**1.** **(a)** Let $\mathcal{B} = \{U_i\}_1^\infty$ be a countable basis for $X$, then for each $U_i$ assign some $x_i \in U_i$. Let $y \in X$, and $V$ a neighborhood of $y$, then since $\mathcal{B}$ contains a neighborhood base for $y$, we have some $x_i \in U_i \subset V$, hence $V \cap \{x_i\}_1^\infty \neq \emptyset$. Since $V$ was arbitrary we can conclude that $y \in \overline{\{x_i\}_1^\infty}$. Since $y$ was arbitrary $X = \overline{\{x_i\}_1^\infty}$. $\square$

**(b)** Let $\{x_i\}_1^\infty$ be a countable dense subset of $X$, now define $\mathcal{B} := \{N_{\frac{1}{n}}(x_i) \mid n, i \in \mathbb{N}\}$, where $N_r(x) := \{y \in X \mid d(x,y) < r\}$. There is an obvious bijection $\mathcal{B} \to \mathbb{N} \times \mathbb{N}$, $N_{\frac{1}{n}}(x_i) \mapsto (n,i)$so in particular $\mathcal{B}$ is countable. Now let $y \in X$, and let $U$ be a neighborhood of $y$. Then $U \supset N_r(y)$ for some $r > 0$. Then there exists some $x_i \in N_r(y)$ by density, by definition of the set, $d(x_i, y) < r$, so that $N_{r-d(x_i,y)}(x_i) \subset N_r(y)$ by the triangle inequality. Since we have $0 < r - d(x_i, y)$, we may choose $N$, such that $\frac{1}{N} < r - d(x_i, y)$ which gives the desired inclusion to prove that $\mathcal{B}$ is a basis.

$$N_{\frac{1}{N}}(x_i) \subset N_{r-d(x_i,y)}(x_i) \subset N_r(y) \subset U \quad \square$$

**2.** **(a)** Since we are in a metric space we may use the $\epsilon - \delta$ definition of continuity, where it is tautological that the function $d(x, \cdot) : X \to X$, since continuity is defined in terms of the distance function. Given this it suffices to show that for $x_1, x_2 \in X$, $|d_F(x_1) - d_F(x_2)| \leq d(x_1, x_2)$ to show continuity. Without loss of generality we may assume that $d_F(x_1) \geq d_F(x_2)$. Note that for any $y \in F$, $d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$, so in particular we have for any $y \in F$ that $d_F(x_1) \leq d(x_1, x_2) + d(x_2, y)$, which of course implies that $d_F(x_1) \leq d(x_1, x_2) + d_F(x_2)$, taken together we get the desired ineqaulity to prove continuity of $d_F$:

$$|d_F(x_1) - d_F(x_2)| = d_F(x_1) - d_F(x_2) \leq d(x_1, x_2) + d_F(x_2) - d_F(x_2) = d(x_1, x_2)$$

$x \in F \implies d_F(x) = 0$ is immediate from $d(x,x) = 0$. Since $F^c$ is open, any $x \in F^c$ has some neighborhood $N_\epsilon(x) \subset F^c$, $\epsilon > 0$ implying that $d_F(x) \geq \epsilon > 0$. $\square$

**(b)** The difference of continuous functions is continuous, we have from part (a) thatb$d_{F_i}$ are continuous which implies continuity of $g$. Now let $x \in F_2 \subset F_1^c$, then by part (a), we have $d_{F_2}(x) = 0, d_{F_1}(x) > 0$, so that $g(x) = d_{F_1}(x) > 0$. If $x \in F_1 \subset F_2^c$, then by part (a) we have $d_{F_1}(x) = 0$ and $d_{F_2}(x) > 0$, implying that $g(x) = -d_{F_2}(x) < 0$. $\square$

**(c)** Metric spaces are Hausdorff and hence $T_1$, so it suffices to show that any two closed sets can be seperated by disjoint open sets. Let $F_1, F_2$ be disjoint closed sets and define $g$ as in (b), we can use from part (b) that $g$ is continuous. This implies that $g^{-1}(0,\infty)$ and $g^{-1}(-\infty, 0)$ are open. Furthermore, we have from part (b) that $F_1 \subset g^{-1}(-\infty, 0)$ and $F_2 \subset g^{-1}(0,\infty)$ we are done since these sets are disjoint:

$$g^{-1}(-\infty,0) \cap g^{-1}(0,\infty) = g^{-1}((-\infty,0) \cap (0,\infty)) = g^{-1}\emptyset = \emptyset \quad \square$$

**3.** This follows simply by rewriting the sets $S$ and $\tilde{S}$.

$$S \stackrel{\text{def}}{=} \{W \cap \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}^{-1}(U_{\alpha,i}) \mid A \text{ is an arbitrary index set, } n \in \mathbb{N}, U_{\alpha,i} \in Y_{\alpha,i}\}$$

$$\tilde{S} \stackrel{\text{def}}{=} \{\bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}|_W^{-1}(U_{\alpha,i}) \mid A \text{ is an arbitrary index set, } n \in \mathbb{N}, U_{\alpha,i} \in Y_{\alpha,i}\}$$

Then we can write

$$W \cap \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}^{-1}(U_{\alpha,i}) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n (f_{\alpha,i}^{-1}(U_{\alpha,i}) \cap W) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n (f_{\alpha,i}^{-1}(U_{\alpha,i} \cap f(W))) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}|_W^{-1}(U_{\alpha,i})$$

So that $S$ and $\tilde{S}$ define the same set. $\square$

**4.** **(a) $\implies$ (b)** This follows from the $f_i$ being continuous on the weak topology they generate. C.f. Notes Prop 1.59. $\square$

**(b) $\implies$ (a)** Suppose for contraposition that $x_\alpha \not\to x$. Then there exists some neighborhood $U$ of $x$, such that for any $\alpha_0 \in A$, there exists some $\alpha \geq \alpha_0$, such that $x_\alpha \notin U$. We may assume without loss of generality that $U$ is an open neighborhood by passing to an open subset containing $x$. By definition of the weak topology, $U = \bigcup_{i \in I} \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$ for open $U_{i_j} \in \mathcal{T}(Y_{i_j})$, $x$ must lie in some $\bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$, so we may once again without loss of generality rechoose $U = \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$. Now suppose for $j = 2, \ldots, N$ there exists some $\alpha_0^j$, such that for any $\alpha \geq \alpha_0^j$ we have $f_{i_j}(x_\alpha) \in U_{i_j}$, then taking $\gamma \geq \alpha_0^j$, $\forall j \in 2, \ldots, N$, we have for any $\alpha_0$, there exists some $\alpha \geq \alpha_0, \gamma$, such that $x_\alpha \notin U$, implying that $x_\alpha \notin \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$, and since $x_\alpha$ lies in $f_{i_2}^{-1}(U_{i_2}), \ldots, f_{i_N}^{-1}(U_{i_N})$ we must have that $x_\alpha \notin f_{i_1}^{-1}(U_{i_1})$, equivalently $f_{i_1}(x_\alpha) \notin U_{i_1}$. But $U_{i_1}$ is open and contains $f_{i_1}(x)$, so we necessarily have $f_{i_1}(x_\alpha) \not\to f_{i_1}(x)$. $\square$

**5. (a)** $\implies$ **(b)** Let $(x, y)$ be an accumulation point of $\{(x, x)\}$, then there is a $(x_\alpha, x_\alpha)_{\alpha \in A} \to (x, y)$, from this we may define the net $(x_\alpha)_{\alpha \in A}$, where we say $x_\alpha \leq x_{\alpha'}$ when $(x_\alpha, x_\alpha) \leq (x_{\alpha'}, x_{\alpha'})$. Then if $U$ is a neighborhood of $x$, $U \times X$ is a neighborhood of $(x, y)$, so for some $\alpha_0$ we have for all $\alpha \geq \alpha_0$, $(x_\alpha, x_\alpha) \in U \times X$, implying that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$, i.e. $x_\alpha \to x$. By the same argument we find that $x_\alpha \to y$, but since $X$ is Hausdorff we know that nets have unique limits, and hence $x = y$, implying that $(x, y) \in \{(x, x)\}$, i.e. $\operatorname{acc}\{(x, x)\} \subset \{(x, x)\}$ is closed. $\square$

**(b)** Let $x \neq y \in X$, then $(x, y) \in \{(x, x)\}^c$ is open. Since $\{U \times V \mid U, V \text{ open in } X\}$ is a basis for the product topology, we have $\{(x, x)\}^c = \bigcup_{\alpha \in A} U_\alpha \times V_\alpha$, then for some $U \times V$, we have $(x, y) \in U \times V$, and furthermore $U \cap V = \emptyset$, since $U \times V \subset \{(x, x)\}^c$. Since $U, V$ are open in $X$ and we have $x \in U, y \in V$ proves that $X$ is Hausdorff.

**6.** Suppose that $x \in \overline{\{x \in X \mid f(x) = g(x)\}}$, equivalently there is some net $x_\alpha \to x$, such that $f(x_\alpha) = g(x_\alpha)$, $\forall \alpha$. By continuity of $f, g$ we have $f(x_\alpha) \to f(x)$, $g(x_\alpha) \to g(x)$. Define a new net $y_\alpha$, where $y_\alpha = f(x_\alpha) = g(x_\alpha)$, such that $y_{\alpha'} \geq y_\alpha$ when $\alpha' \geq \alpha$, then $y_\alpha \to f(x), y_\alpha \to g(x)$, since $Y$ is Hausdorff, nets in $Y$ have unique limits implying that $f(x) = g(x)$, so that $x \in \{x \in X \mid f(x) = g(x)\}$, which suffices to show that $\{x \in X \mid f(x) = g(x)\} = \overline{\{x \in X \mid f(x) = g(x)\}}$. $\square$