1. (a) We know that since for any element $x \in G$ that C_x the centralizer of x is a subgroup of G, by orbit stabilizer $\#C_x \#\mathcal{O}_x = \#G$ where $\mathcal{O}_x$ is the orbit of x under the conjugation action. It follows that listing the distinct orbits $\mathcal{O}_{x_i}$,

$$\#G = \#\bigsqcup_i \mathcal{O}_{x_i} = \sum_i \#\mathcal{O}_{x_i}$$

and each $\mathcal{O}_{x_i}|G$ implying that $\#\mathcal{O}_{x_i} \in \{1, p, p^2\}$, if we assume for contradiction that $Z(G) = 1$, then from the above class equation $1 + \sum_{i\geq 2} \#\mathcal{O}_{x_i} = 1 + \sum_{i\geq 2} k_i p = p^2$, taking this equation modulke p we get a contradiction, so that $\#Z(G) = p$ or p^2 in the p^2 case we are done, and in the other case we have $G/Z(G)$ is cyclic, so that any element of G can be written in the form $x^i a$, $a \in Z(G)$, but $(x^i a)(x^j b) = x^{i+j}ab = x^j x^i ba = x^j b x^i a$ which shows that G is abelian, this contradicts $\#Z(G) = p$, so $\#Z(G) = \#G$ and G is abelian. □

(b) A group of order p is cyclic and generated by any of its nontrivial elements, so that all of its elements aside from 1 have order p. Hence $p-1$ such elements. A group of order p^2 is of the form C_p^2 or C_{p^2} by the classification of abelian groups. In the former case, we use the fact that $o(x, y) = \mathrm{lcm}[x, y]$, so as long as either x or y have order p we have an element of order p, this gives $p^2 - 1$ elements of order p. In the latter case, C_{p^2} is generated by any element k with $\gcd(k, p^2) = 1$ the number of these is $\varphi(p^2) = p(p-1) = p^2 - p$, so there are $p^2 - (p^2 - p) - 1 = p - 1$ elements of order p. □

2. We can use orbit stabilizer with S_9 acting on the pearls, the stabilizer of the necklace BBBBWWWRR is clearly $S_4 \times S_3 \times S_2$ has cardinality $12 \cdot 4!$, so there are $\frac{9!}{12\cdot 4!}$ necklaces. □

3.

$$X = \{(g_1, \ldots, g_p) \mid g_i \in G \text{ and } \prod_1^p g_i = 1\}$$

Then we have an action of $\mathbf{F}_p$ on X via $k \cdot (g_i) = (g_{[k+i]})$ where $[n]$ denotes n mod p. Note that when determining an element of X, the first $p-1$ choices are free, meaning there are n^{p-1} choices for the first $p-1$ coordinates (here $n = \#G$), but the last coordinate is fixed as $\left(\prod_1^{p-1} g_i\right)^{-1}$, so X has n^{p-1} elements. In the case where $g_i = g_j, \forall i, j$ the action is trivial, otherwise the orbit of the action has order p. The cardinality of X is the sum of the cardinality of the orbits, letting r be the number of single element orbits, and q the number of p element orbits we get $n^{p-1} = r + qp$ so that since $p|n$ we get $p|(r + qp)$ which implies $p|r$, but $r \neq 0$ since $(1, \ldots, 1)$ has a one element orbit, hence there is another one element orbit $(g, \ldots, g)$, but this means that $g^p = 1$, so that $o(g)|p$, but $g \neq 1$ means that $o(g) \neq 1$ so $o(g) = p$ as desired.

4. Since the groups are not commutative they must have composite order, write $\#G = \prod_1^r p_i$ where $p_r \geq p_{r-1} \geq \cdots \geq p_1$. Then p_1 connot be 11, so p_1 is at most 7, moreover if $p_1 = 7$, then $G = C_{49}, C_7$ or C_7^2 all of which are abelian, so that p_1 is at most 5, if $p_1 = 5$, then once again we get an abelian group since the only possible factorizations are $p_1 = p_2 = 5$ which is abelian by question 1, or $p_2 = 7$, it follows that the subgroup N of order 7 is normal since it has index 5, the smallest prime dividing the order of the group, so this group can't be simple. This implies that $p_1 \in \{2, 3\}$.

Now we note that no group of order pq for p, q both primes is simple, this is immediate from Sylows theorem since if $q > p$, the number of q sylow subgroups must be one hence normal. Now we can look at the case p^2q, if $p > q$, we are done since the Sylow-p group has to be normal, so assume $q > p$, then there are either p^2 or 1 sylow q subgroups, in the latter case we are done and in the former case, these sylow q subgroups all need to have intersection 1 since they are cyclic so we have $p^2(q-1) = p^2q - p^2$ elements of order q, the remaining elements must all be in the same sylow p subgroup having order p^2, so the sylow p subgroup must be normal in this case, contradicting simplicity.

To rule out all groups with 3 prime factors we are thus left with the groups of order 30 and 42, the group of order 42 is easy since the sylow-7 subgroup must be normal by Sylow 2. For the group of order 30, we need only consider the case where there are 6 sylow-5 subgroups and hence 24 elements of order 5, the sylow 3 subgroup must be normal otherwise there would be 10 sylow 3 subgroups adding 20 elements of order 3 giving too many elements, so at this point we have ruled out all groups with 3 prime factors.

Four prime factors (not all $2, 3$) gives us groups of order 56 and 40 in the 56 case we get the sylow-7 subgroup has index 1 or 8, in the index 8 case we get 48 elements of order 7, so the remaining 8 elements must constitute a single sylow-2 subgroup, which must be normal so this case is null. In the order 40 case the sylow 5 subgroup must be normal.

Now the problem has been reduced to ≥ 4 prime factors all being $2, 3$, for now I will appeal to Burnside's theorem, but I should finish it more satisfyingly later.

5. (a) The types of elements in A_5 are $1, (a\,b\,c), (a\,b)(c\,d), (a\,b\,c\,d\,e)$ The conjugacy classes are each contained in their conjugacy classes in S_5, i.e. cycle types, and hence the orders are divisors. We compute $\#\mathcal{O}_1 = 1$, the normalizer of elements of the form $(a\,b)(c\,d)$ are elements of A_5 sending $\{a, b\} \to \{a, b\}$ and $\{c, d\} \to \{c, d\}$ or $\{a, b\} \to \{c, d\}$ and $\{c, d\} \to \{a, b\}$, of which there are 4 elements

(b) A normal subgroup of A_5 must be a union of conjugacy classes (including 1) with order dividing $\#A_5 = 60$, by reading the conjugacy class sizes in (a), no such union of conjugacy classes exists.

(c)

(d)

(e)

6. (a) Suppose $gx = y$, then for $h \in \text{Stab}_x$ we have $ghg^{-1}y = ghx = gx = y$, so that $ghg^{-1} \in \text{Stab}_y$, by rewriting the equation $g^{-1}y = x$, we see this conjugation is onto since it has inverse $h \mapsto g^{-1}hg$ by the same argument. $\square$

(b) If $F : X \to Y$ is a G-set isomorphism, then suppose $g \in \text{Stab}(x)$, then $gF(x) = F(gx) = F(x)$, moreover if $h \in \text{Stab}(y)$, then $hF(x) = F(x)$