

1. If compactly supported cohomology were homotopy invariant, then we would require $H_c^*(\mathbb{R}^n) \cong H_c^*(\{\text{pt.}\})$ for all $n \in \mathbb{Z}_{\geq 0}$ since $\mathbb{R}^n \simeq \{\text{pt.}\}$ for all n . To further explicate this, Consider for each n , the homotopy

$$\begin{aligned} H : \mathbb{R}^n &\times [0, 1] \rightarrow \mathbb{R}^n \\ (x, t) &\mapsto x(1 - t) \end{aligned}$$

interpolates the maps $1_{\mathbb{R}^n}$ and the zero map. Then we could take $g : \{0\} \hookrightarrow \mathbb{R}^n$, and $f : \mathbb{R}^n \rightarrow \{0\}$, so that $gf = 1_{\{0\}}$, and fg is the zero map which we already showed is homotopy equivalent to $1_{\mathbb{R}^n}$.

Thus if compactly supported cohomology were a homotopy invariant we would have

$$H_c^*(\mathbb{R}^n) \cong H_c^*(\{0\})$$

choosing $n = 1 = *$, we get $\mathbb{R} \cong 0$ (as \mathbb{R} -vector spaces) by the poincare lemma. This is a contradiction. \square

2. (a) Let η_U, η_V be a partition of unity subordinate to U, V , then we can define the following maps making the sequence short exact:

$$0 \longrightarrow \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M) \longrightarrow 0$$

$$\omega \longrightarrow (\eta_V \cdot \omega, \eta_U \cdot \omega)$$

$$(\omega, \nu) \longrightarrow (\eta_U \cdot \omega, \eta_V \cdot \nu)$$

The first map has $\text{supp}(\eta_V \cdot \omega) \subset \text{supp}(\omega) \supset \text{supp}(\eta_U \cdot \omega)$, so there are no issues with the compact support, similarly for the second map $\text{supp}(\eta_U \cdot \omega + \eta_V \cdot \nu) \subset \text{supp}(\omega) \cup \text{supp}(\nu)$ which is a union of two compact sets hence compact.

To see the first map is an injection let $\omega \in \Omega_c^p(U \cap V)$ (for some p) and suppose that $(\eta_V \cdot \omega, \eta_U \cdot \omega) \equiv 0$, then $\eta_V \cdot \omega \equiv 0$ on U and $\eta_U \cdot \omega \equiv 0$ on V , this of course implies $(\eta_V \cdot \omega)|_{U \cap V} \equiv 0$ and $(\eta_U \cdot \omega)|_{U \cap V} \equiv 0$ on $U \cap V$, so the following easy computation shows injectivity,

$$(\eta_V \cdot \omega)|_{U \cap V} + (\eta_U \cdot \omega)|_{U \cap V} = \eta_V|_{U \cap V} \cdot \omega + \eta_U|_{U \cap V} \cdot \omega = (\eta_U|_{U \cap V} + \eta_V|_{U \cap V}) \cdot \omega = \omega$$

Now checking surjectivity of the second map, Let $\omega \in \Omega_c^p(M)$ for some p , then we have $\omega|_U \in \Omega_c^p(U)$ and $-\omega|_V \in \Omega_c^p(V)$, then I claim that the image of $(\omega|_U, -\omega|_V) = \eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega$. To check this, it suffices to check equivalence pointwise, so we can simply check on each of the sets $U \cap V^c$, $V \cap U^c$ and $U \cap V$, to see it on $U \cap V^c$ we have $\eta_V = 0$, and $\eta_U = 1$ so that $\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega|_U$ on this set, but since $U \cap V^c \subset U$, this is the same thing as ω here. Checking on $V \cap U^c$ is similar, finally on $U \cap V$, we have

$$\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = (\eta_U + \eta_V)\omega|_{U \cap V} = \omega|_{U \cap V}$$

which is of course just ω on $U \cap V$, this shows surjectivity.

Finally, we need to check that $\ker((\omega, \nu) \mapsto \eta_U \cdot \omega - \eta_V \cdot \nu) = \text{Im}(\omega \mapsto (\eta_V \cdot \omega, \eta_U \cdot \nu))$, checking the image is a subset of the kernel, is straightforward since composing both maps we get

$$\omega \mapsto \eta_U \cdot \eta_V(\omega - \omega) = 0$$

Now to check that all elements of the kernel are of this form, suppose $(\omega, \nu) \mapsto 0$, then wherever $\eta_V = 0$, we have $\eta_U \cdot \omega - \eta_V \cdot \nu = \omega$, but since we are assuming this is zero we must have $\text{supp } \omega \subset \text{supp } \eta_V$, the same argument shows that $\text{supp } \nu \subset \text{supp } \eta_U$. Now we define the following form α on $U \cap V$

$$\alpha = \begin{cases} \eta_V^{-1} \cdot \omega & \eta_U, \eta_V > 0 \\ \omega & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases}$$

Then $\text{supp } \alpha \subset \text{supp } \omega \cup \text{supp } \eta$ is compact, to verify that α is indeed smooth, note that away from $\eta_V = 0$, this is clear, to see that its smooth near $\eta_V = 0$, we can use the kernel condition to give us $\eta_U \cdot \omega = \eta_V \cdot \nu$, when $\eta_U, \eta_V > 0$ this gives us $\eta_V^{-1} \omega = \eta_U^{-1} \nu$, but this shows that we are smooth near $\eta_V = 0$. Finally, we compute

$$\eta_V \cdot \alpha = \begin{cases} \omega & \eta_V > 0 \\ 0 & \eta_V = 0 \end{cases} = \omega$$

since $\text{supp } \omega \subset \text{supp } \eta_V$, similarly

$$\begin{aligned} \eta_U \alpha &= \begin{cases} \eta_U \eta_V^{-1} \cdot \omega & \eta_V, \eta_U > 0 \\ 0 & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases} = \begin{cases} \eta_V \eta_V^{-1} \cdot \nu & \eta_V, \eta_U > 0 \\ 0 & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases} \\ &= \nu \end{aligned}$$

Once again, the last equality follows from $\text{supp } \nu \subset \text{supp } \eta_U$. This suffices to show exactness of the previous sequence. \square

(b) Since Theorem 1.4, Lecture 22 in the notes holds for any short exact sequence of cochain complexes, it in particular holds for

$$0 \longrightarrow \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M) \longrightarrow 0$$

The induced long exact sequence is exactly the desired one. \square

3. I will denote ρ to be the covering map $\mathbb{R}^2 \rightarrow \mathbb{T}^2$.

(a) Since $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$, we get by problem 2 of homework 6 it is orientable, since its a codimension 1 submanifold of \mathbb{R}^3 , since its orientable it has a trivial determinant bundle so we can just take the section $t(x) = (x, 1)$. Now we can choose an orientation on \mathbb{R}^2 using the section $s : x \mapsto (x, s'(x))$ where $s'(x) = \pi(\det d\rho(1))$ (here π denotes the map throwing out the base point), s is smooth from construction, and is nonvanishing since ρ is a local diffeomorphism, which implies that $\det d\rho$ is an isomorphism on each fiber, but now by construction we have ρ orientation preserving, since $\pi \det d\rho(s) = (\pi \det d\rho(1))^2 > 0$.

Now that we have shown ρ is orientation preserving, we can use the fact that $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is orientation preserving when $\det A = 1$ and orientation reversing when $\det A = -1$ alongside the following diagram, to conclude that the top path is orientation reversing when $\det A = -1$, and orientation preserving when $\det A = 1$ (for extra justification use functoriality of determinant and the fact that $\det \rho > 0$), by commutativity of the diagram, this means that ϕ_A is orientation preserving iff $\det A = 1$.

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \downarrow \rho & & \downarrow \rho \\ \mathbb{T}^2 & \xrightarrow{\phi_A} & \mathbb{T}^2 \end{array}$$

\square

(b) In part (a), we verified that $SL_2(\mathbb{Z}) \rightarrow \Gamma_{1,0}$ via the map, it remains to check its a homomorphism. The following diagram commutes by definition of ϕ_X being induced by X for each $X \in SL_2(\mathbb{Z})$

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 & \xrightarrow{B} & \mathbb{R}^2 \\ \downarrow \rho & & \downarrow \rho & & \downarrow \rho \\ \mathbb{T}^2 & \xrightarrow{\phi_A} & \mathbb{T}^2 & \xrightarrow{\phi_B} & \mathbb{T}^2 \end{array} \rightsquigarrow \begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{BA} & \mathbb{R}^2 \\ \downarrow \rho & & \downarrow \rho \\ \mathbb{T}^2 & \xrightarrow{\phi_B \phi_A} & \mathbb{T}^2 \end{array}$$

But by definition the induced map in the right diagram is ϕ_{BA} , so that $\phi_B \phi_A = \phi_{BA}$, this is exactly the homomorphism property.

(c) If $A = B$, then $\phi_A = \phi_B$, so clearly they are isotopic and we are done. Conversely, assume $\phi_A \simeq \phi_B$, we want to write down how ϕ_A acts on cohomology.

We start by defining 1-forms on \mathbb{T}^2 , consider the standard one forms dx, dy on \mathbb{R}^2 , we get well defined forms from the restriction (it will suffice to check just for dx). To check this, take open sets $U_1, \dots, U_s \subset \mathbb{R}^2$ with $\bigcup_1^s \rho(U_j) = \mathbb{T}^2$, and $\rho|_{U_j}$ is a diffeomorphism to its image, denote the inverse as ρ_j^{-1} . The existence of such sets follows from ρ being a covering map and we can guarantee finitely many since \mathbb{T}^2 is compact. Now let η_j be a partition of unity subordinate to the U_j , then we can define $dx' = (\sum_1^s \eta_j \cdot \rho_j^{-1})^* dx$, then since ρ is a local diffeomorphism any tangent vector in $T\mathbb{T}^2$ is locally of the form $d\rho(v)$ for $v \in T\mathbb{R}^2$, then

$$dx'(d\rho(v)) = \left(\sum_1^s \eta_j \cdot \rho_j^{-1} \right)^* dxd\rho(v) = (dx)(d \sum_1^s \eta_j \cdot \rho_j^{-1})(d\rho)(v) = (dx) \sum_1^s (d\eta_j)(d\rho_j^{-1})(d\rho)(v)$$

Then since dx is translation invariant, and $(d\rho_j^{-1})(d\rho) = d(x \mapsto x+g)$ for some $g \in \mathbb{Z}^2$, there are no issues with what point we are taking dx at, so that the right hand side further simplifies to $(dx)d(\sum_1^s \eta_j)(v) = dx(v)$, so the form descends to \mathbb{T}^2 . Writing $f = \sum_1^s \eta_j \cdot \rho_j^{-1}$, the computation also verifies that $\rho^* f^* dx = dx$, and the same will hold for dy . In order to write down how $[\phi_A^*]$ acts on cohomology, we will first determine how ϕ_A^* acts on dx', dy' , to do so first let $\alpha \partial_x + \beta \partial_y = v \in T\mathbb{R}^2$

$$\begin{aligned} A^* dx(v) &= (dx)(dA)(v) = dx((a\alpha + b\beta)\partial_x + (c\alpha + d\beta)\partial_y) = a\alpha + b\beta \\ A^* dy(v) &= (dy)(dA)(v) = dy((a\alpha + b\beta)\partial_x + (c\alpha + d\beta)\partial_y) = c\alpha + d\beta \end{aligned}$$

So that $A^* dx = adx + bdy$ and $A^* dy = cdx + ddy$. Now, we can use this to compute $\phi_A^* dx'$ and $\phi_A^* dy'$, by noticing that $\phi_A \rho = \rho A$, so that $\rho^* \phi_A^* = A^* \rho^*$ as well as noticing since $\rho f = 1_{\mathbb{T}^2}$ we get $f^* \rho^* = 1_{\mathbb{T}^2}^*$, from here we can compute

$$\begin{aligned} \phi_A^* dx' &= 1_{\mathbb{T}^2}^* \phi_A^* dx' = f^* \rho^* \phi_A^* dx' = f^* A^* \rho^* f^* dx = f^* A^* dx = f^*(adx + bdy) \\ &= a \circ f \cdot f^* dx + b \circ f \cdot f^* dy = adx' + bdy' \\ \phi_A^* dy' &= 1_{\mathbb{T}^2}^* \phi_A^* dy' = f^* \rho^* \phi_A^* dy' = f^* A^* \rho^* dy' = f^* A^* \rho^* f^* dy = f^* A^* dy = f^*(cdx + ddy) \\ &= c \circ f \cdot f^* dx + d \circ f \cdot f^* dy = cdx' + ddy' \end{aligned}$$

Now assuming that $\{[dx'], [dy']\}$ is a basis for $H^1(\mathbb{T}^2)$ what we just computed shows that

$$[\phi_A^*]^1 : H^1(\mathbb{T}^2) \xrightarrow{A^T} H^1(\mathbb{T}^2)$$

with respect to this basis, and therefore if $\phi_A \simeq \phi_B$, then $A^T = [\phi_A^*]^1 = [\phi_B^*]^1 = B^T$, but this of course implies that $A = B$.

Now, checking that indeed $[dx'], [dy']$ is a basis, we use that $S^1 \cong \mathbb{R}/\mathbb{Z}$, and that $d\theta$ is given by dx using the analogous 1-dimensional restriction for what we did on the Torus, we can write the pull-back here as \hat{f} , and the quotient map as q , using the same computations as for the Torus this gives us $\rho^* \hat{f}^* dx = dx$. To see that $H^1(S^1) = [d\theta]$, recall that $H^1(S^1) \cong \mathbb{Z}$, and using stokes theorem we know that if $d\theta$ were exact it would have zero integral since the circle has no boundary. Now it is straightforward to compute

$$\int_{S^1} d\theta = \int_0^1 \rho^* d\theta = \int_0^1 \rho^* \hat{f}^* dx = \int_0^1 dx = 1$$

So that indeed $H^1(S^1) = [d\theta]$. I will denote the constant function $S^1 \rightarrow \{1\}$ as 1, we use the Künneth Theorem (from the notes Lecture 23) to find that the following is an isomorphism:

$$\mathbb{Z}^2 \cong (H^*(S^1) \otimes H^*(S^1))^1 = \langle [d\theta] \otimes [1] \rangle \oplus \langle [1] \otimes [d\theta] \rangle \xrightarrow{[\pi_1^* \wedge \pi_2^*] \oplus [\pi_1^* \wedge \pi_2^*]} H^1(\mathbb{T}^2)$$

Then by construction we get that $[\pi_1^* \wedge \pi_2^*](d\theta \otimes 1) \stackrel{\text{def}}{=} [\pi_1^*(d\theta) \wedge \pi_2^*(1)] = [dx']$ and $[\pi_1^* \wedge \pi_2^*](1 \otimes d\theta) \stackrel{\text{def}}{=} [\pi_1^*(1) \wedge \pi_2^*(d\theta)] = [dy']$, which verifies that these indeed form a basis for the first cohomology. \square