1. Since X is path connected, for any $y \in X$, there is some path γ between x_0 and y, by the path lifting property, $f \circ \gamma : ([0,1],0) \to (B, b_0)$ has a unique lift $\widetilde{f \circ \gamma} : ([0,1],0) \to (E, e_0)$. Then define $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$. To see this is well defined, suppose α, β are two paths connecting x_0 and y, from homework 1, since X is simply connected there is a homotopy h such that $h(t,0) = \alpha(t), h(t,1) = \beta(t)$ and $h(0,s) = x_0, = h(1,s) = y$ for all s. This gives a homotopy between $f \circ \alpha$ and $f \circ \beta$, namely $f \circ h$. By the homotopy lifting property, there is a unique lift $\widetilde{f \circ h}$ to E, then since $\rho \circ \widetilde{f \circ h} = f \circ h$ satisfies $f \circ h|_{\{1\} \times [0,1]} = f(y)$, since ρ is a covering map it is necessarily the case that $\widetilde{f \circ h}|_{\{1\} \times [0,1]} \subset \rho^{-1}(f(y)) = \bigsqcup \{y_\alpha\}_\alpha$, but since $\widetilde{f \circ h}$ is continuous, and $\{1\} \times [0,1]$ is connected $\widetilde{f \circ h}(\{1\} \times [0,1])$ is connected, and hence must be a singleton, thus $\widetilde{f \circ \beta}(1) = \widetilde{f \circ h}(1) = \widetilde{f \circ \alpha}(1)$ implying that the map $\tilde{f}$ is well defined. To prove continuity, it suffices to check locally. I.e. we can check that for every $y \in X$, there exists some open $V_y \supset \{y\}$ such that $\tilde{f}|_{V_y}$ is continuous, since we can write for $U \subset E$ open,

$$\tilde{f}^{-1}(U) = \bigcup_{y \in \tilde{f}^{-1}(U)} \tilde{f}^{-1}(U) \cap V_y = \bigcup_{y \in \tilde{f}^{-1}(U)} \tilde{f}^{-1}|_{V_y}(U)$$

So let $y_0 \in X$, then by the covering property there is some open U with $f(y_0) \subset U$ where $\rho^{-1}(U) = \bigsqcup_I U_i$ with $\rho|_{U_i} : U_i \cong U$. By local path connectivity there is some path connected $V \subset f^{-1}(U)$. Now fixing some arbitrary path γ between x_0, and y_0, we can define a path to any $y \in V$ by taking $\gamma_y : [0,1] \to V$ connecting y_0 to y, and considering $\gamma \cdot \gamma_y$. Now we can notice that $\tilde{f}(y_0) \in U_i$ for some fixed index i, hence since $[\frac{1}{2}, 1]$ is connected, and γ_y lies in V, $\widetilde{f \circ \gamma \cdot \gamma_y}^{-1}([\frac{1}{2}, 1])$ is connected and contained in $\bigsqcup_I U_i$, therefore it must be contained in the same U_i. Since $\rho \circ \tilde{f}(y) = \rho \circ f \circ \tilde{\gamma} \cdot \gamma_y(1) = f(y)$, this implies $\tilde{f}(y) = \rho|_{U_i}^{-1}(f(y))$, now since y was arbitrary, this implies that $\tilde{f}|_V = (\rho|_{U_i}^{-1} \circ f)|_V$, which is a composition of continuous functions restricted to an open set, hence continuous on that open set. $\square$

2. Define $\pi_X : (x,y) \mapsto (x, y_0)$, and $\pi_Y : (x,y) \mapsto (x_0, y)$, then we can define

$$\begin{aligned} \psi : \pi_1(X \times Y, (x_0, y_0)) &\to \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ [\gamma] &\mapsto ((\pi_X)_*[\gamma], (\pi_Y)_*[\gamma]) \end{aligned}$$

Now taking $\iota_X : X \to X \times Y, x \mapsto (x, y_0)$ and $\iota_Y : Y \to X \times Y, y \mapsto (x_0, y)$ we get a second map

$$\begin{aligned} \phi : \pi_1(X, x_0) \times \pi_1(Y, y_0) &\to \pi_1(X \times Y, (x_0, y_0)) \\ ([\alpha], [\beta]) &\mapsto ((\iota_X)_*[\alpha]) \cdot ((\iota_Y)_*[\beta]) \end{aligned}$$

It remains to check that these maps are indeed inverses. In the case of $\psi \circ \phi$, we really only need to trace through the definitions of the maps

$$\psi \circ \phi([\alpha], [\beta]) = \psi[(\alpha, y_0)] \cdot [(x_0, \beta)] = ([\alpha] \cdot [1_X], [\beta] \cdot [1_Y]) = ([\alpha], [\beta])$$

Now for the converse direction $\phi \circ \psi$, we can consider $\gamma : (S^1, 0) \to (X \times Y, (x_0, y_0))$, and decompose it on coordinates as (γ_X, γ_Y), i.e. $\gamma_X = \pi_X \circ \gamma$ and $\gamma_Y = \pi_Y \circ \gamma$ then

$$\phi \circ \psi([\gamma]) = \phi([\gamma_X], [\gamma_Y]) = [\gamma_X] \cdot [\gamma_Y] = [\gamma_X \cdot \gamma_Y]$$

which means that it suffices to show that $\gamma_X \cdot \gamma_Y \sim \gamma$, this is a consequence of the following homotopy between $(\pi_X \circ \gamma) \cdot (\pi_Y \circ \gamma)$ and $\gamma \cdot 1_{X \times Y}$

$$h(t,s) = \begin{cases} (\pi_X \circ \gamma(2t), y_0) & t \leq \frac{1}{2}s \\ (\pi_X \circ \gamma(2t), \pi_Y \circ \gamma(2(t - \frac{1}{2}s))) & t \in (\frac{1}{2}s, \frac{1}{2}] \\ (x_0, \pi_Y \circ \gamma(2(t - \frac{1}{2}s))) & t \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2}s] \\ (x_0, y_0) & t > \frac{1}{2} + \frac{1}{2}s \end{cases}$$

3. Suppose that γ is an even function, then $\gamma(\frac{1}{2}) = 0 = \gamma(0)$, so that $[\gamma|_{[0,\frac{1}{2}]}]$ and $[\gamma|_{[\frac{1}{2},1]}] \in \pi_1(S^1, 0)$. But since $\gamma(x) = \gamma(-x) = \gamma(x + \frac{1}{2})$, we find that $\gamma|_{[0,\frac{1}{2}]} = \gamma|_{[\frac{1}{2},1]}$ (by identifying $[0, \frac{1}{2}]$ with $[\frac{1}{2}, 1]$), this gives us

$$[\gamma] = [\gamma|_{[0,\frac{1}{2}]} \cdot \gamma|_{[\frac{1}{2},1]}] = [\gamma|_{[0,\frac{1}{2}]}] \cdot [\gamma|_{[\frac{1}{2},1]}] = 2[\gamma|_{[0,\frac{1}{2}]}]$$

Now suppose that γ is an odd function, and define $\gamma' : [0, 1] \to S^1, t \mapsto \gamma(\frac{t}{2})$, another way to write γ being an odd function on S^1 is that $\gamma(x + \frac{1}{2}) = \frac{1}{2} + \gamma(x)$ (since on S^1 the antipode of x is $\frac{1}{2} + x$). So letting $\tilde{\gamma'} : [0, 1] \to \mathbb{R}$ be the unique lift of γ' based at 0 given by the unique path lifting property using this description of odd functions implies that $\gamma = \gamma' \cdot (\frac{1}{2} + \gamma')$, if ρ denotes the standard covering map $\mathbb{R} \to S^1$, then we can define

$$\tilde{\gamma} : t \mapsto \begin{cases} \tilde{\gamma'}(2t) & t \in [0, \frac{1}{2}] \\ \tilde{\gamma'}(1) + \tilde{\gamma'}(2(t - \frac{1}{2})) & t \in (\frac{1}{2}, 1] \end{cases}$$

Which is the lift of γ based at zero since $\rho \circ \tilde{\gamma}(t) = \gamma(t)$ and lifts are unique, by the isomorphism used to identify $\mathbb{Z} \cong \pi_1(S^1, 0)$ we have $[\gamma] = \tilde{\gamma}(1) = 2\tilde{\gamma'}(1)$, then since $\rho\tilde{\gamma'}(1) = \gamma'(1) = \gamma(\frac{1}{2}) = \gamma(0) + \frac{1}{2} = \frac{1}{2}$ we know that $\tilde{\gamma'}(1) \in \mathbb{Z} + \frac{1}{2}$, whence $[\gamma] = 2\tilde{\gamma'}(1) \in 2\mathbb{Z} + 1$ is odd. $\square$

4. (a) In order to work explicitly with the Mobius strip, denote $I = [-1, 1]$, then write

$$M = \frac{I^2}{(-1, x) \sim (1, -x)}$$

Now define $\gamma : S^1 \to M$, writing $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ we can define the homotopy

$$h_\gamma(t, s) = (\gamma_1(t), (1 - s)\gamma_2(t))‘'$$

so that $h_\gamma(-, 1) : S^1 \to S^1 = \frac{I}{-1\sim 1} \subset M$. We can define the map $r : M \to S^1$ via $(x, y) \mapsto x$, it is immediate that $\iota \circ r \circ \gamma = h_\gamma(-, 1)$, this homotopy tells us that

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "\iota_*"] \arrow[rr, bend left, "1_*"] & \pi_1(M) \arrow[r, "r_*"] \arrow[rr, bend right, "1_*"'] & \pi_1(S^1) \arrow[r, "\iota_*"] & \pi_1(M) \end{tikzcd}$$

Which suffices to show that $r_* : \pi_1(M) \to S^1 \cong \mathbb{Z}$ is an isomorphism with inverse ι_*.

(b) Assume such a retraction R exists. Notice that $\partial M = S^1$, and consider

$$\gamma : S^1 \to \partial M, \quad t \mapsto \begin{cases} (4t - 1, 1) & t \in [0, \frac{1}{2}] \\ (4(t - 2), -1) & t \in (\frac{1}{2}, 1] \end{cases}$$

Then under the identification of $\pi_1(\partial M) \cong \mathbb{Z}$ we have $[\gamma] = 1$. Using r from part (a) and identifying $\partial M \overset{j}{\hookrightarrow} M$, we get $r_*[\gamma] = [r \circ \gamma] = 2$. Moreover since any $[\alpha] \in \pi_1(M)$ can be written as $k[\gamma]$ for $\gamma \in \mathbb{Z}$, and r_* is a group homomorphism we find that

$$r_*[\alpha] = r_*[k\gamma] = kr_*[\gamma] = 2k$$

Now by assumption of existence of R, we have the following diagram

$$\begin{tikzcd} & \pi_1(S^1) \arrow[d, shift left, "\iota_*"] & \\ \pi_1(\partial M) \arrow[r, "j_*"] \arrow[rr, bend right, "1_*"'] & \pi_1(M) \arrow[r, "R_*"] \arrow[u, shift left, "r_*"] & \pi_1(\partial M) \end{tikzcd}$$

Identifying up to isomorphism, and writing explicitly the compositions this diagram becomes

$$\begin{tikzcd}
& \mathbb{Z} \arrow[d, shift left, "\iota_*"] \arrow[dr, "R_*\iota_*"] & \\
\mathbb{Z} \arrow[ur, "\cdot 2"] \arrow[r, "j_*"] \arrow[rr, bend right, "\cdot 1"'] & \mathbb{Z} \arrow[u, shift left, "r_*"] \arrow[r, "R_*"] & \mathbb{Z}
\end{tikzcd}$$

Then if $R_*\iota_*(1) = k$, then $1 = R_*\iota_*(2) = 2R_*\iota_*(1) = 2k$, but $1 = 2k$ has no integer solutions so this is a contradiction. $\square$