

1. We first find an expression for $\zeta(s)\Gamma(s)$ on $\Re(s) > 1$, to do so we will do a substitution $u = x/n$, valid for any $n \in \mathbb{Z}_{>0}$

$$\Gamma(s) = \int_0^\infty e^{-nu} (nu)^{s-1} n du = n^s \int_0^\infty u^{s-1} (e^{-u})^n du$$

Now since I don't like u I will switch back to x ; Multiplying n^{-s} on both sides and summing over $n \in \mathbb{Z}_{>0}$ yields

$$\zeta(s)\Gamma(s) = \sum_1^\infty \int_0^\infty x^{s-1} (e^{-x})^n dx \stackrel{\text{DCT}}{=} \int_0^\infty x^{s-1} \sum_1^\infty (e^{-x})^n dx = \int_0^\infty x^{s-1} \frac{1}{e^x - 1} dx \quad (1)$$

Where DCT is taken with respect to $|x^{s-1}| \sum_1^\infty e^{-nx}$. We can use $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to rewrite (1):

$$\zeta(s) = \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad (2)$$

Now we deal with the contour integral, letting $C(\epsilon)$ denote the curve described in the problem for fixed $\epsilon \in \mathbb{R}_{>0}$. The expression $e^z - 1$ has no poles away from $\{z \mid e^z = 1\} = \{2\pi ki \mid k \in \mathbb{Z}\}$, whence if we take the branch cut of log away from the non-negative reals the integral is not dependent on ϵ for $\epsilon < 2\pi$, since (for $\delta < \epsilon < 2\pi$) we have the area enclosed between the two curves is a quotient of holomorphic functions with the denominator non-vanishing in between the curves. This independence is a result of taking separating the curve $C(\epsilon) - C(\delta)$ into two curves (see picture), the first of which has integral zero by Cauchy's theorem, and the second being arbitrarily small depending on where we take the cut.

$$\int_{C(\epsilon)} f - \int_{C(\delta)} f = \int_{C(\epsilon, \delta, M)} f + \int_{\gamma_1(\epsilon, \delta, M)} f + \int_{\gamma_2(\epsilon, \delta, M)} f$$

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Then for f holomorphic away from the real line, the $C(\epsilon, \delta, M)$ term vanishes. Notice now that using the standard arclength inequality for large M we have

$$\begin{aligned} \left| \int_{\gamma_j(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| &\leq (\epsilon - \delta) \left| \frac{M^{\Re(s)-1}}{e^M - 1} \right| + \left| \int_M^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx - \int_M^\infty \frac{(-x - i\delta)^{s-1}}{e^{x+i\delta} - 1} dx \right| \\ &\leq (\epsilon - \delta) \left| \frac{M^{\Re(s)-1}}{e^M - 1} \right| + 2 \int_M^\infty \frac{|x + i\epsilon|^{\Re(s)-1}}{|e^x| - 1} dx \end{aligned}$$

The right hand side clearly converges to zero. as $M \rightarrow \infty$ using basic limits of exponentials and DCT. This gives the desired invariance.

$$\left| \int_{C(\epsilon)} \frac{(-z)^{s-1}}{e^z - 1} dz - \int_{C(\delta)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| \leq \left| \int_{\gamma_1(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| + \left| \int_{\gamma_2(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| = 0$$

Now, we can compute the value of the integral along this curve by letting $\epsilon \rightarrow 0$, to get $C(0)$, a ray from ∞ to 0 where $(-z)^{s-1} = x^{s-1}e^{-(s-1)\pi i}$, and a ray from 0 to ∞ where $(-z)^{s-1} = x^{s-1}e^{(s-1)\pi i}$, to see that we can indeed pass to this limit, once again decompose $C = C(\epsilon)$ into three curves, with C_1 the ray in the upper half plane, C_2 the ray in the lower half plane and C_3 the circular portion, then once again using the arc length inequality and the fact that $e^z - 1 = \mathcal{O}(z)$

$$\left| \int_{C_3} \frac{(-z)^{s-1}}{e^z - 1} dz \right| \leq 2\pi\epsilon \sup_{|z|=\epsilon} \frac{|z^{s-1}|}{|e^z - 1|} = 2\pi\epsilon \mathcal{O}(\epsilon^{\Re(s)-2}) = 2\pi\mathcal{O}(\epsilon^{\Re(s)-1}) \xrightarrow{\Re(s)>1} 0$$

We can check convergence on C_1 explicitly

$$\begin{aligned} & \left| \int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx - \int_\epsilon^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx \right| \\ & \leq \left| \int_0^\epsilon \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx \right| + \left| \int_\epsilon^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx - \int_\epsilon^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx \right| \\ & \leq \left| \int_0^\epsilon \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx \right| + \left| \int_\epsilon^\infty \frac{(e^{x+i\epsilon} - 1)x^{s-1} e^{-(s-1)\pi i} - (e^x - 1)(\sqrt{x^2 + \epsilon^2})^{(s-1)} e^{i(s-1) \arctan \frac{\epsilon}{x} - (s-1)\pi i}}{(e^x - 1)(e^{x+i\epsilon} - 1)} dx \right| \end{aligned}$$

Convergence as $\epsilon \rightarrow 0$ of the big ugly term to zero is actually simple from convergence of each of the terms in the two expressions in the product. Convergence of the first term follows from $\Re(s) > 1$, so writing the bounds of integration as $\chi_{(0,\epsilon)}$ we can just apply DCT to the absolute value of the integrand. The proof of convergence for C_2 is similar to C_1 .

Now we finally established that (due to taking the limit in ϵ and invariance with respect to ϵ)

$$\int_C \frac{(-z)^{s-1}}{e^z - 1} dz = - \int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx + \int_0^\infty \frac{x^{s-1} e^{(s-1)\pi i}}{e^x - 1} dx \quad (3)$$

$$= \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin((s-1)\pi) = - \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin(s\pi) \quad (4)$$

Multiplying (2) by $1 = \frac{\int_C \frac{(-z)^{s-1}}{e^z - 1} dz}{- \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin(s\pi)}$ yields the desired equality

$$\zeta(s) = - \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz \quad (5)$$

Now, using the right side of (5) as an analytic continuation, we first compute $(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)$, since we know it has a simple pole at zero, hence is meromorphic this expression just comes from evaluating the systems of equations given for the coefficients for $(a_{-1}z^{-1} + a_0 + a_1z + \dots)(\sum_0^\infty \frac{z^n}{n!}) = 1$. This is everything we need to evaluate $\zeta(0)$. First using a similar separation of curves into two parts as was used for invariance, we find that $\int_C \frac{1}{z(e^z - 1)} dz = \int_\gamma \frac{1}{z(e^z - 1)} dz$ where γ is a piecewise C^1 closed curve, this allows us to use the residue theorem

$$\zeta(0) = \frac{\Gamma(1)}{2\pi i} \int_\gamma \frac{1}{z(e^z - 1)} dz = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2) \right) dz \quad (6)$$

$$= \text{Res} \left(\frac{1}{z} \left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2) \right) \right) = -\frac{1}{2} \quad (7)$$

□

2. (a) Let ϵ be small enough so that $B_{2\epsilon}(z_0) \subset D$, then

$$\int_{\partial D} \frac{f(w)}{w - z_0} dw - \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w - z_0} dw \stackrel{\text{Stokes}}{=} \int_{D \setminus B_\epsilon(z_0)} \frac{\partial}{\partial \bar{w}} \frac{f(w)}{w - z_0} d\bar{w} \wedge dw = 0$$

By Cauchy Riemann since $\frac{f(w)}{w - z_0}$ is holomorphic in $D \setminus B_\epsilon(z_0)$. From this, we can prove the base case which is Cauchy's integral formula.

$$\int_{\partial D} \frac{f(w)}{w - z_0} dw = \int_{\partial B_\epsilon(z_0)} \frac{f(w)}{w - z_0} dw = \int_0^{2\pi} f(z_0 + \epsilon e^{it}) i dt$$

Which holds for arbitrary ϵ , but since f is bounded (on D) and continuous we have

$$\int_{\partial D} \frac{f(w)}{w - z_0} dw = \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} f(z_0 + \epsilon e^{it}) i dt = \int_0^{2\pi} \lim_{\epsilon \rightarrow 0} f(z_0 + \epsilon e^{it}) i dt = 2\pi i f(z_0)$$

Which proves the desired result in the case of $n = 0$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0} dw \quad (8)$$

Now, we can proceed by induction and simply take derivatives.

$$\begin{aligned} 2\pi i \lim_{h \rightarrow 0} \frac{1}{h} (f^{(n)}(z_0 + h) - f^{(n)}(z_0)) &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{\partial D} \frac{f(w)}{(w - z_0 - h)^{n+1}} - \frac{f(w)}{(w - z_0)^{n+1}} dw \\ &= \lim_{h \rightarrow 0} \int_{\partial D} \frac{1}{h} \frac{f(w)(w - z_0)^{n+1} - f(w)(w - z_0 - h)^{n+1}}{(w - z_0 - h)^{n+1}(w - z_0)^{n+1}} dw \\ &= \lim_{h \rightarrow 0} \int_{\partial D} \frac{f(w)(w - z_0)^n}{(w - z_0 - h)^{n+1}(w - z_0)^{n+1}} + \mathcal{O}(h) dw \end{aligned}$$

Then we can clearly apply DCT for $|h| < \frac{1}{2} |w - z_0|$, which gives us

$$\lim_{h \rightarrow 0} \frac{1}{h} (f^{(n)}(z_0 + h) - f^{(n)}(z_0)) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z_0)^{n+2}} dw \quad (9)$$

(b) Let $z \in \mathbb{C}$, then by equation (9) in part (a)

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2} dw$$

Applying the standard arc-length inequality yields

$$|f'(z)| \leq \frac{1}{2\pi} 2\pi R \frac{\sup_{\mathbb{C}} |f|}{R^2} = \frac{\sup_{\mathbb{C}} |f|}{R} \xrightarrow{R \rightarrow \infty} 0 \quad (10)$$

Therefore $f' \equiv 0$, for any two points in \mathbb{C} , to see this implies f is constant we use the fundamental theorem of calculus, if $z_0, z_1 \in \mathbb{C}$ take γ to be the straight line starting at z_0 and ending at z_1 so that

$$|f(z_1) - f(z_0)| = \left| \int_{\gamma} f'(z) dz \right| \leq \ell(\gamma) \sup_{\mathbb{C}} |f'| = 0$$

□

(c) Suppose a polynomial P does not have a root. Then $\frac{1}{P}$ is entire, hence $1/P$ is constant by Liouville's theorem. We conclude any polynomial without a root is of degree zero. □

3. Assume for contradiction that f has an essential singularity at z_0 , but there exists some $a \in \mathbb{C}$ and $\delta > 0$ such that for some $\epsilon > 0$ we have $f(D_{\epsilon}^*)(z_0) \cap B_{\delta}(a) = \emptyset$. Now define a new function $g(z) = \frac{1}{f(z)-a}$, since $|f(z) - a| \geq \delta$ on $D_{\epsilon}^*(z_0)$ we find that g is holomorphic and nonvanishing on this punctured annulus with modulus bound above by $\frac{1}{\delta}$. Then we can define on $D_{\epsilon}(z_0)$

$$(z - z_0)^2 g(z) = \begin{cases} (z - z_0)^2 g(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases} \quad (11)$$

This is holomorphic away from z_0 , thus is holomorphic since

$$\lim_{h \rightarrow 0} \frac{(z_0 + h - z_0)^2 g(z_0 + h)}{h} = \lim_{h \rightarrow 0} h g(z_0 + h) = 0$$

where the last equality is a consequence of g being bounded. Therefore we can write

$$g(z)(z - z_0)^2 = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

which gives the following expression on D_ϵ^* for $b_k = a_{k+2}$

$$g(z) = \sum_{-2}^{\infty} b_k (z - z_0)^k = \sum_0^{\infty} b_k (z - z_0)^k \quad (12)$$

Where $b_{-2} = b_{-1} = 0$ follows from g being bounded near z_0 . Moreover since g is nonvanishing, we must have some $b_N \neq 0$, it follows that

$$\lim_{z \rightarrow z_0} (z - z_0)^{N+1} (f(z) - a) = \lim_{z \rightarrow z_0} \frac{(z - z_0)^{N+1}}{g(z)} \stackrel{(12)}{=} \lim_{z \rightarrow z_0} \frac{z - z_0}{b_N + \mathcal{O}(z - z_0)} = 0 \quad (13)$$

Now we can show directly from the definition that f has a pole

$$\lim_{z \rightarrow z_0} (z - z_0)^{N+1} f(z) = \lim_{z \rightarrow z_0} (z - z_0)^{N+1} (f(z) - a) \stackrel{(13)}{=} 0$$

Which is the desired contradiction. \square

4. Denote the annulus in question as A , the outer disc as D_{r_2} and the inner disc as D_{r_1} , then for $z_0 \in A^\circ$, we can let γ be a ray from D_{r_1} to D_{r_2} not intersecting z_0 . It follows that $\partial D_{r_2} - \gamma + \gamma - \partial D_{r_1}$ is a closed piecewise C^1 curve in A around z_0 , so that by Cauchy's integral formula (8):

$$f(z_0) = \int_{\partial D_{r_2} - \gamma + \gamma - \partial D_{r_1}} \frac{f(w)}{w - z_0} dw = \int_{\partial D_{r_2}} \frac{f(w)}{w - z_0} dw - \int_{\partial D_{r_1}} \frac{f(w)}{w - z_0} dw \quad (14)$$

Now we can apply algebraic manipulations, since on ∂D_{r_2} we have $|w| > |z_0|$ we get the following uniformly convergent power series expansion

$$\frac{f(w)}{w - z_0} = \frac{f(w)}{w} \left(\frac{1}{1 - \frac{z_0}{w}} \right) = \frac{f(w)}{w} \sum_0^{\infty} \left(\frac{z_0}{w} \right)^k \quad (15)$$

Similarly, on ∂D_{r_1} $|z_0| > |w|$, so that there is a uniformly convergent power series expansion given by

$$-\frac{f(w)}{w - z_0} = \frac{f(w)}{z_0} \left(\frac{1}{1 - \frac{w}{z_0}} \right) = \frac{f(w)}{z_0} \sum_0^{\infty} \left(\frac{w}{z_0} \right)^k \quad (16)$$

Combining (14), (15) and (16) along with uniform convergence in (15) and (16) furnishes

$$2\pi i f(z_0) = \int_{\partial D_2} \frac{f(w)}{w} \sum_0^{\infty} \left(\frac{z_0}{w} \right)^k dw + \int_{\partial D_1} \frac{f(w)}{z_0} \sum_0^{\infty} \left(\frac{w}{z_0} \right)^k dw \quad (17)$$

$$= \sum_0^{\infty} z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw + \sum_0^{\infty} \frac{1}{z_0^{k+1}} \int_{\partial D_{r_2}} f(w) w^k dw \quad (18)$$

$$= \sum_0^{\infty} z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw + \sum_{-\infty}^1 z_0^k \int_{\partial D_{r_1}} \frac{f(w)}{w^{k+1}} dw \quad (19)$$

Now letting $r \in (r_1, r_2)$, we can use the same trick with a ray γ used in (14) to apply Cauchy's theorem and find that

$$\sum_0^{\infty} z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw - \sum_0^{\infty} z_0^k \int_{\partial D_{r_2}} \frac{f(w)}{w^{k+1}} dw = 0 \quad (20)$$

$$\text{and } \sum_{-\infty}^1 z_0^k \int_{\partial D_{r_1}} \frac{f(w)}{w^{k+1}} dw - \sum_{-\infty}^1 z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw = 0 \quad (21)$$

Where application of Cauchy's theorem follows by $\sum_0^\infty z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw$ analytic on $B_{r_2}(0)$ and similarly $\sum_{-\infty}^1 z_0^k \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw$ analytic on $\overline{B_{r_1}(0)}^c$. Now we can substitute into (19) using (20) and (21) to obtain

$$f(z_0) = \sum_{-\infty}^{\infty} z_0^k \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(w)}{w^{k+1}} dw \quad (22)$$

□

5. From the chain rule we have $f \circ (w \mapsto 1/w)$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Suppose for contradiction that $f \circ (w \mapsto 1/w)$ has an essential singularity at zero, then by Casorati-Weierstrass $f(\overline{B_\epsilon(0)})^c = f \circ (w \mapsto \frac{1}{w})(B_\epsilon(0) \setminus \{0\})$ is dense in \mathbb{C} , this gives us a contradiction since $f(B_\epsilon(0))$ is open by the open mapping principle, hence $f(B_\epsilon(0)) \cap f(\overline{B_\epsilon(0)})^c \neq \emptyset$ which contradicts injectivity of f . It follows that $f \circ (w \mapsto \frac{1}{w})$ has a pole at 0, and hence a Laurent series. Now write

$$f \circ (w \mapsto \frac{1}{w}) = \sum_{-N}^{\infty} a_k z^k \implies f(w) = \sum_{-N}^{\infty} a_k z^{-k}$$

f being entire implies that $a_j = 0$ for $j \geq 1$, hence $f = \sum_0^N a_{-k} z^k$, since $f'(z) = \sum_1^N a_{-k} z^{k-1}$ is nonvanishing, by the fundamental theorem of algebra f' must be a constant nonzero polynomial, hence $a_{-k} = 0$ for $k > 1$ and $f = a_{-1}z + a_0$ as desired. □

6. (i) Let $[z_0 : \dots : z_n] \in \mathbb{P}^n$, then $(z_0, \dots, z_n) \neq 0$, hence atleast one coordinate is nonzero, label that coordinate j to see that $[z_0 : \dots : z_n] \in U_j$ by definition. To check φ is a homeomorphism we can check it is a continuous and open bijection. Injectivity follows from the map φ_j choosing the unique equivalence class representative with j -th entry equal to one. To see surjectivity let $z = (z_0, \dots, 1, \dots, z_n) \in \mathbb{C}^n$, then $z = \varphi_j([z_0 : \dots : 1 : \dots : z_n])$. By definition of the quotient topology φ_j is continuous if and only if $\varphi_j \circ \pi$ is, same for open. To check this simply write $\varphi_j \circ \pi(z_0, \dots, z_n) = (z_0/z_j, \dots, z_n/z_j)$, since we are on $\{z_j \neq 0\}$ this map is smooth, and its Jacobian has a copy of $\frac{1}{z_j} 1_{n \times n}$, thus this map is a submersion, since submersions are open maps this map is smooth, open and a bijection hence a homeomorphism.

(ii) On $\varphi_j(U_i \cap U_j)$ we have

$$\varphi_i \varphi_j^{-1}(z_0, \dots, 1, \dots, z_n) = \left(\frac{z_0}{z_i}, \dots, \frac{1}{z_i}, \dots, \frac{z_n}{z_i} \right) \quad (23)$$

So each of the coordinate functions is of the form 1 (which is not actually considered in the image by identification with \mathbb{C}^n), $\frac{1}{z_i}$ or z_k/z_i for $k \neq i$. From here it is immediate, since $\frac{1}{z_i}$ is holomorphic on $\{z_i \neq 0\} \supset \varphi_i(U_i \cap U_j)$ we get for $\ell \neq k, i$ that

$$\frac{\partial}{\partial \bar{z}_\ell} \frac{1}{z_i} = \frac{1}{z_i} \frac{\partial}{\partial \bar{z}_\ell} 1 = 0 = z_k/z_i \frac{\partial}{\partial \bar{z}_\ell} 1 = \frac{\partial}{\partial \bar{z}_\ell} z_k/z_i$$

and by holomorphicity or $\frac{1}{z_i}$, as well as z_k

$$\frac{\partial}{\partial \bar{z}_k} \frac{z_k}{z_i} = 1/z_i \frac{\partial}{\partial \bar{z}_k} z_k = 0 \text{ and } \frac{\partial}{\partial \bar{z}_i} \frac{1}{z_i} = 0 = z_k \frac{\partial}{\partial \bar{z}_i} \frac{1}{z_i} = \frac{\partial}{\partial \bar{z}_i} \frac{z_k}{z_i}$$

□

7. I will explain it for π_N , π_S is clearly the same by symmetry (however we flip the convention $(x, y, 0) \rightsquigarrow x - iy$ to ensure holomorphicity). Stereographic projection geometrically is embedding $S^2 \hookrightarrow \mathbb{R}^3$ (i.e. defining S^2 as the equation in question), where we identify \mathbb{R}^2 with \mathbb{C} in the obvious way $(x, y, 0) \rightsquigarrow x + iy$, then we associate to each point (x_1, x_2, x_3) in $S^2 \setminus N$ the unique point where the $\mathbb{C} \rightsquigarrow \mathbb{R}^2 \times \{0\}$ such that the line through N and (x_1, x_2, x_3) intersects \mathbb{C} . The equation for π_N is simply parameterizing in terms of (x_1, x_2, x_3) the point where the line meets \mathbb{C} . Concretely we have the following parameterization

$$\ell(t) = t(x_1, x_2, x_3 - 1) + (0, 0, 1) \quad (24)$$

We get the point of projection by solving for t_0 such that $\ell(t_0) \in \mathbb{R}^2 \times \{0\}$, this is $t_0 = \frac{1}{1-x_3}$, now plugging this into (24) gives

$$\ell(t_0) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0 \right) \rightsquigarrow \frac{x_1 + ix_2}{1-x_3} \stackrel{\text{def}}{=} \pi_N(x_1, x_2, x_3) \quad (25)$$

From this geometric realization we can also compute π_S^{-1} as being the unique point on $S^2 \setminus S$ which intersects the line through $(x, -y, 0) \rightsquigarrow x + iy$, this gives the parameterized line

$$\ell'(t) = t(x, -y, 1) + (0, 0, -1) \quad (26)$$

so solving for t_0 where $\ell'(t_0) \in S^2$ we get

$$t_0^2(x^2 + y^2 + 1) - 2t_0 = 0 \quad (27)$$

Where the zeroes of (27) correspond to S and

$$\pi_S^{-1}(x + iy) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{-2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1} \right) \quad (28)$$

Now we can apply π_N to get

$$\pi_N \pi_S^{-1}(x + iy) = \frac{\frac{2(x-iy)}{x^2+y^2+1}}{1 - \frac{1-x^2-y^2}{1+x^2+y^2}} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \quad (29)$$

Which is indeed holomorphic on $\mathbb{C} \setminus \{0\}$. □

8. Note that the reason we only need to check on the charts $\mathbb{P}^1 \setminus \{[0 : 1]\}$, and $S^2 \setminus N$ are that away from S we have that f can be written in charts as $\phi_{\tilde{S}} \circ f \circ \pi_S^{-1} = 1_{\mathbb{C}^\times}$. To begin with we compute $\phi_{\tilde{N}} \circ f \circ \pi_N^{-1}$ we write $z = x + iy$

$$\phi_{\tilde{N}} \circ f \circ \pi_N^{-1}(x + iy) = \phi_{\tilde{N}} \circ \phi_{\tilde{S}}^{-1} \circ \pi_S \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{-2}{1 + x^2 + y^2} + 1 \right) \quad (30)$$

$$= \phi_{\tilde{N}} \circ \phi_{\tilde{S}}^{-1}(1/z) = \phi_{\tilde{N}}[1/z : 1] = z \quad (31)$$

This of course extends to the identity map from $\mathbb{C} \rightarrow \mathbb{C}$, so if we take $f(S) = [1 : 0]$, the extended map on $S^2 \setminus N \rightarrow \mathbb{P}^1 \setminus \{[1 : 0]\}$ charts is just the identity map $z \mapsto z$ which is holomorphic, so that every point in S^2 is contained in an open set with (the extension of) f holomorphic in coordinates on that open set. Moreover, by the holomorphic inverse function theorem (the extension of) f is biholomorphic, since the derivative in either pair of charts is just given by 1 which is non-vanishing.