{tot} say, of the closed system made up of the body and the loading mechanism. When the transformation occurs in the presence of the external load the increase in the potential energy of the loading mechanism is equal to the work which the surface tractions $\overset{\text{A}}{p_{ij}}n_j$ do on the body as the surface displacements change from $\overset{\text{A}}{u_i}$ to $\overset{\text{A}}{u_i} + \overset{\text{F}}{u_i}$. Thus the increase in the energy of the whole system (inclusion plus matrix plus loading mechanism) is

$$\Delta E_{\text{tot}} = E_{\text{inc}} + E_{\text{int}} \tag{2.36}$$

where

$$E_{\text{int}} = -\int_{S_0} \overset{\text{A}}{p_{ij}}\, \overset{\text{F}}{u_i}\, \mathrm{d}S_j\,. \tag{2.37}$$

We may write

$$E_{\text{tot}} = E_0 + E_{\text{A}} + E_{\text{inc}} + E_{\text{int}} \tag{2.38}$$

where E_0 is the potential energy of the loading mechanism in the absence of the inclusion. In (2.38) the first two terms depend only on the elastic field due to the load and the third only on the field due to the inclusion. The last term, which depends on both these fields, may be appropriately called the interaction energy.

We can now answer the question posed in the preceding paragraph. If the applied stress is chosen so that $\Delta E_{\text{tot}} < 0$ (and this can always be done) it is energetically possible for the transformation to take place spontaneously. Generally ΔE_{tot}, whether it is positive or negative, may be called the energy of formation of the inclusion in the applied stress-field. This concept is familiar in thermodynamics. In fact, if the transformation leading to the formation of the inclusion occurs under adiabatic conditions ΔE_{tot} represents, in thermodynamic language, the increase in the enthalpy of the body, while if the transformation takes place under isothermal conditions ΔE_{tot} is the increase in its Gibbs free energy. This follows at once from the definition of these quantities. The increase in enthalpy associated with some change in the state of a thermally isolated system is equal to the increase in its internal energy plus the work which it does on its environment during the course of the change. For a change at constant temperature the Gibbs free energy is similarly defined, replacing 'internal energy' by 'Helmoltz free energy'. That ΔE_{tot} is the change in enthalpy or Gibbs free energy follows from the fact that, as we have seen, the 'elastic energy' has to be identified with the internal energy in adiabatic processes and with Helmholtz free energy in isothermal processes. Thus ΔE_{tot} is the enthalpy or Gibbs free energy of formation of the inclusion.

The expression (2.39) for the interaction energy can be put into a more useful form by the following artifice. We re-write (2.37) as

$$E_{\text{int}} = -\int_{S_0} (\overset{\text{A}}{p}_{ij}\, \overset{\text{F}}{u}_i - \overset{\text{F}}{p}_{ij}\, \overset{\text{A}}{u}_i)\, \mathrm{d}S_j\,. \tag{2.39}$$

Because $\overset{\text{F}}{p}_{ij}\, \overset{\text{A}}{u}_i = 0$ on S_0 the added term is zero. Consider the divergence of the integrand. It can be split into two terms

$$(\overset{\text{im}}{p}_{ij}\, \overset{\text{A}}{u}_i - \overset{\text{A}}{p}_{ij}\, \overset{\text{im}}{u}_i)_{,j} = \overset{\text{im}}{p}_{ij}\, \overset{\text{A}}{e}_{ij} - \overset{\text{A}}{p}_{ij}\, \overset{\text{im}}{e}_{ij} \tag{2.40}$$

and

$$(\overset{\text{C}}{p}_{ij}\, \overset{\text{A}}{u}_i - \overset{\text{A}}{p}_{ij}\, \overset{\text{C}}{u}_i)_{,j} = \overset{\text{C}}{p}_{ij}\, \overset{\text{A}}{e}_{ij} - \overset{\text{A}}{p}_{ij}\, \overset{\text{C}}{e}_{ij}\,. \tag{2.41}$$

By (1.3) the expression (2.41) is zero both in the matrix and the inclusion and so the image terms make no contribution. On the other hand, (2.40) vanishes in the matrix but not in the inclusion, since $\overset{\text{C}}{p}_{ij}$ is discontinuous across S. Hence in (2.39) we may replace $\overset{\text{F}}{p}_{ij}$, $\overset{\text{F}}{u}_i$ by $\overset{\text{C}}{p}_{ij}$, $\overset{\text{C}}{u}_i$ and carry out the integration over the boundary of the inclusion instead of over the outer surface of the body; that is

$$E_{\text{int}} = \int_{S} (\overset{\text{C}}{p}_{ij}\, \overset{\text{A}}{u}_i - \overset{\text{A}}{p}_{ij}\, \overset{\text{C}}{u}_i)\, \mathrm{d}S_j\,. \tag{2.42}$$

In (2.43) we may replace $\overset{\text{C}}{p}_{ij}$ by $\overset{\text{I}}{p}_{ij}$ since $\overset{\text{C}}{p}_{ij} n_j = \overset{\text{I}}{p}_{ij} n_j$ on S. The integral can then be converted to a volume integral over the inclusion:

$$E_{\text{int}} = \int_{V} (\overset{\text{I}}{p}_{ij}\, \overset{\text{A}}{e}_{ij} - \overset{\text{A}}{p}_{ij}\, \overset{\text{C}}{e}_{ij})\, \mathrm{d}S_j\,.$$

By (1.3) the integrand is equal to $(\overset{\text{I}}{p}_{ij} - \overset{\text{C}}{p}_{ij})\, \overset{\text{A}}{e}_{ij}$ and so by (2.19)

$$E_{\text{int}} = -\int_{V} \overset{\text{T}}{p}_{ij}\, \overset{\text{A}}{e}_{ij}\, \mathrm{d}v = -\int_{V} \overset{\text{A}}{p}_{ij}\, \overset{\text{T}}{e}_{ij}\, \mathrm{d}v\,. \tag{2.43}$$

Evidently to find the interaction energy we need only know the stress-free strain $\overset{\mathrm{T}}{e}_{ij}$; it is unnecessary to solve the elastic problems associated with the determination of $\overset{\mathrm{C}}{u}_i$ or u_i^{im}.

§ 3. The Ellipsoidal Inclusion

3.1. The elastic field

When the inclusion is bounded by the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

the elastic field may be found explicitly. The form of the harmonic potential φ is well-known (KELLOGG [1929]). There is an analogous expression for the biharmonic potential ψ (ESHELBY [1959b]), but in fact all the derivatives of ψ which enter (2.7) can be found in terms of derivatives of φ. Let us compare f_{12} (eq. (2.7)) with the function $g = a^2(x_1\varphi_{,2} - x_2\varphi_{,1})/(a^2 - b^2)$ which plays a role in the hydrodynamic theory of rotating ellipsoids. Each of these functions is harmonic inside and outside the ellipsoid, is continuous across its surface and falls to zero at infinity. Across the surface there is a discontinuity $4\pi x_1 n_2$ in the normal derivative of f_{12} (eq. (2.22)). The corresponding discontinuity for g is $4\pi a^2(x_1 n_2 - x_2 n_1)/(a^2 - b^2)$. This is simply $4\pi x_1 n_2$ in view of the relation $b^2 x_1 n_2 = a^2 x_2 n_1$ which follows from the expressions

$$\begin{aligned} n_1 &= x_1/a^2 h\,, \\ n_2 &= x_2/b^2 h\,, \qquad (h^2 = x^2/a^4 + y^2/b^4 + z^2/c^4) \\ n_3 &= x_3/c^2 h\,, \end{aligned}$$

for the components of the normal to an ellipsoid. Hence f_{12} and g are identical, being harmonic potentials of the same surface distribution. We therefore have

$$f_{12} = x_1\varphi_{,2} - \psi_{,12} = \frac{a^2}{a^2 - b^2}(x_1\varphi_{,2} - x_2\varphi_{,1})$$

and similarly for the other f_{ij} with $i \neq j$. Thus

$$\begin{aligned} \psi_{,12} &= \frac{a^2}{a^2 - b^2}\, x_2\varphi_{,1} + \frac{b^2}{b^2 - a^2}\, x_1\varphi_{,2} \\ \psi_{,23} &= \frac{b^2}{b^2 - c^2}\, x_3\varphi_{,2} + \frac{c^2}{c^2 - b^2}\, x_2\varphi_{,3} \\ \psi_{,31} &= \frac{c^2}{c^2 - a^2}\, x_1\varphi_{,3} + \frac{a^2}{a^2 - c^2}\, x_3\varphi_{,1}\,. \end{aligned}$$

It is more difficult to derive expressions for f_{11}, f_{22}, f_{33} and hence for $\psi_{,11}$, $\psi_{,22}$, $\psi_{,33}$. However, there is no need to obtain them, since all the third derivatives $\psi_{,ijk}$ appearing in (2.8) can be made to depend on φ, $\psi_{,12}$, $\psi_{,23}$, $\psi_{,31}$. We may write, for example,

$$\psi_{,112} = (\psi_{,12})_{,1} \qquad \psi_{,111} = 2\varphi_{,1} - (\psi_{,12})_{,2} - (\psi_{,13})_{,3}\,.$$

The first relation is trivial; the second is obtained by differentiating $\nabla^2\psi = 2\varphi$ with respect to x_1.

Substitution in (2.8) gives

$$\begin{aligned} 8\pi(1-\sigma)u_1^{\mathrm{C}} = {} & \frac{e_{22}^{\mathrm{T}} - e_{11}^{\mathrm{T}}}{a^2-b^2}\frac{\partial}{\partial x_2}(a^2x_2\varphi_{,1} - b^2x_1\varphi_{,2}) + \\ & \frac{e_{33}^{\mathrm{T}} - e_{11}^{\mathrm{T}}}{c^2-a^2}\frac{\partial}{\partial x_3}(c^2x_1\varphi_{,3} - a^2x_3\varphi_{,1}) - \\ & 2\{(1-\sigma)e_{11}^{\mathrm{T}} + \sigma(e_{22}^{\mathrm{T}} + e_{33}^{\mathrm{T}})\}\varphi_{,1} - \\ & 4(1-\sigma)(e_{12}^{\mathrm{T}}\varphi_{,2} + e_{13}^{\mathrm{T}}\varphi_{,3}) + \frac{\partial}{\partial x_1}\bar{\beta} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \bar{\beta} = {} & \frac{2e_{12}^{\mathrm{T}}}{a^2-b^2}(a^2x_2\varphi_{,1} - b^2x_1\varphi_{,2}) + \\ & \frac{2e_{23}^{\mathrm{T}}}{b^2-c^2}(b^2x_3\varphi_{,2} - c^2x_2\varphi_{,3}) + \\ & \frac{2e_{31}^{\mathrm{T}}}{c^2-a^2}(c^2x_1\varphi_{,3} - a^2x_3\varphi_{,1})\,. \end{aligned}$$

u_2^{C} and u_3^{C} are found by cyclic permutation of (1, 2, 3), (a, b, c).

At an internal point

$$\varphi = \tfrac{1}{2}(a^2 - x^2)I_a + \tfrac{1}{2}(b^2 - y^2)I_b + \tfrac{1}{2}(c^2 - z^2)I_c \tag{3.2}$$

where I_a, I_b, I_c are constants depending only on the axial ratios of the ellipsoid. Consequently u_i^{C} is a linear function of the x_i and the strain and stress are uniform within the inclusion, as stated in the Introduction. The constant strains e_{ij}^{C} are linear functions of the e_{ij}^{T} and we may write

$$e_{ij}^{\mathrm{C}} = S_{ijkl}\,e_{kl}^{\mathrm{T}}\,. \tag{3.3}$$

The S_{ijkl} are symmetric in ij and in kl, but in general S_{ijkl} is different from S_{klij}. It is easy to verify that these coefficients vanish unless they are of the form S_{iiii}, S_{iijj} or S_{ijij} ($i \neq j$; no summation). That is, unlike shears are not coupled, and shears are not coupled to extensions. From (3.1) we obtain

$$
\begin{aligned}
8\pi(1-\sigma)S_{1111} &= \frac{a^2 I_a - b^2 I_b}{a^2-b^2} + \frac{a^2 I_a - c^2 I_c}{a^2-c^2} - (1-\sigma) I_a \\
8\pi(1-\sigma)S_{1122} &= -\, b^2 \frac{I_a - I_b}{a^2-b^2} - (1-2\sigma) I_a \\
8\pi(1-\sigma)S_{1212} &= -\frac{1}{2}\,\frac{a^2+b^2}{a^2-b^2}(I_a - I_b) + \tfrac{1}{2}(1-2\sigma)(I_a+I_b)\,;
\end{aligned}
\tag{3.4}
$$

the remaining coefficients may be obtained by cyclic interchange. (For an alternative way of obtaining these coefficients see ESHELBY [1957].) The rotation inside the inclusion is also constant. We have at once from (2.25) and (3.2)

$$
\begin{aligned}
4\pi\omega^{\mathrm{C}}_{12} &= (I_b - I_a)\, e^{\mathrm{T}}_{12}, \\
4\pi\omega^{\mathrm{C}}_{23} &= (I_c - I_b)\, e^{\mathrm{T}}_{23}, \\
4\pi\omega^{\mathrm{C}}_{31} &= (I_a - I_c)\, e^{\mathrm{T}}_{31}.
\end{aligned}
\tag{3.5}
$$

In their role of demagnetising factors I_a, I_b, I_c have been plotted as functions of b/a and c/a by OSBORN [1945] with the notation $I_a = L$, $I_b = M$, $I_c = N$. They may also be found from tables of elliptic integrals $F(\theta,k)$, $E(\theta,k)$ using the relations

$$
\begin{aligned}
I_a &= \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{\frac{1}{2}}}\,(F-E), \\
I_b &= 4\pi - I_a - I_c, \\
I_c &= \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{\frac{1}{2}}}\left\{\frac{b(b^2-c^2)^{\frac{1}{2}}}{ac} - E\right\} \\
k^2 &= \frac{a^2-b^2}{a^2-c^2}, \qquad \theta = \sin^{-1}\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.6}
$$

(Compare (3.7) below.)

For a point outside the ellipsoid the potential takes the form (KELLOGG [1929], MACMILLAN [1958])

$$
\varphi = \frac{2\pi abc}{l^3}\left\{\left[l^2 - \frac{x^2}{k^2} + \frac{y^2}{k^2}\right] F(\theta,k) + \left[\frac{x^2}{k^2} - \frac{y^2}{k^2 k'^2} + \frac{z^2}{k'^2}\right] E(\theta,k) + \frac{l}{k'^2}\left[\frac{C}{AB}\,y^2 - \frac{B}{AC}\,z^2\right]\right\}
\tag{3.7}
$$

where

$$
\left.
\begin{aligned}
A &= (a^2+\lambda)^{\frac{1}{2}}, \qquad B = (b^2+\lambda)^{\frac{1}{2}}, \qquad C = (c^2+\lambda)^{\frac{1}{2}} \\
l &= (a^2-c^2)^{\frac{1}{2}}, \qquad k^2 = 1 - k'^2 = \frac{a^2-b^2}{a^2-c^2}
\end{aligned}
\right\},
\tag{3.8}
$$

$$a^2 > b^2 > c^2 , \tag{3.9}$$

and F, E are elliptic integrals of modulus k and argument

$$\theta = \sin^{-1} (l/A) . \tag{3.10}$$

λ is the greatest (and in fact the only positive) root of

$$x^2/A^2 + y^2/B^2 + z^2/C^2 = 1 . \tag{3.11}$$

Equation (3.7) also gives the potential at an internal point if we put $\lambda = 0$; this gives (3.6).

To carry out the differentiations necessary to find the displacement or stress outside the inclusion one can make repeated use of

$$\partial F/\partial\lambda = -\tfrac{1}{2}l/ABC, \qquad \partial E/\partial\lambda = -\tfrac{1}{2}lB/A^3C$$

$$\partial\lambda/\partial x = 2x/Ah, \ldots, \qquad h^2 = x^2/A^4 + y^2/B^4 + z^2/C^4 .$$

In forming the first (but not the higher) derivatives of φ, λ may be treated as a constant. The condition (3.9) is not really necessary. It ensures that $0 < k^2 < 1$ and $0 < \theta < \frac{1}{2}\pi$. If it is violated $F(\theta,k), E(\theta,k)$ can be made to depend on $F(\theta_1,k_1)$, $E(\theta_1,k_1)$ with $0 < k_1^2 < 1$, $0 < \theta_1 < \frac{1}{2}\pi$ with the help of known transformations. (See, for example, BYRD and FRIEDMAN [1954].) This is useful if it becomes convenient to ignore (3.9) at a late stage in a calculation.

The results we have obtained can only be applied to the sphere after a tedious passage to the limit. However, we may use the following expressions for the potentials of a sphere of radius a:

$$\left.\begin{aligned} \varphi &= \frac{4}{3}\pi a^2 \left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{a^2}\right), && r < a \\ &= \frac{4}{3}\pi a^2 \frac{a}{r}, && r > a \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} \psi &= \frac{4}{3}\pi a^4 \left(\frac{3}{4} + \frac{1}{2}\frac{r^2}{a^2} - \frac{1}{20}\frac{r^4}{a^4}\right), && r < a \\ &= \frac{4}{3}\pi a^4 \left(\frac{1}{5}\frac{a}{r} + \frac{r}{a}\right), && r > a \end{aligned}\right\}. \tag{3.13}$$

The expression for φ is well-known. ψ may be found by integrating

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\psi}{\mathrm{d}r}\right) = 2\varphi$$

and calculating $\psi(0)$ by direct integration:

$$\psi(0) = 4\pi \int_0^a r \cdot r^2 \,\mathrm{d}r = \pi a^4 .$$

For the sphere (3.3) reduces to

$$e^{\mathrm{C}} = \alpha e^{\mathrm{T}}, \qquad 'e^{\mathrm{C}}_{ij} = \beta\, 'e^{\mathrm{T}}_{ij}$$

with

$$\alpha = \frac{1}{3}\,\frac{1+\sigma}{1-\sigma}, \qquad \beta = \frac{2}{15}\,\frac{4-5\sigma}{1-\sigma}. \tag{3.14}$$

KRÖNER [1958b] has given the S_{ijkl} (with the notation $\overset{-1}{w}_{ijkl}$) for prolate and oblate spheroids.

For some purposes it may be unnecessary to deal with the relatively complex field outside the inclusion. A knowledge of the e^{T}_{ij} alone is enough to give the field far from the inclusion (2.23) or the interaction energy with an applied field (2.43). When the numerical coefficients S_{ijkl} have been computed we can find the elastic field inside the inclusion, and also, with the help of (2.27), the field at points in the matrix immediately outside the inclusion.

The displacement (3.1) at points external to the ellipsoid may be regarded as the solution to one or other of the following boundary-value problems:

(i) To find the elastic field falling to zero at infinity and having the displacement

$$u_i = (e^{\mathrm{C}}_{ij} + \omega^{\mathrm{C}}_{ij})x_j \qquad (e^{\mathrm{C}}_{ij},\ \omega^{\mathrm{C}}_{ij}\ \text{constant})$$

over the surface of an ellipsoid.

(ii) To find the elastic field falling to zero at infinity and giving surface tractions

$$T_i = p^{\mathrm{I}}_{ij}\, n_j \qquad (p^{\mathrm{I}}_{ij}\ \text{constant})$$

on the surface of an ellipsoid.

The solution (3.1) is designed to give the constrained elastic field directly when the stress-free strain e^{T}_{ij} is known. Thus in using it to solve (i) or (ii) the first step would have to be the solution of (3.3), or (2.19) and (3.3) to find the e^{T}_{ij} appropriate to the prescribed u^{C}_i or p^{I}_{ij}.

Regarded as a solution of (i) the solution (3.1) is closely related to one already given by DANIELE [1911]. Again, if a constant stress-field $p^{\mathrm{A}}_{ij} = -p^{\mathrm{I}}_{ij}$ is superimposed on the solution of (ii) the ellipsoidal surface is free of stress and we are left with a solution representing an ellipsoidal cavity perturbing a uniform stress p^{A}_{ij}. The solution has been given by SADOWSKY and STERNBERG [1949] for the case where $p^{\mathrm{A}}_{12} = p^{\mathrm{A}}_{23} = p^{\mathrm{A}}_{31} = 0$.

Daniele determined the displacements in an infinite medium outside an ellipsoidal surface over which the displacement is required to be

$$u_i = \xi_{ij} x_j \tag{3.15}$$

with constant coefficients ξ_{ij}. He assumed that there was a solution of the form

$$\begin{aligned} u_i &= \alpha_{ij}\varphi_{,j} + \alpha_{ijkl}x_j\varphi_{,kl} \\ e &= \lambda_{ij}\varphi_{,ij} \end{aligned} \quad (\alpha_{ij}, \alpha_{ijkl}, \lambda_{ij} \text{ constant}) .$$

The α_{ij} and α_{ijkl} were determined by substituting in the equilibrium equations, equating the coefficients of $\varphi_{,ij}$ and $\varphi_{,ijk}$ to zero (subject to the conditions $\nabla^2\varphi_{,ij} = 0$, $\nabla^2\varphi_{,ijk} = 0$) and applying the boundary condition (3.15). Actually Daniele's solution is more general than (3.1) since we cannot prescribe $\overset{\mathrm{C}}{e}_{ij}$ and $\overset{\mathrm{C}}{\omega}_{ij}$ independently. If the $\overset{\mathrm{C}}{e}_{ij}$ are given the $\overset{\mathrm{T}}{e}_{ij}$ follow from (3.3) and the $\overset{\mathrm{C}}{\omega}_{ij}$ are then fixed by (3.5). *This connection between the $\overset{\mathrm{C}}{e}_{ij}$ and $\overset{\mathrm{C}}{\omega}_{ij}$ is due to the fact that there* is no external couple acting on the inclusion. Thus, in physical terms, Daniele's solution gives the field about a rigid embedded inclusion which suffers a homogeneous deformation and is then rotated from its equilibrium position by an external couple. (The case where the ellipsoid is rotated but not deformed was discussed earlier by EDWARDES [1893].) When the requirement of zero couple is imposed Daniele's solution agrees with (3.1).

Sadowsky and Sternberg presented their solution in ellipsoidal coordinates. The problem they had in mind was that of an ellipsoidal cavity perturbing a stress field which at infinity is uniform and has its principal axes parallel to those of the ellipsoid; ROBINSON [1951] and NIESEL [1953] have pointed out that it can be applied to the inclusion problem. Robinson treats the case where $\overset{\mathrm{T}}{e}_{ij}$ is a pure dilatation; Niesel considers the more general case where $\overset{\mathrm{T}}{e}_{11}$, $\overset{\mathrm{T}}{e}_{22}$, $\overset{\mathrm{T}}{e}_{33}$ are unequal and $\overset{\mathrm{T}}{e}_{12}$, $\overset{\mathrm{T}}{e}_{23}$, $\overset{\mathrm{T}}{e}_{31}$ are zero.

Sadowsky and Sternberg's solution is expressed in the form (2.20) with

$$\left.\begin{aligned} B_1 = \mathscr{C}A_1X, \quad B_2 = \mathscr{C}A_2Y, \quad B_3 = \mathscr{C}A_3Z, \\ \beta = \mathscr{C}A_4F_1 + \mathscr{C}A_5F_2 \end{aligned}\right\} . \tag{3.16}$$

(We have introduced the constant $\mathscr{C} = -2(1-\sigma)/\mu$ to make our notation agree with Sadowsky and Sternberg's.) The A_n are numerical coefficients and X, Y, Z, F_1, F_2 are harmonic functions chosen so that the boundary conditions can be satisfied. They are related to the potential φ (eq. (3.7)) as follows:

$$X = -vk\,\varphi_{,1}, \qquad Y = +vkk'^2\,\varphi_{,2}, \qquad Z = -v\,\frac{k'^2}{k}\,\varphi_{,3}$$

$$F_1 = -vk^3 f_{ii} = vk^3(2\varphi - x_i\varphi_{,i})$$

$$F_2 = \gamma_{bc} f_{11} + \gamma_{ca} f_{22} + \gamma_{ab} f_{33}$$

with

$$\gamma_{qr} = \frac{2(p-1)}{3\pi abc}(d^2 - q^2)(d^2 - r^2) \qquad (q,r = a,b,c)$$

$$v = \frac{l^3}{4\pi abc}, \qquad d^2 = a^2 - 2\,\frac{a^2 - b^2}{p(1+k^2)}$$

$$p = 2 + \frac{2(k'^2 + k^4)^{\frac{1}{2}}}{1 + k^2}.$$

For completeness we have given the values of the numerical coefficients connecting φ and X, Y, Z, F_1, F_2. We note that f_{11}, f_{22}, f_{33} (eq. (2.22)) only enter the solution through the two particular linear combinations F_1 and F_2. This is a result of Sadowsky and Sternberg's requirement that F_1 and F_2 (and also X, Y, Z) shall take the form

$$f(\alpha_1)\, g(\alpha_2)\, h(\alpha_3) \tag{3.17}$$

when expressed in terms of their ellipsoidal coordinates $\alpha_1, \alpha_2, \alpha_3$. Thus the A_n cannot be found by direct comparison of (3.17) and (2.21). For example, if we choose A_4 and A_5 so that the coefficients of f_{11} and f_{22} agree with (2.7), the coefficient of f_{33} is fixed and so $\overset{\mathrm{T}}{p}_{33}$ cannot be prescribed at will. This could be remedied by adding to β a further harmonic function, $\mathscr{C}A_5'F_2'$ say, where F_2' is derived from F_2 by changing the sign of the radical in the equation defining p. (Cf. SADOWSKY and STERNBERG [1949] eq. (41).) There is, however, no need to do this. In a representation such as (2.20) it is possible to modify B_i and β simultaneously and leave $\overset{\mathrm{C}}{u}_i$ unchanged. Sadowsky and Sternberg give a matrix relation (their eq. (50)) which enables the A_n to be determined in such a way that the resulting stress annuls the surface tractions on the ellipsoidal surface due to uniform stresses $p_{11} = \sigma_1,\ p_{22} = \sigma_2,\ p_{33} = \sigma_3$.

According to NIESEL [1953] it is convenient to include the solution F_2' in β if we wish to pass to the limiting case of a spheroid. To extend Sadowsky and Sternberg's solution to the case where the principal axes of the stress at infinity are not parallel to the axes of the cavity (or to apply it to the inclusion problem with non-vanishing $\overset{\mathrm{T}}{e}_{12}, \overset{\mathrm{T}}{e}_{23}, \overset{\mathrm{T}}{e}_{31}$) it would be necessary to add to β terms $\mathscr{C}A_6(f_{12} + f_{21})$, $\mathscr{C}A_7(f_{23} + f_{32})$, $\mathscr{C}A_8(f_{31} + f_{13})$. These terms have the required form (3.17) when transcribed into ellipsoidal coordinates.

The use of ellipsoidal coordinates does not seem to offer much advantage. As we have seen, the problem may be set up and solved formally in Cartesian coordinates. To find the elastic field at a given point (x, y, z) outside the ellipsoid we have to solve the cubic (3.11) for λ. We appear to be spared this when ellipsoidal coordinates are used. But, in fact, we must have already prepared an ellipsoidal coordinate network in order to be able to locate points relative to the ellipsoid, and, further, a different network is required for each value of the axial ratio b/a.

LURIE [1952] has given a solution to Sadowsky and Sternberg's problem in the disconcertingly simple form

$$B_1 = (M/a^2)\varphi_{,1}\,, \qquad B_2 = (M/b^2 - N)\varphi_{,2}\,, \qquad B_3 = (M/c^2 - N)\varphi_{,3}$$
$$\beta = N(x_i\varphi_{,i} - 2\varphi) + P\varphi$$

where M, N, P are disposable constants. (Our M, N, P differ from Lurie's by a common multiplicative factor.) Unfortunately the contributions to u_i^{C} from the terms in N and P are both of the form const. $\varphi_{,i}$. (This illustrates the fact mentioned above that different choices of B_i,β can give the same elastic field.) Consequently there are really only two disposable constants, and so only two of p_{11}^{T}, p_{22}^{T}, p_{33}^{T} (or, in the cavity problem, σ_1, σ_2, σ_3) can be prescribed independently.

3.2. THE INHOMOGENEOUS INCLUSION

So far we have been concerned with a homogeneous inclusion, that is, one which has the same elastic constants as the matrix. We consider next the case where the elastic constants of matrix and inclusion are different. As in § 2 we imagine that the inclusion undergoes a transformation specified by a uniform stress-free strain whilst constrained by the matrix, and try to calculate the resulting elastic field. This must be distinguished from the problem considered in § 4 below. There we have, in effect, a *perfectly fitting* inhomogeneous insertion cemented into a cavity in the matrix, and consequently the material is everywhere stress-free in the absence of applied forces.

We have seen that (2.26) still applies when the bulk moduli of matrix and inclusion differ. This seems to be the only simple statement one can make for an inclusion of arbitrary shape. For the ellipsoid, on the other hand, we may solve the general problem very simply by taking advantage of the fact that the stress is constant inside a homogeneous ellipsoidal inclusion (ROBINSON [1951], NIESEL [1953], ESHELBY [1957]).

For brevity let E denote the ellipsoidal inclusion we have been considering hitherto. It has the same elastic constants λ, μ as the matrix, and it has suffered a permanent change of shape characterized by the stress-free strain $\overset{\mathrm{T}}{e}_{ij}$ while embedded in the matrix. The strain $\overset{\mathrm{C}}{e}_{ij}$ relates its final form to its form before transformation. Take a second ellipsoid E^* which to begin with has the same form as E had before its transformation and which has elastic constants λ^*, μ^*. Let E^* undergo a stress-free strain $\overset{\mathrm{T*}}{e}_{ij}$. To E^* apply surface tractions chosen so as to produce a uniform elastic strain $\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{C*}}{e}_{ij}$ in it. It then has precisely the same form as the embedded inclusion E. If this treatment should happen also to produce in E^* stresses identical with those in E we can replace E by E^* without upsetting the continuity of displacement and surface traction across the interface. The condition for this to be possible is (cf. eq. (2.19))

$$\begin{aligned} \overset{\mathrm{I}}{p}_{ij} &= \lambda(\overset{\mathrm{C}}{e} - \overset{\mathrm{T}}{e})\delta_{ij} + 2\mu(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T}}{e}_{ij}) \\ &= \lambda^*(\overset{\mathrm{C}}{e} - \overset{\mathrm{T*}}{e})\delta_{ij} + 2\mu^*(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T*}}{e}_{ij}). \end{aligned} \tag{3.18}$$

When the values of λ^*, μ^* and $\overset{\mathrm{T*}}{e}_{ij}$ for the inhomogeneous inclusion are given we can solve (3.18) for the $\overset{\mathrm{T}}{e}_{ij}$. The elastic field inside and outside the inhomogeneous inclusion is then identical with that of a homogeneous inclusion with the stress-free strain $\overset{\mathrm{T}}{e}_{ij}$.

To find $\overset{\mathrm{T}}{e}_{ij}$ for the equivalent homogeneous inclusion we use (3.3) to express $\overset{\mathrm{C}}{e}_{ij}$ in terms of $\overset{\mathrm{T}}{e}_{ij}$. For the non-diagonal components we have simply

$$\overset{\mathrm{T}}{e}_{12} = \frac{\mu^*}{2(\mu^* - \mu)S_{1212} + \mu}\,\overset{\mathrm{T*}}{e}_{12}, \qquad \overset{\mathrm{T}}{e}_{23} = \ldots .$$

To find $\overset{\mathrm{T}}{e}_{11}$, $\overset{\mathrm{T}}{e}_{22}$, $\overset{\mathrm{T}}{e}_{33}$ we have to solve the set of three simultaneous equations

$$(\lambda^* - \lambda)\overset{\mathrm{C}}{e} + \lambda\overset{\mathrm{T}}{e} + 2(\mu^* - \mu)S_{ijkl}\overset{\mathrm{T}}{e}_{kl} = \lambda^*\overset{\mathrm{T*}}{e} + 2\mu^*\overset{\mathrm{T*}}{e}_{ij}$$
$$(ij = 11, 22, 33)\,.$$

Only $\overset{\mathrm{T}}{e}_{11}$, $\overset{\mathrm{T}}{e}_{22}$, $\overset{\mathrm{T}}{e}_{33}$ enter the kl-summation, and in the first term we have

$$\overset{\mathrm{C}}{e} = \frac{1-2\sigma}{4\pi(1-\sigma)}(I_a\overset{\mathrm{T}}{e}_{11} + I_b\overset{\mathrm{T}}{e}_{22} + I_c\overset{\mathrm{T}}{e}_{33}) + \frac{\sigma}{1-\sigma}\overset{\mathrm{T}}{e} \tag{3.19}$$

by (2.24) and (3.2).

It is a simple matter to calculate the elastic energy E^*_∞ associated with an inhomogeneous (ellipsoidal) inclusion in an infinite matrix. For the equivalent homogeneous inclusion we have from (2.32)

$$E_\infty = -\tfrac{1}{2}V\overset{\mathrm{I}}{p}_{ij}\,\overset{\mathrm{T}}{e}_{ij} \qquad (V = \tfrac{4}{3}\pi abc)\,. \tag{3.20}$$

The energy in the matrix is the same for E and E^*. The internal stress is $\overset{\mathrm{I}}{p}_{ij}$ for both, but the effective strain is $\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T}}{e}_{ij}$ for E and $\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T*}}{e}_{ij}$ for E^*. Thus we have to add $\frac{1}{2}V\overset{\mathrm{I}}{p}_{ij}(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T*}}{e}_{ij}) - \frac{1}{2}V\overset{\mathrm{I}}{p}_{ij}(\overset{\mathrm{C}}{e}_{ij} - \overset{\mathrm{T}}{e}_{ij})$ to (3.20), which gives the simple result (cf. ROBINSON [1951])

$$E^*_\infty = -\tfrac{1}{2}V\overset{\mathrm{I}}{p}_{ij}\,\overset{\mathrm{T*}}{e}_{ij}. \tag{3.21}$$

The results of this section may be adapted to the case of an ellipsoidal cavity containing a fluid under pressure. If the pressure P of the fluid is prescribed it is only necessary to put $\overset{\mathrm{I}}{p}_{ij} = -P\delta_{ij}$ in (3.18) and solve for the $\overset{\mathrm{T}}{e}_{ij}$ or $\overset{\mathrm{T*}}{e}_{ij}$. If instead we are given the excess volume v of fluid introduced (measured at zero pressure) we have to put $e^{\mathrm{T}} = v$ and require $\overset{\mathrm{I}}{p}_{ij}$ to have the form $\frac{1}{3}p^{\mathrm{I}}\delta_{ij}$. The solution of (3.18) then gives p^{I}, $'\overset{\mathrm{T}}{e}_{ij}$ and $\overset{\mathrm{T*}}{e}_{ij}$.

§ 4. The Ellipsoidal Inhomogeneity

4.1. THE ELASTIC FIELD

In this section we shall take up the second problem mentioned in the introduction. It may be formulated as follows:

An infinite solid has elastic constants λ^*, μ^* inside a region bounded by a closed surface S (the 'inhomogeneity') and elastic constants λ, μ in the region outside S (the 'matrix'). To find the elastic field everywhere when the strain is required to reduce to the constant value $\overset{\mathrm{A}}{e}_{ij}$ far from S.

Although the problem can only be solved in detail for an ellipsoidal inhomogeneity it is convenient to start from the case where the form of the inhomogeneity is arbitrary. The problem can be reduced to the determination of the elastic field produced by a certain layer of body-force distributed over S. To see this, suppose that the strain $\overset{\mathrm{A}}{e}_{ij}$ is impressed throughout the medium. The displacement is then

$$\overset{\mathrm{A}}{u}_i = \overset{\mathrm{A}}{e}_{ij}x_j$$

plus an inessential rigid-body displacement. The equilibrium equations are satisfied inside and outside S. The traction on the inner boundary of the matrix is $(\lambda e^{\mathrm{A}}\delta_{ij} + 2\mu\overset{\mathrm{A}}{e}_{ij})n_j$, but the traction on the outer surface of the inhomogeneity is $(\lambda^* e^{\mathrm{A}}\delta_{ij} + 2\mu^*\overset{\mathrm{A}}{e}_{ij})n_j$. Consequently the required state of strain can only be maintained if there is a layer of body-force of surface density $\{(\lambda^* - \lambda)e^{\mathrm{A}}\delta_{ij} + 2(\mu^* - \mu)\overset{\mathrm{A}}{e}_{ij}\}n_j$ spread over S. To find the actual elastic field we apply an equal and opposite layer of body force of surface density

$$T_i = \{(\lambda - \lambda^*)\overset{\mathrm{A}}{e}\delta_{ij} + 2(\mu - \mu^*)\overset{\mathrm{A}}{e}_{ij}\}n_j \tag{4.1}$$

and calculate the displacement $\overset{\mathrm{C}}{u}_i$ which it induces in the medium. The final displacement is then

$$u_i = \overset{\mathrm{A}}{e}_{ij}\, x_j + \overset{\mathrm{C}}{u}_i\,.$$

$\overset{\mathrm{C}}{u}_i$ is evidently given by the expression

$$\overset{\mathrm{C}}{u}_i(\boldsymbol{r}) = \{(\lambda - \lambda^*)\overset{\mathrm{A}}{e}\delta_{kj} + 2(\mu - \mu^*)\overset{\mathrm{A}}{e}_{kj}\}\int_{\mathrm{S}} U_{ik}(\boldsymbol{r}, \boldsymbol{r}')\, \mathrm{dS}_j\,, \tag{4.2}$$

where $U_{ik}(\boldsymbol{r}, \boldsymbol{r}')$ is the i-component of the displacement at $\boldsymbol{r}$ when a unit point-force is applied at $\boldsymbol{r}'$ parallel to the x_k-axis. Because the medium is inhomogeneous U_{ik} depends on $\boldsymbol{r}$ and $\boldsymbol{r}'$ separately and not simply on $|\boldsymbol{r} - \boldsymbol{r}'|$ as does the corresponding quantity in (2.4). It is not possible to determine U_{ik} for an arbitrary form of S, and hence a transformation of (9.2) corresponding to the step from (2.6) to (2.7) cannot be made. However, we shall find that the formulation in terms of a layer of force is useful in deriving certain energy relations.

When the inhomogeneity has the form of an ellipsoid the solution can be found from the solution for the ellipsoidal inclusion by making use of the fact that the stress in the inclusion is uniform. For the special case of a cavity ($\lambda^* = 0$, $\mu^* = 0$) the method has already been outlined in § 1. The general ellipsoidal inhomogeneity is handled similarly. On the elastic field of a homogeneous inclusion with stress-free strain $\overset{\mathrm{T}}{e}_{ij}$ superimpose a uniform strain $\overset{\mathrm{A}}{e}_{ij}$. Let

$$\overset{\mathrm{A}}{p}_{ij} = \lambda \overset{\mathrm{A}}{e}\delta_{ij} + 2\mu \overset{\mathrm{A}}{e}_{ij}$$

be the corresponding stress.The stress in the inclusion is now

$$\overset{\mathrm{inc}}{p}_{ij} = \overset{\mathrm{I}}{p}_{ij} + \overset{\mathrm{A}}{p}_{ij} = \overset{\mathrm{C}}{p}_{ij} - \overset{\mathrm{T}}{p}_{ij} + \overset{\mathrm{A}}{p}_{ij}\,,$$

and the strain in the inclusion is

$$\overset{\mathrm{inc}}{e}_{ij} = \overset{\mathrm{C}}{e}_{ij} + \overset{\mathrm{A}}{e}_{ij}\,.$$

On account of the term $-\overset{\mathrm{T}}{p}_{ij}$ in (2.19) (which appears because there is no stress associated with the stress-free transformation strain $\overset{\mathrm{T}}{e}_{ij}$) $\overset{\mathrm{inc}}{p}_{ij}$ and $\overset{\mathrm{inc}}{e}_{ij}$ are not related by Hooke's law for material with elastic constants λ, μ. They are, however, related by Hooke's law for a material with constants λ^*, μ^* provided these satisfy

$$\overset{\mathrm{inc}}{p}_{ij} = \lambda^* \overset{\mathrm{inc}}{e}\, \delta_{ij} + 2\mu^* \overset{\mathrm{inc}}{e}_{ij}\,,$$

that is, if

$$(\lambda-\lambda^*)(e^{\mathrm{C}}+e^{\mathrm{A}})\delta_{ij}+2(\mu-\mu^*)(e_{ij}^{\mathrm{C}}+e_{ij}^{\mathrm{A}}) = (\lambda e^{\mathrm{T}}\delta_{ij}+2\mu e_{ij}^{\mathrm{T}})\equiv p_{ij}^{\mathrm{T}}. \quad (4.3)$$

An ellipsoid made of material with these constants can be used to replace the transformed inclusion with continuity of stress and displacement across the interface provided that in its unstressed state this ellipsoid coincides in shape and size with the untransformed inclusion. This replacement does not alter the stresses inside or outside the ellipsoid; they remain the same as the stresses due to a homogeneous transformed inclusion with stress-free strain e_{ij}^{T}, together with the original applied stress p_{ij}^{A}.

The argument has been presented as if e_{ij}^{A} and e_{ij}^{T} were given and λ^*, μ^* were to be found. In the actual inhomogeneity problem λ^*, μ^* and e_{ij}^{A} (or p_{ij}^{A}) are given and we have to determine e_{ij}^{T}. To do this we express p_{ij}^{C} and e_{ij}^{C} in terms of S_{ijkl} and e_{kl}^{T} and substitute in (4.3). This gives

$$(\lambda^*-\lambda)S_{mmkl}\,e_{kl}^{\mathrm{T}}\,\delta_{ij}+2(\mu^*-\mu)S_{ijkl}\,e_{kl}^{\mathrm{T}}+\lambda e^{\mathrm{T}}\delta_{ij}+2\mu e_{ij}^{\mathrm{T}}$$
$$= (\lambda-\lambda^*)e^{\mathrm{A}}\delta_{ij}+2(\mu-\mu^*)e_{ij}^{\mathrm{A}}. \quad (4.4)$$

As in the case of (3.18) the solution for the non-diagonal e_{ij}^{T} is immediate,

$$e_{12}^{\mathrm{T}} = \frac{\mu-\mu^*}{2(\mu^*-\mu)S_{1212}+\mu}\,e_{12}^{\mathrm{A}}, \qquad e_{23}^{\mathrm{T}} = \ldots,$$

while for e_{11}^{T}, e_{22}^{T}, e_{33}^{T} we have the three simultaneous equations

$$(\lambda^*-\lambda)e^{\mathrm{C}}+2(\mu^*-\mu)S_{ijkl}\,e_{kl}^{\mathrm{T}}+\lambda e^{\mathrm{T}}+2\mu e_{ij}^{\mathrm{T}} = (\lambda-\lambda^*)e^{\mathrm{A}}+2(\mu-\mu^*)e_{ij}^{\mathrm{A}} \quad (4.5)$$

(ij = 11, 22, 33) with the value (3.19) for e^{C}.

For a sphere (4.4) reduces to

$$e^{\mathrm{T}} = Ae^{\mathrm{A}}, \qquad {}'e_{ij}^{\mathrm{T}} = B\,{}'e_{ij}^{\mathrm{A}} \quad (4.6)$$

where

$$A = A\{\varkappa, \varkappa^*\} = \frac{\varkappa^*-\varkappa}{(\varkappa-\varkappa^*)\alpha-\varkappa},$$
$$B = B\{\mu, \mu^*\} = \frac{\mu^*-\mu}{(\mu-\mu^*)\beta-\mu} \quad (4.7)$$

with the values (3.14) for α, β.

The e_{ij}^{T} found in this way is the stress-free strain of an 'equivalent homogeneous inclusion' from which the elastic field can be calculated. Note that e_{ij}^{T} goes to zero with e_{ij}^{A}. Consequently the solution does correspond to a perfectly fitting inhomogeneous ellipsoid in a body which is stress-free when not acted on by external forces.

The displacement inside and outside the ellipsoid is

$$u_i = u_i^{\mathrm{A}} + u_i^{\mathrm{C}} . \tag{4.8}$$

The term u_i^{A} represents the unperturbed displacement; it is equal to $e_{ij}^{\mathrm{A}} x_j$ plus an arbitrary rigid-body displacement. The term u_i^{C} represents the perturbation due to the presence of the ellipsoid; it is calculated from (3.1) with the e_{ij}^{T} of the equivalent inclusion. The stress is

$$p_{ij} = p_{ij}^{\mathrm{A}} + p_{ij}^{\mathrm{C}}$$

outside the ellipsoid and

$$p_{ij} = p_{ij}^{\mathrm{A}} + p_{ij}^{\mathrm{I}}$$

inside it. The form

$$p_{ij} = p_{ij}^{\mathrm{A}} + \lambda e^{\mathrm{C}} \delta_{ij} + 2\mu e_{ij}^{\mathrm{C}}$$

is valid inside and outside the ellipsoid; in calculating the interior field we use the 'wrong' elastic constants λ, μ in place of λ^*, μ^* and this compensates for the change from p_{ij}^{C} to p_{ij}^{I}.

It is perhaps worth mentioning that the results of this section and of § 3.2 for the inhomogeneous inclusion can be generalized to the case where the matrix is isotropic but the interior of the ellipsoid is anisotropic and has elastic constants $\overset{*}{c}_{ijkl}$, say. For the transformation problem it is only necessary to replace (3.18) by

$$\lambda(e^{\mathrm{C}} - e^{\mathrm{T}})\delta_{ij} + 2\mu(e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}}) = \overset{*}{c}_{ijkl} \, (e_{ij}^{\mathrm{C}} - e_{ij}^{\mathrm{T}*}) . \tag{4.9}$$

For the inhomogeneity problem (4.4) becomes

$$\lambda(S_{mmkl} \, e_{kl}^{\mathrm{T}} - e^{\mathrm{T}} + e^{\mathrm{A}})\delta_{ij} + 2\mu(S_{ijkl} \, e_{kl}^{\mathrm{T}} - e_{ij}^{\mathrm{T}} + e_{ij}^{\mathrm{A}})$$
$$= \overset{*}{c}_{ijpq} \, (S_{pqkl} \, e_{kl}^{\mathrm{T}} + e_{pq}^{\mathrm{A}}) . \tag{4.10}$$

By solving (4.9) or (4.10) the e_{ij}^{T} of the equivalent inclusion can be found.

The case of an anisotropic ellipsoid in an isotropic medium may seem rather artificial. It finds an application, however, in the theory of aggregates of anisotropic crystals (cf. § 6).

The argument leading to the expression (2.6) for the displacement u_i^{C} due to a transformed inclusion of any shape applies equally to an anisotropic material. To obtain an explicit expression for u_i^{C} analogous to (2.7) we should have to know the form of the displacement due to a point force in an anisotropic material. Unfortunately an expression explicit enough for our purpose cannot be obtained (FREDHOLM [1900], LIFSHITZ and ROSENZWEIG [1947], KRÖNER [1953b]). It is, however, possible to carry the analysis far enough to show that the stress is uniform inside a transformed anisotropic ellipsoid in an anisotropic matrix (ESHELBY [1957]).

We may give a picturesque interpretation to (4.8) by saying that the applied field 'induces' an inclusion in the inhomogeneity, with a stress-free strain e_{ij}^{T} proportional to the applied stress (cf. KRÖNER [1958a]).

If the inhomogeneity is in a finite body we may write the perturbing field of the inclusion in the form

$$u_i^{\mathrm{F}} = u_i^{\mathrm{C}} + u_i^{\mathrm{im}}, \qquad p_{ij}^{\mathrm{F}} = p_{ij}^{\mathrm{C}} + p_{ij}^{\mathrm{im}}$$

as in § 2.1, with the image terms chosen so that the boundary conditions on the outer surface S_0 of the body are satisfied. They are not, of course, quite identical with the image terms for the equivalent inclusion in a homogeneous medium.

4.2. Energy relations

It is sometimes necessary to compare the elastic behaviour of a body containing an inhomogeneity with the behaviour of a similar body which is homogeneous. In such cases it is convenient to imagine that we have a single body and that the inhomogeneity may be introduced or removed at will.

When an inhomogeneity is introduced into a body already stressed by some external mechanism there will be a change in its elastic energy. At the same time there may be a change in the potential energy of the loading mechanism. As in § 2.2 we can define a change in total energy (or in enthalpy or Gibbs free energy). We shall calculate this quantity for an ellipsoidal inhomogeneity in a body subjected to two types of loading, rigidly imposed surface displacements and constant surface tractions.

Consider first the case where a constant displacement is imposed on the outer boundary by a perfectly rigid external mechanism, producing uniform stress and strain p_{ij}^{A}, e_{ij}^{A} in the absence of the inhomogeneity.

The perturbing field when the inhomogeneity is introduced must thus satisfy the condition

$$u_i^{\mathrm{F}} = u_i^{\mathrm{C}} + u_i^{\mathrm{im}} = 0 \quad \text{on} \quad \mathrm{S}_0 \,. \tag{4.11}$$

The elastic energy E_{A} in the medium when the inclusion is absent is found by integrating the (constant) energy density over the whole volume of material. Let the inhomogeneity be introduced and, as in § 4.1, let the material be held in a state of uniform strain e_{ij}^{A} by a layer of body force $-T_i$ (eq. (4.9)). The elastic energy is found by replacing λ, μ by λ^*, μ^* in that part of the volume integral for E_{A} which refers to the interior of S. Thus at this stage the elastic energy is $E_{\mathrm{A}} + W_1$ where

$$W_1 = \tfrac{1}{2}V[(\lambda^* - \lambda)e^{\mathrm{A}}\delta_{ij} + 2(\mu^* - \mu)e_{ij}^{\mathrm{A}}]e_{ij}^{\mathrm{A}} \,.$$

If the layer of force $-T_i$ is relaxed to zero each element dS of S suffers a displacement u_i^F and an amount of energy

$$W_2 = \tfrac{1}{2}\int_{\mathrm{S}} T_i u_i^{\mathrm{F}} \mathrm{dS}$$

is withdrawn from the medium. By Gauss's theorem and (4.11) this can be put in the form

$$W_2 = \tfrac{1}{2}V[(\lambda - \lambda^*)e^{\mathrm{A}}\delta_{ij} + 2(\mu - \mu^*)e_{ij}^{\mathrm{A}}]e_{ij}^{\mathrm{C}} - E_{\mathrm{im}}$$

where

$$E_{\mathrm{im}} = \tfrac{1}{2}[(\lambda - \lambda^*)e^{\mathrm{A}}\delta_{ij} + 2(\mu - \mu^*)e_{ij}^{\mathrm{A}}]\int_V e_{ij}^{\mathrm{im}}\,\mathrm{d}v \,.$$

The increase in the elastic energy when the inhomogeneity is introduced is thus

$$\Delta E_{\mathrm{el}} = W_1 - W_2 \,.$$

This is also the increase in the total energy, ΔE_{tot}, since the rigid loading mechanism does no work. With the help of (4.3) we have

$$W_1 - W_2 = -\tfrac{1}{2}Vp_{ij}^{\mathrm{A}}e_{ij}^{\mathrm{T}} + E_{\mathrm{im}} \,.$$

We consider next the case where the body is loaded by constant surface tractions $p_{ij}^{\mathrm{A}}n_j$. The perturbing field due to the introduction of the inhomogeneity must now satisfy

$$p_{ij}^{\mathrm{F}}n_j = (p_{ij}^{\mathrm{C}} + p_{ij}^{\mathrm{im}})n_j = 0 \quad \text{on} \quad \mathrm{S}_0 \,.$$

The elastic energy when the inhomogeneity is introduced and the layer of force $-T_i$ is present is again given by $E_{\mathrm{A}} + W_1$, and W_2 still represents the energy removed on relaxing the layer of force. But in addition there will now be a movement of the outer boundary in which the surface tractions do an amount of work

$$W_3 = \int_{S_0} \overset{\mathrm{A}}{p}_{ij} \overset{\mathrm{F}}{u}_i \, \mathrm{d}S_j$$

on the body. Thus

$$\Delta E_{\mathrm{el}} = W_1 - W_2 + W_3 .$$

If we are interested in the change in total energy we do not need to know the value of W_3; evidently W_3 also represents the decrease in the energy of the loading mechanism and so

$$\Delta E_{\mathrm{tot}} = \Delta E_{\mathrm{el}} - W_3 = W_1 - W_2 .$$

It is in fact not difficult to establish the relation

$$W_3 = 2W_2 - W_1 .$$

We make use of the same device as in (2.39) and write

$$W_3 = \int_{S_0} (\overset{\mathrm{A}}{p}_{ij} \overset{\mathrm{F}}{u}_i - \overset{\mathrm{F}}{p}_{ij} \overset{\mathrm{A}}{u}_i) \mathrm{d}S_j , \tag{4.12}$$

change the surface of integration from S_0 to S and convert to a volume integral over the inhomogeneity. The relation (4.12) then follows if we note that in the inhomogeneity

$$\overset{\mathrm{A}}{p}_{ij} = \lambda \overset{\mathrm{A}}{e} \delta_{ij} + 2\mu \overset{\mathrm{A}}{e}_{ij}$$

by definition, but that, contrary to the general rule laid down in § 1

$$\overset{\mathrm{im}}{p}_{ij} = \lambda^* \overset{\mathrm{im}}{e} \delta_{ij} + 2\mu^* \overset{\mathrm{im}}{e}_{ij}$$

since $\overset{\mathrm{im}}{p}_{ij}$, $\overset{\mathrm{im}}{e}_{ij}$ represent the actual image field in the inhomogeneity.

If we introduce the notation

$$E_{\mathrm{int}} = -\tfrac{1}{2} V \overset{\mathrm{A}}{p}_{ij} \overset{\mathrm{T}}{e}_{ij} + E_{\mathrm{im}} \tag{4.13}$$

these results may be summarised as follows:
(i) for a rigidly-held boundary

$$\Delta E_{\mathrm{tot}} = E_{\mathrm{int}}, \qquad \Delta E_{\mathrm{el}} = \Delta E_{\mathrm{tot}} , \tag{4.14}$$

(ii) for a boundary subject to constant loads

$$\Delta E_{\mathrm{tot}} = E_{\mathrm{int}}, \qquad \Delta E_{\mathrm{el}} = -\Delta E_{\mathrm{tot}} . \tag{4.15}$$

The difference in sign between (4.14) and (4.15) may be illustrated by considering the case where the inhomogeneity takes the form of a crack, that is, a narrow region in which the elastic constants are zero. Introduction of the crack into a body strained by rigidly-imposed surface displacements will obviously reduce the elastic energy. On the other hand the presence of the crack makes the body more 'flexible'. Consequently given applied loads will deform it more, do more work on it and so increase its elastic energy.

The first term in (4.13) is precisely half the interaction energy (2.43) for the equivalent homogeneous inclusion. The image term E_{im} will have a different value in (i) and (ii) since e_{ij}^{im} is derived from different boundary conditions. If the inhomogeneity is far from the boundaries it will be small compared with $-\frac{1}{2}Vp_{ij}^{\mathrm{A}}e_{ij}^{\mathrm{T}}$, and we may say that provided the initial stress p_{ij}^{A} is the same in each case, the total energy changes in cases (i) and (ii) are the same, but that the changes in elastic energy are equal and opposite. It is possible to extend the analysis to the mixed case where a constant displacement is imposed on part of S_0 and a constant load on the remainder. As one might expect, ΔE_{tot} is still given by (4.14) (with the appropriate e_{ij}^{im} inserted in E_{im}), but no general statement can be made about the relation between ΔE_{el} and ΔE_{tot}. This is no drawback since it is ΔE_{tot} which is the physically important quantity.

§ 5. Relation to the Theory of Dislocations

There is a close relation between the inclusion problem and the theory of dislocations, more particularly with the general type of dislocation introduced by SOMIGLIANA [1914, 1915]). It will be convenient, however, to begin with the more familiar type of dislocation which plays a part in the physical theory of plasticity (NABARRO [1952], SEEGER [1955]). For want of a better name we may refer to these as VOLTERRA [1907] dislocations, though in fact they only correspond to the first three of his six classes. For our purposes a Volterra dislocation may be defined as a state of self-stress in which the displacement has discontinuity b_i, the Burgers vector, across a surface bounded by an open or closed curve, the dislocation line. If the dislocation line forms a closed loop and lies in a plane, the dislocation is characterised by giving the form and orientation of the loop and the value of the Burgers vector.

The displacement at large distances from a Volterra dislocation loop of area A, Burgers vector b_i and normal n_i situated at the origin is

$$u_i = \frac{A b_j n_k g_{ijk}}{8\pi(1-\sigma)r^2} \tag{5.1}$$

with the notation of (2.23) (BURGERS [1939], NABARRO [1951]). This can be put in the alternative form

$$u_i = \frac{A b_j n_k}{8\pi(1-\sigma)} \vartheta_{ijk}\, r \tag{5.2}$$

where

$$\vartheta_{ijk} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} - \left\{\sigma \delta_{jk} \frac{\partial}{\partial x_i} + (1-\sigma)\delta_{ij} \frac{\partial}{\partial x_k} + (1-\sigma)\delta_{ik} \frac{\partial}{\partial x_j}\right\} \nabla^2 . \tag{5.3}$$

The limiting process

$$A \to 0, \quad b_j \to \infty, \quad A b_j \to s_j \tag{5.4}$$

gives an elementary infinitesimal dislocation loop of strength s_j and normal n_k. The displacement at any distance from it is given by (5.1) or (5.2) if we identify $A b_j$ with s_j.

Comparison of (5.1) with (2.23) shows that the remote field of a (finite) dislocation loop is the same as the remote field of an inclusion of arbitrary shape whose volume and stress-free strain satisfy

$$V e_{ij}^{\mathrm{T}} = \tfrac{1}{2}(b_i n_j + b_j n_i) . \tag{5.5}$$

To an elementary dislocation loop there corresponds an elementary inclusion. It is natural to imagine it as a platelet coinciding with the loop. Its thickness and stress-free strain may be so chosen that in the limit (5.5) agrees with (5.4). A finite plane dislocation loop may be formed by spreading elementary loops over a plane surface. When each loop is replaced by an equivalent elementary inclusion we obtain an inclusion in the form of a thin disc, and the elastic field which it produces is the same as that of the dislocation. It is, in fact, possible to build up directly an expression for the elastic field of a finite plane dislocation loop starting from an inclusion in the form of a disc of small (and ultimately vanishing) thickness. The quantity B_i (2.7) is then the potential of a plane lamina, and the f_{ij} (2.22) are the potentials of certain double layers. The discontinuities in potential and attraction on crossing them can be found by elementary electrostatics, and the e_{ij}^{T} may then be so adjusted that there is a discontinuity b_i in the displacement u_i^{C} between points on opposite faces of the disc. This procedure, however, does not completely determine the e_{ij}^{T}; it is necessary to impose further conditions to ensure, for example, that there is not a line of dilatation running round the rim of the disc.

We turn now to the more fruitful connection between inclusions and the general Somigliana dislocation. A Somigliana dislocation can be constructed as follows. Make a cut over a surface S (open or closed) and give the two faces of the cut an arbitrary small relative displacement, removing material where there would be interpenetration. Fill in any gaps and weld the material together again. Let $b_i(\boldsymbol{r})$ be the relative displacement at the point $\boldsymbol{r}$ on the cut. There are several ways of calculating the resulting state of self-stress. The simplest is to regard the Somigliana dislocation as equivalent to a distribution of elementary dislocations over S, the strength of the loop associated with the element of area $\mathrm{dS}(\boldsymbol{r})$ being $b_j(\boldsymbol{r})\mathrm{dS}$. By (5.2) the displacement is

$$u_i^{\mathrm{D}} = \frac{1}{8\pi(1-\sigma)} \vartheta_{ijk} I_{jk} \tag{5.6}$$

where

$$I_{jk} = \int_{\mathrm{S}} b_j(\boldsymbol{r}')\,|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{dS}_k.$$

This can be shown to be equivalent to the expression given by Somigliana [1915].

To see the connection with the inclusion problem suppose that a cut is made over the surface S of the Somigliana dislocation. The faces spring apart leaving a gap $b_i(\boldsymbol{r})$ (in some places the 'gap' may in fact be an interpenetration of material). As we saw in § 2.1 the inclusion behaves in just this fashion. If we make a cut over the interface a gap

$$b_j(\boldsymbol{r}) = e_{jl}^{\mathrm{T}} x_l \tag{5.7}$$

appears. (To (5.7) we might possibly add a rigid body displacement; for the moment we ignore it.) Consequently the inclusion is equivalent to a Somigliana dislocation in which S is a closed surface coinciding with the interface and having the discontinuity (5.7). We should expect the displacement calculated from (5.7) and (5.6) to be not u_i^{C} but rather the quantity which, in anticipation, we called u_i^{D} in § 2.1. That is, it should coincide with u_i^{C} outside the inclusion and differ from u_i^{C} by the displacement $e_{il}^{\mathrm{T}} x_l$ inside the inclusion. This may be verified analytically. With (5.7) we have, using Gauss's theorem,

$$\begin{aligned} I_{jk} &= e_{jl}^{\mathrm{T}} \int_{\mathrm{S}} x_l' \,|\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{dS}_k = e_{jl}^{\mathrm{T}} \int_V \frac{\partial}{\partial x_k'} \{x_l' |\boldsymbol{r}-\boldsymbol{r}'|\}\,\mathrm{d}v \\ &= e_{jk}^{\mathrm{T}}\, \psi + e_{jl}^{\mathrm{T}} X_{lk} \end{aligned}$$

where ψ is the biharmonic potential (2.9) and

$$X_{lk} = \int_V x_l' \frac{\partial}{\partial x_k'} |\boldsymbol{r}-\boldsymbol{r}'|\,\mathrm{d}v\,.$$

When inserted in (5.6) the term in ψ gives u_i^{C} in the form (2.8). It can be verified that

$$\vartheta_{ijk} X_{lk} = 2(1-\sigma)\delta_{ij} \nabla^2 \int_V \frac{x_l'}{|r-r'|} \mathrm{d}v \,. \tag{5.8}$$

The integral is the harmonic potential of matter of density x_l filling S, and so its Laplacian is $-4\pi x_k$ inside S and zero outside. Consequently

$$\begin{aligned} \vartheta_{ijk} X_{lk} &= -8\pi(1-\sigma)\delta_{ij}\, x_l \quad \text{inside} \quad \mathrm{S} \\ &= 0 \quad \text{outside} \quad \mathrm{S} \end{aligned} \tag{5.9}$$

and so

$$\begin{aligned} u_i^{\mathrm{D}} &= u_i^{\mathrm{C}} - e_{il}^{\mathrm{T}}\, x_l \quad \text{inside} \quad \mathrm{S} \\ &= u_i^{\mathrm{C}} \quad \text{outside} \quad \mathrm{S} \end{aligned} \tag{5.10}$$

as expected. If we had included a rigid-body rotation $\omega_{jl}^{\mathrm{T}} x_l$ in (5.7) I_{jk} would have contained the additional terms $\omega_{jk}^{\mathrm{T}}\psi + \omega_{jl}^{\mathrm{T}} X_{lk}$. The first contributes nothing to u_i^{D}, since ϑ_{ijk} is symmetric in jk. According to (5.8) the second term leaves u_i^{D} unchanged outside S. Inside S it gives a rigid-body rotation which in a structureless medium produces no observable effect.

It is clear that treatment in terms of a Somigliana dislocation offers no computational advantages when applied to an inclusion which has undergone a homogeneous stress-free strain. However, we may use the formula (5.6) to extend our results to the case where the inclusion suffers any small change of shape, not necessarily a homogeneous deformation.

Suppose that the inclusion undergoes a permanent change of form which, if the matrix were absent, would not be associated with stress. To specify this 'stress-free change of form' it is only necessary to give the displacement of each point of its surface,

$$u_i^{\mathrm{T}} = u_i^{\mathrm{T}}(r), \qquad r \quad \text{on} \quad \mathrm{S}\,. \tag{5.11}$$

Alternatively, to maintain the analogy with § 2.1, we could introduce a variable stress-free strain $e_{ij}^{\mathrm{T}}(r)$ defined throughout the interior of the inclusion and giving a displacement agreeing with (5.11) on S. (We recall (SOKOLNIKOFF [1946]) that specification of a (compatible) strain throughout a region fixes the displacement in it to within a rigid body displacement.) Since e_{ij}^{T} would be largely arbitrary it seems better to work with u_i^{T} in the general case.

If we try to put the transformed inclusion into the matrix there will be a misfit u_i^{T} at each point of S. When this misfit is removed by suitably straining the material and welding together corresponding points on either side of S we are left with a Somigliana dislocation for which $b_i = u_i^{\mathrm{T}}$. Thus to find the elastic field when the inclusion is constrained by the matrix we have only to put $b_j(\boldsymbol{r}') = u_j^{\mathrm{T}}(\boldsymbol{r}')$ in (5.6). The resulting u_i^{D} is the displacement measured from an initial state in which the matrix is unstressed and the inclusion is transformed and stress-free. Consequently the stress is given by

$$p_{ij}^{\mathrm{D}} = \lambda u_{m,m}^{\mathrm{D}}\,\delta_{ij} + \mu(u_{i,j}^{\mathrm{D}} + u_{j,i}^{\mathrm{D}})$$

both inside and outside the inclusion.

This problem can also be solved by the method of § 2.1. Let

$$u_i^{\mathrm{T}} = u_i^{\mathrm{T}}(\boldsymbol{r}), \ \ \boldsymbol{r} \text{ inside or on S} \ \ ,$$

be any convenient continuous displacement which coincides with (5.11) on S, and put

$$p_{ij}^{\mathrm{T}} = \lambda u_{m,m}^{\mathrm{T}}\,\delta_{ij} + \mu(u_{i,j}^{\mathrm{T}} + u_{j,i}^{\mathrm{T}}) \ .$$

Remove the transformed inclusion from the matrix and apply surface tractions $-p_{jk}^{\mathrm{T}} n_j$ and a distribution of body force of amount $+p_{jk,k}^{\mathrm{T}}$ per unit volume. This produces a displacement $-u_i^{\mathrm{T}}$ and so restores the inclusion to its untransformed shape. Cement it back in the matrix and relax the unwanted forces. This gives the displacement (cf. (2.6))

$$u_i^{\mathrm{C}} = \int_{\mathrm{S}} \mathrm{dS}_k\, p_{jk}^{\mathrm{T}}(\boldsymbol{r}')\, U_{ij} - \int_{V} \mathrm{d}v\, \frac{\partial p_{jk}}{\partial x_k'}(\boldsymbol{r}')\, U_{ij}\,.$$

It can be shown by the same sort of analysis as led to (5.10) that $u_i^{\mathrm{C}} = u_i^{\mathrm{D}}$ in the matrix and that $u_i^{\mathrm{C}} = u_i^{\mathrm{D}} + u_i^{\mathrm{T}}$ in the inclusion. Since u_i^{T} is by definition a stress-free displacement the stresses calculated from u_i^{C} and u_i^{D} are identical.

The solution for the elastic field due to an inclusion which has suffered a non-uniform transformation does not seem to have any obvious applications. However, for an ellipsoidal inclusion the solution has a property which enables it to be used to find how an ellipsoidal cavity or inhomogeneity perturbs a non-uniform stress-field of fairly general type. This property generalizes the result that a uniform transformation strain leads to a uniform strain in the constrained ellipsoid. It may be stated thus: if the stress-free transformation displacement is a polynomial in x_1, x_2, x_3 of degree N, then the displacement is also a polynomial of degree N inside the constrained ellipsoid. We shall only outline its derivation and application. The details of the calculation can easily be filled in.

The analysis involves a number of polynomials in x_i or x'_i with constant coefficients. We shall denote them by script capitals and indicate only the argument and degree. Thus, for example, $\mathscr{P}(M, x)$ stands for a polynomial in x_1, x_2, x_3 whose highest term is of degree M. Similarly $\mathscr{G}_{ijk}(M, x')$, $i,j = 1, 2, 3$ denotes a set of twenty-seven polynomials in x'_1, x'_2, x' all of the same degree, and so forth. We shall also use R as an abbreviation for $|\boldsymbol{r} - \boldsymbol{r}'|$.

Let the ellipsoidal inclusion undergo the transformation specified by

$$u_i^{\mathrm{T}}(\boldsymbol{r}) = \mathscr{T}_i\,(N, x) \qquad (\boldsymbol{r} \text{ on S})\,.$$

If we put $b_i = u_i^{\mathrm{T}}$ in (5.6) and use Gauss's theorem we find that I_{jk} has the form

$$I_{jk} = \int_V \mathscr{E}_{jk}(N-1, x')R\,\mathrm{d}v + \int_V \mathscr{T}_i(N, x')\,\frac{\partial R}{\partial x'_k}\,\mathrm{d}v\,. \qquad (5.12)$$

In the first integrand introduce the factor

$$1 = \frac{(x_m - x'_m)(x_m - x'_m)}{|\boldsymbol{r} - \boldsymbol{r}'|^2}$$

and in the second write $\partial R/\partial x'_k$ as $(x'_k - x_k)/R$. After a little re-arrangement (5.12) takes the form

$$I_{jk} = x_m x_m \int_V \mathscr{F}_{jk}(N-1, x')\,\frac{\mathrm{d}v}{R} + x_p \int_V \mathscr{G}_{pjk}(N, x')\,\frac{\mathrm{d}v}{R}$$
$$+ \int_V \mathscr{H}_{jk}(N+1, x')\,\frac{\mathrm{d}v}{R}\,. \qquad (5.13)$$

The integrals are the harmonic potentials of solid ellipsoids whose densities are polynomial functions of the co-ordinates. FERRERS [1877] and DYSON [1891] have discussed the potentials of inhomogeneous ellipoids of this type. Their results show that when the density is of the form $\mathscr{P}(M, x)$ the potential is of the form $\mathscr{Q}(M+2, x)$ inside the ellipsoid. The coefficients of the polynomial $\mathscr{Q}$ can be calculated from the coefficients of $\mathscr{P}$ and the three quantities I_a, I_b, I_c (eq. (3.2)). Outside the ellipsoid the potential is a similar polynomial in x_1, x_2, x_3 but with coefficients which are themselves functions of position. (Compare the relation between (3.2) and (3.7).) These variable coefficients can be expressed in terms of the coefficients of $\mathscr{P}$ and the quantities

A, B, C, F, E of (3.8), (3.10). Consequently, by (5.13) and (5.6) the constrained displacement is a polynomial of the form

$$u_i^{\mathrm{D}}(r) = \mathscr{D}_i(N,x)$$

inside the ellipsoid. Thus the constrained and unconstrained displacements of the inclusion are similar polynomials with calculable relations between their coefficients.

The solution of the cavity problem follows the lines of § 4.1. Superimpose a displacement

$$u_i^{\mathrm{A}}(r) = -\mathscr{D}_i(N,x) \tag{5.14}$$

everywhere. The total displacement $u_i^{\mathrm{A}} + u_i^{\mathrm{D}}$ is zero inside the ellipsoid. Hence the inclusion is unstressed, and it can be removed without disturbing the matrix. The coefficients in $\mathscr{T}_i$ can be chosen so as to make (5.14) any required polynomial of degree N. From these coefficients the coefficients of $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ (eq. (5.13)) can be found. The field $u_i^{\mathrm{A}} + u_i^{\mathrm{D}}$ in the matrix then follows from (5.13) and Ferrers' and Dyson's results. The extension to the ellipsoidal inhomogeneity follows as in § 4.1.

The calculations are not too unwieldy for small N. With $N = 2$ the displacement

$$u_i^{\mathrm{A}} = \alpha_{ij}^{\mathrm{A}} x_j + \beta_{ijk}^{\mathrm{A}} x_j x_k$$

covers the case of an applied stress which is a combination of simple torsion, bending and flexure. Solutions, based on other methods, have been given for a number of such problems involving spheroids and spheres (NEUBER [1958], SEN [1933], DAS [1953]).

§ 6. Applications

The application of the results reviewed here to the actual calculation of stress in and about inclusions and inhomogeneities calls for no special comment. The engineering application of the theory of cavities to the calculation of stress concentrations has been thoroughly treated by NEUBER [1958]. In this section we indicate some of the applications to physical problems.

Expressions for the elastic energy and interaction energy of an inclusion find a use in the theory of martensitic transformations. (For a general review see KAUFMAN and COHEN [1958].) Suppose that a metal can exhibit two crystal structures, γ and α, the former stable at high temperatures and the latter at low temperatures. Ideally we might expect that on cooling the whole of a single crystal of the metal would undergo a uniform stress-free strain in the sense of § 2, the e^T_{ij} being the deformation which carries the γ-lattice into the α-lattice. In a martensitic transformation, however, the low-temperature phase first appears in the form of inclusions of α embedded in the γ matrix. Thus, in considering the thermodynamics of the transformation, one must take into account the elastic energy of the misfitting inclusions of α and also, if there is an externally applied stress, their interaction energy with the latter.

The Gibbs free energy change associated with the formation of a martensitic inclusion may be written in the form

$$\Delta G = \Delta G_{\text{chem}} + \Delta G_{\text{surf}} + E_{\text{inc}} + E_{\text{int}}\,. \tag{6.1}$$

ΔG_{chem} is the 'chemical' contribution, the free energy change which would occur if the inclusion underwent its stress-free strain in the absence of the matrix. The theory of elastic inclusions can tell us nothing about it. ΔG_{surf} is a contribution due to the interface between matrix and inclusion. It may be estimated by using the theory of dislocations. E_{inc} is the elastic energy associated with the inclusion, and E_{int} is the interaction energy with any externally applied stress which may be present. According to the discussion in § 2.2 the terms E_{inc} and E_{int} taken together make up the elastic contribution to the change of Gibbs free energy.

It will usually be accurate enough to identify E_{int} with E_∞, the value in an infinite medium, and to ignore the difference between the elastic constants of matrix and inclusion. If the inclusion can be considered to be some form of ellipsoid E_∞ can be calculated from (3.3) and (2.32) when the transformation strain e^T_{ij} is known. Suppose, for example, that the inclusion takes the form of a plate in the x_1x_2-plane and that the transformation is made up of a shear parallel to this plane through an angle s, a uniform dilatation Δ and an extension ξ perpendicular to the plane of the plate. Then

$$e^T_{13} = e^T_{31} = \tfrac{1}{2}s, \qquad e^T_{11} = e^T_{22} = \tfrac{1}{3}\Delta\,, \qquad e^T_{33} = \xi + \tfrac{1}{3}\Delta \tag{6.2}$$

and the remaining components are zero. If the plate has the form of a thin oblate spheroid with $c \ll a = b$ we have

$$\frac{1-\sigma}{\mu V}E_\infty = \frac{1}{8}\pi(2-\sigma)\frac{c}{a}s^2 + \frac{2}{9}(1+\sigma)\Delta^2 + \frac{1}{4}\pi\frac{c}{a}\xi^2 + \frac{1}{3}\pi(1+\sigma)\frac{c}{a}\xi\Delta \tag{6.3}$$

(CHRISTIAN [1958]). There is also a fairly simple expression for E_∞ when the inclusion is a flat ellipsoid with $a > b \gg c$ (ESHELBY [1957]). The expression (6.3) has been used by CHRISTIAN [1958, 1959] and KAUFMAN [1959] to discuss the nucleation of martensite.

The interaction energy may be found from (2.43). In an actual experiment the applied stress will usually be a uniaxial tension; $\overset{A}{p}_{ij}$ then takes the form $\tau n_i n_j$, where τ is the magnitude of the tension and n_i is its direction. If the transformation strain is given by (6.2) we have

$$E_{\text{int}} = -\tau C V \tag{6.4}$$

with

$$C = s \cos\beta \sin\beta + \xi \cos^2\beta + \tfrac{1}{3}\Delta$$

where β is the angle between the x_3-axis and the direction of the tension, and the latter is assumed to lie in the x_1x_3-plane. A result equivalent to (6.4) was first obtained (with $\Delta = 0$) by PATEL and COHEN [1953] (cf. also MACHLIN and WEINIG [1953]). FISHER and TURNBULL [1953] have also discussed the effect of an applied stress on the free energy change associated with martensite formation. In effect, they take the interaction energy to be the cross-term in the elastic energy between the applied field and the field due to the inclusion. As we have seen (§ 2.2), this quantity should be zero. Since they only integrate the energy density over the immediate neighbourhood of the inclusion they obtain a finite result which, however, is not quite correct numerically and which, in addition, has the wrong sign. To correct this they have to make an arbitrary reversal of sign in the relation between applied stress and applied strain.

When a crystal having the high-temperature γ-structure is cooled, ΔG decreases. According to elementary thermodynamics a martensitic inclusion of the α-phase can form when a temperature is reached at which ΔG is zero. In fact a nucleation barrier has to be overcome and the transformation will occur when ΔG reaches some finite negative value, say ΔG_{nuc}. Let T_s be the temperature at which $\Delta G = \Delta G_{\text{nuc}}$ in the absence of external stress, i.e. with $E_{\text{int}} = 0$ in (6.1). When there is an applied stress the term E_{int} will alter the temperature at which $\Delta G_{\text{nuc}} = \Delta G$ to, say, $T_s + \Delta T_s$. If ΔG_{nuc} is independent of temperature and ΔG_{chem} is the only strongly temperature-dependent term in (6.1) we must have

$$\frac{\mathrm{d}\Delta G_{\text{chem}}}{\mathrm{d}T}\Delta T_s + E_{\text{int}} = 0\,.$$

and so from (6.4) we obtain the expression

$$\frac{\mathrm{d}T_s}{\mathrm{d}\tau} = C \Big/ \frac{1}{V} \frac{\mathrm{d}\Delta G_{\mathrm{chem}}}{\mathrm{d}T} \tag{6.5}$$

for the rate of change of transformation temperature with applied tensile stress. C can be found from crystallographic measurements, and the rate of change of chemical free energy with temperature may be obtained from thermodynamic data. Patel and Cohen found excellent agreement between theory and experiment for certain iron-nickel and iron-nickel-carbon alloys. They also showed how (6.5) should be modified when ΔG_{nuc} varies with temperature.

For martensitic inclusions the transformation strain is essentially a pure shear. For particles of precipitate formed by diffusion it is essentially a dilatation. For an ellipsoidal precipitate the strain energy can be found from (3.21) with $e_{ij}^{\mathrm{T}*} = \frac{1}{3}e^{\mathrm{T}}\delta_{ij}$. ROBINSON [1951] has given a detailed treatment. In these calculations it is assumed that the inclusion is coherent with the matrix. NABARRO [1940] and KRÖNER [1954] have treated the problem of finding the minimum strain energy due to an inclusion which has broken away from the matrix. The displacement need not be continuous across the interface, but the matrix and inclusion are supposed to be in contact everywhere. The volume misfit is prescribed, but the stress-free shape of the inclusion has to be determined so as to minimize the energy. According to KRÖNER [1953a, 1954] the energy is a minimum for the state in which the stress in the inclusion is purely hydrostatic. (Matrix and inclusion may be made of different anisotropic materials.)

The theory of cracks, i.e. empty cavities one of whose dimensions is evanescent, has played a part in Griffiths' treatment of rupture (see, for example, SNEDDON [1951]) and, more recently, in the theory of the brittle and ductile fracture of metals. The crack is supposed to have associated with it a surface energy γA, where A is its surface area and γ is a constant representing true surface energy or an effective surface energy associated with plastic deformation. According to the discussion in § 4.2 the total energy of the system made up of the body containing the crack and the external loading mechanism is

$$E_{\mathrm{int}} + \gamma A + \mathrm{const.}$$

By studying the way in which E_{int} and A vary when the form of the crack is altered slightly one can find whether it is energetically favourable for the crack to spread or contract.

We may find the necessary properties of an elliptical crack from the results of §§ 4.1, 4.2 by putting $\lambda^* = \mu^* = 0$ and letting the c-axis tend to zero. The equivalent stress-free strain e_{ij}^{T} approaches infinity for a fixed applied stress p_{ij}^{A}, but the products Ve_{ij}^{T} (and hence also the potentials φ, ψ) remain finite.

There exist a number of calculations for the interaction energy of a circular or two-dimensional crack (INGLIS [1913], STARR [1928], SACK [1946], SEGEDIN [1951]). They may be deduced from the following results for the elliptical crack

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\,.$$

(i) The applied stress is a pure tension p_{33}^{A} normal to the plane of the crack (tensions p_{11}^{A}, p_{22}^{A} evidently have no effect). Then

$$E_{\text{int}} = -\frac{2\pi(1-\sigma)ab^2}{3E(k)}\,\frac{(p_{33}^{\mathrm{A}})^2}{\mu}\,. \tag{6.6}$$

This is easily deduced from the results of GREEN and SNEDDON [1949].
(ii) The applied stress is a pure shear p_{13}^{A} in the plane of the crack. Then (ESHELBY [1957])

$$E_{\text{int}} = -\frac{2\pi ab^2}{3\eta}\,\frac{(p_{13}^{\mathrm{A}})^2}{\mu}$$

where

$$\eta = E(k) + \frac{\sigma}{1-\sigma}\,\frac{K(k)-E(k)}{k^2}\,. \tag{6.7}$$

In (6.6) and (6.7) E, K are complete elliptic integrals of modulus $k = (1 - b^2/a^2)^{\frac{1}{2}}$. Eq. (6.7) is valid for both $a > b$ and $a < b$. In the latter case the elliptic integrals may be reduced to real form with the help of the relation

$$K(k_1) = (1-k^2)^{\frac{1}{2}}K(k), \qquad E(k_1) = E(k)/(1-k^2)^{\frac{1}{2}},$$
$$\text{where} \quad k_1 = ik/(1-k^2)^{\frac{1}{2}}\,.$$

(Compare the remark following (3.6).) For the circular crack $a = b$, $E = \frac{1}{2}\pi$, $\eta = \pi(2-\sigma)/4(1-\sigma)$. STROH [1958] has calculated the interaction energy for two-dimensional cracks in anisotropic materials.

The concept of the interaction energy of an inhomogeneity with an applied stress simplifies the calculation of the bulk elastic constants of elastically inhomogeneous aggregates. Suppose, for example, that we wish to calculate the effective elastic constants of a material of bulk modulus $\varkappa$ and shear modulus μ containing a dispersion of spherical inhomogeneities with elastic constants $\varkappa^*$, μ^* (ESHELBY [1957]). In the absence of the inhomogeneities unit volume of the material has elastic energy

$$E_0 = \frac{1}{2}\left(\frac{1}{9\varkappa} p^{\mathrm{A}} p^{\mathrm{A}} + \frac{1}{2\mu} {}'p_{ij}^{\mathrm{A}}\, {}'p_{ij}^{\mathrm{A}}\right) \tag{6.8}$$

if it is subjected to the uniform stress

$$p_{ij}^{\mathrm{A}} = \tfrac{1}{3} p^{\mathrm{A}} \delta_{ij} + {}'p_{ij}^{\mathrm{A}}$$

(for the notation see § 1). If the spheres are introduced the elastic energy becomes

$$E = E_1 - \Sigma E_{\mathrm{int}} \tag{6.9}$$

by (4.15), where Σ denotes summation over all the spheres. If we ignore the image term in (4.13) and also the interaction between the spheres (so that we limit ourselves to a dilute dispersion) (6.9) may be written as

$$E = \frac{1}{2}\left\{\frac{1}{9\varkappa}(1 + Av) p^{\mathrm{A}} p^{\mathrm{A}} + \frac{1}{2\mu}(1 + Bv)\, {}'p_{ij}^{\mathrm{A}}\, {}'p_{ij}^{\mathrm{A}}\right\}$$

where v is the fraction of the volume of material occupied by the inhomogeneous spheres and A, B are given by (4.7). Comparing with (6.8) we see that the effective elastic constants are

$$\varkappa_{\mathrm{eff}} = \varkappa/(1 + Av), \qquad \mu_{\mathrm{eff}} = \mu/(1 + Bv)$$

or, since the analysis is only valid for $v \ll 1$,

$$\varkappa_{\mathrm{eff}} = \varkappa(1 - Av), \qquad \mu_{\mathrm{eff}} = \mu(1 - Bv)\,. \tag{6.10}$$

These expressions have been obtained in another manner by HASHIN (REINER [1958]).

Much attention has been given to a related problem: namely, to determine the macroscopic elastic constants, $\overset{0}{c}_{ijkl}$, $\overset{0}{s}_{ijkl}$ say, of a polycrystalline aggregate whose actual constants c_{ijkl}, s_{ijkl} vary from grain to grain, not because the material is inhomogeneous, but because the crystal orientation varies. The estimates $\overset{0}{c}_{ijkl} = \bar{c}_{ijkl}$ and $\overset{0}{s}_{ijkl} = \bar{s}_{ijkl}$ are associated with the names of Voigt and Reuss. Here the bar denotes an average over all crystal orientations, weighted if necessary according to the relative frequency of each orientation. HILL [1952] has shown that

$$\bar{s}'_{ijkl} \leqslant \overset{0}{s}_{ijkl} \leqslant \bar{s}_{ijkl}$$

$$\bar{c}'_{ijkl} \leqslant \overset{0}{c}_{ijkl} \leqslant \bar{c}_{ijkl}$$

where $\bar{c}'_{ijkl}$ and $\bar{s}'_{ijkl}$ are, respectively, the tensors inverse to $\bar{s}_{ijkl}$ and $\bar{c}_{ijkl}$. For an aggregate of cubic crystals the upper and lower estimates of the bulk modulus coincide and we have exactly

$$\varkappa = \tfrac{1}{3}(c_{11} + 2c_{12}) . \tag{6.11}$$

Hill's relation seems to mark the limit of what can be proved precisely. To go further it is necessary to make assumptions which may be physically plausible, but cannot be proved conclusively. Most of the attempts in this direction are rather outside the scope of this article. HERSHEY's [1954] and KRÖNER's [1958b] calculations, however, make use of the concept of an anisotropic inhomogeneity in an isotropic matrix. We shall not give their physical arguments, but instead present a simplified treatment which involves essentially the same mathematics.

We confine ourselves to an aggregate of cubic crystals which is macroscopically isotropic. Let the effective elastic constants be $\varkappa$, μ. Imagine that the material of each grain is replaced by isotropic material with constants $\varkappa$, μ, but that the boundaries between grains can still be distinguished. We thus obtain, trivially, an 'equivalent isotropic aggregate' with the same overall elastic constants as the polycrystalline aggregate. It seems clear that this equivalent aggregate must pass the following test: if any representative sample of its grains have their original anisotropic constants restored the bulk elastic constants are unaltered. In applying the test we may take the collection of re-transformed grains to be so far apart that the effect of the perturbing field of one grain on another may be neglected. Of course the isotropic aggregate must also pass the more severe test that its bulk elastic constants are unchanged when *every* grain is restored to its original anisotropic state. However, we may hope that if the weaker test gives definite values for $\varkappa$ and μ, then these will be the same as those which would result from applying the severer test.

We shall suppose that the set of test grains have their crystal axes oriented at random and that they may be treated as spheres. This second assumption is hard to justify rigorously, but without it not much progress can be made. In the presence of a uniform applied strain $\overset{\mathrm{A}}{e}_{ij}$ the $\overset{\mathrm{T}}{e}_{ij}$ of the equivalent inclusion for one of the grains can be written in the form

$$\overset{\mathrm{T}}{e}_{ij} = D_{ijkl}\, \overset{\mathrm{A}}{e}_{kl} ;$$

the D_{ijkl} depend on the orientation of the grain. When the anisotropic test grains are introduced into the equivalent isotropic material its elastic constants are changed by an amount proportional to the sum of their interaction energies with the applied field, by the same argument as led to (6.9). Thus the test requires that

$$\Sigma\,(\overset{\mathrm{A}}{p}_{ij}\,\overset{\mathrm{T}}{e}_{ij}) = \Sigma\,(\overset{\mathrm{A}}{p}_{ij}\,\overset{\mathrm{A}}{e}_{kl}\,D_{ijkl}) = 0 \tag{6.12}$$

where Σ implies summation over all the re-transformed grains. If we choose fixed coordinate axes, $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ is the same for each term of the summation but D_{ijkl} varies from term to term. It is therefore more convenient to choose for each term axes parallel to the crystal axes of the grain in question. Then D_{ijkl} is the same for each term, but $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ is not. The calculation of the sum $\Sigma\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl}$ could be carried out as follows. Assign fixed values to the principal strains of $\overset{\mathrm{A}}{e}_{ij}$, form $\overset{\mathrm{A}}{p}_{ij}\,\overset{\mathrm{A}}{e}_{kl}$, average the result over all orientations of the principal axes of the strain and multiply by the number of terms. The resulting expression will have the form of the most general isotropic tensor which has the symmetry of the suffixes in $\overset{\mathrm{A}}{p}_{ij}\overset{\mathrm{A}}{e}_{kl} = \overset{\mathrm{A}}{p}_{kl}\,\overset{\mathrm{A}}{e}_{ij}$. Consequently it must have the same form as the elastic constant tensor c_{ijkl} of an isotropic medium, that is

$$\Sigma\,\overset{\mathrm{A}}{p}_{ij}\,\overset{\mathrm{A}}{e}_{kl} = c_1\delta_{ij}\delta_{kl} + c_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with arbitrary c_1, c_2. Inserting this in (6.12) we get $D_{iijj} = 0$ and $D_{ijij} = 0$, or in view of the symmetry (spherical grains, cubic crystal referred to its principal axes)

$$D_{1111} + 2D_{1122} = 0 \tag{6.13}$$

and

$$D_{1111} + 2D_{1212} = 0\,. \tag{6.14}$$

The values of the D_{ijkl} are easily calculated from (4.6) and (4.7) if we bear in mind that when extended along its axes a cubic crystal behaves like an isotropic material with $\varkappa^* = \frac{1}{3}(c_{11} + 2c_{12})$, $\mu^* = \frac{1}{2}(c_{11} - c_{22})$ and that for a shear of the type e_{12} it behaves like an isotropic material with $\mu^* = c_{44}$. Eq. (6.14) gives $A\{\varkappa, \frac{1}{3}(c_{11} + 2c_{12})\} = 0$ or $\varkappa = \frac{1}{3}(c_{11} + 2c_{12})$ in agreement with the already known result (6.11). The expression for the shear modulus is more interesting. Eq. (6.14) gives

$$\tfrac{2}{3}B\{\mu, \tfrac{1}{2}(c_{11} - c_{12})\} - B\{\mu, c_{44}\} = 0\,.$$

On inserting the value of β (eq. (3.14)) and clearing of fractions we are left with a cubic equation:

$$16\mu^3 + 2(5c_{11} + 4c_{12})\mu^2 - 2c_{44}(7c_{11} - 4c_{12})\mu - c_{44}(c_{11} - c_{12})(c_{11} + 2c_{12}) = 0 \tag{6.15}$$

for determining μ. It has only one real positive root.

HERSHEY [1954] obtained a quartic equation for μ which in fact is what one obtains on multiplying (6.15) by $8\mu + 9\varkappa$. Evidently this introduces no new positive root. KRÖNER [1958b] obtained both the quartic and cubic and gave an argument to show that their respective positive roots are upper and lower bounds for μ, and that since these roots coincide the value so obtained is exact apart from statistical uncertainties due to the fact that the aggregate contains only a finite number of grains.

GOODIER [1936] has emphasized that there is a useful analogy between problems in the slow motion of viscous liquids and elastic problems for incompressible solids. In general special care has to be taken in extrapolating elastic solutions to the incompressible case $\sigma = \frac{1}{2}$, since we have, in effect, to make the simultaneous transition $u_{k,k} \to 0$, $\lambda \to \infty$. However, in the case of the solution (2.8) to the general inclusion problem there is no trouble. With $\sigma = \frac{1}{2}$ it satisfies

$$\mu\nabla^2\boldsymbol{u}^{\mathrm{C}} = 0\,, \qquad \operatorname{div} \boldsymbol{u}^{\mathrm{C}} = 0$$

outside the inclusion. These are just the Stokes equations governing the velocity $\boldsymbol{u}^{\mathrm{C}}$ in an incompressible fluid of viscosity μ when inertial effects may be neglected and the hydrostatic pressure, p_0, say, is independent of position. Consequently (2.8) with $\sigma = \frac{1}{2}$ and u_i^{C} interpreted as a velocity represents a certain state of viscous flow. The associated stress is

$$p_{ij} = \mu(u^{\mathrm{C}}_{i,j} + u^{\mathrm{C}}_{j,i}) - p_0\delta_{ij}\,.$$

Eq. (2.8) describes the flow about a solid which is deforming in such a way that each point $\boldsymbol{r}$ of its surface S has a velocity $u_i^{\mathrm{C}}(\boldsymbol{r})$. Since we cannot prescribe $u_i^{\mathrm{C}}(\boldsymbol{r})$, but only the constants e_{ij}^{T} the analogy is not of much use in the general case. Eq. (3.1), however, (with $\sigma = \frac{1}{2}$) describes the viscous flow around an ellipsoid which is undergoing a change of shape specified by the constant rate-of-strain tensor e_{ij}^{C}, and the e_{ij}^{T} appropriate to prescribed e_{ij}^{C} can be found be solving (3.3).

The solution to the problem of an ellipsoidal inhomogeneity perturbing a uniform strain e_{ij}^{A} has a simple viscous interpretation when the ellipsoid is perfectly rigid and incompressible ($\lambda^* \to \infty$, $\mu^* \to \infty$). Eq. (4.3) gives $e_{ij}^{\mathrm{C}} + e_{ij}^{\mathrm{A}} = 0$ on S or equivalently

$$u_i^{\mathrm{C}} + u_i^{\mathrm{A}} = 0 \quad \text{on} \quad \mathrm{S}\,. \tag{6.16}$$

Translated into terms of velocity this is the condition that the liquid shall adhere to the surface of the ellipsoid, and so the field $u_i = u_i^{\mathrm{C}} + u_i^{\mathrm{A}}$ gives the velocity about a solid ellipsoid immersed in the uniform flow specified by the rate-of-strain tensor e_{ij}^{A}. The elastic energy density translates into half the rate of dissipation of energy per unit volume. (In the one case we are concerned with the familiar half stress times strain, in the other with stress times rate of deformation.) Consequently $2E_{\mathrm{int}}$ (eq. (4.13)) is equal to the additional rate of dissipation of energy, $\dot{E}_{\mathrm{diss}}$ say, when a solid is introduced into a viscous liquid at whose boundary constant velocities are maintained.

If the image term may be neglected in (4.13) the interaction energy for an inhomogeneity is half the interaction energy for the equivalent transformed inclusion, and it follows that $\dot{E}_{\mathrm{diss}}$ is given by the right hand side of (2.42), or, in view of (6.16), by

$$\dot{E}_{\mathrm{diss}} = -\int_{\mathrm{S}} (p_{ij}^{\mathrm{C}} + p_{ij}^{\mathrm{A}}) u_i^{\mathrm{C}} \mathrm{dS}_j$$
$$= \int_{\mathrm{S}} \boldsymbol{p}_n \cdot \boldsymbol{u}^{\mathrm{C}} \, \mathrm{dS}$$

where $\boldsymbol{p}_n$ is the actual surface traction on the solid (with the sign convention usual in hydrodynamics) and $\boldsymbol{u}^{\mathrm{C}}$ is the perturbation of velocity due to the presence of the solid. This is identical with a result of BRENNER's [1958]. The expression (6.10) for the effective shear modulus of a dispersion of spheres gives the Einstein viscosity formula $\mu_{\mathrm{eff}} = (1 + \frac{5}{2}v)\mu$ when we put $\mu^* = \infty$, $\sigma = \frac{1}{2}$. The difference in sign between (4.14) and (4.15) has the following interpretation in terms of viscosity. The viscosity is always increased by the introduction of solid particles. Consequently a viscometer working at constant load produces a lower rate of deformation and so less energy dissipation, while a viscometer working at constant speed has to work harder to maintain the prescribed rate of strain.

The applications considered above have all been essentially macroscopic. Elastic inclusions and inhomogeneities have also been found useful as models of lattice defects in crystals. (For a general account see, for example, FRIEDEL [1956] or ESHELBY [1956].)

Suppose that one atom of a crystal lattice is replaced by a foreign atom. The foreign atom will generally have a different size from the host atoms. As a simple elastic model we may take a spherical hole in an isotropic continuum into which a misfitting elastic sphere is inserted (BILBY [1950]). Since the elastic constants of the foreign atom (in so far as one can speak of them for a single atom) will differ from those of the host atoms the misfitting elastic sphere will in general have to be assigned elastic constants differing from those of the matrix. It is useful to consider two limiting cases. (i) A pure inclusion; the sphere and inclusion have the same elastic constants, and the misfit gives rise to a permanent elastic field. (ii) A pure inhomogeneity; there is no misfit, but the elastic constants of sphere and matrix differ. There is no permanent elastic field, but a field can be 'induced' by an applied field. The general case, intermediate between (i) and (ii), corresponds to the inhomogeneous inclusion of § 3.2.

It is not at all obvious that such a crude model will be of any use in discussing the behaviour of lattice defects. We shall discuss some of the consequences of taking the model seriously and then try to indicate the reasons for its success.

The elastic field of a pure inclusion is given by (2.26) with the value (3.12) for φ and e^{T} equal to the fractional volume misfit between hole and sphere. The stress falls off as r^{-3} and its hydrostatic component is zero. The field of a pure inhomogeneity in an applied field e_{ij}^{A} is found by calculating the equivalent e_{ij}^{T} from (4.6) and inserting them in (2.8) with the values (3.12), (3.13) for φ and ψ.

It may be shown that the expressions (2.43) and (4.13) for the interaction energies of inclusions and inhomogeneities remain valid when p_{ij}^{A}, e_{ij}^{A} refer to the field produced by some source of internal stress, another inclusion or a dislocation for example. Thus the interaction energy between two pure inclusions is proportional to $p_{ij}^{\mathrm{A}} e_{ij}^{\mathrm{T}}$, where p_{ij}^{A} is the stress produced by one inclusion and e_{ij}^{T} is the stress-free strain associated with the other. Since $e_{ij}^{\mathrm{T}} = \frac{1}{3}e^{\mathrm{T}}\delta_{ij}$ the interaction energy is proportional to p^{A}. But neither defect produces a hydrostatic pressure, and so the interaction energy is zero (BITTER [1931]). Again, let p_{ij}^{A} refer to the field of a pure inclusion and let e_{ij}^{T} be the equivalent stress-free strain which it induces in a pure inhomogeneity at a distance r from it. Since the p_{ij}^{A} are proportional to r^{-3} and the e_{ij}^{T} are proportional to the p_{ij}^{A} it follows that the interaction energy between a pure inclusion and a pure inhomogeneity is proportional to the inverse sixth power of the distance between them. The interaction energy between two defects each of which is represented by a misfitting and inhomogeneous sphere is also proportional to r^{-6}. Its numerical value may be found by a detailed calculation (ESHELBY [1958]).

The stress-field at a distance r from a dislocation line is proportional to r^{-1}. It follows by the same kind of argument as before that the dislocation has an interaction energy proportional to r^{-1} with a pure inclusion but an interaction energy proportional to r^{-2} with a pure inhomogeneity. This difference allows one to distinguish experimentally between the two types of defect. Since the interaction energy is a function of position there is an effective force

$$\boldsymbol{F} = -\operatorname{grad} E_{\mathrm{int}}(\boldsymbol{r}) \tag{6.17}$$

on the defect. If the defect is capable of diffusing through the lattice, a drift velocity

$$\boldsymbol{v} = D\boldsymbol{F}/kT$$

is superimposed on its random motion, where D is the diffusion coefficient, k is Boltzmann's constant and T is the absolute temperature. Thus, if dislocations are introduced into a crystal containing defects, the latter will be attracted to the dislocations. The resulting depletion of defects in the bulk of the material can be detected by measuring suitable physical properties (electrical resistance, internal friction). *Calculation shows that for an r^{-1} interaction the number of defects* drawn into the dislocations is proportional to $t^{\frac{2}{3}}$, where t is the time since the interaction between defects and dislocations began. If, on the other hand, there is an r^{-2} interaction the expression $t^{\frac{2}{3}}$ is replaced by $t^{\frac{1}{2}}$ (FRIEDEL [1959]).

For a defect which can be represented by an inserted sphere which is both misfitting and inhomogeneous the interaction energy with a dislocation will evidently have the form $Ar^{-1} + Br^{-2}$. Even if the coefficient B is relatively large the A-term will dominate at large distances and we should expect the $t^{\frac{2}{3}}$ law to be most nearly obeyed. The $t^{\frac{2}{3}}$ law has, in fact, been verified for carbon and nitrogen diffusing in iron. We might, perhaps, regard a vacant lattice site as a pure inhomogeneity ($A = 0$). However, such calculations as have been made (e.g. TEWORDT's [1958] for copper) indicate that there is an appreciable displacement of the atoms bordering the vacant site. On the elastic model this means that there is a stress-field associated with the vacancy even in the absence of an applied field, and so $A \neq 0$.

Rather surprisingly WINTENBERGER's [1957] measurements on vacancies in aluminium follow the $t^{\frac{1}{2}}$ law, which indicates that, in aluminium at least, lattice vacancies behave as pure inhomogeneities.

We shall now try to indicate why the simple misfitting sphere model has been relatively successful. In the first place it is reasonable to suppose that sufficiently far from a lattice defect the crystal can be treated as an elastic continuum. The displacement representing the disturbance due to the defect can be expanded in ascending inverse powers of r (the distance from the defect) each provided with a suitable angular factor. If we treat the material as isotropic the leading term of the expansion has precisely the form (2.23) with arbitrary symmetric e^{T}_{ij}. Thus, if the material is isotropic, (2.23) gives the elastic field at large distances from the most general type of point defect. In many of the common metals the elastic field of the defect will have cubic symmetry. This physical condition, combined with the artificial limitation to isotropy, requires that e^{T}_{ij} shall be of the form $\frac{1}{3}e^{\mathrm{T}}\delta_{ij}$. According to (2.26) the remote field is independent of the shape of the inclusion and we may take it to be a sphere. In this way we recover the misfitting sphere model. In some cases (in particular, interstitial carbon and nitrogen in iron) the assumption that e^{T}_{ij} is a pure dilatation is inadequate, but the displacement may still be taken to have the form (2.23) with suitably chosen values for the e^{T}_{ij}.

The interaction energy (2.42) can be re-written in the form

$$E_{\text{int}} = \int_{\mathrm{S}'} (p^{\mathrm{C}}_{ij}\, u^{\mathrm{A}}_{i} - p^{\mathrm{A}}_{ij}\, u^{\mathrm{C}}_{i})\mathrm{dS}_{j} \tag{6.18}$$

where S′ is any surface drawn in the material and enclosing the inclusion. (The expressions (2.42) and (6.18) are identical because the divergence of the integrand is zero between S and S′.) The expression for the interaction energy was derived on the assumption that the infinitesimal theory could be applied everywhere. This is obviously not true near an inclusion representing an atomic defect. However, it may be shown that for (6.18) to be correct it is only necessary that the infinitesimal theory be valid in the neighbourhood of the surface S′, a much less severe requirement. The following is an outline of the argument (cf. ESHELBY [1959a]). We may form an expression for the x_l-component of the effective force (6.17) by subtracting from (6.18) the corresponding expression with p^{C}_{ij}, u^{C}_{i} replaced by $p^{\mathrm{C}}_{ij} + p^{\mathrm{C}}_{ij,l}\,\varepsilon$, $u^{\mathrm{C}}_{i} + u^{\mathrm{C}}_{i,l}\,\varepsilon$, dividing by ε and letting ε approach zero. The result is

$$F_l = \int_{\mathrm{S}'} (p^{\mathrm{C}}_{ij,l}\, u^{\mathrm{A}}_{i} - p^{\mathrm{A}}_{ij}\, u^{\mathrm{C}}_{i,l})\mathrm{dS}_{j}\,. \tag{6.19}$$

It is possible (ESHELBY [1956]) to derive a general expression for F_l which is valid for an arbitrary non-linear stress-strain relation and for finite deformation. Like (6.19) this expression takes the form of an integral over a surface surrounding the defect on which the force is to be calculated. If it is permissible to apply the infinitesimal theory on this surface the general expression reduces to (6.19). Hence (6.19), (6.18) or the simple formula (2.43) can be used whenever it is possible to draw a surface S′ enclosing the defect and no other source of internal stress and far enough from it for the strains on S′ to be reasonably small. Applied to the interaction of two point defects, for example, this means that (2.42) can be used if the linear theory is approximately obeyed halfway between them.

The treatment of lattice defects by continuum methods sometimes gives useful results even in cases too extreme for the above considerations to apply. FRIEDEL [1954] has shown, for example, that in some cases it is possible to get a reasonable estimate for the energy of solution of atoms of a metal X in a metal Y by associating with each dissolved X-atom a strain energy (3.21), taking for the volumes of hole and inclusion the atomic volumes of X and Y and for (λ, μ), (λ^*, μ^*) the ordinary elastic constants of the metals Y and X. An amusing example is provided by JACOBS' [1954] calculation of the effect of hydrostatic pressure on the frequency of the absorption band of an F-centre in an alkali halide crystal. An F-centre is an electron trapped at a negative-ion vacancy. An empirical rule of MOLLWO's [1933] states that in passing from one alkali halide crystal with the sodium chloride structure to another the product νa^2 remains constant; ν is the frequency of the maximum of the F-absorbtion band and a is the lattice parameter. It is reasonable to suppose that the same relation will govern the behaviour of a given alkali halide when its lattice parameter is changed by compression. We should then have the relation $-(\mathrm{d}\nu/\nu)/(\mathrm{d}a/a) = 2$. In place of 2, Jacobs' experiments gave values between 4.4 and 3.4. This discrepancy can be reconciled if we admit that the characteristics of an F-centre are determined by the positions of the atoms bordering the vacancy. Then Mollwo's rule must be interpreted as $\nu R^2 = \text{const.}$, where R is, say, the distance of a neighbouring atom from the centre of the vacancy, so that

$$-\frac{\mathrm{d}\nu/\nu}{\mathrm{d}R/R} = 2\ . \tag{6.20}$$

On passing from one type of crystal to another $(\mathrm{d}R/R)/(\mathrm{d}a/a)$ is unity, since R/a depends only on the lattice geometry. However, when a given crystal is compressed, $\mathrm{d}a/a$ and $\mathrm{d}R/R$ are not identical. In fact, if we idealize the vacancy as a spherical hole in an isotropic continuum we have

$$\frac{\mathrm{d}R/R}{\mathrm{d}a/a} = \frac{e^{\mathrm{A}} + e^{\mathrm{C}}}{e^{\mathrm{A}}} \tag{6.21}$$

in the notation of (4.6), with $\varkappa^* = 0$ in A. For the alkali halides we may put $\sigma = \frac{1}{4}$; the ratio (6.21) is then 2.25, and from (6.20) and (6.21) we have

$$-\frac{\mathrm{d}\nu/\nu}{\mathrm{d}a/a} = 4.5$$

in much better agreement with experiment. The above is admittedly rather a travesty of Jacobs' argument, but in his more rigorous calculation he also found it necessary to introduce the magnification factor (6.21).

References

BILBY, B. A., 1950, Proc. Phys. Soc. A **63** 191.
BITTER, F., 1931, Phys. Rev. **37** 1526.
BRENNER, H., 1958, The Physics of Fluids **1** 388.
BURGERS, J. M., 1939, Proc. Acad. Sci. Amsterdam **42** 293.
BYRD, P. F. and M. D. FRIEDMAN, 1954, Handbook of Elliptic Integrals (Springer-Verlag, Berlin 1954).
CHRISTIAN, J. W., 1958, Acta Met. **6** 377.
CHRISTIAN, J. W., 1959, Acta Met. **7** 218.
DANIELE, E., 1911, Nuovo Cimento [6] **1** 211.
DAS, S. C., 1953, Bull. Calcutta Math. Soc. **45** 55.
DYSON, F. W., 1891, Q.J. Pure Appl. Math. **25** 259.
EDWARDES, D., 1893, Q.J. Pure Appl. Math. **26** 70.
ESHELBY, J. D., 1951, Phil. Trans. Roy. Soc. A **244** 87.
ESHELBY, J. D., 1956, in Solid State Physics (ed. Seitz and Turnbull) **3** 79.
ESHELBY, J. D., 1957, Proc. Roy. Soc. A **241** 376.
ESHELBY, J. D., 1958, Ann. Physik [7] **1** 116.
ESHELBY, J. D., 1959a, in Internal Stresses and Fatigue in Metals (ed. Rassweiler and Grube; Elsevier, Amsterdam, 1959) 41.
ESHELBY, J. D., 1959b, Proc. Roy. Soc. A **252** 561.
FERRERS, N. M., 1877, Q.J. Pure Appl. Math. **14** 1.
FISHER, J. C. and D. TURNBULL, 1953, Acta Met. **1** 310.
FREDHOLM, I., 1900, Acta Math. **23** 1.
FRIEDEL, J., 1954, Advances in Physics **3** 446.
FRIEDEL, J., 1956, Les Dislocations (Gauthier-Villars, Paris, 1956).
FRIEDEL, J., 1959, in Internal Stresses and Fatigue in Metals (ed. Rassweiler and Grube; Elsevier, Amsterdam, 1959) 220.
GOODIER, J. N., 1936, Phil. Mag. **22** 678.
GOODIER, J. N., 1937, Phil. Mag. **23** 1017.
GREEN, A. E. and I. N. SNEDDON, 1949, Proc. Camb. Phil. Soc. **46** 159.
HERSHEY, A. V., 1954, J. Appl. Mech. **21** 236.
HILL, R., 1952, Proc. Phys. Soc. A **65** 349.
INGLIS, C. E., 1913, Trans. Inst. Naval Arch. **55** 219.
JACOBS, I. S., 1954, Phys. Rev. **93** 993.
KAUFMAN, L., 1959, Acta Met. **7** 216.
KAUFMAN, L. and M. COHEN, 1958, Progress in Metal Physics **7** 165.
KELLOGG, O. D., 1929, Potential Theory (Springer-Verlag, Berlin, 1929).
KRÖNER, E., 1953a, Diplomarbeit (Stuttgart, 1953).

KRÖNER, E., 1953b, Z. Phys. **136** 404.
KRÖNER, E., 1954, Acta Met. **2** 302.
KRÖNER, E., 1958a, Kontinuumstheorie der Versetzungen und Eigenspannungen (Springer-Verlag, Berlin, 1958).
KRÖNER, E., 1958b, Z. Phys. **151** 504.
LIFSHITZ, I. M. and N. ROSENZWEIG, 1947, J. Eksp. Teor. Fiz. **17** 783.
LOVE, A. E. H., 1954, Mathematical Theory of Elasticity (Cambridge University Press, 1954).
LURIE, A. I., 1952, Doklady Akad. Nauk SSSR **87** 709.
MACHLIN, E. S. and S. WEINIG, 1953, Acta Met. **1** 480.
MACMILLAN, W. D., 1958, The Theory of the Potential (Dover Publications, New York, 1958) 175.
MOLLWO, E., 1933, Z. Phys. **85** 56.
NABARRO, F. R. N., 1940, Proc. Roy. Soc. A **175** 519.
NABARRO, F. R. N., 1951, Phil. Mag. **42** 1224.
NABARRO, F. R. N., 1952, Advances in Physics **1** 269.
NEUBER, H., 1958, Kerbspannungslehre (Springer-Verlag, Berlin, 1958).
NIESEL, W., 1953, Inauguraldissertation, Karlsruhe.
OSBORN, J. A., 1945, Phys. Rev. **67** 351.
PATEL, J. R. and M. COHEN, 1953, Acta Met. **1** 531.
PEACH, M. O., 1951, J. Appl. Phys. **22** 1359.
POINCARÉ, H., 1899, Théorie du Potentiel Newtonien (Carré et Naud, Paris, 1899) 118.
REINER, M., 1958, Encyclopedia of Physics VI (Springer-Verlag, Berlin, 1958) 528.
ROBINSON, K., 1951, J. Appl. Phys. **22** 1045.
SACK, R. A., 1946, Proc. Phys. Soc. **58** 729.
SADOWSKY, M. A. and E. STERNBERG, 1949, J. Appl. Mech. **16** 149.
SEEGER, A., 1955, Encyclopedia of Physics VII (1) 383.
SEGEDIN, C. M., 1951, Proc. Camb. Phil. Soc. **47** 396.
SEN, B., 1933, Bull. Calcutta Math. Soc. **25** 107.
SNEDDON, I. N., 1951, Fourier Transforms (McGraw-Hill Book Company, New York, 1951).
SOKOLNIKOFF, I. S., 1946, Mathematical Theory of Elasticity (McGraw-Hill Book Company, New York, 1946).
SOMIGLIANA, C., 1914, R. C. Accad. Lincei [5] **23** (1) 463.
SOMIGLIANA, C., 1915, R. C. Accad. Lincei [5] **24** (1) 655.
STARR, A. T., 1928, Proc. Camb. Phil. Soc. **24** 489.
STERNBERG, E., 1958, Appl. Mech. Rev. **11** 1.
STROH, A. N., 1958, Phil. Mag. **30** 623.
TEWORDT, L., 1958, Phys. Rev. **109** 61.
TIMOSHENKO, S. and J. N. GOODIER, 1951, Theory of Elasticity (McGraw-Hill Book Company, New York, 1951) 425.
VOLTERRA, V., 1907, Ann. École Norm. Super. [3] **24** 400.
WINTENBERGER, M., 1957, C. R. Acad. Sci. Paris **244** 2800.

CHAPTER IV

PLASTIC WAVES

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CONTENTS

§ 1. Introduction

The mathematical theory of plastic waves is an attempt to explain, in terms of the concepts of plasticity, the propagation in metals of stresses large enough to cause yielding and inelastic behaviour. Originally the theory was concerned with problems of protection against ballistic or explosive attack, but a developed theory would be of much wider application, and might be expected to cast light on such diverse problems as fatigue failure on the one hand and the mechanism of earthquakes on the other.

In fact, as the following work shows, the theory is very far from complete and there remain elementary questions, as for example the importance or otherwise of strain-rate effects in an element, on which no clear decision is yet possible. The crucial difficulty in the way of further understanding of the problem is that there remains a gap between theory and experiment.

Most of the experimental work available on the propagation of plastic waves has aimed at producing longitudinal waves in wires, strips or bars. In such experiments it is possible to make detailed measurements of the longitudinal strain on the surface as a function of time, but, as will be explained in the sequel, a uni-dimensional analysis of the results is not satisfactory, and a complete mathematical analysis is very difficult. It is therefore almost impossible to deduce any general properties of materials at high rates of strain from experiments of this kind. On the other hand, the few problems for which a complete mathematical analysis is possible, for example the problem of plane waves in an unbounded solid or of radially expanding torsional waves in a plate, are either very difficult to reproduce in practice or not susceptible of detailed measurement.

At this stage, then, it is for the theoretician to attempt to apply simple constitutive equations to situations which might conceivably be reproduced in a laboratory, and for the experimentalist to suggest possible experiments in which there is sufficient symmetry for a mathematical analysis of the situation. Attempts to explain experimental results by postulating new forms of constitutive equations are premature, and cannot add to the understanding of the subject, unless complete mathematical analysis accompanies them.

The selection from the literature of topics to be included in this chapter has been made in accordance with the foregoing arguments. Three points about the treatment of the subject may need explanation.

Firstly, the assumption has been made that there is no significant difference in the behaviour of a metal at different rates of strain. The equations relative to quasi-static plasticity have been amended only by the inclusion of inertia terms. This assumption has the practical advantage of simplicity, and accords with the general observation that time-dependent effects (creep) in metals require a fairly long time scale. It is the contention of the author that, at the present time, there is no conclusive experimental evidence of the existence of visco-elastic effects in metals at high rates of strain; however, if such effects should be important, they could be included by modifying the theory given here in fairly obvious ways suggested by the theory of visco-elasticity.

Secondly, in spite of the deficiency of uni-dimensional analysis, a large space has been devoted to problems of uniaxial stress. Partly, this reflects the concentration on such problems in the literature, but it also seems advisable because it affords the simplest illustration of the methods which must be used in more realistic problems.

Thirdly, the number of references to the literature in the body of the chapter has been kept to a minimum. More comprehensive lists of papers of relevance to the subject of each section are given at the ends of the sections.

Finally, the author acknowledges his debt to the monograph by N. CRISTESCU [1958], which has been used as a source book, and from which most of the references to Russian literature have been taken.

§ 2. Equations of Plasticity

It will be assumed throughout this chapter that strains are sufficiently small, so that second order effects may be neglected. A Cartesian strain tensor ε_{ij} may then be defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.1}$$

where u_i, $i = 1, 2, 3$, are the Cartesian components of the displacement of the point with initial coordinates x_i. A superposed dot will be used when necessary to denote the partial derivative with respect to time. The rate of strain tensor is then

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right). \tag{2.2}$$

When an Eulerian description is preferred it is more convenient to define the velocity vector, v_i, and to use a rate of strain

$$\frac{\mathrm{D}\varepsilon_{ij}}{\mathrm{D}t} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2.3}$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion. The strain may then be obtained by integration of (2.3). Notice, however, that this leads to a logarithmic strain, while (2.1) defines engineering strain. For small strains the difference is of second order and may be ignored†. Similarly, a stress tensor σ_{ij} may be defined relative to either the strained or unstrained state of the material, and rate of stress $\dot{\sigma}_{ij}$ follows by partial differentiation.

It is convenient to emphasize from the start that dilatational and deviatoric strains must be treated differently, just as in classical elasticity, so write

$$\Delta = \varepsilon_{kk}\,, \qquad \varepsilon'_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\Delta\,\delta_{ij}\,, \tag{2.4}$$

where Δ is the dilatation, ε'_{ij} the deviatoric or reduced strain, δ_{ij} the unit tensor, and the repeated suffix is summed. In a similar manner write

$$3p = -\sigma_{kk}\,, \qquad \sigma'_{ij} = \sigma_{ij} + p\delta_{ij}\,, \tag{2.5}$$

where p may conveniently be called the hydrostatic stress and σ'_{ij} the reduced stress.

2.1. Elasticity

In classical elasticity the relation between stress and strain is linear,

$$-p = \varkappa\Delta\,, \qquad \sigma'_{ij} = 2G\varepsilon'_{ij} \tag{2.6}$$

and the constants $\varkappa$, G are the bulk modulus and the shear modulus

† The whole discussion is only strictly applicable to first order strains and rotations. Considerable complications arise when the motion is large. See, for example, Green and Rivlin [1960].

or modulus of rigidity. In terms of Lamé's elastic constants λ, μ or Young's modulus E and Poisson's ratio ν, they are expressed by

$$\varkappa = \lambda + \tfrac{2}{3}\mu = \frac{E}{1-2\nu}, \qquad G = \mu = \frac{E}{2(1+\nu)}. \tag{2.7}$$

For small strains and rotations, where rates of stress and strain are obtained as partial derivatives with respect to time, (2.6) may be replaced by

$$-\dot{p} = \varkappa\dot{\Delta}, \qquad \dot{\sigma}'_{ij} = 2G\dot{\varepsilon}'_{ij}. \tag{2.8}$$

It is convenient to extend the idea of elastic behaviour to cases where (2.8) hold but are not necessarily integrable to (2.6). Any region of a metal where the stress rate and strain rate obey (2.8) is then called an elastic region.

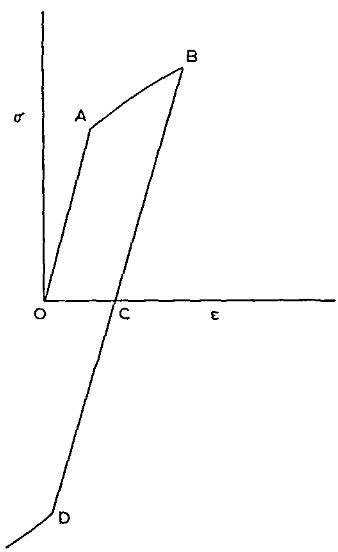

Fig. 2.1. Stress-strain curve.

2.2. Stress-strain curves for metals

If a piece of metal is subjected to a gradually increasing stress, for example in a tension testing machine, its behaviour may be described by a curve in which the stress, σ, is plotted against the strain, ε. The stress-strain curve will commonly be of one of three types, illustrated in Figs. 2.1 and 2.2. In Fig. 2.1 the curve is straight for stresses up to a certain limit (point A), but the slope then changes, more or less suddenly, and as the stress is further increased the strain increases more rapidly. In such circumstances A is called the yield point of the material, and the behaviour symbolised by the curve beyond A is called plastic. If the material is unloaded, by slowly reducing the stress, from a point B beyond A, the curve follows a line BC parallel to OA cutting the axis $\sigma = 0$ at a point C which corresponds to permanent strain. If now the stress changes sign, the linear behaviour persists to a point D, after which plastic strain again occurs. If $|\sigma_D| < \sigma_B$, the material is said to exhibit a Bauschinger effect. The curves in Fig. 2.2 show two possible departures from the above behaviour. In (a) plastic behaviour begins as soon as the stress is applied, but unloading nevertheless gives a linear relation (BC) at a slope equal to that of the tangent to the curve OB at O. In (b) there is some reduction of stress for increasing strain, followed by a gradual increase of stress. The material is then said to exhibit upper and lower yield points.

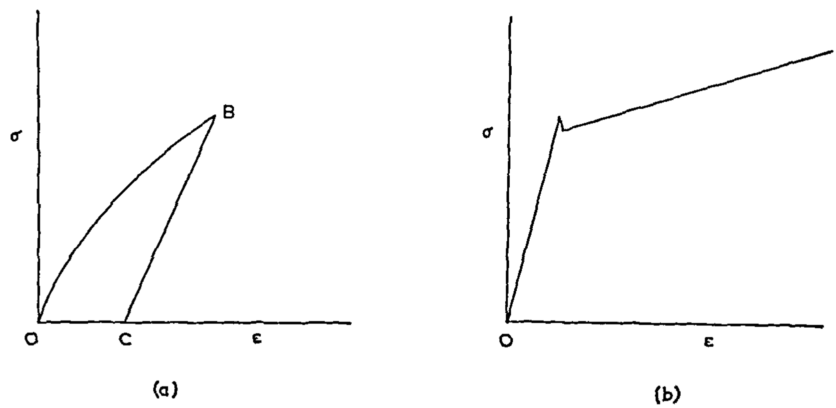

Fig. 2.2. Stress-strain curves.

In order to simplify the analysis, it is common to simplify the curve of Fig. 2.1 by one or both of two approximations. In the first, the rigid-plastic approximation, elastic strains are neglected so that the lines OA and BCD are replaced by vertical lines, corresponding to constant strain. The second approximation, of perfect plasticity, is to assume that the curve AB is straight and parallel to the strain axis, so that plastic strain takes place at constant stress.

2.3. Yield surface

Before the theory of plasticity can be applied to general situations, the results of experiments on test specimens must be expressed in terms applicable to a general stress tensor. In place of the yield stress, define a yield surface.

Let

$$f(\sigma_{ij}) = k \tag{2.9}$$

represent a closed surface, in the nine space with coordinates σ_{ij}, which contains the origin. Then, if the material behaves elastically, according to (2.8), for $f < k$, and plastically for $f \geqslant k$, the surface is called the yield surface for the material. Note that in general f, k may depend on the previous history of deformation (and temperature) of the material, and may differ from point to point. The most important types of material are (i) perfectly plastic materials, where f is a definite function of its arguments and k a constant, and (ii) work-hardening materials for which f is again a definite function of its arguments, but k increases as work is done on the material. It is usual to ignore the Bauschinger effect, but if it is required to include it the whole surface may be made to move relative to the origin as strain takes place (PRAGER [1955]).

The functional form of f is considerably restricted if isotropy is assumed. It is then possible to transform, at a point of the material, to principal axes, so that $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{33} = \sigma_3$ and the other components of stress are zero. Then

$$f = f(\sigma_1, \sigma_2, \sigma_3)$$

and it is convenient to label the axes so that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. A full discussion of the equations of plasticity is given by HILL [1950], chapter II.

For the present purpose two simple forms of yield surface will be assumed. The first, due to TRESCA [1864], is given by

$$f = |\sigma_1 - \sigma_3| \tag{2.10}$$

and the second, suggested by VON MISES [1913] is

$$f = \sigma_{ij}\,\sigma_{ij} = \tfrac{2}{3}(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2)\,. \tag{2.11}$$

Note that, for both these special cases, the yield surface remains unchanged when an arbitrary hydrostatic stress is superimposed. This is an assumption which is valid for metals, but not for some other materials (e.g. soils).

2.4. *Stress-strain relations in the plastic zone*

The yield surface, as defined above, provides a limit on the stresses for which the material behaves elastically, but gives no direct information on the plastic behaviour. Stress-strain relations must be the subject of further assumptions. The commonest of these is that of a plastic potential which coincides with the yield function f (HILL, loc. cit. p. 33 et seq.). According to this assumption, the strain may be divided into two tensors, the elastic strain e_{ij} and the plastic strain η_{ij}, such that

$$\varepsilon_{ij} = e_{ij} + \eta_{ij}\,, \tag{2.12}$$

$$e_{ij} = \frac{-p}{3\varkappa}\,\delta_{ij} + \frac{1}{2\sigma}\,\sigma'_{ij} \tag{2.13}$$

and

$$\dot{\eta}_{ij} = \dot{\Lambda}\,\frac{\partial f}{\partial \sigma_{ij}} \tag{2.14}$$

where $\dot{\Lambda}$ is a factor of proportionality which will in general be a function of time and position. It is written in the form of a time derivative in order that (2.14) may be seen to be homogeneous linear in time derivatives. This again implies that the stress and strain in a material may depend on the loading history, but not on the rate at which the loading programme is carried out. Note also that, if f is independent of hydrostatic stress, the plastic strain takes place without change of volume.

2.5. Hardening and perfect plasticity

It is not necessary to the work of this chapter to adopt a theory of the mechanism of hardening, by which the yield stress increases as plastic flow proceeds (see, for example, MOTT [1952, 1953]). It is sufficient to remark that hardening occurs, even when the plastic strains alternate in sign. The simplest mathematical treatment of the phenomenon is given by retaining the forms of f in (2.10) or (2.11) and writing

$$k = k(W)\,, \tag{2.15}$$

where

$$W = \sigma_{ij}\,\dot{\eta}_{ij} \tag{2.16}$$

is the rate at which plastic work is done by external forces. If, now, the equality $f = k$ is accepted as a condition for plastic flow, and (2.14), (2.15) and (2.16) are combined:

$$\frac{\partial f}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} = \dot{k} = \frac{\mathrm{d}k}{\mathrm{d}W}\,\dot{W} = \dot{\Lambda}\,\frac{\mathrm{d}k}{\mathrm{d}W}\,\sigma_{ij}\,\frac{\partial f}{\partial \sigma_{ij}}$$

and $\dot{\Lambda}$ becomes a determinate quantity.

For a perfectly plastic material, on the other hand, the quantity k is constant and Λ remains indeterminate. In fact Λ may then be eliminated from (2.14) to give five equations, but it is often more convenient to retain the symmetrical form.

2.6. Unloading

It will be assumed throughout that unloading proceeds elastically. Then, whenever

$$f < k \quad \text{or} \quad f = k\,, \quad \dot{f} < 0\,,$$

the plastic strain remains constant, $\dot{\eta}_{ij} = 0$ and

$$\dot{\varepsilon}_{ij} = \dot{e}_{ij}\,.$$

2.7. Influence of hydrostatic stress

Both the Tresca and von Mises forms for the yield criterion are independent of the hydrostatic stress p, and so depend only on the reduced stress σ'_{ij}. The corresponding relations between stress and strain, (2.14), show the dilatation in the plastic strain as zero. It follows that at all times the dilatation is related to the hydrostatic stress by the elastic equation

$$-p = \varkappa \Delta\,. \tag{2.17}$$

This equation is satisfactory for most of the problems discussed in this chapter, where the hydrostatic stress is small. However, for large hydrostatic stress, such as may occur when stress waves are initiated by an explosion, a better representation of the properties of a real material may be needed. Bridgman [1931], chapter VI suggests that for large hydrostatic stress equation (2.17) should be replaced by

$$\Delta = -ap + bp^2\,, \tag{2.18}$$

where a, b are constant and $a/b = O(10^5)$ kg/cm^2. The compressibility thus decreases as the pressure increases. There seems to be no evidence to suggest that the strain implied by (2.18) is not reversible, so the equation may be used both for loading and unloading. Note that (2.18) does not come under the heading of plastic strain, but is only a non-linear elastic behaviour. Later work by Bridgman [1952], pp. 67–71, suggests that both the form of the yield function and the magnitude of the yield stress may depend on p, for large pressures, but the effect is fairly small, and may reasonably be neglected in the first instance.

2.8. Rate of strain effects

In the previous sections no allowance has been made for strain-rate effects. It is true that the stress-strain relation (2.14) contains the rate of strain, but the equation is homogeneous in the time derivatives, and therefore remains unchanged if the time-scale is altered in an arbitrary manner. It follows that the equations here suggested do not allow of visco-elastic behaviour, creep or any other time dependent phenomenon. At the time of writing, no method of including plastic and visco-elastic effects in a single tensor theory has been generally agreed, though considerable discussion has taken place in connexion with special problems, mostly uni-dimensional. The theory of visco-elastic waves without plasticity is the subject of a survey by Hunter [1960].

At this point it may be well to remark also on the so-called deformation theories of plasticity. In such theories plastic effects are discussed essentially in terms of the elastic equations (2.6) but with variable moduli $\varkappa$, G. A critical appraisal of the inadequacy of such theories is given by Hill [1950], p. 46 et seq., in connexion with quasi-static problems. As far as dynamic problems are concerned, the deformation theories are no simpler to work with than the flow theory, and so will not be further discussed here.

2.9. Standard plastic models

For easy reference the equations for the most commonly used theories of plasticity are collected below.

For a perfectly *rigid-plastic* material the yield condition is

$$f(\sigma_{ij}) = k\,, \quad \text{constant}\,, \tag{2.19}$$

and the flow law

$$\dot{\varepsilon}_{ij} = \dot{\eta}_{ij} = \Lambda\,\frac{\partial f}{\partial \sigma_{ij}}\,. \tag{2.20}$$

When the von Mises criterion for plasticity is used these reduce to

$$\sigma'_{ij}\sigma'_{ij} = 2k^2\,, \qquad \dot{\varepsilon}_{ij} = \Lambda\sigma'_{ij}\,. \tag{2.21}$$

Equations (2.21) specify a *St. Venant-Mises* material.

The corresponding *elastic-plastic* material, with

$$\dot{\eta}_{ij} = \Lambda\sigma'_{ij}\,, \qquad \varepsilon_{ij} = e_{ij} + \eta_{ij}\,, \tag{2.22}$$

is called a *Prandtl-Reuss* material. In either case *work-hardening* is most easily included by writing

$$k = k(W), \qquad W = \sigma_{ij}\dot{\eta}_{ij}\,. \tag{2.23}$$

Note that the special case of linear hardening (that is, a linear stress-strain curve in the plastic region) is given by $k\,\mathrm{d}k/\mathrm{d}W = \text{const.}$

Significant simplifications sometimes result from the use of the Tresca criterion of plasticity,

$$\sigma_1 - \sigma_3 = k\,. \tag{2.24}$$

The associated flow rule is

$$\dot{\eta}_1 + \dot{\eta}_2 + \dot{\eta}_3 = 0\,, \tag{2.25}$$

with

$$\dot{\eta}_2 = 0\,, \quad \sigma_1 > \sigma_2 > \sigma_3\,,$$

and (for example)

$$\dot{\eta}_1 - \dot{\eta}_3 > \dot{\eta}_2 - \tfrac{1}{2}(\dot{\eta}_1 + \dot{\eta}_3) > 0\,, \quad \sigma_1 = \sigma_2\,. \tag{2.26}$$

§ 3. Waves of Uniaxial Stress

At first sight the simplest way to investigate wave motion in an elastic-plastic material is by producing longitudinal waves in a thin wire. One may then replace the stress tensor by a single term, corresponding to the mean stress over the cross-section of the wire, and assume that this is related to the mean strain in the same way as in a quasi-static loading of the wire. This approach has certain limitations, which are discussed in section 3.8, but it has also considerable advantages, in that it illustrates the main properties of plastic waves without undue algebraic complication, and so makes a good introduction to the subject.

The basic theory presented here was first published by RAKHMATULIN [1945a, 1945b], who used an analysis of Lagrangian type, and considered both longitudinal and transverse waves in a flexible string. A similar theory was suggested by VON KARMAN [1942] in a report for restricted circulation, but was presented in less detail. An equivalent theory in terms of Eulerian coordinates was formulated by TAYLOR [1940, 1942] also in official reports.

It is assumed throughout this section that the stress tensor reduces to a single term $\sigma_{xx} = \sigma$, representing the longitudinal tension in the wire, and that for continued loading there is a stress-strain relation

$$\sigma = F(\varepsilon) \tag{3.1}$$

where ε is written for ε_{xx} and F is a non-decreasing function of ε, independent of the rate of loading.

The same assumptions may be applied to compressive stress in a cylinder of uniform cross-section, but the assumption of uniform stress over a cross-section is then of more doubtful validity. In both tension and compression cases it is assumed that unloading, and reloading up to a previously reached limit, take place according to the elastic equation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = E ,$$

where E is Young's modulus.

3.1. Lagrangian method for longitudinal, loading waves

Let $x + \xi$ be the coordinate at time t of the point of the wire initially at x. Then the (engineering) strain is

$$\varepsilon = \frac{\partial \xi}{\partial x} \tag{3.2}$$

and the velocity at the point is

$$u = \frac{\partial \xi}{\partial t} . \tag{3.3}$$

The equation of motion of an element is

$$\varrho \frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x} = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} \frac{\partial \varepsilon}{\partial x} , \tag{3.4}$$

where ϱ is the density in the initial state. Also, (3.2) and (3.3) give

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial u}{\partial x} . \tag{3.5}$$

Eqs. (3.4) and (3.5) are a pair of non-linear hyperbolic equations for u, ε. When there is no unloading, (3.1) holds and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = F'(\varepsilon) . \tag{3.6}$$

The characteristic coordinates appropriate to (3.4) and (3.5) are then α, β where

$$\delta\alpha = \delta x + c(\varepsilon)\delta t , \tag{3.7}$$

and

$$\delta\beta = \delta x - c(\varepsilon)\delta t , \tag{3.8}$$

with

$$\varrho c^2(\varepsilon) = F'(\varepsilon) , \tag{3.9}$$

and (3.4) and (3.5) may be replaced by

$$\frac{\partial u}{\partial \alpha} - c\,\frac{\partial \varepsilon}{\partial \alpha} = \frac{\partial u}{\partial \beta} + c\,\frac{\partial \varepsilon}{\partial \beta} = 0 , \tag{3.10}$$

or

$$\frac{\partial}{\partial \alpha}(u - \theta) = \frac{\partial}{\partial \beta}(u + \theta) = 0 , \tag{3.11}$$

where

$$\theta = \int c \, \mathrm{d}\varepsilon , \tag{3.12}$$

and this substitution is valid because c does not depend on u. The general solution of (3.11) is

$$u - \theta = f(\beta) , \qquad u + \theta = g(\alpha) , \tag{3.13}$$

but this solution is of less practical value than might be supposed because the characteristic curves (3.7) and (3.8) depend on the solution, and to find them explicitly is equivalent to solving the original equations.

Simple waves

In the case of a simple wave, in which there is propagation in only one direction along the wire, the solution can be expressed directly in terms of x, t. Assume

$$g(\alpha) = \text{constant} \tag{3.14}$$

so that u, θ and ε depend only on β,

$$u = u(\beta) , \qquad \theta = \theta(\beta) , \qquad \varepsilon = \varepsilon(\beta) . \tag{3.15}$$

Solve the last of (3.15) to give $\beta = \beta(\varepsilon)$. Then, with a change of notation for the functions,

$$\theta = \theta(\varepsilon) = \varrho^{-\frac{1}{2}} \int [F'(\varepsilon)]^{\frac{1}{2}} \, \mathrm{d}\varepsilon \tag{3.16}$$

and

$$u(\varepsilon) = g - \theta(\varepsilon) . \tag{3.17}$$

(3.8) then shows that any particular characteristic, $\beta = \beta_0 = \beta(\varepsilon_0)$, is a straight line of slope

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(\varepsilon_0) = \varrho^{-\frac{1}{2}}[F'(\varepsilon_0)]^{\frac{1}{2}} , \tag{3.18}$$

and u, ε are constant on this line.

As an example, consider an infinite wire $x > 0$, initially unstrained, and subject to the boundary condition $\varepsilon = G(t)$, $G'(t) > 0$, at $x = 0$. Then, if $\varepsilon_0 = G(t_0)$, (3.18) integrates to

$$x = \varrho^{-\frac{1}{2}} [F'(\varepsilon_0)]^{\frac{1}{2}} (t - t_0) . \tag{3.19}$$

This simple wave solution is the one given by VON KARMAN [1942], but the full characteristic theory is due to RAKHMATULIN [1945a]. The lines (3.19) are of increasing slope if $\mathrm{d}^2\sigma/\mathrm{d}\varepsilon^2 > 0$, and commence with one of gradient $x/t = c_0$ appropriate to elastic waves (see Fig. 3.1). On each line u, ε, σ are constants, equal to their values at $x = 0$. Note that, if there is a finite initial yield stress Y, all the characteristic lines on which $\sigma < Y$ have the same slope c_0. If $\mathrm{d}^2\sigma/\mathrm{d}\varepsilon^2 \leqslant 0$ it may be necessary to introduce shock waves (see section 3.4).

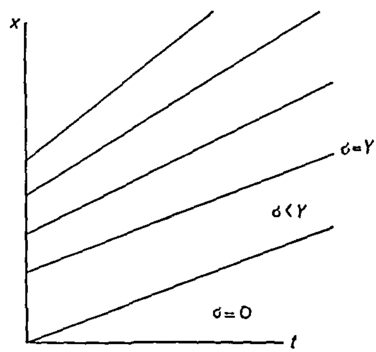

Fig. 3.1. Characteristic curves in a simple wave.

The above theory was extended by G. S. SHAPIRO [1946, 1952] to apply when the cross-section of the wire varies along the length, assuming that the stress is nevertheless uniaxial to a sufficient approximation.

3.2. GENERAL SOLUTIONS FOR WAVES WITHOUT UNLOADING

When the boundary conditions in a problem are such that no unloading of any element of the material occurs, but waves propagate in both directions, eqs. (3.4) and (3.5) may be solved by the kind of approximate procedures which have long been familiar in the theory of the Cauchy problem for hyperbolic differential equations in two variables (see, for example, COURANT, FRIEDRICHS and LEWY [1928] or FRIEDRICHS [1948]). It is sufficient here to summarize the basic principles of three of the most useful methods.

3.2.1. *Use of the characteristic curves in the physical plane*

Suppose the values of u, ε are given on a smooth curve Γ (Fig. 3.2), nowhere tangent to a characteristic curve ($\mathrm{d}x/\mathrm{d}t \neq \pm c$). Choose near points of Γ, say P, Q, and construct straight lines PR, QR of slopes $c(\mathrm{P})$, $-c(\mathrm{Q})$ to meet in R. Then PR, QR may be taken as first

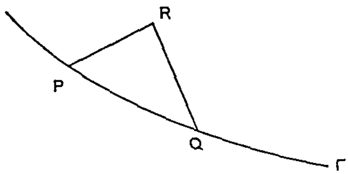

Fig. 3.2. Characteristic arcs.

approximations to arcs of characteristic curves, α constant, β constant, and the values of u, ε at R may then be deduced from those at P, Q, by use of (3.10). The method may be refined by using the approximate value of c at R to correct the slopes or curvatures of the arcs PR, QR. By successive applications of this method the values of u, ε may be found on a line near to Γ, and thence, by stages, in the whole region in which they are uniquely determined by the values on Γ. The modifications to this procedure when Γ is either not smooth, or is somewhere tangent to a characteristic curve, are explained in the above references (or see HILL, loc. cit., p. 113 et seq., p. 136 et seq.).

3.2.2. *Use of characteristic curves in the u, ε plane*

A variation in the above method consists in interchanging the dependent and independent variables so that eqs. (3.4) and (3.5) are replaced by

$$\varrho\,\frac{\partial x}{\partial \varepsilon} = F'(\varepsilon)\,\frac{\partial t}{\partial u}\,, \qquad \frac{\partial t}{\partial \varepsilon} = \frac{\partial x}{\partial u}\,.$$

The characteristic coordinates for these equations are defined by

$$\delta\alpha' = \delta u + c\,\delta\varepsilon = \delta u + \delta\theta\,,$$
$$\delta\beta' = \delta u - c\,\delta\varepsilon = \delta u - \delta\theta\,,$$

and so are

$$\alpha' = u + \theta\,, \qquad \beta' = u - \theta\,. \tag{3.20}$$

Then

$$\frac{\partial x}{\partial \alpha'} = c\,\frac{\partial t}{\partial \alpha'}, \qquad \frac{\partial x}{\partial \beta'} = -c\,\frac{\partial t}{\partial \beta'} \tag{3.21}$$

where $c = c(\alpha' - \beta')$ is a known function.

The Cauchy problem of (3.21) now refers to a curve Γ' in the u, ε plane on which x, t are given, the characteristic arcs PR, QR are known from (3.20) and the values of x, t for given u, ε follow by direct quadrature, with no need to use successive approximations to the characteristics.

3.2.3. *The direct use of finite differences*

A third method of numerical integration of (3.4) and (3.5) is to choose an arbitrary rectangular grid of lines

$$x = x_0 + i\delta x\,, \qquad t = t_0 + j\delta t\,, \qquad i,j = 1,2,3\,\ldots\,,$$

and to replace the equations directly by the finite difference approximations based on this grid. Write

$$u_{ij} = u(x_0 + i\delta x\,,\, t_0 + j\delta t)\,, \qquad \varepsilon_{ij} = \varepsilon(x_0 + i\delta x\,,\, t_0 + j\delta t)\,.$$

Then (3.4) and (3.5) are replaced by

$$\varrho\,\frac{(u_{i,j+1} - u_{i,j})}{\delta t} = F'(\varepsilon')\,\frac{(\varepsilon_{i+1,j} - \varepsilon_{i,j})}{\delta x}\,, \tag{3.22}$$

and

$$\frac{u_{i+1,j} - u_{i,j}}{\delta x} = \frac{\varepsilon_{i,j+1} - \varepsilon_{i,j}}{\delta t} \tag{3.23}$$

where

$$\varepsilon' = \tfrac{1}{4}(\varepsilon_{i+1,j+1} + \varepsilon_{i+1,j} + \varepsilon_{i,j+1} + \varepsilon_{i,j})\,.$$

The equations (3.22) and (3.23), for a grid covering as much of the x, t plane as may be necessary, may now be solved either by matrix methods or by iteration to give an approximate solution of the differential equations. A discussion of the legitimacy of this procedure in general cases may be found in the paper by FRIEDRICHS [1948], already referred to, or see O'BRIEN, HYMANS and KAPLAN [1951].

For the problem most relevant to the present discussion, that of obtaining u, ε for the case of an initially stationary unstrained medium under increasing end loads, the procedure is legitimate provided that $\delta t < c^{-1}\delta x$.

3.3. Longitudinal waves with unloading

It is assumed in plasticity theory that unloading, or subsequent reloading to a value of σ not higher than the previous maximum, takes place on a stress-strain curve whose slope is that appropriate to elastic behaviour. Suppose that the highest previous stress reached was σ_m, with strain ε_m, then

$$\sigma = \sigma_m - E(\varepsilon_m - \varepsilon) \tag{3.24}$$

for $\sigma < \sigma_m$. Substitute in (3.4) then

$$\varrho \frac{\partial u}{\partial t} = E \frac{\partial \varepsilon}{\partial x} + \frac{\partial \sigma_m(x)}{\partial x} - E \frac{\partial \varepsilon_m(x)}{\partial x} . \tag{3.25}$$

Eq. (3.5) still holds:

$$\frac{\partial u}{\partial x} = \frac{\partial \varepsilon}{\partial t} , \tag{3.26}$$

and the solution of (3.25) and (3.26) is

$$\varepsilon = \varepsilon_0(x) + g_1(x + c_0 t) + g_2(x - c_0 t) , \tag{3.27}$$

$$u = c_0 g_1(x + c_0 t) - c_0 g_2(x - c_0 t) , \tag{3.28}$$

where $\varepsilon_0(x) = \varepsilon_m(x) - E^{-1}\sigma_m(x)$ is the permanent strain left behind at x by previous plastic loading.

Eqs. (3.27) and (3.28) can be fitted into any of the approximate methods already mentioned, though of course some care is necessary to ensure that at each stage of a calculation the appropriate set of equations is used, according as the previous maximum of stress is or is not exceeded.

3.4. Shock waves

In the above analysis it has been assumed that, except perhaps at a boundary, u, ε are differentiable functions of x, t. The possibility

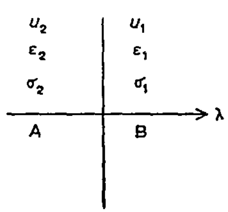

Fig. 3.3. Shock wave.

of discontinuities in u, ε must also be considered (White and Griffis [1947]). Such discontinuities are conveniently called shock waves by analogy with the usage of hydrodynamics. Let a shock wave move along the wire with a velocity λ in the direction of x increasing. Let the values of the particle velocity, strain and stress immediately in front of the shock wave be u_1, ε_1, σ_1, and those immediately behind it u_2, ε_2, σ_2, Fig. 3.3. Consider two points A, B with AB $= \delta x$, during the time interval $\delta t = \lambda^{-1}\delta x$ when the shock wave lies between them. Then the velocities differ by $u_1 - u_2$, so the change in length in the time δt is $(u_1 - u_2)\delta t$, and the change in strain is

$$(u_1 - u_2)\lambda^{-1} = \varepsilon_2 - \varepsilon_1 \,. \tag{3.29}$$

Again, the tensions at A,B are σ_2, σ_1 and the change in velocity of AB in the time is $u_2 - u_1$, so $\varrho(u_2 - u_1)\delta x = (\sigma_1 - \sigma_2)\delta t$, and

$$\lambda\varrho(u_2 - u_1) = \sigma_1 - \sigma_2 = \lambda^2\varrho(\varepsilon_1 - \varepsilon_2) \tag{3.30}$$

using (3.29).

For elastic conditions $\sigma_1 - \sigma_2 = E(\varepsilon_1 - \varepsilon_2)$ so (3.30) is satisfied when

$$\lambda^2 = \frac{E}{\varrho} = c_0{}^2 \tag{3.31}$$

and the shock waves have the characteristic value for infinitesimal disturbances. For plastic behaviour (3.30) determines a shock velocity only when σ_1, σ_2 can be expressed in terms of ε_1, ε_2 respectively. For example, if it is assumed that the states of the material given by the suffixes 1 and 2 both lie on the loading curve $\sigma = \sigma(\varepsilon)$,

$$\varrho\lambda^2 = \frac{\sigma(\varepsilon_2) - \sigma(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} \,. \tag{3.32}$$

This equation is sometimes called the Hugoniot equation, from its similarity to the corresponding equation in the theory of flow of compressible fluids. It is necessary to distinguish three cases.

(i) Linear strain-hardening

If the plastic loading curve has the form

$$\sigma = P\varepsilon \,,$$

with constant P, then (3.32) leads to $\varrho\lambda^2 = P$, and the shock velocity coincides with the characteristic velocity, just as under elastic conditions.

(ii) Increasing hardening

If $d^2\sigma/d\varepsilon^2 > 0$, the stress-strain curve is concave to the stress axis, and (3.32) determines $\varrho\lambda^2$ as the slope of a chord to the curve, with λ greater than the characteristic velocity at the lower-stress end of the chord. This implies that a shock wave (of increasing stress) propagates faster than infinitesimal disturbances. Again, infinitesimal waves travel faster, the higher the stress they carry, so one would expect high stresses to overtake low, and a shock wave to result from the coalescing of infinitesimal wavelets. In these circumstances, then, shock waves must be regarded as possible.

(iii) Decreasing hardening

When $\mathrm{d}^2\sigma/\mathrm{d}\varepsilon^2 < 0$, the slope of a chord to the stress-strain curve is less than that of the tangent at its lower end, infinitesimal loading waves travel faster than shock waves, and one would expect any shock wave in initial or boundary conditions to be dispersed into a continuous solution.

Summing up: shock waves are to be expected only for linear hardening or increasing hardening. (The arguments above can be replaced by more rigorous arguments based on thermodynamical principles; see MORLAND [1959].)

It will be observed that with linear or increasing strain-hardening for increasing stress, and elastic behaviour for decreasing stress, both loading and unloading shock waves are possible. Such waves may meet, or overtake each other, with consequent reflection and refraction. The analysis of such conditions is straightforward, but tedious and will not be repeated here. Details can be found in the papers already referred to in this section.

3.5. EXAMPLES

It is possible to construct an exact solution of the foregoing equations by an inverse method due to RAKHMATULIN [1945a]. Consider an infinite region $x > 0$, initially at rest and unstrained. Use an elastic-plastic theory with $\sigma = f(\varepsilon)$ on the initial loading curve and Young's modulus E in any elastic part. Assume as boundary condition a stress $\sigma = S(t)$ at $x = 0$. First take $S = S_0$, constant for $t > 0$. Then the solution is of the simple wave type, and may be illustrated by drawing the characteristic lines as in Fig. 3.4. On each characteristic $x/t = c(\varepsilon)$, $\sigma = f(\varepsilon)$ and $u = -\int_0^\varepsilon [f'(\varepsilon)]^{\frac{1}{2}}\mathrm{d}\varepsilon = -\,\psi(\varepsilon)$ say. The stress is zero for $x > c_0 t$ and constant, equal to S_0, for $x/t < c(f^{-1}[S_0])$, where f^{-1} denotes the function inverse to f.

Next assume that the stress changes suddenly at $t = 0$ to S_0, but then decreases towards zero in such a way that

$$S = S(t) > 0\,, \quad \dot{S}(t) < 0\,, \quad t > 0\,, \quad S(0) = S_0\,. \tag{3.33}$$

Suppose that the unloading due to the reduction in S affects only the

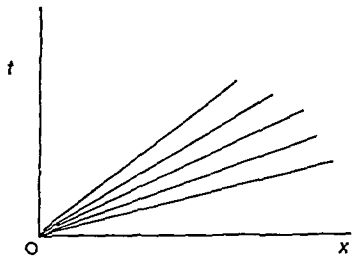

Fig. 3.4. Characteristic lines.

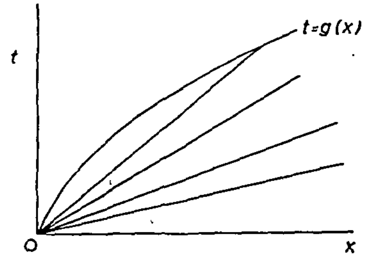

Fig. 3.5. Plastic-elastic boundary.

region $t > g(x)$, with $g'(x) > c_0$, $g''(x) < 0$, as shown in Fig. 3.5. Then the solution (3.33) remains valid for $t < g(x)$. Assume also that no plastic strain takes place for $t > g(x)$ and that the strain is continuous across $t = g(x)$. Then the maximum strain reached, $\varepsilon_{\mathrm{m}}(x)$, is that at x, $g(x)$ and

$$g(x) = x/c(\varepsilon_{\mathrm{m}}) \tag{3.34}$$

where strictly $\varepsilon_{\mathrm{m}} = \varepsilon_{\mathrm{m}}(x)$, but the argument is suppressed for ease of notation. Now, for $t > g(x)$, (3.27) and (3.28) give

$$\varepsilon = \varepsilon_{\mathrm{m}}(x) - E^{-1}\sigma_{\mathrm{m}}(x) + F_1(x + c_0t) + F_2(x - c_0t)\,,$$
$$u = c_0F_1(x + c_0t) - c_0F_2(x - c_0t)\,.$$

Apply these equations at $t = g(x)$. Then

$$\varepsilon_{\mathrm{m}}(x) = \varepsilon_{\mathrm{m}}(x) - E^{-1}\sigma_{\mathrm{m}}(x) + F_1(x + c_0t) + F_2(x - c_0t)$$

and

$$-\,\psi(\varepsilon_{\mathrm{m}}) = c_0F_1(x + c_0t) - c_0F_2(x - c_0t)$$

with $t = g(x)$. Thus

$$2F_1(x + c_0g[x]) = [E^{-1}\sigma_{\mathrm{m}}(x) - c_0^{-1}\psi(\varepsilon_{\mathrm{m}})]\,, \tag{3.35}$$
$$2F_2(x - c_0g[x]) = [E^{-1}\sigma_{\mathrm{m}}(x) + c_0^{-1}\psi(\varepsilon_{\mathrm{m}})]\,. \tag{3.36}$$

If now $\varepsilon_{\mathrm{m}}(x)$ is assumed known, $g(x)$ follows from (3.34), and (3.35) and (3.36) determine the functions F_1 and F_2. These in turn may be used to find the strain, and hence the stress, at $x = 0$, and so $S(t)$ is given in terms of $\varepsilon_{\mathrm{m}}(x)$. A comparison of the results above and below the curve $t = g(x)$ shows that the first derivative of strain is discontinuous across the curve.

RAKHMATULIN [1946] extended this technique to the propagation of stress due to a load, unload, reload sequence. The only significant alteration is that necessary to allow for the variation of yield stress, in the reloading phase, due to the different maxima of stress in the previous loading.

The above solution suffers from the usual disadvantage of inverse methods, namely that the boundary conditions cannot be arbitrarily assigned. RAKHMATULIN [1952] later gave a method for solving for given boundary conditions, by using a series expression for $g(x)$, and determining the coefficients in terms of $S(t)$. However, the analysis is very cumbersome, and the mathematical restrictions necessary are onerous. A more reliable way of attacking the general problem is to use one of the numerical methods suggested in section 3.2, keeping a check on the stress at each stage to determine whether the previous maximum stress is or is not exceeded, and using plastic or elastic theory accordingly. The most instructive example of this technique is that due to LEE [1953], who assumed a finite rod, with given velocity at one end and freedom from stress at the other. The results show the interesting property that the plastic flow takes place in two distinct stages, separated by a region of unloading. This result emphasizes the danger of trying to extend the analytic method of Rakhmatulin, summarized above, to problems other than simple waves.

3.6. Transverse waves in flexible strings

There is no difficulty in extending the above analysis to allow for transverse motion of a string, provided that the stress remains uniaxial (RAKHMATULIN [1945b]). For example, consider a thin, perfectly flexible wire, elastic or plastic, and neglect shear stress, bending moment and kinetic energy of rotation. Let a point of the wire, initially at $(x, 0, 0)$ move to the point $(x + \xi, \eta, \zeta)$ at time t. Take $(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$ as the direction cosines of the tangent to the wire and $r = 1 + \varepsilon$, as the ratio of the stretched to the unstretched length, where ε is the small strain but the angles are not restricted to be small.

Then

$$1 + \frac{\partial\xi}{\partial x} = r\cos\theta\,, \quad \frac{\partial\eta}{\partial x} = r\sin\theta\cos\varphi\,, \quad \frac{\partial\zeta}{\partial x} = r\sin\theta\sin\varphi\,. \quad (3.37)$$

The stress-strain relation gives

$$\sigma = \sigma(\varepsilon) \tag{3.38}$$

and the equations of motion are

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial x} (\sigma \cos \theta)\,, \tag{3.39}$$

$$\varrho \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x} (\sigma \sin \theta \cos \varphi)\,, \tag{3.40}$$

$$\varrho \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial}{\partial x} (\sigma \sin \theta \sin \varphi)\,, \tag{3.41}$$

where ϱ is the density. Differentiate with respect to x, and substitute for $\partial\xi/\partial x$, $\partial\eta/\partial x$ and $\partial\zeta/\partial x$ from (3.37). Eliminate second derivatives of each pair of r, θ, φ in turn. Then

$$\varrho\left\{\frac{\partial^2 r}{\partial t^2} - r\left(\frac{\partial\theta}{\partial t}\right)^2 - r\sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2\right\} = \frac{\partial^2\sigma}{\partial x^2} - \sigma\left(\frac{\partial\theta}{\partial x}\right)^2 - \sigma\sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2, \tag{3.42}$$

$$\varrho\left\{r\sin\theta\frac{\partial^2\varphi}{\partial t^2} + 2\frac{\partial}{\partial t}(r\sin\theta)\frac{\partial\varphi}{\partial t}\right\} = \theta\sin\sigma\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial}{\partial x}(\sigma\sin\theta)\frac{\partial\varphi}{\partial x}, \tag{3.43}$$

$$\varrho\left\{r\frac{\partial^2\theta}{\partial t^2} + 2\frac{\partial r}{\partial t}\frac{\partial\theta}{\partial t} - r\sin\theta\cos\theta\left(\frac{\partial\varphi}{\partial t}\right)^2\right\}$$

$$= \sigma\frac{\partial^2\theta}{\partial x^2} + 2\frac{\partial\sigma}{\partial x}\frac{\partial\theta}{\partial x} - \sigma\sin\theta\cos\theta\left(\frac{\partial\varphi}{\partial x}\right)^2. \tag{3.44}$$

Eqs. (3.42), (3.43) and (3.44) are quasi-linear second-order hyperbolic equations with the same characteristics as the set

$$\varrho\frac{\partial^2 r}{\partial t^2} = \varrho\frac{\partial^2\varepsilon}{\partial t^2} = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\frac{\partial^2\varepsilon}{\partial x^2}\,, \tag{3.45}$$

$$\varrho r\frac{\partial^2\varphi}{\partial t^2} = \sigma\frac{\partial^2\varphi}{\partial x^2}\,, \tag{3.46}$$

$$\varrho r\frac{\partial^2\theta}{\partial t^2} = \sigma\frac{\partial^2\theta}{\partial x^2}\,. \tag{3.47}$$

Thus, the longitudinal waves travel with the same velocity, $(\varrho^{-1}\mathrm{d}\sigma/\mathrm{d}\varepsilon)^{\frac{1}{2}}$, as in the previous analysis, and infinitesimal transverse waves travel with the velocity $(\sigma/\varrho)^{\frac{1}{2}}$, which would be expected from the classical theory of transverse waves in an inextensible string.

Transverse shock waves, in which there are sudden changes in the direction of the tangent to the string, may be treated by an obvious extension of the theory of section 3.4. The result of carrying out the analysis is that transverse shocks are found to travel at the same speed as infinitesimal transverse waves, and to involve no change in tension, while sharp changes in tension travel with the same speed, relative to the string, as in the case of pure longitudinal motion. When a transverse shock and a longitudinal shock meet, there are, of course, reflected and transmitted waves of both kinds.

RAKHMATULIN [1945b, 1947, 1952] also considered various cases of the impact of smooth or rough solids against elastic-plastic strings, including equations of relative motion over the surfaces of the solids.

3.7. USE OF EULERIAN COORDINATES

When large displacements are envisaged, there are certain advantages in the use of Eulerian coordinates, as suggested by TAYLOR [1940]. For simplicity, the discussion here will be confined to motion in two dimensions.

Let s, ψ be intrinsic coordinates on the deformed string, and U, V the tangential and normal components of velocity. Use $\mathrm{D}\varepsilon/\mathrm{D}t$ for the rate of strain referred to current length (so that integration following the motion gives ε as the logarithmic strain). Then there are two geometrical relations,

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial U}{\partial s}, \qquad \frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial V}{\partial s}, \tag{3.48}$$

and two dynamic equations,

$$\varrho \frac{\mathrm{D}U}{\mathrm{D}t} = \frac{\partial \sigma}{\partial s}, \quad \varrho \frac{\mathrm{D}V}{\mathrm{D}t} = \sigma \frac{\partial \psi}{\partial s}. \tag{3.49}$$

Let $\sigma = \sigma(\varepsilon)$ represent the relation between stress and logarithmic strain, then

$$\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = \frac{\partial U}{\partial s}, \qquad \varrho^{-1} \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} \frac{\partial \varepsilon}{\partial s} = \frac{\mathrm{D}U}{\mathrm{D}t} \tag{3.50}$$

are a pair of hyperbolic equations for U, ε, and

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial V}{\partial s}, \qquad \frac{\sigma}{\varrho} \frac{\delta \psi}{\partial s} = \frac{\mathrm{D}V}{\mathrm{D}t} \tag{3.51}$$

are a pair of hyperbolic equations for ψ, V. For small motion these equations are equivalent to those of the previous section. When the strain is large, the equations will still be equivalent to those of section 2.6 if the stress-strain curve is expressed in terms of logarithmic strain instead of natural strain.

3.8. Validity of the approximation of uniaxial stress

The work reproduced in this section represents a more or less complete solution of the equations of motion of an elastic-plastic wire, or of a specimen in the form of a cylinder, under the assumption of uniaxial stress. It is now necessary to consider whether, and under what circumstances, the assumption may be valid.

Consider first a perfectly elastic material. Then, except in the unnatural case where Poisson's ratio is zero or negative, any longitudinal extension of a cylinder implies lateral contraction. If the extension changes with time, it follows that there is lateral motion, and this again implies the existence of shear stress, so that the stress cannot be exactly uniaxial. Now for an infinite, perfectly elastic, circular cylinder, the motion with the curved surface free from traction may be found exactly by the methods of Pochhammer [1876] and Chree [1886] (see Love [1927] p. 287 et seq.). It can then be shown that the velocity, v, of propagation of a sinusoidal disturbance along a cylinder depends on the ratio of the wavelength, say λ to the diameter, a, of the cylinder, and that

$$v \to \left(\frac{E}{\varrho}\right)^{\frac{1}{2}} \quad \text{as} \quad \frac{\lambda}{a} \to \infty .$$

It is this circumstance which makes it possible to obtain reasonably accurate solutions to wave propagation problems in elastic cylinders by the assumption of uniaxial stress, provided that the rate of change of stress is small. Rapid changes of stress are propagated with velocity different from $(E/\varrho)^{\frac{1}{2}}$, but this does not mean that the equations of elasticity depend on rate of strain, but only that the assumption of uniaxial stress is not of universal validity.

In the theory of plasticity no solution analogous to the Pochhammer-Chree solution has yet been found, and there is therefore no means of checking the validity of the approximation of uniaxial stress. However, it is reasonable, by analogy with the elastic case, to assume that the results will be fairly accurate when the rates of change of stress and strain are small.

The above argument has considerable bearing on the important question, whether or not strain-rate effects must be allowed for in metallic plasticity. The existence of such effects has often been postulated (e.g. MALVERN [1951a, b]), but the experimental evidence brought forward in support of the postulate is invariably concerned with situations in which uniaxial stress has been assumed, and, as KARMAN and DUWEZ [1950] remarked, "It is not logical to use tension impact tests to study the influence of rate of strain on the properties of metals."

Further references

For further work on longitudinal waves the reader may consult CEBAN [1953], CRISTESCU [1957], JUHASZ [1949], LAZUTKIN [1952], LEE and TUPPER [1954], who include a discussion based on neglect of elastic strains, LENSKII [1949], MOCHALOV [1955], WHITE [1949], ZVEREV [1950], who includes the effect of radial inertia in the equations, and SHOI-YEAN HWANG and N. DAVIDS [1960].

Writers on transverse waves include CRISTESCU [1951, 1954], CRAGGS [1954], GRIGORYAN [1949], NEZHENTSEV [1956], REABOVA [1953] and SHAPIRO [1952].

§ 4. Propagation of Plane Waves

In view of the considerations outlined in section 3.8, the use of thin wire experiments to determine dynamic stress-strain curves is not satisfactory. A situation which can be more accurately analysed theoretically, whatever assumptions are made about the constitutive equations, is that in which the stress and strain propagate in plane waves.

4.1. UNIAXIAL STRAIN

Consider, first, a rectangular block of material, $0 < x < a$, $-b < y < b$, $-c < z < c$, loaded by a force suddenly and uniformly applied over the face $x = 0$. Then, if the ratios a/b and a/c are small, the motion may be approximated by the assumption that the velocity reduces to a single component, u, in the direction of the x axis, and that it is independent of y and z. For an isotropic material the assumption gives the exact solution at points near the line $y = z = 0$, for times less than the travel time of the fastest possible wave over the shorter of the two distances $2b$, $2c$, and the approximation will be good for longer times if there is appreciable attenuation of stress waves with distance travelled. The assumption is conveniently labelled by the term uniaxial strain (WOOD [1952]) since the strain tensor has only one non-zero component,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \varepsilon \,. \tag{4.1}$$

The stress tensor has three non-zero components, of which two are equal from symmetry, so put

$$\sigma_{xx} = \sigma_1 \,, \qquad \sigma_{yy} = \sigma_{zz} = \sigma_2 \,. \tag{4.2}$$

Assume a known relation

$$-3p = f(\varDelta) \tag{4.3}$$

between the hydrostatic stress and the dilatation. Then

$$\sigma_1 + 2\sigma_2 = f(\varDelta) = f\left(\frac{\partial u}{\partial x}\right) . \tag{4.4}$$

Assume also a yield stress k, then according to either the von Mises or the Tresca criterion,

$$\sigma_1 - \sigma_2 = -k \,, \tag{4.5}$$

where k differs by a numerical factor from the yield parameter introduced in § 2. Again, with another slight change in notation for this simple problem, strain-hardening may be allowed by the equation

$$k = k(W_{\mathrm{T}}) \,, \tag{4.6}$$

where W_{T} is used for the total work done in shear in continual loading, so that

$$\dot{W}_{\mathrm{T}} = \tfrac{2}{3}(\sigma_1 - \sigma_2)\dot{\varepsilon} \,. \tag{4.7}$$

Finally, the equation of motion is

$$\varrho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_1}{\partial x} \,, \tag{4.8}$$

where it will be sufficient in most cases to use the initial density of the material since the dilatation is small.

Now write $\partial u/\partial t = v$ for the velocity; then from (4.7), (4.5) and (4.1)

$$\frac{\partial W_{\mathrm{T}}}{\partial t} = -\tfrac{2}{3}k \frac{\partial \varepsilon}{\partial t} \,, \tag{4.9}$$

which may be integrated directly since W_T may be regarded as a function of k only. Continuity gives

$$\frac{\partial v}{\partial x} = \frac{\partial \varepsilon}{\partial t}, \tag{4.10}$$

and (4.8), with (4.5) and (4.4) leads to

$$\varrho \frac{\partial v}{\partial t} - \left\{ \tfrac{1}{3} k \frac{\mathrm{d}k}{\mathrm{d}W_T} + \tfrac{1}{3} f'(\varepsilon) \right\} \frac{\partial \varepsilon}{\partial x} = 0 . \tag{4.11}$$

Now (4.10) and (4.11) are a pair of quasi-linear hyperbolic equations for v and ε with the usual wave properties, based on a wave velocity c given by

$$9\varrho c^2 = 4 \frac{\mathrm{d}k}{\mathrm{d}W_T} + 3f'(\varepsilon) . \tag{4.12}$$

Shock waves

When the pressure-dilatation relation is that suggested in (2.18):

$$\Delta = -ap + bp^2 , \qquad a > 0 , \qquad b > 0 ,$$

the stress-strain curve is of the form appropriate to the formation of shock waves, and the equations (4.10) and (4.11) of continuous motion must be replaced in certain circumstances by the equations for shock waves, obtained by the methods of section 3.5. Note, however, that if unloading occurs according to the same law, unloading waves will not involve shocks.

The problem of most immediate practical interest is that in which a block $0 < x < a$ is subject to given stress at $x = 0$, with free boundary conditions at $x = a$. This problem has been exhaustively discussed by MORLAND [1959], using analysis basically the same as that suggested in § 3.

4.2. GENERAL PLANE WAVE

The method of plane waves may be applied to problems of more generality than that of uniaxial strain.

Use the suffix notation of § 2, with Cartesian coordinates x_i and displacement components u_i. Suppose u_i represents a set of displacements which lead to strains ε_{ij} depending only on x_1 and t. (Thus u_i may be linear in u_2 and u_3, and the axes of strain are not restricted to coincide with the coordinate axes.) Assume a general reversible compressibility relation,

$$p = p(\Delta) , \tag{4.13}$$

together with the von Mises plasticity condition

$$\sigma'_{ij}\sigma'_{ij} = 2k^2\,, \tag{4.14}$$

work-hardening

$$k = k(W)\,, \tag{4.15}$$

with

$$\dot{W} = \sigma'_{ij}\dot{\eta}_{ij} \tag{4.16}$$

and the proportionality of reduced stress and plastic strain-rate

$$\dot{\eta}_{ij} = \Lambda\sigma'_{ij} = \frac{1}{4k^3}\frac{\mathrm{d}W}{\mathrm{d}k}\sigma'_{lm}\dot{\sigma}'_{lm}\sigma'_{ij}\,, \tag{4.17}$$

where the last step depends on (4.16) and (4.14). Here η_{ij} is the plastic strain, obtained by subtracting the reversible elastic strain from the total strain, so the total rate of strain is given by

$$\dot{\varepsilon}_{ij} = \dot{\eta}_{ij} + (2G)^{-1}\dot{\sigma}'_{ij} - \frac{1}{9}\left(\frac{\mathrm{d}p}{\mathrm{d}\Delta}\right)^{-1}\dot{\sigma}_{kk}\delta_{ij} \tag{4.18}$$

using (4.13). When the strain-rate tensor is expressed in terms of the velocity vector $\dot{u}_i = \partial u_i/\partial t$ (4.18) becomes

$$\frac{1}{2}\left(\frac{\partial\dot{u}_i}{\partial x_j} + \frac{\partial\dot{u}_j}{\partial x_i}\right) = \frac{1}{4k^3}\frac{\mathrm{d}W}{\mathrm{d}k}\sigma'_{lm}\dot{\sigma}'_{lm}\sigma'_{ij} + \frac{1}{2G}\dot{\sigma}'_{ij} - \frac{1}{9}\frac{\mathrm{d}\Delta}{\mathrm{d}p}\dot{\sigma}_{kk}\delta_{ij}\,. \tag{4.19}$$

Three further equations connecting velocity and stress components are given by the equations of motion

$$\varrho\frac{\partial\dot{u}_i}{\partial t} = \frac{\partial\sigma_{ij}}{\partial x_j}\,. \tag{4.20}$$

Now, for a given material, specified by the value of the elastic shear modulus G and equations (4.13) and (4.15), equations (4.19) and (4.20) are a complete set of first order, quasi-linear equations for σ_{ij} and $\dot{u}_i$. To find plane wave solutions, choose a particular coordinate, x_1, and assume

$$\sigma_{ij} = \sigma_{ij}(\omega)\,, \qquad u_i = u_i(\omega) \tag{4.21}$$

where

$$\omega = x_1 - \lambda t\,. \tag{4.22}$$

Now write

$$\tau_{ij} = \frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}\omega}\,, \qquad v_i = \frac{\mathrm{d}u_i}{\mathrm{d}\omega}\,. \tag{4.23}$$

Then eqs. (4.19) and (4.20) are a set of linear algebraic equations for τ_{ij} and v_i, compatible only for certain values of λ, which correspond to the wave velocities for infinitesimal plane waves. Written separately, with $k^* = \mathrm{d}k/\mathrm{d}W$ and $p^* = -\mathrm{d}p/\mathrm{d}\Delta$, these equations are

$$\left.\begin{aligned}
v_1 &= \frac{\lambda}{4k^3k^*}\sigma'_{11}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{11} + \frac{\lambda}{9p^*}\tau_{ii} \\
\tfrac{1}{2}v_2 &= \frac{\lambda}{4k^3k^*}\sigma_{12}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{12} \\
\tfrac{1}{2}v_3 &= \frac{\lambda}{4k^3k^*}\sigma_{13}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{13} \\
0 &= \frac{\lambda}{4k^3k^*}\sigma'_{22}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{22} + \frac{\lambda}{9p^*}\tau_{ii} \\
0 &= \frac{\lambda}{4k^3k^*}\sigma'_{33}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{33} + \frac{\lambda}{9p^*}\tau_{ii} \\
0 &= \frac{\lambda}{4k^3k^*}\sigma'_{23}\sigma'_{ij}\tau_{ij} + \frac{\lambda}{2G}\tau'_{23}\,,
\end{aligned}\right\} \tag{4.24}$$

and

$$\varrho\lambda v_1 = \tau_{11}\,, \qquad \varrho\lambda v_2 = \tau_{12}\,, \qquad \varrho\lambda v_3 = \tau_{13}\,. \tag{4.25}$$

Now $\sigma'_{ii} = \tau'_{ii} = 0$, so addition of the first, fourth and fifth of (4.24) gives

$$3p^*v_1 = \lambda\tau_{ii}\,. \tag{4.26}$$

Again, $\sigma'_{ij}\sigma'_{ij} = 2k^2$, so addition of the equations (4.24) after multiplication by σ_{11}, $2\sigma_{12}$, $2\sigma_{13}$, σ_{22}, σ_{33}, $2\sigma_{23}$ respectively, gives

$$\sigma'_{11}v_1 + \sigma_{12}v_2 + \sigma_{13}v_3 = \lambda\left(\frac{1}{4kk^*} + \frac{1}{2G}\right)\sigma'_{ij}\,\tau_{ij}\,. \tag{4.27}$$

Now, in the first of (4.24) write $\tau'_{11} = \tau_{11} - \frac{1}{3}\tau_{ii}$ and substitute for τ_{ii} from (4.26) and for $\sigma'_{ij}\tau'_{ij}$ from (4.27). Then

$$v_1 = \frac{G\sigma'_{11}\,(\sigma'_{11}v_1 + \sigma_{12}v_2 + \sigma_{13}v_3)}{2k^2(G + kk^*)} + \frac{\lambda}{2G}\tau_{11} + \frac{2G - 3p^*}{6G}v_1$$

which, from (4.25), is equivalent to

$$\left\{\frac{4G + 3p^* - 3\varrho\lambda^2}{6G} - \frac{G\sigma'^2_{11}}{2k^2(G + kk^*)}\right\} - \frac{G\sigma'_{11}(\sigma_{12}v_2 + \sigma_{13}v_3)}{2k^2(G + kk^*)} = 0. \tag{4.28}$$

Similar treatment of the second and third of (4.24) leads to

$$v_2\left\{1-\frac{\varrho\lambda^2}{G}-\frac{G\sigma_{12}^2}{k^2(G+kk^*)}\right\}-\frac{G\sigma_{12}(\sigma_{11}'v_1+\sigma_{13}v_3)}{k^2(G+kk^*)}=0 \qquad (4.29)$$

and

$$v_3\left\{1-\frac{\varrho\lambda^2}{G}-\frac{G\sigma_{13}^2}{k^2(G+kk^*)}\right\}-\frac{G\sigma_{13}(\sigma_{11}'v_1+\sigma_{12}v_2)}{k^2(G+kk^*)}=0\,. \qquad (4.30)$$

Now write

$$K=1+G^{-1}kk^* \qquad (4.31)$$

and

$$C=p^*+\tfrac{4}{3}G\,. \qquad (4.32)$$

Then the condition of compatibility of equations (4.28), (4.29) and (4.30) is

$$(G-\varrho\lambda^2)\Big[\varrho^2\lambda^4-G\varrho\lambda^2\Big\{1+\frac{C}{G}-\frac{1}{k^2K}(\sigma_{11}'^2+\sigma_{12}^2+\sigma_{13}^2)\Big\}$$
$$+G^2\Big\{\frac{C}{G}-\frac{1}{k^2K}\Big(\sigma_{11}'^2+\frac{C}{G}\sigma_{12}^2+\frac{C}{G}\sigma_{13}^2\Big)\Big\}\Big]=0\,. \qquad (4.33)$$

The square bracket in (4.33), treated as a quadratic expression for $(\varrho\lambda^2/G)$, has discriminant

$$\Big\{1-\frac{C}{G}+\frac{1}{k^2K}(\sigma_{11}'^2+\sigma_{12}^2+\sigma_{13}^2)\Big\}^2+\frac{4\sigma_{11}'^2(\sigma_{12}^2+\sigma_{13}^2)}{k^2K} \qquad (4.34)$$

which is certainly positive, since C, G and K are essentially positive. It follows that (4.33) has three real roots for $\varrho\lambda^2$. Again, $C/G>1$, from (4.32) and $K\geqslant 1$ if the material hardens, so the roots for $\varrho\lambda^2$ are all positive, and there are three real characteristic velocities.

Special cases

The nature of the plane waves corresponding to the roots of eq. (4.33) is most easily demonstrated by the use of special coordinate axes. Rotate the axes at a point about the direction of x_1 to make $\sigma_{12}=0$. Then (4.29) gives

$$v_2\left(1-\frac{\varrho\lambda^2}{G}\right)=0\,, \qquad (4.35)$$

which implies that transverse waves with the velocity derivative in the direction of zero shear stress over the plane of the wave travel with the velocity $\lambda = (G/\varrho)^{\frac{1}{2}} = c_2$ characteristic of elastic shear waves (the secondary waves of seismological theory).

Again, with the same axes, (4.28) and (4.30) reduce to

$$v_1 \left\{\frac{C - \varrho\lambda^2}{G} - \frac{\sigma_{11}'^2 + \sigma_{13}^2}{k^2 K}\right\} - \frac{\sigma_{11}'\sigma_{13}}{k^2 K} v_3 = 0 \tag{4.36}$$

and

$$\frac{\sigma_{11}'\sigma_{13}v_1}{k^2 K} - \left\{1 - \frac{\varrho\lambda^2}{G} - \frac{\sigma_{13}^2}{k^2 K}\right\} v_3 = 0\,. \tag{4.37}$$

In general the roots of (4.33) are not zeros of the coefficients of v_1, v_3 in (4.36) and (4.37). This means that the true plastic waves, given by these two equations, each involve both transverse and longitudinal components. There are, however, two special cases, in which purely transverse and purely longitudinal waves can occur.

First suppose that $\sigma_{13} = 0$ $(= \sigma_{12})$. Then (4.37) reduces to the same equation as (4.35), with v_2 replaced by v_3, and (4.36) shows that pure longitudinal waves are possible, and travel with velocity λ_1 given by

$$\lambda_1 = \left\{\frac{C}{\varrho} - \frac{G\sigma_{11}'^2}{\varrho k^2 K}\right\}^{\frac{1}{2}}. \tag{4.38}$$

Secondly, suppose that $\sigma_{11}' = 0$. Then (4.36) gives pure longitudinal waves with velocity

$$\lambda_1 = \left\{\frac{C}{\varrho} - \frac{G\sigma_{13}^2}{\varrho k^2 K}\right\}^{\frac{1}{2}} \tag{4.39}$$

and (4.37) gives pure transverse waves with velocity

$$\lambda_2 = \left\{\frac{G}{\varrho} - \frac{G\sigma_{13}^2}{\varrho k^2 K}\right\}^{\frac{1}{2}}. \tag{4.40}$$

Note that it is possible to have $\sigma_{11}' = \sigma_{12} = \sigma_{13} = 0$, and that then (4.38) and (4.39) are identical and give $\lambda_1 = c_1 = (C/\varrho)^{\frac{1}{2}}$, which is the velocity of longitudinal elastic waves (the primary waves of seismology). So, even in a plastic material, there may be waves travelling as fast as the fastest waves in elasticity. It is interesting to note, too, the minimum wave velocities. These are given by $\sigma_{13}^2 = k^2$, and (4.39) then gives

$$\varrho\lambda_1^2 = C - \frac{G}{K} > p + \tfrac{1}{3}G\,,$$

from (4.31) and (4.32). But for a metal it is certainly true that the bulk modulus $p > \frac{2}{3}G$ (see for example (2.7) for the elastic case), so in all cases $\lambda_1 > c_2$.

[In fact it is easy to show that under all circumstances one of the wave velocities given by (4.33) must be greater than c_2. One of the plastic waves must always have a speed exceeding the speed of the slower elastic wave.] Again, put $\sigma_{13}^2 = k^2$ in (4.40), then

$$\lambda_2^2 = \frac{G}{\varrho}\left(1 - \frac{1}{K}\right) = \frac{Gk\mathrm{d}k/\mathrm{d}W}{\varrho(G + k\mathrm{d}k/\mathrm{d}W)} = c_3^2\,. \tag{4.41}$$

This corresponds to a shear wave in a medium in pure shear. In such a situation (4.17) and (4.18) imply

$$\dot{\varepsilon}_{13} = \frac{\dot{\sigma}_{13}}{2G} + \frac{\dot{\sigma}_{13}}{2k\mathrm{d}k/\mathrm{d}W} = \frac{\dot{\sigma}_{13}(G + k\mathrm{d}k/\mathrm{d}W)}{2Gk\mathrm{d}k/\mathrm{d}W}\,,$$

so c_3 is the velocity corresponding to the tangent modulus, the slope of the stress-strain curve, in pure shear. Note that this case, of a shear wave propagating in a medium in pure shear, is the only case where one can expect the 'tangent modulus' to give the plastic wave velocity.

A little algebra based on (4.33) easily shows that, quite generally, there are two plastic waves, with velocities λ_1, λ_2 such that

$$c_1^2 \geqslant \lambda_1^2 \geqslant c_2^2 \geqslant \lambda_2^2 \geqslant c_3^2 \tag{4.42}$$

(compare CRAGGS [1957]). There is some similarity between these results and those found by MUSGRAVE [1954] for plane waves in aeleotropic elasticity.

§ 5. Spherical and Cylindrical Waves

There is a partial analogue to the theory of § 4 in the theory of spherical and cylindrical waves. However, the solutions are more restricted than in elasticity theory, since in general the waves generated by a disturbance at a point of a plastic medium will not spread out as spherical waves. The use of spherical and cylindrical waves is therefore restricted to problems in which the appropriate symmetry exists.

5.1. SPHERICAL WAVES

The symmetry of a spherical wave implies purely radial displacement, u, and that the principal axes of stress are radial and circumferential, with the circumferential stresses equal. The non-zero components of stress and strain are therefore

$$\varepsilon_{11} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{22} = \varepsilon_{33} = \frac{u}{r}, \qquad \sigma_{11} = \sigma_r, \qquad \sigma_{22} = \sigma_{33} = \sigma_\theta .$$

The equation of radial motion is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\theta)}{r} = \varrho \frac{\partial^2 u}{\partial t^2} . \tag{5.1}$$

The hydrostatic stress and the reduced stress components are given by

$$3p = \sigma_r + 2\sigma_\theta \tag{5.2}$$

and

$$3\sigma'_{11} = 2(\sigma_r - \sigma_\theta) = -6\sigma'_{22} = -6\sigma'_{33} , \tag{5.3}$$

and the equation of plasticity is

$$(\sigma_r - \sigma_\theta)^2 = 3k^2 . \tag{5.4}$$

It follows that the stress-strain relations (4.19) must be replaced by

$$\frac{\partial^2 u}{\partial r \partial t} = \left\{\frac{1}{k}\frac{\mathrm{d}W}{\mathrm{d}k} + \frac{1}{G}\right\} \frac{\partial}{\partial t}\left(\frac{\sigma_r - \sigma_\theta}{3}\right) - \frac{\mathrm{d}\Delta}{\mathrm{d}p}\frac{\partial}{\partial t}\left(\frac{\sigma_r + 2\sigma_\theta}{9}\right) \tag{5.5}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial t} = -\left\{\frac{1}{k}\frac{\mathrm{d}W}{\mathrm{d}k} + \frac{1}{G}\right\} \frac{\partial}{\partial t}\left(\frac{\sigma_r - \sigma_\theta}{6}\right) - \frac{\mathrm{d}\Delta}{\mathrm{d}p}\frac{\partial}{\partial t}\left(\frac{\sigma_r + 2\sigma_\theta}{9}\right) . \tag{5.6}$$

Elimination of σ_r, σ_θ leads to the following second order equation for $v = \partial u/\partial t$:

$$\frac{\partial}{\partial r}\left\{p^*\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right) + \frac{4kk^*}{K}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right\} + \frac{4kk^*}{K}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = \varrho \frac{\partial^2 v}{\partial t^2} . \tag{5.7}$$

Note that it is not, in general, legitimate to integrate (5.7) into an equation for u, and that in performing the differentiation on the right p^* and k^* must be regarded as variables depending on the strain. However, in the special case of perfect plasticity and constant compressibility, $p^* = \varkappa$, $k^* = 0$, $K = 1$, and (5.7) reduces to

$$\varkappa \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right) = \varrho \frac{\partial^2 v}{\partial t^2} . \tag{5.8}$$

This equation is equivalent to that given by HUNTER [1957] but emphasizes, as Hunter's integrated equation does not, that the velocity and strain propagate with standard wave velocity $c_p = (\varkappa/\varrho)^{\frac{1}{2}}$. In applying the theory to any special problem, it will be necessary to integrate v for the displacement, and to include such a function of r (only) as may be necessary to ensure continuity of displacement as a function of time. This brings the method into line with that of Hunter.

In general, the equation (5.7) is a non-linear hyperbolic equation with characteristics given by $\mathrm{d}r/\mathrm{d}t = \pm \lambda$, where

$$\varrho\lambda^2 = p^* + \frac{4kk^*}{K}. \tag{5.9}$$

Since now $3\sigma_{11}'^2 = 4k^2$, eq. (5.9) gives the same wave velocity as (4.38).

For details of applications to problems of spherical symmetry the reader may consult the article by HOPKINS [1960].

5.2. CYLINDRICAL WAVES

The third classical special case of wave propagation is that corresponding to cylindrical symmetry. Use r, θ, z as cylindrical polar coordinates and assume stress, strain and velocity to be independent of θ and z. Take displacement components as u_1, u_2 and u_3. Then in a static problem the symmetry would allow u_3 linear in z, but in the dynamic case this is compatible with the equations of motion only if $\partial u_3/\partial t = 0$, and there is no significant loss of generality in assuming $u_3 = 0$. Also u_1 and u_2 depend only on r and t. The relevant stress components are $\sigma_{11} = \sigma_r$, $\sigma_{22} = \sigma_\theta$, $\sigma_{12} = \tau$, $\sigma_{33} = \sigma_z$. The equations of motion are

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = \varrho\frac{\partial^2 u_1}{\partial t^2} \tag{5.10}$$

and

$$\frac{\partial\tau}{\partial r} + \frac{2\tau}{r} = \varrho\frac{\partial^2 u_2}{\partial t^2}, \tag{5.11}$$

and eqs. (4.19) are replaced by

$$\frac{\partial\dot{u}_1}{\partial r} = \frac{1}{4k^3k^*}\sigma_{ij}'\dot{\sigma}_{ij}'\sigma_{11}' + \frac{1}{2G}\dot{\sigma}_{11}' + \frac{1}{9p^*}\dot{\sigma}_{ii}, \tag{5.12}$$

$$\frac{\dot{u}_1}{r} = \frac{1}{4k^3k^*}\sigma_{ij}'\dot{\sigma}_{ij}'\sigma_{22}' + \frac{1}{2G}\dot{\sigma}_{22}' + \frac{1}{9p^*}\dot{\sigma}_{ii}, \tag{5.13}$$

$$\frac{1}{2}\left(\frac{\partial\dot{u}_2}{\partial r} - \frac{\dot{u}_2}{r}\right) = \frac{1}{4k^3k^*}\sigma_{ij}'\dot{\sigma}_{ij}'\sigma_{12} + \frac{1}{2G}\dot{\sigma}_{12}, \tag{5.14}$$

and

$$0 = \frac{1}{4k^3k^*}\sigma_{ij}'\dot{\sigma}_{ij}'\sigma_{33}' + \frac{1}{2G}\dot{\sigma}_{33}' + \frac{1}{9p^*}\dot{\sigma}_{ii}. \tag{5.15}$$

The characteristics of eqs. (5.12) to (5.15) may be found by arguments exactly parallel to those of section 4.2, and the wave velocities are given by λ, where

$$\varrho^2\lambda^4 - G\varrho\lambda^2\left\{1 + \frac{C}{G} - \frac{1}{k^2K}\,(\sigma_r'^2 + \tau^2)\right\} + G^2\left\{\frac{C}{G} - \frac{1}{k^2K}\,(\sigma_r'^2 + \tau^2)\right\} = 0\ . \tag{5.16}$$

This equation contains the result of CRISTESCU [1956].

Pure torsion

Special cases again arise for purely radial displacement and for pure torsion. In the case of pure radial displacement the analysis is similar to that for spherical waves, except for the inevitable complications involved in dealing with cylindrical waves. However, the present author has found no detailed treatment of the problem in the literature.

For pure torsion, eqs. (5.10) to (5.15) reduce to

$$\frac{\partial v}{\partial r} - \frac{v}{r} = \left(\frac{1}{kk^*} + \frac{1}{G}\right)\frac{\partial \tau}{\partial t} \tag{5.17}$$

with

$$\varrho\,\frac{\partial v}{\partial t} = \frac{\partial \tau}{\partial r} + \frac{2\tau}{r} \tag{5.18}$$

where $v = \partial u_2/\partial t$ is the circumferential velocity.

The analysis for given boundary conditions (traction or displacement at given radii) is similar to that for uniaxial stress. The problem was first studied by RAKHMATULIN [1948], for the simple wave. Further solutions have been given by BAKHSHIYAN [1948a] and KOCHETKOV [1950].

§ 6. Bending of Beams

A problem which has attracted much attention is that of the dynamic bending of an initially straight beam. The problem is of considerable interest, because it is one where fairly complete experimental results can be obtained by high speed photography. It is then possible, in principle, to deduce the complete curvature-time history at every point of the beam. The assumptions made in the theoretical analysis are similar to those of § 3, in that only the stress resultant over the cross-section is used, but are more drastic, in that the shear stress and the rotational inertia are neglected. The theory is due to DUWEZ, CLARK and BOHNENBLUST [1950].

6.1. Basic equations

Use a Lagrangian description of the motion, taking x for the coordinate along the (undisturbed) length of the beam and y for the transverse displacement. Let Q be the shearing force and M the bending moment, with sign conventions such that

$$Q = \frac{\partial M}{\partial x} \tag{6.1}$$

and

$$\sigma \frac{\partial^2 y}{\partial t^2} = \frac{\partial Q}{\partial x} \tag{6.2}$$

where σ is the mass per unit length of the beam and (6.1) implies neglect of rotational inertia, and (6.2) neglects the slope of the bent beam. Assume a known relation

$$M = M(\varkappa) \tag{6.3}$$

between the bending moment and the curvature,

$$\varkappa = \frac{\partial^2 y}{\partial x^2}, \tag{6.4}$$

where again the slope is assumed small. The relation (6.3) is supposed independent of rate of strain and may be found either from static bending tests or by deduction from a (static) tension test (NADAI [1931], p. 165). Then

$$\sigma \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial M}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\mathrm{d}M}{\mathrm{d}\varkappa}\frac{\partial^3 y}{\partial x^3}\right) \simeq \frac{\mathrm{d}M}{\mathrm{d}\varkappa}\frac{\partial^4 y}{\partial x^4}, \tag{6.5}$$

neglecting a second-order term. When the beam is elastic, with $M = EI\varkappa$, where E is Young's modulus and I the second moment of area of the cross-section, (6.5) implies no new approximation and reduces to Boussinesq's equation

$$\sigma \frac{\partial^2 y}{\partial t^2} = EI \frac{\partial^4 y}{\partial x^4}, \tag{6.6}$$

(BOUSSINESQ [1885]).

6.2. Solutions with dynamic similarity

Eq. (6.5) is not a wave equation, for it is not, and cannot be reduced to, a hyperbolic equation, and no general theory analogous to the characteristics theory of the wave equation exists. It would also be false terminology to speak of a simple wave solution, but an analogue to the simple wave solution of the wave equation does exist, in the form of a solution suggested by Boussinesq for (6.6). This may be called a solution with dynamic similarity.

Following Duwez, Clark and Bohnenblust, put

$$\eta = \frac{1}{4a^2}\frac{x^2}{t} \tag{6.7}$$

with

$$a^2 = \left(\frac{EI}{\sigma}\right)^{\frac{1}{2}} \tag{6.8}$$

and assume a solution of (6.5) with

$$y = t f(\eta) . \tag{6.9}$$

Note that this implies that at $x = 0$, where $\eta = 0$, $y = Vt$ for some fixed V. Then

$$2a^2\varkappa = f'(\eta) + 2\eta f''(\eta) \tag{6.10}$$

which shows that $\varkappa$ and therefore M are functions of η only; and

$$\frac{\partial^2 y}{\partial t^2} = \frac{\eta^2}{t} f''(\eta) .$$

It is convenient to avoid a fourth-order equation by working with a non-dimensional shear stress, S, defined by

$$EIS = 2a^3 t^{\frac{1}{2}} Q = 2a^2 \eta^{\frac{1}{2}} \frac{\mathrm{d}M}{\mathrm{d}\eta} \tag{6.11}$$

from (6.1). Then (6.2) reduces to

$$\eta^{\frac{3}{2}} f = S' \tag{6.12}$$

where dashes denote derivatives with respect to η, and differentiation of (6.10) and use of (6.11) and (6.12) leads to

$$S'' + EIS \frac{\mathrm{d}\varkappa}{\mathrm{d}M} = 0 . \tag{6.13}$$

But from (6.11),

$$M = \frac{-EI}{2a^2} \int_{\eta}^{\infty} \frac{S(\eta)}{\eta^{\frac{1}{2}}} \,\mathrm{d}\eta , \tag{6.14}$$

where the upper limit is obtained from the boundary condition, $M \to 0$ as $t \to 0$, $x > 0$. Also $\varkappa$ is a function of M only, and $\mathrm{d}\varkappa/\mathrm{d}M$

may be expressed as a function of M so (6.13) becomes an ordinary non-linear integro-differential equation for S.

Now the Boussinesq elastic solution corresponds to $EI\mathrm{d}\varkappa/\mathrm{d}M = 1$, and gives

$$S = A \cos \eta \,, \tag{6.15}$$

so, from (6.14), it is seen that the quantity $\mathrm{d}M/\mathrm{d}\eta$ alternates in sign, and the curvature of the beam alternately increases and decreases. The same kind of behaviour may be expected in the plastic problem, and it is therefore necessary to assume a relation between $\varkappa$ and M in (6.13) which allows for loading and unloading. Duwez, Clark and Bohnenblust use a linear strain-hardening curve as shown in Fig. 6.1, with elastic unloading or reloading, and no Bauschinger

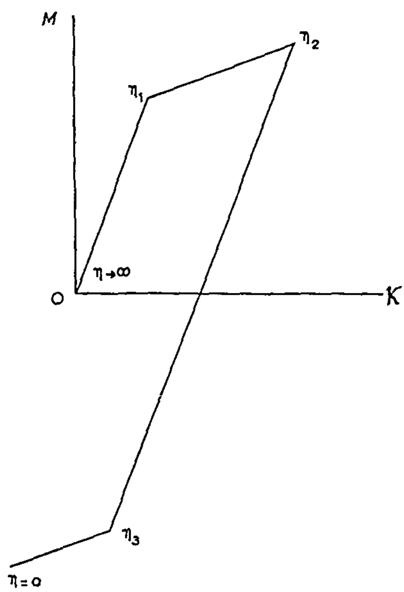

Fig. 6.1. Moment-curvature curve.

effect. Let the slopes in the elastic and plastic regions be $\mathrm{d}M/\mathrm{d}\varkappa = EI$ and $EI\alpha^2$, respectively, where α is positive less than one. Then (6.13) has solutions of the forms

$$S = A \cos (\eta + \varepsilon)$$

in elastic loading or unloading regions, and

$$S = A \cos \left(\frac{\eta}{\alpha} + \varepsilon\right)$$

in plastic loading regions. In a typical solution the bending moment

M increases elastically as η decreases from infinity to some value η_1, increases plastically as η decreases from η_1 to some value η_2, decreases with elastic unloading as η decreases further to a value η_3, and then decreases with reverse yielding to $\eta = 0$.

6.3. Rigid-plastic solutions

The above analysis cannot easily be extended, either to boundary conditions other than that of uniform imposed velocity at $x = 0$, or to include boundary conditions corresponding to the other end of a finite beam. With a non-linear differential equation, superposition of solutions cannot be allowed, and no other method of solving an equation of the Boussinesq type has been developed.

The use of rigid-plastic analysis, however, leads to a very much simpler theory. The method is due to Lee and Symonds [1952], who showed how the idea of plastic hinges, already familiar from static theory of limit analysis of beams, may be extended to dynamic problems. The method depends on the assumption of a non-hardening, rigid-plastic, material. Then the bending moment is related to the curvature by

$$\varkappa = \text{constant}, \qquad M \neq \pm M_0\,, \tag{6.16}$$

with

$$\dot{\varkappa} \geqslant 0\,, \qquad M = M_0\,, \qquad \dot{\varkappa} \leqslant 0\,, \qquad M = -M_0\,,$$

where the dot denotes the time derivative of the curvature at a fixed point of the beam. Now consider a finite section of the beam over which $M = M_0$. Then $\partial M/\partial x = 0$, so from (6.1) the shearing force Q is zero. The section of the beam can therefore only move as a rigid body. Again, any section in which $-M_0 < M < M_0$ must move as a rigid body. Thus, bending of the beam occurs only at isolated points, which are called plastic hinges. Suppose, at any such hinge, $M = M_0$. Then $\partial M/\partial x = Q$ must have opposite signs on the two sides of the hinge (since M_0 is the greatest possible bending moment). Thus, unless there is a finite external force acting at the hinge, the shearing force there must be zero. When the external forces are known, therefore, the moment and resultant force on the rigid part of the beam between any two plastic hinges is known, and the motion may be found by the usual formulae for motion of a rigid body.

For the best introduction to the theory the reader should consult the paper by Lee and Symonds already mentioned.

Further references

The use of rigid-plastic analysis seems to have been first suggested by CONROY [1952]. Applications have been given by ALVERSON [1956], CONROY [1956], COTTER and SYMONDS [1955], DAVIES and NEAL [1959], HOPKINS [1955], MENTEL [1955], PARKES [1955], SALVADORI and DIMAGGIO [1953], SALVADORI and WEIDLINGER [1957], SEILER and SYMONDS [1954], SEILER, COTTER and SYMONDS [1956], and SYMONDS and LETH [1954].

§ 7. Plates and Shells

It is a natural step forward from the problems considered in § 3 and 6 to problems involving the plastic deformation of thin plates and shells. Analysis by means of stress resultants and bending moments is, of course, subject to the same limitations as in the theory of wires or beams. This implies that predictions of shock waves, for example, will be unreliable, but the analysis may nevertheless be expected to give reasonable good answers to such questions as the size of the collapse loads or the amount of permanent deformation left behind after loading and release.

7.1. Axially symmetric, co-planar deformation of a plate

The simplest problem of deformation of a plate is that of the axially symmetric deformation of a plate in its own plane (FREIBERGER [1952]). Assume that an existing circular hole in an infinite thin plate is enlarged by radial pressure on the sides of the hole. Use Eulerian coordinates, with U for the radial velocity at radius r and time t. Let h be the (possibly variable) thickness of the plate, and use σ_r, σ_θ for the radial and circumferential stress components. These are two of the principal stresses, the third, the direct stress across the plate, is zero by hypothesis. The equation of motion is

$$\frac{\partial}{\partial r}(h\sigma_r) + \frac{h}{r}(\sigma_r - \sigma_\theta) = \varrho h \frac{\mathrm{D}U}{\mathrm{D}t}, \tag{7.1}$$

where ϱ is the density, and

$$\frac{\mathrm{D}U}{\mathrm{D}t} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} \tag{7.2}$$

is the derivative following the motion. The principal rates of strain are

$$\dot{\varepsilon}_r = \frac{\partial U}{\partial r}, \qquad \dot{\varepsilon}_\theta = \frac{U}{r}, \qquad \dot{\varepsilon}_z = \frac{1}{h}\frac{\mathrm{D}h}{\mathrm{D}t}. \tag{7.3}$$

7.1.1. *Elastic solution*

In an elastic region the above equations are supplemented by the stress-strain relations,

$$E\varepsilon_r = \sigma_r - \nu\sigma_\theta, \qquad E\varepsilon_\theta = \sigma_\theta - \nu\sigma_r. \tag{7.4}$$

Now let u be the displacement of a point from its original radius, and write $f = u/r$. Then the strain components are

$$\varepsilon_\theta = \frac{u}{r-u} = \frac{f}{1-f} \approx f, \qquad \varepsilon_r \approx \frac{\partial u}{\partial r} = f + r\frac{\partial f}{\partial r}, \tag{7.5}$$

and U is $\delta r/\delta t$ subject to $r-u$ constant, so

$$U\left(1 - f - r\frac{\partial f}{\partial r}\right) = r\frac{\partial f}{\partial t}. \tag{7.6}$$

Substitute (7.5) in (7.4), express σ_θ and σ_r in terms of f and substitute for σ_θ, σ_r and U in (7.1), and neglect terms of second order in f. Then (7.1) becomes

$$\frac{\partial^2 f}{\partial r^2} + \frac{3}{r}\frac{\partial f}{\partial r} = \frac{\varrho(1-\nu^2)}{E}\frac{\partial^2 f}{\partial t^2}. \tag{7.7}$$

This is the equation for elastic waves and may be solved by the usual characteristics methods. However, for the present application only the simple wave solution is needed, and this may be found by solving an ordinary differential equation for $f = f(r/t)$ as

$$f = A\left[\frac{ct(c^2t^2 - r^2)^{\frac{1}{2}}}{r^2} - \log\left\{\frac{ct + (c^2t^2 - r^2)^{\frac{1}{2}}}{r}\right\}\right] \tag{7.8}$$

where A is an arbitrary constant and $\varrho c^2 = E/(1-\nu^2)$. The corresponding stresses are

$$(1-\nu^2)\sigma_r = -AE\left[\frac{(1-\nu)ct(c^2t^2 - r^2)^{\frac{1}{2}}}{r^2} + (1+\nu)\log\left\{\frac{ct + (c^2t^2 - r^2)^{\frac{1}{2}}}{r}\right\}\right]$$

and

$$(1-\nu^2)\sigma_\theta = AE\left[\frac{(1-\nu)ct(c^2t^2 - r^2)^{\frac{1}{2}}}{r^2} - (1+\nu)\log\left\{\frac{ct + (c^2t^2 - r^2)^{\frac{1}{2}}}{r}\right\}\right]$$

and the velocity is

$$U = \frac{cA(c^2t^2 - r^2)^{\frac{1}{2}}}{r}.$$

This analysis is from CRAGGS [1952]. Freiberger assumed an elastic solution $\sigma_r = -\sigma_\theta = A(t)/r^2$. This is equivalent to letting $c \to \infty$, and so is appropriate only in rigid-plastic work, or when inertia terms are negligible.

7.1.2. *Plastic solutions* (*Prandtl-Reuss material*)

For a Prandtl-Reuss material with work-hardening, the yield criterion becomes

$$\tfrac{1}{2}\sigma'_{ij}\sigma'_{ij} = \tfrac{1}{3}(\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2) = k^2(W) \tag{7.9}$$

with

$$\sigma'_r = \tfrac{1}{3}(2\sigma_r - \sigma_\theta)\,, \qquad \sigma'_\theta = \tfrac{1}{3}(2\sigma_\theta - \sigma_r)\,, \qquad \sigma'_z = -\tfrac{1}{3}(\sigma_r + \sigma_\theta) \tag{7.10}$$

and

$$\begin{aligned}\frac{\mathrm{D}W}{\mathrm{D}t} = \sigma'_{ij}\dot{\eta}_{ij} &= \sigma'_r\{\dot{\varepsilon}_r - E^{-1}(\dot{\sigma}_r - \nu\dot{\sigma}_\theta)\}\\ &\quad + \sigma'_\theta\{\dot{\varepsilon}_\theta - E^{-1}(\dot{\sigma}_\theta - \nu\dot{\sigma}_r)\}\\ &\quad + \sigma'_z\{\dot{\varepsilon}_z + \nu E^{-1}(\dot{\sigma}_r + \dot{\sigma}_\theta)\}\,,\end{aligned}$$

and the flow rule gives

$$\begin{aligned}\frac{\dot{\varepsilon}_r - E^{-1}(\dot{\sigma}_r - \nu\dot{\sigma}_\theta)}{\sigma'_r} &= \frac{\dot{\varepsilon}_\theta - E^{-1}(\dot{\sigma}_\theta - \nu\dot{\sigma}_r)}{\sigma'_\theta}\\ = \frac{\dot{\varepsilon}_z + \nu E^{-1}(\dot{\sigma}_r + \dot{\sigma}_\theta)}{\sigma'_z} &= \frac{\sigma'_r\dot{\sigma}_r + \sigma'_\theta\dot{\sigma}_\theta}{2k^3\mathrm{d}k/\mathrm{d}W}.\end{aligned} \tag{7.11}$$

Eqs. (7.1), (7.2) and (7.11) form a non-linear hyperbolic set with characteristics $\mathrm{d}r/\mathrm{d}t = \alpha$ where

$$\begin{aligned}&\frac{\varrho}{E}(\alpha - U)^2\left(\frac{1-\nu^2}{E} + \frac{(\sigma'_r)^2 + (\sigma'_\theta)^2 + 2\nu\sigma'_r\sigma'_\theta}{2k^3\mathrm{d}k/\mathrm{d}W}\right)\\ &= \frac{1}{E}\left(1 - \frac{\nu(1+\nu)\sigma_r}{E}\right) + \frac{(\sigma'_\theta)^2 - (E^{-1}\sigma_r)[(\sigma'_r)^2 + (1+2\nu)(\sigma'_\theta)^2 + \sigma'_r\sigma'_\theta]}{2k^3\mathrm{d}k/\mathrm{d}W}.\end{aligned} \tag{7.12}$$

Such equations may be easily solved by the numerical methods suggested in § 3, but no solution seems yet to have been published except for the case of a simple wave.

7.1.3. *Elastic-plastic solutions*

Freiberger used the Tresca criterion, which leads to some slight simplification in this problem. He observes that in the elastic region, (I), σ_θ and σ_r are of opposite sign, and deduces that this region merges into a plastic region in which $\sigma_\theta > \sigma_z > \sigma_r$. The appropriate form for the Tresca criterion in such a region is $\sigma_\theta - \sigma_r = Y$, where Y is a (fixed) yield stress. The flow equation becomes

$$\dot{\eta}_r + \dot{\eta}_\theta = 0\,, \qquad \dot{\eta}_z = 0 \tag{7.13}$$

where $\dot{\eta}$ represents the plastic strain rate. Then variations in h may be neglected in (7.1) so

$$\frac{\partial \sigma_r}{\partial r} + \frac{Y}{r} = \varrho \frac{\mathrm{D}U}{\mathrm{D}t} \tag{7.14}$$

and (7.13) with $\sigma_\theta - \sigma_r = Y$, $\sigma_z = 0$ gives

$$\frac{\partial U}{\partial r} + \frac{U}{r} = 0 \tag{7.15}$$

leading to

$$rU = g(t)\,. \tag{7.16}$$

Eq. (7.14) then gives

$$\sigma_r + Y \log r = a_0 + \varrho[g'(t) \log r + \tfrac{1}{2}U^2]\,. \tag{7.17}$$

The values of the function $g(t)$ and the constant a_0 may be deduced from considerations of continuity at the elastic-plastic boundary, though Freiberger in fact preferred to neglect the inertia terms and to join the resulting solution to the elasto-static one already mentioned.

With the Tresca criterion, different equations are needed when σ_z is no longer the intermediate principal stress. In Freiberger's analysis, σ_θ decreases as r decreases until the situation is described by

$$\sigma_r = -Y\,, \qquad \sigma_\theta = \sigma_z = 0\,. \tag{7.18}$$

He then assumes that for the remainder of the plate $\sigma_\theta = 0$, and calls the part where this holds region III. The equation of motion in region III reduces to

$$\frac{\mathrm{D}U}{\mathrm{D}t} + \frac{Y}{\varrho}\left(\frac{1}{r} + \frac{1}{h}\frac{\mathrm{D}h}{\mathrm{D}t}\right) = 0 \tag{7.19}$$

and, as the stress is constant throughout the region, there is no change in elastic strain, also plastic strain is equivoluminal, so

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{h}\frac{\mathrm{D}h}{\mathrm{D}t} = 0\,. \tag{7.20}$$

Eqs. (7.19) and (7.20) are a hyperbolic set, with characteristics given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U \pm \left(\frac{Y}{\varrho}\right)^{\frac{1}{2}} \tag{7.21}$$

and Freiberger solves these equations by a numerical method, allowing for the formation of shock waves where appropriate.

The validity of the assumption (7.18) may perhaps be questioned, but the results of ALEXANDER and FORD [1954] for the corresponding problem without inertia terms suggest that it is a reasonable simplification to make. (See also the discussion of this point by HODGE and SANKARANARAYANAN [1957].)

7.1.4. *Shear stress*

RAKHMATULIN [1948] discussed the propagation of torsional waves in a thin sheet. Let the circumferential displacement be $r\varphi$, and the shear stress $\sigma_{r\theta} = \tau$, and use Lagrangian coordinates. Then the equation of motion is

$$\varrho r\,\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial\tau}{\partial r} + \frac{2\tau}{r} \tag{7.22}$$

and it is legitimate for small motion of a work-hardening plastic material to write

$$\tau = k(W)$$

replacing (2.15). Then

$$\begin{aligned}\dot{\tau} &= 2\,\frac{\mathrm{d}k}{\mathrm{d}W}\,\tau\left(\dot{\varepsilon}_{r\theta} - \frac{\dot{\tau}}{2G}\right)\\ &= 2\,\frac{\mathrm{d}k}{\mathrm{d}W}\,\tau\left(\tfrac{1}{2}r\,\frac{\partial\dot{\varphi}}{\partial r} - \frac{\dot{\tau}}{2G}\right),\end{aligned} \tag{7.23}$$

using the dot for a time derivative. The characteristics of (7.22) and (7.23) are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\pm\, Gk\, \mathrm{d}k/\mathrm{d}W}{G + k\, \mathrm{d}k/\mathrm{d}W} \tag{7.24}$$

and the wave speeds correspond to the tangent modulus in shear strain. Note that, when the motion is not assumed small, eq. (7.23) must be rewritten to allow for rotation of the principal axes of stress relative to the material as the motion proceeds.

7.2. Transverse deflection of a thin plate

The transverse motion of a plate which is sufficiently thin so that the bending moments may be neglected may be studied by a simple extension of the above analysis.

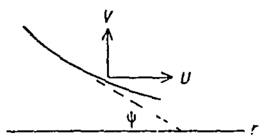

Fig. 7.1. Variables used in thin plate problem.

Let the transverse velocity be V and the angle between the tangent to the middle surface and the original plane of the plate be ψ (Fig. 7.1). Use σ_s for the direct stress component along the plate. Then there is a geometrical equation

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = -\cos\psi\left(\frac{\partial V}{\partial r}\cos\psi + \frac{\partial U}{\partial r}\sin\psi\right), \tag{7.25}$$

and the equations of motion are

$$\varrho h\,\frac{\mathrm{D}V}{\mathrm{D}t} = -\cos\psi\,\frac{\partial}{\partial r}(r\sigma_s h\sin\psi) \tag{7.26}$$

and

$$\varrho h\,\frac{\mathrm{D}U}{\mathrm{D}t} = \cos\psi\,\frac{\partial}{\partial r}(r\sigma_s h\cos\psi) - h\sigma_\theta\,. \tag{7.27}$$

These equations imply that ψ is determined by a pair of hyperbolic equations with characteristics

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U \pm \frac{\sigma_s}{\varrho}\cos^2\psi\,. \tag{7.28}$$

Eq. (7.28) is the familiar equation for the velocity of transverse waves in a plate under stress. The set of equations may now be completed by adding the equations of elasticity or plasticity. CRAGGS [1952] considered simple wave solutions of these equations, including shock waves.

7.3. DEFORMATION OF A THICK PLATE

In the above analysis it was assumed that the shearing stress and bending moment in the plate were negligible, compared with the membrane stresses. The converse assumption, of a plate sufficiently thick so that membrane stresses are negligible, also leads to comparatively simple results. Two methods of studying the transverse motion have been used, the first, due to BAKHSHIYAN [1948b] is based on the assumption that the deformation is basically due to shear, and the second, due to HOPKINS and PRAGER [1954], assumes the deformation to be mainly due to bending.

7.3.1. *Shear deformation*

Bakhshiyan assumes that the plate deforms by shear across cylindrical surfaces. Let w be the displacement, perpendicular to the original plane of the plate, and let τ be the shear stress. Then the equation of motion is

$$\varrho r \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial r}(r\tau)\,. \tag{7.29}$$

Now the mean shear strain is

$$\varepsilon = \frac{1}{2}\frac{\partial w}{\partial r} \tag{7.30}$$

so the displacement satisfies the hyperbolic equation

$$2\varrho \frac{\partial^2 w}{\partial t^2} = \frac{\mathrm{d}\tau}{\mathrm{d}\varepsilon}\frac{\partial^2 w}{\partial r^2} + \frac{\tau}{r} \tag{7.31}$$

in which τ, $\mathrm{d}\tau/\mathrm{d}\varepsilon$ are known functions of $\varepsilon = \frac{1}{2}\partial w/\partial r$, given by the stress-strain curve in shear.

This theory is a reasonable one when the plate is deformed by a load applied over a circle of radius comparable with the thickness of the plate, and so is applicable to the armour-piercing problem of a thick plate attacked by a projectile of diameter comparable with the thickness. (See also KOCHETKOV [1950].)

7.3.2. *Deformation by bending*

When a load is applied to a plate over an area large compared to the thickness it is more natural to make the assumptions of classical thin plate theory, neglecting shear deformation by comparison with bending. Use w for the transverse displacement, Q for the shearing force per unit arc at radius r, and M, N for the bending moments caused by radial and circumferential stress respectively. Then the equation of motion under a transverse applied force p per unit area of plate is

$$\mu r \frac{\partial^2 w}{\partial t^2} = pr + \frac{\partial}{\partial r}(rQ)\,, \tag{7.32}$$

where μ = mass per unit area. Rotational inertia is neglected, so

$$rQ + N - \frac{\partial}{\partial r}(rM) = 0\,, \tag{7.33}$$

and Q can therefore be eliminated from (7.32). The curvatures of the plate corresponding to M, N are $\varkappa$, λ where

$$\varkappa = -\frac{\partial^2 w}{\partial r^2}\,, \qquad \lambda = -\frac{1}{r}\frac{\partial w}{\partial r}\,. \tag{7.34}$$

Hopkins and Prager assume a rigid-plastic material without strain-hardening. Then curvature can only occur when the whole plate is plastic, and, if the yield criterion may be written in the form

$$f(\sigma_r, \sigma_\theta) = k\,, \tag{7.35}$$

the corresponding condition for the plate is

$$f(M,N) = \tfrac{1}{4}kh^2 = M_0 \tag{7.36}$$

where h is the thickness of the plate.

The authors assume the Tresca flow criterion, with the associated flow rule, and observe that in the problems they consider this condition takes one of the three forms

$$\begin{aligned}
&M = N = M_0\,, && && \dot\varkappa = 0\,, \quad \dot\lambda \geqslant 0 \\
&0 < M < M_0\,, && N = M_0\,, && \dot\varkappa \geqslant 0\,, \quad \dot\lambda \geqslant 0 \\
\text{or}\quad &M = 0\,, && N = M_0\,, && 0 \geqslant \dot\varkappa \geqslant -\dot\lambda\,.
\end{aligned} \tag{7.37}$$

An analysis similar to the rigid-plastic analysis of beams can then be formulated, with hinge circles and rigid annuli replacing the hinges and rigid lengths of section 6.3.

For details the reader should consult HOPKINS and PRAGER [1953, 1954], WANG [1955], or WANG and HOPKINS [1954]. (See also COX and MORLAND [1959].)

7.3.3. *A semi-empirical approach*

A different approach to the problem of penetration of a thin plate has been suggested by ZAID and PAUL [1957, 1958, 1959]. They consider the attack of a thin plate by a projectile of given momentum, and equate the momentum lost by the projectile to that gained by the plate. The kinetics of the motion of the plate is then deduced from photographic evidence.

7.4. CYLINDRICAL SHELLS

Symmetrical deformations of cylindrical shells may be divided into four types. First, longitudinal deformation symmetrical about the axis of the shell (which may be treated as merely a special case of the theory of § 2), secondly pure torsion about the axis, thirdly radial deformation, and lastly some combination of the three. Torsion and radial motion are discussed in the next two subsections. There seems to be nothing significant in the literature on combined stresses.

7.4.1. *Torsion*

Consider a shell of internal radius a, thickness h and density ϱ. Use cylindrical polar coordinates r, θ, z, and assume the displacements

$$u_r = u_z = 0\,, \qquad u_\theta = r\varphi(r, z, t)\,. \tag{7.38}$$

Then the strain components are

$$\varepsilon_{\theta z} = \tfrac{1}{2}r\,\frac{\partial \varphi}{\partial z}\,, \qquad \varepsilon_{r\theta} = \tfrac{1}{2}r\,\frac{\partial \varphi}{\partial r}\,, \tag{7.39}$$

and rates of strain follow by differentiation with respect to time. The equation of motion is

$$\varrho r\,\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z}\,. \tag{7.40}$$

For a Prandtl-Reuss material the flow equation is

$$\frac{r(\partial^2\varphi/\partial t\partial z) - G^{-1}(\partial\sigma_{\theta z}/\partial t)}{\sigma_{\theta z}} = \frac{r(\partial^2\varphi/\partial t\partial z) - G^{-1}(\partial\sigma_{r\theta}/\partial t)}{\sigma_{\theta r}}$$

$$= \frac{\partial k/\partial t}{k^2 \mathrm{d}k/\mathrm{d}w}\,, \tag{7.41}$$

with

$$\sigma_{\theta z}^2 + \sigma_{\theta r}^2 = K^2 . \tag{7.42}$$

Equations (7.40), (7.41) and (7.42) are sufficient for a complete solution of the problem of propagation of plastic strain, but no accurate solution has been given. An approximate solution was suggested by WOLF [1949], who neglected $\sigma_{\theta r}$. This approximation, which is in the spirit of thin shell theory, leads to

$$\frac{\partial \varphi}{\partial r} = 0 ,$$

$$r \frac{\partial^2 \varphi}{\partial t \partial z} = \frac{G + k \mathrm{d}k/\mathrm{d}W}{Gk \mathrm{d}k/\mathrm{d}W} \frac{\partial \sigma_{\theta z}}{\partial t} , \tag{7.43}$$

where the coefficient on the right is the inverse of the tangent modulus of the stress strain curve in torsion. Then from (7.40) and (7.43),

$$\varrho \frac{G + k \mathrm{d}k/\mathrm{d}W}{Gk \mathrm{d}k/\mathrm{d}W} \frac{\partial^3 \varphi}{\partial t^3} = \frac{\partial^3 \varphi}{\partial z^2 \partial t} , \tag{7.44}$$

and it is evident that waves are propagated with the speed corresponding to the tangent modulus. Note that this solution is exact, for a tube of any thickness, when the stress-strain relation is linear. This has an important bearing on the design of experiments to determine dynamic stress-strain curves, for the approximation involved in using (7.44) for a thin tube will be valid when the curvature of the stress-strain curve is small, independently of the rate of strain. This contrasts with the case of uniaxial tension, where the approximations leading to the tangent modulus are valid only at low rates of strain.

7.4.2. *Radial displacement*

The propagation of radial displacement along a thin tube presents a more difficult problem, in which progress has so far been made only by the use of crude approximations to the yield criterion. Assume a stress system with

$$\tau = \int_a^{a+h} \sigma_{rz} \mathrm{d}r , \qquad \sigma_\theta = \int_a^{a+h} \sigma_{\theta\theta} \mathrm{d}r , \qquad M = \int_a^{a+h} r \sigma_{zz} \mathrm{d}r ,$$

$$\int_a^{a+h} \sigma_{zz} \mathrm{d}r = 0 , \qquad \sigma_{r\theta} = \sigma_{\theta z} = 0 , \qquad \sigma_{rr}/\sigma_{\theta\theta} \text{ small} ,$$

and allow a radial force p, positive outwards, on the shell. Then, with the usual approximations of shell theory,

$$\frac{\partial M}{\partial z} = \tau\,, \qquad \frac{\partial \tau}{\partial z} + p - \frac{\sigma_\theta}{a} = \varrho\,\frac{\partial^2 u}{\partial t^2} \tag{7.45}$$

where u is the radial displacement and ϱ the mass per unit area of shell.

Now (7.45) must be supplemented by a yield criterion and flow rule. HODGE [1955] suggested an approximation as follows. For a perfectly rigid-plastic material the maximum absolute value for any direct stress component is the yield stress Y, so $|\sigma_\theta| \leqslant Yh$, $|M| \leqslant \frac{1}{4}Yh^2$. Again, for the Tresca criterion $\sigma_{\theta\theta} = Y$ does not imply any lower limit for σ_{zz}, so $\sigma_\theta = Yh$ is compatible with $|M| \leqslant \frac{1}{4}Yh^2$ and similarly $M = \frac{1}{4}Yh^2$ is compatible with $|\sigma_\theta| \leqslant Yh$. Hodge therefore suggests plasticity equations

$$\begin{aligned}
&\sigma_\theta = \pm\, Yh\,, && |M| < \tfrac{1}{4}Yh^2\,, && \frac{\partial^3 u}{\partial z^2 \partial t} = 0\,,\\
&M = \pm\, \tfrac{1}{4}Yh^2\,, && |\sigma_\theta| < Yh\,, && \frac{\partial u}{\partial t} = 0\,,\\
&\sigma_\theta = \pm\, Yh\,, && M = \pm\, \tfrac{1}{4}Yh^2, && \frac{\partial u}{\partial t} \geqslant 0,\ \pm\frac{\partial^3 u}{\partial z^2 \partial t} \leqslant 0\,,
\end{aligned}$$

where in the last line the same signs must be chosen for σ_θ and $\partial u/\partial t$ and for M and $\partial^3 u/\partial z^2 \partial t$.

Problems have been worked out on this theory by HODGE [1955] and EASON and SHIELD [1956].

§ 8. Experimental Methods

In order to justify the use of a mathematical model of a physical phenomenon, it is necessary either to show that the model includes all the relevant physical properties, or to confirm the predictions of the mathematical theory by reference to the results of experiments. In the case of plastic waves, the former method is unlikely to succeed, for the mechanism of plasticity is still only imperfectly understood. The second possibility, of verifying the theory by reference to experiments, requires the choice of particular experiments for which a complete mathematical analysis, according to the model, may be made. Approximate methods of solving the mathematical equations need not be rejected, as long as the effect of the approximations can be estimated, but to use approximate equations is more serious, for it is equivalent to changing the model in an undefined way. Obviously it is advantageous to use experiments for which the mathematical solution is as simple as possible.

Now the problem of propagation of plastic waves presents two main difficulties, one in the theoretical analysis and the other on the experimental side. The difficulty of the theory is that of analysing the motion of a medium in two or more dimensions without the usual mathematical tools of Fourier analysis or complex variable. The practical difficulties in the design of experiments are to separate the plastic waves, which theory suggests should be comparatively slow, from elastic waves, which travel much faster and may reflect from the surfaces of the medium, and to separate the longitudinal and transverse waves. In this section these points are discussed with reference to existing experimental work, and some tentative suggestions for more crucial experiments are made. No attempt is made to give a complete survey of the literature on the experimental side, and most of the early experiments are passed over, because though they were valuable in showing the type of difficulty to be overcome, are too difficult to analyse mathematically, and so can give only a vague idea of the truth or otherwise of a mathematical theory.

8.1. Measurement of uniaxial strain

One of the few special problems for which a complete mathematical analysis is possible under fairly general constitutive equations is that of uniaxial strain (section 4.1). Moreover, if waves are caused to propagate through the thickness of a material, in a slab of width large compared with the thickness, and if the load is applied uniformly over the surface of the slab, such waves do involve only uniaxial strain, at least near the centre of the slab, and for times small enough to avoid the reflection of elastic waves from the edges of the slab. The first experiments designed to take advantage of this fact were performed by Pack, Evans and James [1948], and subsequent workers, Allen [1953], Allen, Mapes and Mayfield [1955], Mallory [1955], Minshall [1955] and Broberg [1955], have used developments of their methods. So far it would seem that none of the techniques used has been sufficiently refined to trace the elastic wave reflected from the back face of the slab, or to record the changing amplitude of the plastic wave as it reacts with this wave. Further experiments which would provide good evidence on these points would be extremely valuable.

8.2. USE OF WAVES OF PLASTIC STRAIN SUPERIMPOSED ON AN EXISTING STRAIN

Suppose a strain-hardening material could be held indefinitely at a given state of stress and strain, corresponding to a point on the loading curve, $f(\sigma_{ij}) = k$. Then an additional plastic stress would be propagated with the velocity appropriate to a plastic wave, and the complication of a reflected elastic wave could be avoided. Two possible methods suggest themselves.

8.2.1. *Waves in a strip*

STERNGLASS and STUART [1953] used a standard testing machine to extend a metal strip beyond the yield limit, and then traced the propagation of the waves due to longitudinal impact, on an anvil attached to the strip. They were unable to find any evidence of plastic waves slower than elastic waves, and drew the conclusion that a small sharp pulse travels always with the velocity of an elastic wave. However, their experiments, admirable as they were in the skill with which they were performed, suffered from three basic faults, and the conclusions cannot therefore be regarded as final.

The first, and unavoidable, defect of the technique lies in the implicit assumption that there is uniaxial stress, for, as was explained in section 3.8, this assumption is not adequate to deal with pulse propagation. Moreover, the detailed analysis of pulse propagation in a strip is extremely difficult, and without such analysis the experiment can never provide definitive information about the properties of the material. In spite of this fact, the experiment could, in principle, give valuable guidance, for it is at least unlikely that the effect of a pulse in a plastic strip should be exactly the same as in an elastic one. However, in the experiments referred to, there were two other complications. One of these is the unknown effect of creep. The testing machine was strain-controlled, and the added load was small. There was therefore a possibility that, due to creep while the various measuring circuits were tested, the strip was not in fact at the elastic limit when the pulse was applied. The third defect was that the strip used was of commercial cold-rolled material, which might well be expected to suffer from differential hardening across the section, so that even though the load-extension curve was not straight at the stress at which the experiment was performed, the surface of the strip might yet have been below its elastic limit.

A repetition of the experiment, but with dead load and a carefully annealed strip would possibly be much more informative.

8.2.2. *Torsion of a thin tube*

A situation in which a complete elastic-plastic analysis is available is that of torsion of a thin tube. This depends on the assumption that the curvature of the stress-strain curve is small, but this is not a very important restriction if the tube is thin (section 7.4.1). Some useful information could perhaps be gleaned even from a simple loading programme, but to get the best possible results out of the experiment the torsion should increase slowly until the tube is plastic, and then suddenly to show the propagation of a plastic wave. Members of the Plasticity Division, National Engineering Laboratory, are at present engaged on such an experiment (GREEN, PUGH and WANG, private communication).

8.3. SURFACE WAVES

There is one other problem which can in principle be analysed mathematically, namely the problem of a surface wave. The existence of such waves in an elastic-plastic material can be established, and surface waves are particularly suitable for measuring purposes. It seems likely that the published results of BELL [1956, 1959] may be explained on these lines, but this is a matter of current research.

8.4. OTHER EXPERIMENTS

Many of the papers quoted in previous sections contain reference to experimental work. Other papers of some relevance are CAMPBELL and DUBY [1956], DAVIES [1949], DURELLI and RILEY [1957], HUGHES and MAURETTE [1956], JOHNSON, WOOD and CLARK [1953], LENSKII [1949] and PIRONNEAU [1953]. DAVIES [1953] gives a very good bibliography.

Notation

The following list includes the symbols most frequently used. Other symbols are explained in the text as introduced.

c, c_1, c_2	wave velocities
$\mathrm{D}/\mathrm{D}t$	derivative following the motion
e_{ij}	elastic part of strain tensor
E	Young's modulus
G	modulus of rigidity
I	second moment of area
k	yield stress in shear
M	bending moment
p	hydrostatic pressure
Q	shearing force
u, u_i	displacement
v, v_i	velocity
W	plastic work
x_i	Cartesian coordinates
Y	yield stress in tension
δ_{ij}	unit tensor
Δ	dilatation
$\varepsilon, \varepsilon_{ij}$	strain, strain tensor
$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon'_{ij}$	principal strains, strain deviator
$\varkappa$	bulk modulus; curvature
λ, μ	Lamé elastic moduli
ν	Poisson's ratio
σ, σ_{ij},	stress, stress tensor
$\sigma_1, \sigma_2, \sigma_3, \sigma'_{ij}$	principal stresses, stress deviator
η_{ij}	plastic part of strain tensor

References

ALEXANDER, J. M. and H. FORD, 1954, Proc. Roy. Soc. London A **226** 543.
ALLEN, W. A., 1953, J. Appl. Phys. **24** 1180.
ALLEN, W. A., J. M. MAPES and E. B. MAYFIELD, 1955, J. Appl. Phys. **26** 1173.
ALVERSON, R. C., 1956, J. Appl. Mech. **23** 411.
BAKHSHIYAN, F. A., 1948, Prik. Math. Mech. **12** 47, 281.
BELL, J. F., 1956, J. Appl. Phys. **27** 1109.
BELL, J. F., 1959, J. Appl. Phys. **30** 196.
BOUSSINESQ, J., 1885, Applications des Potentiels (Paris).
BRIDGMAN, P. W., 1931, The Physics of High Pressure (Bell, London).
BRIDGMAN, P. W., 1952, Studies in Large Plastic Flow and Fracture (McGraw-Hill Book Co., New York).
BROBERG, K. B., 1955, J. Appl. Mech. **22** 317.

CAMPBELL, J. D. and J. DUBY, 1956, Proc. Roy. Soc. London A **236** 24.
CEBAN, V. G., 1953, Prik. Math. Mech. **17** 200.
CHREE, C., 1886, Qu. J. Math. **21**.
CONROY, M. F., 1952, J. Appl. Mech. **19** 465.
CONROY, M. F., 1956, J. Appl. Mach. **23** 239.
COTTER, B. A. and P. S. SYMONDS, 1955, Proc. American Soc. Civil Eng. **81** 21.
COURANT, R., K. FRIEDRICHS and H. LEWY, 1928, Math. Ann. **100** 32.
COX, A. D. and L. W. MORLAND, 1959, J. Math. Phys. Solids **7** 229.
CRAGGS, J. W., 1952, Proc. Roy. Soc. Edinburgh **53** 359.
CRAGGS, J. W., 1954, J. Math. Phys. Solids **2** 286.
CRAGGS, J. W., 1957, J. Math. Phys. Solids **5** 115.
CRISTESCU, N., 1951, Com. Acad. R. P. Roumania **1** 439.
CRISTESCU, N., 1954, Prik. Math. Mech. **18** 257.
CRISTESCU, N., 1956, Com. Acad. R. P. Roumania **6** 19.
CRISTESCU, N., 1957, Prik. Math. Mech. **21** 486.
CRISTESCU, N., 1958, Probleme Dinamice in Teoria Plasticitatii (Bucharest, Editura Tech.).
DAVIES, G. and B. G. NEAL, 1959, Proc. Roy. Soc. London A **253** 542.
DAVIES, R. M., 1948, Phil. Trans. Roy. Soc. London A **240** 375.
DAVIES, R. M., 1949, Proc. Roy. Soc. London A **197** 416.
DAVIES, R. M., 1953, Appl. Mech. Rev. **6** 1.
DURELLI, A. J. and W. F. RILEY, 1957, J. Appl. Mech. **24** 69.
DUWEZ, P. E., D. S. CLARK and H. F. BOHNENBLUST, 1950, J. Appl. Mech. **17** 27.
EASON, G. and R. T. SHIELD, 1956, J. Mech. Phys. Solids **4** 53.
FREIBERGER, W., 1952, Proc. Cambridge Phil. Soc. **48** 135.
FRIEDRICHS, K., 1948, American J. Math. **70** 555.
GREEN, A. E. and R. RIVLIN, 1960, Archiv. Rat. Mech. Anal. **4** 387.
GRIGORYAN, D. M., 1949, Prik. Math. Mech. **13** 277.
HILL, R., 1950, The Mathematical Theory of Plasticity (Clarendon Press, Oxford).
HODGE, P. G., 1955, J. Mech. Phys. Solids **3** 176.
HODGE, P. G. and R. SANKARANARAYANAN, 1957, unpublished.
HOPKINS, H. G., 1955, J. Math. Phys. Solids **4** 38.
HOPKINS, H. G., 1960, Progress in Solid Mechanics, Vol. I (North-Holland Publ. Co., Amsterdam) p. 85.
HOPKINS, H. G. and W. PRAGER, 1953, J. Mech. Phys. Solids **2** 1.
HOPKINS, H. G. and W. PRAGER, 1954, Z. Ang. Math. Phys. **5** 317.
HUGHES, D. S. and C. MAURETTE, 1956, J. Appl. Phys. **27** 1184.
HUNTER, S. C., 1957, Proc. Conference: Properties of Materials at High Rates of Strain (Inst. Mech. Engineers, London, 1957) p. 147.
HUNTER, S. C., 1960, Progress in Solid Mechanics, Vol. I (North-Holland Publishing Co., Amsterdam) p. 3.
JOHNSON, J. E., D. S. WOOD and D. S. CLARK, 1953, J. Appl. Mech. **20** 523.
JUHASZ, K., 1949, J. Franklin Inst. **248** 15.
KARMAN, T. VON, 1942, unpublished.
KARMAN, T. VON and P. E. DUWEZ, 1950, J. Appl. Phys. **21** 987.
KOCHETKOV, A. M., 1950, Prik. Math. Mech. **14** 203.
LAZUTKIN, D. F., 1952, Prik. Math. Mech. **16** 94.
LEE, E. H., 1953, Qu. Appl. Math. **10** 335.
LEE, E. H. and P. S. SYMONDS, 1952, J. Appl. Mech. **19** 308.
LEE, E. H. and S. J. TUPPER, 1954, J. Appl. Mech. **21** 63.
LENSKII, V. S., 1949, Prik. Math. Mech. **13** 165.
LOVE, A. E. H., 1927, The Mathematical Theory of Elasticity (4th Ed.) (Cambridge University Press).

MALLORY, H. D., 1955, J. Appl. Phys. **26** 555.
MALVERN, L. E., 1951a, Qu. Appl. Math. **8** 405.
MALVERN, L. E., 1951b, J. Appl. Mech. **18** 203.
MENTEL, T. J., 1955, Canadian J. Technology **33** 237.
MINSHALL, S., 1955, J. Appl. Phys. **26** 463.
MISES, R. VON, 1913, Göttinger Nachrichten (Math. Phys. Klasse) 582.
MOCHALOV, S. D., 1955, Uch. Zap. Tomskogo In-Ta **25** 59.
MORLAND, L. W., 1959, Phil. Trans. Roy. Soc. A **251** 341.
MOTT, N. F., 1952, Phil. Mag. **43** 1151.
MOTT, N. F., 1953, Phil. Mag. **44** 742.
MUSGRAVE, M. J. P., 1954, Proc. Roy. Soc. London A **226** 239.
NADAI, A., 1931, Plasticity (McGraw-Hill Book Co., New York).
NEZHENTSEV, P. I., 1956, Trud. Nikolaevskogo, Korablestroit In-ta **8** 221.
O'BRIEN, G., M. A. HYMANS and S. KAPLAN, 1951, J. Math. Phys. **29** 223.
PACK, D. C., W. M. EVANS and H. J. JAMES, 1948, Proc. Phil. Soc. London **61** 1.
PARKES, E. W., 1955, Proc. Roy. Soc. London A **228** 462.
PIRONNEAU, Y., 1953, Comptes Rendus Paris **236** 46.
POCHHAMMER, L., 1876, J. F. Math. (Crelle) **81** 287.
PRAGER, W., 1955, Proc. Inst. Mech. Eng. **169** 41.
RAKHMATULIN, K. A., 1945a, Prik. Math. Mech. **9** 91.
RAKHMATULIN, K. A., 1945b, Prik. Math. Mech. **9** 449.
RAKHMATULIN, K. A., 1946, Prik. Math. Mech. **10** 333.
RAKHMATULIN, K. A., 1947, Prik. Math. Mech. **11** 379.
RAKHMATULIN, K. A., 1948, Prik. Math. Mech. **12** 39.
RAKHMATULIN, K. A., 1952, Prik. Math. Mech. **16** 23.
REABOVA, E. V., 1953, Vest. M.G.U. (Moscow) **10** 85.
SALVADORI, M. G. and F. DIMAGGIO, 1953, J. Appl. Mech. **20** 223.
SALVADORI, M. G. and P. WEIDLINGER, 1957, Proc. American Soc. Civil Eng. **83**.
SEILER, J. A., B. A. COTTER and P. S. SYMONDS, 1956, J. Appl. Mech. **23** 515.
SEILER, J. A. and P. S. SYMONDS, 1954, J. Appl. Phys. **25** 556.
SHAPIRO, G. S., 1946, Prik. Math. Mech. **10** 597.
SHAPIRO, G. S., 1952, Prik. Math. Mech. **16** 335.
SHAPIRO, G. S., 1959, Prik. Math. Mech. **23** 234.
SHOI-YEAN HWANG and N. DAVIDS, 1960, J. Mech. Phys. Solids **8** 52.
STERNGLASS, E. J. and D. A. STUART, 1953, J. Appl. Mech. **20** 427.
SYMONDS, P. S. and C. LETH, 1954, J. Mech. Phys. Solids **2** 92.
TAYLOR, G. I., 1940, 1942, unpublished.
TRESCA, H., 1864, Comptes Rendus Paris **59** 754.
WANG, A. J., 1955, J. Appl. Mech. **22** 375.
WANG, A. J. and H. G. HOPKINS, 1954, J. Mech. Phys. Solids **3** 22.
WHITE, M. P., 1949, J. Appl. Mech. **16** 39.
WHITE, M. P. and L. VAN GRIFFIS, 1947, J. Appl. Mech. **14** 337.
WOLF, H., 1949, unpublished.
WOOD, D. S., 1952, J. Appl. Mech. **19** 521.
ZAID, M. and B. PAUL, 1957, J. Franklin Inst. **264** 117.
ZAID, M. and B. PAUL, 1958, J. Franklin Inst. **265** 317.
ZAID, M. and B. PAUL, 1959, J. Franklin Inst. **266** 24.
ZVEREV, I. N., 1950, Prik. Math. Mech. **14** 295.

CHAPTER V

THE MEASUREMENT OF DYNAMIC ELASTIC PROPERTIES

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CONTENTS

§ 1. Introduction

The term *dynamic* elastic properties implies a certain restriction on the type of measurement. Although no test procedure can be truly described as static, it is usual to limit dynamic measurements to those taking place in less than one second or to cylic measurements occurring with repetition rates greater than one per second. These limits are accepted in this chapter so that methods of measurement based on the analysis of creep or stress relaxation are excluded.

The description of the elastic properties of a material is best achieved by stating the values of the constants describing a generalized stress-strain relation. For example, if the relation is a linear one, a generalized Hooke's Law with stress tensor σ and strain tensor ε is applicable; such a relation is given by:

$$\sigma_i = c_{ij}\varepsilon_j \tag{1}$$

where c_{ij} represent the elastic constants. By reasons of symmetry only 21 independent constants are required in the most anisotropic case; for a more isotropic body, this number is reduced considerably and for a completely isotropic body there are only two independent constants required and all values of c_{ij} are functions of these two (LOVE [1927]). However, if a simple linear system cannot be assumed, then the description of the elastic properties of a material in general terms becomes very complex. HEARMON [1956] has discussed the requirements of crystal systems when higher order terms are included in the general stress-strain relation. The completely anisotropic body requires 56 constants and in the case of isotropic bodies 3 independent coefficients are required (HEARMON [1953]). When materials are elastically deformed dynamically so that relative motion occurs between different parts of the material, some of the elastic energy is inevitably converted to heat and is lost from the elastic system. This phenomenon is referred to as internal friction. The simplest way of defining the loss is to express it in terms of the energy loss, ΔW, as a fraction of the total energy W stored in the specimen at maximum strain. When referred to a cyclic stress or strain this ratio $\Delta W/W$ is called the "specific damping capacity" or "specific loss". The value of the specific loss varies very widely for different materials, but for a large range of materials at room temperature it is quite small. It is, however, no longer possible to use eq. (1) to define the elastic behaviour of the material. The simplest addition to the equation that will introduce the concept of a loss dependent on velocity was made by VOIGT [1892], while MAXWELL [1890], deriving his equation from the nature of viscosity, introduced a term proportional to rate of change of stress with time. However neither simple treatment adequately accounts for the full behaviour of even simple materials and a combination of the two has been widely discussed. This has been referred to as the standard linear solid by ZENER [1948] and its stress-strain equation can be written

$$a_1\sigma + a_2 \frac{\partial \sigma}{\partial t} = b_1 \varepsilon + b_2 \frac{\partial \varepsilon}{\partial t}. \tag{2}$$

This contains three independent constants and can be more conveniently written

$$\sigma + \tau_1 \frac{\partial \sigma}{\partial t} = M\left(\varepsilon + \tau_2 \frac{\partial \varepsilon}{\partial t}\right). \tag{3}$$

The constants τ_1 and τ_2 are termed relaxation times and M can be termed the elastic modulus.

A further complication is introduced by the behaviour of materials that show markedly time-dependent elastic properties. The largest class of these materials is probably that included in the generic name "plastics". No completely satisfactory way of describing the behaviour of such materials has yet been found. Recourse has been made to model systems formed by combinations of springs and dashpots but unfortunately, without extending the concept to include non-linear springs and dashpots with non-Newtonian viscosity, no exact description can be found. The suggestion of BLAND and LEE [1955] that the stress-strain equation be written with complex operator coefficients

$$P_i\left(\frac{\partial}{\partial t}\right)\sigma_i = Q_j\left(\frac{\partial}{\partial t}\right)\varepsilon_j, \tag{4}$$

where $P_i(\partial/\partial t)$ and $Q_j(\partial/\partial t)$ are polynomials in $(\partial/\partial t)$ with constant coefficients, is probably one of the most comprehensive.

The simplest method of measuring elastic constants is to submit the sample to a known stress and to observe the strain or to deform the sample by a known amount and to measure the necessary stress. Although some success has been achieved by such direct methods applied to dynamic testing, there are very great difficulties in providing suitable equipment because of inertia effects and these difficulties have led to the use of indirect methods. Here the ease of experimental work is offset by the difficulties in interpreting the results in terms of elastic constants. Two main divisions of these indirect methods can be distinguished: methods using resonance techniques, which depend on vibration theory, and acoustic methods based on wave propagation theory.

§ 2. Resonance Methods

The specimen, or some complex unit containing the specimen, is set into oscillation in a known mode of vibration, and observations are made of the response of the unit to changes of frequency. Alternatively, the specimen may be set into free vibration and the value of the frequency obtained at which the unit vibrates naturally. It is assumed that vibrations are in some form of simple harmonic motion. This assumption is generally validated by the low amplitude of the vibrations that are observed. The most important measurement is one of frequency and with modern electronic techniques no difficulty is experienced in obtaining accuracies of better than 1 in 10^6 (PURSEY and PYATT [1954]). The limitations on accuracy therefore arise from measurements of the dimensions of the vibrating unit and from doubts about the exact theoretical relation between the elastic constants and the observed vibrations.

The most straightforward equation of motion of particles of a body subjected to vibration is

$$P = M\ddot{\xi} + \eta\dot{\xi} + E\xi\,. \tag{5}$$

P is the impressed force and is equal to the sum of a term involving mass M and acceleration $\ddot{\xi}$, a term proportional to velocity and an elastic restoring force proportional to displacement, the last two constants of proportionality being η and E. In the case of free resonance $P = 0$ and it can be shown that a solution of eq. (5) then becomes

$$\xi = K \exp\left(\frac{-\eta t}{2M}\right) \cos\,(pt + \beta) \tag{6}$$

where $p^2 = E/M - \eta^2/4M^2$; K and β depend on boundary conditions. The frequency of the free oscillations is equal to $p/2\pi$ and the ratio of the amplitude of successive oscillations is $\exp(\eta\pi/Mp)$. The logarithm of this ratio is referred to as the logarithmic decrement Δ and is thus given by

$$\Delta = \frac{\eta\pi}{Mp}. \tag{7}$$

If the damping is small then $p \to E/M$ and $\Delta = p\pi\eta/E$. Furthermore, since the elastic energy stored in the system is proportional to the square of the amplitude of the displacement, the specific loss $\Delta W/W$ per cycle is therefore given by

$$\frac{\Delta W}{W} = \frac{\xi_1^2 - \xi^2}{\xi_1^2} \approx \frac{2(\xi_1 - \xi)}{\xi_1} \approx 2 \log \frac{\xi}{\xi_1} = 2\Delta\,. \tag{8}$$

Thus, for small values of specific loss, measurements of the log. decrement of free oscillations will equal half the specific loss and the frequency of the free oscillations can be used to find the elastic constant.

For forced oscillations, returning to eq. (5) we have $P = P_0 \sin pt$ and the steady state solution is

$$\xi = \frac{P_0 \sin(pt - \delta)}{pZ} \tag{9}$$

where

$$Z^2 = \left(\frac{E}{p} - Mp\right)^2 + \eta^2 \tag{10}$$

and

$$\tan\delta = \frac{p\eta}{E - Mp^2}. \tag{11}$$

These equations are analogous to electrical circuit theory and Z is called the mechanical impedance of the vibrating system. From eq. (9) clearly the amplitude is a maximum when pZ is a minimum and hence

$$p^2 = \frac{E}{M} - \frac{\eta^2}{2M^2}. \tag{12}$$

This resonance frequency can therefore be used to determine the elastic constant E, if the specific loss is small enough to be neglected.

More information, however, can be obtained from the shape of the resonance curve (i.e. amplitude plotted against frequency). A useful parameter of this shape is the half width of the curve, i.e. values of p at which the amplitude ξ is half the value at resonance. If these values are p_1 and p_2 it can be shown that

$$p_1^2 - p_2^2 = \frac{2\sqrt{3}\eta p}{M}. \tag{13}$$

Again, if the specific loss is small and ΔN and N are the half width in frequency and resonant frequency, respectively, then

$$\frac{\Delta N}{N} \approx \frac{p_1 - p_2}{p} \approx \frac{p_1^2 - p_2^2}{2p^2}. \tag{14}$$

Then, from (13) and (12),

$$\frac{\Delta N}{N} \approx \frac{\sqrt{3}\eta}{Mp}. \tag{15}$$

From eqs. (7) and (8)

$$\frac{\Delta N}{N} = \frac{\sqrt{3}}{2\pi}\frac{\Delta W}{W}. \tag{16}$$

Thus, if the specific loss is low, the half width of the frequency curve can be used to determine the specific loss. The term often used in electrical circuits to define sharpness of resonance is "Q". This is defined as

$$\frac{1}{Q} = \frac{1}{\sqrt{3}}\frac{\Delta N}{N}$$

and hence

$$Q = \frac{\pi}{\Delta} = \frac{2\pi}{\Delta W/W}. \tag{17}$$

No specific vibration systems have been discussed in this section. The following section is devoted to free oscillation systems and 2.2 to forced vibration systems.

2.1. Free vibrations

The specimen or composite unit containing the specimen is set in vibration and then left free from external force, whilst the frequency and the decay of the oscillations are observed. Although the solution of (5) for free oscillations [eq. (6)], is given as an oscillation, more detailed examination shows that an essential condition for this is that η^2 is less than $4EM$, i.e. the total damping must be kept small. Great care must be taken to isolate the specimen completely, and it is usual to evacuate the specimen container, if measurements of the decay of oscillations are required. The method is of limited application since, by definition, the sample cannot be coupled directly to any sensitive detector. The initiation of the vibrations must be arranged so that the transient constraint of the free mode of oscillation is rapidly eliminated and a sufficiency of free vibrations remain to allow for accurate measurement of frequency. In the case of a composite oscillator the sample under test and some known oscillator, usually quartz, are bonded together and set in oscillation. From the known elastic constants of the quartz those of the sample can be calculated.

The commonest application of this method is to the measurement of the torsional oscillations of thin rods or wires. KÊ [1947] has used the method for metal wires, LETHERSICH [1950] for polymers, MEREDITH [1954] for textile fibres and BENBOW [1953] for organic glasses. The equipment used by Benbow illustrates the general method. The sample in the form of a rod 2 to 6 inch long $\frac{1}{8}$ to $\frac{1}{4}$ inch diameter is mounted vertically with the upper end rigidly held. The lower end is securely bonded to a very light, single open loop of copper foil. Very light flexible leads connect this coil to a variable frequency oscillator, and the coil on the specimen is located between the poles of a powerful horse-shoe magnet. A mirror is attached at the end of the specimen so that oscillations can be followed. By adjusting the frequency of the oscillator to resonance and then arranging to isolate the coil by a high speed switch, the decay of free oscillations of the rod can be followed. The frequency at which the measurements are carried out depends on the dimensions and shear modulus of the sample. Benbow used samples 14.8 cm and 6.6 cm in length and obtained results at frequencies of 1200–1600 c/s and 2800–3300 c/s depending on temperature within each range. At these frequencies the free oscillations of the specimen are best observed by converting the light signal by a suitable photocell circuit into an electrical signal which can be displayed on an oscilloscope and then photographed. At lower frequencies, as in Meredith's work, simple observation or the direct photographic recording of a reflected light spot on a moving film (drum camera) is sufficient.

The chief disadvantage of most resonance methods is the limited range of frequency that can be covered with one specimen. In a free oscillation method devised by BODNER and KOLSKY [1958] this disadvantage has been overcome. The specimen is in the form of a long bar. The bar is suspended and fitted with coils at each end. These coils are placed in magnetic fields, one coil acts as driving transducer to set the bar into oscillation in the required mode; the other coil acts as detector. By arranging the direction of the magnetic field appropriately, the bar is vibrated in longitudinal or flexural modes in fundamental and higher harmonics. Once the bar has been set into vibration in the required mode, power is switched from the driving oscillator and the decay of free oscillations followed by the detector transducer. Since the bar is excited in only longitudinal or flexural modes, the complex Young's modulus is the relevant elastic constant in both modes and this constant has been measured over a frequency range of 17–4500 c/s for a sample of lead.

2.2. FORCED VIBRATIONS

By coupling some form of driving oscillator to the specimen and observing the forced resonance of the system, a much more powerful

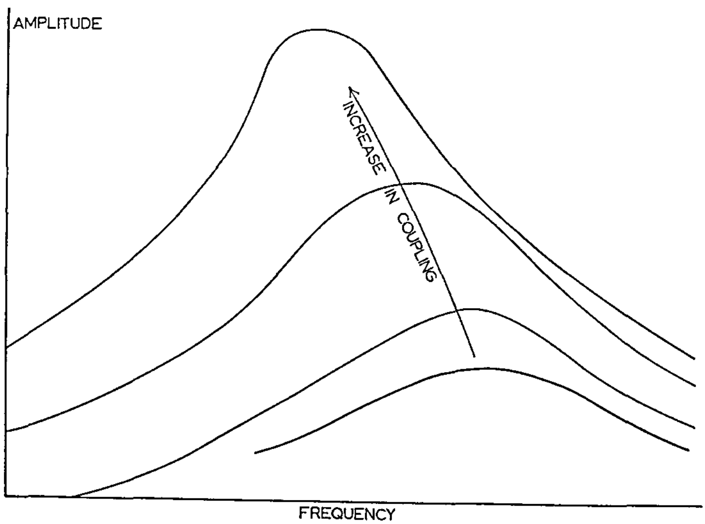

Fig. 1. Effect of coupling on the resonance curve.

and versatile method of investigating elastic constants is available. The degree of coupling can vary over very wide limits. If the coupling is tight, the specimen and driver must be treated as a composite oscillator; it is possible, however, to make the coupling loose enough so that errors introduced by neglecting the composite nature of the system are a small fraction of the experimental error. The nature of the change in the resonance curve as a system is progressively decoupled is illustrated in Fig. 1 taken from a paper by HILLIER [1951]. The simplest system is furnished by the longitudinal oscillations of a bar, generally of circular section. Such oscillations can be excited in a bar by using a piezoelectric transducer, a magnetostriction unit or an electrostatic unit. The original design is due to QUIMBY [1925] who used a quartz crystal driving unit. The specimen of similar dimensions to the quartz crystal is bonded to it by a suitable cement or the mating faces are ground smooth and then wrung together with a trace of liquid (oil). The composite unit is then treated as a resonating system and resonance curves of amplitude of oscillation, or more usually changes in electrical impedance of the driving circuits, are obtained. Further curves are found for the quartz crystal alone and the performance of the specimen deduced from these observations. If M_1 is the mass of the crystal, M_2 the mass of the specimen, f_1 and f_2 the resonant frequencies of the crystal and specimen, if vibrating independently, and f the observed resonant frequency of the composite unit, then

$$M_1 f_1 \tan\left(\frac{\pi f}{f_1}\right) + M_2 f_2 \tan\left(\frac{\pi f}{f_2}\right) = 0\,. \tag{18}$$

From eq. (18) f_2 can be found and from the length of the specimen l the velocity of longitudinal waves (c_1) in the specimen can be calculated:

$$c_1 = 2lf_2\,. \tag{19}$$

Several corrections must be made to this simple theory. TERRY [1957] has discussed errors that can arise if there is a difference in the cross sectional areas of the driving oscillator and the specimen. If the cross sectional areas are A_1 and A_2, the velocities c_1 and c_2, and densities ϱ_1 and ϱ_2, then eq. (18) can be replaced more exactly by the equation

$$\varrho_1 c_1 A_1 \tan\left(\frac{\pi f}{f_1}\right) + \varrho_2 c_2 A_2 \tan\left(\frac{\pi f}{f_2}\right) = 0\,. \tag{20}$$

A further error is introduced by the presence of a cement; this can be regarded as a third element, i.e. if ϱ_3, c_3 are constants for the cement equation (20) can be rewritten

$$\varrho_1 c_1 A_1 \tan\left(\frac{\pi f}{f_1}\right) + \varrho_2 c_2 A_2 \tan\left(\frac{\pi f}{f_2}\right) + \varrho_3 c_3 A_3 \tan\left(\frac{\pi f}{f_3}\right) = 0\,. \tag{21}$$

Terry has shown that the fractional error in the resonant frequency of the specimen $R = \delta f_2/f_2$ is given by

$$R = -\frac{M_3}{M_2}\left[\cos^2\theta_2 + \left(\frac{\varrho_2 c_2}{\varrho_3 c_3}\right)^2 \sin^2\theta_2\right]\left[\frac{2\theta_2}{2\theta_2 - \sin 2\theta_2}\right] \tag{22}$$

where θ_i are written for $\pi f/f_i$, $(i = 1, 2, 3)$. If, however, the specimen size is chosen so that $f_1 \approx f_2 \approx 2f$ then $\theta_2 = \frac{1}{2}\pi$ and eq. (6) can be shown to reduce to

$$R = \frac{l_3 K_2}{l_2 K_3} \tag{23}$$

l_3 being the thickness of the cement film.

A further error may be introduced by the neglect of lateral motion. This, however, will only be important if short rods of large section are used. Boyle and Sproule [1929] have shown that if $l > 2d$ (d = diameter of bar) the lateral motion is negligible.

Marx [1951] has described a refinement to this method, which has advantages if the loss in the material is to be measured. The composite

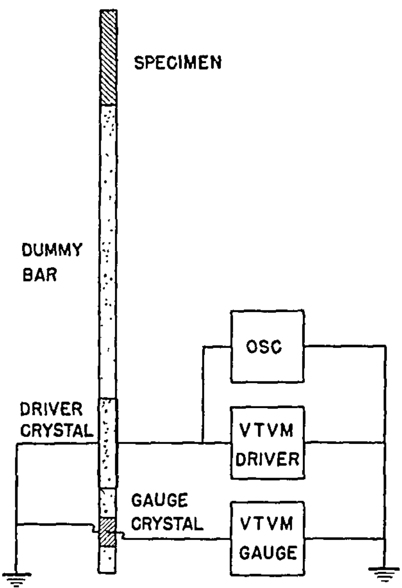

Fig. 2. Composite resonator for longitudinal vibrations (Marx).

resonator is made in three parts, the driving crystal, the specimen and a detecting crystal. The two crystals are matched and are first coupled without the specimen; a calibration factor can then be obtained which is used later when the specimen is inserted. The circuit arrangement is shown in Fig. 2 and the advantages arise from the ease with which the decrement and actual strain amplitude can be measured from the readings of the valve voltmeters recording the peak amplitude of both driver and detector or gauge crystal. If δ is the loss coefficient equal to internal friction α divided by frequency then

$$\delta = \frac{K}{mf^2} \frac{V_\mathrm{d}}{V_\mathrm{g}} \tag{24}$$

where K is calibrated crystal response parameter, m mass of composite resonator, f the resonant frequency, and V_d and V_g the maximum (peak to peak) voltages at resonance of the driving and gauge crystals respectively. This method has been used extensively and in particular MARX and SIVERSTEN [1953] have determined the elastic constants of glass over a wide range of temperature. It can be shown that, provided δ is not greater than 0.1, the component values of δ in a composite resonating unit are given by

$$m_\mathrm{t}\delta_\mathrm{t} = m_\mathrm{a}\delta_\mathrm{a} + m_\mathrm{b}\delta_\mathrm{b} + \text{etc.} \tag{25}$$

If suffix s identifies specimen and q quartz crystals then

$$\delta_\mathrm{s} = \frac{m_\mathrm{t}\delta_\mathrm{t} - m_\mathrm{q}\delta_\mathrm{q}}{m_\mathrm{s}}. \tag{26}$$

From (24) and (26), therefore, the decrement of the specimen can be calculated.

In the arrangement used by Marx and Siversten the specimen of glass formed part of a rod $n\lambda$ in length which was compared with one $(n-1)\lambda$ in length, only the end section being in the furnace. Then

$$\delta_\mathrm{s} = \frac{m_n\delta_n - m_{n-1}\delta_{n-1}}{m_\mathrm{s}}.$$

Thus values of δ for bars having values 4λ and 5λ would enable decrement δ_s of the specimen at the high temperature to be ascertained without subjecting the crystal transducers to high temperature. BAKER [1957] further modified the method to permit the specimen to be placed under static stress. His equipment is shown in Fig. 3. The equipment incorporates two additional steel resonators either side of the specimen of length $\frac{1}{2}\lambda$ with flanges at the centre forming nodal sections. The static forces on the specimen are applied through these flanges and the elastic constants measured in the normal way, the damping being obtained from application of (25) using the observed decrements of the sections.

Although the modification used by Marx and Siversten described above overcomes the difficulties of using quartz crystals at high

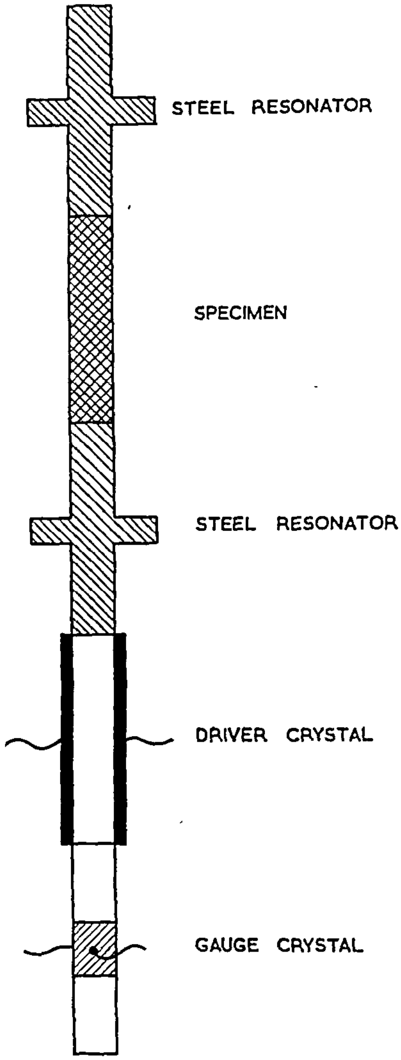

Fig. 3. Composite resonator for specimens under static compression (Baker).

temperatures, it is not always possible to obtain the appropriate lengths of specimen, and the advantages of electrostatic transducers become apparent. The refinements of this technique are described by PURSEY and PYATT [1954], who used the method for measurement of metal crystals, and by Fine [1957], who used it to make measurements to 800 °C on small samples of aluminium and tungsten.

The specimen is suitably fixed by supports located at the nodal section and the ends ground flat or suitably metalled (in the case of a non-conductor) to provide the earth plate of a condenser. The 'live' plate is so mounted that its position relative to the specimen end can be controlled by a micrometer screw, and is also sufficiently massively held to be regarded as static when the sample is vibrating. It is preferable to polarise the condenser by a suitable D.C. potential and to apply the driving e.m.f. through isolating capacitors. It is convenient (as in the arrangement by Pursey and Pyatt) to monitor the resonance by measuring the changes in capacity (caused by changes in the distance between the plates, hence the amplitude of the vibrations) by placing the condenser in an oscillating circuit of much higher frequency than the driving force. For example, the specimen resonances can be in the range 2–250 kc/s and the high frequency detector operate at 20–60 Mc/s. The high frequency circuit suffers frequency modulation by the specimen resonance, the amplitude of which indicates the state of resonance. The detection of this f.m. signal provides the primary measurement, together with the modulation frequency, of the elastic behaviour of the specimen. The method is applicable only to specimens having relatively low damping, since the efficiency of the coupling is low; it finds greatest application in the metal field.

Among the more recent experimental methods of determining elastic coefficients by using longitudinal oscillations is that due to Thompson and Glass [1958]. It can be shown that for small values of the internal friction such that δ is not greater than 3×10^{-2}, the following relation exists between A the maximum amplitude of oscillation, F the driving force, ω_0 the pulsatance at resonance:

$$A = \frac{\sqrt{3}F}{m\omega_0^2\delta}. \tag{27}$$

If the amplitude of oscillation is *maintained constant* throughout the resonance curve, then clearly from eq. (27) F is proportional to δ and hence measurements of F will give values of δ. This system was used to measure the changes in elastic constants of materials whilst subjected to atomic radiation, and extra complexity in the auxiliary circuits was derived from the need to telemeter information from the specimen to the outside of the atomic pile used for irradiation. The basic circuit is shown in Fig. 4. The essential element is the inverse automatic gain control which is arranged to maintain the amplitude of the oscillations constant despite the resonance. The driving force is provided by an eddy current transducer. The oscillations of the sample, as in Pursey and Pyatt's equipment, are detected by a capacity type R.F. distance meter which is monitored by the A.G.C. unit. Feedback from this

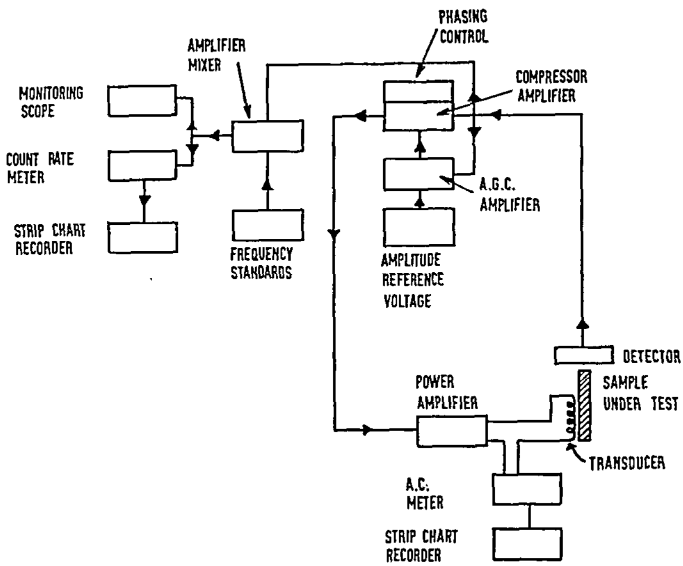

Fig. 4. Equipment for the accurate measurement of elastic constants of irradiated samples.

unit is arranged to modify the driving oscillator, so that the *input* signal to the A.G.C. unit is maintained at a constant preset level — A in (27). It is apparent that the time constant of the A.G.C. feedback loop must be shorter than the decay time of oscillations in the specimen if the circuit is not to 'hunt' or become unstable. The accuracy achieved by Thompson and Glass is very high, measurements of the resonant frequency can be made to 1 in 150 000 and the absolute elastic moduli are available to the accuracy of measurement of sample dimensions.

Flexural oscillations are very frequently used when measurements are being made on specimens available as thin rods or bars. The mode of oscillation is simple to generate and a large displacement of the specimen is available for detection with only a low strain amplitude. Among the systems using flexural vibrations a considerable amount of work has been carried out on the vibrating cantilever, i.e. a bar clamped at one end. The linear elastic theory of this mode of oscillation was adequately treated by LORD RAYLEIGH [1894] and LOVE [1927]. The exact equations of motion are complex, involving correction terms for the rotary inertia and for shearing forces (KOLSKY [1953]). However, provided the cross section dimensions of the bar are small compared with the length and only the lowest modes are studied, then the corrections become negligible and the motion is given by the simple equation

$$\varrho A \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4} \tag{28}$$

where ϱ is density of bar, A is cross section area, I moment of inertia, E Young's modulus, and y is displacement of particle on axis at distance x from one end.

A solution of eq. (28) can be obtained for the boundary conditions relevant to the experimental arrangement since the bar is oscillating with simple harmonic motion. If the length of the rod is l, cross sectional area A and density ϱ, the frequency of the fundamental mode f_0 is given by the equation

$$\frac{k}{2\pi}\left(\frac{E}{\varrho}\right)^{\frac{1}{2}} = \frac{l^2 f_0}{z} \tag{29}$$

where k is the radius of gyration of the cross section perpendicular to its plane of motion and z is given by (PRESCOTT [1946])

$$\frac{1 + \cosh z \cos z}{\cosh z \sin z - \sinh z \cos z} = cz\,. \tag{30}$$

If the simple elastic theory is not applicable, the treatment of the vibrating cantilever becomes difficult, probably the most comprehensive theory is that already referred to by Bland and Lee. The generalized stress strain relation given in eq. (2) has been applied by them to the vibrating cantilever. The method for the extraction of the elastic constants from the observed forced motion of the rod is an iterative one. By considering the analogy between electrical and mechanical systems NOLLE [1948] has derived a simpler method which is applicable to a certain range of values of internal friction such that δ values (equivalent to $1/Q$) are less than 0.2. HILLIER [1951] and HORIO and ONOGI [1951] independently showed that above this value of δ the elastic modulus as well as the value of δ would be in error, if the simple theory were used.

Experimentally the method involves the excitation of the oscillations and the observations of their amplitude and possibly phase. Equipment designed by DAVIES and JAMES [1934] for use with metals, later modified for use with polythene by Hillier, depends for the excitation of the vibrations on a small insert of ferromagnetic material at the end of the specimen distant from the nodal clamp. This also forms a convenient marker for observing the amplitude of the oscillations. A combined D.C. and alternating field is applied to this insert by suitable electromagnetic coils in order to excite vibrations, the D.C. field polarising the field and preventing the oscillations from occurring at twice the alternating field frequency. This method introduces a high degree of coupling and it is necessary to carry out experiments at different values of coupling, easiest arranged by varying the applied D.C. field, and extrapolating the resonance frequency to a value for zero coupling.

For this reason other methods of exciting the oscillations have been adopted. The simplest experimental system is that due originally to Nolle but copied extensively (e.g. ROBINSON [1955]).

The specimen in the form of a thin rod or bar has one end clamped rigidly to the vibrating element of an electro-mechanical transducer, the other end is free and motion of this free end is observed. Care must be taken to ensure that only transverse modes are being excited as any composite mode will cause interpretation of the results to become unreliable. Robinson discusses precautions that can be taken to reduce the possibilities of such composite modes. The complete unit is very small and by choosing a suitable form of transducer a very wide temperature range can be covered. Observation of the free end can be made optically, by using a photocell system or by using a capacity unit similar in principle to that employed by Pursey and Pyatt.

A third approach to the experimental method for using transverse oscillations is due to FÖRSTER [1937], who devised the method for metals, and to KLINE [1956], who adapted it for use with polymers. In this method a 'free-free' bar is employed. The specimen in the usual thin rod form is suspended from two light strings as in Fig. 5. One string is used to feed oscillations into the bar and the other used to detect them. A wide range of temperature can be covered since both the electro-mechanical transducer used to drive the bar and the transducer used to detect oscillations can be outside the temperature enclosure around the specimen.

The advantages of both the vibrating cantilever driven through the clamp and Kline's method arise from the low degree of coupling between driver and vibrating specimen. The resonant frequency observed during the test can be used to calculate the elastic constants directly, no extrapolation to zero coupling is necessary. The accuracy that can be achieved by the method does not appear to be high.

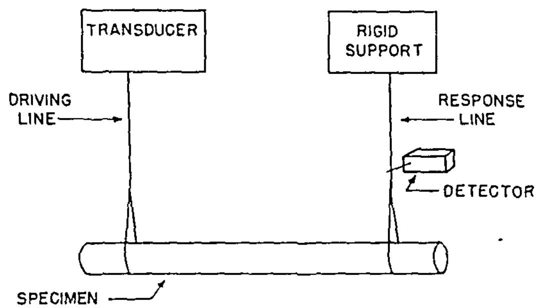

Fig. 5. Vibrating bar with free-free ends.

Kline quotes values of the resonant frequency f_0 accurate to $\pm 1\%$ and values of δf to $\pm 3\%$. Although comparative values taken on the same specimen as, for example, when exploring the variation in elastic constants with temperature, can probably be obtained with somewhat better accuracies, the method nevertheless is not the most precise available. The advantages lie in the lack of experimental difficulties and the method has been very widely used in investigations of the change in elastic constants with temperature. In particular it has been used for polymers where theories of the molecular interpretation of the elasticity and internal friction have been tested by reference to such investigations (WOODWARD and SAUER [1959], FERRY [1956]). The problem in this type of work is not so much to get accurate values as to map out the approximate values over a wide range of temperature and chemical composition.

In all the work so far described, no consideration has been given to the possible variation in elastic properties with strain. It is possible by suitably controlling the amplitude and energy of the driving oscillator to cover a narrow range with equipment so far described but a wide range would call for very sensitive detectors. MASON [1957] has devised a strain intensifier based on an exponential horn transducer unit. This unit is shown in Fig. 6. A barium titanate element, in the form of a hollow cylinder 2.5 inch outer diameter 4.75 inch long and with a wall thickness of 0.25 inch with a triple electrode system, form the basic vibrator; one set of electrodes drives the element in longitudinal vibrations (along the axis), the second set forms a guard electrode system and the third set is used for detection.

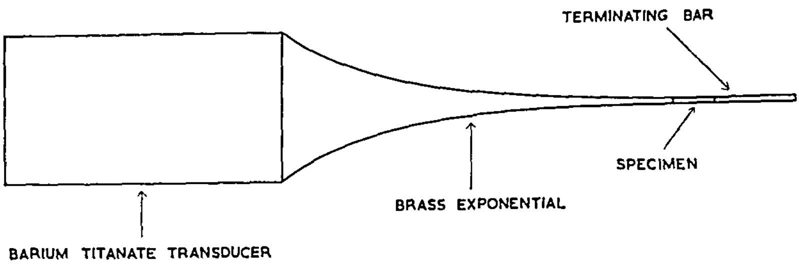

Fig. 6. Equipment using strain intensifier for measurements at high levels of strain (Mason).

Non-Resonant Forced Vibrations

By careful design it is possible to include a sample in an electromechanical circuit and to measure the performance of such a circuit over a range of frequency without forcing the specimen into resonance. The usual method is to measure the impedance of the network containing the sample so that both real and imaginary components of the elastic modulus are obtained. It is necessary to know the impedances of all other parts of the network which cannot be assumed to be negligible in this type of experiment. Usually a calibrating method is employed whereby the specimen is replaced by a sample of known properties, e.g. high elasticity and small internal friction compared with the materials being investigated. One of the most accurate designs utilizing these principles is due to FITZGERALD and FERRY [1953]. The equipment is illustrated in Fig. 7. The sample is in the form of two circular discs and they are vibrated in shear. The apparatus consists of a rigid metal tube wound with driving and detecting coils suspended in a radial magnetic field so that the tube can move axially. In the centre of the tube is a heavy cylindrical mass. The two similar specimen discs are placed between the central mass and the tube. By passing alternating current through the driver coils the specimens are vibrated in shear. Great care is taken in the construction so that no instrument resonances occur over the whole range 25 c/s to 6 kc/s. This is checked by replacing the specimens by a rigid connection between the coil tube and the centre mass. The specimen impedance is then infinite and any nonlinear electrical response of the network indicates some unwanted resonance. The equivalent circuit of the apparatus is shown in Fig. 8. Z_{m0} is the mechanical impedance of the driving coil and former, Z_{m1} is the impedance of the specimen and Z_{m2} is the impedance of the floating mass and wire supports. The specimen, in the form of two discs, is clamped by means of wedges between

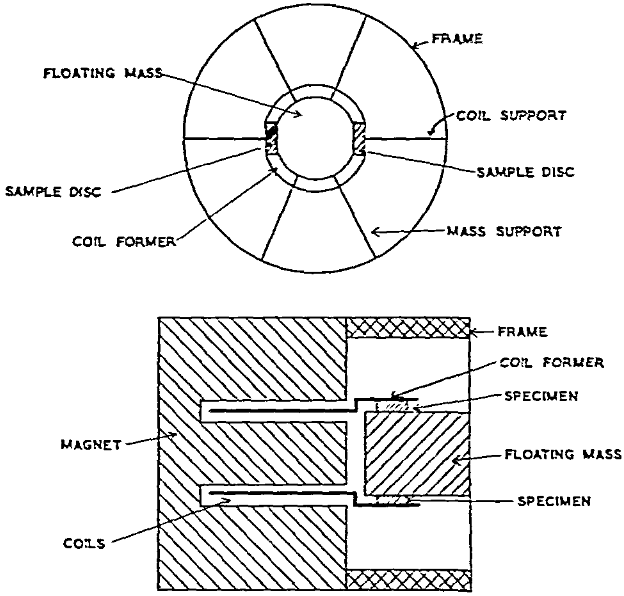

Fig. 7. Shear resonance equipment (Fitzgerald and Ferry).

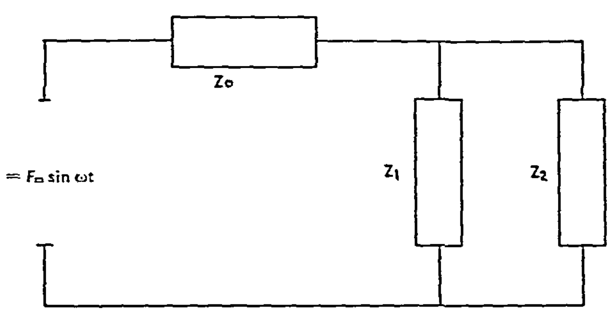

Fig. 8. Equivalent circuit of Fitzgerald equipment.

the driving tube and the floating mass, and, depending on the stiffness and dimensions of the specimens some distortion (barrelling) of the shape will occur. This is minimized by choosing suitably shaped specimens so that the thickness of the disc is much less than the radius.

The equivalence of the mechanical impedance with the electrical impedance of the coil system depends on the elimination of mutual inductance coupling between the driving and detector coils. This is arranged by fitting guard coils which are used in the electrical bridge circuit to balance out such coupling. It is considered that the measurement of the electrical impedance of the network can be made to within 0.5 % over the range 25 c/s to 5 kc/s. The impedance of the specimen however depends on the subtraction of the terms due to the driving tube impedance and the smaller floating mass impedance. The specimen thickness is therefore chosen so that at each frequency and temperature being studied these other terms are minimized and no final result has an error greater than $\pm$ 2 %. The equipment will measure to this accuracy over the temperature range –50°C to 150°C, frequency range 25 c/s to 5 kc/s and the range of elasticity covered is from shear modulus values 10^5 to 10^{10} dyns/cm^2. The complex part of the modulus can also be determined to the same accuracy. The equipment has been used to study the elastic behaviour of polymers (FITZGERALD [1957]).

§ 3. Wave Propagation Methods

3.1. CONTINUOUS WAVE PROPAGATION

The propagation constants, velocity and attenuation, of elastic or visco-elastic waves depend on the elastic constants and on the density of the material as well as on the type of wave and the dimensions of the specimen. By measuring the propagation constants therefore an estimate can be made of the elastic constants at the frequency of the waves. The advantage of wave propagation methods lies in the wide frequency range that can be covered with a single specimen. If continuous waves are used, the amplitude of the particle displacement must be small to prevent power absorption and consequent heating and a lower frequency limit is imposed by the need to contain more than one wavelength of the oscillations in the specimen.

The simplest system employing continuous wave propagation is shown in Fig. 9. A variable frequency oscillator of sufficient power drives a suitable electromechanical transducer which is coupled to the specimen. The specimen is in the form of a thin, long filament or strip. Longitudinal oscillations are generated by the transducer in the specimen and travel down with a characteristic velocity and if the material is viscoelastic, with a certain attenuation. A detector unit is moved along the filament so arranged to pick up the amplitude and phase of the particle motion at the point of contact. The electrical output from the detector (usually a piezoelectric crystal) is amplified and the phase shift and amplitude relative to the input signal fed to the filament are measured. These measurements of phase shift and

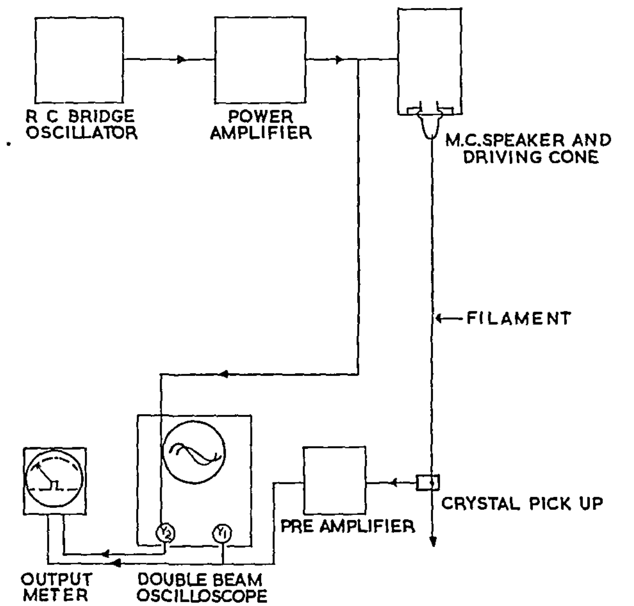

Fig. 9. Apparatus for the measurement of longitudinal sonic waves in monofilaments.

amplitude are obtained as a function of the distance along the specimen. By having a suitably long specimen or by carefully terminating the far end of the specimen no significant reflection will take place. However, it cannot be assumed that no reflection occurs at the detector unit and the theory is developed assuming that partial reflection occurs at the detector. It is further assumed that such reflected waves do not interfere with the driving source. A comprehensive treatment is given in papers by BALLOU and SMITH [1949] and by HILLIER and KOLSKY [1949]. If the attenuation constant is α, the wavelength of the vibration λ, then the particle displacement ξ is given by

$$\xi = A_1 e^{-\alpha x} e^{i(\omega t - 2\pi x/\lambda)} + A_2 e^{\alpha x} e^{i(\omega t + 2\pi x/\lambda)} . \tag{31}$$

If the complex reflection coefficient at the detector, i.e. both phase and amplitude, is given by $R \exp(\mathrm{i}\Phi)$ then the displacement at the detector is

$$\xi_l = \frac{[1 - 2R \cos \Phi + R^2]\, \xi_0\, \mathrm{e}^{-\alpha l}\, \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{e}^{\mathrm{i}\omega t}}{[1 - 2R\mathrm{e}^{-2\alpha l} \cos (\Phi - 4\pi l/\lambda) + R^2 \mathrm{e}^{-4\alpha l}]^{\frac{1}{2}}} \,. \tag{32}$$

In the type of detector used $\Phi = 0$ and θ the phase shift relative to the source is given by

$$\tan \theta = \left(\tan \frac{2\pi l}{\lambda}\right) \frac{1 + R\mathrm{e}^{-2\alpha l}}{1 - R\mathrm{e}^{-2\alpha l}} \,. \tag{33}$$

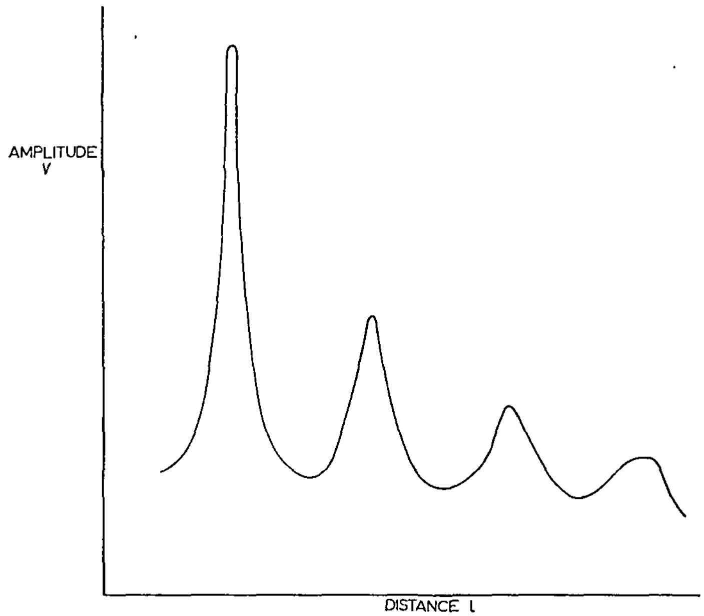

Fig. 10. Amplitude-distance relation for partially reflected longitudinal waves.

The amplitude of the signal received is proportional to ξ_l and is

$$V = \frac{K\mathrm{e}^{-\alpha l}}{[1 - 2R\mathrm{e}^{-2\alpha l} \cos (\Phi - 4\pi l/\lambda) + R^2 \mathrm{e}^{-4\alpha l}]^{\frac{1}{2}}} \,. \tag{34}$$

If α is greater than 0.23 and l greater than 10 cm then

$$V \approx K\mathrm{e}^{-\alpha l} \tag{35}$$

a result obtained with rubber samples by NOLLE [1948]. If, however, α is not large the form of (34) is shown in Fig. 10. It is convenient to consider the maximum, $V_{\max}$ and minimum, $V_{\min}$, values of this function which are separated approximately by values of l given by

$$\cos (\Phi - 4\pi l/\lambda) = \pm 1 \,. \tag{36}$$

If $R = e^{-2\beta}$ then

$$\begin{aligned} V_{\max} &= \frac{K'}{2 \sinh (\alpha l + \beta)} \\ V_{\min} &= \frac{K'}{2 \cosh (\alpha l + \beta)} \end{aligned} \tag{37}$$

and hence

$$\frac{V_{\max}}{V_{\min}} = \tanh (\alpha l + \beta) . \tag{38}$$

It is therefore possible to calculate α by obtaining the ratio of the envelope curves $V_{\max}/V_{\min}$ and plotting $\tanh^{-1}(V_{\max}/V_{\min})$ against l when the slope of the resulting straight line is equal to α and the intercept β. From eq. (33) it can be seen that, for large values of l, $Re^{-2\alpha l} \to 0$ and thus

$$\theta \approx \frac{2\pi l}{\lambda} . \tag{39}$$

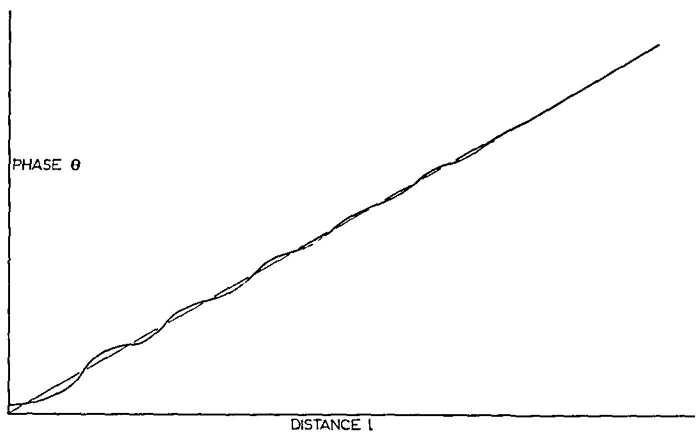

Fig. 11. Phase-distance relation for partially reflected longitudinal waves.

A typical plot of θ against l for polythene which has a moderate value of α at the frequency used (around 2 kc/s) is given in Fig. 11. It can be seen that the relation between θ and l, after initial oscillations at low values of l, becomes a straight line as l increases with a slope of $2\pi/\lambda$.

The method can be used for any materials which can be produced in the form of thin wires or filaments, provided that the attenuation factor is neither too high (when no progressive wave is propagated) nor too low ($\alpha < 0.001$) when reflections at the detector and at the end of the specimen interfere. An accuracy of $\pm 1\%$ can be obtained for the propagation velocity and about $\pm 5\%$ for the attenuation coefficient.

In the study of the propagation of continuous waves, instead of using piezo-electric detectors it is possible to utilise optical phenomena associated with the passage of elastic waves through transparent solids. If the wavelength of the elastic waves is less than the width of a coherent light beam, then the alternate rarefactions and compressions caused by the elastic wave in the material, act in the same way as a

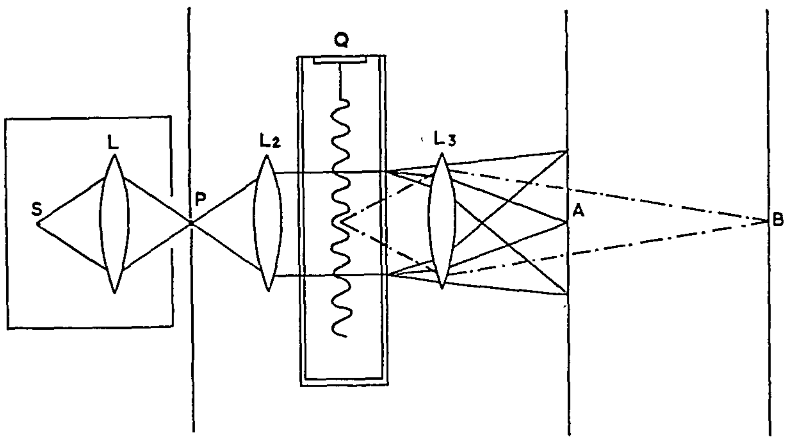

Fig. 12. Optical equipment used to measure the propagation constants of ultrasonic waves.

grating and a diffraction pattern is formed. This pattern can be measured and its dimensions can be used to obtain the elastic wave constants of the material.

In the simplest case the elastic solid is an isotropic transparent material, e.g. ordinary glass. The apparatus is shown in Fig. 12. A condenser lens L produces an image of the source S on a pinhole P. From this secondary source the lens L_2 produces a parallel beam of light which is passed through the specimen. A further lens L_3 is used to produce an image of the pinhole at A. The specimen is excited with high frequency vibrations at right angles to the light beam, Q. In the steady state a system of three-dimensional standing waves will exist in the specimen, caused by multiple reflections from the boundaries. On the screen at A two diffraction rings are observed due to shear waves and bulk waves respectively. If r is the radius of this ring, A is the distance between the centre of the specimen and the screen, λ_0 the wavelength of the light, ϱ the density of the specimen, and f

the frequency of vibrations, then μ the shear modulus is given by the equation

$$\mu = \left(\frac{A\lambda_0 f}{r}\right)^2 \varrho \,. \tag{40}$$

BERGMANN [1954] discusses the more elaborate patterns observed when anisotropic materials are used.

If the specimen is so arranged that no large reflections take place, and continuous progressive waves are passing through, then a further elaboration of the technique is necessary. This is also shown in Fig. 12. The screen is moved to position B and the lens L_3 adjusted to provide on the screen an image of the wave pattern in the specimen. The undiffracted beam is blocked at the plane A. This arrangement is called a Schlieren method and, if stroboscopic light source is used synchronised with the source of elastic waves, a pattern corresponding to these waves is seen on the screen (GIACOMINI [1947]).

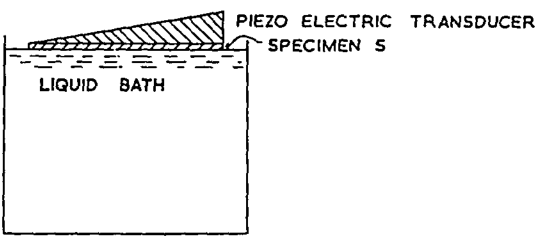

Fig. 13. Equipment for the measurement of the elastic constants of crystals by ultrasonic waves (Bhagavantam).

An indirect method is preferred which eliminates the need for stroboscopic light source but uses a shaped specimen. BHAGAVANTAM [1955] has used such a method in a study of the elastic constants of crystals. His method is illustrated in Fig. 13. The piezoelectric generator is shaped as a wedge of gradually increasing thickness, so that on applying a variable frequency oscillating voltage to the surface electrodes, only the point of the crystal corresponding to the correct thickness will be excited (together with harmonic sections). The sample, S, being measured is cut in the form of a thin plate of the appropriate orientation required and cemented or oiled on the wedge. The composite unit is then mounted so that the lower face of the sample is just immersed in a liquid bath. The beam of ultrasonic waves from the transducer Q passing through the sample is then transmitted through the liquid. The wave pattern in the liquid is observed by the diffraction method and used to calculate the propagation in the crystal sample. Bhagavantam claims that the method yields results accurate to 5% in the frequency range 1–12 Mc/s. The method can be used on polycrystalline material if it can be compressed into a non-porous block. There are limitations on the method imposed by the use of a liquid which must not attack the substance under test, and an upper limit exists on the temperature of testing, since the liquid must be free from convection currents in order to preserve the definition of the diffraction pattern.

3.2. Pulse propagation methods

If, instead of sending continuous waves through a specimen and observing the relative phase shift as a function of distance, a short pulse of high frequency oscillations is transmitted, the transit time of the pulse can be used to determine its velocity of propagation. The attenuation of the amplitude of the pulse will measure the loss in the material.

In the interior of an elastic solid two types of wave propagation are possible: these are normally referred to as dilation and distortional waves. In a linearly elastic isotropic body the velocity of propagation of these two types of waves, denoted by C_1 and C_2 respectively, can be expressed in terms of the Lamé constants λ and μ and the density ϱ:

$$
\begin{aligned}
C_1 &= \left(\frac{\lambda + 2\mu}{\varrho}\right)^{\frac{1}{2}} \\
C_2 &= \left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{41}
$$

In anisotropic materials the propagation velocities are considerably more complex and will vary with the direction of wave front relative to the axes of the crystal. Hearmon [1956] considers this problem and reviews the work reported on the relations between wave propagation velocities and elastic constants. In any specimen that is used for measurement it is not possible to consider the propagation as occurring in an infinite solid of the material. The boundaries of the specimen must be considered and the effect that these have on the propagation of waves is considerable. Davies [1948] and Kolsky [1954] have discussed this and give expressions for the propagation constants for specimens in the form of cylindrical bars.

The equipment for measuring transit times and attenuation of elastic pulses is derived from that developed intensively during the Second World War for radar and sonar detection. As a result of this war-time activity, very reliable and accurate electronic circuits have become available for obtaining transit times in the range 1–10^4 microseconds. A block diagram of a typical design of equipment for measuring the propagation constants is shown in Fig. 14. The display

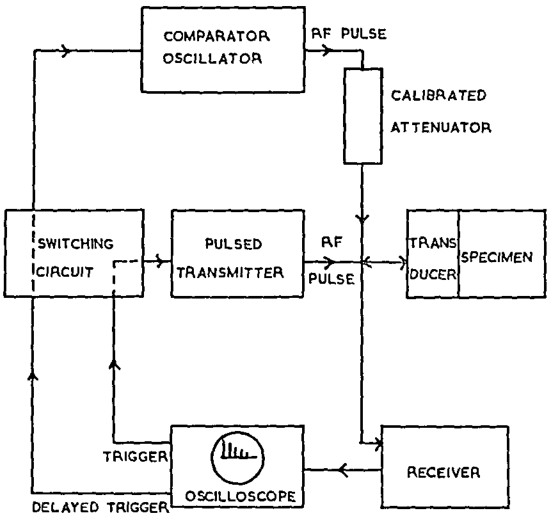

Fig. 14. Standard pulse equipment for determining transit times and attenuation of short pulses.

oscilloscope provides trigger pulses at a regular low frequency through a switching unit to the pulse transmitter, and this sends short, steep-fronted pulses of energy to the transducer, generally a piezoelectric element. These pulses cause the element to vibrate for short times and to transmit short trains of elastic waves into the specimen. The reflected waves from the specimen are converted in the transducer to electrical signals, which, after amplification in the receiver, are displayed on the oscilloscope. Circuits are incorporated to suppress the generator pulse, which might otherwise swamp the receiver, and also to provide time marker signals from an accurate crystal controlled oscillator. In Fig. 14 other auxiliary circuits are shown to provide a comparison pulse, which is displayed alongside the pulse received via the specimen. By using the calibrated attenuator shown, the comparison pulse is matched in amplitude with the specimen pulse, and hence the attenuation occurring in the specimen can be accurately measured. Sometimes a separate receiving transducer is employed, so that transmission through the specimen without reflection is studied.

One of the great advantages of this method derives from the compact nature of the equipment so that very small specimens can be measured. It has therefore been widely used to measure the elastic constants of single crystals. A disadvantage arises from the readiness with which complex modes of transmission may occur. This is caused in part by the generation of two waves, one dilatational, the other distortional at any reflection with a boundary. The transducer generating the wave packet in the specimen can be regarded as a piston source, so that both plane and spherical waves are produced (RODERICK and TRUELL [1952]). The tracing of the subsequent path of these, until they affect the receiving transducer or return to the original one, can become difficult, and the interpretation of the results on the oscilloscope may be ambiguous. Some of these difficulties can be overcome by changing the geometrical form of the specimen, thus altering the path length of the various reflected waves and therefore changing the transit times.

By a suitable choice of generating transducer either distortional (shear) waves or dilatational waves can be sent into the specimen. If shear waves are required, however, the transducer must be coupled to the specimen in such a way that shear stresses can be transmitted across the interface. This is normally achieved by using a cement or adhesive film of some form, but MASON [1947] has shown that highly viscous liquids such as heavy oils or liquid polymeric materials can be used. The interface between the transducer and the specimen presents most of the experimental difficulties associated with the use of this method. Care must be taken to grind flat mating surfaces on the transducer and the specimen and thin films of liquid, or a free flowing cement, must be placed between the surfaces. Even so, reflections will occur at the interface and attenuation of the pulse will take place. This attenuation is not readily measured and is not very reproducible. The easiest way of providing relatively reproducible interface conditions is to immerse the whole unit in a suitable liquid bath. This serves two other functions: it provides efficient insulation for the high voltage pulses driving the crystal transducer, and it makes temperature control of the specimen easier. Care must be taken to remove all the gas bubbles from the surfaces and the temperature range must be restricted to avoid bubbling in the liquid. Recently, MCSKIMIN [1959] has described equipment to increase the temperature range by using molten metal as the coupling fluid. Fig. 15 shows the form of the equipment.

By taking advantage of the propagation of low loss materials such as quartz, the specimen can be isolated from the transducer, though care must be taken to reduce extra reflections which would confuse the signal received. McSKIMIN [1959] has described the use of a silica buffer rod for this purpose. This rod is cut with a screw

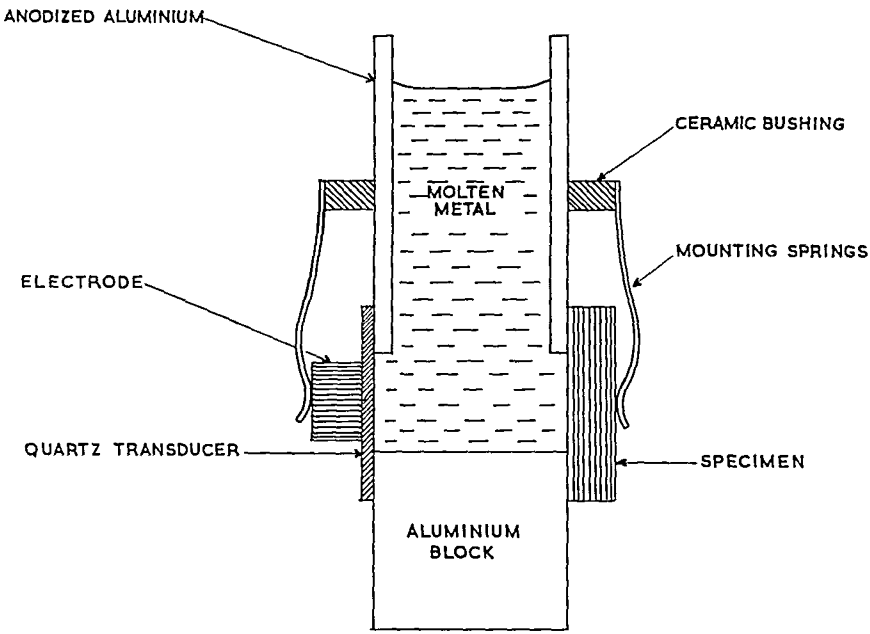

Fig. 15. Modification of pulse propagation equipment for use at high temperatures using molten metal as coupling medium.

thread on its surface so that mode conversion reflections are presented, because the incidence angles are greater than the critical values. Furthermore, the rod is tapered on to the specimen which considerably reduces the reflection at the buffer rod/specimen interface. Fig. 16 shows a diagram of the arrangement. Using a silicone oil as the coupling medium, this buffer rod technique has been effectively used up to temperatures of 350°C. By combining the use of molten metal with this buffer rod technique, it is expected that temperatures above 550°C will be attained.

By specially shaping the specimen and taking advantage of the fact that a dilatational, or longitudinal, wave generates both a dilatational and a distortional wave on reflection, mode conversion can be used to enable shear elasticities to be obtained without the need for elaborate coupling fluids or cements. This avoids uncertainty in transit times arising from such interfaces. ARENBERG [1948] has developed the

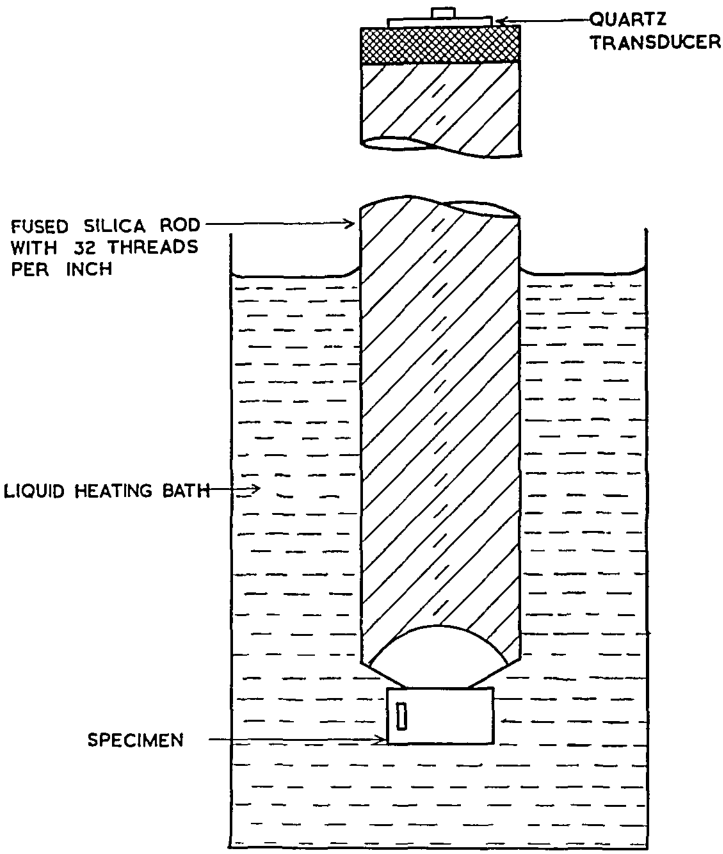

Fig. 16. Modification of pulse propagation equipment for use at high temperatures using silica rod as coupling medium.

theory and techniques have been described by BARONE [1951] and McSKIMIN [1959]. In Fig. 17 the use of a specimen cut to a parallelogram section to measure the velocity of shear waves is shown. If V_l is velocity of longitudinal waves and V_s that of shear waves then

$$\left(\frac{V_l}{V_s}\right)^2 = \sin^2 \alpha + \left(\frac{V_l(n_0 - \Phi/2\pi)}{2\omega f_0} - \frac{l}{\omega} - \cos \alpha\right)^2. \qquad (42)$$

In eq. (42) f_0 is the pulse repetition frequency at which echoes traversing the specimen and reflected from the first face are in phase,

Φ_0 is the total phase shift suffered by the beam at all reflections. If Φ_0 is independent of f then

$$n_0 - \frac{\Phi_0}{2\pi} = \frac{f_0}{\Delta f} \tag{43}$$

where Δf is the frequency difference between f_0 and the next pulse

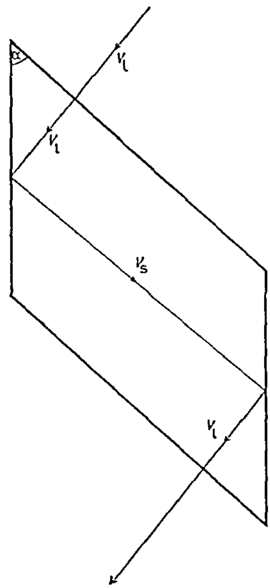

Fig. 17. Shaped specimen for obtaining values of the transverse velocity without coupling media.

repetition frequency at which phase coincidence occurs. A second method is illustrated in Fig. 18 where the specimen is cut in the form of a right angled prism. The longitudinal input wave at an angle θ to the hypotenuse is split into two shear waves in the prism, the refracted longitudinal beam being of zero amplitude. By choosing θ correctly the shear waves strike the opposite faces of the prism normally and are returned along their own path. The specimen size can be very small and it is easy to cut the prism in such a way that propagation in specific crystal directions can be studied, e.g. (100) or (110) in a cubic crystal.

Accurate measurements of the transit time depend on a clear indication of some characteristic point on the oscilloscope display of the pulse. It is not difficult to design electronic circuits to reshape the the reflected pulse so that a clear and definite point is available.

EROS and REITZ [1958] have shown however that this process only reduces the amount of necessary information that can be obtained from a careful study of the pulse shape after suffering transmission and reflection. Fig. 19 shows the type of degeneration of the pulse shape

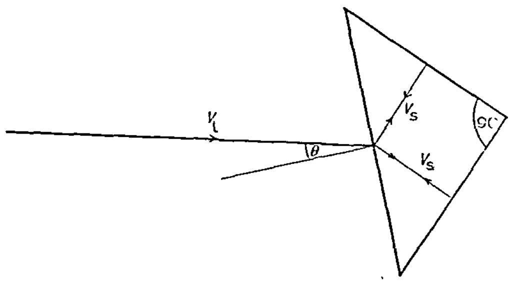

Fig. 18. Shaped specimen for obtaining values of the transverse velocity without coupling media.

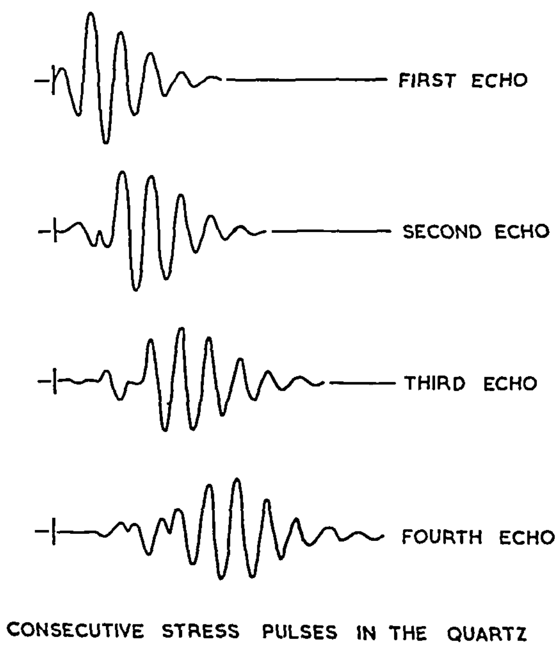

Fig. 19. Change in shape of pulse signal after multiple passages in quartz.

that occurs. In their work on the evaluation of the dynamic elastic constants of single crystals of potassium chloride and sodium iodide, they obtained multiple echoes and used very wide band detector amplifiers to preserve as much as possible of the change in shape of the pulse. They claim accuracies of measurement of $\pm$ 0.5%.

Although absolute measurements of the elastic constants are not obtained so accurately, the pulse propagation method will yield very high accuracy if used for comparative measurements on the same specimen, e.g. over a range of temperature. In this type of equipment, sometimes erroneously called an interferometer, two matched units are used. One specimen is placed in a constant temperature enclosure and the other is varied as required. Signals from both units are added and rectified. The changes in velocity $\Delta v/v_0$ are obtained from measurements of the time delay between the pulses. The measurements depend rather critically on having exact matching of the units; in particular, the specimens must be plane parallel to within 5×10^{-5} radians, of the same thickness to 10^{-4} cm and be mounted on the crystals by similar cement films. The accuracy of the comparative measurements is then very high; changes in velocity of propagation as small as 5×10^{-6} cm/sec can be detected (ESPINOLA and WATERMAN [1958]).

These pulse propagation methods have been very widely used to measure the elastic constants of single crystals, especially of metals. GRANATO and LÜCKE [1956] summarized the data existing at that time and they concluded that the method was generally more accurate than the definition of the crystal being measured. That is, the direction of wave propagation in the specimen is not sufficiently well defined and the state of the specimen with regard to annealing etc. is not sufficiently reproducible to take advantage of the accuracy of the elastic measurements. FISHER and MCSKIMIN [1958] have made the same point in their paper on measurements made on α-uranium in the frequency range 50–180 Mc/s. Values for c_{11}, c_{13} and c_{22} (see eq. (1)) are given at 25°C as $2.147 \pm 0.14\%$, $0.218 \pm 1.5\%$ and $1.986 \pm 0.14\%$ all units being dyne/cm^2. They point out that with accuracies of this order the method could be used to calculate the orientation of crystal faces with greater precision than the usual Laue method based on X-ray diffraction.

The small compact size of the equipment necessary to make measurements by pulse propagation methods makes it very suitable for estimates of elastic constants on samples subjected to static stress. BERGMANN and SHAHBENDER [1958], for example, have measured the change in the propagation constants for longitudinal and shear waves in aluminium subjected to high static stress. The sample of aluminium is set in the loading jig and suitable crystal transducers are fixed to the specimen to generate longitudinal and shear waves both perpendicular to and in a direction parallel to the applied static stress.

In order to improve the accuracy of the comparative measurements, a similar aluminium sample with equivalent transducers is used unloaded to provide reference propagation times. Similarly SINGH and NOLLE [1959] have studied the effect of hydrostatic pressure on the propagation of elastic waves in the polymer poly-isobutylene. They transmitted pulses through an immersion fluid which did not swell the polymer in the range of temperatures (—20°C to 80°C) studied. The thin samples were placed in the liquid and reflections from the faces of the specimen used to obtain the velocity of longitudinal waves. The attenuation was also measured. The unit was so built that hydrostatic pressure to 9000 lb/sq.in. could be applied. It was found that such pressures affected the propagation constants in the polymer.

3.3. LARGE AMPLITUDE PULSE METHODS

If a sufficiently large strain pulse can be generated in a specimen, the propagation of this pulse can be followed by more direct methods. If the sample is transparent and exhibits photoelastic properties, equipment can be designed, using polarised light and high speed photography, to display the passage of the waves (CHRISTIE [1952]). Suitably intense waves are most readily generated by the detonation of small high-explosive charges. The detonation is arranged to start the high speed photographic equipment with a suitable delay to allow the elastic waves to travel into the specimen. Christie used a multiple spark camera after the design of SCHARDIN [1950] and the timing of the spark light sources was recorded on an oscillograph by means of a photocell. The time intervals were recorded accurately to 0.1 microsecond, so that the method affords a fairly accurate method of measuring the propagation of stress waves in transparent materials. FEDER [1956] has described similar work though using a high speed cine camera to record the passage of the waves. Christie observed the dilatational wave travelling through the material with a spherical wave front followed by slower transverse waves generated by distortions of the top edge of the specimen near the explosive charge. Feder used different shaped specimens in which one or other modes of propagation would dominate. Fig. 20(a) shows the long narrow specimen used to observe longitudinal waves and 20(b) the broad specimen used for the observation of transverse waves. In the second arrangement a dilatational wave must precede the transverse wave but most of the energy is contained in the transverse component.

If the material is not transparent, the passage of elastic waves can normally only be followed by the surface displacements, though SCHALL [1958] has described the use of X-rays for this purpose. The changes of surface displacements can be observed directly by optical magnification, by using bonded wire strain gauges or by using capacity displacement meters, often called condenser microphones. KOLSKY [1954] has described equipment in which specimens in the form of

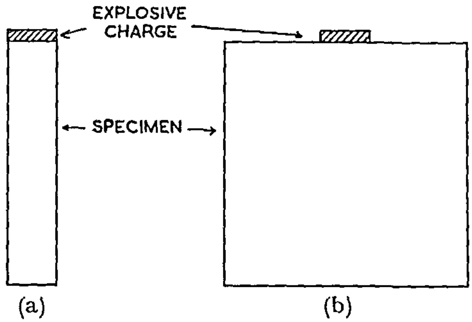

Fig. 20. Shapes of specimens used in observing (a) longitudinal and (b) transverse stress waves arising from detonation of an explosive charge.

cylindrical bars are used. Suitable waves are generated at one end of the bar by the detonation of a small explosive charge whilst a condenser microphone at the other end records the successive arrival of pulses travelling in the specimen in the several modes. The fastest wave travels as a pure dilatation wave. If the ratio of length of cylinder to radius is low, about unity, this first pulse can be easily distinguished from the second pulse which is reflected once from the surface of the cylinder and is still purely dilatational throughout. The third pulse is that which travels partly as a distortional, or shear wave; this pulse has the same path length as the second pulse being derived by the shear wave generated with a dilatational wave at the reflection at the cylindrical surface. HUGHES, PONDROM and MIMS [1949] used a similar method generating the pulse however by piezoelectric crystals.

§ 4. Direct Stress-Strain Measurements

It was mentioned in § 1 that direct methods of obtaining stress-strain data in dynamic tests were difficult. These difficulties can be attributed to inertia effects in the measuring apparatus. As the rate of loading is increased, the acceleration of any moving parts of the equipment requires forces comparable, or even greater than those necessary to deform the specimen, so that the determination of the true applied stress is not accurately known. TAYLOR [1946] has discussed these difficulties and described techniques by which they could be surmounted. The basic unit employed in these methods is a long cylindrical bar hung as a ballistic pendulum. VOLTERRA [1948] used a photographic method for recording stress and strain in the specimen, whereas later workers have used electrical methods. In the work of Taylor and Volterra one bar was hung freely with the specimen under test in the form of a short cylinder of the same diameter as the bar in contact with one face. A second bar suspended in the same manner as the first was swung against the specimen which was thus compressed between the two bars. The specimen length changes were obtained from a photographic record made with a high speed camera and the stress-time relation obtained from the motion of the steel bar initially at rest. The stress-time curve is obtained by double differentation of the displacement-time curve of the end of the bar in contact with the specimen, obtained from the photographic record. Thus stress-strain curves for the loading cycle can be obtained. SENIOR and WELLS [1946] and later FROCHT and FLYNN [1956] have used photoelastic methods employing transparent bars and photographing the passage of the stress wave by means of spark or streak cameras. Certain assumptions about the nature of the stress-optic law are made in this work especially the invariance of the photoelastic constant with time.

KOLSKY [1949], using a method similar in principle to that of Taylor and Volterra, has improved considerably on the precision of the measurements. His equipment is shown in Fig. 21.

A pressure pulse is produced by a detonator fired against a replaceable hardened steel anvil on one end of the long steel bar. The pressure pulse travels down the bar and compresses the disc-shaped specimen placed between the end of the bar and a second, shorter extension bar. A well-fitted collar holds the specimen and extension bar in position. The surfaces of the bars are ground flat and the flat surfaces of the specimen are lubricated with a thin layer of grease to allow free lateral movement. The amplitude of the pulse before reaching the specimen is measured with a cylindrical condenser microphone. The pulse, after travelling through the specimen and the extension bar, is measured by a parallel plate condenser microphone. The inertia switch preceding the microphones triggers the time base of the double beam oscilloscope. The output from the two microphones, suitably amplified with high gain pulse amplifiers, is displayed on the oscilloscope together with time marking traces. Since only a single sweep of the trace occurs occupying a few microseconds, the oscilloscope trace is recorded photographically. The microphone at the end of the extension bar gives a record of displacement ξ_2 as a function of time, $\xi_2(t)$. From this the stress $\sigma_2(t)$ can be calculated from

$$\sigma_2(t) = \tfrac{1}{2}\varrho c \frac{d\xi_2}{dt}. \tag{44}$$

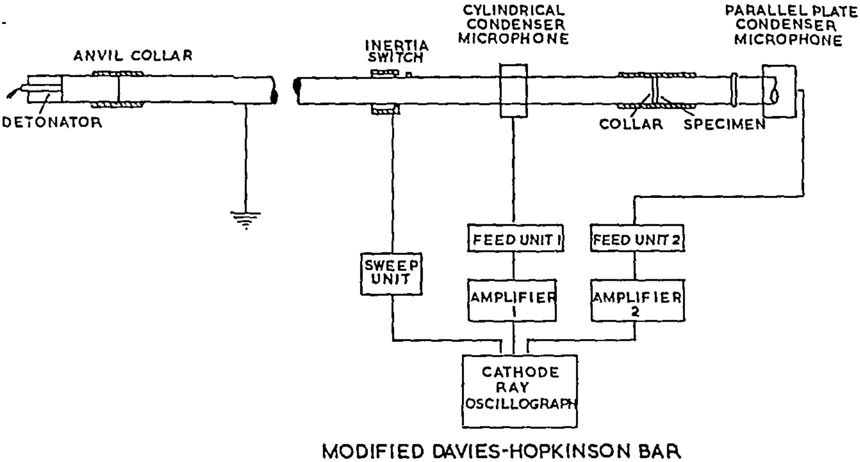

Fig. 21. Davies-Hopkinson bar as used by Kolsky in measurements of stress/strain properties at very high rates of loading.

The stress-time relation for the main bar is $\sigma_1(t)$, which is derived from the cylindrical microphone.

A more exact measure can be obtained from subsidiary experiments without either specimen or extension bar and with the parallel plate microphone on the end of the main bar. The end of the specimen in contact with the main bar suffers a displacement due to the incident pulse, and also due to the reflected pulse returning from the end of the extension bar. These two components are given in equations 45(a) and (b) respectively while 45(c) gives the final displacement-time relation for this face:

$$\frac{1}{\varrho c}\int_0^t \sigma_1(t)dt \tag{45a}$$

$$\frac{1}{\varrho c}\int_0^t [\sigma_1(t) - \sigma_2(t)]dt \tag{45b}$$

$$\frac{1}{\varrho c}\int_0^t [2\sigma_1(t) - \sigma_2(t)]dt. \tag{45c}$$

The displacement of the face of the specimen in contact with the extension bar derives only from the pulse in that bar and is therefore

$$\frac{1}{\varrho c}\int_0^t \sigma_2(t)\mathrm{d}t\,.$$

The change in dimension of the specimen is thus

$$\frac{2}{\varrho c}\int_0^t [\sigma_1(t) - \sigma_2(t)]\mathrm{d}t\,. \tag{46}$$

If the specimen can be regarded as short so that the stress is the same throughout at any instant, then from eq. (44) and writing

$$\xi_1(t) = \frac{2}{\varrho c}\int_0^t \sigma_1(t)\mathrm{d}t$$

the strain-time relation for the specimen is given by

$$\frac{1}{b}[\xi_1(t) - \xi_2(t)] \tag{47}$$

where b is the thickness of the specimen.

There is a correction to be made for the radial kinetic energy of the specimen and this is discussed by KOLSKY [1949]. Direct stress-strain relationships can be obtained for a variety of materials in the form of thin discs. It has been used by Kolsky to determine the effect of rate of loading on stress-strain behaviour for several high polymer materials, polythene, rubber etc. and also for copper and lead. By varying the thickness of the specimen a small range of rate of strain application can be obtained. The time of loading is normally about 20–40 microseconds. Although Kolsky and Davies (DAVIES [1956]) used capacity displacement gauges, other workers have used wire resistance gauges and optical means. KRAFFT, SULLIVAN and TIPPER [1954] have used the arrangement shown in Fig. 22. One steel bar is used to transmit a pulse into a steel anvil on to which is bonded a resistance wire strain gauge bridge network. The specimen is placed between this anvil and a second similar one also carrying strain gauges. This second anvil is followed by a further long bar which prevents rapid reflection of the pulse. ½ inch diameter bars were used and pulses about 65 microseconds were generated. In the first five microseconds the stress builds up; it is constant for about 55 microseconds and decreases for 5 microseconds. Means were provided for varying the temperature over the range —200°C to 100°C and the dynamic stress-strain curves of mild steel were studied. In a slight variant of this method ALTER and CURTIS [1956] studied the properties of lead at high rates of loading. Steel bars were used for generating the stress pulse and to remove the transmitted pulse from the specimen. The resistance wire gauge was placed on the specimen itself. The impact bar producing the stress pulse was constructed with a diameter step in it so that a double pulse was applied to the specimen. It is therefore possible by using the arrangement to investigate the effect of strain amplitude on the properties of the material.

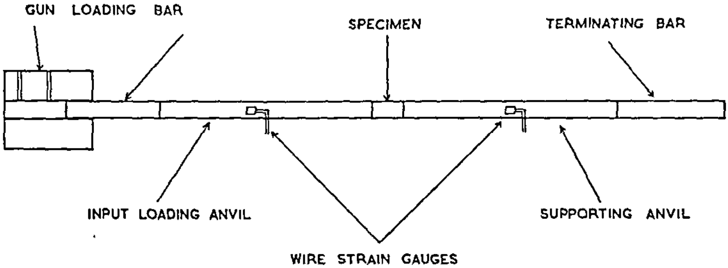

Fig. 22. Bar as used by Krafft *et al.* using wire strain gauges.

J. F. BELL [1956] has devised an optical method of amplifying the displacement of the stress bar. The material that he studied was annealed aluminium. A grating was ruled on the surface of the bar with a line spacing of 8300 lines/inch and a total length of 0.03 inch. The experimental arrangement of the detecting system is shown in Fig. 23. The grating was illuminated by monochromatic light from a filtered mercury arc source through a 'V' shaped slit. A cylindrical lens with the axis of the lens parallel to the grating was used to focus the diffracted beam on to a photocell. By using a cylindrical lens in this way only angular changes of the surface of the aluminium bar were detected by the photocell; linear displacements not being focused by the lens. The stress pulse was provided by projecting a shorter, larger diameter bar at the specimen bar. The photocell amplifiers of suitable wide frequency response were fed, in the usual way, into an oscilloscope. A modification of this method has been suggested by Bell where the surface of the bar can not be ruled directly. Small replica gratings can be mounted on the specimen bar in the same way as resistance wire gauges are mounted. Such gratings are of no larger size than gauges and the use of such replica gratings would widen the scope of the method considerably. In the work reported by Bell he claimed that the total error in the dynamic constants would be not greater than ± 5 %. Two advantages can be attributed to this optical method. As with the use of capacity gauges no secondary effect is utilised, e.g. the change in resistance with strain, so that no possible time dependance of this property is neglected. The second advantage claimed by Bell is the readiness with which the method can be used for large strains (up to 8 %). Both capacity and resistance wire gauges

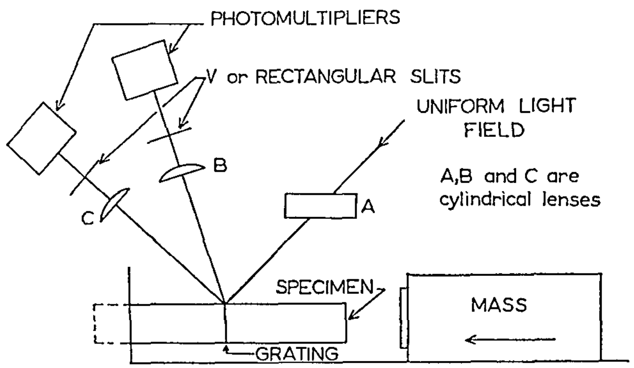

Fig. 23. Bar employed by Bell using optical gratings.

would be difficult to design to cope with such a large strain. Another method of detecting the passage of stress pulses propagated axially in long bars of the test material has been described by RAMBERG and IRWING [1957]. This method depends on electromagnetic induction and can only be applied to non-magnetic materials. A small coil of wire is bonded to the bar and is arranged to be near one pole of a hollow cylindrical bar magnet. Resistance wire strain gauges are bonded on the bar to provide records of strain and strain gradient in the bar. The bonded coil is used to record the particle acceleration by recording the induced e.m.f. Ramberg and Irwin reported great difficulty in maintaining the equipment stable and reproducible and the method cannot be recommended. MIKLOWITZ and NISEWANGER [1957] studied the propagation of stress pulses in aluminium alloy rods 1 inch diameter. They employed a 'shock' tube to generate the pressure pulse. This arrangement consists of a hollow tube at the end of the specimen rod in which a gas mixture is ignited electrically. Conditions of gas pressure and mixture are chosen so that the advancing gas flame front becomes more abrupt and reaches the specimen boundary as a shock pulse. This method of generating a pulse in the specimen bar is extremely reproducible if care is taken to control the gas mixture. Miklowitz and Nisewanger were able to measure axial strains in the bar in one set of experiments using bonded wire strain gauges and radial strains in a second set using a cylindrical condenser microphone. The experimental arrangement is thus simplified and dangers of cross modulation between amplifier channels avoided. In the experiments performed by Kolsky and his associates specially prepared electric detonators were used in which the explosive charge had been carefully weighed and packed in small increments. The reproducibility between experiments though quite high could not have been relied on to the extent that the shock tube could.

Several experimenters have obtained dynamic elastic constants by using conventional methods with suitable precautions to avoid spurious results caused by inertia in the measuring equipment. Schiefer and his colleagues (STONE, SCHIEFER and FOX [1955]) have successfully tested textile yarns at strain rates to 8000 %/second. The yarn specimen was about 0.67 metre in length and was fastened into a metal head mass and a smaller tail mass. The yarn was positioned in an extended form in a transparent tube and equipment was arranged to provide high speed cine records of the motion of both head and tail masses, so that the images of each appeared on the same frame of the film against suitable scales. The head mass was set into motion by impact with a heavy flywheel travelling at high rotational speed and the subsequent displacement-time data for both ends of the yarn specimen were obtained from the cine records. By double differentiation this data could be converted to acceleration-time records and thus from the known weight of the masses the stresses could be calculated. Stress-strain curves for specimen in both loading and unloading could be obtained. At the high speeds of the head mass (40 m/s) there is an appreciable time lag between the application of the impulse and the reaction at the tail end. This is caused by the propagation of the stress wave along the specimen. A correction to the stress/time relation for the yarn must be made to allow for this, and even so the results really describe only the average behaviour of the material over its whole length. In the experiments reported, the error in the maximum average strain by ignoring wave propagation amounted to only 2–3 %.

A direct method of a different type is that described by MARVIN, ALDRICH and SACH [1954]. They were interested in the dynamic bulk modulus of rubber and subjected the rubber sample to an alternating hydrostatic pressure while measuring the corresponding changes in volume. This is most conveniently carried out in a fluid contained in a very rigid vessel which it fills completely. The fluid must have low viscosity and very low or zero shear modulus at the frequency of the test. Light paraffin oil fulfils these conditions. The generator and detector are piezoelectric crystals cut to oscillate with net volume

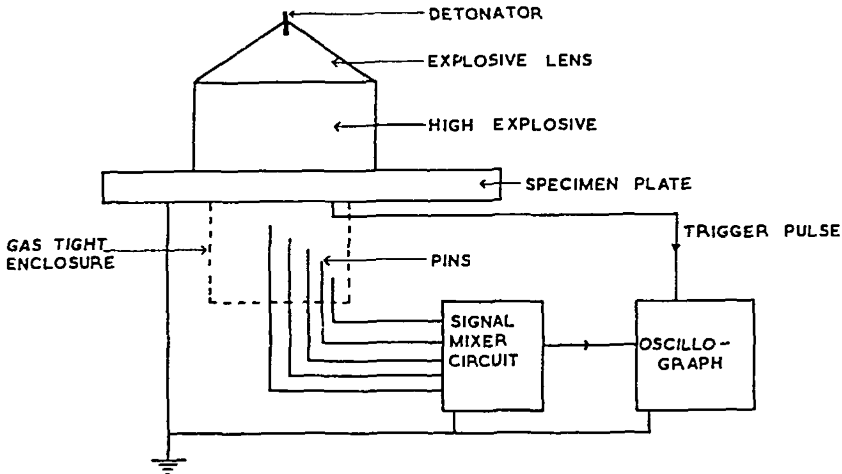

Fig. 24. Equipment using plane waves generated by shaped explosive charges to measure compressibility (Goranson).

change in the cycle. Difficulties arise from gas bubbles which are most conveniently eliminated by placing the whole vessel under a static hydrostatic pressure of about 100 kg/cm^2. The apparatus is calibrated by inserting a specimen of steel into the fluid and assuming that the value of bulk modulus is the same at the frequency employed, 10 000 c/s, as when measured at low frequency. The sample of rubber is then inserted and measurements of bulk stress and strain at the high frequency obtained. Marvin, Aldrich and Sach report the existence of a hysteresis loop thus indicating the presence of a complex bulk modulus. Another method of tackling dynamic compressibility is that due to GORANSON [1955]. In this method a special shaped explosive charge is used to provide a plane stress wave on the surface of a specimen plate. A piezo crystal transducer is used to measure the pressure profile and the displacement of the plate is measured by timing the motion of the back surface with a spaced pin array in a non-ionizing gas (propane). The equipment is shown diagrammatically in Fig. 24. The method has been used to provide data on armour steel plate to pressures of 0.3 megabars. It is claimed that the velocity of the plate surface can be measured to $\pm 0.5\%$ accuracy by the spaced pin array.

§ 5. Conclusions

In this article a very wide range of methods and materials has been considered. No one method can be considered to be outstanding but with careful attention to detail methods are available for most materials that will provide values for dynamic elastic constants with accuracies better than 1 %. This accuracy is sufficient to indicate generally the major errors in the theoretical interpretation of the results and at the present time it appears that experimental data is in advance of comprehensive theory. It has also been shown that in many cases, especially with metals and crystals, existing methods of measuring dynamic elastic constants provide the most accurate method of defining the orientation and comparative physical state of the specimen. Here again further advances would appear to be needed in the specification of the specimen.

References

ALTER, B. E. K. and C. W. CURTIS, 1956, J. Appl. Phys. **27** 1079.
ARENBERG, D. L., 1948, J. Acoust. Soc. Amer. **20** 1.
BAKER, G. S., 1957, J. Appl. Phys. **28** 734.
BALLOU, J. W. and J. C. SMITH, 1949, J. Appl. Phys. **20** 493.
BARONE, A., 1951, Ric. Sci. **21** 513.
BELL, J. F., 1956, J. Appl. Phys. **27** 1109.
BENBOW, J. J., 1953, J. Sci. Instr. **30** 412.
BERGMANN, L., 1954, Der Ultraschall und Seine Anwendung in Wissenschaft und Technik (Verlag, Zurich).
BERGMANN, R. H. and R. A. SHAHBENDER, 1958, J. Appl. Phys. **29** 1736.
BHAGAVANTAM, S., 1955, Proc. Indian Acad. Sci. A **41** 72.
BLAND, D. R. and E. H. LEE, 1955, J. Appl. Phys. **26** 1497.
BODNER, S. R. and H. KOLSKY, 1958, Proc. 3rd U.S. Nat. Cong. Appl. Mech. p. 495.
BOYLE and SPROULE, 1929, Nature **123** 13.
CHRISTIE, D. G., 1952, Trans. Soc. Glass Tech. **36** 74.
DAVIES, R. M., 1948, Phil. Trans. A **240**, 375.
DAVIES, R. M., 1956, Brit. J. Appl. Phys. **7**, 203.
DAVIES, R. M. and F. G. JAMES, 1934, Phil. Mag. **18** 1023.
EROS, S. and J. R. REITZ, 1958, J. Appl. Phys. **29** 683.
ESPINOLA, R. P. and P. C. WATERMAN, 1958, J. Appl. Phys. **29** 718.
FEDER, J., 1956, Proc. Soc. Exptl. Stress Anal. **14** 1, 109.
FERRY, J. D., 1956, Die Physik der Hochpolymeren **4** (Springer, Berlin) p. 373.
FINE, M. E., 1957, Rev. Sci. Instr. **28** 643.
FISHER, E. S. and H. J. MCSKIMIN, 1958, J. Appl. Phys. **29** 1473.
FITZGERALD, E. R., 1957, J. Chem. Phys. **27** 1180.
FITZGERALD, E. R. and J. D. FERRY, 1953, J. Colloid Sci. **8** 1.
FÖRSTER, W., 1937, Z. Metallk. **29** 109.
FROCHT, M. M. and L. FLYNN, 1956, J. Appl. Mech. **23**, 116.

GIACOMINI, G., 1947, R. C. Acad. Naz. Lincei **2** 791.
GORANSON, R. W., D. BANCROFT *et al.*, 1955, J. Appl. Phys. **26** 1472.
GRANATO, A. and K. LÜCKE, 1956, J. Appl. Phys. **27** 789.
HEARMON, R. F. S., 1953, Acta Cryst. **6** 331.
HEARMON, R. F. S., 1956, Phil. Mag. Suppl. **5** 323.
HILLIER, K. W., 1951, Proc. Phys. Soc. B **64** 998.
HILLIER, K. W. and H. KOLSKY, 1949, Proc. Phys. Soc. B **62** 111.
HORIO, M. and S. ONOGI, 1951, J. Appl. Phys. **22**, 977.
HUGHES, D. S., W. L. PONDROM and R. L. MIMS, 1949, Phys. Rev. **75** 1552.
KÊ, T. S., 1947, Phys. Rev. **71** 533; **72** 41.
KLINE, D. E., 1956, J. Poly. Sci. **22** 455.
KOLSKY, H., 1949, Proc. Phys. Soc. B **62** 676.
KOLSKY, H., 1953, Stress Waves in Solids (Clarendon Press, Oxford).
KOLSKY, H., 1954, Phil. Mag. **7, 45** 712.
KRAFFT, J. M., A. M. SULLIVAN and G. F. TIPPER, 1954, Proc. Roy. Soc. A **221** 114.
LETHERSICH, W., 1950, Brit. J. Appl. Phys. **1** 294.
LOVE, A. E. H., 1927, The Mathematical Theory of Elasticity (University Press, Cambridge).
MARVIN, R. S., R. ALDRICH and H. S. SACH, 1954, J. Appl. Phys. **25** 1212.
MARX, J. W., 1951, Rev. Sci. Instr. **22** 503.
MARX, J. W. and J. M. SIVERSTEN, 1953, J. Appl. Phys. **24** 81.
MASON, W. P., 1957, 9th Internat. Cong. Appl. Mech. Proc. Brussels **5** 379.
MASON, W. P. and H. J. MCSKIMIN, 1948, J. Appl. Phys. **19** 940.
MAXWELL, C., 1890, Scientific Papers, Vol. ii (University Press, Cambridge) p. 26.
MCSKIMIN, H. J., 1959, J. Acoust. Soc. Amer. **31** 287.
MEREDITH, R., 1954, J. Text. Inst. T **45** 489.
MIKLOWITZ, J. and C. R. NISEWANGER, 1957, J. Appl. Mech. **24** 240.
NOLLE, A. W., 1948, J. Appl. Phys. **19** 753.
PRESCOTT, J., 1946, Applied Elasticity (Dover Publications, New York) Ch. IX.
PURSEY, H. and E. C. PYATT, 1954, J. Sci. Instr. **31** 248.
QUIMBY, S. L., 1925, Phys. Rev. **25** 558.
RAMBERG, L. K. and W. IRWING, 1957, Proc. 9th Int. Conf. Appl. Mech. Brussels **8** 480.
RAYLEIGH, LORD, 1894, Theory of Sound, Vol. I (MacMillan, London).
ROBINSON, D. W., 1955, J. Sci. Instr. **32** 2.
RODERICK, R. L. and R. TRUELL, 1952, J. Appl. Phys. **23** 267.
SCHALL, R., 1958, Proc. 3rd Int. Congress on High Speed Photog. (Butterworth).
SCHARDIN, H., 1950, Glastechn. Ber. **23** 1, 67, 325.
SENIOR, D. and A. WELLS, 1946, Phil. Mag. **7, 37** 463.
SINGH, H. and A. W. NOLLE, 1959, J. Appl. Phys. **30** 337.
STONE, W. K., H. F. SCHIEFER and G. FOX, 1955, Text. Res. J. **25** 520, 701.
TAYLOR, G. I., 1946, J. Inst. Civil Engrs. **26** 486.
TERRY, N. B., 1957, Brit. J. Appl. Phys. **8** 270.
THOMPSON, D. O. and F. M. GLASS, 1958, Rev. Sci. Instr. **29** 1034.
VOIGT, W., 1892, Ann. Physik **47** 671.
VOLTERRA, E., 1948, Nuovo Cimento **4** 1.
WOODWARD, A. E. and J. A. SAUER, 1958, Fortsch. Hochpolymer. Forsch. **1** 114.
ZENER, C., 1948, Elasticity and Anelasticity of Metals (University Press, Chicago).

CHAPTER VI

DISCONTINUITY RELATIONS IN MECHANICS OF SOLIDS

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CONTENTS

§ 1. Introduction

An essential component of techniques for handling boundary-value problems in continuum mechanics is the study of surfaces of discontinuity in the dependent variables and their derivatives. These surfaces may be isolated or may be characteristic in the full sense; they may be stationary or propagate as waves. The subject is fascinating, yet difficult and with far-reaching ramifications. It will undoubtedly become of major importance in the immediate future with the advent of new branches of solid mechanics and the re-organization of old ones.

However, incisive analytic tools for treating general discontinuities are far from wellknown, nor does there appear to be any comprehensive recent account. The main sources are Hadamard's monumental classic of 1903, now largely forgotten; an attractive, though limited, little book by Levi-Civita in 1932; and an extended series of valuable but not easily read papers by Thomas from 1953 onwards.

The present moment calls urgently for a unified and detailed exposition of the basic analysis. It is the purpose of this article to provide one, in as simplified a form as the intrinsic difficulty of the subject allows. A variety of applications in elastic and plastic solids are also surveyed in order to illustrate methods of attack. Much of this is new and has not been published before: for example, the flexible treatment of derivatives of discontinuous functions (section 3); a more detailed discussion of kinematic relations at a shock wave (section 4); the use of the unsymmetric tensor of nominal stress in connexion with discontinuities of stress-rate (section 5); a general formula, (33a,b,c), relating the jump in nominal traction-rate with the jump in stress and the speed of propagation of the discontinuity surface; a formula, (36), for the jump in strain across arbitrary dislocations in isotropic elastic solids; an improved and generalized analysis of quasi-static discontinuities in rigid/plastic solids with arbitrary anisotropic convex potentials (section 7), including an extended uniqueness theorem allowing for stress jumps.

§ 2. Hadamard's Compatibility Relations

The standard work on discontinuity surfaces in continuum mechanics is HADAMARD's superb "Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique" [1903]. Some essential ideas had been supplied earlier by Riemann and Hugoniot, but the main development, elegant invariant formulation, and pioneering applications to elastic solids and perfect fluids are due to Hadamard. Since his analysis cannot even now be considered wellknown, we begin with a résumé of certain results relevant to the present article.

2.1. DERIVATIVES OF A FUNCTION DEFINED ON A SURFACE

Let $f(x^j)$ be a function of curvilinear coordinates x^j $(j = 1, 2, 3)$ which is required to satisfy the following conditions on a (curved) surface Σ and in a certain open neighbourhood N on one side: (a) f is constant on Σ, (b) f is continuous in N and on Σ, (c) the first partial derivatives of f exist in N and tend to finite limits as Σ is approached. Then, necessarily, the first (one-sided) derivatives exist on Σ itself and are equal to the respective limits. We enquire to what extent the restrictions (a), (b) and (c) leave undetermined the values of the first derivatives on Σ, or equivalently the vector gradient of f there. The answer is obvious: the normal component of the gradient is undetermined, the components tangential to Σ vanishing since f is constant on Σ. Expressed invariantly, the gradient vector of f is directed along the normal and has arbitrary magnitude, or

$$\frac{\partial f}{\partial x^j} = \lambda \nu_j \tag{1}$$

on Σ where ν_j is the covariant unit normal to Σ and λ is an arbitrary scalar function of position (it is in fact $\partial f/\partial \nu$, the normal space derivative). A formal proof, trivial here but lending itself to subsequent generalization, is that by (b) and (c) $(\partial f/\partial x^j)\mathrm{d}x^j = \mathrm{d}f$ on Σ and by (a) this vanishes for all $\mathrm{d}x^j$ such that $\nu_j \mathrm{d}x^j = 0$ (summation convention).

Suppose, next, that a vector $\boldsymbol{u}$ is given, constant on Σ, each of whose components satisfies (b) and (c). Since $\mathrm{d}\boldsymbol{u} = 0$ for variations on Σ, $u_{i,j}\mathrm{d}x^j = 0$ for all $\mathrm{d}x^j$ such that $\nu_j \mathrm{d}x^j = 0$, where the comma denotes covariant differentiation. Hence

$$u_{i,j} = \lambda_i \nu_j \tag{2}$$

where $\boldsymbol{\lambda}$ is an arbitrary vector function on Σ ($= \partial \boldsymbol{u}/\partial \nu$). Alternatively, this formula could be proved first in rectangular coordinates (and then generalized) by applying (1) in turn to each component u_i, constant on Σ for such coordinates. $\boldsymbol{\lambda}$ was called by Hadamard the 'characteristic segment' on account of the following suggestive geometrical interpretation [op. cit., § 56], which has been neglected (with loss of intuitive insight) in accounts by other writers. Regard $\boldsymbol{u}$ as the finite displacement of a particle in a continuum with initial (Lagrangian) coordinates x^j. Then (2) gives the possible values of the displacement gradients at a surface Σ composed of particles all of which undergo the same translation. Equivalently, the distortion in the neighbourhood of Σ is such that

$$\delta \boldsymbol{u} = \boldsymbol{\lambda} \delta \nu + O(\delta \nu)^2 \tag{3}$$

is the relative displacement vector between a particle, initially distant $\delta \nu$ from Σ, and the nearest particle on Σ. A characteristic segment tangential to Σ would thus correspond locally to a simple shear parallel to the surface, while a segment normal to Σ would correspond to a pure extension or compression normal to the surface.

A variant of Hadamard's geometrical interpretation, even more suggestive for what follows, is to regard $\boldsymbol{u}$ as an infinitesimal displacement, or better as a velocity, with x^j as current (Eulerian) coordinates. Then the strain-rate and rate of spin on Σ, corresponding to the velocity field, are

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(\lambda_i \nu_j + \lambda_j \nu_i)\,, \\ \omega_{ij} &= \tfrac{1}{2}(\lambda_i \nu_j - \lambda_j \nu_i)\,. \end{aligned} \tag{4}$$

With the segment decomposed into parts normal and tangential to Σ, viz. $\boldsymbol{\lambda} = \varepsilon \boldsymbol{\nu} + 2\gamma \boldsymbol{\mu}$ where $\boldsymbol{\mu}$ is a unit vector tangential to Σ,

$$\varepsilon_{ij} = \varepsilon \nu_i \nu_j + \gamma(\mu_i \nu_j + \mu_j \nu_i)\,, \qquad \mu_k \nu_k = 0\,. \tag{5}$$

The two parts of ε_{ij} are instantly recognizable (either via (3) or at once from the tensor transformation rule) as an extension-rate ε normal to Σ and a (tensor) shear-rate γ over Σ in the direction $\boldsymbol{\mu}$. Equivalently, taking at the considered point on Σ a special choice of rectangular axes such that x^3 is along the normal, (5) states that the components ε_{11}, ε_{12}, ε_{22} 'in the surface' vanish, while ε_{13}, ε_{23}, ε_{33} are undetermined; this, of course, is also obvious directly, without specializing (5), from the elementary geometrical meanings of the individual components.

[It is important not to overlook that the distortion (5) is also compatible with *non*-uniform velocities on Σ for which the distortion in the surface vanishes.]

We come now to the generalization of (1) to a function which, together with its derivatives up to order $(n-1)$, is continuous on Σ and in N, while the $(n-1)$th derivatives in *rectangular coordinates* are constant on Σ. Then (HADAMARD [op. cit., § 59]), if the derivatives of order n exist in N and tend to finite limits as Σ is approached, the nth derivatives exist on Σ and their possible values are

$$\frac{\partial^n f}{\partial x^i \partial x^j \partial x^k \ldots} = \lambda v_i v_j v_k \ldots \qquad (n \text{ factors}), \qquad (i,j,k \ldots = 1,2 \text{ or } 3) \tag{6}$$

in *rectangular coordinates* where λ is an arbitrary scalar function of position. Correspondingly, the generalization of (3) is

$$\delta u = \lambda(\delta r)^n/n! + O(\delta r)^{n+1} \tag{7}$$

when u and its derivatives up to order $(n-1)$ are given *zero* on Σ.

To prove (6) start with $n = 2$, grad f being constant on Σ. Replacing f in (1) by its first derivatives in turn, we see that in rectangular coordinates $\partial^2 f/\partial x^i \partial x^j$ is proportional to v_j for each fixed i and to v_i for each fixed j, and hence to the product $v_i v_j$ for any i and j; that is, $\partial^2 f/\partial x^i \partial x^j = \lambda v_i v_j$, or in general coordinates

$$f_{,ij} = \lambda v_i v_j \,. \tag{8}$$

Alternatively, this result could be reached from (2), with $u_i = f_{,i}$ and a similar equation obtained by interchanging i and j; together these give $\lambda_i v_j = \lambda_j v_i$ and hence $\lambda_i = \lambda v_i$ with λ written for $\lambda_k v_k$. The result (6) can now be reached by a process of induction.

Notice that (6) really expresses no more than that one nth derivative, namely with respect to x^3, is undetermined and all others are zero, when local coordinates are taken with x^3 along the normal. However, the invariant form is indispensable in applications for a concise and symmetric treatment.

2.2. DISCONTINUITIES IN DERIVATIVES OF A CONTINUOUS FUNCTION

By a slight extension of viewpoint the previous results can immediately be re-phrased in terms of discontinuities.

Let Σ be a surface separating two open regions N_+ and N_- in each of which f satisfies conditions (b) and (c) preceding (1) but not necessarily (a). Then f is continuous across Σ but the one-sided first derivatives can be discontinuous. The jumps in these quantities from N_- to N_+ will be denoted by $[\partial f/\partial x^j]$. Imagine the values of f in N_-, denoted by f_-, analytically continued across Σ into N_+ in any way such that the new function and its first derivatives are continuous. Denote this constructed function in N_+ also by f_-; in general it will differ from f_+, the actual value of f in N_+. However, the difference $f_+ - f_-$ vanishes on Σ, satisfying (a), and also (b) and (c) in N_+; hence, by (1), $\partial(f_+ - f_-)/\partial x^j = \lambda\nu_j$ on Σ at the N_+ side. But, by construction, $\partial f_-/\partial x^j$ is continuous across Σ. Hence

$$\left[\frac{\partial f}{\partial x^j}\right] = \lambda\nu_j\,. \tag{9}$$

To be definite, in this and all similar formulae, the sense of the unit normal is arbitrarily chosen to point in the direction of the jump, namely into N_+.

Two remarks in passing are in order. First, it must be remembered that when f varies on Σ the one-sided derivatives do not satisfy relations of type (1); the actual relations are obtained later in section 3. Second, (9) contains (1); for, given any f in N_+ satisfying (a), (b), and (c), (1) follows from (9) on arbitrarily defining f constant in N_- equal to its value on Σ.

For a vector $\boldsymbol{u}$, not necessarily constant on Σ, whose components satisfy (b) and (c) in each of N_+ and N_-,

$$[u_{i,j}] = \left[\frac{\partial u_i}{\partial x^j}\right] = \lambda_i\nu_j\,. \tag{10}$$

This can be proved by applying (9) to each component u_i in turn. The covariant derivative can be used in place of the partial derivative since their jumps are equal, $\boldsymbol{u}$ being continuous across Σ (and naturally the Christoffel symbols also). Equally, $\lambda_i = [\partial u_i/\partial\nu]$ can replace $\boldsymbol{\lambda} = [\partial \boldsymbol{u}/\partial\nu]$.

HADAMARD's own derivation of (9) [op. cit., § 73] was based on the remark that on Σ, for all $\mathrm{d}x^j$ satisfying $\nu_j\mathrm{d}x^j = 0$, $[(\partial f/\partial x^j)\mathrm{d}x^j] = [\mathrm{d}f] = \mathrm{d}[f] = 0$ since f is continuous across Σ. However, the additional flexibility of the present approach is needed for later developments. Actually, HADAMARD gives elsewhere [op. cit., § 78] a geometrical interpretation of (10), closely akin in spirit to the method by which we have here derived (9).

Finally, corresponding to (8), the possible jumps in the second derivatives of a function whose first derivatives are continuous across Σ (implying that $[f]$ itself is at most a constant) are

$$[f_{,ij}] = \left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \lambda \nu_i \nu_j \,. \tag{11}$$

The difference between the covariant and partial second derivatives in general coordinates is immaterial here since its jump vanishes (grad f and the Christoffel symbols being continuous). Again, (11) can be proved in various ways; as these should by now be sufficiently obvious, details will not be given. The general formula corresponding to (6) was also given by HADAMARD [op. cit., § 74].

Explicit formulae, such as (9), (10) and (11), have considerable advantages compared with what is still the usual treatment of discontinuities (despite Hadamard and a few later writers), for example in the standard theory of characteristic surfaces. There the jumps themselves are regarded as the unknowns (or equivalently the actual derivatives on one side of the surface are, if the more usual 'uniqueness' viewpoint of Cauchy is taken). The compatibility restrictions on these are taken in implicit form along with the field equations in the continuum, and the solution often involves the reduction of a high-order determinant. The order can be reduced to the number of variables by the artifice of choosing local axes at the point under consideration, so as to separate out the required 'exterior' derivatives from the 'interior' derivatives on Σ (e.g. LEVI-CIVITA [1932, pp. 12 and 13]). Hadamard's method is essentially the symmetric and invariant form of this, not involving a special choice of axes. The compatibility relations are eliminated at the outset by substituting the explicit formulae (9), (10), (11), etc., into the difference of the field equations on either side; the characteristic scalars (λ), vectors (λ_i), tensors (λ_{ij}), etc., then become the unknowns, considerably less in number than the derivatives themselves. Several examples of this technique for waves in fluids and elastic solids are to be found in Hadamard's treatise, and for electromagnetic waves in LEVI-CIVITA's monograph [op. cit., pp. 78 *et seq.*]. Recently, THOMAS [1953, 1956, 1957a,b, 1958a,b,c, 1959] has revived and exploited the technique; his work is referred to in more detail later.

§ 3. Extended Compatibility Relations

The function f itself is now allowed to be discontinuous across Σ. This situation was not treated by Hadamard, and the first detailed discussion is apparently that of THOMAS [1957a]. The present approach is believed to be more flexible and structurally simpler, with advantages in applications.

3.1. JUMPS IN FIRST DERIVATIVES OF A DISCONTINUOUS FUNCTION

Let f be the considered function which, together with its gradient, is continuous in each open neighbourhood N_+ and N_- and tends to finite limits as Σ is approached from either side (the values of f on Σ itself need not be defined). $[f]$, the jump in the limiting values of f from N_- to N_+, is a given continuous and continuously differentiable function of position on Σ. The possible jumps in the limits of the first derivatives, $[\partial f/\partial x^j]$, are to be examined.

Introduce a function ϕ, continuous together with its gradient on Σ and in one neighbourhood (say N_+), otherwise arbitrary except that on Σ it is equal to $[f]$. Construct the auxiliary function F, equal to f in N_- and to $f - \phi$ in N_+, so that $[F] = 0$. By (9) $[\partial F/\partial x^j] = \lambda \nu_j$ and so

$$\left[\frac{\partial f}{\partial x^j}\right] = \lambda \nu_j + \frac{\partial \phi}{\partial x^j}\,. \tag{12a}$$

One choice of ϕ in N_+ suffices with λ an arbitrary function on Σ; it is enough, for example, to take $\phi \equiv 0$ when $[f] = 0$, or ϕ constant along the normals when $[f]$ varies. Another choice would, according to (1), merely have the effect of changing $\lambda = [\partial f/\partial \nu] - \partial \phi/\partial \nu$.

In passing, it is remarked that any jump formula such as this contains the compatibility relations appropriate to a continuous function (say f_+) defined on Σ and a neighbourhood on one side (say N_+). For, taking (12a) in illustration, we can apply it to the discontinuous function which is equal to f_+ in N_+ and on Σ and vanishes in N_-. This gives relations for $\partial f_+/\partial x^j$ analogous to the right side of (12a) with $\phi = f_+$ on Σ.

An alternative form of (12a) is obtained by simply omitting the λ term, on the understanding that ϕ is an *undetermined* function in the class defined:

$$\left[\frac{\partial f}{\partial x^j}\right] = \frac{\partial \phi}{\partial x^j}\,. \tag{12b}$$

Again, (12a) could be replaced by

$$\left[\frac{\partial f}{\partial x^j}\right] = \lambda \nu_j + \begin{cases} (\delta_j^k - \nu_j \nu^k)\,\partial \phi/\partial x^k\,, \\ (\partial/\partial x^j - \nu_j\, \partial/\partial \nu)\,\phi\,, \end{cases} \qquad \lambda = \left[\frac{\partial f}{\partial \nu}\right] \tag{12c}$$

where $\lambda - \partial\phi/\partial\nu$ has been written for the previous λ. The operator on ϕ is just the tangential part of the vector gradient. In this version of the formula, therefore, the values assigned to ϕ in N_+ not only do not affect the expression as a whole but not even the individual terms in it.

This dependence solely on the surface values $\phi = [f]$ can indeed be exhibited explicitly since

$$\left(\frac{\partial}{\partial x^j} - \nu_j \frac{\partial}{\partial \nu}\right) \phi = \left(g_{jk}\, g^{\varrho\sigma} \frac{\partial x^k}{\partial \theta^\varrho} \frac{\partial}{\partial \theta^\sigma}\right) \phi \tag{13}$$

where g_{jk} is the covariant metric tensor for the x^j coordinates and $g^{\varrho\sigma}$ ($\varrho, \sigma = 1, 2$) is the contravariant surface metric tensor for the surface coordinates θ^ϱ. This formula in effect resolves the left-hand surface vector into components in the directions tangential to the surface coordinates. On combining (12c) and (13):

$$\left[\frac{\partial f}{\partial x^j}\right] = \lambda\nu_j + g_{jk}\, g^{\varrho\sigma} \frac{\partial x^k}{\partial \theta^\varrho} \frac{\partial [f]}{\partial \theta^\sigma}, \qquad \lambda = \left[\frac{\partial f}{\partial \nu}\right]. \tag{12d}$$

This is THOMAS's form [1957a, equation (14), derived otherwise; also 1959, equation (3.3)].

When applied to the components u_i of a vector all the above formulae are valid as they stand when the x^j coordinates are rectangular (with g_{jk} = unit tensor). If, however, they are curvilinear, partial differentiation of u_i and ϕ_i by x^j must be replaced by covariant differentiation, and in (13) $\partial\phi_i/\partial\theta^\sigma$ must be replaced by $\phi_{i,l}\, \partial x^l/\partial\theta^\sigma$ and similarly for the derivative of $[u_i]$ in (12d). Also $\boldsymbol{\lambda} = [\partial \boldsymbol{u}/\partial\nu]$ but $\lambda_i \neq [\partial u_i/\partial\nu]$ since $\boldsymbol{u}$ is discontinuous.

Still another variant of (12) is

$$\left[\frac{\partial f}{\partial x^j}\right] = a_j \tag{12e}$$

where $\boldsymbol{a}$ is an undetermined vector function, needing definition on Σ only and such that

$$\mathrm{d}[f] = a_j \mathrm{d}x^j \text{ for } \mathrm{d}x^j \text{ in } \Sigma\,.$$

This result follows directly from $\mathrm{d}[f] = [\mathrm{d}f] = [(\partial f/\partial x^j)\mathrm{d}x^j] = a_j \mathrm{d}x^j$ for variations $\mathrm{d}x^j$ in Σ; a_j is undetermined to the extent of an additive part $\lambda\nu_j$. In particular, (12b) is recovered when $\boldsymbol{a}$ is assigned the surface values of grad ϕ, with ϕ as previously defined.

3.2. JUMPS IN SECOND DERIVATIVES OF A DISCONTINUOUS FUNCTION

In addition to the conditions on f laid down in 3.1, permitting jumps in f and its gradient, it is now required that second derivatives exist in

N_+ and N_- tending to finite (different) limits as Σ is approached. $[f]$ and $[\partial f/\partial \nu]$, and hence $[\operatorname{grad} f]$, are given functions on Σ, continuous and continuously differentiable.

Introduce a function ϕ, continuous together with its first and second derivatives on Σ and in one neighbourhood (say N_+), otherwise arbitrary except that on Σ $\phi = [f]$ and $\partial\phi/\partial\nu = [\partial f/\partial\nu]$. Construct the auxiliary function F, equal to f in N_- and to $f - \phi$ in N_+, so that $[F] = 0$, $[\partial F/\partial\nu] = 0$. By (11) $[\partial^2 F/\partial x^i \partial x^j] = \lambda\nu_i\nu_j$ and so

$$\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \lambda\nu_i\nu_j + \frac{\partial^2\phi}{\partial x^i\partial x^j}. \tag{14a}$$

Any *one* choice of ϕ in N_+ suffices with λ arbitrary (for example, a linear variation along the normals with gradient $[\partial f/\partial\nu]$). Another choice with the given gradient on Σ would, according to (8), merely change λ.

Alternatively, the λ term can be dropped, on the understanding that ϕ is an *undetermined* function in the class defined:

$$\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \frac{\partial^2\phi}{\partial x^i\partial x^j}. \tag{14b}$$

More generally, if only $[f]$ is given on Σ and not $[\partial f/\partial\nu]$, ϕ is only required to satisfy $\phi = [f]$, $\partial\phi/\partial\nu$ being arbitrary (automatically equal to the undetermined $[\partial f/\partial\nu]$, as can be seen by integrating (14b) in Σ).

For some purposes it may be desirable to show explicitly the dependence on $[\partial f/\partial\nu]$. To this end split ϕ arbitrarily into $\chi + \psi$, where $\psi = 0$, $\partial\psi/\partial\nu = [\partial f/\partial\nu]$ and $\chi = [f]$, $\partial\chi/\partial\nu = 0$ on Σ. Then, using *rectangular coordinates* for simplicity,

$$\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \mu\nu_i\nu_j + \left[\frac{\partial f}{\partial\nu}\right]\frac{\partial^2 p}{\partial x^i\partial x^j} + \left(\nu_i\frac{\partial}{\partial x^j} + \nu_j\frac{\partial}{\partial x^i}\right)\frac{\partial\psi}{\partial\nu} + \frac{\partial^2\chi}{\partial x^i\partial x^j} \tag{14c}$$

where p is the perpendicular distance to Σ. The μ term can be dropped on the understanding that either ψ or χ is arbitrary subject to the boundary conditions. Note also (again in rectangular coordinates) that

$$\left(\frac{\partial}{\partial x^j} - \nu_j\frac{\partial}{\partial\nu}\right)\nu_i = \left(\frac{\partial}{\partial x^i} - \nu_i\frac{\partial}{\partial\nu}\right)\nu_j = \frac{\partial^2 p}{\partial x^i\partial x^j} \tag{15}$$

where $\boldsymbol{\nu}$ here is any differentiable vector field in N_+ continuous with the unit normal on Σ (for instance $\boldsymbol{\nu} = \operatorname{grad} p$). The tensor (15) vanishes when the surface is plane, and only then.

To establish (14c) we begin by observing that grad $\psi = \boldsymbol{\nu}\, \partial\psi/\partial\nu$ on Σ since ψ is constant there. Hence, by the remark in the paragraph following (12a), applied to the function grad ψ,

$$\frac{\partial^2\psi}{\partial x^i \partial x^j} = \lambda_i \nu_j + \frac{\partial}{\partial x^j}\left(\nu_i \frac{\partial\psi}{\partial\nu}\right) = \lambda_j \nu_i + \frac{\partial}{\partial x^i}\left(\nu_j \frac{\partial\psi}{\partial\nu}\right).$$

Multiply the right-hand equation by ν_j, sum over j, and substitute the resulting expression for λ_i into the left-hand equation. Equation 14(c) follows with the use of (15), when μ is written for $\boldsymbol{\lambda\nu} - \partial^2\psi/\partial\nu^2$.

As in (12c) a variant of (14c) is obtained by retaining the μ term and modifying the operator on $\partial\psi/\partial\nu$, which can then (if desired) be written explicitly as an operator on $[\partial f/\partial\nu]$ with the help of (13). Adding to this the observation that, in rectangular coordinates, the tensor (15) is equal to

$$-g^{\varrho\sigma} g^{\gamma\delta} b_{\varrho\gamma} \frac{\partial x^i}{\partial\theta^\sigma} \frac{\partial x^j}{\partial\theta^\delta}$$

where $b_{\varrho\sigma}$ are (in the accepted notation) the coefficients of the second fundamental form for Σ, we recover THOMAS's expression for $\partial^2\psi/\partial x^i\partial x^j$ [1957a, equation (29), derived otherwise]. He also gives what is, in effect, a replacement for $\partial^2\chi/\partial x^i\partial x^j$ in terms of the first and second derivatives of $[f]$ with respect to the surface coordinates θ^ϱ. However, the complexity of the result (unless Σ is plane) makes its usefulness doubtful.

HADAMARD treated only the case when f is continuous [op. cit., § 119], for which χ can be put zero in (14c). His form is, effectively,

$$\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\right] = \left[\frac{\partial f}{\partial\nu}\right] \frac{\partial^2 p}{\partial x^i \partial x^j} + (\nu_i b_j + \nu_j b_i) \tag{16}$$

where $\boldsymbol{b}$ is a vector function defined only on Σ, undetermined to the extent of an additive part $\mu\boldsymbol{\nu}$, such that

$$b_j \mathrm{d}x^j = \mathrm{d}\left[\frac{\partial f}{\partial\nu}\right] \quad \text{for } \mathrm{d}x^j \text{ in } \Sigma.$$

§ 4. Kinematic Relations

When the surface Σ moves, jumps in space derivatives are usually accompanied by jumps in time derivatives. The corresponding restrictions on possible values of the latter are known as 'kinematic relations'; these will now be derived.

It is first necessary to define the speed of propagation of Σ; we take here a standpoint more general than usual. Let ξ^α ($\alpha = 1,2,3$) parametrize a convected system of curvilinear coordinate surfaces embedded in the continuum and deforming with it. Any material particle is thereby specified by a triad of ξ^α's, constant in time. At any moment t $\Sigma(t)$ contains particles such that $\Phi(\xi^\alpha, t) = 0$ say. Relative to the ('current' or 'final') configuration of the convected coordinates at time t this equation defines the position of Σ in physical space. But we may also regard the equation as defining a surface, $\Sigma_0(t)$ say, relative to the ('initial' or 'reference') configuration of the convected coordinates at an arbitrarily chosen fixed instant t_0. $\Sigma_0(t)$ coincides with the initial positions of those particles that momentarily lie on $\Sigma(t)$ at time t, and as $\Sigma(t)$ moves in physical space so does $\Sigma_0(t)$ (the two coinciding only when $t = t_0$, generally speaking, so that $\Sigma_0(t_0) = \Sigma(t_0)$). If, now, $c_0\delta t$ denotes the (numerical) perpendicular distance between the positions of $\Sigma_0(t)$ and $\Sigma_0(t + \delta t)$ at any considered point, c_0 is the local speed of Σ_0 normal to itself in physical space. It is also a measure of the rate at which Σ moves relative to the continuum, and is therefore called the 'speed of propagation'; however, by the manner of its definition, it depends on the choice of the reference state (unless the speed is zero).

Consider an infinitesimal area $\mathrm{d}\Sigma(t)$ of the surface at time t, $\mathrm{d}\Sigma_0(t)$ being the area occupied at time t_0 by the same particles. Let m be the rate at which mass crosses $\Sigma(t)$ per unit area, and let ϱ_0 be the density of this material at time t_0. Since mass is conserved,

$$\varrho_0 c_0 \,\mathrm{d}\Sigma_0(t) = m\,\mathrm{d}\Sigma(t)$$

and so the left-hand quantity is invariant with respect to choice of reference state.†

Let V be the actual speed of Σ in space in the normal direction $\boldsymbol{\nu}$ (pointing into N_+) and let v_+^ν, v_-^ν be the normal components of material velocity in space (the neighbourhoods are labelled so that material crosses Σ from N_+ to N_-). Take $t_0 = t$, thereby choosing the reference state as that at the considered time t itself. Σ_0 momentarily coincides with Σ as it crosses from N_-, where its speed as $t \to t_0-$ is

$$c_- = V - v_-^\nu$$

† This is closely connected with Nanson's theorem; e.g. see HILL [1959 equation (31)].

to N_+, where its speed jumps to

$$c_+ = V - v_+^{\nu}$$

as $t \to t_0+$. With these special values the above conservation equation gives

$$\varrho_+ (V - v_+^{\nu}) = \varrho_- (V - v_-^{\nu}) = \varrho_0 c_0 \, \mathrm{d}\Sigma_0(t)/\mathrm{d}\Sigma(t) = m \,, \qquad (17)$$

which includes

$$[v_\nu] + m[1/\varrho] = 0 \,.$$

In particular, if there is no jump in normal velocity and density, there is a definite speed of propagation $V - v^{\nu}$ relative to the material in its current state. This speed will be denoted by

$$c = m/\varrho \,.$$

Consider, now, continuous functions $f(x^j, t)$ satisfying condition (c) in section 2 in each of N_+ and N_- throughout a certain time interval, and further such that $(\partial f/\partial t)_x$ exists at time t and is continuous in N_+ and N_-. Such a function can also be expressed in the Lagrangian coordinates ξ^α; to simplify the notation the same symbol f will still be used, and correspondingly the time derivative with the values of the ξ^α's held fixed (i.e. following a particle) will be written $(\partial f/\partial t)_\xi$. The rate of change following $\Sigma_0(t)$ in the direction of its normal $\boldsymbol{\nu}_0$ is

$$\left(\frac{\partial}{\partial t}\right)_\xi + c_0 \left(\frac{\partial}{\partial \nu_0}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\Sigma_0} , \quad \text{say} \,.$$

Since f remains continuous across Σ_0, and by reason of the analytic conditions imposed on the derivatives, $(\mathrm{d}f/\mathrm{d}t)_{\Sigma_0}$ is the same whether evaluated in N_+ or N_-. Hence its jump vanishes and

$$\left[\left(\frac{\partial f}{\partial t}\right)_\xi\right] = -\, c_0 \left[\frac{\partial f}{\partial \nu_0}\right]. \qquad (18)$$

Writing this with (9) applied to $\Sigma_0(t)$:

$$\left[\frac{\partial f}{\partial \xi^\alpha}\right]_{\Sigma_0} = \mu \nu_{0\alpha} \,, \qquad \left[\left(\frac{\partial f}{\partial t}\right)_\xi\right] = -\, \mu c_0 \,, \qquad \mu = \left[\frac{\partial f}{\partial \nu_0}\right]_{\Sigma_0} , \qquad (19a)$$

where $\nu_{0\alpha}$ are the covariant components of $\boldsymbol{\nu}_0$ with respect to the initial configuration of the convected coordinates (HADAMARD [op. cit., § 101] gives this when the coordinates are rectangular).

In particular, letting $t \to t_0+$,

$$\lim_{\Sigma_0 \to \Sigma+} \left[\frac{\partial f}{\partial \xi^\alpha}\right] = \mu_+ \nu_\alpha \,,$$

$$\left[\left(\frac{\partial f}{\partial t}\right)_\xi\right] = -\mu_+ c_+ \,, \tag{19b}$$

$$\mu_+ = \lim_{\Sigma_0 \to \Sigma+} \left[\frac{\partial f}{\partial \nu_0}\right] ,$$

with a similar equation for $t \to t_0-$.

From the Eulerian standpoint with coordinates x^j the operator

$$\left(\frac{\partial}{\partial t}\right)_x + V\left(\frac{\partial}{\partial \nu}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_\Sigma , \qquad \text{say} ,$$

is the rate of change following $\Sigma(t)$ in the direction of its normal $\boldsymbol{\nu}$, and so

$$\left[\frac{\partial f}{\partial x^j}\right]_\Sigma = \lambda \nu_j \,, \qquad \left[\left(\frac{\partial f}{\partial t}\right)_x\right] = -\lambda V \,, \qquad \lambda = \left[\frac{\partial f}{\partial \nu}\right]_\Sigma . \tag{20}$$

Reconciliation with (19b) is effected with the help of

$$\left(\frac{\partial}{\partial t}\right)_\xi = \left(\frac{\partial}{\partial t}\right)_x + (\boldsymbol{v}\ \mathrm{grad})$$

together with

$$c_+ (\lambda - \mu_+) = c_+ \left\{ \left(\frac{\partial f}{\partial \nu_0}\right)_{\Sigma_0 -} - \left(\frac{\partial f}{\partial \nu}\right)_{\Sigma -} \right\} = [\boldsymbol{v}]\,(\mathrm{grad}\, f)_{\Sigma -}$$

as $t \to t_0+$. In this limiting process it must be borne in mind that $\partial f/\partial \nu_0$ on the negative side of Σ_0 is formed in the evanescent strip in the ξ^α space between $\Sigma_0(t_0)$ and $\Sigma_0(t)$ as $t \to t_0+$. (Alternatively, and this seems to be HADAMARD's standpoint [op. cit., §§ 89 and 100], we could imagine the state of deformation in N_+ continued smoothly into N_- to form the reference configuration there.)

When the velocity vector is continuous, however, all subscripts can be dropped in (19b), which becomes simply

$$\left[\frac{\partial f}{\partial x^j}\right] = \lambda \nu_j \,, \qquad \left[\left(\frac{\partial f}{\partial t}\right)_\xi\right] = -\lambda c \,, \qquad \lambda = \left[\frac{\partial f}{\partial \nu}\right] . \tag{21}$$

When f is discontinuous, then in conjunction with (12c) or (12d) applied to $\Sigma_0(t)$,

$$\left[\left(\frac{\partial f}{\partial t}\right)_\xi\right] = -\mu c_0 + \left[\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_{\Sigma_0}\right] . \tag{22}$$

Equivalent formulae corresponding to the other variants of (12) can easily be written down. Again, instead of the second of (20),

$$\left[\left(\frac{\partial f}{\partial t}\right)_x\right] = -\lambda V + \left[\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_\Sigma\right], \tag{23}$$

a version due to THOMAS [1957a, equation (21)].

Corresponding to (19a) formulae for the jumps in the mixed and second time-derivatives of a continuous function with continuous first derivatives are (HADAMARD [op. cit., § 103], where also are given the analogous nth order formulae):

$$\left[\frac{\partial^2 f}{\partial \xi^\alpha \partial \xi^\beta}\right] = \mu v_{0\alpha} v_{0\beta}, \qquad \left[\frac{\partial^2 f}{\partial \xi^\alpha \partial t}\right] = -\mu c_0 v_{0\alpha}, \qquad \left[\frac{\partial^2 f}{\partial t^2}\right] = \mu c_0^2. \tag{24}$$

Similar formulae can be written corresponding to (20). The second and third of (24) follow immediately from (18) applied first to $\partial f/\partial \xi^\alpha$ and then to $\partial f/\partial t$ in place of f.

Second-order kinematic relations when f is discontinuous have been derived by THOMAS [1957a].

§ 5. Stress-Rate Discontinuity

For materials with constitutive laws involving stress-rate the kinematic relations for the stress tensor are of special concern. In the form required here, however, they do not follow straightforwardly from the preceding results and so are handled separately in this section.

We note first the wellknown dynamical equation

$$[\boldsymbol{F}] + m[\boldsymbol{v}] = 0 \tag{25}$$

where $[\boldsymbol{F}]$ is the jump in true traction on a plane element parallel to Σ (with the convention that $\boldsymbol{F}$ is exerted by the material on the $+$ side).

In the present section the material velocity will be assumed continuous when $m \neq 0$. The traction is then continuous whether Σ is a wave ($m \neq 0$) or a stationary discontinuity ($m = 0$), in Hadamard's terminology. Subject to this, jumps can occur in the tensor components of stress. However, we begin with the situation where all components are continuous.

5.1. STRESS CONTINUOUS

Let σ^{ij} (Roman indices) be the contravariant components of true stress associated with fixed Eulerian coordinates x^j, and let $\sigma^{\alpha\beta}$ (Greek indices) be those associated with convected coordinates ξ^α coinciding with the x^j coordinates at the instant t under consideration.

By applying (21) to each component σ^{ij}, and replacing partial derivatives by covariant derivatives in view of the assumed continuity of stress and velocity:

$$\left[\frac{\partial\sigma^{ij}}{\partial x^k}\right] = [\sigma^{ij}_{,k}] = \mu^{ij}\nu_k\,,$$

$$\left[\left(\frac{\partial}{\partial t} + v^k\frac{\partial}{\partial x^k}\right)\sigma^{ij}\right] = \left[\frac{\mathrm{D}\sigma^{ij}}{\mathrm{D}t}\right] = -c\mu^{ij}\,, \tag{26}$$

where

$$\frac{\mathrm{D}\sigma^{ij}}{\mathrm{D}t} = \frac{\partial\sigma^{ij}}{\partial t} + v^k\sigma^{ij}_{,k}$$

is the 'intrinsic derivative of stress'. It is the rate of change of components referred to the x^j coordinates translating rigidly with the considered particle. μ^{ij} is an undetermined symmetric tensor function on Σ, equal to $[\partial\sigma^{ij}/\partial\nu]$.

If, now, the material at both sides of Σ is in equilibrium in the x^j frame of reference, with body-forces continuous across Σ,

$$[\sigma^{ij}_{,j}] = 0 \qquad \text{and so} \qquad \mu^{ij}\nu_j = 0\,. \tag{27}$$

These restrictions on μ^{ij} are analogous to those satisfied by the stress tensor on a surface free from traction. From this analogy we recognize at once that *μ^{ij} is a biaxial tensor*, with one principal axis normal to Σ and the associated principal value zero. Moreover, it is *orthogonal to any tensor of type* (2) *or* (4), in the sense that the scalar product

$$\mu^{ij}\cdot\lambda_i\nu_j \equiv 0$$

for arbitrary λ_i, in view of (27). This is a fundamental observation, pivotal in applications. It can be remembered intuitively through the interpretations of μ^{ij} above and of ε_{ij} following (5): with the special choice of coordinates relative to Σ the non-zero components of μ^{ij} and ε_{ij} are exactly complementary and the orthogonality is apparent.

The intrinsic derivative of stress does not, however, produce a concise treatment of general questions in continuum mechanics;

for that the unsymmetric 'nominal stress-rate', $\dot{s}^{\alpha\beta}$, is needed (HILL [1959]). In this the reference state for the nominal stress $s^{\alpha\beta}$ is that at the considered instant t itself. Explicitly, at any general time, the components $s^{\alpha\beta}$ are such that the traction on an arbitrary surface element embedded in the material has contravariant components

$$F^{\beta} = \nu_{\alpha} s^{\alpha\beta}$$

with respect to the *initial* convected coordinates, per unit *initial* area with covariant unit normal ν_{α}. (Of course, at the instant t the nominal and true stresses or tractions coincide.) The nominal traction-rate, i.e. the rate of change of the components F^{β} following the same element, is therefore

$$\dot{F}^{\beta} = \nu_{\alpha} \dot{s}^{\alpha\beta}$$

since ν_{α} is fixed. It can be shown that

$$\dot{s}^{\alpha\beta} = (\text{intrinsic derivative})^{\alpha\beta} + v^{\gamma}_{,\gamma}\sigma^{\alpha\beta} - v^{\alpha}_{,\gamma}\sigma^{\beta\gamma}.$$

Now, by (10), $[v^{\alpha}_{,\beta}] = \lambda^{\alpha}\nu_{\beta}$ and so from (26)

$$[\dot{s}^{\alpha\beta}] = -c\mu^{\alpha\beta} + (\lambda^{\gamma}\sigma^{\alpha\beta} - \lambda^{\alpha}\sigma^{\beta\gamma})\nu_{\gamma} \tag{28}$$

where ν_{α} now refers to an element of Σ, not to an arbitrary element. In passing, note that (28) implies the continuity of the nominal traction-rate, since $\nu_{\alpha}[\dot{s}^{\alpha\beta}] = 0$ by (27).

Finally, jump formulae for certain stress-rates independent of rigid-body spin are set down. First, take the 'convected derivative of true stress', $\dot{\sigma}^{\alpha\beta}$ (introduced by OLDROYD [1950]), which is the rate of change of components associated with the ξ^{α} coordinates deforming with the element. From (26) and (OLDROYD [op. cit., equation (24)]),

$$[\dot{\sigma}^{\alpha\beta}] = -c\mu^{\alpha\beta} - (\lambda^{\alpha}\sigma^{\beta\gamma} + \lambda^{\beta}\sigma^{\alpha\gamma})\nu_{\gamma}\,. \tag{29}$$

The jump in the convected derivative of the Kirchhoff stress (HILL [1959, p. 219]) is obtained by adding $\sigma^{\alpha\beta}\lambda^{\gamma}\nu_{\gamma}$ to the right of this equation. Second, take the 'rigid-body derivative of true stress', $\mathscr{D}\sigma^{ij}/\mathscr{D}t$ (terminology of HILL [op. cit., p. 222]), which is the rate of change of components associated with the coordinates x^j translating with the element and rotating rigidly with its spin

$$\omega_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i})\,.$$

We note, in passing,

$$[\omega_{ij}] = \tfrac{1}{2}(\lambda_i\nu_j - \lambda_j\nu_i)\,. \tag{30}$$

From (29) and HILL [op. cit., equation (45)], or from

$$\frac{\mathscr{D}\sigma^{ij}}{\mathscr{D}t} = \frac{\mathrm{D}\sigma^{ij}}{\mathrm{D}t} + \omega_k^{\cdot i}\sigma^{jk} + \omega_k^{\cdot j}\sigma^{ik} ,$$

$$\left[\frac{\mathscr{D}\sigma^{ij}}{\mathscr{D}t}\right] = -c\mu^{ij} + \tfrac{1}{2}\{(\lambda_k\nu^i - \lambda^i\nu_k)\sigma^{jk} + (\lambda_k\nu^j - \lambda^j\nu_k)\sigma^{ik}\} . \qquad (31)$$

A formula equivalent to this was given by THOMAS [1956, equation (31)], who uses the term 'absolute time derivative of stress'.

5.2. Stress discontinuous

We shall not trouble to derive the jump formulae for the individual components of stress-rate when the stress itself is discontinuous; they can be obtained when needed from (26) modified on the lines of (22). It is enough for present purposes to have jump formulae for the nominal traction-rate only, when the traction is continuous. The result, (33) below, is attractive and seems to be new.

To be able to apply (18) to the components of nominal traction on an element momentarily situate on Σ, there must first be associated continuously with this traction a vector field continuous across Σ during a time interval. To this end we refer once again to the family of surfaces $\Sigma_0(t)$ in the space of the Lagrangian coordinates (section 4). Regard this family as coinciding with the positions at reference time t_0 of surfaces embedded in the continuum; at any general time t the nominal traction on these material surfaces is the constructed vector field.

With this interpretation placed on the field $\boldsymbol{F}$ and similarly on the field of the unit normal, (18) gives

$$[\dot{\boldsymbol{F}}] = -c_0\left[\frac{\partial \boldsymbol{F}}{\partial \nu_0}\right]$$

or

$$[\dot{F}^\beta] = -c_0\left\{[s^{\alpha\beta}]\frac{\partial \nu_{0\alpha}}{\partial \nu_0} + \nu_{0\alpha}\left[\frac{\partial s^{\alpha\beta}}{\partial \nu_0}\right]\right\}$$

where, for simplicity, the convected coordinates at the reference time t_0 are taken as *rectangular*. Now the local perpendicular distance between two infinitesimally near surfaces of the Σ_0 family is proportional to c_0, and so

$$c_0\frac{\partial \nu_{0\alpha}}{\partial \nu_0} + \left(\frac{\partial}{\partial \xi^\alpha} - \nu_{0\alpha}\frac{\partial}{\partial \nu_0}\right)c_0 = 0 \qquad (\text{rect. } \xi^\alpha) . \qquad (32)$$

(This equation could be derived from (18) and a formula given by THOMAS [1957a, equation (60)], but is best shown directly by geometry or as below.) With the use also of $[\partial s^{\alpha\beta}/\partial\xi^{\alpha}] = 0$, from the equilibrium equations in rectangular coordinates assuming continuous body-forces,

$$[\dot{F}^{\beta}] = \left[\left(\frac{\partial}{\partial\xi^{\alpha}} - \nu_{0\alpha}\frac{\partial}{\partial\nu_{0}}\right)(c_0 s^{\alpha\beta})\right] \qquad (\text{rect. } \xi^{\alpha}) . \qquad (33a)$$

This formula is especially convenient since it effectively involves only surface data. Explicitly, as in (13),

$$[\dot{F}^{\beta}] = g^{\varrho\sigma}\frac{\partial\xi^{\alpha}}{\partial\theta^{\varrho}}\frac{\partial}{\partial\theta^{\sigma}}(c_0[s^{\alpha\beta}]) . \qquad (33b)$$

Alternatively, in the spirit of (12b), one could write

$$[\dot{F}^{\beta}] = \left(\frac{\partial}{\partial\xi^{\alpha}} - \nu_{0\alpha}\frac{\partial}{\partial\nu_{0}}\right)\phi^{\alpha\beta} , \qquad (33c)$$

where $\phi^{\alpha\beta} = c_0[s^{\alpha\beta}]$ on Σ_0, and $\phi^{\alpha\beta}$ is arbitrary off Σ_0 (in particular, for example, constant along normals). Notice that, for each β, $\phi^{\alpha\beta}$ is the α component of a *surface vector* (i.e. no normal component) since the jump in traction $\nu_{0\alpha}[s^{\alpha\beta}] = 0$.

The result (33) is so fundamental as to deserve an alternative proof from 'first principles'. Consider the material represented in Lagrangian space by a thin curved slice, with arbitrary finite area and varying thickness $c_0\delta t$, between the surfaces $\Sigma_0(t)$ and $\Sigma_0(t + \delta t)$; this material is traversed by the discontinuity during the interval δt. Subtract the equations of overall equilibrium of the slice at the instants t and $t + \delta t$. The curvature terms cancel, being the same to first order in both equations, leaving

$$\iint [\dot{F}^{\beta}]\, \mathrm{d}\Sigma_0 = \oint c_0 n_{0\alpha}[s^{\alpha\beta}]\, \mathrm{d}s_0 \qquad (\text{rect. } \xi^{\alpha}) \qquad (34)$$

where the surface integral extends over the arbitrary area of Σ_0 and the line integral round its boundary, to which $n_{0\alpha}$ is the outward unit normal tangential to Σ_0. Now apply the surface form of the Gauss divergence theorem for a surface vector, $\boldsymbol{w}$ say:

$$\iint \{\nabla\boldsymbol{w} - \boldsymbol{\nu}_0(\boldsymbol{\nu}_0\nabla)\boldsymbol{w}\}\mathrm{d}\Sigma_0 = \oint (\boldsymbol{n}_0\boldsymbol{w})\,\mathrm{d}s_0 , \qquad \boldsymbol{\nu}_0\boldsymbol{w} = 0 .$$

[This can be deduced from the ordinary 3-dimensional divergence theorem applied to a slice of vanishing uniform thickness, or from Stokes's theorem applied to $\boldsymbol{\nu}_0 \times \boldsymbol{w}$, noting that on the boundary

$(\boldsymbol{\nu}_0 \times \boldsymbol{w})\mathrm{d}s_0 = (\boldsymbol{n}_0\boldsymbol{w})\mathrm{d}s_0$.] Identify $w^\alpha = c_0[s^{\alpha\beta}]$ for each fixed β and replace the line integral in (34) by the equivalent surface integral, finally equating integrands to recover (33a).

As a variant one could start with the equation of overall equilibrium of the non-uniform slice at time t, and form its jump:

$$\iint c_0\left[\frac{\partial F^\beta}{\partial \nu_0}\right]\mathrm{d}\Sigma_0 + \oint c_0 n_{0\alpha}\,[s^{\alpha\beta}]\,\mathrm{d}s_0 = 0 \qquad \text{(rect. } \xi^\alpha)$$

where the continuity of the traction has been used. Combine this with the kinematic relations to obtain (34) and proceed as before.

At this point it is convenient to intercalate the deferred proof of (32). Apply to the non-uniform slice the observation that the projection of any closed surface S is zero, or $\iint \boldsymbol{\nu}\mathrm{d}S = 0$. In the limit

$$\iint c_0\,\frac{\partial}{\partial \nu_0}\,(\boldsymbol{\nu}_0\,\mathrm{d}\Sigma_0) + \oint c_0\boldsymbol{n}_0\,\mathrm{d}s_0 = 0\,.$$

But, by transformation,

$$\iint \boldsymbol{\nabla} c_0\,\mathrm{d}\Sigma_0 - \iint \boldsymbol{\nu}_0\,\frac{\partial}{\partial \nu_0}\,(c_0\,\mathrm{d}\Sigma_0) = \oint c_0\boldsymbol{n}_0\,\mathrm{d}s_0\,.$$

Eliminate the line integral and equate to zero the resulting integrand:

$$c_0\,\frac{\partial \boldsymbol{\nu}}{\partial \nu_0} + \{\boldsymbol{\nabla} - \boldsymbol{\nu}_0(\boldsymbol{\nu}_0\,\boldsymbol{\nabla})\}c_0 = 0\,,$$

of which (32) gives the components in rectangular coordinates.

A useful corollary to (33) is now derived. Let $t_0 = t$, the material velocity being assumed continuous. Since

$$\left[\left(\frac{\partial}{\partial \xi^\beta} - \nu_\beta\,\frac{\partial}{\partial \nu}\right)F^\beta\right] = 0$$

on Σ, the jump in the normal component of nominal traction-rate is

$$\nu_\beta\,[\dot{F}^\beta] = -\,c\nu_\beta\left[\frac{\partial F^\beta}{\partial \nu}\right] = -\,c\left[\frac{\partial F^\beta}{\partial \xi^\beta}\right].$$

Put $F^\beta = \nu_\alpha s^{\alpha\beta}$ in the right-hand side and use $[\partial s^{\alpha\beta}/\partial \xi^\beta] = 0$, which follows from the equilibrium equations since the nominal stress is symmetric at the reference time. Then

$$\nu_\beta\,[\dot{F}^\beta] = -\,c[s^{\alpha\beta}]\,\left(\frac{\partial}{\partial \xi^\beta} - \nu_\beta\,\frac{\partial}{\partial \nu}\right)\nu_\alpha \qquad (35)$$

where the term in $\partial \nu_\alpha / \partial \nu$ has been incorporated, even though it contributes nothing, in order to show explicitly the dependence on surface data only (formula (15) is also recalled).

Note, finally, some general implications of (33) and (35). (a) If the stress is continuous so is the nominal traction-rate (noted also after (28)). (b) If the speed of propagation and the jump in stress are uniform over the surface, the traction-rate is continuous. (c) If the surface is plane, at least the normal component of nominal traction-rate is continuous (with reference to the current configuration). Of course this sometimes also happens when the surface is curved, the two biaxial tensors in product in (35) being orthogonal, for example in torsion of a prismatic rigid/plastic bar.

§ 6. Discontinuities in Classical Elastic Solids

In illustration of the use of the basic theory we take first the classical elastic solid undergoing infinitesimal strain and rotation. It is supposed that the material is isotropic, with smoothly varying moduli, and under continuously distributed body-force.

6.1. Dislocations

Consider a stationary non-uniform discontinuity of displacement across a surface Σ separating two regions consisting of the same material initially unstressed and finally in mechanical contact. We examine what jumps in strain these conditions alone leave possible, nothing further being specified (for instance, whether or not the two regions are connected parts of a single body).

Let u_i be the infinitesimal displacement from the unstressed configuration and let x^j denote position in curvilinear coordinates. Then, analogously to (12a),

$$[u_{i,j}] = \eta_i \nu_j + \phi_{i,j}$$

where $\boldsymbol{\eta}$ is undetermined and $\boldsymbol{\phi}$ is any vector function equal on Σ to the given jump in displacement, $[\boldsymbol{u}]$. The strain jump is thus

$$[e_{ij}] = \tfrac{1}{2}(\eta_i \nu_j + \eta_j \nu_i) + \tfrac{1}{2}(\phi_{i,j} + \phi_{j,i}) \, .$$

But the regions are finally in contact and so the traction is continuous:

$$\lambda [e^k_k] \nu_j + 2\mu [e^i_j] \nu_i = 0 \, ,$$

where λ and μ are the Lamé constants. These are three equations determining $\boldsymbol{\eta}$ in terms of the 'strain' in the chosen $\boldsymbol{\phi}$ field ($\lambda + 2\mu \neq 0$), whence

$$[e_{ij}] = \left\{(\delta_i^k - \nu^k \nu_i)(\delta_j^l - \nu^l \nu_j) - \frac{\lambda}{\lambda + 2\mu}(g^{kl} - \nu^k \nu^l)\nu_i \nu_j\right\} \times \tfrac{1}{2}(\phi_{k,l} + \phi_{l,k}) \,. \tag{36}$$

The strain jump is thereby obtained by a linear transformation (the operator in curly brackets) of the 'strain' in any 'displacement' field $\boldsymbol{\phi}$ equal to $[\boldsymbol{u}]$ on Σ. This formula appears to be new. From the Lamé relations the corresponding stress jump is given by

$$[\sigma_{ij}] = \left\{(\delta_i^k - \nu^k \nu_i)(\delta_j^l - \nu^l \nu_j) + \frac{\lambda}{\lambda + 2\mu}(g^{kl} - \nu^k \nu^l)(g_{ij} - \nu_i \nu_j)\right\} \times \tfrac{1}{2}(\phi_{k,l} + \phi_{l,k}) \,. \tag{37}$$

The jump in rotation is

$$\tfrac{1}{2}[u_{i,j} - u_{j,i}] = \tfrac{1}{2}(\phi_{i,j} - \phi_{j,i}) + \nu^k(\delta_j^l \nu_i - \delta_i^l \nu_j) \cdot \tfrac{1}{2}(\phi_{k,l} + \phi_{l,k}) \,. \tag{38}$$

All these formulae are independent of the particular field $\boldsymbol{\phi}$. A different choice of $\boldsymbol{\phi}$ changes the derivatives $\phi_{i,j}$ by a tensor of type $c_i \nu_j$ and changes $\boldsymbol{\eta}$ by $-\boldsymbol{c}$; consequently the jumps in the displacement gradients are unaffected. Or one can check directly that any such change in the derivatives has no effect in the formulae. In particular, the operator in (36) annihilates any tensor of type $c_k \nu_l$ and therefore also $\tfrac{1}{2}(c_k \nu_l + c_l \nu_k)$.

Equally, the strain jump itself vanishes when (and only when) the distribution of $[\boldsymbol{u}]$ over Σ is such that the $\boldsymbol{\phi}$ fields all give rise to strains of type $\tfrac{1}{2}(c_k \nu_l + c_l \nu_k)$ there. But, as in section 2, such a strain is just the most general *inextensional* distortion of a membrane coinciding with Σ, and this property or its absence can be recognized from the given $[\boldsymbol{u}]$ values alone (without needing to construct a $\boldsymbol{\phi}$ field). The ultimate conclusion can be expressed as follows: the strain tensor is necessarily continuous across two surfaces in mechanical contact when the surfaces (regarded as membranes) are equally strained (i.e. the $\widehat{11}$, $\widehat{12}$, $\widehat{22}$ components are equal when x^3 is normal to Σ). In particular, the strain is continuous across a Volterra dislocation, $[\boldsymbol{u}] = \boldsymbol{a} + \boldsymbol{\theta} \times \boldsymbol{x}$ where $\boldsymbol{a}$, $\boldsymbol{\theta}$ are constant; here the surfaces are not only equally strained, considered as membranes, but are also identical in shape in the unstressed state, where the same pairs of particles could be brought into contact by relatively displacing the surfaces rigidly. Furthermore, the jump in the rotation vector is equal to $\boldsymbol{\theta}$ itself, or $[u_{i,j}] = \theta_{ij}$, the associated antisymmetric tensor (as is apparent from (38) by choosing ϕ equal to the same rigid-body movement extended off Σ).

The results in the last paragraph can easily be established directly, without recourse to the general formulae. It is simplest to take a special choice of local rectangular axes with x^3 along the normal, as was first done by SOMIGLIANA in a little-known paper [1914]†. He does not give the invariant form (36), only its specialization for this choice of axes††. For convenience, a surface across which the displacement is discontinuous but the strain tensor is continuous will be called a Somigliana dislocation (some authors use this nomenclature for an arbitrary dislocation). This emphasizes that the property of strain continuity is not solely possessed by a Volterra dislocation, which is an impression liable to be gained from the literature.

To illustrate the application of (36) let us calculate the strain jump when $[u_i] = \xi_{ij}x^j$ where ξ_{ij} is a constant symmetric tensor. Obviously this same linear form is the most convenient choice for the field ϕ, leading straight to $[e_{ij}] = \{\ldots.\}\xi_{kl}$ where $\{\ldots.\}$ is the operator in (36). With Σ interpreted as the interface between an inclusion and surrounding matrix with identical properties, and ξ_{ij} as a uniform transformation strain in the inclusion, a result due to ESHELBY is recovered [1961, equation (2.27), derived otherwise; note that Eshelby's $[\overset{\mathrm{C}}{e}_{kl}] = [e_{kl}] - \xi_{kl}$, cf. (5.10)].

We show, next, that the strain derivatives, $\partial e_{ij}/\partial x^k$, are continuous across a Volterra dislocation. It is enough to prove this in rectangular coordinates. Apply (14a) to $\boldsymbol{u}$, remembering that $[\partial u_i/\partial x^j] = \theta_{ij}$ as already proved:

$$\left[\frac{\partial^2 \boldsymbol{u}}{\partial x^j \partial x^k}\right] = \boldsymbol{\eta}\nu_j\nu_k$$

where $\boldsymbol{\eta}$ is undetermined. The second derivatives of the function ϕ vanish when it is chosen as the linear form $\boldsymbol{a} + \boldsymbol{\theta} \times \boldsymbol{x}$ which makes

† I have been unable to discover a reference to these results of Somigliana. After deriving (36), etc., my attention was drawn to Somigliana's paper by Dr. J. D. Eshelby, who also kindly lent his photostat copy.

†† When Σ is curved it is necessary to put a favourable interpretation on Somigliana's quantities $\partial[u_i]/\partial x_j$ (in the present notation) which he does not adequately define, $[u_i]$ having no meaning off Σ. The uncertainty is even more marked in a subsequent paper [1915], where second derivatives of the jumps appear.

$\phi = [u]$, $\partial\phi/\partial\nu = \boldsymbol{\theta} \times \boldsymbol{\nu} = \theta_{ij}\nu_j = [\partial u/\partial\nu]$ on Σ as required. Whence, by substituting in the jump of Navier's equations,

$$\mu\boldsymbol{\eta} = -(\lambda + \mu)(\boldsymbol{\eta}\boldsymbol{\nu})\boldsymbol{\nu} .$$

When $\lambda + 2\mu \neq 0$ this implies $\boldsymbol{\eta} = 0$ and therefore no jump in the strain gradient.

Note, on the other hand, that a jump does occur across a general Somigliana dislocation. Indeed, as Weingarten deduced from Cesaro's identity (LOVE [1927], p. 223), a dislocation is necessarily of Volterra type when both strain and strain gradient are continuous. This follows by 'strain geometry' alone, assuming merely that the displacement is twice continuously differentiable in the two regions.

6.2. GENERAL IMPOSSIBILITY OF STATIONARY DISCONTINUITIES

Dislocations and shock waves are now excluded from consideration, so that the displacement is continuous along with the traction. From 6.1 there follows the continuity also of all components of stress and displacement gradient. The question arises whether stationary discontinuities are possible in the gradient of stress and second derivatives of displacement. This was answered by implication by HADAMARD [op. cit., § 260] in showing that such jumps are necessarily propagated as waves. For completeness it seems worthwhile to present a concise version of the proof here, though it has been given at length by several writers (e.g., Lampariello†, an account of whose method conveniently appears in the French edition of LEVI-CIVITA's monograph [op. cit., pp. 71–78]).

By (24), with the unstressed state as reference,

$$\left[\frac{\partial^2 \boldsymbol{u}}{\partial x^j \partial x^k}\right] = \boldsymbol{\eta}\nu_j\nu_k , \qquad [\boldsymbol{f}] = c^2\boldsymbol{\eta} ,$$

where $\boldsymbol{f}$ denotes particle acceleration. Form the jump in Navier's equations

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \mu\nabla^2\boldsymbol{u} = \varrho(\boldsymbol{f} - \boldsymbol{g}) ,$$

remembering that the density ϱ and body force $\boldsymbol{g}$ per unit mass are continuous:

$$(\varrho c^2 - \mu)\boldsymbol{\eta} = (\lambda + \mu)(\boldsymbol{\eta}\boldsymbol{\nu})\boldsymbol{\nu} .$$

† LAMPARIELLO [1931] also gave the formally similar analysis proving the impossibility of stationary second-order discontinuities of velocity in slow flow of a Newtonian viscous fluid. HADAMARD's theory of waves in compressible elastic solids *finitely* strained [op. cit., §§ 264–267] has been extended to incompressible materials by ERICKSEN [1953].

Hence either (a) $\boldsymbol{\eta}\boldsymbol{\nu} = 0$, $\varrho c^2 = \mu$, which is called a transverse wave, the characteristic segment $\boldsymbol{\eta}$ being tangential to Σ; or (b) $\boldsymbol{\eta} = \eta\boldsymbol{\nu}$, $\varrho c^2 = \lambda + 2\mu$ (if positive), which is called a longitudinal wave, the segment being normal to Σ.

Similarly, the impossibility of stationary discontinuities of any order can easily be proved by induction, assuming the existence and continuity of all displacement derivatives. Suppose the nth derivatives ($n \geqslant 1$) are continuous (and therefore the $(n-1)$th derivatives of density). Then, from (6) and its associated kinematic relations,

$$\left[\frac{\partial^{n+1}\boldsymbol{u}}{\partial x^j \partial x^k \partial x^l \ldots}\right] = \boldsymbol{\eta}\nu_j\nu_k(\nu_l \ldots)\,,$$

$$\left[\frac{\partial^{n+1}\boldsymbol{u}}{\partial x^l \ldots \partial t^2}\right] = c^2\boldsymbol{\eta}\;(\nu_l \ldots)$$

where $(\nu_l \ldots)$ contains $n-1$ factors. Apply the operator $[\partial^{n-1}/\partial x^l \ldots]$ to Navier's equations. Then, provided the inhomogeneity in the two regions is such that the Lamé constants have derivatives of all orders continuous across Σ, the same equation for $\boldsymbol{\eta}$ as before is obtained. Consequently, discontinuities of all orders are propagated in the same two ways. In particular, if the nth derivatives of displacement are continuous there can be no stationary discontinuity in the $(n+1)$th derivatives. The required result follows by induction since the first derivatives are continuous under the conditions stated.

6.3. Shock waves

Finally, we give an analysis of shocks, due except in details to Thomas [1957b]. By a shock is meant a propagating discontinuity in velocity, accompanied by one in displacement gradient, the displacement itself being continuous. Thus,

$$\left[\frac{\partial \boldsymbol{u}}{\partial x^j}\right] = \boldsymbol{\eta}\nu_j\,, \qquad |\boldsymbol{\eta}| \ll 1\,,$$

the analysis for a classical elastic solid necessarily being restricted to shocks having characteristic segments of infinitesimal magnitude. To this approximation spatial derivatives in the Lagrangian coordinates on either side of Σ can be replaced by x^j derivatives. Then, from (19b),

$$[\boldsymbol{v}] = -c\boldsymbol{\eta}$$

where, in such a formula, c_+ and c_- need not be distinguished since $[c]/c = -[v^\nu]/c = \boldsymbol{\eta}\boldsymbol{\nu} \ll 1$. From (25)

$$[\boldsymbol{F}] = -\varrho c[\boldsymbol{v}] = \varrho c^2\boldsymbol{\eta}\,.$$

Use of the Lamé relations now gives

$$(\varrho c^2 - \mu)\boldsymbol{\eta} = (\lambda + \mu)(\boldsymbol{\eta}\boldsymbol{\nu})\boldsymbol{\nu} ,$$

the same equation as in 6.2.

There are therefore just two kinds of shock. (a) $\varrho c^2 = \lambda + 2\mu$ (if positive), $\boldsymbol{\eta} = \eta\boldsymbol{\nu}$. The jumps in traction and velocity are purely normal, and there is no jump in rotation. (b) $\varrho c^2 = \mu$, $\boldsymbol{\eta}\boldsymbol{\nu} = 0$. The jumps in velocity and traction are purely tangential, and there is no jump in density. For other details of the deformation undergone in crossing Σ in (a) and (b) one may conveniently recall the geometrical interpretation of (3) and (5) in section 2.

§ 7. Discontinuities in Rigid/Plastic Solids

The particular class of rigid/plastic solids contemplated here is such that in a plastic state of stress the strain-rate ε_{ij} satisfies

$$\varepsilon_{ij} = \lambda n_{ij} \qquad \text{with} \qquad \lambda \geqslant 0 , \tag{39}$$

where λ is merely an undetermined factor of proportionality and n_{ij} is the unit outward normal to the current yield surface in stress space at the point representing the stress. So long as the stress point is inside the current yield surface the material stays rigid. It is further supposed that the yield surface is *convex*: that is, any line segment cuts it in at most two points or, exceptionally, lies in the surface. The surface may be open at infinity or closed, and can have singular points where there is no unique normal; at such a point it is to be understood that n_{ij} can have any of the range of values to be found on a vanishingly small convex cap that would remove the singularity. For present purposes no other restrictions need be imposed (except in 7.1 below): the material can be anisotropic and continuously inhomogeneous, while n_{ij} can vary in any manner during deformation and need not be deviatoric (plastic volume changes not being excluded).

7.1. Velocity jumps

Since stationary discontinuities in velocity are tantamount to rupture in solids, only waves can properly be treated within the ordinary framework of continuum theory. Shocks in the strict sense (m finite) will not be considered here, only wave discontinuities of velocity whose speed of propagation is made arbitrarily small ($m \to 0$) by quasi-static variation of the loads and geometric constraints applied to the body surface. (The dimensional homogeneity in time of the above constitutive law permits the possibility, though by no means guarantees it). Inertial terms are then negligible in the equations of motion, and the traction jump can be treated as zero ($\varrho c^2/\sigma \to 0$, where σ is the order of magnitude of the stress field).

A velocity jump in the hypothetical rigid/plastic solid is *arbitrarily* regarded as a vanishingly thin layer Σ of intense strain-rate, throughout which the stress and velocity vary continuously and satisfy all field equations. By this extraneous consideration the approximation to an actual elastic/plastic solid is preserved; moreover, certain fundamental theorems for rigid/plastic solids remain valid when the body contains discontinuities of this restricted kind. Suppose, then, that $[\boldsymbol{v}]$ is the jump in velocity. We are to regard this as defining in direction the characteristic segment $\boldsymbol{\lambda}$ of a finite strain (3) smoothly developed as a material element traverses the layer, the strain-rate remaining predominantly in the direction (4) (which is possible at least when the volume change is infinitesimal). The stress within the layer, therefore, must correspond to the normal direction

$$n_{ij} \propto \nu_i[v_j] + \nu_j[v_i]\,. \tag{40}$$

For equilibrium it is necessary that the fixed-axes components of this stress should remain constant as the layer is traversed, and therefore unaffected by the progressive rotation and strain.

In passing, note that, if the shape of the yield surface is such that normal directions of type (40) do not exist for certain velocity jumps, these particular discontinuities are impossible. For example, if yielding is not influenced by the hydrostatic part of the stress, so that n_{ij} is deviatoric, only tangential jumps in velocity are admissible.

Now the differences between the stress in the layer and the stresses just outside can be at most *biaxial* states parallel to Σ, since the traction is continuous. Moreover, any such biaxial tensor is *orthogonal* to the tensor (40); note the remarks following (27) which are relevant also in this analogous situation. But the postulated convexity implies that in general all line segments from a particular point on the yield surface, perpendicular to the normal there, fall completely outside. No such biaxial state can exist therefore.

Consequently, there is *no stress jump* at the boundaries of the layer, nor therefore in the limit across Σ. The material at both sides is thus plastically stressed, (40) being the associated normal direction, and momentarily deforms (if at all) with the respective strain-rates in that direction; it follows that $[\varepsilon_{ij}] = [\lambda]n_{ij}$ (unless the stress point is singular).

Exceptionally, the stress can be discontinuous when the yield surface contains line segments perpendicular to the normal direction (40); yielding is then unaffected by addition of the corresponding biaxial states. It is now only the normal n_{ij} which undergoes no jump.

Previous work on velocity discontinuities in rigid/plastic solids has been almost entirely confined to tangential jumps, because it is customary to assume no volume changes. In soil mechanics more general jumps have recently been admitted (originating apparently with DRUCKER and PRAGER [1952]), though they are treated as 'sliding' discontinuities (i.e. stationary) and not as waves. Also, in all previous work (except HILL [1950, p. 160; 1952, p. 24]), the stress has been *assumed* to be continuous automatically. As the above proof makes clear, however, this assumption would be incorrect for a yield surface which is not strictly convex, and *a fortiori* if it is locally concave. The proof of stress continuity is therefore essential. In effect, it eliminates the possibility of a velocity jump sandwiched between two stress jumps (themselves the limit of layers of intense stress gradient in a homogeneous elastic/plastic solid), or even of a stress jump sandwiched between two velocity jumps.

7.2. STRESS JUMP (TRACTION CONTINUOUS)

Here the yield surface is restricted to be *strictly* convex, so that by 7.1 the velocity is continuous across the stress jump. We consider, now, the possibility of a jump in strain-rate, which can at most be

$$[\varepsilon_{ij}] = \tfrac{1}{2}(\eta_i \nu_j + \eta_j \nu_i) \tag{41}$$

where η_i is undetermined. As noted previously, $[\sigma^{ij}]$ can only be a biaxial state parallel to Σ and therefore orthogonal to $[\varepsilon_{ij}]$:

$$[\sigma^{ij}][\varepsilon_{ij}] = 0\,.$$

On the other hand, when at least one side is deforming, strict convexity implies

$$[\sigma^{ij}][\varepsilon_{ij}] > 0$$

since an internal line segment directed *towards* a point on the yield surface always has a positive product with the outward normal there (or, if the point is singular, with any normal in the permitted range). To avoid a contradiction the *strain-rate tensor must vanish on both* *sides*. The gradient of strain-rate, however, need not be continuous.

Statements of this result in the literature are for special cases only, and do not explicitly recognize the orthogonality property which is the key to the generalization (e.g. HILL [1952, p. 26], for a general yield surface and plane stress; a proof on similar lines to this reference is given independently for plane strain by GEIRINGER [1953, p. 276, repeated in 1958, p. 348]; PRAGER [1954, p. 26] applies the Hill-Geiringer method to a jump in the generalized stresses in a bent plate with Mises yield condition).

An inadequate treatment of stress jumps was given in the uniqueness theorem for rigid/plastic solids of type (39) (HILL [1956, 1957]), but can now be rectified with the help of (33). Without going into details it is enough to recall that the proof involves

$$\iint \Delta[\dot{F}^j]\Delta v_j \mathrm{d}\Sigma$$

taken over the entire discontinuity surface, where the prefix Δ attached to any quantity signifies the difference in its values in supposedly distinct solutions for the velocity and stress-rate fields, the existing field of stress and Σ itself being given. By (33), and since the strain-rate vanishes, this transforms into

$$\oint n_i[\sigma^{ij}]\Delta c \Delta v_j \mathrm{d}s$$

taken round the boundary of Σ, the reference state being that at the current instant. Various possibilities can arise: on a boundary of Σ inside the body, $[\sigma^{ij}] = 0$ *ipso facto*; on a boundary of Σ in the body surface either (a) $[\sigma^{ij}] = 0$ where the loading is continuous, or (b) $\Delta v_j = 0$ where the velocity is prescribed, or (c) $\Delta c = 0$ where there is a jump in traction moving in a prescribed way over the surface, since c is uniquely determined by the given orientation of Σ and the prescribed component speed. In all these cases the integrand of the line integral vanishes, and therefore the integral itself. Consequently, the original proof of the uniqueness theorem is unaffected by the presence of a stress jump.

7.3. STRAIN-RATE JUMP (STRESS AND VELOCITY CONTINUOUS)

The possibility of jumps in strain-rate, when the velocity is continuous, is now examined.† The stress is *necessarily* continuous by

† Attention is drawn to related work on waves in elastic/plastic non-hardening Mises solids by THOMAS [1956, 1958b,c].

the converse of 7.2, or by direct proof on similar lines. This appears as an *assumption* in most treatments, with the single exception of ERICKSEN [1955] who proved it for the Mises yield surface, though by an *ad hoc* algebraic method.

It follows that the strain-rates on both sides of Σ correspond to the same normal on the yield surface, differing only in magnitude. They are therefore themselves of type (41), with parallel segments $\boldsymbol{\eta}$, being the same combination of a pure shear strain-rate over Σ and a uniaxial strain-rate normal to it. A strain-rate jump can thus only exist where the stress permits this type of distortion. For the Mises yield condition this result has been obtained by CRAGGS [1954], PRAGER [1954, p. 25], and ERICKSEN [1955]. Here the jump is of tangential type and the stress deviator is a pure shear over Σ.

7.4. STRESS GRADIENT JUMP (STRESS, VELOCITY, AND STRAIN-RATE CONTINUOUS)

Appealing once again to the remarks after (27), the characteristic tensor μ^{ij} of the contemplated jump is orthogonal to that of the jump in strain-rate gradient:

$$\left[\frac{\partial \varepsilon_{ij}}{\partial x^k}\right] = \tfrac{1}{2}(\eta_i \nu_j + \eta_j \nu_i)\nu_k \, .$$

μ^{ij} is also orthogonal to the unit normal n_{ij} (assuming the yield function has continuous space derivatives). Now, by differentiating (39) and forming afterwards the jump:

$$\tfrac{1}{2}(\eta_i \nu_j + \eta_j \nu_i) = \left[\frac{\partial \lambda}{\partial \nu}\right] n_{ij} + \lambda \frac{\partial n_{ij}}{\partial \sigma^{kl}} \mu^{kl} \, .$$

Hence

$$\lambda \frac{\partial n_{ij}}{\partial \sigma^{kl}} \mu^{ij} \mu^{kl} = 0 \, . \tag{42}$$

But, when the yield surface is strictly convex, this implies $\lambda = 0$ when $\mu^{ij} \neq 0$, so that the *strain-rate must vanish on* Σ. This further implies

$$\left[\frac{\partial \lambda}{\partial \nu}\right] n_{ij} = \tfrac{1}{2}(\eta_i \nu_j + \eta_j \nu_i) \, . \tag{43}$$

Apart from this, no jump in stress gradient is possible (unless the strain-rate is discontinuous). For the Mises solid this was first stated by CRAGGS [1954].

If, however, the surface is *merely convex* it is possible additionally to have $\lambda \neq 0$ for certain $\mu^{ij} \neq 0$. For example, suppose the hydrostatic part of the stress does not influence yielding. Then (42) vanishes for $\mu^{ij} = \mu g^{ij}$, for arbitrary μ. Equation (43) still holds, but now $\eta\nu = 0$ since n_{ij} is deviatoric. Thus, it is necessary that the strain-rate be just a pure shear over Σ (supposing that $[\partial\lambda/\partial\nu] \neq 0$) which determines the stress deviator uniquely. For the Mises solid this additional possibility was first demonstrated in rather different fashion by THOMAS [1953, pp. 346–351].

References

CRAGGS, J. W., 1954, Qu. J. Mech. Appl. Math. **7** 35.
DRUCKER, D. C. and W. PRAGER, 1952, Qu. Appl. Math. **10** 157.
ERICKSEN, J. L., 1953, J. Rat. Mech. Anal. **2** 329.
ERICKSEN, J. L., 1955, J. Maths. Phys. **34** 74.
ESHELBY, J. D., 1961, Progress in solid mechanics **2** (N.-Holland Pub. Co.).
GEIRINGER, H., 1953, Advances in applied mechanics **3** 197.
GEIRINGER, H., 1958, Handbuch d. Phys. **6** 322.
HADAMARD, J., 1903, Leçons sur la propagation des ondes et les équations de l'hydrodynamique (Paris).
HILL, R., 1950, Mathematical theory of plasticity (Clarendon Press).
HILL, R., 1952, J. Mech. Phys. Solids **1** 19.
HILL, R., 1956, J. Mech. Phys. Solids **4** 247.
HILL, R., 1957, J. Mech. Phys. Solids **5** 153.
HILL, R., 1959, J. Mech. Phys. Solids **7** 209.
LAMPARIELLO, G., 1931, Rend. Accad. Lincei **13** 688.
LEVI-CIVITA, T., 1932, Caractéristiques des systèmes différentiels et propagation des ondes (Paris).
LOVE, A. E. H., 1927, Mathematical theory of elasticity (4th ed., Cambridge).
OLDROYD, J. G., 1950, Proc. Roy. Soc. A **200** 523.
PRAGER, W., 1954, Proc. 2nd U.S. Nat. Cong. Appl. Mech. 21.
SOMIGLIANA, C., 1914, Rend. Accad. Lincei **23** 61.
SOMIGLIANA, C., 1915, Rend. Accad. Lincei **24** 655.
THOMAS, T. Y., 1953, J. Rat. Mech. Anal. **2** 339.
THOMAS, T. Y., 1956, J. Rat. Mech. Anal. **5** 251.
THOMAS, T. Y., 1957a, J. Maths. Mech. **6** 311.
THOMAS, T. Y., 1957b, J. Maths. Mech. **6** 455.
THOMAS, T. Y., 1958a, J. Maths. Mech. **7** 141.
THOMAS, T. Y., 1958b, J. Maths. Mech. **7** 291.
THOMAS, T. Y., 1958c, J. Maths. Mech. **7** 893.
THOMAS, T. Y., 1959, J. Maths. Mech. **8** 1.

CHAPTER VII

THE STABILITY OF ELASTIC-PLASTIC STRUCTURES

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I. GENERAL PRINCIPLES

II. METHODS OF ANALYSIS

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I. GENERAL PRINCIPLES

§ 1. The Conditions for Any Theoretical State

Any theory of the static behaviour of structures is an elaboration, in one form or another, of the three requirements that have to be satisfied by any theoretical state, namely the requirements of (1) equilibrium, (2) compatibility and (3) conformity with the stress-strain relations of the materials involved.

The equilibrium requirement states that the internal stresses shall be in statical equilibrium with external loads. The compatibility condition states that the displacements of the structure shall be geometrically compatible with the internal strains, the internal strains being in turn compatible with each other. Finally, the internal stresses and the strains are related according to the characteristics of the materials, due account being taken where necessary of initial or 'locked-up' stresses and of strains due to temperature. The second and third conditions (compatibility and conformity with stress-strain relations) have on occasion been taken together when dealing with elastic structures, but it is desirable for inelastic structures always to treat them as separate requirements.

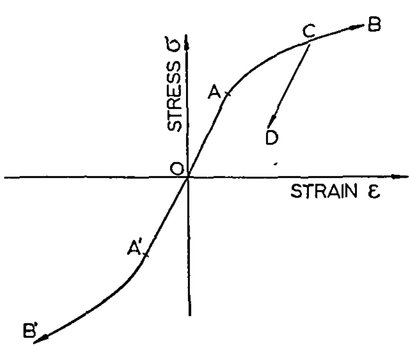

Fig. 1.

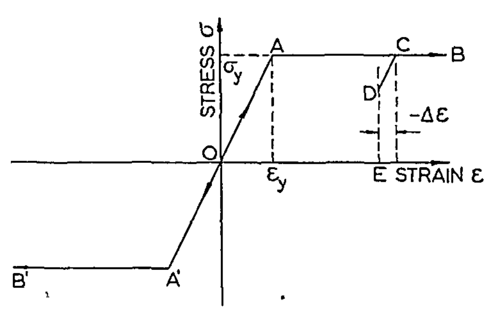

Fig. 2.

A structure is elastic provided the strains and stresses are linearly and reversibly related. Thus, for direct tension or compression, the stress-strain relation shown in Fig. 1 is linear and reversible in the ranges OA, OA′ (elastic), while beyond A and A′ the relation becomes non-linear and irreversible (inelastic). If, from some point C in the inelastic range, the stress is decreased, the relationship between change of stress and change of strain again becomes elastic, the straight line CD being parallel to the initial elastic line OA.

A particular type of inelastic behaviour is the elastic-plastic relation, shown in Fig. 2. Beyond the limit of elasticity at A or A′, the material becomes purely plastic in behaviour, the strain increasing to a large value with no increase of stress. Much of what follows will be concerned exclusively with such behaviour, which is typical of mild steel.

§ 2. The Problem of Uniqueness

The above considerations are sufficient to establish whether any given state of a structure is a theoretically possible one, dynamic forces being absent. The external loads acting on the free surface of a structure, and the displacements imposed at any given points such as points of support, constitute the boundary conditions for that structure. The question then arises: given the initial state of the structure and the initial and final boundary conditions, under what circumstances is it possible to find a unique solution for the final state?

It is first evident that, if the stress-strain relations within a structure are non-conservative, then the final state of the structure will not be uniquely determined by the conditions stated. Stress-strain relations are non-conservative when the strain is not a single-valued function of the stress, the actual relationship being dependent on the history of the stresses at any particular point. This behaviour is typical of stress-strain relations when they extend, during some part of the loading sequence, into the inelastic range. Secondly, the final state of the structure will not be unique when the applied loads are non-conservative in relation to the corresponding displacements, even though the structure may remain elastic. In general, when either the stress-strain or the load-displacement relations are non-conservative, the final state may only be derived from the initial state if account is taken of the sequence of loads, displacements, stresses and strains throughout the structure. In other words, the complete history of the boundary conditions and the corresponding states of the structure must be considered. If this is not done, there is in general a continuous set of possible solutions for the end state. Thus, consider a material which, in the virgin state, has stress-strain relations for uniaxial tension and compression given by OAB, OA′B′ in Fig. 2. If now a fibre is strained in tension to some point C with continuously increasing strain, and the strain is then reduced by $\Delta\varepsilon$, the stress reduces elastically to the value at D. The condition represented by point D has been reached as the result of the given strain history, but a differing strain history resulting in the same final strain OE would usually result in a differing final stress. The final stress may in fact lie anywhere within the range bounded by the yield stresses in tension and compression.

The condition that both the stress-strain and the load-displacement relations must be conservative is thus certainly necessary if the final state is to be uniquely determined by the initial state and the final boundary conditions. This condition is not, however, sufficient, since more than one configuration may be possible for the deformed state of an elastic structure under loads of given magnitudes. Thus, for example, the elastic, fixed-base frame in Fig. 3(a), with a vertical

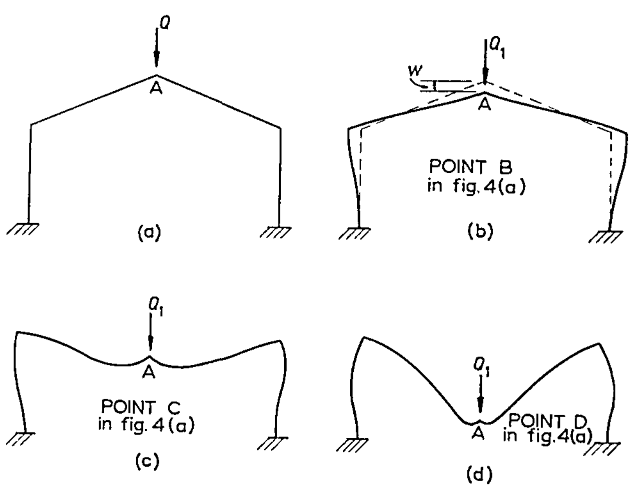

Fig. 3.

load Q at the apex A, may have the three configurations in Figs. 3(b), (c) and (d) for the same value of $Q = Q_1$. If the apex A is moved vertically downwards by an external agency, the graph of vertical force versus deflection will have the form shown in Fig. 4(a), the points B, C and D corresponding to the states of the structure (b), (c) and (d) respectively in Fig. 3. Lack of uniqueness of this sort is concerned with a finite number of non-adjacent states, and it is found for almost all practical structures (with the notable exception of

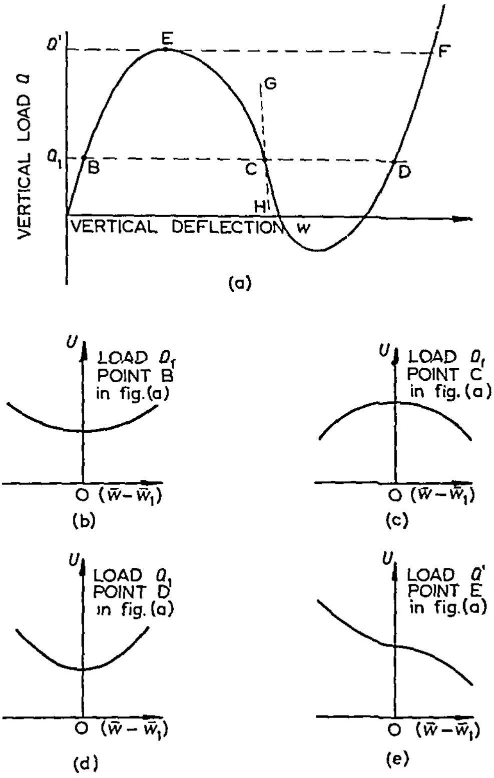

Fig. 4.

structures containing thin plates) that not more than one *elastic* state of the structure exists to support a given set of loads. Thus, any practicable portal frame structure would not be elastic under the gross deformations represented by Fig. 3(c).

The complexities of following step-by-step the behaviour of a non-conservative structure as the loading conditions are varied are so formidable that it is usual when dealing with elastic-plastic structures to make such assumptions and approximations that the structure may be treated as though it were conservative. Thus, as far as loading is concerned, the structure is assumed to be subjected to a monotonically varying load system, the intensity increasing continuously until the maximum capacity of the structure is reached. It is assumed that the structure is initially elastic (usually unstressed), and that once yield has occurred in any fibre the strain at any point continues to increase always in the same direction. Assuming elastic-pure plastic behaviour, the relation between stress and strain anywhere in the structure will be on B'A'OAB in Fig. 2. With these assumptions, a structure loaded beyond the elastic limit may be treated as though it were conservative. Actually some degree of reversal of straining almost invariably occurs in elastic-plastic structures, even when subjected to monotonically increasing loads. It appears that if such structures are assumed to behave in a conservative manner according to the stress-strain relation B'A'OAB (Fig. 2), the resulting collapse loads are on the safe (low) side compared with the theoretically correct values (RODERICK and HORNE [1948], BAKER *et al.* [1949]). This, however, is a subject in need of further investigation.

§ 3. The Stability Condition

We have dealt with the requirements (namely equilibrium, compatibility, and conformity with stress-strain relations) that have to be satisfied for any theoretical state, and the conditions under which a unique solution is possible. We now enquire whether, for a structure in a given equilibrium state, any infinitely small disturbing forces will cause the structure to depart a finite amount from that state. If the infinitely small disturbing force causes only an infinitely small disturbance, the structure is said to be stable, and this is a necessary condition if any theoretical equilibrium state is to be one in which the structure could in practice remain under the given boundary conditions. The principles enunciated in § 1 are sufficient to establish theoretically the conditions for stability, since, by considering the effect of a small disturbance, it is possible to determine whether the resulting changes in the internal resistance of the structure are statically sufficient to balance the changes in equilibrium conditions due to the incremental deflections and external forces. While simple problems may be treated thus, it is usually more convenient to use the concept of total potential energy for a criterion of stability. The total potential energy must be a minimum with respect to any incremental state of deformation if the structure is to be stable. If the potential energy is not a minimum, then an infinitely small disturbing force will produce changes in deformation that result in a net release of energy, and the structure will acquire finite velocities. The condition for stability, namely that infinitely small disturbing forces produce only infinitely small deflections will then be violated.

It may be noted that the concept of minimum total potential energy, as usually expressed, is applicable only to conservative structures. The potential energy of the external loads is then definable in terms of the positions of their points of application, while the internal potential energy consists of elastic strain energy definable in terms of the state of strain in the structure. The total potential energy is the sum of these terms. In non-conservative structures, care has to be taken over the method of stating the principle of minimum potential energy, since energy is continuously being expended irreversibly, the amount of energy thus expended depending on the strain cycle for all parts of the structure. By adopting a modified potential energy concept it is, however, possible to stipulate a form of the minimum energy criterion for stability that is applicable to non-conservative structures.

§ 4. Dynamic Instability

The above discussion is based on the assumption that dynamic forces are absent. HOFF [1949, 1951] has maintained that, for non-conservative structures, a quasi-statical treatment of the stability problem is inadequate. It is readily seen that, under many circumstances, the stability problem must become a dynamic one. Thus, if the Q applied to the frame in Fig. 3 is a dead load, and is gradually increased to the peak value Q' [point E in Fig. 4(a)], then finite velocities occur as soon as the load Q' is reached. Since the static load-deflection curve in Fig. 4(a) refers to an elastic structure, the point F will be reached when the subsequent oscillations of the structure have died down, but if the structure were in the process to become partially plastic, the final state reached could only be derived by following the motion of the structure in detail. If it is desired to follow the static curve EC in Fig. 4(a), then the load Q must be applied to the frame in Fig. 3 by a straining device which is sufficiently rigid to render the composite structure consisting of frame together with straining device stable at all stages of loading. Suppose that, when the straining device is adjusted to apply load Q_1 at point C in Fig. 4(a), the relationship between force Q and deflection w for the straining device is given by the line GH. Then it may be shown that the complete arrangement will be stable at the load Q_1 provided GH has a negative slope greater than the tangent to the static load-deflection curve for the frame at this point.

It might appear from the above argument that, by making the straining device sufficiently rigid, *it should always be possible to* follow the static load-deflection curve experimentally. The structure has, however, an infinity of degrees of freedom, and even if the loads on a structure are all applied by infinitely stiff straining devices, there may be insufficient control to prevent dynamic behaviour according to some mode of deformation. The point maintained by Hoff that dynamic behaviour should be considered must therefore be conceded if we are wishing to deal correctly with behaviour beyond the state at which the structure ceases to be stable. Since we shall not be concerned with any accurate study of behaviour beyond this state, there is no need to consider dynamic forces. The point of limiting stability [point E in Fig. 4(a)] may be derived, even for non-conservative structures, by a consideration of static behaviour only.

§ 5. The Bifurcation of Equilibrium, and the Tangent and Reduced Modulus Loads

Finally, mention should be made of a particular form of 'lack of uniqueness' connected with the stability of symmetrically loaded members in structures. It may happen that, for the elastic frame

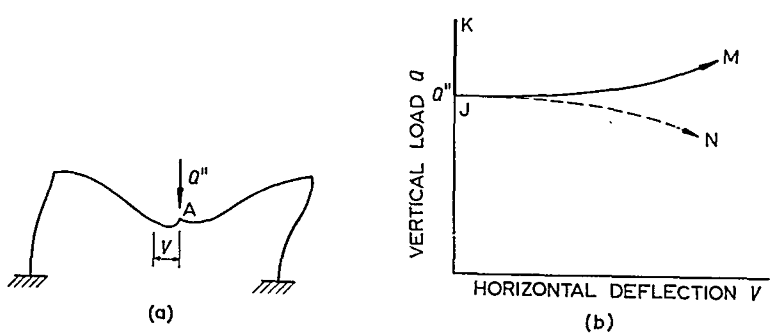

Fig. 5.

in Fig. 3, a tendency arises quite abruptly at some value Q'' of the load Q for an unsymmetrical mode of deformation to develop [Fig. 5(a)]. While it may be theoretically possible *for the relationship* between horizontal deflection at A and vertical load to continue with zero deflection as the load Q increases beyond Q'' [JK, Fig. 5(b)], the structure may also deflect sideways, following a load-deflection relation such as JM or JN, according to the particular characteristics of the structure. The phenomenon reached at load Q'', whereby the structure may follow either JK or JM, is known as a 'bifurcation of equilibrium'. Since the smallest imperfection would cause the structure to follow JM or JN rather than JK, the theoretical possibility of the higher relationship JK is of no practical importance. The equilibrium of a centrally compressed bar shows a similar 'bifurcation' at the Euler load, when the bar may theoretically either deflect in the form of a sine wave or remain straight.

Bifurcation of equilibrium can also occur in inelastic members. Thus, if a centrally compressed bar is composed of material with a stress-strain relation in compression of the form Oaef in Fig. 6(a), the lateral deflection will unambiguously remain zero up to some load P_t [point A in Fig. 6(b)]. Beyond this load, a bifurcation of equilibrium is possible. If the bar remains straight, there is a uniform increase of both stress and strain, and between A and B in Fig. 6(b) the stress and strain across the section change from AA′ to BB′ in Fig. 6(c). The bar may, however, bend and, in this case, the load-deflection relation moves to C [Fig. 6(b)], while stress and strain change from AA′ to AC′ [Fig. 6(c)]. Since all strains are increasing in the compressive direction, the relationship between the changes of stress and strain are according to the slope of the stress-strain curve at the mean stress σ_t corresponding to the load P_t. The load P_t is therefore calculated by reference to the slope of the tangent at the mean stress σ_t in Fig. 6(a), and is thus known as the tangent modulus load.

If the lower equilibrium path ACD is followed, there is initially an increase of deflection only as the axial load continues to increase. If, on the other hand, the higher equilibrium path AE is followed, a load P_r is reached at which lateral deflections occur without any increase of load. The changes of stress and strain across the section when the deflection increases to point F are shown in Fig. 6(d). The fibres on the concave face as the bar bends undergo further compression according to the slope of the stress-strain curve at the corresponding mean stress σ_r [Fig. 6(a)], while the stresses in the fibres on the convex face decrease according to the linear stress-strain relation e.g. parallel to the elastic line Oh. The load P_r is called the reduced modulus load.

The slightest imperfection in an inelastic bar in compression will induce it to start deflecting laterally at the tangent modulus load, but the bar will not become unstable until some load P_f is reached [Fig. 6(b)]. Despite this, the tangent modulus load is frequently used

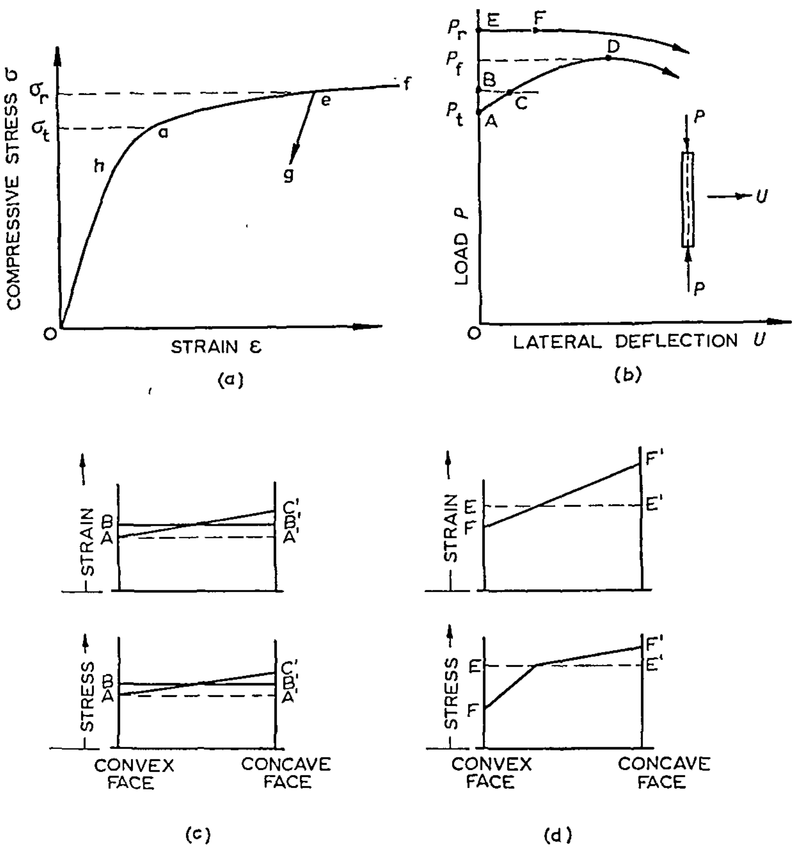

Fig. 6.

as an estimate of the failure load. Because of the presence of initial imperfections, this empirical procedure frequently gives results reasonably consistent with experiment, but the correlation is largely fortuitous.

It is interesting to note that the possibility of a bifurcation of equilibrium at the tangent modulus load was for a long time ignored, until SHANLEY [1947] discussed the problem. If symmetrical loading conditions are assumed, it is very easy, both in elastic and elastic-plastic structures, to make serious errors by failure to notice when a bifurcation of equilibrium is possible. Since perfect symmetry is in any case a fictitious condition, it is both unwise and unrealistic to confine attention to symmetrical conditions. It is unfortunate that the problem of instability in the plastic range has so often been dominated in the literature by the assumed condition of central loading. The buckling problem of a centrally loaded bar has only limited significance for the stability of realistically loaded structures in the plastic range. In structures containing imperfections and loaded in an arbitrary manner, the stability problem is not dominated by the phenomenon of bifurcation of equilibrium as it is for idealised structures. The imperfections and loading will define a specific behaviour such that two identical structures with identical imperfections and subjected to the same loading will follow the same load-deformation path. Thus a uniquely defined condition is reached at which the structure will either collapse completely or make a 'dynamic jump' to an alternative state at which stable equilibrium is again possible [e.g. a jump from E to F in Fig. 4(a)]. The problem of bifurcation of equilibrium loses the centrality to the problem which has sometimes been ascribed to it, and since it does not affect the general treatment will not be considered further.

II. METHODS OF ANALYSIS

§ 6. Introduction

The principles enunciated in Part I are theoretically sufficient to determine the state of a structure, whether elastic or inelastic, in static equilibrium with a given set of loads. An embarrassing number of devices and derivative principles have, however, been enunciated as aids to analysis such as slope-deflection equations, stability functions, and the principles of virtual work, stationary potential and complementary energy, and minimum complementary energy. Before entering the discussion of inelastic stability given in Part III, these aids to analysis will be summarised. Although the devices and principles are mostly well known, it is unfortunate that their real significance is easily hidden when attention is confined to elastic structures, as is usually the case. This may result in mistaken application to inelastic problems.

Certain aids to analysis which refer only to elastic structures are first summarised in § 7. General theorems relating to the equilibrium and compatibility conditions for any state are dealt with in § 8. § 9 contains a discussion of the potential energy criterion for elastic structures, and the implications of the modified energy criterion for inelastic structures are discussed in § 10. The use of the virtual work equations in the stability analysis of both elastic and inelastic structures is considered in § 11.

§ 7. Aids to the Analysis of Elastic Structures

The most elementary aids for elastic structures are those which describe the behaviour of single members. Provided no axial load is present, and flexure is the only important deformation, the relations between the terminal shear forces, bending moments, displacements and rotations are linear, and involve the equilibrium conditions

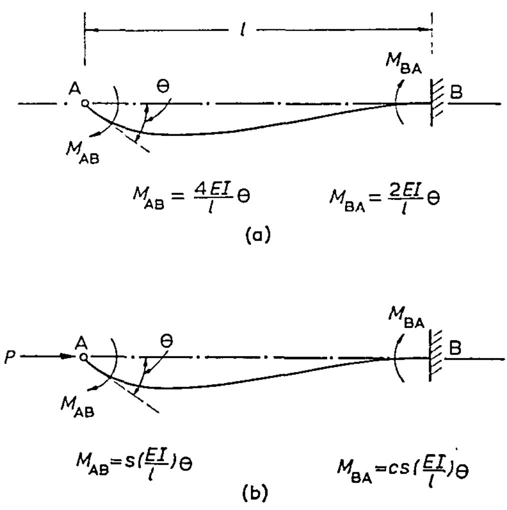

Fig. 7.

together with the 'slope-deflection' equations which describe the flexural behaviour of the member. Thus, if the initially straight member AB in Fig. 7(a) is of uniform cross-section with flexural rigidity EI, and has a length l, the basic slope-deflection relations are

$$\begin{aligned} M_{AB} &= 4\left(\frac{EI}{l}\right)\theta \\ M_{BA} &= 2\left(\frac{EI}{l}\right)\theta . \end{aligned} \tag{1}$$

The total elastic behaviour of a member under any combination of terminal conditions may be derived from these equations by superposition. The effect of transverse loads within the span of the member may be allowed for by the introduction of the 'fixed-end moments' into the equations, these being functions of the intensity and distribution of the load (see, for example, PIPPARD and BAKER [1957]).

When an axial load P is present [Fig. 7(b)], the relationship may still be expressed linearly, provided the necessary changes are made in the stiffnesses in the slope-deflection equation. Thus equations (1) become

$$\begin{aligned} M_{AB} &= s\left(\frac{EI}{l}\right)\theta \\ M_{BA} &= cs\left(\frac{EI}{l}\right)\theta \end{aligned} \tag{2}$$

where the 'stability functions' s and c are transcendental functions of the quantity P/P_E, P_E being the Euler critical load for the member AB treated as a pin-ended strut. Stability functions of various sorts have been tabulated by a number of writers, the first being those due to BERRY [1919]. A recent presentation has been given by LIVESLEY and CHANDLER [1956]. The slope-deflection equations are not, of course, linear with respect to the axial load P, and non-linear changes occur in the distance between the ends of the member. These changes are of order $l\theta^2$.

The total behaviour of a structure composed of initially straight members may be derived by assembling the slope-deflection equations for individual members, using also the conditions of continuity and the requirements of equilibrium. When the displacements are everywhere small, displacements of order $l\theta^2$ may be neglected compared with the general displacements of order $l\theta$. We note that the basic slope-deflection equation may be interpreted 'applied moment equals stiffness times end rotation'. Similarly, provided θ is everywhere small, we may assemble the equations for the whole structure in the matrix form

$$\xi = Y \cdot \Delta \tag{3}$$

where ξ is a load vector, Y the stiffness matrix and Δ the displacement vector for the structure. The above is the notation used by LIVESLEY [1956]. It is to be noted that

1. The load vector ξ depends only on the undeflected form of the structure and the distribution and intensity of the loads;

2. The stiffness matrix depends only on the undeflected form of the structure and the stiffnesses of the individual members as modified by the axial loads they carry;
3. The displacement vector $\mathbf{\Delta}$ depends only on the undeflected form of the structure and the displacements of the joints, of order $l\theta$.

If the axial loads in the members of the structure remained constant, equation (3) would give the deflections as linear functions of the applied loads. The deflections would then become infinite if the stiffness matrix became zero. Since infinite deflections correspond to a condition of elastic instability, the structure on this argument becomes unstable when the axial loads are such as to lead to a zero stiffness matrix. Equation (3) has been used in this sense to evaluate elastic instability loads (LIVESLEY [1956]). It should, however, be noted that the condition of a zero stiffness matrix gives in general only an approximation to the elastic instability load. First, there is the difficulty of estimating the axial loads in the members at any assumed external loading, since these depend on the state of deformation of the structure as well as on the values of the external loads themselves. The accurate estimation of axial loads is therefore very difficult. Second, the dependence of the terms in the stiffness matrix on the axial loads, and therefore on the state of deformation, means that the stiffness matrix is not invariant with respect to deformations. The condition governing changes of deflection under constant external loads becomes, from equation (3),

$$0 = \dot{Y}\mathbf{\Delta} + Y\dot{\mathbf{\Delta}} \,. \tag{4}$$

The condition $Y = 0$ represents accurately a state of limiting stability if the deflections of the structure are zero. For many structures, particularly multi-storey frames carrying predominantly vertical loads, the axial loads in the columns are little affected by the state of deformation, and $Y = 0$ gives a close estimate of the elastic instability loads, but for a structure such as that in Fig. 3(a) $Y = 0$ gives a poor estimate. For the particular case in which the loading produces only direct axial loads in all the members, as for example a multi-storey frame under vertical loads applied at the beam to column joints only, the condition $Y = 0$ is accurate for limiting elastic stability provided the effect of direct axial compression can be ignored.

Further restrictions on the use of $Y = 0$ as a stability criterion arise from an inconsistency in the assembling of equation (3). The matrix Y differs from its value when all axial loads are zero only when the flexural shortenings of the compression members (of order $l\theta^2$) are of significance in the work equation. The effect of deformations of this order is, however, ignored in the displacement vector $\mathbf{\Delta}$, and the general position regarding second order deformation terms in elastic stability analysis is unsatisfactory.

§ 8. The Equilibrium and Compatibility Conditions

This section is concerned with theorems relating to the equilibrium and compatibility conditions for both elastic and inelastic structures. In discussing the state of stress and strain in a structure, it will for the sake of simplicity be assumed that uni-directional stress and strain only are involved. Let σ denote the direct stress at any point, and ε the corresponding direct strain. Let any load applied to a point in the structure be denoted by Q, and let the displacement of that point in the direction of Q be w. The loads Q are taken to include any body forces present.

According to the *virtual work* relation, if any set of external loads Q are in equilibrium with internal stresses σ, and if any set of *small* displacements Δw are compatible with strains $\Delta\varepsilon$, then

$$\sum \sigma\Delta\varepsilon - \sum Q\Delta w = 0\,. \tag{5}$$

This applies to non-conservative as well as conservative structures. It may be used to establish an equilibrium condition for actual loads Q^* and corresponding stresses σ^*, making use of *any* set of displacements Δw and corresponding compatible strains $\Delta\varepsilon$. Conversely, a compatibility condition may be established between actual displacements Δw^* and strains $\Delta\varepsilon^*$ by using *any* set of loads Q that are known to be in equilibrium with stresses σ.

The principle of *stationary potential energy* is a statement of the equilibrium condition for conservative structures. Let U denote the total potential energy of a structural system, that is, the potential energy of the applied loads together with the internal strain energy. We consider the displacements w_i of the structure and the corresponding compatible strains ε_j to be related functionally to a single displacement $\bar{w}$, so that

$$w_i = F_i(\bar{w})$$

and

$$\varepsilon_j = f_j(\bar{w})\,.$$

We take a given state of the structure corresponding to $\bar{w} = \bar{w}_1$, and consider a small deformation of the structure from the given state to an adjacent state, corresponding to a change in the displacement $\bar{w}$ from $\bar{w}_1$ to $\bar{w}_1 + \delta\bar{w}$. By changing the nature of the functional relationships $F_i(\ )$ and $f_j(\ )$ we may obtain any desired set of displacements and compatible strains. The principle of stationary potential energy states that, for any set of functional relationships,

$$\frac{\partial U}{\partial \bar{w}} = 0\,. \tag{6}$$

The equation so obtained is identical with that obtained from the virtual work relation by associating the actual loads and stresses in the structure with the set of infinitely small displacements

$$\delta w_i = (F_i'(\bar{w}))_{\bar{w}=\bar{w}_1}\,\delta\bar{w}$$

and strains

$$\delta\varepsilon_j = (f_j'(\bar{w}))_{\bar{w}=\bar{w}_1}\,\delta\bar{w}\,.$$

Although the virtual work relation is applicable to non-conservative structures, the principle of stationary potential energy cannot be so applied because the total potential energy cannot be expressed as a function of the displacements and strains. A stationary principle can, however, be formulated, using a modified total energy U_{N} which includes, as well as the total potential energy, the energy absorbed in plastic deformation as a result of the actual loading path. It is imagined that the structure has reached a given state defined by $\bar{w} = \bar{w}_1$, and then undergoes the further displacements

$$\delta w_i = (F_i'(\bar{w})_{\bar{w}=\bar{w}_1})\;\delta\bar{w}$$

and strains

$$\delta\varepsilon_j = (f_i'(\bar{w})_{\bar{w}=\bar{w}_1})\;\delta\bar{w}\,,$$

the change in the modified energy U_{N} being calculated accordingly. Then it is an equilibrium requirement of the given state that

$$\frac{\partial U_{\mathrm{N}}}{\partial \bar{w}} = 0\,. \tag{7}$$

§ 9. The Stability Condition for Elastic Structures

If, for a structure in a given state, an infinitely small disturbing force causes only an infinitely small departure from the given state, then the structure will be capable of remaining in that state without external restraint. This condition will be satisfied if, in order to cause a departure from the given state of deformation into any adjacent theoretical state, additional work has to be supplied. Hence a conservative structure will be stable in a given state provided the total potential energy is a minimum with respect to any change of deformation which satisfies the conditions of compatibility. In a non-conservative structure, the modified total energy U_{N} is similarly required to be a minimum as a necessary and sufficient condition of stability.

It was shown in the previous section that, since the structure is in equilibrium in the given state, the total energy U or U_{N} is required to be stationary [equations (6) and (7)]. For stable equilibrium, we have the additional requirement that, for any mode of deformation,

$$\frac{\partial^2 U}{\partial \bar{w}^2} > 0 \tag{8}$$

or

$$\frac{\partial^2 U_{\mathrm{N}}}{\partial \bar{w}^2} > 0\,. \tag{9}$$

Conversely, if there exists any state of deformation which can result in a decrease in U or U_{N}, then the structure will be unstable. In this section we consider the stability of elastic structures.

The stability criterion for the portal frame in Fig. 3(a) in the states depicted successively in Figs. 3(b), (c) and (d) is illustrated in Figs. 4(b), (c) and (d) respectively. The load Q_1 is supported in a stable manner in states (b) and (d) [points B and D in Fig. 4(a)], because the total potential energy U is a minimum with respect to any deformation from the corresponding state. It is assumed that bifurcation of equilibrium, as described in § 5, does not occur. For the state (c) in Fig. 3 [point C in Fig. 4(a)], if the load is maintained constant at Q_1 while a departure from the deformed state is made according to the increments of deformation actually obtained at C in Fig. 4(a), then the total potential energy U will appear as a maximum, as shown in Fig. 4(c). The structure is then in an unstable condition.

Consider now the state of the structure when it attains its maximum load Q' at E in Fig. 4(a). If the load is maintained at Q' and departures in deformation from the state at E are made according to the deflections of the actual structure, then the potential energy U varies as shown in Fig. 4(e). Provided the first differential of the stress-strain relation with respect to strain is a continuous function (as it is for an elastic structure, being in fact the elastic modulus and therefore constant), then the curve of total potential energy in Fig. 4(e) must be a continuous function as far as its second differential with respect to deflections. This follows from the fact that strain energy is given by the area under the stress-strain relation. Since $\partial^2 U/\partial\bar{w}^2 > 0$ when $\bar{w} < \bar{w}_1$, and $\partial^2 U/\partial\bar{w}^2 < 0$ when $\bar{w} > \bar{w}_1$, then $\bar{w} = \bar{w}_1$ is a point of inflexion and

$$\frac{\partial^2 U}{\partial \bar{w}^2} = 0 \,. \tag{10}$$

This equation represents the condition of limiting stability, and defines the peak load Q' called the elastic critical load. For the structure in Fig. 3(a), and for most states of loading in any structure, bending deflections occur as soon as any load is applied, and the exact determination of the elastic critical load is extremely laborious. In structures subjected to certain load systems, however, the deflections remain negligible until a critical load is reached, the only deflections below the critical load being those associated with direct axial strain. When the critical load is reached, but not below that load, deflections of a different order of magnitude become possible. The calculation of the critical loads of structures subject to such types of loading is comparatively easy. Any one structure subject to a pattern of loads which produces no bending deformation at arbitrary levels will have a series of critical load intensities with their corresponding modes of deformation ('critical modes'), all satisfying equation (10).

In discussing the fixed-base frame in Fig. 3(a), it has been stated that the structure will be stable in rising parts of the load-deflection curve, and unstable in falling parts. Although this statement may appear intuitively obvious, it requires some justification, since any load-deflection curve represents information relating to one particular deflection in the structure only.

Suppose Fig. 8(a) represents the load-deflection curve for a structure, the loads being expressed in terms of a typical load $\bar{Q}$ and the deflections in terms of a typical deflection $\bar{w}$. It is assumed that the structure is monotonically loaded. Let the strain energy of the structure be denoted by V, and let the total work done by the structure on the loading system be W. Then, since the structure is at all stages of loading in equilibrium with the applied loads,

$$\mathrm{d}V = -\mathrm{d}W$$

and

$$V + W = 0\,. \tag{11}$$

If the individual loads are denoted by $\lambda_i\bar{Q}$ where the λ_i are constant coefficients, and the corresponding deflections at the loading points are denoted by w_i, then

$$\mathrm{d}V = -\,\mathrm{d}W = \sum_i \lambda_i \bar{Q}\mathrm{d}w_i = \sum_i \lambda_i \bar{Q}\left(\frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}}\right)\mathrm{d}\bar{w}\,. \tag{12}$$

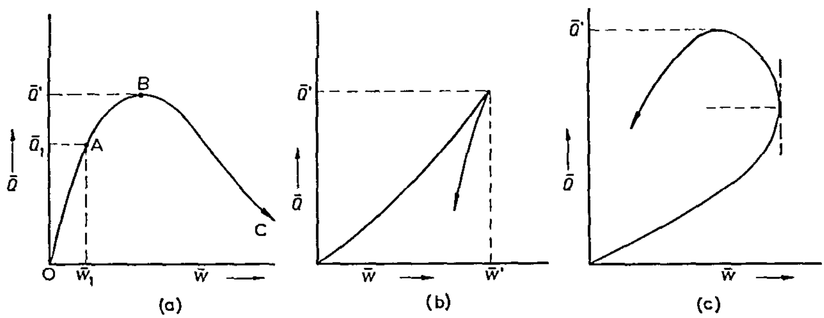

Fig. 8.

Hence, considering the behaviour of the structure up to the stage at which $\bar{w}$ reaches some value $\bar{w}_1$,

$$V = -W = \sum_i \lambda_i \int_0^{\bar{w}_1} \bar{Q}\left(\frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}}\right)\mathrm{d}\bar{w}\,. \tag{13}$$

Consider now the state of the structure at some point A in Fig. 8(a), when the load $\bar{Q}$ and the deflection $\bar{w}$ have the particular values $\bar{Q}_1$ and $\bar{w}_1$. Suppose the loads on the structure are imagined to remain constant at the values $\lambda_i\bar{Q}_1$ while the deflections change from w_i to $w_i + \mathrm{d}w_i$. The change in the potential energy of the external loads $\lambda_i\bar{Q}_1$ is then $\mathrm{d}W_1$ where

$$\mathrm{d}W_1 = -\sum_i \lambda_i \bar{Q}_1 \mathrm{d}w_i = -\bar{Q}_1 \sum_i \lambda_i \left(\frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}}\right)\mathrm{d}\bar{w}\,. \tag{14}$$

Hence, measuring the potential energy W_1 of the loads $\lambda_i \bar{Q}_1$ from the deflections of the structure when the loads are zero,

$$W_1 = -\bar{Q}_1 \sum_i \lambda_i \int_0^{\bar{w}_1} \left(\frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}}\right) \mathrm{d}\bar{w} \,. \tag{15}$$

The internal energy of the structure when the deflections are the w_i corresponding to $\bar{w} = \bar{w}_1$ is given by equation (13). Hence the total potential energy U becomes

$$U = V + W_1 = \sum_i \lambda_i \int_0^{\bar{w}_1} (\bar{Q} - \bar{Q}_1) \left(\frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}}\right) \mathrm{d}\bar{w} \,. \tag{16}$$

It follows that

$$\frac{\mathrm{d}U}{\mathrm{d}\bar{w}} = \sum_i \lambda_i (\bar{Q} - \bar{Q}_1) \frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}} \,, \tag{17}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}\bar{w}^2} = (\bar{Q} - \bar{Q}_1) \sum_i \lambda_i \frac{\mathrm{d}^2 w_i}{\mathrm{d}\bar{w}^2} + \frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{w}} \sum_i \lambda_i \frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}} \,. \tag{18}$$

When $\bar{Q} = \bar{Q}_1$, equation (17) gives the equilibrium requirement $\mathrm{d}U/\mathrm{d}\bar{w} = 0$, while equation (18) gives

$$\frac{\mathrm{d}^2U}{\mathrm{d}\bar{w}^2} = \frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{w}} \sum_i \lambda_i \frac{\mathrm{d}w_i}{\mathrm{d}\bar{w}} \,. \tag{19}$$

We now assume the positive direction of $\bar{w}$ to be so chosen that $\mathrm{d}\bar{Q}/\mathrm{d}\bar{w}$ is positive when the loads on the structure are zero ($\bar{Q} = 0$). If the structure is stable at zero load, then from equation (19) the term $\sum_i \lambda_i (\mathrm{d}w_i/\mathrm{d}\bar{w})$ is also positive. As the loads are increased, the structure will reach its elastic critical load when either the first or the second term in equation (19) becomes zero. Considering the second possibility, let $\sum_i \lambda_i (\mathrm{d}w_i/\mathrm{d}\bar{w}) = 0$ when $\bar{w} = \bar{w}'$. Then the system of load increments $\lambda_i (\mathrm{d}\bar{Q}/\mathrm{d}\bar{w})_{\bar{w}=\bar{w}'} \,\mathrm{d}\bar{w}$ represents a set of disturbing forces applied to the structure, and since the structure is unstable no such set of incremental loads can be applied without causing infinite displacements. Hence the value of $\bar{w}$ must be stationary with respect to the general deflections of the structure, and the type of relationship between $\bar{Q}$ and $\bar{w}$ obtained is shown in Fig. 8(b). This is, however, only a particular case, and there will only be a finite number of points in the structure which remain stationary at the critical load $\bar{Q}'$. More generally, the elastic critical load will be reached only when

$\mathrm{d}\bar{Q}/\mathrm{d}\bar{w}$ passes through zero. It therefore follows that, in the rising part OAB of the type of curve more usually obtained [Fig. 8(a)], since $\mathrm{d}\bar{Q}/\mathrm{d}\bar{w}$ is positive, the structure is stable ($\mathrm{d}^2U/\mathrm{d}\bar{w}^2 > 0$), while in the falling part of the curve $\mathrm{d}\bar{Q}/\mathrm{d}\bar{w}$ is negative and the structure is unstable ($\mathrm{d}^2U/\mathrm{d}\bar{w}^2 < 0$).

It may be noted that, if the deflection $\bar{w}$ momentarily becomes stationary and then decreases without the structure becoming unstable, then $\mathrm{d}\bar{Q}/\mathrm{d}\bar{w}$ must change from $+\infty$ to $-\infty$, as also does the second term $\sum_i \lambda_i(\mathrm{d}w_i/\mathrm{d}\bar{w})$ in equation (19). The general form of the load-deflection curve ($\bar{Q}$ versus $\bar{w}$) will then be as in Fig. 8(c), the structure not becoming unstable until the attainment of load $\bar{Q}'$ at which $\mathrm{d}\bar{Q}/\mathrm{d}\bar{w} = 0$.

§ 10. The Stability Condition for Inelastic Structures

The application of the concept of modified total energy U_N to the stability of inelastic structures may be discussed in relation to the obliquely loaded cantilever EF in Fig. 9(a). Consider the points A, B and C on the load-deflection curve in Fig. 9(b), the deflections $\bar{w}$ being

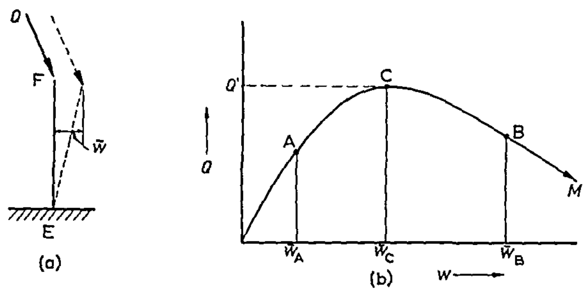

Fig. 9.

the horizontal displacement of the top of the cantilever. If the load Q is kept constant at any one of the points A, B and C while a slight departure is made in the deformed state of the structure, the change in modified total energy U_N may be calculated, U_N consisting of the potential energy of the applied load, the elastic strain energy in the cantilever and the energy absorbed in permanent deformation.

Taking the point A in Fig. 9(b) first, the cantilever is stable, and as a necessary condition, the modified energy U_N must be a minimum with respect to any change in the deflection $\bar{w}$ [Fig. 10(a)]. The cantilever may be imagined to undergo deformation according to any pattern, but, whatever the deformations considered, U_{N} must be a minimum in the state represented by point A in Fig. 9(b). Since changes in U_{N} must be calculated by starting from this point, arrows are inserted in the branches of the energy curve, Fig. 10(a).

The variations of U_{N} with deformations away from the state of the structure at point B in Fig. 9(b) are shown in Fig. 10(b). Many patterns of deformation may be chosen, and for some of these $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2$ will be positive, and for others it will be negative. Since the cantilever is known to be unstable with respect to a change of deformation corresponding to the path BM in Fig. 9(b), there will be at least one curve GK in Fig. 10(b) for which $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2 < 0$.

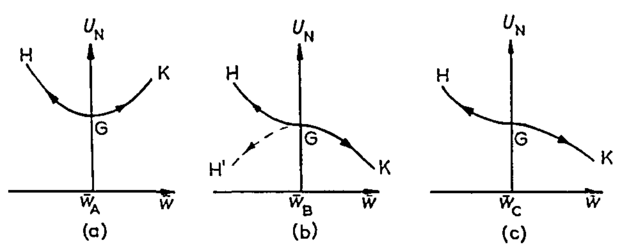

Fig. 10.

At the point C on the load-deflection curve [Fig. 9(b)], when the structure just becomes unstable, changes of U_{N} for changes of deformation are shown in Fig. 10(c). Since $\mathrm{d}Q/\mathrm{d}\bar{w} = 0$ in Fig. 9(b), the curve GK in Fig. 10(c) has $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2 = 0$. Any curve GH for reversed deformations will have $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2 > 0$.

The test of whether an inelastic structure is in a state of stable equilibrium is thus that the modified total energy U_{N} shall be at a stationary minimum with respect to departures in deformation from the given state ($\mathrm{d}U_{\mathrm{N}}/\mathrm{d}\bar{w} = 0$, $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2 > 0$) where $\bar{w}$ is a typical deflection. While the whole of the modified total energy U_{N} must be used as a test of equilibrium, a test of stability may, under certain conditions, be made for elastic-plastic structures by referring only to the sum U^* of the potential energy of the external loads and the elastic strain energy of the structure.

Denote the potential energy of the external loads by W, the elastic strain energy of the structure by V_{E} and the energy absorbed in plastic deformation by V_{P}. Then

$$U_{\mathrm{N}} = W + V_{\mathrm{E}} + V_{\mathrm{P}}\,, \tag{20}$$

$$U^* = W + V_{\mathrm{E}}\,. \tag{21}$$

Suppose the general displacements w and the strains ε throughout the entire structure may be expressed *linearly* in terms of a single displacement $\bar{w}$. Since plastic deformation only proceeds in the structure under constant stress, it then follows that V_{P} contributes nothing to the value of $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2$, and so

$$\frac{\mathrm{d}^2U_{\mathrm{N}}}{\mathrm{d}\bar{w}^2} = \frac{\mathrm{d}^2U^*}{\mathrm{d}\bar{w}^2}\,.$$

Hence the stability condition $\mathrm{d}^2U_{\mathrm{N}}/\mathrm{d}\bar{w}^2 > 0$ may be replaced by

$$\frac{\mathrm{d}^2U^*}{\mathrm{d}\bar{w}^2} > 0\,. \tag{22}$$

The variations in the energy U^* for incremental deflections under constant load at points A, B and C on the load-deflection curve, Fig. 9(b), are shown in Figs. 11(a), (b) and (c) respectively. Usually, defor-

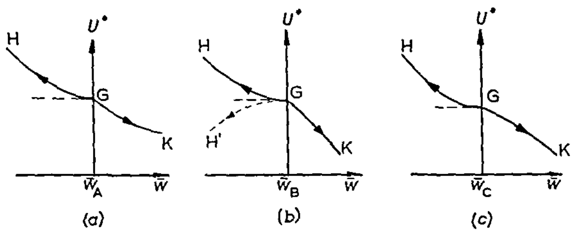

Fig. 11.

mations of the structure in a direction contrary to that taken by the deformations in the load-deflection curve will result in stress reversal in the plastic zones in the structure, so that straining is everywhere elastic and

$$\frac{\mathrm{d}U^*}{\mathrm{d}\bar{w}} = \frac{\mathrm{d}U_{\mathrm{N}}}{\mathrm{d}\bar{w}}\,.$$

With forward deformations, active plastic deformation occurs and the increase in U^* is less than the increase in U_{N}, so that

$$\frac{\mathrm{d}U^*}{\mathrm{d}\bar{w}} < \frac{\mathrm{d}U_{\mathrm{N}}}{\mathrm{d}\bar{w}}\,.$$

Since for an infinitely small change of deformation, equilibrium requires that $\mathrm{d}U_{\mathrm{N}}/\mathrm{d}\bar{w} = 0$, then $\mathrm{d}U^*/\mathrm{d}\bar{w} = 0$ for backward deformation and $dU^*/d\bar{w} < 0$ for forward deformation. Thus, Fig. 11(a), referring to the stable equilibrium state at A in Fig. 9(b), gives $dU^*/d\bar{w} = 0$ and $d^2U^*/d\bar{w}^2 > 0$ for GH at G (backward deformation) and $dU^*/d\bar{w} < 0$ and $d^2U^*/d\bar{w}^2 > 0$ for GK at G (forward deformation). In Fig. 11(b), corresponding to the unstable point B in Fig. 9(b), forward deformation (GK at G) gives $dU^*/d\bar{w} < 0$ and $d^2U^*/d\bar{w}^2 < 0$. In Fig. 11(c), corresponding to the maximum load Q' in Fig. 9(b), forward deformation (GK at G) has $dU^*/d\bar{w} < 0$ and $d^2U^*/d\bar{w}^2 = 0$.

In Fig. 9(b) the load-deflection curve for an inelastic structure has been drawn continuous in $dQ/d\bar{w}$. This is usually true in practice, but analytical simplifications are sometimes made when dealing with elastic-plastic structures, causing discontinuities in $dQ/d\bar{w}$. Thus the moment-curvature relations for a member bent about an axis of

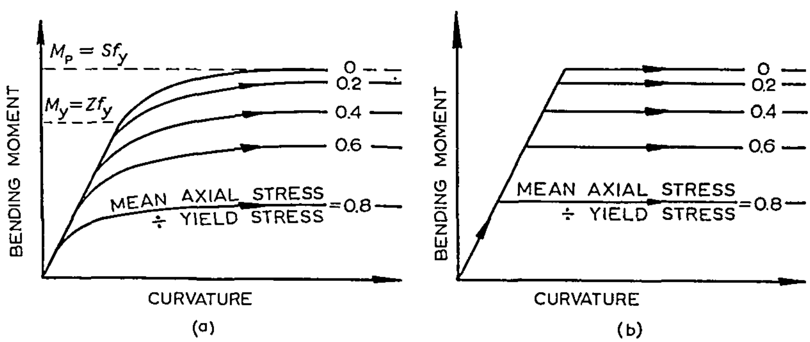

Fig. 12.

symmetry take the general form shown in Fig. 12(a) for various mean axial stresses, but these relations are commonly approximated by straight lines, as shown in Fig. 12(b). The use of such approximate moment-curvature relations leads to load-deflection relations for complete structures of the type shown in Fig. 13.

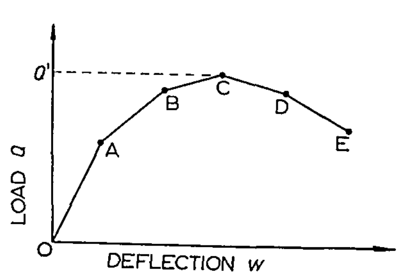

Fig. 13.

The discontinuities of slope occur in Fig. 13 when an additional plastic hinge forms in the structure. If C represents the peak load on the structure and $\bar{w}_c$ the corresponding deflection, then the stability criterion becomes $d^2U_N/d\bar{w}^2 > 0$ for $\bar{w} < \bar{w}_c$ and $d^2U_N/d\bar{w}^2 < 0$ for $\bar{w} > \bar{w}_c$.

§ 11. Use of the Virtual Work Equation to Establish Stability Conditions

When the virtual work relation, equation (5), is used to establish a condition of equilibrium, the incremental displacements and deformations represent any change of deformation from the given state. We may express the displacements w_i, the strains ε_j, together with the corresponding loads Q_i and stresses σ_j as functions of a single typical displacement $\bar{w}$, i.e.

$$w_i = F_i(\bar{w}) , \qquad \varepsilon_j = f_j(\bar{w}) ,$$
$$Q_i = G_i(\bar{w}) , \qquad \sigma_j = g_j(\bar{w}) .$$

In the virtual work equation (5) for the state when $\bar{w} = \bar{w}_1$, we associate the loads Q_i and the stresses σ_i with incremental displacements

$$\Delta w_i = F_i'(\bar{w})_{\bar{w}=\bar{w}_1} \Delta \bar{w}$$

and incremental strains

$$\Delta \varepsilon_j = f_j'(\bar{w})_{\bar{w}=\bar{w}_1} \Delta \bar{w} ,$$

giving

$$\sum_j \sigma_j \Delta \varepsilon_j - \sum_i Q_i \Delta w_i = 0 . \tag{23}$$

This virtual work equation is identical with that obtained by expressing, for elastic structures, the total potential energy U, and for inelastic structures, the modified total energy U_N, as a function of the displacement $\bar{w}$, and equating $\partial U/\partial \bar{w}$ or $\partial U_N/\partial \bar{w}$ to zero for $\bar{w} = \bar{w}_1$.

Consider now the application of the virtual work equation to the adjacent state of the structure defined by $\bar{w} = \bar{w}_1 + \Delta \bar{w}$. If the equilibrium condition is just satisfied, both when $\bar{w} = \bar{w}_1$ and when $\bar{w} = \bar{w}_1 + \Delta \bar{w}$, then the second order terms obtained by differentiating equation (23) must vanish. Hence

$$\sum_j \{\Delta \sigma_j \Delta \varepsilon_j + \sigma_j \Delta^2 \varepsilon_j\} - \sum_i \{\Delta Q_i \Delta w_i + Q_i \Delta^2 w_i\} = 0 \tag{24}$$

where

$$\Delta Q_i = G_i'(\bar{w})_{\bar{w}=\bar{w}_1} \Delta \bar{w} , \qquad \Delta \sigma_j = g_j'(\bar{w})_{\bar{w}=\bar{w}_1} \Delta \bar{w} ,$$
$$\Delta^2 w_i = F_i''(\bar{w})_{\bar{w}=\bar{w}_1} (\Delta \bar{w})^2 , \qquad \Delta^2 \varepsilon_j = f_j''(\bar{w})_{\bar{w}=\bar{w}_1} (\Delta \bar{w})^2 .$$

Equation (24) is identical with that obtained by putting $\partial^2 U/\partial \bar{w}^2 = 0$ or, in the case of inelastic structures, $\partial^2 U_N/\partial \bar{w}^2 = 0$ for $\bar{w} = \bar{w}_1$. When the entire expression in equation (24) is positive, $\partial^2 U/\partial \bar{w}^2 > 0$ or $\partial^2 U_N/\partial \bar{w}^2 > 0$, and the structure is stable with respect to the displacements Δw_i and the strains $\Delta \varepsilon_j$. Conversely, when the expression is negative, $\partial^2 U/\partial \bar{w}^2 < 0$ or $\partial^2 U_N/\partial \bar{w}^2 < 0$ and the structure is unstable with respect to these displacements and strains.

Structures subject to dead loads only have values of Q_i which are independent of deflections. Hence $\Delta Q_i = 0$ and the condition for limiting stability becomes

$$\sum_j \{\Delta\sigma_j \Delta\varepsilon_j + \sigma_j \Delta^2 \varepsilon_j\} - \sum_i Q_i \Delta^2 w_i = 0\,. \tag{25}$$

In view of the difficulties of calculating the modified total energy of an inelastic structure for all conceivable strain paths, it is more convenient to use the condition that equation (24) or (25) shall be positive for all deformation increments away from the given state rather than the condition that the modified total energy U_N shall have a minimum value.

III. REVIEW OF SOLUTIONS OF THE STABILITY PROBLEM FOR ELASTIC-PLASTIC STRUCTURES

§ 12. Introduction

Solutions of instability problems in the elastic range usually refer to single members subjected to concentric or eccentric end loads, comparatively little attention having been given to the difficult problem of continuous structures. The first progress was made by considering elastic-plastic stanchion lengths rigidly fixed to loaded beams which remained elastic (BAKER and RODERICK [1942, 1948a,b], BAKER *et al.* [1949], RODERICK and HORNE [1948]), an attempt here being made at an 'exact' solution. This work has been extended to more general conditions imposed upon the stanchion (EICKHOFF [1955]). The first complete analyses presented for a complete structure were those obtained by FOULKES [1953]† for rigid-jointed triangles. WOOD [1957/58] has recently presented accurate solutions for two four-storey frames, and his results are of great interest.

† See also HORNE [1956], a discussion of LIVESLEY [1956].

The above accurate solutions are mainly based on the elastic-pure plastic stress-strain relation OAB in Fig. 2. The analyses are dominated by the corresponding moment-curvature relations for the members, and these take the general form of the curves in Fig. 12(a). The exact shapes of the curves differ according to the cross-section of the member, differing curves being obtained for different axial loads. If the uppermost curve in Fig. 12(a) represents the relationship at zero axial load, a linear relationship is obtained up to the yield moment $M_y = Zf_y$ where Z is the elastic modulus of the member and f_y is the yield stress. Thereafter the relationship is curved and approaches asymptotically the full plastic moment $M_p = Sf_y$ where S is the plastic modulus. The ratio $M_p/M_y = S/Z = \nu$ where ν is the shape or form factor. For solid rectangular members $\nu = 1.5$, while for rolled I-sections bent about an axis parallel to the flange, $\nu = 1.15$ approximately.

In many studies of elastic-plastic structures, the approximation is made that the moment-curvature relation consists at a given axial load of two straight lines as shown in Fig. 12(b), the horizontal lines corresponding to the full plastic moments at various values of axial load. This corresponds to the assumption that the member consists of two flange plates of negligible thickness and a web of negligible area, bending being about an axis parallel to the flange plates. Such analyses are commonly called 'unit shape factor' or 'unit form factor' analyses, since they correspond to the assumption $S/Z = \nu = 1$. The simplification achieved in the analysis is considerable, since plasticity is confined to plastic hinges. A great many studies of elastic-plastic structures have been made on this basis by SALEM [1958], see also MERCHANT *et al.* [1958], while WOOD [1957/58] has also made use of the approximation in the course of his analysis. Recently, LIVESLEY [1959] has produced a digital computer programme for calculating elastic-plastic collapse loads on the assumption of unit shape factor. MERCHANT [1954, 1958] has discussed the relationship between the failure load and various idealised loads, such as the rigid-plastic collapse load and the elastic critical load. Experimental studies of elastic-plastic instability have been made on model frames from three to seven storeys in height by LOW [1958/59] and ARIARATNAM [1959].

The following section summarises the progress previously reported in studying the elastic-plastic instability of structures, and contains a discussion of the leading ideas presented by the above authors.

§ 13. The First Yield Load, the Elastic Critical Load and the Rigid-Plastic Collapse Load of a Structure

The final purpose of investigations into elastic-plastic instability is usually to obtain some convenient means of estimating the failure loads of structures. It is recognised that, given sufficient patience and the necessary computing facilities, failure loads may be calculated for any given structure once the physical assumptions have been stated. Such particular calculations are, however, so complicated that their *only* real use is to point the way to approximate but more convenient methods of calculation. In such approximate treatments certain idealised loads are important.

Consider a structure of which the material has the stress-strain relation OAB, OA′B′ in Fig. 2. The three most important idealised loads for such a structure are the yield load, the elastic critical load and the rigid-plastic collapse load. It is assumed that the loads on the structure are maintained in a constant ratio each to each, a simple numerical load factor λ then being sufficient to define the intensity of loading. We then have the following definitions.

1. *The load factor at first yield* λ_Y. This is the load at which the most highly stressed fibre is just subjected to the yield stress σ_Y, with the corresponding yield strain ε_Y (Fig. 2).
2. *The elastic critical load factor* λ_C. The stress-strain relation is imagined to be indefinitely elastic [Fig. 14(a)], and at the load factor λ_C, the condition $\partial^2 U/\partial \bar{w}^2 = 0$ is reached [see equation (10)].
3. *The rigid-plastic collapse load factor* λ_P. This is the collapse load calculated by the plastic theory of structures, and is based on the assumption of the rigid-plastic stress-strain relation OAB in Fig. 14(b).

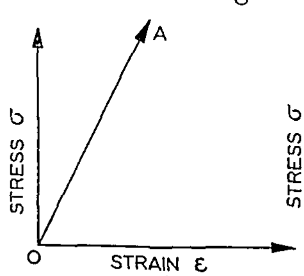

Fig. 14a.
Perfectly elastic material.

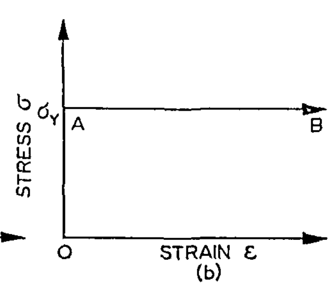

Fig. 14b.
Rigid-plastic material.

The idealised load with the longest history is the load at first yield. This is essentially the basis of many traditional methods of design, although these have usually been expressed in terms of a 'safe stress' rather than the yield stress. It has been shown (BAKER *et al.* [1956], BAKER and RODERICK [1942, 1948a,b]) that the yield load has no consistent significance in the calculation of the failure loads of continuous structures. Localised attainment of the yield stress may have virtually no effect on the ability of a structure to sustain loads of greatly increased intensity without failure.

The estimation of elastic critical loads of complete frameworks has long been recognised as difficult, and remains so despite recent improvements in technique. The calculation of the critical loads of structures loaded so that they undergo bending deformation from the inception of loading is particularly laborious, and accurate solutions for rigid frames have been obtained only for single bay, single storey structures (see for example CHWALLA [1938]). Attention has been confined almost exclusively to the eigenvalue problem, that is the estimation of λ_C for frames so loaded that they remain undeflected except at the critical load. Such loading is resisted in the structure by axial loads in the members only, and may thus be called 'axial loading'. The deformations produced by direct axial compression are assumed to be negligible.

Because of the difficulty of calculating the critical loads of structures in which the loading produces bending moments at arbitrary load factors, it is usual to replace the actual load distribution by what is regarded as 'equivalent axial loading'. Thus for the portal frame in

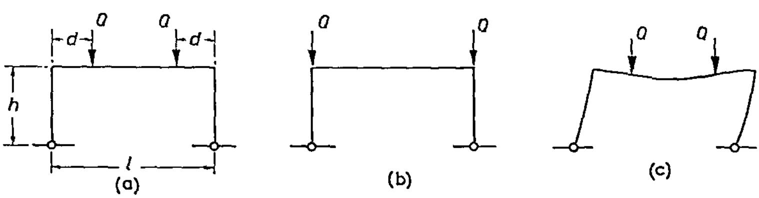

Fig. 15.

Fig. 15(a), the loading shown would, for the purpose of calculating the elastic critical load, be replaced by that in Fig. 15(b). The frame loaded as in Fig. 15(a) becomes unstable due to a sidesway mode, as shown in Fig. 15(c). Since the important factor in the instability analysis is the influence of the axial loads in the columns, the axial load in the beam being unimportant, the error involved in using the loading in Fig. 15(b) to estimate the critical load is small. CHWALLA [1938] investigated this particular case, and for $l = 3h$, found an error of 3.0%. The substitution of axial loading for the actual distribution seems generally to be justified, but difficulties arise in choosing the equivalent axial loading for some cases. Thus the loading in Fig. 16(a) would not be adequately represented by that in Fig. 16(b), nor yet by that in Fig. 16(c), since considerable changes would occur in the axial loads in the rafters because of deformations of the structure.

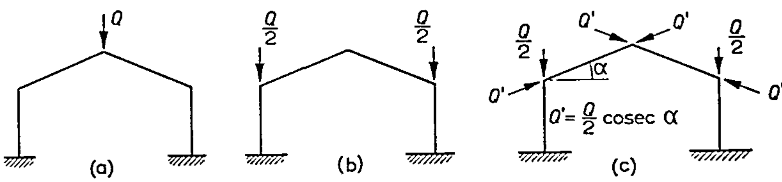

Fig. 16.

The various methods of calculating elastic critical loads for conditions of axial loading have been classified by Bleich†. His classification may be summarised briefly as (1) analytical (2) energy and (3) relaxation. In analytical methods, the homogeneous linear equation governing the characteristic modes is obtained, usually in the form of a determinant equated to zero. In order to form the separate simultaneous equations from which the determinant is derived, slope deflection-relations involving the stability functions described in § 7 are used. Bleich† has summarised the general principles involved in this type of analysis. MERCHANT [1955], SMITH and MERCHANT [1956], BOWLES and MERCHANT [1956] have suggested techniques which facilitate the writing down of the equations for multi-storey frames, at the same time reducing the number of separate equations. LIVESLEY [1956] has programmed the analytical method for a digital computer.

The energy method uses the condition $\partial^2 U/\partial \bar{w}^2 = 0$ (see § 9) governing the state of limiting stability. This leads to a determinant which is required to vanish at the critical load, and the energy method only differs from the analytical in the way in which this determinant is obtained. The energy method is not suitable for extensive structures.

The third group of methods for obtaining elastic critical loads uses

† BLEICH [1952], Ch. 2 and 6.

the relaxation technique for finding the reaction of the structure to an imposed deformation or disturbing force which produces bending moments in the structure. At the critical load, the stiffness of the structure with respect to such an arbitrary deformation or disturbing force becomes zero, and the relaxation process fails to converge. Both these properties have been used as a criterion for obtaining the critical load. The extension of relaxation techniques to bar structures, allowing for axial loads, was given many years ago separately by JAMES [1935] and BAKER and OCKLESTON [1935]. LUNDQUIST [1939] first developed the application of such modified relaxation techniques to the estimation of critical loads, using the convergence and stiffness criteria, and more recent examples have been given by WINTER *et al.* [1948]. BOLTON [1955] has obtained an analytical stiffness criterion for the critical load of trusses, using the relaxation concept, by restricting attention to a few steps only in the relaxation process.

The third idealised load is the rigid-plastic collapse load. Fundamental theorems relating to the estimation of such loads for plane rigid frames have been established, namely the Uniqueness Theorem and the Minimum and Maximum Principles (HORNE [1949/50] and GREENBERG and PRAGER [1952]). Many techniques for estimating rigid-plastic collapse loads, based on these fundamental theorems, are available (BAKER *et al.* [1956], NEAL [1956], HODGE [1959]).

According to the Uniqueness Theorem, if, at any load factor λ, a bending moment distribution may be found satisfying the three conditions of equilibrium, mechanism and yield, then λ is the rigid-plastic collapse load factor λ_P. The equilibrium condition is that the internal moments and accompanying forces shall be in static equilibrium with the external loads. The mechanism condition states that there shall be sufficient plastic hinges to transform the structure, or part of it, into a mechanism. The yield condition states that the full plastic moment shall be nowhere exceeded.

In the case of many structures, particularly single bay portal frames, it is easy to obtain by inspection a bending moment distribution satisfying the three conditions, and some methods of analysis, particularly semi-graphical methods, are based on this approach. In more extensive structures, use is made of the Minimum and Maximum Principles to proceed to a final solution by stages, placing ever narrowing upper and lower bounds on the collapse load factor.

The Minimum Principle states that a load for which a bending moment distribution satisfying the equilibrium and mechanism conditions can be obtained is an upper bound on the collapse load. This is the basis of the 'combination of mechanisms' method due to Neal and Symonds (NEAL and SYMONDS [1952]), the procedure being to combine elementary mechanisms into composite mechanisms in such a way as to minimise the load factor.

The Maximum Principle states that a load for which a bending moment distribution satisfying the equilibrium and yield conditions can be obtained is a lower bound on the collapse load. The 'plastic moment distribution' method, due to HORNE [1954], uses the Maximum Principle, bending moments in equilibrium with the loads at unit load factor being adjusted step by step to make the smallest ratio of full plastic moment to corresponding internal moment as large as possible. The multiplication of loads and bending moments by the factor so obtained then gives a bending moment distribution which satisfies the equilibrium and yield conditions. By combining the minimum and maximum principles in analysis, the rigid-plastic collapse loads of plane frames of any degree of complexity are readily obtained.

§ 14. The Idealised Loads as Parameters in the Estimation of Failure Loads

MERCHANT [1954]† has suggested that it might be possible to consider the failure load factor λ_F of an elastic-plastic structure as some function of various idealised load factors, including the first yield load factor λ_Y, the elastic critical load factor λ_C and the rigid-plastic load factor λ_P. For practical building frames of not more than two or three storeys, the most satisfactory correlation so far established is that of the simple plastic theory, in which we take $\lambda_F = \lambda_P$. Hence any empirical formula which may be suggested should reduce to $\lambda_F \simeq \lambda_P$ in such cases. Merchant† has tested for a large number of theoretical structures the following formula, which may be regarded as a generalisation of Rankine's formula for struts

$$\frac{\lambda_F}{\lambda_P} + \frac{\lambda_F}{\lambda_C} = 1 . \tag{26}$$

Since for practical structures of only one to three storeys, λ_C is usually large compared with λ_P, the requirement $\lambda_F \simeq \lambda_P$ is satisfied. If λ_F/λ_P is plotted vertically and λ_F/λ_C horizontally, equation (26) is simply

† See also MERCHANT *et al.* [1958].

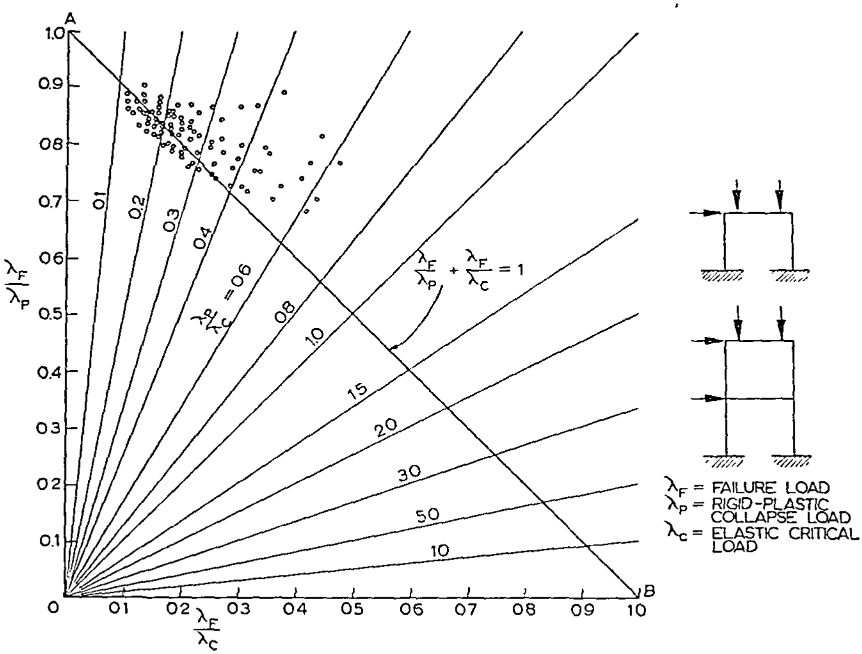

Fig. 17. Theoretical failure loads, one and two-storey frames (Salem [1958]).

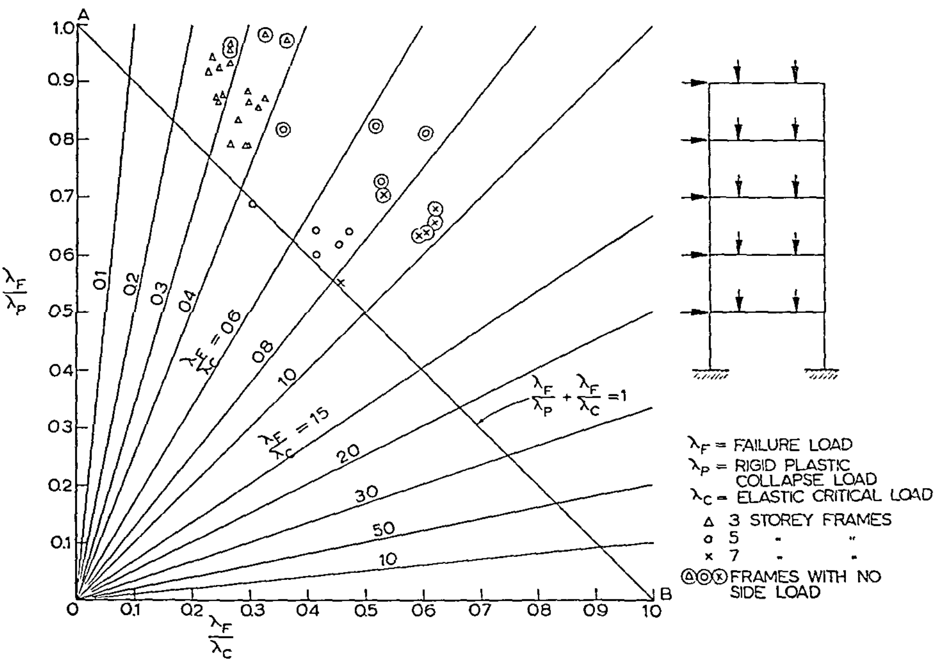

Fig. 18. Experimental failure loads, three, five and seven-storey frames.
(Low [1958/59].)

the straight line AB in Figs. 17 and 18. The points plotted on Fig. 17 were obtained theoretically by SALEM [1958]† for one and two-storey frames loaded as shown. Lines corresponding to various ratios of λ_P/λ_C have been drawn, and it is readily seen that the Rankine formula (26) is most successful when λ_P/λ_C is small and the collapse load is close to the rigid-plastic collapse value. When $\lambda_P/\lambda_C > 0.3$ the scattering of the points away from the Rankine solution is considerable. Fig. 18 shows the experimental results obtained by LOW [1958/59] for miniature single bay frames of three, five and seven storeys. Merchant suggests the Rankine formula as a safe (that is, lower) limit for the collapse load.

Merchant has suggested another theoretical load which may be explained by reference to the fixed-base portal frame shown in Fig. 19(a). The actual relationship between load factor λ and some typical deflection w is given by OHFJ in Fig. 20, the failure load (defined by the load factor λ_F) being reached at F. Suppose the rigid-plastic collapse mechanism is as shown in Fig. 19(b), the corre-

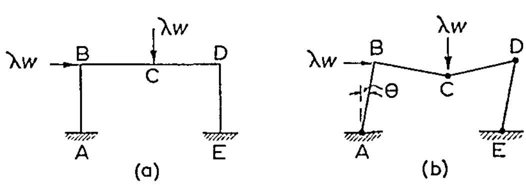

Fig. 19.

sponding load factor being λ_P. The value of λ_P is, as usual, obtained by considering the equation of equilibrium relating to the undeformed state of the frame. We may however recalculate the load factor for some rigid-plastic deformed state, as in Fig. 19(b), making allowance for the change of geometry. We then find that the load factor for equilibrium conditions decreases with increasing deflection according to the curve MN in Fig. 20. The curve MN is called the 'mechanism curve'. Finally, suppose OQ is the load factor versus deflection curve calculated on the assumption that the frame behaves entirely in an elastic manner. The curve OQ is called the 'elastic response curve', and its intersection at G with the mechanism curve MN defines a new load factor λ_G. Merchant suggests that this load factor may be used as a somewhat high estimate of the failure load factor λ_F. In a series of

† See also MERCHANT *et al.* [1958].

tests on triangulated structures, MURRAY [1956, 1958] found that the ratio λ_F/λ_G varied between 0.77 and 0.98.

A rough correlation between λ_F and λ_G is to be expected, since the true load-deflection curve OHFJ in Fig. 20 is identical with OQ up to the first yield load at H, and approaches very close to the mechanism curve after the last plastic hinge has formed in the structure. If the deflection recorded is that of a plastic-hinge, the load deflection curve becomes coincident with the mechanism curve on the formation of the last hinge. There is, however, a lack of precise definition of the load λ_G in Merchant's treatment. We may construct the diagram in Fig. 20 by selecting a particular deflection [for example, the horizontal deflection at D, Fig. 19(a)] and thereby arrive at a particular value for λ_G. The calculation may be repeated, but with reference to some other deflection [such as the vertical deflection at the loading point C, Fig. 19(a)], the horizontal scale in Fig. 20 being adjusted so that the

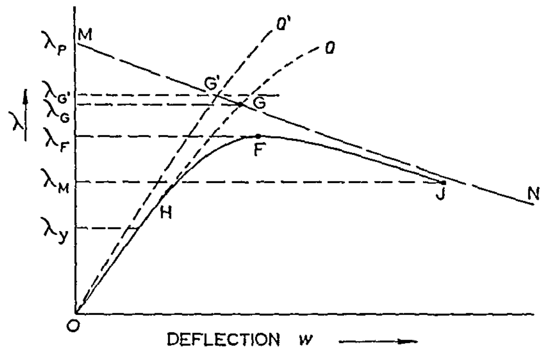

Fig. 20.

mechanism curve MN remains unchanged. This is possible because the deflections in the rigid-plastic mechanism remain proportional to each other. Constructing the new elastic response curve, the curve OQ' is obtained. In general, OQ' will not be coincident with OQ, and the intersection point G' gives a new load factor $\lambda_{G'}$ in place of λ_G. This indeterminacy may not be important in most cases, since the load factor λ_G is proposed merely as a means of obtaining an approximate estimate of the collapse load, but there may be frames in which widely divergent values of λ_G are obtainable according to the particular deflection chosen.

A final criticism of Merchant's approach is that he does not give a definition of the elastic critical load λ_C which he proposes to use as a parameter. It has been stated in § 13 above that the elastic critical loads for the two load distributions Fig. 15(a) and (b) are almost the same. If the critical load for axial loading [Fig. 15(b)] is that actually used in calculations, the error involved in applying this critical load to a frame loaded as in Fig. 15(a) cannot be at *all* significant, and for such a case, the lack of definition in Merchant's treatment is unimportant. In the case of a pitched roof portal frame, Fig. 16(a), neither of the axial loadings in Fig. 16(b) or (c) give a reasonable estimate of the critical load for the original loading pattern, and it appears that the elastic critical load for the true loading must be obtained. The critical loads for such load distributions are tedious to calculate. Moreover, the usefulness of such calculations as a means of estimating failure loads is open to doubt, since the values of the axial loads in the component members of the structure bear no proportionate relation to the axial loads in the real frame when this is on the point of collapse. Not only may the axial loads be disproportionate, but the gross distortion of the frame may render the elastic-critical load irrelevant. This would be so for the wide span portal frame in Fig. 21(a), of which the state when

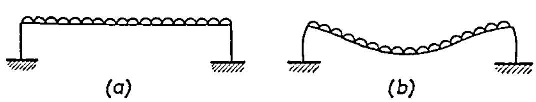

Fig. 21.

in the elastic critical condition might well be truly represented as in Fig. 21(b). For the majority of structures, the failure load is much nearer to the rigid-plastic collapse load than to the elastic critical load. Here a promising approach would seem to be to use the distribution of axial loads obtained at rigid-plastic collapse as the basis for calculating an idealised elastic critical load. Such a critical load is likely to be a more useful parameter in the approximate estimation of failure loads than the elastic critical load as used by Merchant.

§ 15. The Concept of Deteriorated Critical Loads

The work of Wood [1957/58] on the plastic stability of frames is notable in two respects. First, fairly complete results are obtained in the analysis of two four-storey frames, allowance being made for the spread of the plastic zones in actual rolled shapes. Secondly, Wood introduces the idea of deteriorated critical loads as theoretical tests for the stability of elastic-plastic structures.

One of the frames analysed by Wood is shown in Fig. 22(a), with 'working' values of the loads indicated. The members are all I-sections with webs in the plane of the frame. Wood obtained the following values for λ_C, λ_P and λ_F:

Elastic critical load factor	$\lambda_C = 12.9$
Rigid-plastic load factor	$\lambda_P = 2.15$
Theoretical failure load	$\lambda_F = 1.90$

The elastic critical load was calculated for beam loads equally divided between points, as shown in Fig. 22(b). To simplify the complete elastic-plastic analysis, half the distributed beam loads were concentrated at the mid-span of the beams and the other half at the joints. The rigid-plastic collapse mechanism, also derived for this load distribution, is shown in Fig. 22(c). The theoretical state of the elastic-plastic structure at collapse is shown in Fig. 22(d). At the theoretical

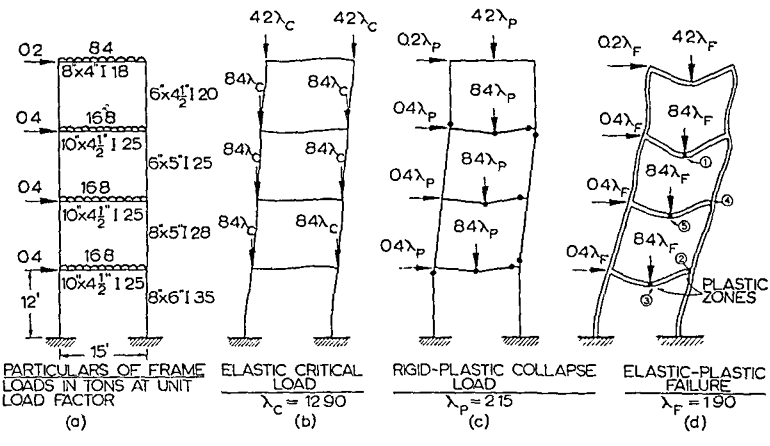

Fig. 22. Four-storey frame analysed by Wood (WOOD [1957/58]).

failure load factor of $\lambda_F = 1.90$, plastic hinges had formed at positions 1, 2, 3 and 4, and a fifth hinge had practically formed at 5. A certain amount of plastic deformation had also occurred at the other sections at which plastic zones are indicated.

Wood maintains that the limiting stability of the frame in the collapse condition (point F in Fig. 20) may be tested by considering the elastic critical load of the *deteriorated structure* under axial loading. The deteriorated structure is obtained by taking only that part of the actual structure which is still behaving elastically. At sections where, in the actual structure, plastic hinges have developed, there is no elastic rigidity with respect to further rotation, and so at the corresponding sections in the deteriorated structure a pin connection is assumed to exist. Wood calculated the elastic critical loads of his frame with pin points assumed at various sections, in order to obtain general guidance in his full analysis. Some of his results are shown in Figs. 23(a) to (e), the circles representing the positions of pin joints.

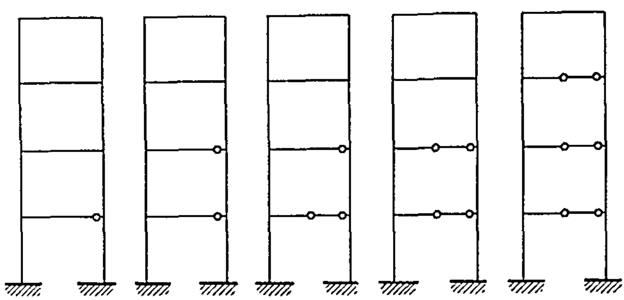

Fig. 23. Deteriorated critical loads of frame analysed by Wood (WOOD [1957/58]).

The load factors quoted represent the elastic critical loads of these deteriorated structures for vertical loads applied at the joints, as in Fig. 23(b).

None of the deteriorated structures correspond exactly to the deteriorated structure at the failure load in the full analysis [$\lambda_F = 1.90$, Fig. 22(d)]. A deteriorated structure intermediate between those in Figs. 23(d) and (e) is however seen to be appropriate [compare Fig. 22 (d)], and the deteriorated critical load factors 2.30 and 1.60 lie either side of the failure load factor 1.90.

It may be shown that Wood's concept of the deteriorated critical load is correct apart from the neglect of certain second order terms which are, for most structures, not very important. The terms that are neglected may be appreciated by reference to equation (3). This equation may be used to express the incremental behaviour of an elastic-plastic structure, provided the stiffness matrix Y allows for the reduced effective stiffness of the member on account of plasticity. Hence we have

$$\dot{\xi} = \dot{Y}\Delta + Y\dot{\Delta}\,. \tag{27}$$

When the structure undergoes an incremental deformation without change of load, $\dot{\xi} = 0$. Wood's concept of the deteriorated critical load is then derived by ignoring the first term on the righthand side, giving $Y = 0$ as the condition for instability. This is a valid approximation provided the deflections $\mathbf{\Delta}$ are small, Y being the deteriorated stiffness matrix for the partially plastic structure.

The work of Wood has provided an illuminating way of conceiving the behaviour of elastic-plastic structures under increasing loads. In the elastic range, the contemporary load factor λ will usually be well below the elastic critical load factor λ_C, and the structure will be stable. As plastic zones and hinges form, the stability of the structure has to be assessed by reference to the corresponding deteriorated structure and deteriorated critical load λ_D. As λ increases and more hinges form, λ_D decreases until, when $\lambda = \lambda_D$, the structure reaches its maximum load and failure occurs.

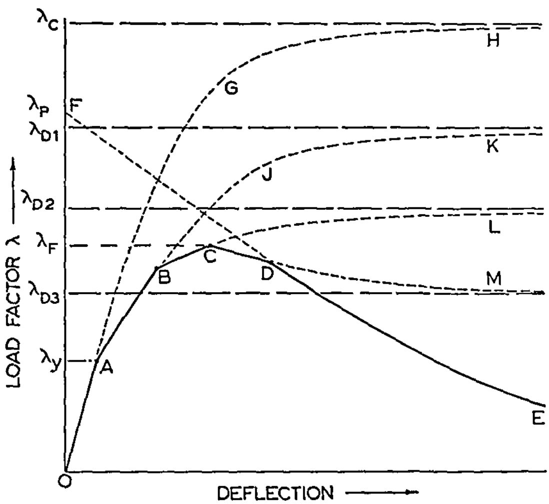

Fig. 24.

It is instructive to follow the load-deflection curve for a structure according to a unit shape factor analysis, interpreting the curve according to the deteriorated load concept. In Fig. 24, OAGH is the curve calculated on the assumption of indefinite elastic behaviour, and rises asymptotically to the line $\lambda = \lambda_C$ where λ_C is the lowest elastic critical load factor. The first plastic hinge is formed at A, defining the yield load factor λ_Y. Thereafter, until the next hinge forms at B, the load-deflection curve follows the curve AJK. This curve rises asymptotically towards the line $\lambda = \lambda_{D_1}$ where λ_{D_1} is the deteriorated elastic critical load. The deteriorated structure has a hinge at the section where, in the actual structure, a plastic hinge has formed. After B, the curve rises towards $\lambda = \lambda_{D_2}$ corresponding to the deteriorated structure with two hinges. In the case illustrated, the formation of the third hinge at C causes the deteriorated critical load factor to fall below the current value, and the load-deflection curve now approaches $\lambda = \lambda_{D_3}$ from above. Hence the point C represents the failure load. If a complete plastic mechanism occurs at the formation of the fourth hinge at D, then, provided the deflection recorded is that of a plastic hinge, D is a point on the mechanism line FE, and the subsequent load-deflection curve follows FE, becoming ultimately asymptotic to the line $\lambda = 0$.

While the concept of the deteriorated critical load is useful in the complete elastic-plastic analysis of a structure, it may be misleading if the attempt is made to use it apart from such an analysis. Thus it will be seen from Figs. 22 and 23 that, while ten plastic hinges are necessary for rigid-plastic collapse at $\lambda = 2.15$ [Fig. 22(c)], four only of these hinges are sufficient to bring about a more than five-fold reduction in the elastic-critical load, namely from 12.9 to 2.3 [Figs. 22(b) and 23(d)]. Judging from these figures only, one might be led to expect that the failure load λ_F would lie very considerably below the rigid-plastic value of 2.15. Actually, λ_F was found to be 1.90, or only 11.5% below.

§ 16. The Last Hinge Method for Estimating Failure Loads

The work of Wood and Merchant has mainly been concerned with exploring the behaviour of structures in which overall stability may be expected to cause large reductions in collapse load as compared with the rigid-plastic collapse load. If, as is very common, overall instability effects are small, a close or, quite frequently, a theoretically correct solution for the collapse load may be obtained by considering only the state of the structure when the theoretical rigid-plastic collapse mechanism has just formed. That such a procedure will always give a safe estimate of the collapse load may be appreciated from Fig. 20. The last hinge of the mechanism forms in this case at J on the load-deflection relation OHFJ, and this point must either be coincident with or below the maximum point on the load-deflection curve. The last hinge point gives the true failure load provided the deteriorated structure is elastically stable at the corresponding load factor. The deteriorated structure has pin joints at all save one of the plastic hinge positions of the rigid-plastic collapse mechanism. The state of the structure when the last plastic hinge just forms is more easily calculated than any state at an earlier stage of loading, since the bending moments are statically determinate for that part of the structure involved in the plastic collapse mechanism. The value of the load factor when the last hinge has just formed will be denoted by λ_M (Fig. 20).

There are essentially two stages in a last hinge analysis: the calculation of the deflected state of the structure and the calculation of the equilibrium requirements when the equilibrium state is known. Strictly speaking, these two stages are interdependent, but the bending moment distribution corresponding to the rigid-plastic collapse *mechanism is sufficiently accurate for a preliminary analysis. It is not* known generally which is the last hinge to form, and any hinge is selected in the first instance. As a result of a flexural analysis in which continuity is assumed at this arbitrarily chosen hinge, the hinge rotations at all the other hinge positions are calculated. If the hinge selected was actually the last to form, then each of the other hinge rotations will be in the same direction as the corresponding hinge rotation in the rigid-plastic mechanism. If one or more hinges have a rotation of the wrong sign, a set of hinge rotations corresponding to the rigid-plastic mechanism are added to the calculated deformation state, the magnitude of the additional mechanism deformation being just sufficient to eliminate total hinge rotations in the wrong direction. The hinge with zero rotation is then the 'last hinge', and the required deformation state has been obtained. Finally, the revised equilibrium conditions are established, enabling the calculation of the 'last hinge load factor' λ_M. The revised bending moments throughout the structure may be used in an iteration of the process until the closing error is sufficiently small, although in most cases in which a last hinge analysis is a reasonably close estimate of the true collapse load no revision of the first calculations is necessary.

HEYMAN [1957] has presented the results of a last hinge analysis for a whole range of pitched roof portal frames subjected to vertical load only, as shown in Fig. 25(a). The rigid-plastic collapse mechanism may theoretically be symmetrical, as in Fig. 25(b). Heyman assumed such a symmetrical mode, and took the apex hinge C as the last hinge in all his calculations. Unfortunately, in not all the frames for which he gave results would the hinge at C be the last to form. Moreover, in many frames, an unsymmetrical mode may develop, giving a lower collapse load than the symmetrical mode of deformation assumed by Heyman. Finally, Heyman did not make a complete equilibrium analysis, and obtained collapse loads which are in some cases very much too high.

The effect of finite deflections on the loads supported by a collapsing

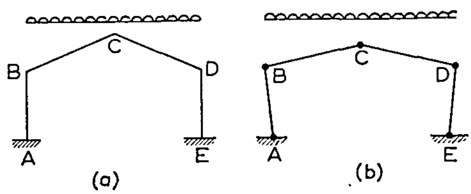

Fig. 25.

structure when it undergoes a finite deformation according to the rigid-plastic collapse mechanism was studied theoretically and experimentally by KING and JENKINS [1956]. The computing methods suggested by these authors are simply the setting down of the equilibrium requirements of each component member of the structure in its rigid-plastic deflected state, and solving as a set of simultaneous equations. The experimental work was carried out by simulating a rigid-plastic structure mechanically with comparatively rigid bars and artificial hinges capable of rotation under an almost uniform resisting moment. This work appears to have little relation to the real problem of elastic-plastic structures.

§ 17. Solution of Elastic-Plastic Structures by Digital Computer

LIVESLEY [1959] has developed a digital computer programme for deriving the elastic-plastic behaviour of rigid frame structures subjected to point loads. Theoretical failure loads may be derived by considering increments of displacement until the stage is reached at which the corresponding load factor begins to decrease. The analysis is subject to the following limitations.

(a) Loads are assumed to be concentrated, and applied only at joints. A member carrying a load at a point along its length must be treated as two members with a joint at the loading point.

(b) With the exception that rigid gusset plates can be allowed for, the position of the maximum moment in a member (and hence the position of a possible plastic hinge) is assumed to be at one or other of the ends.

(c) The spreading of plastic zones either side of a hinge is ignored, or in other words, the analysis is based on unit shape factor.

The effect of limitations (b) and (c) is to eliminate all consideration of local column instability. In frames subject to overall instability, columns are unlikely to fail by the development of plastic zones within their length, so it is safe to restrict attention to plasticity at the ends. Wood [1957/58] has demonstrated that the spreading of plastic zones has negligible effect on overall stability. Hence the limitations in Livesley's analysis are not at all serious.

§ 18. Conclusions

The most thorough studies of the overall elastic-plastic behaviour of individual frames have been made by Wood [1957/58]. This work has clarified ideas on the subject, and reveals the complexity of the problem analytically.

The use of computer programmes, such as that developed by Livesley, may in one sense solve the whole problem of the elastic-plastic behaviour of structures. Automatic computer methods do not necessarily lead, however, to a better understanding (on the part of those using the programmes, as opposed to those who develop them) of the structural significance of the phenomena investigated. Approximate methods, based on demonstrable structural principles, may be more instructive. Merchant's work, based as it is on the idealised rigid-plastic and elastic critical loads, is of interest here. In the main, this approach has been developed intuitively and by analogy, and it is desirable that some of Merchant's ideas should be subjected to a more searching examination than has yet been given.

In the work of most investigators on frame instability, there is some lack of clarity as to the extent to which second order terms are included in the analysis. This is true, for example, of Livesley's treatment of the elastic instability problem described in § 7, and also of Wood's use of the deteriorated critical load concept, described in § 15. Strictly speaking, all second order terms are important in stability problems, since the stability criterion itself may be expressed in terms of the second order terms in the virtual work equation (see § 11). Nevertheless, it is usual to neglect certain second order terms as negligible, and this problem requires investigation, particularly in the partially plastic range in which deflections become large.

References

ARIARATNAM, S. T., 1959, The Collapse Load of Elastic-Plastic Structures (Ph.D. Dissertation, Cambridge).

BAKER, J. F., M. R. HORNE and J. HEYMAN, 1956, The Steel Skeleton **2** (Cambridge University Press).

BAKER, J. F., M. R. HORNE and J. W. RODERICK, 1949, Proc. Roy. Soc. A **198** 493.

BAKER, J. F. and A. J. OCKLESTON, 1935, A.R.C. Technical Reports and Memoranda No. 1667.

BAKER, J. F. and J. W. RODERICK, 1942, Trans. Inst. Welding **5** 97.

BAKER, J. F. and J. W. RODERICK, 1948a, Welding Research **2** 2.

BAKER, J. F. and J. W. RODERICK, 1948b, Welding Research **2** 110.

BERRY, H., 1919, Trans. Roy. Aero. Soc., No. 1.

BLEICH, F., 1952, Buckling Strength of Metal Structures (McGraw-Hill) Ch. 2.

BOLTON, A., 1955, Structural Engineer **33** 90.

BOWLES, R. E. and W. MERCHANT, 1956, Structural Engineer **34** 324.

CHWALLA, E., 1938, Der Bauingenieur **19** 69.

EICKHOFF, K. G., 1955, The Plastic Behaviour of Columns and Beams (Ph.D. Thesis, Cambridge University).

FOULKES, J., 1953, The Analysis and Minimum Weight Design of Ductile Structures (Ph.D. Thesis, Cambridge University).

GREENBERG, H. J. and W. PRAGER, 1952, Trans. Amer. Soc. Civ. Engrs. **117** 447.

HEYMAN, J., 1957, Proc. Inst. Civ. Engr. **8** 119.

HODGE, P. G., 1959, Plastic Analysis of Structures (McGraw-Hill) Ch. 2.

HOFF, N. J., 1949, Research, Engineering Structures Supplement (Butterworths, London) p. 121.

HOFF, N. J., S. V. NARDO and B. ERIKSON, 1951, Proceedings Society for Experimental Stress Analysis **9**, No. 1, p. 201.

HORNE, M. R., 1949/50, J. Inst. Civ. Engrs. **34** 174.

HORNE, M. R., 1954, Proc. Inst. Civ. Engrs. Part III, **3** 51.

HORNE, M. R., 1956, Structural Engineer **34** 294.

JAMES, B. W., 1935, National Advisory Council for Aeronautics, Technical Note 534.

KING, J. W. H. and D. E. JENKINS, 1956, The Engineer **201** 320.

LIVESLEY, R. K., 1956, Structural Engineer **34** 1.

LIVESLEY, R. K., 1959, Symposium on the Use of Electronic Computers in Structural Engineering, University of Southampton.

LIVESLEY, R. K. and D. B. CHANDLER, 1956, Stability Functions for Structural Frameworks (Manchester University Press).

LOW, M. W., 1958/59, Proc. Inst. Civ. Engrs. **13** 287.

LUNDQUIST, E. E., 1939, Proc. 5th Int. Congr. App. Mechanics, Cambridge Mass. 1938 (Wiley, New York) p. 145.

MERCHANT, W., 1954, Structural Engineer **32** 185.

MERCHANT, W., 1955, Structural Engineer **33** 85.
MERCHANT, W., C. A. RASHID, A. BOLTON and A. SALEM, 1958, Proceedings, Fiftieth Anniversary Conference, Institution of Structural Engineers.
MURRAY, N. W., 1956, Proc. Inst. Civ. Engrs. Part III, **5** 213.
MURRAY, N. W., 1958, Proc. Inst. Civ. Engrs. **10** 503.
NEAL, B. G., 1956, The Plastic Methods of Structural Analysis (Chapman and Hall) Ch. 4.
NEAL, B. G. and P. S. SYMONDS, 1952, Proc. Inst. Civ. Engrs. Part III, **1** 58.
PIPPARD, A. J. S. and J. F. BAKER, 1957, The Analysis of Engineering Structures (Arnold).
RODERICK, J. W. and M. R. HORNE, 1948, Report FEI/11, British Welding Research Association.
SALEM, A., 1958, Structural Frameworks (Ph.D. Thesis, Manchester University).
SHANLEY, F. R., 1947, J. Aero. Sci. **14** 261.
SMITH, R. B. L. and W. MERCHANT, 1956, Structural Engineer **34** 285.
WINTER, G., P. T. HSU, B. KOO and M. H. LOH, 1948, Engineering Experimental Station Bulletin No. 36, Cornell University.
WOOD, R. H., 1957/58, Proc. Inst. Civ. Engrs. **11** 69.

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