1. Any field isomorphism must fix the base field, in this case $\mathbb{Q}$, so that $\sqrt{2} \mapsto a + b\sqrt{3},\ b \neq 0$ is necessitated by injectivity. If τ is such a map, then

$$2 = \tau(2) = \tau(\sqrt{2})\tau(\sqrt{2}) = (a + b\sqrt{3})^2 = 3b^2 + 2ab\sqrt{3} + a^2$$

since $\sqrt{3}$ is linearly independent of 1 and $b \neq 0$ we must have $a = 0$, hence $2/3 = b^2$. We may write $b = s/t,\ s, t \in \mathbb{Z}$ coprime, equivalently $2t^2 = 3s^2$, so $2|s^2$ implies $2|s$, so that $4|2t^2$, implies $2|t^2$ implies $2|t$, contradicting s, t being coprime.

Finite dimensional vector spaces are isomorphic when they have the same dimension, $\sqrt{2}$ and 1 are linearly independent in a $\mathbb{Q}$ vector space since $\sqrt{2}$ is irrational (similarly $\sqrt{3}$ and 1 are linearly independent). To see that $(1, \sqrt{2})$ and $(1, \sqrt{3})$ are bases respectively, we use algebraicity of $\sqrt{2}$, $\mathbb{Q}(\sqrt{2}) \simeq \mathbb{Q}[\sqrt{2}] \simeq \mathbb{Q}/(x^2 - 2)$, so by polynomial long division any element can be written as $ax + b \mapsto a\sqrt{2} + b$, for $a, b \in \mathbb{Q}$, hence this is a basis, and the argument is the same for $\mathbb{Q}(\sqrt{3})$.

2. To see that K/L is algebraic, it will suffice to show that t is algebraic over L (hence all elements, due to sums products and innverses preserving algebraicity). Consider the polynomial $g(x)u - f(x)$ in $L[x]$, evaluating at t gives

$$g(t)u - f(t) = g(t)\frac{f(t)}{g(t)} - f(t) = f(t) - f(t) = 0$$

Hence t is algebraic over L, implying that K is algebraic over L. If L/F were algebraic, then by transitivity K/F would be algebraic but $t \in K$ is transcendental over F, so by contrapositive L/F is algebraic. Now we prove that $[K : L] = \max(\deg(f), \deg(g))$, it will suffice to show $\min(t, L) = g(x)u - f(x)$, i.e. that this polynomial is irreducible. We can apply Gauss' lemma, so that it will suffice to show $g(x)u - f(x) \in F[u][x]$ is irreducible. So write $p(u, x)q(u, x) = g(x)u - f(x)$, one of p, q has degree one in u so assume its q (hence $\deg_u(p) = 0$), then $q(u, x) = uh(x) + r(x)$. It follows that $uh(x)p(x) + r(x)p(x) = 1$, hence $p(x)|f(x), g(x)$, implying $p(x) = 1$.

3. Let $u, v \in K,\ r, s \in F$, then since multiplication is distributive and commputative in a field,

$$L_\alpha(ru + sv) = \alpha(ru + sv) = \alpha ru + \alpha sv = r\alpha u + s\alpha v = rL_\alpha(u) + sL_\alpha(v)$$

Since K is finite it is algebraic. Let $m(a) := \min(\alpha, F) = c_n x^n + \cdots + c_0$, then $1, a, \ldots, a^{n-1}$ are a basis for K as a $F-$vectorspace. It follows that $L_a : a^k \mapsto a^{k+1},\ 0 \leq k \leq n-2$, and $a^{n-1} \overset{L_a}{\mapsto} a^n = -c_{n-1}a^{n-1} - \cdots - c_0$. Writing L_a in our a^k basis,

$$L_a = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

Now assume for induction that $\det(L_{a_{k\times k}} - I_{k\times k}x) = x^k + x^{k-1}c_{n-1} + \cdots + c_{n-k}$ (referring to the lower right $k \times k$ submatrix). This is clear for $k = 1$, then

$$\begin{aligned} \det(L_{a_{k+1\times k+1}} - I_{k+1\times k+1}x) &= x\det(L_{a_{k\times k}} - I_{k\times k}x) + (-1)^{k+1}c_{n-k-1}\det(-I_{k+1\times k+1}) \\ &= x^{k+1} + x^k c_{n-1} + \cdots + xc_{n-k} + c_{n-k-1} \end{aligned}$$

The only elements having $\det(Ix - L_\alpha) = \min(\alpha, F)$ are α such that $F(a) = F(\alpha)$ i.e. $\deg(\min(\alpha, F)) = [F(a) : F]$, the above proves $\det(Ix - L_\alpha) = \min(\alpha, F)$ when $F(\alpha) = F(a)$ (just write L_α in α^k basis). For the converse notice that $\deg\det(Ix - L_\alpha) = [F(a) : F]$, so the degree is too large to be the minimum polynomial of α with minimum polynomial of smaller degree.

4. We have the tower of extensions $F(a)/F(a^2)/F$, since a satisfies $x^2 - a^2$ in $F(a^2)[x]$, it is either in $F(a^2)$, or is algebraic of degree 2. Assume the latter, then by multiplicativity of degree, $F(a)/F$ is even, hence by contrapositive $a \in F(a^2)$.

5. It will suffice to show R has no non-trivial proper ideals, first note R is a domain, since K does not have 0 divisors and $R \subset K$. Consider an ideal $0 \neq I \subset R$, if $I \cap F \neq 0$, then $1 \in I = R$, since any non-zero element of F is invertible, so that $k \in I \cap F$ implies that $1 \in kk^{-1} \in IF \subset IR = I$. Otherwise if $I \cap F = 0$, then for some $\alpha \in K \setminus F$, we have $\alpha \in I$, α is algebraic, so has a minimum polynomial in F, $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_0 \neq 0$. It follows that $-a_0^{-1}(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha) = 1 \in I = R$.

To show that $a_0 \neq 0$, assume it were, then take the smallest k, such that $a_k \neq 0$. It follows that $\alpha^k(\alpha^{n-k} + a_{n-1}\alpha^{n-k-1} + \cdots + a_k) = 0$, hence R is a domain implies that $\alpha^k = 0$ or $(\alpha^{n-k} + a_{n-1}\alpha^{n-k-1} + \cdots + a_k) = 0$, contradicting $\min(\alpha, F) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$