Malmhedn's Theorem

Tigran Tadevosyan

May 2019

Abstract

In this document I would try to give a brief introduction to Malmheden's theorem. Latter in the document a python implantation for the algorithm is given.

Contents

1 Introduction

1.1 Definition of Problem

Malmeden's algorithm is an elegant way of solving Dirichlet problem for Laplace equation in an n-dimensional ball. In a figure below you can see the formulation of the problem.

$$\begin{cases} \Delta u(c) = 0, & c \in B(c_0, r) \\ u(c) = \varphi(c), & c \in \partial B(c_0, r) \end{cases}$$

With the help of the Malmeden's algorithm we can find the value for that harmonic function u with any valid boundary condition function φ for ball centered at c_0 and radius r.

1.2 Definition of Malmeden's Theorem

Let's take an n-dimensional ball with radius r. Let's define a function v as the linear interpolation of values obtained by the boundary condition function φ on the border of that ball. If we calculate the average of the integrate of that function over the whole boundary we will obtain the get the harmonic function inside that ball which satisfies the boundary condition which means the solution of the Dirichlet problem.

2 A Proof Malmheden's theorem

For simplicity let's look at the case of Dirichlet problem inside a disk.

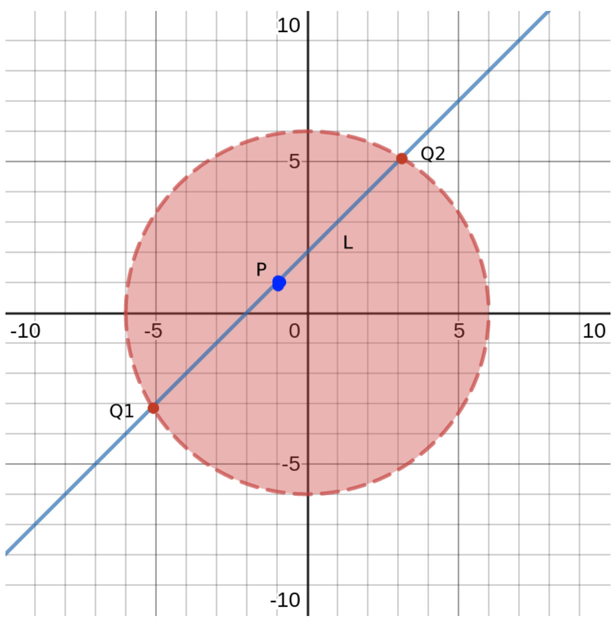

Figure 1: Disk

Let's define Ω as a disk with a boundary Γ. After drawing a chord L in Ω it will intersect Γ in two points' Q_1 and Q_2. Our boundary function φ will obtain values φ_1 and φ_2 at Q_1

and Q_2. Let's define yet another function called υ that will calculate the value of a point inside a disk with linear interpolation. For example υ can interpolate any value on chord L from values φ_1 and φ_2. If we integrate the function υ over the angle θ from 0 to 2π and divide the result by the 2π we will obtain the average of linear interpolation and from the Malmheden's theorem it should be the harmonic function we are seeking. So if we prove that the function we obtain by averaging that integration is harmonic we will prove Malmheden's theorem.

$$\upsilon(P) = \frac{r_1 f_2 + r_2 f_1}{r_1 + r_2}$$

This is what the υ will look like where P is the point we are trying to interpolate and r_1 and r_2 are the distances of P from Q_1 and Q_2 respectively. If we turn to polar coordinates we obtain $\upsilon(P, \theta)$ where θ is the angle of chord makes with the positive direction of $x - axis$. Let's calculate the u function with everything we already got.

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} \upsilon(P, \theta) dx$$

Obviously we obtain that $r_1(\theta) = r_2(\theta + \pi)$ because it is the same chord with opposite direction. So we can say that $f_1(\theta) = f_2(\theta + \pi)$. From that we can update υ function.

$$\upsilon(P, \theta) = 2\frac{r_2 f_1}{r_1 + r_2}$$

Let's define α as the angle between inner normal n to Γ at Q_1 and the chord L. Turns out that if we compare slight changes in arclength s on γ and arclength of the circle of radius $r1$ centered at P we will obtain that $r_1 d\theta = \cos(\alpha) ds$. So after multiplying the top of function υ from last iteration with $\cos(\alpha)ds$ and the bottom $r_1 d\theta$ after some modifications we obtain.

$$u(P) = \frac{1}{\pi} \int_0^c \frac{\cos(\alpha)}{r_1} f_1 ds - \frac{1}{\pi} \int_0^c \frac{\cos(\alpha)}{r_1 + r_2} f_1 ds$$

Where c is the length of Γ.

The second integral is just a constant depending on the radius of the disk and boundary function. If we replace $\frac{\cos(\alpha)}{r_1}$ by $\frac{\partial \log|P-Q_1|}{\partial n_{Q_1}}$ because they are equal we will obtain that the first integral is the "double layer" potential of f_1 which is a harmonic functions. So that means our function u is a harmonic function that satisfies the boundary condition, so it is the solution to Dirichlet problem.

3 Algorithm Implementation

For the implementation of the algorithm I have use python 3 and PyQT4 for UI. All the codes are available on github with this link.