MAT1120 - Lecture 7. Jan 14, 2026

Hall Theorem 4.28, G a compact matrix Lie Group. then every finite dim rep. of G is completely reducible.

Pf. Use H. Weyl's "Unitary trick".

- G a matrix Lie group in $M_n(\mathbb{C})$.
  $\mathfrak{g} := T_e G$. Then for $A \in G$, $T_A G = \{XA \mid X \in \mathfrak{g}\}$

$\dim(\mathfrak{g}) = k$ then take non-zero k-linear alternating form $\alpha_A = (T_A^* G)^n$ where α is defined at e in an arbitrary way, then use right action of A to transport α to every point in G. This gives an orientation for G then $(Y_1, \dots, Y_n)$ basis for $T_K G$ is positively oriented when $\alpha_K(Y_1, \dots, Y_n) > 0$.

- If $f: G \to \mathbb{R}$ a smooth function,
  $\int_G f\alpha = \int_{A \in G} f(A)\alpha(A)$

Letting now V be a fin dim rep of G. And take $\langle -,- \rangle$ on V, then

$\langle v,w \rangle_G = \langle \rho(A)v, \rho(A)w \rangle \alpha(A)$, this is clearly positive. And for $B \in G$,

$$\langle \rho(B)v, \rho(B)w \rangle_G = \int_{A\in G} \langle \rho(A)\rho(B)v, \rho(A)\rho(B)w \rangle \alpha(A)$$

Need compactness to integrate.

$$= \int_G \langle \rho(AB)v, \rho(AB)w \rangle \alpha(A)$$

$= \int \langle v, w \rangle_G$. Then, $\rho(B)$ is unitary wrt. $\langle -,- \rangle_G$.

By prop. 4.2.7 ρ is completely reducible.

~ This is Just Maschke's Theorem.

MAT1120 - Lecture 8

Jan 21, 2026

Continue $SL(2,\mathbb{C})$ rep. Theory.

Recall: Basis for $sl(2,\mathbb{C}) = X, Y, H$

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and by computation: $[H,X] = 2X$, $[H,Y] = -2Y$,
$[X,Y] = H$.

__Theorem.__ (Hall, 4.3.2) All irreps of same dimension are isomorphic. ($SL(2,\mathbb{C})$ reps.)

- Irreps $SL(2,\mathbb{C}) \curvearrowright \{\mathbb{C}[z] \otimes \mathbb{C}[w] \mid \dim p = k\}$
this action is determined by $SL(2,\mathbb{C})$ acting on deg 1 polynomials. $A(z,w) = A \cdot \begin{pmatrix} z \\ w \end{pmatrix}$.

- Recall - ρ a rep of $sl(2,\mathbb{C})$ $\rho(H)$ has eigenvector u and eigenvalue α, then
$\rho(H)\rho(X)u = (\alpha+2)\rho(X)$.

Then $\rho(X)u = 0$, or $\rho(X)u$ is an eigenvector w/ eigenvalue $\alpha+2$ for H.
Similarly, $\rho(H)\rho(Y)u = (\alpha-2)\rho(Y)u$.

Pf (Hall 4.3.2). ρ an irrep of $sl(2, \mathbb{C}) \underset{\rho}{\curvearrowright} V$, a finite dim cpx. vectorspace.

Then $\rho(H)$ has an eigenvalue since $\mathbb{C}$ is alg. closed.

Letting u be an eigenvector for $\rho(H)$, with eigenvalue α, then

$\rho(H)\rho(X)u = (\alpha+2)\rho(X)$.

$$\rho(H)\rho(X)^k u = (\alpha+2k)\rho(X)^k u$$

since $\rho(H)$ has finitely many eigenvalues, this tells us not all $\rho(X)^k$ can be nonzero.

let N be the smallest w/ $\rho(X)^{N+1} = 0$.

$u_0 = \rho(X)^N u$, $\lambda = \alpha + 2N$

$\Rightarrow \rho(H)u_0 = \lambda u_0$ & $\rho(X)u_0 = 0$.

Then $u_k = \rho(Y)^k u_0$

$\rho(H)u_k = (\lambda - 2k)u_k$.

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By induction on k,
$$\rho(X)u_k = k(\lambda-k+1)u_{k-1}$$
$$\rho(Y)u_k = k(\lambda-k-1)u_{k+1}$$
$\rho(H)$ has finitely many eigenvalues, so only finitely many $u_k \neq 0$.

There is some $m \geq 0$, so $u_k = \pi(Y)^k u_0 \neq 0$ $\forall k \leq m$

& $u_{m+1} = 0$.

$$u_{m+1} = 0 \Rightarrow \rho(X)u_{m+1} = 0$$
$$\Rightarrow (m+1)(\lambda-m)u_m = 0$$
$$\Rightarrow \lambda = m.$$

So for all irreps, we have $u_0, \dots, u_m$ w/

$$\rho(H)u_k = (m-2k)u_k$$
$$\rho(Y)u_k = \begin{cases} u_{k+1} & k < m \\ 0 & k \geq m \end{cases}$$
$$\rho(X)u_k = \begin{cases} m(m-k-1)u_{k-1} & k > 0 \\ 0 & k = 0. \end{cases}$$

- These u_k are lin. independent since they are eigenvectors of $\rho(H)$ w/ distinct eigenvalues.

$\Rightarrow$ span$\{u_0, \ldots, u_m\}$ is an invariant subspace of $sl(2,\mathbb{C})$ by irreducibility $V = \text{span}\{u_0, \ldots, u_m\}$ with the action on each u_k described explicitly.

Lie Theory - Lecture 9

Jan 23, 2026

Theorem (Hall 4.34) (π, V) a finite dim. rep of $sl(2, \mathbb{C})$ (not necessarily irred.)

Recall:
$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

a) eigenvalues of $\pi(H)$ are in $\mathbb{Z}$

b) $\text{Eigenspace}(\pi(H)) \subset \{v \mid \pi(X)^n v = 0 \text{ for some } n\}$.

Pf. $\exists N$, $\pi(X)^N v \neq 0$, $\pi(X)^{N+1} v = 0$.

so $\pi(X)^N v$ is an eigenvector with eigenvalue $\lambda + 2N$ a non-negative integer, where $\lambda \in \mathbb{Z}$.

↳ can have $N = 0$.

↪ This seems a bit circular, but remember that we showed irreps have integer eigenvalues last lecture, so all reps must have integer eigenvalues, by decomposing action on $V = V_1 \oplus \cdots \oplus V_n$.

2) Recall any matrix can be written as $A = S + N$, for S diagonalizable & N nilpotent.

moreover, $SN = NS$.

- Then $\pi(H)$ has a basis of generalized eigenvectors.

$$(\pi(H) - (\lambda+2))^k \pi(X) = \pi(X)(\pi(H) - \lambda)^k$$

by induction on k, since $[H, X] = 2X$.

- So if v is a generalized eigenvector for $\pi(H)$ with eigenvalue λ, then $\pi(X) = 0$, or $\pi(X)v$ is a generalized eigenvector with eigenvalue $\lambda+2$.
- By $\pi(H)$ having finitely many eigenvectors (on eigenspace), $\pi(X)$ is nilpotent. Same argument shows $\pi(Y)$ nilpotent. (on eigenspace).

- $S: V \longrightarrow V$ as $e^{\pi(X)} e^{-\pi(Y)} e^{-\pi(X)}$, then $S\pi(H) = \pi(H)S$.

Pf. $$S\pi(H)S^{-1} = e^{\pi(X)} e^{-\pi(Y)} \underbrace{e^{-\pi(X)} \pi(H) e^{\pi(X)}}_{Ad_{e^{-\pi(X)}} \pi(H)} e^{-\pi(Y)} e^{-\pi(X)}$$

- $$Ad_{\exp g} v = \exp(ad_g) v = v + ad_g v + \frac{ad_g^2}{2} v + \dots$$

$$\Rightarrow e^{-\pi(X)} \pi(H) e^{\pi(X)} = \exp(-ad_{\pi(X)}) \pi(H)$$

$$\exp(\mathrm{ad}\,\pi(X))\,\pi(H) = \pi(H) + [\pi(X), \pi(H)]$$

Then:

$$= \pi(H) - 2\pi(X)$$

$$e^{-\mathrm{ad}\,\pi(Y)}\,(\pi(H) - 2\pi(X))$$

$$= \pi(H) - 2\pi(X) - 2\pi(Y) - 2\pi(H) + \frac{1}{4}\pi(Y)$$

$$= -\pi(H) - 2\pi(X)$$

Then finally: $e^{\mathrm{ad}\,\pi(X)}\,(-\pi(H) - 2\pi(X)$

$$= -\pi(H) - 2\pi(X) + 2\pi(X)$$

$$= -\pi(H).$$