1. It suffices to show that Spec(A) satisfies the finite intersection property (FIP) for closed sets, since if given any collection of closed sets $\{V_i\}_{i\in I}$ we have

$$\bigcap_{i\in I} V_i = \emptyset \implies \exists V_{i_1}, \ldots, V_{i_N}, \text{ such that } \bigcap_{j=1}^{N} V_{i_j} = \emptyset$$

Then given any collection $\{U_i\}_{i\in I}$ of open sets we have

$$\bigcup_{i\in I} U_i = \mathrm{Spec}(A) \iff \bigcap_{i\in I} U_i^c = \emptyset \implies \exists U_{i_1}^c, \ldots, U_{i_N}^c, \text{ such that } \bigcap_{j=1}^{N} U_{i_j}^c = \emptyset$$
$$\iff \left(\bigcup_{j=1}^{N} U_{i_j}\right)^c = \emptyset \iff \bigcup_{j=1}^{N} U_{i_j} = \mathrm{Spec}(A)$$

Now let $\{V_i\}_{i\in I}$ be a collection of closed sets, such that $\bigcap_{i\in I} V_i = \emptyset$, then by the characterization of Zariski closed sets, $V_i = V(S_i)$, for some $S_i \subset A$, and $\emptyset = \bigcap_{i\in I} V_i = V(\bigcup_{i\in I} S_i)$. This suffices to show that $\left\langle \bigcup_{i\in I} S_i \right\rangle = A$, since if $\left\langle \bigcup_{i\in I} S_i \right\rangle$ were a proper ideal of A, then there would exist some maximal ideal $\mathfrak{m} \supset \left\langle \bigcup_{i\in I} S_i \right\rangle$, and since maximal ideals are prime we would have $\mathfrak{m} \in V(\bigcup_{i\in I} S_i)$ which is impossible since it is empty. Since $\left\langle \bigcup_{i\in I} S_i \right\rangle = A$, there exist $\{s_k\}_{k=1}^n \subset \bigcup_{i\in I} S_i$ and $\{a_k\}_{k=1}^n \subset A$, such that $\sum_{k=1}^n a_k s_k = 1$, each s_k lies in some S_{i_k} which implies that $\left\langle \bigcup_{k=1}^{N} S_{i_k} \right\rangle = A$, in particular

$$\emptyset = V(A) = V\left(\bigcup_{k=1}^{N} S_{i_k}\right) = \bigcap_{k=1}^{N} V_{i_k}$$

This suffices to show that Spec(A) satisfies the FIP and is hence quasi-compact. □

2. First suppose that Nil(A) is prime, and let $V(S_1), V(S_2)$ be Zariski closed sets, such that $V(S_1) \cup V(S_2) = \mathrm{Spec}(A)$, then since Nil$(A)$ is prime it must be contained in one of the two closed sets, without loss of generality assume that Nil$(A) \subset V(S_1)$, then

$$S_1 \subset \mathrm{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

Implying that $P \in V(S_1)$ for all prime ideals $P \subset A$, but this is equivalent to $V(S_1) = \mathrm{Spec}(A)$, since $V(S_1), V(S_2)$ were arbitrary this suffices to show that Spec(A) is irreducible.

I will prove the converse using the contrapositive. Assume that Nil(A) is not prime, then there are $x, y \in A$, such that $x, y \notin \mathrm{Nil}(A)$, and $xy \in \mathrm{Nil}(A)$. It follows that

$$V((x)) \cup V((y)) = V((xy)) \supset V(\mathrm{Nil}(A)) = \mathrm{Spec}(A)$$

where $V(\mathrm{Nil}(A)) = \mathrm{Spec}(A)$ is proven in the previous part of the problem. So it will suffice to show that $V((x)), V((y)) \subsetneq \mathrm{Spec}(A)$ to conclude that Spec(A) is irreducible. Since

$$x, y \notin \mathrm{Nil}(A) = \bigcap_{\substack{P\subset A \\ P \text{ is a prime Ideal}}} P$$

there are prime ideals $x \notin P_x, y \notin P_y$, so that $P_x \notin V((x)), P_y \notin V((y))$ hence neither can be all of Spec(A). □

3. Lemma. *M is finitely generated implies M satisfies the aescending chain condition (ACC).* Assume that $M = \langle x_1, \ldots, x_n \rangle$, then for any chain of submodules, $(N_i)_I$ we have $\bigcup_I N_i = M$ implies that $N_j = M$ for some $j \in I$ (and hence all $i \geq j$).

Proof of Lemma.

(a)

Let $M \neq 0$ be a finitely generated A-module.

(b) Suppose for contradiction that $N \subsetneq \mathbb{Q}$ is maximal, then by the correspondance theorem $\mathbb{Q}/N$ has no proper submodules. Hence for any $0 \neq x \in \mathbb{Q}/N$ (there is always such an x since N is a proper submodule), it must be the case that $\langle x \rangle = \mathbb{Q}/N$, since $\mathbb{Q}/N$ is generated by a single element as a $\mathbb{Z}$ module, it must be a homomorphic image of $\mathbb{Z}$, furthermore since it is generated by any of its elements as a $\mathbb{Z}$ module, it must be a finite cyclic group- in other words $\mathbb{Q}/N \cong \mathbb{Z}/(p)$ for some prime p. Hence by the first isomorphism theorem, we have a surjective $\mathbb{Z}$ module homomorphism $\varphi : \mathbb{Q} \to \mathbb{Z}/(p)$, with $\ker \varphi = N$. I claim that φ is the zero map, contradicting that it is surjective, as proof, let $x \in \mathbb{Q}$, then $\frac{x}{p} \in \mathbb{Q}$, then

$$0 + (p) = p\varphi(\frac{x}{p}) + (p) = \varphi(p\frac{x}{p}) + (p) = \varphi(x) + (p)$$

Since x was arbitrary this shows that $\varphi(x) = 0$ for all $x \in \mathbb{Q}$