Algebraic operads

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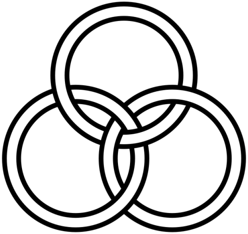
Where do operads appear naturally?

One of the many uses of operads is encoding (perhaps seemingly hidden) "higher structures". Consider the complement X of the Borromean rings in three space:



There are three classes $x_1, x_2, x_3 \in H^*(X)$ corresponding to each of the three rings above, and all the products $x_i x_j$ are zero: pairwise, the rings are not linked.

The Borromean links, however, are linked! Then how can we see this in the ring $H^*(X)$? The answer, due to W. Massey [Mas98], is that there is a ternary non-vanishing operation $\langle x_1, x_2, x_3\rangle$ witnessing this "higher" linking.

This triple Massey product belongs to a hierarchy of "higher" Massey products giving $H^*(X)$ an algebraic structure that refines the usual cup product [Sta70].

In its different incarnations, Massey products have been used by several authors to "detect" these more subtle information contained in $H^*(X)$, see [LS05, Liv15] for examples. We will see that **operads** both explain and organize this kind of computations [BM03].

The work of Jim Stasheff

Let M be a connected topological monoid and suppose that M' is homotopy equivalent to M. What kind of algebraic structure does M' admit?

Since M and ΩBM are homotopy equivalent, we can instead ask: what are the connected topological spaces that are homotopy equivalent to a loop space?

The answer to this question was given by Stasheff and his result can be stated elegantly in terms of operads:

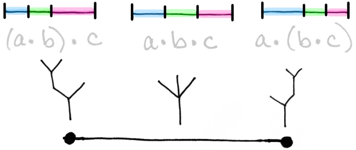
Theorem (Stasheff '63)

A connected space X is of the homotopy type of a loop space if and only if it is an algebra over a certain topological operad $\mathcal{K}$.

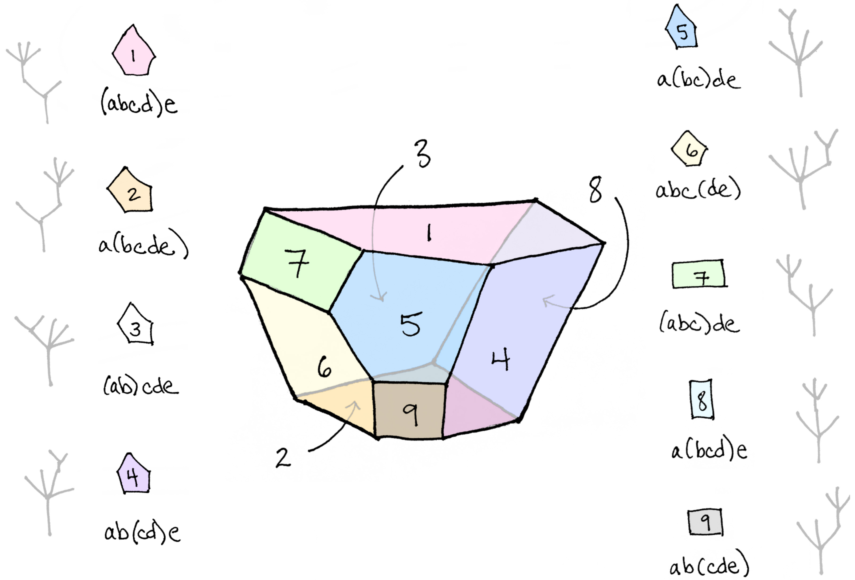
The operad $\mathcal{K}$ is constructed from a sequence of convex polyhedra

$$K_2, K_3, K_4, K_5, \ldots$$

that encode a continuum of operations on X. For example, K_3 encodes the two products $x_1(x_2x_3)$ and $(x_1x_2)x_3$ and the interval of operations that witness the associativity of x_1x_2 up to homotopy, as in the following picture from the blog "Math3ma" of Tai-Danae Bradley:



Here's a lovely picture of K_5, also found in "Math3ma":



The geometry of iterated loop spaces

The ideas of Stasheff were followed by the work of Beck [Bec69], Boardman–Vogt [BV68], Milgram [Mil66] and May [May72]. Although their results were stated in different languages, we can surmise them, as May did, in the language of operads.

Theorem (The recognition principle)

A connected space X is of the homotopy type of a k-loop space if and only if it is an algebra over a certain topological operad $\mathcal{C}_k$.

Why is this result useful? It is usually easier to give a space additional structure than to determine its homotopy type!

The work of D. Quillen and D. Sullivan

Around the same time, Dennis Sullivan [Sul77] and Daniel Quillen [Qui69] developed algebraic tools to perform computations and describe, through simply commutative algebras and connected Lie algebras, the rational homotopy type of simply connected topological spaces.

Theorem (Quillen '69)

There is a very nice adjunction $\mathrm{dgCom}^{\geqslant 1} \Longleftrightarrow \mathrm{dgLie}^{\geqslant 0}$ *between simply connected commutative algebras and connected Lie algebras.*

This is no longer a geometrical statement, so what is the algebraic explanation?

The renaissance of operads

The decade of the 90s was “renaissance era” for the theory of operads, mostly fueled by the development of deformation theory, formality theorems, and quantum field theory. The answer to our last question is the following:

Theorem (Ginzburg–Kapranov ’94)

The operads controlling Lie and commutative algebras are Koszul dual to each other.

Along with the seminal paper of Ginzburg–Kapranov [GK94] came the work of E. Getzler, V. Hinich, J. Jones, M. Kontsevich, J. E. McClure, Y. I. Manin, M. Markl, V. Schechtman, V. Smirnov, J. H. Smith, and D. Tamarkin, among others.

What should an operad look like?

Abstractions generally look like something concrete they model:

1. Groups look like automorphisms of sets or spaces, and a representation of an abstract group is precisely a morphism $G \longrightarrow \mathrm{Aut}(X)$ or $G \longrightarrow \mathrm{GL}(V)$.
2. Algebras look like endomorphisms of vector spaces, and a representation of an abstract algebra is precisely a morphism $A \longrightarrow \mathrm{End}(V)$.

In this sense, groups control symmetries of objects, while algebras control linear operations on vector spaces.

Philosophy: operads (whatever they may be!) control operations with multiple arguments on objects (sets, topological spaces, complexes, vector spaces, say)... so what should an operad look like?

Let X be an object we are interested in, like a set, a vector space, a complex, or a topological space, and let X^j denote the product of X with itself (cartesian or tensor product, depending on our interest!)

Definition

We define a sequence of objects $\mathrm{End}_X = (\mathrm{End}_X(0), \mathrm{End}_X(1), \ldots)$ so that for each $j \in \mathbb{N}$, we have $\mathrm{End}_X(j) = \{$ functions $f : X^j \longrightarrow X$ $\}$. This is called the *endomorphism operad of X.*

We can compose operations: for example $f(x_1, g(x_2, x_3), x_4)$ is an operation obtained by inserting in the second argument of $f : X^3 \longrightarrow X$ an operation $g : X^2 \longrightarrow X$.

The object End_X captures all "reasonable" multi-valued operations on X. Hence, an operad is something that should look like End_X.

Some groupoids

1. Write Fin for the category of finite sets with morphisms the bijections.
2. Write Ord for the category of *ordered* finite sets with morphisms the *order preserving* bijections.

Ord is equivalent to the category with objects $0, 1, 2, 3, \ldots$ and trivial hom-sets.

Fin is equivalent to the category with the same objects and with the only non-trivial hom-sets given by $\hom(n, n) = S_n$.

Ns and symmetric sequences

From now on let us fix a concrete category C, like that of sets, vector spaces, or topological spaces.

Definition

A non-symmetric (ns) sequence in C is a functor $\mathcal{X} : \mathsf{Ord} \longrightarrow \mathsf{C}$. A symmetric sequence is a functor $\mathcal{X} : \mathsf{Fin} \longrightarrow \mathsf{C}$.

In both cases, we can specify such a sequence by the list of its values on the skeleton:

$$\mathcal{X} = (\mathcal{X}(0), \mathcal{X}(1), \mathcal{X}(2), \mathcal{X}(3), \ldots).$$

If $\mathcal{X}$ is symmetric, we also need to give for each $n \in \mathbb{N}$ an action of S_n on the set $\mathcal{X}(n)$.

Examples

Let us consider a few examples: for each $n \in \mathbb{N}$,

1. (sym) let $\mathrm{Com}(n)$ be the one dimensional vector space $\mathbb{k}$.
2. (sym) let $\mathrm{Ass}(n)$ be the group algebra $\mathbb{k}S_n$.
3. (sym) let $\mathrm{RT}(n)$ be the vector space of rooted trees with vertices labeled by $[n]$.
4. (ns) let $\mathcal{K}(n)$ be the collection of points $(t_1, \ldots, t_n)$ in I^n such that for each $j = 1, \ldots n$ we have $t_1 \cdots t_j \leqslant 2^{-j}$.
5. (sym) let $\mathbb{W}_{S^1}(n)$ be the product $(S^1)^n$ of the circle with itself.

Sequences as "analytic" endofunctors

Every ns sequence gives rise to an endofunctor on vector spaces given by the prescription:

$$V \longmapsto \bigoplus_{n \geqslant 0} \mathcal{X}(n) \otimes V^{\otimes n}$$

where we would take invariants over S_n in the symmetric case.

As opposed to an arbitrary endofunctor, these are "analytic": the sequence $\mathcal{X}$ is telling us the series expansion of this endofunctor.

In the same way we prefer working with analytic functions rather than arbitrary functions, we prefer working with analytic endofunctors rather than arbitrary endofunctors.

Some products

Just like a pair of formal power series f and g can be multiplied and composed[1] to yield new formal power series

$$f \cdot g, \quad f \circ g$$

two (ns) sequences $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{Y}(0) = \varnothing$ can be multiplied and composed to give new (ns) sequences

$$\mathcal{X} \otimes \mathcal{Y}, \quad \mathcal{X} \circ \mathcal{Y}.$$

Definition

The product of $\mathcal{X}$ with $\mathcal{Y}$ is given by

$$(\mathcal{X} \otimes \mathcal{Y})(I) = \bigoplus_{S \sqcup T = I} \mathcal{X}(S) \otimes \mathcal{Y}(T).$$

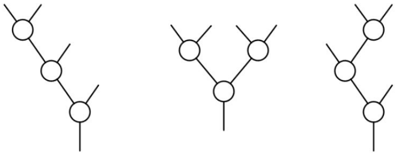
[1]...when $g(0) = 0$. Why do we need $g(0) = 0$?

Definition

The composition of two ns sequences is the ns sequence

$$(\mathcal{X} \circ_{ns} \mathcal{Y})(n) = \bigoplus_{\lambda \vdash n} \mathcal{X}(r) \otimes \mathcal{Y}(\lambda_1) \otimes \cdots \mathcal{Y}(\lambda_r)$$

as λ runs through partitions of n.

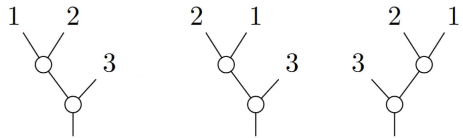


Definition

The symmetric composition of two symmetric sequences is the sequence

$$(\mathcal{X} \circ_\Sigma \mathcal{Y})(I) = \bigoplus_{\lambda \vdash n} \mathcal{X}(r) \otimes_{S_r} \mathcal{Y}(\lambda_1) \otimes \cdots \mathcal{Y}(\lambda_r) \uparrow_{S_\lambda}^{S_r}$$

as λ runs through partitions of n.



Ns and symmetric operads

Definition (Compact form)

A ns operad is a ns collection $\mathcal{P}$ equipped with an associative and unital composition law $\gamma : \mathcal{P} \circ_{ns} \mathcal{P} \longrightarrow \mathcal{P}$.

To obtain the definition of a symmetric operad, we merely need to replace the ns composition with the symmetric composition product.

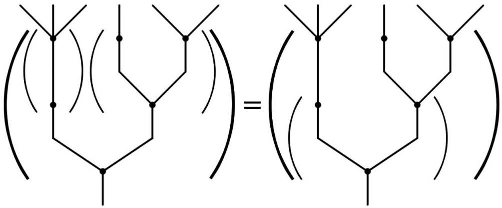
The only caveat is that γ now includes an equivariance axiom with respect to symmetric group actions.

Definition (Explicit form)

A ns operad is a ns collection $\mathcal{P}$ equipped with composition maps

$$\gamma : \mathcal{P}(r) \otimes \mathcal{P}(\lambda_1) \otimes \cdots \mathcal{P}(\lambda_r) \longrightarrow \mathcal{P}(\lambda_1 + \cdots + \lambda_r)$$

which we usually write $\mu(\nu_1, \ldots, \nu_r)$ satisfying the following "generalized associativity" condition:



along with a unit $1 \in \mathcal{P}(1)$ satisfying $\mu(1, \ldots, 1) = 1(\mu) = \mu$.

Definition (Partial compositions)

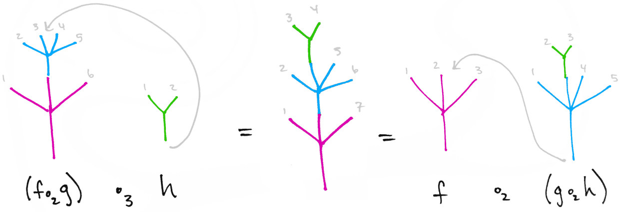
A ns operad is a ns collection $\mathcal{P}$ equipped with partial composition maps

$$\circ_i : \mathcal{P}(r) \otimes \mathcal{P}(s) \longrightarrow \mathcal{P}(s+r-1)$$

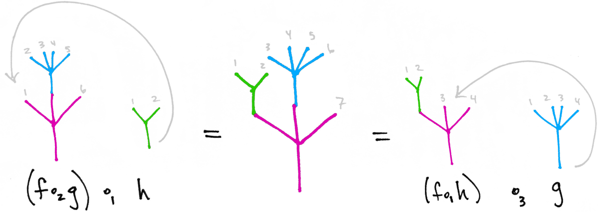
satisfying the following "generalized associativity" conditions[2] known as the sequential and parallel composition axioms, along with a unit $1 \in \mathcal{P}(1)$ satisfying $\mu \circ_i 1 = 1 \circ_1 \mu = \mu$ for any i.

(Since every tree can be decomposed, inserting units when needed, using these two type of compositions, these two axioms are enough to recover the "generalized associativity" of the previous definition.)

[2] Credits for the images go to Tai-Danae Bradley



(Sequential composition axiom)



(Parallel composition axiom)

The three graces of Loday

(Commutative operad) Looking at our examples, let

$$\circ_i : \mathsf{Com}(n) \otimes \mathsf{Com}(m) \longrightarrow \mathsf{Com}(n+m-1)$$

be the unique equivariant map for which $1_n \otimes 1_m \longmapsto 1_{n+m-1}$. Since all the spaces of operations are one dimensional, there is not much of a choice when defining the composition!

(Associative operad) Similarly, let

$$\circ_i : \mathsf{Ass}(n) \otimes \mathsf{Ass}(m) \longrightarrow \mathsf{Ass}(n+m-1)$$

be the unique equivarant map for which $\mathrm{id}_n \otimes \mathrm{id}_m \longmapsto \mathrm{id}_{n+m-1}$. Since the space of n-ary operations is the regular representation, the composition map is determined uniquely by this requirement.

If we think of a permutation as a non-commutative monomial $x_\sigma = x_{\sigma(1)} \cdots x_{\sigma(n)}$, then the composition

$$x_\sigma \circ_i y_\tau$$

corresponds to the monomial where the variable y_i has been replaced by the monomial x_σ, with an appropriate relabeling.

For example, we have that $x_2x_1x_3 \circ_2 x_2x_1 = \underline{x_3x_2}x_1x_4$.

(The fact that every permutation can be written as a product of transpositions shows that *every operation* in Ass can be obtained by partial compositions of the two operations x_1x_2 and x_2x_1.)

(The Lie operad) Finally, let us consider the operation

$$[x_1, x_2] = x_1 x_2 - x_2 x_1$$

in Ass, and consider all possible compositions obtained from it. This produces a subsequence Lie that is, by construction, a suboperad of Ass.

These three operads, baptized as "the three graces" by J.-L. Loday, play central roles in the work of Adams–Hilton, Quillen and Sullivan in rational homotopy theory, and in the classical Poincaré–Birkoff–Witt and Hochschild–Kostant–Rosenberg theorems, for example.

Sanity check!

The analytic endofunctors that we obtain from Ass, Com and Lie are the functors

1. $V \longmapsto T(V)$ defining the (free) tensor algebra on V
2. $V \longmapsto S(V)$ defining the (free) symmetric algebra on V
3. $V \longmapsto \mathbb{L}(V)$ defining the (free) Lie algebra on V

So when we view an operad as an endofunctor, they define their "free algebras" automatically.

Operads coming from monoids

Let M be a monoid, and for each n set $\mathbb{W}_M(n) = M^n$, the collection of words in M of length n, we can define partial composition for $x \in M^n$ and $y \in M^m$ as follows:

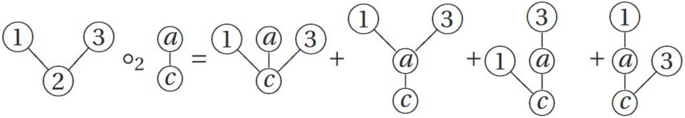
$$x \circ_i y = (x_1, \ldots, x_{i-1}, x_i y_1, \ldots, x_i y_n, x_{i+1}, \ldots, x_n).$$

We call this the word operad associated to the monoids M. These have been considered by V. Dotsenko in [Dot20], we will explain their use in the second part of this talk.

The operad of rooted trees

(PreLie operad) As before, let $\mathrm{RT}(n)$ be the vector space spanned by rooted trees with vertices labelled by n. For two trees T and S, we define $T \circ_i S$ as follows.

Delete the ith vertex of T and replace it by S. Then, reattach the incoming edges of T back to the vertices of S in "all possible ways". For example:



It is a bit less obvious that this gives an associative composition law. See [CL01] and [DFC20] for more.

Types of algebras

If $\mathcal{P}$ is an operad and V is a vector space, we know what to do to think of abstract operations on V coming from $\mathcal{P}$:

Definition

A $\mathcal{P}$-algebra structure on V is the datum of a map of operads

$$\gamma_V : \mathcal{P} \longrightarrow \mathsf{End}_V .$$

Unravelling the definitions, we see that for every operation μ in $\mathcal{P}$ of arity n, we obtain a concrete operation $\mu : V^{\otimes n} \longrightarrow V$ on V.

The requirement that this is a map of operads states that the concrete operations satisfy the abstract relations the operations of $\mathcal{P}$ satisfy.

The usual algebras

One can check that, with this formalism, we recover the usual notions of

1. associative algebras for the operad Ass,
2. commutative algebras for the operad Com,
3. Lie algebras for the operad Lie.

Moreover, the careful study of these operads through this new lens allows us to explain swiftly and coherently a plethora of existing results and constructions for these algebras!

New types of algebras

Let us consider the operad RT of rooted trees. It is not evident, but this operad is generated by a single operation x_1x_2 corresponding to the tree $T_{12} \in \mathrm{RT}$ with its root at 2.

This operation satisfies an interesting relation: the associator

$$A = x_1(x_2x_3) - (x_1x_2)x_3$$

is nonzero, but symmetric in x_2 and x_3.

Definition

An algebra over the operad RT is called a pre-Lie algebra.

These algebras were considered by Gerstenhaber in his seminal work on Hochschild cohomology of associative algebras, and also by Kupershmidt, Vinberg and Koszul. See these notes by Chengming Bai for more details.

So what now?

Following the philosophy of V. Hinich [Hin97], we can now "go up on level" in the study of algebras.

Working with operads, we can prove universal statements and produce universal constructions on their algebras.

For example, operads that are "Koszul" produce a Koszul duality theory for their category of algebras.

Studying morphisms of operads produces statements about functors obtained from this morphism (for example, about universal enveloping algebras of some type, see [DT20, KT20, Kho18, DF20, DFC20].

So what now?

Just like in the case of algebras, one can study operads from various points of view and with different toolsets.

In particular, we can use homological and homotopical methods, coupled with more concrete and algorithmic rewriting methods to study operads.

This, in turn, can be "dropped one level down" to obtain very powerful results about types of algebras. We will discuss this in the next talk!

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