1. (Durrett 1.1.5) $\mathcal{A}$ is not an algebra, hence not a σ-algebra, as proof let A be the even numbers, and B be as defined below

$$B = \bigcup_{n \text{ even}} \{k \mid k \text{ odd and } 2^n \leq k < 2^{n+1}\} \bigcup_{n \text{ odd}} \{k \mid k \text{ even and } 2^n \leq k < 2^{n+1}\}$$

Then it is clear $\theta(A) = \theta(B) = \frac{1}{2}$. Now we want to consider $\theta(A \cup B)$, note that $A \cup B$ contains $\{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$ when n is even, but contains only $\{2^n, 2^n + 1, \dots, 2^{n+1} - 1, 2^{n+1}\} \cap \{\text{even numbers}\}$ for odd n. Then denote $\theta_n = \frac{\#((A \cup B) \cap \{1, \dots, 2^{n+1}\})}{2^{n+1}}$, then the first few terms are $\theta_1 = 1, \theta_2 = \frac{3}{4}, \theta_3 = \frac{7}{8}, \theta_4 = \frac{11}{16}$ and from the definition of A, B we get $\theta_{2n+1} = \frac{\theta_{2n}}{2} + \frac{1}{2}$ and $\theta_{2n+2} = \frac{\theta_{2n+1}}{4} + \frac{1}{4}$, it follows that by induction the subsequences θ_{2n} and θ_{2n+1} are decreasing, then once again by induction using this recurrence we find that $\frac{11}{16} \geq \theta_{2n} \geq \frac{1}{2}$ and $1 \geq \theta_{2n+1} \geq \frac{3}{4}$, but then $\liminf \theta_{2n+1} \geq \frac{3}{4} > \frac{11}{16} \geq \limsup \theta_n$, so these subsequences of $\frac{\#((A \cup B) \cap \{1, \dots, n\})}{n}$ can't possibly converge to the same limit, and hence a limit for the sequence cannot exist and $A \cup B$ does not have an asymptotic density.

2. (Durrett 1.2.3) First note that the left limit of a distribution function is well defined,

$$F(x-) := \lim_{y_n \uparrow x} F(x) = \bigcup_1^\infty P(X \leq y_n) = P(X < x)$$

The last equality following from throwing out y_n such that for some $k < n$, there is $y_k > y_n$ and applying continuity from below.

It follows that for each point of discontinuity of F, we must have $F(x) > F(x-)$, assuming there are uncountably many points of discontinuity for F and denote that set of points as A, we know that since $0 \leq F(x) \leq 1$ is an increasing function that

$$1 = \lim_{x \to \infty} F(x) \geq \sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\}$$

Denote $E_n = \{\alpha \in A \mid F(\alpha) - F(\alpha-) \geq \frac{1}{n}\}$, then since $\bigcup_1^\infty E_n = A$, we must have atleast one E_n is uncountable. This implies that

$$\sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\} \geq \sup_{M \in \mathbb{N}} \frac{M}{n} = \infty$$

Which is a contradiction.

3. (Durrett 1.3.5) If f is not LSC, then there is some x and $y_n \to x$, such that $\lim f(y_n) < f(x)$ (this follows from the negation since we can take a subsequence which gives the liminf). But then let $\epsilon = f(x) - \lim f(y_n)$, if we remove the y_n terms such that $f(y_n) > f(x) + \frac{\epsilon}{2}$ from the sequence then the sequence still converges to x, so we may assume the sequence is uniformly bounded by $f(x) + \frac{\epsilon}{2}$. But then we have a sequence $y_n \in \{t \mid f(t) \leq f(x) - \epsilon/2\}$ which converges to a value x not in the set, so in particular the set is not closed.

Conversely, if for some a, the set $S_a := \{x \mid f(x) \leq a\}$ is not closed, then we get a sequence $y_n \in S_a$ such that $y_n \to x$, but $x \notin S_a$, it follows that $f(x) > a$, but $\liminf_{y \to x} f(y) \leq \lim f(y_n) \leq a < f(x)$ so that f is not LEC.

4. (Durrett 1.3.7) First we note that all simple functions are measurable, and measurable functions are closed under pointwise limits TODO.... Now let f be an arbitrary measurable function on $(\Omega, \mathcal{F})$ mapping to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Then we can write $f = f_+ - f_-$ so it will suffice to show that an arbitrary positive function f is a pointwise limit of simple functions. Let $A_1 = f^{-1}[1, \infty)$, and $\phi_1 = 1_{A_1}$, now we can define the rest of the ϕ_i recursively:

$$A_n = (f - \phi_{n-1})^{-1}[\frac{1}{n}, \infty) \qquad \phi_n = n^{-1} 1_{A_n}$$

TODO... use the harmonic series diverges