Properties of quasi-projective dimension over abelian categories

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Abstract

Quasi-projective dimension was introduced by Gheibi, Jorgensen and Takahashi to generalize the Auslander-Buchsbaum formula and the depth formula in commutative algebra. In this paper, we establish some basic properties of quasi-projective dimensions of objects in abelian categories. Analogous to global dimension of rings, we also introduce the concept of quasi-global dimension for left Noetherian rings, and then compare quasi-global dimension with global dimension for a class of Nakayama algebras. This provides new examples of finite-dimensional algebras with finite quasi-global dimensions but infinite global dimensions.

1 Introduction

In the representation theory of algebras and homological algebra, projective dimensions of objects in abelian categories plays a very important role. In [8], Gheibi, Jorgensen and Takahashi introduced a generalization of projective dimension, called *quasi-projective dimension*. This new dimension not only supplies a general framework to establish the Auslander–Buchsbaum formula and the depth formula in commutative algebra for modules of finite quasi-projective dimension (see [9]), but also gives an equivalent characterization of equipresented, local, complete intersection rings in term of finite quasi-projective dimension (see [3]). Dual to quasi-projective dimensions, quasi-injective dimensions of modules over rings were introduced to provide extensions of the Bass's formula and the Chouinard's formula in commutative algebra for modules of finite quasi-injective dimensions (see [7, 12]).

We start from recalling some elementary properties of quasi-projective dimensions of objects in abelian categories established in [8]. Throughout this section, let \mathcal{A} be an arbitrary abelian category with enough projective objects. For each object $M \in \mathcal{A}$, we denote by $\operatorname{pd}_{\mathcal{A}} M$, $\operatorname{qpd}_{\mathcal{A}} M$ and $\Omega^n(M)$ its projective dimension, quasi-projective dimension and n-th syzygy for $n \ge 0$ in \mathcal{A} , respectively. For the definitions of quasi-projective resolution and quasi-projective dimension, we refer to Definition 2.1.

Theorem 1.1. [8, Propositions 3.3 and 3.6(2)] The following statements hold for an object $M \in \mathcal{A}$.

- (1) $\operatorname{qpd}_{\mathcal{A}}(M \oplus N) \leq \sup \{\operatorname{qpd}_{\mathcal{A}}M, \operatorname{qpd}_{\mathcal{A}}N\}$ for any object $N \in \mathcal{A}$. In particular, if N is projective, then $\operatorname{qpd}_{\mathcal{A}}(M \oplus N) \leq \operatorname{qpd}_{\mathcal{A}}M$.
 - (2) $\operatorname{qpd}_{\mathcal{A}} \Omega(M) \leqslant \operatorname{qpd}_{\mathcal{A}} M$.
 - (3) If M is periodic, that is, there exists a positive integer r such that $\Omega^r(M) \simeq M$, then $\operatorname{qpd}_{\mathfrak{A}} M = 0$.

From Theorem 1.1, we can see some obvious differences between projective dimension and quasi-projective dimension. For example, the inequalities in Theorem 1.1(1) are equalities if quasi-projective dimension is replaced by projective dimension, while a periodic, non-projective object has infinite projective dimension.

The paper is devoted to providing more homological properties for quasi-projective dimensions of objects in abelian categories. Our main result reads as follows.

Theorem 1.2. Let \mathcal{A} be an abelian category with enough projective objects. The following statements hold for an object $M \in \mathcal{A}$.

(1) If $\operatorname{pd}_{\mathfrak{A}} M < \infty$, then $\operatorname{qpd}_{\mathfrak{A}} M = \operatorname{pd}_{\mathfrak{A}} M$.

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²⁰²⁰ Mathematics Subject Classification: Primary 16E10, 18G20, 16G10; Secondary 18E10, 16E35.

Keywords: Quasi-global dimension; Quasi-projective dimension; Nakayama algebra; Self-orthogonal module.

- (2) If $\operatorname{qpd}_{\mathcal{A}} M < \infty$ and $\operatorname{Ext}_{\mathcal{A}}^n(M,M) = 0$ for all $n \ge 2$, then $\operatorname{pd}_{\mathcal{A}} M < \infty$.
- (3) If $0 \to N \to E \to M \to 0$ is an exact sequence in \mathcal{A} such that E is projective-injective, then

$$\operatorname{qpd}_{\mathcal{A}} M \leqslant \operatorname{qpd}_{\mathcal{A}} N + 1.$$

(4) Let P_{\bullet} be a deleted projective resolution of M in \mathcal{A} . Suppose that there is an integer $n \ge 2$ and a chain map $g_{\bullet}: P_{\bullet} \to P_{\bullet}[n]$ of complexes over \mathcal{A} :

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

$$\downarrow^{g_{n+1}} \qquad \downarrow^{g_n} \qquad \downarrow^0 \qquad \downarrow^0$$

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

inducing an isomorphism $\Omega^{n+m}(M) \to \Omega^m(M)$ for some integer $m \geqslant 0$. Then $\operatorname{qpd}_{\mathcal{A}} M \leqslant m$.

Note that Theorem 1.2(1) and Theorem 1.2(2) have been shown for quasi-projective dimensions of finitely generated modules over commutative Noetherian rings by using some techniques from commutative algebra; see [8, Corollary 4.10 and Theorem 6.20]. Theorem 1.2(3)(4) are completely new (even in the case of commutative rings) and provide two effective methods for bounding quasi-projective dimensions (see Example 3.10). Note that Theorem 1.2(3) does not hold when the object E is projective but not injective; see Example 3.7 for a counterexample.

An interesting application of Theorem 1.2(2) is the validity of *the Auslander–Reiten conjecture* for modules of finite quasi-projective dimension over left Noetherian rings. Recall that the conjecture says that if a finitely generated R-module M over a left Noetherian ring R satisfies $\operatorname{Ext}_R^n(M, M \oplus R) = 0$ for all $n \ge 1$, then it is projective.

We formulate the following corollary for left coherent rings which are a generalization of left Noetherian rings.

Corollary 1.3. Let R be a left coherent ring. Suppose that M is a finitely presented left R-module with finite quasi-projective dimension. If $\operatorname{Ext}_{R}^{n}(M,M\oplus R)=0$ for all $n\geqslant 1$, then M is projective.

Analogous to global dimension, we introduce the notion of quasi-global dimension for left Noetherian rings and establish some basic properties of this new dimension in Proposition 4.2. In particular, we show that *Tachikawa's second conjecture holds for any self-injective Artin algebra with finite quasi-global dimension (or equivalently, with quasi-global dimension 0)*; see Corollary 4.3 and Remark 4.4. It is worth mentioning that if two finite-dimensional self-injective algebras over a field are stably equivalent of Morita type or derived equivalent, then they have the same quasi-global dimension (see Corollary 4.6). For other new advances on Tachikawa's second conjecture, we also refer the reader to [4, 6].

Further, using Theorem 1.2, we can also determine the quasi-global dimensions of a class of finite-dimensional Nakayama algebras (see Theorem 4.8 for more details). Consequently, those algebras always have finite quasi-global dimension, although they may have infinite global dimension in some cases.

The paper is organized as follows. In Section 2, we recall some definitions and basic facts on quasi-projective dimensions of objects in abelian categories. In Section 3, we show Theorem 1.2. In Section 4, we discuss quasi-global dimension for left Noetherian rings and calculate this dimension for a class of Nakayama algebras.

2 Preliminaries

In this section, we introduce some standard notation and recall the definitions of quasi-projective resolution and quasi-projective dimensions of objects in abelian categories from [8].

All rings considered in this paper are assumed to be associative and with identity and all modules are unitary left modules.

Let R be a ring. We denote by R-Mod the category of all R-modules and by R-mod the category of all finitely presented R-modules. For an R-module M, let $\operatorname{add}(M)$ (respectively, $\operatorname{Add}(M)$) be the full subcategory of R-Mod consisting of all direct summands of finite (respectively, arbitrary) direct sums of copies of M. In many circumstances, we write R-proj and R-Proj for $\operatorname{add}(R)$ and $\operatorname{Add}(R)$, respectively. When R is *left coherent*, that is, every finitely generated left ideal of R is finitely presented, the category R-mod is an abelian category and has R-proj as its full subcategory consisting of all projective objects.

Let \mathcal{A} be an additive category. By a *complex* $X_{\bullet} := (X_i, d_i^X)_{i \in \mathbb{Z}}$ over \mathcal{A} , we mean a sequence of morphisms d_i^X between objects X_i in \mathcal{A} :

$$\cdots \xrightarrow{d_{i+2}^X} X_{i+1} \xrightarrow{d_{i+1}^X} X_i \xrightarrow{d_i^X} X_{i-1} \xrightarrow{d_{i-1}^X} \cdots,$$

such that $d_i^X d_{i+1}^X = 0$ for $i \in \mathbb{Z}$. For simplicity, we sometimes write $(X_i)_{i \in \mathbb{Z}}$ for X_{\bullet} without mentioning d_i^X . For a fixed integer n, we denote by $X_{\bullet}[n]$ the complex obtained from X_{\bullet} by shifting n degrees, that is, $(X_{\bullet}[n])_i = X_{i-n}$. Let $\mathscr{C}(\mathcal{A})$ be the category of all complexes over \mathcal{A} with chain maps, and let $\mathscr{K}(\mathcal{A})$ be the homotopy category of $\mathscr{C}(\mathcal{A})$. For a chain map $f_{\bullet}: X_{\bullet} \to Y_{\bullet}$ in $\mathscr{C}(\mathcal{A})$, we denoted by $\operatorname{Con}(f_{\bullet})$ the mapping cone of f_{\bullet} . When \mathcal{A} is an abelian category, we also define the i-th cycle, boundary and homology of X_{\bullet} for each i as follows:

$$Z_i(X_{\bullet}) := \operatorname{Ker}(d_i^X), \quad B_i(X_{\bullet}) := \operatorname{Im}(d_{i+1}^X) \quad \text{and} \quad H_i(X_{\bullet}) := Z_i(X_{\bullet})/B_i(X_{\bullet}).$$

Moreover, the supremum, infimum, homological supremum and homological infimum of X_{\bullet} are defined by

$$\sup(X_{\bullet}) := \sup\{i \in \mathbb{Z} \mid X_i \neq 0\}, \quad \inf(X_{\bullet}) := \inf\{i \in \mathbb{Z} \mid X_i \neq 0\},$$

$$\operatorname{hsup}(X_{\bullet}) := \sup\{i \in \mathbb{Z} \mid H_i(X_{\bullet}) \neq 0\}, \quad \operatorname{hinf}(X_{\bullet}) := \inf\{i \in \mathbb{Z} \mid H_i(X_{\bullet}) \neq 0\}.$$

Clearly, $\inf(X_{\bullet}) \leq \inf(X_{\bullet}) \leq \sup(X_{\bullet}) \leq \sup(X_{\bullet})$. We say that X_{\bullet} is *bounded* (respectively, *bounded below*) if $\sup(X_{\bullet}) < \infty$ and $\inf(X_{\bullet}) > -\infty$ (respectively, $\inf(X_{\bullet}) > -\infty$). Let $\mathscr{C}^{-}(\mathcal{A})$ and $\mathscr{C}^{b}(\mathcal{A})$ be the categories of all bounded below and bounded complexes over \mathcal{A} , respectively. Their homotopy categories are denoted by $\mathscr{K}^{-}(\mathcal{A})$ and $\mathscr{K}^{b}(\mathcal{A})$, respectively.

When \mathcal{A} is an abelian category, the bounded below derived category of \mathcal{A} is denoted by $\mathcal{D}^-(\mathcal{A})$, which is the localization of $\mathcal{K}^-(\mathcal{A})$ at all quasi-isomorphisms. Moreover, the *extension* of two full subcategories X and Y in $\mathcal{D}^-(\mathcal{A})$, denoted by $X * \mathcal{Y}$, is defined to be the full subcategory of $\mathcal{D}^-(\mathcal{A})$ consisting of all objects Z such that there exists a distinguished triangle $X \to Z \to Y \to X[1]$ in $\mathcal{D}^-(\mathcal{A})$ with $X \in X$ and $Y \in \mathcal{Y}$. Normally, we identify \mathcal{A} with the full subcategory of $\mathcal{D}^-(\mathcal{A})$ consisting of all stalk complexes concentrated in degree 0.

From now on, let \mathcal{A} be an abelian category with enough projective objects.

Let M be an object of \mathcal{A} . We denote by $\operatorname{pd}_{\mathcal{A}}M$ the projective dimension of M in \mathcal{A} . Let

$$P_{\bullet}: \cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

be a deleted projective resolution of M in \mathcal{A} . For each integer $i \ge 1$, the i-th syzygy of M (with respect to P_{\bullet}) is defined to be $\Omega^i(M) := B_{i-1}(P_{\bullet})$. As usual, we write $\Omega(M)$ for $\Omega^1(M)$, and understand that $\Omega^0(M) = M$. The object M is said to be *periodic* if there exists an integer $r \ge 1$ such that $\Omega^r(M) \simeq M$. When $\mathcal{A} = R$ -Mod, we simply write $pd_R M$ for $pd_{\mathcal{A}} M$.

The following definition is taken from [8, Definition 3.1].

Definition 2.1. (1) A complex $P_{\bullet} \in \mathscr{C}^{-}(\mathcal{A})$ is called a quasi-projective resolution of M if the following two conditions hold:

- (a) all P_i are projective for $i \in \mathbb{Z}$;
- (b) for all integers $j \ge \inf(P_{\bullet})$, there are integers $n_j \ge 0$, not all zero, such that $H_j(P_{\bullet}) \simeq M^{n_j}$. If, in addition, $\sup(P_{\bullet}) < \infty$ (that is, $P_{\bullet} \in \mathscr{C}^b(\mathcal{A})$), then P_{\bullet} is called a finite quasi-projective resolution of M.
 - (2) The quasi-projective dimension of a nonzero object M in \mathcal{A} is defined to be

$$\operatorname{qpd}_{\mathfrak{A}} M := \inf\{\sup(P_{\bullet}) - \operatorname{hsup}(P_{\bullet}) | P_{\bullet} \text{ is a finite quasi-projective resolution of } M\}.$$

We understand that the quasi-projective dimension of the zero object is 0 (compared with [8, Definition 3.1(2)]).

By Definition 2.1, $\operatorname{qpd}_{\mathcal{A}} M = \infty$ if and only if M does not have a finite quasi-projective resolution. Since each deleted projective resolution of M is automatically a quasi-projective resolution, it is clear that $\operatorname{qpd}_{\mathcal{A}} M \leq \operatorname{pd}_{\mathcal{A}} M$. The inequality can be strict, for example, if M is periodic but non-projective (see Theorem 1.1(3)). Moreover, by shifting quasi-projective resolutions, we see that $\operatorname{qpd}_{\mathcal{A}}(M)$ is the infimum of the projective dimensions $\operatorname{pd}_{\mathcal{A}}(\operatorname{Coker}(d_1^P), \operatorname{where} P_{\bullet} := (P_i, d_i^P)_{i \in \mathbb{Z}} \operatorname{runs} \operatorname{through} \operatorname{all} \operatorname{finite} \operatorname{quasi-projective} \operatorname{resolutions} \operatorname{of} M$ with $\operatorname{hsup} P_{\bullet} = 0$. In this paper, for a ring R and any R-module M, we set $\operatorname{qpd}_{R-\operatorname{Mod}} M$.

The following result reveals a close relationship between objects with finite quasi-projective dimensions and objects with finite projective dimensions.

Proposition 2.2. Let \mathcal{P} be the full subcategory of \mathcal{A} consisting of (not necessarily all) projective objects and being closed under direct summands and finite direct sums. Let M be an object of \mathcal{A} such that $\operatorname{qpd}_{\mathcal{A}}(M) < \infty$. Suppose that M admits a projective resolution of the form

$$Q_{\bullet}: \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

with all $Q_i \in \mathcal{P}$ for $i \ge 0$. Then there exists a bounded complex

$$P_{\bullet}: 0 \longrightarrow P_r \xrightarrow{d_r} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots \xrightarrow{d_{-s+1}} P_{-s} \longrightarrow 0$$

in $\mathscr{C}^{\mathsf{b}}(\mathcal{P})$ with $s \geqslant 0$ and hsup $P_{\bullet} = 0$ satisfying the following:

- (a) The complex P_{\bullet} is a quasi-projective resolution of M and $\operatorname{qpd}_{\mathcal{A}} M = \operatorname{pd}_{\mathcal{A}} N$, where $N := \operatorname{Coker}(d_1)$.
- (b) For a right exact, covariant functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, let $L^i F$ denote the i-th left-derived functor of F and let $d := \sup\{i \ge 0 \mid L^i F(M) \ne 0\}$.
- If $0 \le d < \infty$, then $d = \sup\{i \ge 0 | L^i F(N) \ne 0\}$. Furthermore, if $0 < d < \infty$, then $L^d F(H_0(P_{\bullet})) \simeq L^d F(N)$.

Proof. Let \mathcal{X} denote the full subcategory of \mathcal{A} consisting of all projective objects. Then $\mathcal{P} \subseteq \mathcal{X}$. Let $r := \operatorname{qpd}_{\mathcal{A}} M < \infty$. Then M has a finite quasi-projective resolution $S_{\bullet} := (S_i, d_i^S)_{i \in \mathbb{Z}} \in \mathscr{C}^{\mathsf{b}}(\mathcal{X})$ in \mathcal{A} with $\operatorname{hsup}(S_{\bullet}) = 0$ and $r = \operatorname{pd}_{\mathcal{A}}\left(\operatorname{Coker}(d_1^S)\right)$. Set $s := -\operatorname{hinf}(S_{\bullet})$. Then $0 \leqslant s < \infty$. By taking truncations of complexes over \mathcal{A} , we have

$$S_{\bullet} \in H_0(S_{\bullet}) * H_{-1}(S_{\bullet})[-1] * \cdots * H_{-s}(S_{\bullet})[-s] \subset \mathcal{D}^{-}(\mathcal{A}).$$

Clearly, for each i, $H_i(S_{\bullet}) \simeq M^{n_i}$ for some integer $n_i \geqslant 0$. Since $\mathcal{P} \subseteq \mathcal{A}$ is closed under finite direct sums, we can replace M with its projective resolution $Q_{\bullet} \in \mathscr{C}^-(\mathcal{P})$ and construct a complex $T_{\bullet} \in \mathscr{C}^-(\mathcal{P})$ that is isomorphic in $\mathscr{D}^-(\mathcal{A})$ to S_{\bullet} . Consequently, $H^i(T_{\bullet}) \simeq H^i(S_{\bullet})$ for each $i \in \mathbb{Z}$. This implies that T_{\bullet} is also a quasi-projective resolution of M with $\text{hsup}(T_{\bullet}) = 0$. Further, since $\mathscr{K}^-(\mathcal{X})$ is triangle equivalent

to $\mathscr{D}^-(\mathcal{A})$ and $\mathscr{K}^-(\mathcal{P}) \subseteq \mathscr{K}^-(\mathcal{X})$, we have $T_{\bullet} \simeq S_{\bullet}$ in $\mathscr{K}^-(\mathcal{X})$. It follows that there exist contractible complexes X_{\bullet} and Y_{\bullet} in $\mathscr{C}^-(\mathcal{X})$ such that $T_{\bullet} \oplus X_{\bullet} \simeq S_{\bullet} \oplus Y_{\bullet}$ in $\mathscr{C}^-(\mathcal{X})$. In both sides of the isomorphism, we take the cokernels of complexes in degree 1 and obtain

$$\operatorname{Coker}(d_1^T) \oplus \operatorname{Coker}(d_1^X) \simeq \operatorname{Coker}(d_1^S) \oplus \operatorname{Coker}(d_1^Y).$$

Since X_{\bullet} and Y_{\bullet} are contractible, both $\operatorname{Coker}(d_1^X)$ and $\operatorname{Coker}(d_1^Y)$ belong to X. Thus $\operatorname{pd}(\operatorname{Coker}(d_1^T)) = \operatorname{pd}(\operatorname{Coker}(d_1^S))$. By $\operatorname{hsup}(T_{\bullet}) = 0$, the sequence

$$\cdots \longrightarrow T_r \longrightarrow T_{r-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow \operatorname{Coker}(d_1^T) \longrightarrow 0,$$

truncated from the complex T_{\bullet} , is a projective resolution of $\operatorname{Coker}(d_1^T)$. It follows that $\Omega^r(\operatorname{Coker}(d_1^T))$ is projective, and thus isomorphic to a direct summand of T_r . Since \mathcal{P} is closed under direct summands in \mathcal{A} and contains T_r , we have $\Omega^r(\operatorname{Coker}(d_1^T)) \in \mathcal{P}$. Now, let P_{\bullet} be the truncation of T_{\bullet} in degree r:

$$P_{\bullet}: 0 \longrightarrow \Omega^{r}(\operatorname{Coker}(d_{1}^{T})) \longrightarrow T_{r-1} \to T_{r-2} \longrightarrow \cdots \longrightarrow T_{1} \longrightarrow T_{0} \longrightarrow \cdots \longrightarrow T_{-s} \longrightarrow 0.$$

Then $P_{\bullet} \in \mathcal{C}^{b}(\mathcal{P})$ is homotopy equivalent to T_{\bullet} and satisfies (a).

To show (b), suppose $0 \le d < \infty$. If s = 0, then $N \simeq H_0(P_{\bullet}) \simeq M^{n_0}$ and thus (b) holds trivially. In the following, let s > 0. Observe that $H_0(P_{\bullet}) \simeq H_0(S_{\bullet}) \simeq M^{n_0} \neq 0$. Applying the functor $L^{d+j}F$ for $j \ge 0$ to the exact sequence $0 \to M^{n_0} \to N \to \operatorname{Im}(d_0) \to 0$ in $\mathcal A$ yields an exact sequence in $\mathcal B$:

$$L^{d+j+1}F(\operatorname{Im}(d_0)) \longrightarrow L^{d+j}F(M^{n_0}) \longrightarrow L^{d+j}F(N) \longrightarrow L^{d+j}F(\operatorname{Im}d_0).$$

Consequently, if $L^{d+j}F(\operatorname{Im}(d_0)) = 0$ for all j > 0, then

$$0 = \left(L^{d+j}F(M)\right)^{n_0} \simeq L^{d+j}F(M^{n_0}) \simeq L^{d+j}F(N) \quad \text{and} \quad 0 \neq L^dF(M^{n_0}) \hookrightarrow L^dF(N),$$

and therefore $d = \sup\{i \ge 0 \mid L^i F(N) \ne 0\}$.

It remains to show $L^{d+j}F(\operatorname{Im}(d_0))=0$ for all j>0.

Now, we fix j > 0. Since $L^{d+k}F(M) = 0$ for all k > 0 and $H_{i-1}(P_{\bullet}) \simeq M^{n_{i-1}}$ for $i \in \mathbb{Z}$, we apply the functor $L^{d+j}F$ to the exact sequence $0 \to \operatorname{Im}(d_i) \to \operatorname{Ker}(d_{i-1}) \to H_{i-1}(P_{\bullet}) \to 0$ and obtain

$$L^{d+j}F(\operatorname{Im}(d_i)) \simeq L^{d+j}F(\operatorname{Ker}(d_{i-1}))$$
 and $L^dF(\operatorname{Im}(d_i)) \hookrightarrow L^dF(\operatorname{Ker}(d_{i-1}))$.

Since $\operatorname{Ker}(d_{i-1}) = \Omega(\operatorname{Im}(d_{i-1}))$ by $P_{i-1} \in \mathcal{P}$, it follows that $L^m F(\operatorname{Ker}(d_{i-1})) \simeq L^{m+1} F(\operatorname{Im}(d_{i-1}))$ for all m > 0. Thus $L^{d+j} F(\operatorname{Im}(d_i)) \simeq L^{d+j+1} F(\operatorname{Im}(d_{i-1}))$, and $L^d F(\operatorname{Im}(d_i)) \hookrightarrow L^d F(\operatorname{Ker}(d_{i-1})) \simeq L^{d+1} F(\operatorname{Im}(d_{i-1}))$ for d > 0. This implies a series of isomorphisms:

$$L^{d+j}F(\text{Im}(d_0)) \simeq L^{d+j+1}F(\text{Im}(d_{-1})) \simeq L^{d+j+2}F(\text{Im}(d_{-2})) \simeq \cdots \simeq L^{d+j+s}F(\text{Im}(d_{-s})) = 0,$$

where the last equality follows from the vanish of the map $d_{-s}: P_{-s} \to 0$.

Suppose d > 0. Then

$$L^dF(\operatorname{Im}(d_0)) \hookrightarrow L^{d+1}F(\operatorname{Im}(d_{-1})) \simeq L^{d+2}F(\operatorname{Im}(d_{-2})) \simeq \cdots \simeq L^{d+s}F(\operatorname{Im}(d_{-s})) = 0,$$

and therefore $L^dF(\operatorname{Im}(d_0)) = 0$. From the exact sequence

$$0 = L^{d+1}F(\operatorname{Im}(d_0)) \longrightarrow L^dF(M^{n_0}) \longrightarrow L^dF(N) \longrightarrow L^dF(\operatorname{Im}(d_0)) = 0,$$

we see that $L^dF(H_0(P_{\bullet})) \simeq L^dF(M^{n_0}) \simeq L^dF(N)$. This shows (b).

Similarly, we can also show that if $d = -\infty$ (that is, $L^i F(M) = 0$ for all $i \ge 0$), then

$$\sup\{i \in \mathbb{N} \mid L^i F(N) \neq 0\} = \begin{cases} 0, & \text{if } F(N) \neq 0; \\ -\infty, & \text{if } F(N) = 0, \end{cases}$$

where the supreme is $-\infty$ means that $L^i F(N) = 0$ for all $i \ge 0$.

A direct consequence of Proposition 2.2 is the following result which generalizes [8, Proposition 3.4].

Corollary 2.3. Let R be a left coherent ring and M a finitely presented R-module with finite quasi-projective dimension. Then:

- (1) There exists a bounded complex $P_{\bullet} \in \mathcal{K}^b(R\text{-proj})$ that is a quasi-projective resolution of M such that $\operatorname{qpd}_R(M) = \sup(P_{\bullet}) \operatorname{hsup}(P_{\bullet})$.
- (2) There exists a finitely presented R-module N such that $pd_R(N) = qpd_R(M)$ and there is an injection $M \hookrightarrow N$ of R-modules.

Proof. Since R is left coherent, R-mod is an abelian category and has R-proj as its full subcategory consisting of all projective objects. In Proposition 2.2, we take $\mathcal{A} = R$ -mod and $\mathcal{P} = R$ -proj. By Proposition 2.2(a), there exists a bounded complex $P_{\bullet} := (P_i, d_i^P)_{i \in \mathbb{Z}} \in \mathscr{C}^b(R$ -proj) satisfying (1) and hsup(P_{\bullet}) = 0. Let $N := \operatorname{Coker}(d_1^P)$. Then $N \in R$ -mod and $\operatorname{pd}_R(N) = \operatorname{qpd}_R(M)$. Since there is an exact sequence $0 \to H_0(P_{\bullet}) \to N \to \operatorname{Im}(d_0^P) \to 0$ of R-modules and $H_0(P_{\bullet}) \simeq M^{n_0}$ for some integer $n_0 > 0$, we obtain an injection $M \hookrightarrow N$ in R-mod.

Corollary 2.4. Let \mathcal{A} be a Frobenius abelian category. Then $\operatorname{qpd}_{\mathcal{A}}(M) = 0$ or ∞ for any $M \in \mathcal{A}$.

Proof. Since \mathcal{A} is a Frobenius category, projective objects and injective objects of \mathcal{A} coincide. This implies that objects of \mathcal{A} with finite projective dimensions are exactly projective objects. By Proposition 2.2(a), if $\operatorname{qpd}_{\mathcal{A}}(M) < \infty$, then $\operatorname{qpd}_{\mathcal{A}}(M) = 0$.

Finally, we give an application of Proposition 2.2 to generalize the classical depth formula (see [2, Theorem 1.2]). The following definition is standard in commutative algebra.

Definition 2.5. Let (R, \mathfrak{m}) be a commutative Noetherian local ring with the maximal ideal \mathfrak{m} . The depth of a finitely generated R-module M is defined as

$$\operatorname{depth}_{R}(M) := \min\{i \geq 0 \mid \operatorname{Ext}_{R}^{i}(R/\mathfrak{m}, M) \neq 0\}.$$

Theorem 2.6. [9, Theorem 4.2] Let R be a commutative Noetherian local ring, and let M and N be non-zero finitely generated R-modules. Define $q := \sup\{i \ge 0 \mid \operatorname{Tor}_i^R(M,N) \ne 0\}$. Suppose that $q < \infty$ and $\operatorname{qpd}_R(M) < \infty$. If q = 0 or $\operatorname{depth}(\operatorname{Tor}_q^R(M,N)) \le 1$, then

$$\operatorname{depth}(N) = \operatorname{depth}(\operatorname{Tor}_q^R(M,N)) + \operatorname{qpd}_R(M) - q.$$

Proof. The case q = 0 in Theorem 2.6 (that is, $\operatorname{Tor}_{i}^{R}(M, N) = 0$ for all i > 0) is a direct consequence of [8, Theorems 4.4 and 4.11].

For the case q > 0, we provide a new proof that is different from the one given in [9, Theorem 4.2].

Let $F := - \otimes_R N : R\text{-Mod} \to R\text{-Mod}$, a covariant and right exact functor. It is known that, for each $i \in \mathbb{N}$, the i-th left-derived functor L^iF of F is exactly the functor $\text{Tor}_i^R(-,N)$. This implies

$$q = \sup\{i \geqslant 0 \mid L^{i}(F)(M) \neq 0\} < \infty.$$

Since $\operatorname{qpd}_R(M) < \infty$, it follows from Proposition 2.2(*b*) that there exists a finite quasi-projective resolution $(P_{\bullet}, d_i)_{i \in \mathbb{Z}}$ of ${}_RM$ with $\operatorname{hsup} P_{\bullet} = 0$ and $C := \operatorname{Coker}(d_1)$ satisfying

$$q=\sup\{i\geqslant 0\ |\ \operatorname{Tor}_i^R(C,N)\neq 0\},\ \operatorname{pd}_R(C)=\operatorname{qpd}_R(M)<\infty\ \text{ and } \operatorname{Tor}_q^R(C,N)\simeq\operatorname{Tor}_q^R(H_0(P_\bullet),N).$$

Note that $H_0(P_{\bullet}) \simeq M^{n_0}$ for some $n_0 > 0$. This implies depth $(\operatorname{Tor}_q^R(C,N)) = \operatorname{depth}(\operatorname{Tor}_q^R(M,N)) \leqslant 1$. Now, we apply *Auslander's depth formula* (see [2, Theorem 1.2]) directly to the pair (C,N) of R-modules, and obtain the formula

$$\operatorname{depth}(N) = \operatorname{depth}(\operatorname{Tor}_q^R(C, N)) + \operatorname{pd}_R(C) - q.$$

Thus
$$\operatorname{depth}(N) = \operatorname{depth}(\operatorname{Tor}_q^R(M,N)) + \operatorname{qpd}_R(M) - q$$
.

3 New properties of quasi-projective dimensions

In this section, we develop some general methods to determine the quasi-projective dimensions of objects in abelian categories. In particular, we give a proof of Theorem 1.2.

Throughout this section, \mathcal{A} stands for an abelian category with enough projective objects. Denote by \mathcal{P} the full subcategory of \mathcal{A} consisting of *all* projective objects.

Our first result generalizes [8, Corollary 4.10] that is focused on finitely generated modules over commutative Noetherian rings and is a corollary of the *Auslander–Buchsbaum formula* for modules of finite quasi-projective dimension established in [8, Theorem 4.4].

Proposition 3.1. Let M be an object of \mathcal{A} with $\operatorname{pd}_{\mathcal{A}} M < \infty$. Then $\operatorname{qpd}_{\mathcal{A}} M = \operatorname{pd}_{\mathcal{A}} M$.

Proof. Clearly, $\operatorname{qpd}_{\mathcal{A}} M \leq \operatorname{pd}_{\mathcal{A}} M < \infty$. It suffices to show $\operatorname{pd}_{\mathcal{A}} M \leq \operatorname{qpd}_{\mathcal{A}} M$. Let N be the object given in Proposition 2.2(a). Then $\operatorname{qpd}_{\mathcal{A}} M = \operatorname{pd}_{\mathcal{A}} N$. We claim $\operatorname{pd}_{\mathcal{A}} M \leq \operatorname{pd}_{\mathcal{A}} N$.

In fact, for each object $X \in \mathcal{A}$, the contravariant functor $F_X := \operatorname{Hom}_{\mathcal{A}}(-,X) : \mathcal{A} \to \mathbb{Z}$ -Mod can be regarded as a right exact, covariant functor from \mathcal{A} to $(\mathbb{Z}\operatorname{-Mod})^{\operatorname{op}}$ (that is the opposite category of $\mathbb{Z}\operatorname{-Mod}$). Then, for each $i \geq 0$, the i-th left-derived functor L^iF_X of F_X is exactly the i-th extension functor $\operatorname{Ext}^i_{\mathcal{A}}(-,X)$. Let $d_X := \sup\{i \geq 0 \mid \operatorname{Ext}^i_{\mathcal{A}}(M,X) \neq 0\}$. By Proposition 2.2(*b*), if $0 \leq d_X < \infty$, then $d_X = \sup\{i \geq 0 \mid \operatorname{Ext}^i_{\mathcal{A}}(N,X) \neq 0\}$. Let $n := \operatorname{pd}_{\mathcal{A}} M < \infty$. Then $\operatorname{Ext}^i_{\mathcal{A}}(M,-) = 0$ for i > n and there exists an object $X \in \mathcal{A}$ such that $\operatorname{Ext}^n_{\mathcal{A}}(M,X) \neq 0$. This implies $d_X = n$. Thus $\operatorname{Ext}^n_{\mathcal{A}}(N,X) \neq 0$ which forces $\operatorname{pd}_{\mathcal{A}} N \geqslant n$.

Lemma 3.2. Let M be an object of \mathcal{A} with $\operatorname{Ext}_{\mathcal{A}}^n(M,M) = 0$ for $2 \le n \le k+1$ with $k \in \mathbb{N}$. Suppose that $X_{\bullet} \in \mathcal{D}^b(\mathcal{A})$ satisfies that $H_i(X_{\bullet}) \in \operatorname{add}(M)$ for $0 \le i \le k$ and $H_i(X_{\bullet}) = 0$ for i > k or i < 0. Then $X_{\bullet} \simeq \bigoplus_{i=0}^k H_i(X_{\bullet})[i]$ in $\mathcal{D}^b(\mathcal{A})$.

Proof. We take induction on k to show Lemma 3.2. Clearly, Lemma 3.2 holds trivially for k = 0. Let k > 0. Taking the canonical truncation of X_{\bullet} at degree k yields a distinguished triangle in $\mathcal{D}^{b}(\mathcal{A})$:

$$\tau_{\geqslant k}X_{\bullet} \longrightarrow X_{\bullet} \longrightarrow \tau_{\leqslant k-1}X_{\bullet} \xrightarrow{f} (\tau_{\geqslant k}X_{\bullet})[1]$$

where $\tau_{\geqslant k}X_{\bullet} \simeq H_k(X_{\bullet})[k] \in \operatorname{add}(M[k])$ and $H_i(\tau_{\leqslant k-1}X_{\bullet}) \simeq H_i(X_{\bullet})$ for $i \leqslant k-1$ but $H_i(\tau_{\leqslant k-1}X_{\bullet}) = 0$ for $i \geqslant k$. In particular, $H_i(\tau_{\leqslant k-1}X_{\bullet}) \in \operatorname{add}(M)$ for $0 \leqslant i \leqslant k-1$. Now, the induction on k-1 implies that

$$\tau_{\leqslant k-1}X_{\bullet} \simeq \bigoplus_{i=0}^{k-1} H_i(\tau_{\leqslant k-1}X_{\bullet})[i] \simeq \bigoplus_{i=0}^{k-1} H_i(X_{\bullet})[i] \in \operatorname{add}(\bigoplus_{i=0}^{k-1} M[i]).$$

Since $\operatorname{Ext}_{\mathcal{A}}^n(M,M)=0$ for $2\leqslant n\leqslant k+1$, we have $\operatorname{Hom}_{\mathscr{D}^b(\mathcal{A})}(\bigoplus_{i=0}^{k-1}M[i],M[k+1])=0$. This forces $\operatorname{Hom}_{\mathscr{D}^b(\mathcal{A})}(\tau_{\leqslant k-1}X_{\bullet},(\tau_{\geqslant k}X_{\bullet})[1])=0$. Thus f=0 and $X_{\bullet}\simeq \tau_{\geqslant k}X_{\bullet}\oplus \tau_{\leqslant k-1}X_{\bullet}\simeq \bigoplus_{i=0}^k H_i(X_{\bullet})[i]$ in $\mathscr{D}^b(\mathcal{A})$.

Theorem 3.3. Suppose that $M \in \mathcal{A}$ has a finite quasi-projective resolution P_{\bullet} . If $\operatorname{Ext}_{\mathcal{A}}^{n}(M,M) = 0$ for $2 \leq n \leq \operatorname{hsup}(P_{\bullet}) - \operatorname{hinf}(P_{\bullet}) + 1$, then $\operatorname{pd}_{\mathcal{A}}(M) < \infty$.

Proof. Let $t := \text{hsup}(P_{\bullet})$ and $s := \text{hinf}(P_{\bullet})$. Then t and s are finite with $s \le t$. Since P_{\bullet} is a finite quasi-projective resolution of M, we see that $P_{\bullet} \in \mathscr{C}^{b}(\mathcal{P})$, $H_{i}(P_{\bullet}) \simeq M^{a_{i}}$ for $s \le i \le t$ and $H_{i}(P_{\bullet}) = 0$ for i > t or i < s, where a_{i} are nonnegative integers and not all zero. Suppose that $\text{Hom}_{\mathscr{D}^{-}(\mathcal{A})}(M,M[n]) = 0$ for $2 \le n \le t - s + 1$. By Lemma 3.2, $P_{\bullet} \simeq \bigoplus_{i=s}^{t} H_{i}(P_{\bullet})[i] \simeq \bigoplus_{i=s}^{t} M^{a_{i}}[i]$ in $\mathscr{D}^{-}(\mathcal{A})$. Now, let Q_{\bullet} be a deleted projective resolution of M. Then there is an isomorphism $P_{\bullet} \simeq \bigoplus_{i=s}^{t} Q_{\bullet}^{a_{i}}[i]$ in $\mathscr{D}^{-}(\mathcal{A})$, where both sides are bounded below complexes of projective objects. Since the localization functor $\mathscr{K}^{-}(\mathcal{P}) \to \mathscr{D}^{-}(\mathcal{A})$ is a triangle equivalence, it follows that $P_{\bullet} \simeq \bigoplus_{i=s}^{t} Q_{\bullet}^{a_{i}}[i]$ in $\mathscr{K}^{-}(\mathcal{P})$. Note that $\mathscr{K}^{b}(\mathcal{P})$ is closed under direct summands in $\mathscr{K}^{-}(\mathcal{P})$. Since $P_{\bullet} \in \mathscr{K}^{b}(\mathcal{P})$ and at least one a_{i} is not zero, we have $Q_{\bullet} \in \mathscr{K}^{b}(\mathcal{P})$. This implies $pd_{\mathcal{A}}(M) < \infty$.

Theorem 3.3 implies the following result.

Corollary 3.4. Let $M \in \mathcal{A}$. If $\operatorname{qpd}_{\mathcal{A}}(M) < \infty$ and $\operatorname{Ext}_{\mathcal{A}}^n(M,M) = 0$ for $n \ge 2$, then $\operatorname{pd}_{\mathcal{A}}(M) < \infty$.

We present two direct corollaries of Corollary 3.4.

Corollary 3.5. Let R be a left coherent ring and let M be a finitely presented R-module with finite quasi-projective dimension such that $\operatorname{Ext}_R^n(M,M\oplus R)=0$ for all n>0. Then M is projective.

Proof. Let $\mathcal{A} := R$ -mod. This is an abelian category with enough projective objects since R is left coherent. By Corollary 3.4, $\operatorname{pd}_R(M) < \infty$. Let $m := \operatorname{pd}_R(M)$. We claim m = 0.

In fact, since $_RM$ is finitely presented, it has a projective resolution $0 \to P_m \xrightarrow{f_m} P_{m-1} \to \cdots \to P_0 \to M \to 0$ with all $P_i \in R$ -proj for $0 \le i \le m$. If m > 0, then the assumption $\operatorname{Ext}_R^m(M,R) = 0$ implies $\operatorname{Ext}_R^m(M,P_m) = 0$, and therefore f_m is a split injection. This leads to $\operatorname{pd}_R(M) < m$, a contradiction. Thus m = 0 and M is projective.

Recall that a ring *R* is *quasi-Frobenius* if all projective left *R*-modules are injective, or equivalently, all injective left *R*-modules are projective. Note that if *R* is quasi-Frobenius, then it is left and right Noetherian and Artinian, and both *R*-Mod and *R*-mod are Frobenius abelian categories.

Corollary 3.6. Let R be a quasi-Frobenius ring and let M be an R-module with finite quasi-projective dimension. If $\operatorname{Ext}_R^n(M,M)=0$ for $n\geqslant 2$, then M is projective.

Proof. By Corollary 3.4, $\operatorname{pd}_R(M) < \infty$. Since *R* is quasi-Frobenius, each *R*-module of finite projective dimension is projective. \square

Corollary 3.4 provides a method for constructing modules with infinite quasi-projective dimension.

Example 3.7. Let A be an algebra over a field K presented by the quiver

$$1 \xrightarrow{\alpha} 2 \widehat{\beta}$$

with relations: $\beta\alpha = 0 = \beta^2$, where the composition of arrows is always taken from right to left. Let S_1 and S_2 be simple A-modules corresponding to the vertices 1 and 2, respectively. It can be checked that $\Omega(S_1) = S_2$ and $\Omega(S_2) = S_2$. This implies $\operatorname{pd}_A(S_1) = \infty = \operatorname{pd}_A(S_2)$ and $\operatorname{Ext}_A^i(S_1, S_1) = 0$ for i > 0 (in fact, S_1 is injective). Thus $\operatorname{qpd}_A(S_1) = \infty$ by Corollary 3.4. Moreover, since S_2 is periodic, $\operatorname{qpd}_A(S_2) = 0$ by Theorem 1.1(3).

Note that there exists an exact sequence $0 \to S_2 \to P_1 \to S_1 \to 0$ of A-modules, where P_1 is the projective cover of S_1 . Obviously, the inequality $\operatorname{qpd}_A(S_1) \leqslant \operatorname{qpd}_A(S_2) + 1$ does not hold. But we always have $\operatorname{pd}_A(M) \leqslant \operatorname{pd}_A(\Omega(M)) + 1$ for any $M \in \mathcal{A}$. This also shows that quasi-projective dimension is different from projective dimension.

Next, we provide a sufficient condition for the inequality $qpd_{\mathcal{A}}(M) \leq qpd_{\mathcal{A}}(\Omega(M)) + 1$ to hold.

Proposition 3.8. Let $0 \to N \to E \to M \to 0$ be an exact sequence in \mathcal{A} such that E is projective-injective. Then $\operatorname{qpd}_{\mathcal{A}}(M) \leqslant \operatorname{qpd}_{\mathcal{A}}(N) + 1$.

Proof. The inequality in Proposition 3.8 automatically holds if $\operatorname{qpd}_{\mathcal{A}}(N) = \infty$. So, we assume $\operatorname{qpd}_{\mathcal{A}}(N) = m < \infty$. Then N admits a quasi-projective resolution of the form

$$P_{\bullet}: \quad 0 \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} \cdots \xrightarrow{d_{-r+1}} P_{-r} \xrightarrow{d_{-r}} 0,$$

where $P_m \neq 0$, hsup $(P_{\bullet}) = 0$, $H_{-r}(P_{\bullet}) \neq 0$, and for each $i \in \mathbb{Z}$, there is an isomorphsim $g_i : H_i(P_{\bullet}) \to N^{a_i}$ for some integers $a_i \in \mathbb{N}$ (not all zero). Let $0 \to N \xrightarrow{f} E \longrightarrow M \to 0$ be the exact sequence in Proposition 3.8. Since E is injective, the composition

$$\operatorname{Ker}(d_i) \twoheadrightarrow H_i(P_{\bullet}) \xrightarrow{g_i} N^{a_i} \xrightarrow{f^{a_i}} E^{a_i}$$

can be lifted to a morphism $h_i: P_i \longrightarrow E^{a_i}$ in \mathcal{A} . This is illustrated by the following commutative diagram

$$P_{i+1} \xrightarrow{d_{i+1}} P_i$$
 $\downarrow h_i$
 $\downarrow h$

Clearly, $h_i d_{i+1} = 0$. Let

$$E_{\bullet} := 0 \longrightarrow E^{a_m} \stackrel{0}{\longrightarrow} \cdots \stackrel{0}{\longrightarrow} E^{a_0} \longrightarrow \cdots \stackrel{0}{\longrightarrow} E^{a_{-r}} \longrightarrow 0$$

be a bounded complex over \mathcal{A} with all differentials zero. Since E is projective, we have $E_{\bullet} \in \mathscr{C}^{b}(\mathcal{P})$. Set $h_{\bullet} := (h_{i})_{i \in \mathbb{Z}} : P_{\bullet} \longrightarrow E_{\bullet}$. Then h_{\bullet} is a chain map in $\mathscr{C}^{b}(\mathcal{P})$. Now, we extend the chain map to a distinguished triangle $P_{\bullet} \xrightarrow{h_{\bullet}} E_{\bullet} \longrightarrow \operatorname{Con}(h_{\bullet}) \longrightarrow P_{\bullet}[1]$ in $\mathscr{K}^{b}(\mathcal{P})$, where $\operatorname{Con}(h_{\bullet})$ denotes the mapping cone of h_{\bullet} . By [13, Corollary 10.1.4], there is a long exact sequence

$$\cdots \longrightarrow H_i(P_{\bullet}) \xrightarrow{H_i(h_{\bullet})} H_i(E_{\bullet}) \longrightarrow H_i(\operatorname{Con}(h_{\bullet})) \longrightarrow H_{i-1}(P_{\bullet}) \xrightarrow{H_{i-1}(h_{\bullet})} \cdots$$

Clearly, $H_i(h_{\bullet}) = f^{a_i}g_i$. Since f is a monomorphism and g_i is an isomorphism, we obtain a short exact sequence

$$0 \longrightarrow H_i(P_{\bullet}) \xrightarrow{H_i(h_{\bullet})} H_i(E_{\bullet}) \longrightarrow H_i(\operatorname{Con}(h_{\bullet})) \longrightarrow 0.$$

It follows from $\operatorname{Coker}(f) \simeq M$ that $H_i(\operatorname{Con}(h_{\bullet})) \simeq M^{a_i}$. Thus $\operatorname{Con}(h_{\bullet})$ is a finite quasi-projective resolution of M. By $\operatorname{hsup}(P_{\bullet}) = 0$, we have $a_i = 0$ for i > 0 and $a_0 \neq 0$. This forces $\operatorname{hsup}(\operatorname{Con}(h_{\bullet})) = 0$. Note that $\operatorname{sup}(\operatorname{Con}(h_{\bullet})) = m+1$ beacuse the (m+1)-th term of the complex $\operatorname{Con}(h_{\bullet})$ is exactly P_m . Therefore $\operatorname{qpd}_{\mathcal{A}}(M) \leqslant \operatorname{sup}(\operatorname{Con}(h_{\bullet})) - \operatorname{hsup}(\operatorname{Con}(h_{\bullet})) = m+1 = \operatorname{qpd}_{\mathcal{A}}(N) + 1$.

Corollary 3.9. Let \mathcal{A} be a Frobenius abelian category. For any $M \in \mathcal{A}$, $P \in \mathcal{P}$ and $i \in \mathbb{Z}$,

$$\operatorname{qpd}_{\mathcal{A}}(M\oplus P)=\operatorname{qpd}_{\mathcal{A}}(M)=\operatorname{qpd}_{\mathcal{A}}(\Omega^{i}(M)).$$

Proof. By Theorem 1.1(1), $\operatorname{qpd}_{\mathcal{A}}(M \oplus P) \leqslant \operatorname{qpd}_{\mathcal{A}}(M)$. By Proposition 3.8, $\operatorname{qpd}_{\mathcal{A}}(M) \leqslant \operatorname{qpd}_{\mathcal{A}}(\Omega(M)) + 1$. Since $\Omega(M) = \Omega(M \oplus P)$, we have $\operatorname{qpd}_{\mathcal{A}}(\Omega(M)) = \operatorname{qpd}_{\mathcal{A}}(\Omega(M \oplus P)) \leqslant \operatorname{qpd}_{\mathcal{A}}(M \oplus P)$ by Theorem 1.1(2). Thus $\operatorname{qpd}_{\mathcal{A}}(M \oplus P) < \infty$ if and only if $\operatorname{qpd}_{\mathcal{A}}(M) < \infty$ if and only if $\operatorname{qpd}_{\mathcal{A}}(\Omega(M)) < \infty$. By Corollary 2.4, the objects $M \oplus P$, M and M (and thus also M (and thus also M (and thus also M (but the same quasi-projective dimension.

Proof of Theorem 1.2. The statements (1)-(3) are Proposition 3.1, Corollary 3.4 and Proposition 3.8, respectively. It remains to show (4).

For each $s \ge 0$, we denote by $P_{\bullet}^{\le s}$ the brutal truncation at degree s, that is,

$$P_i^{\leqslant s} = \begin{cases} P_i & \text{if } i \leqslant s; \\ 0 & \text{if } i > s. \end{cases}$$

The monomorphism $\Omega^{n+m}(M) \longrightarrow P_{n+m-1}$ induces a distinguished triangle in $\mathscr{D}^-(\mathcal{A})$:

$$\Omega^{n+m}(M)[n+m-1] \longrightarrow P_{\bullet}^{\leqslant n+m-1} \longrightarrow M \longrightarrow \Omega^{n+m}(M)[n+m],$$

where M is identified with the complex $0 \to \Omega^{n+m}(M) \to P_{n+m-1} \to P_{n+m-2} \to \cdots \to P_1 \to P_0 \to 0$, up to isomorphism. Similarly, there is a distinguished triangle in $\mathcal{D}^-(\mathcal{A})$:

$$\Omega^m(M)[m-1] \longrightarrow P_{\bullet}^{\leq m-1} \longrightarrow M \longrightarrow \Omega^m(M)[m].$$

Shifting this triangle by n steps yields a new distinguished triangle

$$\Omega^m(M)[n+m-1] \longrightarrow P_{\bullet}^{\leqslant m-1}[n] \longrightarrow M[n] \longrightarrow \Omega^m(M)[n+m].$$

Further, since the chain map $g_{\bullet}: P_{\bullet} \to P_{\bullet}[n]$ induces an isomorphism $\Omega^{n+m}(M) \to \Omega^m(M)$, we can construct the following commutative diagram in $\mathcal{D}^-(\mathcal{A})$:

where all columns and rows are distinguished triangles by the Octahedral axiom. Clearly, $\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})$ lies in $\mathscr{C}^{\mathsf{b}}(\mathcal{P})$. By $n \geqslant 2$, we see that $H_i(\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})) \simeq M$ if i=1 or n, and $H_i(\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})) = 0$ for other i. It follows that $\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})$ is a finite quasi-projective resolution of M. Thus $\operatorname{qpd}_{\mathcal{A}}(M) \leqslant \sup(\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})) - \operatorname{hsup}(\operatorname{Con}(g_{\bullet}^{\leqslant n+m-1})) = n+m-n=m$.

Example 3.10. let A be an algebra over a field K presented by the quiver

$$\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
\delta \uparrow & & \downarrow \beta \\
4 & \longleftarrow & 3
\end{array}$$

with relations: $\delta\gamma\beta\alpha = \beta\alpha\delta\gamma = 0$. Denote by *J* the Jacobson radical of *A*, and by e_i the primitive, idempotent element corresponding to the vertex *i* for $1 \le i \le 4$. Then there exists an obvious algebra automorphism $\Phi: A \to A$ sending $e_1 \mapsto e_3$, $e_2 \mapsto e_4$, $e_3 \mapsto e_1$ and $e_4 \mapsto e_2$.

We will show that

- (1) $\operatorname{qpd}_A(Ae_1/Je_1) = \operatorname{qpd}_A(Ae_3/Je_3) = \operatorname{qpd}_A(Ae_2/J^2e_2) = \operatorname{qpd}_A(Ae_4/J^2e_4) = 2.$
- (2) $\operatorname{qpd}_A(Ae_2/J^3e_2) = \operatorname{qpd}_A(Ae_4/J^3e_4) = 1.$

In fact, A is a Nakayama algebra, and thus is of finite representation type. By [1, Theorem V.3.5], every indecomposable A-module is isomorphic to P/J^tP for some indecomposable projective A-module P and for some integer t with $1 \le t \le 5$. Moreover, the indecomposable projective modules of A are determined by their composition sequences:

$$Ae_1 = \frac{1}{3},$$
 $Ae_2 = \frac{4}{4},$ $Ae_3 = \frac{4}{1},$ $Ae_4 = \frac{2}{2}.$ $\frac{3}{4}$

Clearly, Ae_2 and Ae_4 are projective-injective, and indecomposable modules of projective dimension 1 are only simple A-modules Ae_2/Je_2 and Ae_4/Je_4 . Thus, for any $X \in A$ -mod, $pd_A(X) \leq 1$ if and only if $X \in add(A \oplus (Ae_2/Je_2) \oplus (Ae_4/Je_4))$. This implies that if $pd_A(X) \leq 1$, then the socle soc(X) of AX is a direct sum of finitely many copies of Ae_2/Je_2 and Ae_4/Je_4 . Let

$$X \in \{Ae_1/Je_1, Ae_3/Je_3, Ae_2/J^2e_2, Ae_4/J^2e_4\}.$$

Then soc(X) is a direct sum of finitely many copies of simple A-modules Ae_1/Je_1 and Ae_3/Je_3 . It follows that X can't be embedded into any A-module of projective dimension at most 1. By Corollary 2.3(2), $qpd_A(X) \ge 2$. To show $qpd_A(X) \le 2$, we construct the following two chain maps of complexes of A-modules:

where each row is a deleted projective resolution of Ae_1/Je_1 ;

where each row is a deleted projective resolution of Ae_2/J^2e_2 . By Theorem 1.2(4), $qpd_A(Ae_1/Je_1) \leq 2$ and $qpd_A(Ae_2/J^2e_2) \leq 2$. Thus $qpd_A(Ae_1/Je_1) = qpd_A(Ae_2/J^2e_2) = 2$. By the isomorphism Φ , we have $qpd_A(Ae_3/Je_3) = 2 = qpd_A(Ae_4/J^2e_4)$.

Finally, we consider the following two exact sequences:

$$0 \rightarrow Ae_1/J^2e_1 \rightarrow Ae_2 \rightarrow Ae_2/J^3e_2 \rightarrow 0, \quad 0 \rightarrow Ae_2/J^3e_2 \rightarrow Ae_4 \rightarrow Ae_4/J^2e_4 \rightarrow 0.$$

Since Ae_1/J^2e_1 is periodic, it follows from Theorem 1.1(3) that $\operatorname{qpd}_A(Ae_1/J^2e_1)=0$. Since Ae_2 is projective-injective, $\operatorname{qpd}_A(Ae_2/J^3e_2)\leqslant 1$ by Theorem 1.2(3). If $\operatorname{qpd}_A(Ae_2/J^3e_2)=0$, then Theorem 1.2(3) implies $\operatorname{qpd}_A(Ae_4/J^2e_4)\leqslant 1$, which is contradictory to $\operatorname{qpd}_A(Ae_4/J^2e_4)=2$. Thus $\operatorname{qpd}_A(Ae_2/J^3e_2)=1$. Similarly, $\operatorname{qpd}_A(Ae_4/J^3e_4)=1$ by the isomorphism Φ .

It is easy to see that all the A-modules listed in (1) and (2) have infinite projective dimension.

4 Quasi-global dimensions of rings

In this section, we introduce the notion of quasi-global dimension for left Noetherian rings by analogy with the classical global dimension. In Section 4.1, we investigate some basic properties of quasi-global dimension. For instance, quasi-global dimension is equal to finitistic dimension whenever the former is finite (Proposition 4.2(1)), and both stable equivalence of Morita type and derived equivalence preserve quasi-global dimension of self-injective algebras (Corollary 4.6). In Section 4.2, we calculate the quasi-global dimensions for a class of Nakayama algebras (Theorem 4.8). It turns out that these algebras have finite quasi-global dimension.

4.1 Left Noetherian rings

Throughout this section, let R be a *left Noetherian* ring. Recall that the *global* and *finitistic dimensions* of R, denoted by gldim(R) and findim(R), respectively, are defined as

$$\operatorname{gldim}(R) := \sup \{ \operatorname{pd}_R M \mid M \in R \operatorname{-mod} \} \text{ and } \operatorname{findim}(R) := \sup \{ \operatorname{pd}_R M \mid M \in R \operatorname{-mod} \text{ with } \operatorname{pd}_R(M) < \infty \}.$$

Clearly, findim $(R) \leq \operatorname{gldim}(R)$. Similarly, we can introduce the quasi-global dimension of R as follows.

Definition 4.1. The quasi-global dimension of R, denoted by qgldim(R), is defined as

$$\operatorname{qgldim}(R) := \sup \{\operatorname{qpd}_R(M) \mid M \in R\operatorname{-mod}\}.$$

Since $\operatorname{qpd}_R(M) \leq \operatorname{pd}_R(M)$ for any R-module M, we have $\operatorname{qgldim}(R) \leq \operatorname{gldim}(R)$. The inequality may be strict (for example, see Theorem 4.8). However, if $\operatorname{gldim}(R) < \infty$, then $\operatorname{qgldim}(R) = \operatorname{gldim}(R)$ by Theorem 1.2(1).

We first collect some basic properties of quasi-global dimension.

Proposition 4.2. *Let R and S be left Noetherian rings. Then:*

- (1) $\operatorname{findim}(R) = \sup \{\operatorname{qpd}_R M \mid M \in R \operatorname{-mod} \operatorname{and} \operatorname{qpd}_R M < \infty \}$. In $\operatorname{particular}$, if $\operatorname{qgldim}(R) < \infty$, then $\operatorname{findim}(R) = \operatorname{qgldim}(R)$.
 - (2) If R and S are Morita equivalent, then qgldim(R) = qgldim(S).
 - (3) $\operatorname{qgldim}(R \times S) = \max{\operatorname{qgldim}(R), \operatorname{qgldim}(S)}.$

Furthermore, if R and S are finite-dimensional algebras over a field K, then

(4) $\operatorname{qgldim}(R \otimes_K S) \geqslant \max{\operatorname{qgldim}(R), \operatorname{qgldim}(S)}$.

Proof. (1) follows from Theorem 1.2(1) and Proposition 2.2(a).

- (2) Suppose that R and S are Morita equivalent. Then there exists an S-R-bimodule P such that both ${}_SP$ and P_R are finitely generated and projective and the tensor functor $P \otimes_R : R$ -Mod $\to S$ -Mod is an equivalence of abelian categories. Since R and S are left Noetherian, this equivalence can be restricted to an equivalence F: R-mod $\to S$ -mod of abelian categories. Let M be a finitely generated R-module with a finite quasi-projective resolution P_{\bullet} . By Corollary 2.3(1), we may assume that P_{\bullet} belongs to $\mathscr{K}^b(R$ -proj). Since F preserves projective modules and commutes with homology functors, it follows that $F(P_{\bullet})$ is a finite quasi-projective resolution of F(M). Now, it is easy to show gldim(R) = gldim(S).
- (3) Without loss of generality, assume $\operatorname{qgldim}(R) \geqslant \operatorname{qgldim}(S)$. We first show the equality $\operatorname{qpd}_R M = \operatorname{qpd}_{R \times S}(M \times S)$ for any $M \in R$ -mod. Let P_{\bullet}^M be a quasi-projective resolution of M. By definition, $H_i(P_{\bullet}^M) \simeq M^{a_i}$ for some $a_i \in \mathbb{N}$. Define $S_{\bullet} := \bigoplus_{i \in \mathbb{Z}} S^{a_i}[a_i]$. Then $P_{\bullet}^M \times S_{\bullet}$ is a quasi-projective resolution of $M \times S$. This implies $\operatorname{qpd}_{R \times S}(M \times S) \leqslant \operatorname{qpd}_R M$. The converse $\operatorname{qpd}_R M \leqslant \operatorname{qpd}_{R \times S}(M \times S)$ follows from the fact that any quasi-projective resolution of the $(R \times S)$ -module $M \times S$ is of the form $Q_{\bullet}^M \times Q_{\bullet}^S$, where Q_{\bullet}^M and Q_{\bullet}^S are quasi-projective resolutions of M and S, respectively. Thus $\operatorname{qpd}_R M = \operatorname{qpd}_{R \times S}(M \times S)$. It follows that $\operatorname{qgldim}(R \times S) \geqslant \max\{\operatorname{qgldim}(R), \operatorname{qgldim}(S)\}$. In particular, if $\operatorname{qgldim}(R) = \infty$, then $\operatorname{qgldim}(R \times S) = \infty$.

It remains to show that $\operatorname{qgldim}(R \times S) \leqslant \max\{\operatorname{qgldim}(R), \operatorname{qgldim}(S)\}$. Suppose $\operatorname{qgldim}(R) < \infty$. Then $\operatorname{qgldim}(S) < \infty$. It follows that finitely generated R-modules and finitely generated S-modules have finite quasi-projective resolutions. Let $M \times N$ be an arbitrary finitely generated $(R \times S)$ -module. Then $M \in R$ -mod and $N \in S$ -mod. We take P_{\bullet} and Q_{\bullet} to be any finite quasi-projective resolutions of M and N, respectively. Then $H_i(P_{\bullet}) \simeq M^{c_i}$ and $H_i(Q_{\bullet}) \simeq N^{d_i}$ for some c_i , $d_i \in \mathbb{N}$. Since P_{\bullet} and Q_{\bullet} are bounded complexes, both c_i and d_i are nonzero for finitely many i. Set

$$F_{\bullet} := \left(\bigoplus_{j \in \mathbb{Z}} (P_{\bullet}[j])^{d_j}\right) \times \left(\bigoplus_{i \in \mathbb{Z}} (Q_{\bullet}[i])^{c_i}\right).$$

Then F_{\bullet} is a finite quasi-projective resolution of $M \times N$. In fact, $H_k(F_{\bullet}) = (M \times N)^{\sum_{i+j=k} c_i d_j}$ for each $k \in \mathbb{Z}$. Note that $\sup F_{\bullet} = \max\{\sup P_{\bullet} + \operatorname{hsup} Q_{\bullet}, \sup Q_{\bullet} + \operatorname{hsup} P_{\bullet}\}$ and $\operatorname{hsup} F_{\bullet} = \operatorname{hsup} P_{\bullet} + \operatorname{hsup} Q_{\bullet}$. This implies $\operatorname{qpd}_{R \times S}(M \times N) \leq \max\{\operatorname{qpd}_R M, \operatorname{qpd}_S N\}$. Thus $\operatorname{qgldim}(R \times S) \leq \max\{\operatorname{qgldim}(R), \operatorname{qgldim}(S)\}$.

(4) The inequality holds trivially if $\operatorname{qgldim}(R \otimes_K S) = \infty$. So, we assume $\operatorname{qgldim}(R \otimes_K S) < \infty$. Let $M \in R$ -mod. Then $M \otimes_K S \in (R \otimes_K S)$ -mod. Let $P_{\bullet} \in \mathscr{C}^b((R \otimes_K S)$ -proj) be any finite quasi-projective resolution of $M \otimes_K S$ with $H_i(P_{\bullet}) \simeq (M \otimes_K S)^{a_i}$ for each i, where $a_i \in \mathbb{N}$ are not all zero. Since S is a finite-dimensional K-algebra, each P_i as an R-module is finitely generated and projective, and $H_i(P_{\bullet}) \simeq M^{a_i \dim_K(S)}$ as R-modules. It follows that $P_{\bullet} \in \mathscr{C}^b(R$ -proj) is a finite quasi-projective resolution of ${}_R M$. Thus $\operatorname{qpd}_R(M) \leqslant \operatorname{qgldim}_R(M \otimes_K S)$, which forces $\operatorname{qgldim}(R) \leqslant \operatorname{qgldim}(R \otimes_K S)$. Similarly, $\operatorname{qgldim}(S) \leqslant \operatorname{qgldim}(R \otimes_K S)$.

Corollary 4.3. Let R be a quasi-Frobenius ring. Then $\operatorname{qgldim}(R) = 0$ or ∞ . If $\operatorname{qgldim}(R) = 0$, then every R-module M with $\operatorname{Ext}^i_R(M,M) = 0$ for $i \ge 2$ is projective.

Proof. Since *R* is quasi-Frobenius, each *R*-module of finite projective dimension is projective. This implies findim(R) = 0. By Proposition 4.2(1), qgldim(R) = 0 or ∞. If qgldim(R) = 0, then each *R*-module *M* has quasi-projective dimension 0. If, in addition, Ext $_R^i(M,M)$ = 0 for all $i \ge 2$, then *M* is projective by Corollary 3.6.

Remark 4.4. Let R be a self-injective Artin algebra. Recall that Tachikawa's second conjecture asserts that a finitely generated R-module M is projective if $\operatorname{Ext}_R^i(M,M)=0$ for $i\geqslant 1$. By Corollary 4.3, Tachikawa's second conjecture holds for a self-injective Artin algebra with finite quasi-global dimension (or equivalently, with quasi-global dimension 0). Note that if R is of *finite representation type*, that is, there are only finitely many isomorphism classes of indecomposable, finitely generated R-modules, then $\operatorname{qgldim}(R)=0$. In this case, each $M\in R$ -mod is periodic and $\operatorname{qpd}_R(M)=0$ by Theorem 1.1(3).

It would be interesting to classify all self-injective Artin algebras with quasi-global dimension 0.

Proposition 4.5. Let A and B be quasi-Frobenius rings. Suppose that $F: A\text{-mod} \to B\text{-mod}$ is an exact functor and induces an equivalence $A\text{-mod} \to B\text{-mod}$ of stable module categories. Then $\operatorname{qgldim}(B) \leqslant \operatorname{qgldim}(A)$.

Proof. Since *F* is an exact functor and induces an equivalence *A*- $\underline{\text{mod}} \to B$ - $\underline{\text{mod}}$, we see that *F* commutes with homology functors, and preserves and detects finitely generated projective modules. It follows that if $P_{\bullet} := (P_i)_{i \in \mathbb{Z}}$ is a quasi-projective resolution of an *A*-module *M*, then $F(P_{\bullet}) := (F(P_i)_{i \in \mathbb{Z}})$ is a quasi-projective resolution of the *B*-module F(M). This implies $\operatorname{qpd}_B(F(M)) \leqslant \operatorname{qpd}_A(M)$. Let $Y \in B$ -mod. As F : A- $\underline{\text{mod}} \to B$ - $\underline{\text{mod}}$ is dense, there exists $X \in A$ -mod and $P_1, P_2 \in B$ -proj such that $Y \oplus P_1 \simeq F(X) \oplus P_2$. Since *A* and *B* are quasi-Frobenius, $\operatorname{qpd}_B(Y) = \operatorname{qpd}_B(F(X))$ by Corollary 3.9. Thus $\operatorname{qpd}_B(Y) \leqslant \operatorname{qpd}_A(X)$. This forces $\operatorname{qgldim}(B) \leqslant \operatorname{qgldim}(A)$. □

Next, we apply Proposition 4.5 to special stable equivalences between self-injective algebras. For the definitions and constructions of stable equivalences of Morita type and derived equivalences, we refer to the survey article [14].

Corollary 4.6. Let A and B be finite-dimensional self-injective algebras over a field K. If A and B are stably equivalent of Morita type or derived equivalent, then they have the same quasi-global dimension.

Proof. Clearly, finite-dimensional self-injective algebras are quasi-Frobenius rings. If A and B are stably equivalent of Morita type, then there exists an A-B-module X and an B-A-module Y such that the tensor functors $Y \otimes_A - : A$ -mod $\to B$ -mod and $X \otimes_B - : B$ -mod $\to A$ -mod are exact and induce equivalences A- $\underline{\text{mod}} \simeq B$ - $\underline{\text{mod}}$ of stable module categories. In this case, $\operatorname{qgldim}(B) = \operatorname{qgldim}(A)$ by Proposition 4.5. If A and B are derived equivalent, then they are stably equivalent of Morita type by [11, Corollary 5.5] and thus $\operatorname{qgldim}(B) = \operatorname{qgldim}(A)$.

Finally, we provide an example of a self-injective algebra with *infinite* quasi-global dimension.

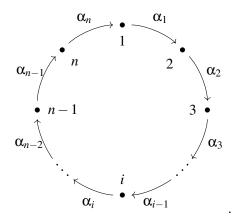
Let K be a field with a non-zero element $q \in K$ that is not a root of unity. The local symmetric K-algebra A, defined by Liu–Schulz (see [10]), is generated by x_0 , x_1 , x_2 with the relations: $x_i^2 = 0$, and $x_{i+1}x_i + qx_ix_{i+1} = 0$ for i = 0, 1, 2.

Proposition 4.7. Let A be Liu–Schulz algebra. Then $qgldim(A) = \infty$.

Proof. Since *A* is a finite-dimensional symmetric *K*-algebra, every non-projective module has infinite projective dimension. By [5, Lemma 6.6], there exists a finitely generated non-projective *A*-module I_0 such that $\operatorname{Ext}_A^i(I_0,I_0)=0$ for all $i\geqslant 2$. It follows from Theorem 1.2(2) that $\operatorname{qpd}_A(I_0)=\infty$. Thus $\operatorname{qgldim}(A)=\infty$.

4.2 Nakayama algebras

In this subsection, we fix an arbitrary field K, an integer $n \ge 2$ and the following quiver Q:



Denote by

$$\rho_i = \begin{cases} \alpha_n \cdots \alpha_1 & \text{for } i = 1, \\ \alpha_{i-1} \cdots \alpha_1 \alpha_n \cdots \alpha_i & \text{for } 2 \leqslant i \leqslant n, \end{cases}$$

the composition of certain arrows in Q. Let m and λ_j be integers with $1 \le m \le n$, $1 \le j \le m$ and $1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_m \le n$. We define

$$A_{n,m} := KQ/\langle \rho_{\lambda_1}, \rho_{\lambda_2}, \dots, \rho_{\lambda_m} \rangle,$$

the quotient algebra of the path algebra KQ modulo the ideal generated by ρ_{λ_i} for all $1 \le i \le m$. Clearly, $A_{n,m}$ is a Nakayama algebra. The main result of this subsection is to compare its quasi-global dimension with its global dimension.

Theorem 4.8. (1) $gldim(A_{n,1}) = qgldim(A_{n,1}) = 2$.

- (2) $\operatorname{gldim}(A_{n,n}) = \infty$ and $\operatorname{qgldim}(A_{n,n}) = 0$.
- (3) If n > 2 and 1 < m < n, then $\operatorname{gldim}(A_{n,m}) = \infty$ and $\operatorname{qgldim}(A_{n,m}) = 2$.

To show Theorem 4.8, we introduce some notation and establish several lemmas.

Let *J* be the Jacobson radical of the algebra $A_{n,m}$. The Loewy length of an $A_{n,m}$ -module *M* is denote by

$$\ell(M) := \min\{s \in \mathbb{N}^+ | J^s M = 0\}.$$

Let S_i and P_i be the simple and indecomposable projective $A_{n,m}$ -module corresponding to each vertex $i \in Q_0$, respectively. For simplicity, we denote by L(i,k) the indecomposable $A_{n,m}$ -module that has top S_i

and Loewy length k. In particular, $L(i,1) = S_i$. Moreover, we set $\Delta := \{\lambda_1, \dots, \lambda_m\}$ and define a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ by

$$\sigma(i) = \begin{cases} i+1 & \text{for } 1 \leq i \leq n-1, \\ 1 & \text{for } i=n. \end{cases}$$

For $r \ge 1$, we denote the compositions $\underbrace{\sigma \cdots \sigma}_{r \text{ times}}$ and $\underbrace{\sigma^{-1} \cdots \sigma^{-1}}_{r \text{ times}}$ by σ^r and σ^{-r} , respectively. As usual, σ^0 stands for the identity map of $\{1, \dots, n\}$. In the following, we always fix $i \in \{1, \dots, n\}$.

Lemma 4.9. The following statements are true.

- (1) For any $A_{n,m}$ -module L(i,k), we have 0 < k < 2n.
- (2) $\ell(P_i) = n + \inf\{r \in \mathbb{N} \mid \sigma^r(i) \in \Delta\}$. In particular, $n \leq \ell(P_i) < 2n$.
- (3) If $n \le k < \ell(P_i)$, then $\ell(P_{\sigma^k(i)}) = \ell(P_i) (k n)$.
- (4) If $0 < k < \ell(P_i)$, then $\Omega(L(i,k)) \simeq L(\sigma^k(i), \ell(P_i) k)$.

Proof. (1) Clearly, k > 0. Conversely, assume $k \ge 2n$. Since L(i,k) is a quotient of P_i , we have $\ell(P_i) \ge k \ge 2n$. This implies that the path $p_i := \alpha_{\sigma^{n-2}(i)} \cdots \alpha_{\sigma(i)} \alpha_i \alpha_{\sigma^{n-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i$ is nonzero in $A_{n,m}$. In the quiver Q, any path of length n (for example, ρ_j for $1 \le j \le n$) is a subpath of p_i . Thus $\rho_j \ne 0$ in $A_{n,m}$, which is a contradiction to the definition of $A_{n,m}$.

(2) By definition of $A_{n,m}$, we have $\ell(P_i) \ge n$. Clearly, $\ell(P_i) = n$ if and only if $i \in \Delta$. Suppose $\ell(P_i) = n + r$ for some r > 0. Then

$$\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma(i)}\alpha_i\neq 0$$
 and $\alpha_{\sigma^{n+r-1}(i)}\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma(i)}\alpha_i=0.$

This implies

$$\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma^r(i)}\neq 0\quad\text{and}\quad\alpha_{\sigma^{n+r-1}(i)}\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma^r(i)}=0.$$

Thus $\sigma^r(i) \in \Delta$.

We now prove that if there exists an integer t > 0 such that $\sigma^t(i) \in \Delta$, then $t \ge r$.

In fact, if t < r, then $\alpha_{\sigma^{n+r-2}(i)} \cdots \alpha_{\sigma^r(i)} \cdots \alpha_{\sigma^t(i)} \neq 0$, and therefore $n+r-2-t+1=n+(r-t)-1 \ge n$, which contradicts $\sigma^t(i) \in \Delta$.

(3) Assume $\ell(P_i) = n + r$ with $r \in \mathbb{N}^+$. Then $\alpha_{\sigma^{n+r-2}(i)} \cdots \alpha_{\sigma(i)} \alpha_i \neq 0$. By (1), r < n. By (2), $\sigma^r(i) \in \Delta$. It follows that

$$\alpha_{\sigma^{n+r-1}(i)}\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma^{n}(i)}\cdots\alpha_{\sigma^{r}(i)}=0.$$

The case k = n is clear. Now assume k = n + s with s > 0. Then s < r. This implies

$$\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma^r(i)}\cdots\alpha_{\sigma^s(i)}\neq 0\quad\text{and}\quad\alpha_{\sigma^{n+r-1}(i)}\alpha_{\sigma^{n+r-2}(i)}\cdots\alpha_{\sigma^r(i)}\cdots\alpha_{\sigma^s(i)}=0.$$

Thus (3) holds.

(4) Note that $\Omega(L(i,k)) = J^k e_i$. Let $f: P_{\sigma^k(i)} \to J^k e_i$ be the $A_{n,m}$ -module homomorphism induced by $e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \alpha_{\sigma^{k-2}(i)} \cdots \alpha_i$. Then $\operatorname{Ker} f = J^{\ell(P_i)-k} e_{\sigma^k(i)}$ and thus (4) holds.

Lemma 4.10. Let
$$n \le k < \ell(P_i)$$
. Then $qpd_{A_{n,m}}(L(i,k)) = pd_{A_{n,m}}(L(i,k)) = 2$.

Proof. Let $\ell(P_i) = n + r$ with r > 0. By Lemma 4.9(2), $i \notin \Delta$. Since $k < \ell(P_i)$, we see from Lemma 4.9(4) that $\Omega(L(i,k)) \simeq L(\sigma^k(i), n + r - k)$. By $k \geqslant n$ and Lemma 4.9(3), $\ell(P_{\sigma^k(i)}) = n + r - (k - n) = 2n + r - k$. Since 2n + r - k > n + r - k, it is clear that $L(\sigma^k(i), n + r - k)$ is non-projective. Thus $\Omega(L(i,k))$ is non-projective. Now, there is a series of isomorphisms:

$$\Omega^{2}(L(i,k)) \simeq \Omega(L(\sigma^{k}(i), n+r-k)) \simeq L(\sigma^{n+r}(i), (2n+r-k) - (n+r-k)) = L(\sigma^{n+r}(i), n).$$

Note that $\sigma^{n+r}(i) = \sigma^r(i)$. By Lemma 4.9(2), $\sigma^r(i) \in \Delta$, which forces $\ell(P_{\sigma^{n+r}(i)}) = n$. This implies that $\Omega^2(L(i,k))$ is projective, and therefore $\operatorname{pd}_{A_{n,m}}(L(i,k)) = 2$. By Proposition 3.1, $\operatorname{qpd}_{A_{n,m}}(L(i,k)) = \operatorname{pd}_{A_{n,m}}(L(i,k))$.

Lemma 4.11. *Let* 0 < k < n. *Then*

$$\operatorname{qpd}_{A_{n,m}}(L(i,k)) \leqslant \begin{cases} 0 & \text{if } i, \sigma^k(i) \in \Delta; \\ 1 & \text{if } i \notin \Delta, \sigma^k(i) \in \Delta; \\ 2 & \text{if } \sigma^k(i) \notin \Delta. \end{cases}$$

Proof. Since 0 < k < n, we have $\Omega(L(i,k)) \simeq L(\sigma^k(i), \ell(P_i) - k)$ by Lemma 4.9(2)(4). In the following, we divide the proof of Lemma 4.11 into four cases.

Case 1. $i, \sigma^k(i) \in \Delta$.

By Lemma 4.9(2), $\ell(P_i) = \ell(P_{\sigma^k(i)}) = n$ and $L(\sigma^k(i), \ell(P_i) - k) = L(\sigma^k(i), n - k)$ which is non-projective. It follows from Lemma 4.9(4) that

$$\Omega^2(L(i,k)) \simeq \Omega(L(\sigma^k(i),n-k)) \simeq L(\sigma^n(i),n-(n-k)) = L(i,k).$$

Thus L(i,k) is periodic. By Theorem 1.1(3), $qpd_{A_{n,m}}(L(i,k)) = 0$.

In the later proof, we will use the following (minimal) projective resolution of the periodic module L(i,k) for several times:

$$\cdots \longrightarrow P_{\sigma^k(i)} \stackrel{g}{\longrightarrow} P_i \stackrel{h}{\longrightarrow} P_{\sigma^k(i)} \stackrel{g}{\longrightarrow} P_i \longrightarrow L(i,k) \longrightarrow 0 \tag{*}$$

where g and h are induced by $e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i$ and $e_i \mapsto \alpha_{\sigma^{n-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^k(i)}$, respectively.

Case 2. $i \notin \Delta$ and $\sigma^k(i) \in \Delta$.

By Lemma 4.9(2), $\ell(P_i) = n + r$ for some $r \in \mathbb{N}^+$ such that $\sigma^r(i) \in \Delta$ and $k \ge r$.

If k = r, then $L(\sigma^k(i), n + r - k) = L(\sigma^r(i), n)$ which is projective by Lemma 4.9(2). In this case, $qpd_{A_{n,m}}(L(i,k)) = pd_{A_{n,m}}(L(i,k)) = 1$, due to Proposition 3.1.

Suppose k > r. Then $\sigma^k(i) \in \Delta$ and $\sigma^{n+r-k}(\sigma^k(i)) = \sigma^{n+r}(i) = \sigma^r(i) \in \Delta$. By Case 1, $L(\sigma^k(i), n+r-k)$ is periodic, and so is $\Omega(L(i,k))$. Applying (*) to $\Omega(L(i,k))$, we can construct a projective resolution of L(i,k) as follows:

$$\cdots \xrightarrow{g_1} P_{\sigma^k(i)} \xrightarrow{h_1} P_{\sigma^r(i)} \xrightarrow{g_1} P_{\sigma^k(i)} \xrightarrow{f} P_i \longrightarrow L(i,k) \longrightarrow 0$$

where f, g_1 and h_1 are induced by

$$e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i, \quad e_{\sigma^r(i)} \mapsto \alpha_{\sigma^{n+r-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^k(i)}, \quad e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \cdots \alpha_{\sigma^{r+1}(i)} \alpha_{\sigma^r(i)},$$

respectively. Let $\beta: P_{\sigma^r(i)} \to P_i$ be the $A_{n,m}$ -module homomorphism induced by $e_{\sigma^r(i)} \mapsto \alpha_{\sigma^{r-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i$. Then we obtain the following commutative diagram:

By Theorem 1.2(4), $qpd_{A_{n,m}}(L(i,k)) \leq 1$.

Case 3. $i \in \Delta$ and $\sigma^k(i) \not\in \Delta$.

By Lemma 4.9(2), $\ell(P_i) = n$ and $\ell(P_{\sigma^k(i)}) = n + r$ for some r > 0 with $\sigma^r(\sigma^k(i)) = \sigma^{r+k}(i) \in \Delta$. Then $\Omega(L(i,k)) \simeq L(\sigma^k(i), \ell(P_i) - k) = L(\sigma^k(i), n - k)$ which is non-projective by n - k < n. It follows that

$$\Omega^2(L(i,k)) \simeq \Omega(L(\sigma^k(i), n-k)) \simeq L(\sigma^n(i), r+k) = L(i, r+k).$$

If L(i,r+k) is projective, then $\operatorname{qpd}_{A_{n,m}}(L(i,k)) = \operatorname{pd}_{A_{n,m}}(L(i,k)) = 2$. Now, assume that L(i,r+k) is non-projective. Since $\ell(P_i) = n$ and L(i,r+k) is a quotient of P_i , we have 0 < r+k < n. Recall that $i \in \Delta$ and $\sigma^{r+k}(i) \in \Delta$. By Case 1, the module L(i,r+k) is periodic. Similarly, by (*), we can construct a projective resolution of L(i,k) as follows:

$$\cdots \xrightarrow{g_2} P_i \xrightarrow{h_2} P_{\sigma^{r+k}(i)} \xrightarrow{g_2} P_i \xrightarrow{f_1} P_{\sigma^k(i)} \xrightarrow{f_0} P_i \longrightarrow L(i,k) \longrightarrow 0,$$

where f_0 , f_1 , g_2 and h_2 are induced by

$$e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i, \quad e_i \mapsto \alpha_{\sigma^{n-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^k(i)},$$

$$e_{\sigma^{r+k}(i)} \mapsto \alpha_{\sigma^{r+k-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i, \quad e_i \mapsto \alpha_{\sigma^{n-1}(i)} \cdots \alpha_{\sigma^{r+k+1}(i)} \alpha_{\sigma^{r+k}(i)},$$

respectively. Further, let $\gamma: P_{\sigma^{r+k}(i)} \to P_{\sigma^k(i)}$ be the $A_{n,m}$ -module homomorphism induced by $e_{\sigma^{r+k}(i)} \mapsto \alpha_{\sigma^{r+k-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^k(i)}$. Then there is a commutative diagram:

By Theorem 1.2(4), $\operatorname{qpd}_{A_{n,m}}(L(i,k)) \leq 2$.

Case 4. $i \notin \Delta$ and $\sigma^k(i) \notin \Delta$.

By Lemma 4.9(2), $\ell(P_i) = n + r_1$ and $\ell(P_{\sigma^k(i)}) = n + r_2$ for $0 < r_1 < n$ and $0 < r_2 < n$ such that $\sigma^{r_1}(i), \sigma^{r_2+k}(i) \in \Delta$. By Lemma 4.9(4), $\Omega(L(i,k)) \simeq L(\sigma^k(i), n + r_1 - k)$.

If either $\Omega(L(i,k))$ or $\Omega^2(L(i,k))$ is projective, then $\operatorname{pd}_{A_{n,m}}(L(i,k)) \leqslant 2$. In this case, $\operatorname{qpd}_{A_{n,m}}(L(i,k)) = \operatorname{pd}_{A_{n,m}}(L(i,k)) \leqslant 2$ by Proposition 3.1.

Now, assume that both $\Omega(L(i,k))$ and $\Omega^2(L(i,k))$ are non-projective. Then $n+r_1-k<\ell(P_{\sigma^k(i)})=n+r_2$. It follows that

$$\Omega^2(L(i,k)) \simeq \Omega(L(\sigma^k(i), n+r_1-k)) \simeq L(\sigma^{n+r_1}(i), r_2-r_1+k) = L(\sigma^{r_1}(i), r_2-r_1+k).$$

By $\sigma^{r_1}(i) \in \Delta$, we have $\ell(P_{\sigma^{r_1}(i)}) = n$ by Lemma 4.9(2). Since $\Omega^2(L(i,k))$ is non-projective, it is clear that $0 < r_2 - r_1 + k < n$. Note that $\sigma^{r_1}(i) \in \Delta$ and $\sigma^{r_2 - r_1 + k}(\sigma^{r_1}(i)) = \sigma^{r_2 + k}(i) \in \Delta$. Thus $L(\sigma^{r_1}(i), r_2 - r_1 + k)$ is periodic by Case 1. Still by (*), we can construct a projective resolution of L(i,k) as follows:

$$\cdots \xrightarrow{g_3} P_{\sigma^{r_1}(i)} \xrightarrow{h_3} P_{\sigma^{r_2+k}(i)} \xrightarrow{g_3} P_{\sigma^{r_1}(i)} \xrightarrow{d_1} P_{\sigma^k(i)} \xrightarrow{d_0} P_i \longrightarrow L(i,k) \longrightarrow 0$$

where d_0, d_1, g_3 and h_3 are induced by

$$e_{\sigma^k(i)} \mapsto \alpha_{\sigma^{k-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_i, \quad e_{\sigma^{r_1}(i)} \mapsto \alpha_{\sigma^{n+r_1-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^k(i)},$$

$$e_{\sigma^{r_2+k}(i)} \mapsto \alpha_{\sigma^{r_2+k-1}(i)} \cdots \alpha_{\sigma^{r_1+1}(i)} \alpha_{\sigma^{r_1}(i)}, \quad e_{\sigma^{r_1}(i)} \mapsto \alpha_{\sigma^{n+r_1-1}(i)} \cdots \alpha_{\sigma^{r_2+k+1}(i)} \alpha_{\sigma^{r_2+k}(i)},$$

Further, let $\delta: P_{\sigma^{r_2+k}(i)} \to P_{\sigma^k(i)}$ and $\epsilon: P_{\sigma^{r_1}(i)} \to P_i$ be the $A_{n,m}$ -module homomorphisms induced by

$$e_{\sigma^{r_2+k}(i)} \mapsto \alpha_{\sigma^{r_2+k-1}(i)} \cdots \alpha_{\sigma^{k+1}(i)} \alpha_{\sigma^{k}(i)} \quad e_{\sigma^{r_1}(i)} \mapsto \alpha_{\sigma^{r_1-1}(i)} \cdots \alpha_{\sigma(i)} \alpha_{i}$$

respectively. Then there is a commutative diagram:

Thus $\operatorname{qpd}_{A_{n,m}}(L(i,k)) \leq 2$ by Theorem 1.2(4).

Proof of Theorem 4.8. (1) Without loss of generality, we can assume $\lambda_1 = 1$. It can be checked that

$$pd_{A_{n,m}}(S_i) = \begin{cases} 2, & \text{if } i = 1, \\ 1, & \text{otherwise.} \end{cases}$$

- (2) $A_{n,n}$ is a self-injective algebra of finite representation type. By Remark 4.4, $\operatorname{qgldim}(A_{n,n}) = 0$.
- (3) Note that $\{L(i,k) \mid 1 \le i \le n, 1 \le k \le \ell(P_i)\}$ is the set of isomorphism classes of indecomposable $A_{n,m}$ -modules. By Theorem 1.1(1),

$$\operatorname{qgldim}(A_{n,m}) = \sup \{\operatorname{qpd}_{A_{n,m}}(L(i,k)) \mid 1 \leqslant i \leqslant n, 0 < k \leqslant \ell(P_i)\}.$$

Combining Lemma 4.10 with Lemma 4.11, we have $\operatorname{qpd}_{A_{n,m}}(L(i,k)) \leqslant 2$ for $1 \leqslant i \leqslant n$ and $0 < k \leqslant \ell(P_i)$. This implies $\operatorname{qgldim}(A_{n,m}) \leqslant 2$. Moreover, since $\Delta \neq \{1,2,\cdots,n\}$, there exists an indecomposable projective $A_{n,m}$ -module P_i such that $\ell(P_i) > n$ by Lemma 4.9(2). It follows from Lemma 4.10 that $\operatorname{qpd}_{A_{n,m}}(L(i,n)) = \operatorname{pd}_{A_{n,m}}(L(i,n)) = 2$. Thus $\operatorname{qgldim}(A_{n,m}) = 2$. Since n > 2 and 1 < m < n, there exists an integer $i \in \Delta$ such that $\sigma^k(i) \in \Delta$ for some integer k with 0 < k < n. According to Case 1 in the proof of Lemma 4.11, the module L(i,k) is periodic but non-projective. This implies $\operatorname{gldim}(A_{n,m}) = \infty$.

Acknowledgements. The research work was partially supported by the National Natural Science Foundation of China (Grant 12031014).

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