Recursion Formula for Siegel Veech Constants
Following Section 8 of Eskin, Masur, Zorich

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1   Review and Vision

We begin by recalling the purpose of our investigations, namely the computation of Siegel
Veech constants. Our approach following EMZ is to compute volumes of Strata. From
which we can recover the Siegel Veech constants.

Recall 1.1. *[Can either skip over this or go over quickly]*

- Given a surface (of genus g) with abelian differential (S, ω), S is given as a surface of
  translation with a finite number of conic singularities $\{z_1, \ldots, z_k\}$, with multiplicities
  $m_1, \ldots, m_k$

- α (a partition of $2g - 2$) is a vector recording the conic angles at the singularities.

- A configuration $\mathcal{C}$ of multiplicity p records the orders of two zeroes z_1 and z_2 joined
  by p saddle connections (recall generically these are homologous since by definition
  they have the same holonomy), $\mathcal{C} = (m_1, m_2, a_1, \ldots, a_{p-1}, a'_1, \ldots, a'_{p-1})$ where m_1 is
  the cone angle at z_1, m_2 is the cone angle at z_2 and the angle between γ_j and γ_{j+1}
  at z_1 is $2\pi(a_j + 1)$ and the angle between them at z_2 is $2\pi(a'_j + 1)$.

- We are working in the strata of the form $\mathcal{H}_1(\alpha)$ which is the subset of the space of
  abelian differentials (S, ω) such that the zeroes of ω have configuration α, with S
  having unit surface area.

- Local coordinates on $\mathcal{H}^1(\alpha)$ are given by choosing a basis $\gamma_1, \ldots, \gamma_n$ for the relative
  homology $H_1(S, \{z_1, \ldots, z_k\}; \mathbb{C})$ for which each basis element is the homology class
  of a saddle connection. Then coordinates are given by:

$$(S, \omega) \mapsto \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega \right) \in \mathbb{C}^n \longleftrightarrow \mathbb{R}^{2n}$$

- The measure μ on $\mathcal{H}_1(\alpha)$ is given by lebesgue measure in these coordinates, normal-
  ized so that $\mu(I^{2n}) = 1$.

- We are interested in counting the asymptotics of the number of saddle connections
  of generic surfaces in $\mathcal{H}_1(\alpha)$. Namely the number of saddle connections having
  configuration $\mathcal{C}$ and length less than L, under the image of the developing map
  taking $\gamma \mapsto \mathrm{hol}(\gamma) \in \mathbb{R}^2$ this set is denoted as $V_{\mathcal{C}}(S) \cap B_L$.

- $\mathcal{H}_1^\epsilon(\alpha)$ is defined as TODO ...

Recall 1.2. • Siegel Veech constants are defined as follows (existence of such a constant is a result of Eskin and Masur)

$$c(\alpha, \mathcal{C}) := \lim_{L \to \infty} \frac{\#\left(V_\mathcal{C}(S) \cap B_L\right)}{\pi L^2}$$

- Last time we saw Richard present the proof of the formula for connected components of stratum

$$c(\alpha, \mathcal{C}) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \frac{\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}$$

Goal 1.1. Further develop this methodology of computing Siegel-Veech constants, namely we would like to understand how to compute $\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$.

2 Approach and Setup

Detail(s) 2.1. • Roughly I will be covering the simplest case for computing $\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$

- We continue to consider connected Strata

- I am only considering saddle connections of multiplicity 1, i.e. $\mathcal{C} = (m_1, m_2)$

- Later we will consider the picture for higher multiplicity?

Example(s) 2.1. Saddle connection of multiplicity 1:
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Saddle Connection of multiplicity > 1:
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Concept 2.1 (Principle Boundary)**.** By shrinking a saddle connection γ (say between z_1 and z_2) on a surface of type $\alpha = (m_1, \ldots, m_k)$, we collapse to a surface of type $\alpha' = (m_1 + m_2, m_3, m_4, \ldots, m_k)$, $\mathcal{H}_1(\alpha')$ is called the principle boundary of $\mathcal{H}_1(\alpha)$.

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Goal 2.1. To understand the substance of **Lemma 8.1 (EMZ)** and its proof.

Roughly the idea is that given a surface with a short saddle connection, we can map it to its principle boundary. Assuming this saddle connection is short, the surface should not change too much, so that (given the data of the saddle connection) we could recover the original surface.

Assuming enough geometric information is preserved we hope to recover $\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$ in terms of $\mathrm{Vol}(\mathcal{H}_1(\alpha'))$.

Theorem 2.1 (EMZ Lemma 8.1 – Imprecise version)**.** Yes, Enough information is preserved.

Namely, up to some error we have

$$\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}) \to \mathcal{H}_1(\alpha') \times B_\epsilon$$
$$(S, m_1, m_2) \mapsto (S', m, \gamma)$$

is a covering map (of degree $m + 1$), and importantly the measure decomposes $\mu = \mu' \times$ Lebesgue

Detail(s) 2.2. The error corresponds to requiring all saddle connections other than the one being collapsed to have length atleast ϵ. This is essential for the covering map to be well defined, since otherwise two stratum could be mapped to the same boundary

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3 Proof of Covering and Precise Restatement

Proof. The covering map is given by collapsing a saddle connection, as previously specified, it is actually easier to understand through looking at the local inverse.

SETUP:

- the order of the zero x we are splitting is $m \geq 2$ and $m_1, m_2 \in \mathbb{Z}_{>0}$ with $m = m_1 + m_2$

- $\gamma \in B_\epsilon \subset \mathbb{R}^2$, with $|\text{hol}(\gamma)| = 2\delta$

- Assume that all other saddle connections and closed geodesics of S' have length atleast 2ϵ [*This corresponds to our assumption ensuring that the covering map is well defined, since the surface in our local inverse will have saddles of length atleast ϵ due to this assumption*]

- Let $\{x, z_1, \ldots, z_\ell\}$ be the zeroes of ω' the differential on S', since we will modify a small neighborhood of x, we take saddle connections whos classes generate the base of our homology $H_1(S, \{x, z_1, \ldots, z_\ell\}; \mathbb{Z})$ avoiding $B_\epsilon(x)$, if x is the only zero this can be done, if there are other zeroes then we require exactly one saddle (β) to have x as its endpoint

- Q: Why can we/do we have to do this? A: Recall since $(S, \{x, z_1, \ldots, z_\ell\})$ is a good pair (Hatcher) the relative homology is equivalent to the homology of the quotient space, in which case each saddle has a nontrivial homology class (e.g. look at the holonomy), and if none of $\gamma_1, \ldots, \gamma_n$ are saddle connections to x, then β will have an independent homology class (once again e.g. the holonomy can be linearly independent). Conversely if we include β, then any other saddle connection including x is homologous to a linear combination of saddle connections avoiding it (Proof - Just draw the 2-simplex).

Proof:

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Glue together $2m + 2$ half disks of radius ϵ, start by gluing the opposing discs along the new saddle connection γ

Considerations:

- This process is ambiguous up to choosing where γ is added, this depends on the choice of disks to be glued along gamma, of which there are $m + 1$ pairs, each pair corresponds to 2 gluings by mirroring, but by marking which point is x_1 and x_2, fixing the holonomy of γ gives one choice. This is where the covering index of $m + 1$ comes from.

- Since we perform this construction locally, the rest of the surface i.e. B_ϵ^c remains unchanged, namely the holonomy is unchanged on the homology basis of S' (which includes the homology basis of S) apart from $\text{hol}(\beta)$ being changed by a factor of $-\gamma/2$. By the triangle inequality all of the lifts lie in $H_1^{\epsilon,\epsilon}(\alpha, m_1, m_2)$

Now we consider the case of collapsing zeroes, here we clearly see the error in the above being a covering map, the cover is ramified and maps only to a "large" subset of $\mathcal{H}^1(\alpha')$

SETUP:

- Let $S \in \mathcal{H}_1^{\epsilon,3\epsilon}(\alpha)$ i.e. S has a saddle connection γ connecting zeroes x_1, x_2 of orders m_1 and m_2, $\text{hol}(\gamma) \in B_\epsilon$, and all other saddle connections of S have length atleast 3ϵ.

- Considering the relative homology $H_1(S, \{x_1, x_2, z_1, \ldots, z_\ell\})$, similarly to adding a saddle connection, we choose our basis for relative homology away from B_ϵ(midpoint of γ), we furthermore take γ in this homology basis, similarly to last time if $\ell > 0$ we will need to take a saddle conection β connecting one of the z_j to x_1 in our basis.

Proof: We break up B_ϵ(midpoint of γ) into $2m+2$ half disks, in the exact same way as breaking up a zero. We then collapse the saddle connection, while leaving the manifold the same outside of the ϵ ball, equivalently we are reversing the breaking up process.
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Considerations:

- Since this construction is local, the holonomy of the basis vectors for $H_1(S', x, z_1, \ldots, z_\ell)$ is unchanged apart from β which gets replaced by a curve to x, hence the holonomy of β is changed by at most $\frac{1}{2}\text{hol}(\gamma)$

- From this we know that the every saddle connection on S' has length atleast 2ϵ

$\square$

4 Example: Σ_3