Notation: Define the following notation for problem 1. Write for $f \in A[[X]]$, $\sum_0^\infty a_i X^i$

$$\begin{aligned}
&\deg : A[[X]] \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \\
&f \mapsto \min\{n \in \mathbb{N} \mid a_n \neq 0\} \\
&TC : A[[X]] \setminus 0 \to A \\
&f \mapsto a_{\deg f}
\end{aligned}$$

finally, if $a|b$ in A, then we can write $b = as$, where we denote s as $\frac{b}{a}$ or $a^{-1}b$, even when a is not invertible.

1. Let I be any ideal in $A[[X]]$, I will show that I is finitely generated. First define $J = \bigcup_I TC(f)$, closure of tail coefficients under addition and multiplication is clear, so that J is an ideal in A. By the Noetherian assumption on A, we may assume that J is finitely generated, and fix generators $\{TC(f_1), \ldots, TC(f_r)\}$.

Now define

$$L_k = \{a \in A \mid \exists f \in I, \text{ such that } \deg f = k \text{ and } a = TC(f)\}$$

once again it is clear that L_k is an ideal in A, so that it is finitely generated, we may fix generators $\{TC(g_1^k), \ldots, TC(g_{N_k}^k)\}$ where each g_i has degree k. Let $m = \max_i \deg f_i$, to make the notation easier later on, replace each f_i with $X^{m - \deg f_i} f_i$, so that each has the same degree.

$$L := (g_1^1, g_2^1, \ldots, g_{N_1}^1, g_1^2, \ldots, g_{N_m}^m) \subset A[[X]]$$

I claim that $I = L + (f_1, \ldots, f_r)$, which suffices to show that I is finitely generated. As proof, let $\varphi \in I$, $\deg \varphi = 0$ means we are done, so assume not. If $\deg \varphi < m$, then there is some $g \in L$, such that $\deg g \leq \deg \varphi$ and $TC(g) \mid TC(\varphi)$, then $\varphi - TC(g)^{-1} TC(\varphi) X^{\deg \varphi - \deg g} g$ is in I with degree strictly larger than φ, thus y repeating this process we may assume φ has degree at least m. Now I claim there are $h_1, \ldots, h_r \in A[[X]]$, such that $\varphi - \sum_1^n h_i f_i = 0$ completing the proof. Construct the h^i as follows, since $\varphi \in I$, we know that $TC(\varphi) \in J$, so that there is some $\{c_i\}_1^r \subset A$, such that $\sum_1^r c_i TC(f_i) = TC(\varphi)$, then take $h_i^1 = c_i X^{\deg \varphi - m}$ for each i. If at any point we have $\varphi - \sum_{i=1}^r \left(\sum_{j=1}^n h_i^j\right) f_i = 0$, then take $h_i = \sum_{j=1}^n h_i^j$, otherwise we can define h_i^{n+1} for each $1 \leq i \leq r$ as follows, since $\varphi - \sum_{i=1}^r \left(\sum_{j=1}^n h_i^j\right) f_i \in I$ we know that $TC\left(\varphi - \sum_{i=1}^r \left(\sum_{j=1}^n h_i^j\right) f_i\right) \in J$, so that there are some $\{c_i\}_1^r \subset A$, such that $\sum_1^r c_i TC(f_i) = TC\left(\varphi - \sum_{i=1}^r \left(\sum_{j=1}^n h_i^j\right) f_i\right)$, then take $h_i^{n+1} = X^{\deg\left(\varphi - \sum_{i=1}^r \left(\sum_{j=1}^n h_i^j\right) f_i\right) - m} c_i$, it is important to note that for each n we have $\deg h_i^n < \deg h_i^{n+1}$, and since they consist only as a single term, $h_i := \sum_{j=1}^\infty h_i^j$ defines an element of $A[[X]]$ for each i. Finally, by construction we get that $\varphi - \sum_{i=1}^r h_i f_i = 0$, so that $\varphi \in L + (f_1, \ldots, f_r) = I$, and since the ideal I was arbitrary, we conclude that $A[[X]]$ is Noetherian.

2. Let $h_1, \ldots, h_m$ be the rows of A in RREF, it is immediate that $(h_1, \ldots, h_m) \subset J$, and that

$$(LM(g_1), \ldots, LM(g_m)) = (LM(h_1), \ldots, LM(h_m))$$

so it will suffice to show the result for the reduced matrix. Here we denote $LM(h_i) = x_{p_i}$ and $P := \{p_1, \ldots, p_m\}$. By Buchburger's criterion, it will suffice to show that for any

$i < j$ we have $\overline{S(h_i, h_j)}^{\{h_1,\ldots,h_m\}} = 0$. Let $i < j$, then

$$S(h_i, h_j) = x_{p_j}h_i - x_{p_i}h_j = \sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_{p_j}x_k - \sum_{\substack{k>p_j \\ k\notin P}} a_{jk}x_{p_i}x_k$$

This is not divisible by x_{p_k} for any $k < i$, so the division algorithm first divides by h_i to produce

$$\begin{aligned}\sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_{p_j}x_k - \sum_{\substack{k>p_j \\ k\notin P}} a_{jk}x_{p_i}x_k + \left(\sum_{\substack{k>p_j \\ k\notin P}} a_{jk}x_k\right) h_i &= \sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_{p_j}x_k + \sum_{\substack{k>p_j \\ k\notin P}} \left(a_{jk}x_k \sum_{\substack{\ell>p_i \\ \ell\notin P}} a_{i\ell}x_\ell\right) \\ &= \sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_{p_j}x_k + \sum_{k,\ell\in\{1,\ldots,m\}\setminus P} c_{k\ell}x_kx_\ell\end{aligned}$$

For $c_{k\ell} \in \mathbb{C}$. It follows once again that no monomial is divisible by x_{p_k} for any $k < j$, so we divide by $h_j$to reduce to

$$\begin{aligned}\sum_{k,\ell\in\{1,\ldots,m\}\setminus P} c_{k\ell}x_kx_\ell + \sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_{p_j}x_k - \left(\sum_{\substack{k>p_i \\ k\notin P}} a_{ik}x_k\right) h_j &= \sum_{k,\ell\in\{1,\ldots,m\}\setminus P} c_{k\ell}x_kx_\ell + \sum_{\substack{k>p_i \\ k\notin P}} \left(a_{ik}x_k \sum_{\substack{\ell>p_j \\ \ell\notin P}} a_{j\ell}x_\ell\right) \\ &= \sum_{k,\ell\in\{1,\ldots,m\}\setminus P} d_{kl}x_kx_\ell\end{aligned}$$

for $d_{kl} \in \mathbb{C}$. Let $F := \sum_{k,\ell\in\{1,\ldots,m\}\setminus P} d_{kl}x_kx_\ell$, this is the remainder of $S(h_i, h_j)$ by division from $\{h_1, \ldots, h_m\}$ since none of the monomial terms are divisible by x_k for $k \in p$. If $F = 0$, then we are done, so assume not for the sake of contradiction.

We know that $F \in J$ by construction (equivalently $(F) \subset J$), by corollary of Hilbert's Nullstellensatz we know that $V((F)) \supset V(J)$, to get the desired contradiction it will suffice to show there is some $\mathbf{x} \in V(J) \setminus V((F))$. Since $F \neq 0$, there is some $d_{k,\ell} \neq 0$, if this is true for some $d_{k,k}$, then take $\mathbf{y} = e_k$, otherwise if all $d_{kk} = 0$, then there is some $k \neq \ell$, such that $d_{kl} = 0$, and we take $\mathbf{y} = e_k + e_\ell$, note that in either case $\mathbf{y}$ has the property that $F(\mathbf{y} + \sum_{s=1}^m c_se_{p_s}) \neq 0$, for any $\{c_1, \ldots, c_m\} \subset \mathbb{C}$. Now we can define $\mathbf{x}$ as follows:

$$\begin{aligned} k = \ell : \qquad & \mathbf{x} = \mathbf{y} + \sum_{s=1}^m -a_{sk}e_{p_s} \\ k \neq \ell : \qquad & \mathbf{x} = \mathbf{y} + \sum_{s=1}^m (-a_{sk} - a_{s\ell})e_{p_s}\end{aligned}$$

since s, k are not pivots, this always defines a solution to the system of equations given by A, hence $\mathbf{x} \in V(J)$, but $\mathbf{x}$ is of the form $\mathbf{y} + \sum_1^m c_se_{p_s}$, implying that $\mathbf{x} \in V(J) \setminus V((F))$ which is a contradiction. $\square$

3. Denote $A = k[X_1, \ldots, X_n]$. $LT(I) \overset{\text{def.}}{=} \left(\bigcup_{f\in I} LT(f)\right)$, since for each $f \in I$, we have $LT(f)$ is in $LT(I)$, and $LT(LT(f)) = LT(f)$, we find that $LT(f) \in LT(LT(I))$ for each $f \in I$, hence $\bigcup_{f\in I} LT(f) \subset LT(LT(I))$, implying that $LT(I) \subset LT(LT(I))$.

Conversely, let $f \in LT(LT(I))$, then $f = \sum_1^r c_i LT(h_i)$, for $c_i \in A$ and $h_i \in LT(I)$, to show that $f \in LT(I)$ it will suffice to show that each $LT(h_i)$ is in $LT(I)$. Let $h \in \{h_1, \ldots, h_r\}$, then $h = \sum_1^m d_i LT(g_i)$ for $d_i \in A$ and $g_i \in I$, then each monomial term in h is divisible by $LT(g_i)$ for some i, so in particular, there is some $i \in \{1, \ldots, m\}$, such that $LT(g_i)|LT(h)$, but this implies that $LT(h) \in LT(I)$, and hence so is f, since f was arbitrary this gives the desired equality of sets. $\square$

4. define $g := xy - y^2$, $h := y^3 + y^2$, then (f_1, f_2, g, h) is a Grobner basis. We first need show that $g, h \in I$, this is straightforward since

$$g = xy - y^2 = (x-1)f_2 - f_1 \in I$$
$$h = y^3 + y^2 = -((xy^2 - y^3 + xy - y^2) - (x^2y^2 + xy^2) + (x^2y^2 - xy)) = -((y+1)g - xf_2 + f_1) \in I$$

Now it will suffice to check Buchburger's criterion for each pair,

$$S(f_1, f_2) = f_1 - xf_2 = -xy^2 - xy$$

long division furnishes $-xy^2 - xy + f_2 = -xy + y^2$, then $-xy + y^2 + g = 0$.

$$S(f_1, g) = f_1 - xyg = xy^3 - xy$$

Dividing by f_2 gives $xy^3 - xy - yf_2 = -xy - y^3$, dividing by g we get $-xy - y^3 + g = -y^3 - y^2$, dividing by h gives $-y^3 - y^2 + h = 0$.

$$S(f_1, h) = yf_1 - x^2h = -x^2y^2 - xy^2$$

dividing by f_1 gives $-x^2y^2 - xy^2 + f_1 = -xy^2 - xy$, dividing by f_2 gives $-xy^2 - xy + f_2 = -xy + y^2$, dividing by g gives $-xy + y^2 + g = 0$.

$$S(f_2, g) = f_2 - yg = y^3 + y^2$$

we cannot divide any of the monomials by f_1, f_2 or g, dividing by h gives $y^3 + y^2 - h = 0$.

$$S(f_2, h) = -xy^2 + y^3$$

dividing by f_2 gives $-xy^2 + y^3 + f_2 = y^3 + y^2$, dividing by h gives $y^3 + y^2 - h = 0$.

$$S(g, h) = y^2g - xh = xy^2 - y^4$$

dividing by f_2 we get $xy^2 - y^4 - f_2 = y^4 - y^2$, dividing by h gives $y^4 - y^2 - (y+1)h = 0$. $\square$

5. I claim that g_1, g_2, g_3, g_4 form a Grobner basis, where the g_i are defined as follows,

$$g_1 := x^2 - y$$
$$g_2 := xy - z$$
$$g_3 := xz - y^2$$
$$g_4 := y^3 - z^2$$

Note that $(g_i)_1^6 \supset I$, since $f_1 = g_1$ and $f_2 = xg_1 + g_2$. To see the reverse inclusion of ideals,

$$\begin{aligned} g_2 &= f_2 - xf_1 \in I \\ g_3 &= yg_1 - xg_2 \in I \\ g_4 &= zg_2 - yg_3 \in I \end{aligned}$$

To check that g_1, g_2, g_3, g_4 form a Grobner basis, we use the Buchburger criterion, (I will do the division algorithm in-line for brevity)

$$\begin{aligned}
S(g_1, g_2) &= x^2y - y^2 - x^2y + xz = -y^2 + xz \overset{-g_3}{\rightsquigarrow} 0 \\
S(g_1, g_3) &= zx^2 - zy - x^2z + xy^2 = xy^2 - zy \overset{-yg_2}{\rightsquigarrow} 0 \\
S(g_1, g_4) &= y^3(x^2 - y) - x^2(y^3 - z^2) = x^2z^2 - y^4 \overset{-z^2g_1}{\rightsquigarrow} -y^4 + yz^2 \overset{+yg_4}{\rightsquigarrow} 0 \\
S(g_2, g_3) &= xyz - z^2 - xyz + y^3 = y^3 - z^2 \overset{-g_4}{\rightsquigarrow} 0 \\
S(g_2, g_4) &= xy^3 - zy^2 - xy^3 + xz^2 = xz^2 - zy^2 \overset{-zg_3}{\rightsquigarrow} 0 \\
S(g_3, g_4) &= xzy^3 - y^5 - xzy^3 + xz^3 = xz^3 - y^5 \overset{-z^2g_3}{\rightsquigarrow} -y^5 + y^2z^2 \overset{+y^2g_4}{\rightsquigarrow} 0 \quad \square
\end{aligned}$$

6. First identify $LM(A) = \{x_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4} \mid d_1 \geq d_2 \text{ and } d_3 \geq d_4\} \cong \{(d_1, d_2, d_3, d_4) \in \mathbb{Z}^4_{\geq 0} \mid d_1 \geq d_2 \text{ and } d_3 \geq d_4\}$. As proof if (d_1, d_2, d_3, d_4) in $LM(A)$, then if $d_2 > d_1$ or $d_4 > d_3$ we can apply $(1\,2)$ or $(3\,4)$ respectively to obtain another term in the polynomial which is strictly larger, contradicting starting with the leading term. Conversely if $d_1 \geq d_2$ and $d_3 \geq d_4$, then we can form the polynomial $f \in A$,

$$f = x_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4} + x_1^{d_3}x_2^{d_2}x_3^{d_1}x_4^{d_4} + x_1^{d_1}x_2^{d_4}x_3^{d_3}x_4^{d_2} + x_1^{d_3}x_2^{d_4}x_3^{d_1}x_4^{d_2}$$

so that $LM(f) = x_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4}$.

Now it remains to show that $\langle (1,0,0,0), (1,1,0,0), (0,0,1,0), (0,0,1,1) \rangle = \{(d_1, d_2, d_3, d_4) \in \mathbb{Z}^4_{\geq 0} \mid d_1 \geq d_2 \text{ and } d_3 \geq d_4\}$ as a monoid. So let $(d_1, d_2, d_3, d_4) \in LM(A)$, then

$$(d_1 - d_2)(1,0,0,0) + d_2(1,1,0,0) + (d_3 - d_4)(0,0,0,1) + d_4(0,0,1,1) = (d_1, d_2, d_3, d_4)$$

as desired. $\square$