AE 504: HW4

Linyi Hou

April 20, 2020

Problem 1

By examining the equations of motion,

$$\begin{bmatrix} \dot{x_h} \\ \dot{x_v} \end{bmatrix} = \begin{bmatrix} v_h \\ v_v \end{bmatrix}, \begin{bmatrix} \dot{v_h} \\ \dot{v_v} \end{bmatrix} = \begin{bmatrix} u_h \\ u_v - g \end{bmatrix} \tag{1}$$

we find that the horizontal and vertical motions of the lander are decoupled. Therefore it is possible to solve for control inputs u_h and u_v separately.

It is also convenient to linearize the EOMs in Equation (1) by combining the position and velocity into one state:

$$\dot{x} = \begin{bmatrix} \dot{x_h} \\ \dot{x_v} \\ \dot{v_h} \\ \dot{v_v} \end{bmatrix} = \begin{bmatrix} v_h \\ v_v \\ u_h \\ u_v - g \end{bmatrix} \tag{2}$$

The problem has a fixed final state and free final time. Thus the Hamiltonian is:

$$H(\lambda_0, \lambda, x, u) = \lambda_0 L(x, u) + \lambda^T f(x, u) \tag{3}$$

where $\int_a^b L(x, u)dt$ is the cost function, and $f(x, u)$ is shown in Equation (2).

Intuition: It is clear from the terrain that the lander must follow a pseudo-parabolic trajectory to avoid all peaks and reach the destination. Initially, the lander should ascend vertically before translating to the right. After passing the tallest peak at $x_v = 50$ the lander will then descend and target the landing zones.

Implementation: A MATLAB simulation was created to determine the control inputs of the lander and simulate its trajectory. A variation of bang-bang control based on the fixed final state, free final time problem was implemented as follows:

a. Define initial and end states $x^{(0)}, x^{(n)}$, and start time $t_0 = 0$.

b. Define any intermediate target points $x^{(i)}$ where $i \in \{1, ..., n-1\}$. Define **segments** as paths between two adjacent states x_i and x_{i+1}.

c. For each segment starting at x_i and ending at x_{i+1}, assume bang-bang control where the control input only switches once. Determine the required time to switch input for u_h and u_v separately, such that the start and end states are satisfied. In this process, two more variables are required: the horizontal throttle value γ, which prevents the horizontal velocity from overshooting, and the total segment time t_i, which governs the total flight time in the current segment.

Results: The resulting control inputs are tabulated below:

t	0	15.8308	16.8413	22.3592	26.5652	28.3475	55.416
u_h	0.13495	-0.13495	0.76224	-0.76224	0.11864	-0.11864	0
t	0	15.1414	16.8413	22.7176	26.5652	31.5468	55.416
u_v	2	0	2	0	0	2	0

The flight profile of the lander is shown in Figure 1. A manually created trajectory is shown in Figure 2 for comparison.

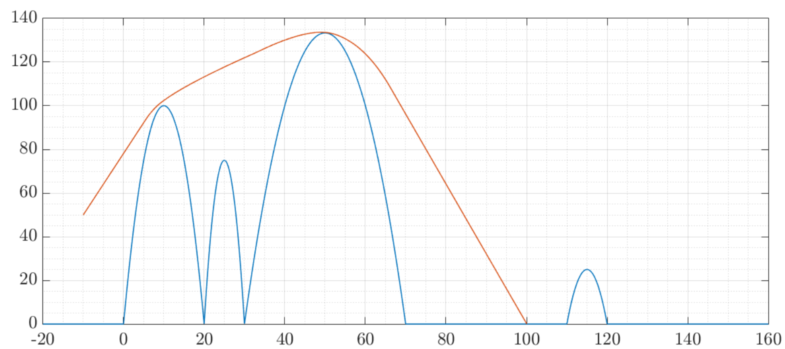

Figure 1: Lander Trajectory (Cost: 102.88)

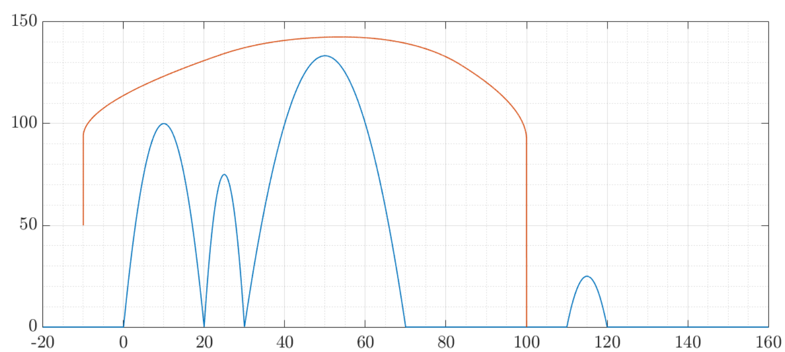

Figure 2: Lander Trajectory, Manual (Cost: 133.67)

Problem 2

The formulation of Problem 1 serves as a good foundation for Problem 2. Intuitively, the lander consumes the least amount of fuel by minimizing its vertical and horizontal burns.

First, the lander should not reach heights beyond the maximum peak height between its starting point and destination.

Next, the lander should never fly over the same horizontal position (assuming there are no caves in the terrain map).

The lander should also smoothly transition between segments of flight. This is a rather vague definition — it loosely correlates to the idea that any abrupt change in motion means extra fuel consumption to redirect the spacecraft's momentum.

Finally, the choice of landing site is governed by the state of the lander after it has reached apoapsis. If the lander cannot slow down quickly enough to reach the nearer landing site, then the farther site is preferred. If the lander cannot reach the farther landing site without requiring extra upward thrust to avoid the final peak, then the nearer site is preferred.

Hypothesis: After eating popcorn I decided what I wrote above is useless. Instead, we hypothesize that a lander trajectory which approximates that of a parabola will minimize fuel consumption. Since the lander will be coasting without fuel expenditure once it is on its parabolic trajectory (until the landing burn), this could in theory minimize fuel costs.

We select three points to define a parabola — the initial lander position, final lander position, and peak terrain position at $x_h = 50$ (this value will later be adjusted up by 1 unit to ensure no collisions with the terrain).

Next, we select two points on either side of the vertex of the parabola as our "injection points". The lander will start from its initial position, inject into the parabola by matching both position and velocity at the injection point, and eject from the parabola at the next point to perform a landing burn. An illustration of this procedure is shown in Figure 3.

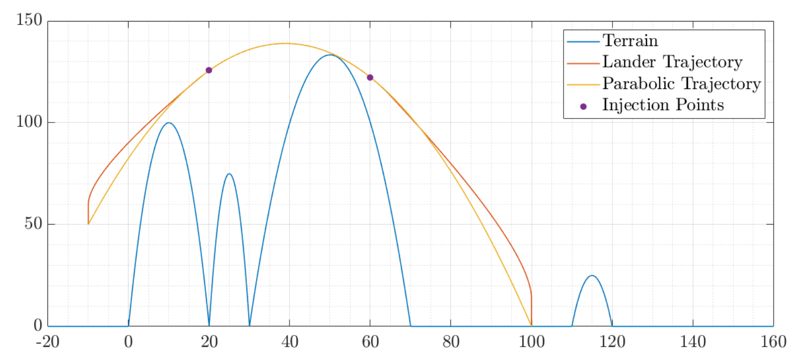

Figure 3: Parabolic Lander Trajectory (Cost: 96.94)

The far landing zone could not be selected due to the parabolic arc intersecting with the terrain. The control inputs to the near landing zone have been compiled below.

t	0	7.1306	19.9891	28.5615	45.7062	54.0825
u_h	0	0.36288	0	-0.27216	0	0
t	0	19.4705	19.9891	28.5615	29.7461	54.0825
u_v	2	0	0	0	2	0

↑
parabolic coast

It was expected that the injection points could not be moved closer to the beginning and end locations, since the thrust limit of the lander prohibits high acceleration to quickly match high velocity values at lower points on the parabola. This is evident from the fact that u_v required maximum thrust for almost the entire time prior to parabolic coasting.

However, it is interesting to note that shifting the injection points inward produced identical fuel consumption values. This makes sense: the lander reached the same apoapsis at the same velocity, and therefore had the same total energy. It follows that the energy/fuel consumption to land in the same location would be identical. Regardless, it is clear that the parabolic trajectory indeed yielded a lower fuel cost than the method used in Problem 1.

Problem 3

(i) It is **not** possible to achieve $C(u_h, u_v) = 0$ for some (u_h, u_v) which satisfies the mission requirements. We know that since u_h is piecewise continuous and $\dot{v_h} = u_h$, v_h must be continuous. Since $\dot{x_h} = v_h$, x_h must also be continuous.

Since $x_{hi} = -10$, $x_{hf} = 100$, x_h is continuous, $v_{hi} = 0$, it follows that

$$\int_{t_i}^{t_f} \int_{t_i}^{\tau} u_h \, dt \, d\tau = x_{hf} - x_{hi} = 110 \neq 0 \tag{4}$$

Thus, $u_h \neq 0$, and $C(u_h, u_v) > 0$. □

(ii) **Intuitively:** Yes, a positive cost is always achievable no matter how small as long as vertical thrust is free. This is because we can always hover above the terrain and apply an infinitesimally small amount of thrust horizontally toward the destination. After an infinite amount of time, reverse thrust infinitesimally and then descend.

Formally: It is always possible to achieve $C(u_h, u_v) < \epsilon$, $\forall \epsilon > 0$. Proof:

Let t_0 be the starting time. Choose t_1 such that $u_v = 2, u_h = 0$ for $t \in [t_0, t_1]$, and $Y = \frac{1}{2}u_v(t_1^2 - t_0^2) > \max(y)$. Let $u_v = g$ for $t \in (t_1, t_2]$. This guarantees no terrain collision will occur on the interval $(t_1, t_2]$. Now, let

$$\begin{cases} u_h = \dfrac{\epsilon}{3} & t \in (t_1, t_1 + 1] \\ u_h = 0 & t \in (t_1 + 1, t_2 - 1] \\ u_h = -\dfrac{\epsilon}{3} & t \in (t_2 - 1, t2] \end{cases} \tag{5}$$

Choose t_2 such that

$$\int_{t_1}^{t_2} \left(v_{h0} + \int_{t_1}^{\tau} u_h \, dt \right) \, d\tau = x_{hf} - x_{h0} \tag{6}$$

Thus at $t = t_2$, $x_h = x_{hf}$, $v_h = 0$. Choose u_v, t_f such that

$$\int_{t_2}^{t_f} \int_{t_2}^{\tau} u_v \, dt \, d\tau = 0 - Y \tag{7}$$

Thus at $t = t_f$, $x_h = x_{hf}$, $v_h = 0$, $x_v = 0$, $v_v = 0$, which satisfies the mission requirements. The total cost is

$$|\frac{\epsilon}{3} \cdot (t_1 + 1 - t_1)| + |\frac{\epsilon}{3} \cdot (t_2 - (t_2 - 1))| = \frac{2\epsilon}{3} < \epsilon \tag{8}$$

□