**1.** This question is handwritten due to having many pictures.

**2.** Define $f : G_2 \to G_1$ via $\mu \mapsto \alpha\beta$, and $\lambda \mapsto \alpha\beta\alpha$, to see that this map gives a well defined group homomorphism we need to check that the relation is satisfied, this simply follows from $f(\lambda)^2 = (\alpha\beta\alpha)^2 = (\alpha\beta)^3 = f(\mu)^3$, so that $f$ gives a well defined group homomorphism. Now we define $g : G_1 \to G_2$ via $g(\alpha) = \mu^2\lambda^{-1}$, and $g(\beta) = \lambda\mu^{-1}$, since $\gamma$ can be written in terms of the relations from $\alpha, \beta$ the value of $g(\gamma)$ is just defined as $g(\alpha)g(\beta)g(\alpha)^{-1} = \mu\lambda\mu^{-2}$. To check that $g$ gives a group homomorphism we just need to check it satisfies the relations of $G_1$ as follows.

$$g(\alpha) = \mu^2\lambda^{-1} = \mu^3\mu^{-1}\lambda^{-1} = \lambda^2\mu^{-1}\lambda^{-1} = \lambda\mu^{-1}\mu\lambda\mu^{-2}\mu\lambda^{-1} = g(\beta)g(\gamma)g(\beta)^{-1}$$

Where we use the relation $\mu^3 = \lambda^2$ in the third equality.

$$g(\beta) = \lambda\mu^{-1} = \lambda^2\lambda^{-1}\mu^{-1} = \mu^3\lambda^{-1}\mu^{-1} = \mu\lambda\mu^{-2}\mu^2\lambda^{-1}\mu^2\lambda^{-1}\mu^{-1} = g(\gamma)g(\alpha)g(\gamma)^{-1}$$

Where again the relation $\mu^3 = \lambda^2$ is used in the third equality, moreover $g$ is defined so as to define the third relation, thus $g$ gives a well defined group homomorphism. To see $G_1 \cong G_2$ it will now suffice to show that $f$ and $g$ are inverse isomorphisms, since they are homomorphisms it suffices to check on generators (note that since $\gamma$ can be written in terms of $\alpha$ and $\beta$ we only need to check for these two generators)

$$gf(\mu) = g(\alpha\beta) = g(\alpha)g(\beta) = \mu^2\lambda^{-1}\lambda\mu^{-1} = \mu$$
$$gf(\lambda) = g(\alpha\beta\alpha) = g(\alpha)g(\beta)g(\alpha) = \mu^2\lambda^{-1}\lambda\mu^{-1}\mu^2\lambda^{-1} = \mu^3\lambda^{-1} = \lambda^2\lambda^{-1} = \lambda$$
$$fg(\alpha) = f(\mu^2\lambda^{-1}) = f(\mu)^2 f(\lambda)^{-1} = \alpha\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta\gamma\beta^{-1}\alpha^{-1} = \alpha\alpha a^{-1} = \alpha$$
$$fg(\beta) = f(\lambda\mu^{-1}) = f(\lambda)f(\mu)^{-1} = \alpha\beta\alpha\beta^{-1}\alpha^{-1} = \alpha\beta(\alpha\beta\alpha^{-1})^{-1} = \alpha\beta\gamma^{-1} = \alpha\beta\alpha^{-1}\alpha\gamma^{-1} = \gamma\alpha\gamma^{-1} = \beta$$

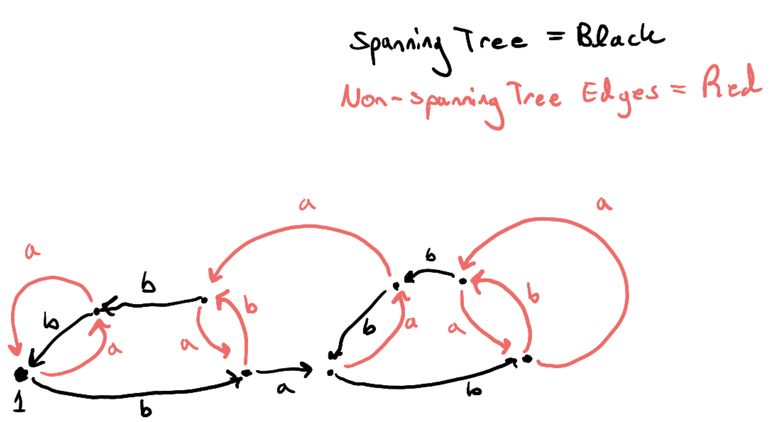
So that indeed $fg = 1_{G_1}$ and $gf = 1_{G_2}$ so that $G_1 \cong G_2$.

**3. (a)** By counting lifts of the wedge point we find the $G$ set has 8 elements, and by reading the space we find $a$ acts via $(12)(3456)(78)$ and $b$ acts via $(1432)(5876)$ □

**(b)** A set of generators is given by:

$$\{ab, b^{-1}a, b^4, b^{-2}ab^{-1}, ba^2ba^{-1}b^{-1}, bab^{-1}ab^2, bab^4a^{-1}b^{-1}, bab^{-2}ab^{-1}a^{-1}b^{-1}, babab^2a^{-1}b^{-1}\}$$

Which corresponds to choosing generators with respect to the following spanning tree: □




**(c)** No, because $S$ is not a normal $G$-set and a covering space is normal exactly when the $G$-set is. To see the $G$-set is not normal note that there is no automorphism permuting 1 and 4 since $a^2 \in \text{stab}(1)$, but $a^2 \notin \text{stab}(4)$. □

**4.** We first check the condition for $\mathrm{Aut}(\rho^{-1}(b_0))$ as a right $G$-set. As a right $G$-set, $\rho^{-1}(b_0) \cong \mathrm{stab}(b_0)\backslash G$, where $\mathrm{stab}(b_0) = \rho_*\pi_1(X) = H$, since these are exactly the paths with action fixing $b_0$. Now let $[\gamma] \in N_G(H)$, since $X$ is connected, $\rho^{-1}(b_0)$ is a transitive $G$-set so we can write any $b \in \rho^{-1}(b_0)$ as $b = b_0 \cdot [\alpha]$ for some $[\alpha]$, this allows us to define the automorphism given by $[\gamma]$ as $\varphi_\gamma(b_0 \cdot [\alpha]) = b_0 \cdot [\gamma][\alpha]$, we need to check this is well defined and indeed defines a right $G$-set automorphism. To see it is well defined, take another representative for $b$, then $b = b_0[\beta]$, but $b_0[\beta][\alpha]^{-1} = b_0$, implying that $[\beta][\alpha]^{-1} \in \mathrm{stab}(b_0) = H$, so that $[\beta] = [h][\alpha]$ for $[h] \in H$, then

$$\varphi_{[\gamma]}(b_0 \cdot [\beta]) = \varphi_{[\gamma]}(b_0 \cdot [h][\alpha]) = b_0 \cdot [\gamma][h][\alpha] = b_0 \cdot [\gamma][h][\gamma]^{-1}[\gamma][\alpha] = [b_0] \cdot [\gamma][\alpha]$$

The last equality following from $[\gamma] \in N_G(H)$ implying that $[\gamma][h][\gamma]^{-1} \in H = \mathrm{stab}(b_0)$ proving that $\varphi_\gamma$ is well defined. Moreover let $[x] \in G$, then once again for any $b \in \rho^{-1}(b_0)$ we write $b = b_0[\alpha]$ for $[\alpha] \in G$

$$\varphi_{[\gamma]}(b) \cdot [x] = b_0 \cdot [\gamma][\alpha][x] = \varphi_{[\gamma]}(b_0 \cdot [\alpha][x]) = \varphi_{[\gamma]}(b \cdot [x])$$

So that $\varphi_{[\gamma]}$ is a morphism of right $G$-sets.

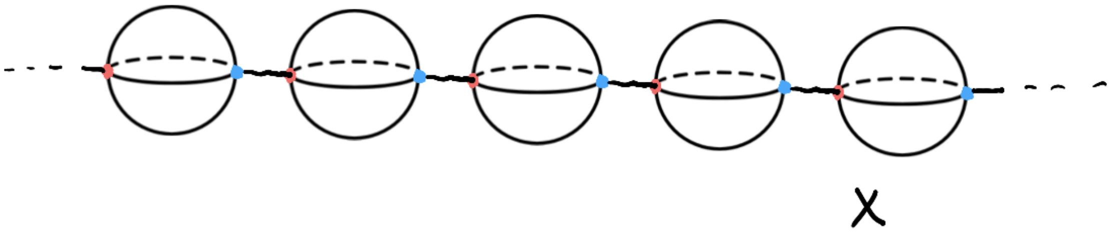
Now suppose that $\varphi \in \mathrm{Aut}(\rho^{-1}(b_0))$, then $\varphi(b_0) = b_0 \cdot [\gamma]$ for some $[\gamma] \in G$, it follows that since $\varphi$ is a morphism of $G$-sets, for $b = b_0 \cdot [\alpha]$ it satisfies $\varphi(b) = \varphi(b_0 \cdot [\alpha]) = \varphi(b_0) \cdot [\alpha] = b_0 \cdot [\gamma][\alpha]$, so that every automorphism is of the form above, now assuming that $\gamma \notin N_G(H)$, we can choose some $[h] \in H$ with $[\gamma][h][\gamma]^{-1} \notin H$, so that for the map $\varphi_{[\gamma]} : b_0 \cdot [\alpha] \to b_0 \cdot [\gamma][\alpha]$ we get

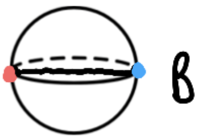
$$\varphi_{[\gamma]}(b_0) \cdot [h][\gamma]^{-1} = b_0 \cdot [\gamma][h][\gamma]^{-1} \neq b_0 = b_0[\gamma][\gamma]^{-1} = \varphi_{[\gamma]}(b_0 \cdot h[\gamma]^{-1})$$

i.e. such a $[\gamma]$ cannot define an automorphism of $\rho^{-1}(b_0)$. Finally to conclude that $\mathrm{Aut}(\rho^{-1}(b_0)) \cong N_G(H)$, we note that $\varphi_{[\alpha]} \circ \varphi_{[\beta]} = \varphi_{[\alpha][\beta]}$, so the bijection respects the group law.

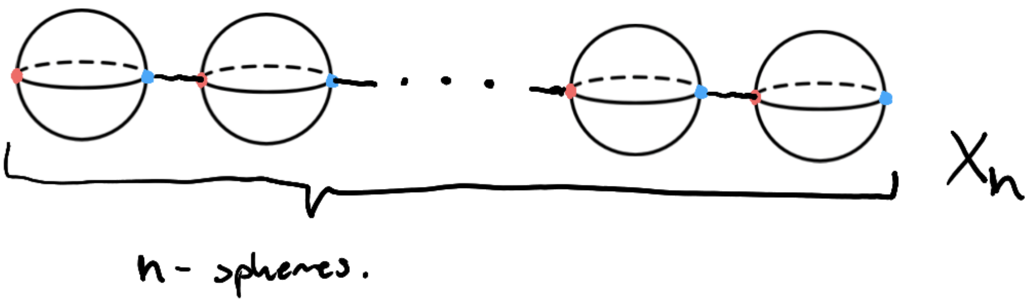
Now that we have shown $\mathrm{Aut}(\rho^{-1}(b_0)) \cong N_G(H)$, it is straightforward to see that the group of deck transformations $\mathrm{Aut}(X) \cong N_G(H)$. To do so we apply the equivalence of categories of right $\pi_1(B, b_0)$-sets and covering spaces $E \to B$. We recal the two functors are $\mathcal{F}$ taking covering spaces to right $G$ sets via $E \xrightarrow{\rho} B \rightsquigarrow \rho^{-1}(b_0)$ the right $G$ set with the path action, and the other functor is given by $\mathcal{G}$ taking right $\pi_1(B, b_0)$ sets and constructing the corresponding covering spaces. Above we computed $\mathrm{Aut}(\mathcal{F}(X \to B))$, then the functor $\mathcal{G}$ takes each automorphism $[\gamma] \in N_G(H)/H$ of right G-sets to an automorphism of covering spaces taking $x \mapsto x \cdot [\gamma]$. So the fact that $\mathrm{Aut}(X) = N_G(H)/H$ simply follows from equivalence of categories preserving hom sets so that $\mathrm{Aut}(X) = \mathcal{G}(\mathrm{Aut}(\mathcal{F}(X)))$. $\square$

**5.** The following picture is the universal cover, where the coloring is used to denote the covering map.




By looking locally, we can see that this is clearly a path connected covering space so all that is left to show is that it is simply connected, as proof note that since $S^1$ is compact, any loop $f : S^1 \to X$ must have image contained within finitely many of the spheres, so we can choose for some $n$, the following space to contain the image of $\gamma$




To see a space of this form is simply connected, we can use induction, and note the base case is already proven since $S^2$ is simply connected, then by assuming $X_{n-1}$ is simply connected we get the open cover given by:

$$U = X_{n-1} \cup_{0\sim x_0} [0, \epsilon) \qquad V = S^2 \cup_{1\sim y_0} (\epsilon/2, 1]$$

Then $U \cap V = (\epsilon/2, \epsilon) \simeq \{*\}$, $U \sim X_{n-1}$ and $V \sim S^2$, so that

$$\pi_1(X_n) \cong \pi_1(X_{n-1}) *_0 \pi_1(S^2) = 0$$

So that the image of gamma is contained in a simply connected subset of $X$, and hence $[\gamma] = 0$ in $\pi_1(X)$, since this holds for an arbitrary $\gamma : S^1 \to X$ we conclude that $\pi_1(X) = 0$, so that $X$ is the universal cover of $B$.

**6.** In problem 1 of homework 2, we showed that if a space $U$ is path connected, simply connected, and locally path connected then for any connected covering space $X \overset{\rho}{\to} B$ and map $f : U \to B$, there is a unique lift $\tilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[d, "\rho"] \\ U \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

We can apply this result directly to the following diagram, where $\rho_U : U \to B$ and $\rho_X : X \to B$ are covering maps, for an arbitrary covering space $\rho_X : X \to B$. The justification of $U$ being locally path connected is that $B$ is locally path connected and $U$ is locally diffeomorphic to $B$.

$$\begin{tikzcd} & X \arrow[d, "\rho_X"] \\ U \arrow[r, "\rho_U"] \arrow[ur, dashed, "\widetilde{\rho_U}"] & B \end{tikzcd}$$

Then commutativity of the diagram is exactly the statement that that the map we get as a result of homework 2 problem 1, $\widetilde{\rho_U}$ is also a covering space morphism.