1. If $V = \mathbf{0}$, then the problem is trivial, so assume not. It will suffice to show that V has a linearly independent subset, and that every linearly independent subset is contained in a basis. To see that V contains a linearly independent subset, let $\mathbf{v} \in V \setminus \mathbf{0}$, then for $a \in F$, $a\mathbf{v} = \mathbf{0} \iff a = 0$, so that $\{\mathbf{v}\}$ is linearly independent. Now suppose that S is a linearly independent set, if $\langle S \rangle = V$ we are done, so assume not. Let $X := \{T \subset V \mid S \subset T \text{ and } T \text{ is linearly independent}\}$, Then $(X, \subset)$ is a poset. Now let C be a chain in X, I claim that $T := \bigcup_{T \in C} T$ is an upper bound. It is immediate that $S \subset T \subset V$, to see that T is linearly independent, let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset T$, such that for $\{a_1, \dots, a_n\} \subset F$ we have $\sum_1^n a_i \mathbf{v}_i = \mathbf{0}$, then each $\mathbf{v}_i \in T_i$ for some $T_i \in C$, since the T_i are in the chain, one of them contains the others, assume without loss of generality $T_1 \supset T_i$, $1 \le i \le n$, then since T_1 is linearly independent it follows that each $a_i = 0$. This proves that T is linearly independent and in particular $T \in X$. Now apply Zorn's lemma to $(X, \subset)$ to furnish a maximal independent set with respect to inclusion M, to see that M must be a basis, assume not. Then there must be some $\mathbf{u} \in V$, such that for any finite subset $\{\mathbf{v}_i\}_{i=1}^n \subset M$, we cannot write

$$\mathbf{u} = \sum_1^n a_i \mathbf{v}_i \quad a_i \in F$$

I claim that $M \cup \{\mathbf{u}\}$ is linearly independent, then $S \subset M \subset M \cup \{\mathbf{u}\}$ implies that $M \cup \{\mathbf{u}\} \in X$ contradicting maximality of M. To see that $M \cup \{\mathbf{u}\}$ is linearly independent, let $\{\mathbf{v}_i\}_{i=1}^n \subset M \cup \{\mathbf{u}\}$, and $a_i \in F$, $1 \le i \le n$, such that

$$\sum_1^n a_i \mathbf{v}_i = \mathbf{0}$$

If $\mathbf{v}_i \neq \mathbf{u}$ for any i, then each $a_i = 0$ by linear independence of M, if some $\mathbf{v}_j = \mathbf{u}$, with $a_j \neq 0$, then

$$\sum_{1 \le i \le n, i \neq j} a_j^{-1} a_i \mathbf{v}_\mathbf{i} = \mathbf{u}$$

contradicts $\mathbf{u} \notin \langle M \rangle$, so that $a_j = 0$, then $a_i = 0$ follows from $\{\mathbf{v}_i\}_{1 \le i \le n, i \neq j} \subset M$ which is linearly independent, contradicting maximality of M hence M must span V, which implies that M is a basis. $\square$

2. (a) Let $P \subset A$ be a non-zero prime ideal. Then since A is a PID, $P = (x)$ for some $x \in A$. Suppose that $I \supset P$ is an ideal, then $I = (y)$ for $y \in A$. By assumption we have for some $a \in A$ that $ay = x$, but since x is prime it follows that $x|y$ or $x|a$. If $x|a$, then $a = xb$ implies that $by = 1$, hence $1 \in I = (y) = A$. If $x|y$, then $y = xb$ so that $ab = 1$, so that

$$(y) = (bx) \subset (x) \text{ and } (x) = (ay) \subset (y) \implies (x) = (y)$$

So that $I = P$ or $I = A$ proving that P is maximal. $\square$

(b) Let k be a field, consider $A = k[X, Y]$, then (X) is prime but not maximal. As proof, note that $(X) \subsetneq (X, Y) \subsetneq k[X, Y]$, so that (X) is not maximal (any non-zero element of (X) has positive degree in X, but $\deg_X Y = 0$, and any element non-zero $f \in (X, Y)$ has $\deg_X f + \deg_Y f > 0$, implying that $1 \notin (X, Y)$) to see that (X) is prime suppose that $fg \in (X)$, if $fg = 0$, then $f = 0$ or $g = 0$ since $k[X, Y]$ is a domain and $0 \in (X)$ so we are done. So assume that $fg \neq 0$, then

$$\deg_X(fg) = \deg_X(f) + \deg_X(g) > 0 \implies \deg_X f > 0 \text{ or } \deg_X g > 0$$

which implies that either $f \in (X)$ or $g \in (X)$ $\square$

3. Consider $X := \{I \subset A \mid I \text{ prime}\}$, along with $\leq$, where for $I, J \in X$ we have $I \leq J$ when $I \supset J$, this defines a partial order (this can easily be seen since the partial order axioms are symmetric and $I \leq J$ when $I \subset J$ is a partial order). Now let C be a chain of ideals in X, and define $P := \bigcap_{I \in C} I$, it will suffice to show that P is prime so that P is an upper bound for C. Suppose that $ab \in P$, if $b \in P$ then we are done, so assume $b \notin P$, then for some $I \in C$ we have $b \notin I$, so for any $J \in C$, such that $J \subset I$, we have $ab \in J$, which is prime implying that $a \in J$, since $b \notin J$. It follows that $a \in J$ for any $J \in C$, since if $J \subset I$ we have $a \in J$, and since $a \in I$ we have $a \in J$ for $J \supset I$ as well, this proves that $a \in P$ so that P is prime and hence an upper bound for C in X. Given that each chain in the poset $(X, \leq)$ has an upper bound, by Zorn's lemma X contains a maximal element $Q \in X$ implies that Q is a prime ideal, and Q maximal with respect to $\geq$ implies that Q is minimal with respect to inclusion. $\square$

4. (a) Suppose that a_0 is a unit, then for some $b_0 \in A$, we have $a_0 b_0 = 1$. Now define

$$b_n := -b_0 \sum_{k=1}^{n} b_{n-k} a_k, \quad g := \sum_{0}^{\infty} b_i X^i$$

then we have $fg = \sum_1^\infty c_i X^i$, by our definition of g we can compute c_n,

$$c_0 = a_0 b_0 = 1$$

$$c_n = \sum_{k=0}^{n} b_{n-k} a_k = \sum_{1}^{n} b_{n-k} a_k + a_0 b_n = \sum_{1}^{n} b_{n-k} a_k - a_0 b_0 \sum_{k=1}^{n} b_{n-k} a_k = \sum_{1}^{n} b_{n-k} a_k - \sum_{k=1}^{n} b_{n-k} a_k = 0, \quad \forall n > 0$$

Hence $fg = 1$ i.e. f is a unit. Conversely suppose that f is a unit, then for some $g = \sum_0^\infty b_i X^i$ we have $fg = 1$, in particular the constant term of fg is equal to 1, the constant term of fg is $a_0 b_0$, so that $a_0 b_0 = 1$ hence proving a_0 is a unit. $\square$

(b) Assume that A is local, then A has a unique maximal ideal I. Define $\mathfrak{m} = \{\sum_0^\infty a_i X^i \in A[[X]] \mid a_0 \in I\}$ which is an ideal (the constant term of a sum or product is the sum or product of the constant terms), $\mathfrak{m}$ is clearly proper since $1 \notin I$ implies that $1 \notin \mathfrak{m}$. Now let $J \neq A[[X]]$ be an ideal, it will suffice to show that $J \subset \mathfrak{m}$. Let $f = \sum_0^\infty a_i X^i \in J$, since $J \neq A[[X]]$ we must have that f is not invertible, implying that by part (a), $a_0 \in A$ is not invertible. But then since (A, I) is local and a_0 is not invertible we must have $a_0 \in I$ so that $f \in \mathfrak{m}$.

Conversely, suppose that $(A[[X]], \mathfrak{m})$ is local, then consider

$$I := \{a \in A \mid a \text{ is the constant term of some } f \in A\}$$

Once again, this is an ideal since the sum of constant terms of formal series is equal to the constant term of their sum and $a \in A, f \in \mathfrak{m}$, then $af \in \mathfrak{m}$, so the set is closed under arbitrary products in A. We can conclude that $1 \notin I$, since if it were, then it would be the constant term of some $f \in \mathfrak{m}$, which would imply f is invertible (by part (a)) which would imply $\mathfrak{m} = A[[X]]$ which is not true. So $I \subsetneq A$. Let $a \in A$, such that a is not invertible, then the formal series $a = a + \sum_1^\infty 0X^i$ is not invertible by part (a), hence $a \in I$, contraposing we can conclude that $a \in A \setminus I$ implies that a is invertible which is equivalent to (A, I) being a local ring. $\square$

5. (a) Let $N = \{\sum_0^\infty a_i X^i \mid a_i \text{ is nilpotent } \forall i\}$, N is an ideal follows from $\mathrm{Nil}(A)$ is an ideal. Suppose for contradiction that that $f \notin N$, and for some n, $f^n = 0$ for $f = \sum_0^\infty a_i X^i \in A[[X]]$.

Let i be the smallest index such that a_i is not nilpotent, and denote $f^n = \sum_0^\infty b_i X^i = 0$, then

$$\mathrm{Nil}(A) = b_{ni} + \mathrm{Nil}(A) = \sum_{\{(j_1,\ldots,j_n) \mid \sum_{k=1}^n j_k = ni\}} a_{j_1} a_{j_2} \cdots a_{j_n} + \mathrm{Nil}(A) = a_i^n + \mathrm{Nil}(A)$$

The last equality follows since all terms of the sum one atleast one $a_j,\ j \neq i$ must also contain some $a_{j'},\ j' \leq i$ (not necessarily distinct from a_j). This is a contradiction, since a_i was taken to not be nilpotent, but a_i^n is nilpotent implying that a_i is nilpotent. Hence proving that the nildradical of $A[[X]]$ is a subset of N as desired. $\square$

(b) To show that the converse is not true, consider $A = k[Y_1, Y_2, \ldots]/(Y_i^{i+1})_{i=1}^\infty$, and define

$$f := \sum_1^\infty Y_i X^i$$

Then for any n, denote $f^n = \sum_0^\infty b_i X^i$, to show $f^n \neq 0$ it will suffice to show atleast one $b_i \neq 0$, consider b_{n^2}, we may write b_{n^2} as a polynomial in Y_n, then

$$b_{n^2} = \sum_{i=0}^n Y_n^i h_i(Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_{n^2})$$

Then in this case $h_n = 1$, and $\deg_{Y_n}(\sum_{i=0}^{n-1} Y_n^i h_i(Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_{n^2})) < n$, so that b_{n^2} has degree n in Y_n and hence must be non-zero. $\square$

6. (a) We first verify that M_a is indeed maximal, by the first isomorphism theorem, $A/M_a \cong h_a(A)$, where $h_a(A) \subset \mathbb{R}$, to see that $h_a(A) = \mathbb{R}$, note that for each $x \in \mathbb{R}$, has some continuous function f_x on $[0,1]$, such that $f(a) = x$, namely the constant function $f : [0,1] \to \{x\}$, this implies that the image of h_a is $\mathbb{R}$ and hence a field implying that M_a was maximal by the correspondance theorem.

Lemma. Given any interval $U \subset \mathbb{R}$, there is some $\chi_U \in A$, such that $\chi_U > 0$ on $U \cap [0,1]$ and $\chi_U = 0$ on $[0,1] \setminus U$. The proof of the lemma is simple, simply take $U = (a,b)$, then $\chi'_U := -(x-a)(x-b)$ satisfies the properties but takes values on $\mathbb{R}$, not $[0,1]$, so we may simply define our function as the restriction, $\chi_U = \chi'_U|_{[0,1]}$.

Let $I \subset A$, supposing that I is not contained in M_a for any a. Then for any $a \in [0,1]$, there exists $f_a \in I$, such that $f_a(a) \neq 0$, we may assume that $f_a(a) > 0$ by multiplying by $\pm 1 \in A$. Then for each f_a, continuity furnishes some $\delta_a > 0$, such that $f_a(a - \delta_a, a + \delta_a) \subset (0, \infty)$, define $U_a = (a - \delta_a, a + \delta_a)$. Then $\{U_a\}_{a \in [0,1]}$ is an open cover for $[0,1]$, so we may take some finite subcover $\{U_{a_1}, \ldots, U_{a_n}\}$ where $f_{a_i}|_{\{U_{a_i}\}} > 0$, by the ideal properties we have that $g = \sum_1^n f_{a_i} \chi_{U_{a_i}} \in I$ (note that each f_{a_i} is only supported on a set where its positive). Now for any $x \in [0,1]$, we have $x \in U_{a_j}$ for some $1 \leq j \leq n$ so that

$$g(x) = \sum_1^n f_{a_i}(x) \chi_{U_{a_i}}(x) \geq f_{a_j}(x) \chi_{U_{a_j}}(x) > 0$$

hence $g(x)$ is non-zero on $[0,1]$, implying that the function $\frac{1}{g(x)}$ is well defined and continuous on $[0,1]$. This implies that $1 = \frac{1}{g} g \in I$, so that $I = A$. This shows that any proper ideal $I \subsetneq A$ is such that $I \subset M_a$ for some $a \in [0,1]$ which allows us to conclude that I being maximal implies that $I = M_a$ for some $a \in [0,1]$. $\square$

(b) Consider $I := \{f \in A \mid \exists \epsilon_f > 0, \text{ such that } f(0, \epsilon_f) = 0\}$. First note that I is an ideal, since if $f, g \in I$, then take $\epsilon = \min\{\epsilon_f, \epsilon_g\}$, so that $x \in (0, \epsilon)$ implies that $f(x) = g(x) = 0$ and hence $(f+g)(x) = 0$ and if $f \in I, g \in A$, then for any $x \in (0, \epsilon_f)$ we have $fg(x) = 0g(x) = 0$. It is also obvious that $1 \notin I$ implying that $I \neq A$. Finally, for any $a \in (0,1)$, the function

$$f : x \mapsto \begin{cases} 0 & x < a/2 \\ x - a/2 & x \geq a/2 \end{cases}$$

is continuous and hence $f \in A$, but $f \notin M_a$, implying that I is not included in M_a for any $a \in (0,1)$, but $I \subsetneq A$ implies that $I \subset J$ for some maximal ideal $J \subset A$, where J cannot be of the form M_a since J includes I. $\square$

(c) Once again there is a maximal ideal not of the form M_a. Consider

$$I := \{f \in A \mid f \text{ has compact support}\}$$

Then I is an ideal, as proof let $f, g \in I$, then we have compact sets U_f, U_g such that $f|_{U_f^c} = 0$ and $g|_{U_g^c} = 0$, since finite unions of compact sets are compact, we have $U_f \cup U_g$ which is compact, and $f + g|_{(U_f \cup U_g)^c} = 0$, the other condition is obvious since if $f \in I, g \in A$, then $\operatorname{supp}(fg) \subset \operatorname{supp}(f)$ which is contained in a compact set. Once again it is obvious that $1 \notin I$ implying that $I \subsetneq A$. Now let $a \in \mathbb{R}$, then we have $f \in I \setminus M_a$, where f is defined as

$$f : x \mapsto \begin{cases} 0 & x \in (-\infty, a-1) \cup (a+1, \infty) \\ (x - (a-1)) & x \in [a-1, a] \\ -(x - (a+1)) & x \in (a, a+1] \end{cases}$$

So same as above we have that $I \not\subset M_a$ for any a, then there exists some maximal ideal $J \supset I$, where J cannot be of the form M_a for any a. $\square$