1. We start by computing the Jacobian of the map $F : x \mapsto \frac{x}{||x||}$

$$\frac{\partial}{\partial x_j}\frac{x_i}{\sqrt{\sum_1^{n+1} x_k^2}} = \frac{\delta_{ij}||x|| - x_i x_j ||x||^{-1}}{||x||^2} = \frac{\delta_{ij}}{||x||} - \frac{x_i x_j}{||x||^3}$$

So the Jacobian looks like

$$\frac{1}{||x||}1 - \frac{1}{||x||^3}xx^T$$

Now beginning the actual proof, let $\{f_1, \dots, f_{n-m}\}$ be a basis for $(\operatorname{Im} A)^\perp$, then define $T : \mathbb{R}^{n-m+1} \to \mathbb{R}^{n+1}$ via $e_i \mapsto f_i$ when $1 \le i \le n-m$ and $e_{n-m+1} \mapsto Ae_1$, then $\overline{T} : \mathbb{RP}^{n-m} \to \mathbb{RP}^n$ is an embedding, so we can refer to its image as the submanifold $X \subset \mathbb{RP}^n$. Now we get that $\operatorname{Im}(\overline{A}) \cap X = q(\operatorname{Im} A \cap \operatorname{Im} T)$, where $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is the quotient. By construction this intersection is $[Ae_1]$, so that once we verify $\overline{A} \pitchfork_{[Ae_1]} X$ we will get that $I_2(\overline{A}, X) = 1$. Now checking transversality, we will use the following maps, where the π maps are the indiced quotients under the action by the discrete group. Note that in particular the π maps are submersions of manifolds of equal dimension and hence local diffeomorphisms by the inverse function theorem.

$$\begin{aligned} \widehat{A} &: S^m \to S^n & \widehat{T} &: S^{m-n} \to S^n \\ v &\mapsto \frac{Av}{||Av||} & v &\mapsto \frac{Tv}{||Tv||} \\ \pi_m &: S^m \to \mathbb{RP}^m & \pi_{n-m} &: S^{n-m} \to \mathbb{RP}^{n-m} \\ \pi &: S^n \to \mathbb{RP}^n \end{aligned}$$

Moreover, the following diagrams commute by definition of $\widehat{A}, \widehat{T}$

$$\begin{tikzcd} S^m \arrow[r, "\widehat{A}"] \arrow[d, "\pi_m"] & S^n \arrow[d, "\pi"] \\ \mathbb{RP}^m \arrow[r, "\overline{A}"] & \mathbb{RP}^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-m} \arrow[r, "\widehat{T}"] \arrow[d, "\pi_{n-m}"] & S^n \arrow[d, "\pi"] \\ \mathbb{RP}^{n-m} \arrow[r, "\overline{T}"] & \mathbb{RP}^n \end{tikzcd}$$

We will verify later that $\operatorname{Im} d_{e_1}\widehat{A} + \operatorname{Im} d_{e_{m-n+1}}\widehat{T} = T_{A(e_1)}S^n$, but assuming it for now we find that (using repeatedly the submersion properties of the projections)

$$\begin{aligned} T_{\overline{A}e_1}\mathbb{RP}^n &= d_{\widehat{A}e_1}\pi(\operatorname{Im} d_{e_1}\widehat{A} + \operatorname{Im} d_{e_{m-n+1}}\widehat{T}) \\ &= \operatorname{Im} d_{e_1}(\pi \circ \widehat{A}) + \operatorname{Im} d_{e_{n-m+1}}(\pi \circ \widehat{T}) \\ &= \operatorname{Im} d_{e_1}(\overline{A} \circ \pi_m) + \operatorname{Im} d_{e_{n-m+1}}(\overline{T} \circ \pi_{n-m}) \\ &= \operatorname{Im}(d_{[e_1]}\overline{A}) + \operatorname{Im}(d_{[e_{n-m+1}]}\overline{T}) \\ &= \operatorname{Im}(d_{[e_1]}\overline{A}) + T_{[Ae_1]}X \end{aligned}$$

This verifies that indeed $\overline{A} \pitchfork_{[Ae_1]} X$, now to complete the proof, note that we have some $[p] \in \mathbb{RP}^n \setminus X$, since $\mathbb{RP}^n$ is connected (therefore path connected), any constant map $\mathbb{RP}^m \to \mathbb{RP}^n$ is homotopic to the map $c : \mathbb{RP}^m \to [p]$, where $I_2(c, X) = 0$ trivially, since intersection number is a homotopy invarient this completes the proof.

(Proof of $\mathbf{d_{e_1}\widehat{A} + d_{e_{m-n+1}}\widehat{T} = T_{A(e_1)}S^n}$): To show this, we will compute the derivatives as maps of $\mathbb{R}^k \setminus \{0\} \to \mathbb{R}^{n+1}$, then use the characterization of the tangent space $T_pS^k = p^\perp \cap T_p\mathbb{R}^{k+1}$. To compute the derivative note that the maps are of the form $\widehat{A} = F \circ A$ and $\widehat{T} = F \circ T$, where we computed the derivative of F prior to tackling the problem, by the chain rule we have

$$\begin{aligned} d_{e_1}\widehat{A} &= \left(\frac{1}{||Ae_1||}1 - \frac{1}{||Ae_1||^3}(Ae_1)\cdot(Ae_1)^T\right)d_{e_1}A \\ d_{e_{n-m+1}}\widehat{T} &= \left(\frac{1}{||Ae_1||}1 - \frac{1}{||Ae_1||^3}(Ae_1)\cdot(Ae_1)^T\right)d_{e_{n+m-1}}T \end{aligned}$$

restricting to the orthogonal compliment of Ae_1, $\frac{1}{||Ae_1||^3}(Ae_1)\cdot(Ae_1)^T \equiv 0$, so that

$$d_{e_1}\widehat{A} \equiv \frac{1}{||Ae_1||}d_{e_1}A \text{ and } d_{e_{n-m+1}}\widehat{T} \equiv \frac{1}{||Ae_1||}d_{e_{n-m+1}}T$$

the derivative should also have restricted domain since these are maps of spheres, restricting the domain of $d_{e_1}\widehat{A}$ to $e_1^\perp$ and $d_{e_{n-m+1}}\widehat{T}$ to $e_{n-m+1}^\perp$ and taking $\rho : \mathbb{R}^{n+1} \to (Ae_1)^\perp$ to be the orthogonal projection we find the images of either differential have respective bases

$$\{\rho(Ae_2), \dots, \rho(Ae_{m+1})\} \text{ and } \{\rho(Te_1), \dots, \rho(Te_{n-m})\}$$

By definition of T, and injectivity of both A and T (which have Ae_1 in their image), this collection of n vectors forms a basis for $T_{Ae_1}S^n$, this is easiest to see by writing it as

$$\rho(\langle Ae_2, \dots, Ae_{m+1}, f_1, \dots, f_{n-m}\rangle)$$

where ρ has no kernel on this subspace, and this space has dimension n by definition of the f_i and injectivity of A. $\square$

2. **(a)** From the Tubular neighborhood theorem we have a diffeomorphism $\phi : NM \to W$ where W is an open neighborhood of M in N, then we take take $s : M \to NM$, where $s(x) = (x, \tilde{s}(x))$, We define the isotopy as follows:

$$e(t, x) = \phi(x, t\tilde{s}(x))$$

Clearly e is smooth, disjointness of $e_1(M)$ from $M = e_0(M)$ follows from injectivity of ϕ, and their disjointness in the normal bundle. Now we only need check that e is an isotopy, but it is clear that for each t, we have $x \mapsto (x, t\tilde{s}(x))$ is an embedding, since it is a smooth section of the normal bundle (so has smooth inverse $\pi_N|_{e_t(M)}$, i.e. is diffeomorphic to its image), then post composing with the diffeomorphism ϕ giving e_t of course still gives an embedding.

(b) From the classification of 1 dimensional manifolds, M must be S^1 Here I will identify points on S^1 with $[-\frac{1}{2}, \frac{1}{2})$ in the obvious way. Then since M has codimension 1, NM is a line bundle for M, so by the classification of line bundles on S^1 must either be the trivial bundle or the Möbius bundle (note that non-vanishing smooth sections are preserved by bundle isomorphisms, so we can work concretely in these cases), in the case of the trivial bundle we are done by taking the $M \times \{1\}$ section. Now I will show the case of the Möbius bundle cannot happen. Take the section family of sections indexed by $\epsilon \in (0, 1]$, $s_\epsilon(x) = (x, \epsilon\sin(\pi x))$, which is a smooth section (and hence an embedding since it has smooth inverse on its image given by the bundle projection), moreover we have $M \cap s_\epsilon(M) = (0, 0)$, to see that $s_\epsilon \pitchfork M$, simply notice that taking $\gamma(t) = t$ has $\frac{d}{dt}|_{t=0}s_\epsilon \circ \gamma = \begin{pmatrix} 1 \\ \epsilon\pi \end{pmatrix}$, and since

$$\begin{pmatrix} 1 \\ \epsilon\pi \end{pmatrix} \notin T_{(0,0)}M \times \{0\} = \operatorname{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The intersection is transverse and thus $I_2(s, M) = 1$. To get a contradiction from this, let g be a riemmannian metric on NM and $\phi : NM \xrightarrow{\cong} W \supset M$ be the diffeomorphism given by the tubular neighborhood theorem. Then since M and $e_1(M)$ are compact and disjoint we have $d(M, e_1(M)) = \delta > 0$, Now take $\phi^{-1}(B_{\delta/2}(M) \cap W)$ where $B_{\delta/2}(M) := \bigcup_{x\in M} B_{\delta/2}(x)$, this is an open neighborhood of M in the normal bundle, so for each $x \in M$, there is some $U_x \supset \{x\}$, and some $\epsilon_x > 0$ with $U_x \times (-\epsilon_x, \epsilon_x) \subset \phi^{-1}(B_{\delta/2}(M) \cap W)$, by compactness there is a finite subcover, and hence some $\epsilon > 0$ so that $B_\epsilon(M) \subset NM$ (where we define $B_\epsilon(M) := \bigcup_{x\in M}\{x\} \times (-\epsilon, \epsilon)$) has the property that $\phi(B_\epsilon(M)) \subset B_{\delta/2}(M) \subset N$, and therefore $I_2(s_{\epsilon/\pi}, e_1(M)) = 0$, since they are disjoint. This of course contradicts intersection number being isotopy invarient. $\square$

3. Suppose $n > 2(r+1)$, and $e : S^r \hookrightarrow \mathbb{R}^n$, I will show that we can extend this to an embedding into some larger Euclidean space, then copy the proof of strong Whitney embedding theorem to show if $N > n$, then any embedding of D^{r+1} into $\mathbb{R}^N$ which restricts to $e : S^r \hookrightarrow \mathbb{R}^n \times \{0\}$ gives rise to an embedding of the disk into $\mathbb{R}^{N-1}$ with the same restriction property. Firstly we let V be a collar for D^{r+1}, i.e. $S^r \subset V$ and $\varphi : V \xrightarrow{\cong} S^r \times [0, 1)$ with $\varphi|_{S^r} = S^r \times \{0\}$, now we can let η, η' be a partition of unity for $[0, 1)$, with η suboordinate to $[0, \frac{1}{2})$, and η' suboordinate to $(\frac{1}{3}, 1)$, this allows us to define η_1, η_2 on D^{r+1}. Then for $x \in V$, we have $\varphi(x) = (s, t)$, so that

$$\eta_1(x) = \begin{cases} \eta(t) & x \in V \\ 0 & x \in V^c \end{cases} \qquad \eta_2(x) = \begin{cases} 1 - \eta(t) & x \in V \\ 1 & x \in V^c \end{cases}$$

So that η_1, η_2 are smooth, $\eta_1 + \eta_2 = 1$ and $\eta_1|_{S^r} = 1$. Now let $F : D^{r+1} \hookrightarrow \mathbb{R}^d$ for some d be given by the Whitney embedding theorem, this allows us to define the following embedding (where once again we can identify $x \in V$ as $(s, t) \in S^r \times [0, 1)$, this is well defined by the support of η_1)

$$\begin{aligned} E : D^{r+1} &\to \mathbb{R}^{n+d+3} \\ x &\mapsto (e(s)\eta_1(x), F(x)\eta_2(x), \eta_2(x), (1-t)\eta_1(x), 1 - \eta_1(x)) \end{aligned}$$

To see this is an embedding, note that on the support of η_1 it is an immersion due to the first n coordinates and second last coordinate, and it is injective on the support of η_1 since if $x \neq y$, but $1 - \eta_1(x) = 1 - \eta_1(y)$, then $e(s_x)\eta_1(x) \neq e(s_y)\eta_1(y)$ by injectivity of e. It is an immersion on the support of η_2 since F is, and is injective since if $x \neq y$ and $\eta_2(x) = \eta_2(y)$, then $F(x)\eta_2(x) \neq F(y)\eta_2(y)$ by injectivity of F, this suffices to show its injective since clearly points in the support of η_1 and η_2 have distinct images, to see that the map is proper simply note that D^{r+1} is compact, hence E is an embedding. Moreover by construction we can see that letting $\iota : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^{d+3}$ be the inclusion into $\mathbb{R}^n \times \{0\}$ we have $E|_{S^n} = \iota \circ e$.

Now we want to turn this into a map into $\mathbb{R}^n$ respecting the original embedding. So suppose that $\ell > 0$, we have $E : D^{r+1} \to \mathbb{R}^{n+\ell}$ be an embedding so that $E|_{S^n} = \iota \circ e$ with $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+\ell}$ the inclusion onto the first n coordinates, I will show this produces a map $E' : D^{r+1} \hookrightarrow \mathbb{R}^{n+\ell-1}$, with the restriction to the sphere the same as including the original embedding e onto the first n coordinates, thus completing the proof by induction. For convenience we will abuse notation a bit and still use ι to denote the inclusion $\mathbb{R}^{n+\ell-1} \to \mathbb{R}^{n+\ell}$ the main labour involved in this proof is to show that for some $x \in S^n$, denoting π_x to be the projection onto $x^\perp$ the following properties hold

1. $\pi_x \circ E$ is injective
2. $\pi_x \circ E$ is an immersion
3. $\pi_x \circ \iota$ is a diffeomorphism $\mathbb{R}^{n+\ell-1} \to \mathbb{R}^{n+\ell-1}$

For now we assume such an x exists, then $\pi_x \circ E$ is an embedding $D^{r+1} \to \mathbb{R}^{n+\ell-1}$ (properness follows from the $r + 1$ disc being compact). This implies that

$$(\pi_x \circ \iota)^{-1} \circ (\pi_x \circ E) : D^{r+1} \to \mathbb{R}^{n+\ell-1}$$

is also an embedding, in fact it is the desired embedding since for $s \in S^r$ we have

$$\pi_x \circ E(s) = \pi_x(e(s), 0) = \pi_x \circ \iota \circ e(s)$$

so that $(\pi_x \circ \iota)^{-1}\pi_x \circ E(s) = e(s) \times \{0\}^{\ell-1}$.

Now we need to verify properties (1-3), it will be easiest to show using Sard's theorem that values of x failing each of these individual properties have measure zero, so that some x satisfying all three must exist. Letting $x \in S^n$ note that $\pi_x \circ \iota$ is a compositon of linear maps, hence linear, by taking the dimensions into account, it suffices to show that it has zero kernel to be an isomorphism (then since the inverse is linear it will be a diffeomorphism), having zero kernel is equivalent to x not being in $\iota(\mathbb{R}^{n+\ell-1})$, but $\iota(\mathbb{R}^{n+\ell-1})$ is measure zero in $\mathbb{R}^{n+\ell}$ by Sard's theorem, so it remains to check properties one and two are fulfilled on sets with compliment measure zero. Property 2 follows directly from the proof of Whitney embedding theorem, taking

$$\begin{aligned} f^{\text{tang}} :& TD^{r+1} \setminus D^{r+1} \times \{0\} \to S^{n+\ell-1} \\ v \mapsto & \frac{dE(v)}{||dE(v)||} \end{aligned}$$

Now if $x \notin \operatorname{Im} f^{\text{tang}}$, then $d(\pi_x \circ E)$ is injective, hence $\pi_x \circ E$ is an immersion, since $\dim TD^{r+1} = 2r+2 < n \leq n+\ell-1$, a point x is a regular value for f^{tang} exactly when it is disjoint from the image, so $\operatorname{Im} f^{\text{tang}}$ has measure zero by Sard's theorem, as desired. Finally, we show condition 1, due to the constraints of manifolds with boundary (i.e. not being able to take products) we will actually need to use multiple maps to do so, define the following maps here we again identify points in D^{r+1} with their polar coordinates (s,t), where $S^r = \{(s,0) \mid s \in S^r\}$

$$\begin{aligned} f_1^{\text{inj}} :& (D^{r+1} \times S^r) \setminus \{(s,0), s\} \to S^{n+\ell-1} \\ (x,y) \mapsto & \frac{E(x) - E(y,0)}{||E(x) - E(y,0)||} \\ f_2^{\text{inj}} :& (D^{r+1} \times S^r \times (0,1)) \setminus \{(s,t), (s,t) \mid s \in S^r, t \in (0,1)\} \to S^{n+\ell-1} \\ (x,y,t) \mapsto & \frac{E(x) - E(y,t)}{||E(x) - E(y,t)||} \\ f_3^{\text{inj}} :& D^{r+1} \setminus \{1\} \to S^{n+\ell-1} \\ x \mapsto & \frac{E(x) - E(1)}{||E(x) - E(1)||} \end{aligned}$$

Now, by construction, if $x \in \operatorname{Im}(f_1^{\text{inj}})^c \cap \operatorname{Im}(f_2^{\text{inj}})^c \cap \operatorname{Im}(f_3^{\text{inj}})^c$, then $\pi_x \circ E$ is injective. The domains of f_i^{inj} have dimensions $2r+1, 2r+2$ and $r+1$ which are all less than $n \leq n+\ell-1$, so the regular values of each of these maps are the points disjoint from their image. Since the critical values of each of these maps have measure zero, the union of their critical values has measure zero as desired, thus property 1 holds everywhere apart from a set of measure zero and we are done. $\square$