Concise AG Notes - UofT MAT1190

Tighe

September 8, 2025

1 Lecture Notes

1.1 Lecture 1 (Sept 3, 2025)

Theorem 1. (Gelfond-Neymark) A compact topological space is determined by its ring of smooth functions. In particular if the ring $C(X) := C(X, \mathbb{R})$ and $C(X) \cong C(Y)$, then $X \cong Y$.

Proposition 1. Each point in X corresponds to a maximal ideal of $C(X)$, moreover if X is compact, then the correspondence is 1-1.

Proof. the evaluation at a point gives a surjective homomorphism $C(X) \to \mathbb{R}$, the image is a field, hence the kernel is a maximal ideal corresponding to the point.

Now in the compact case, (assume X is Hausdorff?), then X is Hausdorff and compact hence normal. We can use Uhrysohn's lemma to get a function vanishing at x but not y. Now suppose that for some maximal ideal $\mathfrak{m} \subset C(X)$ for any point $p \in X$ there is a continuous function with $f(p) \neq 0$, then the set $U_f = \{x \in X \mid f(x) \neq 0\}$ is open , and $\bigcup_{f \in C(X)} U_f = X$, so we get a finite subcover. Take a linear combination of the functions in this subcover to complete the proof. $\square$

definition 1. The Zariski Topology on $\mathrm{Spec}_{\max}(R)$ is the coarsest topology such that when $\mathfrak{m} \leftrightarrow x$ $f : \mathfrak{m} \to f(x)$ is continuous, where the topology on $\mathbb{R}$ is taken as the cofinite topology. The closed sets in this topology are the vanishing loci of $f \in C(X)$.

- Exercise 1, complete Hartshorne exercise 1.4

1.2 Lecture 2 (Sept 5, 2025)

definition 2. For $T \subset R_n := k[x_1, \dots, x_n]$ and $S \subset k^n$ we define

$$V(T) = \{x \in k^n \mid f(x) = 0, \forall f \in T\} \text{ and } I(S) = \{f \in R_n \mid f(x) = 0, \forall x \in S\}$$

Proposition 2. Suppose k is an uncountable field, and L/k is an extension with $[L : k] \leq \#\mathbb{N}$, then $L = k$.

Proof. Suppose not, then let $x \in L \setminus k$, we find that $\{\frac{1}{x-\lambda} \mid \lambda \in k\}$ is uncountable, so that there must be an algebraic relation. Thus there exist $\mu_i \in k$ with $\sum_1^n \frac{\mu_i}{x-\lambda_i} = 0$, so that $\sum_1^n \mu_j \prod_{i \neq j}(x - \lambda_i) = 0$, but then x is algebraic over k, hence $x \in k$, contradiction. $\square$

Theorem 2. (Nullstellensatz - weak form) $V(T) = \emptyset \implies (T) = R_n$

Proof. We assume here that k is uncountable (this is unnecessary- use Noether Normalization). Since $J := (T) \subset R_n$ is an ideal it is contained in a maximal ideal $\mathfrak{m}$. Then $R_n/\mathfrak{m}$ is a field extension of k with countable dimension, by the previous proposition it is equal to k. It follows that each $x_i \mapsto a_i \in k$ when taking the quotient $R_n \to R_n/\mathfrak{m} = k$, it follows that I vanishes on $(a_1, \dots, a_n)$, so I cannot be contained in a maximal ideal. $\square$

Theorem 3. (Nullstellensatz)

$$IV(J) = \sqrt{J}$$

Proof. By Hilbert's basis theorem, we reduce to the finitely generated case. Let $f \in IV(\{f_1, \dots, f_r\})$, then $(1 - tf, f_1, \dots, f_r) \subset R_n[t]$ has no common zero. Then $g_0(1 - tf) + g_1 f_1 + \cdots + g_r f_r = 1$, and let $N = \max_i\{\deg_t g_i\}$. Taking $t = f^{-1}$, we get $\sum_1^r g_i f_i = 1$, so that for $h_i = f^N g_i \in R_n$ we get $\sum_1^r h_i f_i = f^N \in I \implies f \in \sqrt{I}$. $\square$

The Nullstellensatz gives a bijection

$$\{\text{Affine algebraic varieties}\} \longleftrightarrow \{\text{Finitely generated reduced k-algebras}\}$$
$$V(\sqrt{J}) \longleftrightarrow R_n/\sqrt{J}$$

Moreover, this is a categorical equivalence

$$\mathrm{Var}_k \cong \left(\mathrm{Alg}_k^{\text{reduced}}\right)^{\text{op}}$$

1.3 Lecture 3 (Sept 8, 2025)

definition 3. Let $\pi : S \to X$ be a local homeomorphism, then S is called an étalé space, or a sheaf on X.

Example(s).

1. $\to X$
2. $1 : X \to X$
3. I a set with the discrete topology and the projection $X \times I \to X$
4. A covering space, more explicitly the mobious covering
$$S^1 \to S^1$$
$$z \mapsto z^2$$
5. $U \subset X$ an open set, $\iota : U \to X$
6. If $x \in X$ is a closed point, then we can construct the space $X \sqcup_{X\setminus\{x\}} X = X \times \{1,2\}/\sim$ where $(y,1) \sim (y,2)$ when $y \neq x$. This comes with the codiagonal map $\nabla : X \sqcup_{X\setminus\{x\}} X \to X$, where $\nabla|_{X\times\{i\}} = 1_X, i \in \{1,2\}$.
7. $I \neq \emptyset$, then take $\underset{X\setminus\{x\}}{\sqcup_I} X \xrightarrow{\nabla} X$

definition 4. If $U \subset X$ is an open set, then a section on U is a continuous map $s : U \to S$ such that the following commutes:

$$\begin{tikzcd} & S \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "\iota"] & X \end{tikzcd}$$

The set of sections is denoted $S(U)$ or $\Gamma(U, S)$. If $U = X$, then s is called a global section with notation $S(X)$ or $\Gamma(S)$.

Example(s). (Revisited)

1.
$$S(U) = \begin{cases} 1_\emptyset & U = \emptyset \\ \emptyset & \text{else} \end{cases}$$

2.
$$S(U) = \{\iota_U\}$$

3.
$$S(U) = \hom_{\text{set}}(\pi_0(U), I)$$

4.
$$S(U) = \{f : U \to \mathbb{C} \mid f(z^2) = z\}$$

5.
$$S(U) = \begin{cases} \{\iota\} & x \notin U \\ \{1, 2\} & x \in U \end{cases}$$

6.
$$S(U) = \begin{cases} \{\iota\} & x \notin U \\ I & x \in U \end{cases}$$

This particular example is called the "sky-scraper sheaf"

Proposition 3. There is a étalé space $\mathcal{H}$ over $\mathbb{C}_{\text{EUC}}$ with sections corresponding to holomorphic functions on $\mathbb{C}$.

Proof. The construction of $\mathcal{H}$ as a set is given, alongside its topology. Verifying the claim is exercise 2.

$$\mathcal{H} := \bigsqcup_{z_0 \in \mathbb{C}} \left\{ \sum_1^\infty c_n (z - z_0)^n \mid \text{the series converges in some neighborhood of } z_0 \right\}$$

And define the topology on $\mathcal{H}$ as the strongest topology such that for any open set U, and holomorphic $f : U \to \mathbb{C}$ we have the following map is continuous

$$\begin{aligned} \mathcal{H}f : U &\to \mathcal{H} \\ z_0 &\mapsto \text{The Taylor expansion of } f \text{ at } z_0 \end{aligned}$$

□

2 Exercises

exercise 1. (Hartshorne Exercise 1.4) An algebraically closed field is infinite, moreover the zero sets of polynomials are either k or a finite subset of k. Consider the closed set $V(x - y) \subset \mathbb{A}^2$, then it is an infinite set so if $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, then it must be of the form $\mathbb{A}^1 \times F \cup E \times \mathbb{A}^1$, where $E, F \subset \mathbb{A}^1$ are closed. But for a fixed x or y we have $V(x - y)$ has cardinality 1 which makes this impossible. □

Example(s). (Show that $\mathcal{H}(U) = \{\mathbf{f} : \mathbf{U} \to \mathbb{C} \mid \mathbf{f} \text{ holomorphic}\}$) where we define $\mathcal{H}$ in proposition 3.

A Assigned Readings

B Misc.

definition 5. A ring or algebra is called reduced when it has no non-zero nilpotents.