1. Suppose $X = \partial W$ for W compact, and f extends to $F : W \to M$, then we can take $\iota : X \hookrightarrow W$, so that $f = F \circ \iota$. Then

$$\int_X f^*\omega = \int_{\partial W} (F \circ \iota)^*\omega = \int_{\partial W} F^*\omega \overset{\text{Stokes}}{=} \int_W dF^*\omega = \int_W F^*d\omega = \int_W F^*0 = 0$$

$\square$

2. First suppose that ω is exact, then $\omega = df$, so that for any $\gamma : S^1 \to M$,

$$\int_{S^1} \gamma^*\omega = \int_{S^1} \gamma^*df = \int_{S^1} d\gamma^*f \overset{\text{Stokes}}{=} \int_{\emptyset} \gamma^*f = 0$$

For the converse, I will explicitly produce such an f, the assumption is used when showing that f is well defined. It will suffice to prove the connected case, since the proof can be iterated on each connected component i.e. if $M = \bigsqcup M_j$, then write $\omega = \sum \omega_j$ where ω_j is supported on M_j, then by using the proof in the connected case, we get smooth functions f_j on M_j with $df_j = \omega_j$, it follows that $d\sum f_j = \sum df_j = \sum \omega_j = \omega$. So letting $x_0 \in M$, for any other $x \in M$ define $f(x) = \int_{x_0}^x \omega$ (which is more explicitly the integral of the pullback along any path), a path for this integral always exists since we are working in the connected case. To show that the integral is well defined, we need to check that it is path independent, so assuming $\gamma(1) = x = \overline{\gamma}(1)$, we want to show $\int_{[0,1]} \gamma^*\omega = \int_{[0,1]} \overline{\gamma}^*\omega$, so that f is well defined. Since ω is closed, we may assume by homotopy invariance of the integral of closed forms that the paths are constantly x_0 on $[0, \frac{1}{4}]$ and constantly x on $[\frac{3}{4}, 1]$ or otherwise replace them with homotopic paths with this property using bump functions, (it should also be noted that we can use homotopy invariance since $\gamma([0,1]) \cup \overline{\gamma}([0,1])$ is compact, and hence contained in an open neighborhood with compact closure, replacing ω with its product with the component of a partition of unity subordinate to this open reduces to the compact case so homotopy can be used) it follows that we can write the smooth map $\beta : S^1 \to M$ via

$$\beta(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \overline{\gamma}(1-2t) & t \in (\frac{1}{2}, 1] \end{cases}$$

From construction β satisfies

$$\int_{S^1} \beta^*\omega = \int_{[0,1]} \gamma^*\omega - \int_{[0,1]} \overline{\gamma}^*\omega$$

and the integral of the pullback along β is zero by assumption, so the two are equal as desired and f is well defined. It remains to check that $\omega = df$, to do so we can check at each $x \in M$, so let $x \in M$, so that $x \in V$ with (U, V, φ) a chart for M, we may assume for simplicity that $\varphi(0) = x$, then for some small ϵ the paths $\alpha_j : t \mapsto te_j$ (where e_j is the j-th standard basis vector on $\mathbb{R}^n$) satisfy $\alpha_j[-\epsilon, \epsilon] \subset U$, for $1 \le j \le n$, now let $x^j = \alpha_j(-\epsilon)$, then for any $y \in M$, we have writing $c_j := \int_{x_0}^{x^j} \varphi^*\omega$, then (where the integral is once again implicitly the pull back along any smooth path)

$$\int_{\varphi(x^j)}^y \omega = \int_{x_0}^y \omega - \int_{x_0}^{\varphi(x^j)} \omega = f(y) - \int_{x_0}^{\varphi(x^j)} \omega = f(y) - c_j$$

so that $f = f_j + c_j$ since c_j is constant for the sake of differentiating we can take $f(x) = \int_{\varphi(x^j)}^x \omega =: f_j$ for whichever j is convenient. Now we get by definition $df = \sum_1^n \frac{\partial}{\partial x_j} f dx_j = \sum_1^n \frac{\partial}{\partial x_j} f_j dx_j$. We can write

$\varphi^*\omega = \sum_1^n \omega_k dx_k$, where the ω_k are smooth real valued functions,

$$\begin{aligned}
\varphi^* d_x f = d_0 \varphi^* f &= \sum_1^n \frac{\partial}{\partial x_j}\bigg|_0 \varphi^* f dx_j = \sum_1^n \left(\frac{d}{dt}\bigg|_{t=0} \varphi^* f \circ \alpha_j \right) dx_j \\
&= \sum_1^n \left(\frac{d}{dt}\bigg|_{t=0} \int_{x_j}^{\alpha_j(t)} \omega(\varphi \circ \alpha_j'(t)) \right) dx_j \\
&= \sum_1^n \left(\frac{d}{dt}\bigg|_{t=0} \int_{-\epsilon}^{t} \alpha_j^* \varphi^* \omega \right) dx_j = \sum_{j=1}^n \left(\frac{d}{dt}\bigg|_{t=0} \int_{-\epsilon}^{t} \sum_{k=1}^n \alpha_j^* \omega_k dx_k \right) dx_j \\
&= \sum_{j=1}^n \left(\frac{d}{dt}\bigg|_{t=0} \int_{-\epsilon}^{t} \sum_{k=1}^n \omega_k \circ \alpha_j d(x_k \circ \alpha_j) \right) dx_j \\
&= \sum_1^n \left(\frac{d}{dt}\bigg|_{t=0} \int_{-\epsilon}^{t} \omega_j \circ \alpha_j \right) dx_j \overset{\text{FTC}}{=} \sum_1^n \omega_j(x) dx_j = \varphi^*\omega(x)
\end{aligned}$$

Thus we have shown locally that, $\varphi^* df = d\varphi^* f = \varphi^*\omega$, so that $(\varphi^{-1})^*\varphi^* df = (\varphi^{-1})^*\varphi^*\omega$, and on V we have $df = (\varphi^{-1})^*\varphi^* df = (\varphi^{-1})^*\varphi^*\omega = \omega$, but since we are working with an arbitrary chart for M this holds on the whole manifold, we find that $df = \omega$, as desired. $\square$

3. **(a)** Since terms in the product with any repeated forms are zero, and for each $\sigma \in S_n$, there is exactly one form in the product $\bigwedge_1^n dp_{\sigma(j)} \wedge dq_{\sigma(j)}$ before rearranging, we can write

$$\left(\sum_1^n dp_j \wedge dq_j \right)^{\wedge^n} = \sum_{\sigma \in S_n} \bigwedge_1^n dp_{\sigma(j)} \wedge dq_{\sigma(j)}$$

I claim that $\sum_{\sigma \in S_n} \bigwedge_1^n dp_{\sigma(j)} \wedge dq_{\sigma(j)} = n! \bigwedge_1^n dp_j \wedge dq_j$, to check this it suffices to check for each $\sigma \in S_n$, we ahve $\bigwedge_1^n dp_{\sigma(j)} \wedge dq_{\sigma(j)} = \bigwedge_1^n dp_j \wedge dq_j$, to do so it will suffice to notice if ζ is a 2-form, then for any form η, we have $\eta \wedge \zeta = (-1)^{2 \deg \eta} \zeta \wedge \eta = \zeta \wedge \eta$, then

$$\bigwedge_1^n dp_{\sigma(j)} \wedge dq_{\sigma(j)} = \eta \wedge dp_1 \wedge dq_1 \wedge \eta' = dp_1 \wedge dq_1 \wedge \eta \wedge \eta'$$

where $\eta \wedge \eta' = dp_{\sigma(1)} \wedge dq_{\sigma(1)} \wedge \cdots \widehat{(dp_{\sigma(\sigma^{-1}(1))} \wedge dq_{\sigma(\sigma^{-1}(1))})} \wedge \cdots \wedge dp_{\sigma(n)} \wedge dq_{\sigma(n)}$, then we can repeat this process recursively, on the remaining form $\eta \wedge \eta'$, until we get the desired volume form. This is a constant multiple of the standard volume form on $\mathbb{R}^{2n}$ (which is nonvanishing), hence is nonvanishing. $\square$

It follows that $\omega_{\text{std}} = n! \bigwedge_1^n dp_j \wedge dq_j$ is non-vanishing, since any nonvanishing section $s \in \bigwedge^n TM$ can be written as $c \bigwedge_1^n \partial_{x_{p_j}} \wedge \partial_{x_{q_j}}$ for $c \in C^\infty$ nonvanishing, then ω_{std}

(b)

$$\lambda_{\text{std}} = \sum_1^n p_j dq_j, \qquad d\lambda_{\text{std}} = \sum_1^n d(p_j dq_j) = \sum_1^n dp_j \wedge dq_j - p_j \wedge d^2 q_j = \sum_1^n dp_j \wedge dq_j = \omega_{\text{std}}$$

$\square$

(c) Take $\omega' = \sum_1^n dq_j \wedge dp_j$, then $\omega' = -\omega_{\text{std}}$, so these are indeed distinct. It is sraightforward to see it is a symplectic form, since we may simply swap each q_j with p_j in (a) and (b). $\square$