1. We start by computing the Jacobian of the map $F : x \mapsto \frac{x}{||x||}$

$$\frac{\partial}{\partial x_j}\frac{x_i}{\sqrt{\sum_1^{n+1} x_k^2}} = \frac{\delta_{ij}||x|| - x_i x_j ||x||^{-1}}{||x||^2} = \frac{\delta_{ij}}{||x||} - \frac{x_i x_j}{||x||^3}$$

So the Jacobian looks like

$$\frac{1}{||x||}1 - \frac{1}{||x||^3}xx^T$$

Now beginning the actual proof, let $\{f_1, \dots, f_{n-m}\}$ be a basis for $(\operatorname{Im} A)^\perp$, then define $T : \mathbb{R}^{n-m+1} \to \mathbb{R}^{n+1}$ via $e_i \mapsto f_i$ when $1 \le i \le n-m$ and $e_{n-m+1} \mapsto Ae_1$, then $\overline{T} : \mathbb{RP}^{n-m} \to \mathbb{RP}^n$ is an embedding, so we can refer to its image as the submanifold $X \subset \mathbb{RP}^n$. Now we get that $\operatorname{Im}(\overline{A}) \cap X = q(\operatorname{Im} A \cap \operatorname{Im} T)$, where $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is the quotient. By construction this intersection is $[Ae_1]$, so that once we verify $\overline{A} \pitchfork_{[Ae_1]} X$ we will get that $I_2(\overline{A}, X) = 1$. Now checking transversality, we will use the following maps, where the π maps are the indiced quotients under the action by the discrete group. Note that in particular the π maps are submersions of manifolds of equal dimension and hence local diffeomorphisms by the inverse function theorem.

$$\widehat{A} : S^m \to S^n \qquad\qquad \widehat{T} : S^{m-n} \to S^n$$

$$v \mapsto \frac{Av}{||Av||} \qquad\qquad v \mapsto \frac{Tv}{||Tv||}$$

$$\pi_m : S^m \to \mathbb{RP}^m \qquad\qquad \pi_{n-m} : S^{n-m} \to \mathbb{RP}^{n-m}$$

$$\pi : S^n \to \mathbb{RP}^n$$

Moreover, the following diagrams commute by definition of $\widehat{A}, \widehat{T}$

$$\begin{tikzcd} S^m \arrow[r, "\widehat{A}"] \arrow[d, "\pi_m"] & S^n \arrow[d, "\pi"] \\ \mathbb{RP}^m \arrow[r, "\overline{A}"] & \mathbb{RP}^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-m} \arrow[r, "\widehat{T}"] \arrow[d, "\pi_{n-m}"] & S^n \arrow[d, "\pi"] \\ \mathbb{RP}^{n-m} \arrow[r, "\overline{T}"] & \mathbb{RP}^n \end{tikzcd}$$

We will verify later that $\operatorname{Im} d_{e_1}\widehat{A} + \operatorname{Im} d_{e_{m-n+1}}\widehat{T} = T_{A(e_1)}S^n$, but assuming it for now we find that (using repeatedly the submersion properties of the projections)

$$\begin{aligned} T_{\overline{Ae_1}}\mathbb{RP}^n &= d_{\widehat{Ae_1}}\pi(\operatorname{Im} d_{e_1}\widehat{A} + \operatorname{Im} d_{e_{m-n+1}}\widehat{T}) \\ &= \operatorname{Im} d_{e_1}(\pi \circ \widehat{A}) + \operatorname{Im} d_{e_{n-m+1}}(\pi \circ \widehat{T}) \\ &= \operatorname{Im} d_{e_1}(\overline{A} \circ \pi_m) + \operatorname{Im} d_{e_{n-m+1}}(\overline{T} \circ \pi_{n-m}) \\ &= \operatorname{Im}(d_{[e_1]}\overline{A}) + \operatorname{Im}(d_{[e_{n-m+1}]}\overline{T}) \\ &= \operatorname{Im}(d_{[e_1]}\overline{A}) + T_{[Ae_1]}X \end{aligned}$$

This verifies that indeed $\overline{A} \pitchfork_{[Ae_1]} X$, now to complete the proof, note that we have some $[p] \in \mathbb{RP}^n \setminus X$, since $\mathbb{RP}^n$ is connected (therefore path connected), any constant map $\mathbb{RP}^m \to \mathbb{RP}^n$ is homotopic to the map $c : \mathbb{RP}^m \to [p]$, where $I_2(c, X) = 0$ trivially, since intersection number is a homotopy invarient this completes the proof.

(Proof of $\mathbf{d_{e_1}\widehat{A} + d_{e_{m-n+1}}\widehat{T} = T_{A(e_1)}S^n}$): To show this, we will compute the derivatives as maps of $\mathbb{R}^k \setminus \{0\} \to \mathbb{R}^{n+1}$, then use the characterization of the tangent space $T_pS^k = p^\perp \cap T_p\mathbb{R}^{k+1}$. To compute the derivative note that the maps are of the form $\widehat{A} = F \circ A$ and $\widehat{T} = F \circ T$, where we computed the derivative of F prior to tackling the problem, by the chain rule we have

$$\begin{aligned} d_{e_1}\widehat{A} &= \left(\frac{1}{||Ae_1||}1 - \frac{1}{||Ae_1||^3}(Ae_1)\cdot(Ae_1)^T\right) d_{e_1}A \\ d_{e_{n-m+1}}\widehat{T} &= \left(\frac{1}{||Ae_1||}1 - \frac{1}{||Ae_1||^3}(Ae_1)\cdot(Ae_1)^T\right) d_{e_{n+m-1}}T \end{aligned}$$

restricting to the orthogonal compliment of Ae_1, $\frac{1}{||Ae_1||^3}(Ae_1) \cdot (Ae_1)^T \equiv 0$, so that

$$d_{e_1}\widehat{A} \equiv \frac{1}{||Ae_1||} d_{e_1} A \text{ and } d_{e_{n-m+1}}\widehat{T} \equiv \frac{1}{||Ae_1||} d_{e_{n-m+1}} T$$

the derivative should also have restricted domain since these are maps of spheres, restricting the domain of $d_{e_1}\widehat{A}$ to $e_1^\perp$ and $d_{e_{n-m+1}}\widehat{T}$ to $e_{n-m+1}^\perp$ and taking $\rho : \mathbb{R}^{n+1} \to (Ae_1)^\perp$ to be the orthogonal projection we find the images of either differential have respective bases

$$\{\rho(Ae_2), \dots, \rho(Ae_{m+1})\} \text{ and } \{\rho(Te_1), \dots, \rho(Te_{n-m})\}$$

By definition of T, and injectivity of both A and T (which have Ae_1 in their image), this collection of n vectors forms a basis for $T_{Ae_1}S^n$, this is easiest to see by writing it as

$$\rho(\langle Ae_2, \dots, Ae_{m+1}, f_1, \dots, f_{n-m}\rangle)$$

where ρ has no kernel on this subspace, and this space has dimension n by definition of the f_i and injectivity of A. $\square$