

1. (Folland 1.2.1) (a) Let $n \in [2, \infty]$, then

$$\bigcap_1^n E_i = E_1 \cap \bigcap_2^n E_i = E_1 \setminus \bigcup_2^n E_i^c = E_1 \setminus \bigcup_2^n (E_1 \cap E_i^c) = E_1 \setminus \bigcup_2^n (E_1 \setminus E_i)$$

By assumption each of the E_i and $E_1 \setminus E_i$ are in \mathcal{R} . □

(b) Let $E \in \mathcal{R}$, then $E^c = X \setminus E \in \mathcal{R}$. □

(c) Denote the (alleged) sigma algebra in the question as \mathcal{A} . Closure under compliments is immediate from the definition. Now let $\{E_i\}_1^\infty \subset \mathcal{A}$, then

$$\bigcup_1^\infty E_i = \bigcup_{E_i \in \mathcal{R}} E_i \bigcup_{E_i^c \in \mathcal{R}} E_i$$

By definition we have the first union is in \mathcal{R} , the second union is equal to $\left(\bigcap_{E_i^c \in \mathcal{R}} E_i^c\right)^c$, and hence by part (a) it has compliment in \mathcal{R} . This reduces the problem to pairwise unions $E \cup F$ for $E \in \mathcal{R}$ and $F^c \in \mathcal{R}$. In this case $(E \cup F)^c = E^c \cap F^c = F^c \setminus E \in \mathcal{R}$. □

(d) Once again refer to the (alleged) sigma algebra as \mathcal{A} . Suppose $\{E_i\}_1^\infty \subset \mathcal{A}$, then

$$E_1^c \cap F = F \setminus E_1 = F \setminus F \cap E_1 \in \mathcal{R} \text{ and } F \cap \bigcup_1^\infty E_i = \bigcup_1^\infty F \cap E_i \in \mathcal{R}$$

□

2. (Folland 1.2.2) Folland has already showed $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_{\mathbb{R}}$ for all j , and that the open and closed intervals both generate $\mathcal{B}_{\mathbb{R}}$. It will suffice to show that for an arbitrary open interval (a, b) , we have $(a, b) \in \mathcal{M}(\mathcal{E}_j)$ for $j > 2$ since this suffices to show $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(E_1) \subset \mathcal{M}(E_j)$. Note that by closure under compliments we have $\mathcal{E}_5 = \mathcal{E}_8$ and $\mathcal{E}_6 = \mathcal{E}_7$, so we may use sets of both forms in these cases. Below we show that $(a, b) \in \mathcal{M}(\mathcal{E}_3), \mathcal{M}(\mathcal{E}_4), \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$ respectively

$$(a, b) = \bigcup_1^\infty (a, b - 1/n] = \bigcup_1^\infty [a + 1/n, b) = (a, \infty) \cap \bigcup_1^\infty (-\infty, b - 1/n] = (-\infty, a) \cap \bigcup_1^\infty [b + 1/n, \infty)$$

□

3. (Folland 1.2.3) (a) Assume not, then let N be the size of the largest collection of disjoint sets in \mathcal{A} . Now let E_1, \dots, E_N be disjoint, we know that $\{\bigcup_S E_i \mid S \subset \mathcal{P}(\{1, \dots, N\})\}$ is finite, hence since \mathcal{A} is infinite, there is some set $F \in \mathcal{A}$ such that $\emptyset \subsetneq F \cap E_i \subsetneq E_i$ for some i , we may assume without loss of generality $i = 1$. It follows that $F \cap E_1, F^c \cap E_1, E_2, \dots, E_N$ are all disjoint, but this contradicts N being the size of the largest collection of disjoint sets in \mathcal{A} . □

(b) Let $\{E_i\}_1^\infty$ be an infinite sequence of non-empty disjoint sets in \mathcal{A} . Then we have

$$F : \{0, 1\}^{\aleph_0} \rightarrow \mathcal{A} \\ b \mapsto \bigcup_{\{n \mid b_n = 1\}} E_i$$

Then F is injective since if $S_1, S_2 \subset \mathbb{Z}_{>0}$ we have $\bigcup_{S_1} E_i \subset \bigcup_{S_2} E_i$ implies that $S_1 \subset S_2$ by the disjointness of the E_i . This shows that $\mathfrak{c} = \#\{0, 1\}^{\aleph_0} \leq \#\mathcal{A}$. □

4. (Folland 1.2.4) Define $F_n = \bigcup_1^n E_i$, then $F_1 \subset F_2 \subset \dots$ and each $F_i \in \mathcal{A}$. It is immediate that $\bigcup_1^\infty E_i = \bigcup_1^\infty F_i \in \mathcal{A}$. □

5. (Folland 1.2.5) Let $\mathcal{P}_\sigma(\mathcal{E}) = \{\mathcal{F} \in \mathcal{P}(\mathcal{E}) \mid \mathcal{F} \text{ is countable}\}$. Then let $\{E_i\}_1^\infty \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$, so for some $\{\mathcal{F}_i\} \subset \mathcal{P}_\sigma(\mathcal{E})$ we have $E_i \in \mathcal{M}(\mathcal{F}_i)$, hence $E_i^c \in \mathcal{M}(\mathcal{F}_i)$, and $\bigcup_1^\infty E_i \in \bigcup_1^\infty \mathcal{M}(\mathcal{F}_i) \subset \mathcal{M}(\bigcup_1^\infty \mathcal{F}_i)$, since each \mathcal{F}_i is countable we get that $\bigcup_1^\infty \mathcal{F}_i \in \mathcal{P}_\sigma(\mathcal{E})$. Each set in \mathcal{E} is countable, so $\mathcal{E} \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$, since the latter is a sigma algebra containing \mathcal{E} , we get that $\mathcal{M}(\mathcal{E}) \subset \bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F})$. Conversely each $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$, so that $\bigcup_{\mathcal{P}_\sigma(\mathcal{E})} \mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. \square

6. (Classify the sigma algebras on the naturals) The sigma algebras are in bijection to partitions of the naturals, or equivalently equivalence relations on the naturals. For any partition of the naturals $\bigsqcup_1^\infty E_i = \mathbb{N}$ we can form the sigma algebra $\mathcal{M}(\{E_i\}_1^\infty)$ which is the set of disjoint unions of the E_i . Conversely, let \mathcal{A} be a sigma algebra, we can define the equivalence relation $x \sim y$ when for any $E \in \mathcal{A}$, $x \in E \implies y \in E$. Reflexivity and transitivity are obvious. To see symmetry we prove the contrapositive, assume $E \in \mathcal{A}$ with $y \in E$ but $x \notin E$, then $E^c \in \mathcal{A}$ and $x \in E^c$ but $y \notin E^c$. Now let $\{E_i\}_1^\infty$ be the partition corresponding to this equivalence relation. To see that $\mathcal{M}(\{E_i\}_1^\infty) = \mathcal{A}$, we first consider $x \in \mathbb{N}$ and $S = \{E \in \mathcal{A} \mid x \in E\}$ with the partial ordering giving by set inclusion, if $E_1 \supset E_2 \supset \dots$ is a chain in S , then $x \in \bigcap_1^\infty E_i \in \mathcal{A}$ is a lower bound, hence by Zorn's lemma there is a smallest set $E_x \in \mathcal{A}$ with $x \in E_x$. Let i such that $x \in E_i$, then $E_i = \bigcap_{\{E \in \mathcal{A} \mid x \in E\}} E = E_x$, since the E_i are disjoint and every element of \mathbb{N} is in some E_i we have that each E_i is the smallest set in \mathcal{A} containing some natural number, hence $\mathcal{M}(\{E_i\}_1^\infty) \subset \mathcal{A}$. To see that we have equality it will suffice to show that there is no non-empty set in \mathcal{A} that is contained in any of the E_i (this reduction follows immediately from the trick in problem 3(a)), but this is immediate since if there were some nonempty $E \in \mathcal{A}$ with $E \subsetneq E_i$, then there is some $x \in E$, but E_i is the smallest set in \mathcal{A} containing x which is an immediate contradiction, hence $\mathcal{A} \subset \mathcal{M}(\{E_i\}_1^\infty)$ the set of disjoint unions of the E_i . \square