1. I will first prove a lemma, since I will use it multiple times in order to prove homotopy equivalences.

Lemma. If $e : X \hookrightarrow M$ is an embedding for manifolds M, X, and there is a strong deformation retract $H : M \to M$ with $H(M \times \{1\}) = X$, then $M \simeq X$.

Proof. Let $r(x) = H(x, 1)$, then $e|_{e(X)}^{-1} r : M \to X$ is smooth, and since H is a strong deformation retract we have $e|_{e(X)}^{-1} re = 1_X$, from which it suffices to show that $ee|_{e(X)}^{-1} r = r \simeq 1_M$, but $r = H(-, 1)$, so this homotopy is exhibited by H and we are done. □

Let $V_0, \dots, V_n$ be the standard charts on $\mathbb{RP}^n$, now take $V = V_0$, and let pt. $= [0 : 0 : \cdots : 1] \in V_0^c$, then take $U = \mathbb{RP}^n \setminus \{\text{pt.}\}$, the standard chart map ϕ_0 gives us $V \cong \mathbb{R}^n$. Similarly, we find that $U \cap V = V \setminus \{\text{pt.}\}$, so that

$$\phi_0^{-1}|_{U\cap V} : U \cap V \xrightarrow{\cong} \mathbb{R}^n \setminus \{\phi_0^{-1}(\text{pt.})\} \simeq S^{n-1}$$

The homotopy equivalence is given by $\mathbb{R}^n \setminus \{\phi_0^{-1}(\text{pt.})\} \xrightarrow{\cong} \mathbb{R}^n \setminus \{0\}$ via $x \mapsto x - \phi_0^{-1}(\text{pt.})$, then taking the strong deformation retract $H(x, t) = (1 - t)x + t\frac{x}{||x||}$ which gives a homotopy equivalence to S^{n-1}. Now it remains to show $U \simeq \mathbb{RP}^{n-1}$. First we consider the smooth map $\theta : \mathbb{RP}^n \setminus \{\text{pt.}\} \to (0, \pi/2)$ via $[x_0, \dots, x_n] \mapsto \arcsin x_n$, where we take the representative of $[x_0, \dots, x_n]$ with $x_n > 0$, we can do this since we removed the point $x_n = 0$, and smoothness follows by arcsin being smooth on $[0, 1)$, so our map is smooth in coordinates, it follows that points in $\mathbb{RP}^n \setminus \{\text{pt.}\}$, now we can define the homotopy (where once again we define the maps on the representative with $x_n > 0$)

$$H([x], t) = \left[\cos((1 - t)\theta(x))\frac{(x_0, \dots, x_{n-1}, 0)}{||(x_0, \dots, x_{n-1}, 0)||} + \sin\theta(x)\right]$$

Once again, this map is smooth since it is defined to be smooth on coordinates, and $H(\mathbb{RP}^n \times \{1\}) = \{[x] \in \mathbb{RP} \mid x_n = 0\} \cong \mathbb{RP}^{n-1}$, where the diffeomorphism is given by the embedding $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n$ via $[x_0, \dots, x_{n-1}] \mapsto [x_0, \dots, x_{n-1}, 0]$, this map is smooth due to being identity on the charts given by the same coordinate non-vanishing loci. Proper since $\mathbb{RP}^{n-1}$ is compact, is clearly injective, and is an immersion since in appropriate coordinates its given by the identity. Hence the homotopy defined above gives a strong deformation retract from $\mathbb{RP}^n$ to $\{[x] \in \mathbb{RP} \mid x_n = 0\}$ from which we get a homotopy equivalence. This concludes the annoying details and now we can proceed with the algebraic argument.

We first want to show that for $0 < k < n$, we have $H^k(\mathbb{RP}^n) = 0$. Let $q : S^n \to \mathbb{RP}^n$ be the covering map, then since q is locally invertible and $\mathbb{RP}^n$ is compact, we have an open cover $U_1, \dots, U_s$ for $\mathbb{RP}^n$, with associated maps $q_1, \dots, q_s$ satisfying $qq_j = 1_{\mathbb{RP}^n}$ for each j, taking a partition of unity suboordinate to the U_j, we can define $f = \sum_1^s \eta_j \cdot q_j$, with $q \circ f = 1_{\mathbb{RP}^n}$, it follows that $f^*q^* = 1^*_{\mathbb{RP}^n}$. Now we want to show that $[q^*] : H^*(\mathbb{RP}^n) \to H^*(S^n)$ is injective, to do so assume that $[q^*]([\omega]) = [0]$, then $q^*\omega = d\nu$ for some ω representing the class $[\omega]$, and some form ν, now we can use our section to find that

$$\omega = f^*q^*\omega = f^*d\nu = df^*\nu$$

this shows that ω is an exact form, and hence $[\omega] = 0$. This suffices to show that $[q^*]$ is injective, but then for $0 < k < n$, we have $[q^*] : H^k(\mathbb{RP}^n) \hookrightarrow H^k(S^n) = 0$, so that $H^k(\mathbb{RP}^n) = 0$ for $0 < k < n$ as desired.

Since $U \cup V$ is an open cover for $\mathbb{RP}^n$, we get the short exact sequence of chain complexes

$$0 \longrightarrow \Omega^*(\mathbb{RP}^n) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V) \longrightarrow 0$$

Mayer Vietoris gives us a long exact sequence on cohomology, the portion of interest is for $n > 1$

$$\begin{tikzcd}
\cdots & H^n(U) \oplus H^n(V) \arrow[l] & H^n(\mathbb{RP}^n) \arrow[l] \\
H^{n-1}(U \cap V) \arrow[urr, bend right] & H^{n-1}(U) \oplus H^{n-1}(V) \arrow[l] & H^{n-1}(\mathbb{RP}^n) \arrow[l]
\end{tikzcd}$$

Since cohomology is a homotopy invariant, we may substitute in the spaces above to this LES.

$$
\begin{tikzcd}
\cdots & H^n(\mathbb{RP}^{n-1}) \oplus H^n(\mathbb{R}^n) \arrow[l] & H^n(\mathbb{RP}^n) \arrow[l] \\
H^{n-1}(S^{n-1}) \arrow[urr, rounded corners] & H^{n-1}(\mathbb{RP}^{n-1}) \oplus H^{n-1}(\mathbb{R}^n) \arrow[l] & H^{n-1}(\mathbb{RP}^n) \arrow[l]
\end{tikzcd}
$$

Now we know the cohomology for spheres, and euclidean space, $\mathbb{RP}^{n-1}$ is $n-1$ dimensional so that its n-th cohomology is zero and finally we already computed that $H^{n-1}(\mathbb{RP}^n) = 0$. Applying this we get

$$
\begin{tikzcd}
\cdots & 0 \arrow[l] & H^n(\mathbb{RP}^n) \arrow[l] \\
\mathbb{R} \arrow[urr, rounded corners] & H^{n-1}(\mathbb{RP}^{n-1}) \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

Exactness of this sequence gives us that $\mathbb{R} \cong H^{n-1}(\mathbb{RP}^{n-1}) \oplus H^n(\mathbb{RP}^n)$ (the splitting is guaranteed since were working with vector spaces). Now since $\mathbb{RP}^1 \cong S^1$, which has $H^1(S^1) \cong \mathbb{R}$, and the above formula holds for $n > 1$, we find recursively that for $n \geq 1$

$$
H^n(\mathbb{RP}^n) \cong \begin{cases} \mathbb{R} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
$$

From this and the fact that $\mathbb{RP}^n$ is connected giving it 0-th cohomology $\mathbb{R}$, we get the cohomology ring

$$
H^*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{R}[x_n]/(x_n^2) & n \text{ odd} \\ \mathbb{R} & n \text{ even} \end{cases}
$$

since the zero-th cohomology class is a unit with respect to wedge, and x_n represents the n-form $[\omega]$, but $\omega \wedge \omega = 0$ since $H^{2n}(\mathbb{RP}^n) = 0$ by dimension considerations. $\square$