1. (a) Let $x \in \{a \in X | \exists U$ open, such that $a \in U \subset A\}$, then $U \cup A^\circ$ is an open subset of A containing A°, hence by maximality $x \in U \cup A^\circ = A^\circ$. If $a \in A^o$, then a is in a open subset contained in A proving the other set inclusion.

let $x \in \{x \in X | \forall U$ open with $x \in U$ and $U \cap A \neq \emptyset\}^c$, then there exists some open $U \subset A^c$ containing x, so that $A \subset U^c$ is closed this implies $\overline{A} \subset U^c$ and hence $x \notin \overline{A}$. Conversely, if $x \in \overline{A}^c$, then $\overline{A}^c$ is an open set disjoint from A containing x, so that $x \in \{x \in X | \forall U$ open with $x \in U$ and $U \cap A \neq \emptyset\}^c$.

(b) U° is open by definition, so $U^\circ = U$ implies U open. If U is open, then U is an open set contained in U, so that $U \subset U^\circ$ and hence $U = U^\circ$.

$\overline{A}$ is closed, hence $A = \overline{A}$ implies A is closed. Now suppose that A is closed, then A is a closed set containing A, hence $A \supset \overline{A}$, which implies $A = \overline{A}$.

(c) The compliment of A° is closed, and $A^\circ \subset A$ implies that $(A^\circ)^c \supset A^c$, implying that $\overline{A^c} \supset (A^\circ)^c$. Conversely, if $x \in \overline{A^c}$, then by part (a), any open set containing x has non-empty intersection with A^c, hence there does not exist an open set U containing x, such that $U \subset A$, applying (a) again ,this means that $x \notin A^\circ$

$\overline{A}^c$ is an open set contained in A^c, hence $\overline{A}^c \subset (A^c)^\circ$. Conversely, if $x \in \overline{A}$, then from (a), any open set containing x has non-trivial intersection with A, hence applying part (a) again we get that $x \notin (A^c)^\circ$, hence $\overline{A} \subset ((A^c)^\circ)^c$, contraposing this gives the desired equality.

2. Consider the collection $\mathcal{I}$ of closed sets in X, which are not finite unions of irreducibles. Every descending chain being eventually constant is equivalent to every descending chain having a lower bound (i.e. If $\cap_i F_i = F_j$, then F_j is a lower bound on the chain). Thus we can apply Zorn's lemma which furnishes a minimal element Z in $\mathcal{I}$, if Z were not irreducible, then it would need to be a union of closed subsets $Z_1 \cup Z_2$, since Z is not a finite union of irreducibles, the same must apply to one of Z_1 or Z_2, but this contradicts the minimality of $Z \in \mathcal{I}$. It follows that $\mathcal{I} = \emptyset$, so that X is a finite union of irreducible elements.

let $\{Y_i\}_{i=1}^m \neq \{Z_i\}_{i=1}^n$ be two collections of irreducible sets, such that no set is contained in the union of the rest of the collection, and $\bigcup_i Y_i = X = \bigcup_i Z_i$. Then there must exist some Y_i, Z_j, such that $Y_i \cap Z_j \neq \emptyset$ and $Y_i \neq Z_j$ (explicitly choose some $Y_i \notin \{Z_j\}_j$, but $\emptyset \neq Y_i = Y_i \cap \cup_j Z_j = \cup_j Y_i \cap Z_j$ cannot all be empty). We may assume WLOG $Y_i \not\subset Z_j$, but this contradicts the Zarisky condition, since $Y_i = (Y_i \cap Z_j) \cup (Y_i \cap \cup_{i \neq j} Z_i)$ is a union of closed proper subsets of Y_i.

3. (a) Suppose X is not connected, then there exists some $X \neq A \neq \emptyset$ which is clopen, it follows that A^c is also clopen. Let $\pi : \begin{cases} x \mapsto 1 & x \in A \\ x \mapsto 0 & x \in A^c \end{cases}$ map X to $\{0,1\}$ with the discrete topology. This is clearly a map, since the preimage of every set is open. Conversely, suppose there exists a surjective map $\pi : X \to \{0,1\}$, we have that $\pi^{-1}(1), \pi^{-1}(0) = (\pi^{-1}(1))^c$ are open, disjoint and non-empty. Since compliments of open sets are open, these sets are also closed. Hence $\pi^{-1}(1)$ is a clopen set not equal to X or $\emptyset$, since it is non-empty with non-empty compliment.

(b) Let $\pi : X \to \{0,1\}$ be a map, suppose WLOG $(0,1) \overset{\pi}{\mapsto} 1$, it follows that $\pi(\{0\} \times [-1,1]) = 1$, since assuming not we let $\alpha = \sup\{y \in [0,1] | \pi(y) = 0\}$, implying that π is not continuous at α, since any open set U containing α must contain some point $\alpha > \beta, (0,\beta) \overset{\pi}{\mapsto} 0$, and some point $\alpha < \gamma, (0,\gamma) \overset{\pi}{\mapsto} 1$ by the supremum property. Let $\beta = \inf\{x | \pi(x, \sin\frac{1}{x}) = 0\}$, if $\beta = \infty$ we are done, and if $\beta > 0$, then the argument is identical to the case of the line, so assume $\beta = 0$. This implies that $\pi(x, \sin\frac{1}{x}) = 0$ for all $x > 0$, once again by the same argument as for the line. Now consider any open set U containing the point $(0,0)$ and take some neighbourhood $N_\epsilon(0,0) \subset U$, taking N so large that $\frac{1}{N\pi} < \epsilon$, we can see that $(\frac{1}{N\pi}, \sin N\pi) = (\frac{1}{N\pi}, 0) \in U$. $\pi(\frac{1}{N\pi}, 0) = 0$ implies no open set containing $(0,0)$ is a subset of $\pi^{-1}(1)$, contradicting π being a map implying that $\beta = \infty$. This argument results in $\pi(X) = 1$, so that X is connected by (a).

Assume for contradiction there exists a path γ between $(0,0)$ and $(\frac{1}{\pi}, 0)$, assume $(0,y) \in \{0\} \times [0,1]$ is in $\gamma([0,1])$, and let $a \in \gamma^{-1}(y)$. By continuity, we may pick some $\delta > 0$, such that $|a - x| < \delta$ implies $d(\gamma(a), \gamma(x)) < \frac{1}{2}$. Let $x \in N_\delta(a)$ with $\gamma(x) = (x_1, x_2)$ and assume for the sake of contradiction $x_1 \neq 0$. The projection map $\pi : (t,s) \mapsto t$ is continuous since $d(\pi(x), \pi(y)) \leq d(x,y)$ and it is clear that the composition of continuous functions is continuous. We can choose N so large that $\frac{2}{(2N+1)\pi} < \frac{1}{N\pi} < x_1$, IVT guruntees existence of some x', x'' in between a and x, such that $\pi\gamma : x' \mapsto \frac{1}{N\pi}, x'' \mapsto \frac{2}{(2N+1)\pi}$.

$$|a - x''| \text{ and } |a - x'| < \delta \text{ and } d(\gamma(a), \gamma(x'')) + d(\gamma(a), \gamma(x')) \geq d(\gamma(x'), \gamma(x'')) = \sqrt{\left(\frac{1}{N\pi} - \frac{2}{(2N+1)\pi}\right)^2 + 1} \geq 1$$

contradicting $d(\gamma(a), \gamma(x'')), d(\gamma(a), \gamma(x'))$ both being less than $\frac{1}{2}$. Hence $x_1 = 0$, so that $S = \{x \in [0,1] | \pi\gamma(x) = 0\}$ is open. Now suppose that $\pi\gamma(y) > 0$, then there exists some $\delta > 0$, so that $|y - x| < \delta$ implies $|\pi\gamma(x) - \pi\gamma(y)| < \pi\gamma(y)$, so for any $x \in N_\delta(y)$

$$\pi\gamma(x) \geq \pi\gamma(y) - |\pi\gamma(x) - \pi\gamma(y)| > \pi\gamma(y) - \pi\gamma(y) = 0$$

Hence $\pi\gamma(x) > 0$, so that S^c is open, hence $S \subset [0,1]$ is clopen. Since $0 \in S$, and $1 \notin S$, $\emptyset \neq S \neq [0,1]$, but S is connected by the same argument that $\{0\} \times [-1,1]$ is connected, so this is a contradiction and X is not path connected.