Exercises (Lecture 2)

Free operads

Introduction to Operads

Note. We will solve some of these exercises during the exercise sessions. Try to solve at least one exercise you find easy and at east two exercises that you find challenging.

Exercise 1. Let $\mathscr{X}$ be a collection such that $\underline{\mathscr{X}} = \mathscr{X}(2)$. Compute a basis of tree monomials for the free operad over $\mathscr{X}$ in each of the following cases:

1. $\mathscr{X}(2) = \mathbb{C}^2$ is the regular representation of S_2.
2. $\mathscr{X}(2) = \mathbb{C}^-$ is the sign representation of S_2.
3. $\mathscr{X}(2) = \mathbb{C}$ is the trivial representation of S_2.

In all cases, decompose the S_3-module $\mathscr{F}_{\mathscr{X}}(3)$ into irreducible representations.

Exercise 2. In the trivial and sign case above, compute a monomial tree basis, and thus the dimension, of $\mathscr{F}_{\mathscr{X}}(4)$.

Exercise 3. Suppose that $\mathscr{X}$ is an alphabet (in sets) that is finite in each arity and such that $\mathscr{X}(n) = \varnothing$ for $n = 0, 1$. Show that $\mathscr{F}_{\mathscr{X}}$ is finite in each arity.

Exercise 4. Define non-symmetric tree monomials over a ns alphabet $\mathscr{X}$ and thus define the free *non-symmetric* operad over a collection $\mathscr{X}$.

Exercise 5. Read the statement and proof of *Theorem 5.4.2* in 'Algebraic Operads' that the colimit construction briefly described in the lecture notes does give the free operad on a symmetric collection.

Exercise 6. Consider the map from ns collections to symmetric sequences that assigns $\mathscr{X}$ to $\Sigma \otimes \mathscr{X}$ such that $(\Sigma \times \mathscr{X})(n) = \Sigma_n \times \mathscr{X}(n)$ with its corresponding symmetric group action. What is the relation between the free ns operad on $\mathscr{X}$ and the free symmetric operad on $\Sigma \times \mathscr{X}$?

Exercise 7. Let V be an S_2-module, and let $\mathscr{X}$ be the symmetric collection with $\mathscr{X}(2) = V$ and zero everywhere else. Show that $\mathscr{F}_{\mathscr{X}}(3)$ consists of three copies of $V^{\otimes 2}$ and describe explicitly the action of S_3 on it.

Exercise 8. Show that the construction of the free operad we carried out during **Lecture 2** indeed defines the free operad on $\mathscr{X}$ where $i : \mathscr{X} \longrightarrow \mathscr{F}_{\mathscr{X}}$ sends an element $x \in \mathscr{X}(I)$ to the corolla whose unique internal vertex is labeled by x (and whose leaves are labeled by I).

Exercise 9. Follow the lecture notes and read about weight gradings and the canonical weight grading on a free operad.