Recursion Formula for Siegel Veech Constants

Following Section 8 of Eskin, Masur, Zorich

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1 Review and Vision

We begin by recalling the purpose of our investigations, namely the computation of Siegel Veech constants. Our approach following EMZ is to compute volumes of Strata. From which we can recover the Siegel Veech constants.

Recall 1.1. *[Can either skip over this or go over quickly]*

- Given a surface (of genus g) with abelian differential (S, ω), S is given as a surface of translation with a finite number of conic singularities $\{z_1, \dots, z_k\}$, with multiplicities $m_1, \dots, m_k$
- α (a partition of $2g-2$) is a vector recording the conic angles at the singularities.
- A configuration $\mathcal{C}$ of multiplicity p records the orders of two zeroes z_1 and z_2 joined by p saddle connections (recall generically these are homologous since by definition they have the same holonomy), $\mathcal{C} = (m_1, m_2, a_1, \dots, a_{p-1}, a'_1, \dots, a'_{p-1})$ where m_1 is the cone angle at z_1, m_2 is the cone angle at z_2 and the angle between γ_j and γ_{j+1} at z_1 is $2\pi(a_j + 1)$ and the angle between them at z_2 is $2\pi(a'_j + 1)$.
- We are working in the strata of the form $\mathcal{H}_1(\alpha)$ which is the subset of the space of abelian differentials (S, ω) such that the zeroes of ω have configuration α, with S having unit surface area.
- Local coordinates on $\mathcal{H}^1(\alpha)$ are given by choosing a basis $\gamma_1, \dots, \gamma_n$ ($n = 4g+2k-2$) for the relative homology $H_1(S, \{z_1, \dots, z_k\}; \mathbb{C})$ for which each basis element is the homology class of a saddle connection. Then coordinates are given by:

$$(S, \omega) \mapsto \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_n} \omega \right) \in \mathbb{C}^n \leftrightsquigarrow \mathbb{R}^{2n}$$

- The measure μ on $\mathcal{H}_1(\alpha)$ is given by lebesgue measure in these coordinates, normalized so that $\mu(I^{2n}) = 1$.
- We are interested in counting the asymptotics of the number of saddle connections of generic surfaces in $\mathcal{H}_1(\alpha)$. Namely the number of saddle connections having configuration $\mathcal{C}$ and length less than L, under the image of the developing map taking $\gamma \mapsto \mathrm{hol}(\gamma) \in \mathbb{R}^2$ this set is denoted as $V_{\mathcal{C}}(S) \cap B_L$.

- $\mathcal{H}_1^\epsilon(\alpha)$ is defined as the subset of the unit strata with configuration α having a saddle connection of length at most ϵ.

Recall 1.2.
- Siegel Veech constants are defined as follows (existence of such a constant is a result of Eskin and Masur)

$$c(\alpha, \mathcal{C}) := \lim_{L\to\infty} \frac{\#\left(V_{\mathcal{C}}(S) \cap B_L\right)}{\pi L^2}$$

- Last time we saw Richard present the proof of the formula for connected components of stratum

$$c(\alpha, \mathcal{C}) = \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2} \frac{\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}$$

Goal 1.1. Further develop this methodology of computing Siegel-Veech constants, namely we would like to understand how to compute $\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$.

2 Approach and Setup

Detail(s) 2.1.
- Roughly I will be covering the simplest case for computing $\mathrm{Vol}(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C}))$
- We continue to consider connected Strata
- I am only considering saddle connections of multiplicity 1, i.e. $\mathcal{C} = (m_1, m_2)$
- Later we will consider the picture for higher multiplicity?

Example(s) 2.1. Saddle connection of multiplicity 1:

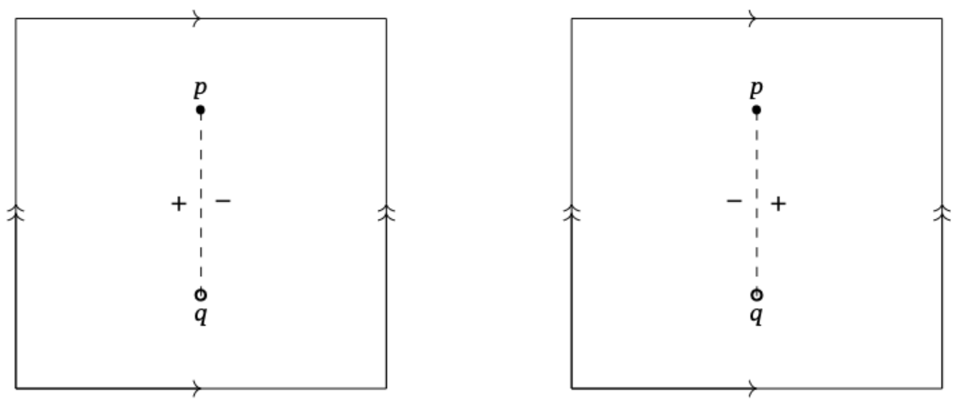
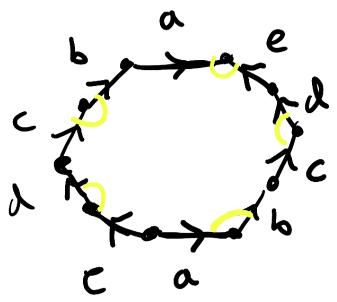


Figure 2.5. A translation surface obtained by gluing two identical square tori along a slit, with each square having opposite sides identified by translation, and the slit gluing indicated by + and −. This yields a genus 2 translation surface with two cone points of angle 4π, at the points p and q.

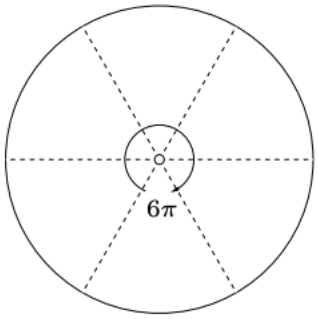
Saddle Connection of multiplicity > 1:

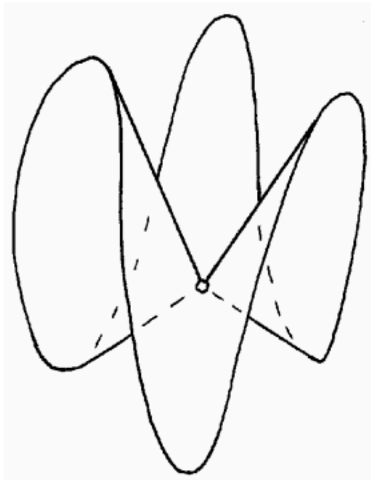
saddle connection multiplicity 5:

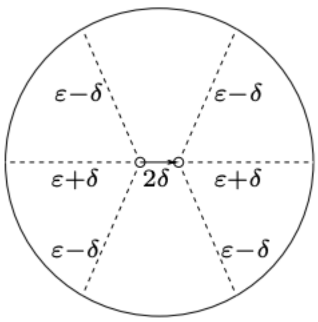


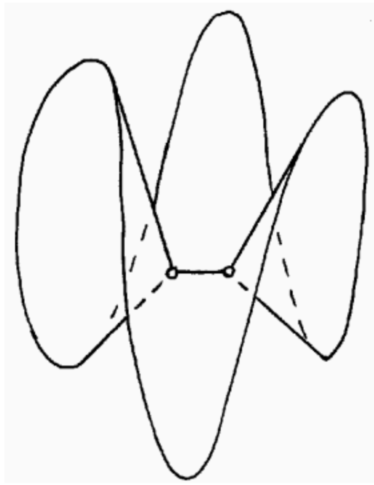
By looking at the yellow, there are two distinct vertices.

Concept 2.1 (Principle Boundary)**.** By shrinking a saddle connection γ (say between z_1 and z_2) on a surface of type $\alpha = (m_1, \ldots, m_k)$, we collapse to a surface of type $\alpha' = (m_1 + m_2, m_3, m_4, \ldots, m_k)$, $\mathcal{H}_1(\alpha')$ is called the principle boundary of $\mathcal{H}_1(\alpha)$.









Goal 2.1. To understand the substance of **Lemma 8.1 (EMZ)** and its proof.

Roughly the idea is that given a surface with a short saddle connection, we can map it to its principle boundary. Assuming this saddle connection is short, the surface should not change too much, so that (given the data of the saddle connection) we could recover the original surface.

Assuming enough geometric information is preserved we hope to recover $\mathrm{Vol}(\mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C}))$ in terms of $\mathrm{Vol}(\mathcal{H}_1(\alpha'))$.

Theorem 2.1 (EMZ Lemma 8.1 – Imprecise version)**.** Yes, Enough information is preserved.

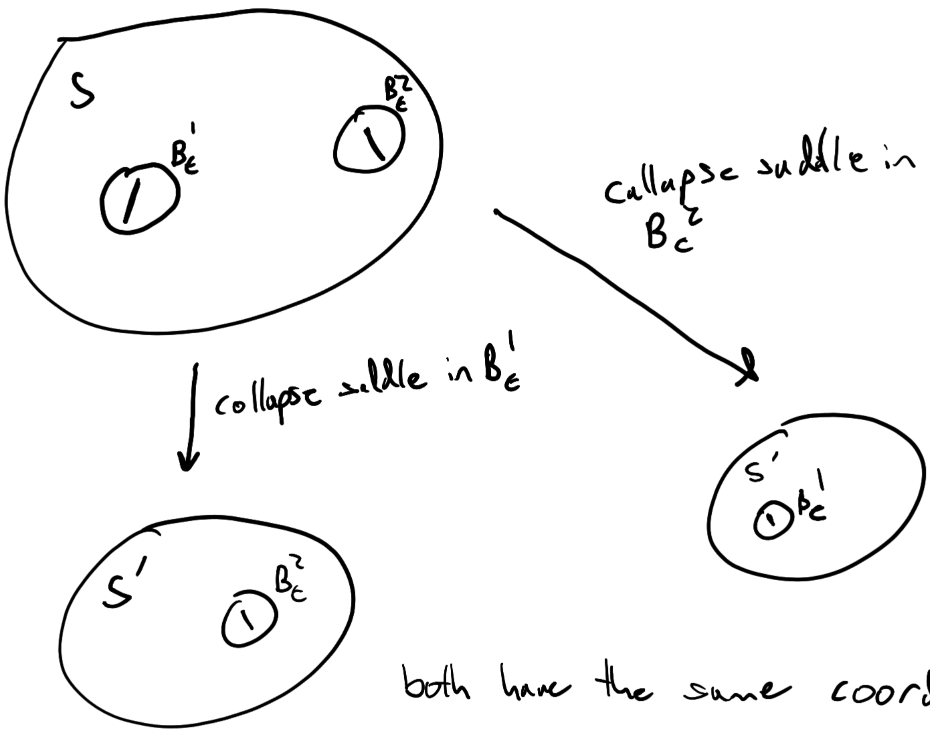
Namely, up to some error we have

$$\begin{aligned} \mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C}) &\to \mathcal{H}_1(\alpha') \times B_{\epsilon} \\ (S, m_1, m_2) &\mapsto (S', m, \gamma) \end{aligned}$$

is a covering map (of degree $m + 1$), and importantly the measure decomposes $\mu = \mu' \times \text{Lebesgue}$

Detail(s) 2.2. The error corresponds to requiring all saddle connections other than the one being collapsed to have length atleast ϵ. This is essential for the covering map to be well defined, since otherwise a stratum could have 2 distinct boundaries

when not in $\mathcal{H}_1^{\epsilon,\epsilon}(\alpha)$,



both have the same coordinates.

3 Proof of Covering and Precise Restatement

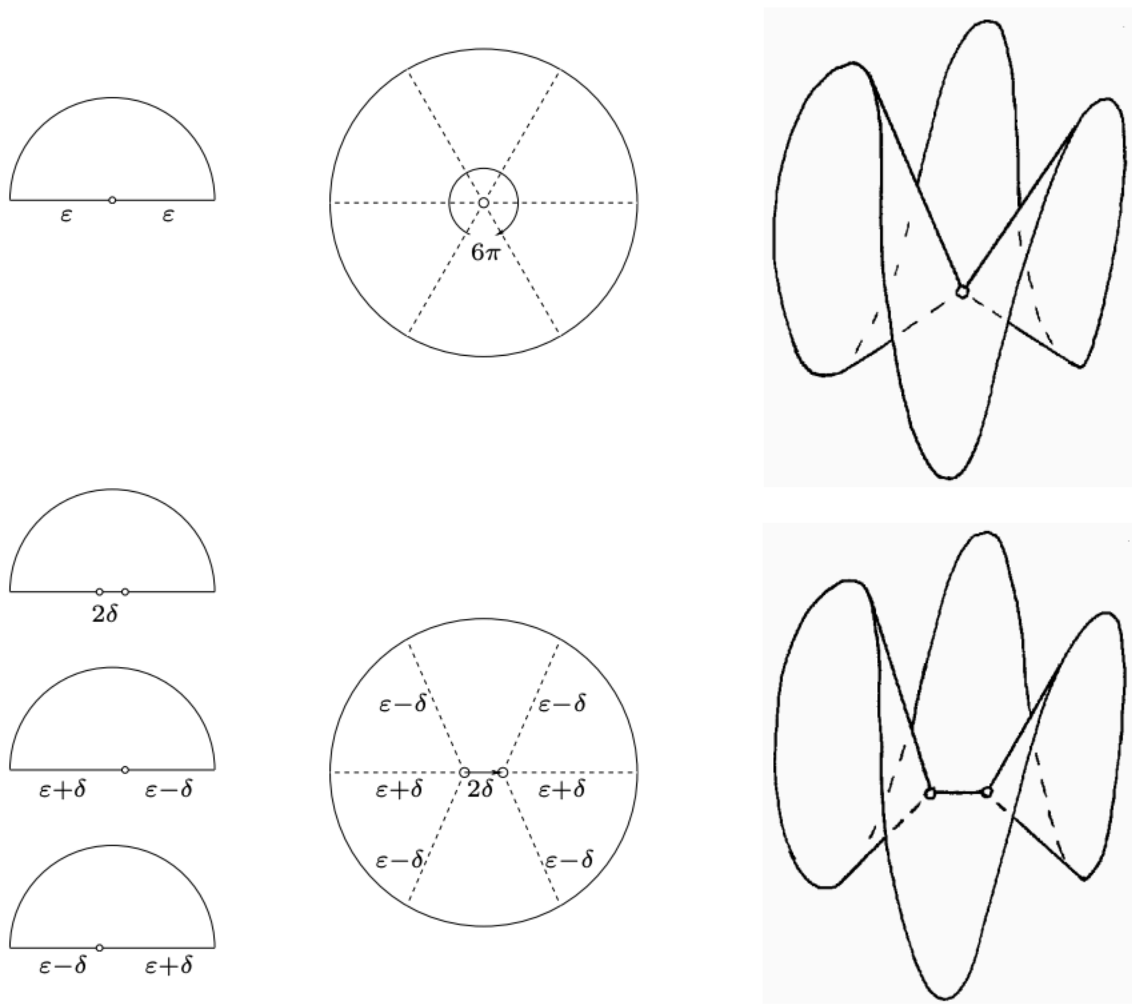
Proof. The covering map is given by collapsing a saddle connection, as previously specified, it is actually easier to understand through looking at the local inverse.

SETUP:

- the order of the zero x we are spliting is $m \geq 2$ and $m_1, m_2 \in \mathbb{Z}_{>0}$ with $m = m_1 + m_2$
- $\gamma \in B_\epsilon \subset \mathbb{R}^2$, with $|\text{hol}(\gamma)| = 2\delta$
- Assume that all other saddle connections and closed geodesics of S' have length atleast 2ϵ [*This corresponds to our assumption ensuring that the covering map is well defined, since the surface in our local inverse will have saddles of length atleast ϵ due to this assumption*]

- Let $\{x, z_1, \ldots, z_\ell\}$ be the zeroes of ω' the differential on S', since we will modify a small neighborhood of x, we take saddle connections whos classes generate the base of our homology $H_1(S, \{x, z_1, \ldots, z_\ell\}; \mathbb{Z})$ avoiding $B_\epsilon(x)$, if x is the only zero this can be done, if there are other zeroes then we require exactly one saddle (β) to have x as its endpoint
- Q: Why can we/do we have to do this? A: Recall since $(S, \{x, z_1, \ldots, z_\ell\})$ is a good pair (Hatcher) the relative homology is equivalent to the homology of the quotient space, in which case each saddle has a nontrivial homology class (e.g. look at the holonomy), and if none of $\gamma_1, \ldots, \gamma_n$ are saddle connections to x, then β will have an independent homology class (once again e.g. the holonomy can be linearly independent). Conversely if we include β, then any other saddle connection including x is homologous to a linear combination of saddle connections avoiding it (Proof - Just draw the 2-simplex).

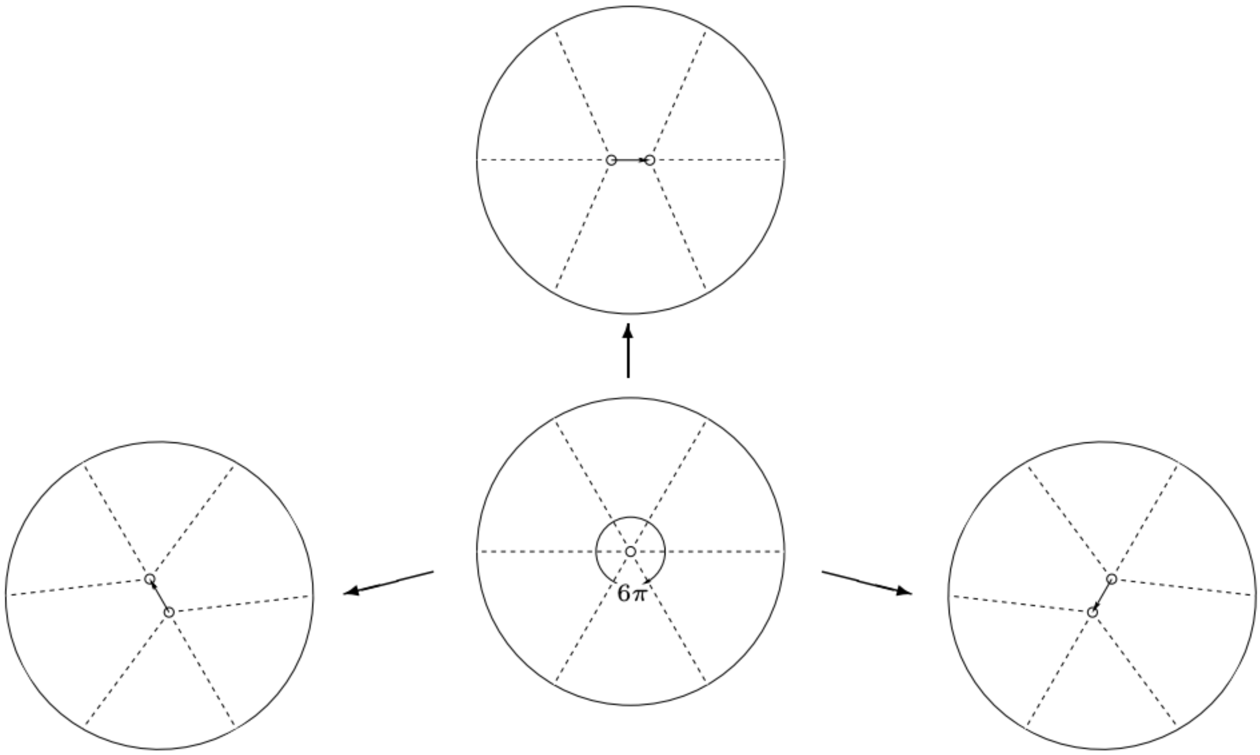
Proof:



Glue together $2m + 2$ half disks of radius ϵ, start by gluing the opposing discs along the new saddle connection γ

Considerations:

- This process is ambiguous up to choosing where γ is added, this depends on the choice of disks to be glued along gamma, of which there are $m+1$ pairs, each pair corresponds to 2 gluings by mirroring, but by marking which point is x_1 and x_2, fixing the holonomy of γ gives one choice. This is where the covering index of $m+1$ comes from.



- Since we perform this construction locally, the rest of the surface i.e. B_ϵ^c remains unchanged, namely the holonomy is unchanged on the homology basis of S' (which includes the homology basis of S) apart from hol(β) being changed by a factor of $-\gamma/2$. By the triangle inequality all of the lifts lie in $H_1^{\epsilon,\epsilon}(\alpha, m_1, m_2)$

Now we consider the case of collapsing zeroes, here we clearly see the error in the above being a covering map, the cover is ramified and maps only to a "large" subset of $\mathcal{H}^1(\alpha')$

SETUP:

- Let $S \in \mathcal{H}_1^{\epsilon,3\epsilon}(\alpha)$ i.e. S has a saddle connection γ connecting zeroes x_1, x_2 of orders m_1 and m_2, hol(γ) $\in B_\epsilon$, and all other saddle connections of S have length atleast 3ϵ.

- Considering the relative homology $H_1(S, \{x_1, x_2, z_1, \ldots, z_\ell\})$, similarly to adding a saddle connection, we choose our basis for relative homology away from B_ϵ(midpoint of γ), we furthermore take γ in this homology basis, similarly to last time if $\ell > 0$ we will need to take a saddle conection β connecting one of the z_j to x_1 in our basis.

Proof: We break up B_ϵ(midpoint of γ) into $2m+2$ half disks, in the exact same way as breaking up a zero. We then collapse the saddle connection, while leaving the manifold the same outside of the ϵ ball, equivalently we are reversing the breaking up process.

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Considerations:

- Since this construction is local, the holonomy of the basis vectors for $H_1(S', x, z_1, \ldots, z_\ell)$ is unchanged apart from β which gets replaced by a curve to x, hence the holonomy of β is changed by at most $\frac{1}{2}\text{hol}(\gamma)$
- From this we know that the every saddle connection on S' has length atleast 2ϵ

$\square$

Failure of this map to be a cover:

- To deal nicely with covering $\mathcal{H}_1(\alpha')$ we will actually work with (and cover) a large (in sense of measure) subset of $\mathcal{H}_1(\alpha')$, so that our coordinates are simple, and our map is surjective on this set.
- We also restricted the image of our mapping into $\mathcal{H}_1^{>2\epsilon}(\alpha')$, which should be accounted for.
- Finally, we would like to work with a singular coordinate system on $\mathcal{H}_1^{>2\epsilon}(\alpha')$, this makes it straightforward to see that the image of the map lies in $\mathcal{H}_1^{>2\epsilon}(\alpha') \times B_\epsilon$
- We can do this on a set $\mathcal{F} \subset \mathcal{H}_1^{>2\epsilon}(\alpha')$ with $\mu(\mathcal{H}_1^{>2\epsilon}(\alpha') \setminus \mathcal{F})$ arbitrarily small.
- First notice that if $\mathcal{F}$ is simply connected, then we can choose a single coordinate chart. Why?
- Choosing a basis for homology in saddle connections (i.e. a chart) at a point, we may extend to an open subset containing that point. We may further extend globally assuming that these choices are compatible, $\mathcal{F}$ being simply connected implies compatibility.
- Well definedness is equivalent to for every path $\gamma : I \to \mathcal{F}$, covering $\gamma(I)$ with compatible charts, we get a consistent choice of chart at $\gamma(1)$. It in fact suffices to check this for closed paths, since if γ and γ' are paths to the same point, and $\gamma\gamma'^{-1}$ does not effect the chart, then the chart on γ' is the same as the chart on $\gamma = \gamma\gamma'^{-1}\gamma'$. The argument for closed loops is as follows: Assume $\gamma : S^1 \to \mathcal{F}$, then by simply connectedness $\gamma \sim 1$ by the homotopy H, extend the finite cover of $\gamma(S^1)$ by compatible charts to one on $H(S^1 \times I)$, the chart at $\gamma(0)$ is invariant under the homotopy and is the original chart when $t = 1$ since H is the constant map.

- How do we take such an arbitrarily large simply connected set? This follows from μ' being a borel measure (it is locally lebesgue so is of course defined by outer measure), so by regularity and finiteness there is a compact set K with $\mu'(\mathcal{H}_1^{>2\epsilon}(\alpha')) - \mu'(K)$ arbitrarily small.
- Now we would like to cover K with finitely many disjoint coordinate charts $U_1, \ldots, U_j$, with $K \setminus \bigcup_1^j U_i$ having arbitrarily small volume, then we may connect these coordinate charts with a spanning tree, since each chart looks like a hollow ball, this is a simply connected subset with volume $> \mu'(K) - \delta$ for any $\delta > 0$.
- To define these coordinate charts U_i take an arbitrary finite open cover by charts $V_1, \ldots, V_r$, taking a suboordinate partition of unity we can define $U_i = \{\eta_i > \eta_\ell\}$ for all ℓ. We can choose the partition of unity so that $\eta_i - \eta_\ell$ is a submersion for each i, ℓ, this implies that $(\bigcup U_i)^c$ is measure zero by Sard's theorem.
- In this case $\mathcal{F} = \bigsqcup U_i \bigcup T$ where T is a spanning tree connecting them is simply connected.
- Since the volume of $\mathcal{F}$ is arbitrarily close to that of $\mathcal{H}_1^{>2\epsilon}(\alpha')$ we have by Lemma 7.1 of EMZ that

$$\mu'(\mathcal{H}_1(\alpha') - \mathcal{F}) = O(\epsilon^2)$$

- Since $\mathcal{H}_1^{\epsilon,3\epsilon}$ maps onto $\mathcal{F} \times B_\epsilon$ we are almost done establishing the covering is of degree $m+1$, however our argumant for degree did not take into account possible ramification points which we address now.
- The ramification points of the cover correspond to the surfaces $(S', \omega') \in \mathcal{F}$ with multiple of the $m+1$ surfaces (S, ω) achieved by the saddle construction being isomorphic.
- The set of points which are images of ramification points has measure zero. Proof being, suppose that S_1, S_2 which both arise from the adding a saddle connection on S' are isomorphic. Then the isomorphism $S_1 \to S_2$ must send the (unique) saddle connection γ_1 of length less than ϵ on S_1 to the short saddle connection γ_2 on S_2, and hence an open ball around γ_1 to an open ball around γ_2, hence the compliments of these open balls are isomorphic. This induces an automorphism on S', but the set of flat surfaces with automorphisms is measure zero.
- Q: Why is this set measure zero? Ans: There are finitely many topological automorphisms (i.e. induce automorphisms on homology) (Hurwitz bound?), a fixed (homnology class of) automorphisms f satisfies for each of the saddle $\int_{\gamma_k} f^*\omega = \int_{\gamma_k} \omega$, so that $\mathrm{hol}(\gamma_k)$ are fixed by f_*, so that the set of automorphic surfaces lie in the eigenspace of f_*, if $f_* \neq 1_*$ this is a proper linear subspace and is thus lower dimensional.
- So we have established that we have a ramified cover $\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha) \to \mathcal{F} \times B_\epsilon$ ramified over a set of measure zero.

4 Measure Theoretic Considerations

Topologically we have defined the covering map we are interested in, but in order to compute volumes we require the measure to decompose nicely over the map. Most of the work in this aspect is done by considering $\mathcal{F} \subset \mathcal{H}_1^{>2\epsilon}(\alpha')$, since we can work in a single chart.

- Taking a homology basis for $\mathcal{F}$, the saddle connections covering some $S' \in \mathcal{F}$ have the same monodromy (apart from $\beta \rightsquigarrow \beta'$). It is easy to see in notation that in coordinates the covering map gives us

$$\left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_{n-1}} \omega, \int_{\beta} \omega, \int_{\gamma} \omega\right) \mapsto \left(\left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_{n-1}} \omega, \int_{\beta'} \omega\right), \int_{\gamma} \omega\right)$$

- The only thing preventing the volume element from splitting is $\beta \rightsquigarrow \beta'$, but we previously controlled this error as $\mathrm{hol}(\beta') = \mathrm{hol}(\beta) - \frac{1}{2}\mathrm{hol}(\gamma)$, so we can check directly the error in the volume computation.

$$\begin{aligned}
\mu(\mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C})) &\overset{\text{Prop 7.1}}{=} \mu(\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha, \mathcal{C})) + \mathcal{O}(\epsilon^4) \\
\mu(\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha, \mathcal{C})) &= \int_{\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha,\mathcal{C})} (\mathrm{hol}(\gamma_1), \dots, \mathrm{hol}(\gamma_{n-1}), \mathrm{hol}(\beta), \mathrm{hol}(\gamma))\, d\mu \\
&= \int_{\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha,\mathcal{C})} \left(\mathrm{hol}(\gamma_1), \dots, \mathrm{hol}(\gamma_{n-1}), \mathrm{hol}(\beta) - \frac{1}{2}\mathrm{hol}(\gamma), \mathrm{hol}(\gamma)\right) d\mu \\
&+ \int_{\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha,\mathcal{C})} \left(\mathrm{hol}(\gamma_1), \dots, \mathrm{hol}(\gamma_{n-1}), \underbrace{\frac{1}{2}\mathrm{hol}(\gamma), \mathrm{hol}(\gamma)}_{\mathcal{O}(\epsilon^4)}\right) d\mu \\
&= (m+1) \int_{\gamma \in B_\epsilon} d\mathrm{Lebesgue}(\gamma) \int_{\mathcal{F}} (\mathrm{hol}(\gamma_1), \dots, \mathrm{hol}(\gamma_{n-1}), \mathrm{hol}(\beta))\, d\mu' + \mathcal{O}(\epsilon^4) \\
&= (m+1)\pi\epsilon^2 \mu'(\mathcal{F}) + \mathcal{O}(\epsilon^4) \\
&= (m+1)\pi\epsilon^2 \left(\mu'(\mathcal{H}_1(\alpha')) + \mu'(\mathcal{H}_1(\alpha') \setminus \mathcal{F})\right) + \mathcal{O}(\epsilon^4)
\end{aligned}$$

Then $\mu'(\mathcal{H}_1(\alpha') \setminus \mathcal{F}) = \mathcal{O}(\epsilon^2)$ by (EMZ Prop. 7.1), so that

$$\mu(\mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C})) = (m+1)\pi\epsilon^2 \mu'(\mathcal{H}_1(\alpha')) + \mathcal{O}(\epsilon^4)$$

This finally resolves the issue of computing the limit,

$$c(\alpha, \mathcal{C}) = \lim_{\epsilon \to 0} \frac{\mu(\mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C}))}{\pi\epsilon^2 \mu(\mathcal{H}_1(\alpha))} = \frac{(m+1)\mu'(\mathcal{H}_1(\alpha'))}{\mu(\mathcal{H}_1(\alpha))}$$

5 One Final Technical Issue

I have actually been cheating this entire time. The covering map/ surgery may not be volume preserving.

- Don't fret too much, all of our previous work is not all for nought. The prior arguments still preserve the fact our map is

$$\mathcal{H}_1^\epsilon(\alpha) \to \mathcal{H}(\alpha') \times B_\epsilon$$

and the corresponding more precise version.

- So topologically our map still makes sense since if the surface which is a result of the surgery is (S', ω'), since we can rescale ω' to $r\omega'$ so that our surface as unit area.
- Unfortunately, this breaks our earlier measure theoretic argument, the conclusion is still true but now we need to integrate in the total space of the strata.
- To recover the unit strata volumes we integrate over the cone, i.e. if $A \subset \mathcal{H}_1(\alpha)$ Then

$$\mathrm{Cone}(A) := \{(S, r\omega) \mid (S,\omega) \in A, r \in (0,1]\}$$

So that for $n = \dim(\mathcal{H}_1(\alpha))$, $n = 4g + 2k - 2$ we get

$$\mu \otimes \mathrm{Lebesgue(Cone(A))} = \int_0^1 r^{n-1} dr \mu(A) = \frac{1}{n}\mu(A)$$

- Repeating the above argument for the cone and taking $\mathcal{F}$ to be its rescaled image in $\mathcal{H}_1(\alpha')$, we get

$$\begin{aligned}
\mu(\mathcal{H}_1^\epsilon(\alpha, \mathcal{C})) &= n \cdot \mu \otimes \mathrm{Lebesgue}(\mathrm{Cone}(\mathcal{H}_1^{\epsilon,3\epsilon}(\alpha, \mathcal{C}))) + \mathcal{O}(\epsilon^4) \\
&= n(m+1)\mu'(\mathcal{F}) \int_0^1 r^{n'-1} \int_{B(\epsilon r)} d\gamma dr + \mathcal{O}(\epsilon^4) \\
&= n(m+1)\mu'(\mathcal{F}) \int_0^1 r^{n'-1} \pi \epsilon r^2 + \mathcal{O}(\epsilon^4) \\
&= \frac{n(m+1)}{\underbrace{n'+2}_{n'=n-2}} \mu'(\mathcal{F}) + \mathcal{O}(\epsilon^4) \\
&= (m+1)\mu'(\mathcal{F}) + \mathcal{O}(\epsilon^4)
\end{aligned}$$

6 Example: Σ_3