1. (Durrett 1.1.5) $\mathcal{A}$ is not an algebra, hence not a σ-algebra, as proof let A be the even numbers, and B be as defined below

$$B = \bigcup_{n \text{ even}} \{k \mid k \text{ odd and } 2^n \leq k < 2^{n+1}\} \bigcup_{n \text{ odd}} \{k \mid k \text{ even and } 2^n \leq k < 2^{n+1}\}$$

Then it is clear $\theta(A) = \theta(B) = \frac{1}{2}$. Now we want to consider $\theta(A \cup B)$, note that $A \cup B$ contains $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ when n is even, but contains only $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1, 2^{n+1}\} \cap \{\text{even numbers}\}$ for odd n. Then denote $\theta_n = \frac{\#((A \cup B) \cap \{1, \ldots, 2^{n+1}\})}{2^{n+1}}$, then the first few terms are $\theta_1 = 1, \theta_2 = \frac{3}{4}, \theta_3 = \frac{7}{8}, \theta_4 = \frac{11}{16}$ and from the definition of A, B we get $\theta_{2n+1} = \frac{\theta_{2n}}{2} + \frac{1}{2}$ and $\theta_{2n+2} = \frac{\theta_{2n+1}}{4} + \frac{1}{4}$, it follows that by induction the subsequences θ_{2n} and θ_{2n+1} are decreasing, then once again by induction using this recurrence we find that $\frac{11}{16} \geq \theta_{2n} \geq \frac{1}{2}$ and $1 \geq \theta_{2n+1} \geq \frac{3}{4}$, but then $\liminf \theta_{2n+1} \geq \frac{3}{4} > \frac{11}{16} \geq \limsup \theta_n$, so these subsequences of $\frac{\#((A \cup B) \cap \{1, \ldots, n\})}{n}$ can't possibly converge to the same limit, and hence a limit for the sequence cannot exist and $A \cup B$ does not have an asymptotic density. $\square$

2. (Durrett 1.2.3) First note that the left limit of a distribution function is well defined,

$$F(x-) := \lim_{y_n \uparrow x} F(x) = \bigcup_{1}^{\infty} P(X \leq y_n) = P(X < x)$$

The last equality following from throwing out y_n such that for some $k < n$, there is $y_k > y_n$ and applying continuity from below.

It follows that for each point of discontinuity of F, we must have $F(x) > F(x-)$, assuming there are uncountably many points of discontinuity for F and denote that set of points as A, we know that since $0 \leq F(x) \leq 1$ is an increasing function that

$$1 = \lim_{x \to \infty} F(x) \geq \sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\}$$

Denote $E_n = \{\alpha \in A \mid F(\alpha) - F(\alpha-) \geq \frac{1}{n}\}$, then since $\bigcup_1^\infty E_n = A$, we must have atleast one E_n is uncountable. This implies that

$$\sup\{\sum_{\alpha \in S} F(\alpha) - F(\alpha-) \mid A \supset S \text{ is finite}\} \geq \sup_{M \in \mathbb{N}} \frac{M}{n} = \infty$$

Which is a contradiction. $\square$

3. (Durrett 1.3.5) If f is not LSC, then there is some x and $y_n \to x$, such that $\lim f(y_n) < f(x)$ (this follows from the negation since we can take a subsequence which gives the liminf). But then let $\epsilon = f(x) - \lim f(y_n)$, if we remove the y_n terms such that $f(y_n) > f(x) + \frac{\epsilon}{2}$ from the sequence then the sequence still converges to x, so we may assume the sequence is uniformly bounded by $f(x) + \frac{\epsilon}{2}$. But then we have a sequence $y_n \in \{t \mid f(t) \leq f(x) - \epsilon/2\}$ which converges to a value x not in the set, so in particular the set is not closed.

Conversely, if for some a, the set $S_a := \{x \mid f(x) \leq a\}$ is not closed, then we get a sequence $y_n \in S_a$ such that $y_n \to x$, but $x \notin S_a$, it follows that $f(x) > a$, but $\liminf_{y \to x} f(y) \leq \lim f(y_n) \leq a < f(x)$ so that f is not LEC. $\square$

4. (Durrett 1.3.7) First we note that all simple functions are measurable, and measurable functions are closed under pointwise limits, closure under pointwise limits follows from $\limsup$ being measurable, and $\lim f_n(x) = \limsup f_n(x)$ at all points x when the limit exists. Now let f be an arbitrary measurable function on $(\Omega, \mathcal{F})$ mapping to $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Then we can write $f = f_+ - f_-$ so it will suffice to show that an arbitrary positive function f is a pointwise limit of simple functions. Let $A_1 = f^{-1}[1, \infty)$, and $\phi_1 = 1_{A_1}$, now we can define the rest of the ϕ_i recursively:

$$A_n = (f - \phi_{n-1})^{-1}[\frac{1}{n}, \infty) \qquad \phi_n = n^{-1} 1_{A_n}$$

Now pointwise the sequences $\phi_n(x)$ are bound above by $f(x)$ and monotone increasing hence convergent, we want to see it converges to $f(x)$, since the harmonic series diverges, for any x, we have some N such that $\sum_1^N \frac{1}{n} > f(x)$, furthermore by construction of ϕ_n we will have $f(x) - \frac{1}{N} \leq \phi_n(x) \leq f(x)$, and moreover for any $k > N$ we also have that $f(x) - \frac{1}{k} \leq \phi_k(x) \leq f(x)$ whence convergence follows immediately. $\square$

5. (Durrett 1.3.8) If $Y = f \circ X$ for $f : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then for any borel set B, we have $Y^{-1}(B) = X^{-1}(f^{-1}(B))$, since f is measurable we know that $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$, so that $X^{-1}(f^{-1}(B)) \in \sigma(X)$ by definition of $\sigma(X)$, which suffices to show all functionsof this form are measurable with respect to $\sigma(X)$. To show that all measurable functions on $\sigma(X)$ are of this form, we can use (Durrett 1.3.7) to check that all pointwise limits of simple functions on $\sigma(X)$ can be written in this form. Consider the simple functions $\phi_k = \sum_1^{N_k} c_i^k 1_{X^{-1}(B_i^k)}$, with $\phi_k \to g : (X, \sigma(X)) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ pointwise. It is immediate that $\phi_k = \varphi_k \circ X$ where $\varphi_k = \sum_1^{N_k} c_i^k 1_{B_i^k}$. It is straightforward to see that φ_k converges pointwise on $\mathbb{R}$, since if $x \in \mathbb{R}$, then $x = X(y)$ for $y \in \Omega$, then sequence $\phi_k \circ X(y)$ is equal to the sequence $\varphi_k(x)$, and hence $\lim_{k\to\infty} \varphi_k(x) = \lim_{k\to\infty} \phi_k \circ X(y) = g(y)$, denoting the pointwise limit of φ_k as f we know that f is measurable by closure of measurable functions under pointwise limits, and moreover, $f \circ X = g$ from construction. $\square$