AE 370: HW3

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Problem 1

1. Let $u \in \mathcal{V}$ be written as $u = f_a + \beta_r$ for some $\beta \in \mathbb{R}$ and $r \in \mathcal{V}$. Since $||f - f_a|| \leq ||f - u||$, the inequality becomes $||f - f_a|| \leq ||f - (f_a + \beta_r)||$. Replacing norms with the inner product:

$$\begin{aligned}(f - f_a, f - f_a) &\leq (f - (f_a + \beta_r), f - (f_a + \beta_r)) &(1)\\ &= ((f - f_a) - \beta_r, (f - f_a) - \beta_r) &(2)\\ &= (f - f_a, f - f_a) - 2\bar{\beta}(f - f_a, r) + \beta^2(r, r) &(3)\end{aligned}$$

Subtracting $(f - f_a, f - f_a)$ from both sides:

$$-2\bar{\beta}(f - f_a, r) + \beta^2(r, r) \geq 0 \tag{4}$$

Since the second term always satisfies the inequality, for the inequality to hold for any u, $(f - f_a, r) = 0$.

Problem 2

1. Evaluating $(\hat{\phi}_i, \hat{\phi}_i)$ for $i = 1, \ldots, n + 1$:

$$\begin{aligned}(\phi_i, \phi_i) &= \left(\frac{\hat{\phi}_i}{\sqrt{(\hat{\phi}_i, \hat{\phi}_i)}}, \frac{\hat{\phi}_i}{\sqrt{(\hat{\phi}_i, \hat{\phi}_i)}}\right) &(5)\\ &= \frac{1}{(\hat{\phi}_i, \hat{\phi}_i)}(\hat{\phi}_i, \hat{\phi}_i) &(6)\\ &= 1 &(7)\end{aligned}$$

2. To show $\phi_1 \perp \phi_2$, we must prove $(\phi_1, \phi_2) = 0$. First, we will demonstrate that proving $(\phi_1, \phi_2) = 0$ is equivalent to proving $(\hat{\phi}_1, \hat{\phi}_2) = 0$:

$$(\phi_1, \phi_2) = \left(\frac{\hat{\phi}_1}{\sqrt{(\hat{\phi}_1, \hat{\phi}_1)}}, \frac{\hat{\phi}_2}{\sqrt{(\hat{\phi}_2, \hat{\phi}_2)}}\right) = \alpha\beta(\hat{\phi}_1, \hat{\phi}_2) \tag{8}$$

where

$$\alpha = \frac{1}{\sqrt{(\hat{\phi_1}, \hat{\phi_1})}}, \beta = \frac{1}{\sqrt{(\hat{\phi_2}, \hat{\phi_2})}} \tag{9}$$

Next, we will prove $(\hat{\phi_1}, \hat{\phi_2}) = 0$:

$$\begin{aligned}
(\hat{\phi_1}, \hat{\phi_2}) &= \left(\frac{1}{(1,1)}, x - \frac{(x,1)}{(1,1)}\right) && (10)\\
&= \left(\frac{1}{(1,1)}, x\right) - \left(\frac{1}{(1,1)}, \frac{(x,1)}{(1,1)}\right) && (11)\\
&= \frac{1}{(1,1)}(1,x) - \frac{(x,1)}{(1,1)} && (12)\\
&= \frac{1}{(1,1)}(1,x) - \frac{1}{(1,1)}(1,x) && (13)\\
&= 0 && (14)
\end{aligned}$$

3. Let

$$\alpha = \frac{1}{\sqrt{(\hat{\phi_1}, \hat{\phi_1})}},\ \beta = \frac{1}{\sqrt{(\hat{\phi_2}, \hat{\phi_2})}},\ \gamma = \frac{1}{\sqrt{(\hat{\phi_3}, \hat{\phi_3})}} \tag{15}$$

   (a) Proof for $\phi_3 \perp \phi_1$:

$$\begin{aligned}
(\phi_3, \phi_1) &= \alpha\gamma(\hat{\phi_1}, \hat{\phi_3}) && (16)\\
&= (\hat{\phi_1}, x\phi_2) - (\hat{\phi_1}, \phi_2)(x\phi_2, \phi_2) - (\hat{\phi_1}, \phi_1)(x\phi_2, \phi_1) && (17)\\
&= (\hat{\phi_1}, x\phi_2) - \beta(\hat{\phi_1}, \hat{\phi_2})(x\phi_2, \phi_2) - \alpha^2(\hat{\phi_1}, \hat{\phi_1})(\hat{\phi_1}, x\phi_2) && (18)
\end{aligned}$$

   Since $\alpha^2 = \frac{1}{(\hat{\phi_1}, \hat{\phi_1})}$, $(\hat{\phi_1}, \hat{\phi_2}) = 0$:

$$\begin{aligned}
(\phi_3, \phi_1) &= (\hat{\phi_1}, x\phi_2) - \beta(\hat{\phi_1}, \hat{\phi_2})(x\phi_2, \phi_2) - \alpha^2(\hat{\phi_1}, \hat{\phi_1})(\hat{\phi_1}, x\phi_2) && (19)\\
&= (\hat{\phi_1}, x\phi_2) - \frac{1}{(\hat{\phi_1}, \hat{\phi_1})}(\hat{\phi_1}, \hat{\phi_1})(\hat{\phi_1}, x\phi_2) && (20)\\
&= (\hat{\phi_1}, x\phi_2) - (\hat{\phi_1}, x\phi_2) && (21)\\
&= 0 && (22)
\end{aligned}$$

   (b) Proof for $\phi_3 \perp \phi_2$:

$$\begin{aligned}
(\phi_3, \phi_2) &= (\hat{\phi_2}, x\phi_2) - (\hat{\phi_2}, \phi_2)(x\phi_2, \phi_2) - (\hat{\phi_2}, \phi_1)(x\phi_2, \phi_1) && (23)\\
&= (\hat{\phi_2}, x\phi_2) - \beta^2(\hat{\phi_2}, \hat{\phi_2})(x\phi_2, \hat{\phi_2}) - \alpha(\hat{\phi_2}, \hat{\phi_1})(\hat{\phi_2}, x\phi_2) && (24)
\end{aligned}$$

Since $\beta^2 = \dfrac{1}{(\hat{\phi_2}, \hat{\phi_2})}$, $(\hat{\phi_1}, \hat{\phi_2}) = 0$:

$$
\begin{aligned}
(\phi_3, \phi_2) &= (\hat{\phi_2}, x\phi_2) - \beta^2(\hat{\phi_2}, \hat{\phi_2})(x\phi_2, \hat{\phi_2}) - \alpha(\hat{\phi_2}, \hat{\phi_1})(\hat{\phi_2}, x\phi_2) && (25)\\
&= (\hat{\phi_2}, x\phi_2) - \frac{1}{(\hat{\phi_2}, \hat{\phi_2})}(\hat{\phi_2}, \hat{\phi_2})(x\phi_2, \hat{\phi_2}) && (26)\\
&= (\hat{\phi_2}, x\phi_2) - (\hat{\phi_2}, x\phi_2) && (27)\\
&= 0 && (28)
\end{aligned}
$$

Problem 3

1. Given the function $f(x)$ and basis $\{\phi_1, \ldots, \phi_{n+1}\} \in \mathbb{V}$, we seek $f_a(x)$ such that $||f_a - f||^2$ is minimized for $f_a \in \mathbb{V}$. Thus $f_a(x)$ must satisfy $(f - f_a, r) = 0 \; \forall r \in \mathbb{V}$.

   Therefore:

$$
(f - \sum_{i=1}^{n+1} c_i\phi_i, r) = 0 \tag{29}
$$

   However, since $\phi \in \mathbb{V}, r \in \mathbb{V}$, we can rewrite the above equation as:

$$
(f - \sum_{i=1}^{n+1} c_i\phi_i, \phi_j) = 0, \; j = 1, \ldots, n+1 \tag{30}
$$

   Ergo,

$$
\begin{aligned}
(f - \sum_{i=1}^{n+1} c_i\phi_i, \phi_j) &= (f, \phi_j) - (\sum_{i=1}^{n+1} c_i\phi_i, \phi_j) && (31)\\
&= (f, \phi_j) - \sum_{i=1}^{n+1} c_i(\phi_i, \phi_j) = 0 && (32)
\end{aligned}
$$

   Finally, we establish the linear equation:

$$
\sum_{i=1}^{n+1} c_i(\phi_i, \phi_j) = (f, \phi_j) \tag{33}
$$

   Which results in the linear system

$$
\begin{bmatrix} (\phi_1, \phi_1) & \cdots & (\phi_{n+1}, \phi_1) \\ \vdots & \ddots & \vdots \\ (\phi_1, \phi_{n+1}) & \cdots & (\phi_{n+1}, \phi_{n+1}) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} (f, \phi_1) \\ \vdots \\ (f, \phi_{n+1}) \end{bmatrix} \tag{34}
$$

2. From the Gram-Schmidt process, we know that:

$$(\phi_j, \phi_k) = \begin{cases} 1 & j = k \\ 0 & \text{else} \end{cases} \tag{35}$$

   Hence only elements on the diagonal of G are equal to 1 while the rest is 0. Thus, $G = I$.

3. Figures and code are attached in Appendices A and B, respectively.

4. The solution does not suffer the same issues as global polynomial interpolants do on uniformly space points. This is because function approximation does not constrain f_a to match the value of interpolation points exactly but instead minimizes the global error. This inherently mitigates the issue of rapidly oscillating functions with global polynomial interpolation.

Appendix A: Figures

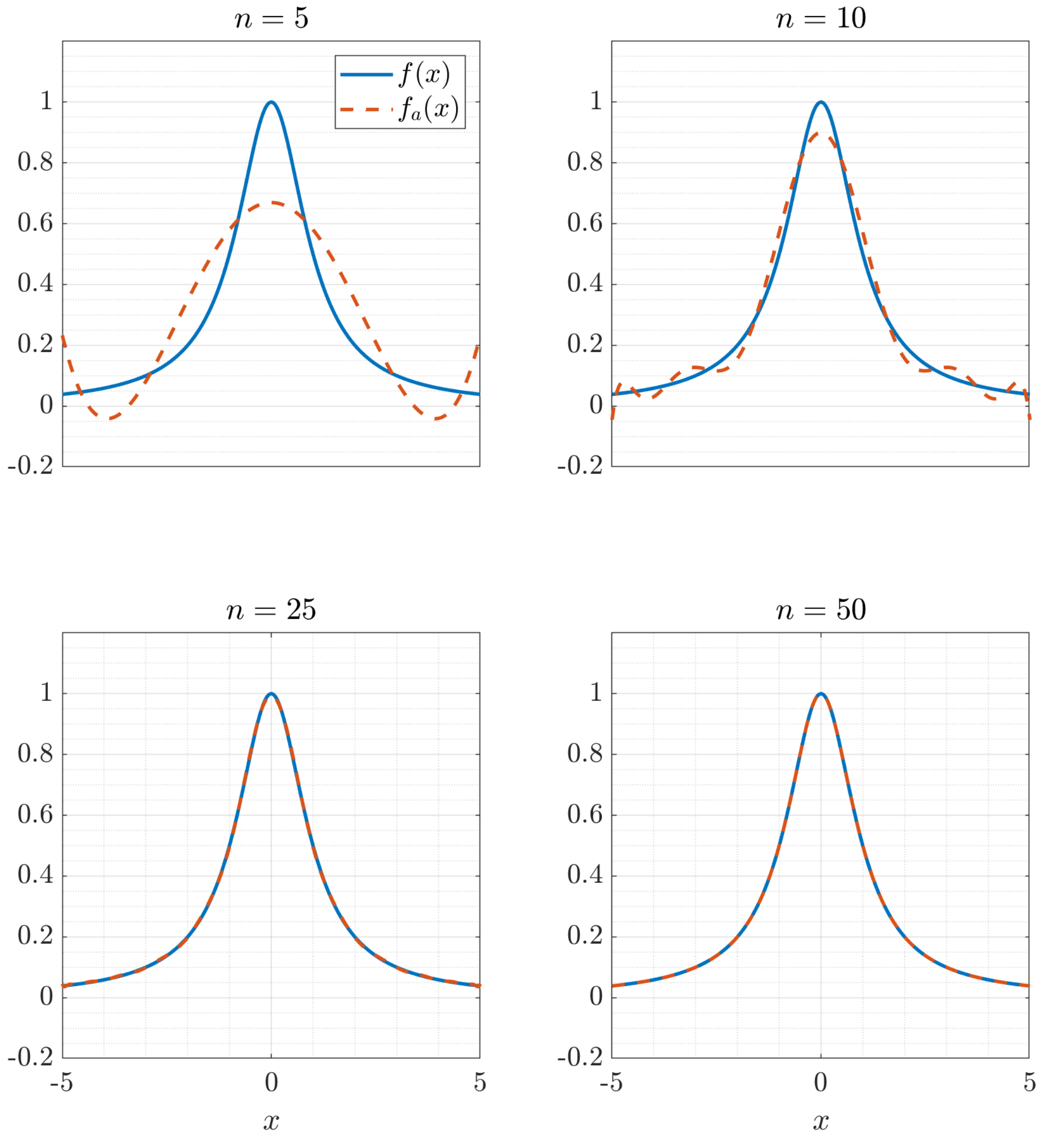

Figure 1: Function Approximation for Problem 3.c

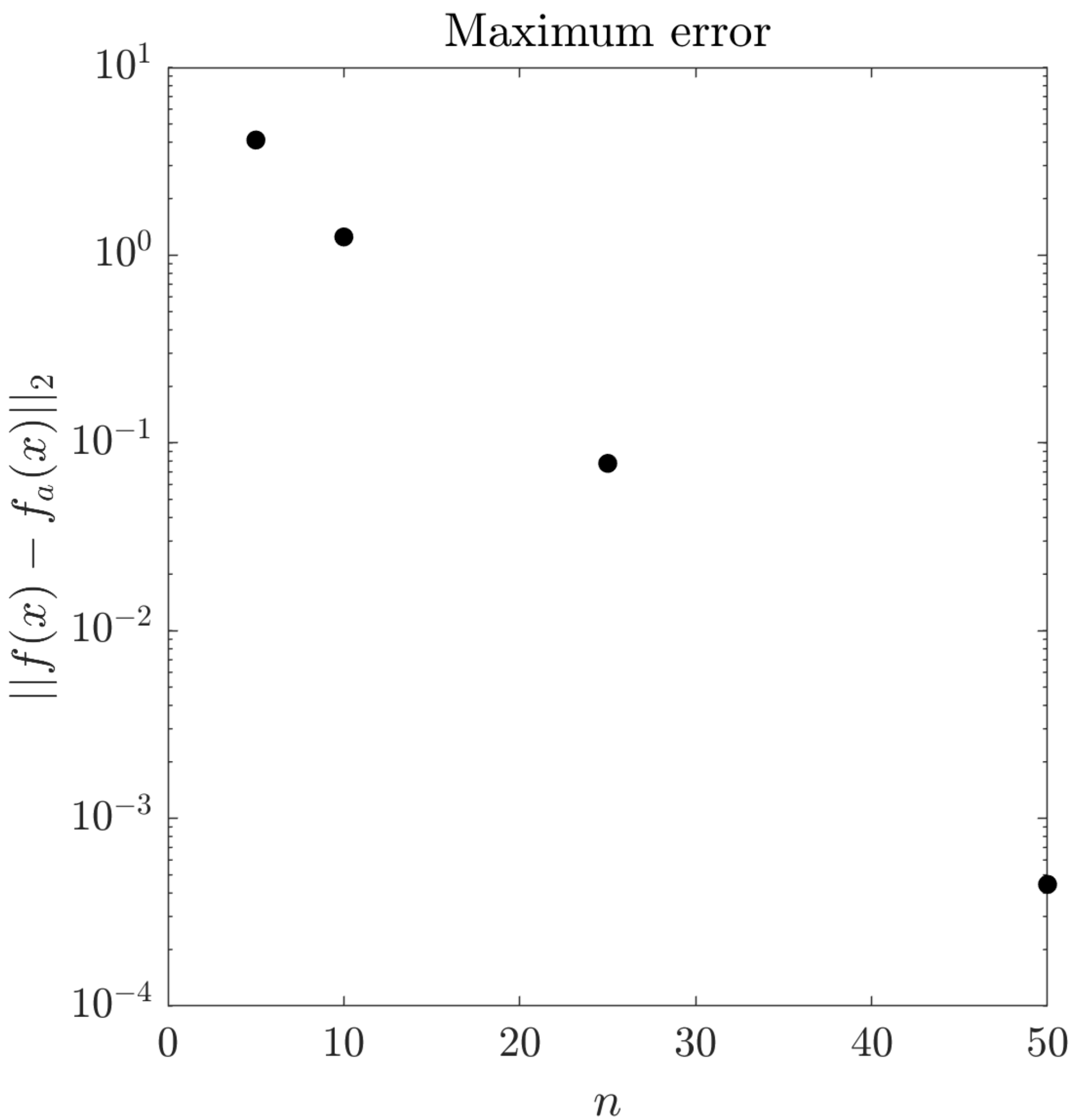

Figure 2: 2-Norm Error for Problem 3.c

Appendix B: Code

```
%% AE 370 HW3
% Linyi Hou (linyih2)

close all hidden
clear;clc

nvect = [5,10,25,50];
% nvect = 3;

fcn = @(x) 1./(1+x.^2);

err = zeros(length(nvect),1);

for j = 1 : length(nvect)

    n = nvect(j);

    xx = linspace(-5,5,1000)';
    grid_ones = ones(size(xx));

    b = zeros(n+1,1);

    % DEFINITIONS
    % phi  - orthogonal basis used to define G
    % b    - R.H.S. of the linear system Gc = b

    %Build first two rows of b and first two terms of polynomial
    %approximant fa
    %(slightly different for k = 1 & 2...)
    phi_1h = grid_ones ./ trapz(xx,grid_ones.*grid_ones);
    phi_1  = phi_1h ./ sqrt(trapz(xx,phi_1h.^2));
    b(1) = trapz(xx,fcn(xx).*phi_1);
    fa = b(1) .* phi_1;

    phi_2h = xx - trapz(xx,xx.*grid_ones)/trapz(xx,grid_ones.*grid_ones);
    phi_2  = phi_2h ./ sqrt(trapz(xx,phi_2h.^2));
    b(2) = trapz(xx,fcn(xx).*phi_2);
    fa = fa + b(2) .* phi_2;
```

```
    %remaining n-1 rows
    phi_km2 = phi_1;
    phi_km1 = phi_2;
    for jj = 3 : n+1

        phi_kh = xx .* phi_km1 - ...
               trapz(xx,xx.*phi_km1.*phi_km1)./trapz(xx,phi_km1.^2).*phi_km1 - ...
               trapz(xx,xx.*phi_km1.*phi_km2)./trapz(xx,phi_km2.^2).*phi_km2;

        phi_k = phi_kh ./ sqrt(trapz(xx,phi_kh.^2));

        b(jj) = trapz(xx,fcn(xx).*phi_k);

        fa = fa + b(jj) .* phi_k;

        phi_km2 = phi_km1;
        phi_km1 = phi_k;
    end

    %Compute error
    err(j) = norm( fcn(xx) - fa );

    figure(1)
    subplot(2,2,j)
    plot( xx, fcn(xx), '-', 'linewidth', 2 ), hold on
    plot( xx, fa, '--', 'linewidth', 2 )
    ylim([-0.2 1.2])

    %make plot pretty
    title( ['$n = ', num2str( n ),'$'] ,'interpreter', 'latex',...
        'fontsize', 16)
    if j == 1
        h = legend( '$f(x)$', '$f_a(x)$');
    end

    if j <= 2
        set( gca, 'XTick', [] )
```

```
    else
        xlabel( '$x$', 'interpreter', 'latex', 'fontsize', 16)

    end
    set(h, 'location', 'NorthEast', 'Interpreter', 'Latex', 'fontsize',
       16 )
    set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )

    set(gcf, 'PaperPositionMode', 'manual')
    set(gcf, 'Color', [1 1 1])
    set(gca, 'Color', [1 1 1])
    set(gcf, 'PaperUnits', 'centimeters')
    set(gcf, 'PaperSize', [25 25])
    set(gcf, 'Units', 'centimeters' )
    set(gcf, 'Position', [0 0 25 25])
    set(gcf, 'PaperPosition', [0 0 25 25])

    grid(gca,'minor')
    grid on

end

figure(1)
print( '-dpng', 'vary_n', '-r200' )

%plot error
figure(100)
semilogy( nvect, err, 'k.', 'markersize', 24, 'linewidth', 2 )

%make plot pretty
title( 'Maximum error' ,'interpreter', 'latex','fontsize', 16)
xlabel( '$n$', 'interpreter', 'latex', 'fontsize', 16)
ylabel( '$||f(x) - f_a(x)||_2$', 'interpreter', 'latex', 'fontsize', 16)

set(gca, 'TickLabelInterpreter','latex', 'fontsize', 16 )

set(gcf, 'PaperPositionMode', 'manual')
set(gcf, 'Color', [1 1 1])
set(gca, 'Color', [1 1 1])
set(gcf, 'PaperUnits', 'centimeters')
```

```
set(gcf, 'PaperSize', [15 15])
set(gcf, 'Units', 'centimeters' )
set(gcf, 'Position', [0 0 15 15])
set(gcf, 'PaperPosition', [0 0 15 15])

svnm = 'error';
print( '-dpng', svnm, '-r200' )
```