1. (Folland 1.3.6) That $\overline{\mu}$ is a measure is clear, since $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$, Im$\overline{\mu}$ = Imμ (this shows it is positive) and if $\{A_i\}_1^\infty$ are disjoint sets in $\overline{M}$, then each A_i can be written as $E_i \cup F_i$ (for F_i) contained in null sets N_i so that $\overline{\mu} \bigcup_1^\infty A_i = \overline{\mu}(\bigcup_1^\infty E_i \bigcup_1^\infty F_i)$, then $\bigcup_1^\infty F_i \subset \bigcup_1^\infty N_i$ is a null set, so

$$\overline{\mu}(\bigcup_1^\infty E_i \bigcup_1^\infty F_i) \overset{\text{defn.}}{=} \mu(\bigcup_1^\infty E_i) = \sum_1^\infty \mu(E_i) = \sum_1^\infty \overline{\mu}(E_i \cup F_i)$$

Suppose $N \in \overline{\mathcal{M}}$ with $\overline{\mu}(N) = 0$ and $F \subset N$, then $N = N_1 \cup N_2$ with $N_1 \in \mathcal{M}$, and $N_2 \subset N_3 \in \mathcal{M}$, so that $N \subset N_1 \cup N_3$ which is a μ-measurable set, hence F is contained in the null set $N_1 \cup N_3$ and is μ-measurable. Finally to see uniqueness of $\overline{\mu}$, suppose μ' is another extension of μ to $\overline{\mathcal{M}}$, then for some $E \cup F \in \overline{M}$ we have $\mu(E) = \overline{\mu}(E \cup F) \neq \mu'(E \cup F)$, hence $\mu'(F) > 0$, but then $F \subset N \in \mathcal{M}$ where N is μ null, so that $0 < \mu'(F) \leq \mu(N) = 0$. $\square$

2. (Folland 1.3.7) Positivity follows from each μ_j and a_j positive, suppose $\{E_i\}_1^\infty$ are disjoint, if any of the $\mu_j(\bigcup_1^\infty E_i) = \infty$ then $\sum_{i=1}^\infty \sum_{j=1}^n a_j \mu_j(E_i) \geq a_j \sum_{i=1}^\infty \mu(E_i) = a_j \mu_j(\bigcup_1^\infty E_i) = \infty$ and additivity is trivial, otherwise we can interchange sums since they converge in absolute value

$$\sum_1^n a_j \mu_j(\emptyset) = \sum_1^n 0 = 0$$
$$\sum_1^n a_j \mu_j(\bigcup_1^\infty E_i) = \sum_{j=1}^n a_j \sum_{i=1}^\infty \mu_j(E_i) = \sum_{i=1}^\infty \sum_{j=1}^n a_j \mu_j(E_i)$$

$\square$

3. (Folland 1.3.8) for any N, we have $\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j \subset E_k$ for all $k \geq N$, hence $\mu(\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j) \leq \liminf \mu(E_k)$. By continuity from below we have $\lim_{N\to\infty} \mu(\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j) = \mu(\liminf E_k)$, but the limit is bounded above by $\liminf \mu(E_k)$, so that $\mu(\liminf E_K) \leq \liminf \mu(E_k)$.

For all N, we have $\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j \supset E_k$ for $k \geq n$, hence $\mu(\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j) \geq \limsup \mu(E_k)$, since $\mu(\bigcup_1^\infty E_i) < \infty$ we can invoke continuity from above to conclude

$$\mu(\limsup(E_k)) = \lim_{n\to\infty} \mu(\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j) \geq \limsup \mu(E_k)$$

$\square$

4. (Folland 1.3.9) We can decompose the sets of interest as follows:

$$E = (E \setminus F) \sqcup (E \cap F), \quad F = (F \setminus E) \sqcup (F \cap E), \quad E \cup F = F \cap E \sqcup (E \setminus F) \sqcup (F \setminus E)$$

The result follows from additivity on disjoint sets,

$$\mu(E) + \mu(F) = \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E) + \mu(F \cap E) = \mu(E \cup F) + \mu(E \cap F)$$

$\square$

5. (Folland 1.3.10) That μ_E is nonnegative follows from μ nonnegative. $= \cap E$ so $\mu_E(\emptyset) = 0$. Finally if $\{A_i\}_1^\infty$ are disjoint sets, then so are $\{A_i \cap E\}_1^\infty$, hence

$$\mu_E(\bigcup_1^\infty A_i) = \mu(E \cap \bigcup_1^\infty A_i) = \mu(\bigcup_1^\infty E \cap A_i) = \sum_1^\infty \mu(E \cap A_i)$$

$\square$

6. (Folland 1.3.11) Suppose $\{E_i\}_1^\infty$ are disjoint sets, then let $F_n = \bigcup_1^n E_i$, it follows that

$$\mu(\bigcup_1^\infty E_i) = \mu(\bigcup_1^\infty F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_1^n \mu(E_i)$$

In the second case, let $K_n = \bigcap_1^n E_n^c$, it follows that $\mu K_1 \leq \mu X$ so we can use continuity from above.

$$\mu(\bigcup_1^\infty E_i) = \mu(X) - \mu(\bigcap_1^\infty E_i^c) = \mu(X) - \mu(\bigcap_1^\infty K_n) = \mu(X) - \lim_{n\to\infty} \mu(K_n) = \mu(X) - \lim_{n\to\infty} \mu\left(\left(\bigcup_1^n E_n\right)^c\right)$$
$$= \mu(X) - \left(\lim_{n\to\infty} \mu(X) - \sum_1^n \mu(E_i)\right) = \lim_{n\to\infty} \sum_1^n \mu(E_i)$$

□

7. (Folland 1.3.12)

(a) $E\Delta F = (E \setminus F) \sqcup (F \setminus E)$, hence $\mu(E \setminus F) = \mu(F \setminus E) = 0$. It follows that

$$\mu(F) \leq \mu(E) + \mu(F \setminus E) = \mu(E) \text{ and } \mu(E) \leq \mu(F) + \mu(E \setminus F) = \mu(F)$$

□

(b) reflexivity follows from $\mu(E\Delta E) = \mu(\emptyset) = 0$, symmetry follows from $E\Delta F = F\Delta E$, finally transitivity follows from the observation $H \setminus F \subset (H \setminus E) \cup (E \setminus F)$, hence $\mu(H\Delta E) = \mu(E\Delta F) = 0$ implies $\mu(H \setminus F) \leq \mu(H \setminus E) + \mu(E \setminus F) = 0$ and $\mu(F \setminus H) \leq \mu(F \setminus E) + \mu(E \setminus H) = 0$ which gives us that $\mu(H\Delta F) = \mu(H \setminus F) + \mu(F \setminus H) = 0$, proving transitivity. □

(c) $\rho(E, F) = 0 \iff E \sim F$, and ρ is nonnegative, symmetry follows from symmetry of Δ, so ρ will define a metric if it satisfies the triangle inequality. But as in the previous question $H \setminus F \subset (H\setminus E)\cup(E\setminus F)$, applying this inequality the other way this implies that $\mu(H\Delta F) \leq \mu(H\Delta E)+\mu(E\Delta F)$, this proves the triangle inequality for ρ. □

8. (Folland 1.3.13) Suppose that μ is not semifinite, then there is some $E \in \mathcal{M}$, such that for all $F \subset E$ we have $\mu(F) = \infty$. Suppose $X = \bigcup_1^\infty E_i$, then $E_i \cap E \neq \emptyset$ for some i, then $\infty = \mu(E_i \cap E) \leq \mu(E_i)$, so that X cannot be a countable union of sets having finite measure. □

9. (Folland 1.3.14) Let $C = \sup\{\mu(F) \mid F \subset E \text{ and } \mu(F) < \infty\}$ and suppose for contradiction that $C < \infty$, then let F_n be a sequence such that $\lim_{n\to\infty} \mu(F_n) = C$, it follows that $\mu(\bigcup_1^n F_j) \geq \mu(F_n)$, hence $\lim_{n\to\infty} \mu(\bigcup_1^n F_j) = C$, using continuity from below we see that in fact $\mu(\bigcup_1^\infty F_n) = C$. Then $\mu(E \setminus \bigcup_1^\infty F_n) = \infty$, so $E \setminus \bigcup_1^\infty F_n$ has some subset A with $0 < \mu(A) < \infty$, but then

$$C \geq \mu(A \bigcup_1^\infty F_n) = \mu(\bigcup_1^\infty F_n) + \mu(A) > \mu(\bigcup_1^\infty F_n) = C$$

□

10. (Folland 1.3.15) (a) $\mu_0 \geq 0$ and $\mu_0(\emptyset) = 0$ are obvious, Now let $\{E_i\}_1^\infty \subset \mathcal{M}$ be disjoint sets, if $\mu_0(E_j) = \infty$ for some j, then $\mu_0(E_j) \leq \mu_0(\bigcup_1^\infty E_i)$ (since every finite measure subset of the former is also a finite measure subset of the latter) and we are done. So assuming each $\mu_0(E_j) < \infty$l, let $\epsilon > 0$, then we can choose for each E_j some μ-measurable $F_j \subset E_j$ such that $\mu_0(E_j) \geq \mu(F_j) \geq \mu_0(E_j) - \epsilon 2^{-j}$, so that

$$\mu_0(\bigcup_1^\infty E_i) \geq \limsup \mu(\bigcup_1^n F_j) \geq \sum_1^\infty \mu_0(E_i) - \epsilon 2^{-i} = -\epsilon + \sum_1^\infty \mu_0(E_i)$$

since ϵ was arbitrary this concludes the inequality. To see the converse inequality, let $F \subset \bigcup_1^\infty E_i$ such that $\mu(F) < \infty$, then $\sum_1^\infty \mu(E_i \cap F) \leq \sum_1^\infty \mu_0(E_i)$, taking a sequence $(F_n)_1^\infty$, such that $\lim_{n\to\infty} \mu(F_n) = \mu_0(\bigcup_1^\infty E_i)$ and $F_n \subset \bigcup_1^\infty E_i$ we conclude that

$$\mu_0(\bigcup_1^\infty E_i) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_1^\infty \mu(F_n \cap E_i) \leq \sum_1^\infty \mu_0(E_i)$$

so that μ_0 is a measure. To check that μ_0 is semifinite, let $E \in \mathcal{M}$ with $\mu_0(E) = \infty$, then by definition there is some $F \subset E$ with $\mu(F) > 0$, it follows that $\mu_0(F) = \mu(F) > 0$. $\square$

(b) Suppose μ is semifinite, then it is clear for a set E of finite measure we have $\mu(E) = \mu_0(E)$ by monotonicity. If E has infinite measure, then by (Folland 1.3.14) we have

$$\mu_0(F) := \sup\{\mu(F) \mid F \subset E \text{ and } \mu(F) < \infty\} = \infty = \mu(F)$$

$\square$

(c) Define ν as follows,

$$\nu(E) := \begin{cases} 0 & E \text{ is } \mu\text{-semi-finite} \\ \infty & \text{else} \end{cases}$$

Once again positivity and $\nu(\emptyset) = 0$ are obvious, to check countable additivity let $\{E_i\}_1^\infty \subset \mathcal{M}$ be disjoint. If $\nu(\bigcup_1^\infty E_i) = 0$, then $\bigcup_1^\infty E_i$ is μ-semi-finite, hence so is every E_i, so that $\sum_1^\infty \nu(E_i) = \sum_1^\infty 0$. If $\nu(\bigcup_1^\infty E_i) = \infty$, then atleast one E_i is not semifinite because otherwise for any $F \subset \bigcup_1^\infty E_i$ with $\mu(F) = \infty$ and $\mu_0(F) = 0$ we have $\infty = \mu(F) = \sum_1^\infty \mu(F \cap E_i)$ and $0 = \mu_0(F) = \sum_1^\infty \mu_0(F \cap E_i)$, so that for some E_i we have $\mu(F \cap E_i) = \infty$, but E_i is semifinite by assumption, so that $\mu_0(F \cap E_i) \neq 0$ a contradiction, this shows that atleast one E_j is not semifinite, so that $\sum_1^\infty \nu(E_i) \geq \nu(E_j) = \infty = \nu(\bigcup_1^\infty E_i)$. This suffices to show that ν is a measure, to see that $\mu = \mu_0 + \nu$, let $E \in \mathcal{M}$, if E is semifinite then either it is finite and $\mu(E) = \mu(E_0)$ by monotonicity, or it has infinite measure in which case we are done by (Folland 1.3.14). If E is not semifinite, then $\infty = \mu(E) = \nu(E) \leq \nu(E) + \mu_0(E)$ and we are done. $\square$

11. (Folland 1.3.16) (a) We can write $X = \bigcup_1^\infty A_i$ with $\mu(A_i) < \infty$ by the sigma finite hypothesis, hence if E is locally measurable , we have $E = E \cap \bigcup_1^\infty A_i = \bigcup_1^\infty (E \cap A_i) \in \mathcal{M}$ by closure under countable unions. $\square$

(b) Let $E \in \tilde{\mathcal{M}}$, then for any A with $\mu(A) < \infty$ we have $E^c \cap A = E \cup A^c = (E \cap A) \cup A^c \in \mathcal{M}$. If $\{E_i\}_1^\infty \subset \mathcal{M}$, then $A \cap \bigcup_1^\infty E_i = \bigcup_1^\infty (A \cap E_i) \in \mathcal{M}$. $\square$

(c) positivity and $\tilde{\mu}(\emptyset) = 0$ are clear. Now let $\{E_i\}_1^\infty \subset \tilde{\mathcal{M}}$ be disjoint, if $\bigcup_1^\infty E_i \notin \mathcal{M}$, then atleast one $E_j \notin \mathcal{M}$, in which case $\sum_1^\infty \tilde{\mu}(E_i) \geq \tilde{\mu}(E_j) = \infty = \tilde{\mu}(\bigcup_1^\infty E_i)$. If $\bigcup_1^\infty E_i \in \mathcal{M}$, and $\mu(\bigcup_1^\infty E_i) = \infty$, then $\bigcup_1^\infty \tilde{\mu}(E_i)$ is infinity in either the case all $E_i \in \mathcal{M}$ or the case some $E_i \notin \mathcal{M}$. Finally, if $\bigcup_1^\infty E_i \in \mathcal{M}$, and $\mu(\bigcup_1^\infty E_i) < \infty$, then $E_i = E_i \cap (\bigcup_1^\infty E_i) \in \mathcal{M}$, so that

$$\tilde{\mu}(\bigcup_1^\infty E_i) = \mu(\bigcup_1^\infty E_i) = \sum_1^\infty \mu(E_i) = \sum_1^\infty \tilde{\mu}(E_i)$$

$\square$

(d) Suppose $F \subset N$ where N is a $\tilde{\mu}$-null set, then $\tilde{\mu}(N) = 0 \neq \infty \implies N \in \mathcal{M}$ and $\mu(N) = \tilde{\mu}(N) = 0$, it follows by completion of μ that $F \in \mathcal{M} \subset \tilde{\mathcal{M}}$. $\square$

(e) It is clear that $\underline{\mu}(\emptyset) = 0$ and $\underline{\mu} \geq 0$. Now let $\{E_i\}_1^\infty \subset \mathcal{M}$ be disjoint, If some $\underline{\mu}(E_j) = \infty$, then so does $\underline{\mu}(\bigcup_1^\infty E_j)$, since the former is a subset of the latter. If this is not the case, then let $\epsilon > 0$ and take $F_i \in \mathcal{M}$ such that $\underline{\mu}(E_i) \geq \mu(F_i) \geq \underline{\mu}(E_i) - \epsilon 2^{-i}$, so that

$$\underline{\mu}(\bigcup_1^\infty E_i) \geq \limsup \mu(\bigcup_1^n F_i) \geq \sum_1^\infty \underline{\mu}(E_i) - \epsilon 2^{-i} = -\epsilon + \sum_1^\infty \underline{\mu}(E_i)$$

since epsilon was arbitrary this gives the inequality. For the converse, if $F \subset \bigcup_1^\infty E_i$ with $\mu(F) < \infty$, then $\sum_1^\infty \mu(E_i \cap F) \leq \sum_1^\infty \underline{\mu}(E_i)$, taking a sequence F_n with $\lim_{n\to\infty} \mu(F_n) = \underline{\mu}(\bigcup_1^\infty E_i)$, and $F_n \subset \bigcup_1^\infty E_i$ (we can do this due to the semifinite assumption), we use the inequality to conclude that

$$\underline{\mu}(\bigcup_1^\infty E_i) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_1^\infty \mu(F_n \cap E_i) \leq \sum_1^\infty \underline{\mu}(E_i)$$

So that indeed $\underline{\mu}$ is a measure. The fact that $\underline{\mu}|_{\mathcal{M}} = \mu$ is directly a consequence of monotonicity. Let E be a $\underline{\mu}$ locally measurable set, and let $A \in \mathcal{M} \subset \tilde{\mathcal{M}}$ be such that $\mu(A) < \infty$, then $\underline{\mu}(A) = \mu(A) < \infty$, so that by locally measurable assumption we have $E \cap A \in \tilde{M}$, hence $E \cap A = (E \cap A) \cap A \in \mathcal{M}$ by definition of $\tilde{\mathcal{M}}$, this suffices to show that E is locally measurable with repect to μ, so that $E \in \tilde{\mathcal{M}}$, i.e. $\underline{\mu}$ is saturated. □

(f) It is clear that $\mu \geq 0$ and $\mu(\emptyset) = \mu_0(\emptyset) = 0$. Now let $\{E_i\}_1^\infty \subset \mathcal{M}$ be disjoint. Then

$$\mu(\bigcup_1^\infty E_i) = \mu_0(X_1 \cap \bigcup_1^\infty E_i) = \mu_0(\bigcup_1^\infty (X_1 \cap E_i)) = \sum_1^\infty \mu_0(X \cap E_i) = \sum_1^\infty \mu(E_i)$$

so that μ is a measure. Note that X_2 (which is not countable or cocountable) is locally finite, since if A has finite measure, then $A \cap X_1$ is finite, hence A^c is uncountable, so that $A \cap X_2$ must be countable, so that $A \cap X_2 \in \mathcal{M}$. Now for any $E \subset X_2$ such that $E \in \mathcal{M}$, we have $\mu(E) = \mu_0(E \cap X_1) = \mu_0(\emptyset) = 0$, so that $\underline{\mu}(X_2) = 0$ (note here that $X_1 \cup X_2$ is obviously semifinite since any set containing infinitely members of X_1 contains a set with finitely many which is also countable or cocountable respectively so that $\underline{\mu}$ is a measure on our space), but $X_2 \notin \mathcal{M}$ so that

$$\tilde{\mu}(X_2) = \infty \neq 0 = \underline{\mu}(X_2)$$

□