

1.

$$\begin{aligned}\Im(\sigma) &= \Im\left(\frac{(a\tau + b)(c\bar{\tau} + d)}{\|c\tau + d\|^2}\right) = \Im\left(\frac{ac\|\tau\|^2 + bd + ad\tau + bc\bar{\tau}}{\|c\tau + d\|^2}\right) \\ &= \frac{1}{\|c\tau + d\|^2} \Im(ad\tau + bc\bar{\tau}) = \frac{1}{\|c\tau + d\|^2} (ad - bc)\Im(\tau) = \frac{1}{\|c\tau + d\|^2} \Im(\tau)\end{aligned}$$

Then since $\|c\tau + d\|^2 > 0$ and $\Im(\tau) > 0$ we conclude that $\Im(\sigma) > 0$.

Now define

$$\begin{aligned}f : \mathbb{C} &\rightarrow \mathbb{C}/(\mathbb{Z} + \sigma\mathbb{Z}) \\ z &\mapsto \frac{z}{c\tau + d} + \mathbb{Z} + \sigma\mathbb{Z}\end{aligned}$$

To see this descends to a holomorphic map on the torus, we need to check it is periodic with respect to $\mathbb{Z} + \tau\mathbb{Z}$, so we want to check that $f(z + n\tau + m) = f(z)$ for any $n, m \in \mathbb{Z}$. Since f is linear it will be sufficient to show that $f(\tau)$ and $f(1)$ both lie in the lattice $\mathbb{Z} + \sigma\mathbb{Z}$. This is just a computation,

$$\tau = (ad - bc)\tau + bd - bd = d(a\tau + b) - b(c\tau + d) \quad (1)$$

$$1 = (ad - bc) + ac\tau - ac\tau = -c(a\tau + b) + a(c\tau + d) \quad (2)$$

So that (1) gives us $f(\tau) = d\sigma - b$ and (2) gives $f(1) = -c\sigma + a$ both are in $\mathbb{Z} + \sigma\mathbb{Z}$, so that f descends to the torus X_τ . To see that f is a biholomorphism just take $\mathbb{C} \rightarrow X_\tau$ via $z \mapsto (c\tau + d)z$, this descends to a holomorphic map on X_σ since $\sigma \mapsto \tau$, and $1 \mapsto c\tau + d$ are both in $\mathbb{Z} + \tau\mathbb{Z}$, moreover this is clearly the inverse of f . \square

2. I will use from class the identification of two copies of \mathbb{C} , the first being labelled (I), and the second (II), so that X is defined by gluing (I) to (II) along the line segments from 0 to 1, and from λ to ∞ along the line through 1 and λ as in lecture. Now on a neighborhood of infinity (more explicitly on $U = \{z \mid |z| > \max\{1, |\lambda|\}\}$ in both (I) and (II)) we can take local coordinate t , so that

$$t^2 = \frac{1}{z} \quad t = \begin{cases} \sqrt{z} & z \in (I) \\ -\sqrt{z} & z \in (II) \end{cases} \quad (3)$$

This allows us to compute

$$2tdt = -\frac{1}{z^2} dz = -t^4 dz \quad (4)$$

$$dz = -2t^{-3} dt \quad (5)$$

Moreover we also have defined $\omega^2 = z(z-1)(z-\lambda)$ with sign conventions

$$\omega = \begin{cases} \sqrt{z(z-1)(z-\lambda)} & z \in (I) \\ -\sqrt{z(z-1)(z-\lambda)} & z \in (II) \end{cases} \quad (6)$$

Then using our local t -coordinates we have

$$\omega^2 = \frac{1}{t^2} \left(\frac{1}{t^2} - 1 \right) \left(\frac{1}{t^2} - \lambda \right) = \frac{1}{t^6} (1-t^2)(1-\lambda t^2)$$

Using the sign conventions for t and ω in (3) and (6) this is consistent with

$$\frac{1}{\omega} = \frac{t^3}{(1-t^2)(1-\lambda t^2)} \quad (7)$$

Substituting (5) into (7) yields the differential in local t -coordinates near infinity.

$$\frac{dz}{\omega} = \frac{-2dt}{(1-t^2)(1-\lambda t^2)} \quad (8)$$

Since $|z| = \frac{1}{|t^2|} > \max\{|1|, |\lambda|\}$ this is a holomorphic form on U , which is clearly non-vanishing. \square

3. (a) Consider an arbitrary meromorphic function $f : X \rightarrow \mathbb{C}$, we may denote the poles of f as P_1, \dots, P_r with multiplicities n_1, \dots, n_r . Away from its poles, f is holomorphic as a map to \mathbb{C} , and hence to \mathbb{P}^1 , so we need only consider the behaviour of f near its poles, we should also note that f is well defined as a set map to \mathbb{P}^1 by taking $f(P_j) = \infty$ for each P_j . Now let (U, ψ) be a chart containing P_1 , by possibly shrinking U we may assume that $0 \notin f(U)$, so that $f(U)$ is contained in the chart of \mathbb{P}^1 containing ∞ , moreover for simplicity we may assume that $0 \in U$ with $\psi(0) = P_1$. Now since f is meromorphic on X with pole P_1 of multiplicity n_1 we have

$$f \circ \psi(z) = \sum_{-n_1}^{\infty} a_k z^k \quad (9)$$

with $a_{-n_1} \neq 0$. Now since the image lies entirely of the chart of \mathbb{P}^1 containing ∞ , to consider this as a map to \mathbb{P}^1 we compose with the coordinate chart φ taking $\mathbb{C} \rightarrow (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$ on \mathbb{P}^1 , which takes $z \mapsto 1/z$, the point ∞ corresponds to 0 in this coordinate chart. This gives the following expression for f around P_1 in coordinates:

$$\varphi^{-1} \circ f \circ \psi : z \mapsto \begin{cases} (\sum_{-n_1}^{\infty} a_k z^k)^{-1} & z \neq 0 \\ \varphi^{-1}(f \circ \psi(0)) = \varphi^{-1}(f(P_j)) = \varphi^{-1}(\infty) = 0 & z = 0 \end{cases} \quad (10)$$

Now we may simplify $(\sum_{-n_1}^{\infty} a_k z^k)^{-1} = \frac{z^{n_1}}{\sum_0^{\infty} a_{k-n_1} z^k}$, which of course takes value 0 at $z = 0$, which simplifies the piecewise expression for f in charts given by (10) to $\frac{z^{n_1}}{\sum_0^{\infty} a_{k-n_1} z^k}$. Since the denominator is a nonvanishing (near zero) holomorphic function convergent near zero, we find that $\varphi^{-1} \circ f \circ \psi$ is a holomorphic map $X \rightarrow \mathbb{P}^1$ in a chart around P_1 , the same argument works for charts around P_2, \dots, P_n so that f is indeed a holomorphic function between X and \mathbb{P}^1 . \square

(b) This question only works if d counts multiplicity and if f is nonconstant, so consider d counting $f^{-1}(p)$ with multiplicity for a nonconstant holomorphic $f : X \rightarrow \mathbb{P}^1$. By compactness, it suffices to show that $f^{-1}(p)$ is discrete to conclude that d is finite (this follows since so long as f is not constant the degree of any single one of the preimages is finite). Moreover, since a finite union of discrete sets is discrete, we may cover X in a finite number of charts $\{(U_j, \varphi_j)\}_1^r$ and check that $f|_{U_j}^{-1}(p)$ is discrete for each j . Since coordinate charts are diffeomorphisms, we can once again reduce the problem to checking each $(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}\{\psi^{-1}(p)\}$ is discrete for some ψ corresponding to a coordinate chart containing p , with $\psi^{-1} \circ f|_{U_j} \circ \varphi_j$ simply being a holomorphic function $\varphi^{-1}(U) \subset \mathbb{C} \rightarrow \mathbb{C}$, this reduces to the fact that the zeroes of a holomorphic function are discrete, and

$$(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}(\psi^{-1}(p)) = \{z \in \varphi^{-1}(U_j) \mid \psi^{-1} \circ f|_{U_j} \circ \varphi_j(z) - \psi^{-1}(p) = 0\}$$

is the set of zeroes of a holomorphic function. Thus by the reductions above, $d < \infty$.

Now to see that d is constant, I will show that d is constant in some open set U containing p , since this will hold for any $p' \in X$ this implies that the degree of a point is a continuous map $X \rightarrow \mathbb{Z}$, so that since X is connected it must be the constant map. To see that d is locally constant, take $\{p_0, \dots, p_r\} = f^{-1}(p)$, so that all together p_1, \dots, p_r have total multiplicity d . Now we can for every $x \in X \setminus \{p_1, \dots, p_r\}$ take some chart (U_x, z_x) with $x \in U_x \subset X \setminus \{p_1, \dots, p_r\}$, and for each p_j we can take disjoint coordinate charts (U_j, z_j) with $p_j \in U_j$. Now let (V, ψ) be a coordinate chart in \mathbb{P}^1 with $\psi(p) = 0$, by possibly taking the intersection with $f^{-1}(V)$ we can assume each of the U_x and U_j map into V under f . Since the degree of a map $\mathbb{C} \rightarrow \mathbb{C}$ is locally constant, we can (once again by possibly shrinking the open sets) assume that for each $x \in X \setminus \{p_1, \dots, p_r\}$ we have $\text{ord}_w(\psi^{-1} \circ f(z_x)) = 0$ for all $w \in U_x$, and for each j , $\text{ord}_w(\psi^{-1} \circ f(z_j)) = \text{ord}_{p_j}(\psi^{-1} \circ f(z_j))$ for all $w \in U_j$. Only now that we have reduced our open sets to be sufficiently nice do we apply the compactness assumption to get a finite subcover $U_{x_1}, \dots, U_{x_N}, U_1, \dots, U_r$ for X , note that since each p_j lies only in U_j , we must have each U_j included in our open cover. By the open mapping theorem we know that each $\psi^{-1} \circ f(z_x)$ (or z_j) is open, so that $Y := \bigcap_1^N \psi^{-1} \circ f(z_x(U_{x_k})) \bigcap_1^r \psi^{-1} f(z_j(U_j))$ is an open set containing since ψ is a homeomorphism we can apply it to get $\psi(Y)$ is an open subset of \mathbb{P}^1 containing p . Moreover the degree of f is locally

constant as a map $X \rightarrow \psi(Y)$, since the degree is zero on each U_k , and locally constant on each U_j from the version of this result on \mathbb{C} . Explicitly, since the U_j are disjoint and $f|_{U_{x_k}} \cap \psi(Y) = \emptyset$, we have for any $p' \in \psi(Y)$ that $f^{-1}(p') = \bigsqcup_1^r f|_{U_j}^{-1}(p')$, and moreover that the number of points with multiplicity d is preserved since the multiplicity of f is constant on each U_j . \square

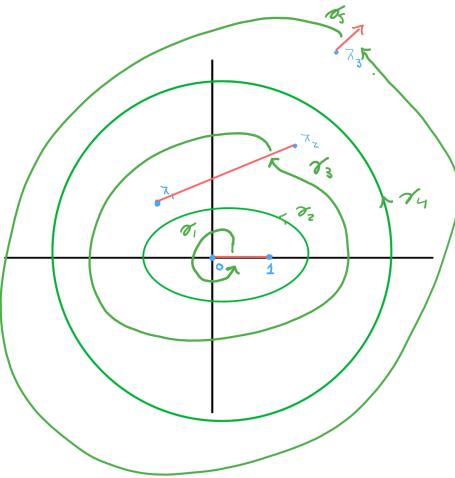
(c) z has degree 2, and w has degree 3. As proof, by parts (a) and (b) it suffices to compute the preimage for a single point in which it is convenient. For z this is very simple, since for any point x away from the identifications $z^{-1}(x)$ consists of two copies of x , one in each glued copy of \mathbb{C} (i.e. choose any value of x on the imaginary line distinct from λ). In the case of w , we count its zeroes, $w^2 = x(x-1)(x-\lambda)$ implies that w vanishes at each of 0, 1 and λ , since these are glued points, we do not get the same double counting as in the z case and this gives degree 3, this is also up to multiplicity, since w has vanishing of order 1 at each of these points, checking at zero (since 1 and λ are similar) we have the local coordinate $t^2 = z$, defined via

$$t = \begin{cases} \sqrt{z} & \text{on (I)} \\ -\sqrt{z} & \text{on (II)} \end{cases}$$

so that $w = t\sqrt{(t^2-1)(t^2-\lambda)}$ indeed has order one vanishing. \square

4. X is a surface of Genus 2. Define line segments $L_1 = [0, 1]$, L_2 the component of the line determined by λ_1 and λ_2 between λ_1 and λ_2 , and L_3 the component of the line segment between λ_2 and λ_3 from λ_3 to ∞ . Here we assume the points $\{0, 1, \lambda_1, \lambda_2, \lambda_3\}$ are arranged so that none of the line segments L_1, L_2 and L_3 intersect. Since X is given as the analytic continuation of $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$, we consider the monodromy of these square root functions around these points

- γ_1 is a curve with winding number 1 around 0, and winding number zero around 1, $\lambda_1, \lambda_2, \lambda_3$
- γ_2 is a curve with winding number 1 around 0 and 1, and winding number zero around $\lambda_1, \lambda_2, \lambda_3$
- γ_3 is a curve with winding number 1 around 0, 1 and λ_1 and winding number zero around λ_2 and λ_3
- γ_4 is a curve with winding number 1 around 0, 1, λ_1 and λ_2 and winding number zero around λ_3
- γ_5 is a curve with winding number 1 around all points



We record the monodromy of each of the square root constituents around these curves:

- On γ_1 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}$, with the $\sqrt{z-1}, \sqrt{z-\lambda_1}, \sqrt{z-\lambda_2}, \sqrt{z-\lambda_3}$ having no monodromy.

- On γ_2 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}$ and $\sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$ with no monodromy for the other factors.
- On γ_3 we get monodromy $\sqrt{z} \rightsquigarrow -\sqrt{z}$, $\sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$, and $\sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}$ with no monodromy for the other factors.
- On γ_4 we get monodromy:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \text{ and } \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}$$

fixing the last factor.

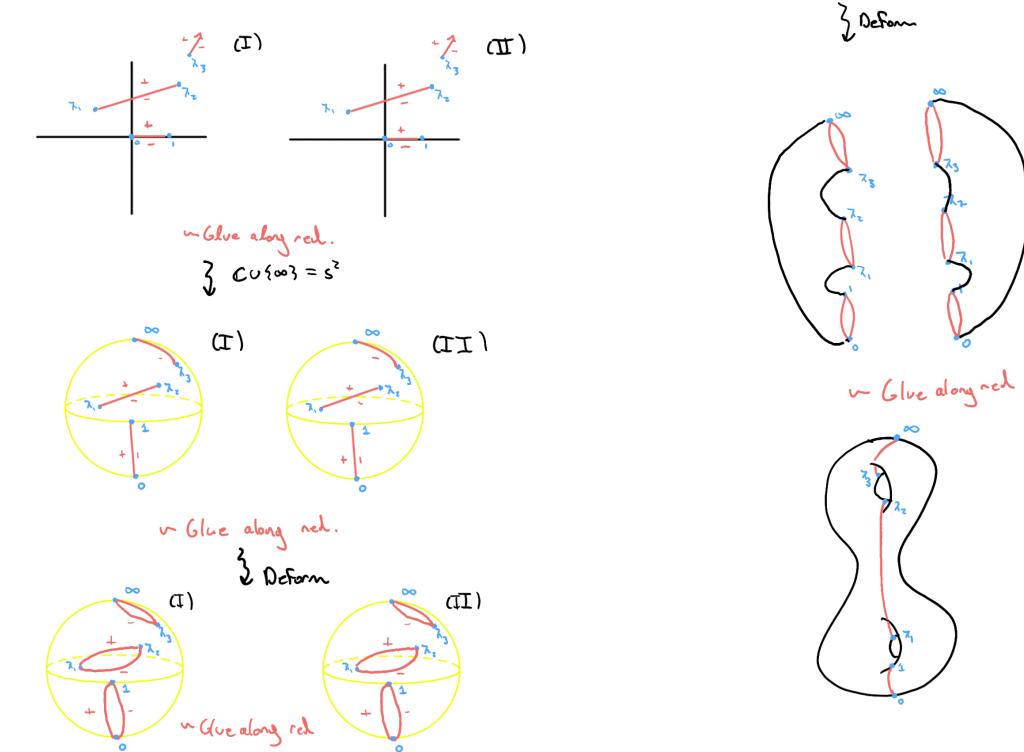
- Finally, on γ_5 we get monodromy on each of the components:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}, \sqrt{z-\lambda_3} \rightsquigarrow -\sqrt{z-\lambda_3}$$

The monodromy's cancel travelling around γ_j for j even, but not for j odd, thus to analytically extend this branch of the square root we once again take two copies of \mathbb{C} , (I) and (II) and gluing along L_1, L_2, L_3 , and analytically continue $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$ by defining function w with $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$ and

$$w = \begin{cases} \sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (I)} \\ -\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (II)} \end{cases} \quad (11)$$

By sketching this glueing we see that X is a surface of genus 2 (please ignore the yellow and pretend that its black SORRY!!).



The meromorphic function z simply projects (I) and (II) coordinates to \mathbb{C} , similarly to part (c) of problem (3), we can pick a point $x \in \mathbb{C}$ away from L_1, L_2, L_3 where the gluing occurs so that $z^{-1}(x)$

is trivially 2 copies of x , one in (I) and one in (II), using question 3 this degree is well defined so z has degree 2. Also similarly to problem (3c), we count calculate the degree of w as $\#w^{-1}(0)$, which has points $0, 1, \lambda_1, \lambda_2$ and λ_3 , to check that each of these zeroes has multiplicity one we check in a local coordinate around 0, since the proof is the same for the other points. So take local coordinate t around zero (the chart being $U = B_\epsilon(0)$ in (I) and (II) with $\epsilon < \min\{1, |\lambda_1|, |\lambda_2|, |\lambda_3|\}$) with $t^2 = z$, and $t = \sqrt{z}$ on (I) and $t = -\sqrt{z}$ on (II), it follows that on U we have $w = t\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$, with $\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$ holomorphic and nonvanishing on U , so that indeed the vanishing of w at 0 is multiplicity one. \square

5. There are four distinct cases, firstly we may have $Q_1 \in \{0, 1, \lambda\}$ and Q_2 generic, secondly we may have Q_1 and $Q_2 \in \{0, 1, \lambda\}$, the other cases deal with the point infinity, namely $Q_1 = \infty$ and Q_2 generic, or $Q_1 \in \{0, 1, \lambda\}$ and $Q_2 = \infty$.

(Case 1) Assume that $Q_1 = 0$, and let $Q_2 \in X \setminus \{0, 1, \lambda, \infty\}$, then we define the meromorphic one form to be

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}z + w}{z \cdot (z - z(Q))} \frac{dz}{w} \quad (12)$$

Near Q , we may work in z charts to get the expression

$$\omega_{0,Q} = \frac{1}{z - z(Q)} \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{dz}{w} \quad (13)$$

so that the residue is just

$$\text{Res}(\omega_{0,Q}, Q) = \left. \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{1}{w} \right|_Q = \frac{2}{z(Q)} \quad (14)$$

To evaluate the residue at zero, we take charts near zero by taking t so that $t^2 = z$ and $t = \pm\sqrt{z}$ on (I) and (II) respectively. we can rewrite the form $\omega_{0,Q}$ given by (12) in t coordinates, this gives the local expression

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}t^2 + t\sqrt{(t^2-1)(t^2-\lambda)}}{t^2(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \quad (15)$$

$$= \frac{1}{t} \left(\frac{\frac{w(Q)}{z(Q)}t + \sqrt{(t^2-1)(t^2-\lambda)}}{(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \right) \quad (16)$$

From this we can calculate the residue,

$$\text{Res}(\omega_{0,Q}, 0) = \frac{\sqrt{\lambda}}{-z(Q)} \frac{2}{\sqrt{\lambda}} = -\frac{2}{z(Q)} \quad (17)$$

Since $\text{Res}(\omega_{0,Q}, 0) = -\text{Res}(\omega_{0,Q}, Q)$ we can simply normalize to get ± 1 . Finally we check these are indeed the only poles of our form, the only points of interest are \tilde{Q} and ∞ where \tilde{Q} is the unique distinct point from Q with $z(\tilde{Q}) = z(Q)$, but plugging in \tilde{Q} the numerator becomes $\frac{w(Q)}{z(Q)}z(\tilde{Q}) + w(\tilde{Q}) = w(Q) - w(Q) = 0$, so that there is indeed no pole at \tilde{Q} , to see there is no pole at infinity recall that dz/w is a nonvanishing holomorphic form, then the expression $\frac{\frac{w(Q)}{z(Q)}z + w}{z(z - z(\tilde{Q}))}$ has denominator $\mathcal{O}(|z|^2)$ and numerator $\mathcal{O}(|z|^{3/2})$ implying there is no pole at infinity.

(Case 2) This time take $Q_1 = 0$ and $Q_2 = 1$, we define

$$\omega_{0,1} = \frac{w}{2z(z-1)} \frac{dz}{w} \quad (18)$$

Similar to the first case, this form only has candidate poles at 0, 1 and infinity, but it is trivial to rule out infinity thus we only need check the residues. Near zero we once again have local coordinates near zero $t = \pm\sqrt{z}$ on (I) and (II) respectively, these local coordinates give the local expression

$$\omega_{0,1} = \frac{2tdt}{2t^2(t^2-1)} = \frac{1}{t} \frac{1}{t^2-1} dt \quad (19)$$

from this we read off $\text{Res}(\omega_{0,1}, 0) = -1$. Similarly we can take local coordinates near one, $t = \pm\sqrt{z-1}$ on (I) and (II) respectively, in these local coordinates

$$\omega_{0,1} = \frac{2tdt}{2(t^2+1)t^2} = \frac{1}{t} \frac{1}{1+t^2} dt \quad (20)$$

Which we can read off as having residue 1.

(Case 3) Take $Q_1 = 0, Q_2 = \infty$. We follow a similar procedure to the first two cases,

$$\omega_{0,\infty} = \frac{w}{2z} \frac{dz}{w} \quad (21)$$

its clear the only candidates for poles are 0 and ∞ , so we check that they are simple and take the residues. Taking local coordinates near infinity we have $t = \pm 1/\sqrt{z}$ on (I) and (II) respectively, in these local coordinates we write

$$\omega_{0,\infty} = \frac{t^2}{2} \frac{-2dt}{t^3} = -\frac{1}{t} dt \quad (22)$$

which is a simple pole with residue -1 . Taking the same coordinates near zero as in the prior cases we find the local expression

$$\omega_{0,\infty} = \frac{1}{2t^2} 2tdt = \frac{1}{t} dt \quad (23)$$

so that we have a simple pole with residue 1 at zero.

(Case 4) Take $Q_1 \in X \setminus \{0, 1, \lambda, \infty\}$ to be a generic point, and $Q_2 = \infty$, then we can define

$$\omega_{Q,\infty} = \frac{1}{2} \frac{w(Q) + w}{z - z(Q)} \frac{dz}{w} \quad (24)$$

The zero of $w(Q) + w$ at \tilde{Q} removes the simple pole at \tilde{Q} (where once again \tilde{Q} is the other lift of $z(Q)$ in the double cover), moreover the pole at Q is clearly simple so we need only check the pole at infinity is simple and compute the residues. To compute the residue at Q , we can simply use z coordinates,

$$\omega_{Q,\infty} = \frac{1}{2} \frac{w(Q) + w}{z - z(Q)} \frac{dz}{w} = \frac{1}{z - z(Q)} \frac{w(Q) + w}{2w} dz \quad (25)$$

So that $\text{Res}(\omega_{Q,\infty}, Q) = \frac{2w(Q)}{2w(Q)} = 1$, to compute the residue at infinity we work in the same local coordinates as in case 3, this gives us the following local expression

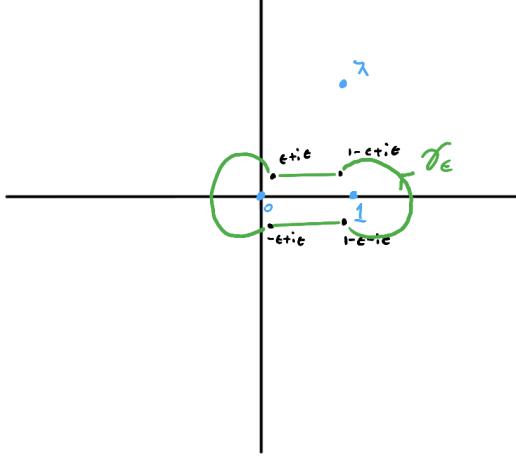
$$\omega_{Q,\infty} = \frac{1}{2} \frac{t^2(w(Q) + w)}{1 - t^2z(Q)} \frac{-2dt}{t^3w} = \frac{-1}{t} \frac{w(Q) + w}{w(1 - t^2z(Q))} \quad (26)$$

From this expression we get the desired value of the residue,

$$\text{Res}(\omega_{Q,\infty}, \infty) = - \left(\frac{w(Q) + w}{w(1 - t^2z(Q))} \right) \Big|_{t=0} = - \left(\frac{w(Q)}{w} + 1 \right) \Big|_{t=0} = 1 \quad (27)$$

Since $w(t)|_{t=0} = \infty$.

Thus for any distinct points $Q_1, Q_2 \in X$ regardless of genericity there is an abelian differential of the third kind ω_{Q_1, Q_2} . \square



6. Since the branch of the square root function used to define Ω is analytic on $\mathbb{C} \setminus L$, we can use the following curve γ_ϵ to define our contour integral (The value of the integral is independent of this choice of curve by Cauchy's integral theorem).

Since Ω is holomorphic on $\mathbb{C} \setminus L$, the area enclosed by γ_ϵ is independent of ϵ so long as ϵ is sufficiently small. Taking the limit $\epsilon \rightarrow 0$, we can show that the circular component of the curve has no contribution using the standard arc length inequality, since near 0 and 1 on γ_ϵ we have $\frac{1}{w} \in \mathcal{O}(\epsilon^{-\frac{1}{2}})$, and the arc length of the circular component is bound above by $2\pi\epsilon$, the integral over the circular component is bound above by $2\pi\epsilon\mathcal{O}(\epsilon^{-\frac{1}{2}}) \in \mathcal{O}(\epsilon^{\frac{1}{2}})$. The horizontal components of the integral then approach the curve on the real line from 1 to 0 from above, and the curve from 0 to 1 from below. From this we get

$$\int_{\gamma} \frac{dz}{w} = \lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} \frac{dz}{w} = \lim_{\epsilon \downarrow 0} \int_{1-\epsilon}^{\epsilon} \frac{dx}{w(x+i\epsilon)} + \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{dx}{w(x-i\epsilon)} \quad (28)$$

$$= \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{1}{w(x-i\epsilon)} - \frac{1}{w(x+i\epsilon)} dx = -2 \int_0^1 \frac{dx}{\lim_{\epsilon \downarrow 0} w(x+i\epsilon)} \quad (29)$$

Where passing to the limit is justified by $\Omega = \frac{dz}{w}$ being a holomorphic differential on X , hence uniformly continuous on a compact set in $\mathbb{C} \setminus L$ containing γ , and $x+i\epsilon$, and $x-i\epsilon$ converging uniformly to x from above and below, finally the sign convention on taking the limit $\frac{1}{w(x-i\epsilon)}$ follows from $\lim_{\epsilon \uparrow 0} w(x+i\epsilon) = -\lim_{\epsilon \downarrow 0} w(x+i\epsilon)$ by the monodromy of w around the branch cut at L . Now if λ is real, then the result is trivial since in this case $\lim_{\epsilon \downarrow 0} w(x+i\epsilon) = \sqrt{x(1-x)(x-\lambda)}$, so that $\int_0^1 \frac{dx}{\sqrt{x(1-x)(x-\lambda)}} \neq 0$ since the integrand is either strictly positive real or strictly positive imaginary depending on $\lambda < 0$ or $\lambda > 1$. For the case of λ not real, I will address $\Im(\lambda) > 0$, since the case of λ in the lower half plane is similar. Now for λ in the upper half plane, first note that explicitly the branch cut of $\sqrt{z(z-1)(z-\lambda)}$ which is analytic away from L is given by $\exp(\frac{1}{2}(\text{Log } z + \text{Log}(z-1) + \text{Log}(z-\lambda)))$, so in order to see that $\int_{\gamma} \frac{dz}{w} = -2 \int_0^1 \frac{dx}{\lim_{\epsilon \downarrow 0} w(x+i\epsilon)} \neq 0$, it suffices to check $\lim_{\epsilon \downarrow 0} w(x+i\epsilon)$ has strictly positive imaginary part for each $x \in [0, 1]$, since this will imply that $(\lim_{\epsilon \downarrow 0} w(x+i\epsilon))^{-1}$ has strictly negative imaginary part, so that $-2(\lim_{\epsilon \downarrow 0} w(x+i\epsilon))^{-1}$ will have strictly positive imaginary component, which implies that $\Im(\int_{\gamma} \frac{dz}{w}) > 0$, so the integral is nonzero.

The verification follows from just computing the limit:

$$\lim_{\epsilon \downarrow 0} \Im(w(x + i\epsilon)) = \Im \left(\exp \left(\frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x + i\epsilon) + \text{Log}(x - 1 + i\epsilon) + \text{Log}(x - \lambda + i\epsilon) \right) \right) \quad (30)$$

$$= \Im \left(\exp \left(\frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x + i\epsilon) \right) \exp \left(\frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x - 1 + i\epsilon) \right) \exp \left(\frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x - \lambda + i\epsilon) \right) \right) \quad (31)$$

$$= \Im(\sqrt{x}(e^{\frac{i\pi}{2}}\sqrt{1-x})(|x-\lambda|^{\frac{1}{2}} e^{\frac{i}{2}\text{Arg}(x-\lambda)})) \quad (32)$$

$$= \sqrt{x(1-x)|x-\lambda|} \Im(e^{\frac{i}{2}(\pi+\text{Arg}(x-\lambda))}) \quad (33)$$

But since $\Im(\lambda) > 0$, $x - \lambda$ lies strictly in the lower half plane giving us $\text{Arg}(x - \lambda) \in (-\pi, 0)$, this implies that $\frac{1}{2}(\pi + \arg(x - \lambda)) \in (0, \frac{\pi}{2})$, so that $\Im(e^{\frac{i}{2}(\pi+\text{Arg}(x-\lambda))}) > 0$. Combining this with (33) we find that

$$\lim_{\epsilon \downarrow 0} \Im(w(x + i\epsilon)) > 0 \quad (34)$$

for all x , as desired. \square

7. To see that $\text{Res}(\Omega, p)$ is well defined, we need to check it is invariant under our choice of coordinate charts, this basically reduces to the compatibility assumptions for holomorphic differentials. Let $(U, z), (V, w)$ be charts for X containing p , then we only need check the two coordinate expressions agree, we may take a loop γ around p with winding number 1, with image contained in $U \cap V$, then γ may be parameterized by either z or w , so that the residue in either chart is given by $\frac{1}{2\pi i} \int_{\gamma(z)} f_U(z) dz$ and $\frac{1}{2\pi i} \int_{\gamma(w)} f_V(w) dw$, we need only check these two expressions are equal.

$$\frac{1}{2\pi i} \int_{\gamma(z)} f_U(z) dz = \frac{1}{2\pi i} \int_{\gamma(z)} f_V(w(z)) w'(z) dz \quad (35)$$

Now we may apply the substitution $w = w(z)$, so that $dw = w'(z) dz$, applying the substitution gives us

$$\frac{1}{2\pi i} \int_{\gamma(z)} f_V(w(z)) w'(z) dz = \frac{1}{2\pi i} \int_{\gamma(w)} f_V(w) dw \quad (36)$$

Combining (35) and (36) gives the desired equality of residue in arbitrary coordinate charts so that $\text{Res}(\Omega, p)$ is indeed well defined. \square