1. (a) Let $0 < \theta < 2\pi$, then using

$$v_1 \sin\theta - v_2 \cos\theta = v_2$$

we use the double angle formulae to get

$$v_1(2\sin(\theta/2)\cos(\theta/2)) = v_2(2\cos^2(\theta/2))$$

On the given domain we have $\sin\theta/2 \neq 0$. If $\cos\theta/2 = 0$ on our domain, then $\theta = \pi$ implying that $v_1 = -v_1$ by the first equation, so that $v_1 = 0$, immediately implying that v_1 satisfies $v_1 = r\cos\theta/2 = 0$. If $\cos\theta/2 \neq 0$, then we get

$$v_1 = v_2 \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \frac{v_2\cos\theta/2}{\sin\theta/2} = r\cos\theta/2$$

The other relation is trivial,

$$r\sin\theta/2 = \frac{\sin\theta/2}{\sin\theta/2} v_2 = v_2$$

Now considering $-\pi < \tilde{\theta} < \pi$. Note that $\cos\tilde{\theta}/2 \neq 0$ on our given domain. Similarly to as above, the double angle formulae and the second equation give

$$v_1(2\sin(\tilde{\theta}/2)\cos(\tilde{\theta}/2)) = v_2(2\cos^2(\tilde{\theta}/2))$$

So that

$$v_2 = v_1 \frac{\sin(\tilde{\theta}/2)}{\cos(\tilde{\theta}/2)} = \rho\sin(\tilde{\theta}/2)$$

Once again, the other relation is trivial,

$$v_1 = \frac{\cos\tilde{\theta}/2}{\cos\tilde{\theta}/2} v_1 = \rho\cos\tilde{\theta}/2$$

(b) First note that $U \cap V = \{e^{i\theta} | \theta \in (0,\pi) \sqcup (\pi, 2\pi)\}$. Now let let $(x,v) = e^{i\theta} \in \pi^{-1}(U \cap V)$, then we may write (x,v) on $\tilde{U} \times \mathbb{R}$ as $(\tilde{\theta}, \rho)$ for $\tilde{\theta} \in (-\pi, 0) \sqcup (0, \pi)$. Then if $(x,v) \in \pi^{-1}(\pi, 2\pi)$ we have $\tilde{\theta} \in (-\pi, 0)$

$$\varphi_1\varphi_2^{-1}(\tilde{\theta}, \rho) = \varphi_1\left(e^{i\tilde{\theta}+2\pi}, \rho\begin{pmatrix}\cos((\tilde{\theta}+2\pi)/2)\\ \sin((\tilde{\theta}+2\pi)/2)\end{pmatrix}\right) = \left(e^{i\tilde{\theta}+2\pi}, r\begin{pmatrix}\cos(\tilde{\theta}/2+2\pi)\\ \sin(\tilde{\theta}/2+2\pi)\end{pmatrix}\right)$$

This implies that $r = \rho\frac{\cos(\tilde{\theta}/2+\pi)}{\cos\tilde{\theta}/2} = -\rho$. Otherwise, if $(x,v) \in \pi^{-1}(0,\pi)$, the coordinate change is the identity, and hence $r = \rho$. Then $\tau_{12} = r/\rho$ implies that

$$\tau_{12}(e^{i\theta}) = \begin{cases} 1 & \theta \in (0,\pi) \\ -1 & \theta \in (\pi, 2\pi) \end{cases}$$

(c) Applying the double angle identities, we get that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and $1 - \cos(\theta) = 2\sin^2(\theta/2)$, the same relations hold of course for $\tilde{\theta}$. We first compute s, let $e^{i\theta} \in U$, then we may write $\theta \in (0, 2\pi)$. Then

$$\begin{pmatrix}\sin\theta \\ 1-\cos\theta\end{pmatrix} = \begin{pmatrix}2\sin(\theta/2)\cos(\theta/2) \\ 2\sin^2(\theta/2)\end{pmatrix} = s(\theta)\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}$$

We find that $s(\theta) = 2\sin(\theta/2)$.

Now to compute $\tilde{s}$, let $e^{i\tilde{\theta}} \in \tilde{U}$, then we may write $\tilde{\theta} \in (-\pi, \pi)$. Then

$$\begin{pmatrix} \sin\tilde{\theta} \\ 1-\cos\tilde{\theta} \end{pmatrix} = \begin{pmatrix} 2\sin(\tilde{\theta}/2)\cos(\tilde{\theta}/2) \\ 2\sin^2(\tilde{\theta}/2) \end{pmatrix} = \tilde{s}(\tilde{\theta}) \begin{pmatrix} \cos\tilde{\theta}/2 \\ \sin\tilde{\theta}/2 \end{pmatrix}$$

We find that $\tilde{s}(\tilde{\theta}) = 2\sin(\tilde{\theta}/2)$.

Now to verify the gluing relation, we have for $e^{i\theta}$, such that $\theta \in (0, \pi)$ that $\theta = \tilde{\theta}$ and hence

$$s(\theta) = 2\sin(\theta/2) = 2\sin(\tilde{\theta}/2) = \tilde{s}(\tilde{\theta}) = \tau_{12}(e^{i\theta})\tilde{s}(\tilde{\theta})$$

And for $e^{i\theta}$, such that $\theta \in (\pi, 2\pi)$ we have $\theta = \tilde{\theta} + 2\pi$, so that

$$s(\theta) = 2\sin(\theta/2) = 2\sin(\tilde{\theta}/2 + \pi) = -2\sin(\tilde{\theta}/2) = -\tilde{s}(\tilde{\theta}) = \tau_{12}(e^{i\theta})\tilde{s}(\tilde{\theta})$$

2. Let $([x], w) \in \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$. Then for any $x \in [x]$, we have for each k, that $\frac{x_k}{x_j}$ is well defined under the equaivalence relation. Hence $w_k/w_j = x_k/x_j$ for each k. Then

$$\Psi_i\Psi_j^{-1}([x], w_j) = \Psi_i([x], (\frac{x_0}{x_j}w_j, \dots, \frac{x_n}{x_j}w_j)) = ([x], \frac{x_i}{x_j}w_j)$$

This gives us how the transition function must be defined,

$$\tau_{ij}([x]) : r \mapsto \frac{x_i}{x_j}r$$

3. From the lectures on submanifolds, we know that the image of an embedding is a submanifold. Firstly, we know that s is smooth, and it is also clear that π is smooth, so that s is smooth with smooth inverse $\pi|_S$. This verifies that s is a homeomorphism. To show that s is an embedding, we need show that Ds_p is injective for any $p \in M$. It will suffice to show it has a left inverse, using the chain rule we get the left inverse

$$\mathbf{1}_{T_pM} = D\mathbf{1}_p = D(\pi \circ s)_p = D\pi_{s(p)}Ds_p$$

So that $s : M \to E$ is an embedding, and S is a submanifold of M. Finally the diffeomorphism property is clear, since $\pi|_S$ is smooth, so that s is smooth with smooth inverse.