Exercises

February 26, 2025

1 Categories and Pre-reqs

exercise - Show the forgetful functor $F : \text{Vect}_k \to \text{Set}$ is a functor.

proof - The identity linear transformation is the identity when regarded as a set mapping, if $T : U \to V, \varphi : V \to S$ as vector spaces, then $F(\varphi) \circ F(T)$ is a well defined set mapping, which maps $U \to S$ as sets.

exercise - $\text{Hom}(-, G)$ is functorial

proof - Note that if $f : H \to K$, then $\text{Hom}(-, G)(f) = f^*$ where $f^* : \text{Hom}(K, G) \to \text{Hom}(H, G)$ via $f^*(k) = k \circ f$. We first check associativity, let $f : H \to K,\ g : K \to N$, and $z \in \text{Hom}(N, G)$, then $(fg)^*(z) = z \circ fg = f^*(z) \circ g = g^* f^* z$.

We can also check that identity is preserved, since if $f : \text{Hom}(H, G) \to \text{Hom}(K, G)$ and $g : \text{Hom}(N, G) \to \text{Hom}(H, G)$, then for $h \in \text{Hom}(H, G)$ and $n \in \text{Hom}(N, G)$

$$f 1_H^* h = f(n \circ 1_H) = f(h) \text{ and } 1_H^* g n = g(n) \circ 1 = g(n) \quad \square$$

2 Chain Complexes

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exercise - Show that $\mathbf{h} = (h_i)$, $h_i : C_i \to C'_{i+1}$, then $h_{i-1} \circ d_i + d'_{i+1} \circ h_i$ is a chain map.

proof - We need to check that $d'_i \circ f_i = f_{i-1} \circ d_i$, in other words we need to show

$$d'_i \circ (h_{i-1} \circ d_i + d'_{i+1} \circ h_i) = (h_{i-2} \circ d_{i-1} + d'_i \circ h_{i-1}) \circ d_i$$

Since we are in a module, we can distribute d'_i on the left hand side, rewriting the condition as

$$d'_i \circ h_{i-1} \circ d_i + d'_i \circ d'_{i+1} \circ h_i = h_{i-2} \circ d_{i-1} \circ d_i + d'_i \circ h_{i-1} \circ d_i$$

So it will suffice to show that

$$d'_i \circ d'_{i+1} \circ h_i = h_{i-2} \circ d_{i-1} \circ d_i$$

But this is trivial since "$d^2 = 0$"

exercise - A homotopy of chains is an equivalence relation

proof - We will prove the following items: reflexivity, symmetry, transitivity

- To see that $f \sim f$, take each to be the zero map, $h_i = 0, \forall i$. In this case the diagram with maps given by $\mathbf{h}$ is immediate.

- Suppose that $f \sim g$, then we have some $\mathbf{h}$, such that $g_i - f_i = h_{i-1} \circ d_i + d'_{i+1} \circ h_i$ for each i. Since h_i are morphisms of modules, so are $-h_i$, so in particular we have $-\mathbf{h} := (-h_i)_i$, so that

$$\begin{aligned} f_i - g_i = -(g_i - f_i) &= -(h_{i-1} \circ d_i + d'_{i+1} \circ h_i) = -h_{i-1} \circ d_i + -d'_{i+1} \circ h_i \\ &= -h_{i-1} \circ d_i + d'_{i+1} \circ -h_i \end{aligned}$$

  The last line follows since d'_{i+1} is linear.

- Suppose that $f \sim g \sim r$, and let $\mathbf{h}, \mathbf{k}$ be respective witnesses of these homotopies. Then we have

$$\begin{aligned} r_i - g_i &= k_{i-1} \circ d_i + d'_{i+1} \circ k_i \\ g_i - f_i &= h_{i-1} \circ d_i + d'_{i+1} \circ h_i \end{aligned}$$

  This furnishes

$$r_i - f_i = k_{i-1} \circ d_i + d'_{i+1} \circ k_i + h_{i-1} \circ d_i + d'_{i+1} \circ h_i = (k_{i-1} + h_{i-1}) \circ d_i + d'_{i+1} \circ (h_i + k_i)$$

  So we have the homotopy $r \sim f$ via $\mathbf{h}+\mathbf{k}$, we are done since we already proved symmetry.

exercise - Show that the following two chain complexes are homotopic:

$$0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{(\cdot, 2\cdot)} \mathbf{Z} \oplus \mathbf{Z} \longrightarrow 0 \oplus \mathbf{Z}/(2) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z} \xrightarrow{2\cdot} \mathbf{Z} \longrightarrow \mathbf{Z}/(2) \longrightarrow 0$$

proof - Let each f_i be the projection of the second coordinate, and each g_i the inclusion into the second coordinate. In thise case we have

$$\mathbf{1}_{C'} - \mathbf{fg} = \mathbf{0}$$

and hence $\mathbf{h} = (0)_i$ witnesses the homotopy. The slightly harder case is the other direction. Define $h_2 = h_0 = h_{-1} = 0$, and $h_1 : (m, n) \mapsto m$. By definition of $\mathbf{f}, \mathbf{g}$, we have $1_{C,i} - g_i f_i : (m, n) \mapsto (m, 0)$, so we just need to check that our given $\mathbf{h}$ satisfies this.

$$\begin{aligned} (d_3 h_2 + h_1 d_2)(m, n) &= 0 + h_1(m, 2n) = (m, 0) \\ (d_2 h_1 + h_0 d_1)(m, n) &= d_2(m, 0) + 0 = (m, 0) \\ (d_1 h_0 + h_{-1} d_0)(0, n) &= 0 + 0 = (0, 0) \end{aligned}$$

This verifies the homotopy.

Theorem - ("*The Fundamental Theorem of Homology For Chain Complexes*") If chain complexes C, C' are homotopic, then they have the same Homology Modules.

proof - Recall that

$$H^i := \frac{\ker(d_i)}{\operatorname{Im}(d_{i+1})}$$

Now let the homotopy be given by $\mathbf{f} : C \to C', \mathbf{g} : C' \to C$, there is a natural induced map of f_i on H^i, given by restricting f_i to $\operatorname{Im}(d_{i+1})$, then taking the unique map from the quotient by the kernel of d_i, which exists and is unique by the first isomorphism theorem. Calling this induced map $f_{i,*}$, we need to check that $f_{i,*}$ maps into H_i'. First it is immediate that $f_i|_{\ker d_i} : \ker d_i \to \ker d_i'$, since $d_i' f_i = f_{i-1} d_i$, where the right hand side is zero when restricted to $\ker(d_i)$, well definition on cosets follows from "$d^2 = 0$". So only need check that $(hd + dh)_* = 0$, $d_{i+1}h \in \operatorname{Im} d_{i+1} = 0$ so it only remains to show that $hd_i = 0$ on $\ker d_i$, this is trivial since $h(0) = 0$ and for any $z \in \ker d_i$ $hd_i(z) = h(0)$. □

3 Homology and Cell Complexes

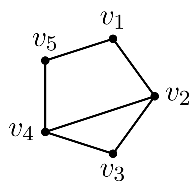
exercise - Let G be a graph, then $\ker d_1 = \{\text{Cycles in } G\}$

proof - We have the map $d_1 : \oplus_i \mathbf{Z} e_i \to \oplus_i \mathbf{Z}\mathbf{v_i}$, then $\sum_i k_i e_i \in \ker d_1 \iff \sum_i k_i(H(e_i) - T(e_i)) = 0$, where $H(e)$ denotes the target of an edge, and $T(e)$ denotes the source of an edge. Then $\sum_i k_i(H(e_i) - T(e_i)) = \sum_j \ell_j v_j = 0 \iff \ell_j = 0, \ \forall j$. Where $\ell_j = \sum_{\{e_i | v_j = H(e_i)\}} k_i - \sum_{\{e_i | v_j = T(e_i)\}} k_i$ so that $\ell_j = 0$ for all j exactly when every vertex has an equal number of in and out edges. □

exercise - Show that for the graph G, with $V(G) = \{x, y\}, E(G) = \{a = (x, y), b = (x, y), c = (x, y)\}$ we have $\ker d_1 = \langle a - b, b - c\rangle$

proof - Here we have $d_1 : a, b, c \mapsto y - x$, then $a - b, b - c \in \ker d_1$. We have $d_1 : n_1 a + n_2 b + n_3 c \mapsto (n_1 + n_2 + n_3)(x - y)$, so that $\ker d_1 = \{na + mb + k\ell \mid n + m + \ell = 0\}$, so fixing $na + mb + k\ell = na + mb + -(n + m)c \in \ker d_1$, we have $na + mb + k\ell = n(a - b) + (m + n)(b - c) \in (a - b, b - c)$ □

exercise - Consider the following Graph:



(1)

Where the edges are oriented with the larger vertex index at the head and smaller at the tail. Compute generators for the cycles, and show that $C_*(G) \simeq C_*(G')$ as chain complexes, where G' is G with (v_1, v_5) replaced with (v_5, v_1). Finally, show that $H_1(G) \cong H_1(G')$

proof - We can write d_1 as a matrix,

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\rightsquigarrow}{\text{Row Reduction}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the kernel has rank 2. From this matrix we can compute generators for the kernel, (and thus the cycles)

$$\ker d_1 = (e_2 - e_4, e_1 + e_5 - e_6)$$

To show that $C_*(G)$, and $C_*(G')$ are chain homotopic, define

$$f : v_i \mapsto v_i', e_i \mapsto e_i', \quad g : v_i' \mapsto v_i, e_i' \mapsto e_i$$

. By construction these are chain maps, since firstly $d_0 = 0, d'_0 = 0$, and $e_i = e'_i$ for each $e_i \neq (v_1, v_5)$, and the values of d_1 agree with d'_1 on each of the equal $e_i = e'_i$. For $e = (v_1, v_5), e' = (v_5, v_1)$, we have $f_1(e) = -e', g_1(e') = -e$, for $e \neq (v_1, v_5), e' \neq (v_5, v_1)$ commutativity is obvious since $d(e) = d'(e')$ in this case. For $e = (v_1, v_5)$ we have $f_0 \circ d(e) = f_0(x-y) = x-y = -(y-x) = -d'_1(e') = d'_1(f_1(e))$, the argument is the exact same for commutativity of the square replacing g for f and e' for e. Then $f \circ g = 1$, so we can just choose our homotopy to be the zero map.

To see that the homology is the same, there are 2 possible solutions both are easy. The first is that they are isomorphic as chain complexes, the second is that d_1 is rewritten by multiplying a row by -1, preserving the row reduced form and hence the generators of the kernel. □

exercise - Let Γ, Γ' be arbitrary graphs such that $\Gamma \simeq \Gamma'$, then $C_*(\Gamma) \simeq C_*(\Gamma')$

proof - In (i) we show that homotopy of graphs is independent of orientation, and in (ii), we show that it is equivalent up to quotienting by an edge between to vertices, this is sufficient since it allows us to show that (since the graphs are homotopic they have the same Euler characteristic)

$$C_*(\Gamma) \simeq C_*(\bigvee_{1 \leq k \leq \chi(\Gamma)} S^1) \simeq C_*(\Gamma')$$

(i) Assume that $\Gamma \setminus e_0 = \Gamma' \setminus e'_0$, such that $e'_0 = -e_0$, we copy the proof in the previous question.

$$f : C_*(\Gamma) \to C_*(\Gamma') \qquad \begin{cases} e_0 \mapsto -e'_0 \\ e_i \mapsto e'_i \\ v_i \mapsto v'_i \end{cases} \quad i \geq 1 \; g : C_*(\Gamma') \to C_*(\Gamma) \qquad \begin{cases} e'_0 \mapsto -e_0 \\ e'_i \mapsto e_i \\ v'_i \mapsto v_i \end{cases} \quad i \geq 1$$

$fg = 1_{C_*(\Gamma)}, gf = 1_{C_*(\Gamma')}$, so they are homotopies so long as they are chain maps. This is trivial except for the "square" where edges map to vertices. Commutativity on generators is trivial apart from e_0, and e'_0, in this case $f \circ d_1(e_0) = f(v_1 - v_0) = v'_1 - v'_0 = -(v'_0 - v'_1) = d'_1(-e'_0) = d'_1 \circ f(e_0)$. The proof is the same for g.

(ii) Suppose we are quotienting the edge $e_0 = (v_0, v_1), v_0 \neq v_1$. Here in order to focus on the most relevant parts of the chain complex we do some book keeping:

$$\begin{aligned} C_*(\Gamma)_2 &= \mathbb{Z}_0^2 \oplus S_2 & C_*(\Gamma)_1 &= \mathbb{Z}_0^1 \oplus \mathbb{Z}_1^1 \oplus S_1 \\ C_*(\Gamma')_2 &= S'_2 & C_*(\Gamma')_1 &= \mathbb{Z}'_0 \oplus S'_1 \end{aligned}$$

Here there are natural identifications of all edges and vertices in S_i and S'_i. There is really only one way to define f if we want to keep the natural mapping $f : S_i \to S'_i$,

$$\begin{aligned} f_1 &: (a_0 e_0, a_1 e_1, \dots, a_m e_m) \mapsto (a_1 \overline{e_1}, a_2 \overline{e_2}, \dots, a_m \overline{e_m}) \\ f_0 &: (a_0 v_0, a_1 v_1, \dots, a_n v_n) \mapsto ((a_0 + a_1)\overline{v_1}, a_2 \overline{v_2}, \dots, a_n \overline{v_n}) \end{aligned}$$

It is clear that f is a chain map from construction. Now we define $g : C_*(\Gamma') \to C_*(\Gamma)$ as follows,

$$g_0 : (a_1 \overline{v_1}, \dots, a_n \overline{v_n}) \mapsto (0, a_1 v_1, \dots, a_n v_n)$$

Then taking the natural map $S'_2 \to S_2$, composition with d gives $e_i \mapsto \sum_{j=0}^n k_j^i v_j$, define $g_1 : \overline{e_i} \mapsto k_0^i e_0 + e_i$. Then we have $\overline{k_1^i} = k_0^i + k_1^i$

$$d_1 g_1(\overline{e_i}) = d_1(k_0^i e_0 + e_i) = k_0^i(v_1 - v_0) + \sum_{j=0}^n k_j^i v_j$$

$$g_0\overline{d_1}(\overline{e_i}) = g_0\left(\sum_{j=1}^n \overline{k_j^i}\overline{v_j}\right) = g_0\left((k_0^i + k_1^i)\overline{v_1} + \sum_{j=2}^n k_j^i\overline{v_j}\right) = \sum_{j=0}^n k_j^i v_j + k_0^i(v_1 - v_0)$$

This implies that g is indeed a chain map. It is immediate that $fg = 1_{C_*(\Gamma')}$, we need to provide a homotopy to show that $gf \simeq 1_{C_*(\Gamma)}$. Explicitly we have

$$1 - g_1 f_1 = \left(a_0 - \sum_1^m a_i k_0^i\right) e_0$$
$$1 - g_0 f_0 = a_0(v_0 - v_1)$$

the map $h : v_0 \mapsto -e_0$ obviously satisfies $d_1 h_1(a_0, \ldots, a_n) = a_0(v_0 - v_1)$. We only need to check that $h_1 d_1 = \left(a_0 - \sum_1^m a_i k_0^i\right) e_0$, but the first coordinate of $d_1(a_0 e_0 + \sum_{i=1}^m a_i e_i)$ is $-a_0 + \sum_{i=1}^m a_i k_0^i$, by definition of k_0^i, and since $e_0 = (v_0, v_1)$. It follows that

$$h_1 \circ d_1(a_0 e_0 + \sum_{i=1}^m a_i e_i) = -(-a_0 + \sum_{i=1}^m a_i k_0^i) e_0 = 1 - g_1 f_1 \quad \square$$

exercise - Using the cell structures for S^2 from $e^2 \cup e^0$, $e^2 \cup e^2 \cup e^1 \cup e^0$ Build $C_*(S^2)$ and compute its Homology.

proof - The first cell structure gives the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

In this case the homology modules are trivially $H_i(C_*(S^2)) = \begin{cases} 0 & i \neq 0, 2 \\ 1 & i = 0, 2 \end{cases}$

The second cell structure gives the sequence

$$0 \longrightarrow \mathbb{Z}e_0^2 \oplus \mathbb{Z}e_1^2 \xrightarrow{d_2} \mathbb{Z}e_0^1 \xrightarrow{d_1} \mathbb{Z}e_0^0 \longrightarrow 0$$

Where $\mathbb{Z}e_0^2 \oplus \mathbb{Z}e_1^2 = \mathbb{Z}\left\langle e_0^2 + e_1^2\right\rangle \oplus \mathbb{Z}\left\langle e_0^2 - e_1^2\right\rangle$, and $d_2 : e_0^2 \mapsto e_0^1, e_1^2 \mapsto -e_0^1$, so that

$$H_2(C_*(S^2)) = \frac{\mathbb{Z}\left\langle e_0^2 + e_1^2\right\rangle \oplus \mathbb{Z}\left\langle e_0^2 - e_1^2\right\rangle}{\mathbb{Z}\left\langle e_0^2 + e_1^2\right\rangle} \cong \mathbb{Z}$$

And $d_1 : e_0^1 \mapsto e_0^0 - e_0^0$, so that $d_1 \equiv 0$ which gives us $H_0(C_*(S^2)) = \mathbb{Z}$, and $H_1(C_*(S^2)) = \mathbb{Z}/\mathbb{Z} \cong 0$, with $H_i = 0$ for $i > 2$. The same result as above.

exercise - Show that "augmentation" still gives a chain complex, i.e. $\epsilon \circ d_1 = 0$ cite Augmentation in notes

proof - It will suffice to check on generators of C_1, let $e_\beta^1 \in C_1$, then

$$\epsilon d_1(e_\beta^1) = \epsilon(e_\alpha^0 - e_\beta^0) = \epsilon(e_\alpha^0) - \epsilon(e_\beta^0) = 1 - 1 = 0$$

exercise - $S^1 \wedge S^1 \simeq S^2$

proof - $S^1 \wedge S^1 \cong \frac{T^2}{S^1 \vee S^1}$, where $T^2 \cong D^2/\sim$, then

$$S^1 \wedge S^1 = \frac{T^2}{S^1 \vee S^1} \cong \frac{D^2/\sim}{\partial D^2} = \frac{D^2}{\partial D^2} \cong S^2$$

exercise - $SX \simeq \sum X$

proof - Here $(X, \{*\})$ is a (pointed but ignore the point for now) CW-complex, we can give $X \times I$ a CW structure, note that we have $\partial e^n = \partial(e^{n-1} \times I) = (\partial e^{n-1} \times I) \cup e^{n-1} \times \partial I$ (we can do this using point set topology and the fact that the sets are closed), then the CW-structure on $(X \times I)^n$ can be taken to be $(X \times I)^0 = X^0 \times \partial I$, then taking φ_α^n to be the gluing map of e_α^n into X^n:

$$(X \times I)^n = (X \times I)^{n-1} \bigcup_{\varphi_\alpha^n} e_\alpha^n \times \partial I \bigcup_{\varphi_\alpha^{n-1} \times I} e_\alpha^{n-1} \times I$$

This gives a CW structure on $X \times I$ with $X \times \partial I$ as a substructure, there is a canonical CW structure on the quotient given by replacing φ_α^n with $\pi|_{X^n} \circ \varphi_\alpha^n$. Furthermore since $* \in X^0$, we have $\{*\} \times I \in (X \times I)^1$, which can be identified with its image in the quotient. This is a contractible subcomplex, so by the CW extension theorem

$$SX = \frac{X \times I}{X \times \partial I} \cong \frac{X \times I}{X \times \partial I \cup \{*\} \times I} = \sum X$$

exercise - $X \wedge S^1 \simeq \sum X$

proof - It is important here that the same point $x \in X$ is used in either quotient. Consider the map $f : I \to S^1, t \mapsto e^{i\pi t}$, then we get the following,

$$\begin{tikzcd} X \times I \arrow[r, "1\times f"] \arrow[d, "\pi"] & X \times S^1 \arrow[d, "\pi'"] \\ \frac{X\times I}{X\times\{1\}\cup X\times\{-1\}\cup\{x\}\times I} \arrow[r, "\overline{1\times f}"] & \frac{X\times S^1}{X\times\{1\}\cup\{x\}\times S^1} \end{tikzcd}$$

The bottom left here is the (reduced) suspension $\sum X$, and the bottom right is $X \wedge S^1$. Here the induced map is clearly bijective, since the quotients here are equivalent to quotienting by the image of the quotients, along with quotienting on the left side $f^{-1}(1) \times \{1\} \sim f^{-1}(-1) \times \{-1\} \sim x$, where f wasn't injective. Continuity of the inverse follows, since we have $\frac{X\times I}{X\times\partial I} \cong X \times S^1$, and we can factor our map through this homeomorphism before taking quotients.

exercise - Use $\tilde{H}_i$ to show that ∂D^n cannot be a retract of D^n

proof - We have that $D^n/\partial D^n \cong S^n$, furthermore $D^n \simeq \{x\}$ for all n, implying that $\tilde{H}_i(D^n) \cong \tilde{H}_i(\{x\}) = 0$, for any $i, n \in \mathbb{Z}_{\geq 0}$. Suppose for contradiction that there were a retract $h : D^n \to \partial D^n$, then denoting $\iota : \partial D^n \hookrightarrow D^n$ we would have that $h \circ \iota \simeq 1_{\partial D^n}$, by functoriality we get the following commutative diagram (also note that $\partial D^n = S^{n-1}$):

$$\begin{tikzcd} \tilde{H}_{n-1}(\partial D^n) \arrow[r, "1"] \arrow[d, "\iota_*"] & \tilde{H}_{n-1}(\partial D^n) \\ \tilde{H}_{n-1}(D^n) \arrow[ur, "h_*"] & \end{tikzcd}$$

Where this diagram is equivalent to

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \\ 0 \arrow[ur] & \end{tikzcd}$$

This is clearly a contradiction since there are no surjective morphisms $0 \to \mathbb{Z}$. $\square$

exercise - Let $f : S^n \to S^n$, show if f is not surjective then $\deg f = 0$

proof - Suppose that f is not surjective, then we can pick some $x \in S^n \setminus f(S^n)$, by the stereographic projection $S^n \setminus \{x\} \cong \mathbb{R}^n \simeq \{*\}$, so we have the following commutative diagram, where the downward map is taken by restricting the codomain of f to $S^n \setminus \{x\}$, and the upper diagonal is the inclusion.

$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[r, "f_*"] \arrow[d] & \tilde{H}_n(S^n) \\ \tilde{H}_n(\{*\}) \arrow[ur] & \end{tikzcd}$$

equivalently,

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_*"] \arrow[d] & \mathbb{Z} \\ 0 \arrow[ur] & \end{tikzcd}$$

so f_* must be the zero map.

exercise - Let f as in the previous exercise, show that $\deg f = \deg \sum_* f$ (remark first show that the suspension $\sum$ is functorial)

proof - given $f : X \to Y$, we can define $\sum_* f$ as the induced map in the following diagram

$$\begin{tikzcd} X \times I \arrow[r, "f \times 1"] \arrow[d, "\pi_X"] & Y \times I \arrow[d, "\pi_Y"] \\ \sum X \arrow[r, "\sum_* f"] & \sum Y \end{tikzcd}$$

Continuity simply follows from continuity of f, and the definition of quotient topology taking opens to opens on the upper half of the square, then commutativity and the universal property of the quotient on the lower square. To get $\sum_* fg = \sum_* f \sum_* g$ simply factor $f \circ g \times 1$ through the following diagram

$$\begin{tikzcd} X \times I \arrow[r, "f \times 1"] \arrow[d, "\pi_X"] & Y \times I \arrow[r, "g \times 1"] \arrow[d, "\pi_Y"] & Z \times I \arrow[d, "\pi_Z"] \\ \sum X \arrow[r, "\sum_* f"] & \sum Y \arrow[r, "\sum_* g"] & \sum Z \end{tikzcd}$$

$\sum_* 1 = 1_{\sum_*}$ is obvious. This suffices to show functoriality.

For the proof of the main result, note that by an identical construction, the canonical C_* is functorial, $f : X \to Y$, then $C_* f : C_{1_X} \to C_{1_Y}$. It suffices to show the following result for SX, since it has the same Homology as $\sum X$ by homotopy equivalence, so factoring f through this equivalence won't affect the degree. We have naturality (identifying $S^n \leftrightarrow S^n \times \{0\} f \leftrightarrow C_* f|_{S^n \times \{0\}}$)

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "f"] & CS^n \arrow[d, "C_* f"] \\ S^n \arrow[r] & CS^n \end{tikzcd}$$

And hence, we get a chain map

$$\begin{array}{ccccccc} H_{n+1}(CS^n) & \longrightarrow & H_{n+1}(CS^n, S^n) & \longrightarrow & H_n(S^n) & \longrightarrow & H_n(CS^n) \\ \big\downarrow{\scriptstyle (C_*f)_*} & & \big\downarrow{\scriptstyle (S_*f)_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (C_*f)_*} \\ H_{n+1}(CS^n) & \longrightarrow & H_{n+1}(CS^n, S^n) & \longrightarrow & H_n(S^n) & \longrightarrow & H_n(CS^n) \end{array}$$

Which we can rewrite as

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{n+1}(S(S^n)) & \longrightarrow & H_n(S^n) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle (C_*f)_*} & & \big\downarrow{\scriptstyle (S_*f)_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (C_*f)_*} \\ 0 & \longrightarrow & H_{n+1}(S(S^n)) & \longrightarrow & H_n(S^n) & \longrightarrow & 0 \end{array}$$

Commutativity suffices to show that the degrees of the maps must be equal. □

Remark 1. Use of the 0 maps on either end of the sequence is a bit of a cheat here so we get for free that $H_{n+1}(S(S^n)) \cong \mathbb{Z}$, and we need to ensure the map $H_{n+1}(S(S^n)) \to H_n(S^n)$ is nonzero since otherwise we cannot conclude about $(\sum_* f)_*(1) = (S_*f)_*(1)$ from the above diagram.

Remark 2. To make sense of degree here we should really show that $\sum S^n \simeq S(S^n) \simeq S^{n+1}$, there is an explicit homeomorphism by simply rescaling by the factor in I in $S^n \times I$, then taking the quotient.

exercise - Compute the Homology of X, where X is the triangulation of the Torus.

proof - Let the upper triangle be e^2_U, the lower triangle be e^2_L, the upper/lower boundary be e^1_α, the left/right boundary be e^1_β, and the diagonal be e^1_γ, there is only one zero-cell, we may call it e^0. Then:

$$\begin{aligned} \pi_\alpha \varphi_U = \pi_\beta \varphi_U = 1 = -\pi_\gamma \varphi_U \\ \pi_\alpha \varphi_L = \pi_\beta \varphi_L = -1 = -\pi_\gamma \varphi_L \\ \pi_0 \varphi_\alpha = \pi_0 \varphi_\beta = \pi_0 \varphi_\gamma = 1 - 1 = 0 \end{aligned}$$

In this case we have the cell structure

$$0 \longrightarrow \mathbb{Z}\left\langle e^2_U \right\rangle \oplus \mathbb{Z}\left\langle e^2_L \right\rangle \xrightarrow{d_2} \mathbb{Z}\left\langle e^1_\alpha \right\rangle \oplus \mathbb{Z}\left\langle e^1_\beta \right\rangle \oplus \mathbb{Z}\left\langle e^1_\gamma \right\rangle \xrightarrow{d_1} \mathbb{Z}\left\langle e^0 \right\rangle \longrightarrow 0$$

Where $\ker d_2 = e^2_U + e^2_L$, $d_1 = 0 = d_0$, rewriting $C_1(X)$ to have $\operatorname{Im} C_2(X)$ as one of its three generators, we get the usual Homology modules for the Torus

$$H_2(X) \cong \mathbb{Z}, \quad H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_0(X) \cong \mathbb{Z}$$

exercise - Show from the Homology axioms that $\tilde{H}_n(X^{n+1}) = \tilde{H}_n(X)$

proof - We have from Axiom 1, the following sequence is exact

$$\tilde{H}_{k+1}(X^{m+1}/X^m) \longrightarrow \tilde{H}_k(X^m) \longrightarrow \tilde{H}_k(X^{m+1}) \longrightarrow \tilde{H}_k(X^{m+1}/X^m)$$

We have the following from construction of a CW complex:

$$X^{n+1}/X^n = \frac{\frac{e^{n+1}_\alpha \bigsqcup_\alpha X^n}{x \sim \varphi_\alpha(x),\, x \in \partial e^{n+1}_\alpha,\, \varphi_\alpha(x) \in X^n}}{X^n} = \frac{\bigsqcup_\alpha e^{n+1}_\alpha}{\bigsqcup_\alpha \partial e^{n+1}_\alpha} \cong \bigvee_\alpha S^{n+1}$$

So for $k \neq m+1, m$ we have

$$0 \longrightarrow \tilde{H}_k(X^m) \longrightarrow \tilde{H}_k(X^{m+1}) \longrightarrow 0$$

by exactness this gives an isomorphism $\tilde{H}_n(X^{n+1}) \cong \tilde{H}_n(X^{n+2}) \cong \cdots \cong \tilde{H}_n(X)$

exercise - Rewrite Homoology axioms (1-3) in the context of the category **Pairs**

proof - First note that functoriality gives homotopy invarience, homotopy equivalent maps $f, g : (X, A) \to (Y, B)$, $f \simeq g$ (here we mean the restrictions induce homotopy equivalences on the smaller spaces) induce the same maps of homology modules $f_* = g_* : H_n(X, A) \to H_n(X, B)$

1. The following sequence is long exact:

$$\cdots H_{n+1}(X, A) \longrightarrow H_n(A, \{*\}) \longrightarrow H_n(X, \{*\}) \longrightarrow H_n(X, A) \longrightarrow \cdots$$

2. For any n, $H_n\left(\bigvee_\alpha (X_\alpha, A_\alpha)\right) = \bigoplus_\alpha H_n(X_\alpha, A_\alpha)$, here we are quotienting an element in A, B resp. when we take $(X, A) \vee (Y, B)$

3. $H_n(S^0, \{*\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$

exercise - Let X be a cell complex with subcomplex A, furthermore let $K \subset A$ be any set such that $\overline{K} \subset A^\circ$, show that

$$\frac{X \setminus K}{A \setminus K} \simeq X/A$$

proof - We get the following commutative diagram, since $\iota : X \setminus K \hookrightarrow X$ is such that $\iota|_{A \setminus K} : A \setminus K \hookrightarrow A$

$$\begin{array}{ccc} X \setminus K & \xrightarrow{\iota} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \frac{X \setminus K}{A \setminus K} & \xrightarrow{\bar{\iota}} & X/A \end{array}$$

The induced map is clearly bijective, since ι is bijective on A^c. Continuity of $\bar{\iota}$ follows from continuity of ι. To see that $\bar{\iota}^{-1}$ is continuous, let U be open in X, if $\{a\} \notin U$ (here $\{a\} \subset A \setminus K$), then $\pi_1^{-1}(U) = \pi_1^{-1}(U) \cap \overline{K}^c$, so $\iota(\pi_1^{-1}(U) \cap \overline{K}^c) = \pi_2^{-1}(\bar{\iota}(U))$ which the former is open in the subspace topology iff it is open in X. Now if $\{a\} \subset U$, then $\pi_2^{-1}\bar{\iota}(U) = \iota\pi_1^{-1}(U) \cup A^\circ$, where $\iota\pi_1^{-1}(U) = V \cap K^c$, where V is open in X, rewriting this we get $\pi_2^{-1}\bar{\iota}(U) = V \cup A^\circ$ which is open in X, so $\bar{\iota}(U)$ is open in the quotient topology. □

exercise - Compute $\tilde{H}_i(\mathbb{RP}^n)$

proof - We compute this using cellular homology and degree. First a lemma,

Lemma. The antipodal map $f : S^n \to S^n, x \mapsto -x$ is such that $\deg f = (-1)^{n+1}$

Proof of lemma: Consider the CW structure $e^n_U \cup e^n_L \cup e^{n-1} \cup e^0$ on S^n, $\tilde{H}_k(S^n) = 0$ unless $k = n$, so we just consider $k = n$. It follows that the map $r : e^n_U \leftrightarrow e^n_L$ on $\tilde{H}_n(S^n) = \mathbb{Z}\langle e^n_U - e^n_L\rangle$ takes $1 \mapsto -1$, so r has degree -1. Now taking one r_i for each coordinate plane in $\mathbb{R}^{n+1}$, we have $f = \prod_{i=1}^{n+1} r_i \simeq r^{n+1}$ (rotations are homotopies), hence $\deg f = \deg r^{n+1} = (\deg r)^{n+1} = (-1)^{n+1}$. □

Continuing with the proof of the theorem, we give $\mathbb{RP}^n$ the usual cell structure, i.e. since

$$\mathbb{RP}^n \cong S^n/\sim \cong \frac{D^n}{\sim_{\partial D^n}} = \frac{D^n}{\sim_{S^{n-1}}}$$

we find that taking φ_n as the identity map $\partial D^n \to S^{n-1}$

$$\mathbb{RP}^n = e^n \cup_{\varphi_n} \mathbb{RP}^{n-1} = e^n \cup_{\varphi_n} e^{n-1} \cup_{\varphi_{n-1}} \cdots \cup_{\varphi_0} e^0$$

This gives a chain complex:

$$\begin{tikzcd} C_{n+1}(\mathbb{RP}^n) \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r] & C_{n-1}(\mathbb{RP}^n) \arrow[r] & C_{n-2}(\mathbb{RP}^n) \arrow[r] & \cdots \arrow[r] & C_0(\mathbb{RP}^n) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

We want to use degree to compute d_k. In this case, we only have one $n-1$-cell and we are attaching only one n-cell, so the we can determine d_n from $d_n(e^n) = (\deg f)e^{n-1}$. Where f is just the composition

$$\begin{tikzcd} \partial e^n \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & \frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^{n-2}} \\ & \mathbb{RP}^{n-1} \arrow[ur, "\pi"'] & \end{tikzcd}$$

Fix a point $y \in \frac{\mathbb{RP}^{n-1}}{\mathbb{RP}^{n-2}}$, then $f^{-1}(y) = \{x, -x\}$, taking neighborhoods as in the definition of local degree, we may take $V = f(U_1)$, and U_2 homeomorphic to V via f by excision on either U_1 or U_2. Then there is a homeomorphism $f|_{U_2} f|_{U_1}^{-1} : U_1 \to U_2$. This map extends to the antipodal map $r^{n+1} : \partial e^n \to \partial e^n$ which has degree $\sigma(n+1)$, it follows that (since $\deg g^{-1} = \deg g$, $\forall g$ invertible) we have $\deg f|_{-x} = (-1)^{n+1} \deg f|_x$. If $\deg f|_x = -1$, then this is actually arbitrary, since we could have chosen opposite generators for $H_n(S^n)$, so we can assume that $\deg f|_x = 1$. This computes d_n

$$d_n(e^n) = \deg f = \deg f|_x + \deg f|_{-x} = 1 + \sigma(n+1)$$

This gives us the chain complex

$$\cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

Now we can directly compute

$$\tilde{H}_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \equiv 1 \mod 2 \\ \mathbb{Z}/(2) & n \geq k \equiv 1 \mod 2 \\ 0 & \text{else} \end{cases}$$

$\square$

exercise - $d^2 = 0$ in simplicial Homology.

proof - Let $\sigma_\alpha : \Delta^n(X) \to X$, as in the definition of simplicial Homology. Then

$$\begin{aligned}
\partial_{n-1}\partial_n\sigma_\alpha &= \partial_{n-1}\sum_{i=0}^{n}(-1)^i\sigma_\alpha|_{[v_0,v_1,\dots,\hat{v_i},\dots,v_n]} = \sum_{i=0}^{n}(-1)^i\partial_{n-1}\sigma_\alpha|_{[v_0,v_1,\dots,\hat{v_i},\dots,v_n]} \\
&= \sum_{i=0}^{n}\sum_{0\le j<i\le n}(-1)^{i+j}\sigma_\alpha|_{[v_0,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n]} + \sum_{i=0}^{n}\sum_{0\le i<j\le n}(-1)^{i+(j-1)}\sigma_\alpha|_{[v_0,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n]} \\
&= \sum_{i=0}^{n}\sum_{0\le j<i\le n}(-1)^{i+j}\sigma_\alpha|_{[v_0,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n]} + \sum_{j=0}^{n}\sum_{0\le i<j\le n}(-1)^{i+j-1}\sigma_\alpha|_{[v_0,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n]}
\end{aligned}$$

Now we may swap the indices i and j in the second sum, which gives us the result that

$$\partial_{n-1}\partial_n\sigma_\alpha = \sum_{i=0}^{n}\sum_{0\le j<i\le n}(-1)^{i+j}\sigma_\alpha|_{[v_0,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n]} - \sum_{i=0}^{n}\sum_{0\le j<i\le n}(-1)^{i+j}\sigma_\alpha|_{[v_0,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n]}$$

So the two summations cancel leaving us with "$\partial^2\sigma_\alpha = 0$" □

4 Cohomology

exercise - $(d^*)^2 = 0$

proof - Let $f \in C^{n+1} = \mathrm{Hom}(C_{n+1}, G)$, then let $x \in C_{n+1}$ be arbitrary, it follows that $d^*_{n+1}d^*_n f = f \circ d_n \circ d_{n+1}$, so that $d^*_{n+1}d^*_n f = f(0) = 0$. □

exercise - (Note the upper sequence is exact) Show the following universal property:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[dr, "0"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl, dashed, "\exists!"] & 0 \\
& & Z & &
\end{tikzcd}$$

proof - $C \overset{f}{\cong} B/A$, so by the first isomorphism theorem, Since the map in question sends $A \to 0$ there exists a unique map $j : B/A \to Z$, such that the diagram commutes with B/A in place of C. It follows that $jf : C \to Z$ makes the diagram commute, and if any other map ι were to make the diagram commute, then $\iota f^{-1} = j$ implies that $\iota = j$ so that uniqueness is satisfied as well. □

exercise - In the proof of the universal coefficient theorem, we define a map

$$f : H^n(C^*) \to \mathrm{Hom}(H_n(C_*), G)$$

To do so, we take a representative of $[\phi] \in H^n(C^*)$, so that $\phi : C^* \to G$, and $d^*\phi = 0$. By choice of ϕ, it follows that the sequence is exact, so we can apply the results of the previous exercise [4]

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r] \arrow[dr, "0"] & Z_n \arrow[r] \arrow[d, "\phi|_{Z_n}"] & Z_n/B_n \arrow[r] \arrow[dl, dashed, "\exists!"] & 0 \\
& & Z & &
\end{tikzcd}$$

We may define $f([\phi])$ to be the unique map making the diagram commute. This is well defined since maps in the equivalence class only differ on boundaries which are mapped to 0. *Show that f is a homomorphism.*

proof - We need to show that $f([\phi]) + f([\psi]) = f([\phi + \psi])$, it will suffice to show that $f([\phi]) + f([\psi])$ satisfies the universal prperty for $(\phi + \psi)|_{Z_n} = \phi|_{Z_n} + \psi|_{Z_n}$, so let $x \in Z_n$ and denote the quotient map as q, then

$$(f([\phi]) + f([\psi]))(q(x)) = f([\phi])(q(x)) + f([\psi])(q(x)) = \phi|_{Z_n}(x) + \psi|_{Z_n}(x) \quad \square$$

exercise - Assume that the following sequence is exact:

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{q} F \longrightarrow 0$$

Show that the following are equivalent:

1. $B \cong A \oplus F$
2. There is a section $s : F \to B$, such that $q \circ s = 1_F$
3. There is a section $s' : B \to A$, such that $s' \circ \iota = 1_A$

Proof - $(1) \implies (2), (3)$ is obvious. Now assume (2), then $s(F) \subset B$ is a submodule, let $x \in s(F) \cap \iota(A)$, then $x = s(y)$ for $y \in F$, it follows that $0 = q(x) = qs(y) = y$ and hence $s(y) = 0$. Now let $x \in B$, then $q(x - sq(x)) = q(x) - q(x) = 0$, hence $x - sq(x) \in \ker q = \text{Im } \iota$, hence $x = sq(x) + \iota(y) \in s(F) + \iota(A)$. Now assume (3), we may define $\varphi : B \to A \oplus F$ via $\varphi : b \mapsto (s'(b), q(b))$, to see that the map is surjective, let $(a, f) \in A \oplus F$, then choose some $x \in q^{-1}(f)$, and consider

$$\varphi(\iota(a - s'(x)) + x) = (s'\iota(a - s'(x)) + s'(x), q\iota(a - s'(x)) + q(x)) = (a - s'(x) + s'(x), f) = (a, f)$$

To check injectivity, suppose that $\varphi(x) = 0$, then $q(x) = 0$ implying that $x \in \iota(A)$, so that $x = \iota(y)$, and $0 = s'(x) = s'\iota(y) = y$ so that $x = \iota(0) = 0$. $\square$

exercise - Suppose that F is free, and the following sequence is exact:

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{q} F \longrightarrow 0$$

Then the sequence splits.

proof - We may write $F = \oplus_I Re_i$, then for each $i \in I$, choose some $b_i \in q^{-1}(e_i)$, it follows that by the universal property of a free module we can define a map by taking $e_i \mapsto b_i$ and extending linearly. This defines a section hence the sequence is split (it satisfies condition 2 of the previous exercise [4]). $\square$

exercise - In the proof of the Universal coefficient theorem, we want to show that f is surjective. to do so it suffices to construct a section $s : \text{Hom}(H_n(C_*), G) \to H^n(C^*)$, such that $fs = 1$. Since B_{n-1} is free, and the following sequence is short exact, there is some $p : C_n \to Z_n$, such that $p|_{Z_n} = 1_{Z_n}$ by the equivalence of definitions of split exact.

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{d_n} B_n \longrightarrow 0$$

To define s, we let $\alpha \in \text{Hom}(H_n(C_*), G)$, then we may take $\overline{\alpha} := \alpha q$, so that the following commutes

$$\begin{tikzcd} C_n \arrow[d, "p"] & \\ Z_n \arrow[r, "q"] \arrow[d, "\overline{\alpha}"] & H_n(C_*) \arrow[dl, "\alpha"] \\ G & \end{tikzcd}$$

By commutativity, of the previous diagram (zero map) the following diagram commutes and the sequence is exact

$$\begin{tikzcd} & & C_n \arrow[d, "p"] & & \\ 0 \arrow[r] & B_n \arrow[r] \arrow[dr, "0"] & Z_n \arrow[r, "q"] \arrow[d, "\overline{\alpha}"] & H_n(C_*) \arrow[r] \arrow[dl, "\alpha"] & 0 \\ & & G & & \end{tikzcd}$$

We may define $s(\alpha) = [\overline{\alpha} \circ p] \in H^n(C^*)$, furthermore commutativity of the above diagram means that for $\overline{\alpha} = \overline{\alpha} \circ p|_{Z_n}$ we have that α satisfies the universal property, so that $fs(\alpha) = f([\overline{\alpha}p]) = \alpha$.
Show that s is a homomorphism

proof - Let $\alpha, \beta \in \text{Hom}(H_n(C_*), G)$, then

$$s(\alpha + \beta) = \overline{\alpha + \beta}p = (\alpha + \beta)qp = \alpha qp + \beta qp = \overline{\alpha}p + \overline{\beta}p = s(\alpha) + s(\beta) \quad \square$$

exercise - Show that if a sequence is split exact, then $\text{Hom}(-, G)$ acts as a (right) exact functor on the sequence.

proof - $\text{Hom}(-, G)$ is always left exact, so it suffices to show right exactness, i.e. given the split exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

Check that $\text{Hom}(B, G) \to \text{Hom}(A, G)$ is surjective in

$$\text{Hom}(A, G) \xleftarrow[f^*]{} \text{Hom}(B, G) \longleftarrow \text{Hom}(C, G) \longleftarrow 0$$

By split exactness, there is some $s : B \to A$, such that $sf = 1$. It follows that $\alpha s \in \text{Hom}(B, G)$, such that $f^*(\alpha s) = \alpha s f = \alpha 1 = \alpha$, this suffices to show surjectivity and hence exactness. $\square$

exercise - We have the split exact sequence of chain complexes

$$0 \longrightarrow \mathbf{Z}_* \longrightarrow \mathbf{C}_* \longrightarrow \mathbf{B}_{*-1} \longrightarrow 0$$

since the sequence is split, it dualizes to an exact sequence of cochain complexes,

$$0 \longrightarrow \mathbf{B}^{*-1} \longrightarrow \mathbf{C}^* \longrightarrow \mathbf{Z}^* \longrightarrow 0$$

By the snake lemma we get a long exact sequence

$$\cdots \longrightarrow Z^{n-1} \xrightarrow{\partial} B^{n-1} \longrightarrow H^n(\mathbf{C}^*) \longrightarrow Z^n \xrightarrow{\partial} B^n \cdots$$

Trace through the snake lemma to show that $\partial = \iota_{n-1}^*$

proof -

exercise - Let H and H' be modules with free resolutions F_* and F'_*, and $\alpha : H \to H'$ be a homomorphism. Define a chain map as follows: Let $e_0 \in F_0$, then define $\alpha_0 : e_0 \mapsto b_0$, where $b_0 \in f_0'^{-1}(\alpha f_0(e_0))$, more generally, take $\alpha_n : e_n \mapsto b_n \in f_n'^{-1}(\alpha_{n-1} f_0(e_n))$, it is immediate by definition that this is a chain map. Now let β_i be another chain map, given by taking possibly different choices of $c_n \in f_n'^{-1}(\alpha_{n-1} f_0(e_n))$ (here we abuse notation that $\alpha = \alpha_{-1}$).

proof - Since $\alpha = \alpha_{-1} = \beta_{-1}$ we should simply choose $h_{-1} = 0$, for the case of h_0, we exploit the fact that F_0 is free, so if $\alpha - \beta = d_1 h_0$ on the generating set, then they are equal on the entire module. So let e be a generator of one of the summands of F_0, then we may simply choose $h_0(e) \in f_1'^{-1}\{\alpha_0(e) - \beta_0(e)\}$ which is a map since F_0 is free. Now we may construct h_n based off of h_{n-1}, once again let e be one of the canonical generators of the free module F_n, then choose $h_n(e) \in f_{n+1}'^{-1}\{\alpha_n(e) - \beta_n(e) - h_{n-1} f_n(e)\}$, once again this is well defined since F_n is free, and it satisfies the property of a chain homotopy by construction. □

exercise - Classify all extensions of $\mathbb{Z}/(2)$ by $\mathbb{Z}/(2)$.

proof - To do so we compute $\mathrm{Ext}(\mathbb{Z}/(2), \mathbb{Z}/(2)) := H^1(F^*; \mathbb{Z}/(2))$, where F^* is a free resolution for $\mathbb{Z}/(2)$. We can take the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

Dualizing we get

$$0 \longleftarrow \mathbb{Z}/(2) \xleftarrow[\cdot 0]{} \mathbb{Z}/(2) \longleftarrow \mathbb{Z}/(2) \longleftarrow 0$$

The first cohomology group is $\frac{\ker \mathbb{Z}/(2) \to 0}{\mathrm{Im}\, 0} \cong \mathbb{Z}/(2)$, so there are two extensions. The extensions are $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ and $\mathbb{Z}/(4)$. It is immediate that these are both extensions, and are non-isomorphic. A second way to see these re the only extensions, is that any extension must have order 4, and these are the only two groups of order 4. □

exercise - When taking a group extension of G by M, i.e. some group E, such that the following is exact

$$0 \longrightarrow M \xrightarrow{j} E \xrightarrow{q} G \longrightarrow 0$$

we may take a set function $s : G \to E$, where $qs = 1_G$, but s is not necessarily a group homomorphism. There is a function $c : G \times G \to E$ which measures the defect of s from being a homomorphism,

$$c : (g_1, g_2) \mapsto s(g_1)s(g_2)s(g_1 g_2)^{-1}$$

Show that $s(1) = c(1, 1)$

proof - Trivial

$$c(1, 1) = s(1)s(1)s(1)^{-1} = s(1) \quad \square$$

exercise - In the previous question, we constructed a function c to measure the deviation of a section from a group law, such a c is determined uniquely by the group extension E. We would like to show that E uniquely determines some $[c] \in H^2(G; M)$, this will suffice to show there is an injection

$$\{E \mid E \text{ is an extension of } G \text{ by } M\} \hookrightarrow H^2(G; M)$$

It is immediate that $c(G\times G)\in\ker q$, so that we may identify c with $j^{-1}c : G\times G\to M$. We still havent specified what $H^2(G;M)$ is, consider the chain complex given by $C^n(G;M)$ as the set of set valued functions $G^n\to M$

$$0 \longrightarrow C^0(G;M)0 \xrightarrow{d_0} C^1(G;M) \xrightarrow{d_1} C^2(G;M) \xrightarrow{d_2} \cdots$$

Where we define

$$df(g_0,g_1,\dots,g_n)=f(g_1,g_2,\dots,g_n)+(-1)^{n+1}f(g_0,g_1,\dots,g_{n-1})+\sum_{k=1}^{n}(-1)^n f(g_0,\dots,g_{i-2},g_{i-1}g_i,g_{i+1},\dots,g_n)$$

in the case of $d_1 : C^1\to C^2$, we get $df(g_1,g_2)=f(g_1)+f(g_2)-f(g_1g_2)$, hence $\ker d$ is the collection of group homomorphisms $G\to M$. It is the case that $d^2=0$, so this defines a chain complex. Since c is uniquely determined by s, it will suffice to show that given another choice $s' : G\to E$ we have that c' differs from c only by a coboundary. *Show that $[c]=[c']$, equivalently for some $f : G\to M$ we have that $c'=c+df$*

proof - Since all of our sections map into $\ker q=\operatorname{Im} j$, we continue to refer to our maps as $G\to M$ by implicity composing with j^{-1}. Now we may define $f=s'-s$, so that $s'=f+s$, it follows that

$$c'(g_1,g_2)=(f(g_1)+s(g_1))+(f(g_2)+s(g_2))-(f(g_1g_2)+s(g_1g_2))=c(g_1,g_2)+df(g_1,g_2)$$

So that $[c]=[c']\in H^2(G,M)$ depends only on isomorphism class of E.

exercise - In the previous exercises we completed verifications to show that

$$\text{Extensions}\hookrightarrow H^2(G;M)$$

We want to show that this is a correspondence, let c be a cocycle, we want to show that c defines a group operation on the set $M\times G$. *Given a cocycle c define a group operation on $M\times G$.*

proof - We define the group law as $(x,y)(z,w)=(x+z+c(y,w),yw)$, (associativity is easy and painful to prove) the identity is $(-c(1,1),1)$, as proof

$$(m,x)(-c(1,1),1)=(m-c(1,1)+c(x,1),x)$$

and since c is a cocycle, we have

$$0=dc(x,1,1)=c(1,1)-c(x,1)-c(x1,1)+c(x,11)=c(1,1)-c(x,1)$$

So that the given identity is indeed the identity. The inverse is $(-c(x,x')-c(1,1)-m,x^{-1})$, which is verified below;

$$\begin{aligned}(m,x)(-c(x,x')-c(1,1)-m,x^{-1})&=(m-m-c(x,x^{-1})-c(1,1)+c(x,x^{-1}),xx^{-1})\\&=(-c(1,1),1)\quad\square\end{aligned}$$