Notes for 2025

January 9, 2025

1 Algebraic Topology

exercise - Show that $\mathbf{h} = (h_i)$, $h_i : C_i \to C'_{i+1}$, then $h_{i-1} \circ d_i + d'_{i+1} \circ h_i$ is a chain map.

proof - We need to check that $d'_i \circ f_i = f_{i-1} \circ d_i$, in other words we need to show

$$d'_i \circ (h_{i-1} \circ d_i + d'_{i+1} \circ h_i) = (h_{i-2} \circ d_{i-1} + d'_i \circ h_{i-1}) \circ d_i$$

Since we are in a module, we can distribute d'_i on the left hand side, rewriting the condition as

$$d'_i \circ h_{i-1} \circ d_i + d'_i \circ d'_{i+1} \circ h_i = h_{i-2} \circ d_{i-1} \circ d_i + d'_i \circ h_{i-1} \circ d_i$$

So it will suffice to show that

$$d'_i \circ d'_{i+1} \circ h_i = h_{i-2} \circ d_{i-1} \circ d_i$$

But this is trivial since "$d^2 = 0$"

exercise - A homotopy of chains is an equivalence relation

proof - We will prove the following items: reflexivity, symmetry, transitivity

- To see that $f \sim f$, take each to be the zero map, $h_i = 0, \forall i$. In this case the diagram with maps given by $\mathbf{h}$ is immediate.
- Suppose that $f \sim g$, then we have some $\mathbf{h}$, such that $g_i - f_i = h_{i-1} \circ d_i + d'_{i+1} \circ h_i$ for each i. Since h_i are morphisms of modules, so are $-h_i$, so in particular we have $-\mathbf{h} := (-h_i)_i$, so that

$$\begin{aligned} f_i - g_i = -(g_i - f_i) = -(h_{i-1} \circ d_i + d'_{i+1} \circ h_i) &= -h_{i-1} \circ d_i + -d'_{i+1} \circ h_i \\ &= -h_{i-1} \circ d_i + d'_{i+1} \circ -h_i \end{aligned}$$

  The last line follows since d'_{i+1} is linear.
- Suppose that $f \sim g \sim r$, and let $\mathbf{h}, \mathbf{k}$ be respective witnesses of these homotopies. Then we have

$$\begin{aligned} r_i - g_i &= k_{i-1} \circ d_i + d'_{i+1} \circ k_i \\ g_i - f_i &= h_{i-1} \circ d_i + d'_{i+1} \circ h_i \end{aligned}$$

  This furnishes

$$r_i - f_i = k_{i-1} \circ d_i + d'_{i+1} \circ k_i + h_{i-1} \circ d_i + d'_{i+1} \circ h_i = (k_{i-1} + h_{i-1}) \circ d_i + d'_{i+1} \circ (h_i + k_i)$$

  So we have the homotopy $r \sim f$ via $\mathbf{h} + \mathbf{k}$, we are done since we already proved symmetry.

exercise - Represent diagrammatically what a homotopy of chain maps means
proof - Here we want $\mathbf{h}$, so that

$$\cdots TODO \cdots$$

exercise - Show that the following two chain complexes are homotopic:

$$0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{(\cdot, 2\cdot)} \mathbf{Z} \oplus \mathbf{Z} \longrightarrow 0 \oplus \mathbf{Z}/(2) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\quad 2\cdot \quad} \mathbf{Z} \longrightarrow \mathbf{Z}/(2) \longrightarrow 0$$

proof - Let each f_i be the projection of the second coordinate, and each g_i the inclusion into the second coordinate. In thise case we have

$$\mathbf{1}_{C'} - \mathbf{fg} = \mathbf{0}$$

and hence $\mathbf{h} = (0)_i$ witnesses the homotopy. The slightly harder case is the other direction. Define $h_2 = h_0 = h_{-1} = 0$, and $h_1 : (m, n) \mapsto m$. By definition of $\mathbf{f}, \mathbf{g}$, we have $1_{C,i} - g_i f_i : (m, n) \mapsto (m, 0)$, so we just need to check that our given $\mathbf{h}$ satisfies this.

$$\begin{aligned}
(d_3 h_2 + h_1 d_2)(m, n) &= 0 + h_1(m, 2n) = (m, 0) \\
(d_2 h_1 + h_0 d_1)(m, n) &= d_2(m, 0) + 0 = (m, 0) \\
(d_1 h_0 + h_{-1} d_0)(0, n) &= 0 + 0 = (0, 0)
\end{aligned}$$

This verifies the homotopy.

CAUTION!! - The connection (this is what we call the diagonal maps often denoted $\mathbf{h}$) diagram need to not commute.

Theorem - If chain complexes C, C' are homotopic, then they have the same Homology Modules.

proof - Recall that

$$H^i := \frac{\ker(d_i)}{\mathrm{Im}(d_{i+1})}$$

Now let the homotopy be given by $\mathbf{f} : C \to C', \mathbf{g} : C' \to C$, there is a natural induced map of f_i on H^i, given by restricting f_i to $\mathrm{Im}(d_{i+1})$, then taking the unique map from the quotient by the kernel of d_i, which exists and is unique by the first isomorphism theorem. Calling this induced map $f_{i,*}$, we need to check that $f_{i,*}$ maps into H'_i.

2 Commutative Algebra

Hilbert's Basis Theorem - if R is Noetherian, then $R[X]$ is Noetherian

proof. Fix an ideal J of $R[X]$, then $J \cap \{\deg = 0\} \subset R$ is finitely generated. It is easy to verify by induction that $J \cap \{\deg \leq m\}$ is finitely generated for any m. Let

$$I = \{b | f \in J \text{ and } f = bx^n + a_{n-1}x^{n-1} + \cdots + a_0\}$$

$I \subset R$ is an ideal, thus is finitely generated, $I = (b_i)_1^n$. Choose for each b_i some f_i where b_i is the leading coefficient. Then letting $m = \max\{\deg_{1\leq i \leq n} f_i\}$it is easy to verify using polynomial division and induction that

$$J = (f_i)_1^n + J \cap \{\deg \leq m\}$$

This proves J is finitely generated. $\square$

It is easy to understand how someone could come up with this proof when we do it by first noticing that $J \cap \{\deg \leq m\}$ is always finitely generated. This motivates the clever choice of ideal of leading coefficients since we want to reduce the degree.

2.1 Tensor Products

Notes -

- $(\cdot) \otimes N : \mathbf{Mod_R} \to \mathbf{Mod_R}$ is a right exact covarient functor.

exercise - $\mathbf{Z}/(10) \otimes \mathbf{Z}/(12) \simeq \mathbf{Z}/(2)$

proof. Define $\varphi : \mathbf{Z}/(10) \otimes \mathbf{Z}/(12) \to \mathbf{Z}/(2),\ a \otimes b \mapsto ab$. This is a homomorphism (mult is bilinear) and its clearly onto. To see the kernel is trivial, suppose $a \otimes b \mapsto 0$, then $a \otimes b = 2(k \otimes \ell)$ by bilinearity. Then

$$2(k \otimes \ell) = 2(k \otimes 25\ell) = 50(k \otimes \ell) = 5(10k \otimes \ell) = 5(0 \otimes \ell) = \mathbf{0}$$

2.2 Localizations

Definition - $S \subset R$ multiplicatively closed, then $S^{-1}R$ has elements a/s, where $a_1/s_1 = a_2/s_2$ exactly when there exists some $s \in S$, such that $s(s_2a_1 - s_1a_2) = 0$ We equip $S^{-1}R$ withh the usual operations for fractions. This comes with the natural embedding

$$R \hookrightarrow S^{-1}R$$

exercise - $R \hookrightarrow S^{-1}R$ is injective if and only if S does not contain zero divisors.

Proof - Suppose that S contains zero divisor s, then for some $r \in R$, $sr = 0$, hence $r \mapsto r/1 = 0/1$ which is the image of zero. Conversely, assume S has no zero divisors, then for any s,

$$s(a1 - r1) = 0 \iff a1 = r1 \iff a = \rho$$

Definition of Localization as a Universal Property - $S^{-1}R$ is initial among A-algebras B, where $S \hookrightarrow B^\times$ (Note an R-Algebra is just a ring containing R thats also an R-Module)

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr] & S^{-1}R \arrow[d, dashed] & S \to B^\times \\ & B & \end{tikzcd}$$

Definition of Localizations of Modules - satisfies the same diagram as localizations of R (where we swap M for R and B for some arbitrary R-module N).

- To show that a localization of an R-module exists we follow the exact same construction as for rings, but here we may relabel $a \in R \rightsquigarrow m \in M$

exercise - Show that $S^{-1}\left(\bigoplus_i M_i\right) = \bigoplus_i S^{-1}M_i$ but not necessarily for infinite direct products

Use the universal property. The explicit details of the proof are obvious (here we have $1/s : a \mapsto b$, such that $sb = a$). To show it does not hold for direct products,

$$(1, 1/2, 1/3, \ldots) \in \prod_1^\infty \mathbf{Q} \setminus \mathbf{Q} \prod_1^\infty \mathbf{Z}$$

Since any element of $\mathbf{Q} \prod_1^\infty \mathbf{Z}$ has a "clearable denomenator"

3 Category Theory

General notes and thoughts -

- Contravarience is naturally present in a pullback.
- A functor is right exact when for

$$M' \to M \to M'' \to 0$$

an exact sequence (i.e. M' maps into the kernel of $M \to M''$, M maps onto M''), then the sequence of induced maps is exact

$$F(M') \to F(M) \to F(M'') \to 0$$

Categorical definition of a product - Here P is the product of X, Y, note that P is determined up to unique isomorphism.

$$\begin{tikzcd} & A \arrow[d, dotted, "\bar{f}"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

Jargon - A (covarient) functor is called faithful when for all A, B, $\mathrm{Mor}(A, B) \to \mathrm{Mor}(F(A), F(B))$ is injective, and full if it is surjective. An example of a full subcategory (under the inclusion map) is finitely generated R-Modules as a subcategory of R-Modules.

exercise - Initial objects are universal.

proof. Suppose that A, B are two initial objects. There exist unique maps between them, when com posed these are maps from the oobjects to themselves, any map from the object to itself must be the identity by uniqueness of maps into an initial object, hence isomorphic. Uniqueness of this isomorphism follows by definition of initial object. *The proof to show final objects are universal is nearly identical.*

example - Initial and final objects in **Set**, **Ring**, **Top**

- In **Set** and **Top** singletons are final.
- In **Ring**, 0 is initial and final.

Definition of fibered product - The following is the definition of the fibered product $(P, h, g) = X \times_Z Y$

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "h"'] & Z \end{tikzcd}$$