1. (Durrett 2.5.3) Define $Y_n = \frac{\sin(n\pi t)}{n} X_n$, then since $X_n \sim N(0,1)$ we get $\operatorname{Var} Y_n = \frac{|\sin(n\pi t)|}{n^2} \leq 1/n^2$. Then since $\sum_1^\infty \operatorname{Var} Y_n < \infty$ and $Y_1, Y_2, \ldots$ are independent, we get from the consequence of Kolmogorov maximal inequality that $\sum_1^\infty Y_n$ converges almost surely. $\square$

2. (Durrett 2.5.6) We can use the Kolmogorov 3-series test. First notice,

$$E[\psi(X_n)] = E\left[|X_n| 1_{|X_n|>1} + X_n^2 1_{|X_n|\leq 1}\right]$$

So in particular by the comparison test,

$$\sum_1^\infty E\left[|X_n| 1_{|X_n|>1}\right] < \infty \qquad \sum_1^\infty E\left[X_n^2 1_{|X_n|\leq 1}\right] < \infty$$

The latter is the series $\sum_1^\infty \operatorname{Var}(X_n 1_{|X_n|\leq 1})$, another of the series is relatively free, namely

$$\sum_1^\infty P(|X_n| > 1) \leq \sum E\left[|X_n| 1_{|X_n|>1}\right] < \infty$$

Finally, we have $|E[X_n 1_{|X_n|>1}]| \leq E\left[|X_n| 1_{|X_n|>1}\right]$, so that $\sum_1^\infty E\left[X_n 1_{|X_n|>1}\right]$ converges, since

$$0 = \sum_1^N E[X_n] = \sum_1^N E\left[X_n 1_{|X_n|>1} + X_n 1_{|X_n \leq 1|}\right] = \sum_1^N E\left[X_n 1_{|X_n|>1}\right] + \sum_1^N E\left[X_n 1_{|X_n \leq 1|}\right]$$

we have

$$\sum_1^\infty E\left[X_n 1_{|X_n \leq 1|}\right] = -\sum_1^\infty E\left[X_n 1_{|X_n|>1}\right] \in \mathbb{R}$$

Thus X_n satisfy the hypothesis of the Kolmogorov 3-series test and converge almost surely. $\square$

3. (Durrett 2.5.8) Write $Y = \log^+|X_1|$ and assume first that $EY = \infty$, then for any $c > 0$, we find that

$$EY = \int P(Y > t)dt = \sum_1^\infty \int_{(n-1)c}^{nc} P(Y > t)dt \leq \sum_0^\infty cP(Y > nc)$$

which in turn implies that $\sum_1^\infty P(\log^+|X_n| > nc) = \sum_1^\infty P(Y > nc) = \infty$, therefore by Borel-Cantelli (ii), we find that $P(X_n > e^{nc} \text{ i.o.}) = 1$, so for any $c \neq 0$ we find that $|X_n|c^n > 1$ i.o. almost-surely. This of course implies that $\sum_1^\infty |X_n|c^n$ has radius of convergence zero almost surely.

Now, conversely suppose $E\log^+|X_n| < \infty$, and let $0 < c < 1$, then choose $\gamma > 0$ so that $e^\gamma c < 1$ we get another layer cake estimate,

$$EY = \int P(Y > t)dt = \sum_1^\infty \int_{(n-1)\gamma}^{n\gamma} P(Y > t)dt \geq \sum_1^\infty \gamma P(Y > n\gamma)$$

Since $\sum_1^\infty \gamma P(Y > n\gamma) = \sum_1^\infty P(\log^+|X_n| > n\gamma)$ by Borel Cantelli, we have $|X_n| > e^{n\gamma}$ only finitely many times. Letting N so that $n \geq N$ implies $|X_n| \leq e^{n\gamma}$, we get

$$\sum_1^\infty c^n|X_n| \overset{\text{a.s.}}{\leq} \sum_1^{N-1} c^n|X_n| + \sum_N^\infty (ce^\gamma)^n < \infty$$

So that the series converges a.s. for any $c < 1$. Now letting $c > 1$, we once again use Borel-Cantelli (ii), we first check the hypothesis

$$\sum_1^\infty P(|X_n|c^n > 1) = \sum_1^\infty P(|X_1| > \frac{1}{c^n}) = \infty$$

Since $\lim_{n\to\infty} P(|X_1| > \frac{1}{c^n}) = P(X_1 \neq 0) > 0$ by assumption, it follows that $P(|X_n|c^n > 1 \text{ i.o.}) = 1$, so the series diverges a.s. for $c > 1$. $\square$

4. (Durrett 3.2.9) We can use a mesh to show that $||F - F_n||_u = 0$. Namely, let $\epsilon > 0$, then we can choose x_0 so that $F(x_1) < \epsilon$ and y so that $F(y) > 1 - \epsilon$, then F is uniformly continuous on $[x_0, y]$, so in particular there is some $\delta > 0$ such that $|w - z| < \delta \implies |F(w) - F(z)| < \epsilon$ on $[x_0, y]$, then take $x_j = x_0 + j\frac{\delta}{2}$ for $j \in \{1, \ldots, N-1\}$, so that $y - \delta/2 < x_{N-1} \leq y$ and denote $y = x_N$. Then take M_j, so that $n \geq M_j \implies |F_n(x_j) - F(x_j)| < \epsilon$ and define $M = \max_{0 \leq j \leq N} M_j$. It follows that for $n \geq N$, we have for $x \in (-\infty, x_0)$,

$$F_n(x) \leq F_n(x_0) < F(x_0) + \epsilon < 2\epsilon$$

and for $x \in (x_N, \infty)$

$$F_n(x) \geq F_n(x_N) > F(x_N) - \epsilon > 1 - 2\epsilon$$

and on these ranges $F(x) < \epsilon$ and $F(x) > 1 - \epsilon$ respectively, meaning $|F(x) - F_n(x)| < 2\epsilon$. Finally, suppose that $x \in [x_{j-1}, x_j]$ for some j, then we have

$$F(x) - 2\epsilon < F(x_{j-1}) - \epsilon \leq F_n(x_{j-1}) \leq F_n(x) \leq F_n(x_j) \leq F(x_j) + \epsilon < F(x) + 2\epsilon$$

So that $||F - F_n||_u < 2\epsilon$ and since ϵ was arbitrary, $||F - F_n||_u \to 0$. □

5. (Durrett 3.2.12) One direction is much easier, so first suppose that $X_n \implies c$, then for any $\epsilon > 0$, we have $P(X_n \leq c - \epsilon) \to P(c \leq c - \epsilon) = 0$, and $P(X_n > c + \epsilon) = 1 - P(X_n \leq c + \epsilon) \to 1 - P(c \leq c + \epsilon) = 0$, so that

$$P(|X_n - c| > \epsilon) \leq P(X_n > c + \epsilon) + P(X_n \leq c - \epsilon) \to 0$$

For the (more general) converse, I should first mention that the DCT applies to convergence in probability (every subsequence has a subsequence converging almost surely which will have the same DCT limit) we can use the Portmanteau lemma, which gives equivalence to weak-* convergence on bounded continuous functions. Namely, we want to show that $\int f d\mu_n \to \int f d\mu$ for all bounded continuous f. Now letting f be bounded and continuous, we have that $E|f| < \infty$ so we can apply DCT, observe from the change of variables formula,

$$\int f d\mu_n = E[f \circ X_n] \xrightarrow{\text{DCT}} E[f \circ X] = \int f d\mu$$

□

6. (Durrett 3.2.13) - Converging Together Lemma

7. (Durrett 3.2.15) We let $Y_j \sim N(0,1)$, and $\hat{X}_n^j = Y_j(\frac{n}{\sum_1^n Y_j^2})$, we will show later that $\hat{X} \overset{\text{dist.}}{=} X$, and that $\hat{X}_n^1 \implies N(0,1)$. Starting with the latter, Y_j^2 are i.i.d. With $E|Y_j^2| = EY_j^2 = \text{var}(Y_j) + (EY_j)^2 = 1$, whence we can apply the strong law of large numbers to $\frac{\sum_1^n Y_j^2}{n} \xrightarrow{\text{a.s.}} 1$ so that $\hat{X}_n^1 \xrightarrow{\text{a.s.}} Y_1 \sim N(0,1)$.

Now to see that $\hat{X} \overset{\text{dist.}}{=} X$, we will show that $\hat{X}$ is uniformly distributed on the sphere of radius $\sqrt{n}$, first note that it indeed takes values on this sphere since $\sqrt{\sum_1^n \left(Y_j\left(\frac{n}{\sum_1^n Y_k^2}\right)\right)^2} = \sqrt{n}$. Now we can use that the uniform distribution is the unique distribution which is preserved by rotation on the $\sqrt{n}$-sphere. Denoting $Y = (Y_1, \ldots, Y_n)$, Y having distribution F can use characteristic functions to check that

$$\varphi_{AY}(t) = \int e^{it^T AY} dF = \int e^{i(A^T t)^T Y} dF = \varphi_Y(A^T t) = e^{||A^T t||^2/2} = e^{||t||^2/2} = \varphi_Y(t)$$

So that $Y = AY$, now $A\hat{X} = \sqrt{n}\frac{AY}{||Y||} = \sqrt{n}\frac{Y}{||Y||}$. □

8. (Durrett 1.7.5) The following is used in the proof of Fourier inversion, namely $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. Moreover the trick $\frac{1}{x} = \int_0^\infty e^{-xy} dy$, then using Fubini is frequently useful for integration. Now on with the proof, Let $I := \int_0^a e^{-xy} \sin x dx$, then

$$I = \left[-\cos x e^{-xy} - y \int \cos x e^{-xy}\right]_{x=0}^a = \left[-\cos x e^{-xy} - y e^{-xy} \sin x - y^2 \int e^{-xy} \sin x\right]_{x=0}^a$$
$$= 1 - \cos a e^{-ay} - y e^{-ay} \sin a + y^2 I$$

So that $I = \frac{1}{1+y^2} - \cos a \frac{e^{-ay}}{1+y^2} - \sin a \frac{ye^{-ay}}{1+y^2}$

Now it is easy to see $\int_0^\infty I dy = \pi/2 - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin a \int_0^\infty \frac{ye^{-ay}}{1+y^2}$. Since Tolleni's theorem always holds for positive functions, we can check the assumptions of Fubini on $|e^{-xy} \sin x|$, using $|\sin x| \leq |x|$

$$\int_0^a \int_0^\infty |e^{-xy} \sin x| dy dx = \int_0^a \frac{|\sin x|}{x} dx \leq a$$

All of the work has been done, so now we can breeze through the proof

$$\int_0^a \frac{\sin x}{x} = \int_0^a \int_0^\infty e^{-xy} \frac{\sin x}{x} dy dx \overset{\text{fubini}}{=} \int_0^\infty \int_0^a e^{-xy} \frac{\sin x}{x} dx dy$$
$$= \int_0^\infty \frac{1}{1+y^2} - \cos a \frac{e^{-ay}}{1+y^2} - \sin a \frac{ye^{-ay}}{1+y^2}$$
$$= \pi/2 - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2} dy - \sin a \int_0^\infty \frac{ye^{-ay}}{1+y^2} dy$$

This gives the desired inequality, namely

$$|\int_0^a \frac{\sin x}{x} - \pi/2| \leq |\cos a| \int_0^\infty e^{-ay} dy + |\sin a| \int_0^\infty ye^{-ay} dy \leq \frac{1}{a} + \frac{1}{a^2} \overset{a \geq 1}{\leq} \frac{2}{a}$$

$\square$

9. (Durrett 3.3.1) Let φ be the characteristic function of X, then let Y be i.i.d. with $-X$, so that

$$\varphi_{X+Y}(t) = E[e^{it(X+Y)}] \overset{\text{ind.}}{=} E[e^{itX}]E[e^{itY}] = \varphi(t)\overline{\varphi(t)} = |\varphi(t)|^2$$

To get that $\Re\varphi$ is a characteristic function, note first that it must be

$$\Re\varphi(t) = E\left[\frac{1}{2}(e^{itX} + e^{-itX})\right]$$

We can actually cheat here using Bochner's theorem with linearity, and $\Re\varphi(0) = 1$, but using Bochner's theorem can feel like bad practice. Instead let Y be a random variable independent of X with $P(Y = 1) = P(Y = -1) = \frac{1}{2}$, then

$$\varphi_{YX}(t) = E[e^{itYX}] = E[e^{itYX}(1_{Y=1} + 1_{Y=-1})] = E[e^{itX}1_{Y=1}] + E[e^{-itX}1_{Y=-1}]$$
$$\overset{\text{indep}}{=} \varphi(t)E[1_{Y=1}] + \overline{\varphi(t)}E[1_{Y=-1}] = \Re\varphi(t)$$

$\square$

10. (Durrett 3.3.2) (i)

$$\int_{-T}^T e^{-ita}\varphi(t)dt = \int_{-T}^T \int e^{it(X-a)} d\mu dt \overset{\text{Fubini}}{=} \int \int_{-T}^T e^{it(X-a)} dt d\mu = \int \int_{-T}^T (\cos + i\sin)(t(X-a)) dt d\mu$$
$$= \int \int_{-T}^T \cos(t(X-a)) dt d\mu = \int \frac{2\sin(T(X-a))}{X-a} d\mu$$

Fubini here is justified, since the integrand has absolute value ≤ 1, and μ is a probability measure, now applying the limit we want to use DCT, which is justified by $\frac{\sin(T(X-a))}{T(X-a)} \leq 1$.

$$\lim_{T\to\infty} \int \frac{2\sin(T(X-a))}{2T(X-a)} d\mu = \int \lim_{T\to\infty} \frac{\sin(T(X-a))}{T(X-a)} d\mu = \int 1_{\{a\}} d\mu = \mu\{a\}$$

(ii) If $x \notin h\mathbb{Z}$, then $P(X = x) = 0$. Now assuming $x \in h\mathbb{Z}$, we find that e^{-itx} is $\frac{2\pi}{h}$ periodic. Moreover

$$\begin{aligned}\varphi(t + 2\pi/h) &= E[\exp(iX(t + 2\pi/h))] = E[1_{h\mathbb{Z}} \exp(iX(t + 2\pi/h))] = \int_{h\mathbb{Z}} \exp(iX(t + 2\pi/h)) d\mu \\ &= \int_{h\mathbb{Z}} e^{itX} e^{X2\pi/h} d\mu = \int_{h\mathbb{Z}} e^{itX} = \varphi(t)\end{aligned}$$

Now, using the previous subpart we find that

$$P(X = x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \exp(-itx)\varphi(t)dt$$

Since the limit exists, it suffices to examing a subsequence, using $2\pi/h$ periodicity, we take $T_n = \frac{2\pi n+\pi}{h}$

$$P(X = x) = \lim_{n\to\infty} \frac{h}{2\pi(2n + 1)} \int_{-T_n}^{T_n} exp(-itx)\varphi(t)dt = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} exp(-itx)\varphi(t)dt$$

(iii) The equality written in the statement is one of the consequences of the definition of characteristic functions. This result is striaghtforward.

11. (Durrett 3.3.3) $\varphi_{X-Y} = |\varphi|^2$, so by the previous problem, and noting $\{x \mid \mu\{x\} \neq 0\}$ is countable,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt &= P(X - Y = 0) = \int 1_{X-Y=0} d\mu d\mu = \int\int 1_{Y=X} d\mu d\mu = \int\int 1_{Y=x} d\mu d\mu(x) \\ &= \int \mu\{x\} d\mu = \int \mu\{x\} 1_{\mu\{x\}\neq 0} = \sum_{\{x|\mu\{x\}\neq 0\}} \mu\{x\} \int 1_{\{x\}} d\mu = \sum_{x} \mu\{x\}^2\end{aligned}$$

□

This exercise has some important consequences, namely if $\varphi(t) \xrightarrow{t\to\infty} 0$, then μ does not have any point masses, moreover the converse is false by the next exercise (3.3.11). There is a partial converse, which says that if μ has a density, the $\varphi(t) \to 0$, to see this if μ has a density f, then $\varphi(t) = \int e^{itx} f(x)dx$, and this is a consequence of the Riemann lebesgue lemma, where we show it for simple functions of intervals, then use that these are dense in $L^1(\mathbb{R})$. On an interval, $|\int_a^b e^{itx}| =\leq 2/|t|$. □

12. Durrett 3.3.11 This problem provides an example of a random variable, with a continuous density, but not satisfying $\lim_{t\to\infty} \varphi(t) = 0$, the random variable is defined using $X_k \sim$ Bernoulli(1/2) and $X = 2\sum_1^\infty X_j/3^j$. The distribution function is continuous, since it is the Cantor-Lebesgue function. Now we compute φ, using exercise 3.3.9, the characteristic function for the Bernoulli distribution and $\varphi_{aX}(t) = \varphi_X(at)$ we get

$$\varphi(t) = \prod_1^\infty \varphi_{2X_i/3^j}(t) = \prod_1^\infty \frac{1}{2}\left(e^{it\frac{2}{3^j}} + 1\right)$$

Now when $t = 3^k\pi$, we get

$$\varphi(t) = \prod_1^\infty \frac{1}{2}\left(e^{it\frac{2\pi}{3^{j-k}}} + 1\right) = 1 \cdot \prod_{k+1}^\infty \frac{1}{2}\left(e^{it\frac{2\pi}{3^{j-k}}} + 1\right) = \varphi(\pi)$$

So as long as we can show $|\varphi(\pi)| > 0$, we are done. We first establish

$$|e^{i2\pi/3^j} - 1| \leq |\sin 2\pi/3^j| + |1 - \cos 2\pi/3^j| \leq \frac{2\pi}{3^j} + \frac{2\pi^2}{3^{2j}} \overset{j>2}{\leq} \frac{4\pi}{3^j}$$

Now taking $C = |\prod_1^{r-1} \frac{1}{2}\left(e^{it\frac{2}{3^j}} + 1\right)|$, where $r > 2$ is chosen so that $2\pi \sum_r^\infty 3^{-j} < \frac{1}{2}$ we find that

$$|\varphi(\pi)| \geq C \prod_r^\infty \frac{1}{2}\left(2 - \frac{4\pi}{3^j}\right) \geq C \prod_r^\infty 1 - \frac{2\pi}{3^j} \geq C - C2\pi \sum_r^\infty 1/3^j \geq C/2 > 0$$

□

13. (Durrett 3.3.4) Take the distribution of a coinflip, with $P(X = 1) = P(X = -1) = \frac{1}{2}$, then

$$\varphi(t) = E[e^{itX}] = \frac{1}{2}(e^{it} - e^{-it}) = \cos t$$

$\cos t \not\to 0$, so it is apparent that $\int_{-\infty}^\infty |\cos t| = \infty$. □

14. (Durrett 3.3.6) Firstly, to solve for the Cauchy distribution, we use that $f(x) = \frac{1}{2\pi}\int e^{-itx}\varphi(t)dt$, in our case we find that $\varphi = e^{-|t|}$, since

$$\begin{aligned}\frac{2}{\pi} f(x) = \frac{2}{1+x^2} = \frac{1}{1+ix} - \frac{1}{1-ix} &= \int_0^\infty e^{-t(1+ix)}dt - \int_0^\infty e^{-t(1-ix)}dt \\ = \int_0^\infty e^{-t(1+ix)}dt + \int_{-\infty}^0 e^{-t(ix-1)}dt &= \int e^{-|t|}e^{-itx}dt = \int \varphi(t)e^{-itx}dt\end{aligned}$$

Then we can use $\varphi_{\frac{\sum_1^n X_k}{n}} = \prod_1^n \varphi_{\frac{X_k}{n}}$, and $\varphi_{\frac{X_k}{n}}(t) = \varphi_{X_k}(t/n)$, so that

$$\varphi_{\frac{\sum_1^n X_k}{n}}(t) = \prod_1^n e^{-|t|/n} = \exp\left(\sum_1^n -|t|/n\right) = e^{-|t|} = \varphi_{X_1}(t)$$

□

15. (Durrett 3.3.7) $X_n \implies X$, hence $\varphi_n \to \varphi$, where $\varphi_n = e^{-\frac{1}{2}\sigma_n^2 t^2}$ and $\varphi = e^{-\frac{1}{2}\sigma^2 t^2}$, noting that $\varphi_n/\varphi \to 1$, we have that $e^{\frac{1}{2}t^2(\sigma^2 - \sigma_n^2)} \to 1$ for all t, now letting $t = \sqrt{2}$, its apparent this only happens when $\sigma_n \to \sigma$. To justify the characteristic functions,

$$E[\exp(itX)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp(itx - \frac{x^2}{2\sigma^2})dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int \cos(tx)e^{-\frac{x^2}{2\sigma^2}}dx$$

Now defining $\varphi(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int \cos(tx)e^{-\frac{x^2}{2\sigma^2}}dx$, we can swap the derivative and integral since our function is absolutely integrable. This gives us

$$\varphi'(t) = \int -xe^{-\frac{x^2}{2\sigma^2}} \sin txdx$$

Now integrating by parts with $u = \sin tx, du = t\cos tx$, and $dv = -xe^{-\frac{x^2}{2\sigma^2}}$, we get $v = \sigma^2 e^{-\frac{x^2}{2\sigma^2}}$ so that

$$\varphi'(t) = -\sigma^2 t\varphi(t) \implies \varphi(t) = e^{-\frac{\sigma^2 t^2}{2}}$$

□

16. (Durrett 3.3.8) We get $\varphi_n^X \to \varphi^X$ and $\varphi_n^Y \to \varphi^Y$, then $\varphi_n^{X+Y} = \varphi_n^X \varphi_n^Y \to \varphi^X \varphi^Y$.

17. (Durrett 3.3.9) For any n, we get that

$$\varphi_{S_n} = \varphi_{S_{n-1}+X_n} = \varphi_{S_{n-1}}\varphi_{X_n}$$

applying this inductively we get that $\varphi_{S_n} = \prod_1^n \varphi_{X_n}$, since $S_n \to S$, the characteristic functions $\varphi_{S_n} \to \varphi_S$ pointwise, since $S_n \implies S$. Then it follows that for each t, $\varphi_S(t) = \lim_{n\to\infty} \prod_1^n \varphi_{X_n}(t)$, and in particular the limit exists.

18. (Durrett 3.3.12) We first Taylor expand $e^{-t^2/2}$,

$$e^{-t^2/2} = \sum_0^\infty (-1)^n \frac{t^{2n}}{n!2^n}$$

Now taking $X \sim N(0,1)$, and using the taylor expansion in the last equality, we find that

$$EX^{2n} = \frac{1}{\sqrt{2\pi}} \int x^{2n} e^{-x^2/2} dx = (-1)^n \varphi^{(2n)}(0) = (-1)^n \left(\frac{d}{dt}\right)^{2n}\Bigg|_0 e^{-t^2/2} = \frac{2n!}{2^n n!}$$

$\square$

19. (Durrett 3.3.14) (i) We can't use an L^2 weak law because we only have access to information on the first two terms of the expansion, it will be hard to use a weak law in general due to only having the characteristic functions, so we need to show it directly, first noting

$$\lim_{n\to\infty} \frac{\varphi(t/n) - \varphi(0)}{t/n} = ia \implies n(\varphi(t/n) - 1) = iat$$

we can rearrange to find that

$$\varphi_{S_n/n}(t) = \varphi(t/n)^n = \left(1 + n\frac{\varphi(t/n) - 1}{n}\right)^n$$

So that $\lim_{n\to\infty} \varphi_{S_n/n} = e^{iat} = \varphi_a$ by continuity of log, since e^{iat} is continuous at zero, we get weak convergence $S_n/n \implies a$.

(ii) Immediate consequence of the continuity theorem.

(iii) TODO

20. (Durrett 3.3.16) If $EX = 0$, and $\mathrm{Var}X = 0$, then $EX^2 = 0$, which implies $X \overset{L^2}{=} 0$, so that $X = 0$ a.s.

Now for the main result, we have from taylors theorem $\lim_{h\to 0} \frac{1-\cos hx}{h^2} = x^2$, this alongside Fatou's lemma gives us

$$\begin{aligned}E[X^2] &\overset{\text{Fatou}}{\leq} 2\liminf_{h\to 0} E\left[\frac{1-\cos(hX)}{h^2}\right] = 2\liminf_{h\to 0} E\left[\frac{1}{h^2} - \frac{e^{ihx} + e^{-ihx}}{2h^2}\right] = \liminf_{h\to 0} \frac{2 - \varphi(h) - \varphi(-h)}{h^2} \\ &= -\limsup_{h\to 0} \frac{\varphi(h) + \varphi(-h) - 2}{h^2} = -2\limsup_{h\to 0} \frac{\varphi(h) - 1}{h^2} < \infty\end{aligned}$$

Then since $E[X^2] < \infty$, we get that

$$\varphi(t) = 1 + itE[X] - t^2E[X^2] + o(t^2)$$

so that $E[X] = 0$ for the above limit to make sense. $\square$

21. (Durrett 3.3.17) One direction is obvious from the continuity theorem, in the other direction $\varphi_n \to \varphi$ where φ is continuous on $(-\delta, \delta)$ so once again by the continuity theorem, $Y_n \implies Y$, where Y has characteristic function φ, we want to show that $\varphi = 1$, so we will be done by the inversion formula. Now since $\lim_{t\downarrow 0} \frac{\varphi(t)-1}{t^2} = 0 > -\infty$, we get from the proof of the previous exercise (3.3.16) $E[Y^2] = 0$, and hence $Y = 0$ a.s. $\square$

22. (Durrett 3.4.1) The histogram correction in this case is $P(S_{180} \leq 24.5)$, applying the approximation given by the CLT,

$$P(S_{180} \leq 24.5) = P\left(\frac{S_{180} - 30}{5} \leq -11/10\right) = \Phi(-1.1) \approx 13.57$$

$\square$

23. (Durrett 3.4.4)