Problem 1. **(a)** *Suppose $f, g : \mathbb{C}^m \to \mathbb{C}$ are holomorphic, and for some open subset $V \subset \mathbb{C}^n$ we have $f|_V = g|_V$, show that $f = g$.*

Proof. The Weierstrass theorem can be interpreted as non-zero holomorphic functions (in one variable) having isolated zeros, thus the result is already proved in 1 dimension and we can proceed by induction. define $h = f - g$, and consider h as a functions of $(\mathbf{u}, z)$ where $\mathbf{u} \in \mathbb{C}^n$, we consider an arbitrary point $(\mathbf{u_0}, z_0) \in \mathbb{C}^{n+1}$. fixing $(\mathbf{u}', z') \in V$ we have that $h(\mathbf{u}', z) \equiv 0$ on $\mathbb{C}$ so in particular $(\mathbf{u}', z_0) = 0$ by the Weierstrass theorem, there is some open neighborhood of this point $U \subset V$, and by the same logic for each $(\mathbf{u}, z') \in U$ we have $h(\mathbf{u}, z_0) = 0$. Thus fixing $z = z_0$ we have that $h(\mathbf{u}, z_0)$ is locally zero on $\mathbb{C}^n$, so by inductive hypothesis it is globaly zero and $h(\mathbf{u_0}, z_0) = 0$. $\square$

(b) *Prove the Maximum Modulus Principle.*

Proof. Consider any closed curve γ containing z_0 with $\gamma \subset U$. It follows that $|f(z_0)| \geq |f(z)|$ on all of γ. Then since $|f(z_0)|$ is constant it is holomorphic, so that

$$0 \leq \int_\gamma |f(z_0)| - |f(z)| = \int_\gamma |f(z_0)| - \int_\gamma |f(z)| \overset{\text{Cauchy's Thm.}}{=} \int_\gamma -|f(z)| \leq 0$$

For any point $w \in U$, there is a closed curve $\gamma \subset U$ passing through w, if $|f(w)| < |f(z)|$, then the above integral would be positive. $\square$

(c) *Prove elliptic regularity, i.e. complex differentiable implies smooth.*

Proof. It will suffice to show that holomorphic functions are equal to their power series, let z_0 be in our domain, and γ be a circle of radius r about p in our domain (with z_0 in the interior of $gamma$), then by Cauchy's integral formula:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - p} \frac{z-p}{z - p - (z_0 - p)} dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - p} \frac{1}{1 - \frac{z_0 - p}{z - p}} dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - p} \sum_0^\infty \left(\frac{z_0 - p}{z - p}\right)^n = \frac{1}{2\pi i}\int_\gamma \sum_0^\infty \frac{f(z)}{(z-p)^{n+1}}(z_0 - p)^n \end{aligned}$$

Since f is continuous and γ is compact, we have that $\sup_\gamma |f| = M$, and it is immediate that $|z_0 - p| < r = |z - p|$ for all $z \in \gamma$. In particular this implies that the sum of functions is abslutely convergent. Swapping the sum and integral gives us

$$f(z_0) = \frac{1}{2\pi i} \sum_0^\infty (z-p)^n \int_\gamma \frac{f(z)}{(z-p)^{n+1}} dz$$

$\square$

Remarks.

- *In (a) the proof technique is familiar to me, it can be used to prove topological properties of varieties. e.g. cartesian products of infinite sets are dense in $\mathbb{A}^n$.*
- *(c) is just copying the methods of proof in GH, the algebra is explicated and uniform convergence, although obvious, is explained more carefully.*

Problem 2. *Let $f(z,w)=\sin(w^2)-z$, find the Weierstrass polynomial g, such that $f=gh$*

Proof. Follow the process of the proof of the Weierstrass preparation theorem, in which case we find that $b_1=\sqrt{\arcsin(z)}, b_2=-\sqrt{\arcsin(z)}$. This implies that $g=w^2-\arcsin(z)$, here we choose the branch of $\arcsin$ with $0\mapsto 0$. Now we can once again take h as in the proof of the theorem, so that

$$h(0,0)=\frac{1}{2\pi i}\int_\gamma \frac{\sin(t^2)-0}{t^2-\arcsin(0)}\frac{1}{t}dt=\frac{1}{2\pi i}\int_\gamma \frac{\sin(t^2)}{t^3}dt \overset{\text{Laurent Series + res. Thm}}{=} 1\neq 0$$

□

Remarks.

- Away from $\sin(w^2)=z$ we have $h(z,w)=\frac{\sin(w^2)-z}{w^2-\arcsin(z)}$, at $\sin(w^2)=z$, we need to be more careful and compute $h(z,w)$ from the definition $h(z,w)=\frac{1}{2\pi i}\int_\gamma\frac{\sin(t^2)-z}{t^2-\arcsin(z)}\frac{1}{t-w}dt$ as above.
- Near 0, we have that $\arctan(z)$ satisfies the differential equation $\frac{d}{dz}\phi(z)=\frac{1}{\sqrt{1+z^2}}$, so we can define it as $\int_0^z\frac{1}{\sqrt{1+t^2}}dt$, this is clearly holomorphic near zero and $\arcsin(z)=\arctan(\frac{z}{\sqrt{1-z^2}})$ is also holomorphic near zero. Alternatively, we know that $\arcsin$ exists and is holomorphic by the *Holomorphic inverse function theorem.*

Problem 3. *Find τ, such that for $\Lambda=\mathbb{Z}\oplus i\tau\mathbb{Z}$ we have $C^\times/z\sim 2z\cong\mathbb{C}/\Lambda$ as complex manifolds.*

Proof. It is intuitive that the equivalence relation on $\mathbb{C}^\times$ defines a torus, chasing the image through the complex logarithm will be sufficient to describe which torus. The following general observation is useful

- Let G be a group of *biholomorphic* maps acting on a complex manifold M (totally discontinuous, free, etc.) suppose $(U_\alpha,\phi_\alpha)_\alpha$ is an atlas for M where the quotient map π is injective (it is straightforward that this always exists), then $(\pi(U_\alpha),\phi_\alpha\circ\pi|_{U_\alpha}^{-1})_\alpha$ is an atlas for M/G (in practice we can remove many sets from this atlas).

It is clear that multiplication by 2^k is holomorphic (a diffeomorphism) on $C^\times$, so it is easy to define charts on the quotient. I claim that $\tau=\frac{2\pi}{\log 2}$. Since multiplication by a constant is holomorphic, it suffices to show that $\mathbb{C}^\times/z\sim 2z\cong\mathbb{C}/\log 2\mathbb{Z}\oplus 2\pi i\mathbb{Z}$. Taking $z\mapsto\log|z|+i\arg z$ it is obvious that this map defines a homeomorphism. We have charts on $\mathbb{C}^\times$ being

$$\begin{aligned}
U_1&=\big(\mathbb{C}^\times\setminus\{\arg z=0\}\big)\cap\{1<|z|<2\}\\
U_2&=\big(\mathbb{C}^\times\setminus\{\arg z=0\}\big)\cap\{3/4<|z|<3/2\}\\
U_3&=\big(\mathbb{C}^\times\setminus\{\arg z=\pi\}\big)\cap\{1<|z|<2\}\\
U_4&=\big(\mathbb{C}^\times\setminus\{\arg z=\pi\}\big)\cap\{3/4<|z|<3/2\}
\end{aligned}$$

and the inclusion into $\mathbb{C}$ as ϕ_i. Now the above note gives a simple definition of charts on $\mathbb{C}^\times/z\sim 2z$. We can also use the standard charts on the Torus.

Now the charts on our quotient manifold, as well as the complex torus given by τ are nicely symmetric, as such we can check on a single chart. Locally on U_1, V_1 the maps are the complex logarithm away from its branch cut, and inversely the complex exponential, both are holomorphic. □

Remarks.

- Not all complex tori are biholomorphic
- Note that a biholomorphic map $\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ induces a biholomorphic map of their universal covering spaces, i.e. $\mathbb{C} \to \mathbb{C}$ with $\Lambda_1 \to \Lambda_2$
- All biholomorphic maps $\mathbb{C} \to \mathbb{C}$ are affine
- Thus Λ_1 and Λ_2 must be related via an affine transformation $z \mapsto az + b$, shifting by b we require that $\Lambda_1 = a\Lambda_2$ for a complex scalar.

Problem 4. *Show that $\mathbb{CP}^n$ is a complex manifold, describe the transition functions.*

Proof. The charts are given by $U_i = \{z_i \neq 0\}$ and $\varphi_i : \mathbf{z} \mapsto (\frac{z_0}{z_i}, \ldots, \hat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i})$. The coordinate change is given by φ_{ij} being multiplication by z_j/z_i, note that although we cannot recover the original coordinates, this ratio is available from the local information. $\square$

Do the same for the tautological line bundle

Proof. Here the charts and coordinate changes are nearly the same, but we pair each U_i with a copy of $\mathbb{C}^{n+1}$, and a trivialization λ, representing $\lambda\overline{z}$, thus the coordinate change on λ is given by multiplication by $\frac{x_i}{x_j}$ as well. $\square$

Remark.

- The sections of the tautological bundle can be identified with the space of -1 degree homogenous polynomials in $n+1$ variables, hence can be written as $\mathcal{O}(-1)$. This also refects that there are no global sections on the tautological line bundle.