1. First let n denote the dimension of M, and let $(U_\alpha, x^i_\alpha)_\alpha$ be a smooth atlas for M, I will denote the coordinate maps as ϕ_α respectively. This induces a smooth atlas $(\pi^{-1}(U_\alpha), (x^i_\alpha, v^i_\alpha))$ for TM, I will denote the coordinate maps as φ_α respectively. Now take charts $(\pi^{-1}(U_\alpha), (x^i_\alpha, v^i_\alpha)), (\pi^{-1}(U_\beta), (x^i_\beta, v^i_\beta))$, (assume that they have non-empty intersection else there is nothing to check) the change of coordinates is given by

$$(x^i_\beta, v^i_\beta) = \varphi_\beta\varphi_\alpha^{-1}(x^i_\alpha, v^i_\alpha) = (\phi_\alpha\phi_\beta(x^i_\alpha), \frac{\partial x^i_\beta}{\partial x^j_\alpha}v^j_\alpha)$$

We want to show that the determinant of the following block diagonal matrix (A) is positive:

$$A := \begin{bmatrix} \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1\le i,j\le n} & \left(\frac{\partial x^i_\beta}{\partial v^j_\alpha}\right)_{1\le i,j\le n} \\ \left(\frac{\partial v^i_\beta}{\partial x^j_\alpha}\right)_{1\le i,j\le n} & \left(\frac{\partial v^i_\beta}{\partial v^j_\alpha}\right)_{1\le i,j\le n} \end{bmatrix}$$

By definition of the coordinate change on TM, it is apparent that

$$\det\left(\frac{\partial x^i_\beta}{\partial v^j_\alpha}\right)_{1\le i,j\le n} = 0$$

since $\phi_\alpha\phi_\beta(x^i_\alpha)$ is independent of each v^i_α. It follows that

$$\det A = \det\left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1\le i,j\le n} \det\left(\frac{\partial v^i_\beta}{\partial v^j_\alpha}\right)_{1\le i,j\le n}$$

But then we can read off of the change of coordinates formula for TM that $\frac{\partial v^i_\beta}{\partial v^j_\alpha} = \frac{\partial x^i_\beta}{\partial x^j_\alpha}$, so that

$$\det A = \left(\det\left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)_{1\le i,j\le n}\right)^2 > 0$$

We can say > 0 here, since the change of coordinates is smooth and invertible with smooth inverse the inverse function theorem tells us its Jacobian is invertible.

2. First we give an orientation on M, consider the coordinate charts

$$U_1 \times U_2, \quad \tilde{U_1} \times U_2, \quad U_1 \times \tilde{U_2}, \quad \tilde{U_1} \times \tilde{U_2}$$

Given by the standard $\theta, \tilde{\theta}$

We equip each copy of S^1 with the standard charts, then let U denote that chart $(0, 2\pi) \times (0, 2\pi)$, with coordinates (θ_1, θ_2). This is an It follows that in U, we have

$$dw = \frac{\partial w}{\partial \theta_1}d\theta_1 + \frac{\partial w}{\partial \theta_2}d\theta_2 = -\sin\theta_1 d\theta_1$$
$$dy = \frac{\partial y}{\partial \theta_1}d\theta_1 + \frac{\partial y}{\partial \theta_2}d\theta_2 = -\sin\theta_2 d\theta_2$$

Using this, we compute $\omega = \sin^2\theta_1 \sin^2\theta_2 d\theta_1 \wedge \theta_2$, so that

$$\int_{T^2}\omega = \int_{T^2\setminus S^1\cup S^1}\omega = \int_U \omega = \int_0^{2\pi}\int_0^{2\pi} \sin^2\theta_1\sin^2\theta_2 d\theta_1 d\theta_2$$
$$= \int_0^{2\pi} \pi\sin^2\theta_2 d\theta_2 = \pi^2$$