AE 504: HW5

Linyi Hou

May 4, 2020

1 Problem 1

The free final state problem has Hamiltonian

$$H = x^2 + u^2 + \lambda u \tag{1.1}$$

with first order necessary conditions

$$0 = \frac{\partial H}{\partial u} = 2u + \lambda \tag{1.2}$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x \tag{1.3}$$
$$\lambda(1) = 0. \tag{1.4}$$

Using the equation of motion $\dot{x} = u$ and Equations (1.2) and (1.3), we find the following:

$$x = -\frac{1}{2}\dot{\lambda} \tag{1.5}$$
$$\dot{x} = -\frac{1}{2}\lambda \tag{1.6}$$

from which we have

$$x = \ddot{x} \tag{1.7}$$

that has solutions of form

$$x = C_1 e^t + C_2 e^{-t} \tag{1.8}$$

From the initial condition $x(0) = 1$ and Equation (1.4), we get a system of equations:

$$1 = C_1 + C_2 \tag{1.9}$$
$$0 = C_1 e^1 - C_2 e^{-1} \tag{1.10}$$

to finally arrive at the formula

$$x = \frac{1}{1+e^2}e^t + \frac{e^2}{1+e^2}e^{-t} \tag{1.11}$$
$$\boxed{u = \frac{1}{1+e^2}e^t - \frac{e^2}{1+e^2}e^{-t}} \tag{1.12}$$

2 Problem 2

The free final state problem has Hamiltonian

$$H = x^2 + u^2 + \lambda u^2 \tag{2.1}$$

with first order necessary conditions

$$\begin{align} 0 = \frac{\partial H}{\partial u} &= 2u + 2\lambda u \tag{2.2} \\ \dot{\lambda} = -\frac{\partial H}{\partial x} &= -2x \tag{2.3} \\ \lambda(1) = 0. \tag{2.4} \end{align}$$

Observe that from Equation (2.2) that $u = 0$ or $\lambda = -1$. Apply Equation (2.4), noting that if $\lambda = -1$ there is a conflicting value of λ, we find that the optimal control is

$$\boxed{u = 0} \tag{2.5}$$

This also makes sense if we look at the dynamics and cost function. Any input of u will increase x, and thus cause both terms in the cost function to increase.

3 Problem 3

The infinite time horizon problem has Hamiltonian

$$H = x^2 + u^2 + \lambda u \tag{3.1}$$

with necessary conditions

$$\begin{align} 0 = \frac{\partial H}{\partial u} &= 2u + \lambda \tag{3.2} \\ \dot{\lambda} = -\frac{\partial H}{\partial x} &= -2x \tag{3.3} \\ H = 0 \tag{3.4} \end{align}$$

From Section 1, we know that the final solution has form

$$x = C_1 e^t + C_2 e^{-t} \tag{Equation (1.8)}$$

Using Equations (3.1), (3.2) and (3.4), we have

$$H = x^2 - u^2 \tag{3.5}$$

Combining with the initial condition $x(0) = 1$ to get a system of equations:

$$1 = C_1 + C_2 \tag{3.6}$$

$$0 = 4C_1C_2 \tag{3.7}$$

which has two solutions: $(C_1, C_2) = (1, 0)$ or $(C_1, C_2) = (0, 1)$. Observe that if $C_1 = 1$ then x grows without bound. Therefore we arrive at the final solution

$$x = e^{-t} \tag{3.8}$$

$$\boxed{u = -e^{-t}} \tag{3.9}$$

4 Problem 4

The infinite time horizon problem has Hamiltonian

$$H = x^2 + u^2 + \lambda x^2 + \lambda u \tag{4.1}$$

with necessary conditions

$$0 = \frac{\partial H}{\partial u} = 2u + \lambda \tag{4.2}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2(1 + \lambda)x \tag{4.3}$$

$$H = 0 \tag{4.4}$$

Substituting $\lambda = -2u$ into Equation (4.4) we get

$$0 = x^2 - 2ux^2 - u^2 \tag{4.5}$$

Solving for u, we find

$$u = -x^2 \pm \sqrt{x^4 + x^2} \tag{4.6}$$

Since choosing the positive root yields x growing without bound, we choose the negative value of u:

$$\boxed{u = -x^2 - \sqrt{x^4 + x^2}} \tag{4.7}$$

5 Problem 5

A maximum does not exist. Define $y(t)$ as follows:

$$\begin{cases} y = \frac{2C}{\epsilon} t \ , & 0 \leq t < \frac{\epsilon}{2} \\ y = -\frac{2C}{\epsilon}(t - \frac{\epsilon}{2}) + C \ , & \frac{\epsilon}{2} \leq t < \epsilon \\ y = 0 \ , & \epsilon < t \end{cases} \tag{5.1}$$

Then the area under y is simply $A = \frac{1}{2}\epsilon C$. Choose $C = \frac{2}{\epsilon}$ to get $A = 1$. The maximum value $C(y) = C$. By inspection, ϵ can be arbitrarily small and positive, and thus conversely C can be arbitrarily large. Thus a maximum $C(y)$ does not exist; the infimum exists, which is $\boxed{C(y) = \infty}$.

6 Problem 6

The free final state problem has Hamiltonian

$$H = \beta(u-1)x + \lambda\alpha x u = ux(\lambda\alpha + \beta) - \beta x \tag{6.1}$$

with necessary conditions

$$\begin{aligned} 0 = \frac{\partial H}{\partial u} &= \beta x + \lambda \alpha x &(6.2)\\ \dot{\lambda} = -\frac{\partial H}{\partial x} &= -\beta(u-1) - \lambda\alpha u &(6.3)\\ \lambda(T) = 0. & &(6.4) \end{aligned}$$

Since $x(0) = N > 0$ and $\dot{x}$ is non-negative by definition, $x \neq 0$. Therefore from Equation (6.2), $\lambda\alpha = -\beta$. Substituting this result into Equation (6.3), we have $\dot{\lambda} = \beta$. Since $\lambda(T) = 0$, we have $\lambda(t) = -\beta T + \beta t$.

To minimize the Hamiltonian, we must choose u to minimize $ux(\lambda\alpha + \beta)$. Thus we arrive at the control law

$$\begin{cases} u = 0, \ , & \lambda \geq -\dfrac{\beta}{\alpha} \\ u = 1, \ , & \lambda < -\dfrac{\beta}{\alpha} \end{cases} \tag{6.5}$$

And finally, rewriting the control law with time:

$$\begin{cases} u = 0, \ , & t \geq T - \dfrac{1}{\alpha} \\ u = 1, \ , & t < T - \dfrac{1}{\alpha} \end{cases} \tag{6.6}$$

7 Problem 7

Observe that $\int_0^1 L_1(t)\, dt$ is a fixed cost and therefore does not need to be accounted for in the Hamiltonian. The Hamiltonian then becomes

$$H = L_2(y) + L_3(y') + \lambda y' \tag{7.1}$$

For a fixed time, free final state problem, H is constant with respect to time:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t} L_2(y) + \frac{\partial}{\partial t} L_3(y') \tag{7.2}$$

$$= \left(\frac{\partial}{\partial t} y\right)\left(\frac{\partial L_2}{\partial t}(y)\right) + \left(\frac{\partial}{\partial t} y'\right)\left(\frac{\partial L_3}{\partial t}(y')\right) = 0 \tag{7.3}$$

Since $y = at + b$, we have $y' = a$. Therefore $\dfrac{\partial y'}{\partial t} = 0$. It follows from Equation (7.3) that $\dfrac{\partial L_2}{\partial t}(y) = 0$, and thus L_2 is a constant. $\square$

8 Problem 8

We can rewrite the the problem formulation with the variable substitutions $x = f(t),\ u = f'(t)$. Thus the problem becomes

$$\min_{x,\,u} \quad \int_0^1 u^2 \, dt \tag{8.1a}$$

$$\text{s.t.} \quad \dot{x} = u, \tag{8.1b}$$

$$x(0) = 0, \tag{8.1c}$$

$$x(1) = 1 \tag{8.1d}$$

Now, we have the fixed time, fixed final state Hamiltonian as

$$H = u^2 + \lambda u + \nu(x - 1) \tag{8.2}$$

where $\nu(x-1)$ is the terminal constraint. We also have necessary conditions

$$0 = \frac{\partial H}{\partial u} = 2u + \lambda \tag{8.3}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \nu \tag{8.4}$$

Thus $\lambda = \nu t + C$, and $u = -\frac{1}{2}\nu t - \frac{1}{2}C$. Using the constraints $x(0) = 1,\ x(1) = 1$, we find $\int_0^1 u\, dt = 1$, and thus $\nu = -2C - 4$. We can now minimize the cost function by integration:

$$\int_0^1 u(t)^2 \, dt = \int_0^1 (-\frac{1}{2}\nu t - \frac{1}{2}C)^2 \, dt \tag{8.5}$$

$$= \frac{1}{12}(C+2)^2 + 1 \tag{8.6}$$

By inspection, the cost is minimized when $C = 2$. Conversely, $\nu = 0$, and $u = 1$.

The minimum is then $\int_0^1 u^2 \, dt = 1$.

9 Problem 9

Since the problem has fixed final time and fixed final state, we define the Hamiltonian as follows, with the variable substitutions $x = f(t),\ u = f'(t)$:

$$H = \sqrt{1+u^2} + \lambda u + \nu(x-0) \tag{9.1}$$

where $\nu(x-0)$ is the terminal state constraint.

We can now obtain the necessary conditions

$$0 = \frac{\partial H}{\partial u} = \frac{u}{\sqrt{1+u^2}} + \lambda \tag{9.2}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \nu \tag{9.3}$$

From Equation (9.2), we find $u = \lambda\sqrt{\frac{1}{1-\lambda^2}}$. Furthermore,

$$\lambda(t) = \nu t + C \tag{9.4}$$

We have now obtained the control law

$$u(t) = (\nu t + C)\sqrt{\frac{1}{1-(\nu t + C)^2}} \tag{9.5}$$

Now we must satisfy the constraints $x(0) = 0,\ x(1) = 0,$ and $\int_0^1 x\ dt = 0.1$. Solving the system yields

$$\nu = -1.0853 \tag{9.6}$$

$$C = 0.543. \tag{9.7}$$

We further observe that the solution path is part of a circle.

10 Problem 10

Define matrices

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \tag{10.1}$$

The infinite horizon, continuous time LQR problem can be solved by numerically solving the Riccati equation:

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q \tag{10.2}$$

where Q, R are cost weighting matrices associated with the cost function

$$\int_0^\infty x^T Q x + u^T R u \, dt \tag{10.3}$$

from which it is easy to see that for our problem, $Q = I_3,\ R = 1$.

Numerically solving Equation (10.2) yields

$$P = \begin{bmatrix} 0.4503 & 0.08753 & 0.3313 \\ 0.08753 & 0.6008 & 0.03228 \\ 0.3313 & 0.03228 & 1.01 \end{bmatrix} \tag{10.4}$$

with controls

$$\boxed{u = -R^{-1}(B^T P)x} \tag{10.5}$$

11 Problem 11

The fixed time, free final state problem has Hamiltonian

$$H = 10I + \alpha + \beta + \lambda_S\left[-(1-\alpha)IS\right] + \lambda_I\left[(1-\alpha)IS - \beta I\right] + \lambda_R\left[\beta I\right] \tag{11.1}$$

with necessary conditions

$$\bar{0} = \begin{bmatrix} \dfrac{\partial H}{\partial \alpha} \\ \dfrac{\partial H}{\partial \beta} \end{bmatrix} = \begin{bmatrix} 1 + (\lambda_S - \lambda_I)IS \\ 1 + (\lambda_R - \lambda_I)I \end{bmatrix} \tag{11.2}$$

$$\dot{\lambda} = \begin{bmatrix} \dot{\lambda_S} \\ \dot{\lambda_I} \\ \dot{\lambda_R} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial H}{\partial S} \\ -\dfrac{\partial H}{\partial I} \\ -\dfrac{\partial H}{\partial R} \end{bmatrix} = \begin{bmatrix} (\lambda_I - \lambda_S)(\alpha - 1)I \\ (\lambda_I - \lambda_S)(\alpha - 1)S + (\lambda_I - \lambda_R)\beta - 10 \\ 0 \end{bmatrix} \tag{11.3}$$

$$\lambda(1) = \bar{0}. \tag{11.4}$$

Using the above conditions we can derive explicit expressions:

$$\lambda_S = \frac{S-1}{IS} \tag{11.5}$$

$$\lambda_I = \frac{1}{I} \tag{11.6}$$

$$\dot{\lambda_S} = \frac{\alpha - 1}{S} \tag{11.7}$$

$$\dot{\lambda_I} = \frac{\alpha + \beta - 1}{I} - 10 \tag{11.8}$$

Note that the values of λ_S and λ_I cannot be zero at the final time. However the Hamiltonian is linear in α, β, so we can still choose α, β to minimize the Hamiltonian at any time. Heuristically, this means

$$\begin{cases} \alpha = 0, & \lambda_I - \lambda_S < (IS)^{-1} \\ \alpha = 1, & \lambda_I - \lambda_S \geq (IS)^{-1} \\ \beta = 0, & \lambda_I < I^{-1} \\ \beta = 1, & \lambda_I \geq I^{-1} \end{cases} \tag{11.9}$$

and thus the control law is bang-bang.

In theory, since we know the initial values of λ and can calculate $\dot{\lambda}$, we could simply propagate the system forward in time to determine the controls. However, it was experimentally determined that this does not yield the optimal results.

Instead, note that for a fixed time, free final state problem, λ at the final time should be zero. Therefore we could try to drive λ to zero by our choice of α and β.

Heuristically:

$$\begin{cases} \alpha = 0, & \lambda_S < 0 \text{ or } \lambda_I < 0 \\ \alpha = 1, & \text{otherwise} \\ \beta = 0, & \lambda_I > 0 \\ \beta = 1, & \text{otherwise} \end{cases} \tag{11.10}$$

The resulting control is $\alpha = \beta = 0$ for all times, with a cost of 0.0172.

However, more interesting cases arise when the time horizon is extended. We choose to examine the case of $t_f = 10$, since this is the approximate time it takes for the entire population to be infected, as shown in Figure 1.

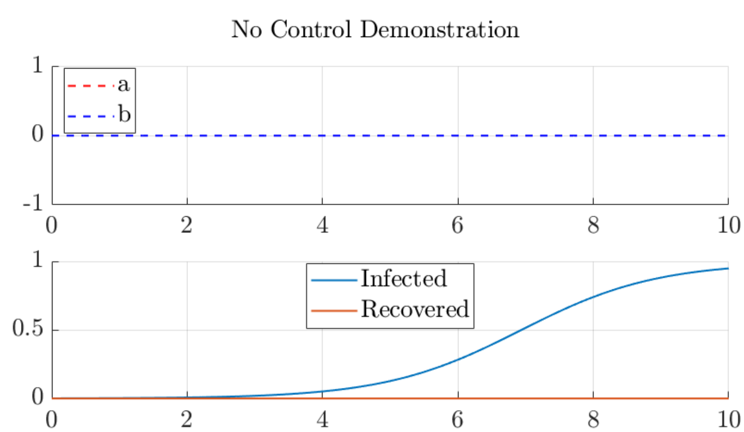

Figure 1: With no control, the entire population becomes infected at $t = 10$

The control method outlined in Equation (11.10) yields the following results in Figure 2:

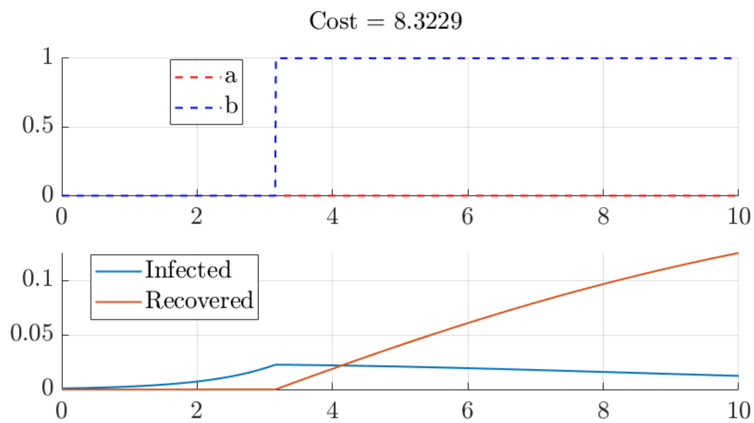

Figure 2: Controls from Equation (11.10) being implemented

The controls are only activated after a certain time has elapsed. This suggests that spending resources to control disease spread from the beginning is ineffective, because the investment acts on only a very small fraction of the population. It is only when the disease has reached some critical density that controls become cost-effective.

This may seem counter-intuitive — shouldn't we try to contain diseases from the get-go? The issue may lie within the cost function, as it only considers the cost up to a certain time, but does not consider the future impact of the current inputs. In other words, **a certain control may be cheap in the short run, but not be able to contain the infection, causing greater damage in the long run**. This can be illustrated using the same setup but with controls $\alpha = \beta = 0.3$. The results are shown in Figure 3.

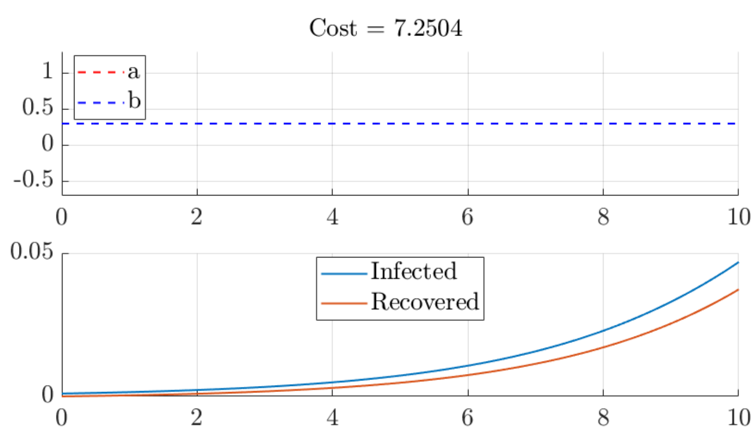

Figure 3: Cheaper control option does not slow down disease spread

While the cost has been reduced from 8.3 to 7.3, the infected population is growing exponentially, whereas the inputs shown in Figure 2 has managed to hold the infected population under control.

Adding penalties to infection growth rate ($\dot{I}$) and total recovered population (R) may result in improved (more realistic) results.