1. (a) We first need to show the homomorphism property, to do so apply the chain rule, where we note that $C_g(e) = e$ for all g. Then

$$d(C_{g_1}C_{g_2})_e(x) = d(C_{g_1})_{C_{g_2}(e)} \circ d(C_{g_2})_e(x) = d(C_{g_1})_e \circ d(C_{g_2})_e(x)$$

This also implies that the pushforward of the identity is the identity (by swapping either of g_1 or g_2 for the identity).

Smoothness follows from lie group multiplication being smooth, and the pushforward of a smooth map is smooth.

(b) We know that e^{tX} is a path satisfying $e^{tX}|_{t=0} = \mathbf{1_n}$ and $\frac{d}{dt}e^{tX}|_{t=0} = X$. Then for $g \in G$ we have

$$d(C_g)_e = \frac{d}{dt}|_{t=0}C_g(e^{tX}) = \frac{d}{dt}|_{t=0}e^{tgXg^{-1}} = gXg^{-1}$$

(c) We take the path $\gamma(t) = e^{tX}$, which satisfies $\gamma(0) = \mathbf{1_n},\ \gamma'(0) = X$, then

$$\begin{aligned} d(Ad)_e(x)(y) &= \frac{d}{dt}|_{t=0}\mathrm{Ad}_\gamma(t)(y) = \frac{d}{dt}|_{t=0}\gamma(t)y\gamma^{-1}(t) \\ &= \frac{d}{dt}|_{t=0}e^{tX}ye^{-tX} = \left(\frac{d}{dt}|_{t=0}e^{tX}y\right)\mathbf{1_n} + \mathbf{1_n}y\frac{d}{dt}|_{t=0}e^{-tX} \\ &= Xy - yX = [X, y] \end{aligned}$$

2. (a) We can simply compute the product of basis elements, first note that $\sigma_i^2 = \mathbf{1}$, and that

$$\begin{aligned} \sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\ \sigma_1\sigma_3 &= -i\sigma_2 = -\sigma_3\sigma_1 \\ \sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \end{aligned}$$

Since each of σ_i are of trace 0, we get for some c_i which turn out not to matter

$$\begin{aligned} \mathrm{Tr}(xy) &= \mathrm{Tr}((x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)(y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3)) \\ &= \mathrm{Tr}(\mathbf{1}(x_1y_1 + x_2y_2 + x_3y_3)) + \mathrm{Tr}(\sum_1^3 c_i\sigma_i) \\ &= \sum_1^3 2x_iy_i + \sum_1^3 c_i\mathrm{Tr}(\sigma_i) = 2\sum_1^3 x_iy_i \end{aligned}$$

So we may conclude that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{2}\mathrm{Tr}xy$$

We already verified in 1a that $\varphi : g \mapsto \mathrm{Ad}(g)$ is a Lie group representation, i.e. This map satisfies the homomorphism property. It remains to show that φ maps into SO(3). Note that orthogonal matrices are characterized by preservation of inner products, i.e. M orthogonal iff for any x, y we have $\langle Mx, My\rangle = \langle x, y\rangle$. To show that $\varphi(g)$ satisfies this property see the following computation:

$$\langle\varphi_g(x), \varphi_g(y)\rangle = \langle gxg^{-1}, gyg^{-1}\rangle = \frac{1}{2}\mathrm{Tr}(gxyg^{-1}) = \frac{1}{2}\mathrm{Tr}(xy) = \langle x, y\rangle$$

Here we use the fact that eigenvalues and hence trace are invarient under conjugation. Finally, we note that $\varphi(\mathbf{1}) = \mathbf{1}$, and for any g, $\varphi(g)$ is orthogonal implies that $\det\varphi(g) = \pm 1$. Note that the continuous map

$$\mathrm{SU}(2) \to S^3$$

$$\begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \mapsto (\mathrm{Re}(u), \mathrm{Im}(u), \mathrm{Re}(v), \mathrm{Im}(v))$$

is a homeomorphism (closed map lemma by SU(2) compact and S^3 Hausdorff). Since S^3 is connected, this implies connectedness of SU(2). It follows that since φ is continuous, that $\varphi(\mathrm{SU}(2))$ is also connected. If we were to have

$$\varphi(\mathrm{SU}(2)) \cap \{X \in O^3 | \det X = -1\} \neq \emptyset \text{ and } \varphi(\mathrm{SU}(2)) \cap \{X \in O^3 | \det X = 1\} \neq \emptyset$$

then this would contradict connectedness of $\varphi(\mathrm{SU}(2))$, and since $\mathbf{1} \in \varphi(\mathrm{SU}(2)) \cap \mathrm{SO}(3) \neq \emptyset$ we must have $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$.

(b) write $g := \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$. Then

$$\begin{aligned}
g\sigma_1 g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-2i\theta} \\ e^{2i\theta} & 0 \end{bmatrix} = \cos 2\theta \sigma_1 + \sin 2\theta \sigma_2 \\
g\sigma_2 g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -ie^{-2i\theta} \\ ie^{2i\theta} & 0 \end{bmatrix} = -\sin 2\theta \sigma_1 + \cos 2\theta \sigma_2 \\
g\sigma_3 g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & -e^{-i\theta} \end{bmatrix} = \sigma_3
\end{aligned}$$

So that $\varphi(g)$ acts as

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on the basis $\{\sigma_i\}_1^3$

(c) First notice that the pushforward of conjugation on a single Pauli matrix is just the bracket as in question 1c ($\mathrm{ad}_x(y)$). Computing this we get (see my mult table in 2a)

$$\begin{aligned}
[\sigma_\alpha/2i, \sigma_j] &= 0 && i = j \\
[\sigma_1/2i, \sigma_2] &= \frac{1}{2i}(i\sigma_3 - (-i\sigma_3)) = \sigma_3 \\
[\sigma_1/2i, \sigma_3] &= \frac{1}{2i}(-i\sigma_2 - i\sigma_2) = -\sigma_2 \\
[\sigma_2/2i, \sigma_1] &= \frac{1}{2i}(-i\sigma_3 - \sigma_3) = -\sigma_3 \\
[\sigma_2/2i, \sigma_3] &= \frac{1}{2i}(i\sigma_1 - (-i\sigma_1)) = \sigma_1 \\
[\sigma_3/2i, \sigma_1] &= \frac{1}{2i}(i\sigma_2 + i\sigma_2) = \sigma_2 \\
[\sigma_3/2i, \sigma_2] &= \frac{1}{2i}(-i\sigma_1 - i\sigma_1) = -\sigma_1
\end{aligned}$$

Using this rule, we can write the matrices.

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) Given $g = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix}$, compute $g^{-1} = \begin{bmatrix} \overline{u} & \overline{v} \\ -v & u \end{bmatrix}$. Then we compute the action of g on each σ_i,

$$g\sigma_1 g^{-1} = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \begin{bmatrix} -v & u \\ \overline{u} & \overline{v} \end{bmatrix} = \begin{bmatrix} -2\mathrm{Re}(uv) & u^2 - \overline{v}^2 \\ \overline{u^2 - \overline{v}^2} & 2\mathrm{Re}(uv) \end{bmatrix} = -2\mathrm{Re}(uv)\sigma_3 + c\sigma_1 + d\sigma_2$$

Where $c = \mathrm{Re}(u)^2 - \mathrm{Re}(v)^2 - \mathrm{Im}(u)^2 + \mathrm{Im}(v)^2$ and $d = -2(\mathrm{Re}(u)\mathrm{Im}(u) + \mathrm{Re}(v)\mathrm{Im}(v))$.

$$\begin{aligned} g\sigma_3 g^{-1} &= \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \begin{bmatrix} \overline{u} & \overline{v} \\ v & -u \end{bmatrix} = \begin{bmatrix} |u|^2 - |v^2| & 2u\overline{v} \\ \overline{2u\overline{v}} & |v|^2 - |u|^2 \end{bmatrix} \\ &= (|u^2| - |v^2|)\sigma_3 + 2(\mathrm{Re}(u)\mathrm{Re}(v) + \mathrm{Im}(u)\mathrm{Im}(v))\sigma_1 + 2(\mathrm{Re}(u)\mathrm{Im}(v) - \mathrm{Im}(u)\mathrm{Re}(v))\sigma_2 \end{aligned}$$

And lastly,

$$g\sigma_2 g^{-1} = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \begin{bmatrix} iv & -iu \\ i\overline{u} & i\overline{v} \end{bmatrix} = \begin{bmatrix} -2\mathrm{Im}(uv) & s + ir \\ s - ir & 2\mathrm{Im}(uv) \end{bmatrix} = s\sigma_1 + r\sigma_2 - 2\mathrm{Im}(uv)\sigma_3$$

Where $r = (\mathrm{Re}(u)^2 - \mathrm{Im}(u)^2 + \mathrm{Re}(v)^2 - \mathrm{Im}(v)^2)$ and $s = 2(\mathrm{Re}(u)\mathrm{Im}(u) - \mathrm{Re}(v)\mathrm{Im}(v))$.

Using this we can write down the matrix representation of $\varphi(g)$, being

$$\varphi(g) = \begin{bmatrix} c & s & 2(\mathrm{Re}(u)\mathrm{Re}(v) + \mathrm{Im}(u)\mathrm{Im}(v)) \\ d & r & 2(\mathrm{Re}(u)\mathrm{Im}(v) - \mathrm{Im}(u)\mathrm{Re}(v)) \\ -2\mathrm{Re}(uv) & -2\mathrm{Im}(uv) & (|u^2| - |v^2|) \end{bmatrix}$$

For g as above, and c, d, r, s defined above.

(e) To compute the kernel, we use the form of the amtrix computed in part (d). Assume that $g \in \ker\varphi$, then using the diagonal entries, (writing $\varphi(g) = (\varphi_{i,j})_{i,j}$). We get the relation

$$4\mathrm{Re}(u)^2 = \sum_1^3 \varphi_{i,i} + |u|^2 + |v|^2 = 4 \iff \mathrm{Re}(u) = \pm 1 \iff u = \pm 1$$

this of course also implies that $v = 0$, plugging in these values for u in we see that both are solutions, so that $\ker\varphi = \mathbf{1}, -\mathbf{1}$.

To show surjectivity, note that since φ is 2:1, with kernel $\pm\mathbf{1}$, we have that $\varphi(g) = \varphi(h) \iff h = -g$, given this if $U \subset \mathrm{SU}(2)$, such that $U \cap -U = \emptyset$, then $\varphi|_U$ is injective. As before, write a general element $g \in \mathrm{SU}(2)$ as $g = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix}$, the projection map $g \mapsto \mathrm{Re}(u)$ is continuous, so that $U := \{g \in \mathrm{SU}(2) | \mathrm{Re}(u) \geq \frac{1}{2}\}$ is closed, and $U \supset V := \{g \in \mathrm{SU}(2) | \mathrm{Re}(u) > \frac{1}{2}\}$ is open. Furthermore, we have $U \cap -U = \emptyset$, since for any $g \in -U$, $\mathrm{Re}(u) \leq -\frac{1}{2}$. This suffices to show that $\varphi|_U$ is injective. Now since $U \subset \mathrm{SU}(2)$ is a closed subset of a compact set (hence compact), and $\mathrm{SO}(3)$ is hausdorff (this is clear since it can be identified as a subspace of $\mathbb{R}^9$) we can conclude by the closed map lemma that $\varphi|_U$ is homeomorphic onto its image, hence is an open map. This implies that $\varphi(V) = \varphi|_U(V)$ is open in $\mathrm{SO}(3)$, and in particular $\mathbf{1} \in V$ implies that $\mathbf{1} \in \varphi(V)$ is an open set in $\mathrm{SO}(3)$ mapped onto by φ containing $\mathbf{1}$. Then we have that $\varphi(\mathrm{SU}(2)) \supset \langle\varphi(V)\rangle = \mathrm{SO}(3)$ proving that the map is onto as desired.