1. Each F-Automorphism of K is an extension of the embedding $F \to F^{\text{alg}}$ which is identity on F to K, and hence $n = \#G \le [K : F]_{\text{sep}} \le [K : F]$. To prove the opposite inequality, consider $\alpha_1, \dots, \alpha_m \in K$ such that $m > n$. Then the system of equations

$$\begin{aligned} &a_1\tau_1(\alpha_1) + a_2\tau_1(\alpha_2) + \cdots + a_m\tau_1(\alpha_m) = 0 \\ &\vdots \\ &a_1\tau_n(\alpha_1) + a_2\tau_n(\alpha_2) + \cdots + a_m\tau_n(\alpha_m) = 0 \end{aligned}$$

With n equations, and m unknowns. Then the equation must have a (non-zero) solution, we wish to show that it has a solution lying in F, which would prove that we cannot have more than n F-linearly independent elements of K. Suppose that $(b_1, \dots, b_m)$ is a solution with the most non-zero terms, WLOG we can take $b_1 \neq 0$, and further take $b_1 = 1$ by dividing the each b_i by b_1. Then for each τ_i, since $\tau_i(0) = 0$, and τ_i permutes the other τ_j since G is a group, we get that applying any τ_i to the system of equations yields another solution, with the same zero terms. But then $(1 - \tau_i(1), \dots, b_m - \tau_i(b_m))$ is another solution, with the same zero coordinates as b, along with the first coordinate being 0 which by our minimality assumption implies that this is the zero vector. Since this holds for each τ_i, it follows that $\tau_i(b_j) = b_j$ for each i, j, so that each b_j is fixed by G and thus lies in F, hence the first equation gives us that $\alpha_1, \dots, \alpha_m$ are F-linearly dependent, so no F-linearly independent set of K may have cardinality greater than n, i.e. $[K : F] \le n$. Since we have proven both inequalities, $[K : F] = n$.

2. Since K/F is finite, we may write it as $K = F(\alpha_1, \dots, \alpha_n)$. It is immediate that since $\{\alpha_1, \dots, \alpha_n\} \subset K \subset KL$, that we can write $KL = L(\alpha_1, \dots, \alpha_n)$, since this field contains F and each α_i it must contain K. Since each α_i satisfies a polynomial with coefficients in $F \subset L$, we know that KL/L is algebraic. To show that its seperable, note that $\min(\alpha_i; L) | \min(\alpha_i; F)$, where $\min(\alpha_i; F)$ contains no repeated roots, proving that each α_i is seperable over L. Finally, since K/F is normal of finite degree, we know that K is the splitting field of some polynomial $f \in F[x]$, it is immediate that $L(\alpha_1, \dots, \alpha_n)$ is the splitting field of f over L, so that KL/L is normal and hence Galois.

Consider the map $\pi : \text{Gal}(KL/L) \to \text{Gal}(K/(K \cap L))$, defined by $\sigma \mapsto \sigma|_K$. It is clear that this map is well defined, and satisfies the homomorphism properties. To check this is an isomorphism, suppose that $\sigma \in \ker \pi$, then $\sigma|_K = 1$, hence $\sigma(\alpha_i) = \alpha_i$ for each i, furthermore σ fixes L by definition. It follows that for any $x \in KL$, we have $x = \sum_i \left(\ell_i \prod_j \alpha_j\right)$, so that

$$\sigma(x) = \sigma\left(\sum_i \left(\ell_i \prod_j \alpha_j\right)\right) = \sum_i \left(\sigma(\ell_i) \prod_j \sigma(\alpha_j)\right) = x$$

implying that $\sigma = 1$, so that this map is injective. To show surjectivity, let $\tau \in \text{Gal}(K/(K \cap L))$, define $\sigma(\alpha_i) = \tau(\alpha_i)$, this is a well defined extension of the identity map on L, since if α_i, α_j are conjugate over $K \cap L$, then they are conjugate over L. This can be seen since

$$\min(\alpha; L) | \min(\alpha; K \cap L) \text{ and } \min(\alpha; L) = (x - \alpha)(x - \beta_1) \cdots (x - \beta_k)$$

for $\beta_i \in K$, then the coefficients of $\min(\alpha; L)$ are the symmetric polynomials in $\alpha, \beta_1, \dots, \beta_k$ so that they also lie in K, so that in particular the coefficients lie in $K \cap L$, this implies that $\min(\alpha; L)$ is a polynomial with coefficients in $K \cap L$ which is satisfied by α, implying that

$\min(\alpha; K \cap L) | \min(\alpha; L)$, so that in particular they are equal. Then by construction, we get $\sigma|_K = \tau$ proving surjectivity. Since this is an ismomorphism between the two Galois groups it is also a bijection, so in particular

$$[KL : L] = \#\mathrm{Gal}(KL/L) = \#\mathrm{Gal}(K/K \cap L) = [K : K \cap L]$$

3. First note that

$$\bigcap_{\sigma \in G} \sigma H \sigma^{-1} = \cap_{\sigma \in G} \mathrm{Gal} K/\sigma(L)$$

This follows since $\tau \mapsto \sigma\tau\sigma^{-1}$ is a bijection from H to $\mathrm{Gal} K/\sigma(L)$.

We first show that $\mathrm{Gal}(K/N) \subset \bigcap_{\sigma \in G} \mathrm{Gal}(K/\sigma(L))$. Proof being, since $N \supset L$ is normal, for any $\sigma \in G$, we have $\sigma(L) \subset \sigma(N) = N$. Thus if $\tau|_N = 1$, then $\tau|_{\sigma(L)} = 1$ for any $\sigma \in G$. This suffices to show that $\tau \in \mathrm{Gal}(K/N)$ implies $\tau \in \bigcap_{\sigma \in G} \mathrm{Gal}(K/\sigma(L))$.

Now we show the other inclusion, namely $\mathrm{Gal}(K/N) \supset \bigcap_{\sigma \in G} \mathrm{Gal}(K/\sigma(L))$. Since N is the compositum $\langle \sigma(L) \rangle_{\sigma \in G}$ (proven below), we have that $\tau \in \cap_{\sigma \in G} \mathrm{Gal} K/\sigma(L)$, then $\tau|_N = 1$, since for any α in N, we can write α as a finite sum/product/quotient of elements in $\bigcup_{\sigma \in G} \sigma(L)$, each of which is fixed by τ, so that τ fixes α by the homomorphism property. It follows that $\tau \in \mathrm{Gal}(K/N)$ proving that indeed $N = \cap_{\sigma \in G} \mathrm{Gal} K/\sigma(L) = \bigcap_{\sigma \in G} \sigma H \sigma^{-1}$.

Proof of $N = \langle \sigma(L) \rangle_{\sigma \in G}$: the right to left inclusion is obvious by normality, then by definition of normal closure, it will suffice to show that $\langle \sigma(L) \rangle_{\sigma \in G}$ is normal. Let $\tau \in G$, then we have that

$$\tau(\langle \sigma(L) \rangle_{\sigma \in G}) = \langle \tau\sigma(L) \rangle_{\sigma \in G} = \langle \sigma(L) \rangle_{\sigma \in G}$$

The second equality follows since $\tau : G \to G$ bijectively. The first equality follows from if α in $\langle \sigma(L) \rangle_{\sigma \in G}$, we can write α as a finite sum/product/quotient of elements in $\bigcup_{\sigma \in G} \sigma(L)$, by the homomorphism property, we apply τ to each element in the sum/product/quotient so that α is a finite sum/product/quotient of elements in $\bigcup_{\sigma \in G} \tau\sigma(L)$

4. Since K/F is Galois, and $K \supset L_0 \supset F$, we have K/L_0 is galois, with galois group $N(H)$. Then since H is normal in $N(H)$, we have L/L_0 is galois. Furthermore, suppose that $L \supset M \supset F$, with L/M Galois implying that H is normal in $\mathrm{Gal}(K/M)$. Then since the normalizer is the largest subgroup R of G, such that $H \subset R$ is normal, we get that $\mathrm{Gal}(K/M) \subset N(H)$, implying that $M \supset L_0$ as desired.

5. We can define the map $\varphi : \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} (\mathbb{Z}/4\mathbb{Z})$ as $\varphi(1) : x \mapsto -x$, this is a well defined automorphism, since $\varphi(1)^2 = \mathbf{1}_{\mathbb{Z}/4\mathbb{Z}} = \varphi(0) = \varphi(1+1)$. Any element $x \in D_4$ can be written in the form of $\sigma^i\tau^j$ using the relation $\sigma\tau = \tau\sigma^{-1}$. So define the map

$$\psi : D_4 \to \mathbb{Z}/4\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$$
$$\sigma^i\tau^j \mapsto (i, j)$$

is an isomorphism. $\mathbf{1} \mapsto (0,0)$ is immediate. And (here I deal with both possible cases $j = 1, 0$ seperately)

$$\psi(\sigma^i\tau\sigma^k\tau^\ell) = \psi(\sigma^{i-k}\tau^{1+\ell}) = (i-k, 1+\ell) = (i + \varphi(1)(k), 1+\ell) = (i,1)(k,\ell) = \psi(\sigma^i\tau)\psi(\sigma^k\tau^\ell)$$
$$\psi(\sigma^i\tau^0\sigma^k\tau^\ell) = \psi(\sigma^{i+k}\tau^\ell) = (i+k, \ell) = (i + \varphi(0)(k), 0+\ell) = (i,0)(k,\ell) = \psi(\sigma^i\tau^0)\psi(\sigma^k\tau^\ell)$$

This proves that ψ is a homomorphism, and

$$\psi(\sigma^i\tau^j) = (0,0) \iff i \equiv 0 \bmod 4 \text{ and } j \equiv 0 \bmod 2 \iff \sigma^i\tau^j = \mathbf{1}$$

proving that $\ker\psi = \mathbf{1}$. Then since $\#D_4 = \#\mathbb{Z}/4\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$ and the map is injective, it must also be surjective.