largely to the low-percentage end of the distribution. Assuming that weather conditions within the antenna beam remain constant during the period required for a measurement at 5° and 20° elevations, the ratios of attenuations at the two elevations should correspond at the extreme abscissa values of the cumulative distribution to those of the slant-path lengths intercepted by the weather conditions of the two models, i.e. they should approach $\sin 20° : \sin 5° = 3{\cdot}9 : 1$ at the extreme high-percentage end of the cumulative distribution curve, and $\cos 20°/\cos 5° = 0{\cdot}94$ at the extreme low-percentage end. The intervening section of the curve could be expected to reflect a transition from one condition to the other, with a mean value near 2·4.

At both 11·75 and 17·0 GHz, the ratios show the suggested trend to a value of 3·9 over the 70–100% time interval. For small percentage times, however, there is the suggestion of a trend to a value of 0·94 only at 17·0 GHz. This may be explained by the events which contribute significantly to the cumulative distribution for such a small percentage time, being insufficient at 11·75 GHz to establish such a trend. For almost 70% of the time, the ratio at each frequency approximates quite closely to a mean value of about 2·5, implying that, taken over the whole observation period, as might be expected, for much of the time a nebulous climate exists in the slant paths with neither of the two simple tropospheric models dominant.

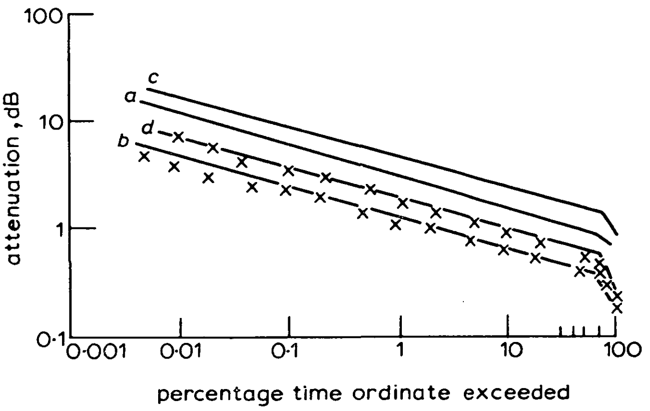

Fig. 2 *Measured and predicted attenuations at 5° and 20° elevations*

× × × = attenuations derived from noise measurements
a Straight-line approximation to measured results of Fig. 1 (curves *a* and *b*) for 5° elevation. Frequency = 11·75 GHz
b Predicted attenuatior at 20° elevation. Frequency = 11·75 GHz
c Straight-line approximation to measured results of Fig. 1 (curves *c* and *d*) for 5° elevation. Frequency = 17 GHz
d Predicted attenuation at 20° elevation. Frequency = 17 GHz

The results suggest that, given the cumulative distributions of attenuation at one elevation, an approximation to the distribution at any other elevation may be deduced through the following relationships, which, for simplicity, assume flat-earth geometry for the first of the tropospheric models and a column of infinite extent for the second. These assumptions are valid if neither very low nor very high angles are considered.

If θ_R is the reference angle of elevation and θ is the elevation at which the distribution is required, then, for values of percentage time $t = 100\%$, $a_\theta = a_{\theta R} \sin\theta_R \operatorname{cosec}\theta$, and, for $0{\cdot}01\% < t < 70\%$, $a_\theta = 2a_{\theta R}/(\cos\theta \sec\theta_R + \sin\theta \operatorname{cosec}\theta_R)$, and for the very small percentages of time assignable to vertical-column model, say $t \leqslant 0{\cdot}001\%$, $a_\theta = a_{\theta R}\cos\theta_R \sec\theta$, although insufficient data are available to establish values of $a_{\theta R}$ for percentages of time less than about 0·1. Predictions of slant-path attenuations at 11·75 GHz and 17·0 GHz derived by these relationships, and referred to the straight-line approximations to the measured cumulative distributions of attenuation for 5° elevation taken from Fig. 1, are given on Fig. 2. The experimental results for 20° elevation are also plotted, and show good agreement with the calculated curve over most of the range. Further measurements are needed at other elevations, and, in particular, extensive measurements for the smaller percentage times are required at all elevations to establish the range of use of these relationships.

D. TURNER

2nd August 1972

Post Office Research Department
Martlesham Heath
Ipswich, Suffolk, England

References

1 TURNER, D., JARVIS, E. G., and HINCHLIFFE, R. V.: 'Aerial noise-temperature measurements at 3·95, 11·75 and 17 GHz' *in* 'Earth station technology'. *IEE Conf. Publ.* 72, 1970, pp. 181–188
2 CRANE, R. K.: 'Propagation phenomena affecting satellite communication systems operating in the centimeter and millimeter wavelength bands', *Proc. Inst. Elec. Electron. Eng.*, 1971, **59**, pp. 173–188
3 IPPOLITO, L. J.: 'Propagation statistics for 15 and 32 GHz earth-space transmissions from the Applications Technology Satellite (ATS5)'. Proceedings of the Fall 1970 URSI Meeting, Columbus, Ohio, USA

LEAST-SQUARE-EXPONENTIAL APPROXIMATION

Indexing term: Least-squares approximations

A method is presented for solving the problem of fitting decay-type experimental data by sums of exponentials. Both the exponents and the coefficients of the exponentials are considered as unknowns. The novel idea presented here is to consider the sum of exponentials as a solution of an integral equation. A step-by-step procedure is given for the solution of the problem in the least-squares sense. An advantage of the proposed method is that the data need not be equidistant.

The problem of fitting decay-type experimental data by sums of exponentials is met frequently in a great variety of scientific studies. Some representative areas where this problem makes its appearance are the life sciences,[1,2] physics,[3] chemistry[4] and numerical analysis.[5,6] In electrical engineering, we meet this problem in time-domain network synthesis.[7,9] There, the problem is formulated as follows: given the desired impulse response $h(t)$ of a network and a positive integer n, find the constants a_i and λ_i of the real function

$$h_n(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t} \qquad (\operatorname{Re}\lambda_i \leqslant 0) \quad \ldots \quad (1)$$

so that the integral

$$\int_0^\infty \{h(t) - h_n(t)\}^2\, dt \quad \ldots \quad (2)$$

is a mimimum. A more general formulation of the exponential-approximation problem is the following: given (*a*) the values of a function $y(t)$ in the interval $0 \leqslant t \leqslant T$ and (*b*) that $y_n(t)$ has the form

$$y_n(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t} \qquad (\operatorname{Re}\lambda_i \leqslant 0) \quad \ldots \quad (3)$$

with n assumed known, find the $2n$ constants a_i and λ_i so that

$$\left\| y(t) - \sum_{i=1}^{n} a_i e^{\lambda_i t} \right\| \quad \ldots \quad (4)$$

is a minimum. The notation $\| \; \|$ indicates norm or distance, and, depending on the choice of a specific norm and the form in which $y(t)$ is given, we can distinguish several classes of exponential-approximation problems. It should be noted that, owing to the unknown exponents λ_i, this is a nonlinear approximation problem and little is known at present about its optimum solution.

In this letter, we present a method of solving a special case of the above problem that is the approximation of a function by a sum of exponentials in the least-squares sense. In what follows, we illustrate the basic idea of the proposed method for two exponentials, thus avoiding notational complexity; generalisation to the case of n exponentials is straightforward. We are, then, concerned here with the solution of the following specific problem: given measured values $y_m(t)$ of a function $y(t)$ in the interval $0 \leqslant t \leqslant T$ and that $y(t)$ has the form

$$y(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

determine the constants a_1, a_2 and λ_1, λ_2 so that the integral

$$I(a_1, a_2, \lambda_1, \lambda_2) \triangleq \int_0^\infty \{y_m(t) - y(t)\}\, dt \quad . \quad . \quad . \quad . \quad (6)$$

assumes its minimum value. Note that the subscript m in $y_m(t)$ is used to signify measured values and that the corresponding values of $y(t)$ and $y_m(t)$ in $0 \leqslant t \leqslant T$ are, in general, different, owing to errors in the measurements. Since differentiation of measured data could greatly increase the errors in the measurements, we make it part of the formulation of the problem that differentiation of $y_m(t)$ is not allowed. The solution proposed here consists mainly of two observations; one which is frequently made and one which constitutes the novel idea of this letter. It has been observed[6] that eqn. 5 can be considered as a solution of the differential equation

$$\frac{dy^2}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

with initial conditions

$$y(0) \triangleq c_0 \qquad \left.\frac{dy}{dt}\right]_{t=0} \triangleq c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This observation implies that the λ_1 and λ_2 of eqn. 5 are the roots of

$$\lambda^2 + b_1 \lambda + b_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and that

$$\left.\begin{aligned} a_1 \quad + a_2 \quad &= c_0 \\ a_1 \lambda_1 + a_2 \lambda_2 &= c_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We now make the observation that eqn. 5 can also be considered as a solution of the equation

$$y(t) - c_0 - c_1 t + b_1 \left\{ \int_0^t y(\tau)\, d\tau - c_0 t \right\} + b_0 \left\{ \int_0^t \int_0^x y(\tau)\, d\tau\, dx \right\} = 0 \quad (11)$$

which is obtained from eqn. 7 by integrating twice with respect to time. With the definitions

$$\int_0^t y(\tau)\, d\tau \triangleq Y_1(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\int_0^t \int_0^x y(\tau)\, d\tau\, dx \triangleq Y_2(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

eqn. 11 can be written as

$$y(t) = c_0 + d_1 t - b_1 Y_1(t) - b_0 Y_2(t) \quad . \quad . \quad . \quad . \quad (14)$$

with

$$d_1 \triangleq c_1 + b_1 c_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

eqn. 6 then becomes

$$\int_0^\infty [y_m(t) - \{c_0 + d_1 t - b_1 Y_1(t) - b_0 Y_2(t)\}]^2\, dt = \text{minimum} \quad (16)$$

The constants c_0, d_1, b_1 and b_0 which minimise this integral are given by the solution of the well known normal equations of least squares[8]

$$\left.\begin{aligned} \langle 1, 1\rangle c_0 + \langle t, 1\rangle d_1 - \langle Y_1, 1\rangle b_1 - \langle Y_2, 1\rangle b_0 &= \langle y_m, 1\rangle \\ \langle 1, t\rangle c_0 + \langle t, t\rangle d_1 - \langle Y_1, t\rangle b_1 - \langle Y_2, t\rangle b_0 &= \langle y_m, t\rangle \\ \langle 1, Y_1\rangle c_0 + \langle t, Y_1\rangle d_1 - \langle Y_1, Y_1\rangle b_1 - \langle Y_2, Y_1\rangle b_0 &= \langle y_m, Y_1\rangle \\ \langle 1, Y_2\rangle c_0 + \langle t, Y_2\rangle d_1 - \langle Y_1, Y_2\rangle b_1 - \langle Y_2, Y_2\rangle b_0 &= \langle y_m, Y_2\rangle \end{aligned}\right\} (17)$$

where the notation $\langle\, ,\rangle$ indicates the inner product of two functions in the interval $[0, T]$

$$\left(\text{e.g.} \quad \langle f(t), g(t)\rangle \triangleq \int_0^T f(t)\, g(t)\, dt\right)$$

The steps in the solution could then be summarised as follows:

(i) from the given $y_m(t)$, compute $Y_1(t)$ and $Y_2(t)$ numerically
(ii) compute the inner products in eqn. 17
(iii) solve eqn. 17 for c_0, d_1, b_1 and b_0
(iv) find c_1 from eqn. 15
(v) find λ_1 and λ_2 from eqn. 9
(vi) find a_1 and a_2 from eqn. 10.

The advantages of the method are that the given data are integrated, thus averaging out errors, and that, if $y_m(t)$ is given in discrete form, its values do not have to be equidistant. Further work is required to study the numerical aspects of the method and the amount of noise that can be tolerated in the data for a given number of exponentials. Numerical experiments are in progress, and their results will be reported in the near future.

J. E. DIAMESSIS

7th August 1972

Department of Electrical Engineering
National Technical University
Athens 147, Greece

References

1 SHEPPARD, C. W.: 'Basic principles of the tracer method' (Wiley, 1962)
2 ROBERTSON, J. S.: 'Theory and use of tracers in determining transfer rates in biological systems', *Physiol. Rev.*, 1957, **37**, p. 133
3 KEEPIN, G. R., WIMETT, T. F., and ZEIGLER, R. K.: 'Delayed neutrons from fissionable isotopes of uranium, plutonium and thorium', *J. Nucl. Energy*, 1957, **6**, pp. 1–21
4 GARDNER, D. G., GARDNER, J. C., LAUSH, G., and MEINKE, W. W.: 'Method for the analysis of multicomponent decay curves', *J. Chem. Phys.*, 1959, **31**, p. 978
5 HAMMING, R. W.: 'Numerical methods for scientists and engineers' (McGraw-Hill, 1962)
6 FRÖBERG, C. E.: 'Introduction to numerical analysis' (Addison-Wesley, 1965), p. 289
7 GUILLEMIN, A. E.: 'Synthesis of passive networks' (Wiley, 1957)
8 RICE, J. R.: 'The approximation of functions' (Addison-Wesley, 1964)
9 SU, K. L.: 'Time-domain synthesis of linear networks' (Prentice-Hall, 1971)

VARIATION OF PULSE DELAY WITH LAUNCH ANGLE IN A LIQUID-FILLED FIBRE

Indexing term: Fibre optics

Pulse measurements on a 400 m length of liquid-filled fibre show increasing delay with launch angle, generally consistent with geometric-ray calculations.

In the study of multimode transmission in optical fibres, variation of pulse delay with angle of launching is a useful clue to the nature of pulse dispersion itself. Although measurements have been reported of both pulse broadening[1] and delay[2] with change of launching conditions, rather short fibre lengths have been used. A longer test length, which is made possible by the advent of low-loss liquid-filled fibres, allows a more meaningful measurement for two reasons: