Thesis

by

Tigran Saidnia

Emission kernels of parton shower in NLO

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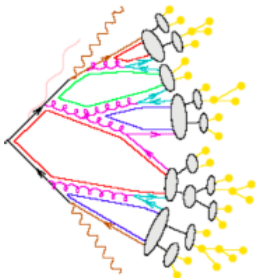

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statement of originality

I hereby confirm that I have written the accompanying thesis by myself, without contributions from any sources other than those cited in the text and acknowledgements. This applies also to all graphics, drawings, maps and images included in the thesis.

Karlsruhe, June 26, 2019

Tigran Saidnia

Abstract

Infrared divergences come in two flavours: soft, due to the massless nature of the radiation (e.g. the massless photon in QED), and collinear, which comes from treating the radiating particle as massless. By the Catani-Seymour dipole factorization a $m+1$-parton matrix element can be written as product of m-parton matrix element and an universal singular splitting function. In this work a mapping for $3 \to 2$ will be outlined and following along those lines one for $m+1 \to m$ is proposed, which is explicitly evaluated for the quadratic matrix elements in terms of the four possible parton splittings in the soft and collinear regions. Furthermore, a general prescription for the simplification of the usage of this algorithm in the next-to-leading order (NLO) level will also be given. For comparison, the known result from the $e^+e^- \to q\bar{q}g$ process is compared with the result of the gluon radiation from a parent quark in the first part of chapter 4. The algorithm is straightforwardly implementable in general purpose Mathematica or Herwig++ [1].

Zusammenfassung

Infrarot-Divergenzen gibt es in zwei Varianten: soft, aufgrund der massenlosen Natur der Emission (z.B. das massenlose Photon in QED), und kollinear, durch die Annahme, dass das strahlende Teilchen selbst massenlos ist. Durch die Catani-Seymour Dipolfaktorisierung kann ein $m+1$-Partonmatrixelement als Produkt aus einem m-Parton Matrixelement und einer universellen singulären Splittingfunktion geschrieben werden. In dieser Arbeit wird eine Mapping für $3 \to 2$ und ebenso für eine $m+1 \to m$ präsentiert, die dann explizit bei der Auswertung der Matrixelemente der vier möglichen Partonsplittungen in den soft und kollinearen Bereichen eingesetzt wird. Vielmehr wird eine allgemeine Prozedur für die Vereinfachung dieses Algorithmus in der Next-to-Leading Order (NLO) Ebene vorgeschlagen. Zum Vergleich wird das bekannte Ergebnis aus $e^+e^- \to q\bar{q}g$ mit dem Ergebnis der Gluonenabstrahlung eines Parent-Quarks aus dem ersten Teil von Kapitel 4 verglichen. Der Algorithmus kann in Mathematica oder Herwig++ [1] implementiert werden.

Contents

1 Introduction

Knowledge is a human need. For thousands of years we have been trying to understand the secrets of the universe. Such riddles fascinated even Johann Wolfgang von Goethe, as he wrote in his book Faust [15]; eine Tragedie, "What holds the world together in its innermost." Almost 400 years before Christ, an ancient Greek philosopher, Democritus, and his teacher Leukipp claimed matter can not be divided at will. Rather, there must be an Atomos (Greek: indivisible) that could no longer be subdivided. Democritus was of the opinion that there were infinitely many atoms with different geometric forms that were in contact in a certain way. He pointed out that a thing has a color, taste or even soul, based on the apparent effect of the composition of these small grains. [4]
This statement of Democritus was first laughed at by the renowned philosopher Aristotiles. It took about 2000 years for a chemist named John Dalton to deal with the subject. Based on various test series, he summarized his conclusion in his book A New System of Chemical Philosophy, that all substances consist of spherical indivisible atoms. The atoms of different elements have different masses and volumes. This was exactly the most striking difference to Democritus's atomic world.[7]
The discovery of the periodic system by D. Mendeleev and P. Meyer enabled us to arrange the atoms according to their mass in such a way that their properties occur in a certain order.[18]
In 1897 Joseph Thompson was able to obtain a stream of particles by heating metals and deflecting them by a magnetic field. This electron beam was 200 times lighter than the lightest atom, hydrogen. His conclusion was that atoms cannot be indivisible. He suggested that each atom consists of an electrically positively charged sphere in which electrically negatively charged electrons are stored - like raisins in a cake.
Furthermore, renowned scientists as well as Marie and Pierre Curie have contributed much to the development of atomic theory by discovering radioactivity, Boltzmann by kinetic gas theory and Plank, the founder of quantum physics. However, one of the most important steps in the atomic model was taken by the British physicist Rutherford. He bombarded a thin aluminium foil with a radioactive sample. If Thompson's cake model were correct, only a few alpha particles would be detected behind the aluminium foil. Surprisingly, many particles were visible, which could only be explained by the assumption that the majority of atoms consisted of empty spaces. Another miracle was that some particles could be seen above or below the target sample. Since it is known, the alpha particles are positively charged, it could be assumed the electric repulsive force of two positive charges. From the ideas of Planck and Rutherford, the Danish physicist Bohr (1885-1962) developed a planetary atomic model. The electrons then move around the nucleus in certain orbits, like planets orbit the sun. The orbits are also called shells. The special

thing about it was that the distances of the electron orbits follow strict mathematical laws.
At first, however, it remained unclear what this core should consist of. [8, 18] In 1912, the Austrian physicist Victor Hess discovered during his balloon flights that the ionization rate of the Earth's atmosphere increases with altitude. This result was not expected because until then the Earth's radioactivity was known as the only source of air ionization. Therefore, he postulated this new type of radiation as cosmic radiation, which must originate outside the Earth's atmosphere [11].
Further investigations two years later confirmed the thesis of a cosmic background of such radiation. After this new discovery, it was discovered that the radiation consists of charged particles. In 1932, the American physicist Carl David Anderson was able to prove the postulated particle of Dirac, the positron, as a component of an air shower through his cloud chamber. For a long time, cosmic rays were the only way to analyse such exotic particles.[2] This changed when particle accelerators were able to generate particles in collisions. But even today, cosmic rays are the only way to study particles of the highest energies, since these energies cannot be reached by today's particle accelerators, such as the LHC. The LHC, the world's largest accelerator at CERN, produces particles with centre-of-mass energy equivalent to a cosmic particle of nearly $10^{17}eV$, with the energy spectrum of cosmic particles reaching up to $10^{20}eV$. However, we can only analyse such exotic particles in detail by increasing the luminosity and precision of the particle accelerators at the nucleus. The discovery of the neutron by Chadwick (1932) showed that atomic nuclei are made up of protons and neutrons. It was also clear that, in addition to gravitation and the electromagnetic force, there should exist two short-range forces in nature; a strong force which binds the nucleons together and a weak force which is responsible for radiation. In the meantime it was agreed that a new theory was needed for the classification and grouping of this particle zoo. This is how the current standard model came into being.
The SLAC experiments indicate that the electrons can be scattered as quasi-free point-like constituents within the proton structure, which actually meant that the protons or neutrons are not point-like and must consist of other constituents. Through the bubble chamber a huge number of previously invisible particles (Gell-Mann's eightfold path) could suddenly be made visible, which represented contradictions to the previous physics. To explain this, the physicist Gell-Mann found basic building blocks from which all previously known atomic particles should be built. The components are later identified with quarks. For the results of particle physics experiments in connection with the strong interaction the perturbation theory is used. In order to make useful and more accurate predictions, the calculations must be carried out at least in next-to-leading order, in order to avoid large uncertainties arising from the non-physical scale dependencies. In this work, the dipole subtraction method will be used for calculating next-to-leading order splitting functions of parton shower in QCD.
The thesis begins with the Feynmann Rules and Colour Algebra in chapter 2 which are later used for the evaluation of the matrix elements. Chapter 3 provides a brief overview of of the general method, describing the subtraction procedure and presenting the dipole formulae. Thereafter, the kinematics for the case of massless partons with the useful prescriptions for the matrix elements evaluation are discussed. The factorization properties of QCD matrix elements in the soft and collinear limits for four possible parton

showers due to parametrisations is outlined in 4. The known result from the $e^+e^- \to q\bar{q}g$ process is compared with the result of the gluon radiation from a parent quark in chapters 4 and 5. Chapter 6 represents a summary of the previous final results that were obtained. MATHEMATICAL TOOLS 6 gives more details, and some examples, of the necessary mathematic formulae for the handling of parametrisations. All detailed steps of the calculations can be found in the appendix 6.

2 Prerequisites

The Standard Model in particle physics encompasses all of the Elementary particles and their interactions. It is a gauge theory spontaneously broken by the Higgs mechanism with the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.
From a theoretical point of view, the Standard Model is a quantum field theory based on local gauge invariance and consists of two parts. The electroweak sector $SU(2)_L \otimes U(1)_Y$ is called GWS (Glashow-Weinberg-Salam) theory and describes the gauge bosons $W^{\pm}$, Z^0, γ, the Higgs sector and its interaction with the leptons and quarks. In contrast to the other gauge bosons, the exchange particles of the weak interaction carry mass, which also affects the properties of the interactions. The colour-charged sector $SU(3)_C$, the chromo dynamics, deals with quarks and contains the eight massless, electrically neutral gluons as gauge bosons. The gauge groups $SU(3)_C$ and $SU(2)_L$ are non-abel gauge theories, more precisely Yang Mills theories. The massive particles, fermions, will be divided into two groups, leptons and quarks. Each group is arranged in 3 generations. Within the leptons there are three electrically neutral neutrinos. The mass of the particles increases from generation to generation [1]. Neutrinos only interact weakly, whereas the charged leptons interact both weakly and electromagnetically. Quarks are characterised by the fact that they can also interact strongly [9].

2.1 Quantum chromo dynamics

Overall, there are four types of interactions.

Interaction	Energy scale	Range [m]	Mediators
Strong	~ 1	10^{-15}	g
Electromagnetic	$\sim 10^{-2}$	∞	γ
Weak	$\sim 10^{-6}$	10^{-18}	$W^{\pm}, Z$
Gravity	$\sim 10^{-38}$	∞	maybe graviton

Nucleons are made up of quark and gluons generally generally referred as partons. Whereby, the gluons are the exchange bosons for this short ranged interaction. To explain the short range of the strong interaction Yukawa (1934) postulated mesons as a mediator for the nuclear force by the exchange of this massive field quanta. Three years later a candidate (π meson) was found in cosmic rays. Later on it was shown massive field quanta

[1]Not sure for neutinos as of yet

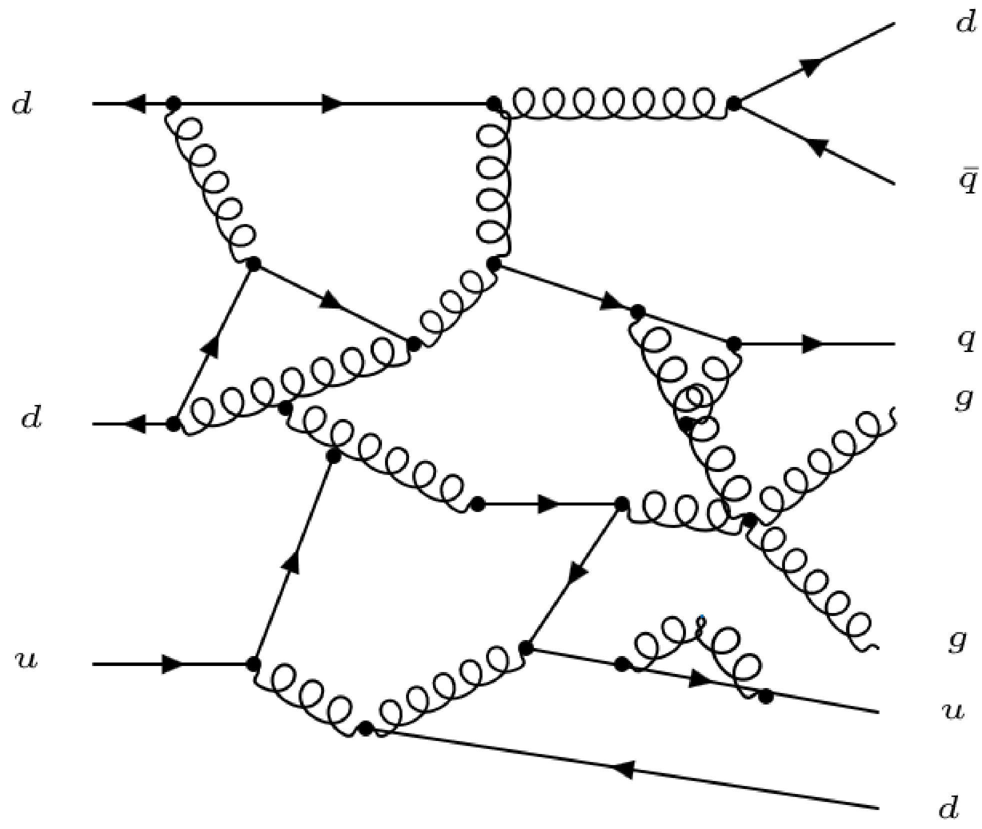

Figure 2.1: A schematic picture of neutron structure. at the left side of the resolution is too low to see. The 3 quarks picture allows us to interpretate the quantum numbers of the neutron in the valence band. In high-resolution picture for a large Q^2 can be obtained a gluon sea and a lot of quarks pairs[3]. The quantum number of a neutron is for each energy scale the same

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break the gauge symmetry so that the mediator must be massless. If it is based on the SU(3) gauge symmetry of the QCD massless Lagrangian how can the strong sector be short range? Another question came from a series of experiments at SLAC. Through high-energy electron-proton scattering there was a search for evidence of the existence of quarks and their behaviour like free particles despite the energetically bound inner proton. The solution to these question was explained by Gross, Politzer and Wilczek through asymptotic freedom. This effect can be proved by the running coupling and anti screening in QCD. For the calculation of the propagator loop correction in QCD both quark loops (negative contribution → screening) and gluon loops (positive contribution → anti screening) have to be taken into account.
The one loop running coupling in QCD is:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) ln(\frac{Q^2}{\mu^2})} \tag{2.1}$$

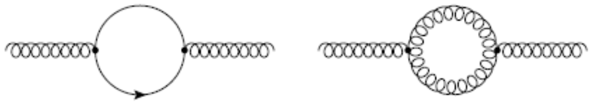
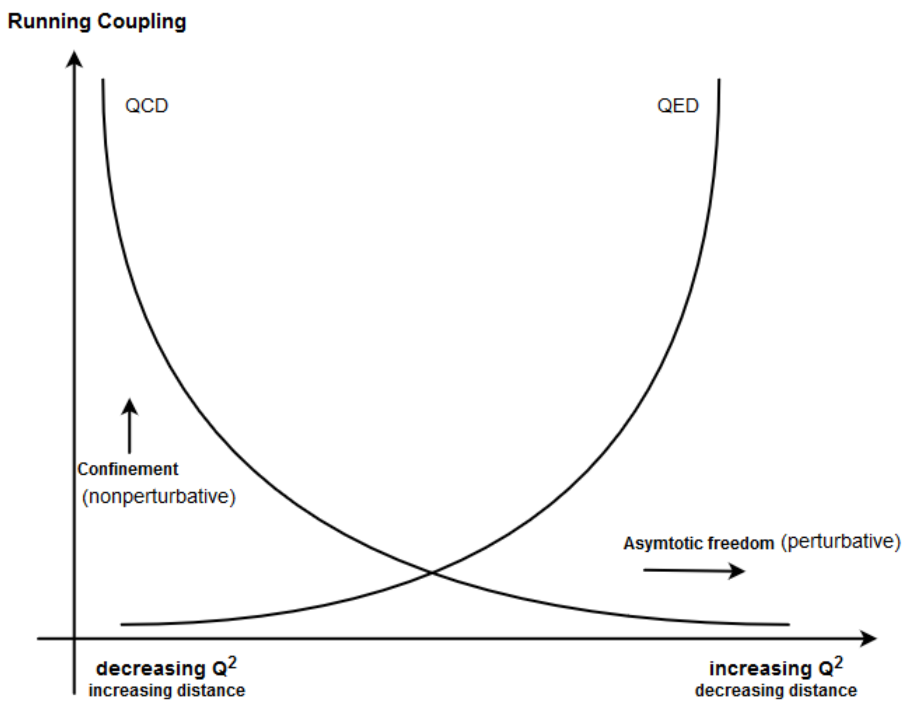


Figure 2.2: Running coupling compared for QED, with a positive and QCD with a negative beta function. The quark loop vacuum polarization diagram gives a negative contribution to $\beta_0 \sim n_f$ and the gluon loop gives a positive contribution to $\beta_0 \sim N_c$. The second contribution is bigger than the first, so that $\beta_0 > 0$ in QCD. The beta function in QED is negative since the second contribution does not exist $N_C = 0$.

Where $\beta_0 = \dfrac{11N_c - 2n_f}{12\pi}$, n_f the number of quarks comes from the first diagram and causes screening. N_c the number of colours is due to anti screening. Obviously, with $n_f = 6$ and $N_c = 3$ in the standard model we will get $\beta_0 > 0$. The Beta function is defined as:

$$\beta(\alpha) = -(\beta_0\alpha^2 + \beta_1\alpha^3 + \beta_2\alpha^4 + ...) = \frac{d\,\alpha(Q^2)}{d\,ln(Q^2)} \tag{2.2}$$

e.g. the first term of the beta-function is negative in contrast to QED with $\beta_0 = -\dfrac{\pi}{3} \rightarrow -(\beta_0\alpha^2) > 0$. The coupling constant in QCD will be increased with reduction of Q^2 (increasing distance), in QED vice versa.



Asymptotic freedom allows us to use perturbation theory. Actually there is need of two more things to make the connection between theory and experiment: either infrared safety or factorisation. That is clarified in the next chapter 3.2. Quarks have not yet been observed as free particles. With increasing separation it will be easier to produce a quark-antiquark pair than to isolate a quark because the coupling between them is too strong. This mechanism is called confinement. Confinement as a non-perturbative theory has been confirmed in Lattice QCD, but not yet mathematically. Quarks prefer to bind into hadrons which can be classified into baryons with three quarks state and mesons with a quark-antiquark state. The wave function of fermions must be antisymmetric according to the Pauli exclusion principle under exchange of two quarks. Interestingly, there are resonance states with spin $\frac{3}{2}$ like Δ^{++}. The spins of the three up quarks are parallel to each other, have the same flavour and orbital angular momentum L=0. This means that an exchange of flavour, spin and space (orbital angular momentum) does not lead to any change. This problem was explained with the additional degree of freedom, the so-called color charge. This additional factor N_C could be determined experimentally in the electron-positron annihilation into hadrons schematically in figure 2.3.

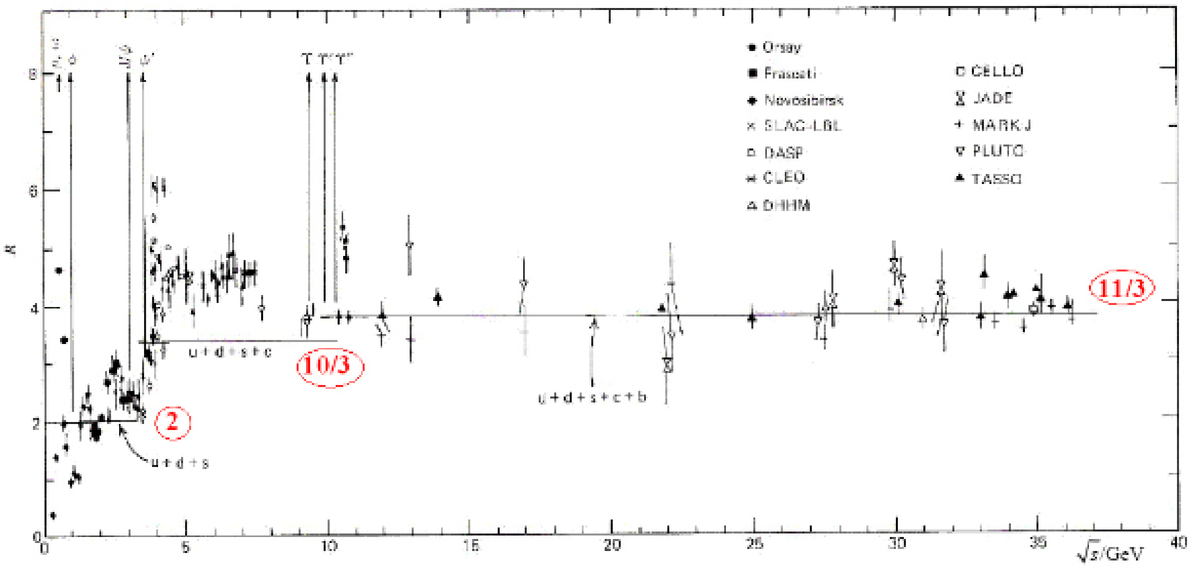


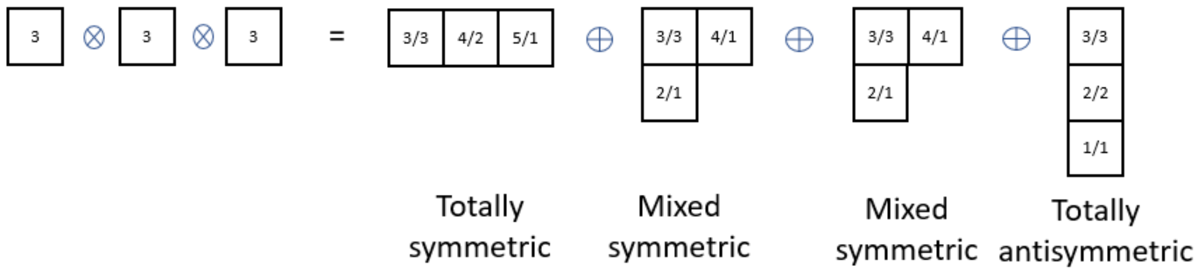
Figure 2.3: Key measurement at lepton collider, $R = \frac{\sigma(e^+e^-\to Hadrons)}{\sigma(e^+e^-\to\mu^+\mu^-)} = N_C\sum_q e_q^2$, evidence for $N_C = 3$ colours of quarks. Without this factor the ration is for u, d, s, u, d, s, c and u, d, s, c, b Respectively $\frac{2}{3}, \frac{10}{9}, \frac{11}{10}$ The experiment showed a third of the expected results [27].

Each quark comes in one of three colours red, green or blue and also anti-colour $\bar{r}, \bar{b}, \bar{g}$ for anti-quarks. The hadrons are colour singlets with regard to the hypothesis and so are invariant under rotations in colour space. The colour hypothesis describes the existence of mesons with $q\bar{q}$ and baryons with qqq.
The total wave function for each particle can be expressed:

$$\begin{aligned} \Psi_{3q} = \psi_{space} \times \chi_{spin} \times \theta_{colour} \times \phi_{flavour} \\ O(3) \quad SU(2) \quad SU(3) \quad SU(6) \end{aligned} \tag{2.3}$$

All possible colour states are described with Young Tableaux [16]. The group theory methods are used to decompose products of irreducible representations into sums for instance the Young Tableaux technique.
The same procedure for SU(2) and SU(6) for spins and flavours of the three quarks turns out:

$$\begin{aligned} 2 \otimes 2 \otimes 2 &= 4 \oplus 2 \oplus 2 \oplus 0 \\ 6 \otimes 6 \otimes 6 &= 56 \oplus 70 \oplus 70 \oplus 20 \end{aligned} \tag{2.4}$$



$= 10 \oplus 8 \oplus 8 \oplus 1$

2.2 QCD Lagrangian

QCD like QED and the weak interaction theory is described by representations of a symmetry group. From the condition that the Lagrangian must be invariant under arbitrary global and local symmetry transformations (Noether's theorem) follows the interaction terms. The Lagrangian of QCD is invariant under $U(1) \times SU(3)$ global transformation. The three Pauli matrices from $SU(2)$ can be replaced by the eight Gell-Mann λ^a with the following Lie algebra:

$$\begin{aligned} &T^a = \frac{1}{2}\lambda^a \\ &[T^a, T^b] = i f^{abc} T^c \qquad \text{fundamental representation} \\ &(T^a{}_{adj})_b c = -i f^{abc} \qquad \text{adjoint representation} \end{aligned} \tag{2.5}$$

To quantize QCD theory, the Faddeev-Popov method [12] is usually used in the path integral to fix a gauge and define a gluon propagator. The Lagrangian is given:

$$\begin{aligned} \mathscr{L} = &\sum_f \bar{\psi}_{if}(i\gamma^\mu \partial_\mu - m_f)\psi^{jf} - \frac{1}{4}F_a{}^{\mu\nu}F^a{}_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a{}_\mu)(\partial^\nu A^a{}_\nu) + (\partial^\mu \chi^{a*})(\partial_\mu \chi^a) \\ &-g_s \bar{\psi}_i T^a{}_{ij}\psi_j \gamma^\mu A^a{}_\mu - \frac{g_s}{2} f^{abc}(\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu)A_b{}^\mu A_c{}^\nu) - \frac{g_s{}^2}{4} f^{abc}(A_b{}^\mu A_c{}^\nu) f^{ade}(A^d{}_\mu A^e{}_\nu) \\ &-g_s f^{abc}(\partial^\mu \chi^{a*})\chi^b A^c{}_\mu \end{aligned} \tag{2.6}$$

The red marked part gives the Langrangian of free particles and the black part takes care of the interactions. $F^a{}_{\mu\nu}$ is field strength tensor for spin-1 gluon field $A^a{}_\mu$ corresponded to a non-Abelian gauge theory with the structure constants f^{abc}.

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - g_s f_{abc} A^b{}_\mu A^c{}_\nu \tag{2.7}$$

The third term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interaction. Here i, j are color indices in the fundamental representation, a, b, c run over 8 colour degrees of freedom of the gluon fields in the adjoint representation of SU(3). f labels the six flavours of the quarks. g_s describes the strong coupling constant. Due to gauge fixing the Langrangian in the non-Abelian case requires the introduction of additional fields. These anti commuting complex scalar fields χ^a are called Faddeev Popov ghosts [23, 29]. Feynman rules follows the Lagrangian. A list of the respective Feynman rules regarding the terms can be found in 6 which will be used for the aims of this work later.

2.3 Colour factor calculation

In this section the calculation of the Casimir operators of the respective diagrams is motivated, which occur later with the evaluation of the matrix elements. The generalization of the Pauli matrices are the so-called Gell-Mann matrices λ^a which is given by [26, 29]

$$T^a = \vartheta^a \equiv \frac{\lambda^2}{2} \tag{2.8}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}$$
$$\lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{2.9}$$

λ^3 and λ^8 are diagonal. These generators satisfy schematically:

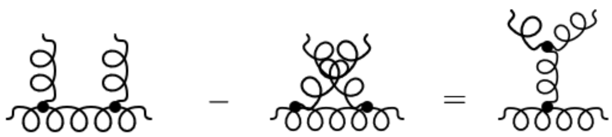
- in the fundamental representation

$[T^a, T^b] = if^{abc}T^c \Rightarrow$.

$T^aT^b \quad - \quad T^bT^a \quad = \quad if^{abc}T^c$

- in the adjoint representation

$[F^a, F^b] = if^{abc}F^c \Rightarrow$.



The most common convention for the normalization of the generators in physics is:

$$\sum_{c,d} f^{acd} f^{bcd} = N\delta^{ab} \tag{2.10}$$

One of the most important equations for the colour factor calculation is the Jaccobi-Identity:

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0 \tag{2.11}$$

In terms of the structure constant:

$$f^{axd} f^{bcx} + f^{cxd} f^{abx} + f^{bxd} f^{cax} = 0 \tag{2.12}$$

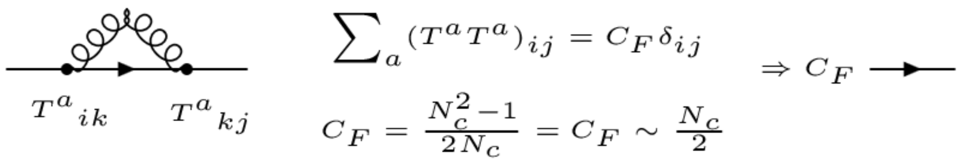
$$f^{abc} = -2i\, tr(T^a[T^b, T^c]) \tag{2.13}$$

Which generalises to:

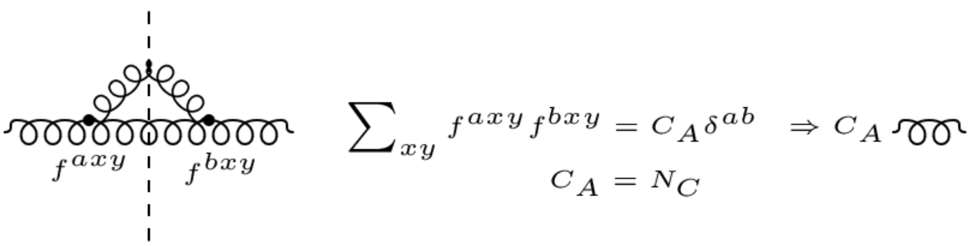
$$f^{abc} f^{xcd} = 4i\, tr(T^a[T^b, [T^c, T^d]]) \tag{2.14}$$

With these relations all Casimir operators can be calculated:

- Fundamental representation Casimir operator



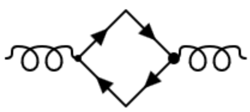
- Adjoint representation Casimir operator



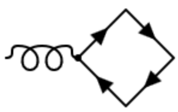
Which means the charge of the gluon is twice that of the quark because:

$$C_A = N_c = 2C_F \sim 2(\frac{N_C}{2}) \tag{2.15}$$

- Trace identities:



$T_f \, \delta^{ab}$

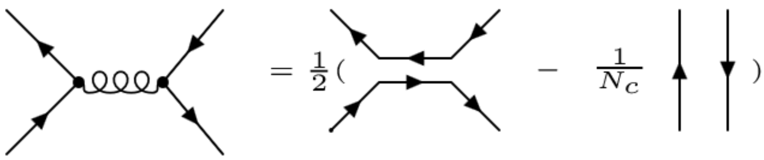


$tr(T^a) = 0$

- Fierz identity

$$\sum_a T_{ij}{}^a T_{kl}{}^a = \frac{1}{2}(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}) \tag{2.16}$$

With this identity is the difference between QED and QCD clarified. The charge transfer in QED takes place along the Fermion line because photons cannot transport charges. On the other hand, the gluons transfer color charges.



Other useful relations for the calculation of casimir operators in SU(N):

$$tr(T^a T^b) = T_{ij}{}^a T_{ji}{}^b = T_F \delta^{ab} \tag{2.17}$$

$$\sum_a (T^a T^a) = C_F \delta^{ij} \tag{2.18}$$

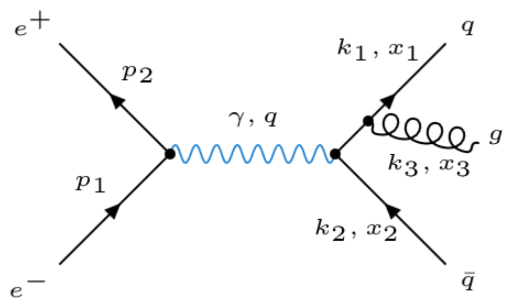
$$f^{acd} f^{bcd} = C_A \delta^{ab} \tag{2.19}$$

With $T_F = \frac{1}{2}$, $C_A = N$ and $C_F = \frac{N^2-1}{2N}$

3 Parton showers based on subtraction method

3.1 IR and Collinar Divergences

Beyond the LO (Leading order) diagrams based on massless particles, singularities occur. Consider first the process $e^-e^+ \to q\bar{q}g$



In order to calculate the cross section of electron-positron annihilation, regard the gluon emission from a parent (anti)quark. The amplitude of this process fig. 3.1 is:

$$\begin{aligned} A =& \frac{\bar{u}(k_1)(-ig_s\gamma^\nu \times T^a)[-i(\not{k}_1 + \not{k}_3)](-iee_q\gamma^\mu)v(k_2)\epsilon_\mu{}^{\lambda_1}\epsilon_\nu{}^{\lambda_2 *}}{(k_1+k_3)^2} \\ &- \frac{\bar{u}(k_1)(-iee_q\gamma^\mu)[i(\not{k}_2 + \not{k}_3)](-ig_s\gamma^\nu \times T^a)v(k_2)\epsilon_\mu{}^{\lambda_1}\epsilon_\nu{}^{\lambda_2 *}}{(k_1+k_3)^2} \\ \Rightarrow A =& -g_sT^a[\frac{\bar{u}\;\not{\epsilon}\,(\not{k}_1 + \not{k}_3)\,\Gamma\,v}{(k_1+k_3)^2} - \frac{\bar{u}\,\Gamma\,(\not{k}_2 + \not{k}_3)\;\not{\epsilon}\,v}{(k_2+k_3)^2}] \end{aligned} \tag{3.1}$$

With $\Gamma = (-iee_q\gamma^\mu)\epsilon_\mu{}^{\lambda_1}$.
Considering that the partons are on-shell, the amplitude can be written:

$$A = -g_sT^a[\frac{\bar{u}\;\not{\epsilon}\,(\not{k}_1 + \not{k}_3)\,\Gamma\,v}{2k_1\cdot k_3} - \frac{\bar{u}\,\Gamma\,(\not{k}_2 + \not{k}_3)\;\not{\epsilon}\,v}{2k_2\cdot k_3}] \tag{3.2}$$

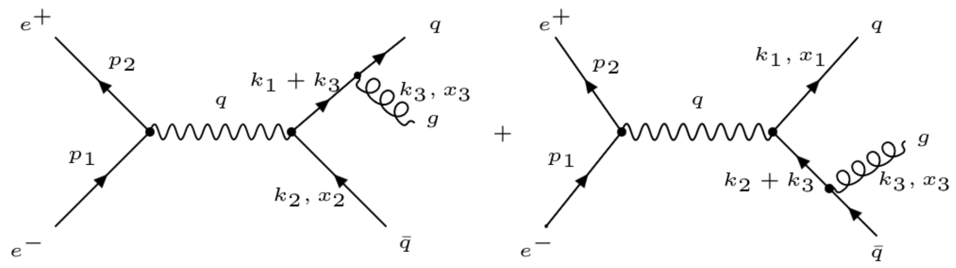


Figure 3.1: Left diagram $e^-e^+ \to qg\bar{q}$ and right $e^-e^+ \to q\bar{q}g$

In the soft limit with $k_0 \to 0$, the amplitude can be factorized in two parts:

$$A = -g_s T^a [\frac{k_1\, \epsilon}{k_1 \cdot k_3} - \frac{k_2\, \epsilon}{k_2 \cdot k_3}] A_{born} \qquad \text{with } A_{born} = \bar{u}\, \Gamma\, v \tag{3.3}$$

Where the left side contains all information about colour and momenta and the other one involves all spin information. The calculation of the cross section results in:

$$\begin{aligned} A = &- C_F g_s^2 \sigma^{born} \int \frac{d^3k}{2k_0(2\pi)^3} 2(\frac{k_1 \cdot k_2}{(k_1 \cdot k_3)(k_2 \cdot k_3)}) \\ &- C_F g_s^2 \sigma^{born} \int dcos\, \theta\, \frac{dk_0}{k_0} \frac{4}{(1 - cos\, \theta)(1 + cos\, \theta)} \end{aligned} \tag{3.4}$$

The energy fraction is defined as:

$$x_i = \frac{2E_i}{\sqrt{s}} = \frac{2q\ \cdot\ k_i}{s} \tag{3.5}$$

It can be shown that $\sum x_i = 2$ and thus, that only two of momenta are independent. The partonic differential cross section with respect to the quark and anti-quark momentum fractions is given by:

$$\frac{d^2\sigma}{dx_1 dx_2} = (\frac{4\pi\alpha}{s}) \sum {e_i}^2 \frac{2\alpha_s}{3\pi} \frac{{x_1}^2 + {x_2}^2}{(1 - x_1)(1 - x_2)} \tag{3.6}$$

There are three singularities within the final result. If the emitted photon is collinear to the outgoing quark or anti-quark ($x_1 \to 1$ or $x_2 \to 1$) and when the emitted gluon is soft ($x_1 \to 1$ and $x_2 \to 1$) 3.2. The singularities come from the quark propagator in each diagram. The denominators contain terms proportional to $\frac{1}{(k_i+k_j)^2}$. Without the neglectable quark mass follows:

$$\frac{1}{(k_i + k_j)^2} = \frac{1}{2k_i \cdot k_j} = \frac{1}{2E_iE_j(1 - cos\theta_{ij})} = \frac{1}{s(1 - x_k)} \tag{3.7}$$

One can show all possibilities for three partons through a triangle:

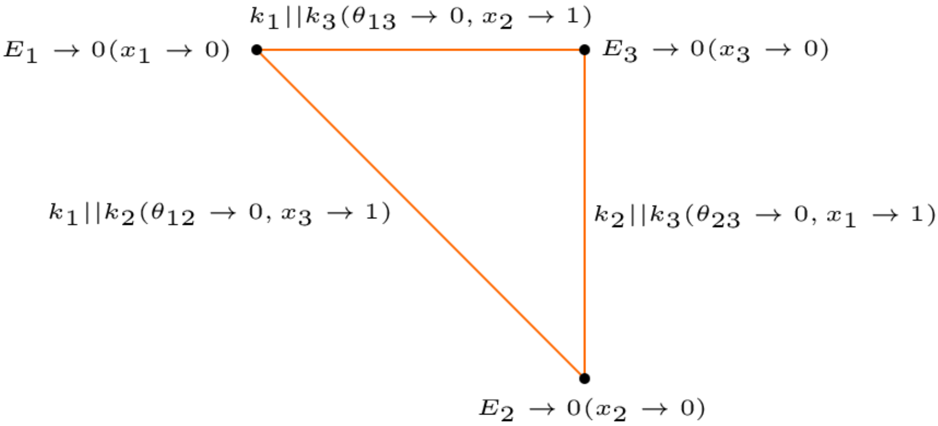


Figure 3.2: three-parton configurations at the boundaries of phase space

The Kinoshita-Lee-Nauenberg (KLN) Theorem assures that a summation over degenerate initial and final states removes all infrared (IR) divergences. The sum of the integrals $\int_R$ and $\int_V$ over the phase space is finite. However, this is not true for the individual contributions. In the next section it is seen how the infrared singularities are absorbed in

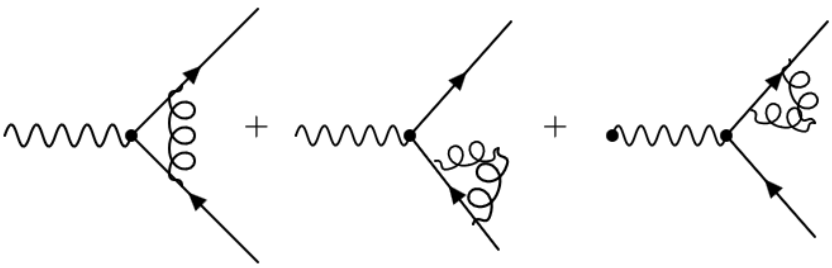


Figure 3.3: Virtual corrections: one-loop corrections to $e^-e^+ \to q\bar{q}$

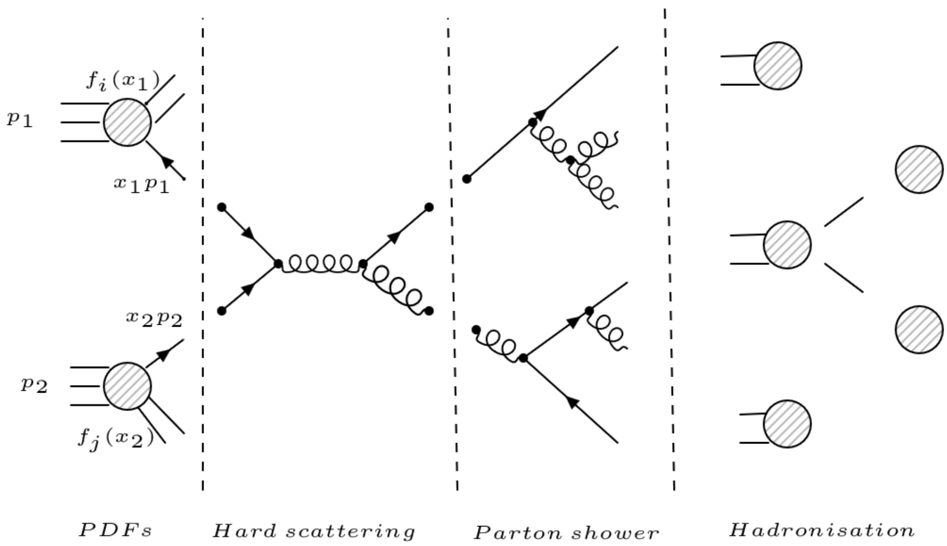
the parton distributions in connection with the deep inelastic scattering (DIS) [3].

3.2 Hard scattering

In the last section it was seen that IR singularities occur in QCD due to the collinearity of two partons or the soft energy of a parton. These divergences happen long time after the initial hard scattering (long distance physics). Fortunately, the infrared-safe observables are calculable in perturbative QCD. But this is not the best way to solve this problem. An interesting way for this aim is the factorization theorem. By this theorem the singularities of long-distance physics are removed from the partonic cross section and absorbed into the parton distribution of the hadrons. This is even feasible for all orders of calculation. The background idea is that the partonic cross section is calculable in perturbation theory. Consider the hadron hadron scattering:

$$\sigma = \sum_{ij} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \sigma_{ij}(x_1, x_2, Q^2/\mu^2 ...) \tag{3.8}$$

With a factorisation scale μ the long and short-distance physics could be separated.



Partons with a greater momentum than μ participate in the hard-scattering and those with less momentum are considered as components of the hadron structure and accordingly absorbed in the parton distribution [22].
The factorisation theorem also applies to deep inelastic scattering, with one of the parton distributions replaced by an $e^+e^-\gamma$ vertex. The DIS cross section can be written as [10]

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4}[(1-y)F_2(x,Q^2) + xy^2F_1(x,Q^2)] \tag{3.9}$$

In this case, the structure function needs to be introduced:

$$F_2{}^{exp}(x) = \sum_i e_i{}^2 x f_i(x) \tag{3.10}$$

It is defined as the charge weighted sum of the parton momentum densities which describe

the probability that the parton carries a momentum fraction between x and ∂x of the proton momentum. The index i denotes the quark flavour. Parton distributions are non-pertubative. Fortunately, they are universal and obtainable from a fit to the data for a particular factorisation scheme and usable in other processes.
The perturbative evolution kernel of a parton distribution due to splitting can be described by the solution of DGLAP evolution equation [31]. It is based on the collinear factorisation property of QCD. Fixing the accuracy of the calculation and the factorisation scheme the evolution kernel is well defined.

$$\frac{\partial f(x,\mu^2)}{\partial\, ln\, \mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} f_j(y,\mu^2) P_{ij}(\frac{x}{y}) + O({\alpha_s}^2) \tag{3.11}$$

This is a system of coupled integral or differential equations. $P_{ij}(\frac{x}{y})$ represents the probability, a daughter parton i with momentum fraction $\frac{x}{y}$ is splits from a parent parton j. The above convolution in compact notation (Mellin Convolution) in the general case:

$$\frac{\partial f_i(x,\mu^2)}{\partial\, ln\, \mu^2} = \sum_{j=-n_f}^{n_f} P_{ij} \otimes f_j(\mu^2) \tag{3.12}$$

The four splittings probabilities are illustrated in the diagram 3.4 and the qq, qg, gg and gq transitions lead to a set of $2n_f + 1$ coupled evolution equations.

$$\frac{\partial}{\partial\, ln\, \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{bmatrix} P_{qq} & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{bmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \tag{3.13}$$

A parton distribution changes when a different parton splits or the parton itself splits. Parton densities can not be analytically determined, but it is possible to predict how they evolve from one scale to another. PDFs are measured in one process and use them as an input for another process.

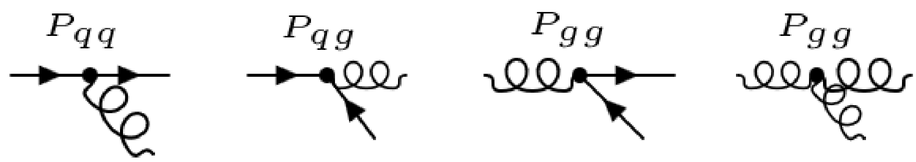


Figure 3.4: The four splitting probabilities

The spin-averaged unregularized Altarelli-Parisi splitting functions in d-dimensions are given by:

$$\left.\begin{aligned} \langle\, \hat{P}_{qq} \rangle &= C_F[\frac{1+z^2}{1-z} - \varepsilon(1-z)] \\ \langle\, \hat{P}_{gq} \rangle &= T_R[1 - \frac{2z(1-z)}{1-\varepsilon}] \\ \langle\, \hat{P}_{qg} \rangle &= C_F[\frac{1+(1-z)^2}{z} - \varepsilon z] \\ \langle\, \hat{P}_{gg} \rangle &= 2C_A[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)] \end{aligned}\right\} \text{ Alterali-Parisi} \tag{3.14}$$

3.3 Subtraction method

The subtraction term is constructed as a sum over all possible dipole configurations, i.e. all possible combinations of two partons are formed to build an emitter while every single one of the remaining partons is considered a spectator. The general quadratic matrix element is defined as:

$$|A|^2 = |A^{(0)}{}_m|^2 + |A^{(0)}{}_{m+1}|^2 + 2Re(A^{(0)*}{}_n A^{(1)}{}_m) \tag{3.15}$$

Where $|A^{(0)}{}_m|^2$ is the tree level contribution (Born sector) from LO and has no divergences, $|A^{(0)}{}_{m+1}|^2 + 2Re(A^{(0)*}{}_m A^{(1)}{}_m)$ comes from NLO and each of them is divergent. The problem in this case is the Integrals which can not be combined due to different phase space dimensions:

$$\sigma^{NLO} = \int_{m+1} \partial\sigma^R + \int_m \partial\sigma^V \tag{3.16}$$

The real and virtual contributions are both IR divergent and need to be regularised in $d = 4 - 2\epsilon$ dim. To tackle this problem one can use the subtraction method with adding and subtracting a local counter term $\partial\sigma^A$ with the same singularity structure as term $\partial\sigma^R$ to the integral. $\partial\sigma^A$ approximates the soft and collinear singularities of $\partial\sigma^R$.

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R - \partial\sigma^A] + \int_m [\partial\sigma^V + \int_1 \partial\sigma^A] \tag{3.17}$$

In this case, one can safely set $\epsilon \to 0$ for $\partial\sigma^R|_{\epsilon\to 0} - \partial\sigma^A|_{\epsilon\to 0}$ under the integral sign in the first term and calculate the integral numerically in 4-dimensions. All the singularities are now associated to the last two terms on the right-hand side. If one is able to carry out analytically the integration of $\partial\sigma^A$ over one parton subspace leading to the ϵ poles, one can combine these poles with those in $\partial\sigma^V$, thus cancelling all the divergences, performing the limit $\epsilon \to 0$ and carrying out numerically the remaining integration over the m-parton phase space [6].

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R|_{\epsilon\to 0} - \partial\sigma^A|_{\epsilon\to 0}] + \int_m [\partial\sigma^V + \int_1 \partial\sigma^A]_{\epsilon\to 0} \tag{3.18}$$

The virtual contribution must be UV-finite:

$$\int_m \partial\sigma^V = \int_m [\int_{loop} \partial\sigma^V{}_{bare} + \sigma^V{}_{Counter\,term}] \tag{3.19}$$

The addition of $\int_1 \partial\sigma^A$ to the $\int_m \partial\sigma^V$ ensures that IR poles are cancelled. The bare and counter contribution are separately divergent and have also different integral dimensions. One can use the same idea with the subtraction method to solve this problem [5]

$$\int_m \partial\sigma^V + \int_{loop} \partial\sigma^L - \int_{loop} \partial\sigma^L = \int_m \int_{loop} [\partial\sigma^V{}_{bare} - \partial\sigma^L] + \int_m [\sigma^V{}_{Counter\,term} + \int_{loop} \partial\sigma^L] \tag{3.20}$$

together one receives:

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R - \partial\sigma^A] + \int_m \int_{loop} [\partial\sigma^V{}_{bare} - \partial\sigma^L] + \int_m [\sigma^V{}_{Counter\,term} + \int_{loop} \partial\sigma^L + \partial\sigma^A] \tag{3.21}$$

Determination of emission kernels

Now we want to introduce the properties of the counter term $\partial\sigma_A$. This term needs to have the same behaviour as $\partial\sigma_R$ in d dimensions. This process and specific observable has to be obtained in a way that is independent of the particular jet observable considered. It must be exactly integrable analytically over one-parton phase space in d and $\partial\sigma_R - \partial\sigma_A$ has to be integrable via Monte Carlo methods. $\partial\sigma_A$ acts as a local counter-term for $\partial\sigma_B$ at this point. One derives improved factorisation formulae which are called dipole or antenna formulae:

$$\partial\sigma_A = \sum_{dipoles} \partial\sigma_B \otimes \partial V_{dipoles} \tag{3.22}$$

Where the sum is over dipoles for all $m+1$ configurations with consideration to a given m-parton state. $\partial\sigma_B$ describes the color/spin projection of the Born-level exclusive cross section. The symbol $\otimes$ describes phase space convolutions and sums over colour and spin indices. $\partial V_{dipoles}$ will be computed and its singular properties matched to the real part. The Dipoles are universal in the sense that they do not depend on the hard scattering. This allows the use of a factorisable mapping from the m + 1-parton phase space to an m-parton subspace. That will be clearer when the parametrisation is used in the next chapter. The integral over all $m+1$ configuration can be written:

$$\int_{m+1} \partial\sigma_A = \int_m \partial\sigma_B \otimes \sum_{dipoles} \int_1 \partial V_{dipoles} \tag{3.23}$$

This is the important result because this can now be written in terms of the known m-sector from LO and the other universal factor which contains all ϵ-poles [6].

Singularity Structure

Before beginning with the collinear limit or soft limit respectively the structure of matrix element from LO has to be introduced:

$$\mathcal{M}_m{}^{c_1,\ldots,c_m;s_1,\ldots,s_m}(p_1,\ldots,p_m) \tag{3.24}$$

c_i, s_i and p_i denote respectively the colour, spin indices and the momenta for each m-parton in the tree level matrix element in the initial/final-state. A common used method here is to define a basis in colour+helicity space.

$$\mathcal{M}_m{}^{c_1,\ldots,c_m;s_1,\ldots,s_m}(p_1,\ldots,p_m) \equiv (< c_i,..,c_m| \otimes < s_1,\ldots,s_m|)|1,..,m>_m \tag{3.25}$$

With $< c_i,..,c_m| \otimes < s_1,\ldots,s_m|)|$ as the basis and $|1,..,m>_m$ as a vector in this space. Thus, for the matrix element squared:

$$\begin{aligned} |\mathcal{M}_m|^2 =& (< c_i,..,c_m| \otimes < s_1,\ldots,s_m|)(|c_i,..,c_m> \otimes |s_1,\ldots,s_m>) < 1,..,m|1,..,m> \\ &= \delta_{c_1c_1},\ldots\delta_{c_mc_m} \otimes \delta_{s_1s_1},\ldots\delta_{s_ms_m}\ {}_m < 1,..,m|1,..,m>\ {}_m \end{aligned} \tag{3.26}$$

Define a colour-charge operator T_i with the emission of a gluon from each parton i:

$$T_i = T_i{}^c|c> \tag{3.27}$$

Its action onto the colour space is defined by:

$$< c_1, .., c_i, ..c_m, c|T_i|b_1, .., b_i, ..b_m >= \delta_{c_1 b_1} ... T^c{}_{c_i b_i} ... \delta_{c_m b_m} \tag{3.28}$$

Where $T^c{}_{c_i b_i}$ is the colour-charge matrix in the adjoint representation for the gluon emission or colour-charge matrix in the fundamental representation for the quark/anti-quark emission case. The following properties must be taken into account:

$$\begin{aligned}
&T_i \cdot T_j = T_j \cdot T_i && \text{if } i \neq j, \text{ commutative property} \\
&T_i^2 = C_i \quad C_i = C_A && \text{for gluon and } C_i = C_F \text{ for (anti)quark} \\
&\sum_{i=1}^{m} T_i|1, ..., m >_m = 0 && \text{for single state}
\end{aligned} \tag{3.29}$$

Thus, the square of colour-correlated tree-amplitudes for the indices I, J referring either to final-state or initial-state partons will be [5, 6]

$$|\mathcal{M}^{I,J}{}_{m,a...}|^2 = {}_{m,a...} < 1, ..., m; a, ...|T_I \cdot T_J|1, ..., m; a, ... > {}_{m,a...} \tag{3.30}$$

Dipole factorisation

Depending on the investigated region, there is a total of three factorisation possibilities, see figures 3.5, 3.6 and 3.7.

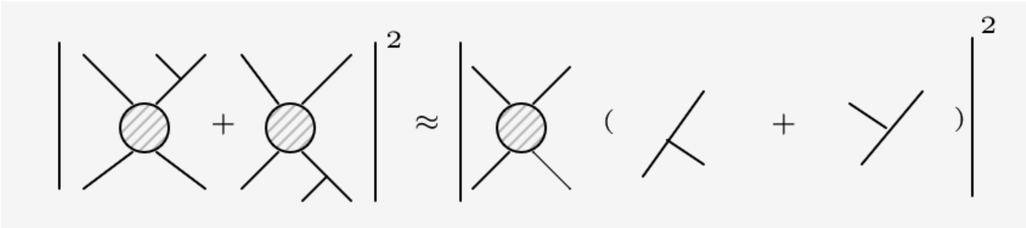


Figure 3.5: Soft factorisation

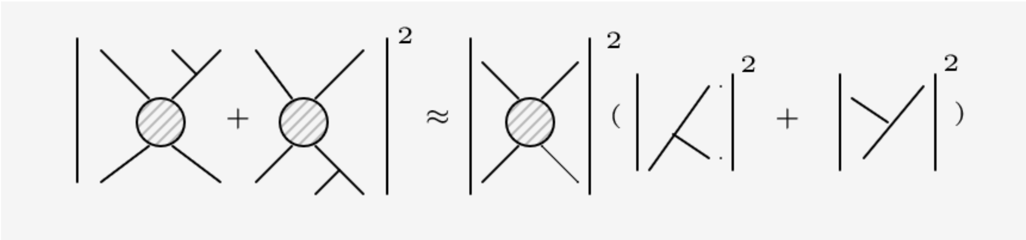


Figure 3.6: Collinear factorisation

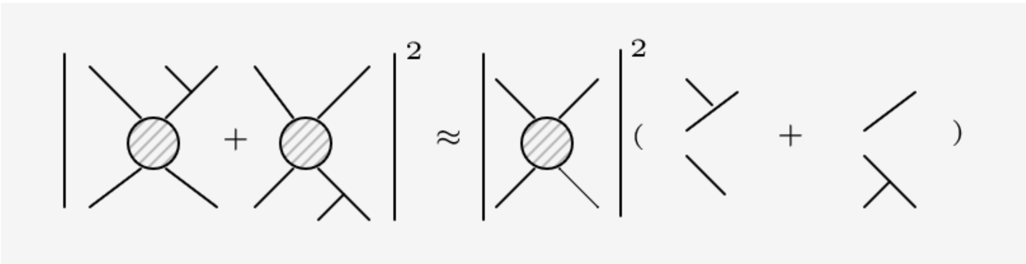


Figure 3.7: Dipole/Antenna factorisation

To explain the meaning of the factorisation consider $(m+1)$-partons with the general matrix element [6, 30]

$$|\mathcal{M}_{m+1}(Q; p_1, ..., p_i, ..., p_j, ..., p_{m+1})|^2 \tag{3.31}$$

In the soft region the momentum p_j can be parametrised with $p_j \to \lambda q, \lambda \to 0$, where q is an arbitrary four vector and λ a scale parameter. The squared matrix element is characterised by $|\mathcal{M}|^2 \sim \frac{1}{\lambda^2}$. If p_i and p_j become collinear, the parametrisation $p_j = \frac{z}{1-z} p_i$ will be chosen. So the matrix element will be $|\mathcal{M}|^2 \sim \frac{1}{p_i \cdot p_j}$. This get covered in more detail in the next chapter. Here a summery of the behaviour of the matrix elements in different regions is given. Based on the Catani-Seymour method for $(m+1)$-partons.

$$|\mathcal{M}_{m+1}|^2 \to \sum |\mathcal{M}_m|^2 \otimes V_{ij,k} \Rightarrow |\mathcal{M}_{m+1}|^2 \to \sum |\mathcal{M}_m|^2 \otimes V_{ij,k} \tag{3.32}$$

$V_{ij,k}$ a singular factor including parton k and its interaction with partons i and j from the m-parton amplitude. This situation can be represented by the diagram 3.8.

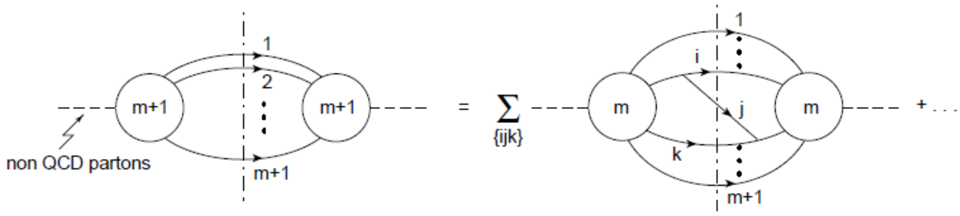


Figure 3.8: Factorisation in dipole formalism
[5]

Here i and j are the emitters and k plays the role of a spectator. The spectator absorbs a longitudinal recoil that arises when a splitting is performed with all participating partons remaining on their mass shells. The blobs denote the tree-level matrix elements and their complex conjugate. The dots on the right-hand side stand for non-singular terms both in the soft and collinear limits. When the partons i and j become soft and/or collinear, the singularities are factorized into the term $V_{ij,k}$ (the dashed box on the right-hand side) which embodies correlations with a single additional parton k.

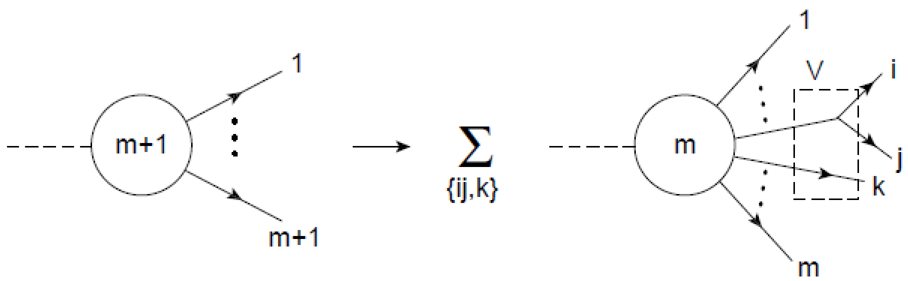
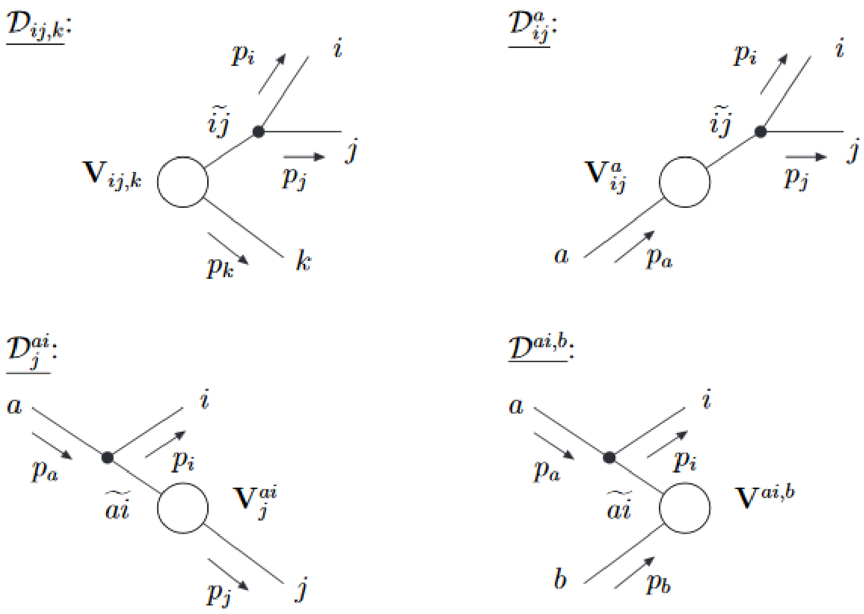


Figure 3.9: Effective diagrams for the different emitter–spectator cases.
[6]

In this context the different dipole factorisation for both initial states and final states shall be presented. All these different possibilities can be seen in the diagram 3.9. $\mathcal{D}_{ij,k}$ describes the final state emitter and final-state spectator (FF), $\mathcal{D}^{a}{}_{ij}$ the final-state emitter and initial-state spectator (FI), $\mathcal{D}^{ai}{}_{k}$ the initial-state emitter and final-state spectator (IF) and $\mathcal{D}^{aj,b}$ the initial-state emitter and initial-state spectator (II). The dipole factorisation formula for all these possibilities is [28]

$$|\mathcal{M}_m + 1|^2 = \sum_{i,j}\sum_{k\neq i,j} \mathcal{D}_{ij,k} + \sum_{i,j}\sum_{a} \mathcal{D}^{a}{}_{ij} + \sum_{a,i}\sum_{k\neq i} \mathcal{D}^{ai}{}_{k} + \sum_{a,i}\sum_{b\neq a} \mathcal{D}^{aj,b} + ... \tag{3.33}$$

In each term i, j and k denote final-state partons and a and b stand for initial-state partons. Note that there are many non-divergent contributions or diagrams, which are marked here with the dots. In this work the first contribution final-state emitter and final-state spectator (FF) is used. Later it becomes clear how to deal with the formula.



The circle in the center of each sub diagram presents the m-partons matrix element and the tilde labels the collinear splitting process for the initial or final states. For this work, the first upper diagram with final-state singularities without initial-state partons, is completely sufficient and is discussed here in detail with its formula. The matrix element for this is written:

$$|\mathcal{M}_{m+1}|^2 =< 1, ..., m+1||1, ..., m+1 >= \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) + \text{finite terms} \tag{3.34}$$

The first term with the sum over dipoles is divergent as $p_i \cdot p_j \to 0$. For the case of final-state emitters with a final-state spectator, for instance, the individual dipole contributions:

$$\mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) = \frac{-1}{2p_i \cdot p_j}{}_m < 1, ..., \tilde{ij}, ..., k, ..., m+1| \frac{T_k \cdot T_{ij}}{{T_{ij}}^2} V_{ij,k}|1, ..., \tilde{ij}, ..., k, ..., m+1 >_m \tag{3.35}$$

Where $T_k \cdot T_{ij}$ are the color charges of spectator and emitter, $V_{ij,k}$ splitting kernel in helicity space of emitter explicit form depends on parton type become proportional to Altarelli-Parisi splitting functions 3.14 and eikonal factors in collinear and soft limits.
The occurring m-parton states are constructed from the original $(m+1)$-particle matrix element by replacing the partons i and j with the new parton $\widetilde{ij}$, and the original parton

k with the $\widetilde{k}$ spectator. In the massless case, their momenta are given by:

$$\begin{aligned} \tilde{p}_{ij}^{\mu} &= {p_i}^{\mu} + {p_j}^{\mu} - \frac{y_{ij,k}}{1-y_{ij,k}} {p_k}^{\mu} \\ \tilde{p}_{k}^{\mu} &= \frac{1}{1-y_{ij,k}} {p_k}^{\mu} \\ y_{ij,k} &= \frac{p_i \cdot p_j}{p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i} \end{aligned} \tag{3.36}$$

Note, that momenta are on-shell. Due to momentum conservation ${p_i}^{\mu} + {p_j}^{\mu} + {p_k}^{\mu} = \tilde{p}_{k}^{\mu} + \tilde{p}_{ij}^{\mu}$.

3.4 Mapping 3 partons to 2 for single emission

At the beginning of this thesis a parametrisation was used, which unfortunately only works for LO. This was recognized later and therefore the old parametrisation needed to be replaced by a new kinematics. In this section, the two parametrisations must be introduced. The important relations just as well the scalar products, Lorenz transformation regarding the new kinematics and a concept for the more efficient calculation of singular terms are presented. This theoretical description will gradually become clearer as the implementation of the mappings begins [14, 24, 25, 26].

3.5 Old mapping

The old parametrisation is defined as follows:

$$\left. \begin{aligned} {q_i}^{\mu} &= z{p_i}^{\mu} + y(1-z){p_j}^{\mu} + \sqrt{zy(1-z)}{m^{\mu}}_{\perp} \\ q^{\mu} &= (1-z){p_i}^{\mu} + yz{p_j}^{\mu} - \sqrt{zy(1-z)}{m^{\mu}}_{\perp} \\ {q_j}^{\mu} &= (1-y){p_j}^{\mu} \end{aligned} \right\} \quad \text{parametrisation} \tag{3.37}$$

Where q is the radiated soft momentum, q_i the momenta of the emitter and q_j the momentum of the spectator is. The variable $y_{ij,k} = \frac{q_i \cdot q}{p_i \cdot p_j}$ need to be introduced, that is zero in the soft and collinear limit. Note that both the emitter and the spectator are on-shell. From the conservation of impulses derives:

$$ {q_i}^{\mu} + q^{\mu} + {q_k}^{\mu} = {p_i}^{\mu} + {p_j}^{\mu} + {m^{\mu}}_{\perp} \tag{3.38}$$

For this mapping it is useful to calculate some common relations which are computed as follow:

$$
\begin{aligned}
&{q_i}^\mu + q^\mu = {p_i}^\mu + y{p_j}^\mu \\
&{q_j}^\mu + q^\mu = (1-z){p_i}^\mu + (1+yz-y){p_j}^\mu - \sqrt{zy(1-z)}m^\mu{}_\perp \\
&q_i \cdot q = y(p_i \cdot p_j) \\
&q_i \cdot q_j = z(1-y)(p_i \cdot p_j) \\
&q_j \cdot q = (1-z)(1-y)(p_i \cdot p_j)
\end{aligned}
\tag{3.39}
$$

For the simplification of the third equation $q_i \cdot q$, the on-shell condition of the emitter $q_i = 0$ was used, consequently:

$$
\begin{aligned}
q_i \cdot q_i = 0 = 2yz(1-z)p_i \cdot p_j + yz(1-z)m^{\mu 2}{}_\perp \\
\Rightarrow m^2{}_\perp = -2p_i \cdot p_j
\end{aligned}
\tag{3.40}
$$

If one uses this result in the equation $q_i \cdot q$, one gets the desired result as $q_i \cdot q = y(p_i \cdot p_j)$.

3.6 Mapping $m+1$ partons to m for multi-emissions

For the general m emission case a new mapping must be defined. The parametrisation of the splitting momenta is formalised as:

$$
\begin{aligned}
&{k_l}^\mu = \alpha_l \alpha \Lambda^\mu{}_\nu {p_i}^\nu + y\beta n^\mu + \sqrt{y\alpha_l\beta_l}n^\mu{}_{\perp,l} \qquad\qquad l = 1, ..., m \\
&{q_i}^\mu = (1 - \sum_{l=1}^{m} \alpha_l)\alpha\Lambda^\mu{}_\nu {p_i}^\nu + y(1 - \sum_{l=1}^{m} \beta_l)n^\mu - \sqrt{y\alpha_l\beta_l}n^\mu{}_{\perp,l} \\
&{q_k}^\mu = \alpha\Lambda^\mu{}_\nu {p_k}^\nu \qquad\qquad k = 1, ..., n \qquad\quad k \neq i
\end{aligned}
\tag{3.41}
$$

$k = 1, ..., n$ labels the emission momenta and is taken to be massless ${k_l}^2 = 0$. Where the label l denotes the count of emissions. In this work just the one-emission kernels is considered. All hard momenta in this mapping are also on-shell, ${p_k}^2 = {q_k}^2 = 0$. n^μ is an auxiliary light-like vector which is necessary to specify the transverse component of $n^\mu{}_{\perp,l}$. To absorb the recoil n^μ is defined as:

$$
n^\mu = Q^\mu - \frac{Q^2}{2p_i \cdot Q}{p_i}^\mu
\tag{3.42}
$$

Whereby Q is the total momentum with:

$$
Q^\mu = {q_i}^\mu + \sum_{l=1}^{m} k_l^\mu + \sum_{k=1}^{m} q_k^\mu = {p_i}^\mu + \sum_{k=1}^{m} p_k^\mu
\tag{3.43}
$$

To fulfil the condition that the emission momenta are massless, the following condition must be taken into account:

$$
\begin{aligned}
n^\mu{}_{\perp,l}\Lambda^\mu{}_\nu {p_i}^\nu = n_{\perp,l} \cdot n = n_{\perp,l} \cdot Q = 0 \\
n^\mu{}_{\perp,l} \cdot p_k \neq 0
\end{aligned}
\tag{3.44}
$$

$n_{\perp,l}^2 = -2\alpha\Lambda^\mu{}_\nu p_i{}^\nu n_\mu$ is not on-shell and in terms of single emission case it follows $n_{\perp,1}^2 = -2p_i \cdot Q$. The parameter y is related to the virtuality of the splitting parton:

$$q_i{}^\mu + \sum_{l=1}^{m} k_l^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu + yn^\mu \tag{3.45}$$

With $\alpha = \sqrt{1-y}$.

Lorenz transformation of momenta

From the parametrisation it is immediately to calculate the respective Lorenz transformation for the Emitters, Spectator and total momentum first. The general definition of the transformation is given by:

$$\begin{aligned}\alpha\Lambda^\mu{}_\nu &= p_i{}^\mu p_{i\nu}\frac{-y^2Q^2}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + p_i{}^\mu Q_\nu \frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}\\ &+ Q^\mu p_{i\nu}\frac{(y^2-y-y\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}\eta^\mu{}_\nu\end{aligned} \tag{3.46}$$

In the collinear limit with $y \to 0, \alpha \to 1$ this transformation reduces to trivial $\eta^\mu{}_\nu$. Finally we are going to compute the Lorenz transformation of the Momenta. The detailed calculation of Lorenz transformation for the Emitters, Spectator and total momentum can be found in the appendix section 6.

$$\boxed{\hat{p_i}^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu = p_i{}^\mu} \tag{3.47}$$

$$\boxed{\hat{Q}^\mu = \tfrac{Q^2}{2p_i\cdot Q} y\, p_i{}^\mu + (1-y)\, Q^\mu} \tag{3.48}$$

$$\boxed{\hat{p_k}^\mu = A_1\, p_i{}^\mu + A_2\, Q^\mu + \sqrt{1-y} p_k{}^\mu} \tag{3.49}$$

with

$$\begin{aligned}A_1 &\equiv \frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}\\ A_2 &\equiv \frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}\end{aligned}$$

3.7 Single emission part

In terms of one emission where $l = 1$ the mapping will be simplified to:

$$\begin{aligned}
{k_1}^{\mu} &= (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))p_i{}^{\mu} + y\beta_1 Q^{\mu} + \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\
{q_i}^{\mu} &= (\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i \cdot Q}))p_i{}^{\mu} + y\alpha_1 Q^{\mu} - \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\
{q_k}^{\mu} &= \alpha\Lambda^{\mu}{}_{\nu} {p_k}^{\nu} \qquad k = 1, ..., n \qquad k \neq i
\end{aligned} \tag{3.50}$$

Short form:

$$\begin{aligned}
{k_1}^{\mu} &= \zeta_1 p_i{}^{\mu} + \lambda_1 Q^{\mu} + \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\
{q_i}^{\mu} &= \zeta_q p_i{}^{\mu} + \lambda_q Q^{\mu} - \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,l} \\
{q_k}^{\mu} &= A_1 p_i{}^{\mu} + A_2 Q^{\mu} + \sqrt{1-y} p_k^{\mu}
\end{aligned}$$

3.8 Common scalar products

For the investigating of the mapping it is useful to determine first the dot products between these four vectors. The often occurring pre-factor products are given in 6.

$$\boxed{k_1 \cdot q_i = y(\alpha_1 + \beta_1)^2 \, p_i \cdot Q = y \, p_i \cdot Q} \tag{3.51}$$

$$\boxed{k_1 \cdot q_k = [\alpha_1(1-y) + y\beta_1(\tfrac{Q^2}{2p_i \cdot Q})] \, p_i \cdot p_k + y\beta_1 \, Q \cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{3.52}$$

$$\boxed{q_i \cdot q_k = [\beta_1(1-y) + y\alpha_1(\tfrac{Q^2}{2p_i \cdot Q})] \, p_i \cdot p_k + y\alpha_1 \, Q \cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{3.53}$$

3.9 Recipe for the usage of the new parametrisation

As in the last chapter mentioned the singularities come from the propagators in each diagram since the denominators contain according Feynmann rules terms with $\sim \frac{1}{2q_a \cdot q_b}$. Whereby a and b label the respective momenta. Since the calculations are sometimes very complicated and confusing, the procedure for eliminating the finite terms is as follows:
In the calculating of the square matrix elements always appear products in the form of $p_a \cdot p_b$ both in the numerator and denominator. The denominator shows which pre-factor causes the singularity. These terms from the numerator with the same prefix can be omitted from the beginning because they appear in both the denominator and the numerator and are therefore finite. This is explicitly shown below for a common denominators. This operation can be performed for the other types of the denominator.

3.9.1 Parametrization in terms of $(k_1 \cdot q_i)(k_1 \cdot q_k)$

$$\boxed{(k_1 \cdot q_i)(k_1 \cdot q_k) \approx y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} \tag{3.54}$$

Here you can quickly see that this term converges for $y \to 0$ and $\beta_1 \to 1$ towards zero. That means, you could ignore all terms with $y(1-\beta_1)$. However, since the equation becomes rather large quickly if we first use all the momenta products and then drop the terms with the pre-factor out of the denominator, this is already done for the scalar products. And this is exactly the biggest simplification in the calculation. The result looks like this:

$$\begin{aligned}
{k_1}^\eta {k_1}^{\eta'} &= [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})^2]{p_i}^\eta {p_i}^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q}){p_i}^\eta Q^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})Q^\eta {p_i}^{\eta'} \\
{k_1}^\eta {q_i}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y{\beta_1}^2 Q^\eta {p_i}^{\eta'} \\
{q_i}^\eta {k_1}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y{\beta_1}^2 {p_i}^\eta Q^{\eta'} \\
{q_i}^\eta {q_i}^{\eta'} &= {\beta_1}^2 {p_i}^\eta {p_i}^{\eta'} \\
{k_1}^\eta {q_k}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i \cdot Q})]\sqrt{1-y}{p_i}^\eta {p_k}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i \cdot Q})A_1\, {p_i}^\eta {p_i}^{\eta'} \\
&\quad - y\beta_1(\frac{Q^2}{2p_i \cdot Q})A_2\, {p_i}^\eta Q^{\eta'} + y\beta_1 A_1\, Q^\eta {p_i}^{\eta'} + y\beta_1 A_2\, Q^\eta Q^{\eta'} + y\beta_1\sqrt{1-y}Q^\eta {p_k}^{\eta'} \\
{q_i}^\eta {q_k}^{\eta'} &= A_1\beta_1 {p_i}^\eta {p_i}^{\eta'} + A_2\beta_1 {p_i}^\eta Q^{\eta'} + \beta_1\sqrt{1-y}{p_i}^\eta {p_k}^{\eta'} \\
{q_k}^\eta {k_1}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i \cdot Q})]\sqrt{1-y}{p_k}^\eta {p_i}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i \cdot Q})A_1\, {p_i}^\eta {p_i}^{\eta'} \\
&\quad - y\beta_1(\frac{Q^2}{2p_i \cdot Q})A_2\, Q^\eta {p_i}^{\eta'} + y\beta_1 A_1\, {p_i}^\eta Q^{\eta'} + y\beta_1 A_2\, Q^\eta Q^{\eta'} + y\beta_1\sqrt{1-y}{p_k}^\eta Q^{\eta'} \\
{q_k}^\eta {q_i}^{\eta'} &= A_1\beta_1 {p_i}^\eta {p_i}^{\eta'} + A_2\beta_1 Q^\eta {p_i}^{\eta'} + \beta_1\sqrt{1-y}{p_k}^\eta {p_i}^{\eta'}
\end{aligned} \tag{3.55}$$

Concept

Before the procedure is explained it should be mentioned the steps are gradually explained in more detail in the next chapter. This only provides a rough overview and can be used as a reference for the computing.

i) Considering a possible splitting. For this one has to ensure that each possible meaningful diagrams have been considered. All M_1, M_2, ${M_1}^\dagger M_2$ and $M_1{M_2}^\dagger$ diagrams need to be indexed independently. To determine the matrix elements, Feynmann rules will be used which is derived in detail in the appendix section 6. Before the kinematics is used, the obtained matrix element should be simplified by matrix algebra, otherwise the calculation becomes clearly more complicated.

ii) Each diagram consists of two emitters and a spectator. The emitter parent with the momentum $q_i + q$ in the old kinematic splits into a daughter-patron with q_i and an another emitter q. One should select the spectator q_j skilfully so that the diagrams are meaningful and calculable in the case of the interference terms, otherwise the final results hast to be changed cause of the unanimity indices. Thus a structure is achieved and the diagrams can be replaced from $M_1, {M_1}^\dagger$, M_2 and ${M_2}^\dagger$ step by step and even their amplitudes can be used for the interference terms.

iii) Before starting to calculate, based on the contracted indices the expected result should be predicted. This is relatively helpful since it can be quickly seen from the square matrix element which terms are relevant for the splitting functions.

iv) It is recommended to use the concepts from the previous section 3.9. Basically, there are four common scalar products from the denominators listed here:

- $(q \cdot q_i)(q \cdot q_i)$
- $(q \cdot q_j)(k_1 \cdot q_j)$
- $(q \cdot q_i)(q \cdot q_j)$
- $(q \cdot q_i)(q_i \cdot q_i)$

For the new parametrisation, the following substitution is used:

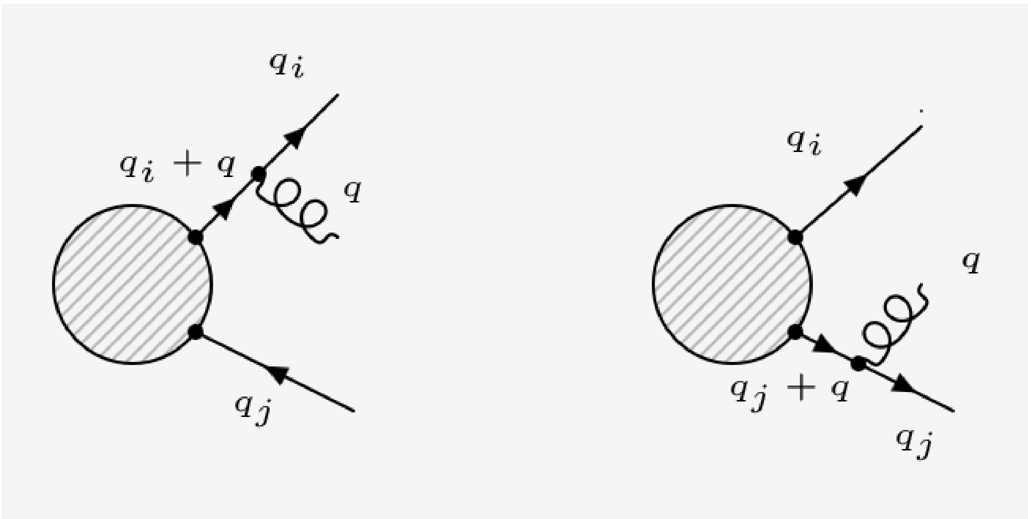
$$\begin{aligned} q_i &\to q_i \\ q &\to k_1 \\ q_j &\to q_k \end{aligned} \tag{3.56}$$

v) finally, to find out whether everything was calculated correctly, the collinear limits checks if the Alterali-Parisi splitting functions 3.14 are output.

vi) In the case of indistinguishable particles in relation to the interference term, the momentum of the particles for the same diagram must be exchanged once.

4 The LO splitting functions

4.1 Gluon emission from a parent quark

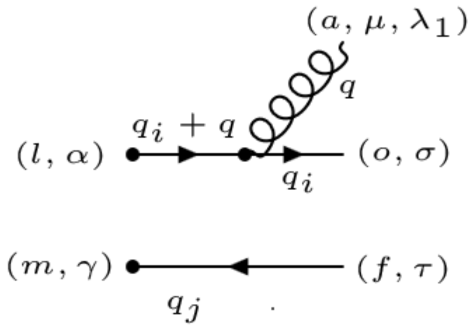


First of all contemplate a quark gluon splitting with an arbitrary spectator like an anti-quark with the momentum q_j. Where $q_j + q$ is the momentum of the parent quark before splitting, q the momentum of the gluon and q_i of daughter quark respectively. The distinction between daughter and parent vanishes, when the gluon becomes soft, and a singularity develops. The other possibility to get a singularity is when the gluon will be collinear to (anti)-quark. The splitting functions are flavour independent since the strong interaction is flavour independent. Furthermore, leading order splitting cannot change the flavour of a quark, thus [19]:

$$\begin{aligned} P_{\bar{q}_i\bar{q}_j} = P_{q_iq_j} &\equiv P_{qq}\delta_{ij} \\ P_{\bar{q}_ig} = P_{q_ig} &\equiv P_{qg} \\ P_{\bar{g}\bar{q}_i} = P_{gq_i} &\equiv P_{gq}\delta_{ij} \end{aligned} \tag{4.1}$$

Since there is no distinction between quark and anti-quark, one can imagine exact the same splitting variation for anti-quark, see the right picture **??** above.

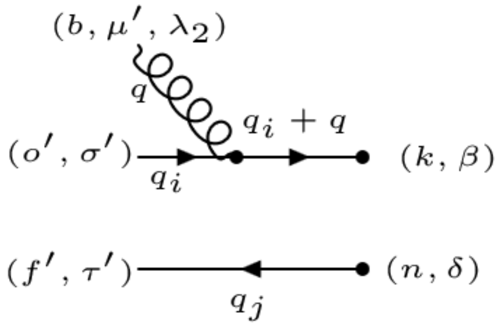
4.1.1 Matrix element of a quark with a gluon radiation $|M_1|^2$



The first step of the procedure was to index the diagrams independently and use the Feynman rules:

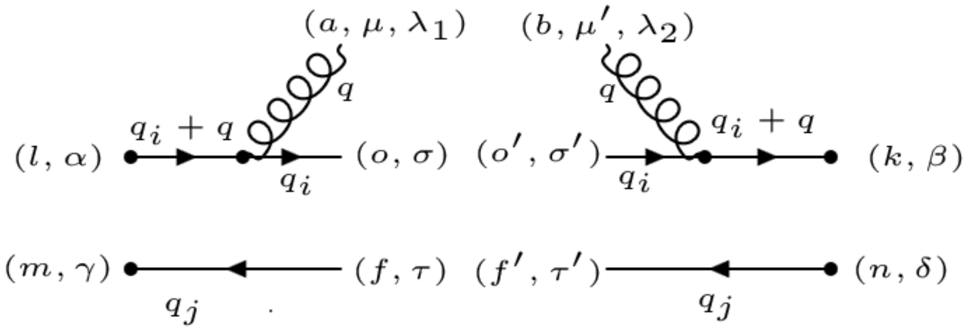
$$M_1 = [\bar{u}_\sigma(q_i)(-ig_s\gamma^\mu \times [T^a]_o{}^l)\frac{i(\not{q}_i + \not{q})}{(q_i+q)^2}\varepsilon^{\lambda_1}{}_\mu(q)]\ [v_\tau(q_j)] \tag{4.2}$$

For the quadratic matrix element the dagger of M_1 need to be calculated.



$$M_1{}^\dagger = [\frac{-i(\not{q}_i + \not{q})}{(q_i+q)^2}\ (ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\ u_{\sigma'}(q_i)\ \varepsilon^{\lambda_2}{}_{\mu'}(q)][\bar{v}_{\tau'}(q_j)] \tag{4.3}$$

Multiplying $M_1{}^\dagger$ and M_1 and putting the diagrams next to each other:



$$
\begin{aligned}
|M_1|^2 = M_1\, {M_1}^\dagger = [\bar{u}_\sigma(q_i)\,(-ig_s\gamma^\mu \times [T^a]_o{}^l)\,\frac{i(\not{q}_i + \not{q})}{(q_i+q)^2}\;\varepsilon^{\lambda_1}{}_\mu(q)][v_\tau(q_j)] \\
[\frac{-i(\not{q}_i + \not{q})}{(q_i+q)^2}\,(ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\,u_{\sigma'}(q_i)\,\varepsilon^{\lambda_2}{}_{\mu'}{}^*(q)][\bar{v}_{\tau'}(q_j)]
\end{aligned}
\tag{4.4}
$$

Connecting those terms which are related to each other:

$$
\begin{aligned}
|M_1|^2 = [\frac{-i(\not{q}_i + \not{q})}{(q_i+q)^2}\;(ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\,\bar{u}_\sigma(q_i)\,u_{\sigma'}(q_i)\,\varepsilon^{\lambda_2}{}^*_{\mu'}(q)\varepsilon^{\lambda_1}{}_\mu(q) \\
\times(-ig_s\gamma^\mu \times [T^a]_o{}^l)\,\frac{i(\not{q}_i + \not{q})}{(q_i+q)^2}][\bar{v}_{\tau'}(q_j)v_\tau(q_j)]
\end{aligned}
\tag{4.5}
$$

Sum over the lorenz index (σ, σ') and (τ, τ') and spin addition relation leads to:

$$
\begin{aligned}
\sum_{\sigma,\sigma'} \bar{u}_\sigma(q_i)\,u_{\sigma'}(q_i) = \not{q}_i\delta^{oo'}, \\
\sum_{\tau,\tau'} \bar{v}_\tau(q_j)\,v_{\tau'}(q_j) = \not{q}_j\delta^{ff'}
\end{aligned}
\tag{4.6}
$$

Sum over polarization index (λ_1, λ_2) :

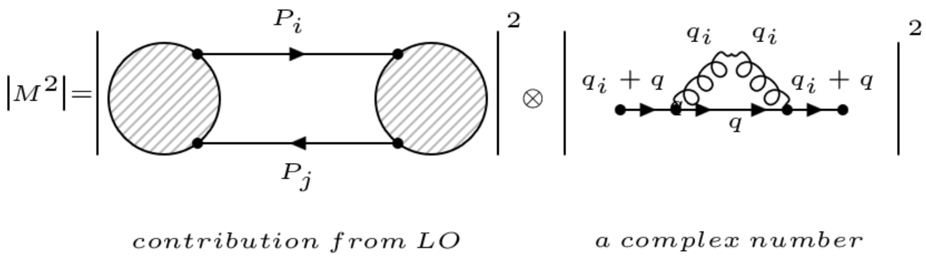
$$
\sum_{\mu,\mu'} \varepsilon^{\lambda_2}{}^*_{\mu'}(q)\varepsilon^{\lambda_1}{}_\mu(q) = -g_{\mu\mu'}\delta^{ab}
\tag{4.7}
$$

The matrix element will be simplified with:

$$
|M_1|^2 = \frac{-g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[(\not{q}_i + \not{q})\;\gamma^{\mu'}\;\;\not{q}_i\,g_{\mu'\mu}\gamma^\mu\,(\not{q}_i + q)][\not{q}_j]
\tag{4.8}
$$

As it was discussed in the procedure the contracted indices lead us to the predict the relevant singular terms:

$$
|M_1|^2 = \frac{-g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[(\not{q}_i + \not{q})\;\gamma^{\mu'}\;\;\not{q}_i\,\gamma_{\mu'}\,(\not{q}_i + q)][\not{q}_j]
\tag{4.9}
$$



From the result the tree level diagram from LO and a number are expected:
Which graphically means:

$$|M_1|^2 = \frac{-g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[\not{P}_i][\not{P}_j] \;\otimes\; (\textit{a complex number}) \tag{4.10}$$

Let's calculate the contribution and compare the final result with this expectation. Calculating the expression in the parentheses from the equation 4.9 separately results in:

$$\begin{aligned} N =: \gamma^{\mu'}\ \not{q}_i\,\gamma_{\mu'} &= q_{i\sigma}\,\gamma^{\mu'}\gamma^{\sigma}\ \gamma_{\mu'} \\ &= q_{i\sigma}\,(\{\gamma^{\mu'},\gamma^{\sigma}\}\ -\gamma^{\sigma}\gamma^{\mu'})\gamma_{\mu'} \\ &= q_{i\sigma}\,2g^{\mu'\sigma}\,\gamma_{\mu'}\ -\ d\,\gamma^{\sigma} \\ &= (2-d)\ \not{q}_i \end{aligned} \tag{4.11}$$

Simplification of the bracket:

$$|M_1|^2 = -(2-d)\ \frac{g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[(\not{q}_i+\ \not{q})\quad \not{q}_i\ \ (\not{q}_i+q)][\not{q}_j] \tag{4.12}$$

$$|M_1|^2 = -(2-d)\ \frac{g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[\not{q}_i\ \not{q}_i\ \not{q}_i+\ \not{q}_i\ \not{q}_i\ \not{q}+\ \not{q}\ \not{q}_i\ \not{q}_i+\ \not{q}\ \not{q}_i\ \not{q}][\not{q}_j] \tag{4.13}$$

With on-shell condition:

$$\begin{aligned} \not{q}_i\ \ \not{q}_i &= {q_i}^2 = {m_i}^2 \\ \not{q}\ \ \not{q} &= q^2 = m^2 \\ \not{q}_j\ \ \not{q}_j &= {q_j}^2 = {m_j}^2 \end{aligned} \tag{4.14}$$

In terms of massless partons:

$$|M_1|^2 = -(2-d)\ \frac{g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(2q_iq)(2q_iq)}[\not{q}\ \not{q}_i\ \not{q}][\not{q}_j] \tag{4.15}$$

The terms in the brackets can be simplified:

$$
\begin{aligned}
L = &\not{q}\ \not{q}_i\ \not{q} = \not{q}[q_{i\sigma} q_\mu\left(\{\gamma^\mu, \gamma^\sigma\} - \gamma^\sigma\gamma^\mu\right)] \\
= &\not{q}[2{q_i}^\mu q_\mu - q_{i\sigma} q_\mu \gamma^\mu \gamma^\sigma \\
= &\not{q}(2q_i q) - q_\mu q_{i\sigma} q_\mu [\gamma^\mu\gamma^\mu\gamma^\sigma] \\
= &\not{q}(2q_i q) - q_\mu q_{i\sigma} q_\mu [\frac{\gamma^\mu\gamma^\mu}{2} + \frac{\gamma^\mu\gamma^\mu}{2}]\gamma^\sigma \\
= &\not{q}(2q_i q) - q_\mu q_{i\sigma} q_\mu [g^{\mu\mu}]\gamma^\sigma \\
= &\not{q}(2q_i q) - q_\mu q_{i\sigma} q^\mu \gamma^\sigma = \not{q}(2q_i q) - q^2\ \not{q}_i \\
= &\not{q}(2q_i q)
\end{aligned}
\tag{4.16}
$$

After inserting the last result of L and simplify the term $(2q_i q)$ from the denominator and nominator, the result becomes:

$$
|M_1|^2 = -(2-d)\,\frac{g_s^2 [T^a]_o{}^k\ [T^a]_o{}^l}{2y(p_i \cdot p_j)}[\not{q}][\not{q}_j] \tag{4.17}
$$

Now the old parametrisation will be used from equation 3.37 to reduce the 3-partons matrix element to 2-partons:

$$
|M_1|^2 = (d-2)\,\frac{g_s^2 [T^a]_o{}^k\ [T^a]_o{}^l}{2y(p_i \cdot p_j)}[(1-z)\ \not{p}_i + zy\ \not{p}_j - \sqrt{zy(1-z)}\ \not{m}_\perp][(1-y)\ \not{p}_j] \tag{4.18}
$$

Multiplying the both sides

$$
\begin{aligned}
|M_1|^2 = (d-2)\,&\frac{g_s^2 [T^a]_o{}^k\ [T^a]_o{}^l}{2y(p_i \cdot p_j)}[(1-z)(1-y)\ \not{p}_i\ \not{p}_j \\
&+ zy(1-y)\ \not{p}_j\ \not{p}_j - (1-y)\sqrt{zy(1-z)}\ \not{m}_\perp\ \not{p}_j]
\end{aligned}
\tag{4.19}
$$

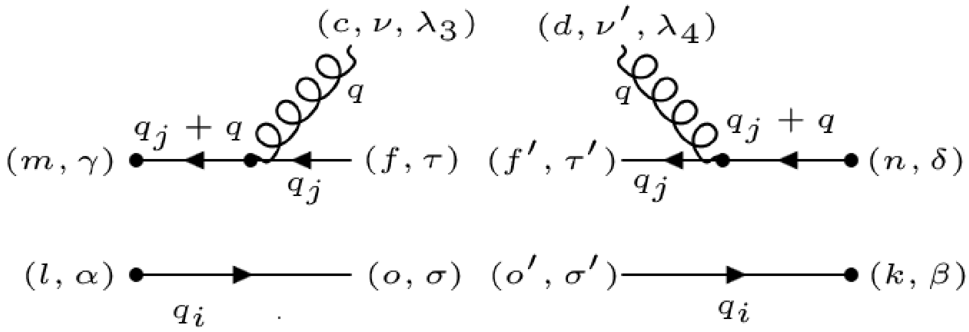
Under consideration of the fact that p_i and p_j are the on-shell momenta of the emitter and spectator partons, the terms with $\not{p}_i\ \not{p}_i$ and $\not{p}_j\ \not{p}_j$ can be eliminated. The $p_i \cdot m_\perp$ and $p_j \cdot m_\perp$ are always 0 because the p_i and p_j are light like, i.e. zero transverse component. So those terms can be neglected.

$$
|M_1|^2 = \frac{g_s^2 [T^a]_o{}^k\ [T^a]_o{}^l}{2y(p_i \cdot p_j)}[\not{p}_i][\not{p}_j] \otimes (d-2)(1-z)(1-y) \tag{4.20}
$$

Expected was a contribution result from the LO and a complex number. The number is just for $y \to 0$ singular and not for $z \to 1$.

4.1.2 Matrix element of an anti-quark with a gluon radiation $|M_2|^2$

The same procedure is used to obtain the matrix element for an anti-quark with a single gluon emission.



$$|M_2|^2 = M_2\, M_2{}^{\dagger} = [\frac{i(\not{q}_j + \not{q})}{(q_j+q)^2}(-ig_s\gamma^{\nu} \times [T^c]_f{}^m)\, v_{\tau}(q_j)\, \varepsilon^{\lambda_3}{}_{\nu}(q)]\,[u_{\sigma}(q_i)] \\ [\bar{v}_{\tau'}(q_j)\,(ig_s\gamma^{\nu'} \times [T^d]_{f'}{}^n)\,\frac{-i(\not{q}_j + \not{q})}{(q_j+q)^2}\,\varepsilon^{\lambda_4}{}_{\nu'}(q)]\,[\bar{u}_{\sigma'}(q_i)] \tag{4.21}$$

$$|M_2|^2 = \frac{g_s^2\,[T^c]_f{}^m\,[T^d]_{f'}{}^n}{4(q_j \cdot q)(q_j \cdot q)}[(\not{q}_j + \not{q})\gamma^{\nu} v_{\tau}\bar{v}_{\tau'}\varepsilon^{\lambda_3}{}_{\nu}\varepsilon^{\lambda_4}{}_{\nu'}\gamma^{\nu'}(\not{q}_j + \not{q})][u_{\sigma}(q_i)]\,[\bar{u}_{\sigma'}(q_i)] \tag{4.22}$$

and after sum over the lorenz and polarization indexes like (σ, σ'), (τ, τ') and (λ_3, λ_4) as well and using the spin addition relation:

$$|M_2|^2 = \frac{g_s^2\,[T^c]_f{}^m\,[T^c]_f{}^n}{(q_j+q)^2(q_j+q)^2}[(\not{q}_j + \not{q})\gamma^{\nu}\;\; \not{q}_j\,(-g_{\nu\nu'})\gamma^{\nu'}(\not{q}_j + \not{q})]\,[\not{q}_i] \tag{4.23}$$

Analogous to the last calculation:

$$|M_2|^2 = (d-2)\frac{g_s^2\,[T^c]_f{}^m\,[T^c]_f{}^n}{(2qq_j)}[\not{q}]\,[\not{q}_i] \tag{4.24}$$

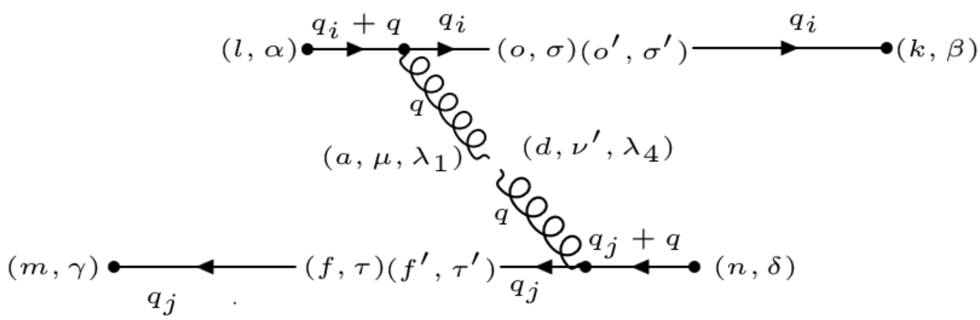
$$
\begin{aligned}
|M_2|^2 = (d-2)\frac{g_s^2\,[T^c]_f{}^m\,[T^c]_f{}^n}{(2qq_j)}&[y(1-z)^2\ \not{p}_i\ \not{p}_j + (1-z)\sqrt{zy(1-z)}\ \not{p}_i\ \not{m}_\perp \\
&+ 2yz^2(p_i\cdot p_j) - yz^2\ \not{p}_i\ \not{p}_j + yz\sqrt{zy(1-z)}\ \not{p}_j\ \not{m}_\perp + z\sqrt{zy(1-z)}\ \not{p}_i\ \not{m}_\perp \\
&+ y(1-z)\sqrt{zy(1-z)}\ \not{p}_j\ \not{m}_\perp + 2zy(1-z)(p_i\cdot p_j)]
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2(1-z)(1-y)(p_i\cdot p_j)}&[y(1-2z)\ \not{p}_i\ \not{p}_j + \sqrt{zy(1-z)}\ \not{p}_i\ \not{m}_\perp \\
&+ y\sqrt{zy(1-z)}\ \not{p}_j\ \not{m}_\perp + 2zy(p_i\cdot p_j)]
\end{aligned} \tag{4.26}
$$

Interestingly, here is a term with y concerning the gluon radiation from an anti-quark. This means that this result cannot contribute to the collinear limit if $y \to 0$.

4.1.3 Interference contribution

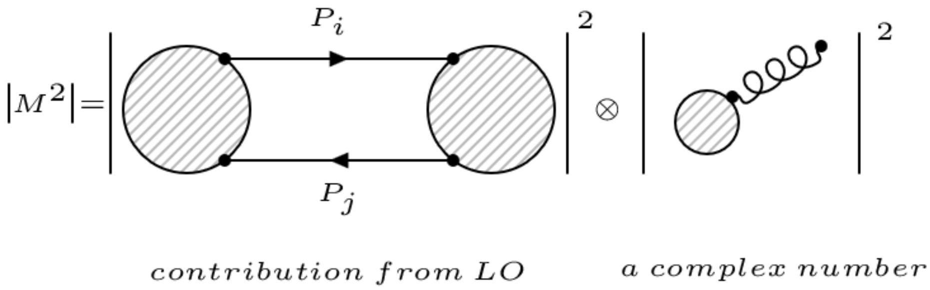
According to step (ii) in the concept to get the interference contribution, the results of M_1 and $M_2{}^\dagger$ need to be put next to each other. And this is exactly the point that was described in the first step of the concept. If a gluon had been chosen as the spectator, the results could not simply be placed next to each other.



$$
\begin{aligned}
M_1\ M_2{}^\dagger = &[\bar{u}_\sigma(q_i)\,(-ig_s\gamma^\mu\times[T^a]_o{}^l)\,\frac{i(\not{q}_i+\not{q})}{(q_i+q)^2}\ \varepsilon^{\lambda_1}{}_\mu(q)][v_\tau(q_j)] \\
&[\bar{v}_{\tau'}(q_j)\,(ig_s\gamma^{\nu'}\times[T^d]_{f'}{}^n)\,\frac{-i(\not{q}_j+\not{q})}{(q_j+q)^2}\ \varepsilon^{\lambda_4}{}_{\nu'}(q)]\,[u_{\sigma'}(q_i)]
\end{aligned} \tag{4.27}
$$

$$
M_1\ M_2{}^\dagger = \frac{-g_s^2[T^a]_o{}^l\,[T^a]_{f'}{}^n}{(2q_iq)(2q_jq)}[\not{q}_i\,\gamma^\mu\,(\not{q}_i+\not{q})\,]\,[\not{q}_j\,\gamma_\mu\,(\not{q}_j+\not{q})] \tag{4.28}
$$

Expectation:



$$
\begin{aligned}
M_1\, M_2{}^{\dagger} = & \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{(4(1-z)y(p_i \cdot p_j)(p_i \cdot p_j)} [(z\; \not{p}_i + y(1-z)\; \not{p}_j + \sqrt{zy(1-z)}\; \not{m}_\perp)\, \gamma^\mu\, (\not{p}_i + y\; \not{p}_j)] \\
& [(1-y)\; \not{p}_j\, \gamma_\mu\, ((1-z)\; \not{p}_i + (1+yz-y)\; \not{p}_j - \sqrt{zy(1-z)}\; \not{m}_\perp)]
\end{aligned}
\tag{4.29}
$$

The singular term in the denominator $y(1-z)$ is used to drop the term with the same pre-factor and after simplification it follows:

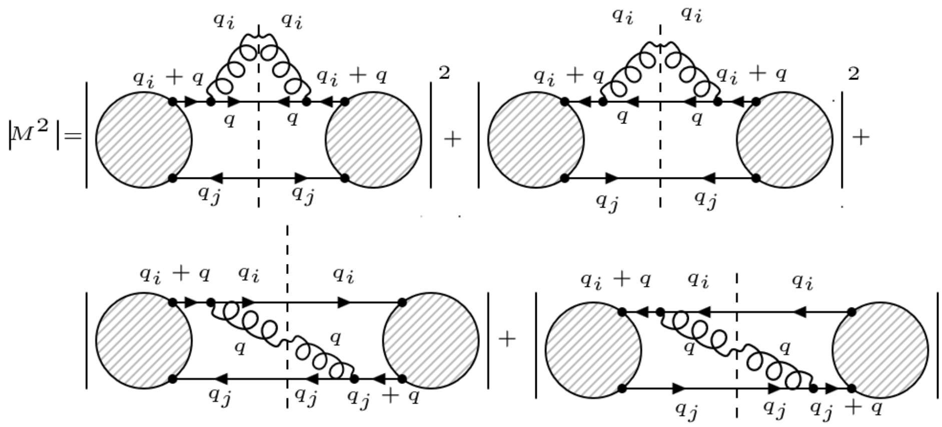
$$
M_1\, M_2{}^{\dagger} = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{y(p_i \cdot p_j)} [\not{p}_i\; \not{p}_j] \otimes \frac{z}{1-z}
\tag{4.30}
$$

4.1.4 Final result

Because the matrix element is a complex number, there is no need to know the ${M_1}^\dagger M_2$ contribution.

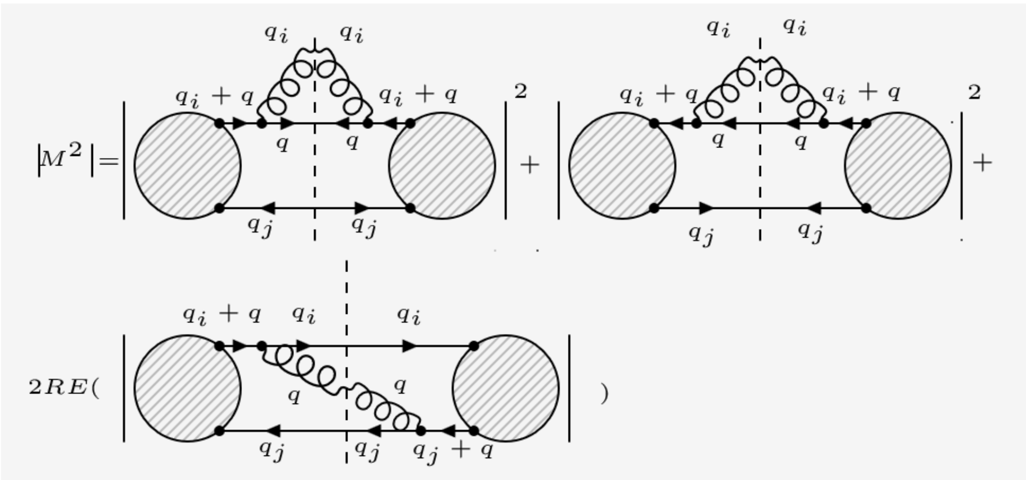
$$\mid M|^2 = \mid M_1|^2 + \mid M_2|^2 + M_1 \, {M_2}^\dagger + {M_1}^\dagger M_2 \tag{4.31}$$

Instead of $M_1 \, {M_2}^\dagger + {M_1}^\dagger M_2$ the double of real part of $M_1 \, {M_2}^\dagger$ will be computed.



$$\mid M|^2 = \mid M_1|^2 + \mid M_2|^2 + 2RE(M_1 \, {M_2}^\dagger) \tag{4.32}$$

finally:



$$\begin{aligned} \mid M|^2 &= (d-2)(1-z)(1-y)\,\frac{g_s^2 {[T^a]_o}^k \, {[T^a]_o}^l}{2y(p_i \cdot p_j)}[\not{p}_i][\not{p}_j] \\ &- (d-2)yz^2 \,\frac{g_s^2 \, {[T^c]_f}^m \, {[T^c]_f}^n}{2(1-z)(1-y)(p_i \cdot p_j)}[\not{p}_i][\not{p}_j] \\ &+ 2RE((\frac{-2z}{z-1})\frac{g_s^2 \; {[T^a]_o}^l \, {[T^a]_f}^n}{2y(p_i \cdot p_j)}[\not{p}_i][\not{p}_j]) \end{aligned} \tag{4.33}$$

The colour factors can be read from the section about the colour algebra 2.3. As an example, the color factor for the interference term is calculated here using the Friez identity 2.16:

$$T^a{}_{o\,k}\,T^a{}_{l\,o} = \frac{1}{2}(\delta_{oo}\delta_{lk} - \frac{1}{N}\delta_{ok}\delta_{lo}) = \frac{1}{2}(N\delta_{lk} - \frac{1}{N}\delta_{lk}) = C_F\delta_{lk} \tag{4.34}$$

After summation over the final colour states and averaging over initial colour states:

$$T^a{}_{o\,k}\,T^a{}_{l\,o} = C_F\delta_{lk} = \frac{1}{N}\sum_{l=1}^{N}\delta_{lk}C_F = C_F \tag{4.35}$$

The splitting function is evaluated in the case of the collinear limes:

$$y \longrightarrow 0 \tag{4.36}$$

$$\mid M|^2 = \frac{g_s^2 C_F}{2y(1-2z+2z^2)(p_i\cdot p_j)}[\not{p}_i][\not{p}_j] \otimes ((d-2)(1-z) - \frac{4z}{z-1}) \tag{4.37}$$

for

$$d = 4 - 2\epsilon \tag{4.38}$$

$$\begin{aligned} \mid M|^2 &= \frac{g_s^2}{y(p_i\cdot p_j)}[\not{p}_i][\not{p}_j] \otimes C_F(\frac{(1+z^2)}{1-z} - \epsilon(1-z)) \\ &= \frac{g_s^2}{q_i\cdot q}[\not{p}_i][\not{p}_j] \otimes \langle\,\hat{P_{qq}}\rangle \end{aligned} \tag{4.39}$$

With $\langle\,\hat{P_{qq}}\rangle$ Alterali-Parisi splitting function 3.14 in the collinear limes, which was mentioned in the fifth step from the concept. This is exactly the confirmation of our calculation that our calculation was actually performed correctly, otherwise we would not have received the same splitting function for soft gluons.

4.1.5 Double-check the results with the new kinematic

With the new kinematics has to test if one gets the same result in the collinear limit. For the new parametrisation, the following substitution is used:

$$\begin{aligned} q_i &\to q_i \\ q &\to k_1 \\ q_j &\to q_k \end{aligned} \tag{4.40}$$

The summarized results are given here:

$|M_1|^2$

$$|M_1|^2 = (d-2)\,\frac{g_s^2 C_F}{(2k_1\cdot q_i)}[\not{k}_1][\not{q}_k] \tag{4.41}$$

$$\begin{aligned} |M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\;\not{p}_i + y\beta_1\;\not{Q} + \sqrt{y\alpha_1\beta_1}\;\not{n}_{\perp,1}] \\ &[A_1\;\not{p}_i + A_2\;\not{Q} + \sqrt{1-y}\;\not{p}_k] \end{aligned} \tag{4.42}$$

$$\begin{aligned} |M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)p_i\cdot Q \\ &+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}p_i\cdot p_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}n_{\perp,1}\cdot p_k] \end{aligned} \tag{4.43}$$

For the collinearity $y \to 0$ we'll get:

$$\begin{aligned} |M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)\;\not{p}_i\;\not{Q} \\ &+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}\;\not{p}_i\;\not{p}_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}\;\not{n}_{\perp,1}\;\not{p}_k] \end{aligned} \tag{4.44}$$

$$|M_1|^2 = (d-2)(1-\beta_1)\sqrt{1-y}\;\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[\not{p}_i\;\not{p}_k] \tag{4.45}$$

$|M_2|^2$

$$|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}[\not{k}_1]\;[\not{q}_i] \tag{4.46}$$

$$\begin{aligned} |M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}&[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\;\not{p}_i + y\beta_1\;\not{Q} + \sqrt{y\alpha_1\beta_1}\;\not{n}_{\perp,1}] \\ &[(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))\;\not{p}_i + y\alpha_1\;\not{Q} - \sqrt{y\alpha_1\beta_1}\;\not{n}_{\perp,l}] \end{aligned} \tag{4.47}$$

Which means:

$$|M_2|^2 \sim (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}y[...]$$
$$|M_2|^2 \to 0 \qquad \text{for} \quad y \to 0 \tag{4.48}$$

$M_1\,M_2{}^\dagger$

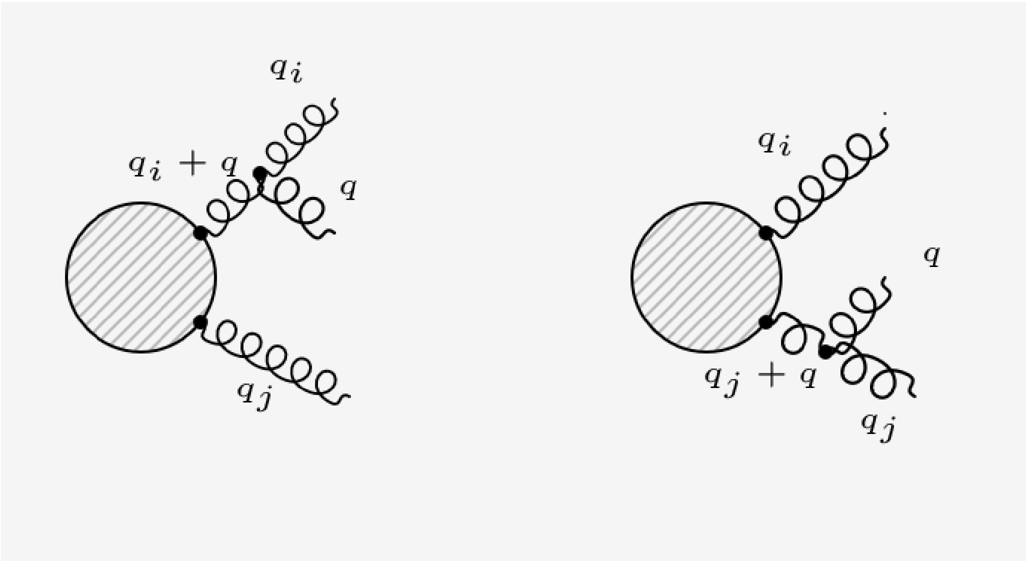
$$M_1\,M_2{}^\dagger = \frac{-g_s^2\,C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}$$
$$4(\beta_1\sqrt{1-y}p_i\cdot p_k)[\beta_1\sqrt{1-y}\;\not{p}_i\;\not{p}_k + (1-\beta_1)\sqrt{1-y}\;\not{p}_i\;\not{p}_k] \tag{4.49}$$

$$M_1\,M_2{}^\dagger = \frac{-g_s^2\,C_F}{y(1-\beta_1)\,(p_i\cdot p_k)(p_i\cdot Q)}\beta_1(p_i\cdot p_k)[\beta_1\;\not{p}_i\;\not{p}_k + (1-\beta_1)\;\not{p}_i\;\not{p}_k] \tag{4.50}$$

$$M_1\,M_2{}^\dagger = \frac{\beta_1}{(1-\beta_1)}\,\frac{-g_s^2\,C_F}{y\,(p_i\cdot Q)}[\,\not{p}_i\;\not{p}_k] \tag{4.51}$$

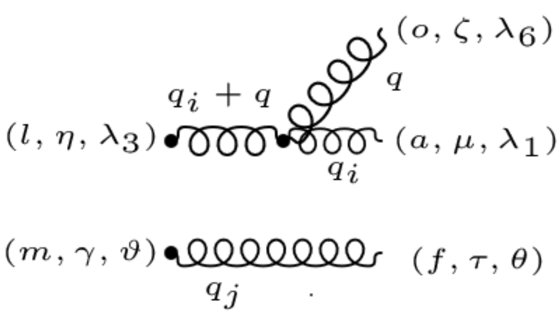
The same result can be reached in each step.

4.2 Gluon radiation from a parent gluon



Now consider a daughter gluon from the splitting of a parent gluon with radiation an another gluon. When the gluon becomes soft, the distinction between daughter and parent vanishes, and a singularity develops. In this chapter we are going to keep the same procedure with a difference that we will not use the old parametrisation since it only works in LO. One of the mainly challenges about this emission kernel is that the calculations are long and complicated.

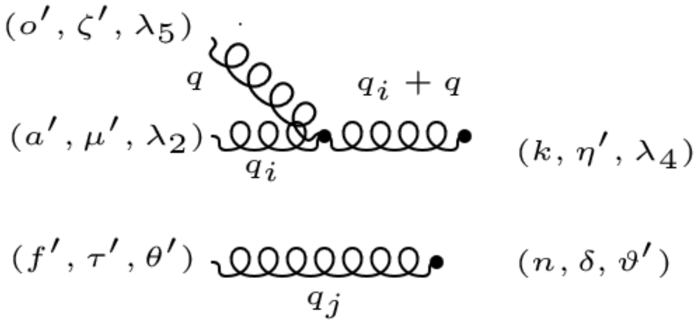
4.2.1 Gluon-Emitter Bubble



Beginning with Feynman rules for this diagram as usual:

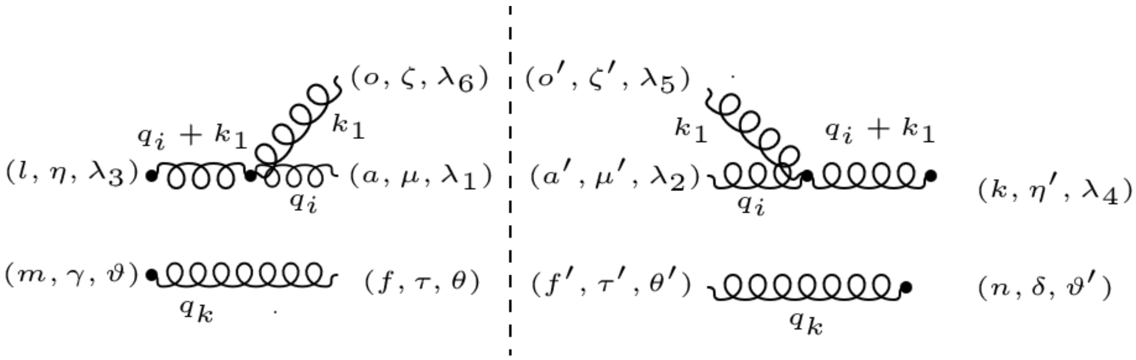
$$
\begin{aligned}
M_1 =& [\frac{-i}{(q_i+q)^2}(-g_s f^{\,a\,o\,l}(g^{\mu\zeta}(q-q_i)^\eta - g^{\zeta\eta}(2q+q_i)^\mu + g^{\eta\mu}(2q_i+q)^\zeta) \\
& \varepsilon^{\lambda_1}{}_\mu(q_i)\varepsilon^{\lambda_6}{}_\zeta(q)][\varepsilon^{\theta}{}_{\tau'}(q_j)]
\end{aligned}
\tag{4.52}
$$

It has to be emphasized that here only the momentum from the parent gluon is incoming and the two further momenta from the vertex are outgoing. This is why the minus signs appear in the equation. If the upper equation is daggered the following diagram will be catch and the corresponding following amplitude:



$$M_1{}^{\dagger} =[\frac{i}{(q_i+q)^2}(-g_s f^{\,a'\,k\,o'}(-g^{\mu'\eta'}(2q_i+q)^{\zeta'} + g^{\eta'\zeta'}(2q+q_i)^{\mu'} + g^{\zeta'\mu'}(q_i-q)^{\eta'}) \\ \varepsilon^{\lambda_2}{}_{\mu'}{}^{*}(q_i)\varepsilon^{\lambda_5}{}_{\zeta'}{}^{*}(q)][\varepsilon^{\theta'}{}_{\tau'}{}^{*}(q_j)] \tag{4.53}$$

Calculating of the matrix element:



$$|M_1|^2 =[\frac{-i}{(q_i+q)^2}(-g_s f^{\,a\,o\,l}(g^{\mu\zeta}(q-q_i)^{\eta} - g^{\zeta\eta}(2q+q_i)^{\mu} + g^{\eta\mu}(2q_i+q)^{\zeta}) \\ \varepsilon^{\lambda_1}{}_{\mu}(q_i)\,\varepsilon^{\lambda_2}{}_{\mu'}{}^{*}(q_i)\varepsilon^{\lambda_6}{}_{\zeta}(q)\,\varepsilon^{\lambda_5}{}_{\zeta'}{}^{*}(q)(-g_s f^{\,a'\,k\,o'}(-g^{\mu'\eta'}(2q_i+q)^{\zeta'} + g^{\eta'\zeta'}(2q+q_i)^{\mu'} \\ + g^{\zeta'\mu'}(q_i-q)^{\eta'})\frac{i}{(q_i+q)^2}]][g^{\gamma\delta}] \tag{4.54}$$

After the summation over the spin- just as well polarization indices can be obtained:

$$|M_1|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[N^{\eta\eta'}][g^{\gamma\delta}] \tag{4.55}$$

The tensor $N^{\eta\eta'}$ appears exactly when all possible indices contract.
The goal of the next step is to evaluate this tensor, which is avoided here and instead of that, it only presents the final result. The more detailed calculation can be found in Appendix section .

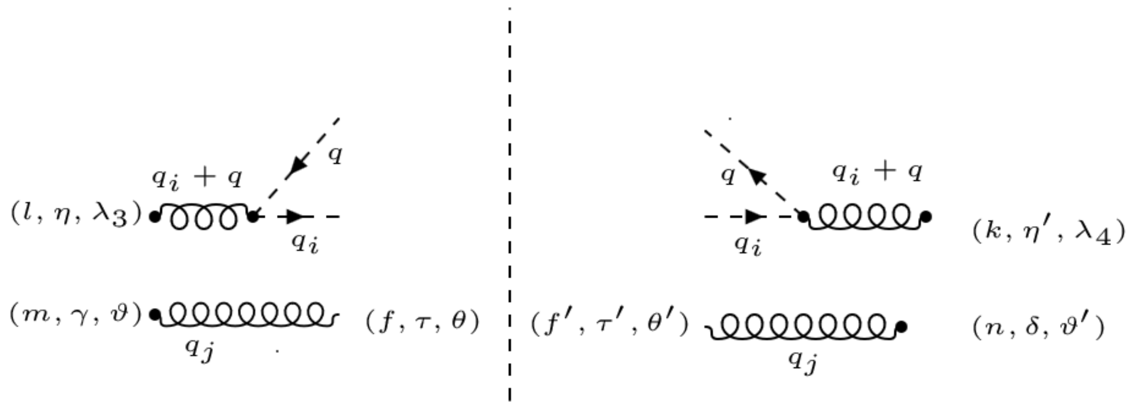
$$\begin{aligned} N^{\eta\eta'} \equiv &[(6-d)q^{\eta}q^{\eta'} + (d+3)q^{\eta}{q_i}^{\eta'} + (d+3){q_i}^{\eta}q^{\eta'} + (6-d){q_i}^{\eta}{q_i}^{\eta'} \\ &- g^{\eta\eta'}(5q^2 + 5{q_i}^2 + 8qq_i)][g^{\gamma\delta}] \end{aligned} \tag{4.56}$$

Replacing this result in the equation:

$$\begin{aligned} |M_1|^2 = &\frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^{\eta}q^{\eta'} + (d+3)q^{\eta}{q_i}^{\eta'} + (d+3){q_i}^{\eta}q^{\eta'} + (6-d){q_i}^{\eta}{q_i}^{\eta'} \\ &- g^{\eta\eta'}(5q^2 + 5{q_i}^2 + 8qq_i)][g^{\gamma\delta}] \end{aligned} \tag{4.57}$$

This Gluon self-energy diagram has to be corrected by ghost Loop to get the complete result.

4.2.1.1 One-loop corrections to the gluon self-energy diagram(Gluon-Emitter Bubble)



In order to get a meaningful result and correct the gluon loop, the same indices must be used. For this the same diagram with a fine difference is used, in which the cut off gluon propagators is replaced with ghost propagators and the rest remain as in the previous diagram.

$$|M_1|^2_{Ghost\,loop} = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[-q_i{}^{\eta} q^{\eta'} - q^{\eta} q_i{}^{\eta'}][g^{\gamma\delta}] \tag{4.58}$$

$$\begin{aligned} |M_1'|^2 = |M_1|^2 + |M_1|^2_{Ghost\,loop} = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^{\eta}q^{\eta'} + (d+3)q^{\eta}q_i{}^{\eta'} \\ + (d+3)q_i{}^{\eta}q^{\eta'} + (6-d)q_i{}^{\eta}q_i{}^{\eta'} - g^{\eta\eta'}(5q^2 + 5q_i{}^2 + 8qq_i) - q_i{}^{\eta}q^{\eta'} - q^{\eta}q_i{}^{\eta'}][g^{\gamma\delta}] \end{aligned} \tag{4.59}$$

After simplification and addition over the same terms:

$$\begin{aligned} |M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^{\eta}q^{\eta'} + (d+2)q^{\eta}q_i{}^{\eta'} + (d+2)q_i{}^{\eta}q^{\eta'} + (6-d)q_i{}^{\eta}q_i{}^{\eta'} \\ - g^{\eta\eta'}(8qq_i)][g^{\gamma\delta}] \end{aligned} \tag{4.60}$$

Implementation the new parametrisation:

$$\begin{aligned} |M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y^2(\alpha_1+\beta_1)^2\,(p_i\cdot Q)\,(p_i\cdot Q)} \\ [(6-d)(\zeta_1 p_i{}^{\eta} + \lambda_1 Q^{\eta} + \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_1 p_i{}^{\eta'} + \lambda_1 Q^{\eta'} + \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ + (d+2)(\zeta_1 p_i{}^{\eta} + \lambda_1 Q^{\eta} + \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_q p_i{}^{\eta'} + \lambda_q Q^{\eta'} - \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ + (d+2)(\zeta_q p_i{}^{\eta} + \lambda_q Q^{\eta} - \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_1 p_i{}^{\eta'} + \lambda_1 Q^{\eta'} + \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ + (6-d)(\zeta_q p_i{}^{\eta} + \lambda_q Q^{\eta} - \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_q p_i{}^{\eta'} + \lambda_q Q^{\eta'} - \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ - 8g^{\eta\eta'}[(\alpha_1{}^2 + \beta_1{}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}] \end{aligned} \tag{4.61}$$

Note that here the short version of the kinematics was used to increase the overview. Now the terms in brackets have to be multiplied and simplified.

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y^2\,(p_i\cdot Q)\,(p_i\cdot Q)}[(6-d)[\zeta_1\zeta_1 {p_i}^{\eta}{p_i}^{\eta'} + \zeta_1\lambda_1 {p_i}^{\eta}Q^{\eta'} + \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_1\zeta_1 Q^{\eta}{p_i}^{\eta'} + \lambda_1\lambda_1 Q^{\eta}Q^{\eta'} + \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} + \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& + \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(d+2)[\zeta_1\zeta_q {p_i}^{\eta}{p_i}^{\eta'} + \zeta_1\lambda_q {p_i}^{\eta}Q^{\eta'} - \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_1\zeta_q Q^{\eta}{p_i}^{\eta'} + \lambda_1\lambda_q Q^{\eta}Q^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} + \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& - \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(d+2)[\zeta_q\zeta_1 {p_i}^{\eta}{p_i}^{\eta'} + \zeta_q\lambda_1 {p_i}^{\eta}Q^{\eta'} + \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_q\zeta_1 Q^{\eta}{p_i}^{\eta'} + \lambda_q\lambda_1 Q^{\eta}Q^{\eta'} + \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& - \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(6-d)[\zeta_q\zeta_q {p_i}^{\eta}{p_i}^{\eta'} + \zeta_q\lambda_q {p_i}^{\eta}Q^{\eta'} - \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_q\zeta_q Q^{\eta}{p_i}^{\eta'} + \lambda_q\lambda_q Q^{\eta}Q^{\eta'} - \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} - \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& + \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}]
\end{aligned}
\tag{4.62}
$$

One replaces now the relations for the often occurring pre-factor products from appendix 6:

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y^2\,(p_i\cdot Q)\,(p_i\cdot Q)}[(6-d)\{({\alpha_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} \\
& + y\alpha_1\beta_1{p_i}^{\eta}Q^{\eta'} + y\beta_1\alpha_1Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} \\
& + (d+2)\{(\alpha_1\beta_1 - y({\alpha_1}^2+{\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\alpha_1}^2{p_i}^{\eta}Q^{\eta'} + y{\beta_1}^2Q^{\eta}{p_i}^{\eta'} \\
& - y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} + (d+2)\{(\beta_1\alpha_1 - y({\beta_1}^2+{\alpha_1}^2)(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} \\
& + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'} + y{\alpha_1}^2Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} \\
& + (6-d)\{({\beta_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\beta_1\alpha_1{p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1Q^{\eta}{p_i}^{\eta'} \\
& + y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}][g^{\gamma\delta}]
\end{aligned}
\tag{4.63}
$$

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y\,(p_i\cdot Q)\,(p_i\cdot Q)}[[8-4d]\beta_1(1-\beta_1){n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q \\
& - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}][g^{\gamma\delta}]]
\end{aligned}
\tag{4.64}
$$

In this step the equation for $d = 4 - 2\epsilon$ was calculated and the value of $(\alpha_1+\beta_1)^2 p_i\cdot Q$ from equation (3.51) was replaced. What here noticeable is at this point that one y from the denominator is abbreviated with one from the nominator. The final result looks like this:

$$
|M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y\,(p_i\cdot Q)}[2[\epsilon-1]\beta_1(1-\beta_1){n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}]
\tag{4.65}
$$

4.2.2 A simplified way within the concept 3.9.1

During the analysis of the evaluation it turned out that the particularly complicated and extensive calculation can be handled with the substitutions conceived below:
The numerator generally consists of the addition of several terms, which mostly consist of scalar products with two four-vectors. This is the reason why these scalar products have to be considered separately. The terms with the same pre-factor are eliminated from the nominator beforehand, since they are finite anyway. In the last chapter, this has already been deduced with regard to the factors in the denominator. For illustration this is calculated once for $|M_1'|^2$ with the denominator $4y^2\,(p_i \cdot Q)\,(p_i \cdot Q)$.
If one looks at this corrected matrix element, with the pre-factor y^2 in the denominator, it can be seen that in the nominator there are terms with y^2 by the multiplication of two four-vectors. Those can be neglected instead of calculating these, since exactly these are the finite terms. This simplifies the results of the scalar products. Under this assumption, the result looks as follows:

$$\begin{aligned}
{k_1}^{\eta}{k_1}^{\eta'} &= ({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1{p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{k_1}^{\eta}{q_i}^{\eta'} &= (\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\alpha_1}^2{p_i}^{\eta}Q^{\eta'} + y{\beta_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{q_i}^{\eta}{k_1}^{\eta'} &= (\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'} + y{\alpha_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{q_i}^{\eta}{q_i}^{\eta'} &= ({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1{p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1}
\end{aligned} \tag{4.66}$$

Inserting these results into N follows:

$$\begin{aligned}
N \equiv\ & (6-d)({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1{p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\alpha_1}^2{p_i}^{\eta}Q^{\eta'} + y{\beta_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'} + y{\alpha_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (6-d)({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1{p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& - 8g^{\eta\eta'}[({\alpha_1}^2 + {\beta_1}^2)p_i \cdot Q - (\beta_1(1-\beta_1))n_{\perp,1} \cdot n_{\perp,1}]
\end{aligned} \tag{4.67}$$

Summary of the equation provides:

$$
\begin{aligned}
N \equiv\, & [(6-d)({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i\cdot Q})) + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2+{\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q})) \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2+{\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q})) + (6-d)({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i\cdot Q}))]{p_i}^{\eta}{p_i}^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 + (d+2)y{\alpha_1}^2 + (d+2)y{\beta_1}^2 + (6-d)y\alpha_1\beta_1]{p_i}^{\eta}Q^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 + (d+2)y{\beta_1}^2 + (d+2)y{\alpha_1}^2 + (6-d)y\alpha_1\beta_1]Q^{\eta}{p_i}^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 - (d+2)y\alpha_1\beta_1 - (d+2)y\alpha_1\beta_1 + (6-d)y\alpha_1\beta_1]n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]
\end{aligned}
\tag{4.68}
$$

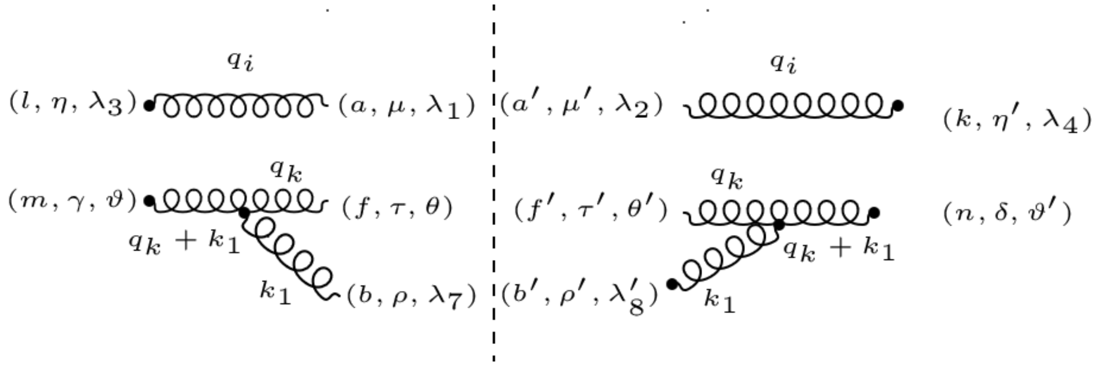
and for the matrix element:

$$
|M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y\,(p_i\cdot Q)}[2[\epsilon-1]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}] \tag{4.69}
$$

conclusion

This concept allows to achieve the same result in just a few steps. This can also be done for further calculations.

4.2.3 Gluon-Spectator Bubble



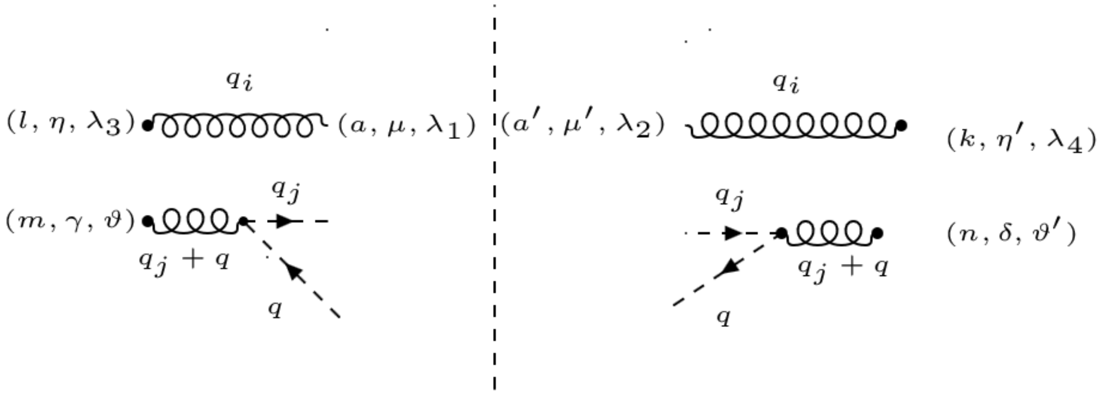
This concept can also be applied to the gluon spectator. The only difference is that a gluon is emitted by a spectator and other indices are used here.
Using the Feynmann rules, the matrix element is evaluated:

$$
\begin{aligned}
|M_2|^2 = {} & \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[(2q+q_j)^\gamma(2q_j+q)^\delta - g^{\delta\gamma}(2q_j+q)^\rho(2q_j+q)_\rho \\
& - (2q_j+q)^\gamma(q-q_j)^\delta - g^{\delta\gamma}(2q+q_j)^\tau(2q+q_j)_\tau + (2q_j+q)^\gamma(2q+q_j)^\delta \\
& + (2q+q_j)^\gamma(q-q_j)^\delta - (q_j-q)^\gamma(2q+q_j)^\delta + (q_j-q)^\gamma(2q_j+q)^\delta \\
& + d(q_j-q)^\gamma(q-q_j)^\delta][g^{\eta\eta'}]
\end{aligned}
\tag{4.70}
$$

It follows:

$$
\begin{aligned}
|M_2|^2 = {} & \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[(3+d)q^\gamma {q_j}^\delta + (6-d)q^\gamma q^\delta + (6-d){q_j}^\gamma {q_j}^\delta \\
& + (3+d){q_j}^\gamma q^\delta - g^{\delta\gamma}(5{q_j}^2 + 5q^2 + 8qq_j)][g^{\eta\eta'}]
\end{aligned}
\tag{4.71}
$$

4.2.3.1 One-loop corrections to the gluon self-energy diagram (Gluon-Spectator Bubble)



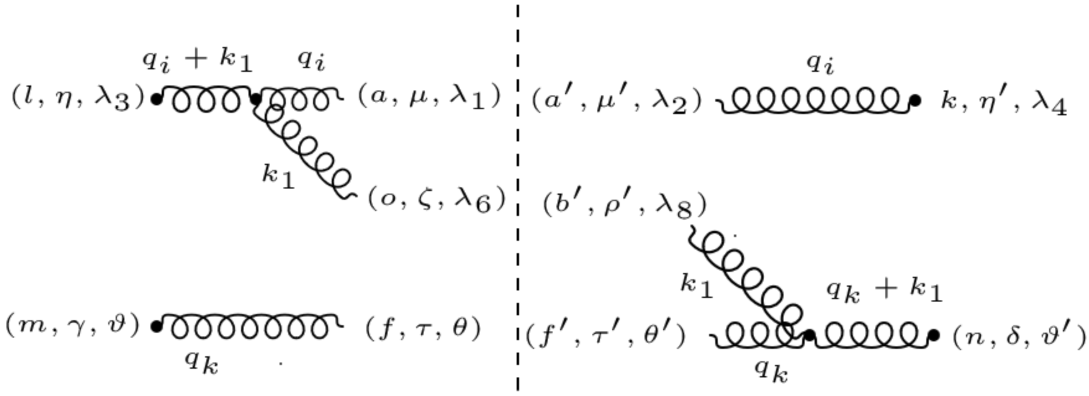
Here not all steps are shown in detail, but only the final result is presented. The reason for this is that all steps can be followed analogously to the last section.

$$|M_2|^2_{Ghost\,loop} = \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[-q_j{}^{\gamma}q^{\delta} - q^{\delta}q_j{}^{\gamma}][g^{\eta\eta'}] \tag{4.72}$$

$$\begin{aligned}|M_2'|^2 = \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}&[(2+d)q^{\gamma}q_j{}^{\delta} + (6-d)q^{\gamma}q^{\delta} \\ &+(6-d)q_j{}^{\gamma}q_j{}^{\delta} + (2+d)q_j{}^{\gamma}q^{\delta} - g^{\delta\gamma}(8qq_j)][g^{\eta\eta'}]\end{aligned} \tag{4.73}$$

$$|M_2'|^2 = \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(1-\beta_1)(1-y)\,(p_i\cdot p_k)}[-2g^{\delta\gamma}][g^{\eta\eta'}] \tag{4.74}$$

4.2.4 Joining the emitter and spectator diagrams together



Analogous to the last two sections, we calculate the quadratic matrix element in the case of the interference term.

$$M_1 M_2{}^{\dagger} = \frac{g_s^2 f^{\,l\,a\,o} f^{\,f'\,b'\,n} \delta^{aa'} \delta^{ob'} \delta^{ff'}}{(q_i+q)^2 (q_j+q)^2} [g_{\mu}{}^{\eta'} g_{\tau\tau'} (g^{\eta\mu}(2q_i+q)^{\zeta} + g^{\mu\zeta}(q-q_i)^{\eta} - g^{\zeta\eta}(2q+q_i)^{\mu})$$
$$g_{\zeta\rho'} (g^{\tau'\rho'}(q_j-q)^{\delta} + g^{\rho'\delta}(2q+q_j)^{\tau'} - g^{\delta\tau'}(2q_j+q)^{\rho'}] \tag{4.75}$$

Multiply the available tensors and summarize all results:

$$\begin{aligned} M_1 M_2{}^{\dagger} = \frac{g_s^2 f^{\,l\,a\,o} f^{\,f\,o\,n}}{4(q\cdot q_i)(q\cdot q_j)} \{ & g^{\eta\eta'} [2q_i{}^{\gamma} q_j{}^{\delta} + 2q_i{}^{\gamma} q^{\delta} + q^{\gamma} q_j{}^{\delta} + q^{\gamma} q^{\delta} + 4q^{\gamma} q_i{}^{\delta} \\ & + 2q^{\gamma} q^{\delta} + 2q_j{}^{\gamma} q_i{}^{\delta} + q_j{}^{\gamma} q^{\delta}] - g^{\eta\eta'} g^{\gamma\delta}(2q\cdot q_j + q\cdot q + 4q_i\cdot q_j + 2q_i\cdot q) \\ & + g^{\gamma\eta'} [q^{\eta} q_j{}^{\delta} - q^{\eta} q^{\delta} - q_i{}^{\eta} q_j{}^{\delta} + q_i{}^{\eta} q^{\delta}] + g^{\eta'\delta} [2q^{\eta} q^{\gamma} + q^{\eta} q_j{}^{\gamma} + q_i{}^{\eta} q^{\gamma} + q_i{}^{\eta} q_j{}^{\gamma}] \\ & - g^{\gamma\delta} [2q^{\eta} q_j{}^{\eta'} + q^{\eta} q^{\eta'} - 2q_i{}^{\eta} q_j{}^{\eta'} - q_i{}^{\eta} q^{\eta'}] - g^{\gamma\eta} [2q^{\eta'} q_j{}^{\delta} - 2q^{\eta'} q^{\delta} + q_i{}^{\eta'} q_j{}^{\delta} - q_i{}^{\eta'} q^{\delta}] \\ & - g^{\eta\delta} [4q^{\eta'} q^{\gamma} + 2q^{\eta'} q_j{}^{\gamma} + 2q_i{}^{\eta'} q^{\gamma} + q_i{}^{\eta'} q_j{}^{\gamma}] + g^{\gamma\delta} [4q_j{}^{\eta} q^{\eta'} + 2q_j{}^{\eta} q_i{}^{\eta'} + q^{\eta} q^{\eta'} + q^{\eta} q_i{}^{\eta'}] \} \end{aligned} \tag{4.76}$$

from the term in the denominator can be seen this is the parametrisation in the case of

(3.54)

$$
\begin{aligned}
{k_1}^{\eta}{k_1}^{\eta'} &= [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]{p_i}^{\eta}{p_i}^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}){p_i}^{\eta}Q^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})Q^{\eta}{p_i}^{\eta'} \\
{k_1}^{\eta}{q_i}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2Q^{\eta}{p_i}^{\eta'} \\
{q_i}^{\eta}{k_1}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'} \\
{q_i}^{\eta}{q_i}^{\eta'} &= {\beta_1}^2{p_i}^{\eta}{p_i}^{\eta'} \\
{k_1}^{\eta}{q_k}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_i}^{\eta}{p_k}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\,{p_i}^{\eta}{p_i}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\,{p_i}^{\eta}Q^{\eta'} \\
&\quad + y\beta_1A_1\,Q^{\eta}{p_i}^{\eta'} + y\beta_1A_2\,Q^{\eta}Q^{\eta'} + y\beta_1\sqrt{1-y}Q^{\eta}{p_k}^{\eta'} \\
{q_i}^{\eta}{q_k}^{\eta'} &= A_1\beta_1{p_i}^{\eta}{p_i}^{\eta'} + A_2\beta_1{p_i}^{\eta}Q^{\eta'} + \beta_1\sqrt{1-y}{p_i}^{\eta}{p_k}^{\eta'} \\
{q_k}^{\eta}{k_1}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\,{p_i}^{\eta}{p_i}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\,Q^{\eta}{p_i}^{\eta'} \\
&\quad + y\beta_1A_1\,{p_i}^{\eta}Q^{\eta'} + y\beta_1A_2\,Q^{\eta}Q^{\eta'} + y\beta_1\sqrt{1-y}{p_k}^{\eta}Q^{\eta'} \\
{q_k}^{\eta}{q_i}^{\eta'} &= A_1\beta_1{p_i}^{\eta}{p_i}^{\eta'} + A_2\beta_1Q^{\eta}{p_i}^{\eta'} + \beta_1\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'}
\end{aligned}
\tag{4.77}
$$

A summarized calculation can be found in the appendix 6. If one is interested in the complete solution which contains both cases, i.e. the collinear and soft region, one has to add the terms. Since the calculation becomes quite long and confusing, we focus on the second term, which is important for the collinear case which is used for the calculation of the splitting function.
The starting point is the second term of the quadratic matrix element:

$$
-g^{\eta\eta'}g^{\gamma\delta}(2q\cdot q_j + q\cdot q + 4q_i\cdot q_j + 2q_i\cdot q) \tag{4.78}
$$

After the replacement of the corresponding Scalar products from 4.77 it results in:

$$
\begin{aligned}
M_1{M_2}^{\dagger} &= g_s^2\,C_A\,g^{\eta\eta'}g^{\gamma\delta}[\frac{1}{2y(p_i\cdot Q)} + \frac{\beta_1(\frac{Q^2}{2p_i\cdot Q})}{2y(1-\beta_1)(1-y)\,(p_i\cdot Q)} \\
&+ \frac{\beta_1\,Q\cdot p_k}{2y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{\beta_1}{y(1-\beta_1)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}]
\end{aligned}
\tag{4.79}
$$

4.2.4.1 Swapping for indistinguishable partons

If now the results from the previous sections are added together, it is noticeable that a part of the final solution falls, which is not demonstrated here. The problem is that the gluons are indistinguishable with respect to the interference term. In fact, it could not be determined which gluon has which momentum. Here exactly the sixth point from the procedure 3.9.1 is happened. This problem is solved by exchanging the momenta for the same diagram once and repeating the steps. Here only the final result is presented and the detailed steps can be looked up in the appendix 6.

$$\begin{aligned} M_1 {M_2^\dagger}' = g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[&\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)} + \frac{1}{2y(p_i\cdot Q)} + \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)} \\ &+ \frac{(1-\beta_1)\, Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}] \end{aligned} \tag{4.80}$$

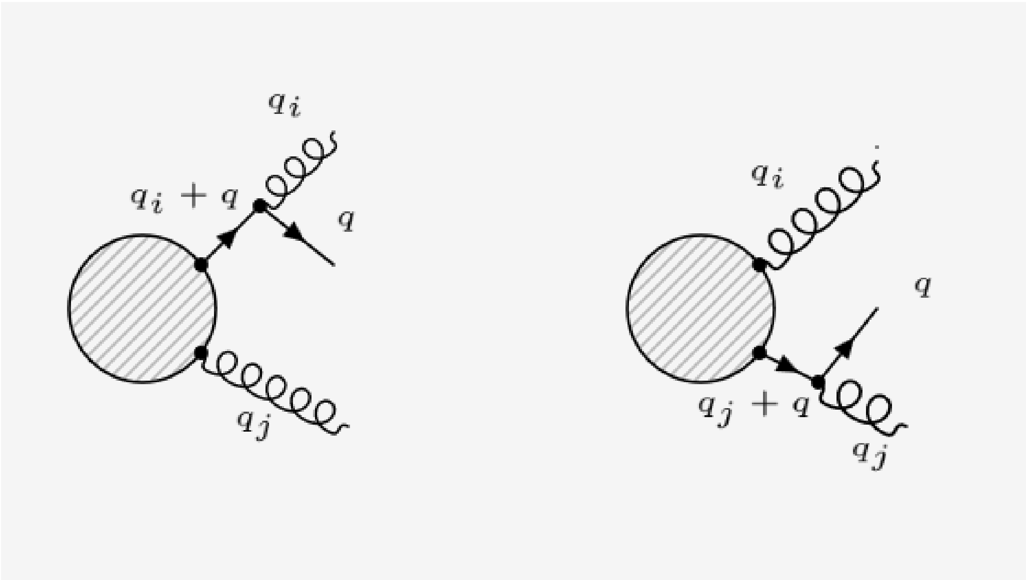
4.2.5 Summary of the results

$$|M|^2 = |{M'}_2|^2 + |{M'}_1|^2 + 2RE(M_1 M_2^\dagger + M_1 {M_2^\dagger}') \tag{4.81}$$

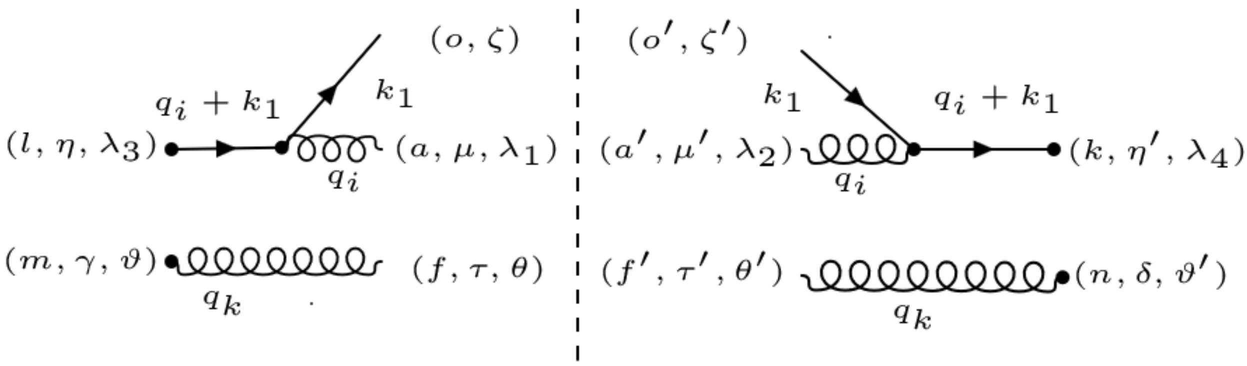
$$\begin{aligned} |M|^2 = g_s^2\, C_A\, g^{\gamma\delta}[&-2(1-\epsilon)\beta_1(1-\beta_1) {n^\eta}_{\perp,1} {n^{\eta'}}_{\perp,1} + \frac{2\beta_1}{y(1-\beta_1)(p_i\cdot Q)} g^{\eta\eta'} + \frac{2(1-\beta_1)}{y\beta_1(p_i\cdot Q)} g^{\eta\eta'} \\ &+ \frac{Q^2}{2y\beta_1(1-y)\,(p_i\cdot Q)(p_i\cdot Q)} g^{\eta\eta'} + \frac{Q\cdot p_k}{y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} g^{\eta\eta'}] \end{aligned} \tag{4.82}$$

Thus, the singular part from the matrix element has full agreement with the splitting function $\langle \hat{P_{gg}} \rangle$ from 3.14. This means that the desired result can be achieve by using the new kinematics in the collinear area.

4.3 A daughter gluon from a parent quark



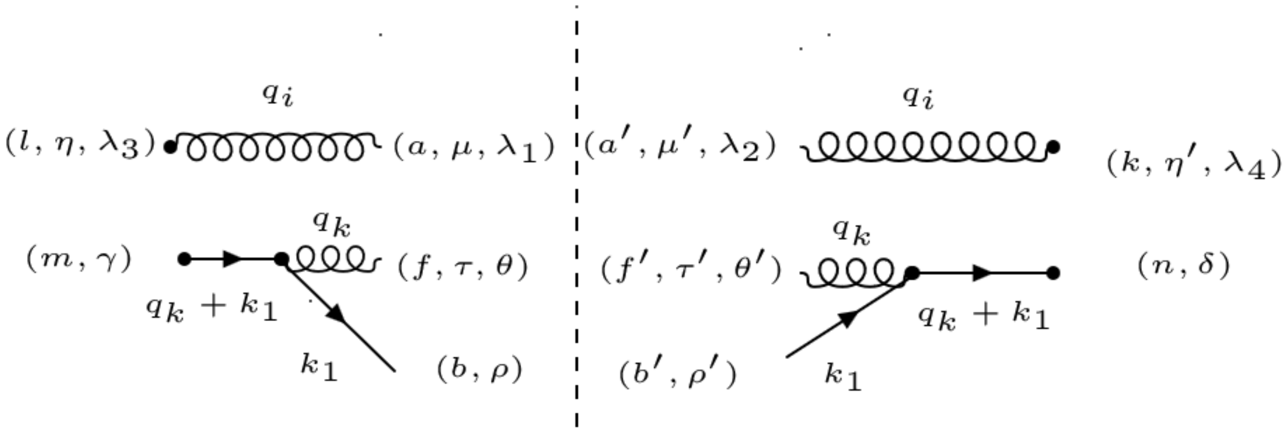
Since the procedure here is the same as in the previous sections, only the final results are presented here.



$$|M_1|^2 = (d-2)(1-\beta_1)\frac{{g_s}^2 C_F}{2y\, p_i \cdot Q}[\not{p}_i][-g^{\delta}{}_{\gamma}] \tag{4.83}$$

$$|M_2|^2 = -\frac{{g_s}^2 C_F}{4(k_1 \cdot q_k)(k_1 \cdot q_k)}[(\not{q}_k + \not{k}_1)\gamma_{\tau'}\, \not{k}_1 \gamma^{\tau}(\not{q}_k + \not{k}_1)][-g^{\eta\eta'}] \tag{4.84}$$

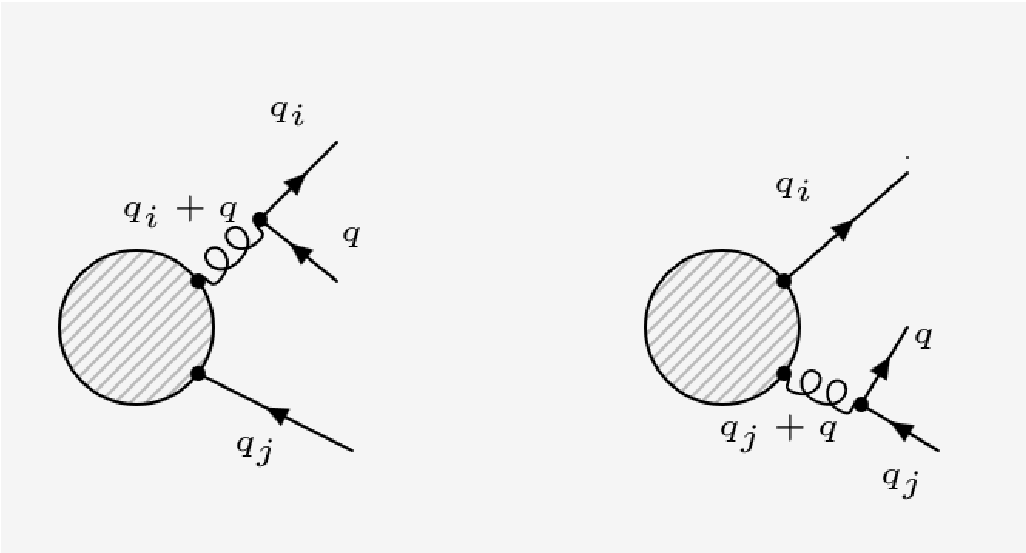
$$|M_2|^2 = (d-2)\frac{{g_s}^2 C_F}{4(k_1 \cdot q_k)}[\not{q}_k][-g^{\eta\eta'}] \tag{4.85}$$



4.3.1 Interpretation of the result

$$|M|^2 = \frac{-{g_s}^2 C_F}{2y(1-y)\,(p_i \cdot Q)}[\not{p}_i][-g^{\delta\gamma}] \otimes [2RE(\frac{2\beta_1}{1-\beta_1}) + (d-2)(1-\beta_1)] \tag{4.86}$$

4.4 A daughter quark from a parent gluon



This case concerns a daughter quark from a parent gluon which splits into a quark-anti-quark pair. Here no singularity develops since daughter and parent can always be distinguished.

5 Example Applications

Our method hast to be illustrated with a simplest examples, namely three-jet production at lepton-collider $e^+e^- \to q + \bar{q} + g$. As in section 3 shown, at first perturbative order, two Feynman diagrams contribute to the matrix element corresponding to the emission of a gluon from either the final-state quark or the anti-quark. The partonic differential cross section with respect to the quark and anti-quark momentum fractions was given by 3.6:

$$\frac{d^2\sigma}{dx_1 dx_2} = \hat{\sigma_0}\frac{\alpha_s}{2\pi}C_F\frac{{x_1}^2 + {x_2}^2}{(1-x_1)(1-x_2)} \tag{5.1}$$

In the parton shower approach, two contributions occur as well. For the final result, those two contributions have to be summed.

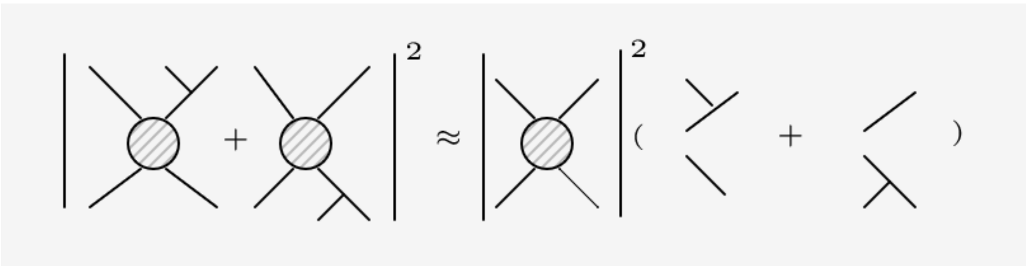


Figure 5.1: Dipole/Antenna factorisation

The dipole factorisation formula is defined by:

$$|\mathcal{M}_m + 1|^2 = \sum_{i,j}\sum_{k\neq i,j}\mathcal{D}_{ij,k} + \sum_{i,j}\sum_{a}\mathcal{D}^a{}_{ij} + \sum_{a,i}\sum_{k\neq i}\mathcal{D}^{ai}{}_k + \sum_{a,i}\sum_{b\neq a}\mathcal{D}^{aj,b} + ... \tag{5.2}$$

The calculation of the subtracted cross section involves just the evaluation of two dipole contributions from the final state emitter and final-state spectator (FF), i. e. $D_{13,2}$ and $D_{23,1}$.

$$\begin{aligned} \mathcal{D}_{13,2}(q_i, q_j, q) &= \frac{-1}{2q_i \cdot q} \;{}_2< 1, \tilde{13}, \tilde{2}|\frac{T_2 \cdot T_{13}}{{T_{13}}^2}V_{13,2}|1, \tilde{13}, \tilde{2} >{}_2 \\ \mathcal{D}_{13,2}(q_i, q_j, q) &= \frac{1}{2p_i \cdot p_j}V_{13,2}|\mathcal{M}_2|^2 \end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathcal{D}_{13,2}(q_i,q_j,q) = \frac{-1}{2p_i\cdot p_j}[&\frac{-(d-2)(1-z)(1-y)}{y} + \frac{(d-2)yz^2}{(1-z)(1-y)} \\ &-(\frac{-2z}{z-1})\frac{1}{y}]|\mathcal{M}_2|^2\end{aligned} \tag{5.4}$$

To find the relationship between the variables, we define a total momentum $Q^\mu = (Q, \overrightarrow{0})$ in the Center of Mass frame. From equation 3.38 can be obtained:

$$\begin{aligned}p_i &= Q/2\,(1, \overrightarrow{0}_\perp, 1) \\ p_i &= Q/2\,(1, \overrightarrow{0}_\perp, 1)\end{aligned} \tag{5.5}$$

On the other hand, the energy fraction x_i is defined as:

$$x_i = \frac{2q_i\cdot Q}{Q^2} \tag{5.6}$$

Therefore:

$$\begin{aligned}x_1 &= z + y(1-z) \\ x_2 &= 1-y \\ x_3 &= (1-z) + yz\end{aligned} \tag{5.7}$$

The addition of x_i is 2 as already mentioned in section 3.1. For convenience, the shower variables are introduced expressed in terms of the x_i:

$$\begin{aligned}\tilde{z_1} &= \frac{x_1+x_2-1}{x_2} \\ y_{13,2} &= 1-x_2\end{aligned} \tag{5.8}$$

For the gluon emission from the quark, after using the expressions it can be obtained [28]

$$\frac{d^2\sigma}{dx_1dx_2}_{PS_q} = \hat{\sigma_0}\frac{\alpha_s}{2\pi}C_F[\frac{1}{1-x_2}(\frac{2}{2-x_1-x_2}-(1+x_1))+\frac{1-x_1}{x_2}] \tag{5.9}$$

For exactly the same calculating for a gluon emission of an anti-quark, it is enough to swap x_1 with x_2:

$$\frac{d^2\sigma}{dx_1dx_2}_{PS_{\bar{q}}} = \hat{\sigma_0}\frac{\alpha_s}{2\pi}C_F[\frac{1}{1-x_1}(\frac{2}{2-x_2-x_1}-(1+x_2))+\frac{1-x_2}{x_1}] \tag{5.10}$$

Thus, the total parton shower cross section will be:

$$\frac{d^2\sigma}{dx_1dx_2}|_{PS} = \frac{d^2\sigma}{dx_1dx_2}|_{PS_q} + \frac{d^2\sigma}{dx_1dx_2}|_{PS_{\bar{q}}} = \hat{\sigma_0}\frac{\alpha_s}{2\pi}C_F[\frac{{x_1}^2+{x_2}^2}{(1-x_1)(1-x_2)}+\frac{1-x_1}{x_2}+\frac{1-x_2}{x_1}] \tag{5.11}$$

Obviously, the parton shower exactly reproduces the soft and collinear singular structure of the matrix element as $x_{1,2} \to 1$.
Our results from the matrix of gluon radiation from an (anti)quark must actually give the same result. Unfortunately, the full calculation can not be obtained because some information from the parametrisation for the evaluation are missing, for instance $\not{p}_i \not{m}^{\mu}{}_{\perp}$, $\not{p}_j \not{m}^{\mu}{}_{\perp}$.

6 Summary and Future outlook

In this thesis a parton shower model based on Catani-Seymour dipole subtraction method was presented. A mapping $3 \to 2$-partons and in this sense one for $m+1 \to m$ in one single emission case was proposed. With the parametrisation, the quadratic matrix elements were explicitly evaluated in terms of the four possible parton splittings in the soft and collinear areas. The kinematics could be used to verify the spin-averaged unregularized Altarelli-Parisi splitting functions in d-dimensions. In addition, a procedure was suggested to show how to concentrate on the important single terms for the calculation of the splitting function in order to waive the considerable and large terms. By this concept it was sufficient to simply sketch the final result, what was expected after the evaluation of the respective matrix elements. Thus, it was completely sufficient to ignore the other terms from the quadratic matrix element and concentrate on a single term. Finally, we took a closer look at a well-known example, namely e^+e^- annihilation, to compare the final result of Gluon emission from a parent (anti)quark 4.1 with the outcome 5.1 from this annihilation process. It was to be noted that in this work information about the behaviour of ${m^\mu}_\perp$ for the evaluation of the full matrix element was missing and moreover the full result from the annihilation process could not reproduce. Nevertheless, both results showed the same properties in the collinear region. For this goal a transformation was proposed so that the result could be shown in the picture of the respective parametrisations of the results from 4.1.4 and the e^+e^- annihilation.

Future outlook

Looking to the future, there are several avenues along which the present work could be continued. As a further procedure one must achieve the full result due to the kinematics and achieve numerically with a contour plot the same three partons configuration with the presented transformation. In this way it can be ensured that both results are in full agreement. For this aim there is a need about the product of ${m^\mu}_\perp$ with the other Momenta p_i, p_j. This information can probably be obtained from the simulation program HERWIG++. Finally one can have a look at the double emissions.

Appendix A

DETAILED CALCULATIONS

Evaluation of the tensor $N^{\eta\eta'}$

$$
\begin{aligned}
&N^{\eta\eta'} \equiv g_{\mu\mu'}g_{\zeta\zeta'}[-g^{\mu\zeta}g^{\mu'\eta'}(q-q_i)^{\eta}(2q_i+q)^{\zeta'} + g^{\mu\zeta}g^{\eta'\zeta'}(q-q_i)^{\eta}(2q+q_i)^{\mu'} \\
&+ g^{\mu\zeta}g^{\zeta'\mu'}(q-q_i)^{\eta}(q_i-q)^{\eta'} + g^{\zeta\eta}g^{\mu'\zeta'}(2q+q_i)^{\mu}(2q_i+q)^{\zeta'} - g^{\zeta\eta}g^{\eta'\zeta'}(2q+q_i)^{\mu}(2q+q_i)^{\mu'} \\
&- g^{\zeta\eta}g^{\zeta'\mu'}(2q+q_i)^{\mu}(q_i-q)^{\eta'} - g^{\eta\mu}g^{\mu'\eta'}(2q_i+q)^{\zeta}(2q_i+q)^{\zeta'} + g^{\eta\mu}g^{\eta'\zeta'}(2q_i+q)^{\zeta}(2q+q_i)^{\mu'} \\
&+ g^{\eta\mu}g^{\zeta'\mu'}(2q_i+q)^{\zeta}(q_i-q)^{\eta'}][g^{\gamma\delta}]
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
N^{\eta\eta'} \equiv [&-(q^{\eta}q^{\eta'} + 2q^{\eta}{q_i}^{\eta'} - {q_i}^{\eta}q^{\eta'} - 2{q_i}^{\eta}{q_i}^{\eta'}) + (2q^{\eta}q^{\eta'} + q^{\eta}{q_i}^{\eta'} - 2{q_i}^{\eta}q^{\eta'} - {q_i}^{\eta}{q_i}^{\eta'}) \\
&+(dq^{\eta}{q_i}^{\eta'} - dq^{\eta}q^{\eta'} - d{q_i}^{\eta}{q_i}^{\eta'} + d{q_i}^{\eta}q^{\eta'}) + (4q^{\eta'}{q_i}^{\eta} + 2q^{\eta'}q^{\eta} + 2{q_i}^{\eta'}{q_i}^{\eta} + {q_i}^{\eta'}q^{\eta}) \\
&-(-2q^{\eta}q^{\eta'} + 2q^{\eta}{q_i}^{\eta'} - {q_i}^{\eta}q^{\eta'} + {q_i}^{\eta}{q_i}^{\eta'}) + (2q^{\eta'}q^{\eta} + q^{\eta'}{q_i}^{\eta} + 4{q_i}^{\eta'}q^{\eta} + 2{q_i}^{\eta'}{q_i}^{\eta}) \\
&+(-q^{\eta}q^{\eta'} + q^{\eta}{q_i}^{\eta'} - 2{q_i}^{\eta}q^{\eta'} + 2{q_i}^{\eta}{q_i}^{\eta'}) - g^{\eta\eta'}(5q^2 + 5{q_i}^2 + 8qq_i)][g^{\gamma\delta}]
\end{aligned}
\tag{2}
$$

Evaluation of the interference term $M_1 M_2{}^\dagger$

Calculation of the first Term

$$
\begin{aligned}
& g^{\eta\eta'}[2\{A_1\beta_1 p_i{}^\gamma p_i{}^\delta + A_2\beta_1 p_i{}^\gamma Q^\delta + \beta_1\sqrt{1-y}p_i{}^\gamma p_k{}^\delta\} \\
& + 2\{[\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]p_i{}^\gamma p_i{}^\delta + y\beta_1{}^2 p_i{}^\gamma Q^\delta\} \\
& + \{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}p_i{}^\gamma p_k{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, p_i{}^\gamma p_i{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, p_i{}^\gamma Q^\delta \\
& + y\beta_1 A_1\, Q^\gamma p_i{}^\delta + y\beta_1 A_2\, Q^\gamma Q^\delta + y\beta_1\sqrt{1-y}Q^\gamma p_k{}^\delta\} \\
& + 3\{[(1-\beta_1)^2 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2]p_i{}^\gamma p_i{}^\delta - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})p_i{}^\gamma Q^\delta - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})Q^\gamma p_i{}^\delta\} \\
& + 4\{[\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]p_i{}^\gamma p_i{}^\delta + y\beta_1{}^2 Q^\gamma p_i{}^\delta\} \\
& + 2\{A_1\beta_1 p_i{}^\gamma p_i{}^\delta + A_2\beta_1 Q^\gamma p_i{}^\delta + \beta_1\sqrt{1-y}p_k{}^\gamma p_i{}^\delta\} \\
& + \{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}p_k{}^\gamma p_i{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, p_i{}^\gamma p_i{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, Q^\gamma p_i{}^\delta \\
& + y\beta_1 A_1\, p_i{}^\gamma Q^\delta + y\beta_1 A_2\, Q^\gamma Q^\delta + y\beta_1\sqrt{1-y}p_k{}^\gamma Q^\delta\}]
\end{aligned}
\tag{3}
$$

Calculation of the second term

$$
-g^{\eta\eta'}g^{\gamma\delta}(2q\cdot q_j + q\cdot q + 4q_i\cdot q_j + 2q_i\cdot q) \tag{4}
$$

$$
\begin{aligned}
& -g^{\eta\eta'}g^{\gamma\delta}[2([\alpha_1(1-y) + y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\, p_i\cdot p_k + y\beta_1\, Q\cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \\
& 4([\beta_1(1-y) + y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\, p_i\cdot p_k + y\alpha_1\, Q\cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \\
& + 2(y\, p_i\cdot Q)]
\end{aligned}
\tag{5}
$$

Calculation of the third term

$$
\begin{aligned}
& + g^{\gamma\eta'}\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}p_i{}^\eta p_k{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, p_i{}^\eta p_i{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, p_i{}^\eta Q^{\eta'} \\
& + y\beta_1 A_1\, Q^\eta p_i{}^\delta + y\beta_1 A_2\, Q^\eta Q^\delta + y\beta_1\sqrt{1-y}Q^\eta p_k{}^\delta - [(1-\beta_1)^2 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2]p_i{}^\eta p_i{}^\delta \\
& - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})p_i{}^\eta Q^\delta - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})Q^\eta p_i{}^\delta] - [A_1\beta_1 p_i{}^\eta p_i{}^\delta + A_2\beta_1 p_i{}^\eta Q^\delta + \beta_1\sqrt{1-y}p_i{}^\eta p_k{}^\delta] \\
& + [\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]p_i{}^\eta p_i{}^{\eta'} + y\beta_1{}^2 p_i{}^\eta Q^{\eta'}\}
\end{aligned}
\tag{6}
$$

Calculation of the fourth term

$$
\begin{aligned}
&+ g^{\eta'\delta}\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q}) - \beta_1]\sqrt{1-y}{p_i}^{\eta}{p_k}^{\gamma} + [2[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]\\
&- y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + A_1\beta_1 + [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]]{p_i}^{\eta}{p_i}^{\gamma}\\
&+ [-2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ + A_2\beta_1 + y{\beta_1}^2]{p_i}^{\eta}Q^{\gamma}\\
&+ [y\beta_1 A_1 \ + 2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]Q^{\eta}{p_i}^{\gamma} + y\beta_1 A_2\, Q^{\eta}Q^{\gamma} + y\beta_1\sqrt{1-y}Q^{\eta}{p_k}^{\gamma}\}
\end{aligned} \tag{7}
$$

Calculation of the fifth term

$$
\begin{aligned}
&- g^{\gamma\delta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] - 2\beta_1]\sqrt{1-y}{p_i}^{\eta}{p_k}^{\eta'}\\
&+ [-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2] - 2A_1\beta_1\\
&- [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\ {p_i}^{\eta}{p_i}^{\eta'}\\
&+ [-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) - y{\beta_1}^2 - 2A_2\beta_1]{p_i}^{\eta}Q^{\eta'}\\
&+ [2y\beta_1 A_1 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\ Q^{\eta}{p_i}^{\eta'} + 2y\beta_1 A_2\, Q^{\eta}Q^{\eta'} + 2y\beta_1\sqrt{1-y}Q^{\eta}{p_k}^{\eta'}\}
\end{aligned} \tag{8}
$$

Calculation of the sixth term

$$
\begin{aligned}
&- g^{\gamma\eta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + \beta_1]\sqrt{1-y}{p_i}^{\eta'}{p_k}^{\delta}\\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 - 2[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]\\
&- [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})] + A_1\beta_1]\ {p_i}^{\eta'}{p_i}^{\delta}\\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ + 2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + A_2\beta_1 - y{\beta_1}^2]{p_i}^{\eta'}Q^{\delta}\\
&+ [2y\beta_1 A_1 + 2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\ Q^{\eta'}{p_i}^{\delta} + 2y\beta_1 A_2\, Q^{\eta'}Q^{\delta} + 2y\beta_1\sqrt{1-y}Q^{\eta'}{p_k}^{\delta}\}
\end{aligned} \tag{9}
$$

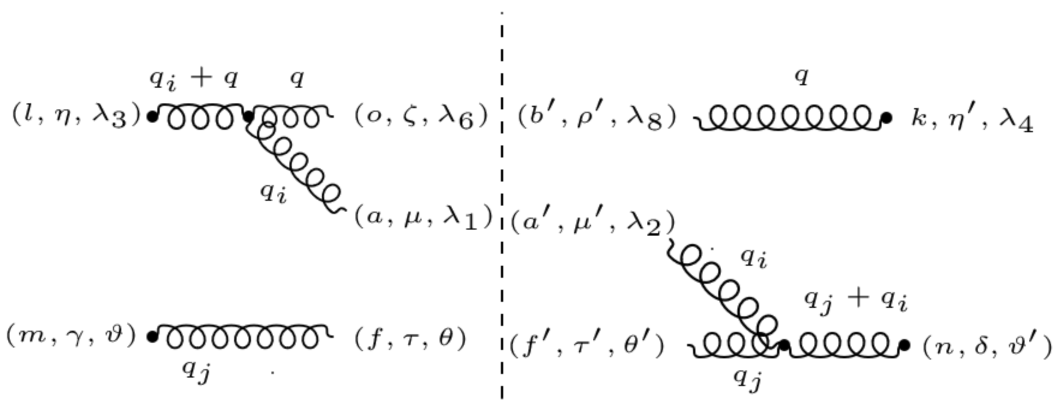
Calculation of the seventh term

$$
\begin{aligned}
&- g^{\eta\delta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + \beta_1]\sqrt{1-y}{p_i}^{\eta'}{p_k}^{\gamma} \\
&[4[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2] - 2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + A_1\beta_1 \\
&+ 2[\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]]{p_i}^{\eta'}{p_i}^{\gamma} \\
&+ [-4y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) - 2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ + 2y{\beta_1}^2 + A_2\beta_1]{p_i}^{\eta'}Q^{\gamma} \\
&+ [-4y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + 2y\beta_1A_1]Q^{\eta}{p_i}^{\eta'} + 2y\beta_1A_2\, Q^{\eta}Q^{\eta'} + 2y\beta_1\sqrt{1-y}Q^{\eta'}{p_k}^{\gamma}\}
\end{aligned} \tag{10}
$$

Calculation of the eighth term

$$
\begin{aligned}
&+ g^{\gamma\delta}\{[4[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + 2\beta_1]\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'} \\
&+ [-4y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + 2A_1\beta_1 + [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})] \\
&+ [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]]\ {p_i}^{\eta}{p_i}^{\eta'} \\
&+ [4y\beta_1A_1 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\ {p_i}^{\eta}Q^{\eta'} + 4y\beta_1A_2\, Q^{\eta}Q^{\eta'} + 4y\beta_1\sqrt{1-y}{p_k}^{\eta}Q^{\eta'} \\
&+ [2A_2\beta_1 - 4y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + y{\beta_1}^2]Q^{\eta}{p_i}^{\eta'}\}
\end{aligned} \tag{11}
$$

Evaluation of the interference term of inverse ${M_1M_2}^{\dagger'}$



$$M_1M_2{}^\dagger = \frac{g_s^2 f^{\,l\,o\,a} f^{\,f'\,a'\,n} \delta^{aa'}\delta^{ob'}\delta^{ff'}}{(q_i+q)^2(q_j+q_i)^2}[g_\zeta{}^{\eta'} g^\gamma{}_{\tau'}(g^{\eta\zeta}(2q+q_i)^\mu + g^{\zeta\mu}(q_i-q)^\eta - g^{\mu\eta}(2q_i+q)^\zeta)$$
$$g_{\mu\mu'}(g^{\tau'\mu'}(q_j-q_i)^\delta + g^{\mu'\delta}(2q_i+q_j)^{\tau'} - g^{\delta\tau'}(2q_j+q_i)^{\mu'}] \tag{12}$$

$$M_1M_2{}^\dagger = \frac{g_s^2 f^{\,l\,o\,a} f^{\,f\,a\,n}}{4(q\cdot q_i)(q_i\cdot q_j)}[g^{\eta\eta'}(2q+q_i)^\gamma(q_j-q_i)^\delta + g^{\eta\eta'}(2q_i+q_j)^\gamma(2q+q_i)^\delta$$
$$- g^{\eta\eta'}g^{\gamma\delta}(2q+q_i)\cdot(2q_j+q_i) + g^{\gamma\eta'}(q_i-q)^\eta(q_j+q_i)^\delta + g^{\eta'\delta}(q_i-q)^\eta(2q_i+q_j)^\gamma \tag{13}$$
$$- g^{\gamma\delta}(q_i-q)^\eta(2q_j+q_i)^{\eta'} - g^{\gamma\eta}(2q_i+q)^{\eta'}(q_j-q_i)^\delta - g^{\eta\delta}(2q_i+q)^{\eta'}(2q_i+q_j)^\gamma$$
$$+ g^{\gamma\delta}(2q_j+q_i)^\eta(2q_i+q)^{\eta'}]$$

Parametrization in terms of $(k_1\cdot q_i)(q_i\cdot q_k)$

$$\boxed{(k_1\cdot q_i)(q_i\cdot q_k) \approx y\beta_1(1-y)\,(p_i\cdot Q)(p_i\cdot p_k)} \tag{14}$$

Calculation of the third term

$$-g^{\eta\eta'}g^{\gamma\delta}\{4k_1\cdot q_j + 2k_1\cdot q_i + 2q_i\cdot q_k\} \tag{15}$$

$$M_1M_2{}^\dagger = \frac{g_s^2 C_A}{4y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} g^{\eta\eta'}g^{\gamma\delta}$$
$$[4([\alpha_1(1-y)+y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\beta_1\,Q\cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \tag{16}$$
$$2([\beta_1(1-y)+y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\alpha_1\,Q\cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1})$$
$$+2(y\,p_i\cdot Q)]$$

$$-g^{\eta\eta'}g^{\gamma\delta}[4([\alpha_1(1-y)+y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\beta_1\,Q\cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1})$$
$$2([\beta_1(1-y)+y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\alpha_1\,Q\cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \tag{17}$$
$$+2(y\,p_i\cdot Q)]$$

$$M_1M_2{}^\dagger = g_s^2\,C_A\,g^{\eta\eta'}g^{\gamma\delta}[\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)} + \frac{1}{2y(p_i\cdot Q)} + \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)}$$
$$+\frac{(1-\beta_1)\,Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}] \tag{18}$$

.0.1 Summary of the results

$$
\begin{aligned}
&|M|^2 = |M'_2|^2 + |M'_1|^2 + 2RE(M_1 M_2{}^{\dagger} + M_1 M_2{}^{\dagger\prime})\\
&|M|^2 = \frac{g_s^2\, C_A}{y\,(p_i\cdot Q)}[2[\epsilon-1]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}]\\
&+ \frac{g_s^2\, C_A}{(1-\beta_1)(1-y)\,(p_i\cdot p_k)}[-2g^{\delta\gamma}][g^{\eta\eta'}]\\
&+ 2Re(g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[\frac{1}{2y(p_i\cdot Q)} + \frac{\beta_1(\frac{Q^2}{2p_i\cdot Q})}{2y(1-\beta_1)(1-y)\,(p_i\cdot Q)}\\
&+ \frac{\beta_1\, Q\cdot p_k}{2y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{\beta_1}{y(1-\beta_1)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}]\\
&+ g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)} + \frac{1}{2y(p_i\cdot Q)} + \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)}\\
&+ \frac{(1-\beta_1)\, Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}])
\end{aligned}
\tag{19}
$$

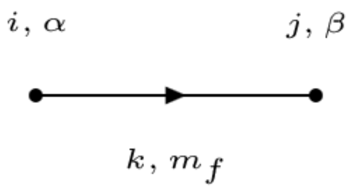
$$
\begin{aligned}
&|M|^2 = g_s^2\, C_A\, g^{\gamma\delta}[-2[1-\epsilon]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} + \frac{\beta_1(\frac{Q^2}{2p_i\cdot Q})}{y(1-\beta_1)(1-y)\,(p_i\cdot Q)}g^{\eta\eta'}\\
&+ \frac{\beta_1\, Q\cdot p_k}{y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}g^{\eta\eta'} + \frac{2\beta_1}{y(1-\beta_1)(p_i\cdot Q)}g^{\eta\eta'} + \frac{2(1-\beta_1)}{y\beta_1(p_i\cdot Q)}g^{\eta\eta'}\\
&+ \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{y\beta_1(1-y)\,(p_i\cdot Q)}g^{\eta\eta'} + \frac{(1-\beta_1)\, Q\cdot p_k}{y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}g^{\eta\eta'}]
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&|M|^2 = g_s^2\, C_A\, g^{\gamma\delta}[-2(1-\epsilon)\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} + \frac{2\beta_1}{y(1-\beta_1)(p_i\cdot Q)}g^{\eta\eta'} + \frac{2(1-\beta_1)}{y\beta_1(p_i\cdot Q)}g^{\eta\eta'}\\
&+ \frac{Q^2}{2y\beta_1(1-y)\,(p_i\cdot Q)(p_i\cdot Q)}g^{\eta\eta'} + \frac{Q\cdot p_k}{y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}g^{\eta\eta'}]
\end{aligned}
\tag{21}
$$

MATHEMATICAL TOOLS

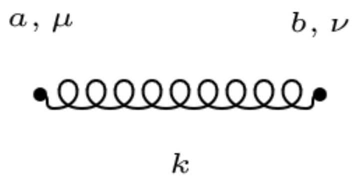
Feynman rules in QCD

$$L_{q,free} = \sum_f \bar{\psi}_{if}(i\gamma^\mu \partial_\mu - m_f)\delta_{ij}\psi^{jf}$$



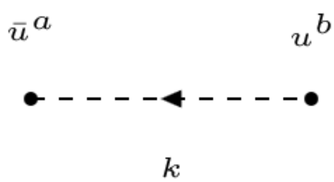
$$= (\frac{i}{\not{k} - m_f})_{\alpha\beta}\, \delta_{ij}$$

$$L_{g,free} = -\frac{1}{4} F_a{}^{\mu\nu} F^a{}_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a{}_\mu)(\partial^\nu A^a{}_\nu)$$



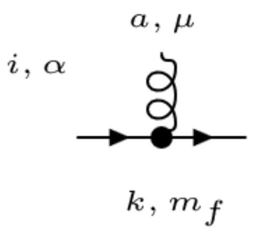
$$= \frac{-i}{k^2}(-g_{\mu\nu} - (1-\zeta)\frac{k_\mu k_\nu}{k^2})\delta^{ab}$$

$$L_{ghost,free} = (\partial^\mu \chi^{a*})(\partial_\mu \chi^a)$$



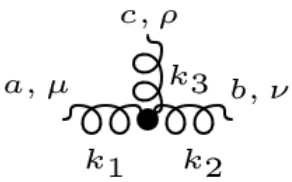
$$= \frac{i}{k^2}\delta^{ab}$$

$$L_{qg\bar{q},int} = g_s \bar{\psi}_i T^a{}_{ij} \psi_j \gamma^\mu A^a{}_\mu$$



$$= i g_s \gamma^\mu \otimes T^a{}_{ij}$$

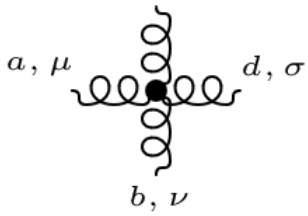
$$L_{ggg,int} = -\frac{g_s}{2} f^{abc} (\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu) A_b{}^\mu A_c{}^\nu$$



$$= -g_s f^{bca} [(k_1 - k_2)^\rho g^{\mu\nu} + (k_2 - k_3)^\mu g^{\nu\rho} + (k_3 - k_1)^\nu g^{\rho\mu}]$$

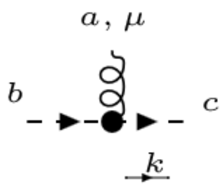
$(all\ Momenta\ are\ incoming!)$

$$L_{gggg,int} = -\frac{g_s{}^2}{4} f^{abc} (A_b{}^\mu A_c{}^\nu) f^{ade} (A^d{}_\mu A^e{}_\nu)$$



$$= -i g_s{}^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]$$

$$L_{\chi g \bar{\chi}} = -g_s f^{abc} (\partial^\mu \chi^{a*}) \chi^b A^c{}_\mu$$



$$= g_s f^{abc} k^\mu$$

Lorenz transformation of momenta $\hat{p_i}^\mu, \hat{p_k}^\mu$ and $\hat{Q}^\mu$

$$\hat{p_i}^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu = p_i{}^\mu p_{i\nu} p_i{}^\nu \frac{-y^2Q^2}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + p_i{}^\mu Q_\nu p_i{}^\nu \frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$
$$+ Q^\mu p_{i\nu} p_i{}^\nu \frac{(y^2-y-y\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}\eta^\mu{}_\nu p_i{}^\nu$$

$$\hat{p_i}^\mu = p_i{}^\mu (Q\cdot p_i)\frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}p_i{}^\mu$$
$$= p_i{}^\mu[\frac{y(1+\sqrt{1-y})}{(2+2\sqrt{1-y}-y)} + \sqrt{1-y}] = p_i{}^\mu$$

$$\boxed{\hat{p_i}^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu = p_i{}^\mu} \tag{6.22}$$

$$\hat{p_k}^\mu = \alpha\Lambda^\mu{}_\nu p_k{}^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}p_k{}^\mu$$

$$\hat{p_k}^\mu = \alpha\Lambda^\mu{}_\nu p_k{}^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+ Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}p_k{}^\mu$$

with

$$A_1 \equiv \frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$
$$A_2 \equiv \frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$

$$\boxed{\hat{p_k}^\mu = A_1\, p_i{}^\mu + A_2\, Q^\mu + \sqrt{1-y}p_k{}^\mu} \tag{6.23}$$

$$\hat{Q}^\mu = \alpha\Lambda^\mu{}_\nu Q^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot Q)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})Q^2}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+ Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot Q)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}Q^\mu$$

with

$$S_1 \equiv \frac{Q^2}{2p_i\cdot Q}[\frac{-y^2}{2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})}{(1+\sqrt{1-y}-\frac{y}{2})}] = \frac{Q^2}{2p_i\cdot Q}y$$
$$S_2 \equiv \frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y} = 1-y$$

$$\boxed{\hat{Q}^\mu = \tfrac{Q^2}{2p_i \cdot Q} y \, {p_i}^\mu + (1-y)\, Q^\mu} \tag{6.24}$$

The often occurring pre-factor products

$$\begin{aligned}
\zeta_1\zeta_1 &= ({\alpha_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}) + y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2) \\
\zeta_1\lambda_1 &= (y\alpha_1\beta_1 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})) \\
\zeta_1\zeta_q &= (\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q}) + y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})^2) \\
\zeta_1\lambda_q &= (y{\alpha_1}^2 - y^2\beta_1\alpha_1(\frac{Q^2}{2p_i\cdot Q})) \\
\zeta_q\zeta_q &= ({\beta_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}) + y^2{\alpha_1}^2(\frac{Q^2}{2p_i\cdot Q})^2) \\
\zeta_q\lambda_1 &= (y{\beta_1}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})) \\
\zeta_q\lambda_q &= (y\beta_1\alpha_1 - y^2{\alpha_1}^2(\frac{Q^2}{2p_i\cdot Q})) \\
\lambda_1\lambda_1 &= y^2{\beta_1}^2 \qquad \lambda_1\lambda_q = y^2\beta_1\alpha_1 \qquad \lambda_q\lambda_q = y^2{\alpha_1}^2
\end{aligned} \tag{6.25}$$

Common scalar products

$$\begin{aligned}
k_1\cdot q_i &= (\zeta_1\lambda_q + \lambda_1\zeta_q)p_i\cdot Q + \lambda_1\lambda_q Q^2 - y\alpha_1\beta_1 {n^2}_{\perp,1} \\
&= [(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))y\alpha_1 + y\beta_1(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))]\, p_i\cdot Q \\
&\quad y^2\beta_1\alpha_1\, Q^2 + 2y\alpha_1\beta_1\, p_iQ \\
\Rightarrow k_1\cdot q_i &= [y{\alpha_1}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}) + y{\beta_1}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})]\, p_i\cdot Q \\
& y^2\beta_1\alpha_1\, Q^2 + 2y\alpha_1\beta_1\, p_iQ
\end{aligned} \tag{6.26}$$

$$\boxed{k_1\cdot q_i = y(\alpha_1+\beta_1)^2\, p_i\cdot Q = y\, p_i\cdot Q} \tag{6.27}$$

$$\begin{aligned}
k_1 \cdot q_k &= (\zeta_1 A_2 + \lambda_1 A_1) p_i \cdot Q + \zeta_1 \sqrt{1-y}\, p_i \cdot p_k + \lambda_1 A_2\, Q^2 + \lambda_1 \sqrt{1-y}\, Q \cdot p_k \\
&+ \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1} \\
&= \{[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))\frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] \\
&+ y\beta_1[\frac{-y^2 Q^2 (p_i \cdot p_k)}{4(p_i \cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]\}\, p_i \cdot Q \\
&+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))\sqrt{1-y}\, p_i \cdot p_k + y\beta_1 \frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} Q^2 \\
&+ y\beta_1 \sqrt{1-y} Q \cdot p_k + \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
k_1 \cdot q_k &= \alpha_1 \frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) - y\beta_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) \\
&+ y\beta_1 \frac{-y^2 Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\beta_1 \frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}\, Q \cdot p_k \\
&+ \alpha_1 \sqrt{1-y}\, p_i \cdot p_k - y\beta_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}\, p_i \cdot p_k \\
&+ y\beta_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\beta_1 \sqrt{1-y}(Q \cdot p_k) \\
&+ \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.29}$$

$$\begin{aligned}
k_1 \cdot q_k &= [\alpha_1 \frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\beta_1 \frac{-y^2 Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \alpha_1 \sqrt{1-y} \\
&- y\beta_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}]\, p_i \cdot p_k + [y\beta_1 \frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\beta_1 \sqrt{1-y}](Q \cdot p_k) \\
&+ \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.30}$$

$$\begin{aligned}
k_1 \cdot q_k &= \{\alpha_1[\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}] \\
&+ y\beta_1(\frac{Q^2}{p_i \cdot Q})[\frac{-y^2}{4(1+\sqrt{1-y}-\frac{y}{2})} - \sqrt{1-y}]\}\, p_i \cdot p_k \\
&+ y\beta_1[\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}](Q \cdot p_k) \\
&+ \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.31}$$

$$\boxed{k_1 \cdot q_k = [\alpha_1(1-y) + y\beta_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\beta_1\, Q \cdot p_k + \sqrt{\alpha_1 \beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{6.32}$$

$$
\begin{aligned}
q_i \cdot q_k &= (\zeta_q A_2 + \lambda_q A_1) p_i \cdot Q + \zeta_q \sqrt{1-y}\, p_i \cdot p_k + \lambda_q A_2\, Q^2 + \lambda_q \sqrt{1-y}\, Q \cdot p_k \\
&- \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1} \\
&= \{[(\beta_1 - y\alpha_1(\frac{Q^2}{2p_i \cdot Q}))\frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] \\
&+ y\alpha_1[\frac{-y^2 Q^2 (p_i \cdot p_k)}{4(p_i \cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]\}\, p_i \cdot Q \\
&+ (\beta_1 - y\alpha_1(\frac{Q^2}{2p_i \cdot Q}))\sqrt{1-y}\, p_i \cdot p_k + y\alpha_1 \frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} Q^2 \\
&+ y\alpha_1 \sqrt{1-y} Q \cdot p_k - \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.33}
$$

$$
\begin{aligned}
q_i \cdot q_k = \beta_1 &\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) - y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) \\
&+ y\alpha_1 \frac{-y^2 Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\alpha_1 \frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}\, Q \cdot p_k \\
&+ \beta_1 \sqrt{1-y}\, p_i \cdot p_k - y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}\, p_i \cdot p_k \\
&+ y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\alpha_1 \sqrt{1-y}(Q \cdot p_k) \\
&- \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.34}
$$

$$
\begin{aligned}
q_i \cdot q_k = [&\beta_1 \frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\alpha_1 \frac{-y^2 Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \beta_1 \sqrt{1-y} \\
&- y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}]\, p_i \cdot p_k + [y\alpha_1 \frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\alpha_1\sqrt{1-y}](Q \cdot p_k) \\
&- \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.35}
$$

$$
\begin{aligned}
k_1 \cdot q_k = \{&\beta_1[\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}] \\
&+ y\alpha_1(\frac{Q^2}{p_i \cdot Q})[\frac{-y^2}{4(1+\sqrt{1-y}-\frac{y}{2})} - \sqrt{1-y}]\}\, p_i \cdot p_k \\
&+ y\alpha_1[\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}](Q \cdot p_k) \\
&- \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.36}
$$

$$
\boxed{q_i \cdot q_k = [\beta_1(1-y) + y\alpha_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\alpha_1\, Q \cdot p_k - \sqrt{\alpha_1 \beta_1 y (1-y)} p_k \cdot n_{\perp,1}} \tag{6.37}
$$

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