1. If compactly supported cohomology were homotopy invariant, then we would require $H_c^*(\mathbb{R}^n) \cong H_c^*(\{\text{pt.}\})$ for all $n \in \mathbb{Z}_{\geq 0}$ since $\mathbb{R}^n \simeq \{\text{pt.}\}$ for all n. To further explicate this, Consider for each n, the homotopy

$$\begin{aligned} H : \mathbb{R}^n \times [0,1] &\to \mathbb{R}^n \\ (x,t) &\mapsto x(1-t) \end{aligned}$$

interpolates the maps $1_{\mathbb{R}^n}$ and the zero map. Then we could take $g : \{0\} \hookrightarrow \mathbb{R}^n$, and $f : \mathbb{R}^n \to \{0\}$, so that $gf = 1_{\{0\}}$, and fg is the zero map which we already showed is homotopy equivalent to $1_{\mathbb{R}^n}$.

Thus if compactly supported cohomology were a homotopy invariant we would have

$$H_c^*(\mathbb{R}^n) \cong H_c^*(\{0\})$$

choosing $n = 1 = *$, we get $\mathbb{R} \cong 0$ (as $\mathbb{R}$-vector spaces) by the poincare lemma. This is a contradiction. $\square$

2. Let η_U, η_V be a partition of unity subordinate to U, V, then we can define the following maps making the sequence short exact:

$$0$$

The first map has $\text{supp}(\eta_V \cdot \omega) \subset \text{supp}(\omega) \supset \text{supp}(\eta_U \cdot \omega)$, so there are no issues with the compact support, similarly for the second map $\text{supp}(\eta_U \cdot \omega + \eta_V \cdot \nu) \subset \text{supp}(\omega) \cup \text{supp}(\nu)$ which is a union of two compact sets hence compact.

To see the first map is an injection let $\omega \in \Omega_c^p(U \cap V)$ (for some p) and suppose that $(\eta_V \cdot \omega, \eta_U \cdot \omega) \equiv 0$, then $\eta_V \cdot \omega \equiv 0$ on U and $\eta_U \cdot \omega \equiv 0$ on V, this of course implies $(\eta_V \cdot \omega)|_{U \cap V} \equiv 0$ and $(\eta_U \cdot \omega)|_{U \cap V} \equiv 0$ on $U \cap V$, so the following easy computation shows injectivity,

$$(\eta_V \cdot \omega)|_{U\cap V} + (\eta_U \cdot \omega)|_{U\cap V} = \eta_V|_{U\cap V} \cdot \omega + \eta_U|_{U\cap V} \cdot \omega = (\eta_U|_{U\cap V} + \eta_V|_{U\cap V}) \cdot \omega = \omega$$

Now checking surjectivity of the second map, Let $\omega \in \Omega_c^p(M)$ for some p, then we have $\omega|_U \in \Omega_c^p(U)$ and $-\omega|_V \in \Omega_c^p(V)$, then I claim that the image of $(\omega|_U, -\omega|_V) = \eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega$. To check this, it suffices to check equivalence pointwise, so we can simply check on each of the sets $U \cap V^c$, $V \cap U^c$ and $U \cap V$, to see it on $U \cap V^c$ we have $\eta_V = 0$, and $\eta_U = 1$ so that $\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = \omega|_U$ on this set, but since $U \cap V^c \subset U$, this is the same thing as ω here. Checking on $V \cap U^c$ is similar, finally on $U \cap V$, we have

$$\eta_U \cdot \omega|_U + \eta_V \cdot \omega|_V = (\eta_U + \eta_V)\omega|_{U\cap V} = \omega|_{U\cap V}$$

which is of course just ω on $U \cap V$, this shows surjectivity.

Finally, we need to check that $\ker((\omega, \nu) \mapsto \eta_U \cdot \omega - \eta_V \cdot \omega) = \text{Im}(\omega \mapsto (\eta_V \cdot \omega, \eta_U \cdot \omega))$, checking the image is a subset of the kernel, is straightforward since composing both maps we get

$$\omega \mapsto \eta_U \cdot \eta_V(\omega - \omega) = 0$$

Now to check that all elements of the kernel are of this form, suppose $(\omega, \nu) \mapsto 0$, then wherever $\eta_V = 0$, we have $\eta_U \cdot \omega - \eta_V \cdot \nu = \omega$, but since we are assuming this is zero we must have $\operatorname{supp} \omega \subset \operatorname{supp} \eta_V$, the same argument shows that $\operatorname{supp} \nu \subset \operatorname{supp} \eta_U$. Now we define the following form α on $U \cap V$

$$\alpha = \begin{cases} \eta_V^{-1} \cdot \omega & \eta_U, \eta_V > 0 \\ \omega & \eta_U = 0 \\ \nu & \eta_V = 0 \end{cases}$$

Then $\operatorname{supp} \alpha \subset \operatorname{supp} \omega \cup \operatorname{supp} \eta$ is compact, to verify that α is indeed smooth,