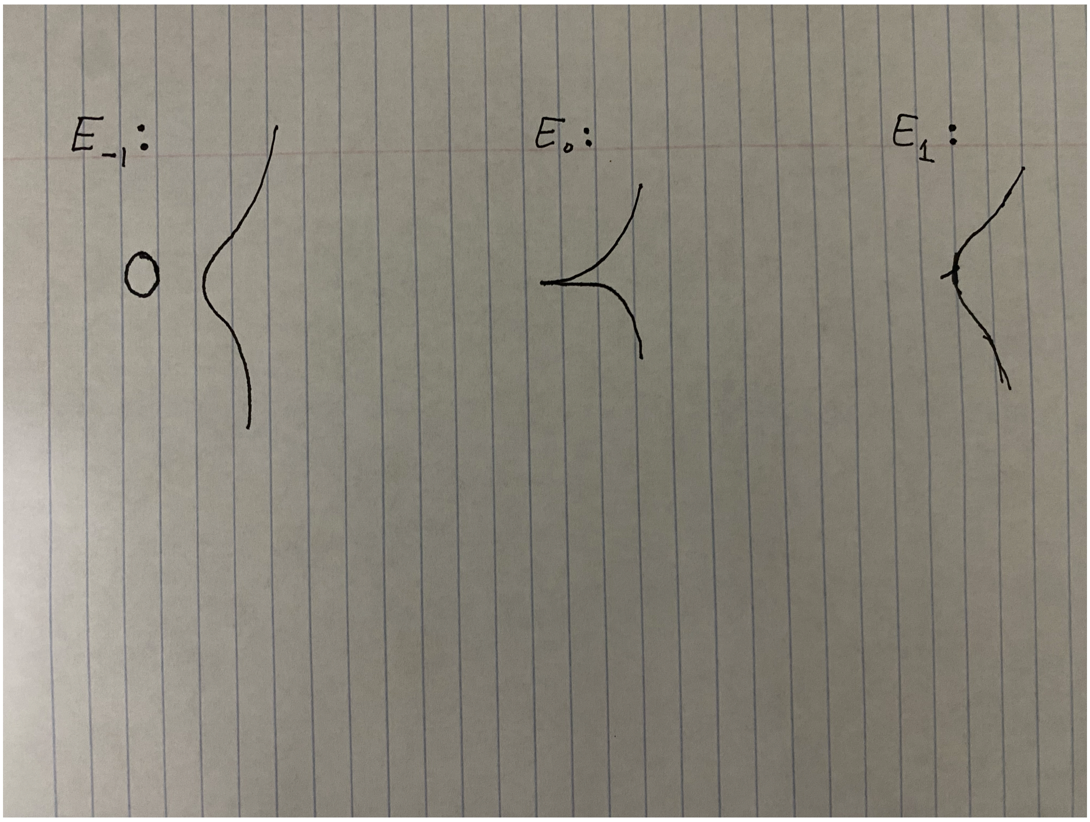
1. (a)



(b) The set $\{y^2 = x^3 + ax\}$ is exactly the zero set of $y^2 - x^3 - ax$, so if $a \neq 0$ we use the submersion theorem for

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto y^2 - x^3 - ax \end{aligned}$$

Then $D_{(x,y)}f = \begin{pmatrix} 2y & 3x^2 - a \end{pmatrix}$, this is surjective so long as it is not equal to $\begin{pmatrix} 0 & 0 \end{pmatrix}$, but this wont happen since any point where $y = 0$ on E_a has $x = 0$, so surjectivity for all (x, y) follows from $a \neq 0$. It follows that $E_a = f^{-1}(0)$ is a submanifold.

E_0 is not a submanifold, we first show that if it were a submanifold it would have dimension 1, note its obvious the dimension can't be zero since E_0 is not a point. Suppose E_0 were to have dimension 2, then if (U, V, ϕ) were a chart for E_0, containing $(1, 1)$ we would have $\phi : E_0 \cap V \stackrel{\cong}{\to} U$, so that $\phi : E_0 \cap V \setminus \{1, 1\} \to U \setminus \{\phi(1, 1)\}$ is also a homeomorphism onto its image, with inverse $\phi^{-1}|_{U \setminus \{\phi(0,0)\}}$, but this is not possible since the former is not connected, and any open subset of $\mathbb{R}^2$ is still connected after removing a point, so that $U \setminus \{\phi(1, 1)\}$ is not connected.

Now we assume for contradiction that E_0 is a one dimensional submanifold of $\mathbb{R}^2$, we apply the theorem that a closed set is a submanifold if and only if it is the image of a proper embedding, so we get a proper embedding $f : E_0 \to \mathbb{R}^2$ with $f(\mathbb{R}) = \{y^2 = x^3\}$, and assuming for convenience that $f(0) = (0, 0)$. To see that E_0 cannot be an embedded submanifold, I will show that $T_0E_0 = 0$, which contradicts E_0 having dimension one. Consider a path $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}$, then $f \circ \gamma : (-\epsilon, \epsilon) \to \{y^2 = x^3\} \subset \mathbb{R}^2$, $t \mapsto (\gamma_1(t), \gamma_2(t))$. By the equation for E_0, we know that $\gamma_1(t) = \gamma_2(t)^{2/3}$, and hence

$$\gamma_1'(0) = \lim_{t \to 0} \gamma_1'(t) = \lim_{t \to 0} \frac{2}{3} \gamma_2^{-\frac{1}{3}}(t) \gamma_2'(t)$$

Since $\gamma_2(0) = 0$, we get that the right hand side diverges unless $\gamma_2'(0) = 0$. Moreover by the equation of E_0, we have $\gamma_1(t) \geq 0$, and hence

$$0 \geq \lim_{t\uparrow 0} \frac{\gamma_1(t)}{t} = \gamma_1'(t) = \lim_{t\downarrow 0} \frac{\gamma_1(t)}{t} \geq 0$$

so that $\gamma_1'(t) = 0$, but since f is an embedding its a diffeomorphism onto its image, so in particular $([\gamma] \mapsto [f \circ \gamma])$ should be an injection on tangent spaces, but by the chain rule the derivative being zero for an arbitrary path implies that the corresponding derivation is zero, and hence $T_0E_0 = \{0\}$ is zero dimensional which contradicts E_0 being a dimension one smooth manifold. $\square$

2. Suppose that $f : \mathbb{R}^n \overset{\cong}{\to} \mathbb{R}^m$ is a diffeomorphism, then denoting $p = f(0)$, we have $d_0f : T_0\mathbb{R}^n \to T_0\mathbb{R}^m$, and $d_pf^{-1} : T_p\mathbb{R}^m \to T_0\mathbb{R}^n$, applying the chain rule we get

$$d_0 1_{\mathbb{R}^n} = d_0(f^{-1} \circ f) = d_pf^{-1}d_0f \text{ and } d_p1_{\mathbb{R}^m} = d_p(f \circ f^{-1}) = d_0fd_pf^{-1}$$

So that d_0f is an isomorphism of tangent spaces. Moreover, we know that $T_0\mathbb{R}^n$ is n-dimensional since $\{\frac{\partial}{\partial x_j}|_0\}_{j=1}^n$ is a basis, similarly, $T_p\mathbb{R}^m$ is m-dimensional since it has basis $\{\frac{\partial}{\partial x_j}|_p\}_{j=1}^m$, and a linear map from an n dimensional vectorspace to an m dimensional vectorspace can only be invertible if $m = n$. $\square$

Verification that $\{\frac{\partial}{\partial x_j}|_p\}_1^n$ is a basis for $T_p\mathbb{R}^n$, and hence the vectorspace is n-dimensional (note since p, n are arbitrary this works for both cases):

First assume that $\sum_1^n a_j \frac{\partial}{\partial x_j}|_p = 0$, then by evaluating at x_i we get $0 = \sum_1^n a_j \frac{\partial}{\partial x_j}|_p(x_i) = a_i$. This shows linear independence, to see that it spans, consider a smooth function on an open set $p \in U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$, then for $x \in U$ we have

$$\begin{aligned} f(x) - f(p) &= \int_0^1 \frac{d}{ds}\bigg|_{s=t} f(p + s(x-p))dt = \int_0^1 \sum_1^n (x_j - p_j)\frac{\partial}{\partial x_j}|_{s=t} f(p + s(x-p))dt \\ &= \sum_1^n \left((x_j - p) \int_0^1 \frac{\partial}{\partial x_j}|_{s=t} f(p + s(x-p)) \right) \end{aligned}$$

So f can be written near p as $f(p) + \sum_1^n (x_j - p_j)f_j(x)$ where the f_j are smooth. Now letting X be a derivation at p, with $X(x_j) = c_j$, we get for any $\overline{f} \in \xi(\mathbb{R}^n, p)$

$$\begin{aligned} \left(X - \sum c_j \frac{\partial}{\partial x_j}|_p\right)(\overline{f}) &= \left(X - \sum c_j \frac{\partial}{\partial x_j}|_p\right)\left(f(p) + \sum_1^n (x_j - p_j)f_j(x)\right) \\ &= \sum_1^n X((x_j - p_j)f_j) - \sum_{1\leq i,j\leq n} c_j \frac{\partial}{\partial x_j}|_p((x_i - p_i)f_i(x)) \\ &= \sum_1^n c_jf_j(p) - \sum_{1\leq i,j\leq n} c_j \frac{\partial}{\partial x_j}|_p((x_i - p_i)f_i(p)) \\ &= \sum_1^n c_jf_j(p) - \sum_1^n c_jf_j(p) = 0 \end{aligned}$$

so that $X \in \langle \frac{\partial}{\partial x_j} \rangle_1^n$, this suffices to show that $\{\frac{\partial}{\partial x_j}\}_1^n$ is a basis. $\square$

3. To avoid confusion with the inclusion maps I will denote my inverse map as *inv*

First to make sense of $T_1G \oplus T_1G$ being the tangent space for $T_{1,1}G \times G$, we provide the isomorphism $v \mapsto (d_{1,1}\pi_1(v), d_{1,1}\pi_2(v))$. Since the dimension is equal on either side it suffices to check that this map is surjective map of finite dimensional vector spaces. So let $(v, w) \in T_1G \oplus T_1G$, then $(v, w) = (v, 0) + (0, w)$, so it suffices to check both of these terms are in the image, this is straightforward since $1_{T_1G} = d_1(\pi_1 \circ \iota_1) = (d_{1,1}\pi)(d_1\iota_1)$, and $d_1\{G \to \{1\}\} = 0 = d_1(\pi_2 \circ \iota_1) = (d_{1,1}\pi_2)(d_1\iota_1)$, which suffices to show $d_1\iota_1 v \mapsto (v, 0)$, similarly $d_1\iota_2 w \mapsto (0, w)$ so this is indeed an isomorphism.

(a) $\mu \circ \iota_1(g) = \mu(g, 1) = g = \mu(1, g) = \mu \circ \iota_2(g)$. $\square$

(b) We first show that $d_1\iota_1 = \begin{pmatrix} 1_{T_1G} \\ 0 \end{pmatrix}$, but this is immediate, since in our identification to the direct sum we are implicitly postcomposing with $d_{1,1}\pi_1 \times d_{1,1}\pi_2 \circ \Delta$, so that as in the above paragraph we get $1_{T_1G} = (d_{1,1}\pi_1)(d_1\iota_1)$ and $0 = (d_{1,1}\pi_2)(d_1\iota_1)$ are the two components.

Writing $d_{1,1}\mu$ in block form $\begin{pmatrix} A & B \end{pmatrix}$, we use the chain rule to compute

$$1_{T_1G} = d_1(1_G) = d_1(\mu \circ \iota_1) = (d_{(1,1)}\mu)(d_1\iota_1) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 1_{T_1G} \\ 0 \end{pmatrix} = A$$

The same computation for $d_1(\mu \circ \iota_2)$ gives $1_{T_1G} = B$. Hence $d_{1,1}\mu : T_1G \oplus T_1G \to T_1G$ acts via $\begin{pmatrix} 1_{T_1G} & 1_{T_1G} \end{pmatrix}$,

$$\begin{pmatrix} 1_{T_1G} & 1_{T_1G} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = v + u$$

$\square$

(c)

$$\mu \circ (1_G \times inv) \circ \Delta(g) = \mu \circ (1_G \times inv)(g, g) = \mu(g, g^{-1}) = gg^{-1} = 1$$

so $\mu \circ (1_G \times inv) \circ \Delta$ is the constant map $G \to \{1\}$. $\square$

(d)

$$\begin{aligned} 0 = d_1(G \to \{1\}) &= d_1(\mu \circ (1_G \times inv) \circ \Delta) = (d_{1,1}\mu)(d_{1,1}(1_G \times inv))(d_1\Delta) \\ &= \begin{pmatrix} 1_{T_1G} & 1_{T_1G} \end{pmatrix} \begin{pmatrix} 1_{T_1G} & 0 \\ 0 & d_1 inv \end{pmatrix} \begin{pmatrix} 1_{T_1G} \\ 1_{T_1G} \end{pmatrix} = 1_{T_1G} + d_1 inv \end{aligned}$$

So that $d_1 inv = -1_{T_1G}$

$$\begin{aligned} -1_{T_1G} : T_1G &\to T_1G \\ v &\mapsto -v \end{aligned}$$

$\square$