Here are some notes meant for reference for my studies in geometric topology. I will only supply proofs when I think doing so will be edifying, these notes are mostly based from Bruno Martelli's Book as well as Do Carmo's book so proofs can be found there.

As convention when writing $C^\alpha(M)$ for a manifold M, $\alpha \geq 1$, but the $\alpha = \infty$ case is most interesting. All manifolds are atleast differentiable. ANotehr convention is to denote $\mathcal{X}(M)$ as the set of C^∞ vector fields on M.

1 Riemannian Geometry

1.1 Vector Fields

Definition 1.1. A vectorfield $X : M \to TM$ can be thought of as a derivation, i.e. if $X(p) = \sum_1^n a_j(p)\frac{\partial}{\partial x_j}$ then $X : C^\alpha(M) \to C^{\alpha-1}(M)$ via

$$X(f)(p) = \sum_1^n a_j(p)\frac{\partial}{\partial x_j} f$$

Proposition 1.1. If $\varphi : M \to M$ is a diffeomorphism, and $v \in T_pM$ with $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$(d_p\varphi(v))(f) = v(f \circ \gamma)(p)$$

Definition 1.2. The Lie bracket of two vector fields X, Y denoted $[X, Y]$ is the unique vectorfield Z satisfying $Z(f) = (XY - YX)(f)$ for all $f \in C^\alpha(M)$

Proposition 1.2. If X, Y are vectorfields on M, with $U \supset \{p\} \subset M$, and φ_t is a flow of X in U, then

$$[X, Y](p) = \lim_{t\to 0} \frac{1}{t}(Y - d\varphi_t Y)(\varphi_t(p))$$

1.2 Riemannian Metrics

Definition 1.3. A Riemannian metric is a positive definite, symmetric bilinear form $g : TM \otimes TM \to \mathbb{R}$

Definition 1.4. A Riemannian Manifold is a pair (M, g) where M is a manifold and g is a Riemannian metric.

Proposition 1.3. A Riemannian metric can be represented in coordinates, given a local frame (X_j), and coframe (σ^j) we have

$$g(-,-) = \sum_{i\leq j} g(X_i, X_j)\sigma^i\sigma^j$$

Definition 1.5. $f : M \to N$ is a isometry between Riemannian manifolds if f is a diffeomorphism and $df^*g_M = g_N$

Definition 1.6. M and N are said to be locally isometric if for any $p \in M$ there is some neighborhood $p \in U$, and some $V \subset N$ with $f : U \to V$ an isometry. Note this is not an equivalence relation.

Example(s) 1.1.

- Euclidean, g_{EUC} has $g_{ij} = \delta_{ij}$
- $f : M \to N$, then we can define $g_M(u, v) = g_N(dfu, dfv)$
- If a group G acts on M smoothly, freely, properly and isometrically then we can define for $\pi : M \to M/G$

$$g_{M/G}(u, v)_p = g_M(d_p\pi^{-1}(u), d_p\pi^{-1}(v))_{\pi^{-1}(p)}$$

The case of $U(1)$ acting on $\mathbb{C}^{n+1} \setminus \{0\}$ is the Fubini-Study metric on $\mathbb{P}^n$

- The product metric $g_{X\times Y}(u,v)_{p,q} = g_X(d\pi_X u, d\pi_X v) + g_Y(d\pi_Y u, d\pi_Y v)$, analogous to dot product.

Definition 1.7. A Riemannian metric on a Lie group is Left (resp. Right) invariant if L_x (resp. R_x) is an isometry for all $x \in G$. It is bi-invariant if it is both left adn right invariant.

Proposition 1.4. All Lie groups have a left invariant Riemannian metric, namely take one on T_eG and define

$$g_{\text{inv}}(u,v)_x = g(dL_{x^{-1}}u, dL_{x^{-1}}v)_e$$

Moreover, if G is compact it has a bi-invariant metric.

Proposition 1.5. g is bi-invariant if and only if for any $U, V, X \in T_eG$ we have

$$g([U,X],V) = -g(U,[V,X])$$

Definition 1.8. $I \subset \mathbb{R}$, and $\gamma : I \to M$, then for any closed interval $[a,b] \subset I$ we can define the length of the curve γ, since g will give a norm. Namely

$$\ell_a^b(\gamma) = \int_a^b \sqrt{g(\gamma'(t),\gamma'(t))}dt$$

Definition 1.9. It may be desirable to define the volume form in terms of Riemannian metrics, in this case we can write locally $x : U \subset M \to V \subset \mathbb{R}^N$ as coordinates and

$$dVol := \sqrt{\det(g_{ij})}dx_1 \cdots dx_n$$

1.2.1 Exercises

Exercise 1.2.1. Put a Riemannian metric on T^n so that $R^n \supset (x_1,\dots,x_n) \overset{f}{\mapsto} (e^{ix_1},\dots,e^{ix_n})$ is a local isometry, show this is isometric to the flat torus.

Proof. We use that the group action of $\mathbb{Z}^n$ by translation has differential 1, so we can just use the metric given in the above examples. This makes f an isometry tautologically. □

Exercise 1.2.2. Give an immersion $f : T^n \to \mathbb{R}^{2n}$ which is isometric to the flat Torus.

Proof.

$$f : (\theta_1,\dots,\theta_n) \mapsto (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$$

Then we cna simply evaluate

$$df|_{(\theta_1,\dots,\theta_n)} = \begin{pmatrix} -\sin\theta_1 & 0 & \cdots & 0 \\ \cos\theta_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin\theta_n \\ 0 & 0 & \cdots & \cos\theta_n \end{pmatrix}$$

The flat metric gives us

$$g_{ij} = \delta_{ij}$$

And the immersion gives us

$$g\left(d_{\underline{\theta}}f(\frac{\partial}{\partial\theta_i}), d_{\underline{\theta}}f(\frac{\partial}{\partial\theta_j})\right)_{f(\underline{\theta})} = g_{\text{EUC}}(-\sin\theta_i e_{2i-1} + \cos\theta_i e_{2i}, -\sin\theta_j e_{2j-1} + \cos\theta_j e_{2j}) = \delta_{ij}$$

□

Exercise 1.2.3. The lie group of Affine transformations can be given by $\mathbb{R} \times (0, \infty)$, with $A(t) = yt + x$, then the lie group structure is composition. Show that $g_{11} = g_{22} = \frac{1}{y^2}$ gives a left invariant metric. Note that using this metric, and identifying $(x, y) \sim x + iy = z$ we get that the $SL(2, \mathbb{R})$ metric acts via isometry on G.

Proof. We simply compute $d_{(x,y)}L_{(A,B)} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$, so that

$$((L_{(A,B)})_* g_{(x,y)})_{11} = ((L_{(A,B)})_* g_{(x,y)})_{22} = B^2/(By)^2 = \frac{1}{y^2}$$
$$((L_{(A,B)})_* g_{(x,y)})_{12} = g_{12} = 0$$

□

Exercise 1.2.4. Show that "Locally isometric" is not an equivalence relation

Proof. Let η be a bump function on $U \subset \mathbb{R}^n$, the define g on $\mathbb{R}^n$ as $g_{\text{EUC}} + \eta g_{\text{EUC}}$, it is clear we can't have a local isometry with flat $\mathbb{R}^n$ on ∂U. Existence in the reverse direction is obvious, since we can just map diffeomorphically away from U. □

Exercise 1.2.5. Show that there exists a Bi-invariant Riemannian metric on any compact connected Lie Group.

Proof. First note that there is a bijection between left invariant n-forms and $(\Lambda^n T_e M)^*$, to see this write down the definition of left invariance:

$$L_x^* \omega = \omega \rightsquigarrow \omega_{L_x(y)}(dL_x v_1, \ldots, dL_x v_n) = \omega_y(v_1, \ldots, v_n)$$

replacing y for e we find

$$\omega_x(dL_x v_1, \ldots, dL_x v_n) = \omega_e(v_1, \ldots, v_n)$$

But since dL_x is invertible for all x, this is a bijection between n-forms and $(\Lambda^n T_e M)^* \cong \mathbb{R}$, so that left invariant forms are unique up to scalar multiples, this also proves existence of left invariant forms. Now, given a left invariant form ω we can observe that for any $a \in G$ we have $R_a^* \omega$ is left invariant, since $(R_a^* \circ R_b^*)(\omega) = R_{ab}(\omega)$, identifying a to the constant multiple given by this process is a continuous group homomorphism to $\mathbb{R}^*$, whence the image is a compact connected subgroup of $\mathbb{R}^*$, but this is only possible if the image is $\{1\}$, so that ω is right invariant.

Now letting ω be a Bi-invariant metric on G, and g a left invariant Riemannian metric, we can define

$$g_{\text{inv}}(u, v)_y = \int_{x \in G} g(d_y R_x u, d_y R_x v)_{yx} \omega$$

Linearity of the integral, along with ω positive on an oriented basis of the identity ensures this is symmetric, bilinear and positive definite and left invariant, thus a left invariant Riemannian metric. The right invariance is simply the reindexing trick from Maschke's theorem. □

1.3 Affine Connections

Definition 1.10. An affine connection ∇ on a C^α manifold M is a map satisfying for $f, g \in C^\infty(M)$

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$
$$(X, Y) \mapsto \nabla_X Y$$

1. $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

Definition 1.11. If M is a manifold with an affine connection and a curve $\gamma : I \to M$, the covariant derivative is the unique correspondance taking a $\mathcal{X}(I) \cap \{V(t) \in \langle\gamma'(t)\rangle\} \to \mathcal{X}(I) \cap \{V(I) \in \langle\gamma'(t)\rangle\}$ satisfying

1. $D/dt(V + W) = D/dtV + D/dtW$
2. $D/dtfV = \frac{df}{dt}V + f\frac{DV}{dt}$
3. $V = Y|_{\gamma(I)}$ for $Y \in \mathcal{X}(M)$, then $\frac{DV}{dt} = \nabla_{\gamma'(t)}Y$

Intuitively the covariant derivative gives the acceleration of the curve, existence and uniqueness follow form writing DV/dt explicitly in local coordinates.