1. I claim that the integral closure of A is $A_0 := F[t]$, to do so I will first show that A_0 is integrally closed then reduce the general case to that of A_0. Let $q(t) \in B$ be integral over A_0, then $q(t) = \frac{r(t)}{g(t)}$ for $r, g \in F[t]$ such that $(r, g) = 1$. By assumption we have some monic polynomial

$$q^n(t) + h_1(t)q^{n-1}(t) + \cdots + h_n(t) = 0$$

which is true if and only if the following identity holds in $F[t]$:

$$g^n(t)(q^n(t) + h_1(t)q^{n-1}(t) + \cdots + h_n(t)) = 0$$

since 0 is in any ideal of $F[t]$, this implies in particular that

$$g^n(t)(q^n(t) + h_1(t)q^{n-1}(t) + \cdots + h_n(t)) \in (g(t)) \implies r^n(t) \in (g(t))$$

(the implication comes from $g(t)$ dividing each other term), since $F[t]$ is a PID, we know that $F[t]/(g(t))$ is a domain, and hence $r^n \in (g)$ implies that $r \in (g)$ so that $g|r$, but $(r, g) = 1$ by assumption, so we can conclude that g is a unit, i.e. $g \in F^\times$, so that $q(t) \in A_0$.

Now to reduce the general case of $A = F[f(t)]$ to that of A_0, note that the integral closure of A contains t, since if the leading coefficient of $f(t)$ is a we find that t satisfies the monic polynomial

$$a^{-1}f(X) - a^{-1}f(t)$$

in A_0, it follows that A is integral over A_0, so that the integral closure of A_0 is the integral closure of A which is A, since it is integrally closed. □

2. Assume for contradiction there is some $b \in B \setminus A$, such that b is integral over A, then there exist $a_1, \ldots, a_n$, such that

$$\begin{aligned} & b^n + a_1 b^{n-1} + \cdots + a_n = 0 \\ \iff & b^n + a_1 b^{n-1} + \ldots b a_{n-1} = -a_n \in A \\ \implies & b^{n-1} + a_1 b^{n-2} + \ldots + a_{n-1} = a_1' \in A \text{ (by closure of } B \setminus A) \\ \implies & b^{n-1} + a_1 b^{n-2} + \ldots + (a_{n-1} - a_1') = 0 \end{aligned}$$

continuing this process recursively, we find that $b \in A$ which is the desired contradiction. □

3. The closure of a set is the intersection of all closed sets containing it, thus it will suffice to show that any Zariski closed set containing $\mathbb{Z}^n$ is the entire space $\mathbb{A}^n$. Let V be a zariski closed set containing $\mathbb{Z}^n$, then by definition, $V = V(I)$ for some $I \subset \mathbb{C}[X_1, \ldots, X_n]$. It will suffice to show that any polynomial $f \in I$ is the zero polynomial. Let $f \in I$, then f vanishes on $\mathbb{Z}^n$, if $n = 1$, then we are done since any nonzero polynomial in $\mathbb{C}[X]$ has finitely many roots. Now assume for $k < n$ that any $f \in \mathbb{C}[X_1, X_2, \ldots, X_k]$ vanishing on $\mathbb{Z}^k$ is the zero polynomial. Since $f(a_1, \ldots, a_{n-1}, X_n)$ has infinitely many roots for any $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$, we find that $f(a_1, \ldots, a_{n-1}, X_n) \equiv 0$ for any such point in $\mathbb{Z}^{n-1}$, in particular, we may write

$$f = X_n^m g_0(X_1, \ldots, X_{n-1}) + X_n^{m-1} g_1(X_1, \ldots, X_{n-1}) + \cdots + g_m(X_1, \ldots, X_{n-1})$$

so that each g_i is zero on $\mathbb{Z}^{n-1}$, by the inductive hypothesis we find that each $g_i = 0$, and hence $f = 0$. □

4. Any finite set of points is compact, since for any open cover we can choose an open set containing each point to furnish a subcover with at most as many open sets as points. Conversely, consider the variety $X \subset \mathbb{C}^n$ and suppose that X has infinitely many points, now define $I = I(X)$. By Noether's normalization we have that $\mathbb{C}[X_1, \ldots, X_n]/I$ is integral over $\mathbb{C}[f_1, \ldots, f_r]$ where the f_i are algebraically independent, there are two cases.

Case $\mathbf{r \geq 1}$. Let $\varphi : X \to \mathbb{A}^r$ be defined as $\varphi : \mathbf{x} \mapsto (f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}))$, then φ is continuous since each f_i is a polynomial function, moreover, φ is onto. As proof, let $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{A}^r$, then the ideal $J_a = (f_1 - a_1, \ldots, f_r - a_r)$ is maximal since $\mathbb{C}[f_1, \ldots, f_r]/J_a \cong \mathbb{C}$ is a field. By the going up theorem, there is some maximal $\mathfrak{m}_a \in \mathbb{C}[X_1, \ldots, X_n]/I$, such that $\mathfrak{m}_\mathfrak{a} \cap \mathbb{C}[f_1, \ldots, f_r] = J_a$. It follows (by maximality) that $V(\mathfrak{m}_a) = \mathbf{x} \in \mathbb{A}^n$ is a point. Now since $\mathfrak{m}_a \supset J_a$ we have $\mathbf{x} = V(\mathfrak{m}_a) \subset V(J_a)$ and hence J_a vanishes on $\mathbf{x}$, i.e. $f_i(\mathbf{x}) - a_i = 0$ for all i, hence $\varphi(\mathbf{x}) = \mathbf{a}$ proving that φ is onto. Since $\mathbb{A}^r$ unbounded, by the Heine Borel theorem it is not compact, the continuous image of a compact set is compact (see below) so that X was not compact.

Case $\mathbf{r = 0}$. It will suffice to show this case cannot happen. If $r = 0$, then $\mathbb{C}[X_1, \ldots, X_n]/I$ is integral over $\mathbb{C}$, it follows that for each X_i, there is some monic polynomial g_i with coefficients in $\mathbb{C}$, such that $g_i(X_i) \in I$, since $\mathbb{C}$ is algebraically closed we may factor each $g_i = \prod_1^{N_i}(X_i - a_j^i)$, it follows that for any $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{A}^n$ we must have $g_i(\mathbf{b}) = g_i(b_i) = 0$, which is only possible if $b_i \in \{a_1^i, \ldots, a_{N_i}^i\}$ for each i. Hence $V(I) = X \subset \prod_{i=1}^n \{a_j^i\}_{j=1}^{N_i}$ so that $\#X \leq \prod_1^n N_i < \infty$, contradicting X being an infinite set. $\square$

Proof That Continuous Image of A Compact Set is Compact. Let K be compact, and f continuous, assume that $\{U_\alpha\}_{\alpha \in A}$ is an open cover for $f(K)$, then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for K, hence admits a finite subcover $\{f^{-1}(U_i)\}_{i=1}^n$, since $K \subset \bigcup_1^n f^{-1}(U_i)$ we have $f(K) \subset f(\bigcup_1^n f^{-1}(U_i)) = \bigcup_1^n U_i$, so that $\{U_i\}_1^n$ is a finite subcover of $f(K)$. $\square$

5. (a) First suppose that $a \in X_1 \cup X_2$, then for any $f \in J_1 J_2$, we can write $f = f_1 f_2$, where $f_i \in J_i$. If $a \in X_1$, then $f_1(a) = 0$ and hence $f(a) = 0$, otherwise, we find that $a \in X_2$, so that $0 = f_2(a) = f(a)$, since f was arbitrary we find that $a \in V(J_1 J_2)$.

Conversely, let $a \in V(J_1 J_2)$, if $a \in X_1$ then we are done, so assume not. Then there is some $f \in J_1$, such that $f(a) \neq 0$, then for any $g \in J_2$ we have $fg \in J_1 J_2$, implying that $f(a)g(a) = 0$, since $f(a)g(a) \in k$ is a field and $f(a) \neq 0$ we conclude that $g(a) = 0$, since this holds for any $g \in J_2$ this proves that $a \in X_2$. $\square$

(b) Since X, Y are algebraic varieties, we may write $X = V(J_X), Y = V(J_Y)$

$$V(J_X) \cap V(J_Y) = V(J_X + J_Y) \implies I(V(J_X) \cap V(J_Y)) = IV(J_X + J_Y) = \sqrt{J_X + J_Y}$$

and since $V(J_X) \cap V(J_Y) = \emptyset$, we find that $1 \in k[X_1, X_2, \ldots, X_n] = I(V(J_X) \cap V(J_Y))$. $1 \in \sqrt{J_X + J_Y}$ it is immediate from definition of radical ideal that this implies $1 \in J_X + J_Y$, so that there is some $f \in J_X$ and $g \in J_Y$, such that $f + g = 1$, it follows that f is the desired polynomial, since $f(x) = 0$ for any $x \in X$ by assumption and $f(y) = 1 - g(y) = 1$ for any $y \in Y$, since $g \in J_Y$ implies that $g(y) = 0$. $\square$

6. No, assume it is the case, then Y satisfies a monic polynomial $Y^n + \sum_1^n f_{n-i}(X) Y^i = 0$ over A. Consider

$$\begin{aligned} \varphi : \mathbb{C}[X, Y] &\to \mathbb{C}[X, \frac{1}{X^2+1}] \\ X &\mapsto X \\ Y &\mapsto \frac{1}{X^2+1} \end{aligned}$$

such a homomorphism exists by the universal property for polynomial rings, moreover $I = (X^2Y + Y - 1) \subset \ker\varphi$. By the first isomorphism theorem, this induces a homomorphism $\overline{\varphi} : B \to \mathbb{C}[X, \frac{1}{X^2+1}]$. It follows that

$$\overline{\varphi}(Y^n + \sum_1^n f_{n-i}(X)Y^i) = 0$$
$$\iff \frac{1}{X^2+1} = \sum_0^{n-1}(X^2+1)^i f_i$$

To see this is a contradiction, note that atleast one $f_i \neq 0$, so that

$$0 = \deg 1 = \deg(X^2+1)\sum_0^{n-1}(X^2+1)^i f_i \geq \deg(X^2+1) = 2 \quad \square$$