1. Notation: let e_{ij} denote the elementary matrix with a 1 in the i, j-th position and zeroes elsewhere
(a) Show that the exponential map for $SL(2, \mathbb{C})$ is not surjective.

Proof. Let $A \in \exp(\mathfrak{sl}(2, \mathbb{C}))$ so that $A = \exp(B)$, then since we are working in $\mathbb{C}$ we can conjugate B to its Jordan canonical form $J = PBP^{-1}$, the jordan canonical form of a matrix with trace zero is one of

$$J_1 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad J_2 = \begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

The exponential of either Jordan canonical form looks like:

$$\exp(J_1) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$$
$$\exp(J_2) = \exp(J_1 + e_{12}) = \exp(J_1)\exp(e_{12}) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}(1 + e_{12}) = \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{-\lambda} \end{pmatrix}$$

So that $\exp(J_2)$ has canonical form $\begin{pmatrix} 1 & 1 \\ 0 & e^{-2\lambda} \end{pmatrix}$. Now since $A = \exp(B) = P\exp(J)P^{-1}$, A must have one of these two normal forms. It follows that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not in the image of exp. □

(b) Show the exponential map for $SL(2, \mathbb{R})$ is not surjective

Proof. Assume it were, then $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ would be in the image of the exponential map on $\mathfrak{sl}(2, \mathbb{R})$, since the exponential map on $\mathfrak{sl}(2, \mathbb{R})$ is the restriction of the exponential map on $\mathfrak{sl}(2, \mathbb{C})$ to real matrices, this contradicts **(a)**. □

2. Let G be a connected Lie group, and U an open neighborhood of the group unit e. Show that any $g \in G$ can be written as a product of elements $g_1 \cdot g_2 \cdots g_n$ with $g_j \in U$.

Proof. Denote inv as the inverse map, and L_g as the map by left multiplication by a group element $g \in G$. Since inv is a diffeomorphism $\mathrm{inv}(U)$ is an open subset of G, moreover $e \in U \cap \mathrm{inv}(U)$, so it will suffice to show that all elements of G can be written as products of elements in $V := U \cap \mathrm{inv}(U)$.

Now, define $S = \{\prod_1^n g_j \mid n \in \mathbb{Z}_{>0} \text{ and } g_j \in V\}$, to see that S is a subgroup of G, we note that $e \in S$, S is closed under products by definition, and if $g \in S$, then $g = \prod_1^n g_j$ with g_j in V, so that $\prod_1^n g_{n+1-j}^{-1} \in S$ since $V = \mathrm{inv}(V)$. Moreover, $S \subset G$ is open, this can easily be seen since $S = \bigcup_{g \in S} L_g(V)$, where each $L_g(V)$ is open since L_g is a diffeomorphism. Now, since S is open to see that $S = G$ it will suffice to prove that S is also closed since G is connected. To see that S is closed, we prove that S^c is open. This is pretty much just an observation, since S is a subgroup its compliment is simply the union of its cosets

$$S^c = \bigcup_{g \in G \setminus S} L_g(S)$$

which is a union of open sets since we have established S is open and L_g is a diffeomorphism. □