1. I will first prove a lemma, since I will use it multiple times in order to prove homotopy equivalences.

Lemma. If $e : X \hookrightarrow M$ is an embedding for manifolds M, X, and there is a strong deformation retract $H : M \to M$ with $H(M \times \{1\}) = X$, then $M \simeq X$.

Proof. Let $r(x) = H(x, 1)$, then $e|_{e(X)}^{-1} r : M \to X$ is smooth, and since H is a strong deformation retract we have $e|_{e(X)}^{-1} re = 1_X$, from which it suffices to show that $ee|_{e(X)}^{-1} r = r \simeq 1_M$, but $r = H(-, 1)$, so this homotopy is exhibited by H and we are done. $\square$

Let $V_0, \dots, V_n$ be the standard charts on $\mathbb{RP}^n$, now take $V = V_0$, and let pt. $= [0 : 0 : \cdots : 1] \in V_0^c$, then take $U = \mathbb{RP}^n \setminus \{\text{pt.}\}$, the standard chart map ϕ_0 gives us $V \cong \mathbb{R}^n$. Similarly, we find that $U \cap V = V \setminus \{\text{pt.}\}$, so that

$$\phi_0^{-1}|_{U\cap V} : U \cap V \xrightarrow{\cong} \mathbb{R}^n \setminus \{\phi_0^{-1}(\text{pt.})\} \simeq S^{n-1}$$

The homotopy equivalence is given by $\mathbb{R}^n \setminus \{\phi_0^{-1}(\text{pt.})\} \xrightarrow{\cong} \mathbb{R}^n \setminus \{0\}$ via $x \mapsto x - \phi_0^{-1}(\text{pt.})$, then taking the strong deformation retract $H(x, t) = (1 - t)x + t\frac{x}{||x||}$ which gives a homotopy equivalence to S^{n-1}. Now it remains to show $U \simeq \mathbb{RP}^{n-1}$. First we consider the smooth map $\theta : \mathbb{RP}^n \setminus \{\text{pt.}\} \to (0, \pi/2)$ via $[x_0, \dots, x_n] \mapsto \arcsin x_n$, where we take the representative of $[x_0, \dots, x_n]$ with $x_n > 0$, we can do this since we removed the point $x_n = 0$, and smoothness follows by arcsin being smooth on $[0, 1)$, so our map is smooth in coordinates, it follows that points in $\mathbb{RP}^n \setminus \{\text{pt.}\}$, now we can define the homotopy (where once again we define the maps on the representative with $x_n > 0$)

$$H([x], t) = \left[\cos((1 - t)\theta(x))\frac{(x_0, \dots, x_{n-1}, 0)}{||(x_0, \dots, x_{n-1}, 0)||} + \sin\theta(x)\right]$$

Once again, this map is smooth since it is defined to be smooth on coordinates, and $H(\mathbb{RP}^n \times \{1\}) = \{[x] \in \mathbb{RP} \mid x_n = 0\} \cong \mathbb{RP}^{n-1}$, where the diffeomorphism is given by the embedding $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ via $[x_0, \dots, x_{n-1}] \mapsto [x_0, \dots, x_{n-1}, 0]$, this map is smooth due to being identity on the charts given by the same coordinate non-vanishing loci. Proper since $\mathbb{RP}^{n-1}$ is compact, is clearly injective, and is an immersion since in appropriate coordinates its given by the identity. Hence the homotopy defined above gives a strong deformation retract from $\mathbb{RP}^n$ to $\{[x] \in \mathbb{RP} \mid x_n = 0\}$ from which we get a homotopy equivalence. This concludes the annoying details and now we can proceed with the algebraic argument.

We first want to show that for $0 < k < n$, we have $H^k(\mathbb{RP}^n) = 0$. Let $q : S^n \to \mathbb{RP}^n$ be the covering map, then since q is locally invertible and $\mathbb{RP}^n$ is compact, we have an open cover $U_1, \dots, U_s$ for $\mathbb{RP}^n$, with associated maps $q_1, \dots, q_s$ satisfying $qq_j = 1_{\mathbb{RP}^n}$ for each j, taking a partition of unity suboordinate to the U_j, we can define $f = \sum_1^s \eta_j \cdot q_j$, with $q \circ f = 1_{\mathbb{RP}^n}$, it follows that $f^*q^* = 1^*_{\mathbb{RP}^n}$. Now we want to show that $[q^*] : H^*(\mathbb{RP}^n) \to H^*(S^n)$ is injective, to do so assume that $[q^*]([\omega]) = [0]$, then $q^*\omega = d\nu$ for some ω representing the class $[\omega]$, and some form ν, now we can use our section to find that

$$\omega = f^*q^*\omega = f^*d\nu = df^*\nu$$

this shows that ω is an exact form, and hence $[\omega] = 0$. This suffices to show that $[q^*]$ is injective, but then for $0 < k < n$, we have $[q^*] : H^k(\mathbb{RP}^n) \hookrightarrow H^k(S^n) = 0$, so that $H^k(\mathbb{RP}^n) = 0$ for $0 < k < n$ as desired.

Since $U \cup V$ is an open cover for $\mathbb{RP}^n$, we get the short exact sequence of chain complexes

$$0 \longrightarrow \Omega^*(\mathbb{RP}^n) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V) \longrightarrow 0$$

Mayer Vietoris gives us a long exact sequence on cohomology, the portion of interest is for $n > 1$

$$\begin{tikzcd}
\cdots & H^n(U) \oplus H^n(V) \arrow[l] & H^n(\mathbb{RP}^n) \arrow[l] \\
H^{n-1}(U \cap V) \arrow[urr, in=0, out=180] & H^{n-1}(U) \oplus H^{n-1}(V) \arrow[l] & H^{n-1}(\mathbb{RP}^n) \arrow[l]
\end{tikzcd}$$

Since cohomology is a homotopy invariant, we may substitute in the spaces above to this LES.

$$\cdots \longleftarrow H^n(\mathbb{RP}^{n-1}) \oplus H^n(\mathbb{R}^n) \longleftarrow H^n(\mathbb{RP}^n) \longleftarrow H^{n-1}(S^{n-1}) \longleftarrow H^{n-1}(\mathbb{RP}^{n-1}) \oplus H^{n-1}(\mathbb{R}^n) \longleftarrow H^{n-1}(\mathbb{RP}^n)$$

Now we know the cohomology for spheres, and euclidean space, $\mathbb{RP}^{n-1}$ is $n-1$ dimensional so that its n-th cohomology is zero and finally we already computed that $H^{n-1}(\mathbb{RP}^n) = 0$. Applying this we get

$$\cdots \longleftarrow 0 \longleftarrow H^n(\mathbb{RP}^n) \longleftarrow \mathbb{R} \longleftarrow H^{n-1}(\mathbb{RP}^{n-1}) \longleftarrow 0$$

Exactness of this sequence gives us that $\mathbb{R} \cong H^{n-1}(\mathbb{RP}^{n-1}) \oplus H^n(\mathbb{RP}^n)$ (the splitting is guaranteed since were working with vector spaces). Now since $\mathbb{RP}^1 \cong S^1$, which has $H^1(S^1) \cong \mathbb{R}$, and the above formula holds for $n > 1$, we find recursively that for $n \geq 1$

$$H^n(\mathbb{RP}^n) \cong \begin{cases} \mathbb{R} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

From this and the fact that $\mathbb{RP}^n$ is connected giving it 0-th cohomology $\mathbb{R}$, we get the cohomology ring

$$H^*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{R}[x_n]/(x_n^2) & n \text{ odd} \\ \mathbb{R} & n \text{ even} \end{cases}$$

since the zero-th cohomology class is a unit with respect to wedge, and x_n represents the n-form $[\omega]$, but $\omega \wedge \omega = 0$ since $H^{2n}(\mathbb{RP}^n) = 0$ by dimension considerations. $\square$

2. Let g be a Riemannian metric for E over M and ω be a representative of $[\mathrm{Th}\,\pi]$ from the definition of vertically compactly supported cohomology, we have that $\pi^{-1}(K) \cap \operatorname{supp} \omega \subset E$ is compact for every $K \subset M$ compact. Now let

$$K_0 \subset U_1 \subset K_1 \subset U_2 \subset K_2 \subset \cdots$$

Be an exuastion for M (i.e. $\bigcup_1^\infty K_j = \bigcup_1^\infty U_j = M$) with K_j compact, and U_j open. It follows that for each K_j, we have $\sup_{(x,v)\in\pi^{-1}(K_j)\cap\operatorname{supp}\omega} g((x,v),(x,0)) = C_j < \infty$, since compact sets are bounded with respect to g. Now let $\{\eta_j\}_1^\infty$ be a partition of unity suboordinate to the U_j, and let $D_j = \inf_{x\in K_j} g(s(x),(x,0)) > 0$ since s is nonvanishing and this is a continuous function on a compact set, hence it attains its infimum. Now denote the vector bundle coordinate of s as s', i.e. if $s(x) = (x,v)$ then $s'(x) = v$, note that this is smooth since its just a projection of s. Now we can define a homotopy

$$H : (x,t) \mapsto \left(x, ts'(x) \sum_1^\infty \eta_j(x) \frac{C_{j+1}+1}{D_j}\right)$$

This is a homotopy from $\iota : M \hookrightarrow M \times \{0\} \subset E$ to $F : (x,t) \mapsto \left(x, s'(x) \sum_1^\infty \eta_j(x) \frac{C_{j+1}+1}{D_j}\right)$ and is clearly smooth since the first coordinate is identity and the second is a product of smooth functions. We want to check that $F^*\omega = 0$, to do so it suffices to check that F maps into $(\operatorname{supp}\omega)^c$, in which case the pullback is clearly zero. By definition of C_j, it will suffice to check that $g(F(x),(x,0)) > C_j$. Since Riemannian metrics are induced by inner products, we get that for fixed x, $|a|\, g((x,0),(x,v)) = g((x,0),(x,av))$ (note here the sum is taken to infinity but only finitely many η_j are nonzero, so all manipulations work out since it is in practicality a finite sum).

$$g((x,0),F(x)) = \sum_1^\infty \eta_j(x)\frac{C_{j+1}+1}{D_j}g((x,0),(x,s'(x))) \geq \sum_1^\infty \eta_j(x)\frac{C_{j+1}+1}{D_j}D_j = \sum_1^\infty \eta_j(x)(C_{j+1}+1)$$

Now note that since our exhaustion is an increasing union, we have that $C_{j+1} \geq C_j$ for all j, and moreover if $x \in K_N \setminus K_{N-1}$ (all $x \in M$ are in some set of this form or M compact and $K_0 = M$, in which case just trake K_0), then $\sup_{\pi^{-1}(x)\cap \operatorname{supp}\omega} g((x,v),(x,0)) \leq C_N$, and all η_j with $j < N-1$ are zero, it follows that

$$g((x,0),F(x)) \geq \sum_{N-1}^\infty \eta_j(x)(C_{j+1}+1) \geq \sum_{N-1}^\infty \eta_j(x)(C_N+1) = (C_N+1)\sum_{N-1}^\infty \eta_j(x) = C_N+1 > C_N$$

This suffices to show that for all x, $F(x) \notin \operatorname{supp}\omega$, so that $F^*\omega = 0$ whence $0 = F^*[\operatorname{Th}\pi] = \iota^*[\operatorname{Th}\pi]$, the second equality of course following from $F \simeq \iota$. □

3. (a) The proof of the case $\alpha' = 0$ is identical to that of $\alpha = 0$, but we show that the rows rather than columns are linearly independent, so assume $v_1 \wedge \cdots \wedge v_p = \alpha \neq 0$. Now, it will suffice to show by induction that if $k < p$, then we can choose ω_{k+1}, so that

$$\left\{ \begin{pmatrix} \langle v_1, \omega_1 \rangle \\ \vdots \\ \langle v_p, \omega_1 \rangle \end{pmatrix}, \begin{pmatrix} \langle v_1, \omega_2 \rangle \\ \vdots \\ \langle v_p, \omega_2 \rangle \end{pmatrix}, \ldots, \begin{pmatrix} \langle v_1, \omega_{k+1} \rangle \\ \vdots \\ \langle v_p, \omega_{k+1} \rangle \end{pmatrix} \right\} \subset \mathbb{R}^p$$

are linearly independent. Since $\alpha = v_1 \wedge \cdots \wedge v_p \neq 0$ we have necessarily that the v_j are linearly independent. Now since $k < p$, we can choose some $(x_1,\ldots,x_p) \in \mathbb{R}^p$ linearly independent from the first k-columns, then $\omega = \omega_{k+1}$ can be constructed as follows, start with $\omega = \frac{x_1 v_1}{||v_1||^2}$ this is the base case, now assume recursively we have $\langle v_1, \omega \rangle = x_1, \ldots, \langle v_j, \omega \rangle = x_j$, then we can take u to be the projection of v_{j+1} to $\operatorname{span}\{v_1,\ldots,v_j\}^\perp$, this is nonzero since $v_{j+1} \notin \operatorname{span}\{v_1,\ldots,v_j\}$. Then we have $\langle v_{j+1}, u \rangle = a \neq 0$ finally denote $\langle v_{j+1}, \omega \rangle = b$, and now take $\omega' = \omega + \frac{x_{j+1}-b}{a}u$, then since u is orthogonal to $v_1,\ldots,v_j$, we still get $\langle v_i, \omega' \rangle = x_i$ for $i = 1,\ldots,j$, but now we also get that

$$\langle v_{j+1}, \omega' \rangle = \langle v_{j+1}, \omega \rangle + \frac{x_{j+1}-b}{a}\langle v_{j+1}, u \rangle = b + \frac{x_{j+1}-b}{a}a = x_{j+1}$$

Continuing this process we get the desired ω_{k+1}, since this holds for any $k < p$, we can always construct some $\omega_1 \wedge \cdots \wedge \omega_p$ with the property that the columns of $(\langle v_i, \omega_j \rangle)_{1\leq i,j\leq p}$ are linearly independent, and hence $\langle \alpha, \omega_1 \wedge \cdots \wedge \omega_p \rangle_p = \det(\langle v_i, \omega_j \rangle)_{1 \leq i,j \leq p} \neq 0$. □

(b) Consider two positively oriented orthonormal bases $e_1,\ldots,e_k$ and $d_1,\ldots,d_k$. Let T be the linear map defined by $T(e_i) = d_i$, and extending linearly, since both bases are positively oriented we get $\det T > 0$, moreover we have $(T^{\mathrm{T}}T)_{ij} = \langle d_i, d_j \rangle = \delta_{ij}$, so that $T^{\mathrm{T}}T = 1_V$ is orthogonal, since $\det T^{\mathrm{T}} = \det T$, this relation gives us $(\det T)^2 = 1$, so $\det T = \pm 1$, but since we have established $\det T > 0$, we get to conclude that $\det T = 1$. Now we are done since

$$d_1 \wedge \cdots \wedge d_k = T(e_1) \wedge \cdots \wedge T(e_k) = (\det T)(e_1 \wedge \cdots \wedge e_k) = e_1 \wedge \cdots \wedge e_k$$

□

(c) We first consider an element of the form $\beta = e_{i_1} \wedge \cdots \wedge e_{i_{k-p}}$ with $i_1 < i_2 < \cdots < i_{k-p}$, now we can denote $\{j_1,\ldots,j_p\} = \{1,\ldots,k\} \setminus \{i_1,\ldots,i_{k-p}\}$ with $j_1 < \cdots < j_p$. It follows that $e_{j_1} \wedge \cdots \wedge e_{j_p} \wedge \beta = (-1)^\ell \omega$ for some ℓ. I claim that $\star\beta = (-1)^\ell e_{j_1} \wedge \cdots \wedge e_{j_p}$ satisfies $\lambda_\beta(\alpha) = \langle \alpha, \star\beta \rangle_p$. We first check this for α of the form $e_{r_1} \wedge \cdots \wedge e_{r_p}$, since if it holds for elements of this form we get general elements of $\Lambda^p(V) = \sum a_i\alpha_i$ for a_i of this form, so that since λ_β is linear we get

$$\lambda_\beta(\sum a_i\alpha_i) = \sum a_i\lambda_\beta(\alpha_i) = \sum a_i\langle \alpha_i, \star\beta \rangle = \langle \sum a_i\alpha_i, \star\beta \rangle$$

so it suffices to check in this simplified case. Now if $\{r_1, \dots, r_p\} \cap \{i_1, \dots, i_{k-p}\} = \{i_z\} \neq \emptyset$, then we get $\alpha \wedge \beta = 0$, hence $\lambda_\beta(\alpha) = 0$, as well as the matrix with determinant $\langle \alpha, \star\beta\rangle_p$ having a row corresponding to $(\langle e_{i_z}, e_{j_1}\rangle, \dots, \langle e_{i_z}, e_{j_p}\rangle) = (0, \dots, 0)$, so that $\langle \alpha, \star\beta\rangle_p = 0$, now in the case that $\{r_1, \dots, r_p\} \cap \{i_1, \dots, i_{k-p}\} = \emptyset$, we get that $r_1, \dots, r_p = \sigma(j_1), \dots, \sigma(j_p)$ for $\sigma \in S_p$, then $e_{r_1} \wedge \cdots \wedge e_{r_p} = \operatorname{sgn}(\sigma) e_{j_1} \wedge \cdots \wedge e_{j_p}$, so that $\alpha \wedge \beta = \operatorname{sgn}(\sigma)(-1)^\ell \omega$, and $\langle \alpha \wedge \beta, \omega\rangle_k = \operatorname{sgn}(\sigma)(-1)^\ell$, moreover $\langle \alpha, \star\beta\rangle = (-1)^\ell \det P_\sigma$ where P_σ denotes the permutation matrix taking $j_1 \mapsto \sigma(j_1)$, of course this is also equal to $(-1)^\ell \operatorname{sgn}(\sigma)$, so we have provided existence of $\star\beta$ for β of the form $e_{i_1} \wedge \cdots \wedge e_{i_{k-p}}$, from this we can establish existence for all β, since any $\beta \in \Lambda^{k-p}(V)$ can be written as $\sum a_i \beta_i$ for β_i of this form, this allows us to define $\star\beta = \sum a_i \star \beta_i$ then for any $\alpha \in \Lambda^p(V)$ we get

$$\begin{aligned}\lambda_\beta(\alpha) &= \langle \alpha \wedge \sum a_i \beta_i, \omega\rangle_k = \langle \sum a_i \alpha \wedge \beta_i, \omega\rangle_k = \sum a_i \langle \alpha \wedge \beta_i, \omega\rangle_k \\ &= \sum a_i \langle \alpha, \star\beta_i\rangle_p = \langle \alpha, \sum a_i \star \beta_i\rangle_p = \langle \alpha, \star\beta\rangle_p\end{aligned}$$

Which suffices to prove existence for any $\beta \in \Lambda^{k-p}(V)$. Now we need to check uniqueness Suppose $\star\beta' = \star\beta$, then $\alpha \mapsto \langle \alpha \wedge (\beta - \beta'), \omega\rangle_k = 0$ for all α. Suppose now that $\beta \neq \beta'$, we can write $\beta = \sum a_i \beta_i$, and $\beta' = \sum b_i \beta_i'$ where β_i, β_i' are of the form $e_{i_1} \wedge \cdots \wedge e_{i_{k-p}}$ for $i_1 < \cdots < k-p$, it follows that the multiplicity of one of these summands must differ between β and β', otherwise the two will be equal. So assume without loss of generality that $\beta_1 = \beta_1'$, but $a_1 \neq b_1$, moreover since one of them must be nonzero we can assume $a_1 \neq 0$. Now denote $\beta_1 = e_{i_i} \wedge \cdots \wedge e_{i_{k-p}}$, and once again define $\{j_1, \dots, j_p\} = \{1, \dots, k\} \setminus \{i_1, \dots, i_{k-p}\}$ with $j_1 < \cdots < j_p$, it follows that for $\alpha = e_{j_1} \wedge \cdots \wedge e_{j_p}$ we have $\alpha \wedge \beta_\ell = 0$ for any $\ell \neq 1$, and same for β_ℓ', since some j_z must appear in the wedge terms of β_ℓ (or respectively β_ℓ') by virtue of β_ℓ (resp. β_ℓ') not being identical to $\beta_1 = \beta_1'$. Moreover, we get $\alpha \wedge \beta_1 = (-1)^r \omega$ for some r. It follows that

$$\begin{aligned}\langle \alpha \wedge (\beta - \beta'), \omega\rangle_k &= \langle \alpha \wedge \beta, \omega\rangle_k - \langle \alpha \wedge \beta', \omega\rangle_k = \sum a_i \langle \alpha \wedge \beta_i, \omega\rangle_k - \sum b_i \langle \alpha \wedge \beta', \omega\rangle_k \\ &= a_1 \langle \alpha \wedge \beta_1, \omega\rangle_k - b_1 \langle \alpha \wedge \beta_1', \omega\rangle_k = (a_1 - b_1)\langle \alpha \wedge \beta_1, \omega\rangle_k \\ &= (-1)^r (a_1 - b_1)\langle \omega, \omega\rangle_k = (-1)^r (a_1 - b_1) \neq 0\end{aligned}$$

Which contradicts $\star\beta = \star\beta'$, so this suffices to show uniqueness.

Now that we have existence and uniqueness, linearity is quite easy. Let $\beta, \gamma \in \Lambda^{k-p}(V)$, then for any α we have

$$\begin{aligned}\langle \alpha, \star(a\beta + b\gamma)\rangle_p &= \langle \alpha \wedge (a\beta + b\gamma), \omega\rangle_k = \langle a\alpha \wedge \beta + b\alpha \wedge \gamma, \omega\rangle_k = a\langle \alpha \wedge \beta, \omega\rangle_k + b\langle \alpha \wedge \gamma, \omega\rangle_k \\ &= a\langle \alpha, \star\beta\rangle_p + b\langle \alpha, \star\gamma\rangle_p = \langle \alpha, a \star \beta + b \star \gamma\rangle\end{aligned}$$

Uniqueness then tells us that $\star(a\beta + b\gamma) = a \star \beta + b \star \gamma$. □

(d) We would like to use (a)-(c) to produce a fiber-wise definition of $\star$, in such a way that we ensure gluing together these fiberwise maps gives a smooth map on TODO

(e) We can use that $\star$ agrees fiberwise with the original fiberwise definition, and in this case identify $dx, dy, dz \leftrightarrow e_1, e_2, e_3$. This in particular means our proof from part (c) shows that if $h \in C^\infty(M, \mathbb{R})$, we get

$$\begin{aligned}\star h dx &= h \star dx = h dy \wedge dz \\ \star h dy &= h \star dy = -h dx \wedge dz \\ \star h dz &= h \star dz = h dx \wedge dy\end{aligned}$$

Now applying this to df, we get

$$\begin{aligned}d \star df &= d \star \frac{\partial}{\partial x} f dx + \frac{\partial}{\partial y} f dy + \frac{\partial}{\partial z} f dz = d(\frac{\partial}{\partial x} f dy \wedge dz - \frac{\partial}{\partial y} f dx \wedge dz + \frac{\partial}{\partial z} f dx \wedge dy) \\ &= \frac{\partial^2}{\partial x^2} f dx \wedge dy \wedge dz - \frac{\partial^2}{\partial y^2} f dy \wedge dx \wedge dz + \frac{\partial^2}{\partial z^2} f dz \wedge dx \wedge dy \\ &= (\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f) dx \wedge dy \wedge dz\end{aligned}$$

□