

1. Suppose $X = \partial W$ for W compact, and f extends to $F : W \rightarrow M$, then we can take $\iota : X \hookrightarrow W$, so that $f = F \circ \iota$. Then

$$\int_X f^* \omega = \int_{\partial W} (F \circ \iota)^* \omega = \int_{\partial W} F^* \omega \stackrel{\text{Stokes}}{=} \int_W dF^* \omega = \int_W F^* d\omega = \int_W F^* 0 = 0$$

□

2. First suppose that ω is closed, then $\omega = df$, so that for any $\gamma : S^1 \rightarrow M$,

$$\int_{S^1} \gamma^* \omega = \int_{S^1} \gamma^* df = \int_{S^1} d\gamma^* f \stackrel{\text{Stokes}}{=} \int_{\emptyset} \gamma^* f = 0$$

For the converse, I will explicitly produce such an f , the assumption is used when showing that f is well defined. It will suffice to prove the connected case, since the proof can be iterated on each connected component. So letting $x_0 \in M$, for any other $x \in M$ define $f(x) = \int_{x_0}^x \omega$ (which is more explicitly the integral of the pullback along any path), a path for this integral always exists since we are working in the connected case. To show that the integral is well defined, we need to check that it is path independent, so assuming $\gamma(1) = x = \bar{\gamma}(1)$, we want to show $\int_{[0,1]} \gamma^* \omega = \int_{[0,1]} \bar{\gamma}^* \omega$, so that f is well defined. I will first prove the result in the case that ω is closed, so we may assume without loss of generality that the paths are constantly x_0 on $[0, \frac{1}{4}]$ and constantly x on $[\frac{3}{4}, 1]$ or otherwise replace them with homotopic paths with this property using bump functions, (it should also be noted that we can use homotopy invariance since $\gamma([0, 1]) \cup \bar{\gamma}([0, 1])$ is compact, and hence contained in an open neighborhood with compact closure, replacing ω with its product with the component of a partition of unity subordinate to this open reduces to the compact case so homotopy can be used) it follows that we can write the smooth map $\beta : S^1 \rightarrow M$ via

$$\beta(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \bar{\gamma}(1-2t) & t \in (\frac{1}{2}, 1] \end{cases}$$

From construction β satisfies

$$\int_{S^1} \beta^* \omega = \int_{[0,1]} \gamma^* \omega - \int_{[0,1]} \bar{\gamma}^* \omega$$

so the two are equal as desired and f is well defined. It remains to check that $\omega = df$, to do so we can check at each $x \in M$, so let $x \in M$, so that $x \in U$ with (U, V, φ) a chart for M , we may assume for simplicity that $\varphi(0) = x$, then for some small ϵ the paths $\alpha_j : t \mapsto te_j$ (where e_j is the j -th standard basis vector on \mathbb{R}^n) satisfy $\alpha_j[-\epsilon, \epsilon] \subset U$, for $1 \leq j \leq n$, now let $x_j = \alpha_j(-\epsilon)$, then for any $y \in M$, we have writing $c_j := \int_{x_0}^{x_j} \omega$

$$\int_{x_j}^y \omega = \int_{x_0}^y \omega - \int_{x_0}^{x_j} \omega = f(y) - \int_{x_0}^{x_j} \omega = f(y) - c_j$$

so that $f = f_j + c_j$ since c_j is constant for the sake of differentiating we can take $f(x) = \int_{x_j}^x \omega =: f_j$ for whichever j is convenient. Now we get that $df = \sum_1^n \frac{\partial}{\partial x_j} f dx_j = \sum_1^n \frac{\partial}{\partial x_j} f_j dx_j$, moreover we have that $\alpha'_j(t) = \frac{\partial}{\partial x_j} \Big|_{\alpha_j(t)}$ (here we implicitly pull everything back along φ , this is done implicitly to simplify

notation)

$$\begin{aligned}
d_x f &= \sum_1^n \frac{\partial}{\partial x_j} \Big|_x f_j dx_j = \sum_1^n \left(\frac{d}{dt} \Big|_{t=0} f \circ \alpha_j \right) dx_j = \sum_1^n \left(\frac{d}{dt} \Big|_{t=0} \int_{x_j}^{\alpha_j(t)} \omega(\alpha'_j(t)) \right) dx_j \\
&= \sum_1^n \left(\frac{d}{dt} \Big|_{t=0} \int_{-\epsilon}^t \alpha_j^* \omega \right) dx_j = \sum_{j=1}^n \left(\frac{d}{dt} \Big|_{t=0} \int_{-\epsilon}^t \sum_{k=1}^n \alpha_j^* \omega_k dx_k \right) dx_j \\
&= \sum_{j=1}^n \left(\frac{d}{dt} \Big|_{t=0} \int_{-\epsilon}^t \sum_{k=1}^n \omega_k \circ \alpha_j d(x_k \circ \alpha_j) \right) dx_j \\
&= \sum_1^n \left(\frac{d}{dt} \Big|_{t=0} \int_{-\epsilon}^t \omega_j \circ \alpha_j \right) dx_j \stackrel{\text{FTC}}{=} \sum_1^n \omega_j(x) dx_j = \omega(x)
\end{aligned}$$

Now it remains to prove the above still holds when ω is not a closed form, the only portion where we used that ω was closed was in concatenating paths to ensure that β was smooth, to do so we work with a neighborhood V of $\gamma([0, 1]) \cup \gamma'([0, 1])$ with compact closure K , moreover we will assume that V lies in a single chart with coordinate map φ , if this is not the case the argument can be broken up into multiple paths (this works since there are finitely many due to compactness), since the integral is invariant under orientation preserving diffeomorphisms, we can work directly with $\gamma : [0, 1] \rightarrow F = \varphi^{-1}(K) \subset \varphi^{-1}(V) = U$, then suppose $\gamma_n : [0, 1] \rightarrow F$ are such that $\varphi^{-1} \circ \gamma_n \rightarrow \varphi^{-1} \circ \gamma$ in C^1 , the convergence is uniform since $[0, 1]$ is compact, we can compute

$$|\varphi^* \omega_{\varphi^{-1}\gamma_n(t)}(\varphi^{-1}\gamma'_n(t)) - \varphi^* \omega_{\varphi^{-1}\gamma(t)}(\varphi^{-1}\gamma'(t))| \quad (1)$$

$$\leq \sup_{t \in [0, 1]} (|\varphi^* \omega_{\varphi^{-1}\gamma_n(t)} - \varphi^* \omega_{\varphi^{-1}\gamma(t)}| |\varphi^{-1}\gamma'_n(t)| + |\varphi^* \omega_{\varphi^{-1}\gamma(t)}| |\varphi^{-1}\gamma'_n(t) - \varphi^{-1}\gamma'(t)|) \quad (2)$$

In order to bound the above, we write explicitly $\varphi^* \omega = \sum a_j dx_j$ for smooth a_j , then each a_j is smooth on F , hence Lipschitz, so that $|a_j(x) - a_j(y)| \leq C_j \|x - y\|$ on F , choosing $C = \max\{C_j \mid 1 \leq j \leq n\}$, we get that

$$|\varphi^* \omega_x(v) - \varphi^* \omega_y(v)| \leq \sum_1^n C \|x - y\| |dx_i v| \leq nC \|v\| \|x - y\|$$

It follows that (here writing the operator norm) $\|\varphi^* \omega_x - \varphi^* \omega_y\| \leq M \|x - y\|$ for $M = nC$, we similarly have $\sup_{x \in F} \|\varphi^* \omega_x\| \leq \sum_1^n \sup_{x \in F} |a_j| < \infty$, we may assume that this is also less than M , by possibly increasing M , we possibly increase M a final time by ensuring it is larger than $\sup_{[0, 1]} \|\varphi^{-1}\gamma'(t)\| + 1$. Let $\epsilon > 0$, then using uniform convergence of $\varphi^{-1}\gamma_n \rightarrow \varphi^{-1}\gamma$, and similarly the derivative, we can choose N large enough so that for any $n \geq N$ and for all $t \in [0, 1]$

$$\|\varphi^{-1}\gamma_n(t) - \varphi^{-1}\gamma(t)\| < \epsilon/2M^2 \text{ and } \|\varphi^{-1}\gamma_n(t) - \varphi^{-1}\gamma(t)\| < \epsilon/2M \text{ and } \|\varphi^{-1}\gamma'_n(t)\| \leq M$$

Then for all $n \geq N$, we get using the inequality between lines (1) and (2),

$$|\varphi^* \omega_{\gamma_n(t)}(\gamma'_n(t)) - \varphi^* \omega_{\gamma(t)}(\gamma'(t))| < \epsilon$$

This allows us to approximate the integral along the pull back of γ by γ_n , since

$$\begin{aligned}
\left| \int_{[0, 1]} \gamma_n^* \omega - \int_{[0, 1]} \gamma^* \omega \right| &= \left| \int_{[0, 1]} (\varphi^{-1}\gamma_n)^* \varphi^* \omega - \int_{[0, 1]} (\varphi^{-1}\gamma)^* \varphi^* \omega \right| \\
&= \left| \int_{[0, 1]} \varphi^* \omega(\varphi^{-1}\gamma'_n(t)) dt - \int_{[0, 1]} \varphi^* \omega(\varphi^{-1}\gamma'(t)) dt \right| \\
&\leq \int_{[0, 1]} |\varphi^* \omega_{\varphi^{-1}\gamma_n(t)}(\varphi^{-1}\gamma'_n(t)) - \varphi^* \omega_{\varphi^{-1}\gamma(t)}(\varphi^{-1}\gamma'(t))| dt \\
&< \epsilon
\end{aligned}$$

The proof will be established once we provide existence of such γ_n which are constant and equal to x on $[1 - \epsilon_n, 1]$, and constant and equal to x_0 on $[0, \epsilon_n]$ for some $\epsilon_n \rightarrow 0$, since then we may simply approximate both paths γ and $\bar{\gamma}$ by such sequences, which can be glued, and with the property that defining β by concatenating $\gamma_n, -\bar{\gamma}_n$ we get a smooth map from S^1 , this will give well definedness, since by this approximation we get that $|\int \gamma^* \omega - \int \bar{\gamma}^* \omega| < \epsilon$ for any $\epsilon > 0$. Letting η_n be smooth bump functions equal to 1 on $[1/n, 1 - 1/n]$ and 0 on $[0, \frac{1}{2n}] \cup [1 - \frac{1}{2n}, 1]$ smoothness of γ guarantees that the following sequence satisfies C^1 convergence of $\varphi^* \gamma_n$

$$\gamma_n(t) = \begin{cases} 1 \end{cases}$$