Hi Paige, you commented last time that my font size was small - I have increased it. Please let me know before the last homework is due if you want me to make the font on that homework even larger.
$\sim$ Tighe

1. (a) Density is immediate, lemma 2.56 of notes states that simple functions (with finite support) are dense in L^p, any simple function with finite support is in $L^1 \cap L^\infty$. To see that $L^1 \cap L^\infty \subset L^p$ for every $p \geq 1$, we first fix $p > 1$ and $f \in L^1 \cap L^\infty$, since $f \in L^1$ it follows that $\mu\{x \in X | \, |f(x)| \geq 1\} = m < \infty$, so that

$$\begin{aligned}\int_X |f|^p &\leq \int_{\{x\in X||f(x)|\geq 1\}} ||f||_\infty^p + \int_{\{x\in X||f|<1\}} |f|^p \\ &\leq m\,|f|_\infty^p + \int_{\{x\in X||f|<1\}} |f| \\ &\leq m||f||_\infty^p + ||f||_1 < \infty\end{aligned}$$

Implying that $f \in L^p$. $\square$

(b) Convergence in L^p implies that $\lim_{n\to\infty}||f - f_n||_p^p = 0$. For $k, n \in \mathbb{Z}_{>0}$ define

$$E_{n,k} := \{x \in X| \mid |f(x) - f_n(x)| > 2^{-k}\}$$

By L^p convergence we find that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$ (*See below*), and hence there exists some N_k such that for any $n \geq N_k$, $\mu(E_{n,k}) < 2^{-k}$. Consider the set $F := \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty E_{N_k,k}$, for any point in $x \in F^c$ there exists some j, such that $x \notin \bigcup_{k=j}^\infty E_{N_k,k}$, it follows that $|f(x) - f_{N_k}(x)| \leq 2^{-k}$ for any $k \geq j$ and hence $f_{N_k}(x) \to f(x)$ on F^c, since for any j we have

$$\mu(F) \leq \mu\left(\bigcup_{k=j}^\infty E_{N_k,k}\right) \leq \sum_{k=j}^\infty 2^{-k} = 2^{1-j}$$

it follows that $\mu(F) = 0$, hence $f_{N_k} \to f$ almost everywhere. $\square$

$\mathbf{L^p}$ convergence implies $\lim_{\mathbf{n}\to\infty} \mu(\mathbf{E_{n,k}}) = \mathbf{0}$: Suppose for contradiction that for some k we have some $\epsilon > 0$, such that for any $N \in \mathbb{Z}_{>0}$ there is some $n > N$ with $\mu(E_{n,k}) \geq \epsilon$, then we have for any $N \in \mathbb{Z}_{>0}$ there is some larger n, such that

$$||f - f_n||_p^p \geq \int_{E_{n,k}} |f - f_n|^p \geq \mu(E_{n,k})2^{-k} \geq \epsilon 2^{-k}$$

contradicting $\lim_{n\to\infty}||f - f_n||_p^p = 0$. $\square$

(c) Define the set in the problem description as S, now suppose that $f \in L^p$, such that a sequence $(f_n)_1^\infty$ in S converges to f in L^p, by part (b) there is a subsequence f_{n_k} converging pointwise almost everywhere. We apply Fatou's lemma to conclude the following,

$$\int |f|^q = \int \liminf |f_{n_k}|^q \leq \liminf \int |f_{n_k}|^q \leq 1 \implies f \in S \quad \square$$

2. (a) By the closed graph theorem, it will suffice to show that the graph of T is closed. Suppose that $T(f_n) \to h$ in L^q, where $f_n \to f$ in L^p, we want to show that $h = fg$. Since $f_n g \to h$ in L^q, by 1(b) there is a pointwise almost everywhere convergent subsequence $f_{n_k} g \to h$, but $f_{n_k} g \to fg$ pointwise, so that $fg = h$ a.e. equivalently $[fg] = [h]$ as desired. $\square$

(b) Since T is bounded, let M such that $||T(f)||_q \leq M||f||_p$ for any $f \in L^p$. First suppose that $p = \infty$, then $||g||_q \leq M||1||_\infty = M < \infty$, so that $g \in L^q$. Now suppose that $p < \infty$ and $q < p$, since $|f|^q\,|g|^q \in L^1$ for any $f \in L^p$, it follows that for any $h \in L^{p/q}$ we have $|h|^{1/q} \in L^p$, and hence $|h|\,|g|^q \in L^1$, where

$$\phi_g(h)^{1/q} = \left(\int |h|\,|g|^q\right)^{1/q} = \left(\int \left(|h|^{1/q}\right)^q |g|^q\right)^{1/q} \leq M\left(\int |h|^{p/q}\right)^{1/p} = M||h||_{p/q}^{1/q}$$
$$\implies \phi_g(h) \leq M||h||_{p/q}$$

This implies that $\phi_{|g|^q}(h) \in (L^{p/q})^*$, and hence by theorem 2.57 of notes, $|g|^q \in L^{(1-\frac{q}{p})^{-1}}$, so that

$$\int |g|^{q(\frac{p}{p-q})} < \infty \implies g \in L^{\frac{pq}{p-q}}$$

Finally if $q = p < \infty$, the solution is the same as the previous case up until $\phi_{|g|^q} \in (L^1)^*$, which implies that (once again by Theorem 2.57) $|g|^q \in L^\infty$, it follows immediately from this fact that $g \in L^\infty$. $\square$

3. (a) We first show linearity which is clear from limit laws, let $a \in \mathbb{C}$ and $x, x' \in \ell^\infty$, then

$$\lim_{n\to\infty} ax_n + x_n' = a\lim_{n\to\infty} x_n + \lim_{n\to\infty} x_n'$$

The equality and existence of these limits follows from limit laws (Rudin- Principles of Mathematical Analysis 3.3). To see that it is bounded, note that for any n, $|x_n| \leq ||x_n||_\infty$, and hence $|f(x)| = |\lim_{n\to\infty} x_n| = \lim_{n\to\infty} |x_n| \leq ||x_n||_\infty$. To see that the operator norm is one, it will suffice to show there is some $x \in \ell^\infty$ with norm 1, such that $|f(x)| = 1$, but this is immediate since $\mathbf{1} := (1, 1, \ldots) \in \ell^\infty$, and $f(\mathbf{1}) = \lim_{n\to\infty} 1 = 1$. $\square$

(b) Since F is an extension of f, it will suffice to show that there is no $a \in \ell^1$, such that $\phi_a|_M = f$. Suppose for contradiction that such an a exists, if $a = 0$, then we are done since in that case $\phi_a(\mathbf{1}) = 0 \neq 1 = f(\mathbf{1})$. If $a \neq 0$, then there is some $n \geq 1$, such that $a_n \neq 0$, then consider $x \in M$ where $x_j = \begin{cases} 1 & j = n \\ 0 & \text{else} \end{cases}$, then $\phi_a(x) = a_n \neq 0$, but $f(x) = \lim_{j\to\infty} x_j = 0$, hence $f(x) \neq \phi_a(x)$. $\square$

4. (a) Assume that $R_f \neq \emptyset$, otherwise we are done. Now suppose that $z \in \mathbb{C}$, such that there is a sequence of points $(x_n)_1^\infty \subset R_f$, such that $x_n \to z$. Now let $\epsilon > 0$, then by convergence, there is some N, such that $|z - x_N| < \epsilon/2$, furthermore, for any $y \in N_{\epsilon/2}(x_N)$, we have that $|y - z| \leq |y - x_n| + |x_n - z| < \epsilon$, so we can conclude that

$$\{w \in X \mid |f(w) - x_N| < \epsilon/2\} \subset \{w \in X \mid |f(w) - z| < \epsilon\}$$

and hence

$$\mu\{w \in X \mid |f(w) - z| < \epsilon\} \geq \mu\{w \in X \mid |f(w) - x_N| < \epsilon/2\} > 0$$

since ϵ was arbitrary this suffices to show that $z \in R_f$ implying that the set is closed. $\square$

(b) Suppose that μ is a nonzero measure, then $\mu(X) = m > 0$, let $S_k := \{x + iy \in \mathbb{C} \mid -k \leq x, y \leq k\}$ denote the rectangle of sidelength $2k$ centered at the origin in $\mathbb{C}$, by continuity from above, $\lim_{k\to\infty} \mu f^{-1}(S_k) = \mu(X)$, hence for some k we have $\mu f^{-1}(S_k) \geq m/2 > 0$. Now given a rectangle S centered at a point $(p_x + ip_y)$, define rectangles S^1, S^2, S^3, S^4, such that

$$\begin{aligned} S^1 &= S \cap \{x+iy \mid x-p_x \geq 0 \text{ and } y-p_y \geq 0\} \\ S^2 &= S \cap \{x+iy \mid x-p_x \leq 0 \text{ and } y-p_y \geq 0\} \\ S^3 &= S \cap \{x+iy \mid x-p_x \leq 0 \text{ and } y-p_y \leq 0\} \\ S^4 &= S \cap \{x+iy \mid x-p_x \geq 0 \text{ and } y-p_y \leq 0\} \end{aligned}$$

Now construct a sequence of rectangles as follows, define $S_1 = S$, for each n, if $\mu f^{-1}(S_n) > 0$, then

$$\sum_{i=1}^{4} \mu f^{-1}(S_n^i) \geq \mu(S_n)$$

so that for some i, we once again have $\mu f^{-1}(S_n^i) > 0$, then define $S_{n+1} = S_n^i$. Since each S_i is compact, we may apply the finite intersection property to find that $\bigcap_1^\infty S_n \neq \emptyset$, and hence we have some $z \in \bigcap_1^\infty S_n$, I claim that $z \in R_f$. Let $\epsilon > 0$, then since $z \in S_n$ for every n, and $\lim_{n\to\infty} \operatorname{diam} S_n = 0$ there is some n, such that $S_n \subset N_\epsilon(z)$, so that

$$\{w \in X \mid |f(w) - z| < \epsilon\} \supset f^{-1}(S_n) \implies \mu\{w \in X \mid |f(w) - z| < \epsilon\} \geq \mu f^{-1}(S_n) > 0$$

since $\epsilon > 0$ was arbitrary, we know that $z \in R_f$. □

(c) Since R_f is closed, it will suffice to show that it is bounded, so we may show that $R_f \subset F := \{z \in \mathbb{C} \mid |z| \leq ||f||_\infty\}$, suppose that $y \in \mathbb{C}$, such that $|y| > ||f||_\infty$, then since F is closed, there is some $\epsilon > 0$, such that $N_\epsilon(y) \subset F^c$. By definition of $||\cdot||_\infty$ we have $\mu f^{-1}(F^c) = 0$, and hence

$$\mu(\{w \in X \mid |y - f(w)| < \epsilon\}) \leq \mu f^{-1}(F^c) = 0$$

and hence $y \notin R_f$ so that $R_f \subset F$ is bounded. The fact that $\sup_{R_f} |z| = \max_{R_f} |z|$ is immediate from compactness, so it suffices to show that $||f||_\infty = \sup_{R_f} |z|$, let $\epsilon > 0$, then

$$E := f^{-1}\{z \in \mathbb{C} \mid ||f||_\infty - \epsilon \leq |z|\}, \quad \mu(E) > 0 \text{ by definition of } ||f||_\infty$$

Since a measurable subset of a measure space defines a measure space, i.e. $(E, \mathcal{M}|_E, \mu|_E)$ is a measure space with $\mu|_E(E) = \mu(E) > 0$ we can simply apply part (b) to this second measure space to conclude that $\emptyset \neq R_{f|_E} \subset R_f$ (where the subset relation is obvious by definition of the essential range), and hence there is some $z \in R_f$ with $|z| \geq ||f||_\infty - \epsilon$, since ϵ was arbitrary and the opposite inequality $\sup_{R_f} |z| \leq ||f||_\infty$ was proven above. We conclude that $||f||_\infty = \sup_{R_f} |z|$, and by compactness we can pass the supremum to the maximum over the set as stated previously. □

5. (a) Let Y denote the collection of linear independent subsets of X, such that for any $e \in Y$ we have $||e|| = 1$ ordered by inclusion. If $X = 0$, then we can take $Y = \emptyset$ and we are done trivially so assume not. It follows that there is some nonzero vector $u \in X$, then $\{\frac{u}{||u||}\} \in Y \neq \emptyset$. Now suppose that C is a chain in Y, I claim that $\bigcup_{E_\alpha \in C} E_\alpha$ is an upper bound for C in Y. To check this it will suffice to show that indeed $\bigcup_{E_\alpha \in C} E_\alpha \in Y$. First let $e \in \bigcup_{E_\alpha \in C} E_\alpha$, then $e \in E_\alpha$ for some α implies that $||e|| = 1$, now suppose that for $\{a_i\}_1^n \subset K$, $\{e_i\}_1^n \subset \bigcup_{E_\alpha \in C} E_\alpha$

we have $\sum_1^n a_i e_i = 0$, it follows that each $e_i \in E_{\alpha_i}$, and since C is a chain we can pick $E_{\alpha_j} = \max\{E_{\alpha_i}\}_{i=1}^n$, then $\sum_1^n a_i e_i = 0$ in the linearly independent set E_{α_j} implying that $a_i = 0$ for each i, this suffices to show that $\bigcup_{E_\alpha \in C} E_\alpha \in Y$. Now we may apply Zorn's lemma to conclude that Y has a maximal element, suppose for contradiction the maximal element $E \in Y$ is not a basis, then there must be some $u \in X \setminus \langle E \rangle$, $0 \in \langle E \rangle$ specifies $u \neq 0$, so we can consider $E \cup \{\frac{u}{||u||}\}$, it is clear that all elements of this new set have norm 1. Since $E \cup \{\frac{u}{||u||}\} \supset E$ which is maximal, it must not be linearly independent, so there exist $\{a_i\}_1^n \subset K$ not all zero and $\{e_i\}_1^{n-1} \subset E$, such that $a_n \frac{u}{||u||} + \sum_1^{n-1} a_i e_i = 0$, linear independence of E implies $a_n \neq 0$, hence $u = \sum_1^{n-1} ||u|| a_n^{-1} a_i e_i$, so that $u \in \langle E \rangle$ a contradiction. Thus we may conclude that E is an algebraic basis with all elements having norm 1. □

(b) Using part (a), let $\{e_\alpha\}_A$ be an algebraic basis for X such that for each $\alpha \in A$ we have $||e_\alpha|| = 1$. Choose some countable subset $\{e_i\}_1^\infty$. Now we can use the universal property of the basis to define f on basis elements and extend linearly, define f as follows:

$$f : \begin{cases} e \mapsto 0 & e \in \{e_\alpha\}_A \setminus \{e_i\}_1^\infty \\ e_i \mapsto i \end{cases}$$

Then it is clear that f is unbounded since for any $M \in \mathbb{R}$, we can pick some $N \in \mathbb{N}$, such that $N > M$, it follows that $f(e_N) = N > M||e_N||$ and since M was arbitrary f is unbounded and hence not continuous. □

(c) Assume for the contradiction that I is countable, then by part (b), we have some linear functional f on X which is not continuous, hence not bounded. Define $E_n = \{x \in X \mid |f(x)| \leq n||x||\}$, to see that $\bigcup_{n=1}^\infty E_n = X$, we first note that $0 \in E_1$, now let $x \in X \setminus \{0\}$, then $||x|| = r > 0$, then choosing $n \geq \frac{|f(x)|}{r}$ we have $x \in E_n$. Now fix $M \in \mathbb{Z}_{>0}$, and assume that E_M contains a non-empty open set U. Let $x \in U$, since U is open we know that $U - x$ is an open set containing the origin, hence there is some $\epsilon > 0$, such that for any w with $||w|| = 1$ we have $\epsilon w \in U - x$. Now let $v \in X$, then $\epsilon \frac{v}{||v||} \in U - x$, hence $x + \epsilon \frac{v}{||v||} \in U$, it follows that

$$\begin{aligned} \frac{\epsilon}{||v||} |f(v)| - |f(x)| &\leq \left| f(x + \epsilon \frac{v}{||v||}) \right| \leq M||x + \epsilon \frac{v}{||v||}|| \leq M||x|| + \epsilon M \\ \implies |f(v)| &\leq ||v|| \left(M \frac{||x||}{\epsilon} + M + \frac{|f(x)|}{\epsilon} \right) \end{aligned}$$

since v was arbitrary and $\left(M \frac{||x||}{\epsilon} + M + \frac{|f(x)|}{\epsilon} \right)$ is independent of v, this implies that f is bounded which is a contradiction, hence E_M does not contain any non-empty open sets, and since M was arbitrary this implies that E_n is nowhere dense for any n. This implies that $\bigcup_1^\infty E_n$ is a countable union of nowhere dense sets, so that $\bigcup_1^\infty E_n = X$ contradicts the Baire Category theorem, thus I cannot be countable. □