

# Concise AG Notes - UofT MAT1190

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## 1 Lecture Notes

### 1.1 Lecture 1 (Sept 3, 2025)

**Theorem 1. (Gelfond-Neymark)** A compact topological space is determined by its ring of smooth functions. In particular if the ring  $C(X) := C(X, \mathbb{R})$  and  $C(X) \cong C(Y)$ , then  $X \cong Y$ .

**Proposition 1.** Each point in  $X$  corresponds to a maximal ideal of  $C(X)$ , moreover if  $X$  is compact, then the correspondence is 1-1.

*Proof.* the evaluation at a point gives a surjective homomorphism  $C(X) \rightarrow \mathbb{R}$ , the image is a field, hence the kernel is a maximal ideal corresponding to the point.

Now in the compact case, (assume  $X$  is Hausdorff?), then  $X$  is Hausdorff and compact hence normal. We can use Urysohn's lemma to get a function vanishing at  $x$  but not  $y$ . Now suppose that for some maximal ideal  $\mathfrak{m} \subset C(X)$  for any point  $p \in X$  there is a continuous function with  $f(p) \neq 0$ , then the set  $U_f = \{x \in X \mid f(x) \neq 0\}$  is open, and  $\bigcup_{f \in C(X)} U_f = X$ , so we get a finite subcover. Take a linear combination of the functions in this subcover to complete the proof.  $\square$

**definition 1.** The Zariski Topology on  $\text{Spec}_{\max}(R)$  is the coarsest topology such that when  $\mathfrak{m} \leftrightarrow x \mapsto f(x)$  is continuous, where the topology on  $\mathbb{R}$  is taken as the cofinite topology. The closed sets in this topology are the vanishing loci of  $f \in C(X)$ .

- Exercise 1, complete Hartshorne exercise 1.4

### 1.2 Lecture 2 (Sept 5, 2025)

**definition 2.** For  $T \subset R_n := k[x_1, \dots, x_n]$  and  $S \subset k^n$  we define

$$V(T) = \{x \in k^n \mid f(x) = 0, \forall f \in T\} \text{ and } I(S) = \{f \in R_n \mid f(x) = 0, \forall x \in S\}$$

**Proposition 2.** Suppose  $k$  is an uncountable field, and  $L/k$  is an extension with  $[L : k] \leq \aleph_0$ , then  $L = k$ .

*Proof.* Suppose not, then let  $x \in L \setminus k$ , we find that  $\{\frac{1}{x-\lambda} \mid \lambda \in k\}$  is uncountable, so that there must be an algebraic relation. Thus there exist  $\mu_i \in k$  with  $\sum_1^n \frac{\mu_i}{x-\lambda_i} = 0$ , so that  $\sum_1^n \mu_j \prod_{i \neq j} (x-\lambda_i) = 0$ , but then  $x$  is algebraic over  $k$ , hence  $x \in k$ , contradiction.  $\square$

**Theorem 2. (Nullstellensatz - weak form)**  $V(T) = \emptyset \implies (T) = R_n$

*Proof.* We assume here that  $k$  is uncountable (this is unnecessary- use Noether Normalization). Since  $J := (T) \subset R_n$  is an ideal it is contained in a maximal ideal  $\mathfrak{m}$ . Then  $R_n/\mathfrak{m}$  is a field extension of  $k$  with countable dimension, by the previous proposition it is equal to  $k$ . It follows that each  $x_i \mapsto a_i \in k$  when taking the quotient  $R_n \rightarrow R_n/\mathfrak{m} = k$ , it follows that  $I$  vanishes on  $(a_1, \dots, a_n)$ , so  $I$  cannot be contained in a maximal ideal.  $\square$

**Theorem 3. (Nullstellensatz)**

$$IV(J) = \sqrt{J}$$

*Proof.* By Hilbert's basis theorem, we reduce to the finitely generated case. Let  $f \in IV(\{f_1, \dots, f_r\})$ , then  $(1 - tf, f_1, \dots, f_r) \subset R_n[t]$  has no common zero. Then  $g_0(1 - tf) + g_1f_1 + \dots + g_rf_r = 1$ , and let  $N = \max_i \{\deg_t g_i\}$ . Taking  $t = f^{-1}$ , we get  $\sum_1^r g_i f_i = 1$ , so that for  $h_i = f^N g_i \in R_n$  we get  $\sum_1^r h_i f_i = f^N \in I \implies f \in \sqrt{I}$ .  $\square$

The Nullstellensatz gives a bijection

$$\{\text{Affine algebraic varieties}\} \longleftrightarrow \{\text{Finitely generated reduced } k\text{-algebras}\}$$

$$V(\sqrt{I}) \longleftrightarrow R_n / \sqrt{I}$$

Moreover, this is a categorical equivalence

$$\text{Var}_k \cong \left( \text{Alg}_k^{\text{reduced}} \right)^{\text{op}}$$

### 1.3 Lecture 3 (Sept 8, 2025)

**definition 3.** Let  $\pi : S \rightarrow X$  be a local homeomorphism, then  $S$  is called an étalé space, or a sheaf on  $X$ .

**Example(s).** 1.  $\rightarrow X$

2.  $1 : X \rightarrow X$

3.  $I$  a set with the discrete topology and the projection  $X \times I \rightarrow X$

4. A covering space, more explicitly the mobious covering

$$S^1 \rightarrow S^1$$

$$z \mapsto z^2$$

5.  $U \subset X$  an open set,  $\iota : U \rightarrow X$

6. If  $x \in X$  is a closed point, then we can construct the space  $X \sqcup_{X \setminus \{x\}} X = X \times \{1, 2\} / \sim$  where  $(y, 1) \sim (y, 2)$  when  $y \neq x$ . This comes with the codiagonal map  $\nabla : X \sqcup_{X \setminus \{x\}} X \rightarrow X$ , where  $\nabla|_{X \times \{i\}} = 1_X, i \in \{1, 2\}$ .

This is a generalization of the line with two origins.

7.  $I \neq \emptyset$ , then take  $\sqcup_I X \xrightarrow{\nabla} X$   
 $X \setminus \{x\}$

**definition 4.** If  $U \subset X$  is an open set, then a section on  $U$  is a continuous map  $s : U \rightarrow S$  such that the following commutes:

$$\begin{array}{ccc} & S & \\ s \nearrow & \downarrow \pi & \\ U & \xrightarrow{\iota} & X \end{array}$$

The set of sections is denoted  $S(U)$  or  $\Gamma(U, S)$ . If  $U = X$ , then  $s$  is called a global section with notation  $S(X)$  or  $\Gamma(S)$ .

**Example(s). (Revisited)**

1.

$$S(U) = \begin{cases} 1_\emptyset & U = \emptyset \\ \emptyset & \text{else} \end{cases}$$

2.

$$S(U) = \{\iota_U\}$$

3.

$$S(U) = \text{hom}_{\text{set}}(\pi_0(U), I)$$

4.

$$S(U) = \{f : U \rightarrow \mathbb{C} \mid f(z^2) = z\}$$

5.

$$S(U) = \begin{cases} \{t\} & x \notin U \\ \{1, 2\} & x \in U \end{cases}$$

6.

$$S(U) = \begin{cases} \{t\} & x \notin U \\ I & x \in U \end{cases}$$

This particular example is called the “sky-scraper sheaf”

**Proposition 3.** There is a étalé space  $\mathcal{H}$  over  $\mathbb{C}_{\text{EUC}}$  with sections corresponding to holomorphic functions on  $\mathbb{C}$ .

*Proof.* The construction of  $\mathcal{H}$  as a set is given, alongside its topology. Verifying the claim is exercise 2.

$$\mathcal{H} := \bigsqcup_{z_0 \in \mathbb{C}} \left\{ \sum_{n=1}^{\infty} c_n (z - z_0)^n \mid \text{the series converges in some neighborhood of } z_0 \right\}$$

And define the topology on  $\mathcal{H}$  as the strongest topology such that for any open set  $U$ , and holomorphic  $f : U \rightarrow \mathbb{C}$  we have the following map is continuous

$$\begin{aligned} \mathcal{H}f : U &\rightarrow \mathcal{H} \\ z_0 &\mapsto \text{The Taylor expansion of } f \text{ at } z_0 \end{aligned}$$

□

## 1.4 Lecture 4 (Sept 10, 2025)

**definition 5.** Let  $\pi : S \rightarrow X$  be étalé, then  $S_x := \pi^{-1}(x)$  is called the stalk of  $x$ .

**Example(s).** 1.  $1 : X \rightarrow X$ ,  $S_x = \{x\}$

2.  $X \times I \rightarrow X$ ,  $S_x \cong I$

3.  $\bigsqcup_{X \setminus \{x\}} X \xrightarrow{\nabla} X$ , then  $S_y \cong \begin{cases} I & y = x \\ \{\overline{y}\} & y \neq x \end{cases}$

4.  $\mathcal{H} \rightarrow \mathbb{C}$   $\mathcal{H}_{z_0}$  is locally convergent power series at  $z_0$ .

**Proposition 4.** If  $\pi : S \rightarrow X$  is étalé and  $y \in \pi^{-1}(x)$ , then there is an open set  $U \ni \{x\}$  and a section  $s : U \rightarrow S$  with  $s(x) = y$ . Moreover, given two sections  $s_i \in \Gamma(U_i, S)$  there is some  $V \subset U_1 \cap U_2$  containing  $x$ , such that  $s_1|_V = s_2|_V$ .

*Proof.* The proof is exercise 3.

□

**Proposition 5.**

$$\varinjlim_{x \in U} \Gamma(U, S) \xrightarrow{s \mapsto s(x)} S_x$$

is a bijection.

*Proof.* This is onto since every element of the stalk has a section mapping to it, and injective by uniqueness of such an element up to the equivalence relation in the colimit. This is essentially restating the previous proposition.  $\square$

**Proposition 6.**  $f : X \rightarrow Y, g : Y \rightarrow Z$  continuous maps, then

- $f$  and  $g$  being local homeomorphisms implies  $g \circ f$  is.
- $g$  and  $f \circ g$  being local homeomorphisms implies  $f$  is.

**definition 6. (The Category of Sheaves on  $X$ )** The objects are étalé spaces  $\pi : S \rightarrow X$ , and the morphisms are  $\varphi : S_1 \rightarrow S_2$  continuous maps where the following commutes:

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi} & S_2 \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & X & \end{array}$$

Note that  $\varphi$  continuous and the diagram commuting implies it is a local homeomorphism, and hence morphisms are actually sheaves on sheaves.

**Proposition 7. (Isomorphism Criterion)** A morphism  $\varphi \in \text{Sh}(X)$  is an isomorphism if and only if the induced map  $(S_1)_x \rightarrow (S_2)_x$  is bijective for all  $x \in X$ .

*Proof.* One direction is easy since stalks get mapped to stalks, and an inverse map must be a bijection. In the converse direction, we know that it must be a local homeomorphism by proposition 6, hence since its invertible as a set function, its inverse is a local homeomorphism.  $\square$

**Proposition 8. (Monomorphism Criterion)** A morphism  $\varphi \in \text{Sh}(X)$  is a monomorphism if and only if the induced map  $(S_1)_x \rightarrow (S_2)_x$  is injective for all  $x \in X$ .

*Proof.* Once again, one direction is easy. For the other direction, if  $\varphi(x_1) = \varphi(x_2)$ , then  $\pi_2 \varphi(x_1) = \pi_2 \varphi(x_2)$ , so that  $\varphi(x_1)$  and  $\varphi(x_2)$  are in the same element of the stalk, by injectivity on stalks we are done.  $\square$

**Proposition 9. (Isomorphism Criterion for Sections)** If  $\varphi : S_1 \rightarrow S_2$  is a morphism in  $\text{Sh}(X)$  such that for any open set the induced map  $\Gamma(U, S_1) \rightarrow \Gamma(U, S_2)$  is a bijection, then  $\varphi$  is an isomorphism. Moreover, the converse is true.

*Proof.* The main thing to check here is that a bijection for all  $U$  gives a bijection on the colimits. Assuming this for now we get that as sets:

$$(S_1)_x \cong \varinjlim_{x \in U} \Gamma(U, S_1) \cong \varinjlim_{x \in U} \Gamma(U, S_2) \cong (S_2)_x$$

So that applying the Isomorphism Criterion we find that  $\varphi$  is an isomorphism. Now if  $\varphi$  is an isomorphism, then  $\varphi : \Gamma(U, S_1) \rightarrow \Gamma(U, S_2)$  via  $s_1 \mapsto \varphi s_1$ , this has map inverse  $\varphi^{-1}$ , so these sets are in bijective correspondence which suffices to prove the converse.  $\square$

Note that the same proof works for injections.

**Warning  $\triangle$ .** If  $\varphi : S_1 \rightarrow S_2$ ,  $\varphi \in \text{Sh}(X)$  is surjective this does not imply that the induced map on  $\Gamma(U, S)$  is in general surjective. A counter example is the Mobius covering of  $S^1$ , i.e.  $X = S = S^1$  with  $\pi_S = \varphi : S \rightarrow X$  via  $z \mapsto z^2$  and  $\pi_X = 1_X$ . Then  $\Gamma(X, S) = \emptyset$ , since there is no globally continuous square root on  $S^1$ . This implies that there is no surjection

$$\emptyset = \Gamma(X, S) \rightarrow \Gamma(X, X) = \{1_X\}$$

The upshot is that local lifts do exist.

**Proposition 10. (Local Lifts)** Let  $S_1, S_2$  be étalé over  $X$ , and  $\varphi : S_1 \rightarrow S_2$  a surjective morphism. Then given a section  $s \in S_2$ , there is an open cover  $\bigcup_I U_i$  with sections  $t_i \in S_1(U_i)$  such that  $\varphi \circ t_i = s|_{U_i}$  for all  $i$ .

*Proof.* Since  $\varphi$  is surjective, it must also be surjective on stalks  $S_x \rightarrow S_x$ . Then for any  $x$ , we have some  $(t_x, V_x)$  so that  $\varphi \circ t_x(x) = s(x)$ , it follows by the existence part of proposition 4 that we can choose a neighborhood  $x \in U_x \subset V_x$  so that  $\nu p : (t_x, U_x) \rightarrow (s, U)$ .  $\square$

**Remark 1. (An abstract perspective on lifts)** Given the setup

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi} & S_2 \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & X & \end{array}$$

and a global section  $s \in \Gamma(X, S_2)$ , we get a sheaf from the fibered product  $s^{-1}S_1 := S_1 \times_{S_2} X$  (this of course means its points are  $\{(t, x) \mid \varphi(t) = s(x)\}$ ). From this perspective,  $s$  having a lift to  $\Gamma(X, S_1)$  is equivalent to  $s^{-1}S_1$  having a global section.

## 1.5 Lecture 5 (Sept 12, 2025)

**definition 7.** If  $f : X \rightarrow Y$  and  $g : Z \rightarrow Y$  are morphisms, then we define the fiber product by the universal property

$$\begin{array}{ccccc} T & & \xrightarrow{\beta} & & Z \\ & \searrow \exists! & & \searrow \pi_Z & \\ & X \times_Y Z & \xrightarrow{\pi_Z} & & Z \\ & \downarrow \pi_X & & \downarrow g & \\ & X & \xrightarrow{f} & & Y \end{array}$$

**Example(s).**

$$\begin{array}{ccc} X \times Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & \{\cdot\} \end{array} \quad \begin{array}{ccc} f^{-1}(y) & \longrightarrow & \{y\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

So that products and fibers are special cases.

## 2 Exercises

**exercise 1. (Hartshorne Exercise 1.4)** An algebraically closed field is infinite, moreover the zero sets of polynomials are either  $k$  or a finite subset of  $k$ . Consider the closed set  $V(x - y) \subset \mathbb{A}^2$ , then it is an infinite set so if  $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ , then it must be of the form  $\mathbb{A}^1 \times F \cup E \times \mathbb{A}^1$ , where  $E, F \subset \mathbb{A}^1$  are closed. But for a fixed  $x$  or  $y$  we have  $V(x - y)$  has cardinality 1 which makes this impossible.  $\square$

**exercise 2.** (Show that  $\mathcal{H}(U) = \{f: U \rightarrow \mathbb{C} \mid f \text{ holomorphic}\}$ ) where we define  $\mathcal{H}$  in proposition 3.

We first check that it is a local homeomorphism, for  $z \in \mathbb{C}$  and  $U \ni z$ , we can take a function  $f$  holomorphic on  $U$ , then  $\mathcal{H}f$  has only one Taylor expansion for  $f$  at each  $z_0$  and is continuous. Since there is only one Taylor expansion at each  $z_0$  the map  $\pi$  taking Taylor series centered at  $z_0$  to  $z_0$  is injective,  $\pi$  is continuous because for an open set  $V$  we have  $\pi^{-1}(V) = \bigsqcup_{z_0 \in V} \mathcal{H}f(z_0)$ , which has open preimage under all of the  $\mathcal{H}f$ . Since  $\mathcal{H}f\pi = 1_S, \pi\mathcal{H}f = 1_X$  we are done this step.

Now  $\pi \circ \mathcal{H}f|_U = \iota_U$  and  $\mathcal{H}f$  continuous suffices to show that every holomorphic function is a section. Conversely, suppose  $g: X \rightarrow \mathcal{H}$  is not induced by a holomorphic function. The first case is  $g$  maps some  $z_1$  to a Taylor expansion around  $z_0 \neq z_1$ , this cannot be a section since then the diagram won't commute. In the second case, there are distinct points  $\{z_\alpha\}_{\alpha \in I}$  in the same connected component of  $U$  each Taylor series  $g(z_\alpha)$  determining a different holomorphic function (near that point)  $f_\alpha$ , denote the set of points that determines  $f_\alpha$  as  $V_\alpha$ , it is immediate that the  $V_\alpha$  are disjoint. We know that  $\mathcal{H}f_\alpha(U)$  is open in  $\mathcal{H}$  for each  $\alpha$  (Check!), but if  $g^{-1}(\mathcal{H}f_\alpha(U)) = V_\alpha$  is open for each  $\alpha \in I$ , then  $U = \bigsqcup V_\alpha$  is not connected, violating our earlier assumption. Hence for some open  $\mathcal{H}f_\alpha(U)$  we have that  $g^{-1}(\mathcal{H}f_\alpha(U))$  is not open and  $g$  is not continuous.  $\square$

Check: If two holomorphic functions have the same Taylor series at a point they are equal so

$$\mathcal{H}\phi^{-1}(\mathcal{H}f_\alpha(U)) = \begin{cases} U \text{ or the domain of definition for } f & \phi = f_\alpha \\ \emptyset & \text{else} \end{cases}$$

In either case the preimage is an open set.

**exercise 3.** (Show the existence and uniqueness of sections for each element in the stalk) Existence is not too bad, since  $\pi$  is a local homeomorphism, hence we can choose some neighborhood  $y \in U$  with  $\pi|_U$  a homeomorphism. Then define  $s: \pi(U) \rightarrow U$  via  $x \mapsto \pi|_U^{-1}(x)$ . Now assume that  $s_1, s_2$  are two such sections associated to open sets  $U_1, U_2$ , then  $y \in \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ , and hence some open set  $y \in \tilde{V} \subset \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$  such that  $\pi|_{\tilde{V}}$  is a homeomorphism. Now let  $V = s_1^{-1}(\tilde{V}) \cap s_2^{-1}(\tilde{V})$  which is nonempty since it contains  $x$ , and open. Since the inclusions are injective, we can say the same about sections, hence  $s_1, s_2: V \rightarrow \tilde{V}$ , and for  $z \in V$  we have  $\pi|_{\tilde{V}} s_i(z) = \iota(z) = z$  and hence  $s_i(z) = \pi|_{\tilde{V}}^{-1}(z)$ .  $\square$

## A Assigned Readings

### B Misc.

**definition 8.** A ring or algebra is called reduced when it has no non-zero nilpotents.

**definition 9.** A map is a monomorphism when it has the left cancellation property  $f g_1 = f g_2 \implies g_1 = g_2$ .