1. First let n denote the dimension of M, and let $(U_\alpha, x^i_\alpha)_\alpha$ be a smooth atlas for M, I will denote the coordinate maps as ϕ_α respectively. This induces a smooth atlas $(\pi^{-1}(U_\alpha), (x^i_\alpha, v^i_\alpha))$ for TM, I will denote the coordinate maps as φ_α respectively. Now take charts $(\pi^{-1}(U_\alpha), (x^i_\alpha, v^i_\alpha)), (\pi^{-1}(U_\beta), (x^i_\beta, v^i_\beta))$, (assume that they have non-empty intersection else there is nothing to check) the change of coordinates is given by

$$(x^i_\beta, v^i_\beta) = \varphi_\beta \varphi_\alpha^{-1}(x^i_\alpha, v^i_\alpha) = (\phi_\alpha \phi_\beta(x^i_\alpha), \frac{\partial x^i_\beta}{\partial x^j_\alpha} v^j_\alpha)$$

We want to show that the determinant of the following block diagonal matrix (A) is positive:

$$A := \begin{bmatrix} \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1 \leq i,j \leq n} & \left(\frac{\partial x^i_\beta}{\partial v^j_\alpha}\right)_{1 \leq i,j \leq n} \\ \left(\frac{\partial v^i_\beta}{\partial x^j_\alpha}\right)_{1 \leq i,j \leq n} & \left(\frac{\partial v^i_\beta}{\partial v^j_\alpha}\right)_{1 \leq i,j \leq n} \end{bmatrix}$$

By definition of the coordinate change on TM, it is apparent that

$$\det \left(\frac{\partial x^i_\beta}{\partial v^j_\alpha}\right)_{1 \leq i,j \leq n} = 0$$

since $\phi_\alpha \phi_\beta(x^i_\alpha)$ is independent of each v^i_α. It follows that

$$\det A = \det \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1 \leq i,j \leq n} \det \left(\frac{\partial v^i_\beta}{\partial v^j_\alpha}\right)_{1 \leq i,j \leq n}$$

But then we can read off of the change of coordinates formula for TM that $\frac{\partial v^i_\beta}{\partial v^j_\alpha} = \frac{\partial x^i_\beta}{\partial x^j_\alpha}$, so that

$$\det A = \left(\det \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1 \leq i,j \leq n}\right)^2 > 0$$

We can say > 0 here, since the change of coordinates is smooth and invertible with smooth inverse the inverse function theorem tells us its Jacobian is invertible. As proof, let $p \in M$, and let f denote the change of coordinates with inverse f^{-1}, then

$$1_{T_pM} = d(f^{-1} \circ f)|_p = (df^{-1}|_{f(p)})(df|_p)$$
$$1_{T_{f(p)}M} = d(f \circ f^{-1})|_{f(p)} = (df|_p)(df^{-1}|_{f(p)})$$

So that df_p is invertible for each $p \in M$.

2. First we give an orientation on M, consider the coordinate charts

$$U_1 \times U_2, \quad \tilde{U}_1 \times U_2, \quad U_1 \times \tilde{U}_2, \quad \tilde{U}_1 \times \tilde{U}_2$$

Given by the product of the standard $\theta, \tilde{\theta}$ coordinates on S^1. Here since $w^2 + x^2 = 1 = y^2 + z^2$, we can associate to any point

$$(w, x, y, z) = (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) \longleftrightarrow (e^{i\theta_1}, e^{i\theta_2})$$

Allowing us to define the charts in the standard way on S^1. Hence if we can show that the coordinate change between U and $\tilde{U}$ on S^1 has jacobian with positive determinant, we will be done since each of the coordinate changes are products of the determinant of the identity and the determinant of this Jacobian. This is straightforward, since $\tilde{\theta} = \theta - 2\pi$, we have $\frac{\partial \tilde{\theta}}{\partial \theta} = 1$ (the reverse coordinate change is the same). Thus this is an orientation.

Let U denote that chart $(0, 2\pi) \times (0, 2\pi)$, with coordinates (θ_1, θ_2). It follows that in U, we have

$$dw = \frac{\partial w}{\partial \theta_1} d\theta_1 + \frac{\partial w}{\partial \theta_2} d\theta_2 = -\sin\theta_1 d\theta_1$$

$$dy = \frac{\partial y}{\partial \theta_1} d\theta_1 + \frac{\partial y}{\partial \theta_2} d\theta_2 = -\sin\theta_2 d\theta_2$$

Using this, we compute $\omega = \sin^2\theta_1 \sin^2\theta_2 d\theta_1 \wedge \theta_2$, so that

$$\int_{T^2} \omega = \int_{T^2 \setminus S^1 \cup S^1} \omega = \int_U \omega = \int_0^{2\pi} \int_0^{2\pi} \sin^2\theta_1 \sin^2\theta_2 d\theta_1 d\theta_2$$

$$= \int_0^{2\pi} \pi \sin^2\theta_2 d\theta_2 = \pi^2$$

3. We first compute $d\mathrm{Vol}_g$. Equip S^2 with the polar coordinate chart given by

$$U = ((0, \pi) \times (0, 2\pi), \phi, \theta)$$

We only need to check for equality of g and ω on this chart, since both are smooth, equality on U also implies equality on $\overline{U} = S^2$. On U we have $(x, y, z) \longleftrightarrow (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, and $\imath : (\phi, \theta) \mapsto (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, so that

$$g = \imath^* g_{\mathrm{EUC}} = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{bmatrix}$$

Where in this case we have

$$A = \begin{bmatrix} \frac{\partial \imath^x}{\partial \phi} & \frac{\partial \imath^y}{\partial \phi} & \frac{\partial \imath^z}{\partial \phi} \\ \frac{\partial \imath^x}{\partial \theta} & \frac{\partial \imath^y}{\partial \theta} & \frac{\partial \imath^z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{bmatrix}$$

This allows us to compute

$$d\mathrm{Vol}_g \overset{\mathrm{def.}}{=} \sqrt{\det \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{bmatrix}} d\phi \wedge d\theta = \sin\phi d\phi \wedge d\theta$$

Now we compute ω on U to check for equality, to simplify the computations, we first compute

$$\imath^* dx = d(x \circ \imath) = d\cos\theta\sin\phi = -\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi$$
$$\imath^* dy = d(y \circ \imath) = d\sin\theta\sin\phi = \cos\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi$$
$$\imath^* dz = d(z \circ \imath) = d\cos\phi = -\sin\phi d\phi$$

So that,

$$\omega = \imath^*(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy) = x\imath^* dy \wedge \imath^* dz + y\imath^* dz \wedge \imath^* dx + z\imath^* dx \wedge \imath^* dy$$
$$= \cos\theta\sin\phi(\cos\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi) \wedge (-\sin\phi d\phi)$$
$$+ \sin\theta\sin\phi(-\sin\phi d\phi) \wedge (-\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi)$$
$$+ \cos\phi(-\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi) \wedge (\cos\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi)$$
$$= -\cos^2\theta\sin^3\phi d\theta \wedge d\phi - \sin^2\theta\sin^3\phi d\theta \wedge d\phi - \cos^2\phi\sin^2\theta\sin\phi d\theta \wedge d\phi - \cos^2\phi\cos^2\theta\sin\phi d\theta \wedge d\phi$$
$$= (-\sin^3\phi - \cos^2\phi\sin\phi)d\theta \wedge d\phi = -\sin\phi d\theta \wedge d\phi = \sin\phi d\phi \wedge d\theta$$

(b) Consider the chart V on B^3, given by $(((0,1) \times (0,\pi) \times (0, 2\pi)), (r, \phi, \theta))$ We may take

$$\imath : V \to \mathbb{R}^3, \quad (r, \phi, \theta) \mapsto (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$$

We remark here that V^c has measure 0, furthermore, B^3 is orientable, and we consider the orientation given by the given coordinate order (r, ϕ, θ). To simplify the computations, we first compute

$$d(x \circ \imath) = \cos\theta \sin\phi dr - r\sin\theta \sin\phi d\theta + r\cos\theta \cos\phi d\phi$$

$$d(y \circ \imath) = \sin\theta \sin\phi dr + r\cos\theta \sin\phi d\theta + r\sin\theta \cos\phi d\phi$$

$$d(z \circ \imath) = \cos\phi dr - r\sin\phi d\phi$$

$$\implies d(x \circ \imath) \wedge d(y \circ \imath) \wedge d(z \circ \imath) = -r^2(\cos^2\theta \sin^3\phi + \sin^2\theta \sin^3\phi + \sin^2\theta \cos^2\phi \sin\phi + \cos^2\theta \cos^2\phi \sin\phi)dr \wedge d\theta \wedge d\phi$$

$$= -r^2 \sin\phi dr \wedge d\theta \wedge d\phi = r^2 \sin\phi dr \wedge d\phi \wedge d\theta$$

$$\int_{S^2} \omega = \int_{\partial B^3} \imath^*\tilde\omega \overset{\text{Stoke's Thm}}{=} \int_{B^3} d\imath^*\tilde\omega = \int_{B^3} \imath^* d\tilde\omega = \int_{B^3} \imath^* d(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$$

$$= \int_{B^3} \imath^*(dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy) = \int_{B^3} \imath^*(3dx \wedge dy \wedge dz)$$

$$= \int_{B^3} 3\imath^*dx \wedge \imath^*dy \wedge \imath^*dz = \int_V d(x \circ \imath) \wedge d(y \circ \imath) \wedge d(z \circ \imath)$$

$$= \int_V 3r^2 \sin\phi dr \wedge d\phi \wedge d\theta = \int_0^{2\pi} \int_0^\pi \int_0^1 3r^2 \sin\phi dr d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^\pi \sin\phi d\phi d\theta = \int_0^{2\pi} -\cos\phi|_0^\pi d\theta = \int_0^{2\pi} 2d\theta = 4\pi$$

(c) Assume that such an α exists, then

$$4\pi = \int_{S^2} \omega = \int_{S^2} d\alpha = \int_{\partial B^3} d\alpha \overset{\text{Stoke's Thm}}{=} \int_{B^3} d^2\alpha = \int_{B^3} 0 = 0$$

Contradiction.