Exercises (Lecture 1)

Basic definitions

Introduction to Operads

Note. We will solve some of these exercises during the exercise sessions. Try to solve at least one exercise you find easy and at east two exercises that you find challenging.

Exercise 1. Follow the lecture notes and read about the partial definition of an operad (and what a Markl operad is). Show that a unital pseudo-operad is the same as a unital May operad.

Exercise 2. Define the category of collections in Vect using the biased approach and the unbiased approach (this requires considering *totally ordered* sets instead of sets, and their order preserving bijections. We will write them with calligraphic letters but use subscripts, so $\mathscr{X}$ has ns components $\mathscr{X}_n\}_{n\geqslant 1}$.

1. Show that it supports a non-symmetric Cauchy product given by
$$(\mathscr{X}\otimes\mathscr{Y})_n = \bigoplus_{i+j=n} \mathscr{X}_i \otimes \mathscr{Y}_j.$$
2. Use this and the unbiased approach to argue that the ns counterpart of a 'subset of I' is an interval: a totally ordered subset of I of the form $[i,j] = \{x \in I : i \leqslant x \leqslant j\}$.
3. Use the previous item to define the non-symmetric composition of ns collections. Define the generating function associated to a collection, and show it behaves well with respect to the products above.

Exercise 3. Since every finite totally ordered set is, in particular, a finite set (and every order preserving function is a fortiori a function) there is a map of categories $\mathsf{FinOrd}^\times \longrightarrow \mathsf{FinSet}^\times$ which induces a map that 'forgets the symmetries' ${}_\Sigma\mathsf{Mod} \longrightarrow \mathsf{Coll}$. Show that there is a functor that assigns a ns sequence $\mathscr{X}$ to the sequence $\mathscr{X}_\Sigma(n) = \Bbbk S_n \otimes \mathscr{X}_n$ which is left adjoint and monoidal.

Exercise 4. Describe the associator for $\circ_\Sigma$ in the category of differential graded collections. In particular, write down the signs explicitly. Explain how this is related to the signs in the parallel composition axiom for *graded operads* that read as follows: for elements f, g and h in an operad (of homogeneous arities) and $\delta = i - j + 1$, we have that

$$(f\circ_j g)\circ_i h = \begin{cases} (-1)^{|g||h|}(f\circ_i h)\circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f\circ_j (g\circ_\delta h) & \delta\in[1,\mathrm{ar}(g)] \\ (-1)^{|g||h|}(f\circ_\delta h)\circ_j g & \delta > \mathrm{ar}(g). \end{cases}$$

Exercise 5. A (unital associative) monoid x in a monoidal category $(\mathscr{C}, \otimes, \alpha, \rho, \lambda, 1)$ is an object along with maps $\mu : x \otimes x \to x$ and $\eta : 1 \longrightarrow x$ such that μ is associative, that is $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)\alpha_{x,x,x}$, and unital for η, that is $\mu(\eta \otimes 1) = \rho_x$ and $\mu(1 \otimes \eta) = \lambda_x$. Show that a Σ-operad is exactly the same as a monoid in $({}_{\Sigma}\mathsf{Mod}, \circ_{\Sigma})$.

Exercise 6. We write End for category of endofunctors of Vect. Show that there is a *monoidal* functor $S : {}_{\Sigma}\mathsf{Mod} \longrightarrow \mathsf{End}$ that assigns $\mathscr{X}$ to $V \longmapsto \bigoplus_{n \geqslant 0} \mathscr{X}(n) \otimes_{\Sigma_n} V^{\otimes n}$. It is called the *Schur functor* associated to $\mathscr{X}$. The endofunctors in the essential image of S are called *analytic*.

Exercise 7. If $\mathscr{X}$ is a symmetric sequence, describe the Σ_n action on $\mathscr{X}^{\otimes n}$ where $\otimes$ is the Cauchy product. Observe that it commutes with the $\mathrm{Aut}(I)$ action on $\mathscr{X}^{\otimes n}(I)$.

Exercise 8. Define ${}_{\Sigma}\mathsf{Mod}(\mathscr{C})$ for any symmetric monoidal category $(\mathscr{C}, \otimes, 1)$ (such as the category of sets, or topological spaces, or chain complexes, among others) along with its *symmetric composition product* $- \circ_{\Sigma} -$.

Exercise 9. Prove that non-unital Markl operads and non-unital May operads differ. To do this, consider the non-unital ns operad $\mathscr{P}$ such that $\mathscr{P}(2)$ and $\mathscr{P}(4)$ are its only non-zero components, and are both one dimensional, and define

$$\gamma : \mathscr{P}(2) \otimes \mathscr{P}(2) \otimes \mathscr{P}(2) \longrightarrow \mathscr{P}(4)$$

to be an isomorphism, and all other maps zero. Check that $\mathscr{P}$ is a May operad, and show that $\mathscr{P}$ is not a Markl operad by exploring the consequences of the equality

$$\mu(\mu, \mu) = (\mu \circ_2 \mu) \circ_1 \mu$$

in any Markl operad.

Exercise 10. Check that examples (1), (2), (4), (5) in page 10 are indeed all operads.

Exercise 11. Follow these steps to construct the Stasheff operad as a sequence of convex polytopes $K_2', K_3', \ldots$ for which the boundary of K_{n+1} is a union of products $K_{r+1} \times K_{s+1}$ with $r + s = n$ indexes by planar rooted trees with two internal vertices.

1. Let us write T_n for the collection of planar rooted *binary* trees with $n+1$ leaves, which we order from left to right. Explain how this gives a total order on the vertices, which we will thus call $1, \ldots, n$.
2. For each $t \in T_n$ and each vertex i of t, let $L(i)$ denote the number of paths from i to a leaf of t going through its left child, and let $R(i)$ denote the the number of paths from i to a leaf of t going through its right child. We define

   $$x(t) = (L(1)R(1), \ldots, L(n)R(n)) \in \mathbb{N}^n.$$

   Show that $x(t)$ always lies in the hyperplane $x_1 + \cdots x_n = \binom{n}{2}$. We write K_{n+2}' for the convex hull of the points $\{x(t) : t \in T_n\}$. *Hint.* Any planar binary rooted tree t decomposes into a left tree L_t and a right tree R_t by looking at the children of the unique child of the root.

Express $W(t) = \sum_{i=1}^{n} x(t)$ in terms of $W(R_t)$ and $W(L_t)$.

3. Show that the polytope K'_{n+2} is of dimension n, and its k-cells for $k \in [n]$ are in bijection with planar rooted trees with $n-k+1$ internal vertices and $n+2$ leaves. Conclude, in particular, that its codimension one faces are in bijection with planar rooted trees with 2 internal vertices and $n+2$ leaves.

4. Suppose that t has $r+1$ leaves and that t' has $s+1$ leaves, and consider the grafting $t \circ_i t'$. We define $x(t) \circ_i x(t')$ by $x(t \circ_i t')$. Show that this defines a map

$$\circ_i : K'_{r+1} \times K'_{s+1} \longrightarrow K'_{r+s+1}.$$

5. Show the maps above give the collection $\{K'_{n+2}\}_{n \geqslant 0}$ the structure of a ns operad.

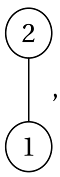
Exercise 12. Suppose that $T \in \mathsf{RT}(n)$ and that $T' \in \mathsf{RT}(m)$, where RT is the operad of rooted trees of Lecture 1, and let $\mathrm{In}(T, i)$ denote the set of incoming edges of T at the vertex labeled i. For each function $f : \mathrm{In}(T, i) \longrightarrow [m]$, define the tree $T \circ_i^f T'$ by replacing vertex i of T by T' and attaching the loose incoming edges of vertex i to the vertices of T' according to the map f: the edge $e \in \mathrm{In}(T, i)$ is attached to vertex $f(e) \in T'$. Finally, define $T \circ_i T'$ by taking the sum through all possible functions f. Show that this gives RT the structure of a unital pseudo-operad, and thus of a usual operad, with unit the tree with no edges and one vertex.

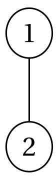
Exercise 13. Describe the operation $T \star T' = S(T, T')$ where S is the rooted tree above in terms of insertions of T' in T and regrafting of incoming edges. Show that it satisfies the following *pre-Lie identity*:

$$(T \star T') \star T'' - T \star (T' \star T'') = (T \star T'') \star T' - T \star (T'' \star T')$$

by explicitly interpreting the left hand side in terms of certain insertions of T' and T'' in T, and showing the resulting sum of trees is symmetric in T' and T''.

Exercise 14. Suppose that $\mathscr{P}$ is an operad and that $\mathscr{X} \subseteq \mathscr{P}$ is a symmetric subsequence. We say $\mathscr{X}$ generates $\mathscr{P}$ if every element of $\mathscr{P}$ is an iterated composition of elements of $\mathscr{X}$. Show that the rooted trees operad RT is generated by the symmetric subsequence given by the two labeled rooted trees with two vertices:

$S =$ 

, $S(12) =$ 

spanning the regular representation of S_2. Follow these steps:

1. Suppose that T is an n-rooted tree and let J be a subset of $[n]$ corresponding to leaves of T that are the children of a vertex $i \in T$. Let T' be the tree obtained by erasing all these leaves and replacing the vertex label by a new symbol $*$, and let T'' be the rooted tree with root i and children labeled by J. Show that $T' \circ_* T'' = T$.

2. Use the above and induction on the number of vertices to show it suffices to prove the

claim for the corollas, that is, trees with one internal root vertex.

3. Let us write T_n for the operation in $\mathsf{RT}(n)$ corresponding to a corolla with root 1, so in particular $T_2 = S$. Show that

$$T_n = T_2 \circ_1 T_{n-1} - \sum_{i=1}^{n-1} (T_{n-1} \circ T_i)\sigma_i$$

where $\sigma_i = (i+1, i+2, \ldots, n) \in S_n$ is a cycle, and use this to conclude.

Note. The operation T_n is usually denoted $\{x_1; x_2, \ldots, x_n\}$ and is called a *symmetric brace*, and the equation above is usually written in the form

$$\{x_1; x_2, \ldots, x_n\} = \{\{x_1; x_2, \ldots, x_{n-1}\}; x_n\} - \sum_{i=1}^{n-1} \{x_1; x_2, \ldots, x_{i-1}, \{x_i; x_n\}, x_{i+1}, \ldots, x_{n-1}\}.$$

Exercise 15. Let $\mathscr{X}$ be a symmetric sequence, and define the derivative $\partial\mathscr{X}$ of $\mathscr{X}$ to be symmetric sequence with $(\partial\mathscr{X})(I) = \mathscr{X}(I^*)$ where $I^* = I \sqcup \{I\}$. Note that S_I acts on I^* fixing the element I. Show that $(\partial\mathscr{X})(n)$ is isomorphic to the restriction of $\mathscr{X}(n+1)$ to $S_n = \mathrm{Fix}(n+1)$, and conclude that

$$\partial_z f_{\mathscr{X}}(z) = f_{\partial\mathscr{X}}(z).$$

Let s be the sequence of singletons and define the pointing of operation by $\mathscr{X}^\bullet = s \otimes_\Sigma \partial\mathscr{X}$. Determine the representation $\mathscr{X}^\bullet(n)$ in terms of $\mathscr{X}(n)$.

Exercise 16. Find two non-isomorphic symmetric sequences with the same generating function.