1. **(a)** Define $F(x) = (\det \mathrm{Hess}_x(f))^2 + \sum_1^k \left(\frac{\partial}{\partial x_j} f(x)\right)^2$, its clear that both the hessian and sum terms are non-negative (since they are squares of real values). Now assume first that $F > 0$ on U, if f has no critical points on U we are done. Now if p is a critical point for f, we have that $d_p f : \mathbb{R}^k \to \mathbb{R}$ is not surjective, where $d_p f = \left(\frac{\partial}{\partial x_1} f \quad \cdots \quad \frac{\partial}{\partial x_k} f\right)$, this matrix is of course surjective so long as atleast one $\frac{\partial}{\partial x_j} f \neq 0$, so at a critical point we get $\frac{\partial}{\partial x_j} f = 0$ for all j whence $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p)\right)^2 = 0$, by our assumption of $F > 0$, this implies that $(\det \mathrm{Hess}_p(f))^2 > 0$, and since p was an arbitrary critical point we can conclude that f is morse. Conversely, if f is morse, then by the computation above, at any regular value, p, we must have some $\frac{\partial}{\partial x_j} f \neq 0$, which implies that $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p)\right)^2 > 0$, which implies $F(p) > 0$ by non-negativity of the hessian term. In the case that p is a critical point, we know that $\det \mathrm{Hess}_p f \neq 0$, so that $(\det \mathrm{Hess}_p f)^2 > 0$, and $F(p) > 0$, since all points are either regular values or critical points we are done. $\square$

(b) Let f be morse, and H a homotopy with $H(x, 0) = f(x)$, moreover we can denote $f_t = H(-, t)$. Then since M is compact we can pick a finite covering by charts $(V_1, U_1, \phi_1), \ldots, (V_n, U_n, \phi_n)$. Now let η_i be a partition of unity suboordinate to these charts, we can define

$$F(x,t) = \sum_1^n \eta_i(x) \left((\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2 \right)$$

F is smooth, due to closure of smooth functions under sums, products and compositions as well as the fact that the hessian and partials vary smoothly with t, this can be seen since the coordinates of these maps are a subset of the coordinates of $H \circ (\phi^{-1}, 1_{[0,1]})$, which of course has smooth partials and hessian. To see that $F(-, 0) > 0$ on M, note that since f_0 is morse and ϕ_i^{-1} are diffeomorphisms, each summand $\left((\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2\right) > 0$ by part (a). Since M is compact, the function $\hat{F}(t) = \inf_{x \in M} F(x, t)$ is continuous, and it has $\hat{F}(0) > 0$, since $F(-, 0)$ is continuous and positive on a compact set (which implies it attains its infimum), then by continuity of $\hat{F}$, there exists some $\delta > 0$, such that $t < \delta$ implies $\hat{F}(t) > 0$. So to show that morse functions are generic, it will suffice to show that $F(-, t) > 0$ implies that f_t is morse, since then we get for $t < \delta$ each $f_t = H(x, t)$ is morse. To check this, note that since the ϕ_i^{-1} are diffeomorphisms, we have that $\mathrm{Hess}_x f_t$ is invertible iff $\mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1}$ for each i (since we have a partition of unity we can always assume $x \in V_i$), moreover f_t is regular if and only if $f_t \circ \phi_i$ is, so at critical points of f_t, we get that for each i, $\sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2$ vanishes. It follows that $(\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 > 0$ for all i, when x is a critical value of f, and since this Hessian is invertible iff the hessian of f_t is, we find that f_t has invertible Hessian at all its critical points, assuming $F(-, t) > 0$ on M which completes the proof. $\square$