Concise AG Notes - UofT MAT1190

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September 12, 2025

1 Lecture Notes

1.1 Lecture 1 (Sept 3, 2025)

Theorem 1. (Gelfond-Neymark) A compact topological space is determined by its ring of smooth functions. In particular if the ring $C(X) := C(X, \mathbb{R})$ and $C(X) \cong C(Y)$, then $X \cong Y$.

Proposition 1. Each point in X corresponds to a maximal ideal of $C(X)$, moreover if X is compact, then the correspondence is 1-1.

Proof. the evaluation at a point gives a surjective homomorphism $C(X) \to \mathbb{R}$, the image is a field, hence the kernel is a maximal ideal corresponding to the point.

Now in the compact case, (assume X is Hausdorff?), then X is Hausdorff and compact hence normal. We can use Uhrysohn's lemma to get a function vanishing at x but not y. Now suppose that for some maximal ideal $\mathfrak{m} \subset C(X)$ for any point $p \in X$ there is a continuous function with $f(p) \neq 0$, then the set $U_f = \{x \in X \mid f(x) \neq 0\}$ is open , and $\bigcup_{f \in C(X)} U_f = X$, so we get a finite subcover. Take a linear combination of the functions in this subcover to complete the proof. $\square$

definition 1. The Zariski Topology on $\mathrm{Spec}_{\max}(R)$ is the coarsest topology such that when $\mathfrak{m} \leftrightarrow x$ $f : \mathfrak{m} \to f(x)$ is continuous, where the topology on $\mathbb{R}$ is taken as the cofinite topology. The closed sets in this topology are the vanishing loci of $f \in C(X)$.

- Exercise 1, complete Hartshorne exercise 1.4

1.2 Lecture 2 (Sept 5, 2025)

definition 2. For $T \subset R_n := k[x_1, \ldots, x_n]$ and $S \subset k^n$ we define

$$V(T) = \{x \in k^n \mid f(x) = 0, \forall f \in T\} \text{ and } I(S) = \{f \in R_n \mid f(x) = 0, \forall x \in S\}$$

Proposition 2. Suppose k is an uncountable field, and L/k is an extension with $[L : k] \leq \#\mathbb{N}$, then $L = k$.

Proof. Suppose not, then let $x \in L \setminus k$, we find that $\{\frac{1}{x-\lambda} \mid \lambda \in k\}$ is uncountable, so that there must be an algebraic relation. Thus there exist $\mu_i \in k$ with $\sum_1^n \frac{\mu_i}{x-\lambda_i} = 0$, so that $\sum_1^n \mu_j \prod_{i \neq j}(x - \lambda_i) = 0$, but then x is algebraic over k, hence $x \in k$, contradiction. $\square$

Theorem 2. (Nullstellensatz - weak form) $V(T) = \emptyset \implies (T) = R_n$

Proof. We assume here that k is uncountable (this is unnecessary- use Noether Normalization). Since $J := (T) \subset R_n$ is an ideal it is contained in a maximal ideal $\mathfrak{m}$. Then $R_n/\mathfrak{m}$ is a field extension of k with countable dimension, by the previous proposition it is equal to k. It follows that each $x_i \mapsto a_i \in k$ when taking the quotient $R_n \to R_n/\mathfrak{m} = k$, it follows that I vanishes on $(a_1, \ldots, a_n)$, so I cannot be contained in a maximal ideal. $\square$

Theorem 3. (Nullstellensatz)

$$IV(J) = \sqrt{J}$$

Proof. By Hilbert's basis theorem, we reduce to the finitely generated case. Let $f \in IV(\{f_1, \ldots, f_r\})$, then $(1 - tf, f_1, \ldots, f_r) \subset R_n[t]$ has no common zero. Then $g_0(1 - tf) + g_1 f_1 + \cdots + g_r f_r = 1$, and let $N = \max_i\{\deg_t g_i\}$. Taking $t = f^{-1}$, we get $\sum_1^r g_i f_i = 1$, so that for $h_i = f^N g_i \in R_n$ we get $\sum_1^r h_i f_i = f^N \in I \implies f \in \sqrt{I}$. □

The Nullstellensatz gives a bijection

$$\{\text{Affine algebraic varieties}\} \longleftrightarrow \{\text{Finitely generated reduced k-algebras}\}$$
$$V(\sqrt{J}) \longleftrightarrow R_n/\sqrt{J}$$

Moreover, this is a categorical equivalence

$$\mathrm{Var}_k \cong \left(\mathrm{Alg}_k^{\text{reduced}}\right)^{\mathrm{op}}$$

1.3 Lecture 3 (Sept 8, 2025)

definition 3. Let $\pi : S \to X$ be a local homeomorphism, then S is called an étalé space, or a sheaf on X.

Example(s).

1. $\to X$
2. $1 : X \to X$
3. I a set with the discrete topology and the projection $X \times I \to X$
4. A covering space, more explicitly the mobious covering
$$S^1 \to S^1$$
$$z \mapsto z^2$$
5. $U \subset X$ an open set, $\iota : U \to X$
6. If $x \in X$ is a closed point, then we can construct the space $X \sqcup_{X\setminus\{x\}} X = X \times \{1,2\}/\sim$ where $(y,1) \sim (y,2)$ when $y \neq x$. This comes with the codiagonal map $\nabla : X \sqcup_{X\setminus\{x\}} X \to X$, where $\nabla|_{X\times\{i\}} = 1_X, i \in \{1,2\}$.

   This is a generalization of the line with two origins.
7. $I \neq \emptyset$, then take $\underset{X\setminus\{x\}}{\sqcup_I} X \xrightarrow{\nabla} X$

definition 4. If $U \subset X$ is an open set, then a section on U is a continuous map $s : U \to S$ such that the following commutes:

$$\begin{tikzcd} & S \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "\iota"] & X \end{tikzcd}$$

The set of sections is denoted $S(U)$ or $\Gamma(U, S)$. If $U = X$, then s is called a global section with notation $S(X)$ or $\Gamma(S)$.

Example(s). (Revisited)

1.
$$S(U) = \begin{cases} 1_\emptyset & U = \emptyset \\ \emptyset & \text{else} \end{cases}$$

2.
$$S(U) = \{\iota_U\}$$

3.
$$S(U) = \hom_{\text{set}}(\pi_0(U), I)$$

4.
$$S(U) = \{f : U \to \mathbb{C} \mid f(z^2) = z\}$$

5.
$$S(U) = \begin{cases} \{\iota\} & x \notin U \\ \{1,2\} & x \in U \end{cases}$$

6.
$$S(U) = \begin{cases} \{\iota\} & x \notin U \\ I & x \in U \end{cases}$$

This particular example is called the "sky-scraper sheaf"

Proposition 3. There is a étalé space $\mathscr{H}$ over $\mathbb{C}_{\text{EUC}}$ with sections corresponding to holomorphic functions on $\mathbb{C}$.

Proof. The construction of $\mathscr{H}$ as a set is given, alongside its topology. Verifying the claim is exercise 2.

$$\mathscr{H} := \bigsqcup_{z_0 \in \mathbb{C}} \left\{ \sum_1^\infty c_n (z - z_0)^n \mid \text{the series converges in some neighborhood of } z_0 \right\}$$

And define the topology on $\mathscr{H}$ as the strongest topology such that for any open set U, and holomorphic $f : U \to \mathbb{C}$ we have the following map is continuous

$$\begin{aligned} \mathscr{H} f : U &\to \mathscr{H} \\ z_0 &\mapsto \text{The Taylor expansion of } f \text{ at } z_0 \end{aligned}$$

□

1.4 Lecture 4 (Sept 10, 2025)

definition 5. Let $\pi : S \to X$ be étalé, then $S_x := \pi^{-1}(x)$ is called the stalk of x.

Example(s).
1. $1 : X \to X$, $S_x = \{x\}$
2. $X \times I \to X$, $S_x \cong I$
3. $\sqcup_{I}\, X \xrightarrow{\nabla} X$ ($X \setminus \{x\}$), then $S_y \cong \begin{cases} I & y = x \\ \{\overline{y}\} & y \neq x \end{cases}$
4. $\mathscr{H} \to \mathbb{C}$ $\mathscr{H}_{z_0}$ is locally convergent power series at z_0.

Proposition 4. If $\pi : S \to X$ is étalé and $y \in \pi^{-1}(x)$, then there is an open set $U \supset \{x\}$ and a section $s : U \to S$ with $s(x) = y$. Moreover, given two sections $s_i \in \Gamma(U_i, S)$ there is some $V \subset U_1 \cap U_2$ containing x, such that $s_1|_V = s_2|_V$.

Proof. The proof is exercise 3. □

Proposition 5.

$$\varinjlim_{x\in U} \Gamma(U,S) \xrightarrow{s\mapsto s(x)} S_x$$

is a bijection.

Proof. This is onto since every element of the stalk has a section mapping to it, and injective by uniqueness of such an element up to the equivalence relation in the colimit. This is essentially restating the previous proposition. □

Proposition 6. $f : X \to Y$, $g : Y \to Z$ continuous maps, then

- f and g being local homeomorphisms implies $g \circ f$ is.
- g and $f \circ g$ being local homeomorphisms implies f is.

definition 6. (The Category of Sheaves on X) The objects are étalé spaces $\pi : S \to X$, and the morphisms are $\varphi : S_1 \to S_2$ continuous maps where the following commutes:

$$\begin{tikzcd} S_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & S_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Note that φ continuous and the diagram commuting implies it is a local homeomorphism, and hence morphisms are actually sheaves on sheaves.

Proposition 7. (Isomorphism Criterion) A morphism $\varphi \in \mathrm{Sh}(X)$ is an isomorphism if and only if the induced map $(S_1)_x \to (S_2)_x$ is bijective for all $x \in X$.

Proof. One direction is easy since stalks get mapped to stalks, and an inverse map must be a bijection. In the converse direction, we know that it must be a local homeomorphism by proposition 6, hence since its invertible as a set function, its inverse is a local homeomorphism. □

Proposition 8. (Monomorphism Criterion) A morphism $\varphi \in \mathrm{Sh}(X)$ is an monomorphism if and only if the induced map $(S_1)_x \to (S_2)_x$ is injective for all $x \in X$.

Proof. Once again, one direction is easy. For the other direction, if $\varphi(x_1) = \varphi(x_2)$, then $\pi_2\varphi(x_1) = \pi_2\varphi(x_2)$, so that $\varphi(x_1)$ and $\varphi(x_2)$ are in the same element of the stalk, by injectivity on stalks we are done. □

Proposition 9. (Isomorphism Criterion for Sections) If $\varphi : S_1 \to S_2$ is a morphism in $\mathrm{Sh}(X)$ such that for any open set the induced map $\Gamma(U,S_1) \to \Gamma(U,S_2)$ is a bijection, then φ is an isomorphism. Moreover, the converse is true.

Proof. The main thing to check here is that a bijection for all U gives a bijection on the colimits. Assuming this for now we get that as sets:

$$(S_1)_x \cong \varinjlim_{x\in U} \Gamma(U,S_1) \cong \varinjlim_{x\in U} \Gamma(U,S_2) \cong (S_2)_x$$

So that applying the Isomorphism Criterion we find that φ is an isomorphism. Now if φ is an isomorphism, then $\varphi : \Gamma(U,S_1) \to \Gamma(U,S_2)$ via $s_1 \mapsto \varphi s_1$, this has map inverse φ^{-1}, so these sets are in bijective correspondence which suffices to prove the converse. □

Note that the same proof works for injections.

Warning ⚠. If $\varphi : S_1 \to S_2$, $\varphi \in \mathrm{Sh}(X)$ is surjective this does not imply that the induced map on $\Gamma(U,S)$ is in general surjective. A counter example is the Mobius covering of S^1, i.e. $X = S = S^1$ with $\pi_S = \varphi : S \to X$ via $z \mapsto z^2$ and $\pi_X = 1_X$. Then $\Gamma(X,S) = \emptyset$, since there is no globally continuous square root on S^1. This implies that there is no surjection

$$\emptyset = \Gamma(X,S) \to \Gamma(X,X) = \{1_X\}$$

The upshot is that local lifts do exist.

Proposition 10. (Local Lifts) Let S_1, S_2 be étalé over X, and $\varphi : S_1 \to S_2$a surjective morphism. Then given a section $s \in S_2$, there is an open cover $\bigcup_I U_i$ with sections $t_i \in S_1(U_i)$ such that $\varphi \circ t_i = s|_{U_i}$ for all i.

Proof. Since φ is surjective, it must also be surjective on stalks $S_x \to S_x$. Then for any x, we have some (t_x, V_x) so that $\varphi \circ t_x(x) = s(x)$, it follows by the existence part of proposition 4 that we can choose a neighborhood $x \in U_x \subset V_x$ so that $vp : (t_x, U_x) \to (s, U)$. □

Remark 1. (An abstract perspective on lifts) Given the setup

$$\begin{tikzcd} S_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & S_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

and a global section $s \in \Gamma(X, S_2)$, we get a sheaf from the fibered product $s^{-1}S_1 := S_1 \times_{S_2} X$ (this of course means its points are $\{(t,x) \mid \varphi(t) = s(x)\}$). From this perspective, s having a lift to $\Gamma(X, S_1)$ is equivalent to $s^{-1}S_1$ having a global section.

2 Exercises

exercise 1. (Hartshorne Exercise 1.4) An algebraically closed field is infinite, moreover the zero sets of polynomials are either k or a finite subset of k. Consider the closed set $V(x-y) \subset \mathbb{A}^2$, then it is an infinite set so if $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, then it must be of the form $\mathbb{A}^1 \times F \cup E \times \mathbb{A}^1$, where $E, F \subset \mathbb{A}^1$ are closed. But for a fixed x or y we have $V(x-y)$ has cardinality 1 which makes this impossible. □

exercise 2. (Show that $\mathscr{H}(U) = \{\mathbf{f : U \to \mathbb{C} \mid f\ holomorphic}\}$) where we define $\mathscr{H}$ in proposition 3.

We first check that it is a local homeomorphism, for $z \in \mathbb{C}$ and $U \supset z$, we can take a function f holomorphic on U, then $\mathscr{H}f$ has only one taylor expansion for f at each z_0 and is continuous. Since there is only one Taylor expansion at each z_0 the map π taking Taylor series centered at z_0 to z_0 is injective, π is continuous because for an open set V we have $\pi^{-1}(V) = \bigsqcup_{z_0 \in V} \mathscr{H}f(z_0)$, which has open preimage under all of the $\mathscr{H}f$. Since $\mathscr{H}f\pi = 1_S, \pi\mathscr{H}f = 1_X$ we are done this step.

Now $\pi \circ \mathscr{H}f|_U = \iota_U$ and $\mathscr{H}f$ continuous suffices to show that every holomorphic function is a section. Conversely, suppose $g : X \to \mathscr{H}$ is not induced by a holomorphic function. The first case is g maps some z_1 to a taylor expansion around $z_0 \neq z_1$, this cannot be a section since then the diagram won't commute. In the second case, there are distinct points $\{z_\alpha\}_{\alpha \in I}$ in the same connected component of U each Taylor series $g(z_\alpha)$ determining a different holomorphic function (near that point) f_α, denote the set of points that determines f_α as V_α, it is immediate that the V_α are disjoint. We know that $\mathscr{H}f_\alpha(U)$ is open in $\mathscr{H}$ for each α (Check!), but if $g^{-1}(\mathscr{H}f_\alpha(U)) = V_\alpha$ is open for each $\alpha \in I$, then $U = \bigsqcup V_\alpha$ is not connected, violating our earlier assumption. Hence for some open $\mathscr{H}f_\alpha(U)$ we have that $g^{-1}(\mathscr{H}f_\alpha(U))$ is not open and g is not continuous. □

Check: If two holomorphic functions have the same Taylor series at a point they are equal so

$$\mathscr{H}\phi^{-1}(\mathscr{H}f_\alpha(U)) = \begin{cases} U \text{ or the domain of definition for } f & \phi = f_\alpha \\ \emptyset & \text{else} \end{cases}$$

In either case the preimage is an open set.

exercise 3. **(Show the existence and uniqueness of sections for each element in the stalk)** Existence is not too bad, since π is a local homeomorphism, hence we can choose some neighborhood $y \in U$ with $\pi|_U$ a homeomorphism. Then define $s : \pi(U) \to U$ via $x \mapsto \pi|_U^{-1}(x)$. Now assume that s_1, s_2 are two such sections associated to open sets U_1, U_2, then $y \in \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$, and hence some open set $y \in \tilde{V} \subset \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ such that $\pi|_{\tilde{V}}$ is a homeomorphism. Now let $V = s_1^{-1}(\tilde{V}) \cap s_2^{-1}(\tilde{V})$ which is nonempty since it contains x, and open. Since the inclusions are injective, we can say the same about sections, hence $s_1, s_2 : V \to \tilde{V}$, and for $z \in V$ we have $\pi|_{\tilde{V}} s_i(z) = \iota(z) = z$ and hence $s_i(z) = \pi|_{\tilde{V}}^{-1}(z)$. □

A Assigned Readings

B Misc.

definition 7. A ring or algebra is called reduced when it has no non-zero nilpotents.

definition 8. A map is a monomorphism when it has the left cancellation property $fg_1 = fg_2 \implies g_1 = g_2$.