1. This question is handwritten due to having many pictures.

2. Define $f : G_2 \to G_1$ via $\mu \mapsto \alpha\beta$, and $\lambda \mapsto \alpha\beta\alpha$, to see that this map gives a well defined group homomorphism we need to check that the relation is satisfied, this simply follows from $f(\lambda)^2 = (\alpha\beta\alpha)^2 = (\alpha\beta)^3 = f(\mu)^3$, so that f gives a well defined group homomorphism. Now we define $g : G_1 \to G_2$ via $g(\alpha) = \mu^2\lambda^{-1}$, and $g(\beta) = \lambda\mu^{-1}$, since γ can be written in terms of the relations from α, β the value of $g(\gamma)$ is just defined as $g(\alpha)g(\beta)g(\alpha)^{-1} = \mu\lambda\mu^{-2}$. To check that g gives a group homomorphism we just need to check it satisfies the relations of G_1 as follows.

$$g(\alpha) = \mu^2\lambda^{-1} = \mu^3\mu^{-1}\lambda^{-1} = \lambda^2\mu^{-1}\lambda^{-1} = \lambda\mu^{-1}\mu\lambda\mu^{-2}\mu\lambda^{-1} = g(\beta)g(\gamma)g(\beta)^{-1}$$

Where we use the relation $\mu^3 = \lambda^2$ in the third equality.

$$g(\beta) = \lambda\mu^{-1} = \lambda^2\lambda^{-1}\mu^{-1} = \mu^3\lambda^{-1}\mu^{-1} = \mu\lambda\mu^{-2}\mu^2\lambda^{-1}\mu^2\lambda^{-1}\mu^{-1} = g(\gamma)g(\alpha)g(\gamma)^{-1}$$

Where again the relation $\mu^3 = \lambda^2$ is used in the third equality, moreover g is defined so as to define the third relation, thus g gives a well defined group homomorphism. To see $G_1 \cong G_2$ it will now suffice to show that f and g are inverse isomorphisms, since they are homomorphisms it suffices to check on generators (note that since γ can be written in terms of α and β we only need to check for these two generators)

$gf(\mu) = g(\alpha\beta) = g(\alpha)g(\beta) = \mu^2\lambda^{-1}\lambda\mu^{-1} = \mu$

$gf(\lambda) = g(\alpha\beta\alpha) = g(\alpha)g(\beta)g(\alpha) = \mu^2\lambda^{-1}\lambda\mu^{-1}\mu^2\lambda^{-1} = \mu^3\lambda^{-1} = \lambda^2\lambda^{-1} = \lambda$

$fg(\alpha) = f(\mu^2\lambda^{-1}) = f(\mu)^2f(\lambda)^{-1} = \alpha\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta\gamma\beta^{-1}\alpha^{-1} = \alpha\alpha\alpha^{-1} = \alpha$

$fg(\beta) = f(\lambda\mu^{-1}) = f(\lambda)f(\mu)^{-1} = \alpha\beta\alpha\beta^{-1}\alpha^{-1} = \alpha\beta(\alpha\beta\alpha^{-1})^{-1} = \alpha\beta\gamma^{-1} = \alpha\beta\alpha^{-1}\alpha\gamma^{-1} = \gamma\alpha\gamma^{-1} = \beta$

So that indeed $fg = 1_{G_1}$ and $gf = 1_{G_2}$ so that $G_1 \cong G_2$.
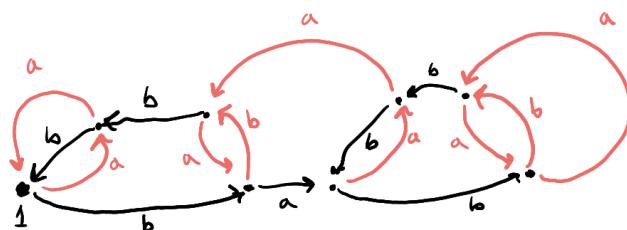
3. (a) By counting lifts of the wedge point we find the G set has 8 elements, and by reading the space we find a acts via $(12)(3456)(78)$ and b acts via $(1432)(5876)$ □
(b) A set of generators is given by:

$$\{ab, b^{-1}a, b^4, b^{-2}ab^{-1}, ba^2ba^{-1}b^{-1}, bab^{-1}ab^2, bab^4a^{-1}b^{-1}, bab^{-2}ab^{-1}a^{-1}b^{-1}, babab^2a^{-1}b^{-1}\}$$

Which corresponds to choosing generators with respect to the following spanning tree: □

(c) No, because S is not a normal G-set and a covering space is normal exactly when the G-set is. To see the G-set is not normal note that there is no automorphism permuting 1 and 4 since $a^2 \in \text{stab}(1)$, but $a^2 \notin \text{stab}(4)$. □

4. We first check the condition for $\mathrm{Aut}(\rho^{-1}(b_0))$ as a right G-set. As a right G-set, $\rho^{-1}(b_0) \cong \mathrm{stab}(b_0)\backslash G$, where $\mathrm{stab}(b_0) = \rho_*\pi_1(X) = H$, since these are exactly the paths with action fixing b_0. Now let $[\gamma] \in N_G(H)$, since X is connected, $\rho^{-1}(b_0)$ is a transitive G-set so we can write any $b \in \rho^{-1}(b_0)$ as $b = b_0 \cdot [\alpha]$ for some $[\alpha]$, this allows us to define the automorphism given by $[\gamma]$ as $\varphi_\gamma(b_0 \cdot [\alpha]) = b_0 \cdot [\gamma][\alpha]$, we need to check this is well defined and indeed defines a right G-set automorphism. To see it is well defined, take another representative for b, then $b = b_0[\beta]$, but $b_0[\beta][\alpha]^{-1} = b_0$, implying that $[\beta][\alpha]^{-1} \in \mathrm{stab}(b_0) = H$, so that $[\beta] = [h][\alpha]$ for $[h] \in H$, then

$$\varphi_{[\gamma]}(b_0 \cdot [\beta]) = \varphi_{[\gamma]}(b_0 \cdot [h][\alpha]) = b_0 \cdot [\gamma][h][\alpha] = b_0 \cdot [\gamma][h][\gamma]^{-1}[\gamma][\alpha] = [b_0] \cdot [\gamma][\alpha]$$

The last equality following from $[\gamma] \in N_G(H)$ implying that $[\gamma][h][\gamma]^{-1} \in H = \mathrm{stab}(b_0)$ proving that φ_γ is well defined. Moreover let $[x] \in G$, then once again for any $b \in \rho^{-1}(b_0)$ we write $b = b_0[\alpha]$ for $[\alpha] \in G$

$$\varphi_{[\gamma]}(b) \cdot [x] = b_0 \cdot [\gamma][\alpha][x] = \varphi_{[\gamma]}(b_0 \cdot [\alpha][x]) = \varphi_{[\gamma]}(b \cdot [x])$$

So that $\varphi_{[\gamma]}$ is a morphism of right G-sets.
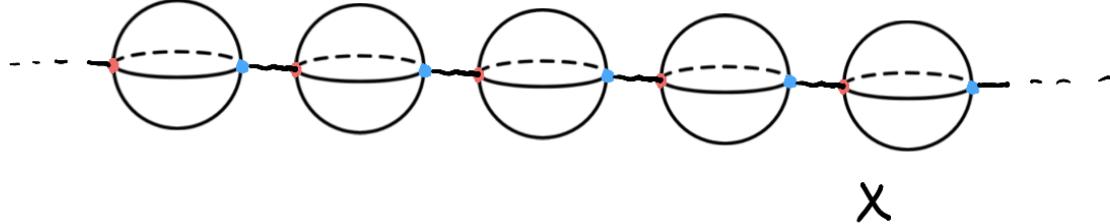
Now suppose that $\varphi \in \mathrm{Aut}(\rho^{-1}(b_0))$, then $\varphi(b_0) = b_0 \cdot [\gamma]$ for some $[\gamma] \in G$, it follows that since φ is a morphism of G-sets, for $b = b_0 \cdot [\alpha]$ it satisfies $\varphi(b) = \varphi(b_0 \cdot [\alpha]) = \varphi(b_0) \cdot [\alpha] = b_0 \cdot [\gamma][\alpha]$, so that every automorphism is of the form above, now assuming that $\gamma \notin N_G(H)$, we can choose some $[h] \in H$ with $[\gamma][h][\gamma]^{-1} \notin H$, so that for the map $\varphi_{[\gamma]} : b_0 \cdot [\alpha] \to b_0 \cdot [\gamma][\alpha]$ we get

$$\varphi_{[\gamma]}(b_0) \cdot [h][\gamma]^{-1} = b_0 \cdot [\gamma][h][\gamma]^{-1} \neq b_0 = b_0[\gamma][\gamma]^{-1} = \varphi_{[\gamma]}(b_0 \cdot h[\gamma]^{-1})$$
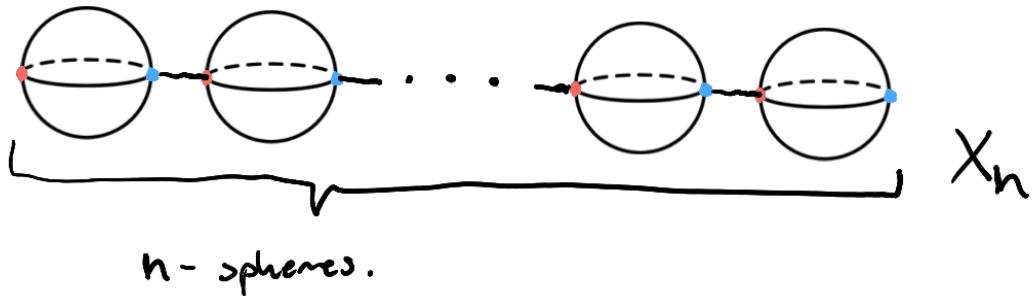
i.e. such a $[\gamma]$ cannot define an automorphism of $\rho^{-1}(b_0)$. Finally to conclude that $\mathrm{Aut}(\rho^{-1}(b_0)) \cong N_G(H)$, we note that $\varphi_{[\alpha]} \circ \varphi_{[\beta]} = \varphi_{[\alpha][\beta]}$, so the bijection respects the group law.

Now that we have shown $\mathrm{Aut}(\rho^{-1}(b_0)) \cong N_G(H)$, it is straightforward to see that the group of deck transformations $\mathrm{Aut}(X) \cong N_G(H)$. To do so we apply the equivalence of categories of right $\pi_1(B, b_0)$-sets and covering spaces $E \to B$. We recal the two functors are $\mathcal{F}$ taking covering spaces to right G sets via $E \xrightarrow{\rho} B \rightsquigarrow \rho^{-1}(b_0)$ the right G set with the path action, and the other functor is given by $\mathcal{G}$ taking right $\pi_1(B, b_0)$ sets and constructing the corresponding covering spaces. Above we computed $\mathrm{Aut}(\mathcal{F}(X \to B))$, then the functor $\mathcal{G}$ takes each automorphism $[\gamma] \in N_G(H)/H$ of right G-sets to an automorphism of covering spaces taking $x \mapsto x \cdot [\gamma]$. So the fact that $\mathrm{Aut}(X) = N_G(H)/H$ simply follows from equivalence of categories preserving hom sets so that $\mathrm{Aut}(X) = \mathcal{G}(\mathrm{Aut}(\mathcal{F}(X)))$. $\qquad\square$

5. The following picture is the universal cover, where the coloring is used to denote the covering map.

X

B

By looking locally, we can see that this is clearly a path connected covering space so all that is left to show is that it is simply connected, as proof note that since S^1 is compact, any loop $f : S^1 \to X$ must have image contained within finitely many of the spheres, so we can choose for some n, the following space to contain the image of γ

X_n

n - spheres.

To see a space of this form is simply connected, we can use induction, and note the base case is already proven since S^2 is simply connected, then by assuming X_{n-1} is simply connected we get the open cover given by:

$$U = X_{n-1} \cup_{0 \sim x_0} [0, \epsilon) \qquad\qquad V = S^2 \cup_{1 \sim y_0} (\epsilon/2, 1]$$

Then $U \cap V = (\epsilon/2, \epsilon) \simeq \{*\}$, $U \sim X_{n-1}$ and $V \sim S^2$, so that

$$\pi_1(X_n) \cong \pi_1(X_{n-1}) *_0 \pi_1(S^2) = 0$$

So that the image of gamma is contained in a simply connected subset of X, and hence $[\gamma] = 0$ in $\pi_1(X)$, since this holds for an arbitrary $\gamma : S^1 \to X$ we conclude that $\pi_1(X) = 0$, so that X is the universal cover of B.

6. Firstly, assume that $X \neq \emptyset$, otherwise the problem is trivial, now proceeding, I assume the definition of covering space as done in class that a covering map need not be surjective. If surjectivity is required I will prove after the problem that a covering space morphism is surjective since X is connected.

In problem 1 of homework 2, we showed that if a space U is path connected, simply connected, and locally path connected then for any connected covering space $X \xrightarrow{\rho} B$ and map $f : U \to B$, there is a unique lift $\tilde{f}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & X \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \rho \\ U & \xrightarrow{f} & B \end{array}$$

We can apply this result directly to the following diagram, where $\rho_U : U \to B$ and $\rho_X : X \to B$ are covering maps, for an arbitrary covering space $\rho_X : X \to B$. The justification of U being locally path connected is that B is locally path connected and U is locally diffeomorphic to B.

$$\begin{array}{ccc} & & X \\ & \overset{\widetilde{\rho_U}}{\nearrow} & \downarrow \rho_X \\ U & \xrightarrow{\rho_U} & B \end{array}$$

Then commutativity of the diagram is exactly the statement that that the map we get as a result of homework 2 problem 1, $\widetilde{\rho_U}$ is also a covering space morphism.

Notice now, that $\widetilde{\rho_U}$ is open, this follows since ρ_U and ρ_X are open since they are local homeomorphisms. ρ_U being open implies that $\rho_X \circ \widetilde{\rho_U}$ is open by commutativity of the diagram. Now, since ρ_X is a local homeomorphism, we know that $\rho_X(U)$ is open if and only if U is, whence $\widetilde{\rho_U}$ must be open, otherwise there would be some open set U for which $\widetilde{\rho_U}(U)$ is not open, and hence $\rho_X \circ \widetilde{\rho_U}(U)$ is not open, contradicting commutativity of the diagram since $\rho_U(U)$ is open.

Now, to see that this makes U a covering space of X, for each point $x \in X$, choose some neighborhood U_x of x so that both $\rho_U^{-1}(U_x) = \bigsqcup_{\mathcal{A}} U_\alpha = U \times D$ for D discrete and $\rho_X^{-1}(U_x) = \bigsqcup_I U_i = U' \times C$ for C discrete where $\rho_U : U \cong U_x$ and $\rho_X : U' \cong U_x$, we can always do this by choosing a neighborhood that works for one and shrinking it to one working for both. Now we can define for $y \in X$, V_y to be the connected component of $\rho_X^{-1}(U_{\rho_X(y)})$ containing y and can consider $\widetilde{\rho_U} : U_y \times D \to V_y \times C$, where $U_y \overset{\rho_U \cong}{\to} U_{\rho_X(y)} \overset{\cong \rho_X}{\leftarrow} V_y$. As a map between products, we may write $\widetilde{\rho_U} = (f(x, d), g(x, d))$ for $f : U_y \times D \to V_y$ continuous, and $g : U_y \times D \to C$ continuous. Since C is discrete and g is continuous, g is independent of the U_y coordinate, thus $\widetilde{\rho_U}|_{U_y \times \{d\}} : U_y \times \{d\} \to V_y \times \{c\}$ for some fixed c. Now if $\widetilde{\rho_U}^{-1}(V_y)$ is empty, then we are done, since the preimage is $\emptyset = V_y \times \emptyset$, so we may assume not, in which case, for some subset $D' \subset D$, that $\widetilde{\rho_U} : U_y \times D' \to V_y$, since for any $d \in D'$ we have $\rho_U|_{U_y \times \{d\}}$ is bijective, by commutativity of the diagram $\widetilde{\rho_U}$ must be injective, and since $\rho_X|_{V_y}$ is also injective, commutativity of the diagram again implies thhat $\rho_U|_{U_y \times \{d\}}$ is surjective. This suffices to show that $\widetilde{\rho_U}^{-1} = \bigsqcup U_y \times D'$, for $\widetilde{\rho_U} : U_y \times \{d\} \overset{\cong}{\to} V_y$ is a covering map.

Finally, if required we show that $\widetilde{\rho_U}$ is surjective, since X is connected it suffices to show that $\widetilde{\rho_U}(U)$ is closed, since we already established the map $\widetilde{\rho_U}$ is open. To see that $\widetilde{\rho_U}(U)$ is closed, suppose that $y \in X \setminus \widetilde{\rho_U}(U)$, then once again take $U_{\rho_X(y)} \subset X$ to be a neighborhood with lift a disjoint union of homeomorphic open sets under the covering map. Assume for contradiction that V_y being the connected component of $\rho_X^{-1}(U_{\rho_X(y)})$ has $\widetilde{\rho_U}(U) \cap V_y \supset \{z\}$, then since ρ_U is surjective, we get some copy of $U_y \subset U$ mapping surjectively onto $U_{\rho_X(y)}$, but then $\widetilde{\rho_U} : U_y \to V_y$, since as above the image is contained in a single connected component, and following the proof above $\widetilde{\rho_U}$ is surjective on this connected component by commutativity of the diagram. This gives the desired contradiction, so that $\widetilde{\rho_U}(U)^c$ is open implying that $\widetilde{\rho_U}(U)$ is closed, so that the map is surjective by the connected hypothesis on X. $\square$