

1. We first find an expression for  $\zeta(s)\Gamma(s)$  on  $\Re(s) > 1$ , to do so we will do a substitution  $u = x/n$ , valid for any  $n \in \mathbb{Z}_{>0}$

$$\Gamma(s) = \int_0^\infty e^{-nu} (nu)^{s-1} n du = n^s \int_0^\infty u^{s-1} (e^{-u})^n du$$

Now since I don't like  $u$  I will switch back to  $x$ ; Multiplying  $n^{-s}$  on both sides and summing over  $n \in \mathbb{Z}_{>0}$  yields

$$\zeta(s)\Gamma(s) = \sum_1^\infty \int_0^\infty x^{s-1} (e^{-x})^n dx \stackrel{\text{DCT}}{=} \int_0^\infty x^{s-1} \sum_1^\infty (e^{-x})^n dx = \int_0^\infty x^{s-1} \frac{1}{e^x - 1} dx \quad (1)$$

Where DCT is taken with respect to  $|x^{s-1}| \sum_1^\infty e^{-nx}$ . We can use  $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$  to rewrite (1):

$$\zeta(s) = \frac{\Gamma(1-s)\sin(\pi s)}{\pi} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad (2)$$

Now we deal with the contour integral, letting  $C(\epsilon)$  denote the curve described in the problem for fixed  $\epsilon \in \mathbb{R}_{>0}$ . The expression  $e^z - 1$  has no poles away from  $\{z \mid e^z = 1\} = \{2\pi ki \mid k \in \mathbb{Z}\}$ , whence if we take the branch cut of log away from the non-negative reals the integral is not dependent on  $\epsilon$  for  $\epsilon < 2\pi$ , since (for  $\delta < \epsilon < 2\pi$ ) we have the area enclosed between the two curves is a quotient of holomorphic functions with the denominator non-vanishing in between the curves. This independence is a result of taking separating the curve  $C(\epsilon) - C(\delta)$  into two curves (see picture), the first of which has integral zero by Cauchy's theorem, and the second being arbitrarily small depending on where we take the cut.

$$\int_{C(\epsilon)} f - \int_{C(\delta)} f = \int_{C(\epsilon, \delta, M)} f + \int_{\gamma_1(\epsilon, \delta, M)} f + \int_{\gamma_2(\epsilon, \delta, M)} f$$

### INSERT ILLUSTRATION HERE

Then for  $f$  holomorphic away from the real line, the  $C(\epsilon, \delta, M)$  term vanishes. Notice now that using the standard arclength inequality for large  $M$  we have

$$\begin{aligned} \left| \int_{\gamma_j(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| &\leq (\epsilon - \delta) \left| \frac{M^{\Re(s)-1}}{e^M - 1} \right| + \left| \int_M^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx - \int_M^\infty \frac{(-x - i\delta)^{s-1}}{e^{x+i\delta} - 1} dx \right| \\ &\leq (\epsilon - \delta) \left| \frac{M^{\Re(s)-1}}{e^M - 1} \right| + 2 \int_M^\infty \frac{|x + i\epsilon|^{\Re(s)-1}}{|e^x| - 1} dx \end{aligned}$$

The right hand side clearly converges to zero. as  $M \rightarrow \infty$  using basic limits of exponentials and DCT. This gives the desired invariance.

$$\left| \int_{C(\epsilon)} \frac{(-z)^{s-1}}{e^z - 1} dz - \int_{C(\delta)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| \leq \left| \int_{\gamma_1(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| + \left| \int_{\gamma_2(\epsilon, \delta, M)} \frac{(-z)^{s-1}}{e^z - 1} dz \right| = 0$$

Now, we can compute the value of the integral along this curve by letting  $\epsilon \rightarrow 0$ , to get  $C(0)$ , a ray from  $\infty$  to 0 where  $(-z)^{s-1} = x^{s-1}e^{-(s-1)\pi i}$ , and a ray from 0 to  $\infty$  where  $(-z)^{s-1} = x^{s-1}e^{(s-1)\pi i}$ , to see that we can indeed pass to this limit, once again decompose  $C = C(\epsilon)$  into three curves, with  $C_1$  the ray in the upper half plane,  $C_2$  the ray in the lower half plane and  $C_3$  the circular portion, then once again using the arc length inequality and the fact that  $e^z - 1 = \mathcal{O}(z)$

$$\left| \int_{C_3} \frac{(-z)^{s-1}}{e^z - 1} dz \right| \leq 2\pi\epsilon \sup_{|z|=\epsilon} \frac{|z^{s-1}|}{|e^z - 1|} = 2\pi\epsilon \mathcal{O}(\epsilon^{\Re(s)-2}) = 2\pi\mathcal{O}(\epsilon^{\Re(s)-1}) \xrightarrow{\Re(s)>1} 0$$

We can check convergence on  $C_1$  explicitly

$$\begin{aligned}
& \left| \int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx - \int_\epsilon^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx \right| \\
& \leq \left| \int_0^\epsilon \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx \right| + \left| \int_\epsilon^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx - \int_\epsilon^\infty \frac{(-x - i\epsilon)^{s-1}}{e^{x+i\epsilon} - 1} dx \right| \\
& \leq \left| \int_0^\epsilon \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx \right| + \left| \int_\epsilon^\infty \frac{(e^{x+i\epsilon} - 1)x^{s-1} e^{-(s-1)\pi i} - (e^x - 1)(\sqrt{x^2 + \epsilon^2})^{s-1} e^{i(s-1) \arctan \frac{\epsilon}{x} - (s-1)\pi i}}{(e^x - 1)(e^{x+i\epsilon} - 1)} dx \right|
\end{aligned}$$

Convergence as  $\epsilon \rightarrow 0$  of the big ugly term to zero is actually simple from convergence of each of the terms in the two expressions in the product. Convergence of the first term follows from  $\Re(s) > 1$ , so writing the bounds of integration as  $\chi_{(0, \epsilon)}$  we can just apply DCT to the absolute value of the integrand. The proof of convergence for  $C_2$  is similar to  $C_1$ .

Now we finally established that (due to taking the limit in  $\epsilon$  and invariance with respect to  $\epsilon$ )

$$\int_C \frac{(-z)^{s-1}}{e^z - 1} dz = - \int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx + \int_0^\infty \frac{x^{s-1} e^{(s-1)\pi i}}{e^x - 1} dx \quad (3)$$

$$= \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin((s-1)\pi) = - \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin(s\pi) \quad (4)$$

Multiplying (2) by  $1 = \frac{\int_C \frac{(-z)^{s-1}}{e^z - 1} dz}{- \int_0^\infty \frac{x^{s-1}}{e^x - 1} 2i \sin(s\pi)}$  yields the desired equality

$$\zeta(s) = - \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz \quad (5)$$

Now, using the right side of (5) as an analytic continuation, we first compute  $(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2)$ , since we know it has a simple pole at zero, hence is meromorphic this expression just comes from evaluating the systems of equations given for the coefficients for  $(a_{-1}z^{-1} + a_0 + a_1z + \dots)(\sum_{n=0}^\infty \frac{z^n}{n!}) = 1$ . This is everything we need to evaluate  $\zeta(0)$ . First using a similar separation of curves into two parts as was used for invariance, we find that  $\int_C \frac{1}{z(e^z - 1)} dz = \int_\gamma \frac{1}{z(e^z - 1)} dz$  where  $\gamma$  is a piecewise  $C^1$  closed curve, this allows us to use the residue theorem

$$\zeta(0) = \frac{\Gamma(1)}{2\pi i} \int_\gamma \frac{1}{z(e^z - 1)} dz = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \left( \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2) \right) dz \quad (6)$$

$$= \text{Res} \left( \frac{1}{z} \left( \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(z^2) \right) \right) = -\frac{1}{2} \quad (7)$$

□