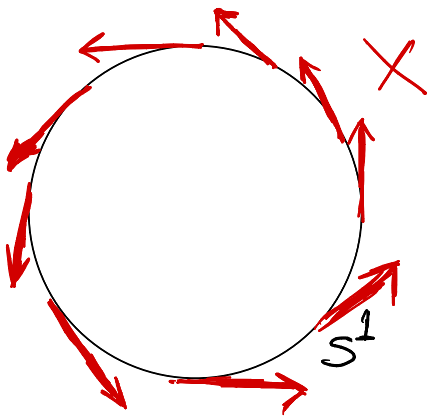

1.

We can rewrite each $x, y \in S^1$ as $(\cos(\theta), \sin(\theta))$ for $\theta \in [0, 2\pi)$ Define $\gamma(t) = (\cos(\theta + t), \sin(\theta + t))$, then we have $\gamma(0) = (\cos(\theta), \sin(\theta)) = (x, y)$ and $\frac{d}{dt}|\gamma(t) = (-\sin(\theta), \cos(\theta)) = (-y, x)$, so this path corresponds to the vector field X.

Now we compute $X|_U$, we get

$$\begin{aligned} X|_U &= \frac{d}{dt}|_{t=0}\gamma(t)(u)\frac{\partial}{\partial u} = \frac{d}{dt}|_{t=0}\frac{\cos(\theta+t)}{1-\sin(\theta+t)}\frac{\partial}{\partial u} = \frac{-(1-\sin(\theta+t))\sin(\theta+t)+\cos^2(\theta+t)}{(1-\sin(\theta+t))^2}\Big|_{t=0}\frac{\partial}{\partial u} \\ &= \frac{-\sin(\theta+t)+\cos^2(\theta+t)+\sin^2(\theta+t)}{1-\sin(\theta+t)^2}\Big|_{t=0}\frac{\partial}{\partial u} = \frac{1}{1-\sin(\theta)}\frac{\partial}{\partial u} = \frac{1}{1-y}\frac{\partial}{\partial u} \end{aligned}$$

On U we have the inverse sterographic projection is $u \mapsto (\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1})$. So we get that

$$X|_U = \frac{1}{1-y}\frac{\partial}{\partial u} = \frac{1}{1-\frac{u^2-1}{u^2+1}}\frac{\partial}{\partial u} = \frac{u^2+1}{2}\frac{\partial}{\partial u}$$

Finally, we compute $X|_{\tilde{U}}$. First note that $\frac{\partial \tilde{u}}{\partial u} = \frac{\partial}{\partial u}\frac{1}{u} = \frac{-1}{u^2} = -\tilde{u}^2$

$$X|_{\tilde{U}} = \frac{\partial \tilde{u}}{\partial u}X = -\tilde{u}^2\frac{u^2+1}{2}\frac{\partial}{\partial \tilde{u}} = -\frac{1+\tilde{u}^2}{2}\frac{\partial}{\partial \tilde{u}}$$

2. Define

$$\Theta_t : (e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i(\theta_1+at)}, e^{i(\theta_2+bt)})$$

To show it is a flow, $\Theta_0(\theta_1, \theta_2) = (\theta_1, \theta_2)$

$$\Theta_t \circ \Theta_s(e^{i\theta_1}, e^{i\theta_2}) = \Theta_t(e^{i(\theta_1+as)}, e^{i(\theta_2+bs)}) = (e^{i(\theta_1+as+at)}, e^{i(\theta_2+bs+bt)}) = \Theta_{t+s}(e^{i\theta_1}, e^{i\theta_2})$$

Equipping each copy of S^1 with the standard charts, gives us 4 charts U_i, smoothness is clear, since in any chart, we have $\Theta_t|_U : (\theta_1, \theta_2) \mapsto (\theta_1 + ta, \theta_2 + tb)$ is smooth. Then in any of the charts, we have

$$\frac{d}{dt}|_{t=0}\Theta_t(\theta_1, \theta_2) = \frac{d}{dt}|_{t=0}\theta_1 + at\frac{\partial}{\partial \theta_1} + \frac{d}{dt}|_{t=0}\theta_2 + bt\frac{\partial}{\partial \theta_2} = a\frac{\partial}{\partial \theta_1} + b\frac{\partial}{\partial \theta_2}$$

3. **(a)** We can compute

$$\hat{\theta_t}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

To show smoothness, we show smoothness between two of the hemisphere charts, the other ones follow similarly. We take the charts $U = \{x > 0\} \cap S^2$ and $\tilde{U} = y > 0 \cap S^2$, then

$$\tilde{\varphi}\hat{\theta}_t\varphi^{-1}(u, v) = \tilde{\varphi}\hat{\theta}_t \begin{bmatrix} \sqrt{1-u^2-v^2} \\ u \\ v \end{bmatrix} = \tilde{\varphi} \begin{bmatrix} \sqrt{1-u^2-v^2} \\ u\cos t + v\sin t \\ -u\sin t + v\cos t \end{bmatrix} = \begin{bmatrix} \sqrt{1-u^2-v^2} \\ -u\sin t + v\cos t \end{bmatrix}$$

Which is smooth since it is infinitely differentiable in u, v (note $u^2 + v^2 \neq 1$ by choice of chart). We check smoothness of $\hat{\theta_t}^{-1}$ on the same charts

$$\tilde{\varphi}\hat{\theta_t}^{-1}\varphi^{-1}(u, v) = \tilde{\varphi}\hat{\theta_t}^{-1} \begin{bmatrix} \sqrt{1-u^2-v^2} \\ u \\ v \end{bmatrix} = \tilde{\varphi} \begin{bmatrix} \sqrt{1-u^2-v^2} \\ u\cos t + -v\sin t \\ u\sin t + v\cos t \end{bmatrix} = \begin{bmatrix} \sqrt{1-u^2-v^2} \\ u\sin t + v\cos t \end{bmatrix}$$

(b) For any point $(x, y, z) \in S^2$, we can compute

$$X = \frac{d}{dt}|_{t=0}\hat{\theta}_t(x,y,z) = \frac{d}{dt}|_{t=0}\begin{bmatrix} x \\ y\cos t + z\sin t \\ -y\sin t + z\cos t \end{bmatrix} = \left.\begin{bmatrix} 0 \\ -y\sin t + z\cos t \\ -y\cos t - z\sin t \end{bmatrix}\right|_{t=0} = \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}$$

Then $X = 0$ only at $N = (1, 0, 0)$ and $S = (-1, 0, 0)$.

4. (a) To show injectivity, assume that $aX + bY + cZ = qX + rY + sZ$, Then we will show $a = q, b = r, c = s$.

$$\begin{aligned} 0 &\equiv (a-q)X + (b-r)Y + (c-s)Z = (a-q)(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}) + (b-r)(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}) + (c-s)(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) \\ &= (-z(b-r) + y(c-s))\frac{\partial}{\partial x} + (z(a-q) - x(c-s))\frac{\partial}{\partial y} + (-y(a-q) + x(b-r))\frac{\partial}{\partial z} \end{aligned}$$

So that setting $(x, y, z) = (1, 0, 0)$ we get $c - s = 0 = b - r$ by linear independence, and setting $(x, y, z) = (0, 1, 0)$ we get $a-q = 0$, hence proving injectivity. We have the correspondance $X \leftrightarrow i$, $Y \leftrightarrow j$ and $Z \leftrightarrow k$. We only need verify that the bracket matches the multiplication rules of cross product, since bilinearity and antisymmetry come from bracket properties, we simply check that $[X, Y] = Z, [Y, Z] = X$ and $[Z, X] = Y$. Computation below:

$$\begin{aligned} [X,Y] &= (z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})x\frac{\partial}{\partial z} + (z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})(-z\frac{\partial}{\partial x}) - \left((x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x})z\frac{\partial}{\partial y} + (x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x})(-y\frac{\partial}{\partial z})\right) \\ &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z \\ [Y,Z] &= (x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x})y\frac{\partial}{\partial x} + (x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x})(-x\frac{\partial}{\partial y}) - \left((y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})x\frac{\partial}{\partial z} + (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})(-z\frac{\partial}{\partial x})\right) \\ &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = X \\ [Z,X] &= (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})z\frac{\partial}{\partial y} + (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})(-y\frac{\partial}{\partial z}) - \left((z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})y\frac{\partial}{\partial x} + (z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z})(-x\frac{\partial}{\partial y})\right) \\ &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} = Y \end{aligned}$$

(b) take $\theta_t : (x, y, z) \mapsto (x\cos t - z\sin t, y, z\cos t + x\sin t)$. Then we can verify immediately that

$$\frac{d}{dt}|_{t=0}\theta_t : (x,y,z) = (-x\sin t - z\cos t)|_{t=0}\frac{\partial}{\partial x} + (-z\sin t + x\cos t)|_{t=0}\frac{\partial}{\partial z} = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} = Y$$

This is clearly smooth, since it is a trig polynomial. To verify it is a flow, we need only verify $\theta_t \circ \theta_s = \theta_{t+s}$ (I will omit the y-coordinate for brevity)

$$\begin{aligned} &\theta_a \circ \theta_b(x,y,z) = \theta_a(x\cos b - z\sin b, z\cos b + x\sin b) \\ &= (\cos a(x\cos b - z\sin b) - \sin a(z\cos b + x\sin b), \cos a(z\cos b + x\sin b) + \sin a(x\cos b - z\sin b)) \\ &= (x(\cos a\cos b - \sin a\sin b) - z(\cos a\sin b + \cos b\sin a), x(\cos a\sin b + \cos b\sin a) + z(\cos a\cos b - \sin a\sin b)) \\ &= (x\cos(a+b) - z\sin(a+b), x\sin(a+b) + z\cos(a+b)) \\ &= \theta_{a+b}(x,y,z) \end{aligned}$$

Here I used the sum of angle formulas $\cos a\cos b - \sin a\sin b = \cos(a+b) = \cos(a+b)$ and $\cos a\sin b + \cos b\sin a = \sin(a+b)$ for the second last equality. Hence this is the unique flow corresponding to Y (uniqueness of flow proven in lecture).