1. Let (a,b) be an arbitrary open interval, then $(-\infty,b)\in\mathcal{B}$, furthermore

$$\begin{aligned}\{(-\infty,a-1/n)\}_{n\in\mathbb{N}}\subset\mathcal{B} &\implies \{[a-1/n,\infty)\}_{n\in\mathbb{N}}\subset\mathcal{B}\\ &\implies (a,\infty)=\bigcup_{\mathbb{N}}[a-1/n,\infty)\in\mathcal{B}\\ &\implies (a,b)=(a,\infty)\cap(-\infty,b)\in\mathcal{B}\end{aligned}$$

Now since each open interval is an open set we have that $\mathcal{B}\subset\mathcal{B}_{\mathbb{R}}$. But since each open set is a countable union of open intervals it follows that each open set is in $\mathcal{B}$, and hence by closure properties we have that the sigma algebra they generate must also be contained in $\mathcal{B}$, so that $\mathcal{B}_{\mathbb{R}}\subset\mathcal{B}$.

2. Countable sets. As proof, assume X is countable, then $\{\{x\}:x\in X\}\subset\mathcal{P}(X)$, and each $\{x\}$ has counting measure 1. It follows by assumption that $\bigcup_{x\in X}\{x\}=X$ is a countable union of sets of finite measure, i.e. σ-finite. Conversely, if X is uncountable then it is not a countable union of countable sets, hence any countable collection of sets $\{X_i\}_{i\in I}$, such that $\bigcup_I X_i=X$ must have atleast one uncountable X_i (so that X_i has infinite counting measure).

3. (a) Suppose that $E\in f_*\mathcal{M}$, then since $f^{-1}(A^c)=f^{-1}(A)^c$,

$$f^{-1}(E)\in\mathcal{M}\implies (f^{-1}(E))^c=f^{-1}(E^c)\in\mathcal{M}\implies E^c\in f_*\mathcal{M}$$

Now suppose that $\{E_i\}_{i\in\mathbb{N}}\subset f_*\mathcal{M}$, then since $\bigcup_{\mathbb{N}}f^{-1}(E_i)=f^{-1}(\bigcup_{\mathbb{N}}E_i)$,

$$\bigcup_{\mathbb{N}}f^{-1}(E_i)\in\mathcal{M}\implies f^{-1}(\bigcup_{\mathbb{N}}E_i)\in\mathcal{M}\implies\bigcup_{\mathbb{N}}E_i\in\mathcal{M}$$

(b) We need only check that $f_*\mu$ is a measure. Since the image of $f_*\mu$ is a subset of the image of μ it is clear that for each E, $0\le f_*(\mu)\le\infty$. It is also immediate that $f_*\mu(\emptyset)=\mu(f^{-1}(\emptyset))=\mu(\emptyset)=0$. To check additivity, consider $\{E_i\}_{\mathbb{N}}\subset f_*\mathcal{M}$, where the E_i are disjoint (note this implies that each $f^{-1}(E_i)$ is disjoint). It follows that

$$f_*\mu(\bigcup_{\mathbb{N}}E_i)=\mu(f^{-1}(\bigcup_{\mathbb{N}}E_i))=\mu(\bigcup_{\mathbb{N}}f^{-1}(E_i))=\sum_{\mathbb{N}}\mu(f^{-1}(E_i))=\sum_{\mathbb{N}}f_*\mu(E_i)$$

(c) The point mass at y_0 let $E\in f_*\mathcal{M}$, then

$$f_*\mu(E)=\begin{cases}\mu(\emptyset)=0 & y_0\notin E\\ \mu(f^{-1}(y_0))=\mu(X) & y_0\in E\end{cases}$$

(d) This measure counts the number of perfect squares in a set E. If $E\subset\mathbb{N}$, then $f_*\mu(E)=\#f^{-1}(E)=\#\{n\in\mathbb{N}:n^2\in E\}$

4. μ is a measure for $j\ge 0$.
Suppose that $j\le -1$, then consider the sets $E_n:=\{n^2\}$.

$$\sum_{\mathbb{N}}\mu(E_n)=\sum_{\mathbb{N}}n^2<\infty=\mu\left(\bigcup_{\mathbb{N}}E_n\right)$$

Conversely, suppose that $j\ge 0$, $\mu(\emptyset)=0$ by definition. If $\{E_i\}_{\mathbb{N}}$ is a countable disjoint family, then we are done immediately if any E_i is infinite, since then $\mu\left(\bigcup_{\mathbb{N}}E_i\right)=\infty=\mu(E_i)\le\sum_{\mathbb{N}}\mu(E_i)$. Similarly, if infinitely many E_i are non-empty, then

$$\mu\left(\bigcup_{\mathbb{N}}E_i\right)=\infty=\sum_{\mathbb{N}}1=\sum_{\mathbb{N}}\sum_{n\in E_i}n^0\le\sum_{\mathbb{N}}\sum_{n\in E_i}n^j$$

Finally, in the case where each E_i is finite, and only finitely many $E_i \neq \emptyset$, we have for some N, $\{E_i\}_{\mathbb{N}} = \{E_i\}_{i=1}^N$, then

$$\mu\left(\bigcup_1^N E_i\right) = \sum_{n\in\bigcup_1^N E_i} n^j = \sum_{i=1}^N \sum_{n\in E_i} n^j = \sum_{i=1}^N \mu(E_i)$$

where the second equality follows from the $\{E_i\}_1^n$ being disjoint.

5. Note, for notational convenience I will use $\ell : (a,b) \mapsto b-a$.

(a) Let $\epsilon > 0$, and consider the finite collection $\{E_i\}_{i=1}^n \subset \mathcal{P}(\mathbb{R})$ and intervals $\{I_i^j\}_{i,j=1}^{n,m}$, so that (note we may pick m independent of n by just choosing m to be the maximum number of intervals associated to any $i \in \{1, ..., n\}$)

$$\bigcup_{j=1}^m I_i^j \supset E_i, \text{ and } \sum_{j=1}^m \ell(I_i^j) < J^*(E_i) + \frac{\epsilon}{n}, \;\; i \in \{1, .., n\}$$

This gives us the desired result,

$$J^*(\bigcup_{i=1}^n E_i) \leq \sum_{i,j=1}^{n,m} \ell(I_i^j) = \sum_{i=1}^n \sum_{j=1}^m \ell(I_i^j) < \sum_{i=1}^n J^*(E_i) + \frac{\epsilon}{n} = \left(\sum_{i=1}^n J^*(E_i)\right) + \epsilon$$

And since ϵ was arbitrary, this proves finite subadditivity. To show that J^* is not countably subbadditive, notice that for any rational number q, $J^*(q) = 0$, since $(q - \epsilon/2, q + \epsilon/2)$ covers q for any $\epsilon > 0$. We may enumerate $\mathbb{Q} \cap [0,1] = \{q_1, q_2, ...\}$, so that applying part (b),

$$J^*(\mathbb{Q} \cap [0,1]) = 1 > 0 = \sum_{i=1}^\infty 0 = \sum_{i=1}^\infty J^*(q_i)$$

(b) Assume for the sake of contradiction that $J^*(\mathbb{Q} \cap [0,1]) < 1$, then there exists some collection $\{I_i\}_{i=1}^n$, $I_i = (a_i, b_i)$ covering $\mathbf{Q} := \mathbb{Q} \cap [0,1]$, such that $\sum_{i=1}^n \ell(I_i) < 1$. We may reindex this collection, so that $a_i \leq a_{i+1}$ (note that this implies $a_1 < 0$), and assume WLOG that $b_i < b_{i+1}$, otherwise $\bigcup_{1\leq j\leq n,\, j\neq i} I_j$ still covers $\mathbf{Q}$, and $\sum_{1\leq j\leq n,\, j\neq i} I_j \leq \sum_{1\leq j\leq n} I_j$, we may also assume $b_1 > 0$ and $a_n < 1$, otherwise $\mathbf{Q} \subset \bigcup_{i=2}^n I_i$ or $\mathbf{Q} \subset \bigcup_{i=1}^{n-1} I_i$. Finally, we have $a_i \leq b_{i-1}$, otherwise $\emptyset \neq (b_{i-1}, a_i) \subset [0,1]$, and since $\mathbb{Q}$ is dense in $\mathbb{R}$,

$$\emptyset \neq \mathbb{Q} \cap (b_{i-1}, a_i) = \mathbf{Q} \cap (b_{i-1}, a_i) \subset (\bigcup_{i=1}^n I_i)^c \implies \mathbf{Q} \not\subset \bigcup_{i=1}^n I_i$$

Using the above results, we get

$$1 > \sum_{i=1}^n b_i - a_i \geq \sum_{i=1}^n b_i - b_{i-1} = b_n - b_1 \geq b_n - a_1 > b_n$$

And hence $1 \in (\bigcup_{i=1}^n I_i)^c$, a contradiction.

(c) Consider any interval I, if $I \not\subset [0,1]$, then $I \not\subset \mathbf{Q} := \mathbb{Q} \cap [0,1]$. If $I \subset [0,1]$, then since the irrationals are dense in $\mathbb{R}$, there exists some $\alpha \in I \cap \mathbb{Q}^c \supset \mathbf{Q}^c$. This proves that there are no non-empty open intervals which are subsets of $\mathbf{Q}$, hence if $\{I_i\}_{i=1}^n$ is a collection of open intervals, such that $\bigcup_1^n I_i \subset \mathbf{Q}$, then each I_i must be empty, implying that $\sum_1^n \ell(I_i) = 0$, so that $J_*(\mathbf{Q}) = 0 \neq 1 = J^*(\mathbf{Q})$, i.e. $\mathbf{Q}$ is not Jordan measureable.

6. First note that from monotonicity, $\mu^*(E) \leq \mu^*(\tilde{E}) = \mu(\tilde{E})$ for any $\tilde{E} \supset E$. Hence we need only show existence of some $\tilde{E} \supset E$, such that $\mu(\tilde{E}) \leq \mu^*(E)$. If $\mu^*(E) = \infty$ we are done trivially with $\tilde{E} = X$, so assume not. Define $I_n := \bigcup_{i=1}^{\infty} A_i \supset E$, such that for each i, $A_i \in \mathcal{A}$ and $\sum_{i=1}^{\infty} \mu_0(A_i) = \sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \frac{1}{n}$; existence of such A_i is guarunteed by the definition of μ^*. It follows that since $\mathcal{A} \subset \mathcal{M}$ (Folland 1.13), each $I_n \in \mathcal{M}$ by the property of $\mathcal{M}$ being an algebra. Then we have

$$
\begin{aligned}
& E \subset \tilde{E} := \cap_{n=1}^{\infty} I_n \in \mathcal{M} \\
\implies & \mu^*(E) \leq \mu^*(\tilde{E}) = \mu(\tilde{E}) \leq \mu(I_n),\ \forall n \\
\implies & \mu^*(E) \leq \mu(\tilde{E}) \leq \mu^*(E) + \frac{1}{n},\ \forall n \\
\implies & \mu(\tilde{E}) = \mu^*(E)
\end{aligned}
$$