1. (Durrett 2.2.2) We assume that r is a real valued function, i.e. that $r(0) \in \mathbb{R}$ is not infinity. Then

$$0 \le E\left[\left(\frac{S_n}{n}\right)^2\right] = \frac{1}{n^2}\left(\sum_1^n EX_j^2 + 2\sum_{\substack{1\le j\le n\\1\le i<j}} EX_iX_j\right) \le \frac{1}{n^2}\left(nr(0) + 2\sum_{\substack{1\le j\le n\\1\le i<j}} |r(i-j)|\right)$$
$$= \frac{r(0)}{n} + \frac{2}{n^2}\left(\sum_{j=1}^{n-1}(n-j)|r(j)|\right)$$

The term $r(0)/n \to 0$, so it suffices to check that $\frac{1}{n^2}\left(\sum_{j=1}^{n-1}(n-j)|r(j)|\right) \to 0$. So take $\epsilon > 0$, then for $N \in \mathbb{N}$ we have $j \ge N$ implies $|r(j)| < \epsilon/2$, then for $M > N$,

$$\frac{1}{M^2}\sum_{j=N}^{M-1}(M-j)|r(j)| < \frac{1}{M^2}(M-N-1)(M-N)\frac{\epsilon}{2} \le \epsilon/2$$

Now we can take M sufficiently large so that $\frac{1}{M}\sum_1^{N-1}|r(j)| < \epsilon/2$, then combining these inequalities, for any $K \ge M$ we get

$$\frac{1}{K^2}\sum_1^{K-1}(K-j)|r(j)| = \frac{1}{K^2}\sum_N^{K-1}(K-j)|r(j)| + \frac{1}{K^2}\sum_1^{N-1}(K-j)|r(j)| < \epsilon/2 + \frac{1}{K}\sum_1^{N-1}|r(j)|$$
$$\le \epsilon/2 + \frac{1}{M}\sum_1^{N-1}|r(j)| < \epsilon$$

So indeed by squeeze theorem we find that $E\left[\left(\frac{S_n}{n}\right)^2\right] \xrightarrow{L^2} 0$, and L^2 convergence implies convergence in probability. □

2. (Durrett 2.2.8) In order to invoke the weak law of traingular arrays, we need to check the conditions.

(i) - $\sum_1^n \mathbf{P}(\mathbf{X_k} > \mathbf{b_n}) \to \mathbf{0}$.

$$P(X_k > b_n) = \sum_{m(n)}^{\infty} p_j = \le 2^{-m(n)}\sum_0^{\infty}\frac{1}{2^j(m(n)+j+1)(m(n)+j)} \le 2^{-m(n)}m(n)^{-3/2}\frac{1}{\sqrt{m(n)}}\sum_0^{\infty}2^{-j}$$
$$< \frac{2}{n\sqrt{m(n)}}$$

So that

$$\sum_1^n P(X_k > b_n) < \frac{2}{\sqrt{m(n)}} \xrightarrow{n\to\infty} 0$$

(ii) - $\frac{1}{\mathbf{b_n^2}}\sum_1^n \mathbf{E}\overline{\mathbf{X}}_{\mathbf{n,k}}^{\mathbf{2}} \to \mathbf{0}$, where $\overline{X}_{n,k} = X_k 1_{\{|X_k|\le b_n\}}$

$$E\overline{X}_{n,k}^2 = \left(\sum_1^{2^{m(n)}} p_k(2^k-1) - p_0\right)^2 = p_0^2 - 2p_0\left(\sum_1^{2^{m(n)}}\frac{1}{k(k+1)} - p_k\right) + \left(\sum_1^{2^{m(n)}} p_k(2^k-1)\right)^2$$
$$\le p_0^2 + \left(\sum_1^{2^{m(n)}}\frac{1}{k(k+1)} - p_k\right)^2 \le p_0^2 + \left(\sum_1^{2^{m(n)}}\frac{1}{k(k+1)}\right)^2 \le 1+1 = 2$$

And hence,

$$\frac{1}{b_n^2}\sum_{1}^{n} E\overline{X}_{n,k}^2 \leq \frac{2n}{b_n^2} = \frac{2n}{2^{-2m(n)}} < \frac{2m(n)^{3/2}}{2^{-m(n)}} \overset{n\to\infty}{\longrightarrow} 0$$

Now we want to obtain workable expressions to explain the asymptotic behaviour, we obtain the following expression for $\sum_1^n E\overline{X}_{n,k}$