1. Let $P \subset A$ be prime, then it will suffice to show that A/P is a field which is equivalent to maximality of P by the correspondence theorem. Consider $0 \neq x \in A/P$, then choose $n \geq 2$ such that $x^n = x$, it follows that $x(1-x^{n-1}) = x - x = 0$, and since P is prime A/P is a domain which implies that $1 - x^{n-1} = 0$, so that $x^{n-1} = 1$ in A/P. $\square$

2. Suppose that M is not flat, then we can fix modules A, B, such that

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact, but

$$0 \longrightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M$$

is not. It follows that there is some $0 \neq \sum_1^n a_i \otimes x_i \in A \otimes M$, such that $f \otimes 1_M(\sum_1^n a_i \otimes x_i) = 0$. I claim that $M_0 = (x_1, \dots, x_n)$ is the desired submodule. To see this, note $(f \times 1)|_{A \times M_0} = f \times 1_{M_0}$, and if j is the map $A \times M \to A \otimes M$ in the definition of the tensor, then $j|_{A \times M_0}$ is equal the map $A \times M_0 \to A \otimes M_0$ in the definition of the tensor. It follows that for any $v \in A \otimes M_0$, $v = j|_{A \times M_0}(u),\ u \in A \times M_0$, so that

$$f \otimes 1_{M_0}(v) = f \otimes 1_{M_0} j|_{A \times M_0}(u) = f \times 1_{M_0}(u) = f \times 1_M(u) = f \otimes 1_M j(u)$$

and hence $f \otimes 1_{M_0} j|_{M_0}(\sum_1^n (a_i, x_i)) = f \otimes 1_M j(\sum_1^n (a_i, x_i)) = f \otimes 1_M(\sum_1^n a_i \otimes x_i) = 0$, where $0 \neq \sum_1^n a_i \otimes x_i = j(\sum_1^n (a_i, x_i)) = j|_{A \times M_0}(\sum_1^n (a_i, x_i))$ which suffices to show that $f \otimes 1_{M_0}$ is not injective, and hence M_0 is not flat, with the following sequence as witness.

$$0 \longrightarrow A \otimes M_0 \xrightarrow{f \otimes 1_{M_0}} B \otimes M_0 \quad \square$$

3. Since $C[X]$ is a PID, it satisfies Bezout's identity. So assume f_1, f_2 are coprime polynomials, it follows that there exist $g, h \in \mathbb{C}[X]$, such that $f_1 h + f_2 g = 1$. Now let $m \otimes n \in M_1 \otimes M_2$, it follows that

$$\begin{aligned} m \otimes n &= (f_1 h + f_2 g)(m \otimes n) = f_1 h(m \otimes n) + f_2 g(m \otimes n) = h(f_1 m \otimes n) + g(m \otimes f_2 n) \\ &= h(0 \otimes n) + g(m \otimes 0) = 0 \end{aligned}$$

Conversely, let $a \in \mathbb{C}$, such that $f_1(a) = f_2(a) = 0$. Let $I = (X - a)$ and consider the map multiplication map

$$m : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X]/(X - a),\ (f, g) \mapsto fg + I$$

To see that this defines a bilinear map $M_1 \times M_2 \to \mathbb{C}[X]/I$ it will suffice to check that m is well defined on cosets so that we can take the induced bilinear map

$$\overline{m} : M_1 \times M_2 \to \mathbb{C}[X]/I,\ (f + (f_1), g + (f_2)) \mapsto fg + I$$

Let $g_1, g_2, h_1, h_2 \in \mathbb{C}[X]$, then

$$m(g_1 + h_1 f_1, g_2 + h_2 f_2) = g_1 g_2 + g_1 h_2 f_2 + g_2 h_1 f_1 + h_1 h_2 f_1 f_2 + I = g_1 g_2 + I$$

the last equality following since both $f_i \in I$. It follows that $\overline{m} : M_1 \times M_2 \to \mathbb{C}(X)/I$ is a nonzero (since $(1,1) \mapsto 1$) bilinear map, so $\overline{m} = \eta j$ where j is the map from the definition of the tensor product and $\eta : M_1 \otimes M_2 \to \mathbb{C}[X]/I$. Since $\overline{m}$ is non-zero, it follows that η is nonzero and hence $M_1 \otimes M_2 \neq 0$ since $\eta \notin \{0\} = \mathrm{Hom}(0, \mathbb{C}[X]/I)$. $\square$

4. Consider the exact sequence of A modules

$$0 \longrightarrow (t) \xrightarrow{\iota} A$$

Where $\iota : t \mapsto t$, injectivity and therefore exactness is clear. To see N is not flat, tensor the above sequence to get

$$0 \longrightarrow (t) \otimes_A N \xrightarrow{\iota_*} A \otimes_A N$$

Here we have

$$\iota_*(t \otimes e_3) = t(1 \otimes e_3) = 1 \otimes te_3 = 1 \otimes 0 = 0$$

So it will suffice to show that $0 \neq t \otimes e_3 \in (t) \otimes_A N$ to conclude that ι_* is not injective. Consider $\phi : N \to N$, $\phi : e_i \mapsto \delta_{i3} e_2$ and extending linearly (here δ_{i3} is the Kronecker delta). Define the map $\varphi : (t) \times N \to N$ via $\varphi : (x, y) \mapsto x\phi(y)$, it is immediate that φ is A-bilinear, hence φ factors through $j : (t) \times N \to (t) \otimes_A N$. We have that $\varphi(t, e_3) = te_2 = e_1 \neq 0$, so that since φ factors through j we have $t \otimes e_3 = j(t, e_3) \neq 0$. □

5. Suppose that $r \leq n$, and $g_1, g_2, \ldots, g_r$ generate I as an A module. It is immediate that I^2 is the ideal generated by all degree 2 monomials of A, it follows that by assumption each monomial in $f_1, \ldots f_m$ is divisible by some element of I^2, and hence $(f_i)_1^m / I^2 = 0$. Furthermore, $\{g_i\}_1^r$ generating I as an A-module implies that $\{g_i + I^2\}_1^r$ generate I/I^2 as an A/I module, since $A/I \otimes_A I \cong I/I^2$ (here the bilinear map inducing isomorphism is multiplication). Applying the third isomorphism theorem,

$$A/I \cong \frac{\mathbb{R}[X_1, \ldots, X_n]/I}{(f_1, \ldots, f_m)/I} \cong \mathbb{R}[X_1, \ldots, X_n]/I \cong \mathbb{R}$$

so that in fact $\{g_i + I^2\}_1^r$ span I/I^2 as an A/I vectorspace. Here

$$I/I^2 \cong \bigoplus_1^n X_i A/I$$

where both the spanning and zero intersection properties are obvious, implying that I/I^2 has dimension n as an A/I vectorspace, since any spanning set must have atleast n elements, we conclude that that $r = n$ □

6. $A[X] = \bigoplus_0^\infty AX^i$ as an A-module, assume for contradiction that $\bigoplus_0^\infty AX^i \cong \bigoplus_0^\infty A$ is not flat, then applying problem 2, there is some finitely generated submodule M_0, such that M_0 is not flat. Since submodules of free modules are free, we know that $M_0 \cong \bigoplus_1^n A$, implying that $\bigoplus_1^n A$ is not flat, but this is a contradiction, since this is only the case if

$$0 \longrightarrow K \xrightarrow{f} L$$

is exact, but the following sequence is not

$$0 \longrightarrow K \otimes \bigoplus_1^n A \xrightarrow{f \otimes 1_{\bigoplus_1^n A}} L \otimes \bigoplus_1^n A$$

but this is equivalent to the following sequence not being exact

$$0 \longrightarrow \bigoplus_1^n K \otimes A \xrightarrow{\bigoplus_1^n f \otimes 1_A} \bigoplus_1^n L \otimes A$$

which once again is equivalent to the following not being exact

$$0 \longrightarrow \bigoplus_1^n K \xrightarrow{\bigoplus_1^n f} \bigoplus_1^n L$$

where $\bigoplus_1^n f$ is injective since f is. $\square$