1. We first find an expression for $\zeta(s)\Gamma(s)$ on $\Re(s) > 1$, to do so we will do a substitution $u = x/n$, valid for any $n \in \mathbb{Z}_{>0}$

$$\Gamma(s) = \int_0^\infty e^{-nu}(nu)^{s-1} n du = n^s \int_0^\infty u^{s-1}(e^{-u})^n du$$

Now since I don't like u I will switch back to x; Multiplying n^{-s} on both sides and summing over $n \in \mathbb{Z}_{>0}$ yields

$$\zeta(s)\Gamma(s) = \sum_1^\infty \int_0^\infty x^{s-1}(e^{-x})^n dx \overset{\text{DCT}}{=} \int_0^\infty x^{s-1} \sum_1^\infty (e^{-x})^n dx = \int_0^\infty x^{s-1} \frac{1}{e^x - 1} dx \tag{1}$$

Where DCT is taken with respect to $\left|x^{s-1}\right| \sum_1^\infty e^{-nx}$. We can use $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to rewrite (1):

$$\zeta(s) = \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{2}$$

Now we deal with the contour integral, letting $C(\epsilon)$ denote the curve described in the problem for fixed $\epsilon \in \mathbb{R}_{>0}$. The expression $e^z - 1$ has no poles away from $\{z \mid e^z = 1\} = \{2\pi k i \mid k \in \mathbb{Z}\}$, whence if we take the branch cut of log away from the non-negative reals the integral is not dependent on ϵ for $\epsilon < 2\pi$, since (for $\delta < \epsilon < 2\pi$) we have the area enclosed between the two curves is a quotient of holomorphic functions with the denominator non-vanishing in between the curves. This independence is a result of taking seperating the curve $C(\epsilon) - C(\delta)$ into two curves (see picture), the first of which has integral zero by Cauchy's theorem, and the second being arbitrarily small depending on where we take the cut.

$$\int_{C(\epsilon)} f - \int_{C(\delta)} f = \int_{C(\epsilon,\delta,M)} f + \int_{\gamma_1(\epsilon,\delta,M)} f + \int_{\gamma_2(\epsilon,\delta,M)} f$$

INSERT ILLUSTRATION HERE

Then for f holomorphic away from he real line, the $C(\epsilon, \delta, M)$ term vanishes. Notice now that using the standard arclength inequality for large M we have

$$\begin{aligned}\left|\int_{\gamma_j(\epsilon,\delta,M)} \frac{(-z)^{s-1}}{e^z - 1} dz\right| &\leq (\epsilon - \delta)\left|\frac{M^{\Re(s)-1}}{e^M - 1}\right| + \left|\int_M^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx - \int_M^\infty \frac{(-x - i\delta)^{s-1}}{e^{x+i\delta} - 1} dx\right| \\ &\leq (\epsilon - \delta)\left|\frac{M^{\Re(s)-1}}{e^M - 1}\right| + 2\int_M^\infty \frac{|x + i\epsilon|^{\Re(s)-1}}{|e^x| - 1} dx\end{aligned}$$

The right hand side clearly converges to zero. as $M \to \infty$ using basic limits of exponentials and DCT. This gives the desired invariance.

$$\left|\int_{C(\epsilon)} \frac{(-z)^{s-1}}{e^z - 1} dz - \int_{C(\delta)} \frac{(-z)^{s-1}}{e^z - 1} dz\right| \leq \left|\int_{\gamma_1(\epsilon,\delta,M)} \frac{(-z)^{s-1}}{e^z - 1} dz\right| + \left|\int_{\gamma_2(\epsilon,\delta,M)} \frac{(-z)^{s-1}}{e^z - 1} dz\right| = 0$$

Now, we can compute the value of the integral along this curve by letting $\epsilon \to 0$, to get $C(0)$, a ray from ∞ to 0 where $(-z)^{s-1} = x^{s-1}e^{-(s-1)\pi i}$, and a ray from 0 to ∞ where $(-z)^{s-1} = x^{s-1}e^{(s-1)\pi i}$, to see that we can indeed pass to this limit, once again decompose $C = C(\epsilon)$ into three curves, with C_1 the ray in the upper half plane, C_2 the ray in the lower half plane and C_3 the circular portion, then once again using the arc length inequality and the fact that $e^z - 1 = \mathcal{O}(z)$

$$\left|\int_{C_3} \frac{(-z)^{s-1}}{e^z - 1}\right| \leq 2\pi\epsilon \sup_{|z|=\epsilon} \frac{|z^{s-1}|}{|e^z - 1|} = 2\pi\epsilon\mathcal{O}(\epsilon^{\Re(s)-2}) = 2\pi\mathcal{O}(\epsilon^{\Re(s)-1}) \overset{\Re(s)>1}{\longrightarrow} 0$$

We can check convergence on C_1 explicitly

$$\left|\int_0^\infty \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx - \int_\epsilon^\infty \frac{(-x-i\epsilon)^{s-1}}{e^{x+i\epsilon}-1}dx\right|$$
$$\leq \left|\int_0^\epsilon \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx\right| + \left|\int_\epsilon^\infty \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx - \int_\epsilon^\infty \frac{(-x-i\epsilon)^{s-1}}{e^{x+i\epsilon}-1}dx\right|$$
$$\leq \left|\int_0^\epsilon \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx\right| + \left|\int_\epsilon^\infty \frac{(e^{x+i\epsilon}-1)x^{s-1}e^{-(s-1)\pi i} - (e^x-1)(\sqrt{x^2+\epsilon^2})^{(s-1)}e^{i(s-1)\arctan\frac{\epsilon}{x}-(s-1)\pi i}}{(e^x-1)(e^{x+i\epsilon}-1)}\right|$$

Convergence as $\epsilon \to 0$ of the big ugly term to zero is actually simple from convergence of each of the terms in the two expressions in the product. Convergence of the first term follows from $\Re(s) > 1$, so writing the bounds of integration as $\chi_{(0,\epsilon)}$ we can just apply DCT to the absolute value of the integrand. The proof of convergence for C_2 is similar to C_1.

Now we finally established that (due to taking the limit in ϵ and invariance with respect to ϵ)

$$\int_C \frac{(-z)^{s-1}}{e^z-1}dz = -\int_0^\infty \frac{x^{s-1}e^{-(s-1)\pi i}}{e^x-1}dx + \int_0^\infty \frac{x^{s-1}e^{(s-1)\pi i}}{e^x-1}dx \tag{3}$$
$$= \int_0^\infty \frac{x^{s-1}}{e^x-1}2i\sin((s-1)\pi) = -\int_0^\infty \frac{x^{s-1}}{e^x-1}2i\sin(s\pi) \tag{4}$$

Multiplying (2) by $1 = \frac{\int_C \frac{(-z)^{s-1}}{e^z-1}dz}{-\int_0^\infty \frac{x^{s-1}}{e^x-1}2i\sin(s\pi)}$ yields the desired equality

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z-1}dz \tag{5}$$

Now, using the right side of (5) as an analytic continuation, we first compute $(e^z-1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)$, since we know it has a simple pole at zero, hence is meromorphic this expression just comes from evaluating the systems of equations given for the coefficients for $(a_{-1}z^{-1} + a_0 + a_1 z + \cdots)(\sum_0^\infty \frac{z^n}{n!}) = 1$. This is everything we need to evaluate $\zeta(0)$. First using a similar separation of curves into two parts as was used for invariance, we find that $\int_C \frac{1}{z(e^z-1)}dz = \int_\gamma \frac{1}{z(e^z-1)}dz$ where γ is a piecewise C^1 closed curve, this allows us to use the residue theorem

$$\zeta(0) = \frac{\Gamma(1)}{2\pi i}\int_\gamma \frac{1}{z(e^z-1)}dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)\right)dz \tag{6}$$
$$= \operatorname{Res}\left(\frac{1}{z}\left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)\right)\right) = -\frac{1}{2} \tag{7}$$

□

2. (a) Let ϵ be small enough so that $B_{2\epsilon}(z_0) \subset D$, then

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw - \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w-z_0}dw \overset{\text{Stokes}}{=} \int_{D\setminus B_\epsilon(z_0)} \frac{\partial}{\partial \overline{w}}\frac{f(w)}{w-z_0}d\overline{w}\wedge dw = 0$$

By Cauchy Riemann since $\frac{f(w)}{w-z_0}$ is holomorphic in $D \setminus B_\epsilon(z_0)$. From this, we cna prove the base case which is Cauchy's integral formula.

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw = \int_{B_\epsilon(z_0)} \frac{f(w)}{w-z_0}dw = \int_0^{2\pi} f(z_0+\epsilon e^{it})idt$$

Which holds for arbitrary ϵ, but since f is bounded (on D) and continuous we have

$$\int_{\partial D} \frac{f(w)}{w-z_0}dw = \lim_{\epsilon\to 0}\int_0^{2\pi} f(z_0+\epsilon e^{it})idt = \int_0^{2\pi} \lim_{\epsilon\to 0} f(z_0+\epsilon e^{it})idt = 2\pi i f(z_0)$$

Which proves the desired result in the case of $n = 0$

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w - z_0}dw \tag{8}$$

Now, we can proceed by induction and simply take derivatives.

$$\begin{aligned}
2\pi i \lim_{h\to 0}\frac{1}{h}(f^{(n)}(z_0+h) - f^{(n)}(z_0)) &= \lim_{h\to 0}\frac{1}{h}\int_{\partial D}\frac{f(w)}{(w-z_0-h)^{n+1}} - \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \lim_{h\to 0}\int_{\partial D}\frac{1}{h}\frac{f(w)(w-z_0)^{n+1} - f(w)(w-z_0-h)^{n+1}}{(w-z_0-h)^{n+1}(w-z_0)^{n+1}}dw \\
&= \lim_{h\to 0}\int_{\partial D}\frac{f(w)(w-z_0)^n}{(w-z_0-h)^{n+1}(w-z_0)^{n+1}} + \mathcal{O}(h)dw
\end{aligned}$$

Then we can clearly apply DCT for $|h| < \frac{1}{2}|w - z_0|$, which gives us

$$\lim_{h\to 0}\frac{1}{h}(f^{(n)}(z_0+h) - f^{(n)}(z_0)) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z_0)^{n+2}}dw \tag{9}$$

(b) Let $z \in \mathbb{C}$, then by equation (9) in part (a)

$$f'(z) = \frac{1}{2\pi i}\int_{|w-z|=R}\frac{f(w)}{(w-z)^2}dw$$

Applying the standard arc-length inequality yields

$$|f'(z)| \le \frac{1}{2\pi}2\pi R\frac{\sup_{\mathbb{C}}|f|}{R^2} = \frac{\sup_{\mathbb{C}}|f|}{R} \xrightarrow{R\to\infty} 0 \tag{10}$$

Therefore $f' \equiv 0$, for any two points in $\mathbb{C}$, to see this implies f is constant we use the fundamental theorem of calculus, if $z_0, z_1 \in \mathbb{C}$ take γ to be the straight line starting at z_0 and ending at z_1 so that

$$|f(z_1) - f(z_0)| = \left|\int_\gamma f'(z)dz\right| \le \ell(\gamma)\sup_{\mathbb{C}}|f'| = 0$$

□

(c) Suppose a polynomial P does not have a root. Then $\frac{1}{P}$ is entire, hence $1/P$ is constant by Liouville's theorem. We conclude any polynomial without a root is of degree zero. □

3. Assume for contradiction that f has an essential singularity at z_0, but there exists some $a \in \mathbb{C}$ and $\delta > 0$ such that for some $\epsilon > 0$ we have $f(D_\epsilon^*)(z_0) \cap B_\delta(a) = \emptyset$. Now define a new function $g(z) = \frac{1}{f(z)-a}$, since $|f(z) - a| \ge \delta$ on $D_\epsilon^*(z_0)$ we find that g is holomorphic and nonvanishing on this punctured annulus with modulus bound above by $\frac{1}{\delta}$. Then we can define on $D_\epsilon(z_0)$

$$(z-z_0)^2 g(z) = \begin{cases}(z-z_0)^2 g(z) & z \ne z_0 \\ 0 & z = z_0\end{cases} \tag{11}$$

This is holomorphic away from z_0, thus is holomorphic since

$$\lim_{h\to 0}\frac{(z_0+h-z_0)^2 g(z_0+h)}{h} = \lim_{h\to 0} h g(z_0+h) = 0$$

where the last equality is a consequence of g being bounded. Therefore we can write

$$g(z)(z-z_0)^2 = \sum_0^\infty a_k(z-z_0)^k$$

which gives the following expression on D_ϵ^* for $b_k = a_{k+2}$

$$g(z) = \sum_{-2}^{\infty} b_k (z - z_0)^k = \sum_{0}^{\infty} b_k (z - z_0)^k \tag{12}$$

Where $b_{-2} = b_{-1} = 0$ follows from g being bounded near z_0. Moreover since g is nonvanishing, we must have some $b_N \neq 0$, it follows that

$$\lim_{z \to z_0} (z - z_0)^{N+1} (f(z) - a) = \lim_{z \to z_0} \frac{(z - z_0)^{N+1}}{g(z)} \overset{(12)}{=} \lim_{z \to z_0} \frac{z - z_0}{b_N + \mathcal{O}(z - z_0)} = 0 \tag{13}$$

Now we can show directly from the definition that f has a pole

$$\lim_{z \to z_0} (z - z_0)^{N+1} f(z) = \lim_{z \to z_0} (z - z_0)^{N+1} (f(z) - a) \overset{(13)}{=} 0$$

Which is the desired contradition. $\square$