1. (a) From the definition of a lie group we know that $\mu : G \times G \to G$ is smooth, then $\mu_g = \mu \circ \iota_g$, where $\iota_g : G \to G \times G$ via $h \mapsto (g, h)$ is the incusion into the product manifold, we have seen previously the inclusion is smooth, so that $\mu_g = \mu \circ \iota_g$ is smooth. Now we can also see that

$$\mu_{g^{-1}}\mu_g = 1_G = \mu_g\mu_{g^{-1}}$$

and $\mu_{g^{-1}}$ is smooth for the same reason μ_g is, so that μ_g is in fact a diffeomorphism, this implies that $d_e\mu_g$ is an isomorphism.

(b)

(Lemma) Let $(\rho, E), (\widehat{\rho}, \widehat{E})$ be two vector bundles on the same base space M, and $F : E \to \widehat{E}$ a smooth bijective map of smooth vector bundles with $F(x, 0) = (x, 0)$ (i.e. F descends to the identity), then F is a diffeomorphism.

Proof. Being a diffeomorphism is a local property, so for a point $x \in M$, let U be an open neighborhood of M where $\rho^{-1}(U)$ admits a local trivialization ζ, moreover there is a second neighborhood $x \in V \subset U$ where $\widehat{\rho}^{-1}(V)$ admits a local trivialization $\widehat{\zeta}$ (since the base manifold is the same by possibly shrinking the neighborhood we can assume that the two bundle charts are equal on $V \times \{0\}$, this is not necessary but removes a lot of bloat from notation). Then $\widehat{\zeta} \circ F \circ \zeta^{-1} : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is smooth, linear on each fiber and bijective on each fiber, so on V, we can write $A(x) = \widehat{\zeta} \circ F \circ \zeta^{-1}(x, -)$. Then on the local trivialization F is given by

$$\widehat{\zeta} \circ F \circ \zeta^{-1}(x, v) = (x, A(x)v)$$

In particular, the Jacobian $D_{(x,v)}(\widehat{\zeta} \circ F \circ \zeta^{-1})$ is given by

$$\begin{pmatrix} 1_n & 0 \\ B(x, v) & A(x) \end{pmatrix}$$

Bijectivity on each fiber implies that $A(x)$ is full rank, so that $\det(D_{(x,v)}\widehat{\zeta} \circ F \circ \zeta^{-1}) = \det A(x) \in \mathbb{R}^\times$, by the inverse function theorem $\widehat{\zeta} \circ F \circ \zeta^{-1}$ has a local smooth inverse, and hence F is a diffeomorphism. $\square$

Since T_eG is n-dimensional, we can identify it with $\mathbb{R}^n$, the following diagram specifies the desired correspondence of vector bundles:

$$\begin{tikzcd} G \times \mathbb{R}^n \arrow[r, shift left, "F"] \arrow[d] & TG \arrow[l, shift left, "T"] \arrow[d] \\ G \arrow[r, "1_G"] & G \end{tikzcd}$$

Where $F(g, v) = (g, d_e\mu_g(v))$,

(when I originally solved the problem I tried to show F and the inverse map T which is not too hard to compute are both smooth, however, after trying to show F, T are smooth for quite some time I did the following computation which allowed me to see that F is smooth, this computation does not generalize easily to T, so the lemma is intended to avoid having to do a similar computation for T).

In order to show F is smooth, it suffices to show that $(g, v) \mapsto d_e\mu_g(v)$ is smooth, here we can use smoothness of μ, and the identification $T(G \times G) \longleftrightarrow TG \oplus TG$ by identifying on each fiber, we have previously computed (last homework) that $d_p\iota_q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ when ι denotes inclusion. We have that $d\mu : T(G \times G) \to TG$ is smooth since μ is a smooth map, then

$$d\mu((g, v), (h, u)) = d_{(g,h)}\mu(v, u)$$
$$d_e\mu_g = d_e(\mu \circ \iota_g)(v, u) = (d_{(g,e)}\mu)(d_e\iota_g)(v, u) = d_{(g,e)}\mu(u)$$

From this computation, we can see that $d_e\mu_g = d_{(g,e)}\mu(0, u)$ is the restriction of $d\mu$ to $TG \times \{e, 0\}$, this is clearly a submanifold directly from the definition of it being a linear subspace given by inclusion on the first $2n$ coordinates. Thus the restriction of $d\mu$ to this submanifold is smooth, and is identified with $d_e\mu_g$. So F is smooth, and we appeal to the lemma to find that T, the set theoretic inverse for F is smooth and hence F is a diffeomorphism.

2. Let $f : X \to \mathbb{R}^m$ be a submersion, where X is a compact smooth manifold. The proof will follow if we can show submersions are open maps, assuming this, since the image of a compact set is compact (by pulling back an open cover along the map) we get that $f(X) \subset \mathbb{R}^m$ is open, but also $f(X) \subset \mathbb{R}^m$ is compact hence closed, so since $X \neq \emptyset$ we have $f(X) = \mathbb{R}^m$, contradicting compactness.

It remains to show that a submersion is open, since f is a submersion, we can cover M, N with charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U'_\beta, V'_\beta, \varphi_\beta)$ respectively with the property that the following commutes (here π is the projection map onto the first n coordinates)

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\phi_\alpha} & V_\alpha \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle f} \\ U'_\beta & \xrightarrow{\varphi_\beta} & V'_\beta \end{array}$$

Now let $E \subset X$ be open, and write $E_\alpha := V_\alpha \cap E$, then

$$f(E) = \bigcup_\alpha f(E_\alpha) = \bigcup_{\alpha,\beta} \varphi_\beta \pi \phi_\alpha^{-1}(E_\alpha)$$

But $\varphi_\beta \pi \phi_\alpha^{-1}$ is a composition of open maps hence open, so that $f(E)$ is open which suffices to show f is open.

3. (a)