Optimal Brachistochrone Spacecraft Controls

AE 504 Final Project | Linyi Hou

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1 Motivation

The goal of this project is to formulate an optimal control strategy for interplanetary spaceflight using high-thrust, high-specific impulse spacecraft, colloquially known as "torchships".

In science-fiction, torchships are spacecraft equipped with advanced propulsion technology that allow them to constantly accelerate at multiple g's for days or even weeks. This significantly reduces the time required for interplanetary travel. The trajectory the spacecraft follows minimizes travel time by using maximum thrust for the entire flight, and is thus know as a brachistochrone trajectory.

However, there is not much discussion on the detailed operations of torchships. Often times the flight profile is described simply as "flip-and-burn", where the spacecraft accelerates toward its target for half the journey, and decelerates for the other half, as shown in Figure 1.

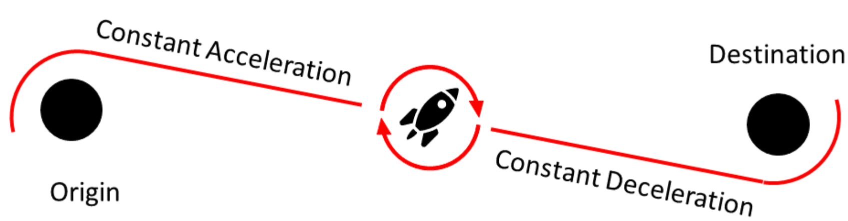

Figure 1: The "flip-and-burn" maneuver

The flip-and-burn maneuver is a simplification of the spacecraft's trajectory. It does not account for the difference in velocity vectors, the effects of gravity, or the thrust vector of the spacecraft. In this project, we will consider all three of the above and attempt to derive an optimal control strategy for brachistochrone spaceflight.

2 Previous Work

As mentioned in Section 1, little literature exists on brachistochrone trajectories for torchships. However, certain elements of the problem have been studied extensively.

The original brachistochrone problem was formulated by Johann Bernoulli in 1696 [1]. In Bernoulli's brachistochrone problem, a particle travels on a path between two points under a gravity field and no friction; the objective was to find the path that minimizes travel time.

Furthermore, continuous thrust trajectories have been analyzed and used substantially in the past. In this project we will be referencing several authors including Professor Prussing [2]. Continuous thrust trajectories are almost always low thrust and seek to minimize fuel consumption, so their applicability to the brachistochrone problem can sometimes be limited.

In the subsequent chapters, we will attempt to consolidate the brachistochrone and continuous thrust trajectories into one optimization problem.

3 Nomenclature

Scalars	H	Hamiltonian
	J	cost functional
	K	final cost
	L	running cost
	t	time
	μ	gravitational parameter of the central body
	Γ	thrust magnitude
Vectors	$\boldsymbol{f}$	equation of motion
	$\boldsymbol{g}$	gravity vector
	$\boldsymbol{r}$	position vector
	$\boldsymbol{v}$	velocity vector
	$\boldsymbol{x}$	state vector
	$\boldsymbol{u}$	thrust unit vector
	$\boldsymbol{\lambda}$	Lagrange multiplier
Matrices	$\boldsymbol{G}$	gravity gradient matrix
	$\boldsymbol{\Phi}$	transition matrix
Subscripts	0	initial state
	f	final state
	max	maximum value

4 Assumptions

Let us consider a spacecraft traveling between Earth and Mars. We assume that restricted two-body dynamics apply, so the spacecraft is only subject to the gravitational pull of the Sun, while the Sun remains stationary in the inertial frame. Gravity is modeled by the inverse-square law:

$$\boldsymbol{g} = -\frac{\mu}{r^3}\boldsymbol{r} \tag{1}$$

and thus the equation of motion that the spacecraft follows is expressed as

$$\dot{\boldsymbol{r}} = \boldsymbol{v}\ ,\ \dot{\boldsymbol{v}} = \boldsymbol{a} = \Gamma\boldsymbol{u} + \boldsymbol{g} \tag{2}$$

To create a first order system, define a state vector and its equation of motion

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{bmatrix}\ ,\ \dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \Gamma\boldsymbol{u} + \boldsymbol{g} \end{bmatrix} \tag{3}$$

For convenience, rewrite Equation (3) as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{4}$$

Furthermore, define an upper bound to the thrust available:

$$\max\Gamma = \Gamma_{max} \tag{5}$$

5 Problem Statement

Out goal is to minimize time-of-flight between two locations in an inverse-square gravitational field using continuous thrust, with fixed initial and final positions and velocities.

The running cost is $L = 1$, and there is no final cost K. Therefore the cost functional

$$J(\boldsymbol{u}) = \int_{t_0}^{t_f} L(t, \boldsymbol{x}(t), \boldsymbol{u}(t))\ dt + K(t_f, \boldsymbol{x_f}) \tag{6}$$

can be simplified to

$$J = \int_{t_0}^{t_f} 1\ dt = t_f - t_0 \tag{7}$$

The spacecraft is subject to initial and final state constraints as well as EOM constraints:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{x}(t_0) &= \boldsymbol{x_0} \\ \boldsymbol{x}(t_f) &= \boldsymbol{x_f} \end{aligned} \tag{8}$$

and a thrust limit

$$|\Gamma| \leq \Gamma_{max} \tag{9}$$

Summarizing, the optimization/minimization problem is therefore defined as

$$\min_{\boldsymbol{x},\boldsymbol{u}} \quad t_f - t_0 \tag{10a}$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{10b}$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{10c}$$

$$\boldsymbol{x}(t_f) = \boldsymbol{x}_f, \tag{10d}$$

$$|\Gamma| \leq \Gamma_{max} \tag{10e}$$

6 Solution Approach

6.1 Hamiltonian

We can define the Hamiltonian as follows:

$$\begin{aligned} H(\lambda, \boldsymbol{x}, \boldsymbol{u}) &= L + \boldsymbol{\lambda}^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ &= 1 + \boldsymbol{\lambda}^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{11}$$

We can further dissect the Lagrange multiplier λ. Let

$$\boldsymbol{\lambda}^T = \left[\boldsymbol{\lambda}_r^T,\ \boldsymbol{\lambda}_v^T\right] \tag{12}$$

Expanding Equation (11) using Equations (3) and (12), we get

$$H = 1 + \boldsymbol{\lambda}_r^T \boldsymbol{v} + \boldsymbol{\lambda}_v^T \left(-\frac{\mu}{r^3}\boldsymbol{r} + \Gamma \boldsymbol{u}\right) \tag{13}$$

with optimality conditions for the free time, fixed final state problem listed as follows:

$$0 = \frac{\partial H}{\partial \boldsymbol{u}} \tag{14}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{15}$$

$$0 = H \tag{16}$$

6.2 Solving for the Control Input

Since the Hamiltonian is linear in $\boldsymbol{u}$ as shown in Equation (14), we cannot set $\frac{\partial H}{\partial \boldsymbol{u}} = \mathbf{0}$ and expect a viable solution. Instead, we must minimize H.

Observe that the only term that $\boldsymbol{u}$ influences is $\boldsymbol{\lambda}_V^T \Gamma u$, so we are able to choose $\boldsymbol{u}$ to always minimize that term. Therefore, choose $\boldsymbol{u}$ such that:

$$\boldsymbol{u} = -\frac{\boldsymbol{\lambda}_v}{\lambda_v} \tag{17}$$

By extension of Equation (17), we know that the term $\boldsymbol{\lambda}_V^T \Gamma u$ is always negative. Thus the choice of Γ that minimizes H is the maximum value, $\Gamma = \Gamma_{max}$.

6.3 Solving for the Lagrange Multiplier

We can compute $\dot{\boldsymbol{\lambda}}$:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}} = -\begin{bmatrix} \boldsymbol{O}_3 & \boldsymbol{G} \\ \boldsymbol{I}_3 & \boldsymbol{O}_3 \end{bmatrix} \boldsymbol{\lambda} \tag{18}$$

where the gravity gradient matrix $\boldsymbol{G}$ can be computed from Equation (19) [2].

$$\begin{aligned} \boldsymbol{G} &= -\frac{\partial H}{\partial \boldsymbol{r}} \\ &= \frac{\mu}{r^5}\left(3\boldsymbol{r}\boldsymbol{r}^T - r^2 \boldsymbol{I}_3\right) \end{aligned} \tag{19}$$

We can express $\boldsymbol{\lambda}$ using the transition matrix as follows:

$$\boldsymbol{\lambda}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\lambda}(t_0) \tag{20}$$

Using results from [3], the transition matrix can be expressed in truncated form as

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{I}_3 + G(\boldsymbol{r}_0)\dfrac{\delta t^2}{2!} & \boldsymbol{I}_3\delta t + G(\boldsymbol{r}_0)\dfrac{\delta t^2}{3!} \\ G(\boldsymbol{r}_0)\delta t & \boldsymbol{I}_3 + G(\boldsymbol{r}_0)\dfrac{\delta t^2}{2!} \end{bmatrix} \tag{21}$$

where $\delta t = t - t_0$. This expression is limited by the growth of truncated terms over time. The maximum allowable time interval is also given by [3]:

$$\delta t_{max} = \frac{r}{2v} \tag{22}$$

In the context of interplanetary flight in the solar system, the value of δt_{max} evaluates to approximately 30 days for an Earth departure orbit. A rough estimate of the minimum Γ_{max} value required fulfill this time limit can be found using the "flip-and-burn" method.

The distance d flown using the "flip-and-burn" method at acceleration Γ_{max} for time t is

$$d = \frac{1}{4}\Gamma_{max}t^2 \leq \frac{1}{4}\Gamma_{max}{\delta t_{max}}^2 \tag{23}$$

note that the coefficient is $\frac{1}{4}$ instead of $\frac{1}{2}$, because deceleration is required for the latter half of the flight. Rearranging the above to solve for Γ_{max}:

$$\Gamma_{max} \geq \frac{4d}{\delta t_{max}{}^2} \tag{24}$$

where d is the separation between the origin and the destination. Substituting in the maximum separation of Earth and Mars yields $\Gamma_{max} \geq 0.23m/s^2$.

6.4 The Optimization Problem

Now that we have established the conditions for optimality in the context of brachistochrone trajectories, it is possible to solve the optimization problem using computational tools.

Since $\boldsymbol{\lambda}(t)$ is known, the control $\Gamma \boldsymbol{u}$ at any time t is also known. It is therefore possible to integrate an initial guess, $\boldsymbol{\lambda}_0$, to an unknown final time t_f, and check whether the final state converges to the desired final state. The formulation is as follows:

$$\begin{aligned}
\text{Solve for: } & [\boldsymbol{\lambda}_0\ ,\ t_f] && (25)\\
\text{with dynamics: } & \boldsymbol{u}(t) = [\boldsymbol{I}_3\ \boldsymbol{0}_3] \cdot \boldsymbol{\lambda}(t) && (26)\\
& \boldsymbol{\lambda}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\lambda}_0 && (27)\\
& \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) && (28)\\
\text{and I.C.s: } & \boldsymbol{x}(t_0) = \boldsymbol{x}_0 && (29)\\
& \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0 && (30)\\
\text{and B.C.s: } & \boldsymbol{x}(t_f) = \boldsymbol{x}_f && (31)\\
& H(t_0) = 1 + \boldsymbol{\lambda}_0^T \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) = 0 && (32)
\end{aligned}$$

For a 3-D case, there are 7 variables to solve for and 7 boundary conditions. Note that the last boundary condition given in Equation (32) comes from Equation (16).

Since state $\boldsymbol{x}$ cannot be expressed analytically due to the changing $\boldsymbol{g}$ vector, the above system must be integrated numerically. This presents a significant challenge to solvers and could possibly lead to no solutions due to the non-linear nature of the solution space.

7 Solution

The formulation shown in Equations (25)–(32) was solved using the MATLAB function fsolve() for an Earth-Mars brachistochrone transfer at constant 1g thrust. For purposes of demonstrating the functionality in three dimensions, the inclination of Mars was significantly increased. The resulting trajectory and controls are shown in Figures 2 and 3, respectively. The time of flight is approximately 10 days.

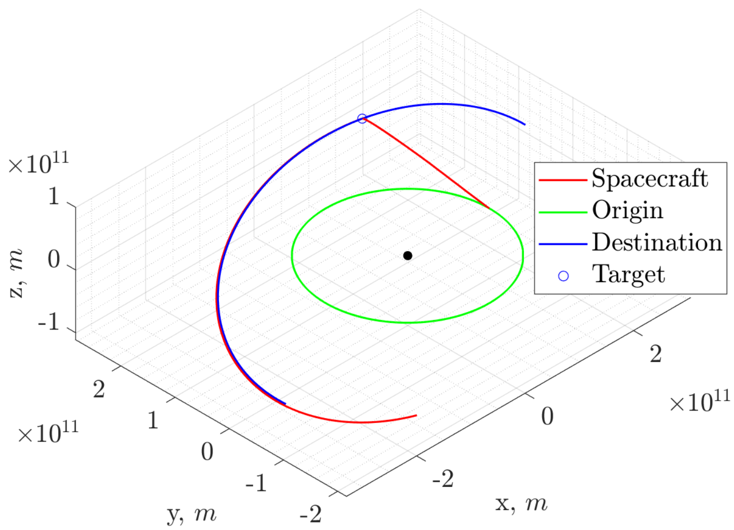

Figure 2: Resulting flight trajectory

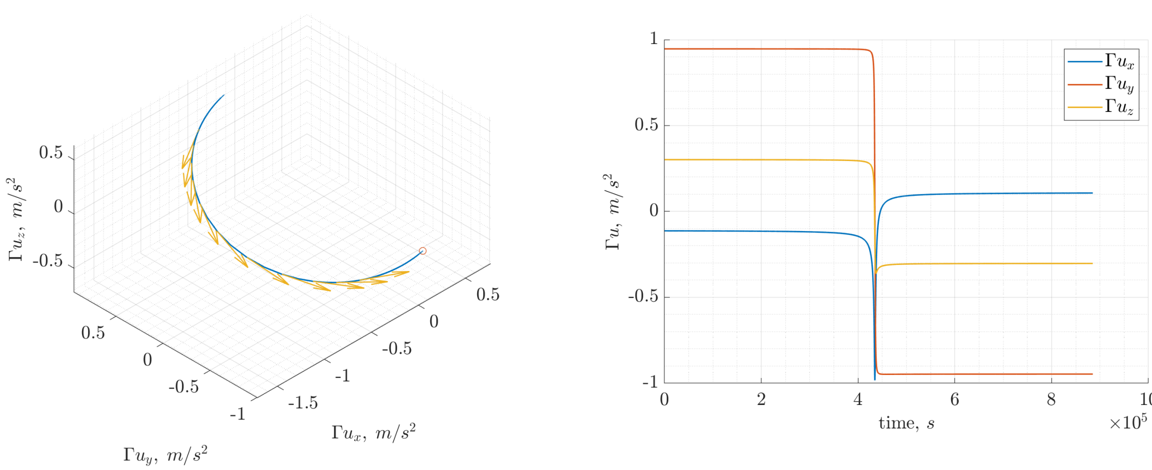

Figure 3: Optimal Control Strategy

8 Results and Discussion

While the trajectory shown in Figure 2 does not look like an orbit whatsoever, we must keep in mind that torchships in science fiction have propulsion capabilities unparalleled by modern technology. The capability of delivering 1g of thrust for days or weeks gives the torchship an incredible amount of ΔV and causes the resulting trajectory to look linear. By examining Figure 3, we see that the resulting thrust solution resembles the "flip-and-burn" control scheme described in Section 1, which gives us some confidence in the solution.

Since the transition matrix $\mathbf{\Phi}$ was created using truncated terms, the control profile is not exactly optimal. However, we know that the resulting time of flight is 10 days, which is shorter than the δt limit established in Equation (22), which is 30 days. Therefore in theory the error contribution from the truncated terms are not significant.

Finally, it should be noted that the effect of the Sun's gravitational field is very small at Earth's distance. It is possible to generate a solution comparable to the one outlined in Section 6 by simply ignoring gravity in the calculations, then adding the force of gravity back into the controls.

References

[1] D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction.* USA: Princeton University Press, 2011.

[2] J. E. Prussing, *Optimal Spacecraft Trajectories.* Oxford University Press, 2018.

[3] J. S. White, "Simplified calculation of transition matrices for optimal navigation," tech. rep., National Aeronautics and Space Administration, 1966.