1. (a) $||\mathbf{1}-T||<1$ implies that the power series $\sum_0^\infty ||\mathbf{1}-T||^n$ converges, since X is Banach this further implies that $\sum_0^\infty(\mathbf{1}-T)^n$ converges in X. For $N \in \mathbb{Z}_{>0}$ we find that

$$T\sum_0^N(\mathbf{1}-T)^n = \sum_0^N(\mathbf{1}-T)^n - \sum_1^{N+1}(\mathbf{1}-T)^n = 1-(\mathbf{1}-T)^{N+1}$$

and hence

$$||\mathbf{1}-T\sum_0^N(\mathbf{1}-T)^n|| = ||\mathbf{1}-T||^{N+1}$$

taking $N \to \infty$ we find that

$$||\mathbf{1}-T\sum_0^\infty(\mathbf{1}-T)^n|| = 0 \implies T\sum_0^\infty(\mathbf{1}-T)^n = 1 \quad \square$$

To see that the inverse is bounded, note that for any $N \in \mathbb{Z}_{>0}$

$$||\sum_0^N(\mathbf{1}-T)^n|| \leq \sum_0^N||\mathbf{1}-T||^n \implies ||\sum_0^\infty(\mathbf{1}-T)^n|| \leq \sum_1^\infty||\mathbf{1}-T||^n < \infty$$

(b) Applying (a), $S^{-1}T$ is invertible with bounded inverse, since

$$||\mathbf{1}-S^{-1}T|| = ||S^{-1}S - S^{-1}T|| \leq ||S^{-1}||||S-T|| < ||S^{-1}||||S^{-1}||^{-1} = 1$$

It is immediate that $\left(S^{-1}T\right)^{-1}S^{-1} = T^{-1}$, since

$$\left(S^{-1}T\right)^{-1}S^{-1}T = \mathbf{1} = SS^{-1}T(S^{-1}T)^{-1}S^{-1} = T(S^{-1}T)^{-1}S^{-1}$$

and T^{-1} is bounded since

$$||T^{-1}|| = ||\left(S^{-1}T\right)^{-1}S^{-1}|| \leq ||\left(S^{-1}T\right)^{-1}||||S^{-1}|| < \infty \quad \square$$

(c) Note that

$$||\mathbf{1}-(\mathbf{1}-\lambda^{-1}T)|| = \lambda^{-1}||T|| < 1 = ||1^{-1}||^{-1}$$

Hence by (b) we find that $\mathbf{1}-\lambda^{-1}T$ is invertible with bounded inverse, multiplying by $-\lambda$ we find that $T-\lambda\mathbf{1}$ is invertible with bounded inverse. $\square$

(d) Let $\lambda \in \rho(T)$ and fix $\delta = ||(T-\lambda\mathbf{1})^{-1}||^{-1}$, then let $\alpha \in N_\delta(\lambda)$, so that $\alpha = \lambda - \beta$ with $||\beta|| < \delta$. It follows that

$$||\mathbf{1}-(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})|| = |\beta|\,||(T-\lambda\mathbf{1})^{-1}|| < 1 = ||\mathbf{1}^{-1}||^{-1}$$

so that $\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1}$ is invertible with bounded inverse. It follows that

$$(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}(T-(\lambda-\beta)\mathbf{1}) = (\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1}) = \mathbf{1}$$

so that $(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}$ is a left sided inverse for $T-\alpha\mathbf{1}=T-(\lambda-\beta)\mathbf{1}$, it is also a right inverse because

$$\begin{aligned}
&(T-(\lambda-\beta)\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(T-\lambda\mathbf{1})^{-1}(T-(\lambda-\beta)\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(T-\lambda\mathbf{1})^{-1}=\mathbf{1}
\end{aligned}$$

so that $(T-\alpha\mathbf{1})^{-1}=(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda)^{-1}\in\mathcal{L}(X,X)$ □

(e) $\sigma(T)\subset\{\lambda\in K\mid|\lambda|\leq||T||\}$ is bounded, and in (d) we showed that $\sigma(T)=\rho(T)^c$ is closed. By the Heine Borel theorem we conclude that $\sigma(T)$ is compact. □

2. (a) We first check that the operator is bounded,

$$||M_g(f)||=||fg||_2=\left(\int_X|fg|^2\right)^{\frac{1}{2}}\leq\left(\int_X|f|^2\,||g||_\infty^2\right)^{\frac{1}{2}}=\sqrt{||g||_\infty^2||f||_2^2}=||g||_\infty||f||_2$$

Let $\epsilon>0$, then by definition of essential supremum, there is some set E of positive measure such that $||g||_\infty-\epsilon\leq|g(x)|$ for any $x\in E$, consider $f:=\frac{1}{\sqrt{\mu(E)}}\chi_E$, it is clear that $||f||_2=1$, and we have that

$$||fg||_2=\left(\int_X\left|\frac{1}{\sqrt{\mu(E)}}\chi_E g\right|^2\right)^{\frac{1}{2}}\geq\left(\int_E\left(\frac{||g||_\infty-\epsilon}{\sqrt{\mu(E)}}\right)^2\right)^{\frac{1}{2}}=(||g||_\infty-\epsilon)\left(\int_E\frac{1}{\mu(E)}\right)^{\frac{1}{2}}=||g||_\infty-\epsilon$$

Since ϵ was arbitrary, we may conclude that $||M_g||\geq||g||_\infty$, where the opposite inequality is provided above, so we conclude that $||M_g||=||g||_\infty$. □