1. Yes, we know that a regular n-gon is constructable exactly when $\varphi(n) = 2^k$ for some k, as such a regular pentagon is constructable ($\varphi(5) = 4 = 2^2$). The angle between the verices of the pentagon is $72°$, hence $72°$ is a constructable angle, hence we can construct $12° = 72° - 60°$.

2. This is a corollary of the more general proof in problem 3, alternatively:

$\Phi_5(x)$ is the irreducible polynomial $x^4 + x^3 + x^2 + x + 1$, with galois group $(\mathbb{Z}/4\mathbb{Z})^\times \simeq C_4$. it follows that if K is the splitting field of Φ_5 over $\mathbb{Q}$, then we may take $K^{C_2} \longleftrightarrow C_2 \subset C_4$ by the Galois correspondence. Then $K/K^{C_2}/\mathbb{Q}$ is a tower of degree two extensions containing all of the roots of Φ_5. Hence the roots of Φ_5 are constructable.

3. We prove a more general statement, a is constructable iff for $f := \min(a; \mathbb{Q})$ we have $\exists k \in \mathbb{N}, \#G_f = 2^k$.

First assume that a is constructable, then a lies in a tower of degree 2 extensions, $F = F_n/F_{n-1}/\ldots/F_1 = \mathbb{Q}$, to show that $G_f = \#2^k$, it will suffice to show that the splitting field of f has cardinality 2^k, and to show this it will suffice to show that if F is obtained from a tower of degree 2 extensions of $\mathbb{Q}$, then F is contained in a normal extension L of $\mathbb{Q}$ degree 2^k for some k, since then the normal closure of $\mathbb{Q}(a)$ must be contained in L, and

$$[L : K][K : \mathbb{Q}] = 2^k \implies \#G_f = [K : \mathbb{Q}] = 2^r, r \leq k$$

We prove this by induction on the height of the tower, if the tower has height 1, this is trivial since degree 2 extensions are normal. Now, assuming its true for $n - 1$, suppose that $F_n = F_{n-1}(\alpha)$ and L' be the normal field containing F_{n-1}. For $\alpha^2 \in F$, let $\{\alpha = \alpha_1, \ldots \alpha_k\}$ be the conjugates of α over $\mathbb{Q}$ it follows that for each α_i, we have σ, such that $\sigma(\alpha) = \alpha_i$ which means that $\sigma(\alpha^2) = \alpha_i^2$, hence all of $\alpha_i^2 \in L'$, defining $L = L'(\alpha_i)_{i=1}^n$, we have

$$L = L'(\alpha_i)_{i=1}^n / L'(\alpha_i)_{i=1}^{n-1} / L'(\alpha)$$

each extension having degree 2 or 1, since $\alpha_i^2 \in L'$, this suffices to show that there is a normal field containing F with extension degree a power of 2 over $\mathbb{Q}$, and hence this also proves that the Galois closure and Galois group must have order 2^r for some r.

Now assume that $\#G_f = 2^k$, we want to show that K (the splitting field of f) can be obtained by a square root tower. We use that if G is a p-group (cardinality p^n), then G contains a tower of normal subgroups

$$H_1 \lhd H_2 \lhd \cdots \lhd H_n = G, \quad \#H_j = p^j$$

applying this to the case $p = 2$, we get the tower from the galois correspondnece

$$1 \lhd H_1 \lhd \cdots \lhd H_k = G_f \longleftrightarrow \mathbb{Q} \subset K_1 \subset \cdots \subset K_k = K$$

where $2 = [H_i : H_{i-1}] = [K_i : K_{i-1}]$.

Apply this to the special cases of A_4, D_4, we see that D_4 has 2^3 order, so if it is isomorphic to the galois group of the minimum polynomial of a, a is indeed constructable. In the second case, if A_4 is isomorphic to the Galois group of the minimum polynomial of a then a cannot be constructable by the above proof ($3|\#A_4 = 12$).

4. Suppose that f has a root $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then denoting τ as complex conjugation and K as the splitting field of f,

$$\tau|_K \in G_f$$

Here we have $\tau|_K(\alpha) \neq \alpha$, so that $\tau|_K \neq 1_K$, and $\tau|_K^2 = 1$ imples that G_f has an element of order 2, hence $G_f \not\cong A_3$ which contains no elements of order 2. Proven by contrapositive.

The converse is false, consider the polynomial

$$f(x) = x^3 - 4x + 2$$

Irreducible by Eisenstein. Furthermore,

$$0 < D(f) = -(4(-4)^3 + 27(2)^2) = 4(4^3 - 27) = 4(37) \notin \mathbb{Q}^2$$

so that f has Galois group S_3 since its discriminant is non-square, and f has real roots since $D(f) > 0$.

5. Yes, $\varphi(257) = 256 = 2^8$. The 70-gon is not constructable since $\varphi(70) = \varphi(5)\varphi(7)\varphi(2) = 24$ which is not a power of 2. The 85-gon is constructable since $\varphi(85) = \varphi(17)\varphi(5) = 4 \cdot 16 = 2^6$