**1. (Folland 1.2.1) (a)** Let $n \in [2,\infty]$, then

$$\bigcap_1^n E_i = E_1 \cap \bigcap_2^n E_i = E_1 \setminus \bigcup_2^n E_i^c = E_1 \setminus \bigcup_2^n (E_1 \cap E_i^c) = E_1 \setminus \bigcup_2^n (E_1 \setminus E_i)$$

By assumption each of the $E_i$ and $E_1 \setminus E_i$ are in $\mathcal{R}$. $\square$

**(b)** Let $E \in \mathcal{R}$, then $E^c = X \setminus E \in \mathcal{R}$. $\square$

**(c)** Denote the (alleged) sigma algebra in the question as $\mathcal{A}$. Closure under compliments is immediate from the definition. Now let $\{E_i\}_1^\infty \subset \mathcal{A}$, then

$$\bigcup_1^\infty E_i = \bigcup_{E_i \in \mathcal{R}} E_i \bigcup_{E_i^c \in \mathcal{R}} E_i$$

By definition we have the first union is in $\mathcal{R}$, the second union is equal to $\left(\bigcap_{E_i^c \in \mathcal{R}} E_i^c\right)^c$, and hence by part (a) it has compliment in $\mathcal{R}$. This reduces the problem to pairwise unions $E \cup F$ for $E \in R$ and $F^c \in R$. In this case $(E \cup F)^c = E^c \cap F^c = F^c \setminus E \in \mathcal{R}$. $\square$

**(d)** Once again refer to the (alleged) sigma algebra as $\mathcal{A}$. Suppose $\{E_i\}_1^\infty \subset \mathcal{A}$, then

$$E_1^c \cap F = F \setminus E_1 = F \setminus F \cap E_1 \in \mathcal{R} \text{ and } F \cap \bigcup_1^\infty E_i = \bigcup_1^\infty F \cap E_i \in \mathcal{R}$$

$\square$

**2. (Folland 1.2.2)** Folland has already showed $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_\mathbb{R}$ for all $j$, and that the open and closed intervals both generate $\mathcal{B}_\mathbb{R}$. It will suffice to show that for an arbitrary open interval $(a,b)$, we have $(a,b) \in \mathcal{M}(\mathcal{E}_j)$ for $j > 2$ since this suffices to show $\mathcal{B}_\mathbb{R} \subset \mathcal{M}(E_1) \subset \mathcal{M}(E_j)$. Note that by closure under compliments we have $\mathcal{E}_5 = \mathcal{E}_8$ and $\mathcal{E}_6 = \mathcal{E}_7$, so we may use sets of both forms in these cases. Below we show that $(a,b) \in \mathcal{M}(\mathcal{E}_3), \mathcal{M}(\mathcal{E}_4), \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$ respectively

$$(a,b) = \bigcup_1^\infty (a, b-1/n] = \bigcup_1^\infty [a+1/n, b) = (a,\infty) \cap \bigcup_1^\infty (-\infty, b-1/n] = (-\infty, a) \cap \bigcup_1^\infty [b+1/n, \infty)$$

$\square$

**3. (Folland 1.2.3) (a)** Assume not, then let $N$ be the size of the largest collection of disjoint sets in $\mathcal{A}$. Now let $E_1, \ldots, E_N$ be disjoint, we know that $\{\bigcup_S E_i \mid S \subset \mathcal{P}(\{1,\ldots,N\})\}$ is finite, hence since $\mathcal{A}$ is infinite, there is some set $F \in \mathcal{A}$ such that $\emptyset \subsetneq F \cap E_i \subsetneq E_i$ for some $i$, we may assume without loss of generality $i = 1$. It follows that $F \cap E_1, F^c \cap E_1, E_2, \ldots, E_N$ are all disjoint, but this contradicts $N$ being the size of the largest collection of disjoint sets in $\mathcal{A}$. $\square$

**(b)** Let $\{E_i\}_1^\infty$ be an infinite sequence of non-empty disjoint sets in $\mathcal{A}$. Then we have

$$\begin{aligned} F : \{0,1\}^{\aleph_0} &\to \mathcal{A} \\ b &\mapsto \bigcup_{\{n \mid b_n = 1\}} E_i \end{aligned}$$

Then $F$ is injective since if $S_1, S_2 \subset \mathbb{Z}_{>0}$ we have $\bigcup_{S_1} E_i \subset \bigcup_{S_2} E_i$ implies that $S_1 \subset S_2$ by the disjointness of the $E_i$. This shows that $\mathfrak{c} = \#\{0,1\}^{\aleph_0} \leq \#\mathcal{A}$. $\square$

**4. (Folland 1.2.4)** Define $F_n = \bigcup_1^n E_i$, then $F_1 \subset F_2 \subset \cdots$ and each $F_i \in \mathcal{A}$. It is immediate that $\bigcup_1^\infty E_i = \bigcup_1^\infty F_i \in \mathcal{A}$. $\square$

**5. (Folland 1.2.5)** Let $\mathcal{P}_\sigma(\mathcal{E}) = \{\mathcal{F} \in \mathcal{P}(\mathcal{E}) \mid \mathcal{F} \text{ is countable}\}$. Then let $\{E_i\}_1^\infty \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$, so for some $\{\mathcal{F}_i\} \subset \mathcal{P}_\sigma(\mathcal{E})$ we have $E_i \in \mathcal{M}(\mathcal{F}_i)$, hence $E_1^c \in \mathcal{M}(\mathcal{F}_1)$, and $\bigcup_1^\infty E_i \in \bigcup_1^\infty \mathcal{M}(\mathcal{F}_i) \subset \mathcal{M}(\bigcup_1^\infty \mathcal{F}_i)$, since each $\mathcal{F}_i$ is countable we get that $\bigcup_1^\infty \mathcal{F}_i \in \mathcal{P}_\sigma(\mathcal{E})$. Each set in $\mathcal{E}$ is countable, so $\mathcal{E} \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$, since the latter is a sigma algebra containing $\mathcal{E}$, we get that $\mathcal{M}(\mathcal{E}) \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$. Conversely each $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$, so that $\bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. $\square$

**6. (Classify the sigma algebras on the naturals)** The sigma algebras are in bijection to partitions of the naturals, or equivalently equivalence relations on the naturals. For any partition of the naturals $\bigsqcup_1^\infty E_i = \mathbb{N}$ we can form the sigma algebra $\mathcal{M}(\{E_i\}_1^\infty)$. Conversely, let $\mathcal{A}$ be a sigma algebra, we can define the equivalence relation $x \sim y$ when for any $E \in \mathcal{A}$, $x \in E \implies y \in E$. Reflexivity and transitivity are obvious. To see symmetry we prove the contrapositive, assume $E \in \mathcal{A}$ with $y \in E$ but $x \notin E$, then $E^c \in \mathcal{A}$ and $x \in E^c$ but $y \notin E^c$. Now let $\{E_i\}_1^\infty$ be the partition corresponding to this equivalence relation. To se that $\mathcal{M}(\{E_i\}_1^\infty) = \mathcal{A}$, we first consider $x \in \mathbb{N}$ and $S = \{E \in \mathcal{A} \mid x \in E\}$ with the partial ordering giving by set inclusion, if $E_1 \supset E_2 \supset \cdots$ is a chain in $S$, then $x \in \bigcap_1^\infty E_i \in \mathcal{A}$ is a lower bound, hence by Zorn's lemma there is a smallest set $E_x \in \mathcal{A}$ with $x \in E_x$. Let $i$ such that $x \in E_i$, then $E_i = \bigcap_{\{E \in \mathcal{A} \mid x \in E\}} E = E_x$, since the $E_i$ are disjoint and every element of $\mathbb{N}$ is in some $E_i$ we have that each $E_i$ is the smallest set in $\mathcal{A}$ containing some natural number, hence $\mathcal{M}(\{E_i\}_1^\infty) \subset A$. To see that we have equality it will suffice to show that there is no non-empty set in $\mathcal{A}$ that is contained in any of the $E_i$ (this reduction follows immediately from the trick in problem 3(a)), but this is immediate since if there were some nonempty $E \in \mathcal{A}$ with $E \subsetneq E_i$, then there is some $x \in E$, but $E_i$ is the smallest set in $\mathcal{A}$ containing $x$ which is an immediate contradiction, hence $\mathcal{A} \subset \mathcal{M}(\{E_i\}_1^\infty)$. $\square$