1. **(a)** Convexity of the interior is a corollary of problem (b). To show that the closure is convex, let $x, y \in \overline{C}$, then there are sequences $x_n \to x$ and $y_n \to y$ such that $x_n, y_n \in C$ for all $n \in \mathbb{N}$. Now let $t \in (0,1)$ and $z = (1-t)y + tx$, by convexity of C, $(1-t)y_n + tx_n \in C$ for each n. Convexity will follow from $(1-t)y_n + tx_n \to z$ which will prove that $z \in \overline{C}$. Let $\epsilon > 0$, then by assumption on x, y there is $N \in \mathbb{N}$, such that $n \geq N$ implies that $||x - x_n|| < \epsilon$ and $||y - y_n|| < \epsilon$. It follows that for any $n \geq N$,

$$\begin{aligned} ||(1-t)y_n + tx_n - z|| &= ||(1-t)y_n + tx_n - (1-t)y - tx|| \leq ||(1-t)(y_n - y)|| + ||t(x_n - x)|| \\ &= (1-t)||y_n - y|| + t||x_n - x|| < (1-t)\epsilon + t\epsilon = \epsilon \quad \square \end{aligned}$$

(b) since $y \in C^\circ$, there is some $r > 0$, such that $N_r(y) \subset C^\circ$. Now let $t \in (0,1)$ and $z = tx + (1-t)y$, I claim that $N_{(1-t)r}(z) \subset C$ implying that $z \in C^\circ$. Let $w \in N_{(1-t)r}(z)$, then $w = z + s$ for some s with $||s|| < (1-t)r$. It follows that $||\frac{s}{1-t}|| < r$, so that $y + \frac{s}{1-t} \in N_r(y)$. Since $N_r(y) \subset C$, convexity implies that

$$(1-t)(y + \frac{s}{1-t}) + tx = z + s = w \in C$$

and since w was arbitrary, we may conclude that $N_{(1-t)r}(z) \subset C$ so that $z \in C^\circ$. $\square$

(c) Let $z \in \overline{C}$, then there is some sequence $(z_n)_1^\infty \subset C$, such that $z_n \to z$. Since C° is nonempty, fix $x \in C^\circ$. Using part (b), we have for $t \in (0,1)$ and $n \in \mathbb{N}$ that $tz_n + (1-t)x \in C^\circ$. Let $t_n = (1 - \frac{1}{n})$, then $t_n z_n + (1-t_n)x \in C^\circ$ for each n. Let $\epsilon > 0$, then choose N large enough that $n \geq N$ implies that $\frac{||x||}{n} < \epsilon/4$ and $\frac{||z||}{n} < \epsilon/4$ and $||z - z_n|| < \epsilon/4$. It follows that for $n \geq N$,

$$\begin{aligned} ||z - t_n z_n + (1-t_n)x|| &= ||z - z_n + \frac{1}{n}z_n + \frac{1}{n}x|| \leq ||z - z_n|| + \frac{1}{n}||z_n|| + \frac{1}{n}||x|| \\ &< \frac{2}{4}\epsilon + \frac{1}{n}||z_n - z + z|| \leq \frac{1}{2}\epsilon + \frac{1}{n}||z_n - z|| + \frac{1}{n}||z|| \\ &< \frac{1}{2}\epsilon + \frac{1}{4n}\epsilon + \frac{1}{4}\epsilon \leq \epsilon \end{aligned}$$

And hence $t_n z_n + (1-t_n)x \to z$, where $t_n z_n + (1-t_n)x \in C^\circ$ for all n. This suffices to show that $z \in \overline{C^\circ}$. $\square$

2. **(a)** Suppose z is an extreme point, since normed vector spaces are T_1, we know that $\{z\}$ is non-empty and closed. Now suppose that $x, y \in C$ and $t \in (0,1)$, such that $tx + (1-t)y = z$. By assumption that z is extreme this implies that $x = y$, so that $z = tx + (1-t)x = x$, so that $y = x = z \in E$. Conversely, suppose that $\{z\}$ is an extreme subset of C, then let $x, y \in C$ and $t \in (0,1)$ such that $tx + (1-t)y = z$, since $\{z\}$ is exreme, we know that $x, y \in \{z\}$ so that $x = y = z$, implying that z is extreme.

(b) We first need to show that B is closed and non-empty, to see that B is closed, let $b \in \overline{B}$, then there is some sequence $(b_n)_1^\infty \subset B$, such that $b_n \to B$, by continuity of f, $\max_{a \in A} f(a) = \lim_{n\to\infty} f(b_n) = f(b) \in B$, now to see that B is non-empty, note that A is a closed subset of a compact set, hence compact so that for some $a \in A$ we have $f(a) = \sup_A f(a)$. Now let $z \in B$, and assume that $x, y \in A$ and $t \in (0,1)$, such that $z = tx + (1-t)y$. Assume WLOG $f(x) \geq f(y)$

$$f(z) = f(tx + (1-t)y) = f(tx) + f((1-t)y) = tf(x) + (1-t)f(y) \leq tf(x) + (1-t)f(x) = f(x)$$

but the other inequality is by assumption of $f(z) = \max_A\{f\}$, so that $f(z) = f(x) \in B$. It follows that

$$f(z) - tf(z) = (1-t)f(y) \implies (1-t)f(z) = (1-t)f(y) \implies f(z) = f(y) \in B \quad \square$$

(c) Let $\mathcal{C} := \{E_\alpha\}_{\alpha \in I} \subset \mathcal{F}(E)$ be a chain, I claim that $\bigcap_I E_\alpha$ is an upper bound for $\mathcal{C}$ in $\mathcal{F}(E)$. Since each E_α is a closed subset of a compact set and hence compact, by the finite intersection property $\bigcap_I E_\alpha \neq \emptyset$, it is also closed since arbitrary intersections of closed sets are closed. To see that it is extreme, let $z \in \bigcap_I E_\alpha$, if $x, y \in C$ and $t \in (0,1)$, such that $z = tx + (1-t)y$, then since E_α is extreme for each α, we have that $x \in E_\alpha$ and $y \in E_\alpha$ for all $\alpha \in I$, thus $x, y \in \bigcap_I E_\alpha$, it is immediate that $E_\alpha \leq \bigcap_I E_\alpha$ for all $\alpha \in I$, and $E \in \mathcal{F}(E) \neq \emptyset$ so by Zorn's lemma a maximal element exists.

Now suppose that $E_0 \in \mathcal{F}(E)$ is maximal, such that there are $x_0, y_0 \in E_0$ with $x_0 \neq y_0$, if $y_0 \in \langle x_0 \rangle$, then we can extend $f : \langle x_0 \rangle \to \mathbb{R}$, $\lambda x_0 \mapsto \lambda ||x_0||$ using the Hahn Banach theorem since $f \leq ||\cdot||$, in this case it follows that since $y_0 \neq x_0$ it must be the case that $f(x_0) \neq f(y_0)$, so in particular we may assume without loss of generality $f(x_0) > y_0$, but in this case $y_0 \notin \{e \in E_0 \mid f(e) = \max_{x \in E_0} f(x)\} \subsetneq E_0$, which is exremal by part (b), contradicting maximality of E_0. Now we may assume that $y_0 \notin \langle x_0 \rangle$, we may define $f : \langle y_0 \rangle \to \mathbb{R}$, $f : \lambda y_0 \mapsto \lambda \inf_{x \in \langle x_0 \rangle} ||y_0 - x||$, it follows that $f \leq d_{\langle x_0 \rangle}$, so once again by the Hahn Banach extension theorem, we find some $F \in X^*$, with $F \leq d_{\langle x_0 \rangle}$ and $F|_{\langle y_0 \rangle} = f$, implying that $F(y_0) > 0 = F(x_0)$, and hence $x_0 \notin \{e \in E_0 \mid f(e) = \max_{x \in E_0} f(x)\} \subsetneq E_0$, which is exremal by part (b), contradicting maximality of E_0. In either case we find that E_0 is not maximal, contradicting our assumption so any maximal set must in fact consist of a single point. $\square$

(d) By definition C is an extremal subset of itself, hence $C \in \mathcal{F}(C) \neq \emptyset$, by the previous problem $\mathcal{F}(C)$ contains a maximal element which is a singleton set $\{z\}$, in part (a) we showed that z is an extremal point. $\square$

(e) $E_C \subset C$ which is closed and convex, so it is trivial that $A = \overline{\operatorname{conv} E_C} \subset C$, it remains to show the reverse inequality. Assume for contradiction that there is some $c \in C \setminus A$, it follows that there is some smaller convex closed convex set $c \notin A \supset E_C$. It follows that $A, \{c\} \subset C$ are closed subsets of a compact set and thus compact, implying that by the Hahn Banach seperation theorem there is a hyperplane strictly seperating A and $\{c\}$. Let f be the functional used in defining the hyperplane (if necessary we may change the sign on f, such that $f(c) > \sup f|_A$), we know that $\{x \in C \mid f(x) = \max_{z \in c} f(z)\} \subset C \setminus A$ is extremal, furthermore by part (c), $\mathcal{F}(\{x \in C \mid f(x) = \max_{z \in c} f(z)\})$ contains a minimal element with respect to inclusion, which must be a singleton $\{z\} \subset C$, by part (a) we know that z is an extreme point, and hence $z \notin A$ implies that A does not contain all extreme points of C, a contradiction. $\square$

3. I will prove both directions by contrapositive. Suppose first that T^* not injective, then there are $f, f' \in Y^*$, such that $fT = f'T$, i.e. $T^*(f - f') = 0$, where $f - f' \neq 0$. Since $0 \neq f - f'$, there is some $y \in Y$, such that $(f - f')(y) = \epsilon > 0$, since $f - f' \in Y^*$ we know they are continuous, and hence there is some $r > 0$, such that $(f - f')|_{N_r(y)} > \frac{\epsilon}{2}$, since $T^*(f - f') = 0$, it follows that $N_r(y) \cap \operatorname{Im} T = \emptyset$, and hence $\operatorname{Im} T$ is not dense in Y.

Conversely, suppose $\operatorname{Im} T$ is not dense in Y, then since $\operatorname{Im} T$ is a subspace of Y, we have that $d_{\operatorname{Im} T} : x \mapsto \inf\{||x - t|| \mid t \in \operatorname{Im} T\}$ is a seminorm on Y. Since $\operatorname{Im} T$ is not dense in Y, we have some non-empty open set $U \subset (\operatorname{Im} T)^c$, fix $y \in U$, it follows that $f : \lambda y \mapsto \lambda \inf_{t \in \operatorname{Im} T} ||y - t||$ is linear of $\langle y \rangle$, and bound above by $d_{\operatorname{Im} T}$. By the Hahn Banach linear extension theorem, there is some $F \in Y^*$, such that $F \leq d_{\operatorname{Im} T}$, and $F|_{\langle y \rangle} = f$. So that $F(y) > 0$ implies that $F \neq 0$, but since $F|_{\operatorname{Im} T} = 0$ we have $T^* F = 0$ thus T^* is not injective. $\square$

4. (a) In this problem we may replace f with another element of its equivalence class, as such assume for convenience that $\sup_X |f| = \operatorname{ess\,sup}_X f$. If $p = q$ we are done trivially with $||i_{q,q}|| = 1$, so assume that $p < q$. First suppose that $q = \infty$, and $f \in L^\infty$, then

$$\left(\int_X |f|^p \right)^{\frac{1}{p}} \leq \left(\int_X \sup_X |f|^p \right)^{\frac{1}{p}} = \mu(X)^{\frac{1}{p}} \sup_X |f| < \infty$$

So that $L^\infty \subset L^p$. $||1||_\infty = 1$, and for any $f \in L^\infty$, $||f||_\infty = 1$, we have that $|f(x)| \leq 1, \forall x \in X$. It follows that

$$\mu(X)^{\frac{1}{p}} = ||1||_q \leq \sup_{f \in L^\infty} ||\iota_{p,\infty} f||_p \leq ||1||_q = \mu(X)^{\frac{1}{p}}$$

so that $\mu(X)^{\frac{1}{p}}$ is the operator norm. Now suppose that $1 \leq p < q < \infty$, then $\frac{p}{q} + \frac{q-p}{q} = 1$, so that for $f \in L^q$ we have

$$\int_X |f|^p \cdot 1 \overset{\text{Holder}}{\leq} \left(\int_X |f|^{p\frac{q}{p}} \right)^{p/q} \left(\int_X 1^{\frac{q}{q-p}} \right)^{\frac{q-p}{q}} = ||f||_q^p \mu(X)^{\frac{q-p}{q}}$$

We can take the p-th root of either side to conclude that

$$||f||_p \leq ||f||_q \mu(X)^{\frac{q-p}{pq}} < \infty$$

so that $L^q \subset L^p$, to see that $||\iota_{p,q}|| = \mu(X)^{\frac{q-p}{pq}}$, we need only provide one $f \in L^q$, with $||f||_q = 1$, such that $||f||_p = \mu(X)^{\frac{q-p}{pq}}$ since the other inequality is proved above, take $f = \frac{1}{\mu(X)^{\frac{1}{q}}}$, then

$$\left(\int_X |f|^q\right)^{1/q} = 1 \text{ and } \left(\int_X |f|^{p/q}\right)^{1/p} = \left(\mu(X)\mu(X)^{-\frac{p}{q}}\right)^{1/p} = \mu(X)^{\frac{1}{p}-\frac{1}{q}} = \mu(X)^{\frac{q-p}{pq}} \quad \square$$

(b) Note the following holds for any p,

$$\left(\int_X |f|^p\right)^{\frac{1}{p}} \leq \left(\int_X ||f||_\infty^p\right)^{\frac{1}{p}} = \mu(X)^{\frac{1}{p}} ||f||_\infty < \infty$$

Now let $\epsilon > 0$, by definition of $||\cdot||_\infty$ we know that for some $E \subset X$ we have $|f|\,|_E > ||f||_\infty - \frac{\epsilon}{2}$ and $0 < m = \mu(E)$. Since $\lim_{p\to\infty} \mu(X)^{\frac{1}{p}} = \lim_{p\to\infty} \mu(E)^{\frac{1}{p}} = 1$, there is N sufficiently large, so that $p \geq N$ implies that $\mu(X)^{\frac{1}{p}} ||f||_\infty < ||f||_\infty + \epsilon$ and $\mu(E)^{\frac{1}{p}}(||f||_\infty - \frac{\epsilon}{2}) > ||f||_\infty - \epsilon$, so that

$$||f||_\infty - \epsilon < \mu(E)^{\frac{1}{p}}\left(||f||_\infty - \frac{\epsilon}{2}\right) \leq \left(\int_E |f|^p\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\right)^{\frac{1}{p}} \leq \left(\int_X ||f||_\infty^p\right)^{\frac{1}{p}} = \mu(X)^{\frac{1}{p}} ||f||_\infty < ||f||_\infty + \epsilon$$

In particular, we find that

$$|||f||_p - ||f||_\infty| < \epsilon$$

which suffices to show $\lim_{p\to\infty} ||f||_p$ exists and is equal to $||f||_\infty$. $\square$

(c) Let $C \in \mathbb{R}_{>0}$ and suppose for contraposition that $f \notin L^\infty$, then there is some $E \subset X$, such that $0 < \delta = \mu(E)$ and $|f|\,|_E \geq 2C$. Since $\lim_{p\to\infty} \mu(E)^{\frac{1}{p}} = 1$, there is N sufficiently large, such that $p \geq N$ implies that $\mu(E)^{\frac{1}{p}} > \frac{1}{2}$. It follows that

$$||f||_p \geq \left(\int_E |f|^p\right)^{\frac{1}{p}} \geq \left(\int_E (2C)^p\right)^{\frac{1}{p}} = \mu(E)^{\frac{1}{p}} 2C > C$$

so that $||f||_p$ is not smaller than C for any $p \geq N$. $\square$

(d) Define f as follows,

$$f = \sum_{i=1}^{\infty} i\chi_{(2^{-i}, 2^{-i+1}]}$$

Immediately by definition we see that $f \notin L^\infty$. Now let $p \in [1, \infty)$, we find that

$$||f||_p = \left(\int_X \sum_1^\infty i^p \chi_{(2^{-i}, 2^{-i+1}]}\right)^{\frac{1}{p}} \overset{\text{MCT}}{=} \left(\lim_{N\to\infty} \int_X \sum_1^N i^p \chi_{(2^{-i}, 2^{-i+1}]}\right)^{\frac{1}{p}} = \left(\lim_{N\to\infty} \sum_1^N i^p 2^{-i}\right)^{\frac{1}{p}}$$

Where $\lim_{i\to\infty} \frac{(i+1)^p 2^{-i-1}}{i^p 2^{-i}} = \frac{1}{2}$, and hence $\sum_1^\infty i^p 2^{-i} < \infty$ by the ratio test. This suffices to show that

$$||f||_p = \left(\sum_1^\infty i^p 2^{-i}\right)^{\frac{1}{p}} < \infty$$

so that $f \in L^p$ and since p was arbitrary we can conclude that $f \in \bigcap_{p\in[1,\infty)} L^p$. $\square$