1. (Durrett 2.5.3) Define $Y_n = \frac{\sin(n\pi t)}{n} X_n$, then since $X_n \sim N(0,1)$ we get $\operatorname{Var} Y_n = \frac{|\sin(n\pi t)|}{n^2} \leq 1/n^2$. Then since $\sum_1^\infty \operatorname{Var} Y_n < \infty$ and $Y_1, Y_2, \dots$ are independent, we get from the consequence of Kolmogorov maximal inequality that $\sum_1^\infty Y_n$ converges almost surely. $\square$

2. (Durrett 2.5.6) We can use the Kolmogorov 3-series test. First notice,

$$E[\psi(X_n)] = E\left[|X_n| 1_{|X_n|>1} + X_n^2 1_{|X_n|\leq 1}\right]$$

So in particular by the comparison test,

$$\sum_1^\infty E\left[|X_n| 1_{|X_n|>1}\right] < \infty \qquad\qquad \sum_1^\infty E\left[X_n^2 1_{|X_n|\leq 1}\right] < \infty$$

The latter is the series $\sum_1^\infty \operatorname{Var}(X_n 1_{|X_n|\leq 1})$, another of the series is relatively free, namely

$$\sum_1^\infty P(|X_n| > 1) \leq \sum E\left[|X_n| 1_{|X_n|>1}\right] < \infty$$

Finally, we have $|E[X_n 1_{|X_n|>1}]| \leq E\left[|X_n| 1_{|X_n|>1}\right]$, so that $\sum_1^\infty E\left[X_n 1_{|X_n|>1}\right]$ converges, since

$$0 = \sum_1^N E\left[X_n\right] = \sum_1^N E\left[X_n 1_{|X_n|>1} + X_n 1_{|X_n \leq 1|}\right] = \sum_1^N E\left[X_n 1_{|X_n|>1}\right] + \sum_1^N E\left[X_n 1_{|X_n \leq 1|}\right]$$

we have

$$\sum_1^\infty E\left[X_n 1_{|X_n \leq 1|}\right] = -\sum_1^\infty E\left[X_n 1_{|X_n|>1}\right] \in \mathbb{R}$$

Thus X_n satisfy the hypothesis of the Kolmogorov 3-series test and converge almost surely. $\square$

3. (Durrett 2.5.8) Write $Y = \log^+|X_1|$ and assume first that $EY = \infty$, then for any $c > 0$, we find that

$$EY = \int P(Y > t)dt = \sum_1^\infty \int_{(n-1)c}^{nc} P(Y > t)dt \leq \sum_0^\infty cP(Y > nc)$$

which in turn implies that $\sum_1^\infty P(\log^+|X_n| > nc) = \sum_1^\infty P(Y > nc) = \infty$, therefore by Borel-Cantelli (ii), we find that $P(X_n > e^{nc} \text{ i.o.}) = 1$, so for any $c \neq 0$ we find that $|X_n|c^n > 1$ i.o. almost-surely. This of course implies that $\sum_1^\infty |X_n|c^n$ has radius of convergence zero almost surely.

Now, conversely suppose $E\log^+|X_n| < \infty$, and let $0 < c < 1$, then choose $\gamma > 0$ so that $e^\gamma c < 1$ we get another layer cake estimate,

$$EY = \int P(Y > t)dt = \sum_1^\infty \int_{(n-1)\gamma}^{n\gamma} P(Y > t)dt \geq \sum_1^\infty \gamma P(Y > n\gamma)$$

Since $\sum_1^\infty \gamma P(Y > n\gamma) = \sum_1^\infty P(\log^+|X_n| > n\gamma)$ by Borel Cantelli, we have $|X_n| > e^{n\gamma}$ only finitely many times. Letting N so that $n \geq N$ implies $|X_n| \leq e^{n\gamma}$, we get

$$\sum_1^\infty c^n|X_n| \overset{\text{a.s.}}{\leq} \sum_1^{N-1} c^n|X_n| + \sum_N^\infty (ce^\gamma)^n < \infty$$

So that the series converges a.s. for any $c < 1$. Now letting $c > 1$, we once again use Borel-Cantelli (ii), we first check the hypothesis

$$\sum_1^\infty P(|X_n|c^n > 1) = \sum_1^\infty P(|X_1| > \frac{1}{c^n}) = \infty$$

Since $\lim_{n\to\infty} P(|X_1| > \frac{1}{c^n}) = P(X_1 \neq 0) > 0$ by assumption, it follows that $P(|X_n|c^n > 1 \text{ i.o.}) = 1$, so the series diverges a.s. for $c > 1$. $\square$

4. (Durrett 3.2.9) We can use a mesh to show that $||F - F_n||_u = 0$. Namely, let $\epsilon > 0$, then we can choose x_0 so that $F(x_1) < \epsilon$ and y so that $F(y) > 1 - \epsilon$