1. To show that F is increasing, first note that μ is nonnegative, so that for any $x > 0$, $y < 0$ we have

$$F(y) = -\mu((y, 0]) \le F(0) = 0 \le \mu((0, x]) = F(x)$$

Then it suffices to show F is increasing on $[-\infty, 0)$ and $(0, \infty]$, both follow directly from monotonicity of μ,

$$0 < x \le y \implies (0, x] \subset (0, y] \implies \mu((0, x]) \le \mu((0, y]) \implies F(x) < F(y)$$
$$y \le x < 0 \implies (x, 0] \subset (y, 0] \implies -\mu((0, y]) \le -\mu((0, x]) \implies F(y) < F(x)$$

Let $x_0 \ge 0$, then $\lim_{x \searrow x_0} F(x) = \lim_{x \searrow x_0} \mu(0, x]$. It suffices to show for an arbitrary decreasing sequence $\{x_i\}_1^\infty$, we have $\lim_{i \to \infty} \mu(0, x_i] = \mu(0, x_0] = F(x_0)$ which follows from continuity from above, so the right sided limit exists and is equal to $F(x_0)$.

Now let $x_0 < 0$, then $\lim_{x \searrow x_0} F(x) = -\lim_{x \searrow x_0} \mu(x, 0]$. It suffices to show for an arbitrary decreasing sequence $\{x_i\}_1^\infty$, we have $\lim_{i \to \infty} -\mu(x_i, 0] = -\mu(x_0, 0] = F(x_0)$, which follows from continuity from below, so the right hand sided limit exists and is equal to $F(x_0)$.

2. (a) F bounded. Proof being, assume $|F| < M$, define sets $\{E_i\}_1^\infty$, where $E_i = (-i, -i + 1] \cup (i - 1, i]$ then we have

$$m^F(\mathbb{R}) = \sum_1^\infty m^F(E_i) = \lim_{n \to \infty} \sum_1^n m^F(E_i)$$
$$= \lim_{n \to \infty} \sum_1^n F(i) - F(i - 1) + F(-i + 1) - F(-i)$$
$$= \lim_{n \to \infty} F(n) - F(-n) \le \lim_{n \to \infty} |F(n)| + |F(-n)| \le 2M$$

Conversely, assume that $m^F(\mathbb{R}) = M < \infty$ (note that $M > 0$), then for any $x \ge 0$, we have $F(x) - F(0) \le m^F(\mathbb{R}) = M$ by monotonicity, and $F(0) < F(x)$. Similarly, if $x < 0$, we have $F(x) \le F(0)$, and by monotonicity $F(0) - F(x) \le m^F(\mathbb{R}) = M$. Taken together for any x we have

$$F(0) - M \le F(x) \le M + F(0)$$

implying F is bounded.

(b) F continuous at x_0. Proof being, assume F is continuous at x_0, then for some sequence $\{\delta_n\}_1^\infty$, we have $|x - x_0| \le \delta_n \implies |F(x) - F(x_0)| < \frac{1}{n}$, then continuity from above implies

$$0 \le m^F(\{x_0\}) = m^F\left(\bigcap_1^\infty (\delta_n, x_0]\right) = \lim_{n \to \infty} m^F((\delta_n, x_0]) \le \lim_{n \to \infty} \frac{1}{n} = 0$$

Conversely we need only show right continuity. Suppose that $m^F(\{x_0\}) = 0$, and let $\epsilon > 0$, then continuity from above implies that

$$\lim_{n \to \infty} m^F(x_0 - \frac{1}{n}, x_0] = m^F\left(\bigcap_1^\infty (x_0 - \frac{1}{n}, x_0]\right) = m^F(\{x_0\}) = 0$$

so in particular, there exists N sufficiently large that $m^F(x_0 - \frac{1}{N}, x_0] = |F(x_0) - F(x_0 - \frac{1}{N})| < \epsilon$, and since F is increasing and right continuous this proves left continuity and hence continuity.

(c) $m^{F,*}$ is the point mass at 0. Proof being, let $0 \in E \subset \mathbb{R}$, then for any collection of half open intervals $\{I_i\}_1^\infty$, we have $0 \in I_n$ for some n. Then we can write $I_n = (a, b]$, for $a < 0$, $b \ge 0$, then $m_0^F(I_n) = 1$, so that $1 \le \sum_1^\infty m_0^F(I_i)$, and since this holds for all such covers of E, we have $1 \le m^{F,*}(E)$, and for the reverse inequality note that $E \subset \mathbb{R} \subset \bigcup_1^\infty (-i, -i + 1] \cup (i - 1, i + 1]$, which is a countable union of half open intervals, all but $(-1, 0]$ having $m_0^F(I) = 0$, so $1 \le m^{F,*}(E) \le m^{F,*}(\mathbb{R}) \le 1$.

Now suppose that $0 \notin E$, then $E \subset (-\infty, 0) \cup (0, \infty)$, but then $(-\infty, 0) \cup (0, \infty) = \bigcup_1^\infty (-n, 1/n) \cup (0, n)$ implies that

$$0 \le m^{F,*}(E) \le m^{F,*}((-\infty, 0) \cup (0, \infty)) \le \sum_1^\infty m_0^F(-n, 1/n) + m_0^F(0, n) = 0$$

Finally note that $m^{F,*}(\emptyset) = 0$ by definition.

I claim that $M_F = \mathcal{P}(\mathbb{R})$, let $A \subset \mathbb{R}$, and $E \subset \mathbb{R}$. First assume that $0 \notin E$, then

$$m^{F,*}(E) = 0 = m^{F,*}(E \cap A) + m^{F,*}(E \cap A^c)$$

since neither of the sets measured on the right hand side of the equation contain 0. Now assume that $0 \in E$, then $0 \in E \cap A$ or $E \cap A^c$ but not both. This suffices to show that

$$m^{F,*}(E) = 1 = m^{F,*}(E \cap A) + m^{F,*}(E \cap A^c)$$

So each $A \in M_F$ is measurable.

(d) m^F counts the number of integers in a set E. Proof being, denote the floor function as F. First apply theorem 1.16 of Folland (since F is increasing and right continuous), then $m^F(a, b] = (F(b) - F(a))$ is a measure, so we may use measure properties. If z is an integer, then we can apply continuity from above:

$$m^F(\{z\}) = m^F\left(\bigcap_1^\infty (z - \frac{1}{n}, z]\right) = \lim_{n \to \infty} m^F(z - \frac{1}{n}, z] = \lim_{n \to \infty} 1 = 1$$

So each integer singleton is a measurable set of measure 1. Now let $E \subset \mathbb{R} \setminus \mathbb{Z}$, then $E \subset \bigcup_{n \in \mathbb{Z}} \bigcup_{k=1}^\infty (n-1, n - \frac{1}{k}]$ is a countable union of sets with measure 0, hence

$$0 \le m^F(E) \le m^F(\bigcup_{n \in \mathbb{Z}} \bigcup_{k=1}^\infty (n-1, n - \frac{1}{k}]) \le \sum_{n \in \mathbb{Z}} \sum_{k=1}^\infty m^F(n-1, n - \frac{1}{k}] = 0$$

Note that singletons are borel sets, then if $E \subset \mathbb{R}$, we can write $E \cap \mathbb{Z} = \{z_i\}_i$, then if there are infinitely many z_i:

$$\infty = m^F\left(\bigcup_i \{z_i\}\right) \le m^F(E)$$

and if there are finitely many z_i:

$$m^F(E) = m^F(E \cap \mathbb{Z}) + m^F(E \cap \mathbb{Z}^c) = m^F\left(\bigcup_{i=1}^n \{z_i\}\right) + 0 = \sum_1^n m^F(\{z_i\}) = n$$

We can once again cite Theorem 1.16 of Folland which says this is the unique measure induced by F.

3. (a) First notice for any d and any $\delta > 0$, given $\{l_j\}_j$ with $0 \le l_j \le \delta$ we have $l_j^d \ge 0$, and hence $0 \le \sum_1^\infty l_j^d$. This suffices to show that $H_d(E) \ge H_{d,\delta}(E) \ge 0$ for any set E.

Consider any d and let $\delta > 0$, (remark we use the convention $0^0 = 0$), then $\emptyset \subset (0, 0+0)$, and $0 \le \delta$, so that $0 \le H_{d,\delta}(\emptyset) \le 0^d = 0$ since δ was arbitrary, this also proves $H_d(\emptyset) = 0$.

Now consider $A \subset B \subset \mathbb{R}$. And consider any d and any $\delta > 0$. Then for any cover of B of the form $B \subset \bigcup_1^\infty (a_j, a_j + l_j)$ $(0 \le l_j \le \delta)$, it is immediate $A \subset \bigcup_1^\infty (a_j, a_j + l_j)$. Hence $H_{\delta,d}(A) \le \sum_1^\infty l_j^d$. And since this holds for an arbitrary cover of B of this form

$$H_{\delta,d}(A) \le \inf\{\sum_1^\infty l_j^d | B \subset \bigcup_1^\infty (a_j, a_j + l_j), 0 \le l_j \le \delta\} = H_{\delta,d}(B)$$

Now since δ was arbitrary, we have for any $\delta > 0$, $H_{\delta,d}(A) \leq H_d(B)$, and since this holds for each δ, $H_d(A) \leq H_d(B)$.

Now to verify subadditivity, let $\epsilon > 0$, then let d be arbitrary and $\delta > 0$, denote $E := \bigcup_1^\infty E^i \subset \mathbb{R}$. For each i we can choose $\{(a_j^i, a_j^i + l_j^i)\}_{j=1}^\infty$ covering E^i, such that $0 \leq l_j^i \leq \delta$ and $\sum_{j=1}^\infty (l_j^j)^d < H_{d,\delta}(E^i) + 2^{-i}\epsilon$. Then $E \subset \bigcup_{i,j}(a_j^i, a_j^i + l_j^i)$, so that

$$H_{\delta,d}(E) \leq \sum_{i,j}(l_j^i)^d < \sum_i H_{\delta,d}(E^i) + 2^{-i}\epsilon = \epsilon + \sum_i H_{\delta,d}(E^i)$$

and since the ϵ was arbitrary, $H_{\delta,d}(E) \leq \sum_i H_{\delta,d}(E^i)$. This implies that for arbitrary $\delta > 0$ we have

$$H_{\delta,d}(E) \leq \sup_{\delta>0} \sum_i H_{\delta,d}(E^i) \leq \sum_i \sup_{\delta>0} H_{\delta,d}(E^i) \overset{\text{def}}{=} \sum_i H_d(E^i)$$

then since δ was arbitrary, $H_d(E) \leq \sum_i H_d(E^i)$.

(b) $H_0(E)$ is the counting measure. Proof being, first assume E is a finite collection of points, i.e. $E = \{z_i\}_1^n$, then for any $\delta > 0$, we have $E \subset \bigcup_1^n (z_i - \delta/2, (z_i - \delta/2) + \delta)$, so that $H_0(E) \leq n$. To show the reverse inequality, let $\delta = \frac{1}{2}\min_{i \neq j}\{|z_i - z_j|\}$. Then we need atleast n intervals of length smaller or equal to δ to cover E, since no two points may lie in the same interval. Hence if $\{(a_j, a_j + l_j)\}_{j \in J}$ is a cover of E, with each $l_j \leq \delta$, we have $\sum_J l_j^0 \geq \sum_1^n l_j^0 = n$, hence

$$n \geq H_0 \geq H_{0,\delta} \geq n$$

Now suppose E is infinite, then E has a countable subset $\{z_i\}_1^\infty$. I will show that for any natural number $H_0(E) > N$, and hence $H_0(E) = \infty$. Let $N \in \mathbb{N}$, then similar to before, let $\delta_N = \frac{1}{2}\min_{i \neq j \text{ and } 1 \leq i,j \leq N}|z_i - z_j|$ then none of $\{z_i\}_1^N$ are in the same interval having length less than or equal to δ_N, hence if $\{z_i\}_1^N \subset E \subset \bigcup_{i \in I}(a_i, a_i + l_i)$ we must have $\#I \geq N$, so that $\sum_{i \in I} l_i^d \geq \sum_1^N l_i^d = N$, which suffices to show $H_{0,\delta_N}(E) \geq N$. Now since $H_0(E) \geq H_{0,\delta_N}(E) \geq N$ for all $N \in \mathbb{N}$ we get $H_0(E) = \infty$ as desired.

(c) Let $d > \frac{\log 2}{\log 3}$, then $3^d > 3^{\frac{\log 2}{\log 3}} = 3^{\log_3 2} = 2$, hence we have $23^{-d} = r < 1$. With this book-keeping out of the way we can move on to solving the problem. Let $\delta > 0$, then for some $N \in \mathbb{N}$ we have for all $n \geq N$, $3^{-n} < \delta$. Then let K_j denote the j-th iteration in the construction of the cantor set, and $\partial_L K_j$ by the left endpoints of the intervals in K_j. Then

$$\mathcal{C} \subset K_{n+1} \subset \bigcup_{x \in \partial_L K_{n+1}} (x - 3^{-(n+1)}, x + 2 \cdot 3^{-(n+1)})$$

Where $\#\partial_L K_{n+1} = 2^{n+1}$ since K_j contains j intervals, note the subset relation follows by the length of the intervals being $3^{-(n+1)}$. It follows that

$$H_{d,\delta}(\mathcal{C}) \leq \sum_1^{2^{n+1}} 3^{-nd} = 2^{n+1}3^{-nd} = 2r^n$$

and since this holds for all n sufficiently large it holds in the limit, proving that

$$H_{d,\delta}(\mathcal{C}) \leq \lim_{n \to \infty} 2r^n = 0$$

4. (a) It is immediate that $\mathcal{C}_{\overrightarrow{\alpha}}$ is closed since it is the intersection of closed sets. Hence it will suffice to show that $\mathcal{C}_{\overrightarrow{\alpha}}$ contains no open sets, and since each open set contains an open interval we can show this for open intervals. Suppose I is an open interval, then it has some length $\ell > 0$, so choose n large enough that $\ell > 2^{-n}$. Let ℓ_j denote the length of the closed intervals making up K_j, then $\ell_0 = 1$ and $\ell_{j+1} = \frac{1}{2}\ell_j(1 - \alpha_{j+1})$, so by induction $\ell_j \leq 2^{-j}$ (this is also the maximum length of an open interval contained in the set), hence $I \not\subset K_n$, and since $\mathcal{C}_{\overrightarrow{\alpha}} \subset K_n$ this implies that $I \not\subset \mathcal{C}_{\overrightarrow{\alpha}}$. And since I was arbitrary it follows that $\mathcal{C}_{\overrightarrow{\alpha}}$ contains no intervals.

(b) Use the same definition of ℓ_j as in (a). Then notice that $m(k_j) = 2^j \ell_j$, then

$$m(k_{j+1}) = 2^{j+1}(\ell_{j+1}) = 2^j \ell_j (1 - \alpha_{j+1}) = m(k_j)(1 - \alpha_{j+1})$$

So continuity from above implies that

$$m(\mathcal{C}_{\vec{\alpha}}) = m\left(\bigcap_1^\infty K_n\right) = \lim_{n \to \infty} m(K_n) = \prod_1^\infty (1 - \alpha_n)$$

Now let $\beta \in [0, 1)$, then we can write $\beta = e^{-x}$ for $x > 0$, then choose $0 < \alpha_i := 1 - e^{-\frac{x}{2^i}} < 1$. It follows that $\log(1 - \alpha_i) = -\frac{x}{2^i}$. Taken together

$$\beta = e^{-x} = \exp(\sum_1^\infty -\frac{x}{2^i}) = \exp(\sum_1^\infty \log(1 - \alpha_i)) = \prod_1^\infty (1 - \alpha_i)$$

As desired.

(c) Assume for contraposition that $E \subset [0, 1]$, with $m(E) = 1$. Then $\overline{E} \subset [0, 1]$ implies that $1 \geq m(\overline{E}) \geq m(E) \geq 1$. Then the compliment of $\overline{E}$ in $[0, 1]$ is open with measure 0, hence empty (since any non-empty open set contains an interval and intervals have non-zero measure). It follows that $\overline{E} = [0, 1]$ contains the interval $(0, 1)$, so E is not nowhere dense. Contraposition completes the proof.

(d) Let $\epsilon > 0$ but as close to 0 as the reader might like, then take $\{q_i\}_1^\infty$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Then take open sets $I_i := N_{\epsilon 2^{-n-1}}(q_i)$, it follows that $I = \bigcup_1^\infty I_i$ is an open set containing all of the rationals in $[0, 1]$, hence I^c is closed containing none of the rationals and hence no interval. Then

$$1 = m([0, 1]) \geq m(I^c \cap [0, 1]) \geq 1 - m(I) \geq 1 - \sum_1^\infty m(I_i) = 1 - \sum_1^\infty 2^{-n} \epsilon = 1 - \epsilon$$

5. (a) $E \subset [0, 1]$ implies that $m^*(E) \leq 1$ by monotonicity. $m^*(E) > 0$, since any set of outer measure 0 is measurable, by the Caratheodory theorem which says that the induced measure by $m^*(E)$ is complete on its measurable sets.

(b) Suppose for contradiction $m(A) > 0$. Then by the Steinhaus theorem $A - A$ contains an interval, but the only rational member of $A - A$ is zero, since either elements are equal or in different rational translate cosets. This proves that $A - A$ cannot contain an interval (since any interval contains countably many rationals), a contradiction.

(c) Consider any countable collection of intervals $\{I_i\}_1^\infty$, such that $[0, 1] \setminus E \subset \bigcup_1^\infty I_i$, then $[0, 1] \cap (\bigcup_1^\infty I_i)^c \subset E$, and is a borel set hence measurable, so it must have measure (by caratheodory theorem this is equal to outer measure) 0 by the previous subpart.

$$m^*\left([0, 1] \cap (\bigcup_1^\infty I_i)^c\right) + m^*\left([0, 1] \cap (\bigcup_1^\infty I_i)\right) = m\left([0, 1] \cap (\bigcup_1^\infty I_i)^c\right) + m\left([0, 1] \cap (\bigcup_1^\infty I_i)\right)$$
$$= m[0, 1] = 1$$

Implying that $m^*\left(\bigcup_1^\infty I_i\right) \geq m^*\left([0, 1] \cap (\bigcup_1^\infty I_i)\right) = 1$, hence by definition $m^*([0, 1] \setminus E) \geq 1$. To show equality, let $\epsilon > 0$, then $[0, 1] \setminus E \subset (-\frac{\epsilon}{2}, 1 + \frac{\epsilon}{2})$ implies that $m^*([0, 1] \setminus E) \leq 1 + \epsilon$, and since ϵ was arbitrary,

$$1 \leq m^*([0, 1] \setminus E) \leq 1$$

as desired.