1. It suffices to show that $\operatorname{Spec}(A)$ satisfies the finite intersection property (FIP) for closed sets, since if given any collection of closed sets $\{V_i\}_{i\in I}$ we have

$$\bigcap_{i\in I} V_i = \emptyset \implies \exists V_{i_1},\dots,V_{i_N}, \text{ such that } \bigcap_{j=1}^{N} V_{i_j} = \emptyset$$

Then given any collection $\{U_i\}_{i\in I}$ of open sets we have

$$\begin{aligned}\bigcup_{i\in I} U_i = \operatorname{Spec}(A) \iff \bigcap_{i\in I} U_i^c = \emptyset \implies \exists U_{i_1}^c,\dots,U_{i_N}^c, \text{ such that } \bigcap_{j=1}^{N} U_{i_j}^c = \emptyset \\ \iff \left(\bigcup_{j=1}^{N} U_{i_j}\right)^c = \emptyset \iff \bigcup_{j=1}^{N} U_{i_j} = \operatorname{Spec}(A)\end{aligned}$$

Now let $\{V_i\}_{i\in I}$ be a collection of closed sets, such that $\bigcap_{i\in I} V_i = \emptyset$, then by the characterization of Zariski closed sets, $V_i = V(S_i)$, for some $S_i \subset A$, and $\emptyset = \bigcap_{i\in I} V_i = V(\bigcup_{i\in I} S_i)$. This suffices to show that $\langle \bigcup_{i\in I} S_i \rangle = A$, since if $\langle \bigcup_{i\in I} S_i \rangle$ were a proper ideal of A, then there would exist some maximal ideal $\mathfrak{m} \supset \langle \bigcup_{i\in I} S_i \rangle$, and since maximal ideals are prime we would have $\mathfrak{m} \in V(\bigcup_{i\in I} S_i)$ which is impossible since it is empty. Since $\langle \bigcup_{i\in I} S_i \rangle = A$, there exist $\{s_k\}_{k=1}^n \subset \bigcup_{i\in I} S_i$ and $\{a_k\}_{k=1}^n \subset A$, such that $\sum_{k=1}^n a_k s_k = 1$, each s_k lies in some S_{i_k} which implies that $\left\langle \bigcup_{k=1}^N S_{i_k} \right\rangle = A$, in particular

$$\emptyset = V(A) = V\left(\bigcup_{k=1}^{N} S_{i_k}\right) = \bigcap_{k=1}^{N} V_{i_k}$$

This suffices to show that $\operatorname{Spec}(A)$ satisfies the FIP and is hence quasi-compact. $\square$

2. First suppose that $\operatorname{Nil}(A)$ is prime, and let $V(S_1), V(S_2)$ be Zariski closed sets, such that $V(S_1) \cup V(S_2) = \operatorname{Spec}(A)$, then since $\operatorname{Nil}(A)$ is prime it must be contained in one of the two closed sets, without loss of generality assume that $\operatorname{Nil}(A) \subset V(S_1)$, then

$$S_1 \subset \operatorname{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

Implying that $P \in V(S_1)$ for all prime ideals $P \subset A$, but this is equivalent to $V(S_1) = \operatorname{Spec}(A)$, since $V(S_1), V(S_2)$ were arbitrary this suffices to show that $\operatorname{Spec}(A)$ is irreducible.

I will prove the converse using the contrapositive. Assume that $\operatorname{Nil}(A)$ is not prime, then there are $x, y \in A$, such that $x, y \notin \operatorname{Nil}(A)$, and $xy \in \operatorname{Nil}(A)$. It follows that

$$V((x)) \cup V((y)) = V((xy)) \supset V(\operatorname{Nil}(A)) = \operatorname{Spec}(A)$$

where $V(\operatorname{Nil}(A)) = \operatorname{Spec}(A)$ is proven in the previous part of the problem. So it will suffice to show that $V((x)), V((y)) \subsetneq \operatorname{Spec}(A)$ to conclude that $\operatorname{Spec}(A)$ is irreducible. Since

$$x, y \notin \operatorname{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

there are prime ideals $x \notin P_x, y \notin P_y$, so that $P_x \notin V((x)), P_y \notin V((y))$ hence neither can be all of $\operatorname{Spec}(A)$. $\square$