1. (a) We first need to show the homomorphism property, to do so apply the chain rule, where we note that $C_g(e) = e$ for all g. Then

$$d(C_{g_1}C_{g_2})_e(x) = d(C_{g_1})_{C_{g_2}(e)} \circ d(C_{g_2})_e(x) = d(C_{g_1})_e \circ d(C_{g_2})_e(x)$$

This also implies that the pushforward of the identity is the identity (by swapping either of g_1 or g_2 for the identity).

Smoothness follows from lie group multiplication being smooth, and the pushforward of a smooth map is smooth.

(b) We know that e^{tX} is a path satisfying $e^{tX}|_{t=0} = \mathbf{1_n}$ and $\frac{d}{dt}e^{tX}|_{t=0} = X$. Then for $g \in G$ we have

$$d(C_g)_e = \frac{d}{dt}|_{t=0}C_g(e^{tX}) = \frac{d}{dt}|_{t=0}e^{tgXg^{-1}} = gXg^{-1}$$

(c) We take the path $\gamma(t) = e^{tX}$, which satisfies $\gamma(0) = \mathbf{1_n}$, $\gamma'(0) = X$, then

$$\begin{aligned} d(Ad)_e(x)(y) &= \frac{d}{dt}|_{t=0}\mathrm{Ad}_\gamma(t)(y) = \frac{d}{dt}|_{t=0}\gamma(t)y\gamma^{-1}(t) \\ &= \frac{d}{dt}|_{t=0}e^{tX}ye^{-tX} = \left(\frac{d}{dt}|_{t=0}e^{tX}y\right)\mathbf{1_n} + \mathbf{1_n}y\frac{d}{dt}|_{t=0}e^{-tX} \\ &= Xy - yX = [X,y] \end{aligned}$$

2. (a) We can simply compute the product of basis elements, first note that $\sigma_i^2 = \mathbf{1}$, and that

$$\begin{aligned} \sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\ \sigma_1\sigma_3 &= -i\sigma_2 = -\sigma_3\sigma_1 \\ \sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \end{aligned}$$

Since each of σ_i are of trace 0, we get for some c_i which turn out not to matter

$$\begin{aligned} \mathrm{Tr}(xy) &= \mathrm{Tr}((x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)(y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3)) \\ &= \mathrm{Tr}(\mathbf{1}(x_1y_1 + x_2y_2 + x_3y_3)) + \mathrm{Tr}(\sum_1^3 c_i\sigma_i) \\ &= \sum_1^3 2x_iy_i + \sum_1^3 c_i\mathrm{Tr}(\sigma_i) = 2\sum_1^3 x_iy_i \end{aligned}$$

So we may conclude that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{2}\mathrm{Tr}xy$$

(b) write $g := \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$. Then

$$\begin{aligned} g\sigma_1g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{-2i\theta} \\ e^{2i\theta} & 0 \end{bmatrix} = \cos 2\theta\sigma_1 + \sin 2\theta\sigma_2 \\ g\sigma_2g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -ie^{-2i\theta} \\ ie^{2i\theta} & 0 \end{bmatrix} = -\sin 2\theta\sigma_1 + \cos 2\theta\sigma_2 \\ g\sigma_3g^{-1} &= \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & -e^{-i\theta} \end{bmatrix} = \sigma_3 \end{aligned}$$

So that $\varphi(g)$ acts as

$$\begin{bmatrix}\cos 2\theta & -\sin 2\theta & 0\\ \sin 2\theta & \cos 2\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

on the basis $\{\sigma_i\}_1^3$

(c) First notice that the pushforward of conjugation on a single Pauli matrix is just the bracket as in question 1c. Computing this we get (see my mult table in 2a)

$$\begin{aligned}
&[\sigma_\alpha/2i, \sigma_j] = 0 && i = j\\
&[\sigma_1/2i, \sigma_2] = \frac{1}{2i}(i\sigma_3 - (-i\sigma_3)) = \sigma_3\\
&[\sigma_1/2i, \sigma_3] = \frac{1}{2i}(-i\sigma_2 - i\sigma_2) = -\sigma_2\\
&[\sigma_2/2i, \sigma_1] = \frac{1}{2i}(-i\sigma_3 - \sigma_3) = -\sigma_3\\
&[\sigma_2/2i, \sigma_3] = \frac{1}{2i}(i\sigma_1 - (-i\sigma_1)) = \sigma_1\\
&[\sigma_3/2i, \sigma_1] = \frac{1}{2i}(i\sigma_2 + i\sigma_2) = \sigma_2\\
&[\sigma_3/2i, \sigma_2] = \frac{1}{2i}(-i\sigma_1 - i\sigma_1) = -\sigma_1
\end{aligned}$$

Using this rule, we can write the matrices.

$$E_1 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} \quad E_2 = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix} \quad E_3 = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

(d) Given $g = \begin{bmatrix}u & -\overline{v}\\ v & \overline{u}\end{bmatrix}$, compute $g^{-1} = \begin{bmatrix}\overline{u} & \overline{v}\\ -v & u\end{bmatrix}$. Then we compute the action of g on each σ_i,

$$g\sigma_1 g^{-1} = \begin{bmatrix}u & -\overline{v}\\ v & \overline{u}\end{bmatrix}\begin{bmatrix}-v & u\\ \overline{u} & \overline{v}\end{bmatrix} = \begin{bmatrix}-2\mathrm{Re}(uv) & u^2 - \overline{v}^2\\ \overline{u^2 - \overline{v^2}} & 2\mathrm{Re}(uv)\end{bmatrix} = -2\mathrm{Re}(uv)\sigma_3 + c\sigma_1 + d\sigma_3$$

Where $c = \mathrm{Re}(u)^2 - \mathrm{Re}(v)^2 - \mathrm{Im}(u)^2 + \mathrm{Im}(v)^2$ and $d = -2(\mathrm{Re}(u)\mathrm{Im}(u) + \mathrm{Re}(v)\mathrm{Im}(v))$.

$$g\sigma_3 g^{-1} = \begin{bmatrix}u & -\overline{v}\\ v & \overline{u}\end{bmatrix}\begin{bmatrix}\overline{u} & \overline{v}\\ v & -u\end{bmatrix} = \begin{bmatrix}|u|^2 - |v^2| & 2u\overline{v}\\ \overline{2u\overline{v}} & |v|^2 - |u|^2\end{bmatrix} =$$