Exercises (Lecture 0)

Recollections

Introduction to Operads

Note. We will solve some of these exercises during the exercise sessions. Try to solve at least one exercise you find easy and at east two exercises that you find challenging.

A. Symmetric groups. Operads are meant to encode operations on objects *along with their symmetries*, which is done through the representation theory of the symmetric groups. The following exercises will remind you of some basic facts about them.

Exercise 1. Let $I = [n]$ so that $\mathrm{Aut}(I) = S_n$ is the symmetric group on n letters. For each ordered partition $\pi = (\pi_1, \ldots, \pi_k)$, let λ be the ordered partition of n with $\lambda_i = \#\pi_i$ for $i \in [k]$. Show that the permutations of $[n]$ that preserve π determine a subgroup of S_n isomorphic to $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Exercise 2. Consider the subgroup of S_n corresponding to the ordered partition of $[n]$ given by $([1,k],[k+1,n])$, along with the inclusion $S_k \times S_{n-k} \hookrightarrow S_n$. Show that a set of representatives for the cosets of this inclusion in S_n is given by the $(k, n-k)$*-shuffles*, those permutations $\sigma \in S_n$ that preserve the linear order in $[1,k]$ and $[k+1,n]$. Conclude that there are exactly $\binom{n}{k}$ shuffles of type $(k, n-k)$ on $[n]$. Define shuffles associated to other partitions of n.

B. Categories. The language of categories and functors permeates most of modern algebra and geometry, and in particular is useful to work with operads and other combinatorial structures defines by graphs. The following will remind you of some important notions we will use during the course.

Exercise 3. A category $\mathscr{C}$ is the datum of a set of objects $\mathrm{Ob}(\mathscr{C})$, and for each $x, y \in \mathrm{Ob}(\mathscr{C})$ a set $\mathscr{C}(x,y)$ of morphisms from x to y. Moreover, we require the existence of an associative and unital composition law

$$- \circ - : \mathscr{C}(y,z) \times \mathscr{C}(x,y) \longrightarrow \mathscr{C}(x,z).$$

The latter means there are distinguished elements $1_x \in \mathscr{C}(x,x)$ for each object of $\mathscr{C}$ that induce the identity $- \circ 1_x$ an $1_x \circ -$ of any $\mathscr{C}(-,x)$ and $\mathscr{C}(x,-)$. Find examples of categories: sets, finite sets, rings, vector spaces, open subsets, posets, and others.

Exercise 4. A functor $F : \mathscr{C}_1 \longrightarrow \mathscr{C}_2$ is a datum that assigns to each object x of the domain an object $F(x)$ of the codomain, and to each morphism $f : x \to y$ a morphism $F(f)$ such that $F(f \circ g) = F(f) \circ F(g)$ and $F(1_x) = 1_{Fx}$ for each pair of composable arrows f and g and each object x of $\mathscr{C}_1$. Find examples of functors between the examples of categories you found above.

Exercise 5. A monoidal category is a category $\mathscr{C}$ along with the datum of a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ along with an associator and left and right units. A monoidal category is *strict* if the associator and left and right units are identities.

1. Expand on the details of these definitions. Define what a braided monoidal category and what a symmetric monoidal category are.
2. Exhibit monoidal structures the following categories: sets, vector spaces, linear representations of a group G, topological spaces, associative algebras, Lie algebras, and others.

Hint. In the case of Lie algebras, consider the category of Lie groups with its canonical tensor product (the cartesian product) and the functor $G \longmapsto T_e(G)$ to decide what the tensor product of two Lie algebras is.

Exercise 6. If $(\mathscr{V}, \otimes)$ is a monoidal category, we say $\mathscr{C}$ is a $\mathscr{V}$-enriched category if each hom-set $\mathscr{C}(x, y)$ is an object of $\mathscr{V}$ and there is a composition law

$$- \circ - : \mathscr{C}(y, z) \otimes \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z).$$

which consists of morphisms in $\mathscr{V}$, and which is associative and unital. Note that an ordinary category is just a category enriched over the category of sets. A linear category is a category enriched over the category of vector spaces, an additive category is a category enriched over Abelian groups. Expand on what this means. Find about Abelian categories, and ponder over the difference: an additive category is a category with structure, while an Abelian category is a category with additional properties.

Exercise 7. A category $\mathscr{D}$ is skeletal if no two distinct objects in it are isomorphic. We say that $\mathscr{D}$ is the skeleton of $\mathscr{C}$ if:

1. It is a full subcategory of $\mathscr{C}$: for each pair of objects $x, y \in \mathscr{D}$, we have that $\mathscr{D}(x, y) = \mathscr{C}(x, y)$.
2. The inclusion of $\mathscr{D}$ in $\mathscr{C}$ is essentially surjective: every object of $\mathscr{C}$ is isomorphic to an object of $\mathscr{D}$.
3. $\mathscr{D}$ is skeletal.

Show that every small category admits a skeleton, and compute the skeleton of the following categories: sets, finite sets, finite dimensional vector spaces over a field.

Exercise 8. Suppose that x is an object in a symmetric monoidal category $(\mathscr{C}, \tau)$. For each $n \in \mathbb{N}$ and each $i \in [n-1]$ define $\tau_i : x^{\otimes n} \longrightarrow x^{\otimes n}$ by

$$\tau_i = 1^{i-1} \otimes \tau \otimes 1^{n-i-1}.$$

Show that the assignment $(i, i+1) \in S_n \longmapsto \tau_i \in \operatorname{Aut}(x^{\otimes n})$ is a group homomorphism. *Note.* This produces in particular a map $S_2 \longrightarrow \operatorname{Aut}(x^{\otimes 2})$ that sends the transposition $(12) \in S_2$ to the flip $\tau_{x,x} : x \otimes x \longrightarrow x \otimes x$.

Exercise 9. A product and permutation category (abbreviated 'PROP') is a monoidal category $\mathscr{C}$ whose set of objects is $\mathbb{N} = \{0, 1, 2, \ldots\}$ and its tensor product is addition (in particular, it is strict and symmetric). Unravel the definitions:

1. Use that $n = 1 + \cdots + 1$ to show that $\mathscr{C}(m, n)$ is a right S_n-module.
2. Similarly, show that $\mathscr{C}(m, n)$ is also a left S_m-module.
3. Show these two actions are compatible (i.e. they commute).
4. Show that the product $+$ induces a *horizontal* composition rule
$$\mathscr{C}(m, n) \times \mathscr{C}(m', n') \longrightarrow \mathscr{C}(m + m', n + n').$$
5. Interpret the usual categorical product as a *vertical* composition rule
$$\mathscr{C}(n, k) \times \mathscr{C}(m, n) \longrightarrow \mathscr{C}(m, k).$$

Consider the definition of a PROP enriched over a symmetric strict monoidal category, like Vect (these are called $\mathbb{k}$-linear PROPs). Define the category of PROPs.

Note. For each $n \in \mathrm{Ob}(\mathscr{C})$ the object $\mathscr{C}(n, n)$ is a monoid under composition that receives a map $S_n \longrightarrow \mathscr{C}(n, n)$. Under the interpretation above, the image of σ is equal to both the left and the right action of S_n on the identity map $n \to n$. In particular, the twist τ of $\mathscr{C}$ is equal to $(12)\mathrm{id}_2$, and may (or may not) be trivial.

C. Graded spaces and complexes. When studying algebraic structures like operads, it will be necessary to use some tools from homological algebra: graded spaces, chain complexes, differentials, their homology, among others. The following exercises are intended to familiarize you with the elements of homological algebra, but we will look at them in more detail during the course.

Exercise 10. A $\mathbb{Z}$-graded vector space (usually just called a graded vector space) is a vector space V with a direct sum decomposition
$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

If $v \in V_n$ we say that v is homogeneous of degree n. Find out about the category of graded vector spaces, specifically:

- What are its (degree zero) morphisms?
- What are its (homogeneous) morphisms?
- What is the tensor product of two graded spaces?
- What is the natural isomorphism $V \otimes W \longrightarrow W \otimes V$?
- How does the last item relate to the 'Koszul sign rule'?

Exercise 11. A differential graded (dg) vector space, usually called a complex, is a pair (V,d) where V is a graded vector space V and $d : V \to V$ is a homogeneous map of degree -1 such that $d^2 = 0$. Repeat the previous exercise replacing gVect with Ch, the category of complexes of vector spaces.

Exercise 12. If (V,d) is a complex, then $Z(V) = \ker d$ is called its space of cycles, and $B(V) = \operatorname{im} d$ is called its space of boundaries. The quotient $Z(V)/B(V)$ is called the homology of V, and is written $H(V)$. Show that a map of complexes $f : V \to W$ induces a map $Z(V) \to Z(W)$ and in turn a map $H(f) : H(V) \to H(W)$.

Exercise 13. (Leisure) Find a book on homological algebra and read about the *snake lemma* and the *five lemma*. If you are very motivated, read about double complexes and spectral sequences.