Lie Theory Lecture 1.

Defn. (Lie Group) Manifold w/ a gp. structure, $L_g : h \mapsto gh$, $\iota : g \mapsto g^{-1}$ smooth.

Rmk. $G =$ Lie gp. simply connected $\iff$ $\forall \gamma : S^1 \longrightarrow G$, $\exists F : D^2 \longrightarrow G$ with $F(e^{i\theta}) = \gamma(\theta)$.

Defn. $SU(2) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \;\middle|\; (z,w) \in S^3 \right\}$

- can be identified w/ Quaternions. (units)

Defn. The Matrix exponential.

$\exp(X) := \sum_0^\infty \frac{X^j}{j!}$

$\rightsquigarrow$ Convergence follows from $\exists$ a norm. $|X| = \sum_{j,k} |x_{j,k}|^2$, then $e^{|X|} < \infty$, so $\exp(X)$ converges.

Properties:

- $e^0 = 1$
- $(e^X)^* = e^{X^*}$
- $(\exp(X))^{-1} = \exp(X^{-1})$
- $\exp(X+Y) = \exp(X)\exp(Y)$
- $\exp(PXP^{-1}) = P\exp(X)P^{-1}$
- $\frac{d}{dt}\exp(tX) = Xe^{tX}$.

Lie Theory Lecture 1

Defn. The lie algebra of a lie group is $T_e G$.

Defn. X is left invariant if $\forall g \in G$, $(L_g)_* X = X$
$\{ = dg\, X \circ g^{-1} \}$

- There is a bij. correspondence left invariant vectorfields on G, and $T_e G$.

Defn. (Lie Bracket) - for matrices: $[X,Y] = XY - YX$.
-In general: Antisymmetric, bilinear satisfying
$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Lecture 7.

Examples of Lie Algebras: - $U(1) = S^1$, w/ lie algebra $\mathbb{R}$,
exp: $\theta \longmapsto e^{i\theta}$

- $O(n)$, can write $A \in O(n) = 1 + \alpha$, then $(1+\alpha)(1+\alpha^T) = 1$
$\Rightarrow \alpha + \alpha^T = 0$, hence $T_{1_n} O(n) = \{\text{skew symmetric mats}\}$

- $GL(n, \mathbb{R})$ det $\neq 0$ is an open condition $\Rightarrow$
$T_{1_n} GL(n, \mathbb{R}) = M_{n\times n}$

- $SL(n, \mathbb{R})$, $T_{1_n}(SL(n,\mathbb{R})) = \left\{\begin{matrix}\text{Trace } 0 \\ \text{matrices}\end{matrix}\right\}$ since
det $\exp(A) = \exp(\operatorname{Tr} A)$, then
det $\exp(A) = 1 \Leftrightarrow \exp(\operatorname{Tr} A) = 1 \Leftrightarrow \operatorname{Tr} A = 0$

- $U(n)$, $A = 1_n + \beta$, then $A^\dagger A = 1_n \Rightarrow \beta + \beta^\dagger = 0$
$\Rightarrow T_{1_n} U(n) = \{\text{skew sym-hermitian matrices}\}$.

- $SU(n)$ then has $T_{1_n} SU(n) = \left\{\begin{matrix}\text{Skew sym} \\ \text{\& Trace } 0\end{matrix}\right\}$.
- The Lie Algebra has basis Pauli matrices.
$\sigma_1 = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Lecture 2.

- Then $SU(2)$ & $SO(3)$ have isomorphic Lie Algebras, since the map

$$\begin{aligned} SU(2) &\longrightarrow SO(3) \\ A &\longmapsto (\sigma_A : u \longmapsto AuA^{-1}) \end{aligned}$$

is 2-1.

This is easier to understand Geometrically.

$SU(2) \cong S^3 \longrightarrow \mathbb{RP}^3 \cong SO(3)$.

Lecture 3.

- Torus Actions on Mflds.

Ex. $U(1) \circlearrowright T^2$, $u \cdot (z_1, z_2) = (u^{m_1} z_1, u^{m_2} z_2)$

- it is notable this action has no fixed points.

This can be expected since $\chi(T^2) = 1-2+1 = 0$.

- In general we expect $U(1)^n \circlearrowright M$ to have Fixed points unless $\chi(M) = 0$.

Defn. (weight) If $\langle \cdot, \cdot \rangle$ is a lie bracket invariant inner product (i.e. $\langle X, [Y,Z] \rangle = \langle [X,Y], Z \rangle$) Then λ is called a weight of representation ρ if $\exists v$, $\rho(\exp(X)) \cdot v = \langle \lambda, X \rangle \cdot v$ For all X in the Lie Algebra.

Rmk. Weight vectors with distinct weights are L.I.

- We can think of weight spaces as eigenspaces of $G \circlearrowright_{\rho} V$.

- An example of when weights are useful is the Duistermaat - Heckman Formula, ω a symplectic form on M, compact mfld & $F_1, \ldots, F_n$ isolated fixed points of action,

$$\int_M e^{t\mu} \frac{\omega^n}{n!} = \frac{\sum_j e^{t\mu(F_j)}}{\prod_{\text{weights at } F_j} \lambda}$$

, where μ is a hamiltonian fn. w/ hamiltonian generating $U(1)$ action.

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Example of weights appearing in formulae:

Duistermaat-Heckman.

Computing integrals of the form $\int e^{f(x)} dx$

- p_i = critical points (f), then $\int e^{f(x)} dx = e^{f(p_i)} \int e^{-c_i (x - x_i)^2} dx$

near p_i

Recall: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-cx^2} dx = \sqrt{\frac{\pi}{c}}$

"Stationary phase approximation"

Example. Momentum map for $U(1) \curvearrowright S^2$ $f(z, p) = z$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} dz\, e^{ikz} = 2\pi \int_{-1}^{1} e^{ikz} dz = \frac{2\pi}{ik}\left(e^{ik} - e^{-ik}\right)$$

Irreducible Representations. V a rep of the Lie group G, then V is irreducible when V cannot be decomposed as $W_1 \oplus W_2$

Schur's Lemma $\Rightarrow$ Any irreducible complex rep. of a Torus $U(1)^n$ is one dimensional.

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- There are many irreducible reps of nonabelian gps. e.g. $SU(2)$ has an irred. cpx. rep. in each dimension V_n.

- e.g. $\mathbb{C}^2$ is an irred. rep. of $SU(2)$.

- $$\begin{aligned} U(1) &\hookrightarrow SU(2) \\ e^{i\theta} &\longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \end{aligned}$$ } but this $U(1)$ rep is reducible.

<u>Schur's Lemma.</u> 1. V, W irreducible reps of abelian group G, and $\phi: V \to W$ is an intertwining map ($\phi(gv) = g\phi(v)$) then $\phi \equiv 0$ or $\phi: V \xrightarrow{\cong} W$.

~ Does not require alg. closed field.

2. If V is an irreducible rep into $GL(n, \mathbb{C})$ and $\phi: V \to V$ is intertwining (into $GL(n,\mathbb{C})$) then $\phi = \lambda \mathbb{1}_{n\times n}$

3. V, W irreducible $\phi_1, \phi_2 : V \to W$ intertwining, then $\phi_1 = \lambda \phi_2$.

- $U(1)^n \curvearrowright V$ by mult by a scalar λ. Whence every subspace of V is a subrep.

<u>Proof of Schur.</u> 1. $\ker\phi$ is invariant under G action. By irreducibility $\ker\phi = 0$ or $\ker\phi = V$. If $\ker\phi = 0$, then $\phi(V) \overset{\text{subrep.}}{\subset} W \Rightarrow \phi(V) = W$.

2. ϕ has an eigenvalue by satisfying its min. pol. The eigenspace $U \subset V$ is a subrep.

3. $\phi_1 \circ \phi_2^{-1} : W \xrightarrow{\lambda 1} W \quad \Rightarrow \phi_1 = \lambda \phi_2$.

<u>Hall - cor 4.3.0:</u> $A \in Z(G)$, then A acts as $\lambda \in \mathbb{C}$.

<u>ex.</u> - $G = GL(n, \mathbb{C})$ $\quad Z(G) = \{c1 \mid c \in \mathbb{C}\}$

- $G = SL(n, \mathbb{C})$ $\quad Z(G) = \{z1 \mid z^n = 1\}$

Finite center.

(i) = "reductive"

Defn. A Lie Group is semisimple if

(i) it is the complexification of a compact lie group.

(ii) The lie algebra of the center $Z(G) = 0$

$\iff Z(G)$ is finite.

Ex. - $Sl(2,\mathbb{C})$ = complexification $SU(2)$, &

$Z(Sl(2,\mathbb{C})) = \{\pm 1\}$ is finite.

- $GL(2,\mathbb{C})$ is not semisimple, since

$\mathbb{C}1 = Z(G)$.

- $n \geq 2$, $SL(n,\mathbb{C})$ is s.s. $Z(G) = \{\lambda I \mid \det(\lambda I) = 1\}$

cpx. of $SU(n)$. $= \{\zeta_n$ - roots of unity$\}$.

- $GL(n,\mathbb{C})$ = cpx.ification $(U(n))$ is reductive.

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Defn. G a compact Lie group, then a <u>maximal torus</u> for G is a maximal abelian connected subgroup of G.

Theorem. (1) every $x \in G$ is contained in a maximal Torus.
(2). All maximal Tori are conjugate.

Ex. $G = U(n)$, then $U(1)^n$ = diagonal (unitary) matrices.
~ To see (1), recall all unitary matrices are diagonalizable.

Ex. $G = SO(4)$, $T = SO(2) \times SO(2)$

$$= \left\{ \begin{bmatrix} \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \end{bmatrix} \;\middle|\; \begin{array}{l} c_j = \cos\theta_j \\ s_j = \sin\theta_j \end{array} \right\}$$

Ex. $G = SO(3) \overset{\text{Torus}}{=} \left\{ \begin{bmatrix} \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \theta = \text{rotation mtx.} \right\}$

$$\text{Torus}(SO(2n)) = \left\{ \begin{bmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_n \end{bmatrix} \;\middle|\; \theta_j = 2\times 2 \text{ rotation mtx.} \right\}$$

$$SO(2n+1) \overset{\text{Torus}}{=} \left\{ \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_n & \\ & & & 1 \end{bmatrix} \;\middle|\; \theta_j = 2\times 2 \text{ rotation matrix} \right\}$$

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Defn. (Root space). The weight space of the adjoint action of a maximal torus $T \subseteq G$,

$T \underset{ad}{\curvearrowright} NT \subseteq TG$.

$\uparrow$ The orthogonal complement of $\mathrm{Lie}(T)$ in $\mathrm{Lie}(G)$

- Decomposition of $\mathrm{Lie}(G)$ as

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \underbrace{\eta}_{\mathrm{Lie}(T)^\perp}$$

- There is a metric (killing form) on G invariant under conjugation.

Ex. $G = SU(2) = \left\{\begin{pmatrix} z & w \\ -\bar{w} & \bar{z}\end{pmatrix}\right\}$, $T = U(1) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$

$\mathrm{Lie}(G) = \{\text{skew hermitian } 2\times 2 \text{ matrices}\}$

$\uparrow A + A^t = 0$.

$= \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} \right\rangle$ (real basis).

- Look at adjoint action of T on $\mathfrak{g} \otimes \mathbb{C}$

$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ has $\mathbb{C}$-basis $\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$.

$T \overset{adj}{\curvearrowright} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix}$

$= e^{2i\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T \overset{adj}{\curvearrowright} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix}$

$= e^{-2i\theta} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are eigenvectors of $T \underset{adj}{\curvearrowright} \mathfrak{g} \otimes \mathbb{C}$. These are called <u>roots</u>.

<u>Defn.</u> (root) weight of adjoint action $T \curvearrowright \mathfrak{g} \otimes \mathbb{C}$.

<u>Root Spaces</u>: For $GL(n, \mathbb{C})$ $T = U(1)^n$, roots are matrices e_{ij}, $i \neq j$.

- Decomposing Tensor Products of $SL(2,\mathbb{C})$ irreps. (Clebsch - Gordan), Appendix C - B. Hall.

- For every $n \in \mathbb{Z}_{\geq 0}$, $\exists!$ irreducible complex representation of $SL(2,\mathbb{C})$ having dimension $n+1$, denoted V_n.

- V_0 = Trivial rep. $\cong \mathbb{C}$.

- $V_1 \cong \mathbb{C}^2$, representation of $SL(2,\mathbb{C})$ by rank 2 square cpx. matrices.

- $V_2 \cong \mathbb{C}^3$, the adjoint representation.
  $SL(2,\mathbb{C}) \curvearrowright sl(2,\mathbb{C})$ by conjugation

- $V_j \cong \mathbb{C}^{j+1}$, are all irred. representations.

- $V_n \otimes V_m$ is a rep. of $SL(2,\mathbb{C})$ (acting diagonally), this rep. is not irreducible.

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<u>Ex.</u> $V_m \otimes V_1$ has dimension $2(m+1)$ and
$V_m \otimes V_1 \cong V_{m+1} \oplus V_{m-1}$.

<u>special case:</u> $V_1 \otimes V_1 \cong V_2 \oplus V_0$

- X, Y, H basis for $sl(2, \mathbb{C})$ $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,
$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

e_1, e_2 = std. $\mathbb{C}$-basis. are eigenvectors for H.
$He_1 = e_1$, $He_2 = -e_2$.

$H(e_1 \otimes e_1) = 2e_1 \otimes e_1$, $H(e_2 \otimes e_2) = 2e_2 \otimes e_2$.
$\rightsquigarrow$ 2 is largest eigenvalue for H, whence $e_1 \otimes e_1$
is killed by X. (since applying X gives a vector w/ a larger eigenvalue).

- Applying Y to $e_1 \otimes e_2$ gives $e_1 \otimes e_2 + e_2 \otimes e_1$
Applying Y again gives $2e_2 \otimes e_2$
$\quad = 0$.

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- Raising & Lowering Operators.
- $\hat{N}$ a "number operator" $\hat{N} = a^{\dagger} a$,

$$a^{\dagger} |\hat{n}\rangle = \sqrt{n+1}\, |n+1\rangle \quad \text{(raising operator)}$$

$$a |n\rangle = \sqrt{n}\, |n-1\rangle \quad \text{(lowering operator)}.$$

$$\hat{N} |n\rangle = n |n\rangle .$$

__Theorem__ (Clebsch, Gordan; Hall, Appendix C.1).

$V_m \otimes V_n$ decomposes into the following irreps.

$$V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n} = V_m \otimes V_n .$$

__Proof (sketch):__ $\{u_m, u_{m-2}, \ldots, u_{-m}\} = \text{Basis}(V_m)$

- same for V_n. With u_j, v_j eigenvectors for the action of H. $Hu_j = j u_j$, $Hv_k = k v_k$.

in V_m rep. in V_n rep.

Then $\{u_j \otimes v_k\} = \text{Basis}(V_m \otimes V_n)$.

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- H acts via $\rho_m(H) \otimes I \oplus I \otimes \rho_n(H)$.

$\hookrightarrow$ $\{\underbrace{u_j \otimes v_k}_{\text{eigenvalue } j+k}\}$ = eigenbasis for $H \circlearrowleft V_m \otimes V_n$.

- Eigenspace $u_m \otimes v_n$ has dim 1.

Eigenspace of $n+m-2$ has dim 2.

Eigenspace of $n+m-4$ has dim 3

$\vdots$

Eigenspace of $m-n, \cdots, n-m$ have same dim. Then dim decreases by 1 at $n-m+2$. and so on.

Rmk. see table on (B.Hall p. 429).

$\hookrightarrow$ see pf. on pg. 428-429.

$m=4$, $n=2$.

Eigenvalue for H	Basis.
6	$u_4 \otimes v_2$
4	$u_2 \otimes v_2$, $u_4 \otimes v_0$
2	$\}$ 3 eigenstates.
0	
-2	
4	$u_{-4} \otimes v_0$, $u_{-2} \otimes v_{-2}$
6	$u_{-4} \otimes v_{-2}$.

Notes From Book.

Defn. A <u>subalgebra</u> of a Lie algebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t. $H_1, H_2 \in \mathfrak{h} \Rightarrow [H_1, H_2] \in \mathfrak{h}$.

- A subalgebra is an ideal when $[X, H] \in \mathfrak{h}$ for any $X \in \mathfrak{g}$, $\mathfrak{h} \ni H$.
- The center of $\mathfrak{g}$ is $\{X \in \mathfrak{g} \mid [X,Y]=0, \forall Y\}$
- $\phi[X,Y] = [\phi(X), \phi(Y)]$, then ϕ is a hom.

Defn. (Adjoint rep). $ad_X : \mathfrak{g} \longrightarrow \mathfrak{g}$

$$Y \longmapsto [X,Y].$$

- Then $ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$ since

$$ad_{[X,Y]} = [ad_X, ad_Y]$$

by the Jacobi identity.

Defn. A lie algebra is irred if it only has the trivial ideals, and simple if irred and $\dim \mathfrak{g} \geq 2$.

- $sl(2;\mathbb{C})$ is simple.

- The commutator ideal is $([g,g])$.

<u>Defn</u>. The upper central series

$$g = g^0 \supset \underset{\underset{[g,g]}{\parallel}}{g^1} \supset \underset{\underset{[g,g^1]}{\parallel}}{g^2} \supset \underset{\underset{[g,g^2]}{\parallel}}{g^3} \supset \cdots$$

Then g is <u>Nilpotent</u> if $g^j = 0$ for some j.

<u>Defn</u>. The derived series

$g_0 = g$, $g_1 = [g_0, g_0]$, $g_2 = [g_1, g_1]$, $\ldots$. Then g is <u>solvable</u> if $\exists j$, $g_j = 0$.

- Nilpotent $\Rightarrow$ solveable.

<u>Ex</u>. $g = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}$

is a Nilpotent Lie Algebra.

Pf. $\mathfrak{g} = \left\langle \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= X}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{= Y}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= Z} \right\rangle$
and

$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0.$

E.X. $\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}$

is solveable but not nilpotent.

Pf. $\left[\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \right] = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix},$

so $[\mathfrak{g}, \mathfrak{g}]$ has dim 1 $\Rightarrow$ commutative.

$\Rightarrow [\mathfrak{g}_1, \mathfrak{g}_1] = 0.$ But for

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$[H, X] = 2X \Rightarrow \mathrm{ad}_H^n(X)$

$= 2^n X \neq 0.$

The Lie Algebra Associated to a Matrix Lie Group.

<u>Defn</u>. The Lie Algebra $\mathfrak{g}$ of G is $\{X \mid \exp(tX) \in G, \forall t \in \mathbb{R}\}$.

- $\mathfrak{g}$ can be identified with $T_1 G$. $X \in \mathfrak{g}$.
- To see e^X is in the identity component, $\gamma(0) = 1$, $\gamma(1) = e^X$, $\gamma(t) = e^{tX}$.

<u>Prop</u>. $\mathfrak{g}$ is closed under conjugation by elts. of G, addition, scalar mult, and $[\cdot,\cdot]$ ← matrix bracket.

- If G is commutative, then $\mathfrak{g}$ is commutative (the converse is also true).

<u>Pf</u>. $[X,Y] = \frac{d}{dt}\Big|_{t=0} \left(\frac{d}{ds}\Big|_{s=0} e^{tX} e^{sY} e^{-tX} \right)$

$$= \frac{d}{dt}\Big|_{t=0} e^{sY} = 0.$$

Theorem. G, H matrix Lie groups, then given Lie Group hom $\Phi : G \to H$, $\exists!$ real-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ with $\Phi(\exp X) = \exp(\phi(X))$. Moreover

(i) $\phi(AXA^{-1}) = \Phi(A)\,\phi(X)\,\Phi(A^{-1})$, $X \in \mathfrak{g}$, $A \in G$

(ii) $\phi([X, Y]) = [\phi(X), \phi(Y)]$

(iii) $\phi(X) = \frac{d}{dt}\Big|_{t=0} \Phi(\exp(tX))$

- In practice ϕ is computed using the property (iii).