1. (a) Here we first note that on U,

$$(x\circ\imath) : (x,y) \mapsto (x,0,0), \quad (y\circ\imath) : (x,y) \mapsto (0,y,0), \quad (z\circ\imath)(x,y) \mapsto (0,0,\varphi^{-1}(x,y)_z) = (0,0,\sqrt{1-x^2-y^2})$$

This allows us to calculate

$$\begin{aligned}
\imath^* dx &= d(x\circ\imath) = dx \\
\imath^* dy &= d(y\circ\imath) = dy \\
\imath^* dz &= d(z\circ\imath) = d(0,0,\sqrt{1-x^2-y^2}) = \frac{\partial}{\partial x}\sqrt{1-x^2-y^2}dx + \frac{\partial}{\partial y}\sqrt{1-x^2-y^2}dy \\
&= -\frac{x}{\sqrt{1-x^2-y^2}}dx - \frac{y}{\sqrt{1-x^2-y^2}}dy
\end{aligned}$$

(b)

$$\begin{aligned}
\imath^*(xdx + ydy + zdz) &= \imath^*(xdx) + \imath^*(ydy) + \imath^*(zdz) \\
&= (x\circ\imath)(\imath^* dx) + (y\circ\imath)(\imath^* dy) + (z\circ\imath)(\imath^* dz) \\
&= xdx + ydy - z\left(\frac{x}{\sqrt{1-x^2-y^2}}dx + \frac{y}{\sqrt{1-x^2-y^2}}dy\right)dz \\
&= xdx + ydy - (xdx + ydy) = 0
\end{aligned}$$

We could have anticipated that this value would be zero, because the evaluation of a tangent vector at a point by this pull back can be seen as taking the dot product of the vector with the point's position vector in $\mathbb{R}^3$ (see the second line of the calculation above), however, the position vector of a point on the 2-sphere is the radial vector outward from the 2-sphere and thus is perpendicular to the tangent space (plane) of the 2-sphere at that point, hence any vector in the tangent space (plane) of the two-sphere at that point should be orthogonal to the position vector at that point, implying that their dot product should be zero.

2. (a) $\{\alpha,\beta\}$ being the dual frame of $\{X,Y\}$ is equivalent to

$$\alpha(X) = \beta(Y) = 1 \text{ and } \alpha(Y) = \beta(X) = 0$$

We solve for X and Y. Firstly, we may write wlog

$$X = u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} \qquad\qquad Y = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}$$

Then we get a system of equations for X,

$$\alpha(X) = 1 \iff xu_1 + yu_2 = 1 \tag{1}$$
$$\beta(X) = 0 \iff yu_1 + xu_2 = 0 \tag{2}$$

taking $x(1) - y(2)$ we get $x^2u_1 - y^2u_1 = x$, so that $u_1 = \frac{x}{x^2-y^2}$. Substituting this into (2) yields $u_2 = \frac{-y}{x^2-y^2}$. Similarly, we solve for Y with the equations

$$\alpha(Y) = 0 \iff xv_1 + yv_2 = 0 \tag{3}$$
$$\beta(Y) = 1 \iff yv_1 + xv_2 = 1 \tag{4}$$

Then taking $y(4) - x(3)$ we get $(y^2-x^2)v_1 = y$ so that $v_1 = \frac{y}{y^2-x^2}$, and finally substituting back into (3), $v_2 = \frac{-x}{x^2-y^2}$. The result is

$$X = \frac{x}{x^2-y^2}\frac{\partial}{\partial x} - \frac{y}{x^2-y^2}\frac{\partial}{\partial y} \qquad\qquad Y = \frac{y}{y^2-x^2}\frac{\partial}{\partial x} - \frac{x}{x^2-y^2}\frac{\partial}{\partial y}$$

It is immediate that by construction the vector fields X, Y satisfy the condition $\alpha(X) = \beta(Y) = 1$ and $\alpha(Y) = \beta(X) = 0$. Furthermore, these are vector fields on $\mathbb{R}^2 \setminus \{(x,y)|x = \pm y\}$, since each expression is infinitely differentiable on $\{(x,y)|x = \pm y\}^c$.

(b) We first compute that

$$dh = 2xydx + x^2dy \text{ and } dg = y\cos(xy)dx + x\cos(xy)dy$$

Then we have that

$$\begin{aligned} dh \wedge dg &= 2xy^2\cos(xy)dx \wedge dx + 2x^2y\cos(xy)dx \wedge dy + x^2y\cos(xy)dy \wedge dx + x^3\cos(xy)dy \wedge dy \\ &= (2x^2y\cos(xy) - x^2y\cos(xy))dx \wedge dy = x^2y\cos(xy)dx \wedge dy \end{aligned}$$

3. (a) The regular level set theorem, to check that c is indeed a regular value, we have that for any $p \in f^{-1}(c)$, that

$$\begin{bmatrix} \frac{\partial}{\partial x}f|_p & \frac{\partial}{\partial y}f|_p & \frac{\partial}{\partial z}f|_p \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Since the z partial is non-zero by assumption. Note that by the regular level set theorem S has dimension $3 - 1 = 2$.

(b) Since S has dimension 2, we have that $\imath^*\omega \in \Omega^2(S)$ is a top form. To see that it is nowhere vanishing, note that $df\imath \equiv 0$ (proof below). Now consider any point $p \in S$, we can take $\{u, v\}$ as a basis for T_pS where $d\imath(u) = (x_1, y_1, z_1)$ and $d\imath(v) = (x_2, y_2, z_2)$, it follows that neither of $(x_i, y_i) = (0,0)$, otherwise

$$df\imath(u) = \frac{\partial}{\partial z}f|_p z_1 \neq 0 \text{ or } df\imath(v) = \frac{\partial}{\partial z}f|_p z_2 \neq 0$$

contradicting that $df\imath \equiv 0$, it follows that $d\imath u = (x_1, y_1, z_1)$ and $d\imath v = (x_2, y_2, z_2)$ such that

$$(0,0) \neq (x_1, y_1) \neq \lambda(x_2, y_2) \neq (0,0) \quad \forall \lambda \in \mathbb{R}$$

since in the case of $(x_1, y_1) = \lambda(x_2, y_2)$ we have $u - \lambda v \neq 0$ (by linear independence of u, v assumed by them being a basis for the two dimensional T_pS), so that $d\imath(u - \lambda v) = (0, 0, z_3)$ for $z_3 \neq 0$, so that for the same reasons as above $df(z_3) \neq 0$, a contradiction. Now given this characterization of T_pS, we compute on T_pS

$$\begin{aligned} \imath^*\omega(u,v) = \omega(d\imath(u), d\imath(v)) &= \omega(x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z}, x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y} + z_2\frac{\partial}{\partial z}) \\ &= dx \wedge dy(x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z}, x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y} + z_2\frac{\partial}{\partial z}) \\ &= x_1y_2 - x_2y_1 = \det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \neq 0 \text{ Since } (x_1, y_1) \notin \text{Span}(x_2, y_2) \end{aligned}$$

Which of course proves that for any p, we have $\imath^*\omega \not\equiv 0$ on T_pS.

Proof that $\mathbf{df\imath \equiv 0}$: Consider any $p \in S$, let $u \in T_pS$, then for some path $\gamma : (-\epsilon, \epsilon) \to S$, we have that $\gamma(0) = p, \gamma'(0) = u$. Then $df\imath_p(u) = \frac{d}{dt}|_{t=0}f\imath(\gamma(t))$, but since $\gamma(-\epsilon, \epsilon) \subset S$, we have $f\imath(\gamma(t)) = c$, $\forall t \in (-\epsilon, \epsilon)$, hence

$$df\imath_p(u) = \frac{d}{dt}|_{t=0}f\imath(\gamma(t)) = \frac{d}{dt}|_{t=0}c = 0$$

4. (a) I claim that $\lambda = 2z$, we check this in the $z > 0$ and $x > 0$ hemisphere charts, the rest of the charts work similarly. Let $U = S^2 \cap \{z > 0\}, V = S^2 \cap \{x > 0\}$, then

$$\begin{aligned} d\eta &= d\imath^*\tilde{\eta} = \imath^*d\tilde{\eta} = \imath^*(-dy \wedge dx + dx \wedge dy) = 2\imath^*(dx \wedge dy) = 2(\imath^*dx \wedge \imath^*dy) \\ \imath^*\tilde{\omega} &= (x \circ \imath)\imath^*(dy \wedge dz) + (y \circ \imath)\imath^*(dz \wedge dx) + (z \circ \imath)\imath^*(dx \wedge dy) \\ &= x(\imath^*dy \wedge \imath^*dz) + y(\imath^*dz \wedge \imath^*dx) + z(\imath^*dx + \imath^*dy) \end{aligned}$$

In $(U, (x, y))$ we have coordinates $(x, y, \sqrt{1 - x^2 - y^2}) \mapsto (x, y)$, hence by the same computations as in problem 1, we have

$$\imath^* dx = dx, \quad \imath^* dy = dy, \quad \imath^* dz = -\frac{1}{\sqrt{1 - x^2 - y^2}}(xdx + ydy)$$

So, in this chart $d\eta = 2dx \wedge dy$, and

$$\begin{aligned}
\omega &= x(\imath^* dy \wedge \imath^* dz) + y(\imath^* dz \wedge \imath^* dx) + z(\imath^* dx + \imath^* dy) \\
&= x(dy \wedge -\frac{1}{\sqrt{1 - x^2 - y^2}}(xdx + ydy)) + y(-\frac{1}{\sqrt{1 - x^2 - y^2}}(xdx + ydy) \wedge dx) + \sqrt{1 - x^2 - y^2} dx \wedge dy \\
&= \frac{x^2}{\sqrt{1 - x^2 - y^2}}(dx \wedge dy) + \frac{y^2}{\sqrt{1 - x^2 - y^2}}(dx \wedge dy) + \sqrt{1 - x^2 - y^2} dx \wedge dy \\
&= \frac{1}{\sqrt{1 - x^2 - y^2}} dx \wedge dy
\end{aligned}$$

So that indeed $\lambda = 2z = 2\sqrt{1 - x^2 - y^2}$ satisfies $\lambda\omega = d\eta$. The closed form for η on x and y hemispheres looks a bit different, so we verify equality on V, then equality on the rest of the hemisphere charts follows similarly. On V we use the same computation as problem 1 due to symmetry of the charts, to find that

$$\imath^* dx = -\frac{1}{\sqrt{1 - x^2 - y^2}}(ydy + zdz), \quad \imath^* dy = dy, \quad \imath^* dz = dz$$

So that in V, we have

$$d\eta = \frac{-2}{\sqrt{1 - x^2 - y^2}}(ydy + zdz) \wedge dy = \frac{2z}{\sqrt{1 - x^2 - y^2}} dy \wedge dz$$

And the calculation for ω is symmetric to the one above, so that

$$\omega = \frac{1}{\sqrt{1 - y^2 - z^2}} dy \wedge dz$$

Verifying that $\lambda\omega = d\eta$ on V, the rest of the charts are computed similarly to one of U or V.

(b) First note that in the $z < 0$ hemisphere, we get $\omega = \frac{-1}{\sqrt{1-x^2-y^2}} dx \wedge dy = \frac{1}{z} dx \wedge dy$.

We rewrite $\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = (\frac{\partial z}{\partial x})^{-1}\frac{\partial}{\partial x} + (\frac{\partial z}{\partial y})^{-1}\frac{\partial}{\partial y}$, in either $z > 0$ hemisphere coordinates or $z < 0$ hemisphere coordinates (note we are doing this since z is a function of x, y here) we get that $\frac{\partial}{\partial z} = -\frac{z}{x}\frac{\partial}{\partial x} - \frac{z}{y}\frac{\partial}{\partial y}$. Then

$$\begin{aligned}
X &= \left(-xz - \frac{z(x^2 + y^2)}{x}\right)\frac{\partial}{\partial x} + \left(-yz - \frac{z(x^2 + y^2)}{y}\right)\frac{\partial}{\partial y} \\
&= -z\left(\frac{2x^2 + y^2}{x}\right)\frac{\partial}{\partial x} - z\left(\frac{x^2 + 2y^2}{y}\right)\frac{\partial}{\partial y}
\end{aligned}$$

This allows us to compute $dx(X)$, and $dy(X)$ in either hemispheres coordinates.

$$\begin{aligned}
dx(X) &= -z\left(\frac{2x^2 + y^2}{x}\right) & dy(X) &= -z\left(\frac{x^2 + 2y^2}{y}\right) \\
dx(Y) &= -y & dy(Y) &= x
\end{aligned}$$

Now computing $X \lrcorner\, \omega$ and $Y \lrcorner\, \omega$ by plugging in the above values,

$$\begin{aligned}
X \lrcorner\, \omega &= X \lrcorner\, \frac{1}{z} dx \wedge dy = \frac{1}{z}(dx(X)dy - dy(X)dx) \\
&= \frac{1}{z}(-z\left(\frac{2x^2+y^2}{x}\right)dy + z\left(\frac{x^2+2y^2}{y}\right)dx) \\
&= -\left(\frac{2x^2+y^2}{x}\right)dy + \left(\frac{x^2+2y^2}{y}\right)dx \\
Y \lrcorner\, \omega &= Y \lrcorner\, \frac{1}{z} dx \wedge dy = \frac{1}{z}(dx(Y)dy - dy(Y)dx) \\
&= \frac{1}{z}\left(-ydy - xdx\right)
\end{aligned}$$

Now we compute their wedge product,

$$\begin{aligned}
X \lrcorner\, \omega \wedge Y \lrcorner\, \omega &= -\left(\frac{2x^2+y^2}{x}\right)dy + \left(\frac{x^2+2y^2}{y}\right)dx \wedge \frac{-1}{z}(ydy + xdx) \\
&= \frac{(2x^2+y^2)}{z} dy \wedge dx - \frac{x^2+y^2}{z} dx \wedge dy \\
&= \frac{-3(x^2+y^2)}{z} dx \wedge dy
\end{aligned}$$

This gives us that $\phi = -3(x^2+y^2)$ in either hemispheres coordinates, it is clear that $\phi \in C^\infty(S^2)$. Since $X \lrcorner\, \omega \wedge Y \lrcorner\, \omega = \phi\omega$ on either hemisphere, and forms are continuous, it follows that they are also equal on the boundary of the hemispheres. This implies that ϕ is defined this way on all of S^2, since the definition is valid for $z > 0$, $z < 0$, and their boundary being $z = 0$.