1.

$$h(t,s) = \begin{cases} \gamma(2t) & t \le \frac{1-s}{2} \\ \gamma(1-s) & t \in \left(\frac{1-s}{2}, \frac{1+s}{2}\right) \\ \overline{\gamma}(2(t-\frac{1}{2})) & t \ge \frac{1+s}{2} \end{cases}$$

□

2. ((iii) $\implies$ (i)): Apply problem 3, in particular any element $[f] \in \pi_1(X, x_0)$ satisfies

$$[f] = [\alpha][1_{x_0}][\alpha]^{-1} = [1_{x_0}]$$

((i) $\implies$ (iii)): Apply problem 3, since there is only one conjugacy class of $\pi_1(X, x_0)$ there is only one homotopy class $S^1 \to X$. Alternatively, let $f_1, f_2 : S^1 \to X$, where we parameterize S_1 as $\frac{I}{0 \sim 1}$, since X is path connected, there is some path $\gamma : I \to X$ with $\gamma(0) = f_1(0)$ and $\gamma(1) = f_2(0)$, then by assumption (i), $[f_1] = [1_{f_1(0)}]$ and $[f_2] = [1_{f_2(0)}]$, now we can write the homotopy between $1_{f_1(0)} \sim f_1$ and $1_{f_2(0)} \sim f_2$, which is given by $h(s,t) = \gamma(s)$. □

((ii) $\implies$ (i)): Let $[\gamma] \in \pi_1(X, x_0)$ for some $x_0 \in X$, then we can define new paths

$$\gamma_1 : t \mapsto \gamma(t/2) \qquad \gamma_2 : t \mapsto \gamma\left(\frac{1+t}{2}\right)$$

then $\overline{\gamma_1}$ and γ_2 satisfy the hypotheses of (ii), which entails $\gamma_2 \sim \overline{\gamma_1}$, now since $\gamma = \gamma_1 \cdot \gamma_2$ we find that $\gamma \sim \gamma_1 \cdot \overline{\gamma_1} \sim 1_{x_0}$, whence $[\gamma] = [1_{x_0}]$. Since $[\gamma]$ was arbitrary we conclude $\pi_1(X, x_0) = 0$. □

((i) $\implies$ (ii)): Let $f, g : I \to X$ with $x = f(0) = g(0)$ and $y = f(1) = g(1)$, then from (i) we get a homotopy h witnessing $f\overline{g} \sim 1_x$, moreover there is some homotopy r giving $\overline{g}g \sim 1_y$ satisfying $h(1,s) = h(0,s) = x$ for all s, and $r(1,s) = r(0,s) = y$ for all s. It is clearly sufficient to provide some path γ such that f is homotopic to γ by h' without moving endpoints, and same for g by h'', since then the following homotopy satisfies the desideratum

$$H(t,s) = \begin{cases} h'(t, 2s) & s \in [0, \frac{1}{2}] \\ h''(1-t, 1-2(s-\frac{1}{2})) & s \in (\frac{1}{2}, 1] \end{cases}$$

Taking $\gamma = f\overline{g}g$, we find that h' can be taken to be

$$h'(t,s) = \begin{cases} h(2t, s) & t \in [0, \frac{1}{2}] \\ g(2(t-\frac{1}{2})) & t \in (\frac{1}{2}, 1] \end{cases}$$

and h'' can be taken to be

$$h''(t,s) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ r(s, 2(t-\frac{1}{2})) & t \in (\frac{1}{2}, 1] \end{cases}$$

□

3. Define the map

$$\psi : \pi_1(X, x_0) \to \{S^1 \to X\}/\sim$$
$$[f] \mapsto [f]$$

In words, we forget the base point of f. This is of course well defined up to homotopy. To check it is well defined on conjugacy classes, it suffices to check that $[f] = [\alpha][f][\alpha]^{-1}$ for arbitrary f and α. This is straightforward by choosing the representative with α, f and $\bar{\alpha}$ each being sped up by 3 times, and defining the homotopy

$$h(t,s) = \begin{cases} \alpha(s+3t) & t \le \frac{1-s}{3} \\ f\left(\frac{t - \frac{1-s}{3}}{1 - 2\frac{1-s}{3}}\right) & t \in \left(\frac{1-s}{3}, \frac{2+s}{3}\right) \\ \alpha(1+s-3(t-\frac{2}{3})) & t \ge \frac{2+s}{3} \end{cases}$$

To see that this map is onto, it suffices to check that every map $f : S^1 \to X$ is homotopic to a map f' with $f'(0) = x_0$. Now checking this consider some $f : S^1 \to X$, and let γ be a path between x_0 and $f(0)$, then we can write

$$f'(t) = \begin{cases} \gamma(3t) & t \leq \frac{1}{3} \\ f(3(t - \frac{1}{3})) & t \in (\frac{1}{3}, \frac{2}{3}) \\ \gamma(1 - 3(t - \frac{2}{3})) & t \geq \frac{2}{3} \end{cases}$$

Then we can verify explicitly that $f' \sim f$

$$h'(t,s) = \begin{cases} \gamma(s + 3t) & t \leq \frac{1-s}{3} \\ f(\frac{t - \frac{1-s}{3}}{1 - 2\frac{1-s}{3}}) & t \in (\frac{1-s}{3}, \frac{2+s}{3}) \\ \gamma(1 + s - 3(t - \frac{2}{3})) & t \geq \frac{2+s}{3} \end{cases}$$

Finally, we need to check injectivity. Assume that $\psi([f]) = \psi([g])$, then there is some homotopy $h : S^1 \times I \to X$ witnessing this equivalence, then we can define $\alpha(s) = h(0,s)$, it is clear that $h(0,s) \sim h(1,s)$ and thus indeed $[h(1,-)] = [h(0,-)]^{-1}$ (as proof just take the homotopy $h' : (t,s) \mapsto h(s,t)$). It remains to check that $[\alpha][f][\alpha]^{-1} = [g]$, this equivalence is given by the following (based) homotopy between $[\alpha][f][\alpha]^{-1}$ and $[1_{x_0}][f][1_{x_0}] = [f]$.

$$H(t,s) = \begin{cases} h(0, 3(1-s)t) & t \leq \frac{1}{3} \\ h(3(t - \frac{1}{3}), 1 - s) & t \in (\frac{1}{3}, \frac{2}{3}) \\ h(1, (1-s) - 3(1-s)(t - \frac{2}{3})) & t \geq \frac{2}{3} \end{cases}$$

$\square$