1. We first show that F is bijective. For surjectivity, note that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is surjective, so that $\tan\frac{x}{2} : (-\pi, \pi) \to \mathbb{R}$ is onto. This is equivalent to $F : e^{i\theta} \mapsto [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}] = [1, \tan\frac{\theta}{2}]$ maps $S^1 \setminus \{-1\} \to \mathbb{RP}^1 \setminus [0,1]$, then $F(-1) = F(e^{i\pi}) = [0,1]$, so F is onto. To see injectivity, it is clear that $F^{-1}([0,1]) = -1$, since $e^{i\pi}$ is the only point on S^1 where $cos(\frac{\theta}{2}) = 0$. Then any other point $-1 \neq e^{i\theta}$ maps to the coset of the form $[1, \tan\frac{\theta}{2}]$, so injectivity follows from tan being strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

To show that F is smooth, consider charts $(U, \varphi), (\tilde{U}, \tilde{vp})$ on S^1, where $\varphi : e^{i\theta} \to \theta$ on $U = (0, 2\pi)$ and $\tilde{\varphi} : e^{i\theta} \to \theta$ on $\tilde{U} = (-\pi, \pi)$. Also consider charts $(V, \eta), (\tilde{V}, \tilde{\eta})$ on $\mathbb{RP}^1$, where

$$V = \mathbb{RP}^1 \setminus [0,1], \eta : [x,y] \mapsto y/x \text{ and } \tilde{V} = \mathbb{RP}^1 \setminus [1,0], \tilde{\eta} : [x,y] \mapsto x/y$$

We check that the maps between $\tilde{\varphi}\tilde{U}$ and $\eta(V)$, as well as $\varphi(U)$ and $\tilde{\eta}(\tilde{V})$ are smooth in coordinates to conclude F is smooth.

$$\eta F \tilde{vp}^{-1}(\theta) = \eta[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}] \overset{\theta \in (-\pi,\pi)}{=} \eta[1, \tan\frac{\theta}{2}] = \tan\frac{\theta}{2}$$
$$\tilde{\eta} F \varphi^{-1}(\theta) = \tilde{\eta}[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}] \overset{\theta \in (0,2\pi)}{=} \tilde{\eta}[\cot\frac{\theta}{2}, 1] = \cot\frac{\theta}{2}$$

Where $\tan\frac{\theta}{2}$ is smooth on $(-\pi, \pi)$, while $2\arctan$ is smooth on $\mathbb{R}$ and $\cot\frac{\theta}{2}$ is smooth on $(0, 2\pi)$, while 2arccot is smooth on .$\mathbb{R}$.

2. Equip $S^1 \times S^1$ with the following charts:

$$(U \times U, \varphi_1) \qquad (U \times \tilde{U}, \varphi_2)$$
$$(\tilde{U} \times U, \varphi_3) \qquad (\tilde{U} \times \tilde{U}, \varphi_4)$$

Where $U, \tilde{U}$ are defined in the previous question, and $\varphi_j : e^{i\theta_1} \times e^{i\theta_2} \mapsto (\theta_1, 2\theta_1 - \theta_2)$. Then the coordinates on each chart C_j are $S \cap C_j$ having coordinates $e^{i\theta_1} \times e^{i2\theta_2} \overset{\varphi_j}{\mapsto} (\theta_1, 0)$.

3. **(a)** Let $p \in \mathcal{Z}$, then since α is a regular value of f, f has constant rank 1 on $\mathcal{Y} \supset \mathcal{Z}$. This means we can apply the rank theorem, which gives us a charts U, V, such that $p \in U$, $F(p) \in V$ so that $f(x^1, \dots, x^n) \overset{\text{loc.}}{=} x^1$, this lets us write $F(p) = (p^1, g(p))$. Since (α, β) is a regular value of F, $p \in \mathcal{Z}$ is a regular point of F, i.e. dF_p has full rank.

$$dF_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x^1}\Big|_p & \frac{\partial g}{\partial x^2}\Big|_p & \cdots & \frac{\partial g}{\partial x^n}\Big|_p \end{bmatrix}$$

In order for this matrix to have rank 2, the second row must have rank 1. Now define $G : Y \to \mathbb{R}$, where $G := g|_y$, if p is a regular point of G, then since $p \in \mathcal{Z}$ was arbitrary, β is a regular value of G, implying that $G^{-1}(\beta) = \mathcal{Z}$ is a submanifold of $\mathcal{Y}$ by the regular level set theorem. It remains to show that p is in fact a regular point. But this is straightforward, since

$$dG_p = \begin{bmatrix} \frac{\partial g}{\partial x^1}\Big|_p & \frac{\partial g}{\partial x^2}\Big|_p & \cdots & \frac{\partial g}{\partial x^n}\Big|_p \end{bmatrix}$$

Which has rank one since dF_p has full rank.

(b) Let $\alpha \in (0,1)$ and define $f : \mathbb{R}^4 \to \mathbb{R}$ $(x,y,z,w) \mapsto x^2+y^2+z^2+w^2$, and $g : \mathbb{R}^4 \to \mathbb{R}$ $(x,y,z,w) \mapsto x^2+y^2$. We first show that $(1, \alpha)$ is a regular value of $F := (f,g) : \mathbb{R}^2 \to \mathbb{R}$, note all of these functions are smooth since they are polynomials. To see $(1, \alpha)$ is regular, let $p = (x,y,z,w) \in F^{-1}(1, \alpha)$, then

$$dF_p = \begin{bmatrix} 2x & 2y & 2z & 2w \\ 2x & 2y & 0 & 0 \end{bmatrix}$$

where one of $x, y \neq 0$ since $x^2 + y^2 = \alpha \neq 0$ and one of $z, w \neq 0$, since $z^2 + w^2 = 1 - \alpha \neq 0$. Then

$$\operatorname{rk}(dF_p) = \dim \operatorname{Rowsp}(dF_p) = \dim \operatorname{Rowsp} \begin{bmatrix} 0 & 0 & 2z & 2w \\ 2x & 2y & 0 & 0 \end{bmatrix} = 2$$

so p is a regular point. Note that 1 is a regular value of $x^2 + y^2 + z^2 + w^2$, since for $p \in S^3$, df_p is of the same form as the first row of dF_p with not all of $x, y, z, w = 0$. Together with part (a), this tells us that $(f, g)^{-1}(1, \alpha)$ is a submanifold of $f^{-1}(1) = S^3$, i.e.

$$\{(x, y) | x^2 + y^2 = \alpha\} \cap S^3 = \{(x, y, z, w) | x^2 + y^2 = \alpha,\ z^2 + w^2 = 1 - \alpha\} = h^{-1}(\alpha, 1 - \alpha)$$

is a submanifold of S^3. It remains to show that $h^{-1}(\alpha, 1 - \alpha)$ has rank 2, consider $p = (x, y, z, w) \in h^{-1}(\alpha, 1 - \alpha)$, then

$$dh_p = \begin{bmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2z & 2w \end{bmatrix}$$

this has rank 2, since $x^2 + y^2 = \alpha \neq 0$ implies $(x, y) \neq (0, 0)$ and $z^2 + w^2 = 1 - \alpha \neq 0$ implies $(z, w) \neq (0, 0)$ implying that $(\alpha, 1 - \alpha)$ is regular. So by the constant rank level set theorem, $h^{-1}(\alpha, 1 - \alpha)$ is a submanifold of $\mathbb{R}^4$ having rank 2, since rank is a property of the manifold, $h^{-1}(\alpha, 1 - \alpha) \subset S^3$ has equal rank 2.

(c) The submanifold approaches the unit circle.