1. Let F^{alg} be an algebraic closure of F containing both K and L. Now let σ an extension of an embedding $F \to F^{\text{alg}}$ to KL. Then for any $x \in KL, x = \frac{\sum_1^n k_i\ell_i}{\sum_1^m k_j'\ell_j'}$. For $k_i, k_j' \in K$ and $\ell_i, \ell_j' \in L$. Then by the homomorphism property,

$$\sigma(x) = \sigma(\frac{\sum_1^n k_i\ell_i}{\sum_1^m k_j'\ell_j'}) = \frac{\sum_1^n \sigma(k_i)\sigma(\ell_i)}{\sum_1^m \sigma(k_j')\sigma(\ell_j')} \in \sigma(K)\sigma(L)$$

Furthermore, any $y \in \sigma(K)\sigma(L)$ is of the form $\frac{\sum_1^n \sigma(k_i)\sigma(\ell_i)}{\sum_1^m \sigma(k_j')\sigma(\ell_j')}$, so the σ is onto with range $\sigma(K)\sigma(L)$. In other words we have proven $\sigma(KL) = \sigma(K)\sigma(L)$. But since K, L are normal we have $K = \sigma(K)$ and $L = \sigma(L)$.

The converse is **not** true, since taking $F = \mathbb{Q}, K = \mathbb{Q}(\sqrt[3]{2}), L = \mathbb{Q}(\zeta_3)$ we have $KL = \mathbb{Q}(\sqrt[3]{2}, \zeta)$ is normal over $\mathbb{Q}$ (it is the splitting field of $x^3 - 2$ as shown in the previosu homework), whereas K is not normal over $\mathbb{Q}$ (Take the extension of the identity on $\mathbb{Q}$ mapping $\sqrt[3]{2} \mapsto \zeta_3\sqrt[3]{2}$ which is not in $\text{Aut}K$).

2. Consider the polynomial $f := x^k - n$ in $\mathbf{F_p}$. Since $k = 2$ or 3, f either has a factor of degree one, hence a root or f is irreducible over $\mathbf{F_p}$. If f has a root we are done, so assume not. Then we can consider the extension $\mathbf{F_p}(\alpha)$, where α is a root of f. This extension has degree k, hence $\mathbf{F_p}(\alpha)$ is a finite field of cardinality p^k. Since all finite fields of equal cardinality p^k are $\mathbf{F_p}$ isomorphic, we have an isomorphism $\sigma : \mathbf{F_p} \to F$, so that $0 = \sigma(0) = \sigma(f(\alpha)) = f({}^\sigma\alpha)$ so that ${}^\sigma\alpha \in F$ is a root of $x^k - n$, hence ${}^\sigma\alpha$ satisfies the equation ${}^\sigma\alpha^k = n$.

3.

1. **True** Firstly, F is finite, hence perfect and since K is a finite field, K/F is a finite extension so it is algebraic over a perfect field, hence seperable. Secondly, since K, F are finite, they must have characteristic p. It follows that $\#K = p^n, \#F = p^m, n \geq m$. Then we have that K is the splitting field of $x^{p^n} - x$, so that $K/\mathbf{F_p}$ is normal. Then any extension of a map from $\mathbf{F_p}$ to an algebraic closure extends to an automorphism of K, and since any extension of a map σ from F into an algebraic closure to K is just an extension of $\sigma|_{\mathbf{F_p}}$ to K this extension must also be in $\text{Aut}K$, proving that K/F is normal. This proves the extension is Galois.

2. **True** First let u denote $t^4 + t^{-4}$, then K is the splitting field of the polynomial $f = X^8 - uX^4 + 1$ in $F[X]$. This is easily seen, since $t \in \{t\zeta_4^n, t^{-1}\zeta_4^n\}_{n=1}^4 \subset \mathbb{C}(t)$ are the roots of f.

3. **False** Since K/F is finite, it is algebraic. K/F admits a normal closure $M/K/F$, so that M/F is normal.

4. **False** Consider $K = \mathbf{F_3}$, then $f(K) = \{0, 1\} \neq K$

5. **True** For every element x of K not equal to zero, $p \neq 2 \implies 2x \neq 0 \implies x \neq -x$, so that $f(x) = f(-x)$. Now we need only show that $y \notin \{x, -x\}$ implies that $f(y) \neq f(x)$. As proof note that the polynomial $T^2 - x^2 \in K[T]$ can have at most two roots in K by the factor theorem. Hence f is 1-1 on 0 and 2-1 on each other element. This implies that $\#f(K) = 1 + \frac{\#K-1}{2} = \frac{p^n+1}{2}$.

6. **True** $\mathbb{Q}(S)$ is the splitting field of the collection of polynomials $\{x^2 - p\}_{p \text{ prime}}$, so $K/\mathbb{Q}$ is normal. Furthermore, $\mathbb{Q}$ has characteristic zero, so is perfect hence $K/\mathbb{Q}$ is seperable making it Galois.