1. (a) From the definition of a lie group we know that $\mu : G \times G \to G$ is smooth, then $\mu_g = \mu \circ \iota_g$, where $\iota_g : G \to G \times G$ via $h \mapsto (g, h)$ is the incusion into the product manifold, we have seen previously the inclusion is smooth, so that $\mu_g = \mu \circ \iota_g$ is smooth. Now we can also see that

$$\mu_{g^{-1}}\mu_g = 1_G = \mu_g\mu_{g^{-1}}$$

and $\mu_{g^{-1}}$ is smooth for the same reason μ_g is, so that μ_g is in fact a diffeomorphism, this implies that $d_e\mu_g$ is an isomorphism.

(b) Since T_eG is n-dimensional, we can identify it with $\mathbb{R}^n$, the following diagram specifies the desired correspondence of vector bundles:

$$\begin{tikzcd} G \times \mathbb{R}^n \arrow[r, shift left, "F"] \arrow[d] & TG \arrow[l, shift left, "T"] \arrow[d] \\ G \arrow[r, "1_G"] & G \end{tikzcd}$$

Where $F(g, v) = (g, d_e\mu_g v)$ and $T(g, v) = (g, d_g\mu_{g^{-1}}v)$. Then

$$F \circ T(g, v) = (g, (d_e\mu_g)(d_g\mu_{g^{-1}})v) = (g, d_g 1_G v) = (g, 1_{T_gG}v) = (g, v)$$
$$T \circ F(g, v) = (g, (d_g\mu_{g^{-1}})(d_e\mu_g)v) = (g, d_e 1_G v) = (g, 1_{T_eG}v) = (g, v)$$

So we are done once we verify that F and T are maps of vector bundles, but the linearity and restriction to fibers properties are trivial, and F, T are continuous since

2. Let $f : X \to \mathbb{R}^m$ be a submersion, where X is a compact smooth manifold. The proof will follow if we can show submersions are open maps, assuming this, since the image of a compact set is compact (by pulling back an open cover along the map) we get that $f(X) \subset \mathbb{R}^m$ is open, but also $f(X) \subset \mathbb{R}^m$ is compact hence closed, so since $X \neq \emptyset$ we have $f(X) = \mathbb{R}^m$, contradicting compactness.

It remains to show that a submersion is open, since f is a submersion, we can cover M, N with charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U'_\beta, V'_\beta, \varphi_\beta)$ respectively with the property that the following commutes (here π is the projection map onto the first n coordinates)

$$\begin{tikzcd} U_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"] & V_\alpha \arrow[d, "f"] \\ U'_\beta \arrow[r, "\varphi_\beta"] & V'_\beta \end{tikzcd}$$

Now let $E \subset X$ be open, and write $E_\alpha := V_\alpha \cap E$, then

$$f(E) = \bigcup_\alpha f(E_\alpha) = \bigcup_{\alpha,\beta} \varphi_\beta \pi \phi_\alpha^{-1}(E_\alpha)$$

But $\varphi_\beta \pi \phi_\alpha^{-1}$ is a composition of open maps hence open, so that $f(E)$ is open which suffices to show f is open.

3. (a)