1. (a) $||\mathbf{1}-T||<1$ implies that the power series $\sum_0^\infty ||\mathbf{1}-T||^n$ converges, since X is Banach this further implies that $\sum_0^\infty (\mathbf{1}-T)^n$ converges in X. For $N \in \mathbb{Z}_{>0}$ we find that

$$T\sum_0^N(\mathbf{1}-T)^n = \sum_0^N(\mathbf{1}-T)^n - \sum_1^{N+1}(\mathbf{1}-T)^n = 1-(\mathbf{1}-T)^{N+1}$$

and hence

$$||\mathbf{1}-T\sum_0^N(\mathbf{1}-T)^n|| = ||\mathbf{1}-T||^{N+1}$$

taking $N\to\infty$ we find that

$$||\mathbf{1}-T\sum_0^\infty(\mathbf{1}-T)^n|| = 0 \implies T\sum_0^\infty(\mathbf{1}-T)^n = 1 \quad \square$$

To see that the inverse is bounded, note that for any $N \in \mathbb{Z}_{>0}$

$$||\sum_0^N(\mathbf{1}-T)^n|| \leq \sum_0^N||\mathbf{1}-T||^n \implies ||\sum_0^\infty(\mathbf{1}-T)^n|| \leq \sum_1^\infty||\mathbf{1}-T||^n < \infty$$

(b) Applying (a), $S^{-1}T$ is invertible with bounded inverse, since

$$||\mathbf{1}-S^{-1}T|| = ||S^{-1}S - S^{-1}T|| \leq ||S^{-1}||||S-T|| < ||S^{-1}||||S^{-1}||^{-1} = 1$$

It is immediate that $\left(S^{-1}T\right)^{-1}S^{-1} = T^{-1}$, since

$$\left(S^{-1}T\right)^{-1}S^{-1}T = \mathbf{1} = SS^{-1}T(S^{-1}T)^{-1}S^{-1} = T(S^{-1}T)^{-1}S^{-1}$$

and T^{-1} is bounded since

$$||T^{-1}|| = ||\left(S^{-1}T\right)^{-1}S^{-1}|| \leq ||\left(S^{-1}T\right)^{-1}||||S^{-1}|| < \infty \quad \square$$

(c) Note that

$$||\mathbf{1}-(\mathbf{1}-\lambda^{-1}T)|| = \lambda^{-1}||T|| < 1 = ||1^{-1}||^{-1}$$

Hence by (b) we find that $\mathbf{1}-\lambda^{-1}T$ is invertible with bounded inverse, multiplying by $-\lambda$ we find that $T-\lambda\mathbf{1}$ is invertible with bounded inverse. $\square$

(d) Let $\lambda \in \rho(T)$ and fix $\delta = ||(T-\lambda\mathbf{1})^{-1}||^{-1}$, then let $\alpha \in N_\delta(\lambda)$, so that $\alpha = \lambda - \beta$ with $|\beta| < \delta$. It follows that

$$||\mathbf{1}-(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})|| = |\beta|\,||(T-\lambda\mathbf{1})^{-1}|| < 1 = ||\mathbf{1}^{-1}||^{-1}$$

so that $\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1}$ is invertible with bounded inverse. It follows that

$$(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}(T-(\lambda-\beta)\mathbf{1}) = (\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1}) = \mathbf{1}$$

so that $(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}$ is a left sided inverse for $T-\alpha\mathbf{1}=T-(\lambda-\beta)\mathbf{1}$, it is also a right inverse because

$$\begin{aligned}
&(T-(\lambda-\beta)\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(T-\lambda\mathbf{1})^{-1}(T-(\lambda-\beta)\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda\mathbf{1})^{-1}\\
=&(T-\lambda\mathbf{1})(T-\lambda\mathbf{1})^{-1}=\mathbf{1}
\end{aligned}$$

so that $(T-\alpha\mathbf{1})^{-1}=(\mathbf{1}+\beta(T-\lambda\mathbf{1})^{-1})^{-1}(T-\lambda)^{-1}\in\mathcal{L}(X,X)$ $\square$

(e) $\sigma(T)\subset\{\lambda\in K\mid|\lambda|\le||T||\}$ is bounded, and in (d) we showed that $\sigma(T)=\rho(T)^c$ is closed. By the Heine Borel theorem we conclude that $\sigma(T)$ is compact. $\square$

2. (a) We first check that the operator is bounded,

$$||M_g(f)||=||fg||_2=\left(\int_X|fg|^2\right)^{\frac{1}{2}}\le\left(\int_X|f|^2\,||g||_\infty^2\right)^{\frac{1}{2}}=\sqrt{||g||_\infty^2||f||_2^2}=||g||_\infty||f||_2$$

Let $\epsilon>0$, then by definition of essential supremum, there is some set E of positive measure such that $||g||_\infty-\epsilon\le|g(x)|$ for any $x\in E$, consider $f:=\frac{1}{\sqrt{\mu(E)}}\chi_E$, it is clear that $||f||_2=1$, and we have that

$$||fg||_2=\left(\int_X\left|\frac{1}{\sqrt{\mu(E)}}\chi_E g\right|^2\right)^{\frac{1}{2}}\ge\left(\int_E\left(\frac{||g||_\infty-\epsilon}{\sqrt{\mu(E)}}\right)^2\right)^{\frac{1}{2}}=(||g||_\infty-\epsilon)\left(\int_E\frac{1}{\mu(E)}\right)^{\frac{1}{2}}=||g||_\infty-\epsilon$$

Since ϵ was arbitrary, we may conclude that $||M_g||\ge||g||_\infty$, where the opposite inequality is provided above, so we conclude that $||M_g||=||g||_\infty$. $\square$

(b) First suppose that $\lambda\notin\mathcal{R}_g$, then for some $\epsilon>0$ we have $\mu\{x\in X\mid|g(x)-\lambda|<\epsilon\}=0$, and hence $\frac{1}{g(x)-\lambda}\le\frac{1}{\epsilon}$ almost everywhere. It follows that $\frac{1}{g(x)-\lambda}\in L^\infty$, and it is immediate that $M_{\frac{1}{g-\lambda}}=(M_g-\lambda\mathbf{1})^{-1}$.

Conversely, let $\lambda\in\mathcal{R}_g$, and let $F_n:=\{x\in X\mid|g(x)-\lambda|<\frac{1}{n}\}$, by definition of the essential range we have that $\mu(F_n)>0$ for infinitely many n, thus we may define a subsequence F_{n_k} each having positive measure. If any F_{n_k} have infinite measure, then we replace them with a subset having finite measure (we may do this since we are working in a σ-finite space). Now take $a_{n_k}:=\frac{1}{n_k\sqrt{\mu(F_{n_k})}}$, it follows that $\sum_1^\infty a_{n_k}\chi_{F_{n_k}}\in L^2$, but not in $\operatorname{Im}(M_g-\lambda\mathbf{1})$. It is obvious that it is in L^2 (by convergence of $\sum_1^\infty\frac{1}{n_k^2}\le\sum_1^\infty\frac{1}{n^2}$), assume for contradiction it is in the image, then let $f\in(M_g-\lambda\mathbf{1})^{-1}(\sum_1^\infty a_{n_k}\chi_{F_{n_k}})$, then

$$|f|\,|_{F_{n_k}}\ge a_{n_k}\frac{1}{\sup_{F_{n_k}}|g-\lambda|}=n_k a_{n_k}=\frac{1}{\sqrt{\mu(F_{n_k})}}$$

so that

$$||f||_2\ge\left(\int_X\left(\sum_1^\infty\frac{\chi_{F_{n_k}}}{\sqrt{\mu(F_{n_k})}}\right)^2\right)^{\frac{1}{2}}=\left(\int_X\sum_1^\infty\frac{\chi_{F_{n_k}}}{\mu(F_{n_k})}\right)^{\frac{1}{2}}\overset{\text{MCT}}{=}\left(\sum_1^\infty 1\right)^{\frac{1}{2}}=\infty$$

which contradicts $f \in L^2$, hence $M_g - \lambda\mathbf{1}$ is not surjective for $\lambda \in \mathcal{R}_g$. $\square$

(c) $M_g^\dagger = M_{\overline{g}}$, it is clear that this is in L^∞ since $||g||_\infty = ||\overline{g}||_\infty$, and $g = \overline{g}$ in L^∞ exactly when g is real almost everywhere. Now to prove the main statement, that $M_g^\dagger = M_{\overline{g}}$. Let $f \in L^2$, then $M_g^* \mathfrak{C}(f) = \phi_f M_g$, so that for any $k \in L^2$ we have $M_g^*\mathfrak{C}(f)(k) = \langle kg, f\rangle = \langle k, f\overline{g}\rangle$, and thus $M_g^*\mathfrak{C}(f) = \phi_{f\overline{g}}$, so that

$$M_g^\dagger(f) \overset{\text{def}}{=} \mathfrak{C}^{-1}M_g^*\mathfrak{C}(f) = \mathfrak{C}^{-1}\phi_{f\overline{g}} = f\overline{g} = M_{\overline{g}}(f)$$

and since f was arbitrary we conclude that $M_g^\dagger = M_{\overline{g}}$. $\square$

3. (a) Let λ be in the residual spectrum of T, then there is some non-empty open set $U \subset X$, such that $\mathrm{Im}(T - \lambda\mathbf{1}) \cap U = \emptyset$, since $\mathrm{Im}(T - \lambda\mathbf{1})$ is a subspace of X, it follows that $\mathrm{Im}(T - \lambda\mathbf{1}) \cap tU = \emptyset$, where $tU := \bigcup_{t\in K^\times}\{tu \mid u \in U\}$. Furthermore, we have $d_{(tU)^c} : X \to X$ is a seminorm, fixing $x \in U$, we have that $d_{(tU)^c} : \langle x\rangle \to K$ is linear, so that by the Hahn Banach theorem there is some $f \in \mathcal{L}(X, X)$, such that $|f|$ is bound above by $d_{(tU)^c}$. Since $\mathrm{Im}(T - \lambda\mathbf{1}) \subset (tU)^c$ we have $(T^* - \lambda\mathbf{1}^*)f = f \circ (T - \lambda\mathbf{1}) = 0$, where $f \neq 0$, and hence $f \in \ker T^* - \lambda\mathbf{1}^*$ which suffices to show that $\lambda \in \sigma_p(T^*)$. $\square$

(b) Note that for any $\lambda \in \rho(T) \cup \sigma_c(T)$ we have $\mathrm{Im}(T - \lambda\mathbf{1})$ is dense in X, so it will suffice to show that if $\mathrm{Im}(T - \lambda\mathbf{1})$ is dense in X, then $\lambda \notin \sigma_p(T^*)$. Fix such a λ, now suppose $0 \neq f \in X^*$, so that there is some $x \in X$ and $\epsilon > 0$, such that $|f(x)| = \epsilon > 0$, since f is continuous, there is some open set U containing x, such that $|f||_U > \epsilon/2$. Since $\mathrm{Im}(T - \lambda\mathbf{1})$ is dense in X, it follows that there is some $y \in X$, such that $(T - \lambda\mathbf{1})(y) \in U$, then

$$|(T^* - \lambda\mathbf{1}^*)f(y)| = |f(T - \lambda\mathbf{1})(y)| > \frac{\epsilon}{2}$$

this suffices to show that $f \notin \ker(T^* - \lambda\mathbf{1}^*)$, and since $f \neq 0$ was arbitrary we conclude that $\ker(T^* - \lambda\mathbf{1}^*) = 0$ and hence $\lambda \notin \sigma_p(T^*)$. $\square$

(c) Lemma. $[\sigma_\mathbf{c}(\mathbf{T}^*) \cup \sigma_\mathbf{r}(\mathbf{T}^*) = \overline{\sigma_\mathbf{c}(\mathbf{T}^\dagger) \cup \sigma_\mathbf{r}(\mathbf{T}^\dagger)}]$ Note that conjugation commutes with union. The following completes the proof:

$$\begin{aligned}
\lambda \in \sigma_c(T^*) \cup \sigma_r(T^*) &\iff \exists f \in \mathcal{H}^* \setminus \{0\}, \text{ such that } f \notin \mathrm{Im}(T^* - \lambda\mathbf{1}^*) \\
&\iff \exists x \in \mathcal{H} \setminus \{0\}, \text{ such that } \phi_x \notin (T^* - \lambda\mathbf{1}^*) \text{ (Riesz-Frechet Theorem)} \\
&\iff \exists x \in \mathcal{H} \setminus \{0\}, \forall y, \langle (T - \lambda)(\cdot), y\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists x \in \mathcal{H} \setminus \{0\}, \forall y, \langle T(\cdot), y\rangle - \langle \cdot, \overline{\lambda} y\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists x \in \mathcal{H} \setminus \{0\}, \forall y, \langle \cdot, (T^\dagger - \overline{\lambda}\mathbf{1})(y)\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists x \in \mathcal{H} \setminus \{0\}, \text{ such that } x \notin \mathrm{Im}(T^\dagger - \overline{\lambda}\mathbf{1}) \\
&\iff \overline{\lambda} \in \overline{\sigma_c(T^\dagger) \cup \sigma_r(T^\dagger)} \quad \square
\end{aligned}$$

$[\sigma_\mathbf{p}(\mathbf{T}^*) = \overline{\sigma_\mathbf{p}(\mathbf{T}^\dagger)}]$. Suppose that $\lambda \in \sigma_p(T^*)$, then there is some $f \in \mathcal{H}^*, f \neq 0$, such that $(T^* - \lambda\mathbf{1}^*)(f) = 0$, by the Riesz-Frechet representation theorem $f = \phi_x$ for some $x \in \mathcal{H}$ (note that since $\phi_x \neq 0$ we know that $x \neq 0$). It follows that

$$\begin{aligned}
(T^* - \lambda\mathbf{1}^*)(\phi_x) = 0 &\iff \langle T(y) - \lambda y, x\rangle = 0, \ \forall y \in \mathcal{H} \\
&\iff \langle T(y), x\rangle - \langle y, \overline{\lambda} x\rangle = 0, \ \forall y \in \mathcal{H} \\
&\iff \langle y, T^\dagger(x)\rangle - \langle y, \overline{\lambda} x\rangle = 0, \ \forall y \in \mathcal{H} \\
&\iff \langle y, T^\dagger(x) - \overline{\lambda} x\rangle, \ \forall y \in \mathcal{H} \\
&\iff (T^\dagger - \overline{\lambda}\mathbf{1})(x) = 0
\end{aligned}$$

From which we conclude that $x \in \ker(T^\dagger - \overline{\lambda}\mathbf{1})$, so that $\lambda \in \overline{\sigma_p(T^\dagger)}$. Conversely if $\lambda \in \overline{\sigma_p(T^\dagger)}$, then there is some $x \neq 0$, such that $(T^\dagger - \overline{\lambda}\mathbf{1})(x) = 0$, tracing backwards throught the if and only ifs we find that $(T^* - \lambda\mathbf{1}^*)(\phi_x) = 0$ for $\phi_x \neq 0$ since $x \neq 0$, so we may conclude that $\lambda \in \sigma_p(T^*)$ $\square$

$[\boldsymbol{\sigma_c(T^*)} = \overline{\boldsymbol{\sigma_c(T^\dagger)}}$ **and** $\boldsymbol{\sigma_r(T^*)} = \overline{\boldsymbol{\sigma_r(T^\dagger)}}]$. By the lemma it will suffice to show that $\sigma_r(T^*) = \overline{\sigma_r(T^\dagger)}$. The proof is as follows (here U denotes a non-empty open set in $\mathcal{H}$, since $x \mapsto \phi_x$ is a homeomorphism this is equivalent to the set $\phi_U = \{\phi_x \mid x \in U\}$ being non-empty and open in $\mathcal{H}^*$).

$$\begin{aligned}
\lambda \in \sigma_r(T^*) &\iff \exists \phi_U \neq \emptyset, \text{ open, such that } \phi_U \cap \operatorname{Im}(T^* - \lambda\mathbf{1}^*) = \emptyset \\
&\iff \exists U, \forall x \in U, \forall y, \langle (T-\lambda)(\cdot), y\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists U, \forall x \in U, \forall y, \langle T(\cdot), y\rangle - \langle \cdot, \overline{\lambda}y\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists U, \forall x \in U, \forall y, \langle \cdot, (T^\dagger - \overline{\lambda}\mathbf{1})(y)\rangle \neq \langle \cdot, x\rangle \\
&\iff \exists U, \text{ such that } \forall x \in U, x \notin \operatorname{Im}(T^\dagger - \overline{\lambda}\mathbf{1}) \\
&\iff \overline{\lambda} \in \overline{\sigma_r(T^\dagger)} \quad \square
\end{aligned}$$

4. (a) $S_r^\dagger = S_\ell$, and $S_\ell^\dagger = S_r$. As proof, let $x, y \in \ell^2$, then

$$\begin{aligned}
\langle S_r(x), y\rangle &= \sum_1^\infty x_i y_{i+1} = \langle x, S_\ell(y)\rangle \\
\text{and } \langle S_\ell(x), y\rangle &= \sum_1^\infty y_i x_{i+1} = \langle x, S_r(y)\rangle \quad \square
\end{aligned}$$

(b) Let $\lambda \in \mathbb{C}$, and let $x \in \ell^2$, such that $x \in \ker(S_r - \lambda\mathbf{1})$, it follows that (denoting $x_0 := 0$)

$$0 = ||(S_r - \lambda\mathbf{1})(x)||^2 = \sum_1^\infty |x_{i-1} - \lambda x_i|^2 = 0 \implies |x_{i-1} - \lambda x_i|^2 = 0, \ \forall i$$

Since $x_0 \overset{\text{def}}{=} 0$, we may show by induction that for each n, $x_n = 0$. Suppose for $i < n$ we have $x_i = 0$, then $|x_{n-1} - \lambda x_n|^2$ and hence $|\lambda x_n|^2 = 0$, if $\lambda \neq 0$, then $x_n = 0$ and we are done by induction, if $\lambda = 0$, then $0 = |x_n - \lambda x_{n+1}|^2 = |x_n|^2$ implies that $x_n = 0$. Hence $x_n = 0$ for all n thus $x = 0$. $\square$

(c) fix λ, such that $|\lambda| \leq 1$, it will suffice to show that $(1, 0, 0, \ldots) \notin \operatorname{Im}(S_r - \lambda\mathbf{1})$ to conclude that $S_r - \lambda\mathbf{1}$ is not invertible and hence $\lambda \in \sigma(S_r)$, this is immediate for $\lambda = 0$, since the first coordinate of $S_r(x)$ is zero for any x, so take $\lambda \neq 0$. Suppose for contradiction that we have $x \in \ell^2$, such that

$$(1, 0, 0, \ldots) = (S_r - \lambda\mathbf{1})(x) = (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \ldots)$$

Then, $x_1 = \frac{1}{-\lambda}$, so that $x_2 = \frac{x_1}{\lambda} = \frac{-1}{\lambda^2}$, we can continue inductively to find that $x_n = \frac{-1}{\lambda^n}$, x in ℓ^2 implies that $\lim_{n\to\infty} |x_n| = 0$, but $|x_n| = \frac{1}{|\lambda|^n} \geq 1$ which is a contradicton, implying that $\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} \subset \sigma(S_r)$.

To show the converse inclusion, suppose that $\lambda \in \mathbb{C}$ and $|\lambda| > 1$, since $||S_r(x)|| = ||x||$, it is immediate that S_r has operator norm 1. We apply the criterion of 1(b), namely $\lambda\mathbf{1}$ is invertible, and

$$||\lambda\mathbf{1} - (\lambda\mathbf{1} - S_r)|| = ||S_r|| = 1 < |\lambda| = ||\lambda\mathbf{1}^{-1}||^{-1}$$

so that $\lambda\mathbf{1} - S_r$ is invertible, which suffices to show that $S_r - \lambda\mathbf{1}$ is invertible. $\square$

(d) The point spectrum is $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. It is immediate that 0 is in the point spectrum since $S_\ell(1,0,0,\ldots) = 0$. Suppose that $0 \neq \lambda \in \sigma_p(S_\ell)$, then there is some $0 \neq x \in \ell^2$, such that

$$0 = (S_\ell - \lambda\mathbf{1})(x) = (x_2 - \lambda x_1, x_3 - \lambda x_2, \ldots)$$

It follows that $x_{i+1} = \lambda x_i$ for each i. If $x_1 = 0$, then $x = 0$, so this cannot be the case, this necessitates that $|\lambda| < 1$, since $x \in \ell^2$ implies that $\lim_{n\to\infty} x_n = 0$. Now suppose $0 < |\lambda| < 1$, then $x := (\lambda, \lambda^2, \lambda^3, \ldots) \in \ell^1 \subset \ell^2$, and $(S_\ell - \lambda\mathbf{1})(x) = 0$, hence $\lambda \in \sigma_p(S_\ell)$.

The eigenspaces are determined by $(S_\ell - \lambda\mathbf{1})(x) = (x_2 - \lambda x_1, x_3 - \lambda x_2, \ldots)$, this says that for $\lambda \in \mathbb{C},\ |\lambda| < 1$ we have $(S_\ell - \lambda\mathbf{1})(x) = 0$ when $x_{n+1} = \lambda x_n$ for each n, since this is also a sufficient condition for $x \in \ell^2$ everything of this form is in the eigenspace, in other words:

$$\ker(S_\ell - \lambda\mathbf{1}) = \{(a, \lambda a, \lambda^2 a, \ldots) \mid a \in \mathbb{C}\} \quad \square$$

(e) Lemma. $\sigma_p(S_\ell) = \sigma_p(S_r^*)$, let $\lambda \in \sigma_p(S_\ell)$, then there is $x \neq 0$ (and hence $\phi_x \neq 0$), such that $(S_\ell - \lambda\mathbf{1})(x) = 0$ it follows that for any $y \in \mathcal{H}$ we have

$$\begin{aligned} 0 = \langle y, (S_\ell - \lambda\mathbf{1})x\rangle &= \sum_1^\infty y_i x_{i+1} - \lambda \sum_1^\infty y_i x_i \\ &= \langle (S_r - \lambda)(y), x\rangle = (S_r^* - \lambda\mathbf{1}^*)\langle y, x\rangle = (S_r^* - \lambda\mathbf{1}^*)(\phi_x)(y) \end{aligned}$$

since y was aribtrary, this implies that $(S_r^* - \lambda\mathbf{1}^*)(\phi_x) = 0$ and hence $\lambda \in \sigma_p(S_r^*)$, to show the converse inequality, assume that $\lambda \in \sigma_p(S_r^*)$, then by the Riesz-Frechet representation theorem there is some $x \in \mathcal{H}$, such that $(S_r^* - \lambda\mathbf{1}^*)(\phi_x) = 0$, this implies that for any $y \in \mathcal{H}$ we have $(S_r^* - \lambda\mathbf{1}^*)(\phi_x)(y) = 0$, by the computation above this implies that $\langle (S_\ell - \lambda\mathbf{1})x, (S_\ell - \lambda\mathbf{1})x\rangle = 0$ so that $(S_\ell - \lambda\mathbf{1})x = 0$ implying that $\lambda \in \sigma_p(S_\ell)$. $\square$

Lemma. $\sigma_p(S_r) = \sigma_p(S_\ell^*)$. Let $\lambda \in \sigma_p(S_r)$, then for some $0 \neq x \in \mathcal{H}$ (and hence $\phi_x \neq 0$) we have $(S_r - \lambda\mathbf{1})(x) = 0$, so that for any $y \in \mathcal{H}$ we have

$$0 = \langle y, (S_r - \lambda\mathbf{1})(x)\rangle = \sum_1^\infty y_{i+1}x_i - \lambda y_i x_i = \langle (S_\ell - \lambda\mathbf{1})(y), x\rangle = (S_\ell^* - \lambda\mathbf{1}^*)(\phi_x)(y)$$

since this identity holds for any y we find that $(S_\ell^* - \lambda\mathbf{1}^*)(\phi_x) = 0$, so that $\lambda \in \sigma_p(S_\ell^*)$. To show the converse inequality, let $\lambda \in \sigma_p(S_\ell^*)$, then by the Riesz-Frechet representation theorem there is some $0 \neq x \in \mathcal{H}$, such that $(S_\ell^* - \lambda\mathbf{1}^*)(\phi_x) = 0$, hence $(S_\ell^* - \lambda\mathbf{1}^*)(\phi_x)((S_r - \lambda\mathbf{1})(x)) = 0$, by the above computation this implies that $\langle (S_r - \lambda\mathbf{1})(x), (S_r - \lambda\mathbf{1})(x)\rangle = 0$, so that $(S_r - \lambda\mathbf{1})(x) = 0$, this implies that $\lambda \in \sigma_p(S_r)$ as desired. $\square$

Now proceeding with the proof, $\sigma_r(S_r) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. As proof first take $\lambda \in \mathbb{C}$, such that $|\lambda| < 1$. Now take $\epsilon := \frac{1}{2\sum_0^\infty |\lambda^{2i}|}$, I claim that $\mathrm{Im}(S_r - \lambda\mathbf{1}) \cap N_{\epsilon^2}(1,0,\ldots) = \emptyset$ so that $\lambda \in \sigma_r(S_r)$. Let $y \in N_{\epsilon^2}(1,0,0,\ldots)$, then $y = (1+\delta_1, \delta_2, \delta_3, \ldots)$ for $\sum_1^\infty |\delta_i|^2 < \epsilon$. Suppose for contradiction there is $x \in \ell^2$, such that $(S_r - \lambda\mathbf{1})(x) = y$. We can compute $x_1 = -\frac{1+\delta_1}{\lambda}$, then by induction if $x_n = -\frac{1+\sum_0^{n-1}\delta_{i+1}\lambda^i}{\lambda^n}$, then

$$x_{n+1} = \frac{x_n - \delta_{n+1}}{\lambda} = \frac{-\frac{1+\sum_0^{n-1}\delta_{i+1}\lambda^i}{\lambda^n} - \delta_{n+1}}{\lambda} = -\frac{1+\sum_0^n \delta_{i+1}\lambda^i}{\lambda^{n+1}}$$

Using this closed form of x_n, we find that

$$|x_n| \geq \frac{1 - \left|\sum_0^{n-1} \delta_{i+1}\lambda^i\right|}{|\lambda|^n} \geq \frac{1 - \sum_0^{n-1} \left|\delta_{i+1}\lambda^i\right|}{|\lambda|^n} \overset{\text{Cauchy-Schwartz}}{\geq} \frac{1 - \sqrt{(\sum_1^n |\delta_i|^2)(\sum_0^{n-1} |\lambda^i|^2)}}{|\lambda|^n}$$
$$\geq \frac{1 - \sqrt{(\sum_0^\infty |\lambda^{2i}|)(\sum_1^\infty |\delta_i|^2)}}{|\lambda|^n} = \frac{1 - \sqrt{\frac{1}{2\epsilon}(\sum_1^\infty |\delta_i|^2)}}{|\lambda|^n} > \frac{1 - \sqrt{\frac{1}{2}}}{|\lambda|^n} > 1 - \frac{1}{\sqrt{2}}$$

and hence $\lim_{n\to\infty} x_n \neq 0$, so that $x \notin \ell^2$ a contradiction, this suffices to show that $\lambda \in \sigma_r(S_r)$. To show the converse inclusion, by 3(a) we have $\sigma_r(S_r) \subset \sigma_p(S_r^*)$ which by the lemma is equal to $\sigma_p(S_\ell) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. This suffices to show that $\sigma_r(S_r) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$. To classify the residual spectrum of S_ℓ, I claim that $\sigma_r(S_\ell) = \emptyset$, by 3(a) $\sigma_r(S_\ell) \subset \sigma_p(S_\ell^*)$, which by the lemma is equal to $\sigma_p(S_r)$, in part (b) we proved this is empty. □

(f) In part (c) we showed $\sigma(S_r) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$, in the previous subparts we also showed that $\sigma_r(S_r) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$, and that the point spectrum is empty, so we may compute

$$\sigma_c(S_r) = \sigma(S_r) \setminus (\sigma_r(S_r) \cup \sigma_p(S_r)) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

It remains to show the continuous spectrum of S_ℓ. First notice that $||S_\ell(x)|| \leq ||x||$, and $||S_\ell(0, 1, 0, 0, \dots)|| = 1$, so that $||S_\ell|| = 1$, it follows that if $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, then

$$||\lambda\mathbf{1} - (\lambda\mathbf{1} - S_\ell)|| = ||S_\ell|| = 1 < |\lambda| = ||\lambda\mathbf{1}^{-1}||^{-1}$$

so by 1(b) $\lambda\mathbf{1} - S_\ell$ is invertible, which implies that $S_\ell - \lambda\mathbf{1}$ is invertible and $\lambda \in \rho(S_\ell)$. Furthermore by problem 1, we know that $\sigma(S_\ell)$ is compact, so in particular $\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\} = \overline{\sigma_p(S_\ell)} \subset \sigma(S_\ell)$, but then by the above computation of the resolvent set this is the entire spectrum, in the previous subpart we showed that $\sigma_r(S_\ell) = \emptyset$, so we may compute

$$\sigma_c(S_\ell) = \sigma(S_\ell) \setminus (\sigma_r(S_\ell) \cup \sigma_p(S_\ell)) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \quad \square$$

(g) $\sigma(S_r) = \overline{N_1(0)}$ and $\sigma(S_\ell) = \overline{N_1(0)}$, in particular they both have infinitely many limit points. By theorem 4.17 of the notes this implies that neither are compact. □