1. We first find an expression for $\zeta(s)\Gamma(s)$ on $\Re(s) > 1$, to do so we will do a substitution $u = x/n$, valid for any $n \in \mathbb{Z}_{>0}$

$$\Gamma(s) = \int_0^\infty e^{-nu}(nu)^{s-1} n du = n^s \int_0^\infty u^{s-1}(e^{-u})^n du$$

Now since I don't like u I will switch back to x; Multiplying n^{-s} on both sides and summing over $n \in \mathbb{Z}_{>0}$ yields

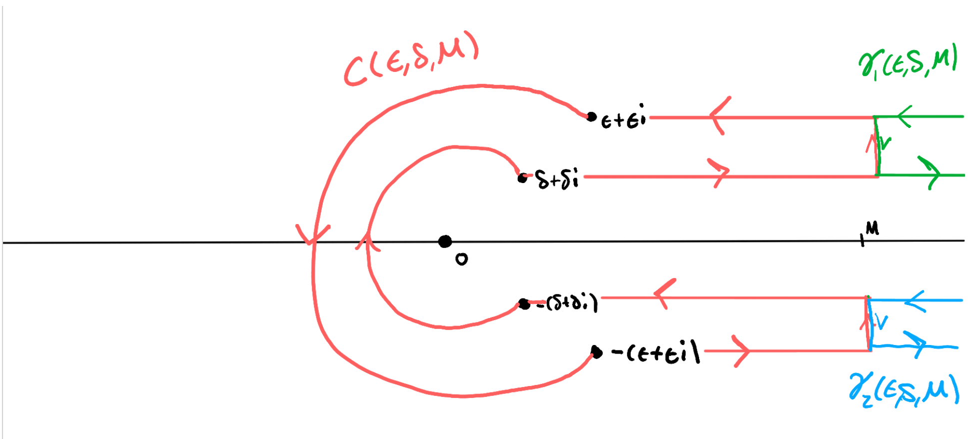
$$\zeta(s)\Gamma(s) = \sum_1^\infty \int_0^\infty x^{s-1}(e^{-x})^n dx \overset{\text{DCT}}{=} \int_0^\infty x^{s-1} \sum_1^\infty (e^{-x})^n dx = \int_0^\infty x^{s-1} \frac{1}{e^x - 1} dx \tag{1}$$

Where DCT is taken with respect to $|x^{s-1}| \sum_1^\infty e^{-nx}$. We can use $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to rewrite (1):

$$\zeta(s) = \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{2}$$

Now we deal with the contour integral, letting $C(\epsilon)$ denote the curve described in the problem for fixed $\epsilon \in \mathbb{R}_{>0}$. The expression $e^z - 1$ has no poles away from $\{z \mid e^z = 1\} = \{2\pi k i \mid k \in \mathbb{Z}\}$, whence if we take the branch cut of log away from the non-negative reals the integral is not dependent on ϵ for $\epsilon < 2\pi$, since (for $\delta < \epsilon < 2\pi$) we have the area enclosed between the two curves is a quotient of holomorphic functions with the denominator non-vanishing in between the curves. This independence is a result of taking seperating the curve $C(\epsilon) - C(\delta)$ into two curves (see picture), the first of which has integral zero by Cauchy's theorem, and the second being arbitrarily small depending on where we take the cut.

$$\int_{C(\epsilon)} f - \int_{C(\delta)} f = \int_{C(\epsilon,\delta,M)} f + \int_{\gamma_1(\epsilon,\delta,M)} f + \int_{\gamma_2(\epsilon,\delta,M)} f$$



Then for f holomorphic away from he real line, the $C(\epsilon,\delta,M)$ term vanishes. Notice now that using the standard arclength inequality for large M we have

$$\begin{aligned}\left|\int_{\gamma_j(\epsilon,\delta,M)}\frac{(-z)^{s-1}}{e^z-1}dz\right| &\leq (\epsilon-\delta)\left|\frac{M^{\Re(s)-1}}{e^M-1}\right|+\left|\int_M^\infty\frac{(-x-i\epsilon)^{s-1}}{e^{x+i\epsilon}-1}dx-\int_M^\infty\frac{(-x-i\delta)^{s-1}}{e^{x+i\delta}-1}dx\right|\\ &\leq (\epsilon-\delta)\left|\frac{M^{\Re(s)-1}}{e^M-1}\right|+2\int_M^\infty\frac{|x+i\epsilon|^{\Re(s)-1}}{|e^x|-1}dx\end{aligned}$$

The right hand side clearly converges to zero. as $M\to\infty$ using basic limits of exponentials and DCT. This gives the desired invariance.

$$\left|\int_{C(\epsilon)}\frac{(-z)^{s-1}}{e^z-1}dz-\int_{C(\delta)}\frac{(-z)^{s-1}}{e^z-1}dz\right|\leq\left|\int_{\gamma_1(\epsilon,\delta,M)}\frac{(-z)^{s-1}}{e^z-1}dz\right|+\left|\int_{\gamma_2(\epsilon,\delta,M)}\frac{(-z)^{s-1}}{e^z-1}dz\right|=0$$

Now, we can compute the value of the integral along this curve by letting $\epsilon\to0$, to get $C(0)$, a ray from ∞ to 0 where $(-z)^{s-1}=x^{s-1}e^{-(s-1)\pi i}$, and a ray from 0 to ∞ where $(-z)^{s-1}=x^{s-1}e^{(s-1)\pi i}$, to see that we can indeed pass to this limit, once again decompose $C=C(\epsilon)$ into three curves, with C_1 the ray in the upper half plane, C_2 the ray in the lower half plane and C_3 the circular portion, then once again using the arc length inequality and the fact that $e^z-1=\mathcal{O}(z)$

$$\left|\int_{C_3}\frac{(-z)^{s-1}}{e^z-1}\right|\leq 2\pi\epsilon\sup_{|z|=\epsilon}\frac{|z^{s-1}|}{|e^z-1|}=2\pi\epsilon\mathcal{O}(\epsilon^{\Re(s)-2})=2\pi\mathcal{O}(\epsilon^{\Re(s)-1})\overset{\Re(s)>1}{\longrightarrow}0$$

We can check convergence on C_1 explicitly

$$\begin{aligned}&\left|\int_0^\infty\frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx-\int_\epsilon^\infty\frac{(-x-i\epsilon)^{s-1}}{e^{x+i\epsilon}-1}dx\right|\\ \leq&\left|\int_0^\epsilon\frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx\right|+\left|\int_\epsilon^\infty\frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx-\int_\epsilon^\infty\frac{(-x-i\epsilon)^{s-1}}{e^{x+i\epsilon}-1}dx\right|\\ \leq&\left|\int_0^\epsilon\frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx\right|+\left|\int_\epsilon^\infty\frac{(e^{x+i\epsilon}-1)x^{s-1}e^{-(s-1)\pi i}-(e^x-1)(\sqrt{x^2+\epsilon^2})^{(s-1)}e^{i(s-1)\arctan\frac{\epsilon}{x}-(s-1)\pi i}}{(e^x-1)(e^{x+i\epsilon}-1)}\right|\end{aligned}$$

Convergence as $\epsilon\to0$ of the big ugly term to zero is actually simple from convergence of each of the terms in the two expressions in the product. Convergence of the first term follows from $\Re(s)>1$, so writing the bounds of integration as $\chi_{(0,\epsilon)}$ we can just apply DCT to the absolute value of the integrand. The proof of convergence for C_2 is similar to C_1.

Now we finally established that (due to taking the limit in ϵ and invariance with respect to ϵ)

$$\int_C\frac{(-z)^{s-1}}{e^z-1}dz=-\int_0^\infty\frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx+\int_0^\infty\frac{x^{s-1}e^{(s-1)\pi i}}{e^x-1}dx \tag{3}$$

$$=\int_0^\infty\frac{x^{s-1}}{e^x-1}2i\sin((s-1)\pi)=-\int_0^\infty\frac{x^{s-1}}{e^x-1}2i\sin(s\pi) \tag{4}$$

Multiplying (2) by $1=\frac{\int_C\frac{(-z)^{s-1}}{e^z-1}dz}{-\int_0^\infty\frac{x^{s-1}}{e^x-1}2i\sin(s\pi)}$ yields the desired equality

$$\zeta(s)=-\frac{\Gamma(1-s)}{2\pi i}\int_C\frac{(-z)^{s-1}}{e^z-1}dz \tag{5}$$

Now, using the right side of (5) as an analytic continuation, we first compute $(e^z-1)^{-1}=\frac{1}{z}-\frac{1}{2}+\frac{z}{12}+\mathcal{O}(z^2)$, since we know it has a simple pole at zero, hence is meromorphic this expression just comes from evaluating the systems of equations given for the coefficients for $(a_{-1}z^{-1}+a_0+a_1z+\cdots)(\sum_0^\infty\frac{z^n}{n!})=1$.

This is everything we need to evaluate $\zeta(0)$. First using a similar separation of curves into two parts as was used for invariance, we find that $\int_C \frac{1}{z(e^z-1)}dz = \int_\gamma \frac{1}{z(e^z-1)}dz$ where γ is a piecewise C^1 closed curve, this allows us to use the residue theorem

$$\zeta(0) = \frac{\Gamma(1)}{2\pi i}\int_\gamma \frac{1}{z(e^z-1)}dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)\right)dz \tag{6}$$

$$= \text{Res}\left(\frac{1}{z}\left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)\right)\right) = -\frac{1}{2} \tag{7}$$

□

2. **(a)** Let ϵ be small enough so that $B_{2\epsilon}(z_0) \subset D$, then

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw - \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w-z_0}dw \overset{\text{Stokes}}{=} \int_{D \setminus B_\epsilon(z_0)} \frac{\partial}{\partial \overline{w}}\frac{f(w)}{w-z_0}d\overline{w} \wedge dw = 0$$

By Cauchy Riemann since $\frac{f(w)}{w-z_0}$ is holomorphic in $D \setminus B_\epsilon(z_0)$. From this, we cna prove the base case which is Cauchy's integral formula.

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw = \int_{B_\epsilon(z_0)} \frac{f(w)}{w-z_0}dw = \int_0^{2\pi} f(z_0+\epsilon e^{it})idt$$

Which holds for arbitrary ϵ, but since f is bounded (on D) and continuous we have

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw = \lim_{\epsilon\to 0}\int_0^{2\pi} f(z_0+\epsilon e^{it})idt = \int_0^{2\pi} \lim_{\epsilon\to 0} f(z_0+\epsilon e^{it})idt = 2\pi i f(z_0)$$

Which proves the desired result in the case of $n = 0$

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z_0}dw \tag{8}$$

Now, we can proceed by induction and simply take derivatives.

$$\begin{aligned}
2\pi i \lim_{h\to 0}\frac{1}{h}(f^{(n)}(z_0+h) - f^{(n)}(z_0)) &= \lim_{h\to 0}\frac{1}{h}\int_{\partial D} \frac{f(w)}{(w-z_0-h)^{n+1}} - \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \lim_{h\to 0}\int_{\partial D} \frac{1}{h}\frac{f(w)(w-z_0)^{n+1} - f(w)(w-z_0-h)^{n+1}}{(w-z_0-h)^{n+1}(w-z_0)^{n+1}}dw \\
&= \lim_{h\to 0}\int_{\partial D} \frac{f(w)(w-z_0)^n}{(w-z_0-h)^{n+1}(w-z_0)^{n+1}} + \mathcal{O}(h)dw
\end{aligned}$$

Then we can clearly apply DCT for $|h| < \frac{1}{2}|w-z_0|$, which gives us

$$\lim_{h\to 0}\frac{1}{h}(f^{(n)}(z_0+h) - f^{(n)}(z_0)) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z_0)^{n+2}}dw \tag{9}$$

(b) Let $z \in \mathbb{C}$, then by equation (9) in part (a)

$$f'(z) = \frac{1}{2\pi i}\int_{|w-z|=R} \frac{f(w)}{(w-z)^2}dw$$

Applying the standard arc-length inequality yields

$$|f'(z)| \leq \frac{1}{2\pi}2\pi R\frac{\sup_{\mathbb{C}}|f|}{R^2} = \frac{\sup_{\mathbb{C}}|f|}{R} \overset{R\to\infty}{\longrightarrow} 0 \tag{10}$$

Therefore $f' \equiv 0$, for any two points in $\mathbb{C}$, to see this implies f is constant we use the fundamental theorem of calculus, if $z_0, z_1 \in \mathbb{C}$ take γ to be the straight line starting at z_0 and ending at z_1 so that

$$|f(z_1) - f(z_0)| = \left|\int_\gamma f'(z)dz\right| \leq \ell(\gamma) \sup_{\mathbb{C}} |f'| = 0$$

$\square$

(c) Suppose a polynomial P does not have a root. Then $\frac{1}{P}$ is entire, hence $1/P$ is constant by Liouville's theorem. We conclude any polynomial without a root is of degree zero. $\square$

3. Assume for contradiction that f has an essential singularity at z_0, but there exists some $a \in \mathbb{C}$ and $\delta > 0$ such that for some $\epsilon > 0$ we have $f(D_\epsilon^*)(z_0) \cap B_\delta(a) = \emptyset$. Now define a new function $g(z) = \frac{1}{f(z)-a}$, since $|f(z) - a| \geq \delta$ on $D_\epsilon^*(z_0)$ we find that g is holomorphic and nonvanishing on this punctured annulus with modulus bound above by $\frac{1}{\delta}$. Then we can define on $D_\epsilon(z_0)$

$$(z - z_0)^2 g(z) = \begin{cases} (z - z_0)^2 g(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases} \tag{11}$$

This is holomorphic away from z_0, thus is holomorphic since

$$\lim_{h\to 0} \frac{(z_0 + h - z_0)^2 g(z_0 + h)}{h} = \lim_{h\to 0} hg(z_0 + h) = 0$$

where the last equality is a consequence of g being bounded. Therefore we can write

$$g(z)(z - z_0)^2 = \sum_0^\infty a_k(z - z_0)^k$$

which gives the following expression on D_ϵ^* for $b_k = a_{k+2}$

$$g(z) = \sum_{-2}^\infty b_k(z - z_0)^k = \sum_0^\infty b_k(z - z_0)^k \tag{12}$$

Where $b_{-2} = b_{-1} = 0$ follows from g being bounded near z_0. Moreover since g is nonvanishing, we must have some $b_N \neq 0$, it follows that

$$\lim_{z\to z_0} (z - z_0)^{N+1}(f(z) - a) = \lim_{z\to z_0} \frac{(z - z_0)^{N+1}}{g(z)} \overset{(12)}{=} \lim_{z\to z_0} \frac{z - z_0}{b_N + \mathcal{O}(z - z_0)} = 0 \tag{13}$$

Now we can show directly from the definition that f has a pole

$$\lim_{z\to z_0} (z - z_0)^{N+1} f(z) = \lim_{z\to z_0} (z - z_0)^{N+1}(f(z) - a) \overset{(13)}{=} 0$$

Which is the desired contradition. $\square$

4. Denote the annulus in question as A, the outer disc as D_{r_2} and the inner disc as D_{r_1}, then for $z_0 \in A^\circ$, we can let γ be a ray from D_{r_1} to D_{r_2} not intersecting z_0. It follows that $\partial D_{r_2} - \gamma + \gamma - \partial D_{r_1}$ is a closed piecewise C^1 curve in A around z_0, so that by Cauchy's integral formula (8):

$$f(z_0) = \int_{\partial D_{r_2} - \gamma + \gamma - \partial D_{r_1}} \frac{f(w)}{w - z_0} dw = \int_{\partial D_{r_2}} \frac{f(w)}{w - z_0} dw - \int_{\partial D_{r_1}} \frac{f(w)}{w - z_0} dw \tag{14}$$

Now we can apply algebraic manipulations, since on ∂D_{r_2} we have $|w| > |z_0|$ we get the following uniformly convergent power series expansion

$$\frac{f(w)}{w - z_0} = \frac{f(w)}{w}\left(\frac{1}{1 - \frac{z_0}{w}}\right) = \frac{f(w)}{w}\sum_0^\infty \left(\frac{z_0}{w}\right)^k \tag{15}$$

Similarly, on ∂D_{r_1} $|z_0| > |w|$, so that there is a uniformly convergent power series expansion given by

$$-\frac{f(w)}{w-z_0} = \frac{f(w)}{z_0}\left(\frac{1}{1-\frac{w}{z_0}}\right) = \frac{f(w)}{z_0}\sum_0^\infty \left(\frac{w}{z_0}\right)^k \tag{16}$$

Combining (14), (15) and (16) along with uniform convergence in (15) and (16) furnishes

$$2\pi i f(z_0) = \int_{\partial D_2} \frac{f(w)}{w}\sum_0^\infty \left(\frac{z_0}{w}\right)^k dw + \int_{\partial D_1} \frac{f(w)}{z_0}\sum_0^\infty \left(\frac{w}{z_0}\right)^k dw \tag{17}$$

$$= \sum_0^\infty z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw + \sum_0^\infty \frac{1}{z_0^{k+1}} \int_{\partial D_{r_2}} f(w) w^k dw \tag{18}$$

$$= \sum_0^\infty z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw + \sum_{-\infty}^1 z_0^k \int_{\partial D_{r_1}} \frac{f(w)}{w^{k+1}} dw \tag{19}$$

Now letting $r \in (r_1, r_2)$, we can use the same trick with a ray γ used in (14) to apply Cauchy's theorem and find that

$$\sum_0^\infty z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw - \sum_0^\infty z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw = 0 \tag{20}$$

$$\text{and } \sum_{-\infty}^1 z_0^k \int_{\partial D_{r_1}} \frac{f(w)}{w^{k+1}} dw - \sum_{-\infty}^1 z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw = 0 \tag{21}$$

Where application of Cauchy's theorem follows by $\sum_0^\infty z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw$ analytic on $B_{r_2}(0)$ and similarly $\sum_{-\infty}^1 z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw$ analytic on $\overline{B_{r_1}(0)}^c$. Now we can substitute into (19) using (20) and (21 to obtain

$$f(z_0) = \sum_{-\infty}^\infty z_0^k \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw \tag{22}$$

□

5. From the chain rule we have $f \circ (w \mapsto 1/w)$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Suppose for contradiction that $f \circ (w \mapsto 1/w)$ has an essential singularity at zero, then by Casorati-Weierstrass $f(\overline{B_\epsilon(0)}^c) = f \circ (w \mapsto \frac{1}{w})(B_{\frac{1}{\epsilon}}(0) \setminus \{0\})$ is dense in $\mathbb{C}$, this gives us a contradiction since $f(B_\epsilon(0))$ is open by the open mapping principle, hence $f(B_\epsilon(0)) \cap f(\overline{B_\epsilon(0)}^c) \neq \emptyset$ which contradicts injectivity of f. It follows that $f \circ (w \mapsto \frac{1}{w})$ has a pole at 0, and hence a Laurent series. Now write

$$f \circ (w \mapsto \frac{1}{w}) = \sum_{-N}^\infty a_k z^k \implies f(w) = \sum_{-N}^\infty a_k z^{-k}$$

f being entire implies that $a_j = 0$ for $j \geq 1$, hence $f = \sum_0^N a_{-k} z^k$, since $f'(z) = \sum_1^n a_{-k} z^{k-1}$ is nonvanishing, by the fundamental theorem of algebra f' must be a constant nonzero polynomial, hence $a_{-k} = 0$ for $k > 1$ and $f = a_{-1} z + a_0$ as desired. □

6. (i) Let $[z_0 : \cdots : z_n] \in \mathbb{P}^n$, then $(z_0, \ldots, z_n) \neq 0$, hence atleast one coordinate is nonzero, label that coordinate j to see that $[z_0 : \cdots : z_n] \in U_j$ by definition. To check φ is a homeomorphism we can check it is a continuous and open bijection. Injectivity follows from the map φ_j choosing the unique equivalence class representative with j-th entry equal to one. To see surjectivity let $z = (z_0, \ldots, 1, \ldots, z_n) \in \mathbb{C}^n$, then $z = \varphi_j([z_0 : \cdots : 1 : \cdots : z_n])$. By definition of the quotient topology φ_j is continuous if and only if $\varphi_j \circ \pi$ is, same for open. To check this simply write $\varphi_j \circ \pi(z_0, \ldots, z_n) = (z_0/z_j, \ldots, z_n/z_j)$, since we are on $\{z_j \neq 0\}$ this map is smooth, and its Jacobian has a copy of $\frac{1}{z_j}1_{n\times n}$, thus this map is a submersion, since submersions are open maps this map is smooth, open and a bijection hence a homeomorphism.

(ii) On $\varphi_j(U_i \cap U_j)$ we have

$$\varphi_i\varphi_j^{-1}(z_0, \dots, 1, \dots, z_n) = \left(\frac{z_0}{z_i}, \dots, \frac{1}{z_i}, \dots, \frac{z_n}{z_i}\right) \tag{23}$$

So each of the coordinate functions is of the form 1 (which is not actually considered in the image by identification with $\mathbb{C}^n$), $\frac{1}{z_i}$ or z_k/z_i for $k \neq i$. From here it is immediate, since $\frac{1}{z_i}$ is holomorphic on $\{z_i \neq 0\} \supset \varphi_i(U_i \cap U_j)$ we get for $\ell \neq k, i$ that

$$\frac{\partial}{\partial \overline{z_\ell}}\frac{1}{z_i} = \frac{1}{z_i}\frac{\partial}{\partial \overline{z_\ell}}1 = 0 = z_k/z_i\frac{\partial}{\partial \overline{z_\ell}}1 = \frac{\partial}{\partial \overline{z_\ell}}z_k/z_i$$

and by holomorphicity or $\frac{1}{z_i}$, as well as z_k

$$\frac{\partial}{\partial \overline{z_k}}\frac{z_k}{z_i} = 1/z_i\frac{\partial}{\partial \overline{z_k}}z_k = 0 \text{ and } \frac{\partial}{\partial \overline{z_i}}\frac{1}{z_i} = 0 = z_k\frac{\partial}{\partial \overline{z_i}}\frac{1}{z_i} = \frac{\partial}{\partial \overline{z_i}}\frac{z_k}{z_i}$$

□

7. I will explain it for π_N, π_S is clearly the same by symmetry (however we flip the convention $(x, y, 0) \leftrightsquigarrow x - iy$ to ensure holomorphicity). Stereographic projection geometrically is embedding $S^2 \hookrightarrow \mathbb{R}^3$ (i.e. defining S^2 as the equation in question), where we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the obvious way $(x, y, 0) \leftrightsquigarrow x+iy$, then we associate to each point (x_1, x_2, x_3) in $S^2 \setminus N$ the unique point where the $\mathbb{C} \leftrightsquigarrow \mathbb{R}^2 \times \{0\}$ such that the line through N and (x_1, x_2, x_3) intersects $\mathbb{C}$. The equation for π_N is simply parameterizing in terms of (x_1, x_2, x_3) the point where the line meets $\mathbb{C}$. Concretely we have the following parameterization

$$\ell(t) = t(x_1, x_2, x_3 - 1) + (0, 0, 1) \tag{24}$$

We get the point of projection by solving for t_0 such that $\ell(t_0) \in \mathbb{R}^2 \times \{0\}$, this is $t_0 = \frac{1}{1-x_3}$, now plugging this into (24) gives

$$\ell(t_0) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0\right) \leftrightsquigarrow \frac{x_1 + ix_2}{1 - x_3} \overset{\text{def}}{=} \pi_N(x_1, x_2, x_3) \tag{25}$$

From this geometric realization we can also compute π_S^{-1} as being the unique point on $S^2 \setminus S$ which intersects the line through $(x, -y, 0) \leftrightsquigarrow x + iy$, this gives the parameterized line

$$\ell'(t) = t(x, -y, 1) + (0, 0, -1) \tag{26}$$

so solving for t_0 where $\ell'(t_0) \in S^2$ we get

$$t_0^2(x^2 + y^2 + 1) - 2t_0 = 0 \tag{27}$$

Where the zeroes of (27) correspond to S and

$$\pi_S^{-1}(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{-2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1}\right) \tag{28}$$

Now we can apply π_N to get

$$\pi_N\pi_S^{-1}(x + iy) = \frac{\frac{2(x-iy)}{x^2+y^2+1}}{1 - \frac{1-x^2-y^2}{1+x^2+y^2}} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z} \tag{29}$$

Which is indeed holomorphic on $\mathbb{C} \setminus \{0\}$. □

8. Note that the reason we only need to check on the charts $\mathbb{P}^1 \setminus \{[0:1]\}$, and $S^2 \setminus N$ are that away from S we have that f can be written in charts as $\phi_{\tilde{S}} \circ f \circ \pi_S^{-1} = 1_{\mathbb{C}^\times}$. To begin with we compute $\phi_{\tilde{N}} \circ f \circ \pi_N^{-1}$ we write $z = x + iy$

$$\phi_{\tilde{N}} \circ f \circ \pi_N^{-1}(x+iy) = \phi_{\tilde{N}} \circ \phi_{\tilde{S}}^{-1} \circ \pi_S \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{-2}{1+x^2+y^2} + 1 \right) \tag{30}$$

$$= \phi_{\tilde{N}} \circ \phi_{\tilde{S}}^{-1}(1/z) = \phi_{\tilde{N}}[1/z : 1] = z \tag{31}$$

This of course extends to the identity map from $\mathbb{C} \to \mathbb{C}$, so if we take $f(S) = [1:0]$, the extended map on $S^2 \setminus N \to \mathbb{P}^1 \setminus \{[1:0]\}$ charts is just the identity map $z \mapsto z$ which is holomorphic, so that every point in S^2 is contained in an open set with (the extension of) f holomorphic in coordinates on that open set. Moreover, by the holomorphic inverse function theorem (the extension of) f is biholomorphic, since the derivative in either pair of charts is just given by 1 which is non-vanishing. $\square$