

1. (1) It goes without saying what the objects are. As per morphisms if $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ and $0 \rightarrow X_* \rightarrow Y_* \rightarrow Z_* \rightarrow 0$ are two short exact sequences of chain complexes, define a morphism f to be morphisms of chain complexes $(f^\alpha)_n^{\alpha=A,B,C}$ so that $f^A : A_* \rightarrow X_*$ and analogously for B, C . Moreover we require commutativity of the following:

$$\begin{array}{ccccc} A_* & \xrightarrow{\iota} & B_* & \xrightarrow{q} & C_* \\ \downarrow f & & \downarrow f & & \downarrow f \\ X_* & \xrightarrow{\iota} & Y_* & \xrightarrow{q} & Z_* \end{array}$$

(2) Since long exact sequences are chain complexes, we can simply view them as a subcategory.

(3) As mentioned in the problem statement the functor on objects has been constructed, namely it is the snake lemma. We need to define the action on morphisms and functoriality. Simply take $f \mapsto f_* : H_n(A) \rightarrow H_n(X)$ for all n , and likewise for $H_n(B), H_n(C)$. Since homology is functorial our construction satisfies identity and inverse properties, but we still need to show that the following commutes

$$\begin{array}{ccccccc} \cdots & H_n(A) & \xrightarrow{\iota_*} & H_n(B) & \xrightarrow{q_*} & H_n(C) & \xrightarrow{\delta} & H_{n-1}(A) & \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots & H_n(X) & \xrightarrow{\iota_*} & H_n(Y) & \xrightarrow{q_*} & H_n(Z) & \xrightarrow{\delta} & H_{n-1}(X) & \end{array}$$

For the squares other than the one with δ , this follows directly from the commutativity conditions on f and functoriality of homology. For the square with the δ -s we need to check. Consider $[c] \in H_n(C)$, then we use the proof of snake lemma to show that $[x] = f_*\delta([c])$ is equal to $\delta(f_*[c])$.

Recall from the proof that $\delta([c])$ is defined by $[a]$ where for (an arbitrary) b such that $q(b) = c$ we have $a = \iota^{-1}(d(b))$. Now considering such a b , by