

I collaborated with Justin Wan on problem 2.

**1. (a)** Let  $(x, y) \sim (w, z)$ , then  $w = \lambda x, z = \lambda^{-1}y$ , so that  $wz = \lambda\lambda^{-1}xy = xy$ . Let  $\pi : \mathbb{R}^2 \setminus \{(0, 0)\}/(\mathbb{R} \setminus \{0\})$  be the quotient. Then  $g(x, y) = f\pi(x, y) = xy$ , is a polynomial function hence continuous. To show  $f$  is continuous, let  $U \in \mathbb{R}$  be open, then  $f^{-1}(U)$  is open iff  $\pi^{-1}f^{-1}(U)$  by definition of quotient, but this is exactly  $g^{-1}(U)$  which is open since  $g$  is continuous.

**(b)**  $\#f^{-1}(t) = 1$ ,  $t \neq 0$  and  $\#f^{-1}(0) = 0$ . Proof being  $xy = 0 \iff x = 0$  or  $y = 0$ , so the preimages of 0 are  $\overline{(1, 0)}$  and  $\overline{(0, 1)}$ . If  $t \neq 0$ , then  $t = xy = zw$ , we may write  $z = \lambda x$ , where  $\lambda = \frac{z}{x} \neq 0$ , then  $xy = \lambda xw$ , so that  $w = \lambda^{-1}y$ , proving that  $\overline{(x, y)} = \overline{(z, w)}$ .

**(c)** Let  $\overline{(1, 0)} \in U, \overline{(0, 1)} \in V$ , for open sets  $U, V$ . Then by definition of the quotient  $\pi^{-1}(U)$  is open, hence by the local definition of open sets (from homework 1) we have some neighborhood of  $(1, 0)$  contained in  $\pi^{-1}(U)$ . This implies that for some  $\epsilon_x > 0$ ,  $\{(1, r) | r < \epsilon\} \subset \pi^{-1}(U)$ . Similarly, there exists some  $\epsilon_y > 0$ , such that  $\{(r, 1) | r, \epsilon\} \subset \pi^{-1}(V)$ . Now choose  $r = \frac{\min(\epsilon_x, \epsilon_y)}{2}$ , so that  $\overline{(1, r)} \in \pi\pi^{-1}(U) = U$ , and  $\overline{(r, 1)} \in \pi\pi^{-1}(V) = V$ . Then  $(r, 1) \sim (r1, r^{-1}r) = (1, r)$  implies that  $\overline{(r, 1)} \in U \cap V$ . This proves that  $X$  is not hausdorff, since  $\overline{(1, 0)}$  and  $\overline{(0, 1)}$  do not satisfy the Hausdorff condition.

**(d)** Consider the maps

$$\begin{aligned}\varphi : X &\rightarrow Y \\ \overline{(x, y)} &\mapsto \begin{cases} (xy, 0) & y \neq 0 \\ (xy, 1) & x \neq 0 \end{cases} \\ \tilde{\varphi} : Y &\rightarrow X \\ (s, t) &\mapsto \begin{cases} \overline{(s, 1)} & s \neq 0 \\ \overline{(0, 1)} & s = t = 0 \\ \overline{(1, 0)} & s = 0, t = 1 \end{cases}\end{aligned}$$

To check that this  $\varphi$  is injective (it is well defined by (a)), we only need check that  $\overline{(1, 0)}, \overline{(0, 1)}$  map to separate points in  $Y$ , since part (c) guarantees the other elements are 1-1, so since these points map to 0 in the first coordinate, away from all other points, and map to separate points in  $Y$  the map is injective. To check surjectivity,  $(0, 0)$  and  $(0, 1)$  are mapped onto, so we can check the other points.  $(x, 1) \mapsto (x, 0)$  shows surjectivity. Similarly, we check for  $\tilde{\varphi}$ , it is well defined since it just extends to equivalence classes. It is onto since  $(0, 1), (1, 0)$  are in the image, and any  $(x, y) \sim (xy, 1)$  (for  $x, y \neq 0$ ) has its equivalence class in the image. Injectivity is also clear since  $(x, 1) \sim (y, 1)$  iff  $x = y$  and  $\overline{(1, 0)}$  is only mapped onto by one point. To see that these are inverse maps, it is immediate they are inverses in the case of  $(0, 1), (0, 0) \in Y$  and  $(0, 1), (1, 0) \in X$ . Checking this for  $x, y, s \neq 0$  we have  $\tilde{\varphi}\varphi(\overline{(x, y)}) = \overline{(xy, 1)} \sim (x, y)$  and  $\varphi\tilde{\varphi}(s, 0) = (s, 0)$ . It remains to show continuity of  $\varphi$  and  $\varphi^{-1} = \tilde{\varphi}$ .

Continuity of  $\varphi$ : Let  $U$  be open in  $Y$ , then  $U$  is of the form  $\pi(V \times \{0\} \cup W \times \{1\})$ , for  $V, W \subset \mathbb{R}$  open. Hence we can write it in the form of  $((V \setminus \{0\}) \cup W \setminus \{0\}) \times \{0\}) \cup \chi_V \cup \chi_W$ ,

$$\chi_V = \begin{cases} \{0, 0\} & 0 \in V \\ \emptyset & 0 \notin V \end{cases} \quad \chi_W = \begin{cases} \{0, 1\} & 0 \in W \\ \emptyset & 0 \notin W \end{cases}$$

Now since  $\{(0, 0)\}^c$  and  $\{(0, 1)\}^c$  are open in  $Y$  (they are images of their complements in  $\mathbb{R} \times \mathbb{Z}$ , where points are closed since T1 follows from hausdorff), it follows that  $V \setminus \{0\} \cup W \setminus \{0\}$  is open. Then  $(\pi_X \varphi)^{-1}((V \setminus \{0\}) \cup W \setminus \{0\}) \times 0$  is just  $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy \in V \setminus \{0\} \cup W \setminus \{0\}\}$  but this is the preimage of an open set in  $\mathbb{R}$  of the continuous polynomial function  $(x, y) \mapsto xy$ , this proves continuity of  $\varphi$  by definition of the quotient space in the case of  $\chi_V = \emptyset = \chi_W$ . Now in the case where atleast one of  $\chi_V, \chi_W$  is non-empty, assume WLOG  $\chi_V \neq \emptyset$ , then since  $V$  is an open set in  $\mathbb{R}$  containing 0, it must contain some open set  $J$  containing 0. Then  $(\pi_X \varphi)^{-1}((J \setminus \{0\}) \times \{0\})$  is the set  $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy \in J \setminus \{0\}\}$ , this is an open set since  $J \setminus \{0\}$  is open, so for some  $\epsilon$  it contains a set of the form  $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} | 0 < xy < \epsilon\}$ . Now we take

$$(\pi_X \varphi)^{-1}(0, 1) = \{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy = 0\} \setminus \{(0, y) | y \neq 0\} \quad (\pi_X \varphi)^{-1}(0, 0) = \{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy = 0\} \setminus \{(x, 0) | x \neq 0\}$$

Note that  $\{(0, y) | y \neq 0\}, \{(x, 0) | x \neq 0\}$  are closed in  $\mathbb{R}^2 \setminus \{0\}$  since their complements are open. Now in the case where  $(0, 0) \in U$ ,  $(\pi_X \varphi)^{-1}(U)$  contains the open set  $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy < \epsilon\} \setminus \{(x, 0) | x \neq 0\}$  containing  $(\pi_X \varphi)^{-1}(0, 0)$ . Similarly if  $(0, 1) \in U$ , then  $(\pi_X \varphi)^{-1}(U)$  contains the open set  $\{(x, y) \in \mathbb{R}^2 \setminus \{0\} | xy < \epsilon\} \setminus \{(0, y) | y \neq 0\}$  containing  $(\pi_X \varphi)^{-1}(0, 1)$ . But since  $(\pi_X \varphi)^{-1}(U) \supset (\pi_X \varphi)^{-1}((V \setminus \{0, 0\}) \cup W \setminus \{0, 1\}) \times \{0\})$  is an open set containing every other point  $(\pi_X \varphi)^{-1}(U)$  is open by the local definition of continuity. This proves that  $\varphi$  is continuous by definition of the quotient map.

Continuity of  $\tilde{\varphi}$ : Let  $U$  be an open set in  $X$  now let  $q$  be any point in  $U$ , but not  $\overline{(1, 0)}$ . Then we can write  $q = \overline{(p, 1)}$  for some  $p$ . Then since  $U$  is open,  $\pi_X^{-1}(U)$  is open containing  $(p, 1)$ , so for some  $\epsilon > 0$  (where if  $p \neq 0$

we can choose  $\epsilon < |p|$ , it contains  $(p + t, 1)$  for  $t$  such that  $|t| < \epsilon$ . Then in the first case where  $p \neq 0$  we have  $(\pi_Y \tilde{\varphi})^{-1}\{(p, 1)\} = \{(p, 0), (p, 1)\}$  is contained in the open set  $\{(p + t, s) | t < \epsilon, s \in \{0, 1\}\} \subset (\pi_Y \tilde{\varphi})^{-1}(U)$  here  $\epsilon < |p|$  guarantees we have  $(p + t, 0)$  and  $(p + t, 1)$  in the preimage dealing with both points at once. Now in the second case where  $p = 0$ , we still have that  $\{(p + t, 0) | t < \epsilon\} \subset (\pi_Y \tilde{\varphi})^{-1}(U)$ , so the preimage still contains an open set containing  $(0, 0)$ . This proves continuity for any  $U$  not containing  $\overline{(1, 0)}$ . If  $U$  does contain  $\overline{(1, 0)}$ , then  $\pi_X^{-1}(U)$  is an open set containing  $(1, 0)$  hence for some  $\epsilon > 0$  it contains  $(1, t)$  for all  $t$ , such that  $|t| < \epsilon$ , this means that  $\pi_X(\pi_X^{-1}(U))$  contains each  $(1, t) = \overline{(t, tt^{-1})} = \overline{(t, 1)}$ . This implies that there is some open set containing  $\overline{(1, 0)}$  contained in  $(\pi_Y \tilde{\varphi})^{-1}(U)$ , namely

$$(\pi_Y \tilde{\varphi})^{-1}(U) \supset (\pi_Y \tilde{\varphi})^{-1}(\{(1, t) | |t| < \epsilon\}) \supset \{(t, 1) | |t| < \epsilon\}$$

This implies by the local definition that  $(\pi_Y \tilde{\varphi})^{-1}(U)$  is open, so by definition of the quotient  $\tilde{\varphi}^{-1}(U)$  is open. We conclude that  $\tilde{\varphi} = \varphi^{-1}$  is continuous along with  $\varphi$ , making  $\varphi$  a homeomorphism from  $X$  to  $Y$ .

**2.** Take  $\mathbb{R}^3 \setminus (0, 0)$ , and  $S^2$  the unit sphere centered at the origin, then  $H(x, t) = \frac{x}{1+t(|x|-1)}$  is a strong deformation retract of  $\mathbb{R}^3$  onto  $S^2$ , hence  $\mathbb{R}^3 \setminus \{\text{pt}\}$  is homotopic to  $S^2$ .

Let  $J$  be the filled Torus (i.e.  $D^2 \times S^1$ ), and let  $D_{\text{Lat}}, D_{\text{Long}}$  denote the latitudinal and longitudinal discs respectively. Then we may write  $\mathbb{R}^3 \setminus \{\text{pt}\} = T^2 \sqcup (J^\circ \setminus \{\text{pt}\}) \sqcup (J^c)^\circ$ . I will show that  $\mathbb{R}^3 \setminus \{\text{pt}\}$  strong deformation retracts onto  $(J \setminus \{\text{pt}\}) \cup D_{\text{Long}}$ , then show that  $(J \setminus \{\text{pt}\}) \cup D_{\text{Long}}$  strong deformation retracts onto  $T \cup D_{\text{Lat}} \cup D_{\text{Long}}$ , the proof follows by transitivity of homotopy equivalence.

For the first equivalence, we can let  $P$  be the  $x, y$  plane, with  $J \setminus \{\text{pt}\}$  embedded in  $\mathbb{R}^3 \setminus \{\text{pt}\}$  at height zero (wlog the point doesn't have height 0). Then we can strong deformation retract  $\mathbb{R}^3 \setminus \{\text{pt}\}$  by projecting the  $z$ -axis onto  $P \cup (J \setminus \{\text{pt}\})$ . Now given a point  $p = (x_p, y_p, z_p)$ , let  $(x_p, y_p, z_0)$  be the closest point to it in  $P \cup (J \setminus \{\text{pt}\}) \cap \{(x_p, y_p, z) | z \in \mathbb{R}\}$ . the homotopy can be written as  $H((x, y, z), t) = (x, y, z + t(z_0 - z))$  for  $z_0$  continuously depending on  $z$  (continuous since  $P \cup J$  is smooth). Now we can deformation retract  $P \cup (J \setminus \{\text{pt}\})$  onto  $J \setminus \{\text{pt}\} \cup D_{\text{Long}}$ , the retract  $H$  is defined to be constant on  $J \setminus \{\text{pt}\} \cup D_{\text{Long}}$ , then assuming the radius from the origin to the outer edge of the torus is  $R$  we only need to define it on points of  $P \setminus D_R^2$ , where  $D_R^2$  denotes the disc of radius  $R$ . On such points, define  $H(p, t) = \frac{p}{1+tR(|p|)-1/R}$ . Transitivity of homotopy equivalence proves that  $\mathbb{R}^3 \setminus \{\text{pt}\} \simeq_H (J \setminus \{\text{pt}\}) \cup D_{\text{Long}}$ .

Now note to show a strong deformation retract of  $(J \setminus \{\text{pt}\}) \cup D_{\text{Long}}$  onto  $T^2 \cup D_{\text{Lat}} \cup D_{\text{Long}}$ , it will suffice to show one exists from  $J \setminus \{\text{pt}\}$  onto  $T^2 \cup D_{\text{Lat}}$ , since  $\partial D_{\text{Long}} \subset T^2$  implies that  $T^2$  remaining fixed in our homotopy allows us to fix  $D_{\text{Long}}$  in our homotopy. Now we may identify  $J \setminus \{\text{pt}\} = \frac{D^2 \times I \setminus \{\text{pt}\}}{(x, 1) \sim (x, 0)}$ . Considering the cylinder centered at the origin, with origin removed, i.e.  $D^2 \times I \setminus \{(0, 0)\}$ , we can write a homotopy to  $\partial(D^2 \times I)$ , namely for each point  $p$ , let  $q_p$  be the intersection of the ray from the origin through  $p$  with  $\partial(D^2 \times I)$ . It is clear that  $q_p$  varies continuously with respect to  $p$ , so we write the homotopy  $H(p, t) = \frac{p}{1+t(|\frac{p}{q_p}| - 1)}$ . Then since a strong deformation retract of the space induces a strong deformation retract of the quotient space, we get that

$$J \setminus \{\text{pt}\} = \frac{D^2 \times I \setminus \{\text{pt}\}}{(x, 1) \sim (x, 0)} \simeq_H \frac{\partial(D^2 \times I)}{(x, 1) \sim (x, 0)} = \frac{S^1 \times I \cup D \times \{0\}}{(x, 1) \sim (x, 0)} = T^2 \cup D_{\text{Lat}}$$

Now as previously mentioned, since this map is a strong deformation retract, it induces one on  $J \setminus \{\text{pt}\} \cup D_{\text{Long}}$  to  $T^2 \cup D_{\text{Long}} \cup D_{\text{Lat}} = X$ . Meaning by transitivity we have  $S^2 \simeq_H \mathbb{R}^3 \setminus \{\text{pt}\} \simeq_H X$ .

**Proof that strong deformation retract induces strong deformation retract on quotient.** Let  $H$  be a strong deformation retract of the topological space  $X$ , we want to show there exists a strong deformation retract  $\bar{H}$  of  $X/\sim$ , which is the quotient of  $H$ . To do so, define the equivalence relation  $\approx$  on  $H \times I$ , where  $(x, t) \approx (y, s)$  iff  $x \sim y$  and  $t = s$ . Then we can take  $\pi_\sim$  to be the quotient map  $X \rightarrow X_\sim$ , we have that  $\pi_\sim H$  is a map from  $H \times I$  to  $X/\sim$ , which is level on equivalence classes of  $\approx$ , since  $\approx$  induces no relations on  $I$ , and we are taking the quotient by  $\sim$  in the map. Hence by the universal property of quotient maps we have some map  $\bar{H} : \frac{X \times I}{\approx} \rightarrow X/\sim$ , which is equal to  $\pi_\sim H$ , hence if  $H$  was a deformation retract of  $X$  onto  $Y \subset X$ , then  $\bar{H}(\frac{X \times I}{\approx}) \subset Y/\sim$ , and  $Y/\sim$  remains fixed, since  $\bar{H}$  agrees with  $H\pi$ . This is equivalent to saying there exists  $\bar{H}$  making the following diagram commute:

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & X \\ \downarrow & & \downarrow \\ \frac{X \times I}{\approx} & \xrightarrow{\exists \bar{H}} & X/\sim \end{array}$$

then we can identify  $\frac{X \times I}{\approx} = X/\sim \times I$ , so that  $\bar{H}$  is in fact our desired homotopy.

**Lemma.** I will use the following lemma to streamline my proofs for problems 3 and 4.

If  $\psi : X \rightarrow Y$  is a homeomorphism, and  $\sim$  is an equivalence relation on  $X$ , and  $\approx$  a equivalence relation on  $Y$ , such that  $\psi(a) \approx \psi(b) \iff a \sim b$ , then  $X/\sim \simeq Y/\approx$ , this says that homeomorphisms from  $X \rightarrow Y$  induce homeomorphisms to the quotients when the points in the same equivalence classes induced by the quotient on  $Y$  are images of the points in the same equivalence classes induced by the quotient on  $X$ , see the diagram.

$$\begin{array}{ccc} X & \xrightarrow{\quad \psi \quad \cong} & Y \\ \downarrow \pi_{\sim} & & \downarrow \pi_{\approx} \\ X/\sim & \xrightarrow{\quad \bar{\psi} \quad \cong} & Y/\approx \end{array}$$

**proof.** Define  $\bar{\psi} : X/\sim \rightarrow Y/\approx$ , by  $\bar{\psi} : \bar{x} \mapsto \overline{\psi(x)}$ , this is surjective since  $\psi$  is surjective and  $\bar{\psi}$  is well defined/injective by definition of  $\approx$ . We can define  $\bar{\psi}^{-1} : Y/\approx \rightarrow X/\sim$ , in the same way. This is the inverse of  $\bar{\psi}$ , since  $\bar{\psi}$  and  $\bar{\psi}^{-1}$  are just restrictions to equivalence classes of  $\psi$  and  $\psi^{-1}$ . To show  $\bar{\psi}$  is continuous, note that  $\bar{\psi} = \pi_{\approx} \circ \psi \circ \pi_{\sim}$ . Let  $U$  be open in  $Y/\approx$ , then the preimage of  $U$  under  $\pi_{\approx}$  is open by definition, so continuity follows from continuity of  $\psi$ . The proof for continuity of  $\bar{\psi}^{-1}$  is the same.

**Additional Justification for problems 3 and 4.** Once again, to streamline the proofs for 3 and 4, I will explain here why the following map is a homeomorphism.

$$\begin{aligned} C_{1_{S^1}} &\xrightarrow{\psi} D^2 \\ (\theta, t) &\mapsto (\theta, 1-t) \end{aligned}$$

This map is clearly bijective, so that it will suffice to show continuity by the closed map lemma, since  $C_{1_{S^1}}$  is the quotient of a compact space hence compact (Heine Borel theorem on  $S^1 \times I$ ) and  $D^2$  is Hausdorff. To see that the map is continuous, let  $U \subset D^2$  open. If  $U$  does not contain  $(0, 0)$ , then we can just regard  $\psi$  as a continuous map between  $S^1 \times I$  and  $D^2$  since it is unaffected by the quotient. Now examining the case where  $U$  contains  $(0, 0)$ , by the local definition of open it must contain some neighborhood around  $(0, 0)$ , and hence  $\pi^{-1}\psi^{-1}(U)$  contains  $S^1 \times \{t\}$  for  $t$  sufficiently close to 1, so that by definition of the quotient  $\overline{(x, 1)}$  is contained in an open set in  $\psi^{-1}(U)$ . Then since  $D^2$  is Hausdorff, each other point is contained in a neighborhood in  $U$  not containing  $(0, 0)$ , so its preimage is contained in some neighborhood of  $\pi^{-1}\psi^{-1}(U)$  as explained previously, this shows that  $\psi^{-1}(U)$  is open by the local definition of open so we are done.

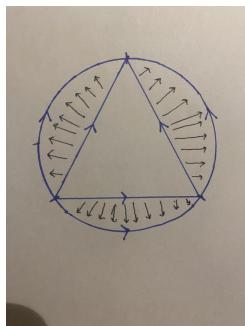
**3.** We use the equivalent definition of  $\mathbb{RP}^2$  as  $D^2/\sim$ , identifying  $e^{ix} \sim e^{-ix}$ . Now writing out the mapping cone,

$$C_f \stackrel{\text{def}}{=} S^1 \times I \sqcup S^1_Y / ((e^{ix}, 0) \approx e^{2ix}, (e^{ix}, 1) \approx (e^{iy}, 1))$$

Now consider  $x, y \in [0, 2\pi)$  we can notice  $(e^{ix}, 0) \approx (e^{iy}, 0) \iff e^{2ix} = e^{2iy}$ . WLOG we can assume  $x < y$ , so that  $y = x + r$ ,  $0 < r < 2\pi$ . Then with these restrictions  $e^{2ix} = e^{2i(x+r)} \iff r = \pi$ , so that the equivalence relation identifies  $e^{ix} \approx e^{ix+\pi} = e^{-ix}$ .

Now define consider the map  $C_{1_{S^1}} \xrightarrow{\psi} D^2$ ,  $(\theta, t) \mapsto (\theta, 1-t)$ , this map is a homeomorphism as explained previously. Additionally, the antipodal points on the boundaries of  $S^1 \times \{0\}$  and  $\partial D^2$  remain antipodal under this map. So the lemma gives us  $D^2/\sim \simeq C_{1_{S^1}}/\approx = C_f$

**4.** Note that the triangle is homeomorphic to the disc. We can inscribe the triangle in a circle with radius  $R$ . Then for each point  $p$ , let  $q$  be the intersection of the ray through  $p$  and the origin with the boundary of the triangle. For each of these points we can map  $p \mapsto \frac{Rp}{|q|}$  this is a homeomorphism since  $q$  varies smoothly with  $p$  and we have inverse  $p \mapsto \frac{|q|p}{R}$ , where  $q$  comes from inscribing the triangle in the circle, which is also continuous. It follows that the equivalence relation induced on  $D^2$  is  $e^{ix} \sim e^{ix}e^{\frac{2\pi}{3}} \sim e^{-ix}$ , which can be seen by the following picture and the lemma. So that the dunce cap can be written as  $D^2/\sim$ .



Now consider the maps  $\mathbf{1}_{S^1}$  and

$$f : S^1 \rightarrow S^1$$

$$e^{ix} \mapsto \begin{cases} e^{3ix} & 0 \leq x < \frac{4\pi}{3} \\ e^{-3ix} & \frac{4\pi}{3} \leq x < 2\pi \end{cases}$$

Take the mapping cone

$$C_f = \frac{S^1 \times I \sqcup S^1}{(x, 0) \sim f(x), (x, 1) \sim (y, 1)}$$

For each  $x$ , we have  $f^{-1}(x) = \{e^{ix/3}, e^{i(x+2\pi)/3}, e^{-ix/3}\}$ , so the equivalence relation induced by  $(x, 0) \sim f$  can be seen to be  $e^{ix/3} \sim e^{i(x+2\pi)/3} \sim e^{-ix/3}$ . We can then take the map  $C_{\mathbf{1}_{S^1}} \xrightarrow{\psi} D^2$ , where  $(x, t) \mapsto (x, 1-t)$ , this is a homeomorphism as explained previously. Since  $C_f$  is a quotient of  $C_{\mathbf{1}_{S^1}}$  by the image of quotients in  $D^2/\sim$  via  $\psi^{-1}(D^2)$ , the lemma implies that  $C_f \simeq D^2/\sim$  the dunce cap.

We have that  $C_{\mathbf{1}_{S^1}}$  is contractible, using the homotopy  $H((x, t), s) = (x, t(1-s))$ , so it will suffice to show that  $C_f \simeq_H C_{\mathbf{1}_{S^1}}$ , and we have proven in class that homotopic maps have homotopic cones. I will show  $f \sim \rho \sim \mathbf{1}_{S^1}$ , where

$$\rho : e^{ix} \mapsto \begin{cases} e^{3ix} & 0 < x < 2\pi/3 \\ 1 & 2\pi/3 \leq x < 2\pi \end{cases}$$

I will provide  $H_1$  for the first equivalence  $f \sim \rho$  and  $H_2$  for the second  $\rho \sim \mathbf{1}_{S^1}$ .

$$H_1(x, t) : \begin{cases} x \mapsto f(x) & x < \frac{2}{3} - \frac{1}{3}t \text{ or } x > \frac{2}{3} + \frac{1}{3}t \\ x \mapsto f(\frac{2}{3} - \frac{1}{3}t) & \frac{2}{3} - \frac{1}{3}t \leq x \leq \frac{2}{3} + \frac{1}{3}t \end{cases}$$

$$H_2(x, t) : \begin{cases} x \mapsto f(\frac{x}{1+2t}) \end{cases}$$