1. (Durrett 1.1.5) $\mathcal{A}$ is not an algebra, hence not a σ-algebra, as proof let A be the even numbers, and B be as defined below

$$B = \bigcup_{n \text{ even}} \{k \mid k \text{ odd and } 2^n \le k < 2^{n+1}\} \bigcup_{n \text{ odd}} \{k \mid k \text{ even and } 2^n \le k < 2^{n+1}\}$$

Then it is clear $\theta(A) = \theta(B) = \frac{1}{2}$. Now we want to consider $\theta(A \cup B)$, note that $A \cup B$ contains $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ when n is even, but contains only $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1, 2^{n+1}\} \cap \{\text{even numbers}\}$ for odd n. Then denote $\theta_n = \frac{\#((A \cup B) \cap \{1, \ldots, 2^{n+1}\})}{2^{n+1}}$, then the first few terms are $\theta_1 = 1, \theta_2 = \frac{3}{4}, \theta_3 = \frac{7}{8}, \theta_4 = \frac{11}{16}$ and from the definition of A, B we get $\theta_{2n+1} = \frac{\theta_{2n}}{2} + \frac{1}{2}$ and $\theta_{2n+2} = \frac{\theta_{2n+1}}{4} + \frac{1}{4}$, it follows that by induction the subsequences θ_{2n} and θ_{2n+1} are decreasing, then once again by induction using this recurrence we find that $\frac{11}{16} \ge \theta_{2n} \ge \frac{1}{2}$ and $1 \ge \theta_{2n+1} \ge \frac{3}{4}$, but then $\liminf \theta_{2n+1} \ge \frac{3}{4} > \frac{11}{16} \ge \limsup \theta_n$, so these subsequences of $\frac{\#((A \cup B) \cap \{1, \ldots, n\})}{n}$ can't possibly converge to the same limit, and hence a limit for the sequence cannot exist and $A \cup B$ does not have an asymptotic density. $\square$

2. (Durrett 1.2.3) First note that the left limit of a distribution function is well defined,

$$F(x-) := \lim_{y_n \uparrow x} F(x) = \bigcup_1^\infty P(X \le y_n) = P(X < x)$$

The last equality following from throwing out y_n such that for some $k < n$, there is $y_k > y_n$ and applying continuity from below.

It follows that for each point of discontinuity of F, we must have $F(x) > F(x-)$, assuming there are uncountably many points of discontinuity for F and denote that set of points as A, we know that since $0 \le F(x) \le 1$ is an increasing function that

$$1 = \lim_{x \to \infty} F(x) \ge \sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\}$$

Denote $E_n = \{\alpha \in A \mid F(\alpha) - F(\alpha-) \ge \frac{1}{n}\}$, then since $\bigcup_1^\infty E_n = A$, we must have atleast one E_n is uncountable. This implies that

$$\sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\} \ge \sup_{M \in \mathbb{N}} \frac{M}{n} = \infty$$

Which is a contradiction. $\square$

3. (Durrett 1.3.5) If f is not LSC, then there is some x and $y_n \to x$, such that $\lim f(y_n) < f(x)$ (this follows from the negation since we can take a subsequence which gives the liminf). But then let $\epsilon = f(x) - \lim f(y_n)$, if we remove the y_n terms such that $f(y_n) > f(x) + \frac{\epsilon}{2}$ from the sequence then the sequence still converges to x, so we may assume the sequence is uniformly bounded by $f(x) + \frac{\epsilon}{2}$. But then we have a sequence $y_n \in \{t \mid f(t) \le f(x) - \epsilon/2\}$ which converges to a value x not in the set, so in particular the set is not closed.

Conversely, if for some a, the set $S_a := \{x \mid f(x) \le a\}$ is not closed, then we get a sequence $y_n \in S_a$ such that $y_n \to x$, but $x \notin S_a$, it follows that $f(x) > a$, but $\liminf_{y \to x} f(y) \le \lim f(y_n) \le a < f(x)$ so that f is not LEC. $\square$

For a more general topological space than a metric space just replace sequences with nets

4. (Durrett 1.3.7) First we note that all simple functions are measurable, and measurable functions are closed under pointwise limits, closure under pointwise limits follows from $\limsup$ being measurable, and $\lim f_n(x) = \limsup f_n(x)$ at all points x when the limit exists. Now let f be an arbitrary measurable function on $(\Omega, \mathcal{F})$ mapping to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Then we can write $f = f_+ - f_-$ so it will suffice to show that an arbitrary positive function f is a pointwise limit of simple functions. Let $A_1 = f^{-1}[1, \infty)$, and $\phi_1 = 1_{A_1}$, now we can define the rest of the ϕ_i recursively:

$$A_n = (f - \phi_{n-1})^{-1}[\frac{1}{n}, \infty) \qquad\qquad \phi_n = n^{-1}1_{A_n}$$

Now pointwise the sequences $\phi_n(x)$ are bound above by $f(x)$ and monotone increasing hence convergent, we want to see it converges to $f(x)$, since the harmonic series diverges, for any x, we have some N such that $\sum_1^N \frac{1}{n} > f(x)$, furthermore by construction of ϕ_n we will have $f(x) - \frac{1}{N} \leq \phi_n(x) \leq f(x)$, and moreover for any $k > N$ we also have that $f(x) - \frac{1}{k} \leq \phi_k(x) \leq f(x)$ whence convergence follows immediately. $\square$

5. (Durrett 1.3.8) If $Y = f \circ X$ for $f : (\mathbb{R}, \mathcal{B}_\mathbb{R}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$, then for any borel set B, we have $Y^{-1}(B) = X^{-1}(f^{-1}(B))$, since f is measurable we know that $f^{-1}(B) \in \mathcal{B}_\mathbb{R}$, so that $X^{-1}(f^{-1}(B)) \in \sigma(X)$ by definition of $\sigma(X)$, which suffices to show all functionsof this form are measurable with respect to $\sigma(X)$. To show that all measurable functions on $\sigma(X)$ are of this form, we can use (Durrett 1.3.7) to check that all pointwise limits of simple functions on $\sigma(X)$ can be written in this form. Consider the simple functions $\phi_k = \sum_1^{N_k} c_i^k 1_{X^{-1}(B_i^k)}$, with $\phi_k \to g : (X, \sigma(X)) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ pointwise. It is immediate that $\phi_k = \varphi_k \circ X$ where $\varphi_k = \sum_1^{N_k} c_i^k 1_{B_i^k}$. It is straightforward to see that φ_k converges pointwise on $\mathbb{R}$, since if $x \in \mathbb{R}$, then $x = X(y)$ for $y \in \Omega$, then sequence $\phi_k \circ X(y)$ is equal to the sequence $\varphi_k(x)$, and hence $\lim_{k\to\infty} \varphi_k(x) = \lim_{k\to\infty} \phi_k \circ X(y) = g(y)$, denoting the pointwise limit of φ_k as f we know that f is measurable by closure of measurable functions under pointwise limits, and moreover, $f \circ X = g$ from construction. $\square$

6. (Durrett 1.6.6)

$$E[Y] = \int Y 1_{Y>0} \overset{\text{Hölder}}{\leq} \left(\int Y^2\right)^{\frac{1}{2}} \left(\int 1_{Y>0}\right)^{\frac{1}{2}}$$

squaring both sides gives us

$$(E[Y])^2 \leq E[Y^2]P(Y > 0)$$

we can rearrange to find

$$\frac{(E[Y])^2}{E[Y^2]} \leq P(Y > 0)$$

$\square$

7. (Durrett 2.1.4) To see that the X_n are uncorrelated (we could recall these functions are orthogonal in $L^2(0,1)$ from basic Fourier analysis– but I'll include some details) note that $\int_0^1 \sin(2\pi nx)dx = 0 = \int_0^1 \cos(2\pi nx)$ for any $n \in \mathbb{Z}$. We can also compute the expectation of the product using this observation, and the trigonometric identity for expressions of the form $\sin(a)\sin(b)$.

$$E[X_n X_m] = \int_0^1 \sin(2\pi nx)\sin(2\pi mx)dx = \int_0^1 \frac{1}{2}\left(\cos(2\pi(n-m)x) - \cos(2\pi(n+m)x)\right)dx = 0$$
$$= E[X_n]E[X_m]$$

This suffices to show the variables are uncorrelated. The collection $\{X_i\}$ is not independent, this can be seen by taking

$$X_1^{-1}(0, \frac{1}{2}) = (0, \frac{1}{12}) \cup (\frac{5}{12}, \frac{1}{2}) \qquad X_2^{-1}(0, \frac{1}{2}) = (0, \frac{1}{24}) \cup (\frac{5}{24}, \frac{1}{4}) \cup (\frac{1}{2}, \frac{13}{24}) \cup (\frac{17}{24}, \frac{3}{4})$$

It is immediate that

$$\frac{1}{24} = P\left(X_1, X_2 \in (0, \frac{1}{2})\right) \neq P\left(X_1 \in (0, \frac{1}{2})\right)P\left(X_2 \in (0, \frac{1}{2})\right) = \frac{1}{36}$$

$\square$

8. (Durrett 2.1.6)

$$P(f \circ X \in A, g \circ Y \in B) = P(X \in f^{-1}(A), Y \in g^{-1}(B)) = P(X \in f^{-1}(A)) \cap P(Y \in g^{-1}(B))$$
$$P(f \circ X \in A)P(g \circ Y \in B) = P(X \in f^{-1}(A))P(Y \in g^{-1}(B))$$

From independence of X, Y we can conclude that

$$P(X \in f^{-1}(A)) \cap P(Y \in g^{-1}(B)) = P(X \in f^{-1}(A))P(Y \in g^{-1}(B))$$

This is the definition of independence for f, g since A, B were arbitrary measurable sets. $\square$

9. (Durrett 2.1.9) Let $\mathcal{A}_1 = \{\emptyset, \Omega, \{1,2\}, \{1,4\}\}$ and $\mathcal{A}_2 = \{\emptyset, \Omega, \{1,3\}\}$, to check independence of these collections, it suffices to check pairwise independence of $\{1,2\}$ and $\{1,3\}$ and $\{1,4\}$ and $\{1,3\}$. This is clear since

$$P(\{1,2\})P(\{1,3\}) = \frac{1}{4} = P(\{1\}) = P(\{1,2\} \cap \{1,3\})$$

We can run the same computation for $\{1,4\}$ and $\{1,3\}$. To see that the generated algebra is not independent, we have $\{1,2\}^c \cap \{1,4\}^c = \{3,4\} \cap \{2,3\} = \{3\} \in \sigma(\mathcal{A}_1)$, but

$$P(\{3\} \cap \{1,3\}) = P(\{3\}) = \frac{1}{4} \neq \frac{1}{8} = P(\{3\})P(\{1,3\})$$

$\square$

10. (Durrett 2.1.14) since $X, Y \geq 0$ we may assume that $z \geq 0$, otherwise $P(XY \leq z) = 0$ trivially, so the distribution is known for those values. To compute the distribution when $z \geq 0$ we can use that independence means that (X, Y) has distribution given by $dF \times dG$ (Durrett Proposition 2.1.11).

$$P(XY \leq z) = \int \int 1_{XY \leq z} dGdF = \int \int 1_{0 < X < z} 1_{Y < \frac{z}{X}} + 1_{X=0} dGdF$$

$$= F(0) + \int 1_{0 < X < z} \int 1_{Y < \frac{z}{X}} dGdF = F(0) + \int 1_{0 < X < z} G(\frac{z}{X}) dF$$

$\square$