1. for $p=2$ we have splitting field $\mathbb{Q}(\sqrt{2})$ which is a degree 2 extension. Now let $p>2$, then the splitting field is $k:=\mathbb{Q}(2^{1/p},\zeta_p 2^{1/p},\dots,\zeta_p^{p-1}2^{1/p})$, then $\frac{\zeta_p 2^{1/p}}{2^{1/p}}=\zeta_p\in k$, hence $k\subset\mathbb{Q}(\zeta_p,2^{1/p})$, and the reverse inclusion is obvious so they are equal. It follows that the degree of this extension is $p(p-1)$, since ζ_p has minimum polynomial $x^{p-1}+x^{p-2}+\cdots+1$ of degree $p-1$ in $\mathbb{Q}[x]$. And $2^{1/p}$ has minimum polynomial x^p-2 (Gauss' lemma to reduce to $\mathbb{Z}$, then irreducible from Eisenstein) of degree p. Implying that $p,p-1|[k:\mathbb{Q}]$, or equivalently $p(p-1)|[k:\mathbb{Q}]$. For equality, note that $[\mathbb{Q}(\zeta_p,2^{1/p}):\mathbb{Q}(\zeta_p)]\leq p$, since $2^{1/p}$ still satisfies x^p-2, so that

$$p(p-1)\leq[k:\mathbb{Q}]=[\mathbb{Q}(\zeta_p,2^{1/p}):\mathbb{Q}(\zeta_p)][\mathbb{Q}(\zeta_p):\mathbb{Q}]\leq p(p-1)$$

Proof that $x^{p-1}+x^{p-2}+\cdots+1$ is irreducible: We first note that $f(x)$ is irreducible in $\mathbb{Q}$ when it is irreducible in $\mathbb{Z}$ by Gauss' lemma. Then $f(x)$ is irreducible if and only if $f(x+1)$ is irreducible, one way to see this is $F:\mathbb{Z}[x]\to\mathbb{Z}[x],x\mapsto x+1$ is a $\mathbb{Z}$-module automorphism. Hence

$$f(x)=g(x)h(x)\iff f(F(x))=F(f(x))=F(g(x))F(h(x))=g(F(x))h(F(x))$$

Then irreducibility follows from Eisensteins criterion after the following computation;

$$\begin{aligned}\sum_{k=0}^{p-1}(x+1)^k=\frac{(x+1)^p-1}{(x+1)-1}&=x^{-1}\sum_{k=1}^{p}\binom{p}{k}x^k=\sum_{1}^{p}\binom{p}{k}x^{k-1}\\&=x^{p-1}+\sum_{k=1}^{p-2}a_kx^k+p,\text{ where } p|a_k\end{aligned}$$

2. I claim that $z:=\zeta_3+2^{1/3}$ is such a number, it will suffice to show that $\deg(\min(z,\mathbb{Q}))=6=[\mathbb{Q}(\zeta_3,2^{1/3}):\mathbb{Q}]$. But then since $\mathbb{Q}(z)$ is a subextension it has degree 2,3 or 6 over $\mathbb{Q}$, so it is sufficient to show that $\deg\min(z,\mathbb{Q})>3$. Then we can the take basis for $\mathbb{Q}(\zeta_3,2^{1/3})/\mathbb{Q}$ to be $\{2^{1/3}\zeta_3,2^{2/3}\zeta_3,2^{1/3}\zeta_3^{-1},2^{2/3}\zeta_3^{-1},\zeta_3,\zeta_3^{-1}\}$. We can see this is a basis, since it contains six elements, and $1=2(\zeta_3+\zeta_3^{-1})$ is sufficient to check that it spans $\mathbb{Q}(\zeta_3,2^{1/3})$ since this allows us to write the rest of the extension in terms of the basis. First we compute the powers of z up to 3.

$$\begin{aligned}&z^2=\zeta_3^{-1}+2\zeta_3 2^{1/3}+2^{2/3} && z^3=3(\zeta_3^{-1}2^{1/3}+\zeta_3 2^{2/3}+2(\zeta_3+\zeta_3^{-1}))\\&1=\zeta_3+\zeta_3^{-1} && z=\zeta_3+2^{1/3}\zeta_3+2^{1/3}\zeta_3^{-1}\end{aligned}$$

Now consider any degree ≤ 3 polynomial evaluated at z,

$$\begin{aligned}p(z)&=az^3+bz^2+cz+d\\&=(2b+c)(2^{1/3}\zeta_3)+3a2^{2/3}\zeta_3+(3a+c)(2^{1/3}\zeta_3^{-1})+(6a+c+d)(\zeta_3)+(6a+b+d)(\zeta_3^{-1})\end{aligned}$$

Then by linear independence of basis elements, $p(z)$ is equal to zero if and only if

$$\begin{aligned}2b+c&=0\\3a+c&=0\\6a+c+d&=0\\6a+b+d&=0\end{aligned}$$

then from the second 2 equations we get $c=-b$, implying from the first equation that $c=b=0$ which implies in the second equation $a=0$, so that $a=b=c=0$, which means d must be zero as well. So that $\deg\min(z,\mathbb{Q})>3$, which implies it must be 6 and we are done.

3. $[K : \mathbb{Q}] \in \{6, 3, 2, 1\}$, since $[K : \mathbb{Q}]|6$. For a more detailed explanation, the largest irreducible factor of $f(x)$ may have degree 1,2 or 3. The first case is the trivial case where f splits over $\mathbb{Q}$, so that $K = Q$ is an extension of degree 1. In the second case $[K : \mathbb{Q}]$ has degree 2, since adjoining a root α of a quadratic polynomial gives a field extension of degree 2 containing the other root. Finally if f itself is irreducible, then $[K : \mathbb{Q}]$ may have degree 3 in the case where the extension is simple, or degree 6 otherwise no other values are possible, since when a root of degree 3 is adjoined, then the polynomial splits in the new field, or is a quadratic, so that the roots are contained in a degree 2 extension of the degree 3 extension, which has degree 6 over the original field.

Examples:

$$[K : \mathbb{Q}] = 1 \quad f(x) = (x-1)^3 \quad K = \mathbb{Q}$$
$$[K : \mathbb{Q}] = 2 \quad f(x) = (x-1)(x^2-2) \quad K = \mathbb{Q}(\sqrt{2})$$
$$[K : \mathbb{Q}] = 3 \quad f(x) = (x-(\zeta_7+\zeta_7^{-1}))(x-(\zeta_7^3+\zeta_7^4))(x-(\zeta_7^2+\zeta_7^5)) = x^3+x^2-2x-1 \quad K = \mathbb{Q}(\zeta_7+\zeta_7^{-1})$$
$$[K : \mathbb{Q}] = 6 \quad f(x) = x^3-2 \quad K = \mathbb{Q}(2^{1/3}, \zeta_3)$$

4. Let L be the algebraic closure containing N and L' be an algebraic closure of N', then we have the embedding by the identity $E \to L'$. By the extension theorem, there exists an extension $\sigma : N \to L'$ which is identity on E and thus F. We first check that ${}^\sigma N$ is normal, if N is the splitting field of polynomials $\{f_i\}_i$, then ${}^\sigma N$ is the splitting field of $\{{}^\sigma f_i\}_i$, since if f_i has roots $\{\alpha_j\}_j$, then $\{{}^\sigma \alpha_j\}_j$ are the roots of ${}^\sigma f_i$. If ${}^\sigma N$ werent the normal closure of E in L', then there would exist $E \subset N'' \subsetneq {}^\sigma N$ normal. Then since σ is invertible on its image, we could take $E \subset {}^{\sigma^{-1}} N'' \subsetneq N$, where once again the image would be normal, but this contradicts N being the normal closure. Hence ${}^\sigma N$ is the normal closure of E in L', this implies that $E \subset {}^\sigma N \cap N' = {}^\sigma N$ since the intersection is normal. Hence $\sigma : N \to N'$ is an F homomorphism.

If $\sigma : N \to N'$ is an F homomorphism, then it must be injective. From the construction above we have identity on E, so that $E \subset {}^\sigma N \subset N'$, but then by hypothesis, there are no subextensions implying that $\sigma(N) = N'$ is an F-isomorphism.

5. E is contained in some algebraicly closed field L, then σ can be seen as an embedding from K into L, so that there exists an embedding $\tau : E \to L$ extending σ by the extension theorem. Since E is normal this embedding is an automorphism on E (one of the three equivalent conditions for normal extensions - NOR 1 in Lang).