Exercises (Lecture 3)

Quadratic operads

Introduction to Operads

Note. We will solve some of these exercises during the exercise sessions. Try to solve at least one exercise you find easy and at east two exercises that you find challenging.

Exercise 1. During Lecture 3 we introduced the associative and commutative operads through binary quadratic presentations. Show that for all $n \geqslant 1$ the space $\mathsf{Ass}(n)$ is the regular representation of S_n, and that for all $n \geqslant 1$ the space $\mathsf{Com}(n)$ is the trivial representation of S_n.

Exercise 2. Use the presentation of the Poisson operad given during Lecture 3 to show that $\mathsf{Poiss}(n) \leqslant n!$ for all $n \geqslant 1$[1].

Exercise 3. Let $x_1 x_2$ be the associative binary generator of Ass and let us consider the operations (which are symmetric and antisymmetric, respectively)

$$x_1 \cdot x_2 = \frac{1}{2}(x_1 x_2 + x_2 x_1), \quad [x_1, x_2] = \frac{1}{2}(x_1 x_2 - x_2 x_1)$$

obtained by 'polarization'. Show that the second is a Lie bracket, and that the second is a commutative (but not associative) product that satisfies the Leibniz rule for $[x_1, x_2]$, and whose associator is equal to $[x_2,[x_1,x_3]]$. This is called the *Livernet–Loday presentation* of the associative operad.

Exercise 4. During Lecture 3, we introduced to operad $t\mathsf{Com}_k$ of totally associative commutative k-ary algebras. It is generated by a single fully symmetric operation μ or arity k subject to the relations $\mu \circ_1 \mu = \mu \circ_i \mu$ for each $i \in [k]$ (and all its symmetric translates). Show that $t\mathsf{Com}_k(n)$ is either the one dimensional trivial representation or zero depending on n. What values must n take so that it is non-zero?

Exercise 5. The permutative operad Perm is generated by a single binary operation $x_1 x_2$ with no symmetries which is associative, and such that

$$x_1(x_2 x_3) = x_2(x_1 x_3).$$

Show that $\mathsf{Perm}(n)$ is of dimension n and is isomorphic as a representation to $\mathrm{Ind}_{S_{n-1}}^{S_n} \mathbb{C}$ where $\mathbb{C}$ is the trivial representation.

We have defined quadratic operads as precisely those operads presented by (homogeneous) quadratic relations on some set of generators. Let us explore how to create maps between them.

[1] There are at least three different ways to show that equality holds.

Exercise 6. Suppose that $(\mathscr{X}, \mathscr{R})$ and $(\mathscr{Y}, \mathscr{Q})$ are quadratic data. Show that a map of sequences $f : \mathscr{X} \longrightarrow \mathscr{Y}$ induces a map on the corresponding quadratic operads if and only if the induced map $F = \mathscr{F}_f$ sends $\mathscr{R}$ to $\mathscr{Q}$.

Exercise 7. Show that there is a map $\mathsf{Ass} \longrightarrow \mathsf{Com}$ induced by the augmentation map $\mathbb{C}S_2 \longrightarrow \mathbb{C}$ (that sends 1 and (12) to 1). What is the interpretation of this at the level of algebras?

Exercise 8. Show that the inclusion map $\mathbb{C}^- \longrightarrow \mathbb{C}S_2$ that assigns 1 to $1-(12)$ induces a map of operads $\mathsf{Lie} \longrightarrow \mathsf{Ass}$. What is the interpretation of this at the level of algebras?

Exercise 9. Show that the inclusion map $\mathbb{C}^- \longrightarrow \mathbb{C}S_2$ that assigns 1 to $1-(12)$ induces a map of operads $\mathsf{Lie} \longrightarrow \mathsf{PreLie}$. What is the interpretation of this at the level of algebras?

Exercise 10. Show that the projection $\mathsf{Ass} \longrightarrow \mathsf{Com}$ actually factors through Perm.