1. We use from the notes the existence of the oriented intersection number, defined when $f : M \to N$, and $Z \subset N$ is a submanifold with $f = f_0 \sim f_1$ and $f_1 \pitchfork Z$, $I(f, Z) = \sum_{p \in f_1^{-1}(Z)}$ orientation $\#(p)$, which is congruent to $I_2(f, Z)$ (I will not give this construction since it was done in class). Now since $\dim M = \dim N$, we can define $\deg f = I(f, \{p\})$ for $p \in M$, we need to show that this is well defined for arbitrary p and that it reduces to $\deg_2 f$.

2. Use the separation theorem to show that NM is orientable, then since $T\mathbb{R}^{k+1} = TM \oplus NM$ and $T\mathbb{R}^{k+1}$ and NM are orientable, we get an orientation on TM (see handwritten notes).

3. Lemma. If G a (finite) discrete group acts on an orientable manifold M such that the action is smooth, free and proper, such that for each $g \in G$ we have $\det(dg) > 0$ on M, then there is an induced orientation on M/G.

Proof. For convenience, take the section $s : M \to \Lambda^n TM$ so that $s > 0$. Let $q : M \to M/G$ be the quotient map induced by the group action, and let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover for M with $q^{-1}(V_\alpha) = \bigsqcup_1^r U_\alpha^i$ and $q|_{U_\alpha^i} : U_\alpha^i \xrightarrow{\cong} V_\alpha$. Now let $\{\eta_\alpha\}_{\mathcal{A}}$ be a partition of unity subordinate to the V_α, we consider the following diagram, and local invertibility of q to define a section $V_\alpha \to \Lambda^n TV_\alpha$ which is either everywhere positive or negative, since $q|_{U_\alpha^i}$ is a diffeomorphism, it induces an isomorphism of tangent bundles $\det dq|_{U_\alpha^1}$, since this is a smooth map which is everywhere non-zero, it is in particular everywhere positive or negative.

$$\begin{array}{ccc} U_\alpha^1 & \xrightarrow{s} & \Lambda^n TU \\ \downarrow{\scriptstyle q|_{U_\alpha^1}} & & \downarrow{\scriptstyle \det(dq|_{U_\alpha^1})} \\ V_\alpha & & \Lambda^n TV_\alpha \end{array}$$

So that we get the section $\overline{s} : x \mapsto \sum_\alpha \eta_\alpha \det dq|_{U_\alpha^1}(s(q|_{U_\alpha^1}^{-1}(x)))$, to check it is everywhere non-zero suppose that $\eta_{\alpha_1}(x), \ldots, \eta_{\alpha_s}(x) > 0$, notice that for each j letting $y_k = q|_{U_{\alpha_j}^1}^{-1}(x)$ we have on some neighborhood of y_j

$$q|_{U_{\alpha_j}^1} = q|_{U_{\alpha_1}^1}\left(q|_{U_{\alpha_1}^1}^{-1} q|_{U_{\alpha_j}^1}\right) = q|_{U_{\alpha_1}^1} g_j$$

for some $g_j \in G$, this gives us that (using functoriality of det)

$$\det d_{y_j} q|_{U_{\alpha_j}^1} = \det d_{y_j} q|_{U_{\alpha_1}^1} g_j = \left(\det d_{y_1} q|_{U_{\alpha_1}^1}\right)\left(\det d_{y_j} g_j\right)$$

Since $\det dg_j > 0$, this implies that each term of the sum $\sum_\alpha \eta_\alpha \det dq|_{U_\alpha^1}(s(q|_{U_\alpha^1}^{-1}(x)))$ is a positive multiple of $\det dq|_{U_{\alpha_1}^1}(s(q|_{U_\alpha^1}^{-1}(x)))$, then since each term is nonzero and has the same sign this suffices to show nowhere vanishing at the point x, and since x was arbitrary, the section is nowhere vanishing. $\square$

Denote $j : S^d \to S^d$ as the antipodal map. When d is odd, we have an isotopy $1_{S^d} \sim j$ via $H((z_1, \ldots, z_{\frac{d+1}{2}}), t) = (e^{i\pi t} z_1, \ldots, e^{i\pi t} z_{\frac{d+1}{2}})$, denoting $H(x, t)$ as $j_t(x)$ we find each induced $\det dj_t : \Lambda^d TS^d \to \Lambda^n TS^d$ is non-vanishing by virtue of being an embedding, moreover by IVT and smoothness in t, we find that $\det dj_t > 0$ on M for all t, and hence the antipodal map satisfies the conditions of the lemma, applying the lemma we find $\mathbb{RP}^d = S^d/(\mathbb{Z}/2\mathbb{Z})$ has an induced orientation for odd n.

In the case that d is even, suppose for the sake of contradiction that $\mathbb{RP}^d$ is orientable. We first check that the antipodal map j is indeed orientation reversing on S^d. Consider S^d as embedded in $\mathbb{R}^{d+1}$ via the standard embedding, we get the decomposition $T\mathbb{R}^{d+1}|_{S^d} = TS^d \oplus (TS^d)^\perp$, since $T\mathbb{R}^{d+1}|_{S^d}$ is trivial, we can fix a section t for it, we can also take nonvanishing the outward normal section $n : S^d \to (S^d)^\perp$ via $n(x) = x$, this gives an orientation on $(S^d)^\perp$ since we have the canonical isomorphism $\Lambda(S^d)^\perp \cong (S^d)^\perp$. We have that j extends to the map $\hat{j} : x \mapsto -x$ on $\mathbb{R}^{d+1}$