**1.**

$$h(t,s) = \begin{cases} \gamma(2t) & t \leq \frac{1-s}{2} \\ \gamma\left(\frac{1-s}{2}\right) & t \in \left(\frac{1-s}{2}, \frac{1+s}{2}\right) \\ \overline{\gamma}(2t) & t \geq \frac{1+s}{2} \end{cases}$$

$\square$

**2. ((iii) $\implies$ (i)):** Let $x_0 \in X$, then $[1_{x_0}] \in \pi_1(X, x_0)$, so for any $[\gamma] \in \pi_1(X, x_0)$ we have $[\gamma] = [1_{x_0}]$ by assumption, whence $\pi_1(X, x_0) = \{[1_{x_0}]\} = 0$. $\square$

**((i) $\implies$ (iii)):** Let $f_1, f_2 : S^1 \to X$, where we parameterize $S_1$ as $\frac{I}{0 \sim 1}$, since $X$ is path connected, there is some path $\gamma : I \to X$ with $\gamma(0) = f_1(0)$ and $\gamma(1) = f_2(0)$, then by assumption (i), $[f_1] = [1_{f_1(0)}]$ and $[f_2] = [1_{f_2(0)}]$, now we can write the homotopy between $1_{f_1(0)} \sim f_1$ and $1_{f_2(0)} \sim f_2$, which is given by $h(s,t) = \gamma(s)$. $\square$

**((ii) $\implies$ (i)):** Let $[\gamma] \in \pi_1(X, x_0)$ for some $x_0 \in X$, then we can define new paths

$$\gamma_1 : t \mapsto \gamma(t/2) \qquad \gamma_2 : t \mapsto \gamma\left(\frac{1+t}{2}\right)$$

then $\overline{\gamma_1}$ and $\gamma_2$ satisfy the hypotheses of (ii), which entails $\gamma_2 \sim \overline{\gamma_1}$, now since $\gamma = \gamma_1 \cdot \gamma_2$ we find that $\gamma \sim \gamma_1 \cdot \overline{\gamma_1} \sim 1_{x_0}$, whence $[\gamma] = [1_{x_0}]$. Since $[\gamma]$ was arbitrary we conclude $\pi_1(X, x_0) = 0$. $\square$

**(() $\implies$ (ii)):**

**3.** Define the map

$$\psi : \pi_1(X, x_0) \to \{S^1 \to X\}/\sim$$
$$[f] \mapsto [f]$$

In words, we forget the base point of $f$. This is of course well defined up to homotopy. To see that this map is onto, it suffices to check that every map $f : S^1 \to X$ is homotopic to a map $f'$ with $f'(0) = x_0$. Now checking this consider some $f : S^1 \to X$, and let $\gamma$ be a path between $x_0$ and $f(0)$, then we can write

$$f'(t) = \begin{cases} \gamma(3t) & t \leq \frac{1}{3} \\ f(3(t - \frac{1}{3})) & t \in (\frac{1}{3}, \frac{2}{3}) \\ \gamma(1 - 3(t - \frac{2}{3})) & t \geq \frac{2}{3} \end{cases}$$

Then we can verify explicitly that $f' \sim f$ by taking first a homotopy of $f'$

$$h'(t,s) = \Big\{ f'()$$