

1.

$$\begin{aligned}\Im(\sigma) &= \Im\left(\frac{(a\tau + b)(c\bar{\tau} + d)}{\|c\tau + d\|^2}\right) = \Im\left(\frac{ac\|\tau\|^2 + bd + ad\tau + bc\bar{\tau}}{\|c\tau + d\|^2}\right) \\ &= \frac{1}{\|c\tau + d\|^2} \Im(ad\tau + bc\bar{\tau}) = \frac{1}{\|c\tau + d\|^2} (ad - bc) \Im(\tau) = \frac{1}{\|c\tau + d\|^2} \Im(\tau)\end{aligned}$$

Then since  $\|c\tau + d\|^2 > 0$  and  $\Im(\tau) > 0$  we conclude that  $\Im(\sigma) > 0$ .

Now define

$$\begin{aligned}f : \mathbb{C} &\rightarrow \mathbb{C}/(\mathbb{Z} + \sigma\mathbb{Z}) \\ z &\mapsto \frac{z}{c\tau + d} + \mathbb{Z} + \sigma\mathbb{Z}\end{aligned}$$

To see this descends to a holomorphic map on the torus, we need to check it is periodic with respect to  $\mathbb{Z} + \tau\mathbb{Z}$ , so we want to check that  $f(z + n\tau + m) = f(z)$  for any  $n, m \in \mathbb{Z}$ . Since  $f$  is linear it will be sufficient to show that  $f(\tau)$  and  $f(1)$  both lie in the lattice  $\mathbb{Z} + \sigma\mathbb{Z}$ . This is just a computation,

$$\tau = (ad - bc)\tau + bd - bd = d(a\tau + b) - b(c\tau + d) \quad (1)$$

$$1 = (ad - bc) + ac\tau - ac\tau = -c(a\tau + b) + a(c\tau + d) \quad (2)$$

So that (1) gives us  $f(\tau) = d\sigma - b$  and (2) gives  $f(1) = -c\sigma + a$  both are in  $\mathbb{Z} + \sigma\mathbb{Z}$ , so that  $f$  descends to the torus  $X_\tau$ . To see that  $f$  is a biholomorphism just take  $\mathbb{C} \rightarrow X_\tau$  via  $z \mapsto (c\tau + d)z$ , this descends to a holomorphic map on  $X_\sigma$  since  $\sigma \mapsto \tau$ , and  $1 \mapsto c\tau + d$  are both in  $\mathbb{Z} + \tau\mathbb{Z}$ , moreover this is clearly the inverse of  $f$ .  $\square$

2. I will use from class the identification of two copies of  $\mathbb{C}$ , the first being labelled (I), and the second (II), so that  $X$  is defined by gluing (I) to (II) along the line segments from 0 to 1, and from  $\lambda$  to  $\infty$  along the line through 1 and  $\lambda$  as in lecture. Now on a neighborhood of infinity (more explicitly on  $U = \{z \mid |z| > \max\{1, |\lambda|\}\}$  in both (I) and (II)) we can take local coordinate  $t$ , so that

$$t^2 = \frac{1}{z} \quad t = \begin{cases} \sqrt{z} & z \in (I) \\ -\sqrt{z} & z \in (II) \end{cases} \quad (3)$$

This allows us to compute

$$2t dt = -\frac{1}{z^2} dz = -t^4 dz \quad (4)$$

$$dz = -2t^{-3} dt \quad (5)$$

Moreover we also have defined  $\omega^2 = z(z-1)(z-\lambda)$  with sign conventions

$$\omega = \begin{cases} \sqrt{z(z-1)(z-\lambda)} & z \in (I) \\ -\sqrt{z(z-1)(z-\lambda)} & z \in (II) \end{cases} \quad (6)$$

Then using our local  $t$ -coordinates we have

$$\omega^2 = \frac{1}{t^2} \left( \frac{1}{t^2} - 1 \right) \left( \frac{1}{t^2} - \lambda \right) = \frac{1}{t^6} (1 - t^2)(1 - \lambda t^2)$$

Using the sign conventions for  $t$  and  $\omega$  in (3) and (6) this is consistent with

$$\frac{1}{\omega} = \frac{t^3}{(1 - t^2)(1 - \lambda t^2)} \quad (7)$$

Substituting (5) into (7) yields the differential in local  $t$ -coordinates near infinity.

$$\frac{dz}{\omega} = \frac{-2dt}{(1 - t^2)(1 - \lambda t^2)} \quad (8)$$

Since  $|z| = \frac{1}{|t^2|} > \max\{1, |\lambda|\}$  this is a holomorphic form on  $U$ , which is clearly non-vanishing.  $\square$

**3. (a)** Consider an arbitrary meromorphic function  $f : X \rightarrow \mathbb{C}$ , we may denote the poles of  $f$  as  $P_1, \dots, P_r$  with multiplicities  $n_1, \dots, n_r$ . Away from its poles,  $f$  is holomorphic as a map to  $\mathbb{C}$ , and hence to  $\mathbb{P}^1$ , so we need only consider the behaviour of  $f$  near its poles, we should also note that  $f$  is well defined as a set map to  $\mathbb{P}^1$  by taking  $f(P_j) = \infty$  for each  $P_j$ . Now let  $(U, \psi)$  be a chart containing  $P_1$ , by possibly shrinking  $U$  we may assume that  $0 \notin f(U)$ , so that  $f(U)$  is contained in the chart of  $\mathbb{P}^1$  containing  $\infty$ , moreover for simplicity we may assume that  $0 \in U$  with  $\psi(0) = P_1$ . Now since  $f$  is meromorphic on  $X$  with pole  $P_1$  of multiplicity  $n_1$  we have

$$f \circ \psi(z) = \sum_{k=-n_1}^{\infty} a_k z^k \quad (9)$$

with  $a_{-n_1} \neq 0$ . Now since the image lies entirely of the chart of  $\mathbb{P}^1$  containing  $\infty$ , to consider this as a map to  $\mathbb{P}^1$  we compose with the coordinate chart  $\varphi$  taking  $\mathbb{C} \rightarrow (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$  on  $\mathbb{P}^1$ , which takes  $z \mapsto 1/z$ , the point  $\infty$  corresponds to 0 in this coordinate chart. This gives the following expression for  $f$  around  $P_1$  in coordinates:

$$\varphi^{-1} \circ f \circ \psi : z \mapsto \begin{cases} (\sum_{k=-n_1}^{\infty} a_k z^k)^{-1} & z \neq 0 \\ \varphi^{-1}(f \circ \psi(0)) = \varphi^{-1}(f(P_1)) = \varphi^{-1}(\infty) = 0 & z = 0 \end{cases} \quad (10)$$

Now we may simplify  $(\sum_{k=-n_1}^{\infty} a_k z^k)^{-1} = \frac{z^{n_1}}{\sum_{k=0}^{\infty} a_{k-n_1} z^k}$ , which of course takes value 0 at  $z = 0$ , which simplifies the piecewise expression for  $f$  in charts given by (10) to  $\frac{z^{n_1}}{\sum_{k=0}^{\infty} a_{k-n_1} z^k}$ . Since the denominator is a nonvanishing (near zero) holomorphic function convergent near zero, we find that  $\varphi^{-1} \circ f \circ \psi$  is a holomorphic map  $X \rightarrow \mathbb{P}^1$  in a chart around  $P_1$ , the same argument works for charts around  $P_2, \dots, P_n$  so that  $f$  is indeed a holomorphic function between  $X$  and  $\mathbb{P}^1$ .  $\square$

**(b)** This question only works if  $d$  counts multiplicity and if  $f$  is nonconstant, so consider  $d$  counting  $f^{-1}(p)$  with multiplicity for a nonconstant holomorphic  $f : X \rightarrow \mathbb{P}^1$ . By compactness, it suffices to show that  $f^{-1}(p)$  is discrete to conclude that  $d$  is finite (this follows since so long as  $f$  is not constant the degree of any single one of the preimages is finite). Moreover, since a finite union of discrete sets is discrete, we may cover  $X$  in a finite number of charts  $\{(U_j, \varphi_j)\}_1^r$  and check that  $f|_{U_j}^{-1}(p)$  is discrete for each  $j$ . Since coordinate charts are diffeomorphisms, we can once again reduce the problem to checking each  $(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}\{\psi^{-1}(p)\}$  is discrete for some  $\psi$  corresponding to a coordinate chart containing  $p$ , with  $\psi^{-1} \circ f|_{U_j} \circ \varphi_j$  simply being a holomorphic function  $\varphi^{-1}(U) \subset \mathbb{C} \rightarrow \mathbb{C}$ , this reduces to the fact that the zeroes of a holomorphic function are discrete, and

$$(\psi^{-1} \circ f|_{U_j} \circ \varphi_j)^{-1}(\psi^{-1}(p)) = \{z \in \varphi^{-1}(U_j) \mid \psi^{-1} \circ f|_{U_j} \circ \varphi_j(z) - \psi^{-1}(p) = 0\}$$

is the set of zeroes of a holomorphic function. Thus by the reductions above,  $d < \infty$ .

Now to see that  $d$  is constant, I will show that  $d$  is constant in some open set  $U$  containing  $p$ , since this will hold for any  $p' \in X$  this implies that the degree of a point is a continuous map  $X \rightarrow \mathbb{Z}$ , so that since  $X$  is connected it must be the constant map. To see that  $d$  is locally constant, take  $\{p_0, \dots, p_r\} = f^{-1}(p)$ , so that all together  $p_1, \dots, p_r$  have total multiplicity  $d$ . Now we can for every  $x \in X \setminus \{p_1, \dots, p_r\}$  take some chart  $(U_x, z_x)$  with  $x \in U_x \subset X \setminus \{p_1, \dots, p_r\}$ , and for each  $p_j$  we can take disjoint coordinate charts  $(U_j, z_j)$  with  $p_j \in U_j$ . Now let  $(V, \psi)$  be a coordinate chart in  $\mathbb{P}^1$  with  $\psi(p) = 0$ , by possibly taking the intersection with  $f^{-1}(V)$  we can assume each of the  $U_x$  and  $U_j$  map into  $V$  under  $f$ . Since the degree of a map  $\mathbb{C} \rightarrow \mathbb{C}$  is locally constant, we can (once again by possibly shrinking the open sets) assume that for each  $x \in X \setminus \{p_1, \dots, p_r\}$  we have  $\text{ord}_w(\psi^{-1} \circ f(z_x)) = 0$  for all  $w \in U_x$ , and for each  $j$ ,  $\text{ord}_w(\psi^{-1} \circ f(z_j)) = \text{ord}_{p_j}(\psi^{-1} \circ f(z_j))$  for all  $w \in U_j$ . Only now that we have reduced our open sets to be sufficiently nice do we apply the compactness assumption to get a finite subcover  $U_{x_1}, \dots, U_{x_N}, U_1, \dots, U_r$  for  $X$ , note that since each  $p_j$  lies only in  $U_j$ , we must have each  $U_j$  included in our open cover. By the open mapping theorem we know that each  $\psi^{-1} \circ f(z_x)$  (or  $z_j$ ) is open, so that  $Y := \bigcap_1^N \psi^{-1} \circ f(z_{x_k}) \cap \bigcap_1^r \psi^{-1} f(z_j(U_j))$  is an open set containing since  $\psi$  is a homeomorphism we can apply it to get  $\psi(Y)$  is an open subset of  $\mathbb{P}^1$  containing  $p$ . Moreover the degree of  $f$  is locally

constant as a map  $X \rightarrow \psi(Y)$ , since the degree is zero on each  $U_k$ , and locally constant on each  $U_j$  from the version of this result on  $\mathbb{C}$ . Explicitly, since the  $U_j$  are disjoint and  $f|_{U_{x_k}} \cap \psi(Y) = \emptyset$ , we have for any  $p' \in \psi(Y)$  that  $f^{-1}(p') = \bigsqcup_1^r f|_{U_j}^{-1}(p')$ , and moreover that the number of points with multiplicity  $d$  is preserved since the multiplicity of  $f$  is constant on each  $U_j$ .  $\square$

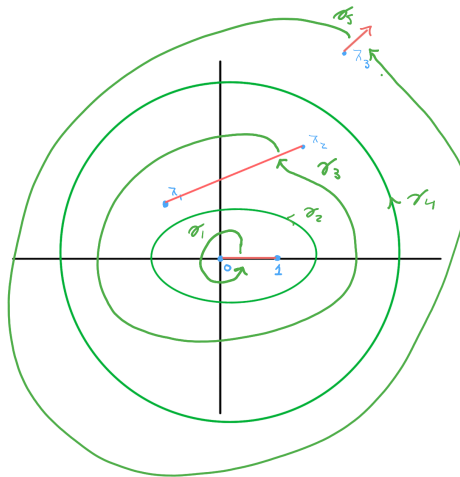
(c)  $z$  has degree 2, and  $w$  has degree 3. As proof, by parts (a) and (b) it suffices to compute the preimage for a single point in which it is convenient. For  $z$  this is very simple, since for any point  $x$  away from the identifications  $z^{-1}(x)$  consists of two copies of  $x$ , one in each glued copy of  $\mathbb{C}$  (i.e. choose any value of  $x$  on the imaginary line distinct from  $\lambda$ ). In the case of  $w$ , we count its zeroes,  $w^2 = x(x-1)(x-\lambda)$  implies that  $w$  vanishes at each of  $0, 1$  and  $\lambda$ , since these are glued points, we do not get the same double counting as in the  $z$  case and this gives degree 3, this is also up to multiplicity, since  $w$  has vanishing of order 1 at each of these points, checking at zero (since  $1$  and  $\lambda$  are similar) we have the local coordinate  $t^2 = z$ , defined via

$$t = \begin{cases} \sqrt{z} & \text{on (I)} \\ -\sqrt{z} & \text{on (II)} \end{cases}$$

so that  $w = t\sqrt{(t^2-1)(t^2-\lambda)}$  indeed has order one vanishing.  $\square$

4.  $X$  is a surface of Genus 2. Define line segments  $L_1 = [0, 1]$ ,  $L_2$  the component of the line determined by  $\lambda_1$  and  $\lambda_2$  between  $\lambda_1$  and  $\lambda_2$ , and  $L_3$  the component of the line segment between  $\lambda_2$  and  $\lambda_3$  from  $\lambda_3$  to  $\infty$ . Here we assume the points  $\{0, 1, \lambda_1, \lambda_2, \lambda_3\}$  are arranged so that none of the line segments  $L_1, L_2$  and  $L_3$  intersect. Since  $X$  is given as the analytic continuation of  $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$ , we consider the monodromy of these square root functions around these points

- $\gamma_1$  is a curve with winding number 1 around 0, and winding number zero around  $1, \lambda_1, \lambda_2, \lambda_3$
- $\gamma_2$  is a curve with winding number 1 around 0 and 1, and winding number zero around  $\lambda_1, \lambda_2, \lambda_3$
- $\gamma_3$  is a curve with winding number 1 around 0, 1 and  $\lambda_1$  and winding number zero around  $\lambda_2$  and  $\lambda_3$
- $\gamma_4$  is a curve with winding number 1 around 0, 1,  $\lambda_1$  and  $\lambda_2$  and winding number zero around  $\lambda_3$
- $\gamma_5$  is a curve with winding number 1 around all points



We record the monodromy of each of the square root constituents around these curves:

- On  $\gamma_1$  we get monodromy  $\sqrt{z} \rightsquigarrow -\sqrt{z}$ , with the  $\sqrt{z-1}, \sqrt{z-\lambda_1}, \sqrt{z-\lambda_2}, \sqrt{z-\lambda_3}$  having no monodromy.

- On  $\gamma_2$  we get monodromy  $\sqrt{z} \rightsquigarrow -\sqrt{z}$  and  $\sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$  with no monodromy for the other factors.
- On  $\gamma_3$  we get monodromy  $\sqrt{z} \rightsquigarrow -\sqrt{z}$ ,  $\sqrt{z-1} \rightsquigarrow -\sqrt{z-1}$ , and  $\sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}$  with no monodromy for the other factors.
- On  $\gamma_4$  we get monodromy:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \text{ and } \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}$$

fixing the last factor.

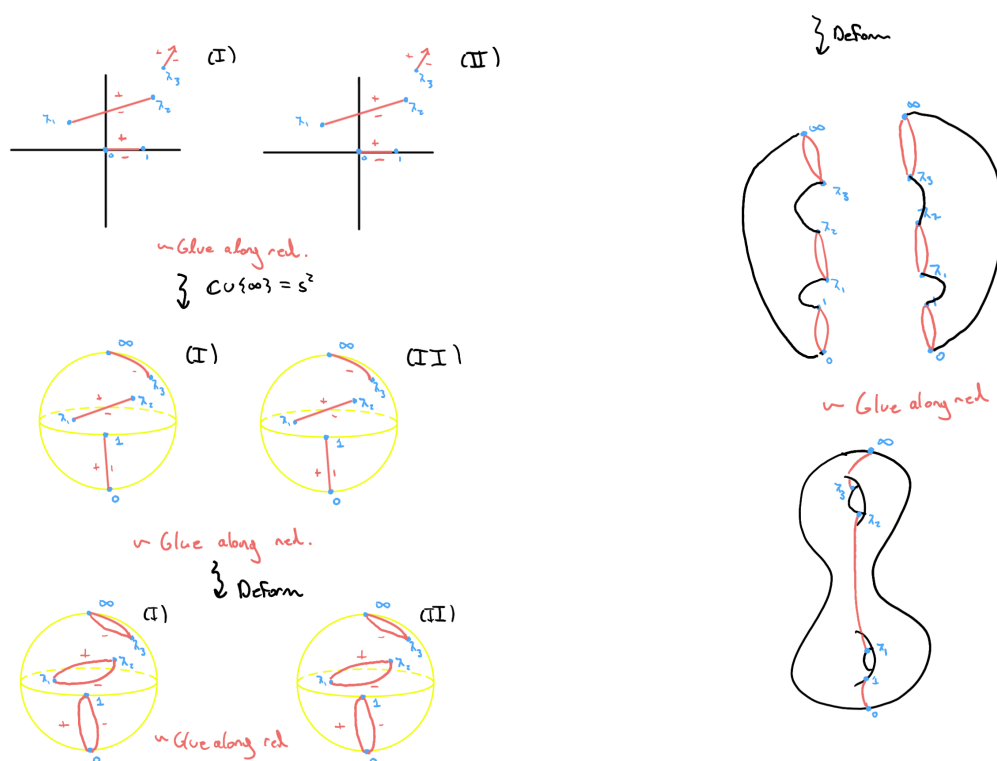
- Finally, on  $\gamma_5$  we get monodromy on each of the components:

$$\sqrt{z} \rightsquigarrow -\sqrt{z}, \sqrt{z-1} \rightsquigarrow -\sqrt{z-1}, \sqrt{z-\lambda_1} \rightsquigarrow -\sqrt{z-\lambda_1}, \sqrt{z-\lambda_2} \rightsquigarrow -\sqrt{z-\lambda_2}, \sqrt{z-\lambda_3} \rightsquigarrow -\sqrt{z-\lambda_3}$$

The monodromy's cancel travelling around  $\gamma_j$  for  $j$  even, but not for  $j$  odd, thus to analytically extend this branch of the square root we once again take two copies of  $\mathbb{C}$ , (I) and (II) and gluing along  $L_1, L_2, L_3$ , and analytically continue  $\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}$  by defining function  $w$  with  $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$  and

$$w = \begin{cases} \sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (I)} \\ -\sqrt{z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} & \text{on (II)} \end{cases} \quad (11)$$

By sketching this glueing we see that  $X$  is a surface of genus 2 (please ignore the yellow and pretend that its black SORRY!!).



The meromorphic function  $z$  simply projects (I) and (II) coordinates to  $\mathbb{C}$ , similarly to part (c) of problem (3), we can pick a point  $x \in \mathbb{C}$  away from  $L_1, L_2, L_3$  where the gluing occurs so that  $z^{-1}(x)$

is trivially 2 copies of  $x$ , one in (I) and one in (II), using question 3 this degree is well defined so  $z$  has degree 2. Also similarly to problem (3c), we can calculate the degree of  $w$  as  $\#w^{-1}(0)$ , which has points  $0, 1, \lambda_1, \lambda_2$  and  $\lambda_3$ , to check that each of these zeroes has multiplicity one we check in a local coordinate around 0, since the proof is the same for the other points. So take local coordinate  $t$  around zero (the chart being  $U = B_\epsilon(0)$  in (I) and (II) with  $\epsilon < \min\{1, |\lambda_1|, |\lambda_2|, |\lambda_3|\}$ ) with  $t^2 = z$ , and  $t = \sqrt{z}$  on (I) and  $t = -\sqrt{z}$  on (II), it follows that on  $U$  we have  $w = t\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$ , with  $\sqrt{(t^2-1)(t^2-\lambda_1)(t^2-\lambda_2)(t^2-\lambda_3)}$  holomorphic and nonvanishing on  $U$ , so that indeed the vanishing of  $w$  at 0 is multiplicity one.  $\square$

**5.** There are four distinct cases, firstly we may have  $Q_1 \in \{0, 1, \lambda\}$  and  $Q_2$  generic, secondly we may have  $Q_1$  and  $Q_2 \in \{0, 1, \lambda\}$ , the other cases deal with the point infinity, namely  $Q_1 = \infty$  and  $Q_2$  generic, or  $Q_1 \in \{0, 1, \lambda\}$  and  $Q_2 = \infty$ .

**(Case 1)** Assume that  $Q_1 = 0$ , and let  $Q_2 \in X \setminus \{0, 1, \lambda, \infty\}$ , then we define the meromorphic one form to be

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}z + w}{z \cdot (z - z(Q))} \frac{dz}{w} \quad (12)$$

Near  $Q$ , we may work in  $z$  charts to get the expression

$$\omega_{0,Q} = \frac{1}{z - z(Q)} \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{dz}{w} \quad (13)$$

so that the residue is just

$$\text{Res}(\omega_{0,Q}, Q) = \frac{\frac{w(Q)}{z(Q)}z + w}{z} \cdot \frac{1}{w} \Big|_Q = \frac{2}{z(Q)} \quad (14)$$

To evaluate the residue at zero, we take charts near zero by taking  $t$  so that  $t^2 = z$  and  $t = \pm\sqrt{z}$  on (I) and (II) respectively. we can rewrite the form  $\omega_{0,Q}$  given by (12) in  $t$  coordinates, this gives the local expression

$$\omega_{0,Q} = \frac{\frac{w(Q)}{z(Q)}t^2 + t\sqrt{(t^2-1)(t^2-\lambda)}}{t^2(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \quad (15)$$

$$= \frac{1}{t} \left( \frac{\frac{w(Q)}{z(Q)}t + \sqrt{(t^2-1)(t^2-\lambda)}}{(t^2 - z(Q))} \cdot \frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \right) \quad (16)$$

From this we can calculate the residue,

$$\text{Res}(\omega_{0,Q}, 0) = \frac{\sqrt{\lambda}}{-z(Q)} \frac{2}{\sqrt{\lambda}} = -\frac{2}{z(Q)} \quad (17)$$

Since  $\text{Res}(\omega_{0,Q}, 0) = -\text{Res}(\omega_{0,Q}, Q)$  we can simply normalize to get  $\pm 1$ . Finally we check these are indeed the only poles of our form, the only points of interest are  $\tilde{Q}$  and  $\infty$  where  $\tilde{Q}$  is the unique distinct point from  $Q$  with  $z(\tilde{Q}) = z(Q)$ , but plugging in  $\tilde{Q}$  the numerator becomes  $\frac{w(Q)}{z(Q)}z(\tilde{Q}) + w(\tilde{Q}) = w(Q) - w(Q) = 0$ , so that there is indeed no pole at  $\tilde{Q}$ , to see there is no pole at infinity recall that  $dz/w$  is a nonvanishing holomorphic form, then the expression  $\frac{\frac{w(Q)}{z(Q)}z + w}{z(z - z(Q))}$  has denominator  $\mathcal{O}(|z|^2)$  and numerator  $\mathcal{O}(|z|^{3/2})$  implying there is no pole at infinity.

**(Case 2)** This time take  $Q_1 = 0$  and  $Q_2 = 1$ , we define

$$\omega_{0,1} = \frac{w}{2z(z-1)} \frac{dz}{w} \quad (18)$$

Similar to the first case, this form only has candidate poles at 0, 1 and infinity, but it is trivial to rule out infinity thus we only need check the residues. Near zero we once again have local coordinates near zero  $t = \pm\sqrt{z}$  on (I) and (II) respectively, these local coordinates give the local expression

$$\omega_{0,1} = \frac{2tdt}{2t^2(t^2-1)} = \frac{1}{t} \frac{1}{t^2-1} dt \quad (19)$$

from this we read off  $\text{Res}(\omega_{0,1}, 0) = -1$ . Similarly we can take local coordinates near one,  $t = \pm\sqrt{z-1}$  on (I) and (II) respectively, in these local coordinates

$$\omega_{0,1} = \frac{2tdt}{2(t^2+1)t^2} = \frac{1}{t} \frac{1}{1+t^2} dt \quad (20)$$

Which we can read off as having residue 1.

**(Case 3)** Take  $Q_1 = 0$ ,  $Q_2 = \infty$ . We follow a similar procedure to the first two cases,

$$\omega_{0,\infty} = \frac{w}{2z} \frac{dz}{w} \quad (21)$$

its clear the only candidates for poles are 0 and  $\infty$ , so we check that they are simple and take the residues. Taking local coordinates near infinity we have  $t = \pm 1/\sqrt{z}$  on (I) and (II) respectively, in these local coordinates we write

$$\omega_{0,\infty} = \frac{t^2-2dt}{2} \frac{1}{t^3} = -\frac{1}{t} dt \quad (22)$$

which is a simple pole with residue  $-1$ . Taking the same coordinates near zero as in the prior cases we find the local expression

$$\omega_{0,\infty} = \frac{1}{2t^2} 2tdt = \frac{1}{t} dt \quad (23)$$

so that we have a simple pole with residue 1 at zero.

**(Case 4)** Take  $Q_1 \in X \setminus \{0, 1, \lambda, \infty\}$  to be a generic point, and  $Q_2 = \infty$ , then we can define

$$\omega_{Q,\infty} = \frac{1}{2} \frac{w(Q) + w}{z - z(Q)} \frac{dz}{w} \quad (24)$$

The zero of  $w(Q) + w$  at  $\tilde{Q}$  removes the simple pole at  $\tilde{Q}$  (where once again  $\tilde{Q}$  is the other lift of  $z(Q)$  in the double cover), moreover the pole at  $Q$  is clearly simple so we need only check the pole at infinity is simple and compute the residues. To compute the residue at  $Q$ , we can simply use  $z$  coordinates,

$$\omega_{Q,\infty} = \frac{1}{2} \frac{w(Q) + w}{z - z(Q)} \frac{dz}{w} = \frac{1}{z - z(Q)} \frac{w(Q) + w}{2w} dz \quad (25)$$

So that  $\text{Res}(\omega_{Q,\infty}, Q) = \frac{2w(Q)}{2w(Q)} = 1$ , to compute the residue at infinity we work in the same local coordinates as in case 3, this gives us the following local expression

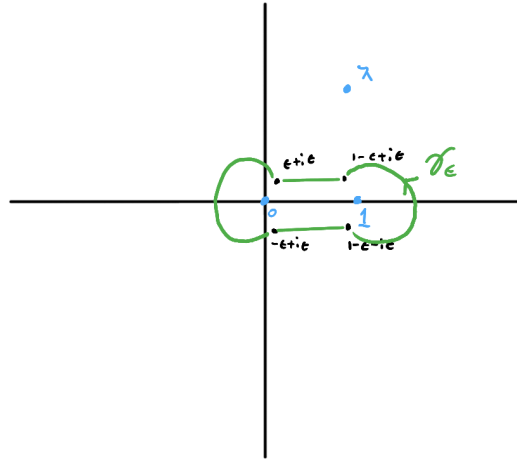
$$w_{Q,\infty} = \frac{1}{2} \frac{t^2(w(Q) + w)}{1 - t^2z(Q)} \frac{-2dt}{t^3w} = \frac{-1}{t} \frac{w(Q) + w}{w(1 - t^2z(Q))} \quad (26)$$

From this expression we get the desired value of the residue,

$$\text{Res}(w_{Q,\infty}, \infty) = - \left( \frac{w(Q) + w}{w(1 - t^2z(Q))} \right) \Big|_{t=0} = - \left( \frac{w(Q)}{w} + 1 \right) \Big|_{t=0} = 1 \quad (27)$$

Since  $w(t)|_{t=0} = \infty$ .

Thus for any distinct points  $Q_1, Q_2 \in X$  regardless of genericity there is an abelian differential of the third kind  $\omega_{Q_1, Q_2}$ .  $\square$



6. Since the branch of the square root function used to define  $\Omega$  is analytic on  $\mathbb{C} \setminus L$ , we can use the following curve  $\gamma_\epsilon$  to define our contour integral (The value of the integral is independent of this choice of curve by Cauchy's integral theorem).

Since  $\Omega$  is holomorphic on  $\mathbb{C} \setminus L$ , the area enclosed by  $\gamma_\epsilon$  is independent of  $\epsilon$  so long as  $\epsilon$  is sufficiently small. Taking the limit  $\epsilon \rightarrow 0$ , we can show that the circular component of the curve has no contribution using the standard arc length inequality, since near 0 and 1 on  $\gamma_\epsilon$  we have  $\frac{1}{w} \in \mathcal{O}(\epsilon^{-\frac{1}{2}})$ , and the arc length of the circular component is bound above by  $2\pi\epsilon$ , the integral over the circular component is bound above by  $2\pi\epsilon\mathcal{O}(\epsilon^{-\frac{1}{2}}) \in \mathcal{O}(\epsilon^{\frac{1}{2}})$ . The horizontal components of the integral then approach the curve on the real line from 1 to 0 from above, and the curve from 0 to 1 from below. From this we get

$$\int_\gamma \frac{dz}{w} = \lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} \frac{dz}{w} = \lim_{\epsilon \downarrow 0} \int_{1-\epsilon}^{\epsilon} \frac{dx}{w(x+i\epsilon)} + \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{dx}{w(x-i\epsilon)} \quad (28)$$

$$= \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \frac{1}{w(x-i\epsilon)} - \frac{1}{w(x+i\epsilon)} dx = -2 \int_0^1 \frac{dx}{\lim_{\epsilon \downarrow 0} w(x+i\epsilon)} \quad (29)$$

Where passing to the limit is justified by  $\Omega = \frac{dz}{w}$  being a holomorphic differential on  $X$ , hence uniformly continuous on a compact set in  $\mathbb{C} \setminus L$  containing  $\gamma$ , and  $x+i\epsilon$ , and  $x-i\epsilon$  converging uniformly to  $x$  from above and below, finally the sign convention on taking the limit  $\frac{1}{w(x-i\epsilon)}$  follows from  $\lim_{\epsilon \uparrow 0} w(x+i\epsilon) = -\lim_{\epsilon \downarrow 0} w(x-i\epsilon)$  by the monodromy of  $w$  around the branch cut at  $L$ . Now if  $\lambda$  is real, then the result is trivial since in this case  $\lim_{\epsilon \downarrow 0} w(x+i\epsilon) = \sqrt{x(1-x)(x-\lambda)}$ , so that  $\int_0^1 \frac{dx}{\sqrt{x(1-x)(x-\lambda)}} \neq 0$  since the integrand is either strictly positive real or strictly positive imaginary depending on  $\lambda < 0$  or  $\lambda > 1$ . For the case of  $\lambda$  not real, I will address  $\Im(\lambda) > 0$ , since the case of  $\lambda$  in the lower half plane is similar. Now for  $\lambda$  in the upper half plane, first note that explicitly the branch cut of  $\sqrt{z(z-1)(z-\lambda)}$  which is analytic away from  $L$  is given by  $\exp(\frac{1}{2}(\text{Log } z + \text{Log}(z-1) + \text{Log}(z-\lambda)))$ , so in order to see that  $\int_\gamma \frac{dz}{w} = -2 \int_0^1 \frac{dx}{\lim_{\epsilon \downarrow 0} w(x+i\epsilon)} \neq 0$ , it suffices to check  $\lim_{\epsilon \downarrow 0} w(x+i\epsilon)$  has strictly positive imaginary part for each  $x \in [0, 1]$ , since this will imply that  $(\lim_{\epsilon \downarrow 0} w(x+i\epsilon))^{-1}$  has strictly negative imaginary part, so that  $-2(\lim_{\epsilon \downarrow 0} w(x+i\epsilon))^{-1}$  will have strictly positive imaginary component, which implies that  $\Im(\int_\gamma \frac{dz}{w}) > 0$ , so the integral is nonzero.

The verification follows from just computing the limit:

$$\lim_{\epsilon \downarrow 0} \Im(w(x + i\epsilon)) = \Im \left( \exp \left( \frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x + i\epsilon) + \text{Log}(x - 1 + i\epsilon) + \text{Log}(x - \lambda + i\epsilon) \right) \right) \quad (30)$$

$$= \Im \left( \exp \left( \frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x + i\epsilon) \right) \exp \left( \frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x - 1 + i\epsilon) \right) \exp \left( \frac{1}{2} \lim_{\epsilon \downarrow 0} \text{Log}(x - \lambda + i\epsilon) \right) \right) \quad (31)$$

$$= \Im(\sqrt{x}(e^{\frac{i\pi}{2}} \sqrt{1-x})(|x - \lambda|^{\frac{1}{2}} e^{\frac{i}{2} \text{Arg}(x-\lambda)})) \quad (32)$$

$$= \sqrt{x(1-x)} |x - \lambda| \Im(e^{\frac{i}{2}(\pi + \text{Arg}(x-\lambda))}) \quad (33)$$

But since  $\Im(\lambda) > 0$ ,  $x - \lambda$  lies strictly in the lower half plane giving us  $\text{Arg}(x - \lambda) \in (-\pi, 0)$ , this implies that  $\frac{1}{2}(\pi + \arg(x - \lambda)) \in (0, \frac{\pi}{2})$ , so that  $\Im(e^{\frac{i}{2}(\pi + \text{Arg}(x-\lambda))}) > 0$ . Combining this with (33) we find that

$$\lim_{\epsilon \downarrow 0} \Im(w(x + i\epsilon)) > 0 \quad (34)$$

for all  $x$ , as desired.  $\square$

**7.** To see that  $\text{Res}(\Omega, p)$  is well defined, we need to check it is invariant under our choice of coordinate charts, this basically reduces to the compatibility assumptions for holomorphic differentials. Let  $(U, z), (V, w)$  be charts for  $X$  containing  $p$ , then we only need check the two coordinate expressions agree, we may take a loop  $\gamma$  around  $p$  with winding number 1, with image contained in  $U \cap V$ , then  $\gamma$  may be parameterized by either  $z$  or  $w$ , so that the residue in either chart is given by  $\frac{1}{2\pi i} \int_{\gamma(z)} f_U(z) dz$  and  $\frac{1}{2\pi i} \int_{\gamma(w)} f_V(w) dw$ , we need only check these two expressions are equal.

$$\frac{1}{2\pi i} \int_{\gamma(z)} f_U(z) dz = \frac{1}{2\pi i} \int_{\gamma(z)} f_V(w(z)) w'(z) dz \quad (35)$$

Now we may apply the substitution  $w = w(z)$ , so that  $dw = w'(z) dz$ , applying the substitution gives us

$$\frac{1}{2\pi i} \int_{\gamma(z)} f_V(w(z)) w'(z) dz = \frac{1}{2\pi i} \int_{\gamma(w)} f_V(w) dw \quad (36)$$

Combining (35) and (36) gives the desired equality of residue in arbitrary coordinate charts so that  $\text{Res}(\Omega, p)$  is indeed well defined.  $\square$