1. (a) Let $\mathcal{B} = \{U_i\}_1^\infty$ be a countable basis for X, then for each U_i assign some $x_i \in U_i$. Let $y \in X$, and V a neighborhood of y, then since $\mathcal{B}$ contains a neighborhood base for y, we have some $x_i \in U_i \subset V$, hence $V \cap \{x_i\}_1^\infty \neq \emptyset$. Since V was arbitrary we can conclude that $y \in \overline{\{x_i\}_1^\infty}$. Since y was arbitrary $X = \overline{\{x_i\}_1^\infty}$. $\square$

(b) Let $\{x_i\}_1^\infty$ be a countable dense subset of X, now define $\mathcal{B} := \{N_{\frac{1}{n}}(x_i) \mid n, i \in \mathbb{N}\}$, where $N_r(x) := \{y \in X \mid d(x,y) < r\}$. There is an obvious bijection $\mathcal{B} \to \mathbb{N} \times \mathbb{N}$, $N_{\frac{1}{n}}(x_i) \mapsto (n,i)$so in particular $\mathcal{B}$ is countable. Now let $y \in X$, and let U be a neighborhood of y. Then $U \supset N_r(y)$ for some $r > 0$. Then there exists some $x_i \in N_r(y)$ by density, by definition of the set, $d(x_i, y) < r$, so that $N_{r-d(x_i,y)}(x_i) \subset N_r(y)$ by the triangle inequality. Since we have $0 < r - d(x_i, y)$, we may choose N, such that $\frac{1}{N} < r - d(x_i, y)$ which gives the desired inclusion to prove that $\mathcal{B}$ is a basis.

$$N_{\frac{1}{N}}(x_i) \subset N_{r-d(x_i,y)}(x_i) \subset N_r(y) \subset U \quad \square$$

2. (a) Since we are in a metric space we may use the $\epsilon-\delta$ definition of continuity, where it is tautological that the function $d(x, \cdot) : X \to X$, since continuity is defined in terms of the distance function. Given this it suffices to show that for $x_1, x_2 \in X$, $|d_F(x_1) - d_F(x_2)| \leq d(x_1, x_2)$ to show continuity. Without loss of generality we may assume that $d_F(x_1) \geq d_F(x_2)$. Note that for any $y \in F$, $d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$, so in particular we have for any $y \in F$ that $d_F(x_1) \leq d(x_1, x_2) + d(x_2, y)$, which of course implies that $d_F(x_1) \leq d(x_1, x_2) + d_F(x_2)$, taken together we get the desired inequality to prove continuity of d_F:

$$|d_F(x_1) - d_F(x_2)| = d_F(x_1) - d_F(x_2) \leq d(x_1, x_2) + d_F(x_2) - d_F(x_2) = d(x_1, x_2)$$

$x \in F \implies d_F(x) = 0$ is immediate from $d(x,x) = 0$. Since F^c is open, any $x \in F^c$ has some neighborhood $N_\epsilon(x) \subset F^c$, $\epsilon > 0$ implying that $d_F(x) \geq \epsilon > 0$. $\square$

(b) The difference of continuous functions is continuous, we have from part (a) that d_{F_i} are continuous which implies continuity of g. Now let $x \in F_2 \subset F_1^c$, then by part (a), we have $d_{F_2}(x) = 0, d_{F_1}(x) > 0$, so that $g(x) = d_{F_1}(x) > 0$. If $x \in F_1 \subset F_2^c$, then by part (a) we have $d_{F_1}(x) = 0$ and $d_{F_2}(x) > 0$, implying that $g(x) = -d_{F_2}(x) < 0$. $\square$

(c) Metric spaces are Hausdorff and hence T_1, so it suffices to show that any two closed sets can be seperated by disjoint open sets. Let F_1, F_2 be disjoint closed sets and define g as in (b), we can use from part (b) that g is continuous. This implies that $g^{-1}(0,\infty)$ and $g^{-1}(-\infty,0)$ are open. Furthermore, we have from part (b) that $F_1 \subset g^{-1}(-\infty, 0)$ and $F_2 \subset g^{-1}(0,\infty)$ we are done since these sets are disjoint:

$$g^{-1}(-\infty,0) \cap g^{-1}(0,\infty) = g^{-1}((-\infty,0) \cap (0,\infty)) = g^{-1}\emptyset = \emptyset \quad \square$$

3. This follows simply by rewriting the sets S and $\tilde{S}$.

$$S \stackrel{\text{def}}{=} \{W \cap \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}^{-1}(U_{\alpha,i}) \mid A \text{ is an arbitrary index set, } n \in \mathbb{N}, U_{\alpha,i} \in Y_{\alpha,i}\}$$

$$\tilde{S} \stackrel{\text{def}}{=} \{\bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}|_W^{-1}(U_{\alpha,i}) \mid A \text{ is an arbitrary index set, } n \in \mathbb{N}, U_{\alpha,i} \in Y_{\alpha,i}\}$$

Then we can write

$$W \cap \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}^{-1}(U_{\alpha,i}) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n \left(f_{\alpha,i}^{-1}(U_{\alpha,i}) \cap W\right) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n \left(f_{\alpha,i}^{-1}(U_{\alpha,i} \cap f(W))\right) = \bigcup_{\alpha \in A} \bigcap_{i=1}^n f_{\alpha,i}|_W^{-1}(U_{\alpha,i})$$

So that S and $\tilde{S}$ define the same set. $\square$

4. (a) $\implies$ (b) This follows from the f_i being continuous on the weak topology they generate, a function is continuous if and only if for any net $x_\alpha \to x \implies f(x_\alpha) \to f(x)$ from Course Notes Prop 1.59. $\square$

(b) $\implies$ (a) Suppose for contraposition that $x_\alpha \not\to x$. Then there exists some neighborhood U of x, such that for any $\alpha_0 \in A$, there exists some $\alpha \geq \alpha_0$, such that $x_\alpha \notin U$. We may assume without loss of generality that U is an open neighborhood by passing to an open subset containing x. By definition of the weak topology, $U = \bigcup_{i \in I} \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$ for open $U_{i_j} \in \mathcal{T}(Y_{i_j})$, x must lie in some $\bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$, so we may once again without loss of generality rechoose $U = \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$. Now suppose for $j = 2, \ldots, N$ there exists some α_0^j, such that for any $\alpha \geq \alpha_0^j$ we have $f_{i_j}(x_\alpha) \in U_{i_j}$ (if no such α_0^j exists for some j, then we are done since then $f_{i_j}(x_\alpha) \not\to f_{i_j}(x_\alpha)$ so we may assume not), then taking $\gamma \geq \alpha_0^j$, $\forall j \in 2, \ldots, N$, we have for any α_0, there exists some $\alpha \geq \alpha_0, \gamma$, such that $x_\alpha \notin U$, implying that $x_\alpha \notin \bigcap_{j=1}^{N} f_{i_j}^{-1}(U_{i_j})$, and since x_α lies in $f_{i_2}^{-1}(U_{i_2}), \ldots, f_{i_N}^{-1}(U_{i_N})$ we must have that $x_\alpha \notin f_{i_1}^{-1}(U_{i_1})$, equivalently $f_{i_1}(x_\alpha) \notin U_{i_1}$. But U_{i_1} is open and contains $f_{i_1}(x)$, so we necessarily have $f_{i_1}(x_\alpha) \not\to f_{i_1}(x)$. $\square$

5. (a) $\implies$ (b) Let (x, y) be an accumulation point of $\{(x, x)\}$, then there is a $(x_\alpha, x_\alpha)_{\alpha \in A} \to (x, y)$, from this we may define the net $(x_\alpha)_{\alpha \in A}$, where we say $x_\alpha \leq x_{\alpha'}$ when $(x_\alpha, x_\alpha) \leq (x_{\alpha'}, x_{\alpha'})$. Then if U is a neighborhood of x, $U \times X$ is a neighborhood of (x, y), so for some α_0 we have for all $\alpha \geq \alpha_0$, $(x_\alpha, x_\alpha) \in U \times X$, implying that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$, i.e. $x_\alpha \to x$. By the same argument we find that $x_\alpha \to y$, but since X is Hausdorff we know that nets have unique limits, and hence $x = y$, implying that $(x, y) \in \{(x, x)\}$, i.e. $\operatorname{acc}\{(x, x)\} \subset \{(x, x)\}$ is closed. $\square$

(b) $\implies$ (a) Let $x \neq y \in X$, then $(x, y) \in \{(x, x)\}^c$ is open. Since $\{U \times V \mid U, V \text{ open in } X\}$ is a basis for the product topology, we have

$$\{(x, x)\}^c = \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} U_\alpha^i \times V_\alpha^i = \bigcup_{\alpha \in A} \left(\bigcap_{i=1}^{N_\alpha} U_\alpha^i \right) \times \left(\bigcap_{i=1}^{N_\alpha} V_\alpha^i \right) = \bigcup_{\alpha \in A} U_\alpha \times V_\alpha$$

Here U_α^i, V_α^i are open sets in X, the last equality simply comes from relabelling since finite products of open sets are open we have U_α, V_α open in X for all $\alpha \in A$. Using this presentation of $\{(x, x)\}^c$ we have for some $\alpha \in A$, $(x, y) \in U_\alpha \times V_\alpha$, and furthermore $U_\alpha \cap V_\alpha = \emptyset$, since $U_\alpha \times V_\alpha \subset \{(x, x)\}^c$. Since U_α, V_α are open in X and we have $x \in U_\alpha, y \in V_\alpha$ the empty intersection proves that X is Hausdorff. $\square$

6. Suppose that $x \in \overline{\{x \in X \mid f(x) = g(x)\}}$, equivalently there is some net $x_\alpha \to x$, such that $f(x_\alpha) = g(x_\alpha)$, $\forall \alpha$. By continuity of f, g we have $f(x_\alpha) \to f(x)$, $g(x_\alpha) \to g(x)$. Define a new net y_α, where $y_\alpha = f(x_\alpha) = g(x_\alpha)$, such that $y_{\alpha'} \geq y_\alpha$ when $\alpha' \geq \alpha$, then $y_\alpha \to f(x), y_\alpha \to g(x)$, since Y is Hausdorff, nets in Y have unique limits implying that $f(x) = g(x)$, so that $x \in \{x \in X \mid f(x) = g(x)\}$, which suffices to show that $\{x \in X \mid f(x) = g(x)\} = \overline{\{x \in X \mid f(x) = g(x)\}}$. $\square$