**1.** From the notes if the roots of $f$ are $\{\theta_i\}_{i=1}^n$, then

$$\text{Disc}(f) = (-1)^{\binom{n}{2}} \prod_1^n f'(\theta_i)$$

In this case, the roots are $\{\zeta_n^i\}_{i=1}^n$, so that

$$\begin{aligned}\text{Disc}(x^n - 1) &= (-1)^{\binom{n}{2}} \prod_1^n n\zeta_n^{-i} = (-1)^{\binom{n}{2}} n^n \prod_1^n \zeta_n^{-i} \\ &= (-1)^{\binom{n}{2}} n^n \overline{\prod_1^n \zeta_n^i} = (-1)^{\binom{n}{2}} n^n \overline{(-1)^n \prod_1^n -\zeta_n^i}\end{aligned}$$

Then we can recognize $\prod_1^n -\zeta_n^i$ as the constant term of $x^n - 1$, since $x^n - 1 = \prod_1^n (x - \zeta_n^i)$ implies that $\prod_1^n -\zeta_n^i = -1$, hence

$$(-1)^n \prod_1^n -\zeta_n^i = (-1)^n(-1) = (-1)^{n-1} = \overline{(-1)^{n-1}} = \overline{(-1)^n \prod_1^n -\zeta_n^i} = (-1)^n(-1)$$

Substituting this in to the original expression, we get

$$\text{Disc}(x^n - 1) = (-1)^{\binom{n}{2}} n^n \overline{(-1)^n \prod_1^n -\zeta_n^i} = (-1)^{\binom{n}{2}+n-1} n^n$$

As desired.

**2.** Denote $f(t)$ as the polynomial in the question, then notice $f(t) = (t+1)^3 - 5$. I claim that the splitting field of $f(t)$ is equal to the splitting field of $p(t) = t^3 - 5 = f(t-1)$. As proof, if $\alpha$ is a root of $p(t)$, then $1 + \alpha$ is a root of $f(t)$, and if $\beta$ is a root of $f(t)$, then $1 - \beta$ is a root of $p(t)$, hence any field containing the roots of one of the two polynomials contains the roots of the other. So it will suffice to find the splitting field of $p(t)$, which is

$$\mathbb{Q}(5^{1/3}, 5^{1/3}\zeta_3, 5^{1/3}\zeta_3^{-1}) = \mathbb{Q}(5^{1/3}, \zeta_3)$$

**3.** We note that by Gauss' lemma, we can check for irreducibility in $\mathbf{F}_p[X,Y][T]$. Assume $f = gh$, then $\deg_{X,Y} g + \deg_{X,Y} h = 1$, so we can assume wlog that $\deg_{X,Y} g = 1$, and $\deg_{X,Y} h = 0$, then $g = Xg_1(T) + Yg_2(T) + g_3(T)$, where $g_i \in \mathbf{F}_p[T]$, then $hg_2 = 1$ implies that $h$ is a unit, so $f$ is irreducible. To see that $[L : k]_s = p$, we write $f(T) = p(T^p)$, where $p(T) = T^p + XT + Y$ (seperable, and irreducible by irreducibility of $f$), so that each root of $f$ has multiplicity $p$, and hence $p$ conjugates. Taking an algebraically closed field $K \supset L$, and an embedding $\sigma : \mathbf{F}_p(X,Y) \to K$, the number of ways to extend $\sigma$ to $k(\alpha)$ is the number of conjuates of $\alpha$, which is equal to $p$, hence by definition $[L : k]_s = p$.

Consider the extension $k(\alpha^p)$, then we have $k(\alpha)/k(\alpha^p)/k$, then $\alpha^p$ is a root of $p(T)$, hence this is a degree $p$ and therefore proper intermediate extension, since $\min(\alpha^p, k)$ is seperable, this is also a seperable extension, equal to the seperable closure of $k$ in $k(\alpha)$ since it has degree equal to $[k(\alpha) : k]_s$. To show this extension is unique, suppose for contradiction there exists some other intermediate extension $k'$. Note that any proper intermediate extension must be such that $[k(\alpha) : k'] = [k' : k] = p$, but then $[k' : k]$ must be purely inseperable, since if it were seperable then it would have to be equal to $k_{\text{sep}} = k(\alpha^p)$, and $[k' : k]_s | p$, implies that $k'/k$ is purely inseperable. Consequently $k(\alpha)/k'$ must be a seperable extension, let $q(T) := \min(\alpha, k')$, then $\deg(q) = p$ and $q|f$, since $f(\alpha) = 0$, and $f(T) \in k'[T]$. Now, since $f$ has $p$ unique factors of multiplicity $p$ and $q$ has $p$ distinct linear factors, it must be the case that $q^p = f$, from the binomial theorem we can see that $q$ must be $T^p + T\sqrt[p]{X} + \sqrt[p]{Y}$. This implies that $\sqrt[p]{X}, \sqrt[p]{Y} \in k'$, so that $k' = k(\sqrt[p]{X})(\sqrt[p]{Y})/k(\sqrt[p]{(X)})/k$ is a tower of degree $p$ extensions, so that $[k' : k] = p^2$, a contradiction.

Then if $L/E/k$ is a tower, such that $E/k$ is not seperable, it must be the case that $E = L$, and in this case, the extension $L/k$ is not purely inseperable.

**4.** 1. No, although every finite extension is algebraic not every finite extension is seperable. As a counter example consider $\mathbf{F}_p(t)(\alpha)/\mathbf{F}_p(t)$, where $\alpha$ is a root of the irreducible polynomial (irreducible by Gauss' Lemma, then Eisensteins criterion) $X^p - t$ in $\mathbf{F}_p(t)[X]$. By construction we have $[\mathbf{F}_p(t)(\alpha) : \mathbf{F}_p(t)] = p$ so the extension is finite, but $\min(\alpha, \mathbf{F}_p(t)(\alpha)) = X^p - t$ has zero derivative so the extension is not seperable.

2. No, the extension $\mathbb{Q}(\sqrt[3]{2})$ is a counterexample. As proof, first note that the extension is seperable since $\mathbb{Q}$ is perfect. However, we have the embedding (fixing $\mathbb{Q}$) $\sigma : \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{C}$, where $\sigma : \sqrt[3]{2} \mapsto \zeta_3\sqrt[3]{2}$, then by the extension theorem this can be extended to an automorphism $\overline{\sigma} : \mathbb{C} \to \mathbb{C}$. $\mathbb{Q}(\sqrt[3]{2})$ is not fixed under this automorphism implying it is not normal.

3. No, in fact every purely inseperable extension is normal. Existence of purely inseperable extensions is proven in the first counter example (for a more in depth proof of why this extension is purely inseperable see the lemma in problem 5). To see they are normal, let $E/F$ be a purely inseperable extension, and $L$ an algebraically closed field containing $E$. Then for any $\alpha \in E$ $\alpha$ is the only root of its minimum polynomial over $F$. Hence if $\sigma$ is an $F$-automorphism of $L$, then we have seen $\sigma(\alpha)$ is a root of ${}^\sigma f = f$, hence $\alpha$ is fixed. Since this holds for all elements of $E$, we have that $\sigma|_E = \mathbf{1}_E \in \mathrm{Aut}(E)$ is an automorphism of $E$ so that $E$ is normal.

**5. Lemma.** A finite extension is purely inseperable if and only if its seperability degree is 1.

As proof let $L, k$ be fields, with $L/k$ and $[L : k] < \infty$, and denote $K$ as an algebraically closed field containing $L$. First assume that $[L : k]_s = 1$, then for any $\alpha \in L$, $\alpha$ has no conjugates, else the embedding $\sigma : k \to K$ could be extended to multiple maps (one for each conjugate) $k(\alpha) \to K$, then by the extension theorem each of these could be extended to a map from $L$ to $K$, which would contradict $[L : k]_s = 1$. Hence $\alpha$ must be a purely inseperable element. Now conversely, assume that $L/k$ is purely inseperable. Then since the extension is finite, we can write it as a tower of simple extensions $L = k(\alpha_1, \dots, \alpha_n)/k(\alpha_1, \dots, \alpha_{n-1})/ \cdots /k(\alpha_1)/k$, it is immediate that each extension is purely inseperable, since for $\beta \in L$, we have $\min(\beta, k(\alpha_1, \dots, k_\ell))| \min(\beta, k)$, which only has one distinct root (similarly for any of the extensions in the tower). Then it will suffice to show that a simple extension by a purely inseperable element has seperable degree 1, but this is clear, since any extension must send $\alpha$ to a root of its minimal polynomial, hence itself. This proves the desired result

$$[L : k]_s = \prod_{i=1}^{n} [k(\alpha_1, \dots, \alpha_i) : k(\alpha_1, \dots, \alpha_{i-1})] = 1$$

In order to show $P$ is a subextension, we only need show $\{a \in E | a \text{ is purely inseperable over } F\}$ is a field since each element of $F$ is purely inseperable by definition. To do so, we will show that if $\alpha, \beta$ are purely inseperable, then so are $\alpha^{-1}, \alpha+\beta$ and $\alpha+\beta$. The case of $\alpha^{-1}$ is obvious since $\alpha$ being purely inseperable over $F$ implies that $[F(\alpha) : F]_s = 1$, since any extension must map $\alpha$ to another root of its minimal polynnomial, hence only itself. This proves that every element of $F(\alpha)/F$ is purely inseperable, in particular $\alpha^{-1}$ is purely inseperable. As per the other two cases, we have

$$[F(\alpha, \beta) : F]_s = [F(\alpha, \beta) : F(\alpha)]_s [F(\alpha) : F]_s = 1$$
$$1 = [F(\alpha, \beta) : F]_s = [F(\alpha, \beta) : F(\alpha + \beta)]_s [F(\alpha + \beta) : F]_s = 1 \implies [F(\alpha + \beta) : F]_s = 1$$
$$1 = [F(\alpha, \beta) : F]_s = [F(\alpha, \beta) : F(\alpha\beta)]_s [F(\alpha\beta) : F]_s = 1 \implies [F(\alpha\beta) : F]_s = 1$$

so that each extension is purely inseperable implying that $\alpha + \beta$ and $\alpha\beta$ are both purely inseperable.

Since purely inseperable extensions are normal and $[P : F] \leq [E : F] < \infty$, we have that $P$ is the splitting field of a polynomial $f$ over $F$. Each of the irreducible factors $\{f_i\}_1^n$ of $f$ must be purely inseperable, hence over $P$ we have

$$f = \prod_1^n (x - \alpha_i)^{p^{k_i}}$$

So we can take the tower of purely inseperable simple extensions (it is easy to see the extensions are purely inseperable since $[P : F]_s = 1$)

$$P = F(\alpha_1, \ldots, \alpha_n)/ \cdots /F(\alpha_1)/F$$

Where each simple extension has $p$-power order by inseperability and hence $P/F$ has $p$-power order by multiplicativity of degree.