1. We can consider the parameterized line defined by N and $\begin{pmatrix} x & y & z \end{pmatrix}$, $l(t) = \begin{pmatrix} tx & ty & 1 + t(z-1) \end{pmatrix}$ Then $t(z-1) = -1$ when $t = \frac{1}{1-z}$. Plugging in this value for t defines φ.

$$\varphi : \begin{pmatrix} x & y & z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x}{1-z} & \frac{y}{1-z} \end{pmatrix}$$

We define $\tilde{\varphi}$ similarly. In this case we get

$$\tilde{\varphi} \begin{pmatrix} x & y & z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x}{1+z} & \frac{y}{1+z} \end{pmatrix}$$

Similar to before, we find φ^{-1} by parameterizing the line through $\big(u, v, 0\big)$ and N, to compute the inverse we check where $\ell(t) := \big(tu, tv, 1-t\big)$ intersects the sphere.

$$(tu)^2 + (tv)^2 + (1-t)^2 = 1 \iff t(t(u^2 + v^2 + 1) - 2) = 0 \iff t = 0 \text{ or } t = \frac{2}{u^2 + v^2 + 1}$$

We can ignore the case of $t = 0$, since this corresponds to N, plugging in this t gives the inverse.

$$\varphi^{-1} : (u, v) \mapsto \begin{pmatrix} \frac{2u}{u^2+v^2+1} & \frac{2v}{u^2+v^2+1} & \frac{u^2+v^2-1}{u^2+v^2+1} \end{pmatrix}$$

Here is the coordinate change:

$$\tilde{\varphi} \circ \varphi^{-1} \begin{pmatrix} u & v \end{pmatrix} = \tilde{\varphi} \begin{pmatrix} \frac{2u}{u^2+v^2+1} & \frac{2v}{u^2+v^2+1} & \frac{u^2+v^2-1}{u^2+v^2+1} \end{pmatrix} = \begin{pmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \end{pmatrix}$$

2.

$$\varphi_0^{-1} : (x_1, \ldots, x_n) \mapsto (1, x_1, \ldots, x_n)$$

Then we have:

$$\begin{aligned} &\varphi_0 \circ \varphi_0^{-1} : (x_1, \ldots, x_n) \mapsto \varphi_0(1, x_1, \ldots, x_n) = (x_1, \ldots, x_n) \\ &\varphi_0^{-1} \circ \varphi_0 : (x_0, x_1, \ldots, x_n) \mapsto \varphi_0^{-1}(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = (1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \sim (x_0, x_1, \ldots, x_n) \end{aligned}$$

Here is the change of coordinates:

$$\varphi_1 \circ \varphi_0^{-1} : (x_i)_{i=1}^n \mapsto (x_1^{-1}, x_2/x_1, x_3/x_1, \ldots, x_n/x_1)$$

3. **(a)** Suppose for the sake of contradiction R_1, R_2 are products of elementary row operations such that

$$R_1 A = \begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \text{ and } R_2 A = \begin{bmatrix} 1 & 0 & y_1 & y_3 \\ 0 & 1 & y_2 & y_4 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

Where $(x_1, x_2, x_3, x_4) \neq (y_1, y_2, y_3, y_4)$ (Note that these two row spaces are subsets of $\text{rowsp}A$, but since they have rank 2 they are equal to eachother and $\text{rowsp}A$). It follows that one of $\mathbf{v}_1 - \mathbf{u}_1$ or $\mathbf{v}_2 - \mathbf{u}_2$ is non-zero (WLOG $\mathbf{w} := \mathbf{v}_1 - \mathbf{u}_1$). But then since $\mathbb{R}\mathbf{v}_1 + \mathbb{R}\mathbf{v}_2 = \mathbb{R}\mathbf{u}_1 + \mathbb{R}\mathbf{u}_2$ it follows that $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2$, $a, b \in \mathbb{R}$. Since $\mathbf{w}_i = 0$, $i \in \{1, 2\}$, and $(a\mathbf{v}_1 + b\mathbf{v}_2)_1 = a$ and $(a\mathbf{v}_1 + b\mathbf{v}_2)_2 = b$ it follows that $a = b = 0$ and hence $\mathbf{w} = \mathbf{0}$, a contradiction.

(b) I will denote $A = (a_{i,j})$ for $i \leq 2, j \leq 4$. Note that the determinant is a continuous function on $(a_{i,j})$, $i, j \leq 2$ (it is a polynomial in these 4 variables and the sum and product of continuous functions is continuous). Furthermore, row operations on A are just row operations on $(a_{i,j})$, $i, j \leq 2$. It follows that if $\det(a_{i,j})$, $i, j \leq 2$ is non-zero, continuity furnishes some $\delta > 0$, such that if $|b_i - a_i| < \delta$ for each i, then $|\det(b_{i,j}) - \det(a_{i,j})| < |\det(a_{i,j})|$ for $i, j \leq 2$, we are done once we apply the reverse triangle inequality

$$|\det(a_{i,j})| - |\det(b_{i,j})| \leq |\det(b_{i,j}) - \det(a_{i,j})| < |\det(a_{i,j})| \implies 0 < |\det(b_{i,j})|$$

Now let $B = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$ be a matrix who's entries are within δ of A's entries. It follows that since $(b_{i,j}),\ i,j \leq 2$ is full rank we get that $\begin{pmatrix}1 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 1\end{pmatrix}$ are in the span of $\mathbb{R}\mathbf{b}_1' + \mathbb{R}\mathbf{b}_2'$ where $\mathbf{b}_i'$ is $\mathbf{b}_i$ restricted to its first two coordinates. It follows that when we dont restrict to the first two coordinates, $\begin{pmatrix}1 & 0 & y_1 & y_2\end{pmatrix}$ and $\begin{pmatrix}0 & 1 & y_1 & y_2\end{pmatrix}$ for some y_i are in the rowspace of B.

(c) If A has full rank, then it has a 2×2 submatrix of full rank, hence by the same procedure as part (b) can be brought to one of these forms by row operations since all 2×2 submatrices are listed.

(d) We can do the computation by using row operations to convert a matrix of form U_2 to U_1. Note we can divide by y_2, since any matrix in $U_1 \cap U_2$ is in U_1, hence has minor corresponding to U_1 having full rank, which implies y_2 is nonzero.

$$\varphi_1 \circ \varphi_2^{-1} : \begin{pmatrix}y_1 & y_2 & y_3 & y_4\end{pmatrix} \mapsto \begin{pmatrix}-y_1/y_2 & 1/y_2 & y_3 - \frac{y_1 y_4}{y_2} & y_4/y_2\end{pmatrix}$$