1. **(a)** Define $F(x) = (\det \mathrm{Hess}_x(f))^2 + \sum_1^k \left(\frac{\partial}{\partial x_j} f(x)\right)^2$, its clear that both the hessian and sum terms are non-negative (since they are squares of real values). Now assume first that $F > 0$ on U, if f has no critical points on U we are done. Now if p is a critical point for f, we have that $d_p f : \mathbb{R}^k \to \mathbb{R}$ is not surjective, where $d_p f = \left(\frac{\partial}{\partial x_1} f \quad \cdots \quad \frac{\partial}{\partial x_k} f\right)$, this matrix is of course surjective so long as atleast one $\frac{\partial}{\partial x_j} f \neq 0$, so at a critical point we get $\frac{\partial}{\partial x_j} f = 0$ for all j whence $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p)\right)^2 = 0$, by our assumption of $F > 0$, this implies that $(\det \mathrm{Hess}_p(f))^2 > 0$, and since p was an arbitrary critical point we can conclude that f is morse. Conversely, if f is morse, then by the computation above, at any regular value, p, we must have some $\frac{\partial}{\partial x_j} f \neq 0$, which implies that $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p)\right)^2 > 0$, which implies $F(p) > 0$ by non-negativity of the hessian term. In the case that p is a critical point, we know that $\det \mathrm{Hess}_p f \neq 0$, so that $(\det \mathrm{Hess}_p f)^2 > 0$, and $F(p) > 0$, since all points are either regular values or critical points we are done. $\square$

(b) Let f be morse, and H a homotopy with $H(x, 0) = f(x)$, moreover we can denote $f_t = H(-, t)$. Then since M is compact we can pick a finite covering by charts $(V_1, U_1, \phi_1), \ldots, (V_n, U_n, \phi_n)$. Now let η_i be a partition of unity suboordinate to these charts, we can define

$$F(x,t) = \sum_1^n \eta_i(x) \left((\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2 \right)$$

F is smooth, due to closure of smooth functions under sums, products and compositions as well as the fact that the hessian and partials vary smoothly with t, this can be seen since the coordinates of these maps are a subset of the coordinates of $H \circ (\phi^{-1}, 1_{[0,1]})$, which of course has smooth partials and hessian. To see that $F(-, 0) > 0$ on M, note that since f_0 is morse and ϕ_i^{-1} are diffeomorphisms, each summand $\left((\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2\right) > 0$ by part (a). Since M is compact, the function $\hat{F}(t) = \inf_{x \in M} F(x, t)$ is continuous, and it has $\hat{F}(0) > 0$, since $F(-, 0)$ is continuous and positive on a compact set (which implies it attains its infimum), then by continuity of $\hat{F}$, there exists some $\delta > 0$, such that $t < \delta$ implies $\hat{F}(t) > 0$. So to show that morse functions are generic, it will suffice to show that $F(-, t) > 0$ implies that f_t is morse, since then we get for $t < \delta$ each $f_t = H(x, t)$ is morse. To check this, note that since the ϕ_i^{-1} are diffeomorphisms, we have that $\mathrm{Hess}_x f_t$ is invertible iff $\mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1}$ for each i (since we have a partition of unity we can always assume $x \in V_i$), moreover f_t is regular if and only if $f_t \circ \phi_i$ is, so at critical points of f_t, we get that for each i, $\sum_{j=1}^k (\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x))^2$ vanishes. It follows that $(\det \mathrm{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 > 0$ for all i, when x is a critical value of f, and since this Hessian is invertible iff the hessian of f_t is, we find that f_t has invertible Hessian at all its critical points, assuming $F(-, t) > 0$ on M which completes the proof. $\square$

2. **(a)** Let $\Delta_{ij} = \{(x_1, \ldots, x_r \mid x_i = x_j)\}$, its clear that by reordering the factors we have each $\Delta_{ij} \cong \Delta_{1,2}$, moreover $\Delta_{12} = \Delta \times M^{r-2}$ where Δ denotes the diagonal of $M \times M$, where we take the convention $M^{r-2} = \emptyset$ if $r - 2 = 0$. Since Δ is a submanifold of $M \times M$, we can realize it as the image of an embedding $e : \Delta \hookrightarrow M \times M$, so that $e \times 1_M^{r-2} : \Delta_{ij} \to M^r$ is an embedding and thus each Δ_{ij} is a submanifold, it is clear from definition that $\Delta = \bigcup_{1 \leq i < j \leq r} \Delta_{ij}$. For ease of notation I will prove the following if N is a manifold with submanifolds $Z_1, \ldots, Z_n$, and X is compact, then the class of maps $f : X \to N$ transverse to all of $Z_1, \ldots, Z_n$ is generic. Of course once this is proved we will be done by replacing the Z_j with Δ_{ij} and M with M^r.

We proceed as in the proof of transversality for a single submanifold, recalling that:

(1) If S is a manifold and $F : X \times S \to N$ has $F \pitchfork Z$, then the set of $t \in S$ such that $f_t = F(-, t)$ has $f_t \pitchfork Z$ has compliment measure zero on S. (Lecture 13 Theorem 1.3)

Now let $f_0 : X \to N$ be smooth, we can start by whitney embedding $N \hookrightarrow \mathbb{R}^\ell$ for some ℓ, and take U to be a regular neighborhood of N in $\mathbb{R}^\ell$ with projection π. Then since U is an open neighborhood, there exists a smooth function $\epsilon : N \to (0, \infty)$, so that if $||x|| < \epsilon(p)$, then $p + x \in U$. From this we get a smooth map

$$\begin{aligned} F :& X \times \mathbb{R}^\ell \to N \\ (p,t) &\mapsto \pi\left(f_0(p) + \epsilon \frac{t}{1+||t||^2}\right) \end{aligned}$$

Since π is a submersion by the regular neighborhood theorem, we just need to check that $(p,t) \mapsto f_0(p) + \epsilon\frac{t}{1+||t||^2}$ is a submersion to get that the composition is. This follows from fixing p, the map is given by a diffeomorphism from $\mathbb{R}^r$ to a ball radius $\epsilon(f_0(p))$, so the matrix giving the jacobian has a rank ℓ minor, and is hence onto mapping to $\mathbb{R}^\ell$. Since F is a submersion it is transverse to each Z_j, and hence the set of $t \in \mathbb{R}^\ell$ such that $F(-,t) \not\pitchfork Z_j$ has measure zero for each j by (1), since the union of finitely many measure zero sets has measure zero, we can conclude that $\{t \in \mathbb{R}^\ell \mid F(-,t) \not\pitchfork Z_j \text{ for some } j\}$ has measure zero. It follows that the set of $t \in \mathbb{R}^\ell$ such that $F(-,t) \pitchfork Z_j$ for all j is dense, now since $F(-,0) = f_0$ we find that F is a sufficient smooth map to prove genericty. □

(b) Let $p = (p_1, \dots, p_r), q = (q_1, \dots, q_r) \in \mathrm{Conf}_r(M)$, then we can identify these points with the same coordinates in M^r. To see that there is a path between p and q, we use path connectedness of M, which gives paths $\gamma_i' : [0,1] \to M$ satisfying $\gamma_i'(0) = p_i$ and $\gamma_i'(1) = q_i$. Now by (1), the class of maps transverse to each of the $Z_1, \dots, Z_n$ each have measure

$$\begin{aligned} \gamma_1(t) &= (\gamma_1'(t), p_2, \dots, p_r) \\ \gamma_2(t) &= (q_1, \gamma_2'(t), p_3, \dots, p_r) \\ &\vdots \\ \gamma_r(t) &= (q_1, \dots, q_{r-1}, \gamma_r'(t)) \end{aligned}$$

We can join these paths together continuously (but not necessarily smoothly) by taking

$$\gamma(t) = \begin{cases} \gamma_1(rt) & t \in [0, \frac{1}{r}) \\ \gamma_2(r(t - 1/r)) & t \in [\frac{1}{r}, \frac{2}{r}) \\ \vdots & \\ \gamma_r((t - \frac{r-1}{r})) & t \in [\frac{r-1}{r}, 1] \end{cases}$$

This gives a continuous path between p and q, which implies the existence of a smooth path, so replace γ with this smooth path. Now since $\Delta_{ij} \cong \Delta \times M^{r-2}$, we have $\dim M^r - \dim \Delta_{ij} \geq \dim M = k \geq 2$, and since $X = [0,1]$ is dimension 1, we have a map $f : X \to M$ is transverse to Δ_{ij} iff $\mathrm{Im}\, f \cap \Delta_{ij} = \emptyset$. So part (a) tells us that there is some homotopy $H : [0,1]^2 \to M^r$ with $H(-,0) = \gamma$, and for any $\epsilon > 0$, there exists some $t < \epsilon$ with $H(-,t) \pitchfork \Delta_{ij}$ for all i,j, but by the dimension argument I just gave, this means that $H(-,t) : [0,1] \to \mathrm{Conf}_r(M)$ for all such t. Now let V_p and V_q be connected open sets in $\mathrm{Conf}_r(M)$ containing p and q respectively (note since $\mathrm{Conf}_r(M)$ is an open subset of M these can be identified with open subsets of M), since H is smooth, we have $H(0,t) \in V_p$ for all $t < \delta_p$ for some $\delta_p > 0$, similarly for $H(1,t) \in V_q$ we get some δ_q. Taking $\delta = \min\{\delta_p, \delta_q\}$, we find that for $t < \delta$ that $H(0,t) \in V_p$, and $H(1,t) \in V_q$, now use genericity to get some t with $t < \delta$ and $H(-,t)$ lying in $\mathrm{Conf}_r M$, denote $p' = H(0,t)$ and $q' = H(1,t)$. Since connected components of manifolds are path connected, we have paths γ_p, γ_q in $\mathrm{Conf}_r M$ connecting p to p' and q' to q. It follows that concatenating gives a smooth path f

$$f(t) = \begin{cases} \gamma_p(3t) & t \in [0, 1/3) \\ \gamma(3(t - 1/3)) & t \in [1/3, 2/3) \\ \gamma(3(t - 2/3)) & t \in [2/3, 1] \end{cases}$$

between p and q in $\mathrm{Conf}_r M$, thus there exists a smooth path between p and q in $\mathrm{Conf}_r M$, since p, q were arbitrary $\mathrm{Conf}_r M$ is path connected.

(c) Part (b) gives the existence of such a path provided that there is some $q \in \operatorname{Conf}_r M \cap U$, i.e. We only need to check that $\operatorname{Conf}_r M \cap U \neq \emptyset$. This is easy to see by Sard's theorem, since $(\operatorname{Conf}_r M)^c$ is a union of finitely many sets (the Δ_{ij}) with dimension strictly less than k^r. Now letting $F : U \xrightarrow{\cong} B_k$ be the diffeomorphism to the unit ball (and hence $F^r : U^r \xrightarrow{\cong} B_k$ given by F in each coordinate is a diffeomorphism). Since $\Delta_{ij} \cap U^r$ is a submanifold of U^r (since U^r has full dimension they are automatically transverse, so their intersection is a submanifold), and once again since U^r has full dimension, we have $\dim \Delta_{ij} = \dim \Delta_{ij} \cap U^r < \dim U^r$, we then get $F^r(U^r \cap \Delta_{ij})$ is a submanifold of B_k^r of strictly smaller dimension, thus having measure 0 by sards theorem, this of course implies that

$$F^r(U^r \cap (\operatorname{Conf}_r M)^c)F^r(U^r \cap \bigcup_{1 \leq i < j \leq r} \Delta_{ij}) = F^r(\bigcup_{1 \leq i < j \leq r} U^r \cap \Delta_{ij}) = \bigcup_{1 \leq i < j \leq r} F^r(U^r \cap \Delta_{ij})$$

is a finite union of measure zero sets, hence measure zero, so that since U^r has positive r-dimensional measure, there does indeed exist some $x \in F^r(U^r \setminus (\operatorname{Conf}_r M)^c) = F^r(U^r \cap \operatorname{Conf}_r M)$, so that we can take $q := F^{-1}(x) \in U^r \cap \operatorname{Conf}_r M$ as desired. $\square$

(d) We proved in the previous subpart, there is a path γ lying in $\operatorname{Conf}_r M$ connecting p to some $q \in U$, this allows us to define an isotopy by taking $e_t(p_i)$ to be the i-th coordinate of $\gamma(t)$, since e_t has a zero dimensional, compact domain we only need to check that its injective in order to being an embedding, but injectivity follows from $\gamma(t) \in \operatorname{Conf}_r M$. Hence e_t is an isotopy of embedding between $e_0 = 1_{\{p_1,\dots,p_r\}}$ and $e_1 : p_i \mapsto q_i \in U$. The isotopy extension theorem gives an isotopy of compactly supported diffeomorphism h_t, with $h_0 = 1$, and $h_t e_0 = e_t$, so that $h_1(p_i) = q_i \in U$. It follows that h_1^{-1} is a compactly supported diffeomorphism satisfying $\{p_1, \dots, p_r\} \in h_1^{-1}(U) \supset \{h_1^{-1}(q_i)\}_1^r$, moreover its isotopic to the identity via $H(x,t) = h_{1-t}^{-1}(x)$, since $1_M^{-1} = 0$. $\square$