

*I collaborated with Sabek, he helped me with a technical detail on 3(d) and I helped him with a technical detail on 1(b).*

1. (a) From the definition of a lie group we know that  $\mu : G \times G \rightarrow G$  is smooth, then  $\mu_g = \mu \circ \iota_g$ , where  $\iota_g : G \rightarrow G \times G$  via  $h \mapsto (g, h)$  is the inclusion into the product manifold, we have seen previously the inclusion is smooth, so that  $\mu_g = \mu \circ \iota_g$  is smooth. Now we can also see that

$$\mu_{g^{-1}}\mu_g = 1_G = \mu_g\mu_{g^{-1}}$$

and  $\mu_{g^{-1}}$  is smooth for the same reason  $\mu_g$  is, so that  $\mu_g$  is in fact a diffeomorphism, this implies that  $d_e\mu_g$  is an isomorphism.

(b)

**(Lemma)** Let  $(\rho, E), (\hat{\rho}, \hat{E})$  be two vector bundles on the same base space  $M$ , and  $F : E \rightarrow \hat{E}$  a smooth bijective map of smooth vector bundles with  $F(x, 0) = (x, 0)$  (i.e.  $F$  descends to the identity), then  $F$  is a diffeomorphism.

*Proof.* Being a diffeomorphism is a local property, so for a point  $x \in M$ , let  $U$  be an open neighborhood of  $M$  where  $\rho^{-1}(U)$  admits a local trivialization  $\zeta$ , moreover there is a second neighborhood  $x \in V \subset U$  where  $\hat{\rho}^{-1}(V)$  admits a local trivialization  $\hat{\zeta}$  (since the base manifold is the same by possibly shrinking the neighborhood we can assume that the two bundle charts are equal on  $V \times \{0\}$ , this is not necessary but removes a lot of bloat from notation). Then  $\hat{\zeta} \circ F \circ \zeta^{-1} : M \times \mathbb{R}^n \rightarrow M \times \mathbb{R}^n$  is smooth, linear on each fiber and bijective on each fiber, so on  $V$ , we can write  $A(x) = \hat{\zeta} \circ F \circ \zeta^{-1}(x, -)$ . Then on the local trivialization  $F$  is given by

$$\hat{\zeta} \circ F \circ \zeta^{-1}(x, v) = (x, A(x)v)$$

In particular, the Jacobian  $D_{(x,v)}(\hat{\zeta} \circ F \circ \zeta^{-1})$  is given by

$$\begin{pmatrix} 1_n & 0 \\ B(x, v) & A(x) \end{pmatrix}$$

Bijectivity on each fiber implies that  $A(x)$  is full rank, so that  $\det(D_{(x,v)}\hat{\zeta} \circ F \circ \zeta^{-1}) = \det A(x) \in \mathbb{R}^\times$ , by the inverse function theorem  $\hat{\zeta} \circ F \circ \zeta^{-1}$  has a local smooth inverse, and hence  $F$  is a diffeomorphism.  $\square$

Since  $T_e G$  is  $n$ -dimensional, we can identify it with  $\mathbb{R}^n$ , the following diagram specifies the desired correspondence of vector bundles:

$$\begin{array}{ccc} G \times \mathbb{R}^n & \xrightleftharpoons[T]{F} & TG \\ \downarrow & & \downarrow \\ G & \xrightarrow{1_G} & G \end{array}$$

Where  $F(g, v) = (g, d_e\mu_g(v))$ ,

*(when I originally solved the problem I tried to show  $F$  and the inverse map  $T$  which is not too hard to compute are both smooth, however, after trying to show  $F, T$  are smooth for quite some time I did the following computation which allowed me to see that  $F$  is smooth, this computation does not generalize easily to  $T$ , so the lemma is intended to avoid having to do a similar computation for  $T$ ).*

In order to show  $F$  is smooth, it suffices to show that  $(g, v) \mapsto d_e\mu_g(v)$  is smooth, here we can use smoothness of  $\mu$ , and the identification  $T(G \times G) \longleftrightarrow TG \oplus TG$  by identifying on each fiber, we have previously computed (last homework) that  $d_p\iota_q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  when  $\iota$  denotes inclusion. We have that  $d\mu : T(G \times G) \rightarrow TG$  is smooth since  $\mu$  is a smooth map, then

$$\begin{aligned} d\mu((g, v), (h, u)) &= d_{(g,h)}\mu(v, u) \\ d_e\mu_g &= d_e(\mu \circ \iota_g)(v, u) = (d_{(g,e)}\mu)(d_e\iota_g)(v, u) = d_{(g,e)}\mu(u) \end{aligned}$$

From this computation, we can see that  $d_e\mu_g = d_{(g,e)}\mu(0, u)$  is the restriction of  $d\mu$  to  $G \times \{0\} \times T_e G$ , this is clearly a submanifold directly from the definition of it being a linear subspace given by inclusion on the first  $n$  coordinates and last  $n$  coordinates. Thus the restriction of  $d\mu$  to this submanifold is smooth, and is identified with  $d_e\mu_g$ . So  $F$  is smooth, and we appeal to the lemma to find that  $T$ , the set theoretic inverse for  $F$  is smooth and hence  $F$  is a diffeomorphism.

**2.** Let  $f : X \rightarrow \mathbb{R}^m$  be a submersion, where  $X$  is a compact smooth manifold. The proof will follow if we can show submersions are open maps, assuming this, since the image of a compact set is compact (by pulling back an open cover along the map) we get that  $f(X) \subset \mathbb{R}^m$  is open, but also  $f(X) \subset \mathbb{R}^m$  is compact hence closed, so since  $X \neq \emptyset$  we have  $f(X) = \mathbb{R}^m$ , contradicting compactness.

It remains to show that a submersion is open, since  $f$  is a submersion, we can cover  $M, N$  with charts  $(U_\alpha, V_\alpha, \phi_\alpha)$  and  $(U'_\beta, V'_\beta, \varphi_\beta)$  respectively with the property that the following commutes (here  $\pi$  is the projection map onto the first  $n$  coordinates)

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\phi_\alpha} & V_\alpha \\ \downarrow \pi & & \downarrow f \\ U'_\beta & \xrightarrow{\varphi_\beta} & V'_\beta \end{array}$$

Now let  $E \subset X$  be open, and write  $E_\alpha := V_\alpha \cap E$ , then

$$f(E) = \bigcup_{\alpha} f(E_\alpha) = \bigcup_{\alpha, \beta} \varphi_\beta \pi \phi_\alpha^{-1}(E_\alpha)$$

But  $\varphi_\beta \pi \phi_\alpha^{-1}$  is a composition of open maps hence open, so that  $f(E)$  is open which suffices to show  $f$  is open.

**3. (a)** From the iterated construction we get

$$\begin{aligned} \overline{[a, b, c, d, e, f, g, h]} &= \overline{[(a, b, c, d, e, f), (-g, -h)]} = \overline{[(a, b, c, d), (-e, -f), (-g, -h)]} \\ &= \overline{[(a, b), (-c, -d), (-e, -f), (-g, -h)]} = [\bar{a}, -b, (-c, -d), (-e, -f), (-g, -h)] \end{aligned}$$

from this taking the  $i$ -th coordinate to be 1 and the others zero we see  $\bar{e}_i = -e_i$ .  $\square$

**(b)** We first note that  $A$  maps real values to real values and imaginary values to imaginary values. This can be seen since if  $\bar{x} = x$  and  $\bar{y} = -y$ , then

$$\overline{A(x)} = A(\bar{x}) = A(x) \quad \overline{A(y)} = A(\bar{y}) = A(-y) = -A(y)$$

And moreover, for arbitrary  $x$ ,  $A(x)A(1) = A(x)$  implies  $A(1) = 1$  so linearity suffices to show that  $A(\alpha) = \alpha$  for  $\alpha \in \mathbb{R}$ , once again by linearity we see that this implies  $\text{Re}(A(x)) = A(\text{Re}(x))$  for all  $x$ . It follows that  $A$  is orthogonal, i.e. preserves the inner product

$$\langle A(x), A(y) \rangle = \text{Re}(A(x)\overline{A(y)}) = \text{Re}(A(x)A(\bar{y})) = \text{Re}(A(x\bar{y})) = \text{Re}(x\bar{y}) = \langle x, y \rangle$$

Since  $A$  preserves the imaginary octonions, it makes sense to restrict  $A$  to acting on  $\text{Im}(\mathbb{O})$ , so that identifying  $A$  with its image in  $O(7)$  is well defined, since the inner product on  $\text{Im}(\mathbb{O})$  induces the standard norm on  $\mathbb{R}^7$  (computation provided below), we know from the polarization identity that the inner products are the same since they can be recovered from the norm, so that the image of  $A$  in  $O(7)$  is still orthogonal in the euclidean sense.

I include here the computation of equivalence of norms using the multiplication table: Consider the octonion given by  $a = (a_i e_i)_0^7$ , of course we are only interested in the case of  $a_0 = 0$ , so we have  $a = (a_i e_i)_1^7$

$$a\bar{a} = \left( \sum_1^7 a_i e_i \right) \left( \sum_1^7 -a_j e_j \right) = \sum_1^7 a_i^2 + \sum_{i < j} a_i a_j e_i e_j + \sum_{i > j} a_i a_j e_i e_j$$

We can read from the off diagonal of the octonions multiplication table that for  $i, j > 0$  and  $i \neq j$  that  $e_i e_j = -e_j e_i$ , this kills the two sums on the right to give us  $\sum_1^7 a_i^2$  the euclidean norm as desired.  $\square$

(c) From the multiplication table we have for any  $i$ ,  $e_i \bar{e}_i = e_0$ , it follows from definitions that  $\langle e_i, e_i \rangle = 1$  for  $i = 1, 2, 4$ . Now reading from the table,

$$e_1 \bar{e}_2 = -e_3, \quad e_1 \bar{e}_4 = -e_5, \quad e_2 \bar{e}_4 = -e_6, \quad (e_1 e_2) \bar{e}_4 = -e_7$$

these all have zero real part, so that by taking the inner product we get zero, this suffices to show its a special triple.  $\square$

(d)

Although not stated in the question,  $V_n(\mathbb{R}^m)$  is orthonormal sets of  $n$  vectors in  $\mathbb{R}^m$ , it is also defined in the notes as a quotient of the orthogonal group by a group action  $O(m)/O(m-n)$ . It follows that  $V_3(\mathbb{R}^7)$  can be realized as  $O(7)/O(4)$ , since  $O(4)$  is a lie group of dimension 6, and  $O(7)$  is a lie group of dimension 21, the quotient  $V_3(\mathbb{R}^7)$  is a  $21 - 6 = 15$  dimensional manifold, its also important here that we are identifying  $O(4)$  as the elements fixing the first second and fourth columns  $e_1, e_2, e_4$ , taking these columns in particular is important for the projection to work out in part (e).

The only additional condition of a special triple that isnt in  $V_3(\mathbb{R}^7)$  is the equation  $\langle xy, z \rangle = 0$ , so it will suffice to check that 0 is a regular value for

$$\begin{aligned} \hat{F} : V_3(\mathbb{R}^7) &\rightarrow \mathbb{R} \\ (x, y, z) &\mapsto \langle xy, z \rangle \end{aligned}$$

where again we are using the identification of the imaginary octonions with  $\mathbb{R}^7$ . The smoothness of  $\hat{F}$  is immediate by multilinearity. It is somewhat hard to deal with  $V_3(\mathbb{R}^7)$ , but we can show that it is a submanifold of  $\mathbb{R}^{21}$ , to do so use the regular value theorem with

$$\begin{aligned} F' : \text{Mat}_{3 \times 7}(\mathbb{R}) &\rightarrow \mathbb{R}^6 \\ (v_1 \quad v_2 \quad v_3) &\mapsto (\|v_1\|^2, \|v_2\|^2, \|v_3\|^2, \langle v_1, v_2 \rangle, \langle v_1, v_3 \rangle, \langle v_2, v_3 \rangle) \end{aligned}$$

$F'$  is polynomial hence smooth. The total derivative looks like (in  $1 \times 7$  blocks)

$$d_{(v_1, v_2, v_3)} F' = \begin{pmatrix} 2v_1 & 0 & 0 \\ 0 & 2v_2 & 0 \\ 0 & 0 & 2v_3 \\ v_2 & v_1 & 0 \\ v_3 & 0 & v_1 \\ 0 & v_3 & v_2 \end{pmatrix}$$

it is straightforward to see that  $(1, 1, 1, 0, 0, 0)$  is a regular value for  $F'$  by the vectors being nonzero at these points (independence of rows then follows easily from orthogonality). This realizes  $V_3(\mathbb{R}^7)$  as a  $15 = 21 - 6$  dimensional manifold  $F'^{-1}\{(1, 1, 1, 0, 0, 0)\}$ . Now we want to show that  $d_{(e_1, e_2, e_4)} \hat{F}$  is surjective, we will check later that using the action of  $G_2$  this gives surjectivity at all points. In order to avoid overloading notation on  $e_i$  I will use  $(f_i)_1^{21}$  to denote the basis on  $\mathbb{R}^{21}$ . Then we can define the path  $\gamma : (-\epsilon, \epsilon) \rightarrow V_3(\mathbb{R}^7)$  via  $\gamma(t) = (e_1, e_2, e_3 \sin t + e_4 \cos t)$ , it is clear by definition that  $\gamma(-\epsilon, \epsilon)$  does indeed lie in  $V_3(\mathbb{R}^7)$ , moreover we have  $\gamma(0) = (e_1, e_2, e_4)$  and  $\gamma'(t) = (0, 0, e_3) = f_{17} \in T_{(e_1, e_2, e_4)} V_3(\mathbb{R}^7)$ . Now we can take

$$\left. \frac{d}{dt} \right|_{t=0} F \circ \gamma(t) = \left. \frac{d}{dt} \right|_{t=0} \langle e_1 e_2, e_3 \sin t + e_4 \cos t \rangle = \left. \frac{d}{dt} \right|_{t=0} e_3 (\bar{e}_3 \cos t + \bar{e}_3 \sin t) = \left. \frac{d}{dt} \right|_{t=0} \cos t + \sin t = 1$$

so that indeed we have  $d_{(e_1, e_2, e_4)} \hat{F} = d_{(e_1, e_2, e_4)} F|_{T_{(e_1, e_2, e_4)} V_3(\mathbb{R}^7)}$  is nonzero, hence surjective. This gives that  $\hat{F}$  is a submersion at the point  $(e_1, e_2, e_4)$ , we need to check it for the rest of  $X = \hat{F}^{-1}(0)$ , here we can use the theorem that for any other point  $(x, y, z) \in X$  we have some  $A \in G_2$  with  $A(e_1, e_2, e_4) = (x, y, z)$ . Now since  $A$  respects products and inner products we have  $\hat{F} \circ A = \hat{F}$ , so that

$$(d_{(x, y, z)} \hat{F})(d_{(e_1, e_2, e_4)} A) = d_{(e_1, e_2, e_4)} (\hat{F} \circ A) = d_{(e_1, e_2, e_4)} \hat{F}$$

So surjectivity of the derivative at the single triple  $(e_1, e_2, e_4)$  gives surjectivity at all triples, this proves that 0 is a regular value.  $\square$

(e) Define the map using the theorem by defining  $A(x, y, z)$  to be the unique map transforming  $(e_1, e_2, e_4) \mapsto (x, y, z)$

$$\begin{aligned}\Phi : X &\rightarrow O(7) \\ (x, y, z) &\mapsto A_{(x, y, z)}\end{aligned}$$

That  $\Phi$  is injective is an immediate consequence of the theorem, surjectivity onto  $G_2$  is also straightforward, since  $G_2$  preserves norms, products and inner products (which are all of the special triple conditions) so all elements of  $G_2$  send special triples to special triples, whence  $G_2$  elements are all in the image of  $X$ , determined by their action on  $e_1, e_2, e_4$ . We can also take  $O(7)$  now with respect to the basis given by the special triple  $(e_1, e_2, e_4)$ , and note that this just amounts to flipping the signs on basis elements  $e_5, e_6$  and  $e_7$ , the reason for doing this is to clean up notation that  $\Phi(e_1, e_2, e_4) = 1$ . We also have that  $A$  is multilinear hence smooth. We check that it indeed defines an immersion, to do so consider the quotient map induced by the action of  $O(4)$  given by  $\pi : O(7) \rightarrow V_3\mathbb{R}^7$ , then  $\pi$  is smooth, and  $\pi \circ \Phi = 1_X$  (recall that  $X$  is a submanifold, then the composition maps into it), it follows that at any special triple we have

$$(d_{A(x, y, z)}\pi)(d_{(x, y, z)}\Phi) = 1_{T_{(x, y, z)}X}$$

injectivity of  $1_{T_{(x, y, z)}X}$  implies injectivity of  $d_{(x, y, z)}\Phi$ , so that  $\Phi$  is an immersion. Finally it only remains to check that  $\Phi$  is proper, but this follows immediately from  $X$  compact. To see that  $X$  is compact, we can identify  $X \subset \overline{B_{0,1}(\mathbb{R}^{21})}$ , where  $X$  is given by the intersection of zero-loci of polynomial equations. This realizes  $X$  as a closed compact subset of Euclidean space, hence compact.  $\square$