1. It suffices to show that Spec(A) satisfies the finite intersection property (FIP) for closed sets, since if given any collection of closed sets $\{V_i\}_{i\in I}$ we have

$$\bigcap_{i\in I} V_i = \emptyset \implies \exists V_{i_1},\dots,V_{i_N},\ \text{such that } \bigcap_{j=1}^{N} V_{i_j} = \emptyset$$

Then given any collection $\{U_i\}_{i\in I}$ of open sets we have

$$\bigcup_{i\in I} U_i = \text{Spec}(A) \iff \bigcap_{i\in I} U_i^c = \emptyset \implies \exists U_{i_1}^c,\dots,U_{i_N}^c,\ \text{such that } \bigcap_{j=1}^{N} U_{i_j}^c = \emptyset$$
$$\iff \left(\bigcup_{j=1}^{N} U_{i_j}\right)^c = \emptyset \iff \bigcup_{j=1}^{N} U_{i_j} = \text{Spec}(A)$$

Now let $\{V_i\}_{i\in I}$ be a collection of closed sets, such that $\bigcap_{i\in I} V_i = \emptyset$, then by the characterization of Zariski closed sets, $V_i = V(S_i)$, for some $S_i \subset A$, and $\emptyset = \bigcap_{i\in I} V_i = V(\bigcup_{i\in I} S_i)$. This suffices to show that $\langle \bigcup_{i\in I} S_i\rangle = A$, since if $\langle \bigcup_{i\in I} S_i\rangle$ were a proper ideal of A, then there would exist some maximal ideal $\mathfrak{m} \supset \langle \bigcup_{i\in I} S_i\rangle$, and since maximal ideals are prime we would have $\mathfrak{m} \in V(\bigcup_{i\in I} S_i)$ which is impossible since it is empty. Since $\langle \bigcup_{i\in I} S_i\rangle = A$, there exist $\{s_k\}_{k=1}^n \subset \bigcup_{i\in I} S_i$ and $\{a_k\}_{k=1}^n \subset A$, such that $\sum_{k=1}^n a_k s_k = 1$, each s_k lies in some S_{i_k} which implies that $\left\langle \bigcup_{k=1}^N S_{i_k}\right\rangle = A$, in particular

$$\emptyset = V(A) = V\left(\bigcup_{k=1}^{N} S_{i_k}\right) = \bigcap_{k=1}^{N} V_{i_k}$$

This suffices to show that Spec(A) satisfies the FIP and is hence quasi-compact. □

2. First suppose that Nil(A) is prime, and let $V(S_1), V(S_2)$ be Zariski closed sets, such that $V(S_1) \cup V(S_2) = \text{Spec}(A)$, then since Nil$(A)$ is prime it must be contained in one of the two closed sets, without loss of generality assume that Nil$(A) \subset V(S_1)$, then

$$S_1 \subset \text{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

Implying that $P \in V(S_1)$ for all prime ideals $P \subset A$, but this is equivalent to $V(S_1) = \text{Spec}(A)$, since $V(S_1), V(S_2)$ were arbitrary this suffices to show that Spec(A) is irreducible.

I will prove the converse using the contrapositive. Assume that Nil(A) is not prime, then there are $x, y \in A$, such that $x, y \notin \text{Nil}(A)$, and $xy \in \text{Nil}(A)$. It follows that

$$V((x)) \cup V((y)) = V((xy)) \supset V(\text{Nil}(A)) = \text{Spec}(A)$$

where $V(\text{Nil}(A)) = \text{Spec}(A)$ is proven in the previous part of the problem. So it will suffice to show that $V((x)), V((y)) \subsetneq \text{Spec}(A)$ to conclude that Spec(A) is irreducible. Since

$$x, y \notin \text{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

there are prime ideals $x \notin P_x, y \notin P_y$, so that $P_x \notin V((x)), P_y \notin V((y))$ hence neither can be all of Spec(A). □

3. Lemma. *M is finitely generated implies M satisfies the aescending chain condition (ACC).* Assume that $M = \langle x_1, \dots, x_n \rangle$, then for any chain of submodules, $(N_i)_I$ we have $\bigcup_I N_i = M$ implies that $N_j = M$ for some $j \in I$ (and hence all $i \geq j$).

Proof of Lemma. Since $\bigcup_I N_i = M$, for each $k = 1, \dots, n$, there is some N_{i_k}, such that $x_k \in N_{i_k}$, since this is a chain of submodules it is totally ordered, implying that there is some $j \in \{i_1, \dots, i_n\}$, such that $N_{i_k} \subset N_j$ for all $k \in \{1, \dots, n\}$, hence $M = \langle x_1, \dots, x_n \rangle \subset N_j$. $\square$

(a) Let $M \neq 0$ be a finitely generated A-module. We can use the ACC proven in the lemma to apply Zorn's lemma. Consider the set $X := \{N \subsetneq M \mid N \text{ is a submodule}\}$, ordered by inclusion. X contains 0, hence is nonempty. Let $(N_i)_{i \in I}$ be a chain in X, then $\bigcup_{i \in I} N_i \neq M$ by the ACC, to see that $\bigcup_{i \in I} N_i$ is a submodule, let $a, b \in A$, $n_1, n_2 \in \bigcup_{i \in I} N_i$. Then $n_1 \in N_{i_1}, n_2 \in N_{i_2}$, and since it is a chain we may assume without loss of generality $N_{i_2} \subset N_{i_1}$ it follows that $an_1 + bn_2 \in N_{i_1} \subset \bigcup_{i \in I} N_i$, thus proving that $\bigcup_{i \in I} N_i \in X$ is an upper bound for the chain. This satisfies the conditions for Zorn's lemma, so there exists some maximal element $N \in X$, so $N \subsetneq M$ is a proper submodule which is not contained in any other proper submodules. $\square$

(b) Suppose for contradiction that $N \subsetneq \mathbb{Q}$ is maximal, then by the correspondance theorem $\mathbb{Q}/N$ has no proper submodules. Hence for any $0 \neq x \in \mathbb{Q}/N$ (there is always such an x since N is a proper submodule), it must be the case that $\langle x \rangle = \mathbb{Q}/N$, since $\mathbb{Q}/N$ is generated by a single element as a $\mathbb{Z}$ module, it must be a homomorphic image of $\mathbb{Z}$, furthermore since it is generated by any of its elements as a $\mathbb{Z}$ module, it must be a finite cyclic group- in other words $\mathbb{Q}/N \cong \mathbb{Z}/(p)$ for some prime p. Hence by the first isomorphism theorem, we have a surjective $\mathbb{Z}$ module homomorphism $\varphi : \mathbb{Q} \to \mathbb{Z}/(p)$, with $\ker \varphi = N$. I claim that φ is the zero map, contradicting that it is surjective, as proof, let $x \in \mathbb{Q}$, then $\frac{x}{p} \in \mathbb{Q}$, then

$$0 + (p) = p\varphi(\frac{x}{p}) + (p) = \varphi(p\frac{x}{p}) + (p) = \varphi(x) + (p)$$

Since x was arbitrary this shows that $\varphi(x) = 0$ for all $x \in \mathbb{Q}$ $\square$

4. Prop 2.4. Let $A = \mathbb{R}$, $M = \oplus_0^\infty \mathbb{R}$, and $\phi : (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$. Now let $n \in \mathbb{N}$, and $a_1, \dots, a_n \in \mathbb{R}$, since these are arbitrary, it will suffice to show that there is some $\mathbf{x} \in \oplus_0^\infty \mathbb{R}$, such that

$$\mathbf{y} = \phi^n(\mathbf{x}) + a_1\phi^{n-1}(\mathbf{x}) + \cdots + a_{n-1}\phi(\mathbf{x}) + a_n\mathbf{x} \neq 0$$

Choosing $\mathbf{x} = (1, 0, \dots)$ we get

$$\mathbf{y}_n = \mathbf{x}_0 + \sum_{i=1}^{n} a_i\mathbf{x}_i = 1 \implies \mathbf{y} \neq 0$$

Cor 2.5. Consider $A = \mathbb{Z}$, $M = \mathbb{Q}$, then $(2)\mathbb{Q} = \mathbb{Q}$, since for any $x \in \mathbb{Q}$, we have $\frac{x}{2} \in \mathbb{Q}$. However for any $k \in \mathbb{Z}$, we have $0 \neq 1 + 2k \in \mathbb{Q}$, it follows that since $\mathbb{Q}$ is a field, for any $x \in \mathbb{Q} \setminus \{0\}$ we have $(1 + 2k)x \neq 0$, it follows that $(1 + 2k)\mathbb{Q} \neq 0$ for any $k \in \mathbb{Z}$.

Prop 2.6. Consider the local ring $(A_2, \mathfrak{m})$, where $A_2 = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n\}$ and $\mathfrak{m} = \{\frac{m}{n} \in A_2 \mid 2|m\}$. We can consider $\mathbb{Q}$ as an A_2 module, i.e. $A = A_2, M = \mathbb{Q}$. Then $\text{Jac}(A)M = \mathfrak{m}M = M$, since for any $x \in \mathbb{Q}$, we have $\frac{x}{2} \in \mathbb{Q}$ and $2 \in \mathfrak{m}$. However, $\mathbb{Q} \neq 0$.

5.

$$M' \xrightarrow{\mu} M \xrightarrow{\nu} M'' \to 0 \tag{1}$$

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{\nu^*} \mathrm{Hom}(M, N) \xrightarrow{\mu^*} \mathrm{Hom}(M', N) \tag{2}$$

$$0 \to N' \xrightarrow{\mu} N \xrightarrow{\nu} N'' \tag{3}$$

$$0 \to \mathrm{Hom}(M, N') \xrightarrow{\mu_*} \mathrm{Hom}(M, N) \xrightarrow{\nu_*} \mathrm{Hom}(M, N'') \tag{4}$$

(1) is exact $\implies$ (2) is exact for all A-Modules N: Let N be an A-module, $\nu^* f = 0 \iff f \circ \nu = 0$, but since ν is surjective by exactness of (1), this implies that $f = 0$ and hence ν^* is injective. Now let $f \in \mathrm{Im}\nu^*$, then $f = g\nu$, so that $\mu^* f = g\nu\mu = g \circ 0 = 0$, since $\ker \nu = \mathrm{Im}\mu$ from exactness of (1), this suffices to show that $\mathrm{Im}\nu^* \subset \ker \mu^*$. Now let $f \in \ker \mu^*$, then $f\mu = 0$, so $\mathrm{Im}\mu = \ker \nu \subset \ker f$, this implies by the first isomorphism theorem that f factors through $\pi : M \to M/\ker \nu$, i.e. $\exists h : M/\ker \nu \to N$, such that $f = h\pi$. By the first isomorphism theorem, there is an isomorphism $s : M/\ker \nu \to M''$, such that $s\pi = \nu$, equivalently $\pi = s^{-1}\nu$, this implies that $f = hs^{-1}\nu = \nu^*(hs^{-1}) \in \mathrm{Im}\nu^*$, hence $\ker \mu^* \subset \mathrm{Im}\nu^*$, so that they are in fact equal, and exactness is proven. □

(2) is exact for all A-Modules N $\implies$ (1) is exact: To show that ν is surjective, let $N = M''/\nu(M)$, we have the quotient map $\pi : M'' \to M''/\nu(M)$, $\pi \in \mathrm{Hom}(M'', N)$, it is immediate that $\nu^*\pi = 0$, which by exactness of (2) means that $\pi = 0$ implying that $M'' = \nu(M)$, so that ν is surjective. To show that $\mathrm{Im}\mu \subset \ker \nu$, let $N = M''$, then $1_{M''} \in \mathrm{Hom}(M'', N)$, by exactness of (2) we have $0 = \mu^*(\nu^* 1_{M''}) = 1_{M''}\nu\mu$, then since $1_{M''}$ is an isomorphism, this implies that $\nu\mu(M') = 0$, and hence $\mathrm{Im}\mu \subset \ker \nu$. To show the opposite inclusion, take $N = M/\mu(M')$, then we have the quotient map $\pi : M \to M/\mu(M')$, $\pi \in \mathrm{Hom}(M, N)$ and $\pi \in \ker \mu^*$, hence $\pi \in \mathrm{Im}\nu^*$ by exactness of (2), so for some $f : M'' \to N$, $\pi = \nu^* f = f\nu$. For any $m \in M$, $\nu(m) = 0 \implies f\nu(m) = f(0) = 0$, hence $\ker \nu \subset \ker f\nu = \mathrm{Im}\mu$, this suffices to show exactness of (1). □

(3) is exact $\implies$ (4) is exact for all A-Modules M: Let M be an arbitrary A module. To see that μ_* is injective, let $f, g \in \mathrm{Hom}(M, N')$ and suppose that $\mu_* f = \mu_* g$, then for any $m \in M$, $\mu f(m) = \mu g(m)$, by exactness of (3), μ is injective so that $f(m) = g(m)$ but then since m was arbitrary $f = g$ proving injectivity of μ_*. Suppose that $f \in \mathrm{Im}\mu_*$, then $f = \mu_* g$, $g \in \mathrm{Hom}(M, N')$. It follows that $\nu_* f = \nu_*\mu_* g = \nu\mu g = 0 \circ g = 0$, so that $\mathrm{Im}\mu_* \subset \ker \nu_*$. Now let $f \in \ker \nu_*$, then $f(M) \subset \ker \nu = \mu(N')$ by exactness of (3), futhermore we know that μ is injective so in particular (taking μ' to be μ with restricted codomain) $\mu' : N' \to \mu(N')$ is inverible, then $f = \mu\mu'^{-1} f = \mu_*(\mu'^{-1} f)$, so that $f \in \mathrm{Im}\mu_*$ which gives us $\ker \nu_* \subset \mathrm{Im}\mu_*$ so that (4) is exact.

(4) is exact for all A-Modules M $\implies$ (3) is exact: We first show $\ker \mu = 0$, let $m \in N'$, and let $M = A$, then consider the map $f : a \to am$, $f \in \mathrm{Hom}(M, N')$, we have that $\mu_* f = 0 \iff \mu_* f(1) = 0 \iff \mu(m) = 0$ then by exactness of (4), $\mu_* f = 0 \iff f = 0 \iff f(1) = 0 \iff m = 0$, so taken together $\mu(m) = 0 \iff m = 0$, since m was chosen arbitrarily this suffices to show that μ is injective. Now let $M = N'$, so that $1_{N'} \in \mathrm{Hom}(M, N')$, then exactness of (4) implies that $\nu_*\mu_* 1_{N'} = \nu\mu = 0$, hence $\mathrm{Im}\mu \subset \ker \nu$. If μ is surjective, then $\ker \nu \subset \mathrm{Im}\mu$ is trivial, so assume not. Let $m \in N \setminus \mu(N')$, then let $M = A$, then the map $f : a \mapsto am$ is such that $f \in \mathrm{Hom}(M, N)$. Then $m \in f(M) \setminus \mu(N')$ implies that $f(M) \not\subset \mu(N')$, so that $f \notin \mathrm{Im}(\mu_*)$, and hence by exactness of (4), $f \notin \ker \nu_*$, it follows that $\nu_* f \neq 0$, and since $\nu_* f(a) = a\nu f(1)$, for any $a \in A$ this implies that $0 \neq \nu^* f(1) = \nu(m)$, hence by choice of m, we have $m \notin \mathrm{Im}\mu \implies m \notin \ker\nu$, contraposing gives the desired result $\ker\nu \subset \mathrm{Im}\mu$ which suffices to show that (3) is exact. $\square$

6. In this problem I will denote $e_\ell \in A^k$ to have ℓ-th coordinate 1, and all other coordinates 0.

(a) Suppose for contradiction that $n > m$, and $f : A^m \to A^n$ is surjective, then consider the module homomorphism

$$\pi : A^n \to A^m, \begin{cases} e_i \mapsto e_i & i \leq m \\ e_i \mapsto 0 & m < i \leq n \end{cases}$$

It is immediate that $\pi \circ f : A^m \to A^m$ is surjective, hence by the second corollary in Nakayama's lemma it must be injective, but this is a contradiction, since by surjectivity of f, for some $0 \neq x \in A^m$, we have $f(x) = e_n$, so that $\pi \circ f(x) = 0$, implying $\ker(\pi \circ f) \neq 0$. $\square$

(b) For the sake of contradiction, assume that $m > n$, and $f : A^m \to A^n$ is injective. Denote the inclusion map $\iota : A^n \hookrightarrow A^m$, so that $\iota \circ f : A^m \to A^m$ is injective, to simplify the notation, define $\varphi = \iota \circ f$. Since $\varphi(A^m) \subset \iota(A^n)$, $m > n$ implies that $ae_m \notin \varphi(A^m)$ for any $a \in A \setminus \{0\}$. We may apply proposition 2.4 from the text, which furnishes $a_1, \ldots a_m$, so that for any $x \in A^m$

$$\varphi^m(x) + a_1\varphi^{m-1}(x) + \cdots + a_{m-1}\varphi(x) + a_m x = 0$$

In particular, this applies for e_m, after applying linearity of φ, this implies that

$$\varphi(\varphi^{m-1}(e_m) + a_1\varphi^{m-2}(e_m) + \cdots + a_{m-1}e_m) = -a_m e_m$$

Hence $-a_m e_m \in \varphi(A^m)$ implies that $a_m = 0$. Hence we can conclude that for all $x \in A^m$,

$$\varphi(\varphi^{m-1}(x) + a_1\varphi^{m-2}(x) + \cdots + a_{m-1}x) = 0 \tag{5}$$

Since both sides of the equation in (5) are in $\varphi(A^m)$, injectivity of φ allows us to apply $\varphi|^{-1}_{\varphi(A^m)}$ to either side of the equation, implying that for any $x \in A^m$

$$\varphi^{m-1}(x) + a_1\varphi^{m-2}(x) + \cdots + a_{m-1}x = 0$$

Applying this argument recursively, we can conclude that $a_{m-1}, \ldots, a_2 = 0$, eventually reaching the desired result

$$\varphi(e_m) + a_1 e_m = 0 \implies a_1 = 0 \implies \varphi(e_m) = 0$$

But this implies that $e_m \in \ker\varphi$, so that φ is not injective, which is a contradiction. $\square$