1. Let $P \subset A$ be prime, then it will suffice to show that A/P is a field which is equivalent to maximality of P by the correspondence theorem. Consider $0 \neq x \in A/P$, then choose $n \geq 2$ such that $x^n = x$, it follows that $x(1-x^{n-1}) = x - x = 0$, and since P is prime A/P is a domain which implies that $1 - x^{n-1} = 0$, so that $x^{n-1} = 1$ in A/P. □

2. Suppose that M is not flat, then we can fix modules A, B, such that

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact, but

$$0 \longrightarrow A \otimes M \xrightarrow{(1\times f)_*} B \otimes M$$

is not. It follows that there is some $0 \neq \sum_1^n a_i \otimes x_i \in A \otimes M$, such that $(1 \times f)_*(\sum_1^n a_i \otimes x_i) = 0$. I claim that $M_0 = (x_1, \ldots, x_n)$ is the desired submodule. To see this, note $(1\times f)|_{A\times M_0} = 1 \times f|_{M_0}$, and if j is the map $A \times M \to A \otimes M_0$ in the definition of the tensor, then $j|_{A\times M_0}$ is equal the map $A \times M_0 \to A \otimes M_0$ in the definition of the tensor. It follows that for any $v \in A \otimes M_0$, $v = j|_{A\times M_0}(u)$, $u \in A \times M_0$, so that

$$(1 \times f|_{M_0})_*(v) = (1 \times f|_{M_0})_* j|_{A\times M_0}(u) = 1 \times f|_{M_0}(u) = 1 \times f(u) = (1 \times f)_* j(u)$$

and hence $(1 \times f|_{A\times M_0})_* j|_{M_0}(\sum_1^n (a_i, x_i)) = (1 \times f)_* j(\sum_1^n (a_i, x_i)) = (1 \times f)_*(\sum_1^n a_i \otimes x_i) = 0$, where $0 \neq \sum_1^n a_i \otimes x_i = j(\sum_1^n (a_i, x_i)) = j|_{A\times M_0}(\sum_1^n (a_i, x_i))$ which suffices to show that $1 \times f|_{M_0}$ is not injective, and hence M_0 is not flat, with the following sequence as witness.

$$0 \longrightarrow A \otimes M_0 \xrightarrow{(1\times f|_{M_0})_*} B \otimes M_0$$

!!! NEED TO DEAL WITH $B \otimes M_0$!!!

3. Since $C[X]$ is a PID, it satisfies Bezout's identity. So assume f_1, f_2 are coprime polynomials, it follows that there exist $g, h \in \mathbb{C}[X]$, such that $f_1 h + f_2 g = 1$. Now let $m \otimes n \in M_1 \otimes M_2$, it follows that

$$\begin{aligned} m \otimes n &= (f_1 h + f_2 g)(m \otimes n) = f_1 h(m \otimes n) + f_2 g(m \otimes n) = h(f_1 m \otimes n) + g(m \otimes f_2 n) \\ &= h(0 \otimes n) + g(m \otimes 0) = 0 \end{aligned}$$

Conversely, let $a \in \mathbb{C}$, such that $f_1(a) = f_2(a) = 0$. Let $I = (X - a)$ and consider the map multiplication map

$$m : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X]/(X - a),\ (f, g) \mapsto fg + I$$

To see that this defines a bilinear map $M_1 \times M_2 \to \mathbb{C}[X]/I$ it will suffice to check that m is well defined on cosets so that we can take the induced bilinear map

$$\overline{m} : M_1 \times M_2 \to \mathbb{C}[X]/I,\ (f + (f_1), g + (f_2)) \mapsto fg + I$$

Let $g_1, g_2, h_1, h_2 \in \mathbb{C}[X]$, then

$$m(g_1 + h_1 f_1, g_2 + h_2 f_2) = g_1 g_2 + g_1 h_2 f_2 + g_2 h_1 f_1 + h_1 h_2 f_1 f_2 + I = g_1 g_2 + I$$

the last equality following since both $f_i \in I$. It follows that $\overline{m} : M_1 \times M_2 \to \mathbb{C}(X)/I$ is a nonzero (since $(1, 1) \mapsto 1$) bilinear map, so $\overline{m} = \eta j$ where j is the map from the definition of the tensor product and $\eta : M_1 \otimes M_2 \to \mathbb{C}[X]/I$. Since $\overline{m}$ is non-zero, it follows that η is nonzero and hence $M_1 \otimes M_2 \neq 0$ since $\eta \notin \{0\} = \mathrm{Hom}(0, \mathbb{C}[X]/I)$. □

4. Consider the exact sequence of $\mathbb{R}$ modules

$$0 \longrightarrow \mathbb{R} \xrightarrow{\iota} A$$

Where $\iota : 1 \mapsto t$, exactness is clear. The following is well defined by extesion of scalars.

$$0 \longrightarrow \mathbb{R} \otimes_A N \xrightarrow{\iota_*} A \otimes_A N$$

Furthermore,

$$\iota_* 1 \otimes e_2 = \iota_* 1 \otimes te_1 = t\iota_* 1 \otimes e_1 = t(t \otimes e_1) = t^2(1 \otimes e_1) = 0$$

Now it will suffice to check that $0 \neq 1 \otimes e_2 \in \mathbb{R} \otimes_A N$. When regarding $\mathbb{R}$ as an A-module, via extension of scalars, we are taking $\mathbb{R} \cong A \otimes_{\mathbb{R}} \mathbb{R} \cong A$, so we define our A-bilinear map on $A \times N$. Simply consider the multiplication map $m : (a, b) \mapsto ab$, then $0 \neq e_2 = m(1, e_2)$, since this map factors is bilinear, it factors through j, so that $1 \otimes e_2 \overset{\text{def}}{=} j(1, e_2) \neq 0$ which suffices to show that $\ker \iota_* \neq 0$ so that the tensored sequence is not exact and hence N is not flat. □

5. Suppose that $r \leq n$, and $g_1, g_2, \ldots, g_r$ generate I as an A module. Let

$$J := (X_1^2, X_1X_2, \ldots, X_1X_n, X_2^2, X_2X_3 \ldots, X_n^2)$$

be the ideal generated by all degree 2 monomials in $\mathbb{R}[X_1, \ldots, X_n]$, it follows that by assumption each monomial in $f_1, \ldots f_m$ is divisible by some element of J, and hence $(f_i)_1^m/J = 0$. It follows that $\overline{g_1}, \ldots, \overline{g_r}$ generate I/J as an A-module, now note that no element of I has a term with degree 0, hence each monomial of g_i (and hence $\overline{g_i}$) has degree atleast one. Since all monomials degree larger than or equal to 2 are annihilated in I/J we may conclude that each $\overline{g_i} = X_j$ up to units. It follows that each $\overline{g_i} = X_j$, and we may reindex without loss of generality so that $(g_1, \ldots, g_r) = (X_1, \ldots, X_r)$.

Applying the third isomorphism theorem,

$$A/I \cong \frac{\mathbb{R}[X_1, \ldots, X_n]/I}{(f_1, \ldots, f_m)/I} \cong \mathbb{R}[X_1, \ldots, X_n]/I \cong \mathbb{R}$$

is a field, and $(X_1, \ldots, X_r)/J = I/J$ as an A-module implies that

$$\bigoplus_1^r A/I \cong (X_1, \ldots, X_r)/J/(X_1, \ldots, X_r)I = I/J/I^2 \cong \bigoplus_1^n A/I$$

since the rank of isomorphic vectorspaces must be equal this implies that $r = n$. □

6. $A[X] = \bigoplus_0^\infty AX^i$ as an A-module, since $A[X^i] \otimes_A M \cong A \otimes_A M \cong M$, it is immediate that AX^i is flat for each i, assume for contradiction that $\oplus_0^\infty AX^i \cong \oplus_0^\infty A$ is not flat, then applying problem 2, there is some finitely generated submodule M_0, such that M_0 is not flat. Since submodules of free modules are free, we know that $M_0 \cong \bigoplus_1^n A$, implying that $\bigoplus_1^n A$ is not flat, but this is a contradiction, since this is only the case if

$$0 \longrightarrow K \xrightarrow{f} L$$

is exact, but the following sequence is not

$$0 \longrightarrow K \otimes \bigoplus_1^n A \xrightarrow{f \otimes 1_{\oplus_1^n A}} L \otimes \bigoplus_1^n A$$

but this is equivalent to the following sequence not being exact

$$0 \longrightarrow \bigoplus_1^n K \otimes A \xrightarrow{\oplus_1^n f \otimes 1_A} \bigoplus_1^n L \otimes A$$

which once again is equivalent to the following not being exact

$$0 \longrightarrow \bigoplus_1^n K \xrightarrow{\oplus_1^n f} \bigoplus_1^n L$$

where $\bigoplus_1^n f$ is injective since f is. $\square$