1. (a)

$$g = \langle \frac{\partial \varphi^{-1}}{\partial x}, \frac{\partial \varphi^{-1}}{\partial x} \rangle = \langle (1, f'(x)), (1, f'(x)) \rangle = 1 + (f'(x))^2$$

Thus the formula for arc length in this context is

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

(b)

$$g = (\varphi^{-1})^* g_{\text{EUC}} = \begin{bmatrix} \frac{\partial (\varphi^{-1})^x}{\partial x} & \frac{\partial (\varphi^{-1})^y}{\partial x} & \frac{\partial (\varphi^{-1})^z}{\partial x} \\ \frac{\partial (\varphi^{-1})^x}{\partial y} & \frac{\partial (\varphi^{-1})^y}{\partial y} & \frac{\partial (\varphi^{-1})^z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial (\varphi^{-1})^x}{\partial x} & \frac{\partial (\varphi^{-1})^x}{\partial y} \\ \frac{\partial (\varphi^{-1})^y}{\partial x} & \frac{\partial (\varphi^{-1})^y}{\partial y} \\ \frac{\partial (\varphi^{-1})^z}{\partial x} & \frac{\partial (\varphi^{-1})^z}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)^2 + 1 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & \left(\frac{\partial f}{\partial y}\right)^2 + 1 \end{bmatrix}$$

So that $\det(g_{ij}) = 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$. It follows that the surface area formula in this context simplifies to

$$\int_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

(c) The solution to this exercise relies on the crucial observation, that on $S \cap V$,

$$F(x, y, z) = 0 = z - f(x, y)$$

With this observation, the exercise is routine, first note that

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(z - f(x, y)) = -\frac{\partial f}{\partial x}$$
$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(z - f(x, y)) = -\frac{\partial f}{\partial y}$$
$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(z - f(x, y)) = 1$$

Now to verify the formula:

$$\int_U \frac{|\nabla F|}{|\frac{\partial}{\partial z} F|} dxdy = \int_U \frac{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}{|\frac{\partial}{\partial z} F|} dxdy$$

$$= \int_U \sqrt{1 + \left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right)^2 + \left(\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right)^2} dxdy$$

$$= \int_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

As in part (b). Now checking the formula:

$$\begin{aligned}
\int_U \frac{|\nabla F|}{|\frac{\partial}{\partial z} F|} dxdy &= \int_U 2\frac{\sqrt{x^2+y^2+z^2}}{2z} dxdy = \int_U \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} dxdy \\
&= \int_U \sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}} dxdy \\
&= \int_U \sqrt{\frac{1}{1-x^2-y^2}} dxdy \\
&= \int_{y=-1}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-x^2-y^2}} dxdy \\
&= \int_{y=-1}^{1} \arctan \frac{x}{\sqrt{1-x^2-y^2}} |_{x\to -\sqrt{1-y^2}}^{x\to\sqrt{1-y^2}} \\
&= \int_{y=-1}^{1} \lim_{x\to\infty} \arctan(x) - \lim_{x\to-\infty} \arctan(x) \\
&= \int_{-1}^{1} \pi dy = 2\pi
\end{aligned}$$

As desired.

2. (a) We compute

$$A = \begin{bmatrix} \frac{\partial}{\partial u} f^x & \frac{\partial}{\partial u} f^y & \frac{\partial}{\partial u} f^z \\ \frac{\partial}{\partial v} f^x & \frac{\partial}{\partial v} f^y & \frac{\partial}{\partial v} f^z \end{bmatrix} = \begin{bmatrix} -b\sin u\cos v & -b\sin u \sin v & b\cos u \\ -(a+b\cos u)\sin v & (a+b\cos u)\cos v & 0 \end{bmatrix}$$

Then in (u,v) coordinates, we have $g = f^* g_{\text{EUC}}$, i.e.

$$g = AA^T = \begin{bmatrix} b^2 & 0 \\ 0 & (a+b\cos u)^2 \end{bmatrix}$$

(b) Write $c_1 = f\circ\gamma_1,\ c_2 = f\circ\gamma_2$. Then we have $\dot\gamma_1^1 = 0, \dot\gamma_1^2 = 1$, so that $|\dot\gamma_1|_g^2 = g_{2,2} = (a+b\cos\pi)^2 = (a-b)^2$. It follows that

$$L(c_1) = \int_0^{2\pi} a - bdt = 2\pi(a-b)$$

Similarly, $\dot\gamma_2^1 = 1, \dot\gamma_2^2 = 0$, so that $|\dot\gamma_1|_g^2 = g_{1,1} = b^2$. It follows that

$$L(c_2) = \int_0^{2\pi} b = 2b\pi$$

(c) $dA = \sqrt{\det g}\, dudv = b(a+b\cos u)dudv$, hence the surface area is given by

$$SA = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos u)dudv = \int_0^{2\pi} uab + b^2\sin u|_0^{2\pi} dv = \int_0^{2\pi} 2\pi ab dv = (2\pi)^2 ab$$