

TRIANGULATED CATEGORIES WITH A COMPACT SILTING OBJECT, BROWN-COMENETZ DUALITY AND BROWN REPRESENTABILITY THEOREMS

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ABSTRACT. In this paper, we establish a dual framework for Neeman’s results concerning triangulated categories with compact silting objects by employing Brown–Comenetz duality. This framework introduces an intrinsic non-compact subcategory, provides its characterization, and demonstrates representability theorems for both the low-bounded and bounded subcategories. Additionally, it elucidates how recollements are restricted to short exact sequences on the dual (non-compact) side. Several localization results and specific applications are also derived.

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1. INTRODUCTION

Recollements and representability theorems lie at the heart of modern homological algebra and the study of derived categories[2, 3, 15]. Neeman’s work on compactly generated triangulated categories[13] clarified how recollements interact with compact objects and produced powerful structural and localization results. Yet a parallel, fully developed theory on the non-compact side—one that treats the duals of compact generators and the categories they generate—has remained incomplete. This paper fills that gap by developing a dual framework based on Brown–Comenetz duality (see Section 3), introducing an intrinsic non-compact subcategory, and proving representability and localization results that mirror and extend Neeman’s compact-object picture[15] and the results of Sun–Zhang[23].

The paper’s principal structural result is a dual Neeman theorem: for a recollement of compactly generated, locally Hom-finite triangulated categories, the third row (the Brown–Comenetz dual side) restricts, up to direct summands, to a short exact sequence of the thick subcategories generated by the Brown–Comenetz duals of the compact generators. This complements Neeman’s original restriction

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on the compact row and yields a full duality between compact and non-compact localization phenomena in recollements.

Theorem 1.1 (Proposition 3.3). *Let the following diagram be a recollement of compactly generated locally Hom-finite k -linear triangulated categories*

$$\begin{array}{ccccc} & i^* & & j_! & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S}. \\ & i^! & & j_* & \end{array}$$

Then the third row restricts to a short exact sequence up to direct summands

$$\mathcal{E}_s \xrightarrow{j_*} \mathcal{E}_t \xrightarrow{i^!} \mathcal{E}_r$$

where $\mathcal{E}_s, \mathcal{E}_t, \mathcal{E}_r$ are the Brown-Comenetz duality of $\mathcal{S}^c, \mathcal{T}^c, \mathcal{R}^c$, respectively.

In particular, for recollements of derived categories of finite-dimensional algebras, the third row restricts to a short exact sequence of homotopy categories of bounded complexes of injectives.

Let \mathcal{T} be an approxiamble triangulated category, in the sense of Neeman [15]. We introduce the intrinsic subcategory \mathcal{T}_c^+ as the formal dual of the subcategory \mathcal{T}_c^- that introduced in [15]. Objects of \mathcal{T}_c^+ are characterized by triangles whose “cohomologically bounded” piece lies in the Brown–Comenetz dual \mathcal{E} of the compact subcategory \mathcal{T}^c . We show that \mathcal{T}_c^+ admits a concrete and useful description: every object of \mathcal{T}_c^+ is a homotopy limit of a strong \mathcal{E} -coapproximating system, and conversely every such homotopy limit lies in \mathcal{T}_c^+ . This homotopy-limit characterization is the technical backbone that allows us to transport finiteness and representability properties from \mathcal{E} to \mathcal{T}_c^+ .

Using the intrinsic description of \mathcal{T}_c^+ and \mathcal{T}_c^b , we can prove two representability theorems below. The key difference between our approach and previous work in [15] lies in the use of Brown-Comenetz duality to construct the dual subcategory \mathcal{T}_c^+ , rather than relying solely on the compact generator. This allows us to extend the results to the non-compact setting. To overcome some technical obstacles, we need to limit our discussion to triangulated categories with compact silting objects. These kind of triangulated categories are approxiamble triangulated categories.

Theorem 1.2 (Theorem 5.18 and Theorem 5.22). *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object.*

(1) *The Yoneda functor*

$$y : \mathcal{T}_c^{+,\text{op}} \longrightarrow \text{Hom}_k(\mathcal{E}, k\text{-Mod}).$$

is full, and the essential image consists of all locally finite \mathcal{E} -homological functors.

(2) *The Yoneda functor*

$$y : \mathcal{T}_c^{b,\text{op}} \longrightarrow \text{Hom}_k(\mathcal{E}, k\text{-Mod}).$$

is fully faithful, and the essential image consists of all finite \mathcal{E} -homological functors.

These results give a clean dual description of representable functors on both the bounded and cohomologically bounded-above sides of an approxiamble triangulated category which has been studied by Neeman in [15, Theorem 9.18, Theorem 9.20]. The dual framework has concrete consequences. We establish localization theorems which concern the two constructed new intrinsic subcategories \mathcal{T}_c^+ and \mathcal{E} , this extends the results in [23].

Theorem 1.3 (Theorem 6.5 and Theorem 6.7). *Let the following diagram be a recollement of locally Hom-finite k -linear triangulated categories, each of which admits a compact silting object.*

$$\begin{array}{ccccc} & i^* & & j_1 & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S} \\ i^! & & & & j_* \end{array}$$

Then

- (1) Suppose that \mathcal{T} has a compact generator G such that there is an integer N with $\text{Hom}_{\mathcal{T}}(G, G[n]) = 0, n < N$. If the recollement can extends one step upwards. The first row induces a short exact sequence

$$\mathcal{S}_c^+ / \mathcal{S}^c \xrightarrow{\overline{j}_1} \mathcal{T}_c^+ / \mathcal{T}^c \xrightarrow{\overline{i}^*} \mathcal{R}_c^+ / \mathcal{R}^c.$$

- (2) The second row induces a short exact sequence

$$\mathcal{R}_c^+ / \mathcal{R}_c^b \xrightarrow{\overline{i}_*} \mathcal{T}_c^+ / \mathcal{T}_c^b \xrightarrow{\overline{j}^*} \mathcal{S}_c^+ / \mathcal{S}_c^b.$$

- (3) The third row induces a short exact sequence

$$\mathcal{S}_c^+ / \mathcal{E}_s \xrightarrow{\overline{j}_*} \mathcal{T}_c^+ / \mathcal{E}_t \xrightarrow{\overline{i}^!} \mathcal{R}_c^+ / \mathcal{E}_r.$$

Finally, we give criteria for constructing adjoints which extends [15, Theorem 13.1] to our dual setting.

Methodologically, the paper combines Brown–Comenetz duality (viewed as a partial Serre functor on compact objects), careful control of t -structures in approximable categories, and the new notion of strong \mathcal{E} -coapproximating systems. These tools let us transfer finiteness, vanishing, and homological stabilization properties across the compact/non-compact divide and produce explicit constructions of objects and adjoints. Compared to previous approaches, which primarily focused on the compact objects and their properties, we provides a complete duality between the compact and non-compact settings. This duality is essential for a full understanding of the structure of approximable triangulated categories and has potential applications in the study of singularity categories, derived equivalences, and other related areas.

The remainder of the paper is organized as follows. Section 2 recalls notation, t -structures, short exact sequences of triangulated categories, and the approximable framework. Section 3 develops Brown–Comenetz duality and proves the dual Neeman theorem. Section 4 defines \mathcal{T}_c^+ , proves its homotopy-limit characterization, and establishes basic finiteness properties. Section 5 contains the representability theorems for \mathcal{T}_c^+ and for \mathcal{E}_t . Section 6 gives localization results for \mathcal{T}_c^+ and \mathcal{E}_t , and their related Verdier quotient categories. Section 7 presents examples and applications, including the restriction to homotopy categories of bounded injectives and constructions of adjoints. An appendix collects basic facts about colimits used in the arguments.

2. PRELIMINARIES

In this section, we collect the notation, definitions, and basic facts that will be used throughout the paper. Throughout, k denotes a field.

2.1. Notation. Let \mathcal{T} be a triangulated category. For two subcategories \mathcal{U}, \mathcal{V} of \mathcal{T} , we set

$$\mathcal{U} * \mathcal{V} := \{T \in \mathcal{T} \mid T \text{ admits a triangle } U \rightarrow T \rightarrow V, U \in \mathcal{U}, V \in \mathcal{V}\}.$$

We also define

$$\mathcal{U}^\perp := \{M \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(U, M) = 0, \forall U \in \mathcal{U}\}$$

as the full subcategory of \mathcal{T} right orthogonal to \mathcal{U} . The left orthogonal to \mathcal{U} is defined dually.

Assume \mathcal{T} has small coproducts and let G be an object of \mathcal{T} . For $a < b \in \mathbb{Z} \cup \{-\infty, +\infty\}$, we denote by $\langle G \rangle^{[a,b]}$ the smallest subcategory of \mathcal{T} which contains $G[-i]$ where $a \leq i \leq b$ and is closed under direct summands, coproducts and extensions. For a positive integer n , the subcategories $\langle G \rangle_n^{[a,b]}$ and $\overline{\langle G \rangle}_n^{[a,b]}$ are defined inductively by

$$\begin{aligned} \langle G \rangle_1^{[a,b]} &= \text{direct summands of finite direct sums of } \{G[-i] \mid a \leq i \leq b\}, \\ \langle G \rangle_n^{[a,b]} &= \text{direct summands of objects in } \langle G \rangle_1^{[a,b]} * \langle G \rangle_{n-1}^{[a,b]}, \\ \overline{\langle G \rangle}_1^{[a,b]} &= \text{direct summands of coproducts of } \{G[-i] \mid a \leq i \leq b\}, \\ \overline{\langle G \rangle}_n^{[a,b]} &= \text{direct summands of objects in } \overline{\langle G \rangle}_1^{[a,b]} * \overline{\langle G \rangle}_{n-1}^{[a,b]}. \end{aligned}$$

We denote $\langle G \rangle := \bigcup_{i \geq 1} \langle G \rangle_i^{[-i,i]}$. Clearly $\langle G \rangle$ is the smallest thick subcategory of \mathcal{T} containing G . For simplicity, we denote $\langle G \rangle_n^{(-\infty,+\infty)}$ by $\langle G \rangle_n$. Dually, for an object $E \in \mathcal{T}$, we define $\overline{\langle E \rangle}_n^{[a,b]}$ by replacing coproducts with products in the definition of $\overline{\langle G \rangle}_n^{[a,b]}$.

Let A be a ring. We denote by $A\text{-Mod}$ (resp., $A\text{-mod}$, $A\text{-proj}$) the category of right A -modules (resp., finitely presented right A -modules, finitely generated projective right A -modules). We write $\mathcal{D}(A)$ (resp., $\mathcal{D}^b(A\text{-mod})$, $\mathcal{D}^-(A\text{-mod})$, $\mathcal{K}^b(A\text{-proj})$, $\mathcal{K}^{-,b}(A\text{-proj})$) for the derived category (resp., bounded derived category, upper-bounded derived category, homotopy category of bounded complexes, homotopy category of upper-bounded complexes with bounded cohomology) of the corresponding module categories.

2.2. t -structures and homological facts for triangulated categories. We briefly recall some basic notions concerning t -structures. A pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories of a triangulated category \mathcal{T} is called a t -structure [3, Definition 1.3.1] if:

- (1) $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$;
- (2) $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}[-1]) = 0$;
- (3) For any $T \in \mathcal{T}$, there exists a triangle

$$U \longrightarrow T \longrightarrow V \longrightarrow U[1],$$

where $U \in \mathcal{T}^{\leq 0}$ and $V \in \mathcal{T}^{\geq 0}[-1]$.

For each $n \in \mathbb{Z}$, we denote

$$\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n] \text{ and } \mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n].$$

The *heart* of the t -structure is $\mathcal{H} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$, which yields a cohomological functor $\mathbf{H}^0 : \mathcal{T} \rightarrow \mathcal{H}$. For $i \in \mathbb{Z}$, we set $\mathbf{H}^i := \mathbf{H}^0 \circ [i]$.

Two t -structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ are said to be *equivalent* if there exists $n \in \mathbb{N}$ such that

$$\mathcal{T}_1^{\leq -n} \subseteq \mathcal{T}_2^{\leq 0} \subseteq \mathcal{T}_1^{\leq n}.$$

When \mathcal{T} has coproducts and a compact generator G , the results of [1, Theorem A.1] and [17, Theorem 2.3] show that G determines a canonical t -structure

$$(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) := (\overline{\langle G \rangle}^{(-\infty, 0]}, (\overline{\langle G \rangle}^{(-\infty, -1)})^\perp),$$

whose equivalence class is called the *preferred equivalence class*.

In a triangulated category with a compact generator G , we consider the subcategories

$$\mathcal{T}^- := \bigcup_{n \in \mathbb{Z}} \mathcal{T}_G^{\leq n}, \quad \mathcal{T}^+ := \bigcup_{n \in \mathbb{Z}} \mathcal{T}_G^{\geq n}, \quad \mathcal{T}^b := \mathcal{T}^- \cap \mathcal{T}^+,$$

and those defined via compact approximations

$$\mathcal{T}_c^- := \bigcap_{i=1}^{\infty} (\mathcal{T}^c * \mathcal{T}_G^{\leq -i}), \quad \mathcal{T}_c^b := \mathcal{T}_c^- \cap \mathcal{T}^b.$$

Definition 2.1. A sequence

$$\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E_1$$

in **Ab** satisfies the *Mittag-Leffler condition* if for each i there exists $j \geq i$ such that the image of $E_s \rightarrow E_i$ equals the image of $E_j \rightarrow E_i$ for all $s \geq j$.

Clearly, if the sequence $\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E_1$ consists of finite-dimensional vector spaces over a field k , then it automatically satisfies the Mittag-Leffler condition.

Lemma 2.2. *If the sequence*

$$\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E_1$$

*in **Ab** satisfies the Mittag-Leffler condition, then there exists an exact sequence in **Ab**:*

$$0 \rightarrow \varprojlim(E_*) \rightarrow \prod_{k=1}^{\infty} E_k \xrightarrow{1-p} \prod_{k=1}^{\infty} E_k \rightarrow 0.$$

Lemma 2.3. *Let \mathcal{T} be a k -linear triangulated category and*

$$\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E_1$$

a sequence in \mathcal{T} . For an object $M \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}(M, E_i)$ is finite-dimensional for all $i > 0$, there is a canonical isomorphism

$$\varprojlim \text{Hom}_{\mathcal{T}}(M, E_*) \simeq \text{Hom}_{\mathcal{T}}(M, \varprojlim(E_*)).$$

Lemma 2.4. *Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t -structure on \mathcal{T} . For $n \in \mathbb{Z}$, if $F \in \mathcal{T}^+$ satisfies $\mathbf{H}^i(F) = 0$ for all $i < n$, then $F \in \mathcal{T}^{\geq n}$.*

Proof. Assume $F \in \mathcal{T}^{\geq k}$ for some $k < n$. Then there exists a triangle

$$\mathbf{H}^k(F)[-k] \rightarrow F \rightarrow F^{\geq k+1} \rightarrow \mathbf{H}^k(F)[-k+1]$$

where $\mathbf{H}^k(F)[-k] = 0 = \mathbf{H}^k(F)[-k+1]$. Thus we have $F \simeq F^{\geq k+1}$, and consequently $F \in \mathcal{T}^{\geq n}$. \square

Lemma 2.5. *Let \mathcal{T} be a compactly generated triangulated category equipped with a t -structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that $\mathcal{T}^{\leq 0}$ is closed under products. Consider a sequence*

$$(E_*, f_*): \cdots \xrightarrow{f_3} E_3 \xrightarrow{f_2} E_2 \xrightarrow{f_1} E_1$$

in \mathcal{T} . If for every integer i there exists $N_i > 0$ such that $\mathbf{H}^i(f_j)$ is an isomorphism for all $j > N_i$, then for all $i \in \mathbb{Z}$, there is a canonical isomorphism

$$\mathbf{H}^i(\varprojlim(E_*)) \simeq \varprojlim(\mathbf{H}^i(E_*)).$$

Proof. By definition of the homotopy limit, there is a triangle

$$\varprojlim(E_*) \longrightarrow \prod_{k=1}^{\infty} E_k \xrightarrow{1-p} \prod_{k=1}^{\infty} E_k \longrightarrow \varprojlim(E_*)[1].$$

Applying $\mathbf{H}^i(-)$ gives the long exact sequence

$$\mathbf{H}^{i-1}\left(\prod_{k=1}^{\infty} E_k\right) \xrightarrow{\mathbf{H}^{i-1}(1-p)} \mathbf{H}^{i-1}\left(\prod_{k=1}^{\infty} E_k\right) \longrightarrow \mathbf{H}^i(\varprojlim(E_*)) \longrightarrow \mathbf{H}^i\left(\prod_{k=1}^{\infty} E_k\right) \xrightarrow{\mathbf{H}^i(1-p)} \mathbf{H}^i\left(\prod_{k=1}^{\infty} E_k\right).$$

Since $\ker(\mathbf{H}^i(1-p)) \simeq \varprojlim(\mathbf{H}^i(E_*))$, it remains to show that $\mathbf{H}^{i-1}(1-p)$ is surjective. Consider the inverse system

$$\dots \longrightarrow \mathbf{H}^{i-1}(E_2) \longrightarrow \mathbf{H}^{i-1}(E_1).$$

Then we obtain a short exact sequence

$$0 \rightarrow \varprojlim(\mathbf{H}^{i-1}(E_*)) \rightarrow \prod_{k=1}^{\infty} \mathbf{H}^{i-1}(E_k) \xrightarrow{\mathbf{H}^{i-1}(1-p)} \prod_{k=1}^{\infty} \mathbf{H}^{i-1}(E_k).$$

For any $H \in \mathcal{H}$, since $H^{i-1}(f_j)$ is an isomorphism for all $j > N_{i-1}$, the inverse system $\mathbf{Hom}_{\mathcal{T}}(H, \mathbf{H}^{i-1}(E_*))$ satisfies the Mittag–Leffler condition in **Ab**. Hence, by Lemma 2.2, applying the functor $\mathbf{Hom}_{\mathcal{T}}(H, -)$ yields the exact sequence:

$$0 \rightarrow \mathbf{Hom}_{\mathcal{T}}(H, \varprojlim(\mathbf{H}^{i-1}(E_*))) \rightarrow \mathbf{Hom}_{\mathcal{T}}\left(H, \prod_{k=1}^{\infty} \mathbf{H}^{i-1}(E_k)\right) \rightarrow \mathbf{Hom}_{\mathcal{T}}\left(H, \prod_{k=1}^{\infty} \mathbf{H}^{i-1}(E_k)\right) \rightarrow 0.$$

Since $\mathcal{T}^{\leq 0}$ is closed under products, the product $\prod_{k=1}^{\infty} \mathbf{H}^{i-1}(E_k)$ lies in \mathcal{H} ; denote it by H . Hence $\mathbf{H}^{i-1}(1-p)$ is a split epimorphism, completing the proof. \square

2.3. Short exact sequences of triangulated categories. In analogy with the notion of short exact sequences in abelian categories, a sequence of triangulated categories

$$\mathcal{R} \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{S}$$

is a *short exact sequence* if it satisfies

- (1) F is fully faithful;
- (2) $GF = 0$; and
- (3) the induced functor $\overline{G} : \mathcal{T}/\mathcal{R} \rightarrow \mathcal{S}$ is an equivalence.

The triangulated analogue of the classical 3×3 lemma [10, Lemma 3.2] will be used frequently.

Lemma 2.6. ([10, Lemma 3.2]) *Let*

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{i|_{\mathcal{U}}} & \mathcal{W} & \xrightarrow{\mathbf{F}|_{\mathcal{S}}} & \mathcal{V} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R} & \xrightarrow{i} & \mathcal{T} & \xrightarrow{\mathbf{F}} & \mathcal{S} \end{array}$$

be a commutative diagram of triangulated categories and triangle functors. Suppose the first row is exact up to direct summands and the second is exact. Then the

diagram can be completed into a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{U} & \xrightarrow{i|_{\mathcal{U}}} & \mathcal{W} & \xrightarrow{\mathbf{F}|_{\mathcal{W}}} & \mathcal{V} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{R} & \xrightarrow{i} & \mathcal{T} & \xrightarrow{\mathbf{F}} & \mathcal{S} \\
 \downarrow & & \downarrow q & & \downarrow \\
 \mathcal{R}/\mathcal{U} & \xrightarrow{\bar{i}} & \mathcal{T}/\mathcal{W} & \xrightarrow{\bar{\mathbf{F}}} & \mathcal{S}/\mathcal{V}
 \end{array}$$

If moreover, \bar{i} is fully faithful, then the third row is exact.

The lemma below provides useful sufficient conditions to detect whether the induced functor \bar{i} in the last lemma is fully faithful or not.

Lemma 2.7. ([12, Lemma 4.7.1], [11, Lemma 10.3]) Let \mathcal{T} be a triangulated category with two full triangulated subcategories \mathcal{R} and \mathcal{S} . Then we put $\mathcal{U} = \mathcal{S} \cap \mathcal{R}$ and we can form the following commutative diagram of exact functors

$$\begin{array}{ccc}
 \mathcal{U} & \longrightarrow & \mathcal{R} \xrightarrow{\text{cano.}} \mathcal{R}/\mathcal{U} . \\
 \downarrow & & \downarrow \\
 \mathcal{S} & \longrightarrow & \mathcal{T} \xrightarrow{\text{cano.}} \mathcal{T}/\mathcal{S}
 \end{array}$$

Assume that either

- (1) every morphism from an object in \mathcal{R} to an object in \mathcal{S} factors some object in \mathcal{U} , or
- (2) every morphism from an object in \mathcal{S} to an object in \mathcal{R} factors some object in \mathcal{U} .

Then the induced functor $\mathbf{J} : \mathcal{R}/\mathcal{U} \rightarrow \mathcal{T}/\mathcal{S}$ is fully faithful.

Definition 2.8. Let \mathcal{T} , \mathcal{R} and \mathcal{S} be triangulated categories. We say that \mathcal{T} is a *recollement* ([3, 1.4.3]) of \mathcal{R} and \mathcal{S} if there is a diagram of six triangle functors

$$\begin{array}{ccccc}
 & i^* & & j_! & \\
 & \swarrow & & \searrow & \\
 \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^* = j_*} & \mathcal{S} \\
 & \downarrow i^! & & \downarrow j_* & \\
 & & & &
 \end{array}$$

such that

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs;
- (2) i_* , j_* and $j_!$ are fully faithful functors;
- (3) $i^! j_* = 0$; and
- (4) for each object $T \in \mathcal{T}$, there are two triangles in \mathcal{T}

$$i_! i^!(T) \rightarrow T \rightarrow j_* j^*(T) \rightarrow i_! i^!(T)[1],$$

$$j_! j^!(T) \rightarrow T \rightarrow i_* i^*(T) \rightarrow j_! j^!(T)[1].$$

A recollement is said to *extend one step downwards* (resp., *upwards*) if additional adjoint functors exist so that the diagram extends accordingly.

2.4. Approximable triangulated categories. We conclude by recalling the notion of approximability introduced in [15, 19], which has become a powerful tool for studying of Bondal–Van den Bergh’s Conjecture and Rouquier’s Strong Generation Conjecture [18], in Rickard’s theorem on derived equivalences [9], and in recent progress on the existence of bounded t -structures [4, 20], and also provides the foundational framework underlying the representability and localization results developed in this paper. Let R be a commutative Noetherian ring and let \mathcal{T} be an R -linear triangulated category with a compact generator G . A functor $\mathbf{H} : (\mathcal{T}^c)^{\text{op}} \rightarrow R\text{-Mod}$ is called \mathcal{T}^c -cohomological if it sends distinguished triangles to long exact sequences. It is said to be *locally finite* (resp. *finite*) if $\mathbf{H}(T)$ is finitely generated for all $T \in \mathcal{T}^c$ and vanishes in sufficiently negative (resp. sufficiently large positive and negative) degrees. The category \mathcal{T} is *locally Hom-finite* if $\text{Hom}_{\mathcal{T}}(X, Y)$ is finitely generated over R for all compact objects $X, Y \in \mathcal{T}^c$.

Definition 2.9. A triangulated category \mathcal{T} with coproducts is called *approximable* if it admits a compact generator G and an integer $n > 0$ such that:

- (i) $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for all $i \geq n$;
- (ii) every object $X \in \mathcal{T}_G^{\leq 0}$ fits into a triangle

$$E \longrightarrow X \longrightarrow D \longrightarrow E[1]$$

with $E \in \overline{\langle G \rangle_n}^{[-n,n]}$ and $D \in \mathcal{T}_G^{\leq -1}$.

This notion is independent of the choice of compact generator and of the representative of the preferred equivalence class of t -structures [15, Facts 1.26].

Theorem 2.10. *Let R be a commutative Noetherian ring, \mathcal{T} a locally Hom-finite approximable R -linear triangulated category, and X an object in \mathcal{T} . Then*

- (1) ([8, Corollary 3.10]) $X \in \mathcal{T}^-$ (resp., \mathcal{T}^+) if and only if there exists an integer N such that $\text{Hom}_{\mathcal{T}}(G[n], X) = 0, n < N$ (resp., $n > N$);
- (2) ([15, Theorem 9.18]) $X \in \mathcal{T}_c^-$ if and only if the functor $\text{Hom}_{\mathcal{T}}(-, X)$ is locally finite \mathcal{T}^c -cohomological;
- (3) ([15, Theorem 9.20]) $X \in \mathcal{T}_c^b$ if and only if the functor $\text{Hom}_{\mathcal{T}}(-, X)$ is finite \mathcal{T}^c -cohomological.

3. BROWN–COMENETZ DUALITY OF COMPACT OBJECTS

In this section, we recall the Brown–Comenetz duality for compact objects in a triangulated category and establish a dual form of Neeman’s classical theorem on recollements. Throughout, let \mathcal{T} be a compactly generated k -linear triangulated category, and let \mathbf{D} denote the k -linear duality $\text{Hom}_k(-, k)$.

For a compact object $G \in \mathcal{T}$, the Brown representability theorem [6, 14] implies that the cohomological functor $\mathbf{D}\text{Hom}_{\mathcal{T}}(G, -)$ is representable. Hence, there exists an object $E \in \mathcal{T}$ and a canonical isomorphism

$$\text{Hom}_{\mathcal{T}}(-, E) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, -).$$

The object E is called the *Brown–Comenetz dual* of G , a notion originating in [7]. Brown–Comenetz duality extends functorially: it induces a triangle functor $\mathbf{S} : \mathcal{T}^c \rightarrow \mathcal{T}$, known as the *partial Serre functor* (see [21, Theorem 3.3]), such that

$$\text{Hom}_{\mathcal{T}}(-, \mathbf{S}(G)) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, -).$$

Let $\text{Im}(\mathcal{S})$ denote the essential image of \mathcal{T}^c under \mathbf{S} , and let \mathcal{E} be the smallest thick subcategory of \mathcal{T} containing $\text{Im}(\mathcal{S})$. We refer to \mathcal{E} as the *Brown–Comenetz dual* of the subcategory \mathcal{T}^c .

A typical example arises from finite-dimensional algebras: if A is a finite-dimensional k -algebra, then $\mathcal{D}(A)$ is a locally Hom-finite triangulated category with compact

generator A . In this case, $\mathcal{T}^c = \mathcal{K}^b(A\text{-proj})$ and its Brown–Comenetz dual is $\mathcal{K}^b(A\text{-inj})$, the homotopy category of bounded complexes of finitely generated injective modules.

Lemma 3.1. *The following statements hold.*

- (1) *If \mathcal{T} admits a compact generator G , then $\mathcal{E} = \langle \mathbf{S}(G) \rangle$.*
- (2) *If \mathcal{T} is locally Hom-finite, then $\mathbf{S} : \mathcal{T}^c \rightarrow \mathcal{E}$ is a triangle equivalence. In particular, $\mathcal{E} = \text{Im}(\mathbf{S})$.*

Proof. (1) In this case, $\mathcal{T}^c = \langle G \rangle$. Since \mathbf{S} is a triangle functor, we have $\text{Im}(\mathbf{S}) \subseteq \langle \mathbf{S}(G) \rangle$. Note that $\langle \mathbf{S}(G) \rangle$ is a thick subcategory of \mathcal{T} . Thus $\mathcal{E} \subseteq \langle \mathbf{S}(G) \rangle$. The inclusion $\langle \mathbf{S}(G) \rangle \subseteq \mathcal{E}$ is clear.

(2) By [21, Observation 3.4], the functor \mathbf{S} in this case is fully faithful. Hence we obtain a triangle equivalence $\mathbf{S} : \mathcal{T}^c \xrightarrow{\sim} \text{Im}(\mathbf{S})$. Thus it suffices to show $\text{Im}(\mathbf{S}) = \mathcal{E}$. By [5, Proposition 2.1.1], \mathcal{T}^c is Karoubian, i.e., every idempotent morphism in \mathcal{T}^c is split in \mathcal{T}^c . Consequently, $\text{Im}(\mathbf{S})$ is also Karoubian and therefore closed under direct summands. Hence $\text{Im}(\mathbf{S})$ is a thick subcategory of \mathcal{T} , and we obtain $\text{Im}(\mathbf{S}) = \mathcal{E}$. \square

We now recall a fundamental result of Neeman concerning recollements of compactly generated triangulated categories.

Lemma 3.2. ([13, Theorem 2.1]) *Let*

$$\begin{array}{ccccc} & & i^* & & \\ & \swarrow & & \searrow & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S} \\ \searrow & & \swarrow & & \\ & i^! & & j_* & \end{array}$$

be a recollement of compactly generated triangulated categories. Then the recollement restricts, up to direct summands, to a short exact sequence

$$\mathcal{S}^c \xrightarrow{j_!} \mathcal{T}^c \xrightarrow{i^*} \mathcal{R}^c.$$

The next proposition provides a dual counterpart of Neeman’s theorem, formulated on the level of Brown–Comenetz duals of compact objects.

Proposition 3.3. *Let*

$$\begin{array}{ccccc} & & i^* & & \\ & \swarrow & & \searrow & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S} \\ \searrow & & \swarrow & & \\ & i^! & & j_* & \end{array}$$

be a recollement of compactly generated, locally Hom-finite k -linear triangulated categories. Then the third row restricts, up to direct summands, to a short exact sequence

$$\mathcal{E}_s \xrightarrow{j_*} \mathcal{E}_t \xrightarrow{i^!} \mathcal{E}_r,$$

where $\mathcal{E}_s, \mathcal{E}_t, \mathcal{E}_r$ denote the Brown–Comenetz duals of $\mathcal{S}^c, \mathcal{T}^c, \mathcal{R}^c$, respectively.

Proof. We claim that the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccccc} \mathcal{S}^c & \xrightarrow{j_!} & \mathcal{T}^c & \xrightarrow{i^*} & \mathcal{R}^c \\ \downarrow \mathbf{s}_s & & \downarrow \mathbf{s}_t & & \downarrow \mathbf{s}_r \\ \mathcal{E}_s & \xrightarrow{j_*} & \mathcal{E}_t & \xrightarrow{i^!} & \mathcal{E}_r, \end{array}$$

where $\mathbf{S}_s, \mathbf{S}_t, \mathbf{S}_r$ are the corresponding partial Serre functors. We verify commutativity for the left square; the right square is analogous.

For $G \in \mathcal{S}^c$, we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(-, \mathbf{S}_t j_!(G)) &\simeq \mathbf{D}\mathrm{Hom}_{\mathcal{T}}(j_! G, -) && (\text{partial Serre functor}) \\ &\simeq \mathbf{D}\mathrm{Hom}_{\mathcal{S}}(G, j^*(-)) && (\text{adjunction}) \\ &\simeq \mathrm{Hom}_{\mathcal{S}}(j^*(-), \mathbf{S}_s(G)) && (\text{partial Serre functor}) \\ &\simeq \mathrm{Hom}_{\mathcal{T}}(-, j_* \mathbf{S}_s(G)) && (\text{adjunction}).\end{aligned}$$

By Yoneda's lemma, this yields a natural isomorphism

$$\mathbf{S}_t j_!|_{\mathcal{S}^c} \cong j_* \mathbf{S}_s|_{\mathcal{S}^c}.$$

Thus the left square commutes up to isomorphism. Since the top row is exact up to direct summands by Lemma 3.2, so is the bottom row. \square

Example 3.4. Let

$$\begin{array}{ccccc} & i^* & & j_! & \\ \mathcal{D}(B) & \xrightarrow{i_* = i_!} & \mathcal{D}(A) & \xrightarrow{j^! = j^*} & \mathcal{D}(C) \\ \curvearrowleft & & \curvearrowright & & \curvearrowleft \\ i^! & & & & j_* \end{array}$$

be a recollement of derived categories of finite-dimensional k -algebras. Neeman's theorem (Lemma 3.2) yields a short exact sequence up to direct summands

$$\mathcal{K}^b(C\text{-proj}) \xrightarrow{j_!} \mathcal{K}^b(A\text{-proj}) \xrightarrow{i^*} \mathcal{K}^b(B\text{-proj}).$$

Our dual result (Proposition 3.3) provides the corresponding sequence on injectives:

$$\mathcal{K}^b(C\text{-inj}) \xrightarrow{j_*} \mathcal{K}^b(A\text{-inj}) \xrightarrow{i^!} \mathcal{K}^b(B\text{-inj}).$$

4. THE INTRINSIC SUBCATEGORY \mathcal{T}_c^+

The aim of this section is to introduce and study a new intrinsic subcategory \mathcal{T}_c^+ of a k -linear triangulated category \mathcal{T} with a compact generator G . This construction should be regarded as the formal dual of the classical subcategory \mathcal{T}_c^- appearing in the theory of approximable triangulated categories. While \mathcal{T}_c^- plays a central role in controlling objects from above via compact approximations, the subcategory \mathcal{T}_c^+ provides a dual mechanism that approximates objects from below using the Brown–Comenetz duality. This dual viewpoint will be essential for the structural results established in later sections.

In this section, let \mathcal{T} be a k -linear triangulated category with a compact generator G , and set $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$. Let E be the Brown–Comenetz dual of G . Define

$$\mathcal{T}_c^+ := \{T \in \mathcal{T} \mid \forall m > 0, \exists \text{ a triangle } D \rightarrow T \rightarrow F \text{ with } F \in \mathcal{E}, D \in \mathcal{T}^{\geq m}\},$$

where \mathcal{E} denotes the Brown–Comenetz dual of \mathcal{T}^c . Note that $\mathcal{E} = \langle E \rangle$ in this situation. (see Lemma 3.1).

Example 4.1. Let A be a finite-dimensional k -algebra and set $\mathcal{T} := \mathcal{D}(A)$. Then the Brown–Comenetz dual of A is $\mathbf{D}(A)$, and in this case one has $\mathcal{T}_c^+ = \mathcal{K}^+(A\text{-inj})$.

Lemma 4.2. *If \mathcal{T} is locally Hom-finite, then*

- (1) $\mathrm{Hom}_{\mathcal{T}}(M, N)$ is finite-dimensional whenever $M \in \mathcal{T}^c$ and $N \in \mathcal{T}_c^+$.
- (2) $\mathrm{Hom}_{\mathcal{T}}(L, K)$ is finite-dimensional whenever $L \in \mathcal{T}_c^+$ and $K \in \mathcal{E}$.

Proof. (1) Since $\mathcal{T}^c = \langle G \rangle$, we have $\mathcal{T}^c \subseteq \mathcal{T}^-$. Let $M \in \mathcal{T}^c$ and $N \in \mathcal{T}_c^+$. There exists $k > 0$ such that $\text{Hom}_{\mathcal{T}}(M, \mathcal{T}^{\geq k}) = 0$. By the definition of \mathcal{T}_c^+ , we may choose a triangle

$$D \longrightarrow N \longrightarrow F \longrightarrow D[1]$$

with $D \in \mathcal{T}^{\geq k+1}$ and $F \in \mathcal{E}$. Applying $\text{Hom}_{\mathcal{T}}(M, -)$ yields an isomorphism

$$\text{Hom}_{\mathcal{T}}(M, N) \simeq \text{Hom}_{\mathcal{T}}(M, F).$$

Thus it suffices to show that $\text{Hom}_{\mathcal{T}}(M, F)$ is finite-dimensional for all $M \in \mathcal{T}^c$ and $F \in \mathcal{E}$.

Define

$$\mathcal{B} := \{Y \in \mathcal{T} \mid \forall M \in \mathcal{T}^c, \text{Hom}_{\mathcal{T}}(M, Y) \text{ is finite-dimensional}\}.$$

Since $\text{Hom}_{\mathcal{T}}(M, E) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, M)$ and \mathcal{T} is locally finite, the right-hand side is finite-dimensional. Thus $E \in \mathcal{B}$ and \mathcal{B} is nonempty. As \mathcal{B} is clearly a thick subcategory of \mathcal{T} , we conclude that $\mathcal{E} \subseteq \mathcal{B}$.

(2) Define

$$\mathcal{C} := \{Z \in \mathcal{T} \mid \forall L \in \mathcal{T}_c^+, \text{Hom}_{\mathcal{T}}(L, Z) \text{ is finite-dimensional}\}.$$

Clearly \mathcal{C} is a thick subcategory of \mathcal{T} . Thus it suffices to show that $E \in \mathcal{C}$. Fix $X \in \mathcal{T}_c^+$. We have

$$\text{Hom}_{\mathcal{T}}(X, E) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, X).$$

By (1), $\text{Hom}_{\mathcal{T}}(G, X)$ is finite-dimensional, and therefore so is $\mathbf{D}\text{Hom}_{\mathcal{T}}(G, X)$. Hence $E \in \mathcal{C}$, completing the proof. \square

Proposition 4.3. *Assume there exists an integer $n > 0$ such that $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \geq n$. Then the following statements hold.*

- (1) $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq 1}, E) = 0$ and $\mathcal{E} \subseteq \mathcal{T}^+$. Moreover, If $E' \in \mathcal{E}$, then there exists $B > 0$ such that $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq B}, E') = 0$;
- (2) \mathcal{T}_c^+ is a thick subcategory of \mathcal{T} , and contains \mathcal{E} .

Proof. (1) Since $G \in \mathcal{T}^{\leq 0}$, we have $\text{Hom}_{\mathcal{T}}(G, \mathcal{T}^{\geq 1}) = 0$. Then we obtain

$$\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq 1}, E) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, \mathcal{T}^{\geq 1}) = 0.$$

For every $i \in \mathbb{Z}$,

$$\text{Hom}_{\mathcal{T}}(G[i], E) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G, G[i]),$$

and hence $\text{Hom}_{\mathcal{T}}(G[i], E) = 0$ for $i \geq n$. By Theorem 2.10(1), we obtain $E \in \mathcal{T}^+$ and hence $\mathcal{E} \subseteq \mathcal{T}^+$.

Now let $E' \in \mathcal{E}$. By Lemma 3.1(1), there exists some $k > 0$ such that $E' \in \langle E \rangle_k^{[-k, k]}$. Since $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq k+1}, E[i]) = 0$ for $-k \leq i \leq k$ and $(\mathcal{T}^{\geq 1+k})^\perp$ is closed under finite direct sums, direct summands and extensions, it follows that $\langle E \rangle_k^{[-k, k]} \subseteq (\mathcal{T}^{\geq 1+k})^\perp$. Consequently, $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq 1+k}, E') = 0$.

(2) The conclusion follows from part (1) together with the strategy of Neeman's proof [15, Proposition 3.10]. \square

Definition 4.4. Let $\mathcal{B} \subseteq \mathcal{T}$ and $(E_*, f_*) : \cdots \xrightarrow{f_3} E_3 \xrightarrow{f_2} E_2 \xrightarrow{f_1} E_1$ be a sequence in \mathcal{B} .

- (1) It is called a *strong \mathcal{B} -coapproximating system* if for each $m \in \mathbb{N}^+$, the morphism $\mathbf{H}^i(f_m)$ is an isomorphism for all integers $i \leq m$.
- (2) Let $F \in \mathcal{T}$. If there exist morphisms $g_j : F \rightarrow E_j$ for all $j \in \mathbb{N}^+$ such that $g_j = f_j g_{j+1}$ for all $j \in \mathbb{N}^+$, and for each $m \in \mathbb{N}^+$, the morphism $\mathbf{H}^i(g_m)$ is an isomorphism for $i \leq m$, then (E_*, f_*) is called a *strong \mathcal{B} -coapproximating system* for F .

Lemma 4.5. *Assume that there exists an integer $n > 0$ such that $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \geq n$ and that $\mathcal{T}^{\leq 0}$ is closed under products, we have:*

- (1) *If (E_*, f_*) is a strong \mathcal{E} -coapproximating system, then it is a strong \mathcal{E} -coapproximating system for $\underline{\text{Holim}}(E_*)$. Moreover, $\underline{\text{Holim}}(E_*) \in \mathcal{T}_c^+$.*
- (2) *Let $F \in \mathcal{T}^+$ and let (E_*, f_*) be a strong \mathcal{E} -coapproximating system for F . Then the induced morphism $F \rightarrow \underline{\text{Holim}}(E_*)$ is an isomorphism.*

Proof. (1) By the definition of the homotopy limit, there is a triangle

$$\underline{\text{Holim}}(E_*) \xrightarrow{g} \prod_{k=1}^{\infty} E_k \longrightarrow \prod_{k=1}^{\infty} E_k \longrightarrow \underline{\text{Holim}}(E_*)[1].$$

Let $p_j: \prod_{k=1}^{\infty} E_k \rightarrow E_j$ be the canonical projection and set $g_j := p_j g$. Then $g_j = f_j g_{j+1}$ for all $j > 0$. For each $m \in \mathbb{N}^+$, Lemma 2.5 shows that $\mathbf{H}^i(g_m)$ is an isomorphism whenever $i \leq m$.

Since $E_1 \in \mathcal{E} \subseteq \mathcal{T}^+$ by Proposition 4.3(1), there exists $n > 0$ such that $E_1 \in \mathcal{T}^{\geq -n}$. By the definition of the system (E_*, f_*) , for any $m \geq 1$ and any $i \leq 1$ we have $\mathbf{H}^i(E_m) \simeq \mathbf{H}^i(E_1)$. Hence $\mathbf{H}^i(E_m) = 0$ for all $i < -n$. Since $E_m \in \mathcal{E} \subseteq \mathcal{T}^+$, Lemma 2.4 implies $E_m \in \mathcal{T}^{\geq -n}$, and therefore $\underline{\text{Holim}}(E_*) \in \mathcal{T}^+$.

Consider the triangle

$$D_m \longrightarrow \underline{\text{Holim}}(E_*) \xrightarrow{g_m} E_m \longrightarrow D_m[1].$$

Clearly $D_m \in \mathcal{T}^+$. Moreover, since $\mathbf{H}^i(g_m)$ is an isomorphism for all $i \leq m$, we obtain $\mathbf{H}^i(D_m) = 0$ for all $i \leq m$. Applying Lemma 2.4 again yields $D_m \in \mathcal{T}^{\geq m+1}$. Since m is arbitrary, the desire result follows.

(2) There exists a morphism $f: F \rightarrow \underline{\text{Holim}}(E_*)$ such that for every $m \geq 1$, the diagram

$$\begin{array}{ccc} F & & \\ f \downarrow & \searrow & \\ \underline{\text{Holim}}(E_*) & \longrightarrow & E_m \end{array}$$

commutes. For each $i \in \mathbb{Z}$, set $N_i := \max\{i + 1, 1\}$, the morphisms $\mathbf{H}^i(F) \rightarrow \mathbf{H}^i(E_{N_i})$ and $\mathbf{H}^i(\underline{\text{Holim}}(E_*)) \rightarrow \mathbf{H}^i(E_{N_i})$ are isomorphisms. By [8, Lemma 3.6], $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a nondegenerate t -structure. Then by [3, Proposition 1.3.7], the morphism f is an isomorphism. \square

Proposition 4.6. *Assume that there exists an integer $n > 0$ such that $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \geq n$ and that $\mathcal{T}^{\leq 0}$ is closed under products, we have:*

$$\mathcal{T}_c^+ = \{\underline{\text{Holim}}(E_*) \mid E_* \text{ is a strong } \mathcal{E}\text{-coapproximable system}\}.$$

Proof. By Lemma 4.5, it suffices to show that every $F \in \mathcal{T}_c^+$ admits a strong \mathcal{E} -coapproximating system. We construct such a system inductively.

By definition, there exists a triangle

$$D_1 \longrightarrow F \xrightarrow{g_1} E_1 \longrightarrow D_1[1]$$

with $D_1 \in \mathcal{T}^{\geq 3}$ and $E_1 \in \mathcal{E}$. Applying the cohomological functor $\mathbf{H}(-)$, we obtain an exact sequence

$$\mathbf{H}^i(D_1) \longrightarrow \mathbf{H}^i(F) \longrightarrow \mathbf{H}^i(E_1) \longrightarrow \mathbf{H}^{i+1}(D_1).$$

For all $i \leq 1$, we have $\mathbf{H}^i(D_1) = 0 = \mathbf{H}^{i+1}(D_1)$ since $D_1 \in \mathcal{T}^{\geq 3}$. Thus $\mathbf{H}^i(F) \simeq \mathbf{H}^i(E_1)$ for all $i \leq 1$.

Now let $n > 1$ and assume that morphisms $g_i: F \rightarrow E_i$ have been constructed for all $1 \leq i \leq n$. By Proposition 4.3(1), there exists $B' > 0$ such that $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq B'}, E_n) = 0$. Set $N = \max\{B', n + 3\}$. By definition, we obtain a triangle

$$D_{n+1} \longrightarrow F \xrightarrow{g_{n+1}} E_{n+1} \longrightarrow D_{n+1}[1]$$

with $D_{n+1} \in \mathcal{T}^{\geq N}$ and $E_{n+1} \in \mathcal{E}$. For all $i \leq n+1$, we still have $\mathbf{H}^i(F) \simeq \mathbf{H}^i(E_{n+1})$. Moreover, the vanishing $\text{Hom}_{\mathcal{T}}(D_{n+1}, E_n) = 0$ induces a morphism $f_n: E_{n+1} \rightarrow E_n$ satisfying $g_n = f_n g_{n+1}$. This completes the inductive step. \square

Lemma 4.7. *Assume that there exists an integer $n > 0$ such that $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \geq n$ and that $\mathcal{T}^{\leq 0}$ is closed under products. Let (E_*, f_*) be a strong \mathcal{E} -coapproximable system. Then there exists a natural transformation*

$$\varphi: \underrightarrow{\text{colim}} \text{Hom}_{\mathcal{T}}(E_*, -) \longrightarrow \text{Hom}_{\mathcal{T}}(\underrightarrow{\text{Holim}}(E_*), -)$$

whose restriction to \mathcal{E} is an isomorphism.

Proof. By the definition of the homotopy limit, there is a triangle

$$\underrightarrow{\text{Holim}}(E_*) \xrightarrow{g} \prod_{k=1}^{\infty} E_k \longrightarrow \prod_{k=1}^{\infty} E_k \longrightarrow \underrightarrow{\text{Holim}}(E_*)[1].$$

Let $p_j: \prod_{k=1}^{\infty} E_k \rightarrow E_j$ be the canonical projection and set $g_j := p_j g: \underrightarrow{\text{Holim}}(E_*) \rightarrow E_j$. Clearly $g_j = f_j g_{j+1}$ for all $j > 0$. Since

$$\mathcal{T}(g_j, -) = \mathcal{T}(g_{j+1}, -) \circ \mathcal{T}(f_j, -)$$

for every $j > 0$, the universal property of the colimit yields a natural transformation

$$\varphi: \underrightarrow{\text{colim}} \text{Hom}_{\mathcal{T}}(E_*, -) \longrightarrow \text{Hom}_{\mathcal{T}}(\underrightarrow{\text{Holim}}(E_*), -).$$

Define

$$\mathcal{B} := \{B \in \mathcal{T} \mid \varphi(B) \text{ is an isomorphism}\}.$$

We claim that if $M \rightarrow N \rightarrow L \rightarrow M[1]$ is a triangle with $M, M[1], L, L[-1] \in \mathcal{B}$, then $N \in \mathcal{B}$. Indeed, consider the commutative diagram

$$\begin{array}{ccccccc} \underrightarrow{\text{colim}} \mathcal{T}(E_*, L[-1]) & \longrightarrow & \underrightarrow{\text{colim}} \mathcal{T}(E_*, M) & \longrightarrow & \underrightarrow{\text{colim}} \mathcal{T}(E_*, N) & \longrightarrow & \underrightarrow{\text{colim}} \mathcal{T}(E_*, L) & \longrightarrow & \underrightarrow{\text{colim}} \mathcal{T}(E_*, M[1]) \\ \downarrow & & \downarrow & & \downarrow \varphi(N) & & \downarrow & & \downarrow \\ \mathcal{T}(\underrightarrow{\text{Holim}}(E_*), L[-1]) & \longrightarrow & \mathcal{T}(\underrightarrow{\text{Holim}}(E_*), M) & \longrightarrow & \mathcal{T}(\underrightarrow{\text{Holim}}(E_*), N) & \longrightarrow & \mathcal{T}(\underrightarrow{\text{Holim}}(E_*), L) & \longrightarrow & \mathcal{T}(\underrightarrow{\text{Holim}}(E_*), M[1]) \end{array}$$

whose rows are exact. By the five lemma, $\varphi(N)$ is an isomorphism, proving the claim. Thus it remains to show that $\langle E \rangle_1 \subseteq \mathcal{B}$. Since \mathcal{B} is closed under finite direct sums and direct summands, it suffices to show that $E[i] \in \mathcal{B}$ for all $i \in \mathbb{Z}$.

From the proof of Lemma 4.5(1), for every $m > 0$ there exists a triangle

$$D_m \longrightarrow \underrightarrow{\text{Holim}}(E_*) \xrightarrow{g_m} E_m \longrightarrow D_m[1]$$

with $D_m \in \mathcal{T}^{\geq m+1}$. Fix $i \in \mathbb{Z}$. As $G[i] \in \mathcal{T}^-$, the morphism

$$\text{Hom}_{\mathcal{T}}(G[i], \underrightarrow{\text{Holim}}(E_*)) \xrightarrow{\mathcal{T}(G[i], g_m)} \text{Hom}_{\mathcal{T}}(G[i], E_m)$$

is an isomorphism for $m \gg 0$. Since $\text{Hom}_{\mathcal{T}}(-, E[i]) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G[i], -)$ for all $i \in \mathbb{Z}$, the morphism

$$\text{Hom}_{\mathcal{T}}(E_m, E[i]) \xrightarrow{\mathcal{T}(g_m, E[i])} \text{Hom}_{\mathcal{T}}(\underrightarrow{\text{Holim}}(E_*), E[i])$$

is an isomorphism for $m \gg 0$. Consequently, $E[i] \in \mathcal{B}$ for all $i \in \mathbb{Z}$. \square

Corollary 4.8. *Assume that there exists an integer $n > 0$ such that $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \geq n$ and that $\mathcal{T}^{\leq 0}$ is closed under products. If $F \in \mathcal{T}_c^+$, then there exists a natural transformation*

$$\varphi : \underline{\text{colim}} \text{Hom}_{\mathcal{T}}(E_*, -) \longrightarrow \text{Hom}_{\mathcal{T}}(F, -)$$

such that the restriction $\varphi|_{\mathcal{E}}$ is an isomorphism, where E_ is a strong \mathcal{E} -coapproximable system for F .*

5. REPRESENTABILITY THEOREMS FOR \mathcal{T}_c^+ AND \mathcal{T}_c^b

This section is to develop representability theorems for the intrinsic subcategory \mathcal{T}_c^+ , which was introduced in the previous section as the formal dual counterparts of the classical subcategory \mathcal{T}_c^- appearing in Neeman's theory of approximable triangulated categories. While Neeman's representability results for \mathcal{T}_c^- [15, Theorem 9.18] rely on homotopy colimits and compact approximations, the dual theory for \mathcal{T}_c^+ involves homotopy limits and Brown–Comenetz duality, and is therefore technically more delicate. In particular, the dual arguments do not mirror the original ones in a straightforward manner.

In this section, let \mathcal{T} be an approximable k -linear triangulated category with a compact generator G , and set $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$. Let E be the Brown–Comenetz dual of G and let \mathcal{E} be the Brown–Comenetz dual of \mathcal{T}^c .

5.1. Homological finite functors. Let $\mathcal{B} \subseteq \mathcal{T}$ be a full subcategory closed under isomorphisms such that $\mathcal{B}[1] = \mathcal{B}$. A \mathcal{B} -homological functor is a k -linear functor $\mathbf{H} : \mathcal{B} \rightarrow k\text{-Mod}$ which sends triangles to long exact sequences (see [15, Definition 1.1]). Let \mathbf{H} be a \mathcal{B} -homological functor. We say that \mathbf{H} is *locally finite* (resp. *finite*) if for every $T \in \mathcal{B}$:

- (1) $\dim_k \mathbf{H}(T) < \infty$, and
- (2) $\mathbf{H}(T[i]) = 0$ for all $i \gg 0$ (resp. for all $|i| \gg 0$).

Remark 5.1. Since $\mathcal{E} = \langle E \rangle$ by Lemma 3.1(1), it follows from [15, Remark 1.2] that an \mathcal{E} -homological functor \mathbf{H} is locally finite (resp. finite) if and only if

$$\dim_k \mathbf{H}(E[j]) < \infty \text{ for each } j \in \mathbb{Z} \quad \text{and} \quad \mathbf{H}(E[i]) = 0 \text{ for } i \gg 0 \text{ (resp. } |i| \gg 0\text{).}$$

Lemma 5.2. *Let*

$$0 \longrightarrow \mathbf{H}_1 \longrightarrow \mathbf{H}_2 \longrightarrow \mathbf{H}_3 \longrightarrow 0$$

be an exact sequence in $\text{Hom}_k(\mathcal{E}, k\text{-Mod})$. If \mathbf{H}_2 and \mathbf{H}_3 are \mathcal{E} -homological functors, then so is \mathbf{H}_1 . Moreover, if \mathbf{H}_2 and \mathbf{H}_3 are locally finite (resp. finite), then \mathbf{H}_1 is also locally finite (resp. finite).

Lemma 5.3. *Assume that \mathcal{T} is locally Hom-finite.*

- (1) *If $X \in \mathcal{T}_c^+$, then $\text{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{E}}$ is a locally finite \mathcal{E} -homological functor.*
- (2) *If $X \in \mathcal{T}_c^+ \cap \mathcal{T}^b$, then $\text{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{E}}$ is a finite \mathcal{E} -homological functor.*

Proof. (1) By Proposition 4.3(1), there exists $m > 0$ such that $\text{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq m}, E) = 0$. Fix $j \in \mathbb{Z}$ and let $m' = \max\{m, m - j\}$. Since $X \in \mathcal{T}_c^+$, there is a triangle

$$D_{m'+1} \longrightarrow X \longrightarrow E' \longrightarrow D_{m'+1}[1]$$

with $D_{m'+1} \in \mathcal{T}^{\geq m'+1}$ and $E' \in \mathcal{E}$. Applying $\text{Hom}_{\mathcal{T}}(-, E[j])$ to this triangle yields an isomorphism

$$\text{Hom}_{\mathcal{T}}(X, E[j]) \simeq \text{Hom}_{\mathcal{T}}(E', E[j]).$$

By Lemma 4.2(2), the latter is finite-dimensional, and hence $\text{Hom}_{\mathcal{T}}(X, E[j])$ is finite-dimensional as well. Since $X \in \mathcal{T}_c^+ \subseteq \mathcal{T}^+$, it also follows from Proposition 4.3(1) that $\text{Hom}_{\mathcal{T}}(X, E[i]) = 0$ for $i \gg 0$. The conclusion then follows from Remark 5.1.

(2) By part (1), it remains to show that $\text{Hom}_{\mathcal{T}}(X, E[i]) = 0$ for all $i \ll 0$. Proposition 4.3(1) gives $E \in \mathcal{T}^+$. Then, since $X \in \mathcal{T}^b \subseteq \mathcal{T}^-$, we immediately obtain $\text{Hom}_{\mathcal{T}}(X, E[i]) = 0$ for all $i \ll 0$. \square

5.2. Preliminary representability theorem. The following lemma is a specialization of [15, Lemma 8.5]. It can also be seen as a dual argument of [15, Remark 9.7].

Lemma 5.4. *Let $f : M \rightarrow N$ be a morphism in \mathcal{T}_c^+ . Then there exists a sequence*

$$N[-1] \longrightarrow L \longrightarrow M \xrightarrow{f} N$$

in \mathcal{T}_c^+ such that

- (1) *The sequence*

$$\text{Hom}_{\mathcal{T}}(N, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(f, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(M, -)|_{\mathcal{E}} \longrightarrow \text{Hom}_{\mathcal{T}}(L, -)|_{\mathcal{E}} \longrightarrow \text{Hom}_{\mathcal{T}}(N[-1], -)|_{\mathcal{E}}$$

is a weak triangle (see [15, Definition 8.2]) in $\text{Hom}_R(\mathcal{E}, k\text{-Mod})$.

- (2) *For all $i \in \mathbb{Z}$, there exists an exact sequence*

$$\cdots \longrightarrow \mathbf{H}^{i-1}(N) \longrightarrow \mathbf{H}^i(L) \longrightarrow \mathbf{H}^i(M) \longrightarrow \mathbf{H}^i(N).$$

- (3) *If M admits a strong $\langle E \rangle_{n_1}$ -coapproximating system and N admits a strong $\langle E \rangle_{n_2}$ -coapproximating system, then L admits a strong $\langle E \rangle_{n_1+n_2}$ -coapproximating system.*

Lemma 5.5. *Let \mathbf{H} be a locally finite $\langle E \rangle_n$ -homological functor. Then there exists a sequence*

$$F_n \xrightarrow{f_{n-1}} F_{n-1} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} F_2 \xrightarrow{f_1} F_1$$

in \mathcal{T}_c^+ such that for each i

- (1) *F_i admits a strong $\langle E \rangle_i$ -coapproximating system.*
- (2) *There exists an epimorphism*

$$\varphi_i : \text{Hom}_{\mathcal{T}}(F_i, -)|_{\langle E \rangle_i} \longrightarrow \mathbf{H}|_{\langle E \rangle_i}$$

making the following diagram commutes

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(F_i, -)|_{\langle E \rangle_i} & \xrightarrow{\mathcal{T}(f_i, -)|_{\langle E \rangle_i}} & \text{Hom}_{\mathcal{T}}(F_{i+1}, -)|_{\langle E \rangle_i} \\ \varphi_i \searrow & & \swarrow \varphi_{i+1}|_{\langle E \rangle_i} \\ & \mathbf{H}|_{\langle E \rangle_i} & \end{array}.$$

- (3) φ_i in (2) satisfies $\ker(\varphi_i|_{\langle E \rangle_1}) = \ker(\mathcal{T}(f_i, -)|_{\langle E \rangle_1})$.

Proof. We argue by induction on n . Let's begin with the case $n = 1$. By definition, $\mathbf{H}(E[j]) \in k\text{-mod}$ for any $j \in \mathbb{Z}$. Fix a finite set of generators $J_j = \{\varphi_{js}\}$ for $\mathbf{H}(E[j])$. By Yoneda's lemma, each φ_{js} corresponds to a morphism $\varphi_{js} : \text{Hom}_{\mathcal{T}}(E[j], -) \rightarrow \mathbf{H}$, which canonically induces a epimorphism

$$\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1} \simeq \bigoplus_{j \in \mathbb{Z}} \bigoplus_{s \in J_j} \text{Hom}_{\mathcal{T}}(E[j], -)|_{\langle E \rangle_1} \rightarrow \mathbf{H}|_{\langle E \rangle_1}.$$

We now prove the isomorphism

$$\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1} \simeq \text{Hom}_{\mathcal{T}}\left(\bigoplus_{j \in \mathbb{Z}} \bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1}.$$

Observe that

$$\text{Hom}_{\mathcal{T}}\left(\bigoplus_{j \in \mathbb{Z}} \bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1} \simeq \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1},$$

which induces a canonical morphism

$$\psi : \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1} \rightarrow \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], -\right)|_{\langle E \rangle_1}.$$

Thus it suffices to verify that ψ is an isomorphism. This reduces to showing that $\psi(E[r])$ is an isomorphism for each $r \in \mathbb{Z}$. Note that

$$\text{Hom}_{\mathcal{T}}\left(\bigoplus_{s \in J_j} E[j], E[r]\right) \simeq \mathbf{D}\text{Hom}_{\mathcal{T}}(G[r], \bigoplus_{s \in J_j} E[j]).$$

By the construction of J_j , we have $\text{Hom}_{\mathcal{T}}(\bigoplus_{s \in J_j} E[j], -)|_{\langle E \rangle_1} = 0$ for $j \gg 0$. Since $G[r] \in \mathcal{T}^-$ and $E \in \mathcal{T}^+$, it follows that

$$\mathbf{D}\text{Hom}_{\mathcal{T}}(G[r], \bigoplus_{s \in J_j} E[j]) = 0 \quad \text{for } j \ll 0.$$

Thus $\text{Hom}_{\mathcal{T}}(\bigoplus_{s \in J_j} E[j], E[r]) = 0$ for $j \ll 0$. Clearly, $\text{Hom}_{\mathcal{T}}(\bigoplus_{s \in J_j} E[j], E[r]) = 0$ for $|j| \gg 0$ implies that $\psi(E[r])$ is an isomorphism. Let $F_1 = \bigoplus_{j \in \mathbb{Z}} \bigoplus_{s \in J_j} E[j]$. We now need to construct a strong $\langle E \rangle_1$ -coapproximating system for F_1 . Since $E \in \mathcal{T}^+$, there is an integer $B < 0$ such that $E[B] \in \mathcal{T}^{\geq 1}$. For each $m > 0$, define

$$E_m = \bigoplus_{j \geq -m+2+B} \bigoplus_{s \in J_j} E[j].$$

The sum is finite by hypothesis, so $E_m \in \langle E \rangle_1$. Clearly, there is a canonical split epimorphism from E_{m+1} to E_m . Moreover, there exists a canonical split epimorphism from F_1 to E_m . It is straightforward to verify that this construction yields a strong $\langle E \rangle_1$ -coapproximating system for F_1 .

Suppose we have established the case $n = r$. Now consider $n = r + 1$, so that \mathbf{H} is a locally finite $\langle E \rangle_{r+1}$ -homological functor. Clearly, $\mathbf{H}|_{\langle E \rangle_r}$ is a locally finite $\langle E \rangle_r$ -homological functor. Thus, there exist $F_r \in \mathcal{T}_c^+$ and an epimorphism $\varphi_r : \text{Hom}_{\mathcal{T}}(F_r, -)|_{\langle E \rangle_r} \rightarrow \mathbf{H}|_{\langle E \rangle_r}$. Let \mathbf{H}' denote the kernel of φ_r . By Lemma 5.3 and Lemma 5.2, \mathbf{H}' is also a locally finite $\langle E \rangle_r$ -homological functor. Thus we obtain $F' \in \mathcal{T}_c^+$ and an epimorphism $\varphi' : \text{Hom}_{\mathcal{T}}(F', -)|_{\langle E \rangle_1} \rightarrow \mathbf{H}'|_{\langle E \rangle_1}$. Moreover, F' admits a strong $\langle E \rangle_1$ -coapproximating system:

$$\cdots \longrightarrow E'_3 \longrightarrow E'_2 \longrightarrow E'_1$$

where we can assume all morphisms in this sequence are split epimorphisms. Denote the composition

$$\text{Hom}_{\mathcal{T}}(F', -)|_{\langle E \rangle_1} \longrightarrow \mathbf{H}'|_{\langle E \rangle_1} \longrightarrow \text{Hom}_{\mathcal{T}}(F_r, -)|_{\langle E \rangle_1}$$

by $\tilde{\varphi}$. Then [15, Lemma 7.8] implies there exists $g_r : F_r \rightarrow F'$ with $\mathcal{T}(g_r, -)|_{\langle E \rangle_1} = \tilde{\varphi}$. By Lemma 5.4, there exists a sequence in \mathcal{T}_c^+ :

$$F'[-1] \longrightarrow F_{r+1} \xrightarrow{f_r} F_r \xrightarrow{g_r} F'$$

such that

$$\text{Hom}_{\mathcal{T}}(F', -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(g_r, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(F_r, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(f_r, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(F_{r+1}, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(F'[-1], -)|_{\mathcal{E}}$$

forms a weak triangle in $\text{Hom}_k(\mathcal{E}, k\text{-Mod})$, and F_{r+1} admits a strong $\langle E \rangle_{r+1}$ -coapproximating system.

Setting $\mathcal{A} = \langle E \rangle_1$, $\mathcal{B} = \langle E \rangle_r$, $\mathcal{C} = \langle E \rangle_{r+1}$, [15, Lemma 8.5] yields a morphism

$$\varphi_{r+1} : \text{Hom}_{\mathcal{T}}(F_{r+1}, -)|_{\langle E \rangle_{r+1}} \longrightarrow \mathbf{H}$$

making the following diagram

$$\begin{array}{ccccc} \text{Hom}_{\mathcal{T}}(F_r, -)|_{\langle E \rangle_r} & \xrightarrow{\mathcal{T}(f_r, -)|_{\langle E \rangle_r}} & \text{Hom}_{\mathcal{T}}(F_{r+1}, -)|_{\langle E \rangle_r} & \xrightarrow{\varphi_{r+1}|_{\langle E \rangle_r}} & \mathbf{H}|_{\langle E \rangle_r}. \\ & \searrow & \curvearrowleft & \nearrow & \\ & & \varphi_r & & \end{array}$$

commutes. Note further that the sequence

$$\text{Hom}_{\mathcal{T}}(F', -)|_{\langle E \rangle_1} \xrightarrow{\mathcal{T}(g_r, -)|_{\langle E \rangle_1}} \text{Hom}_{\mathcal{T}}(F_r, -)|_{\langle E \rangle_1} \xrightarrow{\varphi_r|_{\langle E \rangle_1}} \mathbf{H}|_{\langle E \rangle_1} \rightarrow 0$$

is exact. Thus by Lemma [15, Lemma 8.6],

$$\varphi_{r+1} : \text{Hom}_{\mathcal{T}}(F_{r+1}, -)|_{\langle E \rangle_{r+1}} \longrightarrow \mathbf{H}$$

is an epimorphism. Finally, Lemma [15, Lemma 8.4] implies that the sequence

$$\text{Hom}_{\mathcal{T}}(F', -)|_{\langle E \rangle_1} \xrightarrow{\mathcal{T}(g_r, -)|_{\langle E \rangle_1}} \text{Hom}_{\mathcal{T}}(F_r, -)|_{\langle E \rangle_1} \xrightarrow{\mathcal{T}(f_r, -)|_{\langle E \rangle_1}} \text{Hom}_{\mathcal{T}}(F_{r+1}, -)|_{\langle E \rangle_1}$$

is exact, which completes the proof of (3). \square

Remark 5.6. If $F_n \in \mathcal{T}_c^+$, then by Proposition 4.6 and [15, Lemma 7.8], each $\tilde{\varphi}_n : \text{Hom}_{\mathcal{T}}(F_n, -)|_{\langle E \rangle_n} \rightarrow \mathbf{H}|_{\langle E \rangle_n}$ admits a lift $\varphi_n : \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ such that $\varphi_n|_{\langle E \rangle_n} = \tilde{\varphi}_n$.

Proposition 5.7. *Let \mathbf{H} be a locally finite \mathcal{E} -homological functor. Then there exist $F \in \mathcal{T}$ and an isomorphism $\varphi : \text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}} \simeq \mathbf{H}$.*

Proof. Since for every $n > 0$, the restriction $\mathbf{H}|_{\langle E \rangle_n}$ is a locally finite $\langle E \rangle_n$ -homological functor, Lemma 5.5 yields a sequence

$$(F_*, f_*): \cdots \longrightarrow F_3 \xrightarrow{f_2} F_2 \xrightarrow{f_1} F_1$$

in \mathcal{T}_c^+ and morphisms $\tilde{\varphi}_n : \text{Hom}_{\mathcal{T}}(F_n, -)|_{\langle E \rangle_n} \rightarrow \mathbf{H}|_{\langle E \rangle_n}$. By Remark 5.6, each $\tilde{\varphi}_n$ admits a lift $\varphi_n : \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ satisfying $\varphi_n|_{\langle E \rangle_n} = \tilde{\varphi}_n$. Then [15, Corollary 7.4] implies that for every i , the following diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(F_i, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(f_i, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(F_{i+1}, -)|_{\mathcal{E}} \\ \searrow \varphi_i & & \swarrow \varphi_{i+1} \\ \mathbf{H} & & \end{array}.$$

commutes. This yields a canonical morphism

$$\varphi : \underrightarrow{\text{colim}} \text{Hom}_{\mathcal{T}}(F_*, -)|_{\mathcal{E}} \rightarrow \mathbf{H}.$$

We first prove that φ is an isomorphism, and then establish that

$$\underrightarrow{\text{colim}} \text{Hom}_{\mathcal{T}}(F_*, -)|_{\mathcal{E}} \simeq \text{Hom}_{\mathcal{T}}(\text{Holim } F_*, -)|_{\mathcal{E}}.$$

To prove that φ is an isomorphism, it suffices to show that $\varphi|_{\langle E \rangle_1}$ is an isomorphism. Define $K_n = \ker(\varphi|_{\langle E \rangle_1})$; then by Lemma 5.5(3), we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n & \longrightarrow & \text{Hom}_{\mathcal{T}}(F_n, -)|_{\langle E \rangle_1} & \xrightarrow{\varphi_n|_{\langle E \rangle_1}} & \mathbf{H}|_{\langle E \rangle_1} \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \mathcal{T}(f_n, -)|_{\langle E \rangle_1} & & \parallel \\ 0 & \longrightarrow & K_{n+1} & \longrightarrow & \text{Hom}_{\mathcal{T}}(F_{n+1}, -)|_{\langle E \rangle_1} & \xrightarrow{\varphi_{n+1}|_{\langle E \rangle_1}} & \mathbf{H}|_{\langle E \rangle_1} \longrightarrow 0 \end{array}.$$

By the right exactness of the colimit functor, we obtain an exact sequence

$$0 = \underrightarrow{\text{colim}} K_* \longrightarrow \underrightarrow{\text{colim}} \text{Hom}_{\mathcal{T}}(F_*, -)|_{\langle E \rangle_1} \xrightarrow{\varphi|_{\langle E \rangle_1}} \mathbf{H}|_{\langle E \rangle_1} \longrightarrow 0.$$

Thus $\varphi|_{\langle E \rangle_1}$ is an isomorphism.

We now prove that $\operatorname{colim} \underline{\operatorname{Hom}}_{\mathcal{T}}(F_*, -)|_{\mathcal{E}} \simeq \operatorname{Hom}_{\mathcal{T}}(\underline{\operatorname{Holim}}(F_*), -)|_{\mathcal{E}}$. It suffices to prove that for any $j \in \mathbb{Z}$, $\operatorname{Hom}_{\mathcal{T}}(\underline{\operatorname{Holim}}(F_*), E[j])$ is the colimit in $k\text{-Mod}$ of the sequence $\operatorname{Hom}_{\mathcal{T}}(F_*, E[j])$. Note that

$$\operatorname{Hom}_{\mathcal{T}}(\underline{\operatorname{Holim}}(F_*), E[j]) \simeq \mathbf{D}\operatorname{Hom}_{\mathcal{T}}(G[j], \underline{\operatorname{Holim}}(F_*)).$$

By Lemma 4.2, each $\operatorname{Hom}_{\mathcal{T}}(G[j], F_i)$ is finite-dimensional. Thus, Lemma 2.3 gives

$$\operatorname{Hom}_{\mathcal{T}}(G[j], \underline{\operatorname{Holim}}(F_*)) \simeq \varprojlim \operatorname{Hom}_{\mathcal{T}}(G[j], F_*).$$

If we can show that $\operatorname{Hom}_{\mathcal{T}}(G[j], \underline{\operatorname{Holim}}(F_*))$ is finite-dimensional, then Proposition A.2 implies that $\mathbf{D}\operatorname{Hom}_{\mathcal{T}}(G[j], \underline{\operatorname{Holim}}(F_*))$ is the colimit of the sequence $\mathbf{D}\operatorname{Hom}_{\mathcal{T}}(G[j], F_*)$, completing the proof.

Since $\operatorname{colim} \underline{\operatorname{Hom}}_{\mathcal{T}}(F_*, E[j]) \simeq \mathbf{H}(E[j])$ and $\mathbf{H}(E[j])$ finite-dimensional, the colimit $\operatorname{colim} \underline{\operatorname{Hom}}_{\mathcal{T}}(F_*, E[j])$ is finite-dimensional. For all $i > 0$, we have

$$\operatorname{Hom}_{\mathcal{T}}(F_i, E[j]) \simeq \mathbf{D}\operatorname{Hom}_{\mathcal{T}}(G[j], F_i),$$

so the colimit of the sequence $\mathbf{D}\operatorname{Hom}_{\mathcal{T}}(G[j], F_*)$ is finite-dimensional. By Proposition A.3, $\operatorname{Dcolim} \underline{\operatorname{Hom}}_{\mathcal{T}}(F_*, E[j])$ coincides with the limit of the sequence

$$\mathbf{DD}\operatorname{Hom}_{\mathcal{T}}(G[j], F_*),$$

which is also finite-dimensional. Since \mathbf{D} is the duality functor on $k\text{-mod}$, the limit of the sequences $\mathbf{DD}\operatorname{Hom}_{\mathcal{T}}(G[j], F_*)$ and $\operatorname{Hom}_{\mathcal{T}}(G[j], F_*)$ coincide. Consequently,

$$\operatorname{Hom}_{\mathcal{T}}(G[j], \underline{\operatorname{Holim}}(F_*)) \simeq \varprojlim \operatorname{Hom}_{\mathcal{T}}(G[j], F_*)$$

is finite-dimensional, completing the proof. \square

5.3. Formal representability theorems. The representability result established in Proposition 5.7 shows that every locally finite \mathcal{E} -homological functor is representable by an object of \mathcal{T} . In order to refine this statement and identify the representing objects inside the intrinsic subcategories \mathcal{T}_c^+ and \mathcal{T}_c^b , we now restrict our attention to triangulated categories equipped with compact silting objects. Such categories enjoy additional structural properties that allow us to control the cohomological degrees of the representing objects. This dual representability framework will serve as a foundation for the applications developed in the next section.

Let \mathcal{S} be a triangulated category with coproducts. An object S in \mathcal{S} is called a *silting object* if the pair $(S^{\perp>0}, S^{\perp<0})$ of subcategories forms a *t-structure* on \mathcal{S} , where

$$\begin{aligned} S^{\perp>0} &:= \{M \in \mathcal{S} \mid \operatorname{Hom}_{\mathcal{S}}(S, M[i]) = 0 \text{ for } i > 0\}, \\ S^{\perp<0} &:= \{M \in \mathcal{S} \mid \operatorname{Hom}_{\mathcal{S}}(S, M[i]) = 0 \text{ for } i < 0\}. \end{aligned}$$

Clearly, $S^{\perp>0}$ is closed under products. Moreover, if S is compact, we call it a *compact silting object*. In this case, $S^{\perp>0} = \mathcal{S}_S^{\leq 0}$ and $S^{\perp<0} = \mathcal{S}_S^{\geq 0}$.

In the remainder of this section, we assume that \mathcal{T} is a locally Hom-finite k -linear triangulated category with a compact silting object G . Note that in this case we have $\operatorname{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i > 0$. Then by [15, Remark 5.3], \mathcal{T} is approximable. This setting includes many important examples and provides the necessary control over cohomological degrees to obtain refined representability theorems.

Example 5.8. There are two typical examples.

- (1) Let A be a non-positive DG k -algebra with each $H^i(A)$ finite-dimensional. Then the derived category $\mathcal{D}(A)$ is locally Hom-finite and admits a compact silting object, namely A itself.

- (2) The homotopy category $\text{Ho}(Sp)$ of the spectra is a locally Hom-finite k -linear triangulated categories with a compact silting object given by the sphere spectrum \mathbb{S} .

Lemma 5.9. $\mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq 1} * \text{add}(E) \subseteq \mathcal{T}^{\geq 1} * \langle E \rangle_1^{[-1,1]}$.

Proof. Since $G \in \mathcal{T}^{\leq 0}$ and $E \in \mathcal{T}^{\geq 0}$, let $T \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0}$. By Lemma 4.2, the space $\text{Hom}_{\mathcal{T}}(T, E)$ is finite-dimensional. Choose a basis $\{f_1, \dots, f_m\} \subseteq \text{Hom}_{\mathcal{T}}(T, E)$, and let $f: T \rightarrow E^m$ be the morphism induced by these maps. This gives a triangle

$$E^m[-1] \longrightarrow D \longrightarrow T \xrightarrow{f} E^m,$$

where $D \in \mathcal{T}^{\geq 0}$ and $E^m \in \text{add}(E)$. It suffices to show that $D \in \mathcal{T}^{\geq 1}$.

Since G is silting, we have $\text{Hom}_{\mathcal{T}}(E[-j], E) = 0$ for all $j > 0$. Hence every morphism in $\text{Hom}_{\mathcal{T}}(D, E)$ factors through T . Because f is an $\langle E \rangle$ -approximation, it follows that $\text{Hom}_{\mathcal{T}}(D, E) = 0$, and therefore $\text{Hom}_{\mathcal{T}}(G, D) = 0$.

Moreover, since $D \in \mathcal{T}^{\geq 0}$ and G is silting, we also have $\text{Hom}_{\mathcal{T}}(G[i], D) = 0$ for all $i > 0$. Thus $D \in \mathcal{T}^{\geq 1}$, completing the proof. \square

Note that for any $n \in \mathbb{Z}$ and $T \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq n}$, we have $T[n] \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0}$. This implies the following corollary.

Corollary 5.10. *For any $n \in \mathbb{Z}, m > 0$,*

$$\mathcal{T}_c^+ \cap \mathcal{T}^{\geq n} \subseteq \mathcal{T}^{\geq n+m} * \langle E \rangle_m^{[n-1, n+m]}.$$

Proof. The assertion follows immediately by combining the octahedron axiom with Lemma 5.9. We leave the details to the reader. \square

Lemma 5.11. *Let \mathbf{H} be a locally finite \mathcal{E} -homological functor. Then there exists an integer $A > 0$ such that for every $n > 0$, there exists $K_n \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$ and a morphism $\psi_n : \text{Hom}_{\mathcal{T}}(K_n, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ such that $\psi_n|_{\langle E \rangle_n}$ is an epimorphism.*

Proof. Since \mathbf{H} is locally finite, there exists $A > 0$ such that $\mathbf{H}(E[i]) = 0$ for all $i \geq A - 1$. As \mathbf{H} is \mathcal{E} -homological, it follows that $\mathbf{H}(E') = 0$ for all $E' \in \langle E \rangle^{(-\infty, -A+1]}$.

Fix $n > 0$. By Lemma 5.5 and Remark 5.6, there exist $F_n \in \mathcal{T}_c^+$ and a morphism

$$\varphi_n : \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} \longrightarrow \mathbf{H}$$

such that $\varphi_n|_{\langle E \rangle_n}$ is an epimorphism. Since $\mathcal{T}_c^+ \subseteq \mathcal{T}^+$, there exists $\ell \in \mathbb{Z}$ with $F_n \in \mathcal{T}^{\geq \ell}$. Using the fact that $\mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$, we may assume $-\ell - A + 1 > 0$ and set $m = -\ell - A + 1$. By Corollary 5.10, we obtain a triangle

$$D_m \longrightarrow F_n \xrightarrow{\alpha} E_m \longrightarrow D_m[1]$$

with $D_m \in \mathcal{T}^{\geq -A+1}$ and $E_m \in \langle E \rangle_{-\ell-A+1}^{[\ell-1, -A+1]} \subseteq \langle E \rangle^{(-\infty, -A+1)}$. The composition

$$\text{Hom}_{\mathcal{T}}(E_m, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(\alpha, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} \xrightarrow{\varphi_n} \mathbf{H}$$

is zero. Moreover, E_m admits a strong \mathcal{E} -coapproximating system

$$\dots \rightarrow E \xrightarrow{\text{Id}_E} E \xrightarrow{\text{Id}_E} E.$$

By Lemma 5.4(1), there exists a sequence

$$E_m[-1] \longrightarrow K_n \xrightarrow{\beta} F_n \xrightarrow{\alpha} E_m$$

such that

$$\text{Hom}_{\mathcal{T}}(E_m, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(K_n, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(E_m[-1], -)|_{\mathcal{E}}$$

is a weak triangle in $\text{Hom}_k(\mathcal{E}, k\text{-Mod})$. By [15, Lemma 8.5], there exists a morphism $\psi_n : \text{Hom}_{\mathcal{T}}(K_n, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ and a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(F_n, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\beta, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(K_n, -)|_{\mathcal{E}} \xrightarrow{\psi_n} \mathbf{H} \\ & \searrow & \\ & & \varphi_n \end{array}$$

Since $\varphi_n|_{\langle E \rangle_n}$ is an epimorphism, so is $\psi_n|_{\langle E \rangle_n}$. As A is fixed, it remains to show that $K_n \in \mathcal{T}^{\geq -A}$.

From the triangle

$$D_m \rightarrow F_n \xrightarrow{\alpha} E_m \rightarrow D_m[1],$$

we obtain an exact sequence

$$\mathbf{H}^i(D_m) \rightarrow \mathbf{H}^i(F_n) \xrightarrow{\mathbf{H}^i(\alpha)} \mathbf{H}^i(E_m) \rightarrow \mathbf{H}^{i+1}(D_m).$$

Since $D_m \in \mathcal{T}^{\geq -A+1}$, $\mathbf{H}^i(\alpha)$ is an isomorphism for all $i \leq -A-1$. By Lemma 5.4(2), we have an exact sequence

$$\mathbf{H}^{i-1}(E_m) \rightarrow \mathbf{H}^i(K_n) \rightarrow \mathbf{H}^i(F_n) \xrightarrow{\mathbf{H}^i(\alpha)} \mathbf{H}^i(E_m).$$

Hence $\mathbf{H}^i(K_n) = 0$ for all $i \leq -A-1$. Lemma 2.4 then implies $K_n \in \mathcal{T}^{\geq -A}$, completing the proof. \square

Lemma 5.12. *Let \mathbf{H} be a locally finite \mathcal{E} -homological functor and $A \geq 0$ be an integer. Suppose F_1, F_2 are objects in $\mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$, E' is an object in $\mathcal{E} \cap \mathcal{T}^{\geq -A}$, and $\alpha : F_1 \rightarrow E'$ is a morphism. Assume we are given an integer $m > 0$ and natural transformations*

$$\varphi_1 : \text{Hom}_{\mathcal{T}}(F_1, -)|_{\mathcal{E}} \rightarrow \mathbf{H} \text{ and } \varphi_2 : \text{Hom}_{\mathcal{T}}(F_2, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$$

such that $\varphi_1|_{\langle E' \rangle_m}$ is an epimorphism. Then there exists a commutative diagram

$$\begin{array}{ccccc} & \text{Hom}_{\mathcal{T}}(\tilde{E}, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\tilde{\alpha}, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} & \xleftarrow{\mathcal{T}(\beta, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(F_2, -)|_{\mathcal{E}} \\ & \nearrow \mathcal{T}(\delta, -)|_{\mathcal{E}} & & \searrow \mathcal{T}(\gamma, -)|_{\mathcal{E}} & \\ \text{Hom}_{\mathcal{T}}(E', -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(F_1, -)|_{\mathcal{E}} & \xrightarrow{\varphi_1} & \mathbf{H} \\ & & & \searrow \tilde{\varphi} & \downarrow \varphi_2 \\ & & & & \end{array}$$

such that

- (1) $\tilde{E} \in \mathcal{E} \cap \mathcal{T}^{\geq -A}$ and $\tilde{F} \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$.
- (2) $\mathbf{H}^i(\tilde{\alpha})$ and $\mathbf{H}^i(\gamma)$ are isomorphisms for all $i \leq m-2$.

Proof. Without loss of generality, assume $A = 0$. The proof is divided into three steps.

Step 1: Construct $\tilde{E} \in \mathcal{E} \cap \mathcal{T}^{\geq 0}$, $\delta : \tilde{E} \rightarrow E'$ and $\gamma : F_1 \rightarrow \tilde{E}$.

By Corollary 5.10, there exists a distinguished triangle

$$D_m \xrightarrow{a} F_2 \xrightarrow{b} E'_m \longrightarrow D_m[1]$$

with $D_m \in \mathcal{T}^{\geq m}$ and $E'_m \in \langle E \rangle_m^{[-1, m]}$. Since $E'_m \in \langle E \rangle_m$ and $\varphi_1|_{\langle E \rangle_m}$ is an epimorphism, there exists a morphism $f : F_1 \rightarrow E'_m$ such that the following diagram

commutes:

$$\begin{array}{ccc}
 & \mathbf{Hom}_{\mathcal{T}}(E'_m, -)|_{\langle E \rangle_m} & \\
 & \downarrow \mathcal{T}(b, -)|_{\langle E \rangle_m} & \\
 \mathcal{T}(f, -)|_{\langle E \rangle_m} & \nearrow & \mathbf{Hom}_{\mathcal{T}}(F_2, -)|_{\langle E \rangle_m} \\
 & & \downarrow \varphi_2|_{\langle E \rangle_m} \\
 \mathbf{Hom}_{\mathcal{T}}(F_1, -)|_{\langle E \rangle_m} & \xrightarrow{\varphi_1|_{\langle E \rangle_m}} & \mathbf{H}|_{\langle E \rangle_m}.
 \end{array}$$

Combining [15, Lemma 7.8] with [15, Corollary 7.4], we also have

$$\varphi_1 \circ \mathcal{T}(f, -)|_{\mathcal{E}} = \varphi_2 \circ \mathcal{T}(b, -)|_{\mathcal{E}}.$$

Note that $F_1 \in \mathcal{T}_c^+$. By Proposition 4.6, F_1 admits a strong \mathcal{E} -coapproximating system:

$$\dots \xrightarrow{f_3} E_3 \xrightarrow{f_2} E_2 \xrightarrow{f_1} E_1.$$

Then by Corollary 4.8, the morphism $\binom{\alpha}{f} \in \mathbf{Hom}_{\mathcal{T}}(F_1, E' \oplus E'_m)$ factors through some E_j . Choosing such an E_j with $j \geq m$ and setting $\tilde{E} := E_j$, we obtain commutative diagrams

$$\begin{array}{ccccc}
 F_1 & \xrightarrow{\binom{\alpha}{f}} & E' \oplus E'_m & \xrightarrow{(1,0)} & E' \\
 \searrow \gamma & & \nearrow & & \nearrow \\
 & \tilde{E} & & & \delta
 \end{array}$$

and

$$\begin{array}{ccccc}
 F_1 & \xrightarrow{\binom{\alpha}{f}} & E' \oplus E'_m & \xrightarrow{(0,1)} & E'_m \\
 \searrow \gamma & & \nearrow & & \nearrow \\
 & \tilde{E} & & & \delta'
 \end{array}$$

By the definition of strong \mathcal{E} -coapproximating system, $\mathbf{H}^i(\gamma) : \mathbf{H}^i(F_1) \rightarrow \mathbf{H}^i(\tilde{E})$ is an isomorphism for all $i \leq m$. Since $F_1 \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0}$, we have $\mathbf{H}^i(\tilde{E}) = 0$ when $i < 0$. Then by Lemma 2.4, we obtain $\tilde{E} \in \mathcal{E} \cap \mathcal{T}^{\geq 0}$. The information we have obtained so far yields the following commutative diagram (*)

$$\begin{array}{ccccccc}
 & & \mathbf{Hom}_{\mathcal{T}}(\tilde{E}, -)|_{\mathcal{E}} & \xleftarrow{\mathcal{T}(\delta', -)|_{\mathcal{E}}} & \mathbf{Hom}_{\mathcal{T}}(E'_m, -)|_{\mathcal{E}} & & \\
 & \nearrow \mathcal{T}(\delta, -)|_{\mathcal{E}} & & \searrow \mathcal{T}(\gamma, -)|_{\mathcal{E}} & & \downarrow \mathcal{T}(b, -)|_{\mathcal{E}} & \\
 \mathbf{Hom}_{\mathcal{T}}(E', -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha, -)|_{\mathcal{E}}} & \mathbf{Hom}_{\mathcal{T}}(F_1, -)|_{\mathcal{E}} & \xrightarrow{\varphi_1} & \mathbf{H} & & \\
 & & & & \downarrow \varphi_2 & &
 \end{array}$$

Step 2: Construct $\tilde{F} \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0}$, $\tilde{\alpha} : \tilde{F} \rightarrow \tilde{E}$ and $\beta : \tilde{F} \rightarrow F_2$.

By (*), there is a morphism $(-\delta', b) : \tilde{E} \oplus F_2 \rightarrow E'_m$. Then by Lemma 5.4(1), we have the sequence

$$E'_m[-1] \xrightarrow{\sigma} \tilde{F} \xrightarrow{\binom{\tilde{\alpha}}{\beta}} \tilde{E} \oplus F_2 \xrightarrow{(-\delta', b)} E'_m$$

in \mathcal{T}_c^+ such that

$$\text{Hom}_{\mathcal{T}}(E'_m, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}((- \delta', b), -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(\tilde{E} \oplus F_2, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(\binom{\tilde{\alpha}}{\beta}, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(\sigma, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(E'_m[-1], -)|_{\mathcal{E}}$$

is a weak triangle. This implies $\delta' \tilde{\alpha} = b \beta$. Therefore, we obtain the following commutative diagram (**)

$$\begin{array}{ccccccc}
& & & & \text{Hom}_{\mathcal{T}}(E'_m, -)|_{\mathcal{E}} & & \\
& & & & \swarrow \mathcal{T}(\delta', -)|_{\mathcal{E}} & & \downarrow \mathcal{T}(b, -)|_{\mathcal{E}} \\
\text{Hom}_{\mathcal{T}}(\tilde{E}, -)|_{\mathcal{E}} & \xleftarrow[\mathcal{T}(\tilde{\alpha}, -)|_{\mathcal{E}}]{} & \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} & \xleftarrow[\mathcal{T}(\beta, -)|_{\mathcal{E}}]{} & \text{Hom}_{\mathcal{T}}(F_2, -)|_{\mathcal{E}} & & \\
\uparrow \mathcal{T}(\delta, -)|_{\mathcal{E}} & & \downarrow \mathcal{T}(\gamma, -)|_{\mathcal{E}} & & \downarrow \varphi_2 & & \\
\text{Hom}_{\mathcal{T}}(E', -)|_{\mathcal{E}} & \xrightarrow[\mathcal{T}(\alpha, -)|_{\mathcal{E}}]{} & \text{Hom}_{\mathcal{T}}(F_1, -)|_{\mathcal{E}} & \xrightarrow[\varphi_1]{} & \mathbf{H} & &
\end{array}$$

By Lemma 5.4(2), we can deduce that there is an exact sequence

$$\mathbf{H}^{i-1}(E'_m) \xrightarrow{\mathbf{H}^i(\sigma)} \mathbf{H}^i(\tilde{F}) \xrightarrow{\binom{\mathbf{H}^i(\tilde{\alpha})}{\mathbf{H}^i(\beta)}} \mathbf{H}^i(\tilde{E}) \oplus \mathbf{H}^i(F_2) \xrightarrow{(-\mathbf{H}^i(\delta'), \mathbf{H}^i(b))} \mathbf{H}^i(E'_m)$$

for all $i \in \mathbb{Z}$. Recall that there is a triangle

$$D_m \xrightarrow{a} F_2 \xrightarrow{b} E'_m \longrightarrow D_m[1]$$

with $D_m \in \mathcal{T}^{\geq m}$. Then $\mathbf{H}^i(b)$ is an isomorphism for $i \leq m-2$, which implies $\mathbf{H}^i(\sigma) = 0$ for $i \leq m-1$. Consequently, we have a short exact sequence

$$0 \rightarrow \mathbf{H}^i(\tilde{F}) \xrightarrow{\binom{\mathbf{H}^i(\tilde{\alpha})}{\mathbf{H}^i(\beta)}} \mathbf{H}^i(\tilde{E}) \oplus \mathbf{H}^i(F_2) \xrightarrow{(-\mathbf{H}^i(\delta'), \mathbf{H}^i(b))} \mathbf{H}^i(E'_m) \rightarrow 0$$

for $i \leq m-2$. This yields a commutative diagram

$$\begin{array}{ccc}
\mathbf{H}^i(\tilde{F}) & \xrightarrow{\mathbf{H}^i(\tilde{\alpha})} & \mathbf{H}^i(\tilde{E}) \\
\mathbf{H}^i(\beta) \downarrow & & \downarrow \mathbf{H}^i(\delta') \\
\mathbf{H}^i(F_2) & \xrightarrow[\mathbf{H}^i(b)]{} & \mathbf{H}^i(E'_m)
\end{array}$$

which is a pullback for $i \leq m-2$. Since $\mathbf{H}^i(b)$ is an isomorphism for $i \leq m-2$, it follows that $\mathbf{H}^i(\tilde{\alpha}) : \mathbf{H}^i(\tilde{F}) \rightarrow \mathbf{H}^i(\tilde{E})$ is also an isomorphism for $i \leq m-2$. As $\tilde{E} \in \mathcal{E} \cap \mathcal{T}^{\geq 0}$, we have $\mathbf{H}^i(\tilde{F}) = 0$ for $i < 0$. By Lemma 2.4, this implies $\tilde{F} \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq 0}$.

Step 3: Construct $\tilde{\varphi} : \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$.

By (**), the composition

$$\text{Hom}_{\mathcal{T}}(E'_m, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}((- \delta', b), -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(\tilde{E} \oplus F_2, -)|_{\mathcal{E}} \xrightarrow{(\varphi_1 \circ \mathcal{T}(\gamma, -)|_{\mathcal{E}}, \varphi_2)} \mathbf{H}$$

is zero. Then by [15, Lemma 8.5], we obtain a morphism $\tilde{\varphi} : \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ and a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{T}}(\tilde{E} \oplus F_2, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\binom{\tilde{\alpha}}{\beta}, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(\tilde{F}, -)|_{\mathcal{E}} \xrightarrow{\tilde{\varphi}} \mathbf{H} \\
& \searrow (\varphi_1 \circ \mathcal{T}(\gamma, -)|_{\mathcal{E}}, \varphi_2) & \nearrow
\end{array}$$

This completes the proof. \square

Lemma 5.13. *Let \mathbf{H} be a locally finite \mathcal{E} -homological functor. Then there exists an integer $A > 0$ such that for every $i > 0$, the following hold.*

- (1) *There exists a commutative diagram in $\mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$:*

$$\begin{array}{ccc} E_{i+1} & \xrightarrow{\delta_i} & E_i \\ \gamma_{i+1} \swarrow & & \searrow \alpha_i \\ L_i & & \end{array}$$

where $E_i \in \mathcal{E}$, and a morphism $\eta_i : \text{Hom}_{\mathcal{T}}(L_i, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ such that $\eta_i|_{\langle E \rangle_{i+2}}$ is an epimorphism. Moreover, $H^{\ell}(\gamma_i)$ and $H^{\ell}(\alpha_i)$ are isomorphisms for all $\ell \leq i$.

- (2) *For all $i > 0$, the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(E_{i+1}, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha_{i+1}, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(L_{i+1}, -)|_{\mathcal{E}} \\ \mathcal{T}(\gamma_{i+1}, -)|_{\mathcal{E}} \downarrow & & \downarrow \eta_{i+1} \\ \text{Hom}_{\mathcal{T}}(L_i, -)|_{\mathcal{E}} & \xrightarrow{\eta_i} & \mathbf{H}. \end{array}$$

Proof. By Lemma 5.11, there exists an integer $A > 0$ such that for every $i > 0$, there are objects $K_i \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$ and morphisms $\psi_i : \text{Hom}_{\mathcal{T}}(K_i, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ with $\psi_i|_{\langle E \rangle_i}$ an epimorphism.

Set $L_1 = K_3$ and $\eta_1 = \psi_3$. There is a triangle

$$D_1 \longrightarrow L_1 \xrightarrow{\alpha_1} E_1 \longrightarrow D_1[1]$$

with $D_1 \in \mathcal{T}^{\geq 3}$ and $E_1 \in \mathcal{E}$. Then $\mathbf{H}^i(\alpha_1)$ is an isomorphism for all $i \leq 1$. Moreover, since $\mathbf{H}^i(L_1) = 0$ for all $i \leq -A - 1$, Lemma 2.4 implies $E_1 \in \mathcal{E} \cap \mathcal{T}^{\geq -A}$.

By Lemma 5.12(1), we obtain a commutative diagram

$$\begin{array}{ccccc} & & \text{Hom}_{\mathcal{T}}(E_2, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha_2, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(L_2, -)|_{\mathcal{E}} \leftarrow \text{Hom}_{\mathcal{T}}(K_2, -)|_{\mathcal{E}} \\ & \nearrow \mathcal{T}(\delta_1, -)|_{\mathcal{E}} & & \searrow \mathcal{T}(\gamma_2, -)|_{\mathcal{E}} & \searrow \eta_2 \\ \text{Hom}_{\mathcal{T}}(E_1, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha_1, -)|_{\mathcal{E}}} & \text{Hom}_{\mathcal{T}}(L_1, -)|_{\mathcal{E}} & \xrightarrow{\eta_1} & \mathbf{H} \\ & & & & \downarrow \psi_4 \end{array}$$

with $E_2 \in \mathcal{E} \cap \mathcal{T}^{\geq -A}$ and $L_2 \in \mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$. By Lemma 5.12(2), both $\mathbf{H}^i(\alpha_2)$ and $\mathbf{H}^i(\gamma_2)$ are isomorphisms for all $i \leq 2$. Since $\psi_4|_{\langle E \rangle_4}$ is an epimorphism, so is $\eta_2|_{\langle E \rangle_4}$.

Because $E_1, E_2 \in \mathcal{E}$, we have the commutative diagram

$$\begin{array}{ccc} & E_2 & \xrightarrow{\delta_1} \\ \alpha_2 \nearrow & \nwarrow \gamma_2 & \swarrow \alpha_1 \\ L_2 & & L_1 \end{array}$$

The general case now follows by induction. \square

Proposition 5.14. *Let \mathbf{H} be a locally finite \mathcal{E} -homological functor. Then there exist an object $F \in \mathcal{T}_c^+$ and an epimorphism*

$$\varphi : \text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}} \longrightarrow \mathbf{H}.$$

Proof. By Lemma 5.13(1), there exists an integer $A > 0$ and a commutative diagram in $\mathcal{T}_c^+ \cap \mathcal{T}^{\geq -A}$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_3 & \xrightarrow{\delta_2} & E_2 & \xrightarrow{\delta_1} & E_1 \\ & & \swarrow \gamma_3 & \nearrow \alpha_2 & \swarrow \gamma_2 & \nearrow \alpha_1 & \\ & & L_2 & & L_1 & & \end{array}$$

where each $E_i \in \mathcal{E}$, and $\mathbf{H}^\ell(\gamma_i)$ and $\mathbf{H}^\ell(\alpha_i)$ are isomorphisms for all $\ell \leq i$. Moreover, for each $i > 0$ there exists a morphism $\eta_i : \mathbf{Hom}_{\mathcal{T}}(L_i, -)|_{\mathcal{E}} \rightarrow \mathbf{H}$ such that $\eta_i|_{\langle E \rangle_{i+2}}$ is an epimorphism. It follows that $\mathbf{H}^\ell(\delta_i)$ is an isomorphism for all $\ell \leq i$. Thus (E_*, δ_*) is a strong \mathcal{E} -coapproximating system.

Set $F = \varprojlim(E_*)$. By Lemma 4.5(1), $F \in \mathcal{T}_c^+$ and (E_*, δ_*) is a strong \mathcal{E} -coapproximating system for F . Lemma 4.7 then yields a morphism

$$\varphi : \mathbf{colim} \mathbf{Hom}_{\mathcal{T}}(E_*, -) \longrightarrow \mathbf{Hom}_{\mathcal{T}}(F, -)$$

whose restriction to \mathcal{E} is an isomorphism.

For each $i > 0$, Lemma 5.13 provides the commutative diagram

$$\begin{array}{ccccc} \mathbf{Hom}_{\mathcal{T}}(E_i, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\delta_i, -)|_{\mathcal{E}}} & \mathbf{Hom}_{\mathcal{T}}(E_{i+1}, -)|_{\mathcal{E}} & \xrightarrow{\mathcal{T}(\alpha_{i+1}, -)|_{\mathcal{E}}} & \mathbf{Hom}_{\mathcal{T}}(L_{i+1}, -)|_{\mathcal{E}} \\ & \searrow \mathcal{T}(\alpha_i, -)|_{\mathcal{E}} & \downarrow \mathcal{T}(\gamma_{i+1}, -)|_{\mathcal{E}} & & \downarrow \eta_{i+1} \\ & & \mathbf{Hom}_{\mathcal{T}}(L_i, -)|_{\mathcal{E}} & \xrightarrow{\eta_i} & \mathbf{H}. \end{array}$$

Hence there is a canonical morphism

$$\eta : \mathbf{colim} \mathbf{Hom}_{\mathcal{T}}(E_*, -)|_{\mathcal{E}} \longrightarrow \mathbf{H},$$

induced by the compositions

$$\mathbf{Hom}_{\mathcal{T}}(E_i, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(\alpha_i, -)|_{\mathcal{E}}} \mathbf{Hom}_{\mathcal{T}}(L_i, -)|_{\mathcal{E}} \xrightarrow{\eta_i} \mathbf{H}.$$

It remains to show that η is an epimorphism. Let $E' \in \mathcal{E}$. Then there exists $m > 0$ with $E' \in \langle E \rangle_{m+2}$, so $\eta_m(E') : \mathbf{Hom}_{\mathcal{T}}(L_m, E') \rightarrow \mathbf{H}(E')$ is an epimorphism. By Proposition 4.3, there exists $m' > 0$ such that $\mathbf{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq m'}, E') = 0$. We may assume $m = m'$.

Extend $\alpha_m : L_m \rightarrow E_m$ to a triangle

$$D_m \longrightarrow L_m \xrightarrow{\alpha_m} E_m \longrightarrow D_m[1].$$

Since $\mathbf{H}^i(\alpha_m)$ is an isomorphism for all $i \leq m$, we have $\mathbf{H}^i(D_m) = 0$ for all $i \leq m$. Lemma 2.4 then implies $D_m \in \mathcal{T}^{\geq m+1}$. Thus

$$\mathcal{T}(\alpha_m, E') : \mathbf{Hom}_{\mathcal{T}}(E_m, E') \longrightarrow \mathbf{Hom}_{\mathcal{T}}(L_m, E')$$

is an epimorphism. Therefore the composition

$$\mathbf{Hom}_{\mathcal{T}}(E_m, E')|_{\mathcal{E}} \xrightarrow{\mathcal{T}(\alpha_m, E')|_{\mathcal{E}}} \mathbf{Hom}_{\mathcal{T}}(L_m, E')|_{\mathcal{E}} \xrightarrow{\eta_m(E')} \mathbf{H}(E')$$

is an epimorphism. Hence η is an epimorphism, completing the proof. \square

Definition 5.15. Let \mathcal{S} be a compactly generated triangulated category. A morphism $f : M \rightarrow N$ is called *cophantom* if for every object $C \in \mathcal{E}$, the induced map $\mathcal{S}(f, C) : \mathbf{Hom}_{\mathcal{S}}(N, C) \rightarrow \mathbf{Hom}_{\mathcal{S}}(M, C)$ is zero. We denote by \mathcal{J} the class of all cophantom morphisms.

Lemma 5.16. Let $F \in \mathcal{T}$ and suppose that $\mathbf{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}}$ is a locally finite \mathcal{E} -homological functor. For any $n > 0$, there exists a triangle

$$D_n \xrightarrow{g} F \xrightarrow{f} F_n \longrightarrow D_n[1]$$

with $F_n \in \mathcal{T}_c^+$, $g \in \mathcal{J}^n$, and $\mathcal{T}(f, -)|_{\mathcal{E}}$ is surjective.

Proof. We proceed by induction, beginning with the case $n = 1$. By Proposition 5.14, there exist $F_1 \in \mathcal{T}_c^+$ and an epimorphism $\varphi_1 : \text{Hom}_{\mathcal{T}}(F_1, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}}$. Since $F_1 \in \mathcal{T}_c^+$, it follows from Proposition 4.6 and [15, Lemma 7.8] that there exists a morphism $f_1 : F \rightarrow F_1$ such that $\varphi_1 = \mathcal{T}(f_1, -)|_{\mathcal{E}}$. Extend f_1 to a triangle

$$D_1 \xrightarrow{g_1} F \xrightarrow{f_1} F_1 \longrightarrow D_1[1].$$

For any $C \in \mathcal{E}$, we obtain an exact sequence

$$\text{Hom}_{\mathcal{T}}(F_1, C) \xrightarrow{\mathcal{T}(f_1, C)} \text{Hom}_{\mathcal{T}}(F, C) \xrightarrow{\mathcal{T}(g_1, C)} \text{Hom}_{\mathcal{T}}(D_1, C).$$

Since $\mathcal{T}(f_1, C)$ is surjective, we have $\mathcal{T}(g_1, C) = 0$, hence g_1 is a copphantom morphism. This completes the case $n = 1$.

Now assume the statement holds for all $n \leq k$ with $k \geq 1$, and consider the case $n = k + 1$. By the induction hypothesis, there exists a triangle

$$D_k \xrightarrow{g_k} F \xrightarrow{f_k} F_k \rightarrow D_k[1],$$

with $F_k \in \mathcal{T}_c^+$, $g_k \in \mathcal{J}^k$, and $\mathcal{T}(f_k, -)|_{\mathcal{E}}$ is surjective. Thus the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{T}}(D_k[1], -)|_{\mathcal{E}} \longrightarrow \text{Hom}_{\mathcal{T}}(F_k, -)|_{\mathcal{E}} \xrightarrow{\mathcal{T}(f_k, -)|_{\mathcal{E}}} \text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}}$$

is exact. Since both $\text{Hom}_{\mathcal{T}}(F_k, -)|_{\mathcal{E}}$ and $\text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}}$ are locally finite \mathcal{E} -homological functors, Lemma 5.2 implies that $\text{Hom}_{\mathcal{T}}(D_k[1], -)|_{\mathcal{E}}$ is also a locally finite \mathcal{E} -homological functor. Applying the $n = 1$ case to this functor, we obtain a triangle

$$D_{k+1} \xrightarrow{h} D_k \longrightarrow F' \longrightarrow D_{k+1}[1],$$

with $h \in \mathcal{J}$ and $F' \in \mathcal{T}_c^+$. By the octahedral axiom, we have the following commutative diagram

$$\begin{array}{ccccccc} D_{k+1} & \xrightarrow{h} & D_k & \longrightarrow & F' & \longrightarrow & D_{k+1}[1] \\ \parallel & & \downarrow g_k & & \downarrow & & \parallel \\ D_{k+1} & \xrightarrow{g_k h} & F & \xrightarrow{f_{k+1}} & F_{k+1} & \longrightarrow & D_{k+1}[1] \\ & & \downarrow f_k & & \downarrow & & \\ & & F_k & \xlongequal{\quad} & F_k & & \\ & & \downarrow & & \downarrow & & \\ & & D_k[1] & \longrightarrow & F'[1] & & \end{array}$$

where all rows and columns are triangles. Clearly, $g_k h \in \mathcal{J}^{k+1}$ and $F_{k+1} \in \mathcal{T}_c^+$. Since $\mathcal{T}(f_k, -)|_{\mathcal{E}}$ is surjective, the same holds for $\mathcal{T}(f_{k+1}, -)|_{\mathcal{E}}$. This completes the induction step and the proof. \square

While the following lemma is well-known, we provide a proof for the reader's convenience.

Lemma 5.17. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object G . Let $n > 0$ and $C \in [\mathcal{E}]_n$. For any $f : X \rightarrow Y$ in \mathcal{J}^n , the induced map $\mathcal{T}(f, C) : \text{Hom}_{\mathcal{T}}(Y, C) \rightarrow \text{Hom}_{\mathcal{T}}(X, C)$ vanishes.*

Proof. We proceed by induction, beginning with the case $n = 1$. In this case $f \in \mathcal{J}$, and there exists $C' \in \mathcal{T}$ such that $C \oplus C' \simeq \prod_{i \in I} X_i$ with each $X_i \in \mathcal{E}$. Since

$$\text{Hom}_{\mathcal{T}}(f, \prod_{i \in I} X_i) \simeq \prod_{i \in I} \text{Hom}_{\mathcal{T}}(f, X_i) = 0,$$

we obtain $\text{Hom}_{\mathcal{T}}(f, C) = 0$. This proves the case $n = 1$.

Assume now that the statement holds for all $n \leq k$ with $k \geq 1$, and consider the case $n = k + 1$. Then there exist an object $Z \in \mathcal{T}$, a morphism $g: X \rightarrow Z$ in \mathcal{J} , and a morphism $h: Z \rightarrow Y$ in \mathcal{J}^k such that $f = hg$.

Let $C \in \overline{[\mathcal{E}]}_{k+1}$. Then there exists a triangle

$$C_1 \xrightarrow{s} C \xrightarrow{t} C_2 \longrightarrow C_1[1]$$

with $C_1 \in \overline{[\mathcal{E}]}_1$ and $C_2 \in \overline{[\mathcal{E}]}_k$. For any morphism $\alpha: Z \rightarrow C$, consider the diagram

$$\begin{array}{ccccccc} & & C_1 & & & & \\ & & \downarrow s & & & & \\ X & \xrightarrow{g} & Z & \xrightarrow{h} & Y & \xrightarrow{\alpha} & C \\ & & \downarrow t & & & & \\ & & & & & C_2 & \\ & & & & & \downarrow & \\ & & & & & & C_1[1] \end{array}$$

By the induction hypothesis, $t\alpha h = 0$, so αh factors through C_1 . Hence $\alpha hg = \alpha f = 0$. This completes the proof. \square

Theorem 5.18. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object G . Consider the Yoneda functor*

$$y: \mathcal{T}_c^{+\text{op}} \longrightarrow \text{Hom}_k(\mathcal{E}, k\text{-Mod}).$$

Then y is full, and the essential image consists of all the locally finite \mathcal{E} -homological functors.

Proof. By Lemma 5.3, the functor $\text{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{E}}$ is locally finite whenever $X \in \mathcal{T}_c^+$. Combining Corollary 4.8 with [15, Lemma 7.8], we conclude that y is full.

Now let \mathbf{H} be a locally finite \mathcal{E} -homological functor. It suffices to show that $\mathbf{H} \in \text{Im } y$. By Proposition 5.7, there exist $F \in \mathcal{T}$ and an isomorphism

$$\varphi: \text{Hom}_{\mathcal{T}}(F, -)|_{\mathcal{E}} \longrightarrow \mathbf{H}.$$

Moreover, by the proof of Lemma 5.5 together with Proposition 5.7, we have $F \in \overline{[\mathcal{E}]}_4$. Applying Lemma 5.16, there exists a triangle

$$D_4 \xrightarrow{g} F \xrightarrow{f} F_4 \longrightarrow D_4[1]$$

with $F_4 \in \mathcal{T}_c^+$ and $g \in \mathcal{J}^4$. By Lemma 5.17, the map

$$\text{Hom}_{\mathcal{T}}(g, F): \text{Hom}_{\mathcal{T}}(F, F) \longrightarrow \text{Hom}_{\mathcal{T}}(D_4, F)$$

vanishes. Hence $g = 0$, and therefore F is a direct summand of F_4 . Since $F_4 \in \mathcal{T}_c^+$, it follows that $F \in \mathcal{T}_c^+$ as well. \square

Corollary 5.19. *Let X be an object in \mathcal{T} . Then $X \in \mathcal{T}_c^+$ if and only if the functor $\text{Hom}_{\mathcal{T}}(X, -)$ is locally finite \mathcal{E} -homological.*

Proof. By Theorem 5.18, there exist $X' \in \mathcal{T}_c^+$ and an isomorphism

$$\varphi: \text{Hom}_{\mathcal{T}}(X', -)|_{\mathcal{E}} \longrightarrow \text{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{E}}.$$

Since $X' \in \mathcal{T}_c^+$, Corollary 4.8 together with [15, Lemma 7.8] implies that there exists a morphism $f: X \rightarrow X'$ such that

$$\varphi = \mathcal{T}(f, -)|_{\mathcal{E}}.$$

Extend f to a triangle

$$X \xrightarrow{f} X' \longrightarrow Y \longrightarrow X[1].$$

For every $i \in \mathbb{Z}$, we have $\text{Hom}_{\mathcal{T}}(Y, E[i]) = 0$. Since $E \in \mathcal{E}$ is a cogenerator of \mathcal{T} , it follows that $Y = 0$. Hence $X \simeq X' \in \mathcal{T}_c^+$. \square

Lemma 5.20. *The following facts are given.*

- (1) $\mathcal{T}^b \cap \mathcal{T}_c^- \subseteq \mathcal{T}_c^+$;
- (2) $\mathcal{T}^b \cap \mathcal{T}_c^+ \subseteq \mathcal{T}_c^-$;
- (3) $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^+ = \mathcal{T}_c^+ \cap \mathcal{T}_c^-$.

Proof. Let $X \in \mathcal{T}^b \cap \mathcal{T}_c^-$. To show that $X \in \mathcal{T}_c^+$, it suffices to prove that $\text{Hom}_{\mathcal{T}}(X, -)$ is a locally finite \mathcal{E} -homological functor (Corollary 5.19). Fix $E' \in \mathcal{E}$. Choose a triangle

$$C \xrightarrow{\alpha} X \xrightarrow{\beta} D \longrightarrow C[1]$$

with $C \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -m}$. Applying $\text{Hom}_{\mathcal{T}}(-, E')$ yields an exact sequence

$$\text{Hom}_{\mathcal{T}}(D, E') \longrightarrow \text{Hom}_{\mathcal{T}}(X, E') \longrightarrow \text{Hom}_{\mathcal{T}}(C, E') \longrightarrow \text{Hom}_{\mathcal{T}}(D[-1], E').$$

By Proposition 4.3(1), we have $E' \in \mathcal{T}^+$, hence

$$\text{Hom}_{\mathcal{T}}(D, E') = 0 = \text{Hom}_{\mathcal{T}}(D[-1], E') \text{ for } m \gg 0.$$

Since $\text{Hom}_{\mathcal{T}}(C, E')$ is finite-dimensional by Lemma 4.2(1), it follows that $\text{Hom}_{\mathcal{T}}(X, E')$ is finite-dimensional.

To prove $\text{Hom}_{\mathcal{T}}(X, E'[i]) = 0$ for $i \gg 0$, it suffices to show $\text{Hom}_{\mathcal{T}}(X, E[i]) = 0$ for $i \gg 0$ (Remark 5.1). Since $X \in \mathcal{T}^b \subseteq \mathcal{T}^+$, we have $\text{Hom}_{\mathcal{T}}(X, E[i]) = 0$ for $i \gg 0$ by Proposition 4.3(1).

(2) is proved in the same way as (1).

(3) follows immediately from (1) and (2). \square

Lemma 5.21. *Let $f: F \rightarrow F'$ be a morphism in \mathcal{T}_c^+ with $F \in \mathcal{T}_c^b$. Then $\mathcal{T}(f, -)|_{\mathcal{E}} = 0$ if and only if $f = 0$.*

Proof. The necessity is clear. For sufficiency, since $F \in \mathcal{T}_c^b$, there exists an integer $\ell > 0$ such that $F \in \mathcal{T}^{\leq \ell}$. As $F' \in \mathcal{T}_c^+$, we may choose a triangle

$$D' \longrightarrow F' \xrightarrow{\alpha} E' \longrightarrow D'[1]$$

with $D' \in \mathcal{T}^{\geq \ell+1}$ and $E' \in \mathcal{E}$. The condition $\mathcal{T}(f, -)|_{\mathcal{E}} = 0$ implies $\alpha f = 0$, so f factors through D' . Since $\text{Hom}_{\mathcal{T}}(F, D') = 0$, it follows that $f = 0$. \square

Theorem 5.22. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object G . Consider the composite*

$$\mathcal{T}_c^{b\text{op}} \xrightarrow{i} \mathcal{T}_c^{+\text{op}} \xrightarrow{y} \text{Hom}_k(\mathcal{E}, k\text{-Mod})$$

Then the composite functor $y \circ i$ is fully faithful, and the essential image consists of all the finite \mathcal{E} -homological functors.

Proof. By Lemma 5.20(3), the functor $i: (\mathcal{T}_c^b)^{\text{op}} \rightarrow (\mathcal{T}_c^+)^{\text{op}}$ is well-defined. Combining Theorem 5.18 with Lemma 5.21, we conclude that $y \circ i$ is fully faithful.

Let $X \in \mathcal{T}_c^b = \mathcal{T}_c^+ \cap \mathcal{T}^b$. By Lemma 5.3(2), the functor $\text{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{E}}$ is finite.

Now suppose $Y \in \mathcal{T}$ is such that $\text{Hom}_{\mathcal{T}}(Y, -)|_{\mathcal{E}}$ is a finite \mathcal{E} -homological functor. It suffices to show that $Y \in \mathcal{T}_c^b$. By Corollary 5.19, we already have $Y \in \mathcal{T}_c^+$. For any $i \in \mathbb{Z}$,

$$\text{Hom}_{\mathcal{T}}(Y, E[i]) \simeq \text{DHom}_{\mathcal{T}}(G[i], Y).$$

Since $\text{Hom}_{\mathcal{T}}(Y, -)|_{\mathcal{E}}$ is finite, $\text{Hom}_{\mathcal{T}}(G[i], Y) = 0$ for $|i| \gg 0$. Hence $Y \in \mathcal{T}^b$, and therefore $Y \in \mathcal{T}^b \cap \mathcal{T}_c^+ = \mathcal{T}_c^b$ by Lemma 5.20(3). \square

The proof of the above theorem essentially yields the following corollary.

Corollary 5.23. *Let X an object in \mathcal{T} . Then $X \in \mathcal{T}_c^b$ if and only if the functor $\text{Hom}_{\mathcal{T}}(X, -)$ is finite \mathcal{E} -homological.*

6. LOCALIZATION THEOREMS FOR \mathcal{T}_c^+ AND \mathcal{E}

In this section, we establish localization theorems for the intrinsic subcategories \mathcal{T}_c^+ and \mathcal{E} , extending the framework from [23] to the Brown–Comenetz duality setting. In the classical theory, recollements of triangulated categories induce short exact sequences of Verdier quotients, and these sequences play a central role in understanding how homological and structural information decomposes across the recollement. We show that analogous localization phenomena hold for the dual intrinsic subcategories \mathcal{T}_c^+ and \mathcal{E} , which arise naturally from Brown–Comenetz duality and the representability theory developed in the previous section. These results constitute the dual counterparts of the localization theorems in [23].

Throughout this section, for an approximable triangulated category \mathcal{T} with a compact generator G , we denote $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ by $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.

6.1. A localization theorem for \mathcal{T}_c^+ .

Lemma 6.1. *Let $\mathbf{F}: \mathcal{T} \rightarrow \mathcal{S}$ be a triangle functor between compactly generated k -linear triangulated categories, each of which has a compact generator. If \mathbf{F} preserves objects of \mathcal{E}_t and products, then \mathbf{F} can restrict to $\mathbf{F}: \mathcal{T}_c^+ \rightarrow \mathcal{S}_c^+$.*

Proof. Since \mathcal{T} is a compactly generated triangulated category, the dual Brown representability theorem (see [12, Proposition 5.3.1]) ensures that \mathbf{F} admits a left adjoint $\mathbf{F}' : \mathcal{S} \rightarrow \mathcal{T}$.

Let G_s and G_t be compact generators of \mathcal{S} and \mathcal{T} respectively, and let E_t denote the Brown–Comenetz dual of G_t . Then

$$\begin{aligned} \mathbf{D}\mathsf{Hom}_{\mathcal{T}}(G_t[i], \mathbf{F}'(G_s)) &\simeq \mathsf{Hom}_{\mathcal{T}}(\mathbf{F}'(G_s), E_t[i]) && (\text{Brown–Comenetz dual}) \\ &\simeq \mathsf{Hom}_{\mathcal{S}}(G_s, \mathbf{F}(E_t)[i]) && (\text{adjunction}). \end{aligned}$$

Since \mathbf{F} respects objects in \mathcal{E}_t , we have $\mathbf{F}(E_t) \in \mathcal{E}_s \subseteq \mathcal{S}^+$ by Proposition 4.3(1). Thus $\mathsf{Hom}_{\mathcal{S}}(G_s, \mathbf{F}(E_t)[i]) = 0$ for $i \ll 0$. Hence, $\mathsf{Hom}_{\mathcal{T}}(G_t[i], \mathbf{F}'(G_s)) = 0$ for $i \ll 0$. By Theorem 2.10(1), $\mathbf{F}'(G_s) \in \mathcal{T}^{\leq n}$ for some $n > 0$. For any $M \in \mathcal{T}^{\geq 1}$,

$$\mathsf{Hom}_{\mathcal{S}}(G_s[i], \mathbf{F}(M)) \simeq \mathsf{Hom}_{\mathcal{T}}(\mathbf{F}'(G_s)[i], M) = 0 \quad \text{for } i \geq n.$$

Then $\mathbf{F}(M) \in \mathcal{S}^{\geq -n+1}$ and hence $\mathbf{F}(\mathcal{T}^{\geq 1}) \subseteq \mathcal{S}^{\geq -n+1}$.

Now let $l > 0$, set $m = n + l$, and take $T \in \mathcal{T}_c^+$. Choose a triangle

$$D \longrightarrow T \longrightarrow F \longrightarrow D[1]$$

with $D \in \mathcal{T}^{\geq m}$ and $F \in \mathcal{E}_t$. Applying \mathbf{F} gives a triangle

$$\mathbf{F}(D) \longrightarrow \mathbf{F}(T) \longrightarrow \mathbf{F}(F) \longrightarrow \mathbf{F}(D)[1],$$

where $\mathbf{F}(D) \in \mathcal{S}^{\geq l}$ and $\mathbf{F}(F) \in \mathcal{E}_s$. Hence $\mathbf{F}(T) \in \mathcal{S}_c^+$. \square

Lemma 6.2. *Let $\mathbf{F}: \mathcal{T} \rightarrow \mathcal{S}$ be a triangle functor between locally Hom-finite k -linear triangulated categories, each of which has a compact silting object. If \mathbf{F} admits a right adjoint functor that preserves objects of \mathcal{E}_s , then \mathbf{F} restricts to $\mathcal{T}_c^+ \rightarrow \mathcal{S}_c^+$.*

Proof. Let $X \in \mathcal{T}_c^+$ and $Y \in \mathcal{E}_s$. Assume that \mathbf{F} admits a right adjoint \mathbf{G} . Then for all $i \in \mathbb{Z}$,

$$\mathsf{Hom}_{\mathcal{S}}(\mathbf{F}(X), Y[i]) \simeq \mathsf{Hom}_{\mathcal{T}}(X, \mathbf{G}(Y)[i]).$$

Since $\mathbf{G}(Y) \in \mathcal{E}_t$, Theorem 5.18 implies that $\mathsf{Hom}_{\mathcal{T}}(X, \mathbf{G}(Y))$ is finite-dimensional and

$$\mathsf{Hom}_{\mathcal{T}}(X, \mathbf{G}(Y)[i]) = 0 \quad \text{for } i \gg 0.$$

Hence $\mathsf{Hom}_{\mathcal{S}}(\mathbf{F}(X), Y)$ is finite-dimensional and $\mathsf{Hom}_{\mathcal{S}}(\mathbf{F}(X), Y[i]) = 0$ for $i \gg 0$. Therefore $\mathbf{F}(X) \in \mathcal{S}_c^+$. \square

Lemma 6.3. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object. Suppose that \mathcal{T} admits a compact generator G satisfying $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \ll 0$. Let E be the Brown–Comenetz dual of G . Then $G \in \mathcal{T}_c^b$ and $E \in \mathcal{T}_c^b$. Moreover, G is in \mathcal{T}_c^+ and E is in \mathcal{T}_c^- .*

Proof. The statement $G \in \mathcal{T}_c^b$ follows from [23, Lemma 2.10]. The assertion $E \in \mathcal{T}_c^b$ is an immediate consequence of Theorem 2.10(3). By Lemma 5.20, $\mathcal{T}_c^b = \mathcal{T}_c^+ \cap \mathcal{T}_c^-$, completing the proof. \square

Lemma 6.4. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object G . Then $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ can be restricted to $\mathcal{T}_c^-, \mathcal{T}_c^b$ and \mathcal{T}_c^+ .*

Proof. Let E be the Brown–Comenetz dual of G . By Neeman’s representability theorem [15] and the representability theorem in Section 5, we obtain the following characterizations:

$$\begin{aligned}\mathcal{T}_c^- &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(G, X[i]) \text{ is finite-dimensional for all } i \text{ and vanishes for } i \gg 0\}, \\ \mathcal{T}_c^b &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(G, X[i]) \text{ is finite-dimensional for all } i \text{ and vanishes for } |i| \gg 0\}, \\ \mathcal{T}_c^+ &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X[i], E) \text{ is finite-dimensional for all } i \text{ and vanishes for } i \ll 0\} \\ &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(G, X[i]) \text{ is finite-dimensional for all } i \text{ and vanishes for } i \ll 0\}.\end{aligned}$$

We show that the t -structure restricts to \mathcal{T}_c^- ; the other cases are analogous. For any $X \in \mathcal{T}_c^-$, consider its truncation triangle

$$X^{\leq 0} \longrightarrow X \longrightarrow X^{\geq 1} \longrightarrow X^{\leq 0}[1].$$

Since $G \in \mathcal{T}^{\leq 0}$, we have $\text{Hom}_{\mathcal{T}}(G[i], X^{\leq 0}) \simeq \text{Hom}_{\mathcal{T}}(G[i], X)$ for $i \geq 0$. Because G is silting, by definition we have

$$\mathcal{T}^{\leq 0} = \{M \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(G, M[i]) = 0 \text{ for } i > 0\}.$$

Consequently, $\text{Hom}_{\mathcal{T}}(G[i], X^{\leq 0}) = 0$ for $i < 0$. This implies $X^{\leq 0} \in \mathcal{T}_c^-$. Since \mathcal{T}_c^- is a triangulated subcategory of \mathcal{T} , we also obtain $X^{\geq 1} \in \mathcal{T}_c^-$. Thus $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ can be restricted to \mathcal{T}_c^- . \square

By Lemma 6.4, locally Hom-finite triangulated categories with compact silting objects satisfy both the Noetherian condition and its dual in the sense of [16, Definition 5.1]. This structural control is essential for establishing localization sequences for \mathcal{T}_c^+ .

Theorem 6.5. *Let the following diagram be a recollement of locally Hom-finite k -linear triangulated categories, each of which admits a compact silting object.*

$$\begin{array}{ccccc} & & i^* & & j_! \\ & \swarrow & & \searrow & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j_! = j^*} & \mathcal{S}. \\ & \searrow & & \swarrow & \\ & i^! & & j_* & \end{array}$$

Then

- (1) *Suppose that \mathcal{T} has a compact generator G_t such that there is an integer N with $\text{Hom}_{\mathcal{T}}(G_t, G_t[n]) = 0, n < N$. If the recollement can extends one step upwards. The first row induces a short exact sequence*

$$\mathcal{S}_c^+ / \mathcal{S}^c \xrightarrow{\bar{j}_!} \mathcal{T}_c^+ / \mathcal{T}^c \xrightarrow{\bar{i}^*} \mathcal{R}_c^+ / \mathcal{R}^c.$$

- (2) *The second row induces a short exact sequence*

$$\mathcal{R}_c^+ / \mathcal{R}_c^b \xrightarrow{\bar{i}_*} \mathcal{T}_c^+ / \mathcal{T}_c^b \xrightarrow{\bar{j}^*} \mathcal{S}_c^+ / \mathcal{S}_c^b.$$

Proof. (1) By Lemma 6.3, we have $\mathcal{T}^c \subseteq \mathcal{T}_c^+$. From the proof of [23, Corollary 3.3], the compact generators in \mathcal{R} and \mathcal{S} also satisfy the assumptions of Lemma 6.3, hence $\mathcal{R}^c \subseteq \mathcal{R}_c^+$ and $\mathcal{S}^c \subseteq \mathcal{S}_c^+$. By Lemma 2.6 and Lemma 2.7, to obtain the exact sequence it suffices to verify the following two claims.

Claim 1. Every morphism from \mathcal{T}^c to $j_!(\mathcal{S}_c^+)$ factors through $j_!(\mathcal{S}^c)$.

Claim 2. $j_!(\mathcal{S}_c^+) \cap \mathcal{T}^c = j_!(\mathcal{S}^c)$.

Let $T \in \mathcal{T}^c$, $V \in \mathcal{S}_c^+$, and $f: T \rightarrow j_!(V)$. Since $j_!$ admits a left adjoint $j^\#$, we have an adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(T, j_!(V)) \simeq \mathrm{Hom}_{\mathcal{S}}(j^\#(T), V).$$

Thus f is the composite

$$T \xrightarrow{\eta_T} j_!j^\#(T) \xrightarrow{j_!(g)} j_!(V)$$

for some $g: j^\#(T) \rightarrow V$. Since $j^\#$ preserves compact objects, $j^\#(T) \in \mathcal{S}^c$, and hence f factors through $j_!j^\#(T) \in j_!(\mathcal{S}^c)$. This proves Claim 1. Claim 2 follows from Claim 1.

(2) By Lemma 6.1 and Lemma 6.2, we have a half recollement

$$\begin{array}{ccccc} \mathcal{R}_c^+ & \xrightarrow{i_* = i_!} & \mathcal{T}_c^+ & \xrightarrow{j^! = j^*} & \mathcal{S}_c^+ \\ & \curvearrowleft i^! & & \curvearrowright j_* & \end{array} .$$

By Lemma 2.6 and Lemma 2.7, to obtain the short exact sequence of Verdier quotients it suffices to prove the following two claims.

Claim 1. Every morphism from \mathcal{T}_c^b to $i_*(\mathcal{R}_c^+)$ factors through $i_*(\mathcal{R}_c^b)$.

Claim 2. $i_*(\mathcal{R}_c^+) \cap \mathcal{T}_c^b = i_*(\mathcal{R}_c^b)$.

Let $X \in \mathcal{T}_c^b$, $Y \in \mathcal{R}_c^+$, and $f: X \rightarrow i_*(Y)$. Since $X \in \mathcal{T}_c^b$, there exists m with $X \in \mathcal{T}^{\leq m}$. Because i_* is t -exact, there exists n such that $i_*(\mathcal{R}^{\geq 0}) \subseteq \mathcal{T}^{\geq n}$.

By Lemma 6.4, for $Y \in \mathcal{R}_c^+$ there is a triangle

$$Y_1 \longrightarrow Y \longrightarrow Y_2 \longrightarrow Y_1[1]$$

with $Y_1 \in \mathcal{R}_c^+ \cap \mathcal{R}^{\leq m-n} \subseteq \mathcal{R}_c^b$ and $Y_2 \in \mathcal{R}^{\geq m-n+1}$. Applying i_* gives a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow g & \nearrow & \downarrow f & \\ i_*(Y_1) & \longrightarrow & i_*(Y) & \longrightarrow & i_*(Y_2) \longrightarrow i_*(Y_1)[1] \end{array}$$

and since $\mathrm{Hom}_{\mathcal{T}}(X, i_*(Y_2)) = 0$, the morphism f factors through $g: X \rightarrow i_*(Y_1)$. Thus f factors through $i_*(Y_1) \in i_*(\mathcal{R}_c^b)$, proving Claim 1. Claim 2 follows from Claim 1.

This completes the proof. \square

Corollary 6.6. *Let the following diagram be a recollement of locally Hom-finite k -linear triangulated categories, each of which admits a compact silting object.*

$$\begin{array}{ccccc} & & i^* & & j_! \\ & \swarrow & \curvearrowright & \curvearrowright & \downarrow \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S} \\ & \curvearrowleft i^! & & \curvearrowright j_* & \end{array}$$

Suppose that \mathcal{T} has a compact generator G_t such that there is an integer N with $\mathrm{Hom}_{\mathcal{T}}(G_t, G_t[n]) = 0, n < N$. If the recollement can extends one step downwards.

Then the recollement induces a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{R}_c^b/\mathcal{R}^c & \xrightarrow{\overline{i_*}} & \mathcal{T}_c^b/\mathcal{T}^c & \xrightarrow{\overline{j^*}} & \mathcal{S}_c^b/\mathcal{S}^c \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{R}_c^+/\mathcal{R}^c & \xrightarrow{\overline{i_*}} & \mathcal{T}_c^+/\mathcal{T}^c & \xrightarrow{\overline{j^*}} & \mathcal{S}_c^+/\mathcal{S}^c \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{R}_c^+/\mathcal{R}_c^b & \xrightarrow{\overline{i_*}} & \mathcal{T}_c^+/\mathcal{T}_c^b & \xrightarrow{\overline{j^*}} & \mathcal{S}_c^+/\mathcal{S}_c^b
 \end{array}$$

in which all rows and columns are short exact sequences of quotient categories.

6.2. A localization theorem for \mathcal{E} .

Theorem 6.7. *Let the following diagram be a recollement of locally Hom-finite approximable k -linear triangulated categories*

$$\begin{array}{ccccc}
 & i^* & & j_! & \\
 \mathcal{R} & \xleftarrow{i_* = i_1} & \mathcal{T} & \xleftarrow{j^! = j^*} & \mathcal{S} \\
 & i^! & & j_* &
 \end{array}$$

- (1) Let \mathcal{U}, \mathcal{V} and \mathcal{W} be triangulated subcategories of \mathcal{R}, \mathcal{S} and \mathcal{T} , respectively, such that $\mathcal{E}_r \subseteq \mathcal{U}$, $\mathcal{E}_s \subseteq \mathcal{V} \subseteq \mathcal{S}_c^+$ and $\mathcal{E}_t \subseteq \mathcal{W}$. If the third row of the recollement is restricted to a short exact sequence

$$\mathcal{V} \xrightarrow{j_*} \mathcal{W} \xrightarrow{i^!} \mathcal{U},$$

then it induces a short exact sequence

$$\mathcal{V}/\mathcal{E}_s \xrightarrow{\overline{j_*}} \mathcal{W}/\mathcal{E}_t \xrightarrow{\overline{i^!}} \mathcal{U}/\mathcal{E}_r.$$

- (2) Moreover, if the triangulated categories \mathcal{R}, \mathcal{T} and \mathcal{S} have compact silting objects, then the third row induces a short exact sequence

$$\mathcal{S}_c^+/\mathcal{E}_s \xrightarrow{\overline{j_*}} \mathcal{T}_c^+/\mathcal{E}_t \xrightarrow{\overline{i^!}} \mathcal{R}_c^+/\mathcal{E}_r.$$

Proof. (1) There is the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{E}_{\mathcal{S}} & \xrightarrow{j_*} & \mathcal{E}_t & \xrightarrow{i^!} & \mathcal{E}_{\mathcal{R}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{V} & \xrightarrow{j_*} & \mathcal{W} & \xrightarrow{i^!} & \mathcal{U} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{V}/\mathcal{E}_s & \xrightarrow{\overline{j_*}} & \mathcal{W}/\mathcal{E}_t & \xrightarrow{\overline{i^!}} & \mathcal{U}/\mathcal{E}_r
 \end{array}$$

in which the first row is exact up to direct summands by Proposition 3.3 and the second row is exact by hypothesis. By Lemma 2.6, it suffices to prove the induced functor $\overline{j_*}$ is fully-faithful.

Note that $\overline{j_*}$ is really the composition

$$\mathcal{V}/\mathcal{E}_s \xrightarrow{\tilde{j_*}} j_*(\mathcal{V})/j_*(\mathcal{E}_s) \xrightarrow{i} \mathcal{W}/\mathcal{E}_t.$$

Since j_* is fully faithful, so \tilde{j}^* in the decomposition above is an equivalence. It turns to prove that i is fully-faithful. By Lemma 2.7, to obtain the short exact sequence, it remains to check the following two claims.

Claim 1: Each morphism from $j_*(\mathcal{V})$ to \mathcal{E}_t factors through an object in $j_*(\mathcal{E}_s)$.

Claim 2: $j_*(\mathcal{V}) \cap \mathcal{E}_t = j_*(\mathcal{E}_s)$. It is obvious.

Let V be an object in $\mathcal{V} \subseteq \mathcal{S}_c^+$. V admits a triangle

$$D \longrightarrow V \longrightarrow F \longrightarrow D[1]$$

with $D \in \mathcal{T}^{\geq m}$ and $F \in \mathcal{E}_s$. For sufficiently large m , we have $\text{Hom}_{\mathcal{T}}(j_*(D), E_t) = 0$ by Proposition 4.3. Hence each morphism from $j_*(V)$ to E_t factors through $j_*(F)$, which belongs to $j_*(\mathcal{E}_s)$. This proves Claim 1. Claim 2 follows from Claim 1.

(2) It follows from Lemma 6.1 and Lemma 6.2 that there is an exact sequence

$$\mathcal{S}_c^+ \xrightarrow{j_*} \mathcal{T}_c^+ \xrightarrow{i^!} \mathcal{R}_c^+.$$

By (1), we obtain the short exact sequence of Verdier quotient categories. \square

Corollary 6.8. *Let the following diagram be a recollement of locally Hom-finite k -linear triangulated categories, each of which admits a compact silting object.*

$$\begin{array}{ccccc} & i^* & & j_! & \\ \mathcal{R} & \xrightarrow{i_* = i_!} & \mathcal{T} & \xrightarrow{j^! = j^*} & \mathcal{S}. \\ \swarrow & & \downarrow & & \searrow \\ i^! & & & & j_* \end{array}$$

Then there exists a commutative diagram

$$\begin{array}{ccccc} \mathcal{S}_c^b/\mathcal{E}_s & \xrightarrow{\overline{j}_*} & \mathcal{T}_c^b/\mathcal{E}_t & \xrightarrow{\overline{i}^!} & \mathcal{R}_c^b/\mathcal{E}_r \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_c^+/\mathcal{E}_s & \xrightarrow{\overline{j}_*} & \mathcal{T}_c^+/\mathcal{E}_t & \xrightarrow{\overline{i}^!} & \mathcal{R}_c^-/\mathcal{E}_r \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_c^+/\mathcal{R}_c^b & \xrightarrow{\overline{j}_*} & \mathcal{T}_c^+/\mathcal{T}_c^b & \xrightarrow{\overline{i}^!} & \mathcal{R}_c^+/\mathcal{S}_c^b \end{array}$$

in which all rows and columns are short exact sequences of quotient categories.

7. EXAMPLES AND APPLICATIONS

The representability and localization theorems established in the previous sections provide a dual framework parallel to Neeman's classical theory. To illustrate the scope and concrete behavior of these results, this section presents an explicit example arising from recollements of derived categories of finite-dimensional algebras. We also give an application to the construction of adjoint functors, providing a dual analogue of [15, Theorem 13.1].

7.1. Examples. Let the following diagram be a recollement of derived categories of finite-dimensional algebras over a field k .

$$\begin{array}{ccccc} & i^* & & j_! & \\ \mathcal{D}(B) & \xrightarrow{i_* = i_!} & \mathcal{D}(A) & \xrightarrow{j^! = j^*} & \mathcal{D}(C). \\ \swarrow & & \downarrow & & \searrow \\ i^! & & & & j_* \end{array}$$

The derived categories $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(C)$ are locally Hom-finite k -linear triangulated category with compact silting objects A , B and C , respectively.

(1) Proposition 3.3 implies that the third row of the recollement restricts to a short exact sequence up to direct summands

$$\mathcal{K}^b(C\text{-inj}) \xrightarrow{j_*} \mathcal{K}^b(A\text{-inj}) \xrightarrow{i^!} \mathcal{K}^b(B\text{-inj}).$$

(2) Theorem 5.22 implies that each object in $\mathcal{D}^b(A\text{-mod})$ is represented by a finite \mathcal{E}_A -homological functor.

(3) Theorem 6.5 implies that we have short exact sequences

$$\mathcal{K}^+(C\text{-inj})/\mathcal{K}^b(C\text{-proj}) \xrightarrow{\bar{j}_!} \mathcal{K}^+(A\text{-inj})\mathcal{K}^b(A\text{-proj}) \xrightarrow{\bar{i}^*} \mathcal{K}^+(B\text{-inj})\mathcal{K}^b(B\text{-proj}),$$

$$\mathcal{K}^+(B\text{-inj})/\mathcal{D}^b(B\text{-mod}) \xrightarrow{\bar{i}_*} \mathcal{K}^+(A\text{-inj})\mathcal{D}^b(A\text{-mod}) \xrightarrow{\bar{j}^*} \mathcal{K}^+(C\text{-inj})\mathcal{D}^b(C\text{-mod}).$$

(4) Theorem 6.7 gives us the following short exact sequence

$$\mathcal{K}^+(C\text{-inj})/\mathcal{K}^b(C\text{-inj}) \xrightarrow{\bar{j}_*} \mathcal{K}^+(A\text{-inj})/\mathcal{K}^b(A\text{-inj}) \xrightarrow{\bar{i}^!} \mathcal{K}^+(B\text{-inj})/\mathcal{K}^b(B\text{-inj}).$$

7.2. The construction of adjoints. The representability theorems established in Section 5 allow us to characterize when a triangle functor $\mathbf{G} : \mathcal{T}_c^b \rightarrow \mathcal{S}$ admits a left adjoint. This result should be viewed as the dual analogue of [15, Theorem 13.1], where Neeman provides necessary and sufficient conditions for the existence of a right adjoint.

Proposition 7.1. *Let \mathcal{T} be a locally Hom-finite k -linear triangulated category with a compact silting object G satisfying $\text{Hom}_{\mathcal{T}}(G, G[i]) = 0$ for $i \ll 0$, and \mathcal{S} be a k -linear triangulated category. A k -linear triangle functor $\mathbf{G} : \mathcal{T}_c^b \rightarrow \mathcal{S}$ has a left adjoint if and only if the following three conditions hold:*

(1) $\dim(\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T))) < \infty$ for all $S \in \mathcal{S}$ and $T \in \mathcal{E}$.

(2) For any $S \in \mathcal{S}$, there exists $B > 0$ with $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(\mathcal{T}_c^b \cap \mathcal{T}_G^{\geq B})) = 0$.

(3) For any object $T \in \mathcal{E}$ and any object $S \in \mathcal{S}$, there exists an integer C such that $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T[i])) = 0$ for $i > C$.

Proof. We start with the necessity. Let $\mathbf{F} : \mathcal{S} \rightarrow \mathcal{T}_c^b$ be the left adjoint of \mathbf{G} . By Lemma 5.20, we have $\mathcal{T}_c^b \subseteq \mathcal{T}_c^+$. Then Lemma 4.2(2) implies that $\dim(\text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T)) < \infty$ for all $S \in \mathcal{S}$ and $T \in \mathcal{E}$. Thus $\dim(\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T))) < \infty$ for all $S \in \mathcal{S}$ and $T \in \mathcal{E}$ follows from the adjunction isomorphism. Fix $S \in \mathcal{S}$. Since $\mathbf{F}(S) \in \mathcal{T}_c^b \subseteq \mathcal{T}^b$, we can choose $B > 0$ and $A < B$ such that $\mathbf{F}(S) \in \mathcal{T}_G^{\geq A} \cap \mathcal{T}_G^{\leq B-1}$. Hence

$$\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(\mathcal{T}_c^b \cap \mathcal{T}_G^{\geq B})) \simeq \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), \mathcal{T}_c^b \cap \mathcal{T}_G^{\geq B}) = 0.$$

Now fix $T \in \mathcal{E}$. By Proposition 4.3(1), there is an integer $B' > 0$ such that $\text{Hom}_{\mathcal{T}}(\mathcal{T}_G^{\geq B'}, T) = 0$. Then $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T[i])) \simeq \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T[i]) = 0$ for $i > B' - A$. Setting $C = B' - A$ completes the proof of necessity.

Now we prove the sufficiency. Under the given conditions, $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(-))$ is clearly a finite \mathcal{E} -homological functor for every $S \in \mathcal{S}$. By Theorem 5.22, there exists an object $T_s \in \mathcal{T}_c^b$ such that $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(-))|_{\mathcal{E}} \simeq \text{Hom}_{\mathcal{T}}(T_s, -)|_{\mathcal{E}}$. Also by Theorem 5.22, the functor $\mathcal{T}_c^b \rightarrow \text{Hom}_k(\mathcal{E}, k\text{-Mod})$, $T \mapsto \text{Hom}_{\mathcal{T}}(T, -)|_{\mathcal{E}}$ is fully faithful. Thus we obtain a k -linear triangle functor $\mathbf{F} : \mathcal{S} \rightarrow \mathcal{T}_c^b$, $S \mapsto T_s$ and a natural isomorphism

$$\varphi_{S,T} : \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T)) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T)$$

for all $T \in \mathcal{E}$ and $S \in \mathcal{S}$. Consider the following composite

$$\text{Hom}_{\mathcal{T}}(T, T') \xrightarrow{\mathbf{G}(-)} \text{Hom}_{\mathcal{S}}(\mathbf{G}(T), \mathbf{G}(T')) \xrightarrow{\varphi_{\mathbf{G}(T), T'}} \text{Hom}_{\mathcal{T}}(\mathbf{F}\mathbf{G}(T), T')$$

where $T \in \mathcal{T}_c^b$, $T' \in \mathcal{E}$. This induces a morphism

$$\alpha : \text{Hom}_{\mathcal{T}}(T, -)|_{\mathcal{E}} \rightarrow \text{Hom}_{\mathcal{T}}(\mathbf{F}\mathbf{G}(T), -)|_{\mathcal{E}}$$

in $\text{Hom}_k(\mathcal{E}, k\text{-Mod})$. By Theorem 5.22, there is a unique morphism $\epsilon_T : \mathbf{F}\mathbf{G}(T) \rightarrow T$ such that $\alpha = \mathcal{T}(\epsilon_T, -)|_{\mathcal{E}}$. Hence we obtain a natural transformation $\epsilon : \mathbf{F}\mathbf{G} \rightarrow \text{Id}$. Now for $T \in \mathcal{T}_c^b$ and $S \in \mathcal{S}$, define

$$\psi_{S,T} : \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T)) \xrightarrow{\mathbf{F}(-)} \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), \mathbf{F}\mathbf{G}(T)) \xrightarrow{\mathcal{T}(\mathbf{F}(S), \epsilon_T)} \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T),$$

which is natural in $S \in \mathcal{S}$ and $T \in \mathcal{T}_c^b$. We want to show ψ is a natural isomorphism.

By Lemma 6.3, $\mathcal{E} \subseteq \mathcal{T}_c^b$. We claim that if $S \in \mathcal{S}$ and $T \in \mathcal{E}$, then $\varphi_{S,T} = \psi_{S,T}$. For $f : S \rightarrow \mathbf{G}(T)$, we have the following commutative diagram

$$\begin{array}{ccc}
\text{Id}_T \in \text{Hom}_{\mathcal{T}}(T, T) & & \\
\downarrow \mathbf{G}(-) & \searrow \mathcal{T}(\epsilon_T, T) & \\
\text{Id}_{\mathbf{G}(T)} \in \text{Hom}_{\mathcal{S}}(\mathbf{G}(T), \mathbf{G}(T)) & \xrightarrow{\varphi_{\mathbf{G}(T), T}} & \text{Hom}_{\mathcal{T}}(\mathbf{F}\mathbf{G}(S), T) \\
\downarrow \mathcal{S}(f, \mathbf{G}(T)) & & \downarrow \mathcal{T}(F(f), T) \\
f \in \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T)) & \xrightarrow{\varphi_{S,T}} & \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T).
\end{array}$$

Thus we obtain $\varphi_{S,T}(f) = \epsilon_T \circ \mathbf{F}(f) = \psi_{S,T}(f)$.

Now fix $S \in \mathcal{S}$ and $T \in \mathcal{T}_c^b$. Since $\mathbf{F}(S) \in \mathcal{T}_c^b$, there exists an integer $m > 0$ such that $\mathbf{F}(S) \in \mathcal{T}_G^{\leq m-1}$. By the given conditions, there exists $B > 0$ such that $\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(\mathcal{T}_c^b \cap \mathcal{T}_G^{\geq B})) = 0$. Set $B' = \max\{m, B\}$. Since $T \in \mathcal{T}_c^b \subseteq \mathcal{T}_c^+$, we have a triangle

$$D \longrightarrow T \longrightarrow F \longrightarrow D[1]$$

with $D \in \mathcal{T}_G^{\geq B'+1}$ and $F \in \mathcal{E}$. Note that $F \in \mathcal{E} \subseteq \mathcal{T}_c^b$ and $T \in \mathcal{T}_c^b$ imply $D \in \mathcal{T}_c^b$. This yields the following commutative diagram

$$\begin{array}{ccccccc}
\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(D)) & \longrightarrow & \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(T)) & \longrightarrow & \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(F)) & \longrightarrow & \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(D[1])) \\
\downarrow & & \downarrow \varphi_{S,T} & & \downarrow \psi_{S,F} = \varphi_{S,F} & & \downarrow \\
\text{Hom}_{\mathcal{T}}(\mathbf{F}(S), D) & \longrightarrow & \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), T) & \longrightarrow & \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), F) & \longrightarrow & \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), D[1])
\end{array}$$

where both rows are exact and

$$\text{Hom}_{\mathcal{S}}(S, \mathbf{G}(D)) = 0 = \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), D), \quad \text{Hom}_{\mathcal{S}}(S, \mathbf{G}(D)[1]) = 0 = \text{Hom}_{\mathcal{T}}(\mathbf{F}(S), D[1]).$$

Consequently, $\psi_{S,T}$ is an isomorphism for all $S \in \mathcal{S}$ and $T \in \mathcal{T}_c^b$. \square

APPENDIX A. BASIC PROPERTIES OF COLIMITS IN k -MODULES

This appendix collects several elementary but useful facts concerning colimits and limits in the category $k\text{-Mod}$. Although these results are well known, we include self-contained proofs for completeness, since they play an essential role in Section 5 when analyzing homotopy limits and colimits of sequences of finite-dimensional vector spaces arising from \mathcal{E} -homological functors.

Throughout this appendix, let k be an arbitrary field and consider a direct system

$$(V_*, f_*): \quad V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \longrightarrow \cdots$$

in $k\text{-mod}$. For an object W in $k\text{-mod}$, a *morphism*

$$h_* : V_* \longrightarrow W$$

means a family of maps $h_i : V_i \rightarrow W$ satisfying $h_i = h_{i+1} \circ f_i$ for all $i \geq 1$. Assume that $V \in k\text{-mod}$ is the colimit of (V_*, f_*) in $k\text{-mod}$, with canonical maps $\gamma_i : V_i \rightarrow V$. Our first goal is to show that V is also the colimit of (V_*, f_*) in the larger category $k\text{-Mod}$.

Lemma A.1. *There exists a positive integer N such that γ_i is surjective for all $i \geq N$.*

Proof. Clearly,

$$\text{Im } \gamma_1 \subseteq \text{Im } \gamma_2 \subseteq \text{Im } \gamma_3 \subseteq \cdots \subseteq V.$$

Hence there exists $N > 0$ such that $\text{Im } \gamma_i = \text{Im } \gamma_{i+1}$ for all $i \geq N$. We now show that $\text{Im } \gamma_N = V$.

Assume to the contrary that $\text{Im } \gamma_N \neq V$. Then there exists a nonzero subspace $W \subseteq V$ such that $V = \text{Im } \gamma_N \oplus W$, with canonical projection $p_W: V \rightarrow W$. Note that V is also the colimit in $k\text{-mod}$ of the sequence $V_N \xrightarrow{f_N} V_{N+1} \rightarrow \dots$. However, for every $i \geq N$, the composition

$$V_i \xrightarrow{\gamma_i} V \xrightarrow{p_W} W$$

is zero, contradicting the universal property of the colimit. Thus $\text{Im } \gamma_N = V$, and therefore each γ_i is surjective for all $i \geq N$. \square

Proposition A.2. *V is the colimit of (V_*, f_*) in $k\text{-Mod}$.*

Proof. By Lemma A.1, we may assume without loss of generality that each γ_i is surjective for all $i \geq 1$. Fix $W \in k\text{-Mod}$. By [22, Corollary 3.9], we may write $W = \bigoplus_{j \in J} W_j$ with each $W_j \in k\text{-mod}$. Let $\sigma_j: W_j \rightarrow \bigoplus_{j \in J} W_j$ be the canonical injection. By the universal property of the coproduct, for each $r \in J$ there exists a morphism $\pi_r: \bigoplus_{j \in J} W_j \rightarrow W_r$ such that $\pi_r \sigma_r = \text{Id}_{W_r}$ and $\pi_r \sigma_j = 0$ for $j \neq r$.

Consider a morphism of sequences $h_*: V_* \rightarrow W$, and assume that $h_{i'} \neq 0$ for some $i' \geq 1$. For each $j \in J$, the composition $\pi_j h_i: V_i \rightarrow W_j$ defines a morphism of sequences $V_* \rightarrow W_j$. Since V is the colimit of (V_*, f_*) in $k\text{-mod}$, there exists a unique morphism $g_j: V \rightarrow W_j$ such that

$$g_j \gamma_i = \pi_j h_i \quad \text{for all } i \geq 1.$$

Because each γ_i is surjective, we have $\pi_j h_i \neq 0$ for some (equivalently, every) $i \geq 1$ if and only if $g_j \neq 0$.

We now show that only finitely many $j \in J$ satisfy $g_j \neq 0$. Since V_1 is compact in $k\text{-Mod}$, the morphism $h_1: V_1 \rightarrow W$ factors through a finite direct sum, yielding a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{h_1} & W \\ & \searrow & \swarrow \\ & \bigoplus_{i=1}^n W_{1i} & \end{array}.$$

For each $j \in J$, we have $g_j \neq 0$ if and only if $\pi_j h_1 \neq 0$. By construction of the projections π_j , this implies that the set $\{j \in J : g_j \neq 0\}$ is finite. Without loss of generality, let $\{t_1, \dots, t_m\} \subseteq J$ be precisely the indices with $g_{t_s} \neq 0$. This yields a unique morphism

$$V \longrightarrow \bigoplus_{s=1}^m W_{t_s}.$$

Composing with the canonical inclusion $\bigoplus_{s=1}^m W_{t_s} \hookrightarrow W$, we obtain a morphism $g: V \rightarrow W$. It is straightforward to verify that $g \gamma_i = h_i$ for all $i \geq 1$. Uniqueness of g follows from the surjectivity of the γ_i . \square

Our next aim is to establish the dual statement of Proposition A.2. Denote by

$$(W_*, f_*): \dots \rightarrow W_3 \xrightarrow{f_2} W_2 \xrightarrow{f_1} W_1$$

a sequence in $k\text{-mod}$. For an object W in $k\text{-Mod}$, the notation $\beta_*: W \rightarrow W_*$ means that for each $i \geq 1$, there is a morphism $\beta_i: W \rightarrow W_i$ satisfying $\beta_i = f_i \circ \beta_{i+1}$. We call β_* a morphism from W to W_* .

Proposition A.3. *Given $\beta_*: W \rightarrow W_*$. If for every $V \in k\text{-mod}$ and every $g_*: V \rightarrow W_*$, there exists a unique $h: V \rightarrow W$ such that $g_i = \beta_i \circ h$ holds for all $i > 0$, then W is the $\varprojlim W_*$.*

Proof. Let $U \in k\text{-Mod}$ and let $g_*: U \rightarrow W_*$ be a morphism of sequences. By [22, Corollary 3.9], we may write $U = \bigoplus_{i \in I} U_i$ with each U_i finite-dimensional. Let $\sigma_j: U_j \rightarrow \bigoplus_{i \in I} U_i$ denote the canonical injections. For each j , the morphisms $g_i \circ \sigma_j: U_j \rightarrow W_i$ (for $i > 0$) define a morphism of sequences $U_j \rightarrow W_*$. By hypothesis, there exists a unique morphism $h_j: U_j \rightarrow W$ such that

$$g_i \circ \sigma_j = \beta_i \circ h_j \quad \text{for all } i > 0.$$

By the universal property of the coproduct, these h_j assemble into a unique morphism $h: U \rightarrow W$ satisfying $h \circ \sigma_j = h_j$ for all $j \in I$.

To verify that $\beta_i \circ h = g_i$ for all $i > 0$, it suffices to check equality after precomposing with each σ_j . Indeed,

$$\beta_i \circ h \circ \sigma_j = \beta_i \circ h_j = g_i \circ \sigma_j,$$

so $\beta_i \circ h = g_i$.

For uniqueness, suppose $h': U \rightarrow W$ also satisfies $\beta_i \circ h' = g_i$ for all $i > 0$. Then for each j ,

$$\beta_i \circ (h' \circ \sigma_j) = g_i \circ \sigma_j = \beta_i \circ h_j,$$

and by the uniqueness of h_j , we obtain $h' \circ \sigma_j = h_j$. Hence $h' \circ \sigma_j = h \circ \sigma_j$ for all $j \in I$. This implies $h' = h$. \square

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