

1. (a) Define $F(x) = (\det \text{Hess}_x(f))^2 + \sum_1^k \left(\frac{\partial}{\partial x_j} f(x) \right)^2$, it's clear that both the hessian and sum terms are non-negative (since they are squares of real values). Now assume first that $F > 0$ on U , if f has no critical points on U we are done. Now if p is a critical point for f , we have that $d_p f : \mathbb{R}^k \rightarrow \mathbb{R}$ is not surjective, where $d_p f = \left(\frac{\partial}{\partial x_1} f \quad \cdots \quad \frac{\partial}{\partial x_k} f \right)$, this matrix is of course surjective so long as at least one $\frac{\partial}{\partial x_j} f \neq 0$, so at a critical point we get $\frac{\partial}{\partial x_j} f = 0$ for all j whence $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p) \right)^2 = 0$, by our assumption of $F > 0$, this implies that $(\det \text{Hess}_p(f))^2 > 0$, and since p was an arbitrary critical point we can conclude that f is morse. Conversely, if f is morse, then by the computation above, at any regular value, p , we must have some $\frac{\partial}{\partial x_j} f \neq 0$, which implies that $\sum_1^k \left(\frac{\partial}{\partial x_j} f(p) \right)^2 > 0$, which implies $F(p) > 0$ by non-negativity of the hessian term. In the case that p is a critical point, we know that $\det \text{Hess}_p f \neq 0$, so that $(\det \text{Hess}_p f)^2 > 0$, and $F(p) > 0$, since all points are either regular values or critical points we are done. \square

(b) Let f be morse, and H a homotopy with $H(x, 0) = f(x)$, moreover we can denote $f_t = H(-, t)$. Then since M is compact we can pick a finite covering by charts $(V_1, U_1, \phi_1), \dots, (V_n, U_n, \phi_n)$. Now let η_i be a partition of unity subordinate to these charts, we can define

$$F(x, t) = \sum_1^n \eta_i(x) \left((\det \text{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k \left(\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x) \right)^2 \right)$$

F is smooth, due to closure of smooth functions under sums, products and compositions as well as the fact that the hessian and partials vary smoothly with t , this can be seen since the coordinates of these maps are a subset of the coordinates of $H \circ (\phi^{-1}, 1_{[0,1]})$, which of course has smooth partials and hessian. To see that $F(-, 0) > 0$ on M , note that since f_0 is morse and ϕ_i^{-1} are diffeomorphisms, each summand $\left((\det \text{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 + \sum_{j=1}^k \left(\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x) \right)^2 \right) > 0$ by part (a). Since M is compact, the function $\hat{F}(t) = \inf_{x \in M} F(x, t)$ is continuous, and it has $\hat{F}(0) > 0$, since $F(-, 0)$ is continuous and positive on a compact set (which implies it attains its infimum), then by continuity of \hat{F} , there exists some $\delta > 0$, such that $t < \delta$ implies $\hat{F}(t) > 0$. So to show that morse functions are generic, it will suffice to show that $F(-, t) > 0$ implies that f_t is morse, since then we get for $t < \delta$ each $f_t = H(x, t)$ is morse. To check this, note that since the ϕ_i^{-1} are diffeomorphisms, we have that $\text{Hess}_x f_t$ is invertible iff $\text{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1}$ for each i (since we have a partition of unity we can always assume $x \in V_i$), moreover f_t is regular if and only if $f_t \circ \phi_i$ is, so at critical points of f_t , we get that for each i , $\sum_{j=1}^k \left(\frac{\partial}{\partial x_j} f_t \circ \phi_i^{-1}(x) \right)^2$ vanishes. It follows that $(\det \text{Hess}_{\phi_i^{-1}(x)} f_t \circ \phi_i^{-1})^2 > 0$ for all i , when x is a critical value of f , and since this Hessian is invertible iff the hessian of f_t is, we find that f_t has invertible Hessian at all its critical points, assuming $F(-, t) > 0$ on M which completes the proof. \square

2. (a) Let $\Delta_{ij} = \{(x_1, \dots, x_r \mid x_i = x_j)\}$, it's clear that by reordering the factors we have each $\Delta_{ij} \cong \Delta_{1,2}$, moreover $\Delta_{12} = \Delta \times M^{r-2}$ where Δ denotes the diagonal of $M \times M$, where we take the convention $M^{r-2} = \emptyset$ if $r - 2 = 0$. Since Δ is a submanifold of $M \times M$, we can realize it as the image of an embedding $e : \Delta \hookrightarrow M \times M$, so that $e \times 1_M^{r-2} : \Delta_{ij} \rightarrow M^r$ is an embedding and thus each Δ_{ij} is a submanifold, it is clear from definition that $\Delta = \bigcup_{1 \leq i < j \leq r} \Delta_{ij}$. Now suppose that X is compact **TODO**

(b) Let $p = (p_1, \dots, p_r), q = (q_1, \dots, q_r) \in \text{Conf}_r(M)$, then we can identify these points with the same coordinates in M^r . To see that there is a path between p and q , we use path connectedness of M , which gives paths $\gamma'_i : [0, 1] \rightarrow M$ satisfying $\gamma'_i(0) = p_i$ and $\gamma'_i(1) = q_i$. Now define

$$\begin{aligned} \gamma_1(t) &= (\gamma'_1(t), p_2, \dots, p_r) \\ \gamma_2(t) &= (q_1, \gamma'_2(t), p_3, \dots, p_r) \\ &\vdots \\ \gamma_r(t) &= (q_1, \dots, q_{r-1}, \gamma'_r(t)) \end{aligned}$$

We can join these paths together continuously (but not necessarily smoothly) by taking

$$\gamma(t) = \begin{cases} \gamma_1(rt) & t \in [0, \frac{1}{r}) \\ \gamma_2(r(t - 1/r)) & t \in [\frac{1}{r}, \frac{2}{r}) \\ \vdots \\ \gamma_r((t - \frac{r-1}{r})) & t \in [\frac{r-1}{r}, 1] \end{cases}$$

This gives a continuous path between p and q , which implies the existence of a smooth path, so replace γ with this smooth path. Now since $\Delta_{ij} \cong \Delta \times M^{r-2}$, we have $\dim M^r - \dim \Delta_{ij} \geq \dim M = k \geq 2$, and since $X = [0, 1]$ is dimension 1, we have a map $f : X \rightarrow M$ is transverse to Δ_{ij} iff $\text{Im } f \cap \Delta_{ij} = \emptyset$. So part (a) tells us that there is some homotopy $H : [0, 1]^2 \rightarrow M^r$ with $H(-, 0) = \gamma$, and for any $\epsilon > 0$, there exists some $t < \epsilon$ with $H(-, t) \pitchfork \Delta_{ij}$ for all i, j , but by the dimension argument I just gave, this means that $H(-, t) : [0, 1] \rightarrow \text{Conf}_r(M)$ for all such t . Now let V_p and V_q be connected open sets in $\text{Conf}_r(M)$ containing p and q respectively (note since $\text{Conf}_r(M)$ is an open subset of M these can be identified with open subsets of M), since H is smooth, we have $H(0, t) \in V_p$ for all $t < \delta_p$ for some $\delta_p > 0$, similarly for $H(1, t) \in V_q$ we get some δ_q . Taking $\delta = \min\{\delta_p, \delta_q\}$, we find that for $t < \delta$ that $H(0, t) \in V_p$, and $H(1, t) \in V_q$, now use genericity to get some t with $t < \delta$ and $H(-, t)$ lying in $\text{Conf}_r M$, denote $p' = H(0, t)$ and $q' = H(1, t)$. Since connected components of manifolds are path connected, we have paths γ_p, γ_q in $\text{Conf}_r M$ connecting p to p' and q' to q . It follows that concatenating gives a smooth path f

$$f(t) = \begin{cases} \gamma_p(3t) & t \in [0, 1/3) \\ \gamma(3(t - 1/3)) & t \in [1/3, 2/3) \\ \gamma_q(3(t - 2/3)) & t \in [2/3, 1] \end{cases}$$

between p and q in $\text{Conf}_r M$, thus there exists a smooth path between p and q in $\text{Conf}_r M$, since p, q were arbitrary $\text{Conf}_r M$ is path connected.

(c) Part (b) gives the existence of such a path provided that there is some $q \in \text{Conf}_r M \cap U$, i.e. We only need to check that $\text{Conf}_r M \cap U \neq \emptyset$. This is easy to see by Sard's theorem, since $(\text{Conf}_r M)^c$ is a union of finitely many sets (the Δ_{ij}) with dimension strictly less than k^r . Now letting $F : U \xrightarrow{\cong} B_k$ be the diffeomorphism to the unit ball (and hence $F^r : U^r \xrightarrow{\cong} B_k$ given by F in each coordinate is a diffeomorphism). Since $\Delta_{ij} \cap U^r$ is a submanifold of U^r (since U^r has full dimension they are automatically transverse, so their intersection is a submanifold), and once again since U^r has full dimension, we have $\dim \Delta_{ij} = \dim \Delta_{ij} \cap U^r < \dim U^r$, we then get $F^r(U^r \cap \Delta_{ij})$ is a submanifold of B_k^r of strictly smaller dimension, thus having measure 0 by sards theorem, this of course implies that

$$F^r(U^r \cap (\text{Conf}_r M)^c) F^r(U^r \cap \bigcup_{1 \leq i < j \leq r} \Delta_{ij}) = F^r(\bigcup_{1 \leq i < j \leq r} U^r \cap \Delta_{ij}) = \bigcup_{1 \leq i < j \leq r} F^r(U^r \cap \Delta_{ij})$$

is a finite union of measure zero sets, hence measure zero, so that since U^r has positive r -dimensional measure, there does indeed exist some $x \in F^r(U^r \setminus (\text{Conf}_r M)^c) = F^r(U^r \cap \text{Conf}_r M)$, so that we can take $q := F^{-1}(x) \in U^r \cap \text{Conf}_r M$ as desired. \square

(d) We proved in the previous subpart, there is a path γ lying in $\text{Conf}_r M$ connecting p to some $q \in U$, this allows us to define an isotopy by taking $e_t(p_i)$ to be the i -th coordinate of $\gamma(t)$, since e_t has a zero dimensional, compact domain we only need to check that its injective in order to being an embedding, but injectivity follows from $\gamma(t) \in \text{Conf}_r M$. Hence e_t is an isotopy of embedding between $e_0 = 1_{\{p_1, \dots, p_r\}}$ and $e_1 : p_i \mapsto q_i \in U$. The isotopy extension theorem gives an isotopy of compactly supported diffeomorphism h_t , with $h_0 = 1$, and $h_t e_0 = e_t$, so that $h_1(p_i) = q_i \in U$. It follows that h_1^{-1} is a compactly supported diffeomorphism satisfying $\{p_1, \dots, p_r\} \in h_1^{-1}(U) \supset \{h_1^{-1}(q_i)\}_1^r$, moreover its isotopic to the identity via $H(x, t) = h_{1-t}^{-1}(x)$, since $1_M^{-1} = 0$. \square