I collaborated with Justin Wan on problem 2.

1. **(a)** Let $(x,y) \sim (w,z)$, then $w = \lambda x, z = \lambda^{-1}y$, so that $wz = \lambda\lambda^{-1}xy = xy$. Let $\pi : \mathbb{R}^2 \setminus \{(0,0)\}/(\mathbb{R}\setminus\{0\})$ be the quotient. Then $g(x,y) = f\pi(x,y) = xy$, is a polynomial function hence continuous. To show f is continuous, let $U \in \mathbb{R}$ be open, then $f^{-1}(U)$ is open iff $\pi^{-1}f^{-1}(U)$ by definition of quotient, but this is exactly $g^{-1}(U)$ which is open since g is continuous.

(b) $\#\overline{f^{-1}(t)} = 1,\ t \neq 0$ and $\#f^{-1}(0) = 0$. Proof being $xy = 0 \iff x = 0$ or $y = 0$, so the preimages of 0 are $\overline{(1,0)}$ and $\overline{((0,1))}$. If $t \neq 0$, then $t = xy = zw$, we may write $z = \lambda x$, where $\lambda = \frac{z}{x} \neq 0$, then $xy = \lambda xw$, so that $w = \lambda^{-1}y$, proving that $\overline{(x,y)} = \overline{(z,w)}$.

(c) Let $\overline{(1,0)} \in U, \overline{(0,1)} \in V$, for open sets U, V. Then by definition of the quotient $\pi^{-1}(U)$ is open, hence by the local definition of open sets (from homework 1) we have some neighborhood of $(1,0)$ contained in $\pi^{-1}(U)$. This implies that for some $\epsilon_x > 0$, $\{(1,r)|r < \epsilon\} \subset \pi^{-1}(U)$. Similarly, there exists some $\epsilon_y > 0$, such that $\{(r,1)|r,\epsilon\} \subset \pi^{-1}(V)$. Now choose $r = \frac{\min(\epsilon_x,\epsilon_y)}{2}$, so that $\overline{(1,r)} \in \pi\pi^{-1}(U) = U$, and $\overline{(r,1)} \in \pi\pi^{-1}(V) = V$. Then $(r,1) \sim (r1, r^{-1}r) = (1,r)$ implies that $\overline{(r,1)} \in U \cap V$. This proves that X is not hausdorff, since $\overline{(1,0)}$ and $\overline{(0,1)}$ do not satisfy the Hausdorff condition.

(d) Consider the maps

$$\begin{aligned}
&\varphi : X \to Y \\
&\overline{(x,y)} \mapsto \begin{cases} (xy,0) & y \neq 0 \\ (xy,1) & x \neq 0 \end{cases} \\
&\tilde{\varphi} : Y \to X \\
&(s,t) \mapsto \begin{cases} \overline{(s,1)} & s \neq 0 \\ \overline{(0,1)} & s = t = 0 \\ \overline{(1,0)} & s = 0, t = 1 \end{cases}
\end{aligned}$$

To check that this φ is injective (it is well defined by (a)), we only need check that $\overline{(1,0)}, \overline{(0,1)}$ map to seperate points in Y, since part (c) guarantees the other elements are 1-1, so since these points map to 0 in the first coordinate, away from all other points, and map to seperate points in Y the map is injective. To check surjectivity, $(0,0)$ and $(0,1)$ are mapped onto, so we can check the other points. $\overline{(x,1)} \mapsto (x,0)$ shows surjectivity. Similarly, we check for $\tilde{\varphi}$, it is well defined since it just extends to equivalence classes. It is onto since $\overline{(0,1)}, \overline{(1,0)}$ are in the image, and any $\overline{(x,y)} \sim (xy,1)$ (for $x,y \neq 0$) has its equivalence class in the image. Injectivity is also clear since $(x,1) \sim (y,1)$ iff $x = y$ and $\overline{(1,0)}$ is only mapped onto by one point. To see that these are inverse maps, it is immediate they are inverses in the case of $(0,1),(0,0) \in Y$ and $\overline{(0,1)}, \overline{(1,0)} \in X$. Checking this for $x,y,s \neq 0$ we have $\tilde{\varphi}\varphi(\overline{(x,y)}) = \overline{(xy,1)} \sim (x,y)$ and $\varphi\tilde{\varphi}(s,0) = (s,0)$. It remains to show continuity of φ and $\varphi^{-1} = \tilde{\varphi}$.

Continuity of φ: Let U be open in Y, then U is of the form $\pi(V \times \{0\} \sqcup W \times \{1\})$, for $W, V \subset \mathbb{R}$ open. Hence we can write it in the form of $((V \setminus \{0\} \cup W \setminus \{0\}) \times \{0\}) \cup \chi_V \cup \chi_W$,

$$\chi_V = \begin{cases} \{0,0\} & 0 \in V \\ \emptyset & 0 \notin V \end{cases} \qquad \chi_W = \begin{cases} \{0,1\} & 0 \in W \\ \emptyset & 0 \notin W \end{cases}$$

Now since $\{(0,0)\}^c$ and $\{(0,1)\}^c$ are open in Y (they are images of their compliments in $\mathbb{R} \times \mathbb{Z}$, where points are closed since T1 follows from hausdorff), it follows that $V \setminus \{0\} \cup W \setminus \{0\}$ is open. Then $(\pi_X\varphi)^{-1}((V \setminus \{0\} \cup W \setminus \{0\}) \times 0)$ is just $\{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy \in V \setminus \{0\} \cup W \setminus \{0\}\}$ but this is the preimage of an open set in $\mathbb{R}$ of the continuous polynomial function $(x,y) \mapsto xy$, this proves continuity of φ by definition of the quotient space in the case of $\chi_V = \emptyset = \chi_W$. Now in the case where atleast one of χ_V, χ_W is non-empty, assume WLOG $\chi_V \neq \emptyset$, then since V is an open set in $\mathbb{R}$ containing 0, it must contain some open set J containing 0. Then $(\pi_X\varphi)^{-1}((J \setminus 0) \times \{0\})$ is the set $\{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy \in J \setminus \{0\}\}$, this is an open set since $J \setminus 0$ is open, so for some ϵ it contains a set of the form $\{(x,y) \in \mathbb{R}^2 \setminus \{0\} | 0 < xy < \epsilon\}$. Now we take

$$(\pi_X\varphi)^{-1}(0,1) = \{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy = 0\} \setminus \{(0,y)|y \neq 0\} \quad (\pi_X\varphi)^{-1}(0,0) = \{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy = 0\} \setminus \{(x,0)|x \neq 0\}$$

Note that $\{(0,y)|y \neq 0\}, \{(x,0)|x \neq 0\}$ are closed in $\mathbb{R}^2 \setminus \{0\}$ since their complemets are open. Now in the case where $(0,0) \in U$, $(\pi_X\varphi)^{-1}(U)$ contains the open set $\{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy < \epsilon\} \setminus \{(x,0)|x \neq 0\}$ containing $(\pi_X\varphi)^{-1}(0,0)$. Similarly if $(0,1) \in U$, then $(\pi_X\varphi)^{-1}(U)$ contains the open set $\{(x,y) \in \mathbb{R}^2 \setminus \{0\} | xy < \epsilon\} \setminus \{(0,y)|y \neq 0\}$ containing $(\pi_X\varphi)^{-1}(0,1)$. But since $(\pi_X\varphi)^{-1}(U) \supset (\pi_X\varphi)^{-1}((V \setminus \{0,0\} \cup W \setminus (0,1)) \times \{0\})$ is an open set containing every other point $(\pi_X\varphi)^{-1}(U)$ is open by the local definition of continuity. This proves that φ is continuous by definition of the quotient map.

Continuity of $\tilde{\varphi}$: Let U be an open set in X now let q be any point in U, but not $\overline{(1,0)}$. Then we can write $q = \overline{(p,1)}$ for some p. Then since U is open, $\pi_X^{-1}(U)$ is open containing $(p,1)$, so for some $\epsilon > 0$ (where if $p \neq 0$

we can choose $\epsilon < |p|$), it contains $(p+t,1)$ for t such that $|t| < \epsilon$. Then in the first case where $p \neq 0$ we have $(\pi_Y\tilde{\varphi})^{-1}\{(p,1)\} = \{(p,0),(p,1)\}$ is contained in the open set $\{(p+t,s)|t < \epsilon s \in \{0,1\}\} \subset (\pi_Y\tilde{\varphi})^{-1}(U)$ here $\epsilon < |p|$ guarantees we have $(p+t,0)$ and $(p+t,1)$ in the preimage dealing with both points at once. Now in the second case where $p = 0$, we still have that $\{(p+t,0)|t < \epsilon\} \subset (\pi_Y\tilde{\varphi})^{-1}(U)$, so the preimage still contains an open set containing $(0,0)$. This proves continuity for any U not containing $\overline{(1,0)}$. If U does contain $\overline{(1,0)}$, then $\pi_X^{-1}(U)$ is an open set containing $(1,0)$ hence for some $\epsilon > 0$ it contains $(1,t)$ for all t, such that $|t| < \epsilon$, this means that $\pi_X(\pi_X^{-1})(U)$ contains each $\overline{(1,t)} = \overline{(t,tt^{-1})} = \overline{(t,1)}$. This implies that there is some open set containing $\overline{(1,0)}$ contained in $(\pi_Y\tilde{\varphi})^{-1}(U)$, namely

$$(\pi_Y\tilde{\varphi})^{-1}(U) \supset (\pi_Y\tilde{\varphi})^{-1}(\{(1,t)||t| < \epsilon\}) \supset \{(t,1)||t| < \epsilon\}$$

This implies by the local definition that $(\pi_Y\tilde{\varphi})^{-1}(U)$ is open, so by definition of the quotient $\tilde{\varphi}^{-1}(U)$ is open. We conclude that $\tilde{\varphi} = \varphi^{-1}$ is continuous along with φ, making φ a homeomorphism from X to Y.

2. Take $\mathbb{R}^3 \setminus (0,0)$, and S^2 the unit sphere centered at the origin, then $H(x,t) = \frac{x}{1+t(|x|-1)}$ is a strong deformation retract of $\mathbb{R}^3$ onto S^2, hence $\mathbb{R}^3 \setminus \{\mathrm{pt}\}$ is homotopic to S^2.

Let J be the filled Torus (i.e. $D^2 \times S^1$), and let $D_{\mathrm{Lat}}, D_{\mathrm{Long}}$ denote the latitudinal and longitudinal discs respectively. Then we may write $\mathbb{R}^3 \setminus \{\mathrm{pt}\} = T^2 \sqcup (J^\circ \setminus \{\mathrm{pt}\}) \sqcup (J^c)^\circ$. I will show that $\mathbb{R}^3 \setminus \{\mathrm{pt}\}$ strong deformation retracts onto $(J \setminus \{\mathrm{pt}\}) \cup D_{\mathrm{Long}}$, then show that $(J \setminus \{\mathrm{pt}\}) \cup D_{\mathrm{Long}}$ strong deformation retracts onto $T \cup D_{\mathrm{Lat}} \cup D_{\mathrm{Long}}$, the proof follows by transitivity of homotopy equivalence.

For the first equivalence, we can let P be the x, y plane, with $J \setminus \{\mathrm{pt}\}$ embedded in $\mathbb{R}^3 \setminus \{\mathrm{pt}\}$ at height zero (wlog the point doesn't have height 0). Then we can strong deformation retract $\mathbb{R}^3 \setminus \{\mathrm{pt}\}$ by projecting the z-axis onto $P \cup (J \setminus \{\mathrm{pt}\})$. Now given a point $p = (x_p, y_p, z_p)$, let (x_p, y_p, z_0) be the closest point to it in $P \cup (J \setminus \{\mathrm{pt}\}) \cap \{(x_p, y_p, z) | z \in \mathbb{R}\}$. the homotopy can be written as $H((x,y,z),t) = (x, y, z + t(z_0 - z))$ for z_0 continuously depending on z(continuous since $P \cup J$ is smooth). Now we can deformation retract $P \cup (J \setminus \{\mathrm{pt}\})$ onto $J \setminus \{\mathrm{pt}\} \cup D_{\mathrm{Long}}$, the retract H is defined to be constant on $J \setminus \{\mathrm{pt}\} \cup D_{\mathrm{Long}}$, then assuming the radius from the origin to the outer edge of the torus is R we only need to define it on points of $P \setminus D^2_R$, where D^2_R denotes the disc of radius R. On such points, define $H(p,t) = \frac{p}{1+tR(|p|)-1/R}$. Transitivity of homotopy equivalence proves that $\mathbb{R}^3 \setminus \{\mathrm{pt}\} \simeq_H (J \setminus \{\mathrm{pt}\}) \cup D_{\mathrm{Long}}$.

Now note to show a strong deformation retract of $(J \setminus \{\mathrm{pt}\}) \cup D_{\mathrm{Long}}$ onto $T^2 \cup D_{\mathrm{Lat}} \cup D_{\mathrm{Long}}$, it will suffice to show one exists from $J \setminus \{\mathrm{pt}\}$ onto $T^2 \cup D_{\mathrm{Lat}}$, since $\partial D_{\mathrm{Long}} \subset T^2$ implies that T^2 remaining fixed in our homotopy allows us to fix D_{Long} in our homotopy. Now we may identify $J \setminus \{\mathrm{pt}\} = \frac{D^2 \times I \setminus \{\mathrm{pt}\}}{(x,1) \sim (x,0)}$. Considering the cylinder centered at the origin, with origin removed, i.e. $D^2 \times I \setminus \{(0,0)\}$, we can write a homotopy to $\partial(D^2 \times I)$, namely for each point p, let q_p be the intersection of the ray from the origin through p with $\partial(D^2 \times I)$. It is clear that q_p varies continuously with respect to p, so we write the homotopy $H(p,t) = \frac{p}{1+t(|\frac{p}{q}|-1)}$. Then since a strong deformation retract of the space induces a strong deformation retract of the quotient space, we get that

$$J \setminus \{\mathrm{pt}\} = \frac{D^2 \times I \setminus \{\mathrm{pt}\}}{(x,1) \sim (x,0)} \simeq_H \frac{\partial(D^2 \times I)}{(x,1) \sim (x,0)} = \frac{S^1 \times I \cup D \times \{0\}}{(x,1) \sim (x,0)} = T^2 \cup D_{\mathrm{Lat}}$$

Now as previously mentioned, since this map is a strong deformation retract, it induces one on $J \setminus \{\mathrm{pt}\} \cup D_{\mathrm{Long}}$ to $T^2 \cup D_{\mathrm{Long}} \cup D_{\mathrm{Lat}} = X$. Meaning by transitivity we have $S^2 \simeq_H \mathbb{R}^3 - \{\mathrm{pt}\} \simeq_H X$.

Proof that strong deformation retract induces strong deformation retract on quotient. Let H be a strong deformation retract of the topological space X, we want to show there exists a strong deformation retract $\overline{H}$ of $X/\sim$, which is the quotient of H. To do so, define the equivalence relation $\approx$ on $H \times I$, where $(x,t) \approx (y,s)$ iff $x \sim y$ and $t = s$. Then we can take $\pi_\sim$ to be the quotient map $X \to X_\sim$, we have that $\pi_\sim H$ is a map from $H \times I$ to $X/\sim$, which is level on equivalence classes of $\approx$, since $\approx$ induces no relations on I, and we are taking the quotient by $\sim$ in the map. Hence by the universal property of quotient maps we have some map $\overline{H} : \frac{X \times I}{\approx} \to X/_\sim$, which is equal to $\pi_\sim H$, hence if H was a deformation retract of X onto $Y \subset X$, then $\overline{H}(\frac{X \times I}{\approx}) \subset Y/_\sim$, and $Y/_\sim$ remains fixed, since $\overline{H}$ agrees with $H\pi$. This is equivalent to saying there exists $\overline{H}$ making the following diagram commute:

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & X \\ \downarrow & & \downarrow \\ \frac{X \times I}{\approx} & \xrightarrow{\exists \overline{H}} & X/_\sim \end{array}$$

then we can identify $\frac{X \times I}{\approx} = X/_\sim \times I$, so that $\overline{H}$ is in fact our desired homotopy.

Lemma. I will use the following lemma to streamline my proofs for problems 3 and 4.
If $\psi : X \to Y$ is a homeomorphism, and $\sim$ is an equivalence relation on X, and $\approx$ a equivalence relation on Y, such that $\psi(a) \approx \psi(b) \iff a \sim b$, then $X/_\sim \simeq Y/_\approx$, this says that homeomorphisms from $X \to Y$ induce homeomorphisms to the quotients when the points in the same equivalence classes induced by the quotient on Y are images of the points in the same equivalence classes induced by the quotient on X, see the diagram.

$$\begin{tikzcd} X \arrow[r, "\overset{\psi}{\simeq}"] \arrow[d, "\pi_\sim"] & Y \arrow[d, "\pi_\approx"] \\ X/_\sim \arrow[r, "\overset{\overline{\psi}}{\simeq}"] & Y/_\approx \end{tikzcd}$$

proof. Define $\overline{\psi} : X/_\sim \to Y/_\approx$, by $\overline{\psi} : \overline{x} \mapsto \overline{\psi(x)}$, this is surjective since ψ is surjective and $\overline{\psi}$ is well defined/injective by definition of $\approx$. We can define $\overline{\psi^{-1}} : Y/_\approx \to X/_\sim$, in the same way. This is the inverse of $\overline{\psi}$, since $\overline{\psi}$ and $\overline{\psi^{-1}}$ are just restrictions to equivalence classes of ψ and ψ^{-1}. To show $\overline{\psi}$ is continuous, note that $\overline{\psi} = \pi_\approx \psi$. Let U be open in $Y/_\approx$, then the preimage of U under $\pi_\approx$ is open by definition, so continuity follows from continuity of ψ. The proof for continuity of $\overline{\psi^{-1}}$ is the same.

Additional Justification for problems 3 and 4. Once again, to streamline the proofs for 3 and 4, I will explain here why the following map is a homeomorphism.

$$\begin{aligned} &C_{\mathbf{1}_{S^1}} \xrightarrow{\psi} D^2 \\ &(\theta, t) \mapsto (\theta, 1-t) \end{aligned}$$

This map is clearly bijective, so that it will suffice to show continuity by the closed map lemma, since $C_{\mathbf{1}_{S^1}}$ is the quotient of a compact space hence compact (Heine Borel theorem on $S^1 \times I$) and D^2 is Hausdorff. To see that the map is continuous, let $U \subset D^2$ open. If U does not contain $(0,0)$, then we can just regard ψ as a continuous map between $S^1 \times I$ and D^2 since it is unaffected by the quotient. Now examining the case where U contains $(0,0)$, by the local definition of open it must contain some neighborhood around $(0,0)$, and hence $\pi^{-1}\psi^{-1}(U)$ contains $S^1 \times \{t\}$ for t sufficiently close to 1, so that by definition of the quotient $\overline{(x,1)}$ is contained in an open set in $\psi^{-1}(U)$. Then since D^2 is Hausdorff, each other point is contained in a neighborhood in U not containing $(0,0)$, so its preimage is contained in some neighborhood of $\pi^{-1}\psi^{-1}(U)$ as explained previously, this shows that $\psi^{-1}(U)$ is open by the local definition of open so we are done.

3. We use the equivalent definition of $\mathbb{RP}^2$ as $D^2/_\sim$, identifying $e^{ix} \sim e^{-ix}$. Now writing out the mapping cone,

$$C_f \overset{\text{def}}{=} S^1 \times I \sqcup S^1_Y/((e^{ix}, 0) \approx e^{2ix}_Y, (e^{ix}, 1) \approx (e^{iy}, 1))$$

Now consider $x, y \in [0, 2\pi)$ we can notice $(e^{ix}, 0) \approx (e^{iy}, 0) \iff e^{2ix} = e^{2iy}$. WLOG we can assume $x < y$, so that $y = x + r$, $0 < r < 2\pi$. Then with these restrictions $e^{2ix} = e^{2i(x+r)} \iff r = \pi$, so that the equivalence relation identifies $e^{ix} \approx e^{ix+\pi} = e^{-ix}$.

Now define consider the map $C_{\mathbf{1}_{S^1}} \xrightarrow{\psi} D^2$, $(\theta, t) \mapsto (\theta, 1-t)$, this map is a homeomorphism as explained previously. Additionally, the antipodal points on the boundaries of $S^1 \times \{0\}$ and ∂D^2 remain antipodal under this map. So the lemma gives us $D^2/_\sim \simeq C_{\mathbf{1}_{S^1}}/_\approx = C_f$

4. Note that the triangle is homeomorphic to the disc. We can insribe the triangle in a circle with radius R. Then for each point p, let q be the intersection of the ray through p and the origin with the boundary of the triangle. For each of these points we can map $p \mapsto \frac{Rp}{|q|}$ this is a homeomorphism since q varies smoothly with p and we have inverse $p \mapsto \frac{|q|p}{R}$, where q comes from inscribing the triangle in the circle, which is also continuous. It follows that the equivalence relation induced on D^2 is $e^{ix} \sim e^{ix}e^{\frac{2\pi}{3}} \sim e^{-ix}$, which can be seen by the picture and lemma. So that the dunce cap can be written as $D^2/_\sim$.

Include Images HERE

Now consider the maps $\mathbf{1}_{S^1}$ and

$$\begin{aligned} &f : S^1 \to S^1 \\ &e^{ix} \mapsto \begin{cases} e^{3ix} & 0 \leq x < \frac{4\pi}{3} \\ e^{-3ix} & \frac{4\pi}{3} \leq x < 2\pi \end{cases} \end{aligned}$$

Take the mapping cone

$$C_f = \frac{S^1 \times I \sqcup S^1}{(x,0) \sim f(x), (x,1) \sim (y,1)}$$

For each x, we have $f^{-1}(x) = \{e^{ix/3}, e^{i(x+2\pi)/3}, e^{-ix/3}\}$, so the equivalence relation induced by $(x,0) \sim f$ can be seen to be $e^{ix/3} \sim e^{i(x+2\pi)/3} \sim e^{-ix/3}$. We can then take the map $C_{\mathbf{1}_{S^1}} \xrightarrow{\psi} D^2$, where $(x,t) \mapsto (x, 1-t)$, this is a homeomorphism as explained previously. Since C_f is a quotient of $C_{\mathbf{1}_{S^1}}$ by the image of quotients in $D^2/\sim$ via $\psi^{-1}(D^2)$, the lemma implies that $C_f \simeq D^2/\sim$ the dunce cap.

We have that $C_{\mathbf{1}_{S^1}}$ is contractible, using the homotopy $H((x,t),s) = (x, t(1-s))$, so it will suffice to show that $C_f \simeq_H C_{\mathbf{1}_{S^1}}$, and we have proven in class that homotopic maps have homotopic cones. I will show $f \sim \rho \sim \mathbf{1}_{S^1}$, where

$$\rho : e^{ix} \mapsto \begin{cases} e^{3ix} & 0 < x < 2\pi/3 \\ 1 & 2\pi/3 \leq x < 2\pi \end{cases}$$

I will provide H_1 for the first equivalence $f \sim \rho$ and H_2 for the second $\rho \sim \mathbf{1}_{S^1}$.

$$H_1(x,t) : \begin{cases} x \mapsto f(x) & x < \frac{2}{3} - \frac{1}{3}t \text{ or } x > \frac{2}{3} + \frac{1}{3}t \\ x \mapsto f(\frac{2}{3} - \frac{1}{3}t) & \frac{2}{3} - \frac{1}{3}t \leq x \leq \frac{2}{3} + \frac{1}{3}t \end{cases}$$

$$H_2(x,t) : \begin{cases} x \mapsto f(\frac{x}{1+2t}) \end{cases}$$