1. Let (a, b) be an arbitrary open interval, then $(-\infty, b) \in \mathcal{B}$, furthermore

$$\begin{aligned}\{(-\infty, a - 1/n)\}_{n\in\mathbb{N}} \subset \mathcal{B} &\implies \{[a - 1/n, \infty)\}_{n\in\mathbb{N}} \subset \mathcal{B} \\ &\implies (a, \infty) = \bigcup_{\mathbb{N}}[a - 1/n, \infty) \in \mathcal{B} \\ &\implies (a, b) = (a, \infty) \cap (-\infty, b) \in \mathcal{B}\end{aligned}$$

Now since each open interval is an open set we have that $\mathcal{B} \subset \mathcal{B}_{\mathbb{R}}$. But since each open set is a countable union of open intervals it follows that each open set is in $\mathcal{B}$, and hence by closure properties we have that the sigma algebra they generate must also be contained in $\mathcal{B}$, so that $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}$.

2. Countable sets. As proof, assume X is countable, then $\{\{x\} : x \in X\} \subset \mathcal{P}(X)$, and each $\{x\}$ has counting measure 1. It follows by assumption that $\bigcup_{x\in X}\{x\} = X$ is a countable union of sets of finite measure, i.e. σ-finite. Conversely, if X is uncountable then it is not a countable union of countable sets, hence any countable collection of sets $\{X_i\}_{i\in I}$, such that $\bigcup_I X_i = X$ must have atleast one uncountable X_i (so that X_i has infinite counting measure).

3. **(a)** Suppose that $E \in f_*\mathcal{M}$, then since $f^{-1}(A^c) = f^{-1}(A)^c$,

$$f^{-1}(E) \in \mathcal{M} \implies (f^{-1}(E))^c = f^{-1}(E^c) \in \mathcal{M} \implies E^c \in f_*\mathcal{M}$$

Now suppose that $\{E_i\}_{i\in\mathbb{N}} \subset f_*\mathcal{M}$, then since $\bigcup_{\mathbb{N}} f^{-1}(E_i) = f^{-1}(\bigcup_{\mathbb{N}} E_i)$,

$$\bigcup_{\mathbb{N}} f^{-1}(E_i) \in \mathcal{M} \implies f^{-1}(\bigcup_{\mathbb{N}} E_i) \in \mathcal{M} \implies \bigcup_{\mathbb{N}} E_i \in \mathcal{M}$$

(b) We need only check that $f_*\mu$ is a measure. Since the image of $f_*\mu$ is a subset of the image of μ it is clear that for each E, $0 \leq f_*(\mu) \leq \infty$. It is also immediate that $f_*\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. To check additivity, consider $\{E_i\}_{\mathbb{N}} \subset f_*\mathcal{M}$, where the E_i are disjoint (note this implies that each $f^{-1}(E_i)$ is disjoint). It follows that

$$f_*\mu(\bigcup_{\mathbb{N}} E_i) = \mu(f^{-1}(\bigcup_{\mathbb{N}} E_i)) = \mu(\bigcup_{\mathbb{N}} f^{-1}(E_i)) = \sum_{\mathbb{N}} \mu(f^{-1}(E_i)) = \sum_{\mathbb{N}} f_*\mu(E_i)$$

(c) The point mass at y_0 let $E \in f_*\mathcal{M}$, then

$$f_*\mu(E) = \begin{cases}\mu(\emptyset) = 0 & y_0 \notin E \\ \mu(f^{-1}(y_0)) = \mu(X) & y_0 \in E\end{cases}$$

(d) This measure counts the number of perfect squares in a set E. If $E \subset \mathbb{N}$, then $f_*\mu(E) = \#f^{-1}(E) = \#\{n \in \mathbb{N} : n^2 \in E\}$

4. μ is a measure for $j \geq 0$.

Suppose that $j \leq -1$, then consider the sets $E_n := \{n^2\}$.

$$\sum_{\mathbb{N}} \mu(E_n) = \sum_{\mathbb{N}} n^2 < \infty = \mu\left(\bigcup_{\mathbb{N}} E_n\right)$$

Conversely, suppose that $j \geq 0$, $\mu(\emptyset) = 0$ by definition. If $\{E_i\}_{\mathbb{N}}$ is a countable disjoint family, then we are done immediately if any E_i is infinite, since then $\mu\left(\bigcup_{\mathbb{N}} E_i\right) = \infty = \mu(E_i) \leq \sum_{\mathbb{N}} \mu(E_i)$. Similarly, if infinitely many E_i are non-empty, then

$$\mu\left(\bigcup_{\mathbb{N}} E_i\right) = \infty = \sum_{\mathbb{N}} 1 = \sum_{\mathbb{N}} \sum_{n \in E_i} n^0 \leq \sum_{\mathbb{N}} \sum_{n \in E_i} n^j$$

Finally, in the case where each E_i is finite, and only finitely many $E_i \neq \emptyset$, we have for some N, $\{E_i\}_{\mathbb{N}} = \{E_i\}_{i=1}^N$, then

$$\mu\left(\bigcup_1^N E_i\right) = \sum_{n \in \bigcup_1^N E_i} n^j = \sum_{i=1}^N \sum_{n \in E_i} n^j = \sum_{i=1}^N \mu(E_i)$$

where the second equality follows from the $\{E_i\}_1^n$ being disjoint.