1.

$$\Im(\sigma) = \Im\left(\frac{(a\tau + b)(c\overline{\tau} + d)}{||c\tau + d||^2}\right) = \Im\left(\frac{ac||\tau||^2 + bd + ad\tau + bc\overline{\tau}}{||c\tau + d||^2}\right)$$

$$= \frac{1}{||c\tau + d||^2}\Im(ad\tau + bc\overline{\tau}) = \frac{1}{||c\tau + d||^2}(ad - bc)\Im(\tau) = \frac{1}{||c\tau + d||^2}\Im(\tau)$$

Then since $||c\tau + d||^2 > 0$ and $\Im(\tau) > 0$ we conclude that $\Im(\sigma) > 0$.

Now define

$$f : \mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \sigma\mathbb{Z})$$
$$z \mapsto \frac{z}{c\tau + d} + \mathbb{Z} + \sigma\mathbb{Z}$$

To see this descends to a holomorphic map on the torus, we need to check it is periodic with respect to $\mathbb{Z} + \tau\mathbb{Z}$, so we want to check that $f(z + n\tau + m) = f(z)$ for any $n, m \in \mathbb{Z}$. Since f is linear it will be sufficient to show that $f(\tau)$ and $f(1)$ both lie in the lattice $\mathbb{Z} + \sigma\mathbb{Z}$. This is just a computation,

$$\tau = (ad - bc)\tau + bd - bd = d(a\tau + b) - b(c\tau + d) \tag{1}$$
$$1 = (ad - bc) + ac\tau - ac\tau = -c(a\tau + b) + a(c\tau + d) \tag{2}$$

So that (1) gives us $f(\tau) = d\sigma - b$ and (2) gives $f(1) = -c\sigma + a$ both are in $\mathbb{Z} + \sigma\mathbb{Z}$, so that f descends to the torus X_τ. To see that f is a biholomorphism just take $\mathbb{C} \to X_\tau$ via $z \mapsto (c\tau + d)z$, this descends to a holomorphic map on X_σ since $\sigma \mapsto \tau$, and $1 \mapsto c\tau + d$ are both in $\mathbb{Z} + \tau\mathbb{Z}$, moreover this is clearly the inverse of f. $\square$