I worked with Aaron on problem 1.

1. (a)

$$g = \langle \frac{\partial \varphi^{-1}}{\partial x}, \frac{\partial \varphi^{-1}}{\partial x} \rangle = \langle (1, f'(x)), (1, f'(x)) \rangle = 1 + (f'(x))^2$$

Thus the formula for arc length in this context is

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

(b)

$$g = (\varphi^{-1})^* g_{\text{EUC}} = \begin{bmatrix} \frac{\partial(\varphi^{-1})^x}{\partial x} & \frac{\partial(\varphi^{-1})^y}{\partial x} & \frac{\partial(\varphi^{-1})^z}{\partial x} \\ \frac{\partial(\varphi^{-1})^x}{\partial y} & \frac{\partial(\varphi^{-1})^y}{\partial y} & \frac{\partial(\varphi^{-1})^z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial(\varphi^{-1})^x}{\partial x} & \frac{\partial(\varphi^{-1})^x}{\partial y} \\ \frac{\partial(\varphi^{-1})^y}{\partial x} & \frac{\partial(\varphi^{-1})^y}{\partial y} \\ \frac{\partial(\varphi^{-1})^z}{\partial x} & \frac{\partial(\varphi^{-1})^z}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)^2 + 1 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & \left(\frac{\partial f}{\partial y}\right)^2 + 1 \end{bmatrix}$$

So that $\det(g_{ij}) = 1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$. It follows that the surface area formula in this context simplifies to

$$\int_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx dy$$

(c) Define the map $g : U \to S \cap V$ by $g(x,y) = (x, y, f(x,y))$ and define $G : U \to \mathbb{R}$, $G(x,y) = (F \circ g)(x,y)$. Since g maps onto a level set of F, we have

$$dG = dF \circ dg = 0$$

$$0 = dG = dF \circ dg = \begin{bmatrix} \frac{\partial}{\partial x}F & \frac{\partial}{\partial y}F & \frac{\partial}{\partial z}F \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial}{\partial x}f & \frac{\partial}{\partial y}f \end{bmatrix}$$

$$= \left(\frac{\partial}{\partial x}F + \frac{\partial}{\partial z}F\frac{\partial}{\partial x}f\right)\frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}F + \frac{\partial}{\partial z}F\frac{\partial}{\partial y}f\right)\frac{\partial}{\partial y}$$

Which gives allows us to solve for the f partials in terms of the F partials

$$\frac{\partial}{\partial x}f = -\left(\frac{\partial}{\partial x}F\right)\left(\frac{\partial}{\partial z}F\right)^{-1} \qquad\qquad \frac{\partial}{\partial y}f = -\left(\frac{\partial}{\partial y}F\right)\left(\frac{\partial}{\partial z}F\right)^{-1}$$

Now we can derive this formula from

$$\text{Area}(S \cap V) = \int_U \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dxdy$$

$$= \int_U \sqrt{\left(\frac{\frac{\partial}{\partial z}F}{\frac{\partial}{\partial z}F}\right)^2 + \left(\frac{\frac{\partial}{\partial x}F}{\frac{\partial}{\partial z}F}\right)^2 + \left(\frac{\frac{\partial}{\partial y}F}{\frac{\partial}{\partial z}F}\right)^2} \, dxdy$$

$$= \int_U \frac{\sqrt{\left(\frac{\partial}{\partial x}F\right)^2 + \left(\frac{\partial}{\partial y}F\right)^2 + \left(\frac{\partial}{\partial z}F\right)^2}}{|\frac{\partial}{\partial z}F|} $$

$$= \int_U \frac{|\nabla F|}{|\frac{\partial}{\partial z}F|} \, dxdy$$

As in part (b). Now checking the formula:

$$\int_U \frac{|\nabla F|}{|\frac{\partial}{\partial z}F|} \, dxdy = \int_U 2\frac{\sqrt{x^2 + y^2 + z^2}}{2z} \, dxdy = \int_U \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \, dxdy$$

$$= \int_U \sqrt{1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}} \, dxdy$$

$$= \int_U \sqrt{\frac{1}{1 - x^2 - y^2}} \, dxdy$$

$$= \int_{y=-1}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2}} \, dxdy$$

$$= \int_{y=-1}^{1} \arctan\frac{x}{\sqrt{1 - x^2 - y^2}} \Big|_{x \to -\sqrt{1-y^2}}^{x \to \sqrt{1-y^2}}$$

$$= \int_{y=-1}^{1} \lim_{x \to \infty} \arctan(x) - \lim_{x \to -\infty} \arctan(x)$$

$$= \int_{-1}^{1} \pi \, dy = 2\pi$$

As desired.

2. (a) We compute

$$A = \begin{bmatrix} \frac{\partial}{\partial u}f^x & \frac{\partial}{\partial u}f^y & \frac{\partial}{\partial u}f^z \\ \frac{\partial}{\partial v}f^x & \frac{\partial}{\partial v}f^y & \frac{\partial}{\partial v}f^z \end{bmatrix} = \begin{bmatrix} -b\sin u\cos v & -b\sin u\sin v & b\cos u \\ -(a + b\cos u)\sin v & (a + b\cos u)\cos v & 0 \end{bmatrix}$$

Then in (u, v) coordinates, we have $g = f^* g_{\text{EUC}}$, i.e.

$$g = AA^T = \begin{bmatrix} b^2 & 0 \\ 0 & (a + b\cos u)^2 \end{bmatrix}$$

(b) Write $c_1 = f \circ \gamma_1$, $c_2 = f \circ \gamma_2$. Then we have $\dot\gamma_1^1 = 0, \dot\gamma_1^2 = 1$, so that $|\dot\gamma_1|_g^2 = g_{2,2} = (a+b\cos\pi)^2 = (a-b)^2$. It follows that

$$L(c_1) = \int_0^{2\pi} a - b \, dt = 2\pi(a - b)$$

Similarly, $\dot\gamma_2^1 = 1, \dot\gamma_2^2 = 0$, so that $|\dot\gamma_1|_g^2 = g_{1,1} = b^2$. It follows that

$$L(c_2) = \int_0^{2\pi} b = 2b\pi$$

(c) $dA = \sqrt{\det g}\ dudv = b(a + b\cos u)dudv$, hence the surface area is given by

$$SA = \int_0^{2\pi} \int_0^{2\pi} b(a + b\cos u)dudv = \int_0^{2\pi} uab + b^2 \sin u|_0^{2\pi} dv = \int_0^{2\pi} 2\pi ab\, dv = (2\pi)^2 ab$$