1. a Let $x \in A_1^\circ \cup A_2^\circ$, then without loss of generality it will suffice to show $A_1^\circ \subset (A_1 \cup A_2)^\circ$. We have that $A_1^\circ \subset A_1 \subset A_1 \cup A_2$, which is open, then

$$(A_1 \cup A_2)^\circ \subset (A_1 \cup A_2)^\circ \cup A_1^\circ \subset A_1 \cup A_2$$

is open, and since $(A_1 \cup A_2)^\circ$ is the largest open set contained in $A_1 \cup A_2$, it must be the case that

$$(A_1 \cup A_2)^\circ = (A_1 \cup A_2)^\circ \cup A_1^\circ$$

so that $A_1^\circ \subset (A_1 \cup A_2)^\circ$.

To show that equality need not hold, consider the usual (metric) topology on $\mathbb{R}$, with $A_1 = [-1, 0], A_2 = [0, 1]$, then

$$A_1^\circ \cup A_2^\circ = (-1, 0) \cup (0, 1) \subsetneq (-1, 1) = (A_1 \cup A_2)^\circ$$

(b) Once again, it will suffice to show that $\overline{A_1} \supset \overline{A_1 \cap A_2}$, then $\overline{A_2} \supset \overline{A_1 \cap A_2} \implies \overline{A_1} \cap \overline{A_2} \supset \overline{A_1 \cap A_2}$ will follow by symmetry. Note that $\overline{A_1} \supset A_1 \supset A_1 \cap A_2$ is closed, so that in particular

$$\overline{A_1 \cap A_2} \supset \overline{A_1} \cap \overline{A_1 \cap A_2} \supset A_1 \cap (A_1 \cap A_2) = A_1 \cap A_2$$

is closed, implying that since $\overline{A_1 \cap A_2}$ is smallest closed set containing $A_1 \cap A_2$, we must have $\overline{A_1 \cap A_2} = \overline{A_1} \cap \overline{A_1 \cap A_2}$ which, in particular, implies that $\overline{A_1 \cap A_2} \subset \overline{A_1}$.

To show equality need not hold, consider the usual (metric) topology on $\mathbb{R}$, with $A_1 = (-1, 0), A_2 = (0, 1)$, then

$$\overline{A_1} \cap \overline{A_2} = \{0\} \supsetneq \emptyset = \overline{A_1 \cap A_2}$$

2. An immediately equivalent condition to nowhere density is that the closure contains no (non-empty) open sets. Furthermore, notice that if A is nowhere dense, then so is $\overline{A}$ which follows from $\overline{A} = \overline{\overline{A}}$.

With the above in mind, let U be open, then $V = U \setminus \overline{A} = U \cap \overline{A}^c$ is nonempty by nowhere density of A (and hence $\overline{A}$), and is furthermore the intersection of two open sets thus open. It follows that $V \setminus \overline{B} = V \cap \overline{B}^c \neq \emptyset$ by nowhere density of B. This implies that

$$\begin{aligned} U \setminus \overline{A \cup B} = U \cap (\overline{A} \cup \overline{B})^c = (U \cap \overline{A}^c) \cap \overline{B}^c = V \cap \overline{B}^c \neq \emptyset \\ \implies U \not\subset \overline{A \cup B} \end{aligned}$$

Since U was arbitrary, we can conclude that $\overline{A \cup B}$ contains no non-empty open sets and is thus nowhere dense.

3. $\emptyset, X \in \mathcal{T}(\mathcal{E})$ is immediate. Now let $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}(\mathcal{E})$, if each U_α is empty, then we are done, otherwise if $X = U_{\alpha'}$ for some α' we have

$$\bigcup_{\alpha \in A} U_\alpha = X \bigcup_{\alpha \in A \setminus \alpha'} U_\alpha = X \in \mathcal{T}(\mathcal{E})$$

Now we may assume without loss of generality that $X \neq U_\alpha, \forall \alpha$, and that $U_\alpha \neq \emptyset \forall \alpha$, since this doesn't affect the union, then each of the sets in the union $\bigcup_{\alpha \in A} U_\alpha$ are unions of finite intersections of sets in $\mathcal{E}'$, so we can rewrite

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{\beta \in B_\alpha} U_{\alpha,\beta} = \bigcup_{(\alpha,\beta) \in S} U_{\alpha,\beta}$$

where $U_\alpha = \bigcup_{\beta \in B_\alpha} U_{\alpha,\beta}$, and each $U_{\alpha,\beta} \in \mathcal{E}'$. And where S is defined as $\{(\alpha, \beta) | \alpha \in A, \beta \in B_\alpha\}$ which suffices to show that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}(\mathcal{E})$ by definition of $\mathcal{T}(\mathcal{E})$.