1. I will start by computing the coordinate changes between U and $\tilde{U}$ on $U \cap \tilde{U} = S^1 \setminus \{-1, 1\}$.

$$\tilde{\varphi}\varphi^{-1} : \theta \mapsto \begin{cases} \theta & \theta \in (0,\pi) \\ \theta - 2\pi & \theta \in (\pi, 2\pi) \end{cases} \qquad \varphi\tilde{\varphi}^{-1} : \tilde{\theta} \mapsto \begin{cases} \tilde{\theta} & \tilde{\theta} \in (0,\pi) \\ \tilde{\theta} + 2\pi & \tilde{\theta} \in (-\pi, 0) \end{cases}$$

Then the tangent bundle TS^1 has charts (written in the coordinate form) $(\pi^{-1}(U), (\theta, v))$ and $(\pi^{-1}(\tilde{U}), (\tilde{\theta}, \tilde{v}))$. Where we can call the coordinate maps φ' and $\tilde{\varphi}'$ respectively. Here I will calculate the change of coordinates between $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$,

$$\tilde{v} = v\frac{d\tilde{\theta}}{d\theta} = \begin{cases} v\frac{d}{d\theta}\theta & \tilde{\theta} \in (0,\pi) \\ v\frac{d}{d\theta}(\theta - 2\pi) & \tilde{\theta} \in (-\pi, 0) \end{cases} = v$$
$$v = \tilde{v}\frac{d\tilde{\theta}}{d\tilde{\theta}} = \begin{cases} \tilde{v}\frac{d}{d\tilde{\theta}}\tilde{\theta} & \theta \in (0,\pi) \\ \tilde{v}\frac{d}{d\tilde{\theta}}(\tilde{\theta} + 2\pi) & \theta \in (-\pi, 0) = \tilde{v} \end{cases}$$

So taken together the change of coordinates are

$$(\theta, v) \mapsto \begin{cases} (\theta, v) & \theta \in (0,\pi) \\ (\theta - 2\pi, v) & \theta \in (\pi, 2\pi) \end{cases} \qquad (\tilde{\theta}, \tilde{v}) \mapsto \begin{cases} (\tilde{\theta}, \tilde{v}) & \tilde{\theta} \in (0,\pi) \\ (\tilde{\theta} + 2\pi, \tilde{v}) & \tilde{\theta} \in (-\pi, 0) \end{cases}$$

Now define the map

$$F : TS^1 \to S^1 \times \mathbb{R}$$
$$\left(e^{i\theta}, v\frac{d}{d\theta}\bigg|_{e^{i\theta}}\right) \mapsto (e^{i\theta}, v) \quad e^{i\theta} \in U$$
$$\left(e^{i\tilde{\theta}}, v\frac{d}{d\tilde{\theta}}\bigg|_{e^{i\tilde{\theta}}}\right) \mapsto (e^{i\tilde{\theta}}, v) \quad e^{i\tilde{\theta}} \in \tilde{U}$$

The map is well defined on the intersection $\pi^{-1}(U) \cap \pi^{-1}(\tilde{U})$ since the coordinate change on these points is the identity. Furthermore it is identity so clearly smooth and bijective on the S^1 component, bijectivity is also clear from the tangent space to $\mathbb{R}$, smoothness from the tangent space is also immediate since in coordinates this is the identity map from $\mathbb{R} \to \mathbb{R}$. smoothness of the inverse follows for the same reason.

2. We have from last homework that the coordinate change $U \to \tilde{U}$ is given by $(u_1, u_2) \mapsto (\frac{u_1}{u_1^2+u_2^2}, \frac{u_2}{u_1^2+u_2^2})$. Then we can use symmetry of the expression to simplify computation of the partials into two computations. For $j \neq i$ we have

$$\frac{\partial}{\partial u_i}\frac{u_i}{u_i^2 + u_j^2} = \frac{u_j^2 - u_i^2}{(u_i^2 + u_j^2)^2} \qquad \frac{\partial}{\partial u_i}\frac{u_j}{u_i^2 + u_j^2} = \frac{-2u_i u_j}{(u_i^2 + u_j^2)^2}$$

So the coordinate change is

$$(u_1, u_2, v_1, v_2) \mapsto \left(\frac{u_1}{u_1^2 + u_2^2}, \frac{u_2}{u_1^2 + u_2^2}, \frac{u_2^2 - u_1^2}{(u_2^2 + u_1^2)^2}v_1 - \frac{2u_1u_2}{(u_1^2 + u_2^2)^2}v_2, \frac{-2u_1u_2}{(u_1^2 + u_2^2)^2}v_1 + \frac{u_1^2 - u_2^2}{(u_2^2 + u_1^2)^2}v_2\right)$$

3. Cover S^2 by charts $U_1 = S^2 \setminus \{N\}, U_2 = S^2 \setminus \{S\}$, with stereographic projection coordinates $\varphi_1(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z})$ and $\varphi_2(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z})$. Cover $\mathbb{CP}^1$ in charts $V_1 = \{(z_1, z_2) | z_1 \neq 0\}, V_2 = \{(z_1, z_2) | z_2 \neq 0\}$, with coordinates $\phi_1(z_1, z_2) = \frac{z_2}{z_1}, \phi_2(z_1, z_2) = \frac{z_1}{z_2}$.

For any point p other than S, we can check that F is smooth by looking at the maps in terms of the chart U_2, namely we check that $\psi_1 F\varphi_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth.

$$(u, v) = (0, 0) \implies \psi_1 F\varphi_2^{-1}(0, 0) = \psi_1 F(0, 0, 1) = \psi_1(1, 0) = 0 = (u, -v)$$
$$(u, v) \neq (0, 0) \implies \psi_1 F\varphi_2^{-1}(u, v) = \psi_1 F(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{1 - u^2 - v^2}{u^2 + v^2 + 1})$$
$$= \psi_1\left(\frac{2}{u^2 + v^2 + 1}(u + iv, u^2 + v^2)\right) = \psi_1(1, u - iv) = (u, -v)$$

Then the map $(u,v) \mapsto (u,-v)$ being smooth implies that F is smooth with smooth inverse on each point not equal to S. To check for S, we look in terms of the chart U_1, namely we check $\psi_2 F \varphi_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth. We avoid case work here since $\frac{u^2+v^2-1}{u^2+v^2+1} \neq 1$.

$$\begin{aligned}\psi_2 F \varphi_1^{-1}(u,v) &= \psi_2 F\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \\ &= \psi_2\left(\frac{2}{u^2+v^2+1}(u+iv,1)\right) = \psi_2(u+iv,1) = (u,v)\end{aligned}$$

This is the identity map so it is smooth with smooth inverse. To see F is a bijection, hence a diffeomorphism, notice that $\varphi_i(U_i) = \mathbb{R}^2 = \psi_i(V_i)$, then $\psi_1 F \varphi_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2,\ (u,v) \mapsto (u,-v)$ is a bijection from $\varphi_2(U_2)$ to $\psi_1(V_1)$, so F is a bijection between U_2 and V_1. Similarly $\psi_2 F \varphi_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^2,\ (u,v) \mapsto (u,v)$ is a bijection from $\varphi(U_1)$ to $\psi(V_2)$, so F is a bijection from U_1 to V_2. Now we can prove F is surjective, since $F(S^2) = F(U_1 \cup U_2) = F(U_1) \cup F(U_2) = V_1 \cup V_2 = \mathbb{CP}^1$. To see F is injective, suppose $F(a) = F(b)$, if a, b are both in the same U_i, then this implies $a = b$ by injectivity on U_i. Now suppose without loss of generality $a \in U_1$ and $b \in U_2$, then $F(a), F(b)$ must be in $V_1 \cap V_2 \subset V_2$, so $b \in U_1$ is a contradiction. This implies that F is bijective, and we have already shown it is smooth so F is a diffeomorphism.