1.

$$\Im(\sigma) = \Im\left(\frac{(a\tau+b)(c\overline{\tau}+d)}{||c\tau+d||^2}\right) = \Im\left(\frac{ac||\tau||^2+bd+ad\tau+bc\overline{\tau}}{||c\tau+d||^2}\right)$$
$$= \frac{1}{||c\tau+d||^2}\Im(ad\tau+bc\overline{\tau}) = \frac{1}{||c\tau+d||^2}(ad-bc)\Im(\tau) = \frac{1}{||c\tau+d||^2}\Im(\tau)$$

Then since $||c\tau+d||^2 > 0$ and $\Im(\tau) > 0$ we conclude that $\Im(\sigma) > 0$.

Now define

$$f : \mathbb{C} \to \mathbb{C}/(\mathbb{Z}+\sigma\mathbb{Z})$$
$$z \mapsto \frac{z}{c\tau+d} + \mathbb{Z} + \sigma\mathbb{Z}$$

To see this descends to a holomorphic map on the torus, we need to check it is periodic with respect to $\mathbb{Z}+\tau\mathbb{Z}$, so we want to check that $f(z+n\tau+m) = f(z)$ for any $n, m \in \mathbb{Z}$. Since f is linear it will be sufficient to show that $f(\tau)$ and $f(1)$ both lie in the lattice $\mathbb{Z}+\sigma\mathbb{Z}$. This is just a computation,

$$\tau = (ad-bc)\tau + bd - bd = d(a\tau+b) - b(c\tau+d) \tag{1}$$
$$1 = (ad-bc) + ac\tau - ac\tau = -c(a\tau+b) + a(c\tau+d) \tag{2}$$

So that (1) gives us $f(\tau) = d\sigma - b$ and (2) gives $f(1) = -c\sigma + a$ both are in $\mathbb{Z}+\sigma\mathbb{Z}$, so that f descends to the torus X_τ. To see that f is a biholomorphism just take $\mathbb{C} \to X_\tau$ via $z \mapsto (c\tau+d)z$, this descends to a holomorphic map on X_σ since $\sigma \mapsto \tau$, and $1 \mapsto c\tau+d$ are both in $\mathbb{Z}+\tau\mathbb{Z}$, moreover this is clearly the inverse of f. □

2. I will use from class the identification of two copies of $\mathbb{C}$, the first being labelled (I), and the second (II), so that X is defined by gluing (I) to (II) along the line segments from 0 to 1, and from λ to ∞ along the line through 1 and λ as in lecture. Now on a neighborhood of infinity (more explicitly on $U = \{z \mid |z| > \max\{1, |\lambda|\}\}$ in both (I) and (II)) we can take local coordinate t, so that

$$t^2 = \frac{1}{z} \qquad t = \begin{cases} \sqrt{z} & z \in (I) \\ -\sqrt{z} & z \in (II) \end{cases} \tag{3}$$

This allows us to compute

$$2tdt = -\frac{1}{z^2}dz = -t^4dz \tag{4}$$
$$dz = -2t^{-3}dt \tag{5}$$

Moreover we also have defined $\omega^2 = z(z-1)(z-\lambda)$ with sign conventions

$$\omega = \begin{cases} \sqrt{z(z-1)(z-\lambda)} & z \in (I) \\ -\sqrt{z(z-1)(z-\lambda)} & z \in (II) \end{cases} \tag{6}$$

Then using our local t-coordinates we have

$$\omega^2 = \frac{1}{t^2}\left(\frac{1}{t^2}-1\right)\left(\frac{1}{t^2}-\lambda\right) = \frac{1}{t^6}(1-t^2)(1-\lambda t^2)$$

Using the sign conventions for t and ω in (3) and (6) this is consistent with

$$\frac{1}{\omega} = \frac{t^3}{(1-t^2)(1-\lambda t^2)} \tag{7}$$

Substituting (5) into (7) yields the differential in local t-coordinates near infinity.

$$\frac{dz}{\omega} = \frac{-2dt}{(1-t^2)(1-\lambda t^2)} \tag{8}$$

Since $|z| = \frac{1}{|t^2|} > \max\{|1|, |\lambda|\}$ this is a holomorphic form on U, which is clearly non-vanishing. □

3. (a) Consider an arbitrary meromorphic function $f : X \to \mathbb{C}$, we may denote the poles of f as $P_1, \dots, P_r$ with multiplicities $n_1, \dots, n_r$. Away from its poles, f is holomorphic as a map to $\mathbb{C}$, and hence to $\mathbb{P}^1$, so we need only consider the behaviour of f near its poles, we should also note that f is well defined as a set map to $\mathbb{P}^1$ by taking $f(P_j) = \infty$ for each P_j. Now let (U, ψ) be a chart containing P_1, by possibly shrinking U we may assume that $0 \notin f(U)$, so that $f(U)$ is contained in the chart of $\mathbb{P}^1$ containing ∞, moreover for simplicity we may assume that $0 \in U$ with $\psi(0) = P_1$. Now since f is meromorphic on X with pole P_1 of multiplicity n_1 we have

$$f \circ \psi(z) = \sum_{-n_1}^{\infty} a_k z^k \tag{9}$$

with $a_{-n_1} \neq 0$. Now since the image lies entirely of the chart of $\mathbb{P}^1$ containing ∞, to consider this as a map to $\mathbb{P}^1$ we compose with the coordinate chart φ taking $\mathbb{C} \to (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$ on $\mathbb{P}^1$, which takes $z \mapsto 1/z$, the point ∞ corresponds to 0 in this coordinate chart. This gives the following expression for f around P_1 in coordinates:

$$\varphi^{-1} \circ f \circ \psi : z \mapsto \begin{cases} (\sum_{-n_1}^{\infty} a_k z^k)^{-1} & z \neq 0 \\ \varphi^{-1}(f \circ \psi(0)) = \varphi^{-1}(f(P_j)) = \varphi^{-1}(\infty) = 0 & z = 0 \end{cases} \tag{10}$$

Now we may simplify $(\sum_{-n_1}^{\infty} a_k z^k)^{-1} = \frac{z^{n_1}}{\sum_0^{\infty} a_{k-n_1} z^k}$, which of course takes value 0 at $z = 0$, which simplifies the piecewise expression for f in charts given by (10) to $\frac{z^{n_1}}{\sum_0^{\infty} a_{k-n_1} z^k}$. Since the denomenator is a nonvanishing (near zero) holomorphic function convergent near zero, we find that $\varphi^{-1} \circ f \circ \psi$ is a holomorphic map $X \to \mathbb{P}^1$ in a chart around P_1, the same argument works for charts around $P_2, \dots, P_n$ so that f is indeed a holomorphic function between X and $\mathbb{P}^1$. $\square$

(b) This question only works if d counts multiplicity and if f is nonconstant, so consider d counting $f^{-1}(p)$ with multiplicity for a nonconstant holomorphic $f : X \to \mathbb{P}^1$. By compactness, it suffices to show that $f^{-1}(p)$ is discrete to conclude that d is finite (this follows since so long as f is not constant the degree of any single one of the preimages is finite). Moreover, since a finite union of discrete sets is discrete, we may cover X in a finite number of charts $\{(U_j, \varphi_j)\}_1^r$ and check that $f|_{U_j}^{-1}(p)$ is discrete for each j. Since coordinate charts are diffeomorphisms, we can once again reduce the problem to checking each $(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}\{\psi^{-1}(p)\}$ is discrete for some ψ corresponding to a coordinate chart containing p, with $\psi^{-1} \circ f|_{U_j} \circ \varphi_j$ simply being a holomorphic function $\varphi^{-1}(U) \subset \mathbb{C} \to \mathbb{C}$, this reduces to the fact that the zeroes of a holomorphic function are discrete, and

$$(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}(\psi^{-1}(p)) = \{z \in \varphi^{-1}(U_j) \mid \psi^{-1} \circ f|_{U_j} \circ \varphi_j(z) - \psi^{-1}(p) = 0\}$$

is the set of zeroes of a holomorphic function. Thus by the reductions above, $d < \infty$.

Now to see that d is constant, I will show that d is constant in some open set U containing p, since this will hold for any $p' \in X$ this implies that the degree of a point is a continuous map $\mathbb{P}^1 \to \mathbb{Z}$, so that since $\mathbb{P}^1$ is connected it must be the constant map. To see that d is locally constant, take $\{p_0, \dots, p_r\} = f^{-1}(p)$, so that all together $p_1, \dots, p_r$ have total multiplicity d. Now we can for every $x \in X \setminus \{p_1, \dots, p_r\}$ take some chart (U_x, z_x) with $x \in U_x \subset X \setminus \{p_1, \dots, p_r\}$, and for each p_j we can take disjoint coordinate charts (U_j, z_j) with $p_j \in U_j$. Now let (V, ψ) be a coordinate chart in $\mathbb{P}^1$ with $\psi(p) = 0$, by possibly taking the intersection with $f^{-1}(V)$ we can assume each of the U_x and U_j map into V under f. Since the degree of a map $\mathbb{C} \to \mathbb{C}$ is locally constant, we can (once again by possibly shrinking the open sets) assume that for each $x \in X \setminus \{p_1, \dots, p_r\}$ we have $\mathrm{ord}_w(\psi^{-1} \circ f(z_x)) = 0$ for all $w \in U_x$, and for each j, $\mathrm{ord}_w(\psi^{-1} \circ f(z_j)) = \mathrm{ord}_{p_j}(\psi^{-1} \circ f(z_j))$ for all $w \in U_j$. Only now that we have reduced our open sets to be sufficiently nice do we apply the compactness assumption to get a finite subcover $U_{x_1}, \dots, U_{x_N}, U_1, \dots, U_r$ for X, note that since each p_j lies only in U_j, we must have each U_j included in our open cover. By the open mapping theorem we know that each $\psi^{-1} \circ f(z_x)$ (or z_j) is open, so that $Y := \bigcap_1^N \psi^{-1} \circ f(z_x(U_{x_k})) \bigcap_1^r \psi^{-1} f(z_j(U_j))$ is an open set containing since ψ is a homeomorphism we can apply it to get $\psi(Y)$ is an open subset of $\mathbb{P}^1$ containing p. Moreover the degree of f is locally constant as a map $X \to \psi(Y)$, since the degree is zero on each U_k, and locally constant on each U_j from the version of this result on $\mathbb{C}$. Explicitly, since the U_j are disjoint and $f|_{U_{x_k}} \cap \psi(Y) = \emptyset$, we have for any $p' \in \psi(Y)$ that $f^{-1}(p') = \bigsqcup_1^r f|_{U_j}^{-1}(p')$, and moreover that the number of points with multiplicity d is preserved since the multiplicity of f is constant on each U_j. $\square$

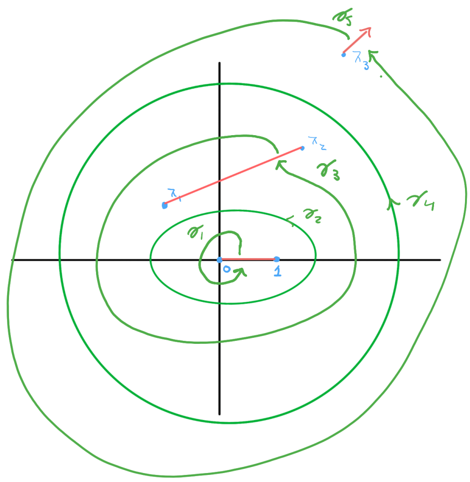
(c) z has degree 2, and w has degree 3. As proof, by parts (a) and (b) it suffices to compute the preimage for a single point in which it is convenient. For z this is very simple, since for any point x away from the identifications $z^{-1}(x)$ consists of two copies of x, one in each glued copy of $\mathbb{C}$ (i.e. choose any value of x on the imaginary line distinct from λ). In the case of w, we count its zeroes, $w^2 = x(x-1)(x-\lambda)$ implies that w vanishes at each of $0, 1$ and λ, since these are glued points, we do not getting the same double counting as in the z case and this gives degree 3, this is also up to multiplicity, since w has vanishing of order 1 at each of these points, checking at zero (since 1 and λ are similar) we have the local coordinate $t^2 = z$, defined via

$$t = \begin{cases} \sqrt{z} & \text{on } (I) \\ -\sqrt{z} & \text{on } (II) \end{cases}$$

so that $w = t\sqrt{(t^2-1)(t^2-\lambda)}$ indeed has order one vanishing. $\square$

4. X is a surface of Genus 2. Define line segments $L_1 = [0,1]$, L_2 the component of the line determined by λ_1 and λ_2 between λ_1 and λ_2, and L_3 the component of the line segment between λ_2 and λ_3 from λ_3 to ∞. Here we assume the points $\{0, 1, \lambda_1, \lambda_2, \lambda_3\}$ are arranged so that none of the line segments L_1, L_2 and L_3 intersect. Since X is given as the analytic continuation of $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$, we consider the monodromy of these square root functions around these points

- γ_1 is a curve with winding number 1 around 0, and winding number zero around $1, \lambda_1, \lambda_2, \lambda_3$
- γ_2 is a curve with winding number 1 around 0 and 1, and winding number zero around $\lambda_1, \lambda_2, \lambda_3$
- γ_3 is a curve with winding number 1 around $0, 1$ and λ_1 and winding number zero around λ_2 and λ_3
- γ_4 is a curve with winding number 1 around $0, 1, \lambda_1$ and λ_2 and winding number zero around λ_3
- γ_5 is a curve with winding number 1 around all points



We record the monodromy of each of the square root constituents around these curves:

- On γ_1 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}$, with the $\sqrt{z-1}, \sqrt{z-\lambda_1}, \sqrt{z-\lambda_2}, \sqrt{z-\lambda_3}$ having no monodromy.

- On γ_2 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}$ and $\sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$ with no monodromy for the other factors.
- On γ_3 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$, and $\sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}$ with no monodromy for the other factors.
- On γ_4 we get monodromy:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \text{ and } \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}$$

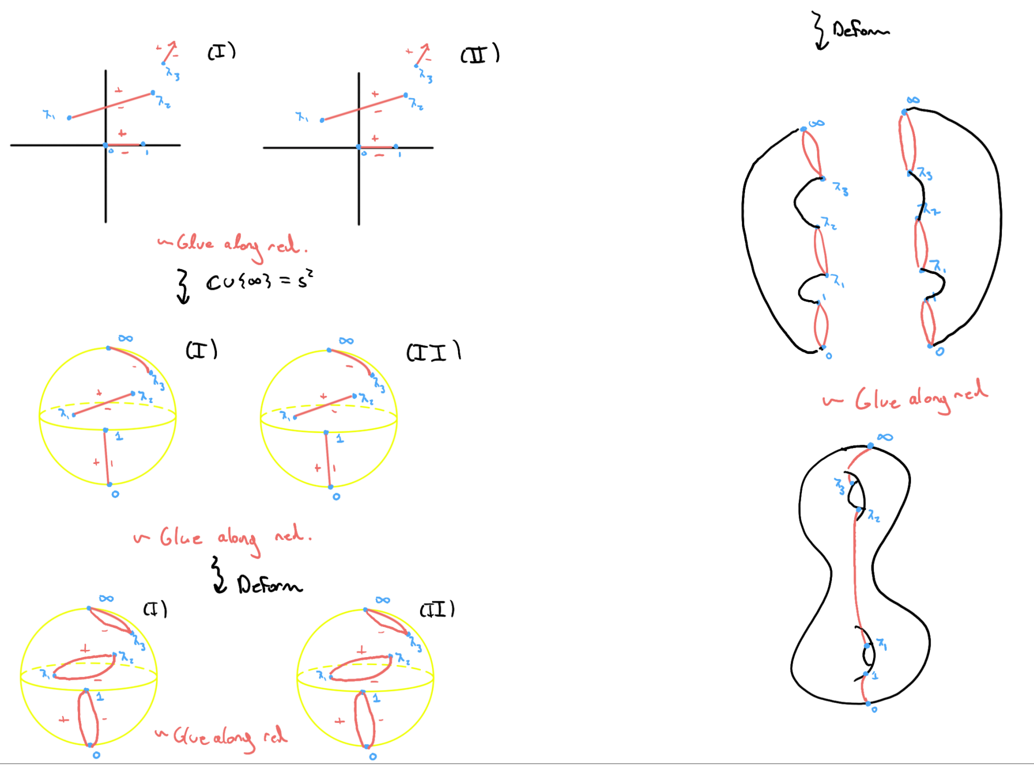
  fixing the last factor.
- Finally, on γ_5 we get monodromy on each of the components:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}, \sqrt{z-\lambda_3} \rightsquigarrow -\sqrt{z-\lambda_3}$$

  The monodromy's cancel travelling around γ_j for j even, but not for j odd, thus to analytically extend this branch of the square root we once agin take two copies of $\mathbb{C}$, (I) and (II) and gluing along L_1, L_2, L_3, and analytically continue $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$ by defining function w with $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$ and

$$w = \begin{cases} \sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (I)} \\ -\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (II)} \end{cases} \tag{11}$$

By sketching this glueing we see that X is a surface of genus 2 (please ignore the yellow and pretend that its black SORRY!!).



The meromorphic function z simply projects (I) and (II) coordinates to $\mathbb{C}$, similarly to part (c) of problem (3), we can pick a point $x \in \mathbb{C}$ away from L_1, L_2, L_3 where the gluing occurs so that $z^{-1}(x)$

is trivially 2 copies of x, one in (I) and one in (II), using question 3 this degree is well defined so z has degree 2. Also similarly to problem (3c), we count calculate the degree of w as $\#w^{-1}(0)$, which has points $0, 1, \lambda_1, \lambda_2$ and λ_3, to check that each of these zeroes has multiplicity one we check in a local coordinate around 0, since the proof is the same for the other points. So take local coordinate t around zero (the chart being $U = B_\epsilon(0)$ in (I) and (II) with $\epsilon < \min\{1, |\lambda_1|, |\lambda_2|, |\lambda_3|\}$) with $t^2 = z$, and $t = \sqrt{z}$ on (I) and $t = -\sqrt{z}$ on (II), it follows that on U we have $w = t\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$, with $\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$ holomorphic and nonvanishing on U, so that indeed the vanishing of w at 0 is multiplicity one. □

5. There are four distinct cases, firstly we may have $Q_1 \in \{0, 1, \lambda\}$ and Q_2 generic, secondly we may have Q_1 and $Q_2 \in \{0, 1, \lambda\}$, the other cases deal with the point infinity, namely $Q_1 = \infty$ and Q_2 generic, or $Q_1 \in \{0, 1, \lambda\}$ and $Q_2 = \infty$.

(Case 1) Assume that $Q_1 = 0$, and let $Q_2 \in X \setminus \{0, 1, \lambda, \infty\}$, then we define the meromorphic one form to be

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}z + w}{z \cdot (z - z(Q))} \frac{dz}{w} \tag{12}$$

Near Q, we may work in z charts to get the expression

$$\omega_{0,Q} = \frac{1}{z - z(Q)} \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{dz}{w} \tag{13}$$

so that the residue is just

$$\operatorname{Res}(\omega_{0,Q}, Q) = \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{1}{w}\bigg|_Q = \frac{2}{z(Q)} \tag{14}$$

To evaluate the reside at zero, we take charts near zero by taking t so that $t^2 = z$ and $t = \pm\sqrt{z}$ on (I) and (II) respectively. we can rewrite the form $\omega_{0,Q}$ given by (12) in t coordinates, this gives the local expression

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}t^2 + t\sqrt{(t^2-1)(t^2-\lambda)}}{t^2(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \tag{15}$$

$$= \frac{1}{t}\left(\frac{\frac{w(Q)}{z(Q)}t + \sqrt{(t^2-1)(t^2-\lambda)}}{(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}}\right) \tag{16}$$

From this we can calculate the residue,

$$\operatorname{Res}(\omega_{0,Q}, 0) = \frac{\sqrt{\lambda}}{-z(Q)} \frac{2}{\sqrt{\lambda}} = -\frac{2}{z(Q)} \tag{17}$$

Since $\operatorname{Res}(\omega_{0,Q}, 0) = -\operatorname{Res}(\omega_{0,Q}, Q)$ we can simply normalize to get ± 1. Finally we check these are indeed the only poles of our form, the only points of interest are $\tilde{Q}$ and ∞ where $\tilde{Q}$ is the unique distinct point from Q with $z(\tilde{Q}) = z(Q)$, but plugging in $\tilde{Q}$ the numerator becomes $\frac{w(Q)}{z(Q)}z(\tilde{Q}) + w(\tilde{Q}) = w(Q) - w(Q) = 0$, so that there is indeed no pole at $\tilde{Q}$, to see there is no pole at infinity recall that dz/w is a nonvanishing holomorphic form, then the expression $\frac{\frac{w(Q)}{z(Q)}z + w}{z(z - z(Q))}$ has denomenator $\mathcal{O}(|z|^2)$ and numerator $\mathcal{O}(|z|^{3/2})$ implying there is no pole at infinity.

(Case 2) This time take $Q_1 = 0$ and $Q_2 = 1$, we define

$$\omega_{0,1} = \frac{w}{2z(z-1)} \frac{dz}{w} \tag{18}$$

Similar to the first case, this form only has candidate poles at 0, 1 and infinity, but it is trivial to rule out infinity thus we only need check the residues. Near zero we once again have local coordinates near zero $t = \pm\sqrt{z}$ on (I) and (II) respectively, these local coordinates give the local expression

$$\omega_{0,1} = \frac{2tdt}{2t^2(t^2-1)} = \frac{1}{t}\frac{1}{t^2-1}dt \tag{19}$$

from this we read off $\text{Res}(\omega_{0,1}, 0) = -1$. Similarly we can take local coordinates near one, $t = \pm\sqrt{z-1}$ on (I) and (II) respectively, in these local coordinates

$$\omega_{0,1} = \frac{2tdt}{2(t^2+1)t^2} = \frac{1}{t}\frac{1}{1+t^2}dt \tag{20}$$

Which we can read off as having residue 1.

(Case 3) Take $Q_1 = 0$, $Q_2 = \infty$. We follow a similar procedure to the first two cases,

$$\omega_{0,\infty} = \frac{w}{2z}\frac{dz}{w} \tag{21}$$

its clear the only candidates for poles are 0 and ∞, so we check that they are simple and take the residues. Taking local coordinates near infinity we have $t = \pm 1/\sqrt{z}$ on (I) and (II) respectively, in these local coordinates we write

$$\omega_{0,\infty} = \frac{t^2}{2}\frac{-2dt}{t^3} = -\frac{1}{t}dt \tag{22}$$

which is a simple pole with residue -1. Taking the same coordinates near zero as in the prior cases we find the local expression

$$\omega_{0,\infty} = \frac{1}{2t^2}2tdt = \frac{1}{t}dt \tag{23}$$

so that we have a simple pole with residue 1 at zero.

(Case 4) Take $Q_1 \in X \setminus \{0, 1, \lambda, \infty\}$ to be a generic point, and $Q_2 = \infty$, then we can define

$$\omega_{Q,\infty} = \frac{1}{2}\frac{w(Q)+w}{z-z(Q)}\frac{dz}{w} \tag{24}$$

The zero of $w(Q) + w$ at $\tilde{Q}$ removes the simple pole at $\tilde{Q}$ (where once again $\tilde{Q}$ is the other lift of $z(Q)$ in the double cover), moreover the pole at Q is clearly simple so we need only check the pole at infinity is simple and compute the residues. To compute the residue at Q, we can simply use z coordinates,

$$\omega_{Q,\infty} = \frac{1}{2}\frac{w(Q)+w}{z-z(Q)}\frac{dz}{w} = \frac{1}{z-z(Q)}\frac{w(Q)+w}{2w}dz \tag{25}$$

So that $\text{Res}(\omega_{Q,\infty}, Q) = \frac{2w(Q)}{2w(Q)} = 1$, to compute the residue at infinity we work in the same local coordinates as in case 3, this gives us the following local expression

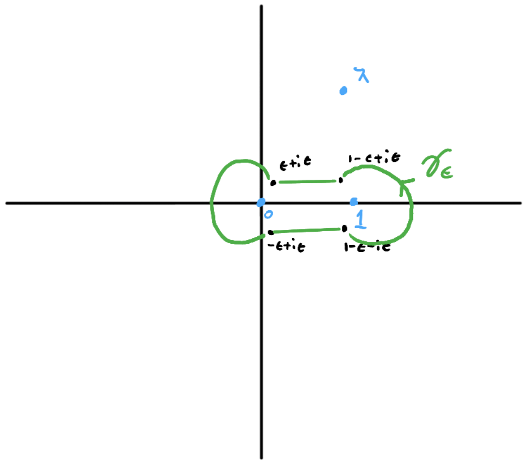
$$w_{Q,\infty} = \frac{1}{2}\frac{t^2(w(Q)+w)}{1-t^2z(Q)}\frac{-2dt}{t^3w} = \frac{-1}{t}\frac{w(Q)+w}{w(1-t^2z(Q))} \tag{26}$$

From this expression we get the desired value of the residue,

$$\text{Res}(w_{Q,\infty}, \infty) = -\left(\frac{w(Q)+w}{w(1-t^2z(Q))}\right)\Bigg|_{t=0} = -\left(\frac{w(Q)}{w}+1\right)\Bigg|_{t=0} = 1 \tag{27}$$

Since $w(t)|_{t=0} = \infty$.

Thus for any distinct points $Q_1, Q_2 \in X$ regardless of genericity there is an abelian differential of the third kind ω_{Q_1,Q_2}. □



6. Since the branch of the square root function used to define Ω is analytic on $\mathbb{C} \setminus L$, we can use the following curve γ_ϵ to define our contour integral (The value of the integral is independent of this choice of curve by Cauchy's integral theorem).

Since Ω is holomorphic on $\mathbb{C} \setminus L$, the area enclosed by γ_ϵ is independent of ϵ so long as ϵ is sufficiently small. Taking the limit $\epsilon \to 0$, we can show that the circular component of the curve has no contribution using the standard arc length inequality, since near 0 and 1 on γ_e we have $\frac{1}{w} \in \mathcal{O}(\epsilon^{-\frac{1}{2}})$, and the arc length of the circular component is bound above by $2\pi\epsilon$, the integral over the circular component is bound above by $2\pi\epsilon\mathcal{O}(\epsilon^{-\frac{1}{2}}) \in \mathcal{O}(\epsilon^{\frac{1}{2}})$. The horizontal components of the integral then approach the curve on the real line from 1 to 0 from above, and the curve from 0 to 1 from below. From this we get

$$\int_\gamma \frac{dz}{w} = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \frac{dz}{w} = \lim_{\epsilon \downarrow 0} \int_{1-\epsilon}^{\epsilon} \frac{dx}{w(x+i\epsilon)} + \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{dx}{w(x-i\epsilon)} \tag{28}$$

$$= \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{1}{w(x-i\epsilon)} - \frac{1}{w(x+i\epsilon)} dx = -2 \int_0^1 \frac{dx}{\lim_{\epsilon \downarrow 0} w(x+i\epsilon)} \tag{29}$$

Where passing to the limit is justified by $\Omega = \frac{dz}{w}$ being a holomorphic differential on X, hence uniformly continuous on a compact set in $\mathbb{C} \setminus L$ containing γ, and $x + i\epsilon$, and $x - i\epsilon$ converging uniformly to x from above and below, finally the sign convention on taking the limit $\frac{1}{w(x-i\epsilon)}$ follows from $\lim_{\epsilon\uparrow 0} w(x+i\epsilon) = -\lim_{\epsilon\downarrow 0} w(x+i\epsilon)$ by the monodromy of w around the branch cut at L. Now if λ is real, then the result is trivial since in this case $\lim_{\epsilon\downarrow 0} w(x+i\epsilon) = \sqrt{x(1-x)(x-\lambda)}$, so that $\int_0^1 \frac{dx}{\sqrt{x(1-x)(x-\lambda)}} \neq 0$ since the integrand is either strictly positive real or strictly positive imaginary depending on $\lambda < 0$ or $\lambda > 1$. For the case of λ not real, I will address $\Im(\lambda) > 0$, since the case of λ in the lower half plane is similar. Now for λ in the upper half plane, first note that explicitly the branch cut of $\sqrt{z(z-1)(z-\lambda)}$ which is analytic away from L is given by $\exp(\frac{1}{2}(\operatorname{Log} z + \operatorname{Log}(z-1) + \operatorname{Log}(z-\lambda)))$, so in order to see that $\int_\gamma \frac{dz}{w} = -2\int_0^1 \frac{dx}{\lim_{\epsilon\downarrow 0} w(x+i\epsilon)} \neq 0$, it suffices to check $\lim_{\epsilon\downarrow 0} w(x+i\epsilon)$ has strictly positive imaginary part for each $x \in [0,1]$, since this will imply that $(\lim_{\epsilon\downarrow 0} w(x+i\epsilon))^{-1}$ has strictly negative imaginary part, so that $-2\,(\lim_{\epsilon\downarrow 0} w(x+i\epsilon))^{-1}$ will have strictly positive imaginary component, which implies that $\Im(\int_\gamma \frac{dz}{w}) > 0$, so the integral is nonzero.

The verification follows from just computing the limit:

$$\lim_{\epsilon\downarrow 0}\Im(w(x+i\epsilon)) = \Im\left(\exp\left(\frac{1}{2}\lim_{\epsilon\downarrow 0}\mathrm{Log}(x+i\epsilon)+\mathrm{Log}(x-1+i\epsilon)+\mathrm{Log}(x-\lambda+i\epsilon)\right)\right) \tag{30}$$

$$= \Im\left(\exp\left(\frac{1}{2}\lim_{\epsilon\downarrow 0}\mathrm{Log}(x+i\epsilon)\right)\exp\left(\frac{1}{2}\lim_{\epsilon\downarrow 0}\mathrm{Log}(x-1+i\epsilon)\right)\exp\left(\frac{1}{2}\lim_{\epsilon\downarrow 0}\mathrm{Log}(x-\lambda+i\epsilon)\right)\right) \tag{31}$$

$$= \Im(\sqrt{x}(e^{\frac{i\pi}{2}}\sqrt{1-x})(|x-\lambda|^{\frac{1}{2}}\, e^{\frac{i}{2}\,\mathrm{Arg}(x-\lambda)})) \tag{32}$$

$$= \sqrt{x(1-x)\,|x-\lambda|}\,\Im(e^{\frac{i}{2}(\pi+\mathrm{Arg}(x-\lambda))}) \tag{33}$$

But since $\Im(\lambda)>0$, $x-\lambda$ lies strictly in the lower half plane giving us $\mathrm{Arg}(x-\lambda)\in(-\pi,0)$, this implies that $\frac{1}{2}(\pi+\arg(x-\lambda))\in(0,\frac{\pi}{2})$, so that $\Im(e^{\frac{i}{2}(\pi+\mathrm{Arg}(x-\lambda))})>0$. Combining this with (33) we find that

$$\lim_{\epsilon\downarrow 0}\Im(w(x+i\epsilon))>0 \tag{34}$$

for all x, as desired. $\square$

7. To see that $\mathrm{Res}(\Omega,p)$ is well defined, we need to check it is invariant under our choice of coordinate charts, this basically reduces to the compatibility assumptions for holomorphic differentials. Let $(U,z),(V,w)$ be charts for X containing p, then we only need check the two coordinate expressions agree, we may take a loop γ around p with winding number 1, with image contained in $U\cap V$, then γ may be parameterized by either z or w, so that the residue in either chart is given by $\frac{1}{2\pi i}\int_{\gamma(z)}f_U(z)dz$ and $\frac{1}{2\pi i}\int_{\gamma(w)}f_V(w)dw$, we need only check these two expressions are equal.

$$\frac{1}{2\pi i}\int_{\gamma(z)}f_U(z)dz = \frac{1}{2\pi i}\int_{\gamma(z)}f_V(w(z))w'(z)dz \tag{35}$$

Now we may apply the substitution $w=w(z)$, so that $dw=w'(z)dz$, applying the subsitiution gives us

$$\frac{1}{2\pi i}\int_{\gamma(z)}f_V(w(z))w'(z)dz = \frac{1}{2\pi i}\int_{\gamma(w)}f_V(w)dw \tag{36}$$

Combining (35) and (36) gives the desired equality of residue in arbitrary coordinate charts so that $\mathrm{Res}(\Omega,p)$ is indeed well defined. $\square$