

1. We first show that F is bijective. For surjectivity, note that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ is surjective, so that $\tan \frac{\theta}{2} : (-\pi, \pi) \rightarrow \mathbb{R}$ is onto. This is equivalent to $F : e^{i\theta} \mapsto [\cos \frac{\theta}{2}, \sin \frac{\theta}{2}] = [1, \tan \frac{\theta}{2}]$ maps $S^1 \setminus \{-1\} \rightarrow \mathbb{RP}^1 \setminus [0, 1]$, then $F(-1) = F(e^{i\pi}) = [0, 1]$, so F is onto. To see injectivity, it is clear that $F^{-1}([0, 1]) = -1$, since $e^{i\pi}$ is the only point on S^1 where $\cos(\frac{\theta}{2}) = 0$. Then any other point $-1 \neq e^{i\theta}$ maps to the coset of the form $[1, \tan \frac{\theta}{2}]$, so injectivity follows from \tan being strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

To show that F is smooth, consider charts $(U, \varphi), (\tilde{U}, \tilde{\varphi})$ on S^1 , where $\varphi : e^{i\theta} \rightarrow \theta$ on $U = (0, 2\pi)$ and $\tilde{\varphi} : e^{i\theta} \rightarrow \theta$ on $\tilde{U} = (-\pi, \pi)$. Also consider charts $(V, \eta), (\tilde{V}, \tilde{\eta})$ on \mathbb{RP}^1 , where

$$V = \mathbb{RP}^1 \setminus [0, 1], \eta : [x, y] \mapsto y/x \text{ and } \tilde{V} = \mathbb{RP}^1 \setminus [1, 0], \tilde{\eta} : [x, y] \mapsto x/y$$

We check that the maps between $\tilde{\varphi}\tilde{U}$ and $\eta(V)$, as well as $\varphi(U)$ and $\tilde{\eta}(\tilde{V})$ are smooth in coordinates to conclude F is smooth.

$$\begin{aligned} \eta F \tilde{\varphi}^{-1}(\theta) &= \eta[\cos \frac{\theta}{2}, \sin \frac{\theta}{2}] \stackrel{\theta \in (-\pi, \pi)}{=} \eta[1, \tan \frac{\theta}{2}] = \tan \frac{\theta}{2} \\ \tilde{\eta} F \varphi^{-1}(\theta) &= \tilde{\eta}[\cos \frac{\theta}{2}, \sin \frac{\theta}{2}] \stackrel{\theta \in (0, 2\pi)}{=} \tilde{\eta}[\cot \frac{\theta}{2}, 1] = \cot \frac{\theta}{2} \end{aligned}$$

Where $\tan \frac{\theta}{2}$ is smooth on $(-\pi, \pi)$, while $2 \arctan$ is smooth on \mathbb{R} and $\cot \frac{\theta}{2}$ is smooth on $(0, 2\pi)$, while $2 \operatorname{arccot}$ is smooth on \mathbb{R} .

2. Equip $S^1 \times S^1$ with the following charts:

$$\begin{array}{ll} (U \times U, \varphi_1) & (U \times \tilde{U}, \varphi_2) \\ (\tilde{U} \times U, \varphi_3) & (\tilde{U} \times \tilde{U}, \varphi_4) \end{array}$$

Where U, \tilde{U} are defined in the previous question, and $\varphi_j : e^{i\theta_1} \times e^{i\theta_2} \mapsto (\theta_1, 2\theta_1 - \theta_2)$. One way to see these are valid coordinates, is given the standard coordinates $U \times U$ above (the other cases are similar), $(\theta_1, \theta_2) \mapsto (\theta_1, 2\theta_1 - \theta_2)$ is a diffeomorphism to its image (which is an open subset of \mathbb{R}^2 , since its the image of an open subset under an open map). To see this, the map is polynomial, hence smooth, and it has inverse on its image being $(\phi_1, \phi_2) \mapsto (\phi_1, -\phi_2 + 2\phi_1)$, which is also a polynomial map hence smooth. Then the coordinates on each chart C_j are $S \cap C_j$ having coordinates $e^{i\theta_1} \times e^{i2\theta_2} \xrightarrow{\varphi_j} (\theta_1, 0)$.

3. (a) Let $p \in \mathcal{Z}$, then since α is a regular value of f , f has constant rank 1 on $\mathcal{Y} \supset \mathcal{Z}$. This means we can apply the rank theorem, which gives us a charts U, V , such that $p \in U$, $F(p) \in V$ so that $f(x^1, \dots, x^n) \stackrel{\text{loc.}}{=} x^1$, this lets us write $F(p) = (p^1, g(p))$. Since (α, β) is a regular value of F , $p \in \mathcal{Z}$ is a regular point of F , i.e. dF_p has full rank.

$$dF_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x^1} \Big|_p & \frac{\partial g}{\partial x^2} \Big|_p & \cdots & \frac{\partial g}{\partial x^n} \Big|_p \end{bmatrix}$$

In order for this matrix to have rank 2, the second row must have rank 1. Now define $G : Y \rightarrow \mathbb{R}$, where $G := g|_Y$, if p is a regular point of G , then since $p \in \mathcal{Z}$ was arbitrary, β is a regular value of G , implying that $G^{-1}(\beta) = \mathcal{Z}$ is a submanifold of \mathcal{Y} by the regular level set theorem. It remains to show that p is in fact a regular point. But this is straightforward, since

$$dG_p = \begin{bmatrix} \frac{\partial g}{\partial x^1} \Big|_p & \frac{\partial g}{\partial x^2} \Big|_p & \cdots & \frac{\partial g}{\partial x^n} \Big|_p \end{bmatrix}$$

Which has rank one since dF_p has full rank.

(b) Let $\alpha \in (0, 1)$ and define $f : \mathbb{R}^4 \rightarrow \mathbb{R} (x, y, z, w) \mapsto x^2 + y^2 + z^2 + w^2$, and $g : \mathbb{R}^4 \rightarrow \mathbb{R} (x, y, z, w) \mapsto x^2 + y^2$. We first show that $(1, \alpha)$ is a regular value of $F := (f, g) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$, note all of these functions are smooth since they are polynomials. To see $(1, \alpha)$ is regular, let $p = (x, y, z, w) \in F^{-1}(1, \alpha)$, then

$$dF_p = \begin{bmatrix} 2x & 2y & 2z & 2w \\ 2x & 2y & 0 & 0 \end{bmatrix}$$

where one of $x, y \neq 0$ since $x^2 + y^2 = \alpha \neq 0$ and one of $z, w \neq 0$, since $z^2 + w^2 = 1 - \alpha \neq 0$. Then

$$\text{rk}(dF_p) = \dim \text{Rowsp}(dF_p) = \dim \text{Rowsp} \begin{bmatrix} 0 & 0 & 2z & 2w \\ 2x & 2y & 0 & 0 \end{bmatrix} = 2$$

so p is a regular point. Note that 1 is a regular value of $x^2 + y^2 + z^2 + w^2$, since for $p \in S^3$, df_p is of the same form as the first row of dF_p with not all of $x, y, z, w = 0$. Together with part (a), this tells us that $(f, g)^{-1}(1, \alpha)$ is a submanifold of $f^{-1}(1) = S^3$, i.e.

$$\{(x, y) | x^2 + y^2 = \alpha\} \cap S^3 = \{(x, y, z, w) | x^2 + y^2 = \alpha, z^2 + w^2 = 1 - \alpha\} = h^{-1}(\alpha, 1 - \alpha)$$

is a submanifold of S^3 . It remains to show that $h^{-1}(\alpha, 1 - \alpha)$ has rank 2, consider $p = (x, y, z, w) \in h^{-1}(\alpha, 1 - \alpha)$, then

$$dh_p = \begin{bmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & 2w \end{bmatrix}$$

this has rank 2, since $x^2 + y^2 = \alpha \neq 0$ implies $(x, y) \neq (0, 0)$ and $z^2 + w^2 = 1 - \alpha \neq 0$ implies $(z, w) \neq (0, 0)$ implying that $(\alpha, 1 - \alpha)$ is regular. So by the constant rank level set theorem, $h^{-1}(\alpha, 1 - \alpha)$ is a submanifold of \mathbb{R}^4 having rank 2, since rank is a property of the manifold, $h^{-1}(\alpha, 1 - \alpha) \subset S^3$ has equal rank 2.

(c) The submanifold approaches the unit circle. Equivalently, one of the free parameters gets arbitrarily small, crushing the circle in the small parameter arbitrarily close to a point.