1.

2.

Lemma. I will use the following lemma to streamline my proofs for problems 3 and 4.
If $\psi : X \to Y$ is a homeomorphism, and $\sim$ is an equivalence relation on X, and $\approx$ a equivalence relation on Y, such that $\psi(a) \approx \psi(b) \iff a \sim b$, then $X/\sim \simeq Y/\approx$
proof. Define $\overline{\psi} : X/\sim \to Y/\approx$, by $\overline{\psi} : \overline{x} \mapsto \overline{\psi(x)}$, this is surjective since ψ is surjective and $\overline{\psi}$ is well defined/injective by definition of $\approx$. We can define $\overline{\psi^{-1}} : Y/\approx \to X/\sim$, in the same way. This is the inverse of $\overline{\psi}$, since $\overline{\psi}$ and $\overline{\psi^{-1}}$ are just restrictions to equivalence classes of ψ and ψ^{-1}. To show $\overline{\psi}$ is continuous, note that $\overline{\psi} = \pi_\approx \psi$. Let U be open in $Y/\approx$, then the preimage of U under $\pi_\approx$ is open by definition, so continuity follows from continuity of ψ. The proof for continuity of $\overline{\psi^{-1}}$ is the same.

3.

4. Note that the triangle is homeomorphic to the disc. We can insribe the triangle in a circle with radius R. Then for each point p, let q be the intersection of the ray through p and the origin with the boundary of the triangle. For each of these points we can map $p \mapsto \frac{Rp}{|q|}$ this is a homeomorphism since q varies smoothly with p and we have inverse $p \mapsto \frac{|q|p}{R}$, where q comes from inscribing the triangle in the circle, which is also continuous. It follows that the equivalence relation induced on D^2 is $e^{ix} \sim e^{ix}e^{\frac{2\pi}{3}} \sim e^{-ix}$, which can be seen by the picture and lemma. So that the dunce cap can be written as $D^2/\sim$.

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Now consider the maps $\mathbf{1}_{S^1}$ and

$$f : S^1 \to S^1$$
$$e^{ix} \mapsto \begin{cases} e^{3ix} & 0 \le x < \frac{4\pi}{3} \\ e^{-3ix} & \frac{4\pi}{3} \le x < 2\pi \end{cases}$$

Take the mapping cone

$$C_f = S^1 \times I/(x,0) \sim f(x), (x,1) \sim (y,1)$$

For each x, we have $f^{-1}(x) = \{e^{ix/3}, e^{i(x+2\pi)/3}, e^{-ix/3}\}$. We can then take the map $C_{\mathbf{1}_{S^1}} \to D^2$, where $(x,t) \mapsto (x, 1-t)$, this is a homeomorphism between the cone and disc, with the quotients in $D^2/\sim$ being the image of quotients of C_f under this map. Hence by the lemma $C_f \simeq D^2/\sim$ the dunce cap.

We have that $C_{\mathbf{1}_{S^1}}$ is contractible, using the homotopy $H((x,t),s) = (x, t(1-s))$, so it will suffice to show that $C_f \simeq C_{\mathbf{1}_{S^1}}$, and we have proven in class that homotopic maps have homotopic cones. I will show $f \sim \rho \sim \mathbf{1}_{S^1}$, where

$$\rho : e^{ix} \mapsto \begin{cases} e^{3ix} & 0 < x < 2\pi/3 \\ 1 & 2\pi/3 \le x < 2\pi \end{cases}$$

I will provide H_1 for the first equivalence $f \sim \rho$ and H_2 for the second $\rho \sim \mathbf{1}_{S^1}$.

$$H_1(x,t) : \begin{cases} x \mapsto f(x) & x < \frac{2}{3} - \frac{1}{3}t \text{ or } x > \frac{2}{3} + \frac{1}{3}t \\ x \mapsto f(\frac{2}{3} - \frac{1}{3}t) & \frac{2}{3} - \frac{1}{3}t \le x \le \frac{2}{3} + \frac{1}{3}t \end{cases}$$
$$H_2(x,t) : \begin{cases} x \mapsto f(\frac{x}{1+2t}) \end{cases}$$