1. If compactly supported cohomology were homotopy invariant, then we would require $H_c^*(\mathbb{R}^n) \cong H_c^*(\{\text{pt.}\})$ for all $n \in \mathbb{Z}_{\geq 0}$ since $\mathbb{R}^n \simeq \{\text{pt.}\}$ for all n. To further explicate this, Consider for each n, the homotopy

$$\begin{aligned} H : \mathbb{R}^n \times [0,1] &\to \mathbb{R}^n \\ (x,t) &\mapsto x(1-t) \end{aligned}$$

interpolates the maps $1_{\mathbb{R}^n}$ and the zero map. Then we could take $g : \{0\} \hookrightarrow \mathbb{R}^n$, and $f : \mathbb{R}^n \to \{0\}$, so that $gf = 1_{\{0\}}$, and fg is the zero map which we already showed is homotopy equivalent to $1_{\mathbb{R}^n}$.

Thus if compactly supported cohomology were a homotopy invariant we would have

$$H_c^*(\mathbb{R}^n) \cong H_c^*(\{0\})$$

choosing $n = 1 = *$, we get $\mathbb{R} \cong 0$ (as $\mathbb{R}$-vector spaces) by the poincare lemma. This is a contradiction. $\square$

2. (a) Let η_U, η_V be a partition of unity subordinate to U, V, then we can define the following maps making the sequence short exact:

$$0 \longrightarrow \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M) \longrightarrow 0$$

$$\omega \longrightarrow (\eta_V \cdot \omega, \eta_U \cdot \omega)$$

$$(\omega, \nu) \longrightarrow (\eta_U \cdot \omega, \eta_V \cdot \nu)$$

The first map has $\text{supp}(\eta_V \cdot \omega) \subset \text{supp}(\omega) \supset \text{supp}(\eta_U \cdot \omega)$, so there are no issues with the compact support, similarly for the second map $\text{supp}(\eta_U \cdot \omega + \eta_V \cdot \nu) \subset \text{supp}(\omega) \cup \text{supp}(\nu)$ which is a union of two compact sets hence compact.

To see the first map is an injection let $\omega \in \Omega_c^p(U \cap V)$ (for some p) and suppose that $(\eta_V \cdot \omega, \eta_U \cdot \omega) \equiv 0$, then $\eta_V \cdot \omega \equiv 0$ on U and $\eta_U \cdot \omega \equiv 0$ on V, this of course implies $(\eta_V \cdot \omega)|_{U \cap V} \equiv 0$ and $(\eta_U \cdot \omega)|_{U \cap V} \equiv 0$ on $U \cap V$, so the following easy computation shows injectivity,

$$(\eta_V \cdot \omega)|_{U \cap V} + (\eta_U \cdot \omega)|_{U \cap V} = \eta_V|_{U \cap V} \cdot \omega + \eta_U|_{U \cap V} \cdot \omega = (\eta_U|_{U \cap V} + \eta_V|_{U \cap V}) \cdot \omega = \omega$$

Now checking surjectivity of the second map, Let $\omega \in \Omega_c^p(M)$ for some p, then we have $\omega|_U \in \Omega_c^p(U)$ and $-\omega|_V \in \Omega_c^p(V)$, then I claim that the image of $(\omega|_U, -\omega|_V) = \eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega$. To check this, it suffices to check equivalence pointwise, so we can simply check on each of the sets $U \cap V^c$, $V \cap U^c$ and $U \cap V$, to see it on $U \cap V^c$ we have $\eta_V = 0$, and $\eta_U = 1$ so that $\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega|_U$ on this set, but since $U \cap V^c \subset U$, this is the same thing as ω here. Checking on $V \cap U^c$ is similar, finally on $U \cap V$, we have

$$\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = (\eta_U + \eta_V)\omega|_{U \cap V} = \omega|_{U \cap V}$$

which is of course just ω on $U \cap V$, this shows surjectivity.

Finally, we need to check that $\ker((\omega, \nu) \mapsto \eta_U \cdot \omega - \eta_V \cdot \omega) = \text{Im}(\omega \mapsto (\eta_V \cdot \omega, \eta_U \cdot \omega))$, checking the image is a subset of the kernel, is straightforward since composing both maps we get

$$\omega \mapsto \eta_U \cdot \eta_V(\omega - \omega) = 0$$

Now to check that all elements of the kernel are of this form, suppose $(\omega, \nu) \mapsto 0$, then wherever $\eta_V = 0$, we have $\eta_U \cdot \omega - \eta_V \cdot \nu = \omega$, but since we are assuming this is zero we must have $\operatorname{supp} \omega \subset \operatorname{supp} \eta_V$, the same argument shows that $\operatorname{supp} \nu \subset \operatorname{supp} \eta_U$. Now we define the following form α on $U \cap V$

$$\alpha = \begin{cases} \eta_V^{-1} \cdot \omega & \eta_U, \eta_V > 0 \\ \omega & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases}$$

Then $\operatorname{supp}\alpha \subset \operatorname{supp}\omega \cup \operatorname{supp}\eta$ is compact, to verify that α is indeed smooth, note that away from $\eta_V = 0$, this is clear, to see that its smooth near $\eta_V = 0$, we can use the kernel condition to give us $\eta_U \cdot \omega = \eta_V \cdot \nu$, when $\eta_U, \eta_V > 0$ this gives us $\eta_V^{-1}\omega = \eta_U^{-1}\nu$, but this shows that we are smooth near $\eta_V = 0$. Finally, we compute

$$\eta_V \cdot \alpha = \begin{cases} \omega & \eta_V > 0 \\ 0 & \eta_V = 0 \end{cases} = \omega$$

since $\operatorname{supp}\omega \subset \operatorname{supp}\eta_V$, similarly

$$\begin{aligned} \eta_U\alpha &= \begin{cases} \eta_U\eta_V^{-1} \cdot \omega & \eta_V, \eta_U > 0 \\ 0 & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases} = \begin{cases} \eta_V\eta_V^{-1} \cdot \nu & \eta_V, \eta_U > 0 \\ 0 & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases} \\ &= \nu \end{aligned}$$

Once again, the last equality follows from $\operatorname{supp}\nu \subset \operatorname{supp}\eta_U$. This suffices to show exactness of the previous sequence. $\square$

(b) Since Theorem 1.4, Lecture 22 in the notes holds for any short exact sequence of cochain complexes, it in particular holds for

$$0 \longrightarrow \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M) \longrightarrow 0$$

The induced long exact sequence is exactly the desired one. $\square$

3. I will denote ρ to be the covering map $\mathbb{R}^2 \to \mathbb{T}^2$.

(a) Since $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$, we get by problem 2 of homework 6 it is orientable, since its a codimension 1 submanifold of $\mathbb{R}^3$, since its orientable it has a trivial determinant bundle so we can just take the section $t(x) = (x, 1)$. Now we can choose an orientation on $\mathbb{R}^2$ using the section $s : x \mapsto (x, s'(x))$ where $s'(x) = \pi(\det d\rho(1))$ (here π denotes the map throwing out the base point), s is smooth from construction, and is nonvanishing since ρ is a local diffeomorphism, which implies that $\det d\rho$ is an isomorphism on each fiber, but now by construction we have ρ orientation preserving, since $\pi \det d\rho(s) = (\pi \det d\rho(1))^2 > 0$.

Now that we have shown ρ is orientation preserving, we can use the fact that $A : \mathbb{R}^2 \to \mathbb{R}^2$ is orientation preserving when $\det A = 1$ and orientation reversing when $\det A = -1$ alongside the following diagram, to conclude that the top path is orientation reversing when $\det A = -1$, and orientation preserving when $\det A = 1$ (for extra justification use functoriality of determinant and the fact that $\det \rho > 0$), by commutativity of the diagram, this means that ϕ_A is orientation preserving iff $\det A = 1$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\rho"] & \mathbb{R}^2 \arrow[d, "\rho"] \\ \mathbb{T}^2 \arrow[r, "\phi_A"] & \mathbb{T}^2 \end{tikzcd}$$

$\square$

(b) In part (a), we verified that $SL_2(\mathbb{Z}) \to \Gamma_{1,0}$ via the map, it remains to check its a homomorphism. The following diagram commutes by definition of ϕ_X being induced by X for each $X \in SL_2(\mathbb{Z})$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\rho"] & \mathbb{R}^2 \arrow[r, "B"] \arrow[d, "\rho"] & \mathbb{R}^2 \arrow[d, "\rho"] & \rightsquigarrow & \mathbb{R}^2 \arrow[r, "BA"] \arrow[d, "\rho"] & \mathbb{R}^2 \arrow[d, "\rho"] \\ \mathbb{T}^2 \arrow[r, "\phi_A"] & \mathbb{T}^2 \arrow[r, "\phi_B"] & \mathbb{T}^2 & & \mathbb{T}^2 \arrow[r, "\phi_B\phi_A"] & \mathbb{T}^2 \end{tikzcd}$$

But by definition the induced map in the right diagram is ϕ_{BA}, so that $\phi_B\phi_A = \phi_{BA}$, this is exactly the homomorphism property.

(c) If $A = B$, then $\phi_A = \phi_B$, so clearly they are isotopic and we are done. Conversely, assume $\phi_A \simeq \phi_B$, we want to write down how ϕ_A acts on cohomology.

We start by defining 1-forms on $\mathbb{T}^2$, consider the standard one forms dx, dy on $\mathbb{R}^2$, we get well defined forms from the restriction (it will suffice to check just for dx). To check this, take open sets $U_1, \ldots, U_s \subset \mathbb{R}^2$ with $\bigcup_1^s \rho(U_j) = \mathbb{T}^2$, and $\rho|_{U_j}$ is a diffeomorphism to its image, denote the inverse as ρ_j^{-1}. The existence of such sets follows from ρ being a covering map and we can guarantee finitely many since $\mathbb{T}^2$ is compact. Now let η_j be a partition of unity subordinate to the U_j, then we can define $dx' = \sum_1^s \eta_j \cdot (\rho_j^{-1})^* dx$, then since ρ is a local diffeomorphism any tangent vector in $T\mathbb{T}^2$ is locally of the form $d\rho(v)$ for $v \in T\mathbb{R}^2$, then

$$dx'(d\rho(v)) = \sum_1^s \eta_j \cdot (\rho_j^{-1})^* dx d\rho(v) = \sum_1^s \eta_j (dx)(d\rho_j^{-1})(d\rho)(v)$$

Then since dx is translation invariant, and $(d\rho_j^{-1})(d\rho) = d(x \mapsto x + g)$ for some $g \in \mathbb{Z}^2$, there are no issues with what point we are taking dx at, so that the sum simplifies to $\sum_1^s \eta_j dx(v) = dx(v)$, so the form descends to $\mathbb{T}^2$.