

Recursion Formula for Siegel Veech Constants

Following Section 8 of Eskin, Masur, Zorich

Tighe McAsey

February 22, 2026

1 Review and Vision

We begin by recalling the purpose of our investigations, namely the computation of Siegel Veech constants. Our approach following EMZ is to compute volumes of Strata. From which we can recover the Siegel Veech constants.

Recall 1.1. *[Can either skip over this or go over quickly]*

- Given a surface (of genus g) with abelian differential (S, ω) , S is given as a surface of translation with a finite number of conic singularities $\{z_1, \dots, z_k\}$, with multiplicities m_1, \dots, m_k
- α (a partition of $2g - 2$) is a vector recording the conic angles at the singularities.
- A configuration \mathcal{C} of multiplicity p records the orders of two zeroes z_1 and z_2 joined by p saddle connections (recall generically these are homologous since by definition they have the same holonomy), $\mathcal{C} = (m_1, m_2, a_1, \dots, a_{p-1}, a'_1, \dots, a'_{p-1})$ where m_1 is the cone angle at z_1 , m_2 is the cone angle at z_2 and the angle between γ_j and γ_{j+1} at z_1 is $2\pi(a_j + 1)$ and the angle between them at z_2 is $2\pi(a'_j + 1)$.
- We are working in the strata of the form $\mathcal{H}_1(\alpha)$ which is the subset of the space of abelian differentials (S, ω) such that the zeroes of ω have configuration α , with S having unit surface area.
- Local coordinates on $\mathcal{H}_1(\alpha)$ are given by choosing a basis $\gamma_1, \dots, \gamma_n$ for the relative homology $H_1(S, \{z_1, \dots, z_k\}; \mathbb{C})$ for which each basis element is the homology class of a saddle connection. Then coordinates are given by:

$$(S, \omega) \mapsto \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_n} \omega \right) \in \mathbb{C}^n \rightsquigarrow \mathbb{R}^{2n}$$

- The measure μ on $\mathcal{H}_1(\alpha)$ is given by lebesgue measure in these coordinates, normalized so that $\mu(I^{2n}) = 1$.
- We are interested in counting the asymptotics of the number of saddle connections of generic surfaces in $\mathcal{H}_1(\alpha)$. Namely the number of saddle connections having configuration \mathcal{C} and length less than L , under the image of the developing map taking $\gamma \mapsto \text{hol}(\gamma) \in \mathbb{R}^2$ this set is denoted as $V_{\mathcal{C}}(S) \cap B_L$.

Recall 1.2. • Siegel Veech constants are defined as follows (existence of such a constant is a result of Eskin and Masur)

$$c(\alpha, \mathcal{C}) := \lim_{L \rightarrow \infty} \frac{\#(V_{\mathcal{C}}(S) \cap B_L)}{\pi L^2}$$

- Last time we saw Richard present the proof of the formula for connected components of stratum

$$c(\alpha, \mathcal{C}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon^2} \frac{\text{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))}{\text{Vol}(\mathcal{H}_1(\alpha))}$$

Goal 1.1. Further develop this methodology of computing Siegel-Veech constants, namely we would like to understand how to compute $\text{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$.

2 Approach and Setup

Detail(s) 2.1. • Roughly I will be covering the simplest case for computing $\text{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$

- We continue to consider connected Strata
- I am only considering saddle connections of multiplicity 1, i.e. $\mathcal{C} = (m_1, m_2)$
- Later we will consider the picture for higher multiplicity?

Example(s) 2.1. Saddle connection of multiplicity 1:

INSERT IMAGE HERE

Saddle Connection of multiplicity > 1:

INSERT IMAGE HERE

Concept 2.1 (Principle Boundary). By shrinking a saddle connection γ (say between z_1 and z_2) on a surface of type $\alpha = (m_1, \dots, m_k)$, we collapse to a surface of type $\alpha' = (m_1 + m_2, m_3, m_4, \dots, m_k)$, $\mathcal{H}_1(\alpha')$ is called the principle boundary of $\mathcal{H}_1(\alpha)$.

INSERT IMAGE HERE

Goal 2.1. To understand the substance of **Lemma 8.1 (EMZ)** and its proof.

Roughly the idea is that given a surface with a short saddle connection, we can map it to its principle boundary. Assuming this saddle connection is short, the surface should not change too much, so that (given the data of the saddle connection) we could recover the original surface.

Assuming enough geometric information is preserved we hope to recover $\text{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$ in terms of $\text{Vol}(\mathcal{H}_1(\alpha'))$.

Theorem 2.1 (EMZ Lemma 8.1). Yes, Enough information is preserved.

Namely,