1. (a) From the definition of a lie group we know that $\mu : G \times G \to G$ is smooth, then $\mu_g = \mu \circ \iota_g$, where $\iota_g : G \to G \times G$ via $h \mapsto (g, h)$ is the incusion into the product manifold, we have seen previously the inclusion is smooth, so that $\mu_g = \mu \circ \iota_g$ is smooth. Now we can also see that

$$\mu_{g^{-1}}\mu_g = 1_G = \mu_g\mu_{g^{-1}}$$

and $\mu_{g^{-1}}$ is smooth for the same reason μ_g is, so that μ_g is in fact a diffeomorphism, this implies that $d_e\mu_g$ is an isomorphism.

(b)

(Lemma) Let $(\rho, E), (\widehat{\rho}, \widehat{E})$ be two vector bundles on the same base space M, and $F : E \to \widehat{E}$ a smooth bijective map of smooth vector bundles with $F(x, 0) = (x, 0)$ (i.e. F descends to the identity), then F is a diffeomorphism.

Proof. Being a diffeomorphism is a local property, so for a point $x \in M$, let U be an open neighborhood of M where $\rho^{-1}(U)$ admits a local trivialization ζ, moreover there is a second neighborhood $x \in V \subset U$ where $\widehat{\rho}^{-1}(V)$ admits a local trivialization $\widehat{\zeta}$ (since the base manifold is the same by possibly shrinking the neighborhood we can assume that the two bundle charts are equal on $V \times \{0\}$, this is not necessary but removes a lot of bloat from notation). Then $\widehat{\zeta} \circ F \circ \zeta^{-1} : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is smooth, linear on each fiber and bijective on each fiber, so on V, we can write $A(x) = \widehat{\zeta} \circ F \circ \zeta^{-1}(x, -)$. Then on the local trivialization F is given by

$$\widehat{\zeta} \circ F \circ \zeta^{-1}(x, v) = (x, A(x)v)$$

In particular, the Jacobian $D_{(x,v)}(\widehat{\zeta} \circ F \circ \zeta^{-1})$ is given by

$$\begin{pmatrix} 1_n & 0 \\ B(x, v) & A(x) \end{pmatrix}$$

Bijectivity on each fiber implies that $A(x)$ is full rank, so that $\det(D_{(x,v)}\widehat{\zeta} \circ F \circ \zeta^{-1}) = \det A(x) \in \mathbb{R}^\times$, by the inverse function theorem $\widehat{\zeta} \circ F \circ \zeta^{-1}$ has a local smooth inverse, and hence F is a diffeomorphism. $\square$

Since T_eG is n-dimensional, we can identify it with $\mathbb{R}^n$, the following diagram specifies the desired correspondence of vector bundles:

$$\begin{tikzcd} G \times \mathbb{R}^n \arrow[r, shift left, "F"] \arrow[d] & TG \arrow[l, shift left, "T"] \arrow[d] \\ G \arrow[r, "1_G"] & G \end{tikzcd}$$

Where $F(g, v) = (g, d_e\mu_g(v))$,

(when I originally solved the problem I tried to show F and the inverse map T which is not too hard to compute are both smooth, however, after trying to show F, T are smooth for quite some time I did the following computation which allowed me to see that F is smooth, this computation does not generalize easily to T, so the lemma is intended to avoid having to do a similar computation for T).

In order to show F is smooth, it suffices to show that $(g, v) \mapsto d_e\mu_g(v)$ is smooth, here we can use smoothness of μ, and the identification $T(G \times G) \longleftrightarrow TG \oplus TG$ by identifying on each fiber, we have previously computed (last homework) that $d_p\iota_q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ when ι denotes inclusion. We have that $d\mu : T(G \times G) \to TG$ is smooth since μ is a smooth map, then

$$d\mu((g, v), (h, u)) = d_{(g,h)}\mu(v, u)$$
$$d_e\mu_g = d_e(\mu \circ \iota_g)(v, u) = (d_{(g,e)}\mu)(d_e\iota_g)(v, u) = d_{(g,e)}\mu(u)$$

From this computation, we can see that $d_e\mu_g = d_{(g,e)}\mu(0, u)$ is the restriction of $d\mu$ to $G \times \{0\} \times T_eG$, this is clearly a submanifold directly from the definition of it being a linear subspace given by inclusion on the first n coordinates and last n coordinates. Thus the restriction of $d\mu$ to this submanifold is smooth, and is identified with $d_e\mu_g$. So F is smooth, and we appeal to the lemma to find that T, the set theoretic inverse for F is smooth and hence F is a diffeomorphism.

2. Let $f : X \to \mathbb{R}^m$ be a submersion, where X is a compact smooth manifold. The proof will follow if we can show submersions are open maps, assuming this, since the image of a compact set is compact (by pulling back an open cover along the map) we get that $f(X) \subset \mathbb{R}^m$ is open, but also $f(X) \subset \mathbb{R}^m$ is compact hence closed, so since $X \neq \emptyset$ we have $f(X) = \mathbb{R}^m$, contradicting compactness.

It remains to show that a submersion is open, since f is a submersion, we can cover M, N with charts $(U_\alpha, V_\alpha, \phi_\alpha)$ and $(U'_\beta, V'_\beta, \varphi_\beta)$ respectively with the property that the following commutes (here π is the projection map onto the first n coordinates)

$$\begin{tikzcd} U_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"] & V_\alpha \arrow[d, "f"] \\ U'_\beta \arrow[r, "\varphi_\beta"] & V'_\beta \end{tikzcd}$$

Now let $E \subset X$ be open, and write $E_\alpha := V_\alpha \cap E$, then

$$f(E) = \bigcup_\alpha f(E_\alpha) = \bigcup_{\alpha,\beta} \varphi_\beta \pi \phi_\alpha^{-1}(E_\alpha)$$

But $\varphi_\beta \pi \phi_\alpha^{-1}$ is a composition of open maps hence open, so that $f(E)$ is open which suffices to show f is open.

3. (a) From the iterated construction we get

$$\begin{aligned} \overline{[a,b,c,d,e,f,g,h]} &= [\overline{(a,b,c,d,e,f)}, (-g,-h)] = [\overline{(a,b,c,d)}, (-e,-f), (-g,-h)] \\ &= [\overline{(a,b)}, (-c,-d), (-e,-f), (-g,-h)] = [\overline{a}, -b, (-c,-d), (-e,-f), (-g,-h)] \end{aligned}$$

from this taking the i-th coordinate to be 1 and the others zero we see $\overline{e_i} = -e_i$. $\square$

(b) We first note that A maps real values to real values and imaginary values to imaginary values. This can be seen since if $\overline{x} = x$ and $\overline{y} = -y$, then

$$\overline{A(x)} = A(\overline{x}) = A(x) \qquad \overline{A(y)} = A(\overline{y}) = A(-y) = -A(y)$$

And moreover, for arbitrary x, $A(x)A(1) = A(x)$ implies $A(1) = 1$ so linearity suffices to show that $A(\alpha) = \alpha$ for $\alpha \in \mathbb{R}$, once again by linearity we see that this implies $\mathrm{Re}(A(x)) = A(\mathrm{Re(x)})$ for all x. It follows that A is orthogonal, i.e. preserves the inner product

$$\langle A(x), A(y)\rangle = \mathrm{Re}(A(x)\overline{A(y)}) = \mathrm{Re}(A(x)A(\overline{y})) = \mathrm{Re}(A(x\overline{y})) = \mathrm{Re}(x\overline{y}) = \langle x, y\rangle$$

Since A preserves the imaginary octonions, it makes sense to restrict A to acting on $\mathrm{Im}(\mathbb{O})$, so that identifying A with its image in $O(7)$ is well defined, since the inner product on $\mathrm{Im}(\mathbb{O})$ induces the standard norm on $\mathbb{R}^7$ (computation provided below), we know from the polarization identity that the inner products are the same since they can be recovered from the norm, so that the image of A in $O(7)$ is still orthogonal in the euclidean sense.

I include here the computation of equivalence of norms using the multiplication table: Consider the octonion given by $a = (a_i e_i)_0^7$, of course we are only interested in the case of $a_0 = 0$, so we have $a = (a_i e_i)_1^7$

$$a\overline{a} = \left(\sum_1^7 a_i e_i\right)\left(\sum_1^7 -a_j e_j\right) = \sum_1^7 a_i^2 + \sum_{i<j} a_i a_j e_i e_j + \sum_{i>j} a_i a_j e_i e_j$$

We can read from the off diagonal of the octonions multiplication table that for $i, j > 0$ and $i \neq j$ that $e_i e_j = -e_j e_i$, this kills the two sums on the right to give us $\sum_1^7 a_i^2$ the euclidean norm as desired. $\square$

(c) From the multiplication table we have for any i, $e_i\overline{e_i} = e_0$, it follows from definitions that $\langle e_i, e_i \rangle = 1$ for $i = 1, 2, 4$. Now reading from the table,

$$e_1\overline{e_2} = -e_3, \quad e_1\overline{e_4} = -e_5, \quad e_2\overline{e_4} = -e_6, \quad (e_1e_2)\overline{e_4} = -e_7$$

these all have zero real part, so that by taking the inner product we get zero, this suffices to show its a special triple. $\square$

(d)

Although not stated in the question, $V_n(\mathbb{R}^m)$ is orthonormal sets of n vectors in $\mathbb{R}^m$, it is also defined in the notes as a quotient of the orthogonal group by a group action $O(m)/O(m-n)$. It follows that $V_3(\mathbb{R}^7)$ can be realized as $O(7)/O(4)$, since $O(4)$ is a lie group of dimension 6, and $O(7)$ is a lie group of dimension 21, the quotient $V_3(\mathbb{R}^7)$ is a $21 - 6 = 15$ dimensional manifold, its also important here that we are identifying $O(4)$ as the elements fixing the first second and fourth columns e_1, e_2, e_4, taking these columns in particular is important for the projection to work out in part (e).

The only additional condition of a special triple that isnt in $V_3(\mathbb{R}^7)$ is the equation $\langle xy, z\rangle = 0$, so it will suffice to check that 0 is a regular value for

$$\begin{aligned} \hat{F} : V_3(\mathbb{R}^7) &\to \mathbb{R} \\ (x, y, z) &\mapsto \langle xy, z\rangle \end{aligned}$$

where again we are using the identification of the imaginary octonions with $\mathbb{R}^7$. The smoothness of $\hat{F}$ is immediate by multilinearity. It is somewhat hard to deal with $V_3(\mathbb{R}^7)$, but we can show that it is a submanifold of $\mathbb{R}^{21}$, to do so use the regular value theorem with

$$\begin{aligned} F' : \mathrm{Mat}_{3\times 7}(\mathbb{R}) &\to \mathbb{R}^6 \\ \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} &\mapsto (||v_1||^2, ||v_2||^2, ||v_3||^2, \langle v_1, v_2\rangle, \langle v_1, v_3\rangle, \langle v_2, v_3\rangle) \end{aligned}$$

F' is polynomial hence smooth. The total derivative looks like (in 1×7 blocks)

$$d_{(v_1,v_2,v_3)}F' = \begin{pmatrix} 2v_1 & 0 & 0 \\ 0 & 2v_2 & 0 \\ 0 & 0 & 2v_3 \\ v_2 & v_1 & 0 \\ v_3 & 0 & v_1 \\ 0 & v_3 & v_2 \end{pmatrix}$$

it is straightforward to see that $(1, 1, 1, 0, 0, 0)$ is a regular value for F' by the vectors being nonzero at these points (independence of rows then follows easily from orthogonality). This realizes $V_3(\mathbb{R}^7)$ as a $15 = 21 - 6$ dimensional manifold $F'^{-1}\{(1, 1, 1, 0, 0, 0)\}$. Now we want to show that $d_{(e_1,e_2,e_4)}\hat{F}$ is surjective, we will check later that using the action of G_2 this gives surjectivity at all points. In order to avoid overloading notation on e_i I will use $(f_i)_1^{21}$ to denote the basis on $\mathbb{R}^{21}$. Then we can define the path $\gamma : (-\epsilon, \epsilon) \to V_3(\mathbb{R}^7)$ via $\gamma(t) = (e_1, e_2, e_3 \sin t + e_4 \cos t)$, it is clear by definition that $\gamma(-\epsilon, \epsilon)$ does indeed lie in $V_3(\mathbb{R}^7)$, moreover we have $\gamma(0) = (e_1, e_2, e_4)$ and $\gamma'(t) = (0, 0, e_3) = f_{17} \in T_{(e_1,e_2,e_4)}V_3(\mathbb{R}^7)$. Now we can take

$$\left.\frac{d}{dt}\right|_{t=0} F \circ \gamma(t) = \left.\frac{d}{dt}\right|_{t=0} \langle e_1e_2, e_3 \sin t + e_4 \cos t\rangle = \left.\frac{d}{dt}\right|_{t=0} e_3(\overline{e_3} \cos t + \overline{e_3} \sin t) = \left.\frac{d}{dt}\right|_{t=0} \cos t + \sin t = 1$$

so that indeed we have $d_{(e_1,e_2,e_4)}\hat{F} = d_{(e_1,e_2,e_4)}F|_{T_{(e_1,e_2,e_4)}V_3(\mathbb{R}^7)}$ is nonzero, hence surjective. This gives that $\hat{F}$ is a submersion at the point (e_1, e_2, e_4), we need to check it for the rest of $X = \hat{F}^{-1}(0)$, here we can use the theorem that for any other point $(x, y, z) \in X$ we have some $A \in G_2$ with $A(e_1, e_2, e_4) = (x, y, z)$. Now since A respects products and inner products we have $\hat{F} \circ A = \hat{F}$, so that

$$(d_{(x,y,z)}\hat{F})(d_{(e_1,e_2,e_4)}A) = d_{(e_1,e_2,e_4)}(\hat{F} \circ A) = d_{(e_1,e_2,e_4)}\hat{F}$$

So surjectivity of the derivative at the single triple (e_1, e_2, e_4) gives surjectivity at all triples, this proves that 0 is a regular value. □

(e) Define the map using the theorem by defining $A(x, y, z)$ to be the unique map transforming $(e_1, e_2, e_4) \mapsto (x, y, z)$

$$\Phi : X \to O(7)$$
$$(x, y, z) \mapsto A_{(x,y,z)}$$

That Φ is injective is an immediate consequence of the theorem, surjectivity onto G_2 is also straightforward, since G_2 preserves norms, products and inner products (which are all of the special triple conditions) so all elements of G_2 send special triples to special triples, whence G_2 elements are all in the image of X, determined by their action on e_1, e_2, e_4. We can also take $O(7)$ now with respect to the basis given by the special triple (e_1, e_2, e_4), and note that this just amounts to flipping the signs on basis elements e_5, e_6 and e_7, the reason for doing this is to clean up notation that $\Phi(e_1, e_2, e_4) = 1$. We also have that A is multilinear hence smooth. We check that it indeed defines an imersion, to do so consider the quotient map induced by the action of $O(4)$ given by $\pi : O(7) \to V_3\mathbb{R}^7$, then π is smooth, and $\pi \circ \Phi = 1_X$ (recall that X is a submanifold, then the composition maps into it), it follows that at any special triple we have

$$(d_{A(x,y,z)}\pi)(d_{(x,y,z)}\Phi) = 1_{T_{(x,y,z)}X}$$

injectivity of $1_{T_{(x,y,z)}X}$ implies injectivity of $d_{(x,y,z)}\Phi$, so that Φ is an immersion. Finally it only remains to check that Φ is proper, but this follows immediately from X compact. To see that X is compact, we can identify $X \subset \overline{B_{0,1}(\mathbb{R}^{21})}$, where X is given by the intersection of zero-loci of polynomial equations. This realizes X as a closed compact subset of Euclidean space, hence compact. □