

1. (Folland 1.3.6) That $\bar{\mu}$ is a measure is clear, since $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$, $\text{Im } \bar{\mu} = \text{Im } \mu$ (this shows it is positive) and if $\{A_i\}_1^\infty$ are disjoint sets in $\bar{\mathcal{M}}$, then each A_i can be written as $E_i \cup F_i$ (for F_i) contained in null sets N_i so that $\bar{\mu}(\bigcup_1^\infty A_i) = \bar{\mu}(\bigcup_1^\infty E_i \cup \bigcup_1^\infty F_i)$, then $\bigcup_1^\infty F_i \subset \bigcup_1^\infty N_i$ is a null set, so

$$\bar{\mu}(\bigcup_1^\infty E_i \cup \bigcup_1^\infty F_i) \stackrel{\text{defn.}}{=} \mu(\bigcup_1^\infty E_i) = \sum_1^\infty \mu(E_i) = \sum_1^\infty \bar{\mu}(E_i \cup F_i)$$

Suppose $N \in \bar{\mathcal{M}}$ with $\bar{\mu}(N) = 0$ and $F \subset N$, then $N = N_1 \cup N_2$ with $N_1 \in \mathcal{M}$, and $N_2 \subset N_3 \in \mathcal{M}$, so that $N \subset N_1 \cup N_3$ which is a μ -measurable set, hence F is contained in the null set $N_1 \cup N_3$ and is μ -measurable. Finally to see uniqueness of $\bar{\mu}$, suppose μ' is another extension of μ to $\bar{\mathcal{M}}$, then for some $E \cup F \in \bar{\mathcal{M}}$ we have $\mu(E) = \bar{\mu}(E \cup F) \neq \mu'(E \cup F)$, hence $\mu'(F) > 0$, but then $F \subset N \in \mathcal{M}$ where N is μ null, so that $0 < \mu'(F) \leq \mu(N) = 0$. \square

2. (Folland 1.3.7) Positivity follows from each μ_j and a_j positive, suppose $\{E_i\}_1^\infty$ are disjoint, if any of the $\mu_j(\bigcup_1^\infty E_i) = \infty$ then $\sum_{i=1}^\infty \sum_{j=1}^n a_j \mu_j(E_i) \geq a_j \sum_{i=1}^\infty \mu(E_i) = a_j \mu_j(\bigcup_1^\infty E_i) = \infty$ and additivity is trivial, otherwise we can interchange sums since they converge in absolute value

$$\begin{aligned} \sum_1^n a_j \mu_j(\emptyset) &= \sum_1^n 0 = 0 \\ \sum_1^n a_j \mu_j(\bigcup_1^\infty E_i) &= \sum_{j=1}^n a_j \sum_{i=1}^\infty \mu_j(E_i) = \sum_{i=1}^\infty \sum_{j=1}^n a_j \mu_j(E_i) \end{aligned}$$

\square

3. (Folland 1.3.8) for any N , we have $\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j \subset E_k$ for all $k \geq N$, hence $\mu(\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j) \leq \liminf \mu(E_k)$. By continuity from below we have $\lim_{N \rightarrow \infty} \mu(\bigcup_{n=1}^N \bigcap_{j=n}^\infty E_j) = \mu(\liminf E_k)$, but the limit is bounded above by $\liminf \mu(E_k)$, so that $\mu(\liminf E_k) \leq \liminf \mu(E_k)$.

For all N , we have $\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j \supset E_k$ for $k \geq n$, hence $\mu(\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j) \geq \limsup \mu(E_k)$, since $\mu(\bigcup_1^\infty E_i) < \infty$ we can invoke continuity from above to conclude

$$\mu(\limsup(E_k)) = \lim_{n \rightarrow \infty} \mu(\bigcap_{n=1}^N \bigcup_{j=n}^\infty E_j) \geq \limsup \mu(E_k)$$

\square

4. (Folland 1.3.9) We can decompose the sets of interest as follows:

$$E = (E \setminus F) \sqcup (E \cap F), \quad F = (F \setminus E) \sqcup (F \cap E), \quad E \cup F = F \cap E \sqcup (E \setminus F) \sqcup (F \setminus E)$$

The result follows from additivity on disjoint sets,

$$\mu(E) + \mu(F) = \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E) + \mu(F \cap E) = \mu(E \cup F) + \mu(E \cap F)$$

\square

5. (Folland 1.3.10) That μ_E is nonnegative follows from μ nonnegative. $= \cap E$ so $\mu_E(\emptyset) = 0$. Finally if $\{A_i\}_1^\infty$ are disjoint sets, then so are $\{A_i \cap E\}_1^\infty$, hence

$$\mu_E(\bigcup_1^\infty A_i) = \mu(E \cap \bigcup_1^\infty A_i) = \mu(\bigcup_1^\infty E \cap A_i) = \sum_1^\infty \mu(E \cap A_i)$$

\square

6. (Folland 1.3.11) Suppose $\{E_i\}_1^\infty$ are disjoint sets, then let $F_n = \bigcup_1^n E_i$, it follows that

$$\mu\left(\bigcup_1^\infty E_i\right) = \mu\left(\bigcup_1^\infty F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_1^n \mu(E_i)$$

In the second case, let $K_n = \bigcap_1^n E_n^c$, it follows that $\mu K_1 \leq \mu X$ so we can use continuity from above.

$$\begin{aligned} \mu\left(\bigcup_1^\infty E_i\right) &= \mu(X) - \mu\left(\bigcap_1^\infty E_i^c\right) = \mu(X) - \mu\left(\bigcap_1^\infty K_n\right) = \mu(X) - \lim_{n \rightarrow \infty} \mu(K_n) = \mu(X) - \lim_{n \rightarrow \infty} \mu\left(\left(\bigcup_1^n E_n\right)^c\right) \\ &= \mu(X) - \left(\lim_{n \rightarrow \infty} \mu(X) - \sum_1^n \mu(E_i)\right) = \lim_{n \rightarrow \infty} \sum_1^n \mu(E_i) \end{aligned}$$

□

7. (Folland 1.3.12)

(a) $E \Delta F = (E \setminus F) \sqcup (F \setminus E)$, hence $\mu(E \setminus F) = \mu(F \setminus E) = 0$. It follows that

$$\mu(F) \leq \mu(E) + \mu(F \setminus E) = \mu(E) \text{ and } \mu(E) \leq \mu(F) + \mu(E \setminus F) = \mu(F)$$

□

(b) reflexivity follows from $\mu(E \Delta E) = \mu(\emptyset) = 0$, symmetry follows from $E \Delta F = F \Delta E$, finally transitivity follows from the observation $H \setminus F \subset (H \setminus E) \cup (E \setminus F)$, hence $\mu(H \Delta E) = \mu(E \Delta F) = 0$ implies $\mu(H \setminus F) \leq \mu(H \setminus E) + \mu(E \setminus F) = 0$ and $\mu(F \setminus H) \leq \mu(F \setminus E) + \mu(E \setminus H) = 0$ which gives us that $\mu(H \Delta F) = \mu(H \setminus F) + \mu(F \setminus H) = 0$, proving transitivity. □

(c) $\rho(E, F) = 0 \iff E \sim F$, and ρ is nonnegative, symmetry follows from symmetry of Δ , so ρ will define a metric if it satisfies the triangle inequality. But as in the previous question $H \setminus F \subset (H \setminus E) \cup (E \setminus F)$, applying this inequality the other way this implies that $\mu(H \Delta F) \leq \mu(H \Delta E) + \mu(E \Delta F)$, this proves the triangle inequality for ρ . □

8. (Folland 1.3.13) Suppose that μ is not semifinite, then there is some $E \in \mathcal{M}$, such that for all $F \subset E$ we have $\mu(F) = \infty$. Suppose $X = \bigcup_1^\infty E_i$, then $E_i \cap E \neq \emptyset$ for some i , then $\infty = \mu(E_i \cap E) \leq \mu(E_i)$, so that X cannot be a countable union of sets having finite measure. □

9. (Folland 1.3.14) Let $C = \sup\{\mu(F) \mid F \subset E \text{ and } \mu(F) < \infty\}$ and suppose for contradiction that $C < \infty$, then let F_n be a sequence such that $\lim_{n \rightarrow \infty} \mu(F_n) = C$, it follows that $\mu(\bigcup_1^n F_j) \geq \mu(F_n)$, hence $\lim_{n \rightarrow \infty} \mu(\bigcup_1^n F_j) = C$, using continuity from below we see that in fact $\mu(\bigcup_1^\infty F_n) = C$. Then $\mu(E \setminus \bigcup_1^\infty F_n) = \infty$, so $E \setminus \bigcup_1^\infty F_n$ has some subset A with $0 < \mu(A) < \infty$, but then

$$C \geq \mu\left(A \bigcup_1^\infty F_n\right) = \mu\left(\bigcup_1^\infty F_n\right) + \mu(A) > \mu\left(\bigcup_1^\infty F_n\right) = C$$

□

10. $\mu_0 \geq 0$ and $\mu_0(\emptyset) = 0$ are obvious,