1. To show that F is increasing, first note that μ is nonnegative, so that for any $x > 0,\ y < 0$ we have

$$F(y) = -\mu((y,0]) \le F(0) = 0 \le \mu((0,x]) = F(x)$$

Then it suffices to show F is increasing on $[-\infty, 0)$ and $(0,\infty]$, both follow directly from monotonicity of μ,

$$\begin{aligned} 0 < x \le y &\implies (0,x] \subset (0,y] \implies \mu((0,x]) \le \mu((0,y]) \implies F(x) < F(y) \\ y \le x < 0 &\implies (x,0] \subset (y,0] \implies -\mu((0,y]) \le -\mu((0,x]) \implies F(y) < F(x) \end{aligned}$$

Let $x_0 \ge 0$, then $\lim_{x\searrow x_0} F(x) = \lim_{x\searrow x_0} \mu(0,x]$. It suffices to show for an arbitrary decreasing sequence $\{x_i\}_1^\infty$, we have $\lim_{i\to\infty} \mu(0,x_i] = \mu(0,x_0] = F(x_0)$ which follows from continuity from above, so the right sided limit exists and is equal to $F(x_0)$.

Now let $x_0 < 0$, then $\lim_{x\searrow x_0} F(x) = -\lim_{x\searrow x_0} \mu(x,0]$. It suffices to show for an arbitrary decreasing sequence $\{x_i\}_1^\infty$, we have $\lim_{i\to\infty} -\mu(x_i,0) = -\mu(x_0,0] = F(x_0)$, which follows from continuity from below, so the right hand sided limit exists and is equal to $F(x_0)$.

2. **(a)** F bounded. Proof being, assume $|F| < M$, define sets $\{E_i\}_1^\infty$, where $E_i = (-i,-i+1] \cup (i-1,i]$ then we have

$$\begin{aligned} m^F(\mathbb{R}) = \sum_1^\infty m^F(E_i) &= \lim_{n\to\infty} \sum_1^n m^F(E_i) \\ &= \lim_{n\to\infty} \sum_1^n F(i) - F(i-1) + F(-i+1) - F(-i) \\ &= \lim_{n\to\infty} F(n) - F(-n) \le \lim_{n\to\infty} |F(n)| + |F(-n)| \le 2M \end{aligned}$$

Conversely, assume that $m^F(\mathbb{R}) = M < \infty$ (note that $M > 0$), then for any $x \ge 0$, we have $F(x) - F(0) \le m^F(\mathbb{R}) = M$ by monotonicity, and $F(0) < F(x)$. Similarly, if $x < 0$, we have $F(x) \le F(0)$, and by monotonicity $F(0) - F(x) \le m^F(\mathbb{R}) = M$. Taken together for any x we have

$$F(0) - M \le F(x) \le M + F(0)$$

implying F is bounded.

(b) F continuous at x_0. Proof being, assume F is continuous at x_0, then for some sequence $\{\delta_n\}_1^\infty$, we have $|x - x_0| \le \delta_n \implies |F(x) - F(x_0)| < \frac{1}{n}$, then continuity from above implies

$$0 \le m^F(\{x_0\}) = m^F\left(\bigcap_1^\infty (\delta_n, x_0]\right) = \lim_{n\to\infty} m^F((\delta_n, x_0]) \le \lim_{n\to\infty} \frac{1}{n} = 0$$

Conversely we need only show right continuity. Suppose that $m^F(\{x_0\}) = 0$, and let $\epsilon > 0$, then continuity from above implies that

$$\lim_{n\to\infty} m^F(x_0 - \frac{1}{n}, x_0] = m^F\left(\bigcap_1^\infty (x_0 - \frac{1}{n}, x_0]\right) = m^F(\{x_0\}) = 0$$

so in particular, there exists N sufficiently large that $m^F(x_0 - \frac{1}{N}, x_0] = |F(x_0) - F(x_0 - \frac{1}{N})| < \epsilon$, and since F is increasing and right continuous this proves left continuity and hence continuity.

(c) $m^{F,*}$ is the point mass at 0. Proof being, let $0 \in E \subset \mathbb{R}$, then for any collection of half open intervals $\{I_i\}_1^\infty$, we have $0 \in I_n$ for some n. Then we can write $I_n = (a,b]$, for $a < 0,\ b \ge 0$, then $m_0^F(I_n) = 1$, so that $1 \le \sum_1^\infty m_0^F(I_i)$, and since this holds for all such covers of E, we have $1 \le m^{F,*}(E)$, and for the reverse inequality note that $E \subset \mathbb{R} \subset \bigcup_1^\infty (-i,-i+1] \cup (i-1,i+1]$, which is a countable union of half open intervals, all but $(-1,0]$ having $m_0^F(I) = 0$, so $1 \le m^{F,*}(E) \le m^{F,*}(\mathbb{R}) \le 1$.

Now suppose that $0 \notin E$, then $E \subset (-\infty, 0) \cup (0, \infty)$, but then $(-\infty, 0) \cup (0, \infty) = \bigcup_1^\infty (-n, 1/n) \cup (0, n)$ implies that

$$0 \leq m^{F,*}(E) \leq m^{F,*}((-\infty, 0) \cup (0, \infty)) \leq \sum_1^\infty m_0^F(-n, 1/n) + m_0^F(0, n) = 0$$

Finally note that $m^{F,*}(\emptyset) = 0$ by definition.

I claim that $M_F = \mathcal{P}(\mathbb{R})$, let $A \subset \mathbb{R}$, and $E \subset \mathbb{R}$. First assume that $0 \notin E$, then

$$m^{F,*}(E) = 0 = m^{F,*}(E \cap A) + m^{F,*}(E \cap A^c)$$

since neither of the sets measured on the right hand side of the equation contain 0. Now assume that $0 \in E$, then $0 \in E \cap A$ or $E \cap A^c$ but not both. This suffices to show that

$$m^{F,*}(E) = 1 = m^{F,*}(E \cap A) + m^{F,*}(E \cap A^c)$$

So each $A \in M_F$ is measurable.

(d) m^F counts the number of integers in a set E. Proof being, denote the floor function as F. First apply theorem 1.16 of Folland (since F is increasing and right continuous), then $m^F(a, b] = (F(b) - F(a))$ is a measure. If z is an integer, then we can apply continuity from above:

$$m^F(\{z\}) = m^F\left(\bigcap_1^\infty (z - \frac{1}{n}, z]\right) = \lim_{n\to\infty} m^F(z - \frac{1}{n}, z] = \lim_{n\to\infty} 1 = 1$$

So each integer singleton is a measurable set of measure 1. Now let $E \subset \mathbb{R} \setminus \mathbb{Z}$, then $E \subset \bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^\infty (n-1, n-\frac{1}{k}]$ is a countable union of sets with measure 0, hence

$$0 \leq m^F(E) \leq m^F(\bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^\infty (n-1, n-\frac{1}{k}]) \leq \sum_{n\in\mathbb{Z}} \sum_{k=1}^\infty m^F(n-1, n-\frac{1}{k}] = 0$$

Note that singletons are borel sets, then if $E \subset \mathbb{R}$, we can write $E \cap \mathbb{Z} = \{z_i\}_i$, then if there are infinitely many z_i:

$$\infty = m^F\left(\bigcup_i \{z_i\}\right) \leq m^F(E)$$

and if there are finitely many z_i:

$$m^F(E) = m^F(E \cap \mathbb{Z}) + m^F(E \cap \mathbb{Z}^c) = m^F\left(\bigcup_{i=1}^n \{z_i\}\right) + 0 = \sum_1^n m^F(\{z_i\}) = n$$

3.