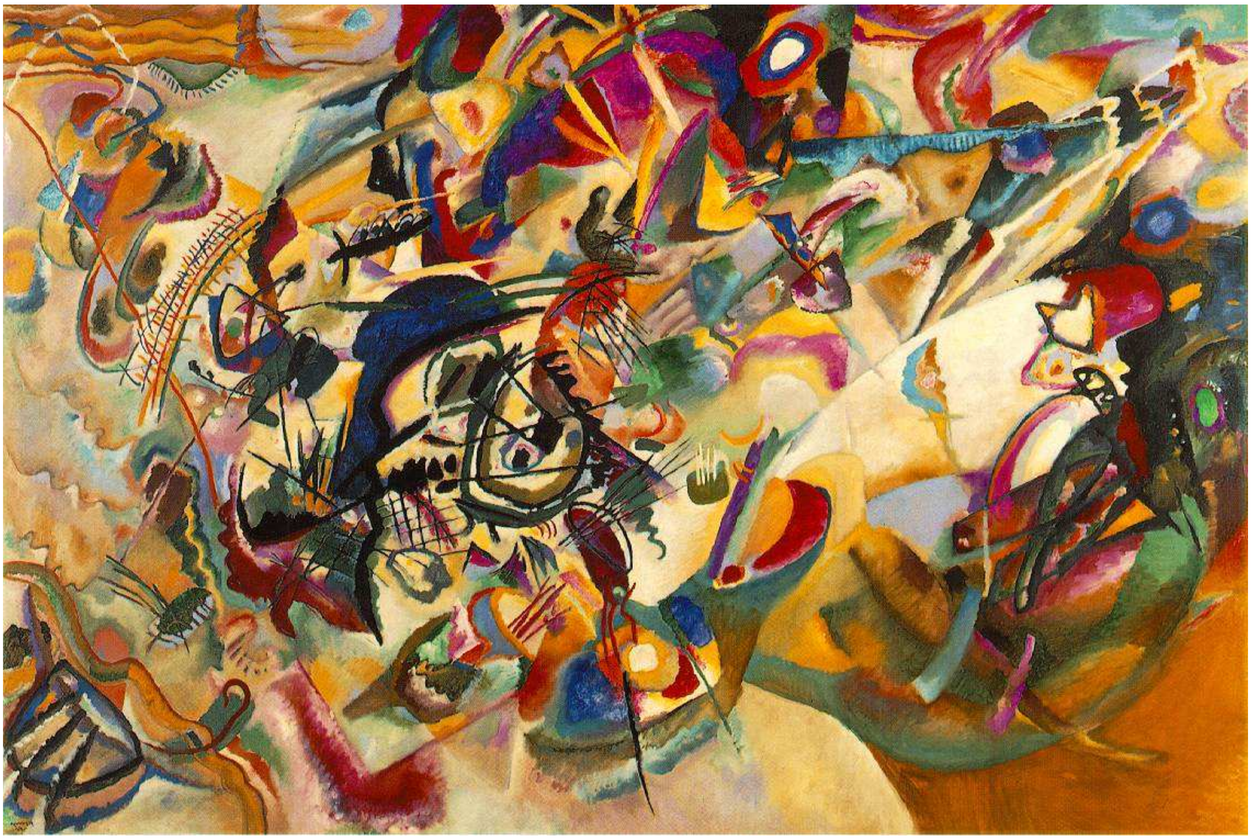
Introduction to perturbative QCD

Notes from Carlo Oleari's lectures



"Composition VII"
Wassily Kandisky
1913, Oil on canvas 200x300cm
Tretyakov Galley, Moscow

DISCLAIMER

These notes have been taken by Simone Alioli, Antonio Amariti, Carlo Alberto Ratti and Emanuele Re during their PhD years.

These notes have **NOT** been corrected or revised by Carlo Oleari, who takes **no responsibility** for the correctness of what follows.

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Chapter 1

Mathematical tools

1.1 The Feynman parametrization

The Feynman parametrization is given by the following formula

$$\prod_{i=1}^{n} \frac{1}{A_i^{c_i}} = \frac{\Gamma(c)}{\prod_{i=1}^{n} \Gamma(c_i)} \int_0^1 \prod_{i=1}^{n} \alpha_i^{c_i - 1} d\alpha_i \, \frac{\delta\left(1 - \sum_{i=1}^{n} \alpha_i\right)}{\left(\sum_{k=1}^{n} \alpha_k A_k\right)^c} \tag{1.1}$$

where

$$c = \sum_{i=1}^{n} c_i \,. \tag{1.2}$$

The proof of this equation is done following a few steps. First of all, we demonstrate it by induction when all the c_i are equal to 1. The case with $n = 2$ is trivial: by a direct inspection

$$\begin{aligned} I_2 &\equiv \int_0^1 d\alpha_1 \, d\alpha_2 \, \frac{\delta(1 - \alpha_1 - \alpha_2)}{(\alpha_1 A_1 + \alpha_2 A_2)^2} = \int_0^1 d\alpha_1 \frac{1}{(\alpha_1 A_1 + (1 - \alpha_1) A_2)^2} \\ &= -\frac{1}{A_1 - A_2} \left[\frac{1}{\alpha_1 A_1 + (1 - \alpha_1) A_2} \right]_0^1 = \frac{1}{A_1 A_2} \end{aligned} \tag{1.3}$$

Supposing now that formula (1.1) is valid for $(n - 1)$

$$\begin{aligned} I_{n-1} &\equiv \frac{1}{A_1 \ldots A_{n-1}} = (n-2)! \int_0^1 \prod_{i=1}^{n-1} d\alpha_i \, \frac{\delta\left(1 - \sum_{i=1}^{n-1} \alpha_i\right)}{\left(\sum_{k=1}^{n-1} \alpha_k A_k\right)^{n-1}} \\ &= \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{(n-2)!}{\left(A_{n-1} + \sum_{k=1}^{n-2} \alpha_k \left(A_k - A_{n-1}\right)\right)^{n-1}} \end{aligned} \tag{1.4}$$

where in the last line

$$0 \leq \sum_{k=1}^{n-2} \alpha_k \leq 1 \tag{1.5}$$

we show, with some algebra, that it is true also for n

$$\begin{aligned} I_n \;\equiv\; \frac{1}{A_1 \dots A_n} &= (n-1)! \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \frac{\delta(1-\sum_{i=1}^{n}\alpha_i)}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^n} \\ &= \int_0^1 \prod_{i=1}^{n-1} d\alpha_i \, \frac{(n-1)!}{\left(A_n + \sum_{k=1}^{n-1}\alpha_k \left(A_k - A_n\right)\right)^n} \end{aligned} \tag{1.6}$$

with

$$0 \leq \sum_{k=1}^{n-1} \alpha_k \leq 1 \,. \tag{1.7}$$

In fact, integrating (1.6) in α_{n-1} between 0 and $\left(1-\sum_{i=1}^{n-2}\alpha_i\right)$, we find

$$\begin{aligned} I_n \;=\; & -\frac{(n-2)!}{(A_{n-1}-A_n)} \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{1}{\left(A_{n-1} + \sum_{k=1}^{n-2}\alpha_k \left(A_k - A_{n-1}\right)\right)^{n-1}} \\ & + \frac{(n-2)!}{(A_{n-1}-A_n)} \int_0^1 \prod_{i=1}^{n-2} d\alpha_i \, \frac{1}{\left(A_n + \sum_{k=1}^{n-2}\alpha_k \left(A_k - A_n\right)\right)^{n-1}} \end{aligned} \tag{1.8}$$

and using (1.4)

$$I_n \;=\; \frac{1}{(A_{n-1}-A_n)\, A_1 \dots A_{n-2}} \left[\frac{1}{A_n} - \frac{1}{A_{n-1}}\right] = \frac{1}{A_1 \dots A_n} \tag{1.9}$$

so equation (1.6) is indeed an identity.

To complete the demonstration of eq. (1.1), we derive c_n times both members of the first line of (1.6) with respect to A_n. On the left-hand side we have

$$\frac{\partial^{c_n}}{\partial A_n^{c_n}} \left(\frac{1}{A_1 \dots A_n}\right) = \frac{(-1)^{c_n}\,(c_n)!}{A_1 \dots A_{n-1} A_n^{c_n+1}} \,, \tag{1.10}$$

while on the right-hand side, the derivation gives

$$\frac{\partial^{c_n}}{\partial A_n^{c_n}} I_n = \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \delta\!\left(1 - \sum_{i=1}^{n}\alpha_i\right) \frac{(-1)^{c_n}\,(n+c_n-1)!\,\alpha_n^{c_n}}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^{n+c_n}} \tag{1.11}$$

Comparing the last two equations one can see that

$$\frac{1}{A_1 \dots A_{n-1} A_n^{c_n}} = \frac{(n+c_n-2)!}{(c_n-1)!} \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \delta\!\left(1 - \sum_{i=1}^{n}\alpha_i\right) \frac{\alpha_n^{c_n-1}}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^{n+c_n-1}} \,. \tag{1.12}$$

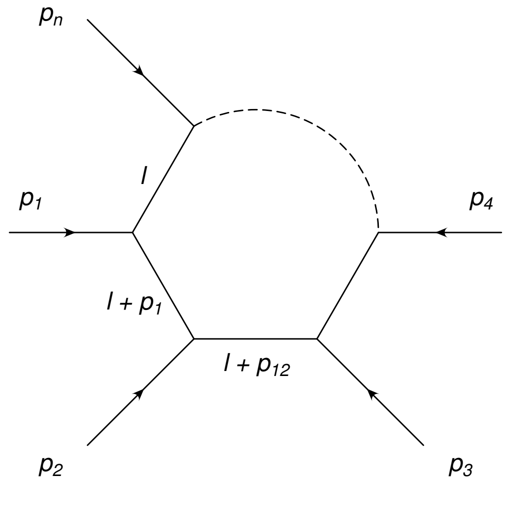
Repeating now the derivation with respect to a generic A_k, we get

$$\prod_{i=1}^{n} \frac{1}{A_i^{c_i}} = \frac{(c-1)!}{\prod_{i=1}^{n} (c_i-1)!} \int_0^1 \prod_{i=1}^{n} \alpha_i^{c_i-1} d\alpha_i \; \frac{\delta\left(1-\sum_{i=1}^{n}\alpha_i\right)}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^c}$$
$$= \frac{\Gamma(c)}{\prod_{i=1}^{n} \Gamma(c_i)} \int_0^1 \prod_{i=1}^{n} \alpha_i^{c_i-1} d\alpha_i \; \frac{\delta\left(1-\sum_{i=1}^{n}\alpha_i\right)}{\left(\sum_{k=1}^{n}\alpha_k A_k\right)^c} . \tag{1.13}$$

This proves eq. (1.1).

1.2 The scalar one-loop integrals

In this section we want to introduce all the principal mathematical tools useful to calculate d-dimensional scalar one-loop Feynman integrals. These integrals are built up with the propagators of n massive particles, with masses m_i, connecting $n+1$ vertexes of interaction with other external particles, each carrying momentum p_i.[1]



The integral can be written in this general form (notice that $\sum_{i=1}^{n} p_i = 0$ for momentum conservation)

$$I = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell+p_1)^2 - m_1^2 + i\eta\right]\left[(\ell+p_{12})^2 - m_2^2 + i\eta\right]\dots\left[(\ell+p_{12\dots n})^2 - m_n^2 + i\eta\right]} , \tag{1.14}$$

[1] All momenta incoming.

where we have introduced a small imaginary part $i\eta$ according to the Feynman prescription for the T–ordered propagator and we have used the shortcut $p_{12} = p_1 + p_2$, and similar ones.

Using the Feynman parametrization (1.1) we can write

$$I = \Gamma\left(n\right) \int_0^1 \prod_{i=1}^n d\alpha_i \int \frac{d^d\ell}{(2\pi)^d} \frac{\delta(1-\sum_{i=1}^n \alpha_i)}{\left(\sum_{k=1}^n \alpha_k A_k\right)^n} \tag{1.15}$$

The sum in the denominator can then be rewritten as

$$\begin{aligned}
\sum_{k=1}^n \alpha_k A_k &= \sum_{k=1}^n \alpha_k \left[(\ell + p_{1\ldots k})^2 - m_k^2 + i\eta\right] = \\
&= \ell^2 + 2\ell \cdot \left(\sum_{k=1}^n \alpha_k p_{1\ldots k}\right) + \sum_{k=1}^n \alpha_k \left(p_{1\ldots k}^2 - m_k^2 + i\eta\right) = \\
&\equiv \ell^2 + 2\ell \cdot P + K^2 + i\eta\,.
\end{aligned} \tag{1.16}$$

The integral (1.15) becomes

$$\begin{aligned}
I &= \Gamma\left(n\right) \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left(\ell^2 + 2\ell\cdot P + K^2 + i\eta\right)^n} \qquad (\ell \to \ell + P) \\
&= \Gamma\left(n\right) \int_0^1 \left[d\alpha\right]_n \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left(\ell^2 - m^2 + i\eta\right)^n}
\end{aligned} \tag{1.17}$$

where we used the shorthand notation

$$\left[d\alpha\right]_n \equiv \prod_{i=1}^n d\alpha_i\, \delta\left(1 - \sum_{i=1}^n \alpha_i\right)\,, \qquad m^2 \equiv P^2 - K^2 \tag{1.18}$$

Notice that in the last line η is not the same one defined previously but it plays the same role again picking the poles away from the path of the integration as the Feynman prescription requires.

The integral over the loop momentum l can be performed once and for all. We first perform the integral over l_0. In Fig. 1.1 we have promoted the real variable l_0 into a complex variable and we have plotted the two poles

$$\ell^2 - m^2 + i\eta \equiv \ell_0^2 - |\ell|^2 - m^2 + i\eta = 0 \qquad \Longrightarrow \qquad l_0 = \pm\sqrt{|\ell|^2 + m^2} \mp i\eta\,. \tag{1.19}$$

The integration over l_0 is along the real axis. Exploiting the fact that the Feynman integrals are analytic functions, we interpret the integration along the real axis as part of the integration over the closed path in the figure. Using the residue theorem, we know that the integral along that closed path is zero, since the poles of the integral are outside the integration path. So we can write

$$0 = \int_{-\infty}^{+\infty} d\ell_0 \ldots + \int_{+\infty}^{-\infty} i d\ell_0^E \ldots \qquad \Longrightarrow \qquad \int_{-\infty}^{+\infty} d\ell_0 \ldots = i \int_{-\infty}^{+\infty} d\ell_0^E \ldots \tag{1.20}$$

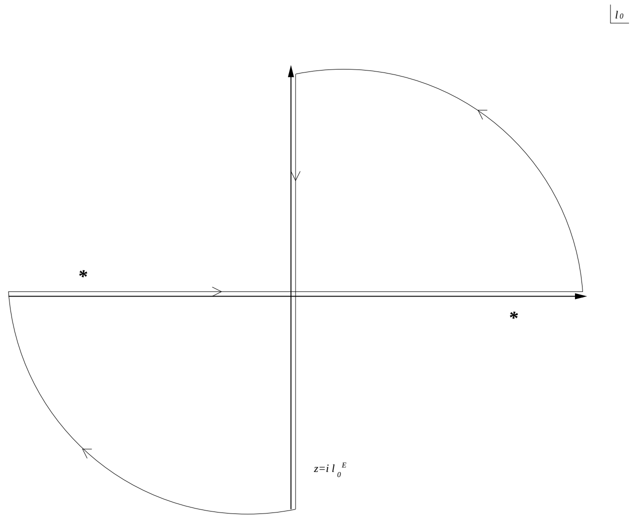


Figure 1.1: Wick rotation.

since the contribution from the circular parts of the path goes to zero as we radius goes to infinity. We have indicated with ℓ_0^E the new integration variable, reminiscent of the fact that now we are using an Euclidean notation and no longer a Minkoskian one. The integral I then becomes

$$I = i\Gamma\left(n\right) \int_0^1 \left[d\alpha\right]_n \int_{-\infty}^{+\infty} \frac{d\ell_0^E \; d^{(d-1)}\ell}{\left(2\pi\right)^d} \frac{1}{\Big(-\left(\ell_0^E\right)^2 - |\ell|^2 - m^2 + i\eta\Big)^n} \tag{1.21}$$

and using spherical coordinates, defining $\ell_E^2 \equiv (\ell_0^E)^2 + |\ell|^2$ (please notice that the integral over the loop momentum is now perfectly defined and we could set $\eta = 0$. We keep it, since it will be useful in the integration over the Feynman parameters α_i, yet to be done)

$$\begin{aligned}
I &= \frac{\left(-1\right)^n i\Gamma\left(n\right)}{\left(2\pi\right)^d} \int_0^1 \left[d\alpha\right]_n \int d^d\Omega \; d\ell_E \, \frac{\left(\ell_E\right)^{d-1}}{\left(\ell_E^2 + m^2 - i\eta\right)^n} \\
&\qquad \Longrightarrow \; t = \frac{\ell_E^2}{m^2} \\
&= \frac{\left(-1\right)^n i\Gamma\left(n\right)\Omega_d}{2\left(2\pi\right)^d} \int_0^1 \left[d\alpha\right]_n \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \int_0^\infty dt \; t^{\frac{d}{2}-1}\left(t+1\right)^{-n} \\
&\qquad \Longrightarrow \; x = \frac{1}{1+t} \\
&= \frac{\left(-1\right)^n i\Gamma\left(n\right)\Omega_d}{2\left(2\pi\right)^d} \int_0^1 \left[d\alpha\right]_n \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \int_0^1 dx \; x^{n-\frac{d}{2}-1}\left(1-x\right)^{\frac{d}{2}-1} \\
&= \frac{\left(-1\right)^n i\Gamma\left(n\right)\Omega_d}{2\left(2\pi\right)^d} \beta\left(\frac{d}{2}, n - \frac{d}{2}\right) \int_0^1 \left[d\alpha\right]_n \; \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \\
&= \frac{\left(-1\right)^n i\Gamma\left(n-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\Omega_d}{2\left(2\pi\right)^d} \int_0^1 \left[d\alpha\right]_n \; \left(m^2 - i\eta\right)^{\frac{d}{2}-n}
\end{aligned} \tag{1.22}$$

where Ω_d is the total angle in d dimensions

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \,. \tag{1.23}$$

Finally, the scalar integral (1.15) takes this form

$$\begin{aligned} I &= (-1)^n \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 [d\alpha]_n \, \left(m^2 - i\eta\right)^{\frac{d}{2}-n} \\ &= (-1)^n \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 [d\alpha]_n \, \left(-\sum_{i>j}^{n} \alpha_i \alpha_j \, p_{j+1\ldots i}^2 + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta\right)^{\frac{d}{2}-n} \end{aligned} \tag{1.24}$$

where in the last line we used

$$\begin{aligned} m^2 &= P^2 - K^2 = \left(\sum_{i=1}^{n} \alpha_i p_{1\ldots i}\right)^2 - \sum_{i=1}^{n} \alpha_i \left(p_{1\ldots i}^2 - m_i^2 + i\eta\right) \\ &= \sum_{i=1}^{n} \alpha_i^2 p_{1\ldots i}^2 + 2\sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} - \sum_{i=1}^{n} \alpha_i p_{1\ldots i}^2 + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i=1}^{n} \alpha_i \sum_{j\neq i} \alpha_j p_{1\ldots i}^2 + 2\sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i}^2 - \sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{1\ldots j} \\ &\quad -\sum_{j>i}^{n} \alpha_i \alpha_j p_{1\ldots i}^2 - \sum_{j>i}^{n} \alpha_j \alpha_i p_{1\ldots j} p_{1\ldots i} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{j+1\ldots i} + \sum_{j>i}^{n} \alpha_i \alpha_j p_{1\ldots i} p_{i+1\ldots j} + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \\ &= -\sum_{i>j}^{n} \alpha_i \alpha_j p_{j+1\ldots i}^2 + \sum_{i=1}^{n} \alpha_i m_i^2 - i\eta \,. \end{aligned} \tag{1.25}$$

In summary

$$\begin{aligned} I &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell + p_1)^2 - m_1^2 + i\eta\right] \left[(\ell + p_{12})^2 - m_2^2 + i\eta\right] \ldots \left[(\ell + p_{12\ldots n})^2 - m_n^2 + i\eta\right]} \\ &= (-1)^n \frac{i}{(4\pi)^{\left(\frac{d}{2}\right)}} \Gamma\left(n - \frac{d}{2}\right) \int_0^1 \frac{[d\alpha]_n}{D^{n-\frac{d}{2}}} \,, \end{aligned} \tag{1.26}$$

where

$$D = -\sum_{i>j} \alpha_i \, \alpha_j \, s_{ij} + \sum_{i=1}^{n} \alpha_i \, m_i^2 - i\eta \,, \tag{1.27}$$

and s_{ij} is the square of the momentum flowing through the i-j cut of the diagram representing I.

1.2.1 The one-point function (tadpole)

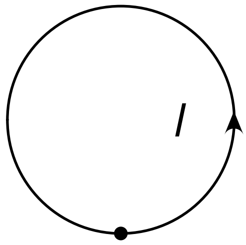


Figure 1.2: One-point function (tadpole).

The one-point function is given by

$$
\begin{aligned}
A_0\big(m^2\big) &= \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{\ell^2 - m^2 + i\eta} = \frac{-i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha\;\delta\left(1-\alpha\right)\big(\alpha m^2 - i\eta\big)^{\frac{d-2}{2}} \\
&= \frac{-i\Gamma\left(\frac{2-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\big(m^2 - i\eta\big)^{\frac{d-2}{2}}
\end{aligned}
\tag{1.28}
$$

where m is the mass of the particle propagating in the loop. Please notice that

$$
m = 0 \qquad \Longrightarrow \qquad A_0 = 0 \tag{1.29}
$$

since if the mass is zero, there are not dimensional variables that carry the dimension of A_0 after the integration over the loop momentum. So the integral must be zero.

If $m \neq 0$, with the usual definition $d = 4 - 2\epsilon$, we have

$$
A_0\big(m^2\big) = \frac{-i\Gamma\left(\epsilon - 1\right)}{(4\pi)^{2-\epsilon}}\big(m^2 - i\eta\big)^{1-\epsilon} = \frac{i}{(4\pi)^2}\,\frac{(4\pi)^{\epsilon}\,\Gamma\left(1+\epsilon\right)}{\epsilon\left(1-\epsilon\right)}\,\big(m^2 - i\eta\big)^{1-\epsilon}\,, \tag{1.30}
$$

that shows that A_0 diverges as $1/\epsilon$ when $\epsilon \to 0$.

1.2.2 The two-point function (bubble) with $m_1 = m_2 = 0$

We now consider the integral corresponding to the two-point function with massless propagators, i.e. $m_1 = m_2 = 0$. The external momentum p must then have $p^2 \neq 0$ otherwise, as for A_0 with $m^2 = 0$, if also the external particles are massless, the integral vanishes. The integral is given by

$$
\begin{aligned}
B_0\big(p^2\big) &= \int \frac{d^d\ell}{(2\pi)^d}\,\frac{1}{\left[\ell^2 + i\eta\right]\left[(\ell+p)^2 + i\eta\right]} = \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 \left[d\alpha\right]_2 \frac{1}{\left(-\alpha_1\alpha_2\,p^2 - i\eta\right)^{\frac{4-d}{2}}} \\
&= \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha_1\,\Big(\alpha_1\left(1-\alpha_1\right)\left(-p^2 - i\eta\right)\Big)^{\frac{d-4}{2}} = \frac{i\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\left(-p^2 - i\eta\right)^{\frac{d-4}{2}}\frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma\left(d-2\right)}
\end{aligned}
\tag{1.31}
$$

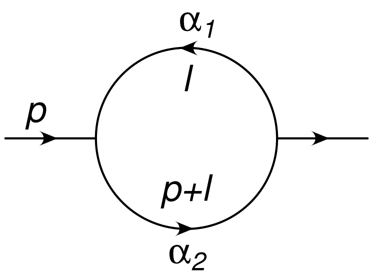


Figure 1.3: Two-point function (bubble).

With $d = 4 - 2\epsilon$ we have

$$B_0\big(p^2\big) \;=\; \frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\left(-p^2-i\eta\right)^{-\epsilon} = \frac{i}{(4\pi)^2}\frac{C_\Gamma}{\epsilon\,(1-2\epsilon)}\left(-p^2-i\eta\right)^{-\epsilon} \tag{1.32}$$

where we have defined

$$C_\Gamma = (4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{1.33}$$

Since we are interested in an expansion in ϵ of B_0, we have to deal with

$$\left(-p^2-i\eta\right)^{-\epsilon} = 1 - \epsilon\log\left(-p^2-i\eta\right) + \mathcal{O}\left(\epsilon^2\right) \tag{1.34}$$

If $p^2 < 0$, then the logarithm is perfectly defined and no imaginary part is needed to give meaning to it. If instead $p^2 > 0$, then $-p^2 - i\eta$ is a complex negative number with a small imaginary part, so that it is below the typical cut for the definition of the logarithm. In this case, we have

$$\left(-p^2-i\eta\right)^{-\epsilon} = 1 - \epsilon\log\left(-p^2-i\eta\right) + \mathcal{O}\left(\epsilon^2\right) = 1 - \epsilon\left[\log\left(p^2\right) - i\pi\right] + \mathcal{O}\left(\epsilon^2\right) \tag{1.35}$$

In the kinematic region $p^2 > 0$ we then have

$$B_0\big(p^2\big) = \frac{i}{(4\pi)^2}\frac{C_\Gamma}{(1-2\epsilon)}\left[\frac{1}{\epsilon} - \log(p^2) + i\pi + \mathcal{O}(\epsilon)\right] \tag{1.36}$$

This integral is divergent as $1/\epsilon$ in the limit $\epsilon \to 0$.

1.2.3 The two-point function (bubble) with $m_1 = m$, $m_2 = 0$

Left as exercise.

1.2.4 The two-point function (bubble) with $m_1 = m_2 = m$

Left as exercise.

Check that you get

$$B_0\left(p^2, m^2, m^2\right) \equiv \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2} \frac{1}{(l+p)^2 - m^2} \tag{1.37}$$

in the kinematic region $p^2 \geq 4m^2$

$$B_0\left(p^2, m^2, m^2\right) = \frac{i}{(4\pi)^2} C_\Gamma \left(m^2\right)^{-\epsilon} \left\{\frac{1}{\epsilon} + 2 + (x_+ - x_-) \log\frac{x_-}{x_+} + i\,\pi\,(x_+ - x_-) + \mathcal{O}(\epsilon)\right\} \tag{1.38}$$

where

$$x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4m^2}{p^2}}\right) \pm i\eta\,. \tag{1.39}$$

1.2.5 The three-point function (triangle) with $m_1 = m_2 = m_3 = 0$

We consider the simplified case where three-point function has all the propagators massless.

Triangle with one external massive leg

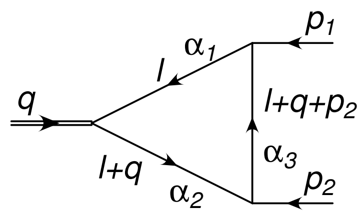


Figure 1.4: The three-point function (triangle). The double line denote the massive leg.

In the notation of Fig. 1.4, we have $q^2 \neq 0$, $p_1^2 = p_2^2 = 0$. We have only one independent invariant, i.e. q^2. Any other relativistic invariant can be written in terms of q^2. The Feynman diagram corresponding to Fig. 1.4 is given by

$$\begin{aligned} C_0\left(q^2\right) &= \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell)^2 + i\eta\right]\left[(\ell+q)^2 + i\eta\right]\left[(\ell+q+p_2)^2 + i\eta\right]} = \\ &= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \frac{[d\alpha]_3}{\left(-\alpha_1\alpha_2\, q^2 - i\eta\right)^{\frac{6-d}{2}}}\,, \end{aligned} \tag{1.40}$$

where we used $p_i^2 = 0$ for $i = 1, 2$. We can integrate over α_3 immediately, using the δ function. This gives $\alpha_3 = 1 - \alpha_1 - \alpha_2$. Since the range of integration of α_3 is from 0 to 1, this means that $0 \leq 1 - \alpha_1 - \alpha_2 \leq 1$, that implies that $\alpha_2 \leq 1 - \alpha_1$. Performing now the integration on the Feynman parameters, we have

$$
\begin{aligned}
C_0\left(q^2\right) &= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2\,\frac{1}{\left(-\alpha_1\alpha_2\,q^2-i\eta\right)^{\frac{6-d}{2}}}\\
&= \frac{-i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}}\left(-q^2-i\eta\right)^{-(1+\epsilon)}\int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2\,\left(\alpha_1\alpha_2\right)^{-(1+\epsilon)}\\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon}\left(-q^2-i\eta\right)^{-(1+\epsilon)}\int_0^1 d\alpha_1\left(\alpha_1\right)^{-(1+\epsilon)}\left[\alpha_2^{-\epsilon}\right]_0^{1-\alpha_1}=\\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon}\left(-q^2-i\eta\right)^{-(1+\epsilon)}\int_0^1 d\alpha_1\left(\alpha_1\right)^{-(1+\epsilon)}\left(1-\alpha_1\right)^{-\epsilon}=\\
&= \frac{i\Gamma\left(1+\epsilon\right)}{(4\pi)^{2-\epsilon}\,\epsilon}\left(-q^2-i\eta\right)^{-(1+\epsilon)}\frac{\Gamma\left(-\epsilon\right)\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)} \qquad (1.41)
\end{aligned}
$$

While this integral can be performed as done before, it is important to keep in mind also the following trick to restore the integration boundaries between 0 and 1. We make the change of variable $\alpha_2=(1-\alpha_1)x$

$$
\begin{aligned}
\int_0^1 d\alpha_1\,\alpha_1^{-(1+\epsilon)}\int_0^{1-\alpha_1} d\alpha_2\,\alpha_2^{-(1+\epsilon)} &= \int_0^1 d\alpha_1\,\alpha_1^{-(1+\epsilon)}\int_0^1 dx\,(1-\alpha_1)(1-\alpha_1)^{-(1+\epsilon)}x^{-(1+\epsilon)}\\
&= \int_0^1 d\alpha_1\,\alpha_1^{-(1+\epsilon)}\,(1-\alpha_1)^{-\epsilon}\int_0^1 dx\,x^{-(1+\epsilon)}\\
&= B(-\epsilon,1-\epsilon)B(-\epsilon,1)=\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(-\epsilon)\Gamma(1)}{\Gamma(1-\epsilon)}\\
&= \frac{1}{\epsilon^2}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \qquad (1.42)
\end{aligned}
$$

where we have used the definition of the B function in eq. (A.9). We finally get

$$
C_0\left(q^2\right)=\frac{i}{(4\pi)^2}\frac{C_\Gamma}{q^2}\left(-q^2-i\eta\right)^{-\epsilon}\frac{1}{\epsilon^2} \qquad (1.43)
$$

where C_Γ is given in equation (1.33). We refer to Sec. 1.2.2 for the expansion of the previous expression in the kinematic regions where $q^2<0$ or $q^2>0$.

Triangle with two external massive legs

Consider now the triangle with two massive external legs. In the notation of Fig. 1.5, we have $p^2=0$, $p_1^2\neq 0$ and $p_2^2\neq 0$. We compute this integral with the further hypothesis that $q_2^2>0$. The sign of q_1^2 is arbitrary. The integral corresponding to this Feynman graph is

$$
\begin{aligned}
C_0\left(q_1^2,q_2^2\right) &= \int\frac{d^d\ell}{(2\pi)^d}\,\frac{1}{\left[(\ell)^2+i\eta\right]\left[(\ell+p)^2+i\eta\right]\left[(\ell+p+q_2)^2+i\eta\right]}=\\
&= \frac{-i\Gamma\left(\frac{6-d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1\frac{[d\alpha]_3}{\left(-\alpha_1\alpha_3q_1^2-\alpha_2\alpha_3q_2^2-i\eta\right)^{\frac{6-d}{2}}}\,. \qquad (1.44)
\end{aligned}
$$

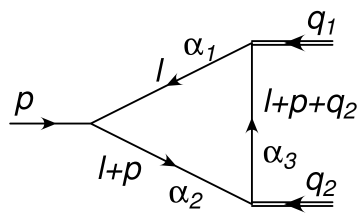


Figure 1.5: The three-point function (triangle). The double line denote the massive leg.

Factorizing out $-q_2^2$ with the right $i\eta$ prescription and using $d = 4 - 2\epsilon$, and defining

$$r = \frac{q_1^2}{q_2^2} + i\eta \tag{1.45}$$

we have

$$\begin{aligned} C_0\left(q_1^2, q_2^2\right) &= \frac{-i\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}} \frac{1}{\left(-1-i\eta\right)^{1+\epsilon}\left(q_2^2\right)^{1+\epsilon}} \int_0^1 \frac{\left[d\alpha\right]_3}{\alpha_3^{1+\epsilon}\left(\alpha_1\, r + \alpha_2\right)^{1+\epsilon}} = \\ &= \frac{-i\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}} \frac{e^{i\pi\epsilon}}{\left(q_2^2\right)^{1+\epsilon}} \int_0^1 d\alpha_3 \int_0^{1-\alpha_3} d\alpha_1 \frac{1}{\alpha_3^{1+\epsilon}\left[\alpha_1\left(r-1\right)+1-\alpha_3\right]^{1+\epsilon}} \end{aligned}$$

Integrating first over α_1 we have

$$\begin{aligned} C_0\left(q_1^2, q_2^2\right) &= \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}\left(q_2^2\right)^{1+\epsilon}} \frac{1}{\epsilon\left(r-1\right)} \int_0^1 \frac{d\alpha_3}{\alpha_3^{1+\epsilon}} \left|\left[\alpha_1\left(r-1\right)+1-\alpha_3\right]^{-\epsilon}\right|_0^{1-\alpha_3} \\ &= \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^{2-\epsilon}\epsilon\left(q_2^2\right)^{1+\epsilon}\left(r-1\right)} \frac{\left(1-r^\epsilon\right)}{r^\epsilon} \int_0^1 d\alpha_3\, \alpha_3^{-(1+\epsilon)}\left(1-\alpha_3\right)^{-\epsilon} \\ &= \frac{-ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^2} \frac{C_\Gamma}{\epsilon^2} \frac{1}{\left(q_2^2\right)^{1+\epsilon}} \frac{\left(1-r^\epsilon\right)}{\left(r-1\right)r^\epsilon} \end{aligned} \tag{1.46}$$

By making a (partial) Laurent expansion in ϵ we have

$$C_0\left(q_1^2, q_2^2\right) = \frac{ie^{i\pi\epsilon}\Gamma\left(1+\epsilon\right)}{\left(4\pi\right)^2} \frac{C_\Gamma}{\epsilon} \frac{\log\left(\dfrac{q_1^2}{q_2^2}+i\eta\right)}{\left(q_1^2-q_2^2\right)\left(q_1^2+i\eta\right)^\epsilon} \tag{1.47}$$

1.2.6 The four-point function (box) with $m_i = 0$

Box with

$$p_1 + p_2 = p_3 + p_4, \qquad p_i^2 = 0, \qquad s = \left(p_1+p_2\right)^2 > 0, \qquad t = \left(p_1-p_3\right)^2 < 0 \tag{1.48}$$

Left as exercise.

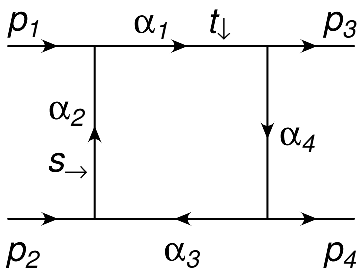


Figure 1.6: Four-point function.

1.3 The tensor one-loop integrals and the Passarino–Veltman reduction formula

We are ready to look at more complicated numerator structures. As previously stated, in QCD (and more in general in the Standard Model), this happens when we have one (or more) fermion legs in the loop or in the presence of triple and quartic gluon vertexes. In the following, we will deal only with massless propagators, to simplify the calculations and the notation. No conceptual problems arise in case of massive propagators.

For example, a massless fermionic n-point loop function is given by

$$\begin{aligned} I_n(\{p_i\}) &= \int \frac{d^d\ell}{(2\pi)^d} \not{\ell}(\not{\ell}+\not{p}_1)(\not{\ell}+\not{p}_1+\not{p}_2)\ldots(\not{\ell}+\not{p}_1+\ldots+\not{p}_{n-1}) \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_1+p_2)^2+i\eta]\ldots[(\ell+p_1+\ldots+p_{n-1})^2+i\eta]} \end{aligned}$$

The gamma matrix structure can be extracted from this integral and we can write $I_n(\{p_i\})$ as

$$\begin{aligned} I_n(\{p_i\}) &= \gamma_{\mu_n}\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_{n-1}} \int \frac{d^d\ell}{(2\pi)^d}\, \ell^{\mu_n}(\ell+p_1)^{\mu_1}(\ell+p_{12})^{\mu_2}\ldots(\ell+p_{1\ldots n-1})^{\mu_{n-1}} \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_{12})^2+i\eta]\ldots[(\ell+p_{1\ldots n-1})^2+i\eta]} \end{aligned}$$

The Feynman integral with tensor components of the loop momentum in the numerator is called **tensor integral**.

$$\begin{aligned} I_n^{\mu_1\mu_2\ldots\mu_k}(\{p_i\}) &\equiv \int \frac{d^d\ell}{(2\pi)^d}\, \ell^{\mu_1}\ell^{\mu_2}\ldots\ell^{\mu_k} \\ &\times \frac{1}{[\ell^2+i\eta]\,[(\ell+p_1)^2+i\eta]\,[(\ell+p_{12})^2+i\eta]\ldots[(\ell+p_{1\ldots n-1})^2+i\eta]} \end{aligned} \tag{1.49}$$

The purpose of this section is to show how to compute this integral. We notice first that all the Lorentz structure of a tensor integral has to be carried by the external momenta $\{p_i\}$ or by the $g^{\mu\nu}$ tensor. The first step is the to write the more general linear combination of tensors of order k constructed with the components of the n external momenta and of the $g^{\mu\nu}$ tensor. The symmetry under permutation of Lorentz indices reduces the allowed tensor structure. In fact, $I_n^{\mu_1\mu_2\cdots\mu_k}(\{p_i\})$ must be totally symmetric with respect to the k indices $(\mu_1, \mu_2, ..., \mu_k)$.

The procedure to compute the tensor integrals has been outlined for the first time by Passarino and Veltman (PV).

We illustrate this procedure with a few examples.

1.3.1 The tensor two-point function $\mathcal{B}^\mu$ ($p^2 \neq 0$)

We start computing $\mathcal{B}^\mu(p)$. Here the tensor decomposition is trivial because only p can bring the index μ of the integral. In order for the integral to be different from zero, we must have $p^2 \neq 0$. We have to compute

$$\mathcal{B}^\mu(p) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\ell^2(\ell+p)^2} = B_{11}\, p^\mu \,. \tag{1.50}$$

In order to compute the coefficient B_{11}, we contract both side of the previous equation with p_μ and use

$$\ell \cdot p = \frac{1}{2}[(\ell+p)^2 - \ell^2 - p^2] \,. \tag{1.51}$$

We have

$$\begin{aligned} p^2\, B_{11} &= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\left[\frac{1}{\ell^2} - \frac{1}{(\ell+p)^2} - \frac{p^2}{\ell^2(\ell+p)^2}\right] \\ &= -\frac{p^2}{2} B_0(p^2) \end{aligned}$$

from which

$$B_{11} = -\frac{1}{2} B_0(p^2) \tag{1.52}$$

This very easy example illustrates the whole strategy of the PV reduction: the first thing to do is to write down the most general linear combination of tensors using the xternal momenta and the metric tensor. Then one has to contract with some tensor structure both sides of this decomposition and, by making use of identities like (1.51), simplify at least one propagator in the denominator. In this way one transforms a tensor integral into a scalar integral or a tensor integral of type I_n to a tensor integral of type I_{n-1}, as we will see in the following. By using different tensor structures to make the contraction, one obtains a set of linear equations[2] to be resolved with respect to the unknown factors B_{ij}, C_{ij},...

[2]The contraction with different elements of a tensor basis ensures to have a set of independent linear equations.

1.3.2 The tensor two-point function $\mathcal{B}^{\mu\nu}$ $(p^2 \neq 0)$

$$\mathcal{B}^{\mu\nu}(p) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+p)^2} = B_{21}\, p^\mu p^\nu + B_{22}\, g^{\mu\nu} \tag{1.53}$$

We can demonstrate that

$$\begin{aligned} B_{21} &= \frac{d}{d-1}\frac{B_0(p^2)}{4}\,, \\ B_{22} &= -\frac{p^2}{d-1}\frac{B_0(p^2)}{4}\,. \end{aligned} \tag{1.54}$$

In fact, contracting eq. (1.53) with $g_{\mu\nu}$ and p_μ, we obtain

$$\begin{aligned} p^2\, B_{21} + d\, B_{22} &= \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{\ell^2(\ell+p)^2} = 0 \\ p^\nu \left(p^2\, B_{21} + B_{22}\right) &= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\nu}{\ell^2(\ell+p)^2}\left[(\ell+p)^2 - p^2 - \ell^2\right] \\ &= -p^\nu \frac{p^2 B_{11}}{2} \end{aligned}$$

The linear system to solve is then

$$\begin{aligned} &B_{21}\, p^2 + B_{22}\, d = 0 \\ &B_{21}\, p^2 + B_{22} = -\frac{p^2}{2} B_{11} \end{aligned}$$

that gives (1.54).

1.3.3 The tensor three-point function $\mathcal{C}^\mu(p_1, p_2)$ $(p_1^2 = p_2^2 = 0, (p_1+p_2)^2 \equiv p_3^2 \neq 0)$

The integral is defined by

$$\mathcal{C}^\mu(p_1, p_2) \equiv \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2} = C_{11}\, p_1^\mu + C_{12}\, p_2^\mu \tag{1.55}$$

and we can easily show that

$$\begin{aligned} C_{11} &= -\frac{B_0((p_1+p_2)^2)}{2p_1 \cdot p_2} - C_0(p_1, p_2) \\ C_{12} &= \frac{1}{2p_1 \cdot p_2} B_0\big((p_1+p_2)^2\big)\ . \end{aligned} \tag{1.56}$$

In fact we can contract eq. (1.55) with p_1^μ and with $(p_1+p_2)^\mu$. We obtain

$$
\begin{aligned}
p_1\cdot p_2\, C_{12} &= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2}\left[(\ell+p_1)^2-\ell^2\right] \\
&= \frac{1}{2}B_0\big((p_1+p_2)^2\big) \\
p_1\cdot p_2\;(C_{11}+C_{12}) &= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{(\ell+p_1+p_2)^2-\ell^2-(p_1+p_2)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2} \\
&= -\frac{(p_1+p_2)^2}{2}C_0(p_1,p_2)
\end{aligned}
$$

and the solution to these two equations is (1.56).

1.3.4 The tensor three-point function $\mathcal{C}^{\mu\nu}(p_1,p_2)$ $(p_1^2=p_2^2=0,(p_1+p_2)^2\equiv p_3^2\neq 0)$

$$
\mathcal{C}^{\mu\nu}(p_1,p_2)\equiv\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2}=C_{21}\,p_1^\mu p_1^\nu+C_{22}\,p_2^\mu p_2^\mu+C_{23}\,p_1^{\{\mu}p_2^{\nu\}}+C_{24}\,g^{\mu\nu} \tag{1.57}
$$

where

$$
p_1^{\{\mu}p_2^{\nu\}}\equiv p_1^\mu p_2^\nu+p_1^\nu p_2^\mu\,.
$$

In fact, $\mathcal{C}^{\mu\nu}$ is symmetric in the exchange $\mu\leftrightarrow\nu$, so that the right-hand side of eq. (1.57) must be symmetric. We get

$$
\begin{aligned}
C_{21} &= C_0(p_1,p_2)+3\frac{B_0((p_1+p_2)^2)}{4(p_1p_2)} \\
C_{22} &= -\frac{B_0((p_1+p_2)^2)}{4(p_1p_2)} \\
C_{23} &= -\frac{d}{d-2}\frac{B_0((p_1+p_2)^2)}{4(p_1p_2)} \\
C_{24} &= \frac{B_0((p_1+p_2)^2)}{2(d-2)}
\end{aligned} \tag{1.58}
$$

Notice that, if one is interested in the behavior of this integral for large integration momenta, i.e. in the UV limit, all the external momenta p_i can be neglected and the only coefficient that survive is C_{24}.

We ca contract both sides of eq. (1.57) with p_1^ν, $(p_1+p_2)^\nu$, $p_1^\mu(p_1+p_2)^\nu$ and $g^{\mu\nu}$. Then by using the identity (for arbitrary k)

$$
\ell\cdot k=\frac{(\ell+k)^2-\ell^2-k^2}{2}
$$

and making use of the mass-shell conditions for p_1 and p_2, we obtain respectively

$$\begin{aligned}
p_1^\mu \left[(p_1p_2)\, C_{23} + C_{24}\right] + p_2^\mu (p_1p_2)\, C_{22} &= \frac{\mathcal{B}^\mu(p_1+p_2)}{2} \\
p_1^\mu \left[(p_1p_2)\,(C_{21}+C_{23}) + C_{24}\right] + p_2^\mu \left[(p_1p_2)\,(C_{22}+C_{23}) + C_{24}\right] &= -(p_1p_2)\, \mathcal{C}^\mu(p_1,p_2) \\
(p_1p_2)\left[C_{22}+C_{23}\right] + C_{24} &= -\frac{B_0(p_1+p_2)}{2} \\
2(p_1p_2)\, C_{23} + d\, C_{24} &= 0
\end{aligned}$$

With some trivial algebra, we can show that the solution of this system is given by eq. (1.58).

Chapter 2

Gauge invariance

2.1 Classical electrodynamics

We start by considering classical electrodynamics. All the information we need to write the equation of motion are inside the field strength tensor $F_{\mu\nu}$, defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

where A_μ is the gauge field.

There is a 1-1 correspondence between physical measurable objects and $F_{\mu\nu}$ ($F^{0i} = -E^i$,$F^{ij} = -\varepsilon^{ijk}B^k$). We prefer to use $F_{\mu\nu}$ because it make the theory manifestly Lorentz covariant. For example, in this formulation Maxwell equations take the form

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= -J^\beta \\ \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= 0 \end{aligned} \tag{2.2}$$

We note that the field A_μ is not uniquely determined by these equations. In fact the physical object ($F_{\mu\nu}$) and Maxwell equations are invariant under the transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\phi \tag{2.3}$$

with ϕ an arbitrary scalar function. This transformation is called a *gauge* transformation.

Using classical electrodynamics as an example, we fix the gauge freedom (2.3). A good choice is the Lorentz condition[1]

$$\partial_\mu A^\mu = 0 \tag{2.4}$$

[1]Later we will see that this is not the only possibility allowed. Anyway this gauge is very useful because it leaves the theory explicitly covariant. For this reasons these types of gauge fixings are also called *covariant gauges*.

The physics does not change with this choice while Maxwell equations become simpler as can be easily seen:

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = \partial_\alpha \partial^\alpha A^\beta = -J^\beta \tag{2.5}$$

Indeed, the term $\partial_\alpha \partial_\beta A_\alpha = \partial_\beta \partial_\alpha A_\alpha$ is zero in this gauge frame.

Despite this choice, the gauge has not been entirely fixed. In fact we still have a residual gauge freedom: if we consider as gauge parameter in (2.3) a function ϕ such that $\partial_\mu \partial^\mu \phi = 0$, we see that A_μ still satisfies Lorentz gauge condition (2.4), so there is again a redundancy in the A_μ definition. Potentials for which $\partial_\mu \partial^\mu \phi = 0$ are said to satisfy the Lorentz gauge.

The problem in the discussion above stays in the link between gauge symmetry and the evaluation of the physical degrees of freedom of the field A_μ. We will now see this in details, counting correctly the physical component of the gauge field. We will find that the residual gauge freedom plays an important role in this task.

In the vacuum $J_\beta = 0$, so in the the Lorentz gauge the equation of motion for the field A_μ (see (2.5)) becomes

$$\partial_\mu \partial^\mu A^\alpha = 0 \tag{2.6}$$

which has the solution

$$A_\alpha = \varepsilon_\alpha e^{ikx} + \varepsilon^*_\alpha e^{-ikx} \tag{2.7}$$

if and only if $k^\mu k_\mu = k^2 = 0$. The next step is to determine ε_α, which is up to now a quadri–vector whose components are generic functions of the momentum k_μ. First of all one has to impose the gauge fixing condition $\partial^\alpha A_\alpha = 0$. In momentum space this condition constraints ε to be Lorentz perpendicular to k:

$$k^\alpha \varepsilon_\alpha = 0 \tag{2.8}$$

This means that now we have not four but three independent components of ε_α to determine. The point is now that we know from experimental evidence (or, using a theoretical approach, from Lorentz group representation theory) that photons have only two degrees of freedom. So we have not yet fixed the A_α fields to be completely physical, but we have again the residual gauge invariance that can help us.

If we use the residual gauge redundancy related to the gauge freedom, we can still make the transformation $A_\mu \to A_\mu + \partial_\mu \phi$ with $\partial_\mu \partial^\mu \phi = 0$. For example we can choose

$$\phi = iae^{ikx} + h.c. \quad , \quad k^2 = 0 \tag{2.9}$$

Using this ϕ in the gauge transformation the potential A_μ takes the form

$$A_\mu \to A'_\mu = \left[(\varepsilon_\mu - ak_\mu)e^{ikx} + h.c.\right] \equiv \left[\varepsilon'_\mu e^{ikx} + h.c.\right] \tag{2.10}$$

where a is completely arbitrary. This last fact can be used to obtain a zero value for one component of the ε'_μ and therefore to remain with only two free components as the physics requires. In fact, choosing one component labeled by μ_0, we can fix $\varepsilon_{\mu_0} = 0$ putting $a = \frac{\varepsilon_{\mu_0}}{k_{\mu_0}}$.

At this point of the discussion is convenient to give an explicit example. We choose

$$k_\mu = k_0(1,0,0,1) \quad \varepsilon_\alpha = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{2.11}$$

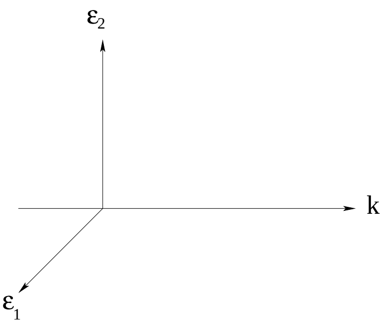
This choice is consistent with the constraint $k^2 = 0$, but we still have to impose $k \cdot \varepsilon = 0$.

$$k^\alpha \; \varepsilon_\alpha = k_0 \; \varepsilon_0 - k_0 \; \varepsilon_3 = 0 \Rightarrow \varepsilon_0 = \varepsilon_3 \tag{2.12}$$

This means that $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_0)$. Then we use the arbitrariness of the parameter a and the residual gauge freedom to cancel, for example, the first and the last component of ε by virtue of the following transformation:

$$\varepsilon_3' = \varepsilon_3 - ak_0 = 0 \Rightarrow a = \frac{\varepsilon_3}{k_0} = \frac{\varepsilon_0}{k_0} \tag{2.13}$$

In this way it is simple to understand that only two of the four component of the vector ε_α are physical, since we obtained $\varepsilon_\alpha = (0, \varepsilon_1, \varepsilon_2, 0)$. Now ε_1 and ε_2 are completely arbitrary and they can't be removed anymore because there is no more residual gauge freedom. With this result for ε_α we can choose a basis made by the relevant physical components orthogonal to k. For example we can choose $\left(\varepsilon_{(1)}\right)_\alpha = (0,1,0,0)$ and $\left(\varepsilon_{(2)}\right)_\alpha = (0,0,1,0)$, obtaining a picture like



What is the physical meaning of this construction? Let's try to see what happens if we make a rotation. One can image that rotation of a θ angle along the direction of the z axis give us informations about the angular momentum along that axis. The associated matrix is

$$R_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

We observe that k does not change after a rotation of a θ angle: $R_{\mu\nu}k^\mu = k_\nu$. The same does not happens for ε_α, since we obtain $\varepsilon_\alpha^{'} = R_{\alpha\beta}\; \varepsilon^\beta$. If we now want to make the physical properties of this system more evident we can change the basis. Calling $\varepsilon^\alpha_{(1)}$ and $\varepsilon^\alpha_{(2)}$ the old basis

$$\begin{aligned} \varepsilon^\alpha_{(1)} &= (0, \varepsilon_1, 0, 0) = (0,1,0,0) \\ \varepsilon^\alpha_{(2)} &= (0,0,\varepsilon_2,0) = (0,0,1,0) \end{aligned} \tag{2.15}$$

we define the new basis as

$$\begin{aligned}\varepsilon_{+}^{\alpha} &= \varepsilon_{(1)}^{\alpha} - i\varepsilon_{(2)}^{\alpha} \\ \varepsilon_{-}^{\alpha} &= \varepsilon_{(1)}^{\alpha} + i\varepsilon_{(2)}^{\alpha}\end{aligned} \tag{2.16}$$

This new basis is more helpful to understand the physics of this system being the basis of eigenvectors of the rotation matrix $R_{\alpha\beta}$. In fact, after the rotation we have

$$\begin{array}{l}\varepsilon_{+}^{'\,\alpha} = (0, \cos\theta - i\sin\theta, -\sin\theta - i\cos\theta, 0) = e^{-i\theta}\varepsilon_{+}^{\alpha} \\ \varepsilon_{-}^{'\,\alpha} = (0, \cos\theta + i\sin\theta, -\sin\theta + i\cos\theta, 0) = e^{+i\theta}\varepsilon_{-}^{\alpha}\end{array} \tag{2.17}$$

So we have a plane wave that describes A_α, which under rotation transforms such that its polarization vector $\varepsilon_\alpha \to \varepsilon_\alpha^{'} = e^{ih\theta}\varepsilon_\alpha$. The h defines a new property of the field A_α, called helicity. In addition, for the helicity we obtained two possible values (± 1): they correspond to the two degrees of freedom of the classical electromagnetic field and in the quantized theory they will be interpreted as the two polarization states of the carrier of the interaction, the photon.

2.2 QED

2.2.1 Sum over polarizations

In the previous section we saw that a photon has two helicities, which means that we have two physical polarizations. We can ask what are the consequences of this fact when we calculate squared amplitudes in quantum electrodynamics.

We will skip all the difficulties of the "second quantization" of the fermionic field ψ and of the field A_μ: we directly suppose to know what are the Feynman rules, i.e. the way to calculate transition amplitude in QED.

Given an amplitude $M = M_\mu \varepsilon^\mu$, if we want to calculate the squared modulus we have

$$|M|^2 = \sum_{pol} M_\mu M_\nu^* \varepsilon^\mu \varepsilon^{*\nu} \tag{2.18}$$

where the sum is only on physical polarizations. Now the question is: what tensor do we have to use when we meet $\sum_{pol} \varepsilon_\mu \varepsilon_\nu^*$?

Since we have two different polarization vectors, the piece written as $\sum_{pol} \varepsilon^\mu \varepsilon^{*\nu}$ is a sum over ε_1^μ and ε_2^μ. For simplicity, choosing the photon momenta k along the z direction, the sum gives

$$\sum_{pol} \varepsilon_\mu \varepsilon_\nu^* = \varepsilon_\mu^1 \varepsilon_\nu^{*1} + \varepsilon_\mu^2 \varepsilon_\nu^{*2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.19}$$

This matrix represents the sum over the physical polarizations. A question immediately arises: why one usually use $-g^{\mu\nu}$ instead of (2.19)? In order to answer this question we will point our attention on the different possibilities we have to fix the gauge.

Up to now we used the Lorentz gauge $\partial_\mu A^\mu = 0$. In this case the propagator has a simple form. On the other hand we are propagating non physical degrees of freedom. A different choice is given by the axial gauge. Axial gauges are defined fixing an axis such that $\eta_\mu A^\mu = 0$ and we will see in section 2.3.3 how to modify the Lagrangian to make this. Different choices of η gives us different gauges (for example we have temporal gauge if $A_0 = 0$, or radiation gauge if $\vec{\nabla} \cdot \vec{A} = 0$).

For example, fixing the z axis along the propagation of the plane wave, we have $k = k_0(1,0,0,1)$ and we can choose $\eta = (1,0,0,-1)$ so that $\eta^2 = 0$ but $\eta \cdot k \neq 0$. The condition $\eta_\mu A^\mu = 0$ then becomes $\eta \cdot \epsilon = 0$, that together with $\partial_\mu A^\mu = 0 \Longrightarrow k \cdot \epsilon = 0$ imply

$$\eta \cdot \epsilon = 0 \Longrightarrow \epsilon_0 + \epsilon_3 = 0 \tag{2.20}$$
$$k \cdot \epsilon = 0 \Longrightarrow \epsilon_0 - \epsilon_3 = 0 \tag{2.21}$$
$$\tag{2.22}$$

so that we obtain $\epsilon_0 = \epsilon_3 = 0$, and we propagate only the two degrees of freedom.

We want now to analyze the sum over the polarizations in the axial gauge. We start writing the more general formula for the sum over polarizations:

$$\Sigma^{\mu\nu} \equiv \sum_{\varepsilon=\varepsilon_1,\varepsilon_2} \varepsilon^\mu \varepsilon^{*\nu} = A g^{\mu\nu} + B k^\mu k^\nu + C k^\mu \eta^\nu + D \eta^\mu k^\nu + E \eta^\mu \eta^\nu \tag{2.23}$$

In this gauge $\eta_\mu A^\mu$ is fixed to be zero and since $A_\mu = \varepsilon_\mu e^{ikx} + h.c.$, we have $\eta_\mu \varepsilon^\mu = 0$. Also we have again $k_\mu \varepsilon^\mu = 0$ because physical photons are transverse. We have now to calculate the coefficients A, B, C, D, E. This can be done contracting $\Sigma^{\mu\nu}$ with k_μ, k_ν, η_μ and η_ν. We have

$$k_\mu \Sigma^{\mu\nu} = 0 \qquad\qquad k_\nu \Sigma^{\mu\nu} = 0 \tag{2.24}$$

When we expand the first equation in (2.24) we observe that the terms which multiply B and C are zero since $k^2 = 0$. For the second equation *ibidem* the same idea applies to the terms with B and D. We have

$$\begin{aligned} A k^\nu + D(\eta \cdot k) k^\nu + E(\eta \cdot k)\eta^\nu = 0 \\ A k^\mu + C(\eta \cdot k) k^\mu + E(\eta \cdot k)\eta^\mu = 0 \end{aligned} \tag{2.25}$$

Since η is still arbitrary we use this freedom to fix $E = 0$. This give us the relation

$$C = D = -\frac{A}{k \cdot \eta} \tag{2.26}$$

Since also $\eta^\mu \varepsilon_\mu = 0$, if we now contract Σ with η we obtain

$$\begin{aligned} A\eta^\nu + B(\eta \cdot k)k^\nu + C(\eta \cdot k)\eta^\nu + D\eta^2 k^\nu &= 0 \\ A\eta^\mu + B(\eta \cdot k)k^\mu + C\eta^2 k^\mu + D(\eta \cdot k)\eta^\mu &= 0 \end{aligned} \tag{2.27}$$

Thanks to (2.26) there are cancellations between A and C in the first equation and between A and D in the second. These cancellations lead to a new relation between the coefficients

$$B = -\frac{D\eta^2}{\eta \cdot k} = \frac{A\eta^2}{(\eta \cdot k)^2} \tag{2.28}$$

The sum over the polarizations $\Sigma^{\mu\nu}$ can be written now as

$$\Sigma^{\mu\nu} = A\left(g^{\mu\nu} + \frac{\eta^2}{k \cdot \eta}k^\mu k^\nu - \frac{1}{k \cdot \eta}\left(k^\mu \eta^\nu + k^\nu \eta^\mu\right)\right) \tag{2.29}$$

We have now to calculate the value of the A coefficient. This is easily done by multiplying with the metric $g^{\mu\nu}$ both sides of the previous relation and using $g_\mu{}^\mu = 4$. We obtain the equation

$$-2 = A(4 + 0 - 2) \Rightarrow A = -1 \tag{2.30}$$

The final form for $\Sigma^{\mu\nu}$ is then

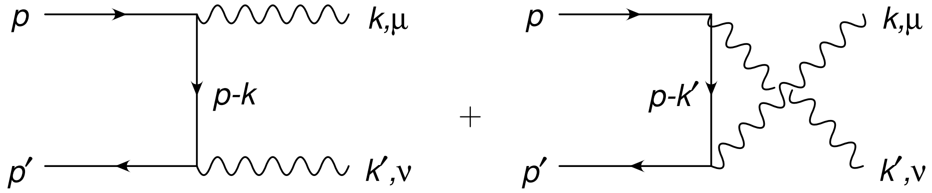
$$\Sigma^{\mu\nu} = -g^{\mu\nu} + \frac{1}{k \cdot \eta}\left(k^\mu \eta^\nu + k^\nu \eta^\mu\right) - \frac{\eta^2}{k \cdot \eta}k^\mu k^\nu \tag{2.31}$$

and this is the most general form for $\Sigma^{\mu\nu}$ in the axial gauge[2].

We are now ready to answer the previous question, which was: why in QED the sum can be taken to be $-g^{\mu\nu}$ and not (2.19)? We need to calculate the amplitude for the process $q\bar{q} \to \gamma\gamma$ in QED.

2.2.2 Gauge invariance in $q(p)\bar{q}(p') \to \gamma(k)\gamma(k')$

The two lowest order graphs contributing to this process



[2]With the choice $k = k_0(1, 0, 0, 1)$ and $\eta = k_0(1, 0, 0, -1)$, we recover exactly the result we have written in (2.19).

give the structure

$$M^{\mu\nu} \;=\; \bar{v}\,(p')\left\{(-ie\gamma^{\nu})\,\frac{i}{\not{p}-\not{k}}\,(-ie\gamma^{\mu})+(-ie\gamma^{\mu})\,\frac{i}{\not{k}-\not{p}'}\,(-ie\gamma^{\nu})\right\}u\,(p) \tag{2.32}$$

and the amplitude is obtained contracting with physical polarizations:

$$M = M^{\mu\nu}\varepsilon_{\mu}(k)\;\varepsilon'_{\nu}(k'). \tag{2.33}$$

A gauge transformation in x-space

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\phi \tag{2.34}$$

become in k-space

$$\epsilon_{\mu} \to \epsilon_{\mu} + \text{const} \times k_{\mu} \tag{2.35}$$

Now, if we want to verify gauge invariance, we can contract $M^{\mu\nu}$ with k_{μ}, since this contraction corresponds to the usual $\partial_{\mu}M^{\mu\nu} = 0$ in coordinate space. In other words we are checking if the Ward identity is satisfied.

$$\begin{aligned} k_{\mu}(M^{\mu\nu}) &= -ie^{2}\;\bar{v}\,(p')\left\{\gamma^{\nu}\frac{\not{p}-\not{k}}{-2\;p\cdot k}\;\not{k}+\not{k}\frac{\not{k}-\not{p}'}{-2\;p'\cdot k}\;\gamma^{\nu}\right\}u\,(p) \\ &= -ie^{2}\;\bar{v}\,(p')\left\{\gamma^{\nu}\frac{\not{p}\;\not{k}}{-2\;p\cdot k}-\frac{\not{k}\;\not{p}'}{-2\;p'\cdot k}\;\gamma^{\nu}\right\}u\,(p) \\ &= -ie^{2}\;\bar{v}\,(p')\left\{-\gamma^{\nu}+\gamma^{\nu}\right\}u\,(p) \\ &= 0. \end{aligned} \tag{2.36}$$

This comes from **Noether theorem** and **conservation of electromagnetic currents**.

In addition, notice that the index ν has not been contracted with the corresponding photon momentum. So, gauge invariance holds independently from what the other photon does. The gauge invariance identity (we have put brackets around the "dummy" index ν since it plays no role)

$$k_{\mu}M^{\mu(\nu)} = 0 \tag{2.37}$$

in a reference frame quer the z axis is align along the direction of motion of the photon implies that

$$k_{0}M^{0(\nu)} - k_{3}M^{3(\nu)} = k_{0}(M^{0(\nu)} - M^{3(\nu)}) = 0 \Longrightarrow M^{0(\nu)} = M^{3(\nu)} \tag{2.38}$$

This then implied that we can use $-g_{\mu\mu'}$ to sum over the photon polarization, only the transverse modes give a contribution.

In this way we have shown that the amplitude is gauge invariant for every emitted photon independently from the other ones. This result is already implicit in the conservation law $\partial_{\mu}J^{\mu} = 0$: the photon emission does not affect the charge of the fermionic current, just because photon does not take away any charge. Thus charge conservation in QED is equivalent to gauge invariance which is equivalent to 2 only physical degrees of freedom.

So we have proved that in QED one can safely use

$$\sum_{pol} \varepsilon^{\mu} \varepsilon^{\nu *} = -g^{\mu\nu} \tag{2.39}$$

and this choice is clearly simpler than (2.31).

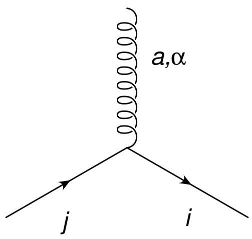
Now we can move to non abelian gauge theories and try to work out all the consequences that the requirement of gauge invariance by itself will force.

2.3 QCD

2.3.1 Gauge invariance in $q\bar{q} \to gg$

The question is now: what happens if a non abelian charge is supported by the vertices of the theory?

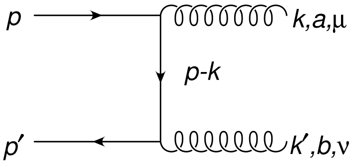
In this case we have to image that the vertex constructed with two fermions and a gluon, the non abelian equivalent of the photon, contains a non abelian current that flows out carrying the non abelian charge. We have to show what happens imposing the gauge invariance in this situation. The usual relation $\partial_\mu J^\mu = 0$ and the conservation of the non abelian charge impose that the new vertex between the gluon and the fermions is



$$= \quad -ig\gamma^{\alpha} t^{a}_{ij} \tag{2.40}$$

where g is the gauge coupling, and $(t^a)_{ij}$ are the generators of the non abelian group .

Gauge invariance in this situation has to be investigated analyzing the process $q\bar{q} \to gg$, the non abelian generalization of $q\bar{q} \to \gamma\gamma$. Following the previous QED calculation, the relevant Feynman graphs for this process are



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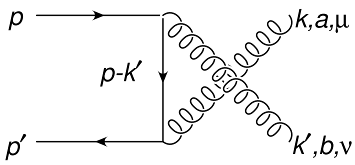


(2.41)

In this case the amplitude is

$$M^{\mu\nu}_{(1)} = \bar{v}(p') \left(\left(-ig\gamma^{\nu} t^{b}\right) \frac{i}{\not{p} - \not{k}} \left(-ig\gamma^{\mu} t^{a}\right) + \left(-ig\gamma^{\mu} t^{a}\right) \frac{i}{\not{p} - \not{k}'} \left(-ig\gamma^{\nu} t^{b}\right) \right) u(p) \tag{2.42}$$

The check for gauge invariance of this process is fulfilled if $k_\mu M^{\mu\nu} = 0$. However if we contract the amplitude with k_μ we do not obtain zero as in (2.36):

$$k_\mu M^{\mu\nu}_{(1)} = -ig^2(t^a t^b - t^b t^a)\, \bar{v}(p')\, \gamma^\nu\, u(p) \tag{2.43}$$

This expression is not zero since $[t^a, t^b] = if^{abc}t^c \neq 0$, i.e. the current does not carry an abelian charge.

If we are asking for gauge invariance we have to conserve the current, which means that we want $k_\mu M^{\mu\nu}_{(1)}$ to be zero. This means that we have to cancel the non zero term given by the commutator of the generators. The way out of this problem can be found adding new interactions and new diagrams. The particles involved in these new interactions can be read from the color structure of the amplitude (2.43) we want to cancel. In particular only a cubic interaction between gluons gives exactly the structure constants f^{abc} that are in (2.43).

With similar considerations on the process $gg \to gg$ we will see later in section 2.3.5 that the theory needs also a quartic boson vertex. But for now let us focus on the amplitude $q\bar{q} \to gg$.

2.3.2 The ggg vertex

Every new interaction term one wants to add to a theory, to be accepted must respect the symmetries of the theory. For example, just looking at the color representation of (2.43) we were able to understand that we were looking for a cubic gluon vertex. In the same way, the Lorentz symmetry strongly constraints the vertex one can write.

Moreover, to build up a renormalizable field theory, there is another constraint from the fact that all the couplings coming from gauge symmetry must be dimensionless.

Last but not least, every interaction term involving identical bosonic or fermionic particles must be completely symmetric or anti-symmetric in the exchange of these particles in order to respect the statistics.

In this section we will point out that the physical constraints here listed are sufficient to completely fix the interaction vertices we are looking for.

We start with the cubic bosonic vertex: in order to have a dimensionless coupling, we must require that the vertex has the dimension of a momentum. Since the only dimensional physical quantities we can use to build up the vertex are the three momenta p_i carried by the bosons, the vertex must be linear in the momenta.

Consider now the Lorentz symmetry: we are writing the vertex for three vector bosons, so it must have three Lorentz vector indexes one of which is carried by the momentum we must use. So we can write this ansatz:

$$V^{\mu_1\mu_2\mu_3} \propto \sum_{i\neq j\neq k} g^{\mu_i\mu_j}\; A_{kl}\; p_l^{\mu_k} + \epsilon^{\mu_1\mu_2\mu_3}{}_{\alpha} \sum_i B_i\; p_i^\alpha \tag{2.44}$$

Here, A_{kl} and B_i are respectively a 3×3 matrix and a 3–vector of coefficients, $g^{\mu\nu}$ is the Minkowskian metric and $\epsilon^{\mu_1\mu_2\mu_3}{}_\alpha$ is the Ricci symbol. The two addenda in the expression above seem proportional since we have forgotten the color dependence, i.e. the structure constants. These are instead very important when we impose on equation (2.44) the Bose statistic: this is the last constraint we must satisfy. Since inside V there is this structure constant f_{abc} which is anti–symmetric, also the right side of (2.44) must be anti–symmetric with respect to the boson indexes i, j, k. This means that the matrix A_{ij} must be anti–symmetric and that $B_i = B$. This last condition, with the conservation of the momentum incoming in the vertex, implies that the term proportional to $\epsilon^{\mu_1\mu_2\mu_3}{}_\alpha$ vanishes.

Bose statistic reduce thus (2.44) to be

$$V^{\mu_1\mu_2\mu_3} = A f^{abc} \sum_{i>j>k} g^{\mu_i\mu_j} \left(p_i - p_j\right)^{\mu_k} \tag{2.45}$$

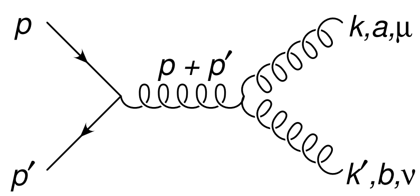
where A is a constant that can be fixed by imposing that the new graph from this vertex for the process $q\bar{q} \to gg$ cancels the problematic (for the gauge symmetry) result (2.43). This calculation will be performed in the next paragraph and will have other problematic aspects that will make manifest the necessity of new quanta in the theory, the so–called ghosts.

However, here we anticipate the right result that will be used in the following: $A = -g$, so that the triple vertex is

$$V^{\mu_1\mu_2\mu_3} = -g f^{abc} \sum_{i>j>k} g^{\mu_i\mu_j} \left(p_i - p_j\right)^{\mu_k} \tag{2.46}$$

2.3.3 More comments on gauge invariance

Now that we have constructed by hand the gluon cubic vertex, we return to the problem of gauge invariance in the process $q(p)\,\bar{q}(p') \to g(k)\,g(k')$. The new graph to be added to (2.41) coming from the triple gluonic vertex is



This graph gives a contribution $M^{\mu\nu}_{(2)}$ to the amplitude, which is

$$\begin{aligned} M^{\mu\nu}_{(2)} &= \bar{v}(p')\left(-ig\gamma^\delta t^d\right)u(p)\frac{i\,\delta^{dc}}{(p+p')^2}\left(-g_{\delta\gamma} + (1-\lambda)\frac{(p+p')_\delta(p+p')_\gamma}{(p+p')^2}\right) \\ &\quad \times g f^{bca}\left[g^{\nu\gamma}\left(-k'-p-p'\right)^\mu + g^{\gamma\mu}\left(p+p'+k\right)^\nu + g^{\mu\nu}\left(k'-k\right)^\gamma\right] \end{aligned} \tag{2.47}$$

If we want now to test the gauge invariance of this process we have to sum $M^{\mu\nu}_{(1)}$ and $M^{\mu\nu}_{(2)}$ and contract the result with $k^\mu \varepsilon'^\nu$:

$$k^\mu \varepsilon'^\nu (M_{\mu\nu}) = \left[k^\mu \varepsilon'^\nu (M^{(1)}_{\mu\nu})\right] + \left[k^\mu \varepsilon'^\nu (M^{(2)}_{\mu\nu})\right] \tag{2.48}$$

In QED we have already showed that this contraction is zero, also if we don't require the second gauge bosons to be physical[3]. Unfortunately here things are more involved.

The first contraction in (2.48) gives

$$\begin{aligned}
k^\mu \varepsilon'^\nu (M^{\mu\nu}_{(1)}) &= k^\mu \varepsilon'^\nu \bar{v}(p') \left(\left(ig\gamma^\nu t^b\right) \frac{i}{\not{p} - \not{k}} \left(ig\gamma^\mu t^a\right) - \left(ig\gamma^\mu t^a\right) \frac{i}{\not{p} - \not{k}'} \left(ig\gamma^\nu t^b\right) \right) u(p) \\
&= -ig^2 \bar{v}(p') \left(t^b t^a \not{\varepsilon}' \frac{\not{p} - \not{k}}{-2p\cdot k} \not{k} + t^a t^b \not{k} \frac{\not{k} - \not{p}'}{-2p'\cdot k} \not{\varepsilon}' \right) u(p) \\
&= -ig^2 \bar{v}(p') \left(t^b t^a \not{\varepsilon}' \frac{-\not{k}\not{p} + 2p\cdot k}{-2p\cdot k} + t^a t^b \frac{\not{p}'\not{k} - 2k\cdot p'}{-2p'\cdot k} \not{\varepsilon}' \right) u(p) \\
&= -ig^2 \, \left[t^a, t^b\right] \, \bar{v}(p') \not{\varepsilon}' u(p)
\end{aligned} \tag{2.49}$$

while the second term gives

$$\begin{aligned}
k^\mu \varepsilon'^\nu (M^{\mu\nu}_{(2)}) &= \bar{v}(p') \left(-ig\gamma^\delta t^d\right) u(p) \frac{i\delta^{cd}}{(p+p')^2} \left(-g^{\delta\gamma} + (1-\lambda) \frac{(p+p')^\delta \, (p+p')^\gamma}{(p+p')^2} \right) \\
&\quad \times \left(-gf^{bca}\right) \Big\{ g^{\nu\gamma} \left(-k' - p - p'\right)^\mu + g^{\gamma\mu} \left(p + p' + k\right)^\nu + g^{\mu\nu} \left(k' - k\right)^\gamma \Big\} k^\mu \varepsilon'^\nu \\
&= -ig^2 \, \bar{v}(p') \gamma^\delta u(p) \, t^d \frac{i\delta^{cd}}{(p+p')^2} \, g^{\delta\gamma} f^{bca} \Big\{ g^{\nu\gamma} \left(-k' - p - p'\right)^\mu \\
&\qquad\qquad + g^{\gamma\mu} \left(p + p' + k\right)^\nu + g^{\mu\nu} \left(k' - k\right)^\gamma \Big\} k^\mu \varepsilon'^\nu \\
&= -\frac{ig^2}{(p+p')^2} \, \left[t^a, t^b\right] \, \bar{v}(p') \Big\{ -2\not{\varepsilon}' \left(k\cdot k'\right) + \not{k} \left(2k + k'\right)\cdot \varepsilon' + \left(\not{k}' - \not{k}\right) k\cdot \varepsilon' \Big\} u(p) \\
&= -\frac{ig^2}{2k\cdot k'} \left[t^a, t^b\right] \bar{v}(p') \Big\{ -2\not{\varepsilon}' (k\cdot k') + \varepsilon'\cdot k' \not{k} + (\not{k} + \not{k}') k\cdot \varepsilon' \Big\} u(p) = \\
&= -ig^2 \left[t^a, t^b\right] \bar{v}(p') \left(\frac{\varepsilon'\cdot k'}{2k\cdot k'} \not{k} - \not{\varepsilon}' \right) u(p)
\end{aligned} \tag{2.50}$$

where computations are done without using $\varepsilon' \cdot k' = 0$. If we now sum the two contributions (2.49) and (2.50) we do not obtain zero:

$$k^\mu \varepsilon'^\nu (M_{\mu\nu}) = -ig^2 \left[t^a, t^b\right] \bar{v}(p') \left(\frac{\varepsilon'\cdot k'}{2k\cdot k'} \not{k} \right) u(p) \tag{2.51}$$

In fact, the second term in (2.50) does cancel the contribution coming from $M_{(1)}$, but the first term remains. We have thus verified by an example that a QCD amplitude is gauge invariant if and only if all the other gluons are physical, i.e. if their polarizations are transverse ($\varepsilon' \cdot k' = 0$). In QED any photon is gauge invariant by itself[4], as eq. (2.36) shows, while here

[3] In fact we did only the contraction with k^μ, leaving the other index free.

[4] This important property of Green functions is known as Ward identity.

the situation is different. This is a crucial difference between an abelian theory (QED) and QCD which is not abelian. Some consideration on the consequence of this fact are necessary.

First, one can ask if this result depends on the gauge fixing or it's a consequence only of the non-abelianity of the theory. The answer is the latter one: in any gauge frame the contraction of an amplitude with more than one external gluon with one polarization vector substituted by the corresponding momentum is zero only if all the other gluons are physical, i.e. if $\varepsilon_\mu^{(i)}$ are such that $\varepsilon^{(i)} \cdot k^{(i)} = 0$, for all i's. For example in the axial gauge, where $\mathcal{L}_{\text{G.F.}} = -\frac{1}{2\lambda}(\eta \cdot A^a)^2$, the gluon propagator is[5]

$$\delta^{ab} \, \frac{i}{p^2 + i\epsilon} \left(-g^{\alpha\beta} + \frac{p^\alpha \eta^\beta + \eta^\alpha p^\beta}{p \cdot \eta} - \lambda \eta^2 \frac{p^\alpha p^\beta}{(p \cdot \eta)^2} \right) \tag{2.52}$$

Equation (2.49) remaina the same while (2.50) becomes

$$k^\mu \varepsilon'^\nu (M^{\mu\nu}_{(2)}) = -ig^2 \, [t_a, t_b] \, \bar{v}(p') \left[\left(\frac{\not{\eta}}{\eta \cdot (k + k')} + \frac{\not{k}}{k \cdot k'} \right) \frac{(\varepsilon' \cdot k')}{2} - \not{\varepsilon}' \right] u(p) \tag{2.53}$$

so the sum again is not zero and again only if we impose also the second gluon to be physical ($\varepsilon' \cdot k' = 0$) we recover gauge invariance (and also the term that depends explicitly from the gauge choice vanishes).

This result makes us suspicious that problems may arise also when we will need to sum over physical polarizations. In fact, it was by virtue of gauge invariance of eq. (2.36) that in QED we can use $-g^{\mu\nu}$ instead of the complicated tensor $\Sigma^{\mu\nu}$.

So the second question is now: are we obliged to use the correct but complicated sum $\Sigma_{\mu\nu}$ over the polarizations, or may we try to use $-g^{\mu\nu}$ again? The answer, that we will explain in the next section, is that we can still use $-g^{\mu\nu}$, but we have to find a mechanism that cancels out the unphysical polarizations, without relying on Ward identity. In order to do that we will have to add some new particles, called ghosts, which will cancel the longitudinal and the temporal components of the squared amplitude.

2.3.4 Ghosts and sum over polarizations

The derivation of the correct ghost term in the Lagrangian and of the associated Feynman rules is clear when one uses functional formalism and path integral. In brief, in addition to the gauge fixing term, in non–abelian Lagrangians it is needed to add a new term that looks like the following:

$$\mathcal{L}_{\text{F.P.}} = \bar{\chi}^a (-\partial_\mu \partial^\mu \delta^{ac} - g \partial^\mu f^{abc} A^b_\mu) \chi^c \tag{2.54}$$

where F.P. stands for Faddeev-Popov. This term is necessary to integrate functionally only over inequivalent gauge configurations.

[5]For simplicity we choose $\eta^2 = 0$.

Given (2.54), new particles appear and new Feynman rules follow. We have two new color–octets degrees of freedom, the fields χ^a and $\bar{\chi}^a$: from the Lorentz point of view these fields are scalar but they instead are anticommuting. From this we deduce that they can not be the quanta of real particles because they violate spin–statistic theorem. Moreover they will appear as internal particles or will be pair–produced in final states and, being anticommuting, loops made only of ghosts will take the usual fermionic minus sign. As (2.54) shows, ghosts couple only to gluons giving the vertex represented in section 2.3.6, where an outgoing (ingoing) arrow identifies a ghost (anti-ghost).

With these new rules we can come back to our previous question. In what follows we will show that in the Lorentz gauge it is again possible to use $-g^{\mu\nu}$ for the sum over physical polarizations, but only if we use ghosts: in other words, ghosts will cancel unphysical polarizations.

Before the computations, we need to fix some notations for the amplitude M. With $M_{\mu\nu}$ it will be denoted the total amplitude for the process $q(p)\bar{q}(p') \to g(k)g(k')$ as calculated in the Lorentz gauge and before contracting with the external gluon polarization vectors while $M(v_1, v_2)$ will instead denote the quantity $M_{\mu\nu}\, v_1^\mu\, v_2^\nu$ where v_1 and v_2 are generic four vectors. Also for the following it will be useful the shortcut notation $v_1^{\{\mu}\, v_2^{\nu\}} = v_1^\mu\, v_2^\nu + v_2^\mu\, v_1^\nu$ and $(v_1 v_2) = v_1 \cdot v_2$.

First of all we need to calculate the squared amplitude by making use of (2.31). Being Σ tensors exactly the sum over physical transverse polarizations, we will denote this squared amplitude by $|M|^2_T$. We will not use ghost particles because we are using only physical states.

$$|M|^2_T = M_{\mu\nu} M^*_{\rho\sigma}\ \Sigma^{\mu\rho}(k,\eta)\Sigma^{\nu\sigma}(k',\eta') \tag{2.55}$$

where (we suppose for simplicity that the axis η and η' are light-like)

$$\Sigma^{\mu\rho}(k,\eta) = -g^{\mu\rho} + \frac{1}{(k\eta)} k^{\{\mu}\eta^{\rho\}} \tag{2.56}$$

Expanding only one of the tensor Σ in (2.55), the second one for example, and remembering that $\Sigma^{\mu\rho}(k,\eta)$ corresponds exactly to the sum over the physical polarizations of the first gluon, we have

$$|M|^2_T = M_{\mu\nu} M^*_{\rho\sigma} \left[\left(\sum_{pol} \varepsilon^\mu \varepsilon^{*\rho} \right) (-g^{\nu\sigma}) + \left(\sum_{pol} \varepsilon^\mu \varepsilon^{*\rho} \right) \frac{k'^{\{\nu}\eta'^{\sigma\}}}{(k'\eta')} \right]$$

The second term is zero because it contains the contraction of $M_{\mu\nu}$ with $\varepsilon^\mu k'^\nu$ and of $M^*_{\rho\sigma}$ with $\varepsilon^{*\rho} k'^\sigma$ and these terms vanish because of gauge invariance, as discussed in paragraph (2.3.3). We stress that these terms vanish because the sum are exactly over the only physical polarizations, and this due to the fact we are using Σ tensors in the computation.

With simple manipulation and using the same idea, we obtain the following chain of equalities:

$$
\begin{aligned}
|M|^2_T &= M_{\mu\nu}M^*_{\rho\sigma}\left(\sum_{pol}\varepsilon^{\mu}\varepsilon^{*\rho}\right)(-g^{\nu\sigma}) \\
&= M_{\mu\nu}M^*_{\rho\sigma}\left[(-g^{\mu\rho})(-g^{\nu\sigma})+\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}(-g^{\nu\sigma})\right] \\
&= M_{\mu\nu}M^*_{\rho\sigma}\left[(-g^{\mu\rho})(-g^{\nu\sigma})+\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\Sigma^{\nu\sigma}(k',\eta')-\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\frac{k'^{\{\nu}\eta'^{\sigma\}}}{(k'\eta')}\right] \\
&= M_{\mu\nu}M^*_{\rho\sigma}\left[(-g^{\mu\rho})(-g^{\nu\sigma})+\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\left(\sum_{pol}\varepsilon'^{\nu}\varepsilon'^{*\sigma}\right)-\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\frac{k'^{\{\nu}\eta'^{\sigma\}}}{(k'\eta')}\right] \\
&= M_{\mu\nu}M^*_{\rho\sigma}\left[(-g^{\mu\rho})(-g^{\nu\sigma})-\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\frac{k'^{\{\nu}\eta'^{\sigma\}}}{(k'\eta')}\right]
\end{aligned}
$$

Expanding further only the second term in the previous equation, we have

$$
\begin{aligned}
&M_{\mu\nu}M^*_{\rho\sigma}\frac{k^{\{\mu}\eta^{\rho\}}}{(k\eta)}\frac{k'^{\{\nu}\eta'^{\sigma\}}}{(k'\eta')} \\
&\quad = M_{\mu\nu}M^*_{\rho\sigma}\frac{1}{(k\eta)(k'\eta')}\left[k^{\mu}\eta^{\rho}k'^{\nu}\eta'^{\sigma}+k^{\mu}\eta^{\rho}\eta'^{\nu}k'^{\sigma}+\eta^{\mu}k^{\rho}k'^{\nu}\eta'^{\sigma}+\eta^{\mu}k^{\rho}\eta'^{\nu}k'^{\sigma}\right]
\end{aligned}
$$

The first and the last term in this relation are zero, as can easily see from (2.51) with the replacement $\varepsilon' \to k'$. Thus at the end we are left with

$$
\begin{aligned}
|M|^2_T &= M_{\mu\nu}M^*_{\rho\sigma}(-g^{\mu\rho})(-g^{\nu\sigma})-\frac{1}{(k\eta)(k'\eta')}\left(M(k,\eta')M^*(\eta,k')+M(\eta,k')M^*(k,\eta')\right) \\
&= M_{\mu\nu}M^*_{\rho\sigma}(-g^{\mu\rho})(-g^{\nu\sigma})-\frac{1}{(k\eta)(k'\eta')}\left(M(k,\eta')M^*(\eta,k')+h.c.\right) \qquad (2.57)
\end{aligned}
$$

Using again (2.51) with the replacement $\varepsilon' \to \eta'$ we obtain:

$$
M(k,\eta') = -ig^2\frac{(k'\eta')}{2(kk')}\bar{v}(p')\left([t^a,t^b]\not{k}\right)u(p)
$$

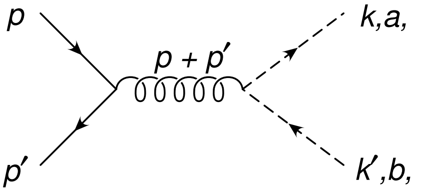
$$
M(\eta,k') = -ig^2\frac{(k\eta)}{2(kk')}\bar{v}(p')\left([t^a,t^b]\not{k}'\right)u(p)
$$

that substituted in (2.57) gives at the end

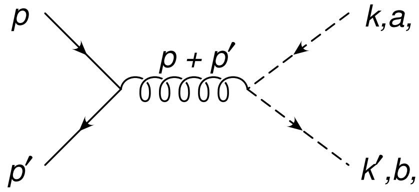
$$
\begin{aligned}
|M|^2_T &= M_{\mu\nu}M^*_{\rho\sigma}(-g^{\mu\rho})(-g^{\nu\sigma})-2\left|\frac{g^2}{2(pp')}f^{abc}\bar{v}(p')\not{k}\,t^c u(p)\right|^2 \\
&= M_{\mu\nu}M^*_{\rho\sigma}(-g^{\mu\rho})(-g^{\nu\sigma})-2\left(\frac{g^2}{2(pp')}\right)^2 f^{cda}f^{cdb}\bar{v}(p')\left(\not{k}\,t^a\right)u(p)\,\bar{u}(p)\left(\not{k}\,t^b\right)v(p')
\end{aligned}
\qquad (2.58)
$$

Now we want to calculate the squared amplitude for the same process trying to use $-g_{\mu\rho}$ for the sum over polarizations: this will obviously give the first term of (2.58). In the following we will calculate the squared amplitude for ghosts pair production and show that this is exactly equal to the second term in (2.58) with a minus sign. In this way we will finally answer the question at the end of the previous section.

The amplitude for producing a ghost of momentum k and an anti–ghost of momentum k' reads



$+$



(2.59)

$$\begin{aligned} M(q\bar{q} \to \chi\bar{\chi}) &= \bar{v}(p')(-igt^d\gamma^\delta)u(p)\frac{-ig_{\delta\gamma}\delta^{dc}}{2(pp')}(gf^{cba}k^\gamma) \\ &= -\frac{g^2}{2(pp')}f^{cba}\bar{v}(p')(\not{k}t^c)u(p) \end{aligned} \tag{2.60}$$

and easily it can be shown that the amplitude for producing $\chi(k')\bar{\chi}(k)$ is exactly the same. If we square each of this contribution and sum them[6], we are left with

$$|M(q\bar{q} \to \chi\bar{\chi})|^2 = 2\left(\frac{g^2}{2(pp')}\right)^2 f^{cda}f^{cdb}\left[\bar{v}(p')\left(\not{k}t^a\right)u(p)\bar{u}(p)\left(\not{k}t^b\right)v(p')\right] \tag{2.61}$$

From eq. (2.61) we see that subtracting (2.61) from $M_{\mu\rho}M^*_{\nu\sigma}(-g^{\mu\nu})(-g^{\rho\sigma})$ gives exactly (2.58) and this answers the question about sum over polarizations: we learnt that in QCD we can still use $-g_{\mu\nu}$ for the sum over polarizations but we have also to add to the result so obtained the squared amplitude for producing ghosts (eq. (2.61)) with a minus sign.

One may wonder if this spoils any of the postulate of the theory, being in presence of a negative probability. We know that ghost fields violate spin-statistic theorem, so in principle there is neither postulate nor physical reason that should prevent some strange thing to happen. From this calculation we evince that ghosts are a mathematical tool that permit us to work in covariant gauges also in QCD, reconstructing the sum over physical polarizations.

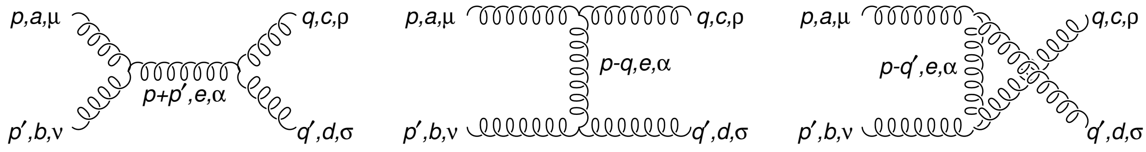
At a deeper level, one can convince himself that ghosts are fundamental to restore the unitarity of the theory in covariant gauge fixing frames. Having not added their contribution with the correct minus sign, we would have obtained a probability of transition $q\bar{q} \to gg$ larger then the physical one ($|M|^2_T$), clearly implying unitarity violation.

Finally, this computation shows that ghosts are required not only as intermediate states to cancel non–physical states propagation in loop diagrams, but also as external particles when one use $\sum_{pol}\varepsilon^\mu\varepsilon^{*\nu} = -g^{\mu\nu}$.

[6]We have two contribution because if ghosts were physical particles, a ghost and an antighost would be distinguishable for example because they carry a colour charge. Note also that obviously the interference between the two graph is not permitted.

2.3.5 The $gggg$ vertex: $gg \to gg$

As already stated at the end of section 2.3.2, the triple gluonic vertex makes possible a new scattering process: the elastic scattering of two gluons. At this point, the Feynman graphs that give contributions to this process are



(2.62)

For this new process, we can follow exactly what we did for the process $qq \to gg$, checking if the amplitude coming from these graphs respect gauge invariance. If this will not be the case (as it will not be), in order to preserve the gauge symmetry we will be forced to admit a new interaction vertex. The form of this vertex will be determined as in the precedent case, using Lorentz invariance, Bose statistic and requiring that the theory stays renormalizable.

In order to check gauge invariance, we calculate the scalar product between the amplitude $M^{\mu\nu\rho\sigma}$, given by the graphs (2.62), and one external momentum, say p_μ. Moreover, in order to simplify the calculations and to avoid all the non trivial subtleties of non abelian gauge theories discussed in the previous section, we will fix the Lorentz gauge and require also that the other external gluons are transverse, contracting the amplitude with their polarization vectors (respectively, $\varepsilon_b^\nu(p')$, $\varepsilon_c^\rho(q)$ and $\varepsilon_d^\sigma(q')$).

It is also convenient to distinguish three different contribution to the amplitude $M^{\mu\nu\rho\sigma}$, coming respectively from the s, t and u–channel. We start writing the contribution from the s–channel: using the form of the triple vertex (2.46) one has

$$\begin{aligned} M_s^{\mu\nu\rho\sigma} &= c\left[g^{\mu\alpha}\left(2p+p'\right)^\nu - g^{\alpha\nu}\left(p+2p'\right)^\mu + g^{\mu\nu}\left(p'-p\right)^\alpha\right] \\ &\quad \times \left[g^{\alpha\rho}\left(2q+q'\right)^\sigma - g^{\alpha\rho}\left(q+2q'\right)^\rho + g^{\rho\sigma}\left(q'-q\right)^\alpha\right] \end{aligned} \tag{2.63}$$

where

$$c = g^2 f^{aeb} f^{ecd} \frac{-i}{\left(p+p'\right)^2} \tag{2.64}$$

Contracting (2.63) with the momentum p_μ and with the polarization vectors one obtain

$$\begin{aligned}
M_s &= M_s^{\mu\nu\rho\sigma}\, p_\mu\, \varepsilon_{b,\nu}\, \varepsilon_{c,\rho}\, \varepsilon_{d,\sigma} \\
&= c\,\Bigg\{ (2p\cdot\varepsilon_b)\Bigg[(p\cdot\varepsilon_c)\,(2q\cdot\varepsilon_d) - (p\cdot\varepsilon_d)\,(2q'\cdot\varepsilon_c) + (\varepsilon_c\cdot\varepsilon_d)\Big(p\cdot(q'-q)\Big)\Bigg] \\
&\quad - (2p'\cdot p)\Bigg[(\varepsilon_b\cdot\varepsilon_c)\,(2q\cdot\varepsilon_d) - (\varepsilon_b\cdot\varepsilon_d)\,(2q'\cdot\varepsilon_c) + (\varepsilon_c\cdot\varepsilon_d)\Big(\varepsilon_b\cdot(q'-q)\Big)\Bigg] \\
&\quad + (p\cdot\varepsilon_b)\Bigg[\Big(\varepsilon_c\cdot(p'-p)\Big)\,(2q\cdot\varepsilon_d) - \Big(\varepsilon_d\cdot(p'-p)\Big)\,(2*****'\cdot\varepsilon_c)\Bigg] \\
&\quad + (p\cdot\varepsilon_b)\Bigg[(\varepsilon_c\cdot\varepsilon_d)\Big((p'-p)\cdot(q'-q)\Big)\Bigg]\Bigg\} \\
&= c\,\Bigg\{2\,(p\cdot\varepsilon_b)\Bigg[\Big((p+p')\cdot\varepsilon_c\Big)\,(q\cdot\varepsilon_d) - \Big((p+p')\cdot\varepsilon_d\Big)\,(q'\cdot\varepsilon_c)\Bigg] \\
&\quad -s*****\Bigg[2\,(\varepsilon_b\cdot\varepsilon_c)\,(q\cdot\varepsilon_d) - 2\,(\varepsilon_b\cdot\varepsilon_d)\,(q'\cdot\varepsilon_c) + (\varepsilon_c\cdot\varepsilon_d)\Big(\varepsilon_b\cdot(q'-q)\Big)\Bigg]\Bigg\}
\end{aligned} \tag{2.65}$$

Using now the conservation of the momentum and the fact that we are working in the Lorentz gauge, it is easy to verify that the terms not proportional to the Mandelstam variable s cancel each other. Using also the definition of c (2.64), we find the result

$$M_s = ig^2 f^{aeb} f^{ecd}\Bigg\{2\,(\varepsilon_b\cdot\varepsilon_c)\,(q\cdot\varepsilon_d) - 2\,(\varepsilon_b\cdot\varepsilon_d)\,(q'\cdot\varepsilon_c) + (\varepsilon_c\cdot\varepsilon_d)\Big[\varepsilon_b\cdot(q'-q)\Big]\Bigg\} \tag{2.66}$$

With similar tricks one finds the contribution from the other channels. From the t–channel we have:

$$M_t = -ig^2 f^{ace} f^{edb}\Bigg\{2\,(\varepsilon_c\cdot\varepsilon_d)\,(q'\cdot\varepsilon_b) + 2\,(\varepsilon_b\cdot\varepsilon_c)\,(p'\cdot\varepsilon_d) - (\varepsilon_b\cdot\varepsilon_d)\Big[\varepsilon_c\cdot(p'+q')\Big]\Bigg\} \tag{2.67}$$

while from the u–channel one gets

$$M_u = -ig^2 f^{ade} f^{ecb}\Bigg\{2\,(\varepsilon_c\cdot\varepsilon_d)\,(q\cdot\varepsilon_b) + 2\,(\varepsilon_b\cdot\varepsilon_d)\,(p'\cdot\varepsilon_c) - (\varepsilon_b\cdot\varepsilon_c)\Big[\varepsilon_d\cdot(p'+q)\Big]\Bigg\} \tag{2.68}$$

In order to sum these three contributions, we make use of the Jacobi identity

$$f^{aeb} f^{ecd} + f^{ade} f^{ecb} - f^{ace} f^{edb} = 0 \tag{2.69}$$

and we change the color factor of the last term of each channel in color factors of the other two channels. Using again the conservation of the momentum and the gauge choice we did, we can write

$$
\begin{aligned}
M &= M_s + M_t + M_u \\
&= ig^2 \Big\{ f^{aeb} f^{ecd} \Big[(\varepsilon_b \cdot \varepsilon_c)\,(p \cdot \varepsilon_d) - (\varepsilon_b \cdot \varepsilon_d)\,(p \cdot \varepsilon_c) \Big] \\
&\qquad + f^{ace} f^{edb} \Big[(\varepsilon_b \cdot \varepsilon_c)\,(p \cdot \varepsilon_d) - (\varepsilon_c \cdot \varepsilon_d)\,(p \cdot \varepsilon_b) \Big] \\
&\qquad + f^{ade} f^{ecb} \Big[(\varepsilon_b \cdot \varepsilon_d)\,(p \cdot \varepsilon_c) - (\varepsilon_c \cdot \varepsilon_d)\,(p \cdot \varepsilon_b) \Big] \Big\}
\end{aligned} \tag{2.70}
$$

This result means that the amplitude described by the Feynman graphs (2.62) does not satisfy the requirement of gauge invariance. So, exactly as for the process $qq \to gg$, we are forced to introduce a new vertex. From the color structure of equation (2.70), one can argue that the vertex needed is a contact term of four gluons. Then, renormalizability tells us that the vertex does not contain any dimensionful quantity and Lorentz symmetry says that the vertex must support four vectorial indexes. This means that we must consider a vertex

$$
V^{\mu_1\mu_2\mu_3\mu_4} \propto \sum_{i \neq j \neq k \neq l} A_{ijkl}\, g^{\mu_i\mu_j}\, g^{\mu_k\mu_l} + B\, \epsilon^{\mu_1\mu_2\mu_3\mu_4} \tag{2.71}
$$

With A_{ijkl} and B we denoted generic numerical coefficients. Even here, the proportionality between the two addenda is due to the fact that we have forgotten the color structure: this must be exactly that given in expression (2.70). So, we should reproduce three times the structure given in (2.71), one for each couple of structure constants. We consider the first couple $f^{aeb} f^{ecd}$ and impose the Bose symmetry on the exchange of gluons indexes[7]: we see immediately that we are forced to require that the coefficients A_{ijkl} and B reproduce a structure antisymmetric in the exchange of $a \leftrightarrow b$ and of $c \leftrightarrow d$ and symmetric in the exchanges of pairs $ab \leftrightarrow cd$. With similar consideration on the other two couples of structure constants, we can write the vertex in this form:

$$
\begin{aligned}
V^{\mu_1\mu_2\mu_3\mu_4} &= A_1 f^{aeb} f^{ecd} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) \\
&\quad + A_2 f^{ace} f^{edb} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) \\
&\quad + A_3 f^{ade} f^{ecb} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4} \right)
\end{aligned} \tag{2.72}
$$

The constants A_i are determined requiring that this new vertex gives a contribution opposite in sign to that of the amplitude (2.70). An easy inspection shows that this requirement fixes $A_i = ig^2$.

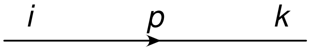
So, the quartic contact vertex can be written in this form:

$$
\begin{aligned}
V^{\mu_1\mu_2\mu_3\mu_4} &= ig^2 \Big\{ f^{aeb} f^{ecd} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) \\
&\quad + f^{ace} f^{edb} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) \\
&\quad + f^{ade} f^{ecb} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4} \right) \Big\}
\end{aligned} \tag{2.73}
$$

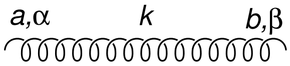
and the gauge invariance is restored again.

[7] We link color index a with Lorentz index μ_1 and so on.

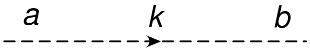
2.3.6 QCD Feynman rules



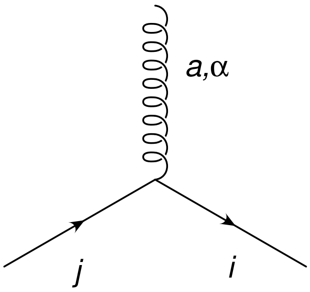
$$= \quad \delta^{ik} \, \frac{i}{\not{p} - m + i\epsilon}$$



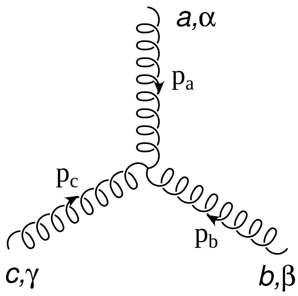
$$= \quad \delta^{ab} \, \frac{i}{k^2 + i\epsilon} \left(-g^{\alpha\beta} + (1 - \lambda) \, \frac{k^\alpha k^\beta}{k^2} \right)$$



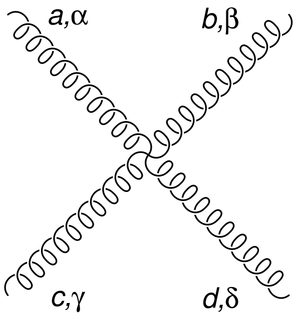
$$= \quad \delta^{ab} \, \frac{i}{k^2 + i\epsilon}$$



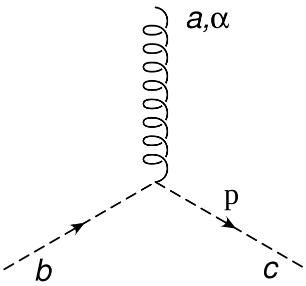
$$= \quad -ig\gamma^\alpha t^a_{ij}$$



$$= \quad -gf^{abc} \left[g^{\alpha\beta} \left(p_a - p_b\right)^\gamma + g^{\beta\gamma} \left(p_b - p_c\right)^\alpha + g^{\gamma\alpha} \left(p_c - p_a\right)^\beta \right]$$



$$\begin{aligned} = \quad -ig^2 \Big[& f^{eac} f^{ebd} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta} \right) + f^{ead} f^{ebc} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} \right) \\ & + f^{eab} f^{ecd} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right) \Big] \end{aligned}$$



$$= \quad gf^{abc} p^\alpha$$

Chapter 3

Color algebra

3.1 SU(3) algebra

The $SU(3)$ group is the group of 3×3 unitary matrices U with unit determinant

$$U^\dagger U = UU^\dagger = 1, \qquad \det U = e^{\mathrm{Tr}\{\log U\}} = 1. \tag{3.1}$$

One can always write

$$U = e^{i\omega_a t^a}, \qquad a = 1, \ldots, N^2 - 1 \tag{3.2}$$

with ω_a reals and matrices t^a hermitian and traceless

$$t^a = (t^a)^\dagger, \qquad \mathrm{Tr}\{t^a\} = 0 \tag{3.3}$$

Quark fields ψ are in the fundamental representation ($\mathbf{3}$), anti-quarks in the anti-fundamental ($\bar{\mathbf{3}}$) and gluons in the adjoint ($\mathbf{8}$). Matter fields transform under $SU(3)$ according to

$$\psi'(x) = U(x)\psi(x) \tag{3.4}$$
$$\bar{\psi}'(x) = \bar{\psi}(x)U(x)^\dagger, \tag{3.5}$$

color singlets can thus be formed out of a quark-antiquark pair via

$$\sum_i \psi_i^* \psi_i \to \sum_{i,j,k} U_{ij}^* \psi_j^* U_{ik} \psi_k = \sum_{j,k} \left(\sum_i U_{ji}^\dagger U_{ik} \right) \psi_j^* \psi_k = \sum_k \psi_k^* \psi_k \tag{3.6}$$

but it's also possible to form color singlet from three quarks (or anti quarks) using

$$\sum_{i,j,k} \epsilon^{ijk} \psi_i \psi_j \psi_k \to \sum_{i,j,k,l,m,n} \epsilon^{ijk} U_{il} U_{jm} U_{kn} \psi_l \psi_m \psi_n = \sum_{l,m,n} \det U \epsilon^{lmn} \psi_l \psi_m \psi_n \tag{3.7}$$

In this way one can accommodate all observed hadrons and mesons in color invariant states. Furthermore, since in a system with n_q quarks and $n_{\bar{q}}$ antiquarks it's possible to form color singlet only if

$$n_q - n_{\bar{q}} \mod 3 = 0, \tag{3.8}$$

it is easy to see that all these invariant states must have integer electric charge, provided the usual charges assignments : $\frac{2}{3}e$ for up type quarks and $-\frac{1}{3}e$ for down type ones. With these choices the QCD Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{G.F.} + \mathcal{L}_{F.P.} + \mathcal{L}_F \tag{3.9}$$

where the pure gauge Lagrangian is

$$\mathcal{L}_G = -\frac{1}{4} F^a_{\mu\nu} F^{a\ \mu\nu}, \ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{3.10}$$

the gauge-fixing part is

$$\mathcal{L}_{G.F.} = -\frac{1}{2\lambda} \left(\partial^\mu A^a_\mu\right)^2 \tag{3.11}$$

and the Faddeev-Popov one is

$$\mathcal{L}_{F.P.} = \partial^\mu \bar{\chi}^a D^{ab}_\mu \chi^b \quad \text{with} \quad D^{ab}_\mu = \delta^{ab}\partial_\mu + igf^{abc}A^c_\mu. \tag{3.12}$$

Finally the fermion Lagrangian reads

$$\mathcal{L}_F = \sum_{flavour} \bar{\psi}^i_f \left(i\not{D}^{ij}_\mu - m_f \delta^{ij} \right) \psi^j_f \quad \text{with} \quad D^{ij}_\mu = \delta^{ij}\partial_\mu + igt^a_{ij} A^a_\mu \tag{3.13}$$

where the $SU(3)$ algebra tells us that

$$\left[t^a, t^b\right] = i\ f^{abc} t^c \tag{3.14}$$

and we chose the convention

$$\mathrm{Tr}\{t^a t^b\} = T_{\mathrm{F}} \delta^{ab}, \quad T_{\mathrm{F}} = \frac{1}{2}. \tag{3.15}$$

One can show that in this way the structure constants f are always reals and antisymmetric. For example taking the complex conjugate of (3.14) one has

$$-i\left(f^{abc}\right)^* (t^c)^\dagger = \left[\left(t^b\right)^\dagger, (t^a)^\dagger\right] = -\left[t^a, t^b\right] \tag{3.16}$$

because of hermiticity of t's. Thus $\left(f^{abc}\right)^* = f^{abc}$. In the same way taking the trace of

$$if^{abc} t^c t^d = \left[t^a, t^b\right] t^d \tag{3.17}$$

one gets

$$if^{abc} T_{\mathrm{F}} \delta^{cd} = \mathrm{Tr}\{\left[t^a, t^b\right] t^d\} \tag{3.18}$$
$$f^{abc} = -2i\mathrm{Tr}\{\left[t^a, t^b\right] t^c\}. \tag{3.19}$$

that shows that f is antisymmetric.

We generalize now to the $SU(n)$ group: the generic hermitian $n \times n$ matrix M can be written as

$$M = n^a t^a + n^0 \ \mathbb{I}_{n\times n} \tag{3.20}$$

with n^0 fixed by the trace to be $n^0 = \mathrm{Tr}\{M\}/n$. In the same way

$$M t^b = n^a t^a t^b + n^0 t^b \tag{3.21}$$

with n^a now fixed to $n^a = 2\mathrm{Tr}\{M t^a\}$. Thus

$$M = 2\mathrm{Tr}\{M t^a\} t^a + \frac{1}{n}\mathrm{Tr}\{M\}\mathbb{I}_{n\times n} \tag{3.22}$$

Taking $M = \left[t^a, t^b\right]$ one can re-derive the formula for f^{abc}

$$\left[t^a, t^b\right] = 2\mathrm{Tr}\{\left[t^a, t^b\right] t^c\} t^c \tag{3.23}$$
$$i f^{abc} t^c = 2\mathrm{Tr}\{\left[t^a, t^b\right] t^c\} t^c \tag{3.24}$$
$$f^{abc} = -2i\mathrm{Tr}\{\left[t^a, t^b\right] t^c\}. \tag{3.25}$$

Using Jacobi identities it's also possible to define the adjoint representation by means of matrices T, made by structure constants

$$\left(T^b\right)_{ac} = i f^{abc} \tag{3.26}$$

such that they satisfy $\left[T^a, T^b\right] = i f^{abc}\ T^c$. Defining now

$$T^2_{ik} = T^a_{ij} T^a_{jk} \tag{3.27}$$

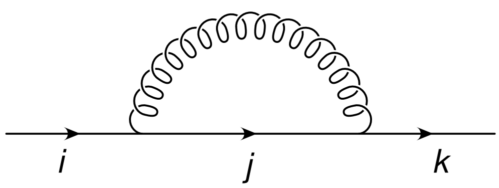
one can show that

$$\left[T^b, T^2\right] = T^b T^a T^a - T^a T^a T^b = -\left[T^a, T^b\right] T^a - T^a \left[T^a, T^b\right] \tag{3.28}$$
$$= -i f^{abc} T^c T^a - T^a i f^{abc} T^c = -i f^{abc}\{T^c, T^a\} = 0 \tag{3.29}$$

T^2 is a Casimir of the representation and by Schur's lemma it must be proportional to the identity.

3.2 Color coefficients

Provided that the most important difference between QCD and QED is the non abelianity of the former, it worths to separate the non abelian part evaluating color coefficients of sequences of t matrices and then proceed as in usual QED computations. For example the color coefficient for the fermion self energy correction is defined to be



$$t^a_{ij} t^a_{jk} \equiv C_{\mathrm{F}} \mathbb{I}_{ik} \tag{3.31}$$

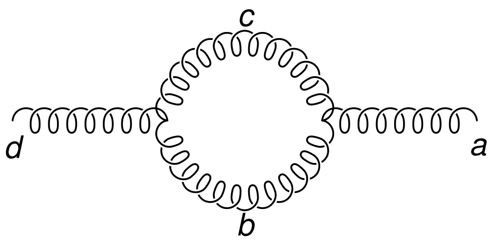
One can easily evaluate it taking the trace of the previous relation

$$\mathrm{Tr}\{t^a_{ij}t^a_{jk}\} = \mathrm{Tr}\{C_{\mathrm{F}}\mathbb{I}_{ik}\} \tag{3.31}$$

$$\frac{1}{2}\delta^a_a = C_{\mathrm{F}} n \tag{3.32}$$

$$C_{\mathrm{F}} = \frac{n^2-1}{2n} \tag{3.33}$$

The gluon self energy diagram



$$\begin{aligned} if^{abc}if^{cbd} &= \left(T^b\right)_{ac}\left(T^b\right)_{cd} \\ &= C_{\mathrm{A}}\ \mathbb{I}_{ad} \end{aligned} \tag{3.34}$$

allows the definition of a new constant C_{A}, but this will be evaluate later on since it is a little bit involved.

Let's instead apply (3.22) to a set of generic hermitian matrices

$$M^i_k = \delta^i_{(j)}\delta^{(l)}_k \tag{3.35}$$

obtaining, after the trace

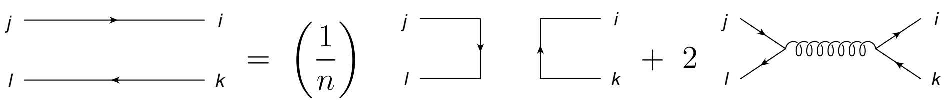
$$\mathrm{Tr}\{M\} = \delta^i_{(j)}\delta^{(l)}_i = \delta^{(l)}_{(j)} \tag{3.36}$$

$$\mathrm{Tr}\{Mt^a\} = \delta^i_{(j)}\delta^{(l)}_k t^a_{ki} = t^a_{(l)(j)}. \tag{3.37}$$

Thus

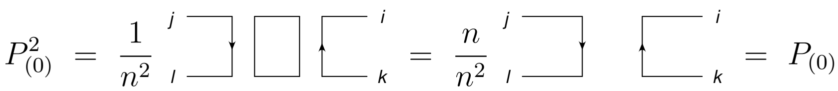
$$M^i_k = \frac{1}{n}\delta^{(l)}_{(j)}\delta^i_k + t^a_{(l)(j)}t^a_{ik}. \tag{3.38}$$

Diagrammatically this is equivalent to

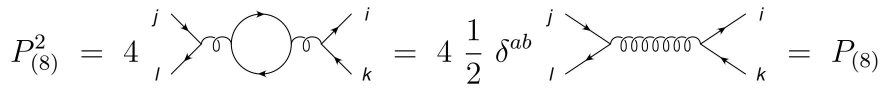


(3.39)

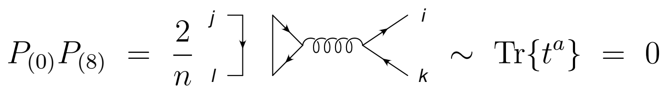
where the $n^2 = 9$ components of the left hand side are shared between the one component of the singlet projector $P_{(0)}$ and the eight ones of the octect projector $P_{(8)}$. Let's show that they are true projectors:



(3.40)

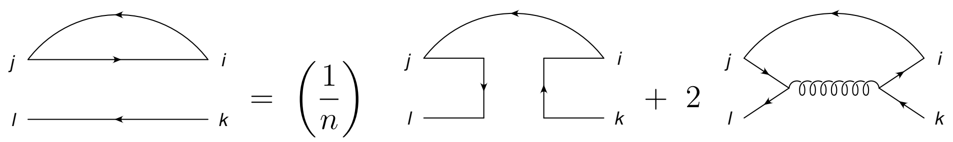


(3.41)



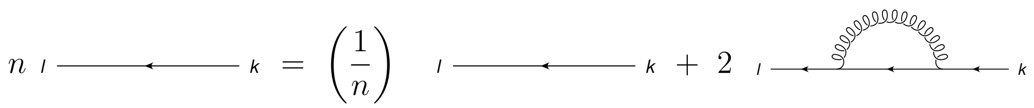
(3.42)

Now using these projectors it's easy to evaluate colour factor. For example closing the quark line $i - j$ in (3.39) as shown below one gets



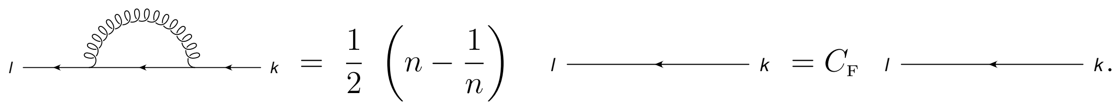
(3.43)

which equals



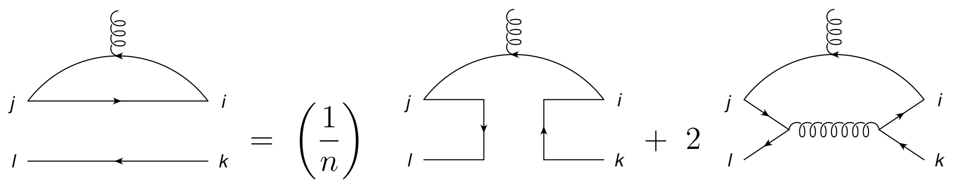
(3.44)

so that



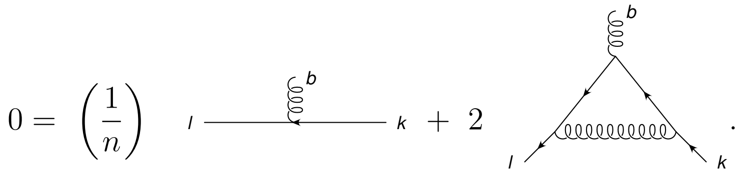
(3.45)

Furthermore, adding one extra gluon, one can recover that



(3.46)

or, in other words,

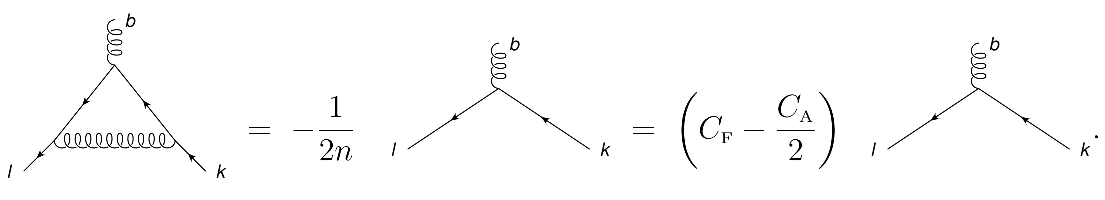


(3.47)

Thus, assumming that b is the color index of the emitted gluon, one gets the relation

$$t^a t^b t^a = -\frac{1}{2n} t^b \tag{3.48}$$

which corresponds to

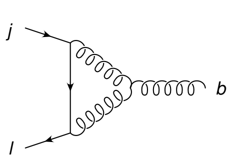


(3.49)

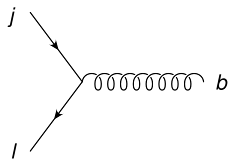
The last equality will be verified as soon as we will find the value of $C_{\rm A}$. But we're now ready to perform this calculation: considering the following relation

$$
\begin{aligned}
t^a t^c i f^{abc} &= \left(\frac{1}{2}\left[t^a, t^c\right] + \frac{1}{2}\{t^a, t^c\}\right) i f^{abc} \\
&= \frac{1}{2}\left[t^a, t^c\right] i f^{abc} \\
&= \frac{1}{2} i f^{acd} t^d i f^{abc} \\
&= \frac{C_{\rm A}}{2} t^b
\end{aligned}
\tag{3.50}
$$

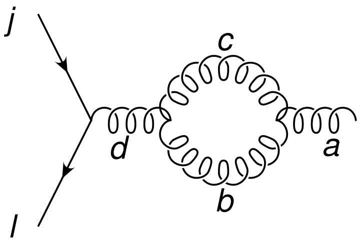
which has the graphical meaning



$$= \frac{C_{\rm A}}{2} \tag{3.51}$$



applied to the following diagram

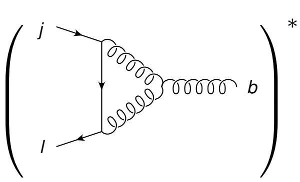


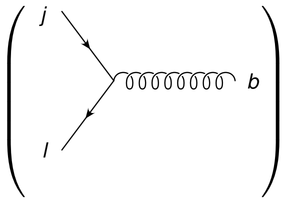
$$
\begin{aligned}
&= i f^{abc} i f^{cbd} t^d \;=\; i f^{abc}\left(t^c t^b - t^b t^c\right) \\
&= \left[t^a, t^b\right] t^b + i f^{acb} t^b t^c \\
&= \left[t^a, t^b\right] t^b + \left[t^a, t^c\right] t^c \\
&= 2\ \left[t^a, t^b\right] t^b \;=\; C_{\rm A} t^a \\
&= 2\ \left(t^a t^b t^b - t^b t^a t^b\right) \\
&= 2 t^a \frac{n^2 - 1}{2n} - 2\left(-\frac{1}{2n}\right) t^a \;=\; n t^a
\end{aligned}
\tag{3.52}
$$

One finds

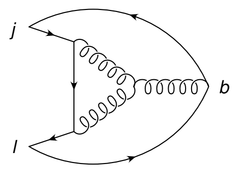
$$C_{\rm A} = n \tag{3.53}$$

With these rules one can compute the colour factor for the generic graph. As an example let's try to evaluate the color factor of the interference term

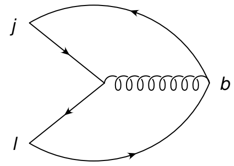




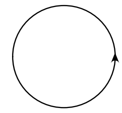
$$=$$



$$= \frac{C_{\mathrm{A}}}{2}$$



$$= \frac{C_{\mathrm{A}}}{2} \; C_{\mathrm{F}}$$



$$= \frac{C_{\mathrm{A}}}{2} \; C_{\mathrm{F}} \; n \quad . \tag{3.54}$$

Chapter 4

QED renormalization

1. To be done even if the quantum corrections were **finite**!
2. the **same procedure** cancels all the divergences at **all orders**

In the first chapters we analyzed many aspects of gauge theories. Time is ready to put all those informations together and to look at the most important consequences that a quantum field theory has on our knowledge of physics.

In this section we start the analysis of perturbative corrections to amplitudes in the simplest context of QED. The bare Lagrangian is

$$\mathcal{L} = \bar{\psi}_B i\partial\!\!\!/ \psi_B - \frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B - g_B \bar{\psi}_B A\!\!\!/_B \psi_B - m_B \bar{\psi}_B \psi_B$$

The abelianity of the theory implies, as we have seen, that there is only one kind of vertex. This point, though simplifying many calculations, does not exclude the possibility to fix the principles governing renormalization and its main consequences.

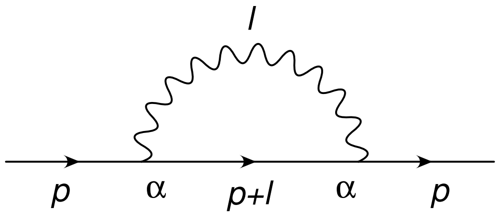
As we have just seen in section 1, as soon as loop integrals are concerned, one has to use a regularization technique in order to prevent the amplitude to diverge. In gauge theories, dimensional regularization, though very mathematical, is a natural choice since it preserves both gauge and Lorentz invariance. Other techniques are more physically based. Here we adopt one of these, consisting in the introduction of an UV cutoff Λ in the loop integral. This cutoff, in the Wilsonian way of thinking at quantum field theories, can be seen as the last energy scale at which our theory is valid: we can look at the theory as an effective field theory that makes sense up to Λ scale.

Dimensional regularization will be used in the next chapter, when we will study 1-loop corrections to QCD amplitudes.

The basic blocks we need to *renormalize the theory* (at one loop), i.e. to extract finite predictions from mathematical divergent quantities, are the computation of the fundamental divergent Feynman diagrams: the fermion and the photon self energy and the vertex corrections. Moreover, in all the computations we will assume for simplicity massless fermions.

4.1 Fermion propagator

We consider the first order corrections to the propagator of a fermion. The Feynman integral we have to calculate is The corresponding value[1], in the Feynman gauge and neglecting the



fermion mass, is given by

$$\begin{aligned} M &= \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4}\, \frac{-i}{\ell^2}\, (-ie_B\gamma^\alpha)\, \frac{i}{\not p + \not \ell}\, (-ie_B\gamma_\alpha) \\ &= -e_B^2 \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2}\, \gamma^\alpha \frac{1}{\not p + \not \ell}\, \gamma_\alpha \\ &= 2e_B^2 \int^{\Lambda} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2}\, \frac{1}{\not p + \not \ell} \end{aligned} \tag{4.1}$$

where Λ is the cutoff. In the last line we used the identity

$$\gamma^\alpha\, \gamma_\beta\, \gamma_\alpha = -2\gamma_\beta \tag{4.2}$$

The proof is a direct consequence[2] of the Clifford algebra $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$.

We observe that M has the dimension of an energy. Having fixed the mass m of the fermion to be 0, the only dimensionful parameter in the integral (apart the cutoff, that plays a different role) is the momentum p_μ and since M is Lorentz-invariant, we can write

$$M = A\not p \tag{4.3}$$

The parameter A comes from the result of the loop integral and, since from (4.3) it has to be dimensionless, we expect it to diverge at most logarithmically with Λ. We also note that (4.1) does not diverge in the infrared region ($\ell \to 0$) because there is a $\not p$ in the denominator. In order to find A, we derive (4.1) and (4.3) with respect to p_μ. Using the relation

$$\partial_{p_\mu}\left(\frac{1}{\not p + \not \ell}\right) = -\frac{1}{\not p + \not \ell}\, \gamma_\mu\, \frac{1}{\not p + \not \ell} \tag{4.4}$$

[1] Having to calculate an amplitude, there would be the usual spinors $\bar{u}(p)$ and $u(p)$ at the extrema of our expression. Actually we are not calculating an amplitude but a self-energy diagram so we do not need to saturate polarization indexes with spinors.

[2] Note that (4.2) as it stands is true in four-dimension Minkowski spacetime.

which is a consequence of $s^{-1}s = 1$, with $s = (\not{p} + \not{\ell})$, we approde to the identity

$$\begin{aligned}
A\gamma_\mu &= -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{\not{p}+\not{\ell}} \gamma_\mu \frac{1}{\not{p}+\not{\ell}} \\
&= -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{(p+\ell)^\alpha (p+\ell)^\beta}{\ell^2 \left[(p+\ell)^2\right]^2} \gamma_\alpha \gamma_\mu \gamma_\beta
\end{aligned} \tag{4.5}$$

Up to now the computation is exact. At this point we neglect all the p momentum dependence in the integral ($\ell \gg p$) since we are now interested in the high momentum behavior of the theory: in other words we want to extract the leading singularity of the integral in the UV limit. After this assumptions the previous formula becomes

$$\begin{aligned}
A\gamma_\mu &\simeq -2e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\alpha \ell^\beta}{(\ell^2)^3} \gamma_\alpha \gamma_\mu \gamma_\beta \\
&= -\frac{e_B^2}{2} \gamma_\alpha \gamma_\mu \gamma^\alpha \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2}
\end{aligned} \tag{4.6}$$

where the Lorentz dependence of the integrand can be extracted by replacing, under the integral, $\ell^\alpha \ell^\beta$ with $\ell^2 g^{\alpha\beta}/4$. Using again (4.2) on the right hand side of (4.6) we have

$$A \simeq e_B^2 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2} \tag{4.7}$$

Perform this integral is now an easy task: passing in Euclidean time and remembering that the surface of a 4–dimensional sphere of radius one is $2\pi^2$ (see equation (1.23)), we get:

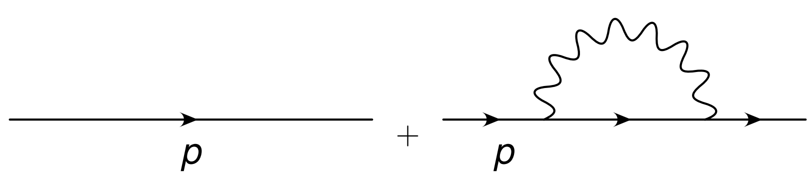
$$\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2)^2} = i \frac{\pi^2}{(2\pi)^4} \log\left(\frac{\Lambda^2}{\mu^2}\right) \tag{4.8}$$

So A becomes

$$A = \frac{i\alpha_B}{4\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) \tag{4.9}$$

We observe that A diverges logarithmically with Λ as expected. Moreover, we have introduced an arbitrary new energy scale μ. At this level the use of μ is required only to maintain the argument of the logarithm dimensionless. Strictly speaking, the integral (4.8), as it stands, would diverge also in the IR region but, as we pointed out earlier, this expression comes from an integral that was free of IR divergences. For this reason the scale μ^2 has not a meaning deeper than that of being a generic scale obtained from the external momentum p.

We now consider the sum of the graphs relative to the fermion propagator and its first order correction. They are given by



$+ \ \mathcal{O}\left(\alpha_B^2\right)$ (4.10) and the related expression M is

$$\begin{aligned} M &= \frac{i}{\not p} + \frac{i}{\not p} A \not p \frac{i}{\not p} \\ &= \frac{i}{\not p}(1+iA) \\ &\equiv \frac{i}{\not p} Z_2 \end{aligned} \tag{4.11}$$

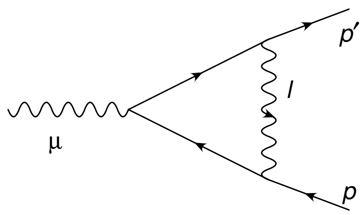
where the renormalization constant Z_2 (or the 1–loop correction to the propagator $\delta_2 \equiv Z_2 - 1$) is defined by

$$\begin{aligned} Z_2 &= 1 - \frac{\alpha_B}{4\pi} \log \frac{\Lambda^2}{\mu^2} + \mathcal{O}\left(\alpha_B^2\right) \\ \delta_2 &= -\frac{\alpha_B}{4\pi} \log \frac{\Lambda^2}{\mu^2} + \mathcal{O}\left(\alpha_B^2\right) \end{aligned} \tag{4.12}$$

Please notice that eq. (4.11) implies that the photon remains **massless**.

4.2 Vertex corrections

We now consider the first order correction to the QED vertex.



(4.13)

Doing computation as before in the Feynman gauge, the graph of fig. (4.13) corresponds to the following[3]:

$$\begin{aligned} M^\mu &= \int^\Lambda \frac{d^4\ell}{(2\pi)^4} (-ie_B\gamma^\alpha) \frac{i}{\not p' + \not \ell} (-ie_B\gamma^\mu) \frac{i}{\not p + \not \ell} (-ie_B\gamma^\beta) \frac{-ig_{\alpha\beta}}{\ell^2} \\ &= -e_B^3 \int^\Lambda \frac{d^4\ell}{(2\pi)^4} \gamma^\alpha \frac{1}{\not p' + \not \ell} \gamma^\mu \frac{1}{\not p + \not \ell} \gamma_\alpha \frac{1}{\ell^2} \end{aligned} \tag{4.14}$$

As in the fermion propagator loop, since we are interested in the UV behavior of the amplitude, we can neglect p and p' in the fermion propagators: collecting all the gamma matrices outside the integral, we are left with

$$\begin{aligned} M^\mu &\simeq -e_B^3 \gamma^\alpha\gamma^\gamma\gamma^\mu\gamma^\delta\gamma_\alpha \left[\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{\ell_\gamma \ell_\delta}{(\ell^2)^3} \right] \\ &= -e_B^3 \gamma^\alpha\gamma^\gamma\gamma^\mu\gamma^\delta\gamma_\alpha \left[\int^\Lambda \frac{d^4\ell}{(2\pi)^4} \frac{g_{\gamma\delta}}{4} \frac{1}{(\ell^2)^2} \right] \end{aligned} \tag{4.15}$$

[3]In (4.13) the momenta p and p' flow out of the graph.

where the Lorentz dependence of the integral within squared brackets can be extracted multiplying it with $g^{\gamma\delta}$. By repeated use of (4.2) we write

$$M^{\mu} = -e_B^3\gamma^{\mu}\int^{\Lambda}\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2)^2}$$

The integral is like that in the previous computation, so at the end we have

$$M^{\mu} = -i\frac{e_B^3}{16\pi^2}\gamma^{\mu}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

The divergence is logarithmic, as power counting shows in eq. (4.14) and the meaning of μ^2 is the same discussed in the previous paragraph.

In order to calculate the vertex renormalization constant Z_1 at 1–loop, we have to sum this graph with the tree level vertex, obtaining

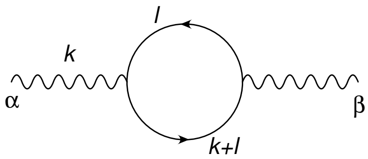
$$-ie_B\gamma^{\mu}Z_1^{-1} \equiv (-ie_B\gamma^{\mu}) + \left(-i\frac{e_B^3}{16\pi^2}\gamma^{\mu}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right)$$

from which at the end we read ($Z_1 \equiv 1+\delta_1$)

$$\begin{aligned} Z_1^{-1} &= 1+\frac{\alpha_B}{4\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)+\mathcal{O}(\alpha_B^2) \\ \delta_1 &= -\frac{\alpha_B}{4\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)+\mathcal{O}(\alpha_B^2) \end{aligned} \tag{4.16}$$

4.3 Photon propagator

The one loop contribution to the photon propagator in the Feynman gauge is



$$= -\left(-ie_B\right)^2 \mathrm{Tr}\int^{\Lambda}\frac{d^4\ell}{(2\pi)^4}\gamma^{\alpha}\frac{i}{\not\ell}\gamma^{\beta}\frac{i}{\not\ell+\not k} \tag{4.17}$$

which seems to diverge quadratically. We will show that this divergence is instead logarithmic. The contribution to the propagator coming out (4.17) is only transverse, due to Ward identity, which means that if we contract this integral with k_{α} or k_{β} we do obtain zero. This means that the structure of the integral can be summarized as follows

$$-\left(-ie_B\right)^2 \mathrm{Tr}\int^{\Lambda}\frac{d^4\ell}{(2\pi)^4}\gamma^{\alpha}\frac{i}{\not\ell}\gamma^{\beta}\frac{i}{\not\ell+\not k} = B\left(k^{\alpha}k^{\beta}-k^2g^{\alpha\beta}\right) \tag{4.18}$$

Since we want to calculate (4.17) this is equivalent to calculate B in (4.18). We also note that since B is dimensionless it can only depend on the ratio Λ^2/k^2. We contract with $g^{\alpha\beta}$ and use the identity (4.2) finding

$$2e_B^2\mathrm{Tr}\int^{\Lambda}\frac{d^4\ell}{(2\pi)^4}\frac{1}{\not\ell}\frac{1}{\not\ell+\not k} = -3Bk^2 \tag{4.19}$$

The term which gives the dimensionful external scale is k and it appears in both the two sides of (4.19). If we would now to simplify the calculation of the integral we should eliminate k in the second side leaving it only in the integral. This can be achieved acting on the equation with the derivative ∂_{k_α} and ∂_{k_β}, using (4.4) we find

$$\begin{aligned}
-3B\partial_{k_\alpha}\partial_{k_\beta}k^2 &= 2e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\partial_{k_\alpha}\partial_{k_\beta}\frac{1}{\not{\ell}+\not{k}} \\
-6Bg^{\alpha\beta} &= 2e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\gamma_\beta\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}+\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}\gamma_\beta\frac{1}{\not{\ell}+\not{k}}\right)
\end{aligned}$$

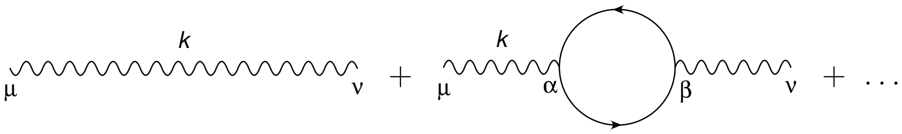
At this point we contract the equation with $g_{\alpha\beta}$ obtaining

$$\begin{aligned}
-24B &= 4e_B^2\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\gamma^\alpha\frac{1}{\not{\ell}+\not{k}}\gamma_\alpha\frac{1}{\not{\ell}+\not{k}}\right) \\
B &= \frac{e_B^2}{3}\mathrm{Tr}\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{\not{\ell}}\left(\frac{1}{\not{\ell}+\not{k}}\right)^3
\end{aligned} \tag{4.20}$$

We can again make the approximation $\ell \gg k$. Recalling that $\not{\ell}^{-1}\not{\ell}^{-1} = 1/(\ell^2)$ and after the trace, one gets

$$B = \frac{4}{3}e_B^2\int^\Lambda \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2)^2} \tag{4.21}$$

where the integral is again the same as in the previous calculations. This one loop integral contributes to the photon propagator when we sum it to the first piece. We have the structure



(4.22)

which corresponds to

$$-i\frac{g^{\mu\nu}}{k^2}+\left(-i\frac{g^{\mu\alpha}}{k^2}\right)B(k_\alpha k_\beta - k^2 g_{\alpha\beta})\left(-i\frac{g^{\beta\nu}}{k^2}\right)+\ldots \tag{4.23}$$

Having instead made the calculation in Lorentz gauge we would have used the tree level propagator with terms containing $k^\mu k^\nu(1-\lambda)$ and this would have given in (4.23) terms proportional to $k^\mu k^\nu$. Nevertheless, the result would be the same since these terms do not give any contribution because of gauge invariance. In fact the sum (4.21) will be connected to an external conserved current or to a polarization vector. In both case we will hit the $k_\mu\ k_\nu$ term with a current or a polarization vector and in both cases this will give a null contribution. We thus obtain from (4.23)

$$-i\frac{g^{\mu\nu}}{k^2}(1+iB) \equiv -i\frac{g^{\mu\nu}}{k^2}\,Z_3 \tag{4.24}$$

Using (4.8) and applying it to (4.21) we get ($Z_3 \equiv 1 + \delta_3$)

$$\begin{aligned} Z_3 &= 1 - \frac{4}{3}\frac{\pi^2}{(2\pi)^4}\, e_B^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) = 1 - \frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right) \\ \delta_3 &= -\frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right) \end{aligned} \tag{4.25}$$

For the photon propagator it is important to take a look also at the result that we obtain if we try to sum all the radiative corrections. This corresponds to sum all the graphs made by connecting more and more 1–particle irreducible Feynman graphs. We obtain a geometric series like

$$-i\frac{g^{\mu\nu}}{k^2}\sum_{n=0}^{+\infty}(iB)^n = -i\frac{g^{\mu\nu}}{k^2}\frac{1}{1-iB} \tag{4.26}$$

where B is the 1–PI graph and at one loop it is exactly our old B.

Despite its obviousness, equation (4.26) shows a fundamental property of QED: the photon remains exactly massless even after higher order quantum corrections are considered. In fact the pole is again at $k^2 = 0$, i.e. it is not displaced by radiative corrections. This is is an example of a fundamental property of Quantum Field Theories: an exact symmetry of the Lagrangian (here the gauge symmetry) has deep consequences also on the way radiative corrections manifest themselves. As we have just seen the local $U(1)$ of QED forbids the photon to acquire a mass after quantum corrections, forcing the propagator structure to be transverse and lowering the degree of divergence from 2 to 0. Another well known example is the (global) chiral symmetry of the Dirac massless Lagrangian that forces the fermion field to stay massless even after loop corrections: from this argument follows that if the mass is present in $\mathcal{L}_{Dirac}$ the self energy has to bee proportional to the mass itself, forcing again the divergences of the one loop self energy to be logarithmical and not linear, as power counting would tell.

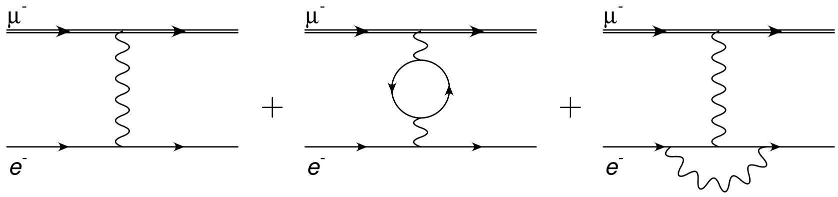
4.4 The Lehmann–Symanzik–Zimmermann (LSZ) formula

4.5 The running of the coupling constant

We begin analyzing the physical meaning of previous calculations, by recalling the definition of renormalization constants Z_i (see (4.12), (4.16) and (4.25))

$$Z_1 \equiv 1 + \delta_1 \tag{4.27}$$
$$Z_2 \equiv 1 + \delta_2 \tag{4.28}$$
$$Z_3 \equiv 1 + \delta_3 \tag{4.29}$$

If we consider $e^-\mu^- \to e^-\mu^-$ scattering and compute higher order virtual corrections to the amplitude, at order e_B^4, we have to sum the following graphs



(4.30)

Box corrections (that are not UV divergent) and the vertex correction at the muon vertex are not shown. This last correction will play a role when considering the renormalization of the muon charge. We then considered only the radiative corrections on the lower half of the diagrams (on the electron part of the amplitude) that means that in the following we will think only to the prediction of the theory for the physical measurable electron charge e_P. From this it follows also that the second graph will contribute with a one half factor (or square root factor). We thus have that the sum of the graphs goes like

$$\begin{aligned} &\sim \; e_B \left(1 + \frac{1}{2}\,\delta_3 - \delta_1 + \delta_2 + \delta_2\right) \\ &\sim \; e_B\, Z_3^{1/2}\, Z_2^2\, Z_1^{-1} \end{aligned} \tag{4.31}$$

In addition, a factor $\left(Z_2^{1/2}\right)^2$ has been added, to comply with the LSZ formula[4]. In this case, we're dealing with external fermions (electrons), so we must multiply twice by $Z_2^{-1/2}$. We note that this gives exactly the same result that one would obtain adding only connected diagrams shorn of self energy corrections on external legs and multiplying this result with a $Z_i^{1/2}$ factor for every external leg of type "i", as in the standard LSZ formula. In both cases the correct answer for the amplitude is

$$M \sim e_B\, Z_3^{1/2}\, Z_2\, Z_1^{-1} = e_B\, Z_3^{1/2} \tag{4.32}$$

where we have used the fact that at the first order our calculation gives $Z_1 = Z_2$ [5]. Equation (4.32) also suggests us that in some sense the renormalization of QED is related only to the correction of the photon self energy (Z_3): we will come back on this at the end of this section.

From all these considerations, we are now left with something proportional to $e_B\sqrt{Z_3}$ and this will be our definition for the physical electron charge e_P, since the cross section we would obtain from the amplitude contains a $(e_B\sqrt{Z_3})^2$ factor and the cross section is the link between theory and measurable quantities. Thus we define[6]

$$e_P = \sqrt{Z_3}\, e_B \tag{4.33}$$

[4] The usual conventions are $i = 2$ for fermions and $i = 3$ for gauge bosons.

[5] As we shall see later on, this equality holds at all orders, by virtue of Ward identities.

[6] Instead of e_P, usually one calls this quantity the *renormalized* charge e_R but in this part we will continue to use e_P in order to remind that this is the value that in the theory has the meaning of measurable, physical electron charge.

From the previous equation it is now easy to see how the physical charge varies with the energy scale μ: at the scale μ^2 its value is

$$e_P(\mu^2) = e_B \left\{ 1 - \frac{2}{3} \frac{\alpha_B}{2\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) \right\}^{1/2} \simeq e_B \left\{ 1 - \frac{1}{3} \frac{\alpha_B}{2\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) \right\} \tag{4.34}$$

while at the scale μ_0^2

$$e_P(\mu_0^2) = e_B \left\{ 1 - \frac{1}{3} \frac{\alpha_B}{2\pi} \log\left(\frac{\Lambda^2}{\mu_0^2}\right) \right\}. \tag{4.35}$$

The difference is thus no more dependent on the cutoff scale Λ

$$e_P(\mu^2) - e_P(\mu_0^2) = e_B \frac{1}{3} \frac{\alpha_B}{2\pi} \left\{ \log\left(\frac{\Lambda^2}{\mu_0^2}\right) - \log\left(\frac{\Lambda^2}{\mu^2}\right) \right\} = \frac{e_B^3}{24\pi^2} \log\left(\frac{\mu^2}{\mu_0^2}\right). \tag{4.36}$$

Now we are free to replace the bare charge e_B with the physical (renormalized) one e_P in the right hand side of the previous equation, up to terms of higher order, finding

$$e_P(\mu^2) - e_P(\mu_0^2) = \frac{e_P^3}{24\pi^2} \log\left(\frac{\mu^2}{\mu_0^2}\right) + \mathcal{O}\left(e_P^4\right). \tag{4.37}$$

The running of the coupling constant is thus

$$e_P(\mu^2) = e_P(\mu_0^2) + \frac{e_P^3}{24\pi^2} \log\left(\frac{\mu^2}{\mu_0^2}\right) + \mathcal{O}\left(e_P^4\right). \tag{4.38}$$

The previous formula is very important since, given the value of the physical charge at one fixed scale μ_0, one can extrapolate the new e_P value at any other scale μ, keeping in mind that one have to remain in the perturbative regime.

Before going on, a remark on the way we introduced the scale μ is due: μ^2 was a scale of the order of the external momenta of the legs of which we calculated the radiative corrections. In particular, looking at the graphs we added (eq. (4.30)), for the photon self energy and the vertex corrections we can think at μ^2 as the off–shellness of the virtual exchanged photon, i.e. the typical scale of the process.

Keeping in mind all these observations, eq. (4.38) tells us a fundamental unexpected thing: if we make two measurements for a process involving the electron charge at different energies and we want to predict the correct result, we have to use different values for the electron charge itself. In this sense we can also say that the constant e_P is no longer a constant but it runs in a way predicted by the theory. It is also clear that for the theory to be predictive it is needed to fix the value of the constant at one scale[7] and then use (4.38) to extract the corresponding value at another scale and use it in the computation.

To obtain the running of the renormalized coupling α_R one can proceed in a slightly different way: it is useful to see how it works because we will use the following argument to

[7]Typically in QED one fix the fine-structure constant α to be equal to the low energy measured value $\approx 1/137$ at the scale $\mu^2 = m_e^2$.

obtain the running of the QCD coupling using dimensional regularization. Obviously, since the theory is the same, the results will be equal to that expressed in (4.38).

The starting point is to consider that if we square (4.33) and then extract the value of α_B as a function of α_R we obtain

$$\alpha_B \;=\; Z_3^{-1}\alpha_R \tag{4.39}$$

The left hand side of this equation can not depend on the renormalization scale[8] μ because e_B was a completely free parameter in the initial Lagrangian. Furthermore, since Z_3 depends on μ, also α_R has to be a function of μ in order to have a meaningful equation. Taking the full derivative of the previous equation with respect to $\log\mu^2$ we obtain

$$\begin{aligned} 0=\frac{d\alpha_B}{d\log\mu^2} &= \frac{d}{d\log\mu^2}\left\{\left[1+\frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right]\alpha_R\left(\mu^2\right)\right\} \\ &= -\frac{2}{3}\frac{\alpha_B}{2\pi}\alpha_R\left(\mu^2\right)+\left[1+\frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right]\frac{d\alpha_R\left(\mu^2\right)}{d\log\mu^2}\,. \end{aligned} \tag{4.40}$$

Solving the previous equation and recalling the definition of the beta function

$$\beta(\alpha_R(\mu^2))=\frac{d\alpha_R\left(\mu^2\right)}{d\log\mu^2} \tag{4.41}$$

one gets the QED beta function at the leading order

$$\beta(\alpha_R(\mu^2))=\frac{2}{3}\frac{\alpha_B}{2\pi}\left[1+\frac{2}{3}\frac{\alpha_B}{2\pi}\log\left(\frac{\Lambda^2}{\mu^2}\right)\right]^{-1}\alpha_R\left(\mu^2\right)=\frac{2}{3}\frac{\alpha_R^2\left(\mu^2\right)}{2\pi}+\mathcal{O}\left(\alpha_R^3\right). \tag{4.42}$$

4.6 Ward identities

The Ward–Takahashi identity states that there is a relation between the vertex $-i\Gamma^\mu$ and the inverse fermion propagator S_F^{-1}, holding at every order in perturbation theory

$$-ik_\mu\Gamma^\mu(p+k,p)=S_F^{-1}(p+k)-S_F^{-1}(p) \tag{4.43}$$

In the limit of soft photon momentum $k\to 0$, we have

$$-i\Gamma^\mu \;=\; (-i\gamma^\mu)Z_1^{-1} \tag{4.44}$$

while the all order propagator are

$$S_F(p) \;=\; \frac{i}{\not p-m}Z_2 \tag{4.45}$$

so we can prove $Z_1=Z_2$ at all order by simply substituting into (4.43), obtaining

$$-i\not k Z_1^{-1}=-iZ_2^{-1}\left[\not p+\not k-m-\not p+m\right]\;\Rightarrow\; Z_1=Z_2. \tag{4.46}$$

[8]The only scale from which α_B can depend is the scale Λ that in some sense was present from the beginning, being the scale up to which the theory is valid. It was the maximum scale at which the Lagrangian is supposed to be correct.

4.7 QED renormalization in short

A straightforward way to understand QED renormalization (and in particular the fact that it depends only on the renormalization constantZ_3) is to consider the bare Lagrangian

$$\mathcal{L} \;=\; \bar{\psi}_B i\partial\!\!\!/ \psi_B - \frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B - e_B \bar{\psi}_B A\!\!\!/_B \psi_B - m_B \bar{\psi}_B \psi_B$$

and look how it is changed after the following rescaling ...

Chapter 5

QCD renormalization

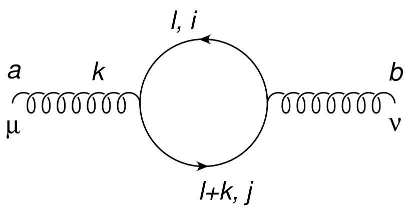
As in QED we proceed through the usual fields strength renormalization, which means that we introduce a set of renormalization constants Z_i such that bare fields are related to renormalized ones. Before doing that we shall show how each divergent contribution obtained using bare fields can be correctly evaluated in the context of dimensional regularization. This regularization prescription consists in doing all the calculations after having analytically continued the number of spacetime dimensions to a value in which integrals does converge. In doing so, in order to maintain the coupling constants dimensionless, one is forced to introduce a dimensionful parameter μ which takes account for the extra dimensions keeping the action dimensionless.

In a d-dimension space-time, the dimension of the Lagrangian is d. It follows from the analysis of the kinetic terms, that the dimension of the fermionic field ψ is $(d-1)/2$, and the dimension of the gauge field A^ν is $(d-2)/2$. The coupling constant than has dimension of $(4-d)/2$. For this reason, in order to deal with a dimensionless coupling constant, one explicitly add a mass parameter μ every time the coupling constant appears. In $d = 4 - 2\epsilon$, one then replaces

$$g \;\to\; g\mu^\epsilon \,. \tag{5.1}$$

5.1 Gluon self-energy

There can be four possible contributions to the gluon self energy at one loop. Let's analyze them separately. The first one is the fermion loop contribution, which gives, after the sum over the n_f flavors that can run in the loop,



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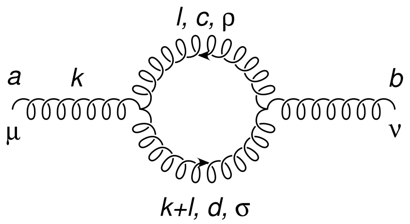
$$
\begin{aligned}
&= -n_f \int \frac{d^d\ell}{(2\pi)^d} \mathrm{Tr}\left[\left(-ig\mu^\epsilon t^a_{ij}\right)\gamma^\mu \frac{i}{\not\ell + \not k}\left(-ig\mu^\epsilon t^b_{ji}\right)\gamma^\nu \frac{i}{\not\ell}\right] \\
\ldots &= -g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ \mathrm{Tr}\{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\} \int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell+k)^\alpha \ell^\beta}{\ell^2 (\ell+k)^2} \\
&= -g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ 4\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right]\left[\mathcal{B}^{\alpha\beta}(k) + k^\alpha \mathcal{B}^\beta(k)\right] \\
&= -g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ 4\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right] \\
&\qquad \left[\frac{dB_0(k)}{4(d-1)}k^\alpha k^\beta - \frac{k^2 B_0(k)}{4(d-1)}g^{\alpha\beta} - \frac{B_0(k)}{2}k^\alpha k^\beta\right] \\
&= g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ \frac{B_0(k)}{(d-1)}\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right]\left[k^2 g^{\alpha\beta} + (d-2)\, k^\alpha k^\beta\right] \\
&= g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ \frac{B_0(k)}{(d-1)} 2\,(d-2)\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \qquad (5.2)
\end{aligned}
$$

If we now put $d = 4 - 2\epsilon$ and expand $B_0(k)$ around it's pole at $\epsilon = 0$ using (1.32) we find

$$
\begin{aligned}
\ldots &= g^2\mu^{2\epsilon} n_f\ T_{\mathrm{F}}\ \delta^{ab}\ \frac{2\,(2-2\epsilon)}{(3-2\epsilon)} \frac{i}{(4\pi)^2} \frac{C_\Gamma\ e^{i\pi\epsilon}}{\epsilon\,(1-2\epsilon)} \left(k^2\right)^{-\epsilon}\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \\
&\approx i\frac{\alpha_{\mathrm{S}}}{4\pi}\delta^{ab}\left(-\frac{4}{3} n_f T_{\mathrm{F}} \frac{1}{\epsilon}\right)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) + O\left(\epsilon^0\right) \qquad (5.3)
\end{aligned}
$$

This is exactly the same result, except for the color factor C_{F} and the Kronecker's δ, one would have find in QED for the photon propagator, if he would have performed the calculation within dimensional regularization scheme.

The second contribution comes from a gluon loop of this kind



$=$

$$
\begin{aligned}
= \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d} &\left(-g\mu^\epsilon f^{acd}\left[g^{\mu\rho}(k-\ell)^\sigma + g^{\rho\sigma}(2\ell+k)^\mu + g^{\sigma\mu}(-2k-\ell)^\rho\right]\right) \cdot \\
&\cdot\left(-g\mu^\epsilon f^{d'c'b}\left[g^{\sigma'\rho'}(k+2\ell)^\nu + g^{\rho'\nu}(-\ell+k)^{\sigma'} + g^{\nu\sigma'}(-2k-\ell)^{\rho'}\right]\right) \cdot \\
&\cdot \frac{-ig_{\sigma\sigma'}\delta^{dd'}}{(k+\ell)^2}\ \frac{-ig_{\rho\rho'}\delta^{cc'}}{\ell^2}
\end{aligned}
$$

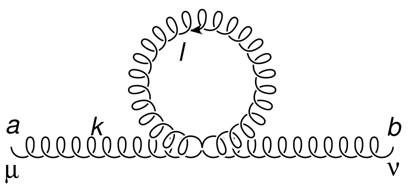
where the meaning of primed indexes, that are not graphically represented, is trivial because they are related to internal gluon propagator lines. The one half factor in front of the integral comes instead from the symmetry factor of the Feynman graph. One has a factor $\left(\frac{1}{3!}\right)^2$ for vertices orientations, a factor 3 for possible contraction between one external leg and the first vertex, a factor 2 for vertices interchange and a final factor $3 \cdot 2$ for internal vertex contractions. All this factors cancels out in the product, but there is also the $\frac{1}{2!}$ term from the Taylor expansion of the functional integral, which remains. Using the same argument it's trivial to show that the symmetry factor for fermion or even ghost loops is 1. There, indeed, there is only the $\frac{1}{2!}$ factor from Taylor expansion to be multiplied by the 2 possible vertices interchange, since fermion (ghost) or antifermion (antighost) internal legs are distinguishable. Returning to our calculation one has

$$\begin{aligned}
\ldots &= \frac{1}{2} g^2 \mu^{2\epsilon} C_{\mathrm{A}} \delta^{ab} \int \frac{d^d\ell}{(2\pi)^d} \left[g^{\mu\rho} (k-\ell)^\sigma + g^{\rho\sigma} (2\ell+k)^\mu + g^{\sigma\mu} (-2k-\ell)^\rho \right] \cdot \\
&\quad \cdot \left[g_{\sigma\rho} (k+2\ell)^\nu + g_\rho{}^\nu (-\ell+k)_\sigma + g_\sigma{}^\nu (-2k-\ell)_\rho \right] \frac{1}{\ell^2 (\ell+k)^2} \\
&= g^2 \mu^{2\epsilon} \frac{C_{\mathrm{A}}}{2} \delta^{ab} \int \frac{d^d\ell}{(2\pi)^d} \left[(d-6) k^\mu k^\nu + 5k^2 g^{\mu\nu} + (2d-3) k^\mu \ell^\nu + \right. \\
&\quad \left. + (2d-3) k^\nu \ell^\mu + (4d-6) \ell^\mu \ell^\nu + 2g^{\mu\nu} k_\alpha \ell^\alpha + 2g^{\mu\nu} \ell^2 \right] \frac{1}{\ell^2 (\ell+k)^2} \\
&= g^2 \mu^{2\epsilon} \frac{C_{\mathrm{A}}}{2} \delta^{ab} \left[B_0(k) \left((d-6) k^\mu k^\nu + 5k^2 g^{\mu\nu} \right) + (2d-3) k^\mu \mathcal{B}^\nu(k) + \right. \\
&\quad \left. + (2d-3) k^\nu \mathcal{B}^\mu(k) + (4d-6) \mathcal{B}^{\mu\nu}(k) + 2g^{\mu\nu} k_\alpha \mathcal{B}^\alpha(k) + 2g^{\mu\nu} A_0 \right] \\
&= g^2 \mu^{2\epsilon} \frac{C_{\mathrm{A}}}{2} \delta^{ab} B_0(k) \left[(d-6) k^\mu k^\nu + 5k^2 g^{\mu\nu} - (2d-3) k^\mu k^\nu - k^2 g^{\mu\nu} + \right. \\
&\quad \left. + \frac{4d-6}{4(d-1)} \left(d\, k^\mu k^\nu - k^2 g^{\mu\nu} \right) \right] \\
&= g^2 \mu^{2\epsilon} \frac{C_{\mathrm{A}}}{2} \delta^{ab} B_0(k) \left[-\frac{7d-6}{2(d-1)} k^\mu k^\nu + \frac{6d-5}{2(d-1)} k^2 g^{\mu\nu} \right]. \\
&= g^2 \mu^{2\epsilon} C_{\mathrm{A}} \delta^{ab} \frac{B_0(k)}{4(d-1)} \left[(6-7d) k^\mu k^\nu + (6d-5) k^2 g^{\mu\nu} \right]. \qquad (5.4)
\end{aligned}$$

At this point one proceed exactly as before, substituting $d = 4 - 2\epsilon$ and expanding around the pole

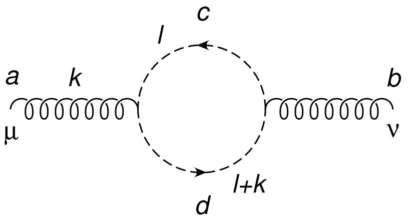
$$\begin{aligned}
\ldots &= g^2 \mu^{2\epsilon} \frac{C_{\mathrm{A}}}{4} \delta^{ab} \frac{i}{(4\pi)^2} \frac{C_\Gamma \, e^{i\pi\epsilon}}{\epsilon (1-2\epsilon)} \left(k^2\right)^{-\epsilon} \left[-\frac{22-14\epsilon}{3-2\epsilon} k^\mu k^\nu + \right. \\
&\quad \left. \frac{19-12\epsilon}{3-2\epsilon} k^2 g^{\mu\nu} \right] \\
&\approx i \frac{\alpha_{\mathrm{S}}}{4\pi} C_{\mathrm{A}} \delta^{ab} \, \frac{1}{\epsilon} \left[\frac{19}{12} k^2 g^{\mu\nu} - \frac{11}{6} k^\mu k^\nu \right] + O\left(\epsilon^0\right) \qquad (5.5)
\end{aligned}$$

There's another graph involving gluons loop but it's contribution is zero in the massless limit since it's proportional to A_0 and there is no scale involved (see (1.29)).



$$\sim \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2} = A_0 = 0 \qquad (5.6)$$

The last, but not the least, graph to be considered is the ghost loop



$$=$$

$$
\begin{aligned}
&= -\int \frac{d^d\ell}{(2\pi)^d}\, g\mu^\epsilon f^{acd}\,(\ell+k)^\mu\, g\mu^\epsilon f^{bdc}\ell^\nu \frac{i}{\ell^2}\frac{i}{(\ell+k)^2} \\
&= -g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu + k^\mu\ell^\nu}{\ell^2\,(\ell+k)^2} \\
&= -g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}\left[\mathcal{B}^{\mu\nu}(k) + k^\mu\mathcal{B}^\nu(k)\right] \\
&= -g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}B_0(k)\left[\frac{d\; k^\mu k^\nu}{4\,(d-1)} - \frac{k^2 g^{\mu\nu}}{4\,(d-1)} - \frac{k^\mu k^\nu}{2}\right] \\
&= g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}\frac{B_0(k)}{4\,(d-1)}\left[k^2 g^{\mu\nu} + (d-2)\,k^\mu k^\nu\right] \qquad (5.7)
\end{aligned}
$$

Expanding eq. (5.7) around $\epsilon = 0$, in $d = 4-2\epsilon$ dimensions, we get

$$
\begin{aligned}
\ldots &= g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}\frac{1}{4\,(3-2\epsilon)}\left[k^2 g^{\mu\nu} + (2-2\epsilon)\,k^\mu k^\nu\right]\frac{i}{(4\pi)^2}\frac{C_\Gamma\, e^{i\pi\epsilon}}{\epsilon\,(1-2\epsilon)}\left(k^2\right)^{-\epsilon} \\
&\approx i\frac{\alpha_{\rm S}}{4\pi}C_{\rm A}\delta^{ab}\,\frac{1}{\epsilon}\left[\frac{1}{12}k^2 g^{\mu\nu} + \frac{1}{6}k^\mu k^\nu\right] + O\left(\epsilon^0\right) \qquad (5.8)
\end{aligned}
$$

The first important observation that can be made at this point is that by summing the contributions from gluons and ghosts loop one obtain a propagator that is purely transverse. By adding the gluon-loop contribution of eq. (5.4) to the ghost one, we get

$$\ldots = g^2\mu^{2\epsilon}C_{\rm A}\delta^{ab}B_0(k)\frac{2-3d}{2\,(d-1)}\left[k^\mu k^\nu - k^2 g^{\mu\nu}\right]. \qquad (5.9)$$

In an expansion in ϵ this becomes[1]

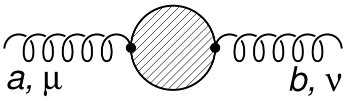
$$(5.5) + (5.8) \;=\; i\frac{\alpha_{\rm S}}{4\pi}C_{\rm A}\delta^{ab}\,\frac{1}{\epsilon}\,\frac{5}{3}\,\left[k^2 g^{\mu\nu} - k^\mu k^\nu\right] + O\left(\epsilon^0\right) \qquad (5.10)$$

Going on and summing the first contribution, i.e. eq. (5.3), one gets the one loop correction to gluon propagator

$$(5.3) + (5.5) + (5.8) \;=\; i\frac{\alpha_{\rm S}}{4\pi}\delta^{ab}\,\frac{1}{\epsilon}\,\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\left[\frac{5}{3}\,C_{\rm A} - \frac{4}{3}n_f T_{\rm F}\right] + O\left(\epsilon^0\right) \qquad (5.11)$$

[1]Strictly speaking one should sum also (5.6), but is is zero.

The full propagator at one loop is thus



$$= \frac{i}{k^2}\delta^{ab}\left(-g^{\mu\nu}\right) + \frac{i}{k^2}\delta^{aa'}\left(-g^{\mu\alpha}\right)\cdot$$

$$\cdot\left[i\frac{\alpha_{\rm S}}{4\pi}\delta^{a'b'}\,\frac{1}{\epsilon}\,\left(k^2 g_{\alpha\beta} - k_\alpha k_\beta\right)\left(\frac{5}{3}\,C_{\rm A} - \frac{4}{3}n_f T_{\rm F}\right)\right]\frac{i}{k^2}\delta^{b'b}\left(-g^{\beta\nu}\right)$$

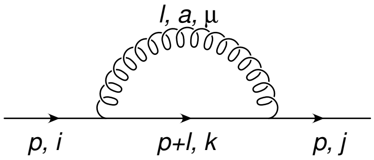
$$= -\frac{i}{k^2}\delta^{ab}g^{\mu\nu}\left[1 + \frac{\alpha_{\rm S}}{4\pi}\,\frac{1}{\epsilon}\left(\frac{5}{3}\,C_{\rm A} - \frac{4}{3}n_f T_{\rm F}\right)\right]. \tag{5.12}$$

The last equality is possible since terms containing k^α or k^β always give zero when the propagator is contracted with a conserved current, as happens in QED. We can now define the renormalization constant Z_3 at one loop as the content of square bracket of (5.12)

$$Z_3 = 1 + \frac{\alpha_{\rm S}}{4\pi}\,\frac{1}{\epsilon}\left(\frac{5}{3}\,C_{\rm A} - \frac{4}{3}n_f T_{\rm F}\right) \tag{5.13}$$

5.2 Quark self-energy

As in the QED case, the one loop self energy of the quark is given only by the following graph:



$$\begin{aligned}
&= \int \frac{d^d\ell}{(2\pi)^d}\left(-ig\mu^\varepsilon t^a_{jk}\gamma^\mu\right)\frac{i}{\not\ell + \not p}\left(-ig\mu^\varepsilon t^a_{ki}\gamma_\mu\right)\frac{-i}{\ell^2} \\
&= -g^2\mu^{2\varepsilon}t^a_{jk}t^a_{ki}\int \frac{d^d\ell}{(2\pi)^d}\gamma^\mu\frac{1}{\not\ell + \not p}\gamma_\mu\frac{1}{\ell^2} \\
&= (d-2)g^2\mu^{2\varepsilon}t^a_{jk}t^a_{ki}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\not\ell + \not p}\frac{1}{\ell^2}
\end{aligned}$$

where in the last equality we have used $\gamma^\mu\gamma^\alpha\gamma_\mu = (2-d)\gamma^\alpha$ that is the d–dimensional generalization of (4.2). Using the properties of colour matrices algebra and the results of the corresponding section, we have

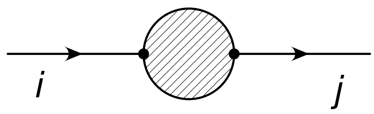
$$\begin{aligned}
\ldots &= (d-2)g^2\mu^{2\varepsilon}\delta_{ij}C_{\rm F}\gamma_\mu\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu + p^\mu}{\ell^2(\ell+p)^2} \\
&= (d-2)g^2\mu^{2\varepsilon}\delta_{ij}C_{\rm F}\gamma_\mu\left[\mathcal{B}^\mu(p) + p^\mu B_0(p)\right] \\
&= (d-2)g^2\mu^{2\varepsilon}\delta_{ij}C_{\rm F}\gamma_\mu\left[-\frac{B_0(p)}{2}p^\mu + p^\mu B_0(p)\right] \\
&= g^2\mu^{2\varepsilon}\delta_{ij}C_{\rm F}\not p\left(\frac{d-2}{2}\right)B_0(p)
\end{aligned}$$

Before going on, we stress that to proceed we have to assume that the quark momenta is such that $p^2 \neq 0$, otherwise this graph would vanish, since massless bubbles are zero in dimensional regularization, as showed in the first section.

As before, we can now expand around the pole at $\varepsilon = 0$:

$$\begin{aligned}
\ldots &= g^2\mu^{2\varepsilon}\delta_{ij}C_{\mathrm{F}}\not{p}(1-\varepsilon)\frac{i}{(4\pi)^2}\frac{C_\Gamma e^{i\pi\varepsilon}}{\varepsilon(1-2\varepsilon)}(p^2)^{-\varepsilon} \\
&\approx g^2\delta_{ij}C_{\mathrm{F}}\not{p}\frac{i}{\varepsilon(4\pi)^2} + O\left(\epsilon^0\right) \\
&= \frac{1}{\varepsilon}\,\frac{\alpha_S}{4\pi}\;C_{\mathrm{F}}\;\delta_{ij}\;i\;\not{p} + O\left(\epsilon^0\right)
\end{aligned}$$

As usual, the final step consists in extracting from this result the right expression for the renormalization constants. We have to sum this virtual correction to the bare propagator, obtaining



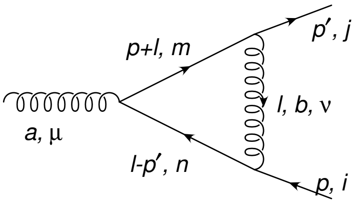
$$\begin{aligned}
&= \delta_{ij}\frac{i}{\not{p}} + \left(\delta_{ik}\frac{i}{\not{p}}\right)\left[\frac{\alpha_S}{4\pi}C_{\mathrm{F}}\delta_{kl}\frac{i\not{p}}{\varepsilon}\right]\left(\delta_{lj}\frac{i}{\not{p}}\right) \\
&= \delta_{ij}\;\frac{i}{\not{p}}\left[1-\frac{1}{\varepsilon}\;\frac{\alpha_S}{4\pi}\;C_{\mathrm{F}}\right]
\end{aligned}$$

From the last equation, we are left with

$$Z_2 = 1 - \frac{\alpha_S}{4\pi}\frac{1}{\varepsilon}C_{\mathrm{F}} \tag{5.14}$$

5.3 Quark-gluon vertex corrections

The one loop corrections to the vertex gqq are given by two diagrams. The first one is



$$\begin{aligned}
&= \int\frac{d^d\ell}{(2\pi)^d}\left(-ig\mu^\epsilon t^b_{jm}\gamma^\nu\right)\frac{i}{\not{\ell}-\not{p}\,'}\cdot \\
&\quad\cdot\left(-ig\mu^\epsilon t^a_{mn}\gamma^\mu\right)\frac{i}{\not{\ell}+\not{p}}\left(-ig\mu^\epsilon t^b_{ni}\gamma_\nu\right)\left(-\frac{i}{\ell^2}\right) \\
&= -\left(C_{\mathrm{F}}-\frac{C_{\mathrm{A}}}{2}\right)t^a_{ji}\;g^3\mu^{3\epsilon}\int\frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\nu\left(\not{\ell}-\not{p}\,'\right)\gamma^\mu\left(\not{\ell}+\not{p}\right)\gamma_\nu}{\ell^2\left(\ell-p\,'\right)^2\left(\ell+p\right)^2}
\end{aligned}$$

where the contribution of the color factors in the integral is estimated using (3.48). From now on, we will make some simplifications, since we're interested only in the UV behavior of the theory. For this reason the momenta p and $p\,'$ will be neglected. Such a dramatic simplification proves to give the correct result anyhow, provided that one keeps in mind that some dimensionful quantity that was present at the beginning is now missing. Neglecting p and $p\,'$ one thus gets

$$-g^3\mu^{3\epsilon}t^a_{ji}\left(C_{\rm F}-\frac{C_{\rm A}}{2}\right)\int\frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\nu\not{\ell}\gamma^\mu\not{\ell}\gamma_\nu}{(\ell^2)^3} \tag{5.15}$$

To calculate the integral in (5.15) we have to make the replacement, valid only inside the integral,

$$\ell_\delta\ell_{\delta'}=\frac{g_{\delta\delta'}}{d}\ell^2, \tag{5.16}$$

obtaining

$$-g^3\mu^{3\epsilon}t^a_{ji}\left(C_{\rm F}-\frac{C_{\rm A}}{2}\right)\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{d}\frac{\gamma^\nu\gamma^\delta\gamma^\mu\gamma_\delta\gamma_\nu}{(\ell^2)^2} \tag{5.17}$$

The product of the five gamma matrices can be reduced to

$$\gamma^\nu\gamma^\delta\gamma^\mu\gamma_\delta\gamma_\nu=(2-d)\gamma^\nu\gamma^\mu\gamma_\nu=(d-2)^2\gamma^\mu \tag{5.18}$$

The resulting integral is thus

$$-g^3\mu^{3\epsilon}t^a_{ji}\left(C_{\rm F}-\frac{C_{\rm A}}{2}\right)\gamma^\mu\frac{(d-2)^2}{d}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2)^2}. \tag{5.19}$$

The last integral is not zero! Actually, unlike the case of A_0, it is not true that it does not carry any physical dimension, it is dimensionless only because of our approximation of neglecting external momenta. Had we performed the full calculation we would have obtained that the integral does depend on a dimensionful parameter Q, made by combination of external momenta. For this reason, in order to reduce the integral to one that it's easily evaluable, we can reintroduce that scale substituting

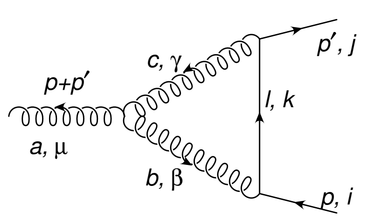
$$\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2)^n}\;\to\;\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-m^2)^n}=(-1)^n\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}(m^2)^{\frac{d}{2}-n} \tag{5.20}$$

In doing so we have recovered the missing dimension in such a way that the divergent behavior of the integral is not altered. This last point is crucial to obtain the correct result.

If we now put $d=4-2\epsilon$, and work in the limit $\epsilon\to 0$ we obtain the final result

$$-igt^a_{ji}\gamma^\mu\frac{\alpha_{\rm S}}{4\pi}\frac{1}{\epsilon}\left(C_{\rm F}-\frac{C_{\rm A}}{2}\right) \tag{5.21}$$

The second diagram gives instead the contribution



$$=\int\frac{d^d\ell}{(2\pi)^d}\left(-g\mu^\epsilon f^{abc}\right)\left[g^{\mu\beta}\left(-2p-p\,'+\ell\right)^\gamma\;+\right.$$

$$+g^{\beta\gamma}\left(-2\ell+p-p\,'\right)^{\mu}+g^{\gamma\mu}\left(\ell+2p\,'+p\right)^{\beta}\Big]$$
$$\frac{-i}{(\ell-p)^2}\frac{-i}{(\ell+p\,')^2}\left(-ig\mu^{\epsilon}\gamma_{\gamma}t^{c}_{jk}\right)\frac{i}{\not\ell}\left(-ig\mu^{\epsilon}\gamma_{\beta}t^{b}_{ki}\right)$$
$$= \int\frac{d^d\ell}{(2\pi)^d}\left(-g\mu^{\epsilon}f^{abc}\right)\left[g^{\mu\beta}\left(-2p-p\,'+\ell\right)^{\gamma}+g^{\beta\gamma}\left(-2\ell+p-p\,'\right)^{\mu}+\right.$$
$$\left.+g^{\gamma\mu}\left(\ell+2p\,'+p\right)^{\beta}\right]\frac{-i}{(\ell-p)^2}\frac{-i}{(\ell+p\,')^2}\left(-ig\mu^{\epsilon}\gamma_{\gamma}t^{c}_{jk}\right)\frac{i}{\not\ell}\left(-ig\mu^{\epsilon}\gamma_{\beta}t^{b}_{ki}\right) \tag{5.22}$$

Since, as shown in (3.50), $if^{abc}t^ct^b=\frac{C_{\mathrm{A}}}{2}t^a$ in front of the integral we have a factor

$$-g^3\mu^{3\epsilon}\frac{C_{\mathrm{A}}}{2}t^a_{ji} \tag{5.23}$$

The momenta p and $p\,'$ can be neglected even in this computations of the integral, provided that the dimensionful parameter that we're casting away will be reintroduced later. So one has

$$\ldots \;=\; -g^3\mu^{3\epsilon}\frac{C_{\mathrm{A}}}{2}t^a_{ji}\int\frac{d^d\ell}{(2\pi)^d}\frac{\left(g^{\mu\beta}\ell^{\gamma}-2g^{\beta\gamma}\ell^{\mu}+g^{\gamma\mu}\ell^{\beta}\right)\gamma_{\gamma}\gamma_{\rho}\gamma_{\beta}\ell^{\rho}}{(\ell^2)^3} \tag{5.24}$$

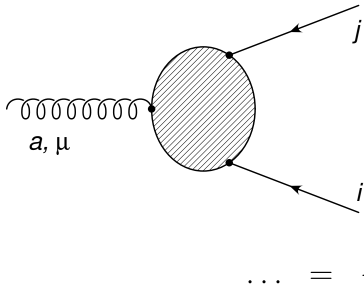
We can now exploit again the replacement (5.16) and also $\gamma^\mu\gamma_\mu=d$. This implies

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\mu+2\frac{(d-2)}{d}\gamma^\mu+\gamma^\mu}{(\ell^2)^2}=\frac{4d-4}{d}\gamma^\mu\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2)^2} \tag{5.25}$$

At this point we proceed exactly as before, substituting the divergent integral as in (5.20), since the problem is the same. The final result is

$$-igt^a_{ji}\gamma^\mu\;\frac{\alpha_{\mathrm{S}}}{4\pi}\;\frac{1}{\epsilon}\;\frac{3}{2}C_{\mathrm{A}} \tag{5.26}$$

Summing now the tree level and the one loop contributions we obtain the gqq vertex at one loop



$$= \;-igt^a_{ji}\gamma^\mu-igt^a_{ji}\gamma^\mu\;\frac{\alpha_{\mathrm{S}}}{4\pi}\;\frac{1}{\epsilon}\left(C_{\mathrm{F}}-\frac{C_{\mathrm{A}}}{2}+\frac{3}{2}C_{\mathrm{A}}\right)$$
$$\ldots \;=\; -igt^a_{ji}\gamma^\mu\left(1+\frac{1}{\epsilon}\;\frac{\alpha_{\mathrm{S}}}{4\pi}(C_{\mathrm{F}}+C_{\mathrm{A}})\right) \tag{5.27}$$

We can now define the renormalization constant Z_1 via

$$Z_1^{-1}\;=\;1+\frac{\alpha_{\mathrm{S}}}{4\pi}\;\frac{1}{\epsilon}\;(C_{\mathrm{F}}+C_{\mathrm{A}}) \tag{5.28}$$

At this point let's take a breath and analyze how the Lagrangian is affected by this higher order corrections. The bare qqg vertex reads

$$g_B \bar{\psi} \not{A} \psi \tag{5.29}$$

where, since we are in $D = 4 - 2\epsilon$, the bare coupling constant g_B is dimensionful. Now, replacing bare fields with renormalized ones in the Lagrangian by virtue of

$$\psi_B = Z_2^{1/2} \psi_R, \qquad A_B^\mu = Z_3^{1/2} A_R^\mu \tag{5.30}$$

we get

$$g_B \; Z_2 \; Z_3^{1/2} \bar{\psi}_R \not{A}_R \psi_R. \tag{5.31}$$

In principle one can extend this argument even at the others vertices [2] of the theory, finding a combination of multiplicative factor for each one of them. Nevertheless these factor are not arbitrarily free, since the BRS symmetry constrain them to combine in such a way that the Lagrangian remains gauge invariant. To cut a long story short, one can demonstrate that there are a set of relations between these vertex correction factors (Slavnov–Taylor identities) by virtue of which one can introduce a single renormalization factor and use a single gauge coupling instead of different ones for different vertices. Calling this common renormalization of the bare gauge coupling Z_g and extracting the dimensionful parameter μ in such a way that the renormalized coupling g_R becomes dimensionless

$$g_B \;\; = \;\; Z_g \, g_R \, \mu^\varepsilon \tag{5.32}$$

one gets that the quark–gluon vertex in the Lagrangian becomes now

$$g_R \, \mu^\varepsilon \, Z_g \, Z_2 \, Z_3^{1/2} \bar{\psi}_R \not{A}_R \psi_R. \tag{5.33}$$

But, from the evaluation of one loop corrections, we know that the qqg vertex gets Z_1^{-1} as a correction factor. Hence, since what we can actually measured can only be the vertex, we ask that all the divergent factors obtained by rescaling fields must be cancelled by the multiplicative vertex correction factor Z_1^{-1} just computed. In this way when one extract a physical quantity from a measurement he gets a finite number. This correspond in this case at the definition

$$Z_1 \;\; = \;\; Z_g \, Z_2 \, Z_3^{1/2} \tag{5.34}$$

Returning now to the Z_g definition we find that, using (5.13), (5.14) and (5.28) the renormalization constant for the gauge coupling is

$$\begin{aligned} Z_g \;\; = \;\; \frac{Z_1}{Z_2 Z_3^{1/2}} &= 1 - \frac{\alpha_{\mathrm{S}}}{4\pi} \frac{1}{\epsilon} (C_{\mathrm{F}} + C_{\mathrm{A}}) + \frac{\alpha_{\mathrm{S}}}{4\pi} \, \frac{1}{\epsilon} C_{\mathrm{F}} - \frac{1}{2} \, \frac{\alpha_{\mathrm{S}}}{4\pi} \frac{1}{\epsilon} \left(\frac{5}{3} C_{\mathrm{A}} - \frac{4}{3} n_F T_f \right) \\ &= 1 - \frac{\alpha_{\mathrm{S}}}{4\pi} \frac{1}{\epsilon} \left(\frac{11}{6} C_{\mathrm{A}} - \frac{4}{6} n_f T_f \right) \equiv 1 - \frac{\alpha_{\mathrm{S}}}{\epsilon} \, \frac{b_0}{2} \end{aligned} \tag{5.35}$$

where

$$b_0 = \frac{1}{12\pi} \left(11 C_{\mathrm{A}} - 4 n_f T_f \right) \tag{5.36}$$

[2] In QCD these are the 3–gluon, 4–gluon and ghost-gluon vertex

5.4 The running coupling constant and the β function

After the long calculations performed in previous sections, it's time to point out, as done in the QED case, what is the underlying physical meaning.

First of all, we resume here the results just found separating the one loop corrections as δ's (see equations (5.13), (5.14), and (5.28)):

$$
\begin{aligned}
Z_1 &= 1+\delta_1 & \delta_1 &= -\frac{\alpha_{\rm S}}{4\pi}\,\frac{1}{\varepsilon}\left(C_{\rm F}+C_{\rm A}\right) \\
Z_2 &= 1+\delta_2 & \delta_2 &= -\frac{\alpha_{\rm S}}{4\pi}\,\frac{1}{\varepsilon}\,C_{\rm F} \\
Z_3 &= 1+\delta_3 & \delta_3 &= \frac{\alpha_{\rm S}}{4\pi}\,\frac{1}{\epsilon}\left(\frac{5}{3}\,C_{\rm A}-\frac{4}{3}n_f T_{\rm F}\right)
\end{aligned}
\tag{5.37}
$$

Moreover, at the end of the last section it was shown that the coupling renormalization constant $Z_g = (1+\delta_g)$ is related to the others Z's through (5.35), so that

$$
\delta_g = -\frac{\alpha_{\rm S}}{\epsilon}\,\frac{b_0}{2}
\tag{5.38}
$$

Knowing the form of renormalization constant for the coupling Z_g, it's easy to write the relation between bare and renormalized α's as

$$
\alpha_B = Z_g^2\,\mu^{2\varepsilon}\,\alpha_R
\tag{5.39}
$$

A very important observation that can be made at this point is that α_B must be blind with respect to a change in renormalization scale μ. This result is somehow expected since μ is a parameter introduced in order to keep α_R dimensionless, while in general α_B can depend on some other physical scale Λ. This implies that $\alpha_R = \alpha_R\left(\mu^2\right)$ must depend on this parameter μ that can be thought as the scale at which we are studying a process or we are making a measure in an experiment.

In order to work out the μ dependence of α_R we compute the beta–function. Deriving (5.39) with respect to $\log(\mu^2)$ one gets:

$$
\begin{aligned}
0 &= 2\mu^2\alpha_R\frac{dZ_g}{d\mu^2} + \epsilon Z_g\alpha_R + \mu^2 Z_g\frac{d\alpha_R}{d\mu^2} \\
&= \left(1+2\,\frac{\alpha_R}{Z_g}\,\frac{dZ_g}{d\alpha_R}\right)\beta\left(\alpha_R\right) + \epsilon\alpha_R
\end{aligned}
\tag{5.40}
$$

where the beta–function is defined by

$$
\beta\left(\alpha_R\right) = \mu^2\frac{d\alpha_R}{d\mu^2}.
\tag{5.41}
$$

With easy algebra and using (5.39) we find that at one loop order:

$$\beta\left(\alpha_R\right)=-b_0\,\alpha_R^2+\mathcal{O}\left(\alpha_R^3\right) \tag{5.42}$$

Solving this differential equation one has:

$$-b_0\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}=\int_{\alpha_R\left(\mu_0^2\right)}^{\alpha_R\left(\mu^2\right)}\frac{d\alpha_R}{\alpha_R^2} \tag{5.43}$$

which implies

$$\alpha_R\left(\mu^2\right)=\frac{\alpha_R\left(\mu_0^2\right)}{1+\alpha_R\left(\mu_0^2\right)b_0\log\left(\frac{\mu^2}{\mu_0^2}\right)} \tag{5.44}$$

Several remarks are now necessary. First of all, in (5.44) we can read the relation between the value of the coupling constant at different scales in the perturbative regime, but it's important to remember that the particular functional form of such a relation (5.44) is obtained as a first order approximation, hence it's validity is limited at that order.

A second observation concerns the role of the regularization parameter ϵ. In QED, in order to regularize the theory, we introduced an UV cut–off Λ that disappeared as soon as one look at physical quantities such the coupling constant. This permitted us to take the limit $(\Lambda\to\infty)$ safely. Remembering that, it's not surprising that the same thing happens to ϵ parameter in dimensional regularization: though the renormalization constants Z's strongly depends on ϵ, this parameter disappears when we look for physical quantities.

Last but not least, one can use (5.44) to find the limit of validity of calculations performed: it is clear that if at the scale μ one has $\alpha_R\left(\mu^2\right)\sim 1$, next to leading corrections become as important as leading order ones, hence perturbative expansion loses sense. So, it is crucial to understand how $\alpha_R\left(\mu^2\right)$ varies with μ^2. But the sign of the beta–function gives us these informations: let's suppose to fix $\alpha_R\left(\mu_0^2\right)$ at a scale $\mu_0^2<\mu^2$ from a physical measurement. Then (5.44) tells us that $\alpha_R\left(\mu^2\right)>\alpha_R\left(\mu_0^2\right)$ if b_0 is negative, i.e. if $\beta\left(\alpha_R\right)$ has positive sign. This is analogue to what happens in QED.

In a non abelian gauge theory with N_f flavors, instead, we see from (5.35) that this only happens if $N_f>\frac{11}{2}N_c$. In this regime the theory has an IR fixed point and perturbative calculations can be made only in this energy region since the coupling constant grow up with the scale.

In the opposite regime, characterized by $N_f<\frac{11}{2}N_c$, the beta–function is negative and we have an UV fixed point. This phenomenon is called asymptotic freedom and it means that the perturbative series can be trusted only at high energy while at low energy the theory is strongly coupled, so that any PT expansion loses sense.

The third possibility is that $N_f=\frac{11}{2}N_c$. In this case the beta–function is always zero and the theory exhibits conformal invariance. In fact in this regime $\alpha_R\left(s\mu^2\right)=\alpha_R\left(\mu^2\right)$ where

s is a scaling factor. So, if we are in weak coupling regime at the scale μ^2, we can apply perturbative theory to any other energy scale.

Experiments show up that a theory that aims to describe strong force effects must be strongly coupled at low energy and weakly coupled at high energy. Fortunately, this happens in QCD: here we have $N_c = 3$ colors and $N_f = 6$ flavors, so that $N_f < \frac{11}{2} N_c$ and we have an UV asymptotically free and IR strongly coupled theory.

At this point of the analysis it is important to fix an energy scale with respect to which define the UV and IR phases. We take this scale as the scale μ_0^2 at which the coupling constant $\alpha_R(\mu_0^2)$ diverges and we call it Λ^2_{QCD}. From (5.44) we see that

$$\alpha_R(\mu^2) = \left[b_0 \, \log \left(\frac{\mu^2}{\Lambda^2_{\text{QCD}}} \right) \right]^{-1} \tag{5.45}$$

Using this relation and measuring $\alpha_R(\mu^2)$ at an arbitrary scale μ one can extract a value, valid as one loop result, for $\Lambda_{\text{QCD}} \simeq 250$ MeV

Notice that this argument is not theoretically self-consistent since we are using some results obtained by means of PT expansion in a regime in which the theory it's strongly coupled. Anyway it helps us to fix approximately the ideas on what IR and UV means in QCD.

A similar calculation could be performed also in QED, in fact taking $C_{\text{A}} = 0$ and $T_f = 1$ we can derive from (5.38) that in QED $b_0 = -\frac{1}{3\pi}$ (we set $N_f = 1$ too). So, from (5.45) and taking $(\mu^2 = m_e^2)^3$, one has at one loop:

$$\begin{aligned} \Lambda^2_{\text{QED}} &= m_e^2 e^{\frac{3\pi}{\alpha_e(m_e^2)}} \simeq m_e^2 \, e^{215} \\ &\simeq 10^{554} \, \text{GeV}^2 \end{aligned} \tag{5.46}$$

That is why nobody talks about Λ_{QED}: it is a scale bigger than Plank scale. At such a scale gravity effects becomes non negligible and this fact must be accounted for if one wants a gauge field theory that could be correct even at that scale.

5.5 Strong coupling renormalization-scheme dependence

The $\overline{\text{MS}}$ renormalization of the strong coupling constant at one loop is given by

$$\alpha_0 = \alpha_{\text{S}} \left(1 - \frac{c_\Gamma}{\epsilon} \, b_0 \, \frac{\alpha_{\text{S}}}{2\pi} \right) \tag{5.47}$$

where

$$b_0 = \frac{11 \, C_{\text{A}} - 4 \, n_{\text{F}} \, T_{\text{F}}}{6}. \tag{5.48}$$

[3] Remember that at the electron mass scale $\alpha_e(m_e^2) = 1/137$

The $\overline{\mathrm{MS}}$ ultraviolet counterterm for the scattering amplitude at one loop that should be added to the unrenormalized result is then

$$\frac{n}{\epsilon}\left[-c_\Gamma\, b_0\, \frac{\alpha_\mathrm{S}}{2\pi}\, \mathcal{A}^{\mathrm{tree}}\right], \tag{5.49}$$

where n is the order of the tree-level amplitude in α_S.

*** Missing all the discussion on the renormalization-scheme dependence ***

5.6 The Callan–Symanzik equations

Is there a general lesson about renormalizability that we can learn from the previous paragraphs of this section? The answer is obviously positive. Consider a general local[4] operator $\mathcal{O}$. One can write that the effects of renormalization on $\mathcal{O}$ are to introduce a renormalization constant $Z_\mathcal{O}$ that relates the bare operator $\mathcal{O}_0$ to the renormalized one $\mathcal{O}_R$, so that this last quantity does not depend on the regularization parameter, like the cut–off Λ or the ϵ parameter in dimensional regularization, that one must introduce in order to keep control of UV divergences.

The drawback of this is to introduce a dependence in $\mathcal{O}_R$ from a new scale μ, called renormalization scale, that it's nothing but the point where the subtraction that cancels the infinities coming from loop integrals takes place.

In one formula, for a cutoff regularized theory, one has:

$$\mathcal{O}_R\left(k, g_R\left(\mu\right), \mu\right) = Z_\mathcal{O}^{-1}\left(g_R\left(\mu\right), \frac{\Lambda}{\mu}\right)\mathcal{O}_0\left(k, g_0, \Lambda\right) \tag{5.50}$$

or, conversely,

$$\mathcal{O}_0\left(k, g_0, \Lambda\right) = Z_\mathcal{O}\left(g_0, \frac{\Lambda}{\mu}\right)\mathcal{O}_R\left(k, g_R\left(\mu\right), \mu\right) \tag{5.51}$$

We explicited the dependence of $\mathcal{O}$'s both from the bare and renormalized coupling constants keeping in mind that it's always possible the express one of these quantities with to respect the other: $g_0 = g_0\left(g_R, \frac{\mu}{\Lambda}\right)$ and vice versa.

It's important to stress that $Z_\mathcal{O}$ has to depend both from Λ and from μ. The dependence from Λ is required because of the dependence of $\mathcal{O}_0$ from this scale, in order to form an object Λ–independent, but since it is adimensional, as we want $\mathcal{O}_R$ and $\mathcal{O}_0$ with the same dimension, we are forced to introduce on the right side of (5.50) the renormalization scale μ and use it to keep all argument of $Z_\mathcal{O}$ dimensionless.

[4] To avoid any misunderstanding, we prefer to stress that local has the meaning that $\mathcal{O}$ is in general a function of only the difference between two spacetime point $|x-y|$, i.e. is a function of a single momentum k.

Let us now fix the cut–off Λ and the bare coupling g_0: for example we can safely take the limit $\Lambda \to \infty$ since equation (5.50) assure us that in this limit its left hand side is well defined.[5]

If we now derive both side of (5.51) with respect to $\log \mu$ we get zero on left hand side, since $\mathcal{O}_0$ does not depend on the renormalization scale, while on the right side we have:

$$\begin{aligned} \frac{d}{d \log \mu}\left[Z_{\mathcal{O}}\left(g_0\left(\Lambda\right), \frac{\mu}{\Lambda}\right) \mathcal{O}_R\left(k_i, g_R\left(\mu\right), \mu\right)\right] &= \\ = \left(\frac{\partial}{\partial \log \mu} + \frac{d g_R}{d \log \mu}\frac{\partial}{\partial g_R}\right)\left[Z_{\mathcal{O}}\ \mathcal{O}_R\right] &= 0. \end{aligned} \tag{5.52}$$

Defining as usual

$$\beta\left(g_R\right) = \frac{d g_R}{d \log \mu} \tag{5.53}$$

equation (5.52) takes the form

$$\left(\frac{\partial}{\partial \log \mu} + \beta \frac{\partial}{\partial g_R}\right)\left[Z_{\mathcal{O}}\ \mathcal{O}_R\right] = 0. \tag{5.54}$$

Noticing that $Z_{\mathcal{O}}$ does depend on μ but not on g_R, we can rewrite the previous equation as

$$\left(\frac{\partial}{\partial \log \mu} + \beta \frac{\partial}{\partial g_R} - \gamma_{\mathcal{O}}\right) \mathcal{O}_R = 0 \tag{5.55}$$

where we have introduced a new function $\gamma_{\mathcal{O}}$ defined by

$$\gamma_{\mathcal{O}}\left(g_R\right) = -\frac{\partial \log Z_{\mathcal{O}}}{\partial \log \mu} \tag{5.56}$$

This is called renormalization group (RG) equation or Callan–Symanzik equation[6]. The

[5] Strictly speaking the independence of $\mathcal{O}_R$ from Λ in the massless limit it's not true for scalar field theories (Weinberg, Phys.Rev. D 8-10, 1973)

[6] What shown it's just a particular case of the general Callan-Symanzik equation for the T-ordered product ($< \ldots >$) of a string of n fields plus the insertion of a local operator $\mathcal{O}$. Such an object, that may be defined by

$$G^{\ (n,1)}(p1, \ldots, p_n; k) = < \phi(p_1) \ldots \phi(p_n)\ \mathcal{O}_R(k) > \tag{5.57}$$

is related to a Green's function of bare fields via the rescaling

$$G^{\ (n,1)}(p1, \ldots, p_n; k) = Z^{-n/2}(\mu)\ Z_{\mathcal{O}}^{-1}(\mu)\ \ < \phi(p_1) \ldots \phi(p_n)\ \mathcal{O}_0(k) > \tag{5.58}$$

Now, defining the beta-function as usual, eq. (5.53), and introducing the gamma-function $\gamma = -\frac{\partial \log Z}{\partial \log \mu}$, one can shown that Green's functions containing a local operator $\mathcal{O}$ do obey a CS equation of the form

$$\left(\frac{\partial}{\partial \log \mu} + \beta \frac{\partial}{\partial g_R} - n\gamma - \gamma_{\mathcal{O}}\right) G^{\ (n,1)} = 0 \tag{5.59}$$

which reduces to (5.55) for n=0.

problem one has to face now is how to find a solution to the Callan–Symanzik equation (5.55). In general one can keep two different approaches: the first one is to try to find a solution in a perturbative expansion with respect to the coupling constant g, while the second one is more formal and gives a solution valid to all orders. Since the Callan-Symanzik equation does not rely on perturbative approach it's important to realize that the first approach is more limited than the second one, although it can be more intuitive.

Because of this last aspect, let's start from the first approach. Consider an observable R which is adimensional and depends on the external momenta in such a way that they produce only one relevant dimensionful scale s.

$$R = R\left(g\left(\mu\right), \frac{s}{\mu^2}\right) \tag{5.60}$$

where μ is the renormalization scale. In order to avoid misunderstandings, all the quantities we are considering are physical ones ($g = g_R, \ldots$). With an abuse of notation we forget the index R from now on.

Just to give an example, R may be thought as the ratio between the cross section for scattering ($e^+e^- \to hadrons$) and ($e^+e^- \to \mu^+\mu^-$):

$$R = \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} \tag{5.61}$$

This R is adimensional and depends on the external momenta through the Mandelstam variable $s = \left(p_{e^+} + p_{e^-}\right)^2$, with canonical dimension $[s] = 2$.

Since we're talking about a physical quantity, not a local operator nor a Green's function and there is only a single scale in the problem, the only γ's that can appear in the CS equation are those related to fields strength renormalization. But physical quantities cannot depend on an arbitrary shift in the value of the fields, so

$$\left(\frac{\partial}{\partial \log \mu^2} + \beta \frac{\partial}{\partial \alpha}\right) R = 0 \tag{5.62}$$

A better argument to show that R can't be renormalized by a Z factor it's simply to look at it as the product of conserved currents for whom every $Z_j = 1$.

Since we're following the perturbative approach, let's take now $\alpha \ll 1$ and expand all the terms appearing in (5.62) in power series of α:

$$\begin{aligned} R &= \sum_{n=0}^{\infty} f_n \, \alpha^n \left(\mu^2\right) \\ \beta &= -\sum_{n=0}^{\infty} b_n \, \alpha^{n+2} \left(\mu^2\right) \end{aligned} \tag{5.63}$$

The expansion of the beta–function is in accord with the general result for a non–abelian gauge theory that we found in equation (5.42).

Substituting back one gets that equation (5.62) becomes

$$\begin{aligned} 0 &= \sum_{m=0}^{\infty} \frac{\partial f_m}{\partial \log \mu^2} \alpha^m - \sum_{n=0}^{\infty} b_n \alpha^{n+2} \sum_{k=0}^{\infty} k f_k \alpha^{k-1} \\ &= \sum_{m=0}^{\infty} \frac{\partial f_m}{\partial \log \mu^2} \alpha^m - \sum_{n=0}^{\infty} \alpha^{n+1} \sum_{k=0}^{n} k f_k b_{n-k} \end{aligned} \tag{5.64}$$

From the last formula it's easy to extract the generic order term in α. Let us consider the first ones: at order α^0 and α^1 one gets contribution only from the first sum in (5.64). This means that f_0 and f_1 must be independent from the renormalization scale μ. So we can choose $f_0 = 1$ and $f_1 = a_1$. Nevertheless this two terms have very different meaning: f_0 comes from tree integrals and it is not strange that it does not feel any renormalization effect. Instead f_1 comes from one–loop integrals. So Callan–Symanzik equation tells us that at one–loop all UV divergences has to cancel out without any further renormalization, otherwise f_1 would take dependence on the renormalization scale!

At the generic order α^n, with $n \geq 2$, from (5.64) one has instead

$$\frac{\partial f_n}{\partial \log \mu^2} = \sum_{k=1}^{n-1} k f_k b_{n-k-1} \tag{5.65}$$

Resolving the equation for f_n gives:

$$\begin{aligned} f_2 &= a_1 b_0 \log \frac{\mu^2}{s} + a_2 \\ f_3 &= a_1 \left[b_0^2 \log^2 \frac{\mu^2}{s} + b_1 \log \frac{\mu^2}{s} \right] + 2 a_2 b_0 \log \frac{\mu^2}{s} + a_3 \\ f_4 &= a_1 \left[b_0^3 \log^3 \frac{\mu^2}{s} + \frac{5}{2} b_0 b_1 \log^2 \frac{\mu^2}{s} + b_2 \log \frac{\mu^2}{s} \right] + \\ & \quad + a_2 \left[3 b_0^2 \log^2 \frac{\mu^2}{s} + 2 b_1 \log \frac{\mu^2}{s} \right] + 3 a_3 b_0 \log \frac{\mu^2}{s} + a_4 \\ f_5 &= \cdots \end{aligned} \tag{5.66}$$

where a_i are integration constants. In general, keeping for every f_i the highest terms in $\log \frac{\mu^2}{s}$, one can write

$$f_n = a_1 \left(b_0 \log \frac{\mu^2}{s} \right)^{n-1} + \cdots \tag{5.67}$$

Using this relations in (5.63) the power series expansion for R can be expressed in the form

$$\begin{aligned} R &= 1 + a_1 \alpha\left(\mu^2\right) \left[\sum_{n=0}^{\infty} \left(\alpha\left(\mu^2\right) b_0 \log \frac{\mu^2}{s} \right)^n \right] + a_2 \alpha^2 \left(\mu^2\right) + \cdots \\ &= 1 + a_1 \frac{\alpha\left(\mu^2\right)}{1 + \alpha\left(\mu^2\right) b_0 \log \frac{s}{\mu^2}} + a_2 \alpha^2 \left(\mu^2\right) + \cdots \end{aligned} \tag{5.68}$$

If we now use the running of the coupling constant (5.44) we arrive to an important result[7] :

$$R = 1 + a_1\,\alpha\,(s) + a_2\,\alpha^2\left(\mu^2\right) + \cdots \tag{5.69}$$

Let us resume what we have done: we started expanding R by using the coupling α at the renormalization scale μ^2. Then RG equation implies that if one is able to resum contributions containing a single logarithm for every power of α coming from every order, i.e. $(\alpha \log(\frac{\mu^2}{s}))^n$, the physical observable R looses its dependence on μ^2 at first order, depending then only on the physical scale s. This is just the statement that the running of the coupling constant effectively resums all the leading logarithms.

If one wants to extend this argument to the term proportional to a_2 he has to consider also the next to leading logarithms in (5.66), i.e. he has to consider also b_1 terms. So it is no more possible to use for α_s the formula in (5.44) since it was derived using only the leading term in the expansion of the beta–function (see equation 5.42).

We remark again that $\alpha\,(s)$ in equation (5.69) comes from the sum of terms to all order in $\alpha\,(\mu^2)$: this is a crucial point and has many consequences. First of all it implies that all truncated power expansions depend on the renormalization scale μ^2. Only considering contributions to all powers in $\alpha\,(\mu^2)$ the dependence from μ^2 is lost and the sum of the series is supposed to depend only on s. The previous calculation shows that it happens at the first order in $\alpha(s)$. Since usually one knows only the first terms of a perturbative expansion, the choice of the renormalization scale becomes fundamental.

The example we have here studied shows that if the physical quantity one is considering depends on only one scale, it is clever to use this scale as renormalization scale in the expansion. In more complicated cases the right choice is not so clear. This is the so called renormalization scale dependence problem.

From an experimental point of view this can be dramatic. However it happens that the dependence of physical quantities is weaker and weaker as the perturbative order of the expansion is increased. Actually, what is usually done is to vary the renormalization scale μ up by a multiplicative factor 2 and down by one half and then consider this band as an error band giving in some sense the order of magnitude of the next order correction.

As anticipated before, one can derive a formal solution of the CS equation without the explicit use of perturbation theory. In this final part we will see how this works for a physical adimensional observable in the case of massless particles. We will see how the logarithm resummation can be expressed in a single formula: in particular the argument will show in a compact way the fact that the better choice for the argument of the running coupling is the typical scale of the process.

Let's start with a massless theory. For example we can think again at the observable R

[7] In the case that R is the ratio defined in (5.61) there would be an extra factor $\left(n_C\ \sum_f q_f^2\right)$ in front of the series (5.69)

as the ratio defined in (5.61) as calculated in massless QCD but the argument is the same for all observables as the one defined in (5.60). The equation R must obey is (5.62) that we recall here for convenience, with the definition $t = \log\left(s/\mu^2\right)$:

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial \alpha}\right] R = 0 \tag{5.70}$$

The general solution to this equation must have the following behavior:

$$R\left(\alpha\left(\mu^2\right), \frac{s}{\mu^2}\right) = R\left(\alpha\left(s\right), 1\right) \tag{5.71}$$

The proof goes as follows. From the definition of the beta function the following equality holds:

$$t = \int_{\alpha}^{\bar{\alpha}} \frac{d\alpha'}{\beta(\alpha')} \tag{5.72}$$

where $\bar{\alpha} = \alpha(\mu^2 = s)$ is a function of s. Taking the derivative of (5.72) with respect to t one obtains

$$1 = \frac{1}{\beta(\bar{\alpha})}\frac{\partial \bar{\alpha}}{\partial t} \;\Rightarrow\; \frac{\partial \bar{\alpha}}{\partial t} = \beta(\bar{\alpha})$$

while deriving with respect to α gives

$$0 = \frac{1}{\beta(\bar{\alpha})}\frac{\partial \bar{\alpha}}{\partial \alpha} - \frac{1}{\beta(\alpha)}\frac{\partial \alpha}{\partial \alpha} \;\Rightarrow\; \frac{\partial \bar{\alpha}}{\partial \alpha} = \frac{\beta(\bar{\alpha})}{\beta(\alpha)}$$

Then every function of $\bar{\alpha}$ is a solution of (5.70):

$$\begin{aligned}\left[-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial \alpha}\right] F(\bar{\alpha}) &= \left[-\frac{\partial \bar{\alpha}}{\partial t}\frac{\partial}{\partial \bar{\alpha}} + \beta(\alpha)\frac{\partial \bar{\alpha}}{\partial \alpha}\frac{\partial}{\partial \bar{\alpha}}\right] F(\bar{\alpha}) \\ &= \left[-\beta(\bar{\alpha})\frac{\partial}{\partial \bar{\alpha}} + \beta(\bar{\alpha})\frac{\partial}{\partial \bar{\alpha}}\right] F(\bar{\alpha}) = 0 \end{aligned} \tag{5.73}$$

We used the chain rule and the two relation just obtained. But now if we require $R\left(\alpha\left(\mu^2\right), s/\mu^2\right)$ to be a solution of (5.70) then it has to be equal to the general solution obtained as a function of $\bar{\alpha}$ only, and this closes the proof.

From equation (5.71) it is clear that if one wants to avoid at all order in the result the presence of large logarithms of the ratio s/μ^2, i.e. if one wants to resum them in a consistent way, then the correct choice for the renormalization scale is $\mu^2 = s$. We have recovered the result obtained with a perturbative analysis.

Chapter 6

Infrared-safe variables and jets

6.1 Infrared divergencies

6.2 $e^+e^- \to q\bar{q}$ at NLO in QCD

6.3 Soft emission in QED

Let's now concetrate on QED case. Infrared divergences in QED with masses can arise only from photons with soft momenta. The contributions to these divergences originate from real photons with energy lower than some experimental treshold for resolvability, i.e. some detector lower cut-off E_e, and virtual photons with $k_2^2 < E_e^2$ after Wick rotation.

We start by studing the multiple emission of n soft photons from an external leptonic leg, without caring if these photons are real or virtual.

Calling p' the momentum of the lepton and $k_1, \ldots, k_n$ the momenta of the photons we can study the Dirac structure of this diagram. In the soft limit multiple emissions factorize, but each propagator before an emission gives a divergent contribution

$$\frac{1}{(p'+k_i)^2 - m^2} = \frac{1}{2p' \cdot k_i} \to \infty \tag{6.1}$$

Thus the amplitude becomes

$$\begin{aligned}
&\bar{u}(p')(-ie\gamma_{\mu_1}))\frac{i(\not{p}' + \not{k}_1 + m)}{2p' \cdot k_1}(-ie\gamma_{\mu_2})\frac{i(\not{p}' + \not{k}_1 + \not{k}_2 + m)}{2p' \cdot (k_1 + k_2) + O(k^2)} \cdots \\
&\ldots (-ie\gamma_{\mu_n})\frac{i(\not{p}' + \sum_{i=1}^{n} \not{k}_i + m)}{2p' \cdot \sum_{i=1}^{n} k_i + O(k^2)}\,(i\mathcal{M}_{\text{hard}}) \ldots
\end{aligned} \tag{6.2}$$

where, since the photons are soft, we can neglect the k_i in the numerators and the $O(k^2)$

terms in the denominator. Simplifying the Dirac structure one gets

$$
\begin{aligned}
&\bar{u}(p')\gamma^{\mu_1}(\not{p}' + m)\gamma^{\mu_2}(\not{p}' + m)\cdots = \bar{u}(p')\gamma^{\mu_1}\left(\gamma^{\mu_A}{p'}_{\mu_A} + m\right)\ldots \\
&= \bar{u}(p')\left(2g^{\mu_1\mu_A}{p'}_{\mu_A} - \gamma^{\mu_A}{p'}_{\mu_A}\gamma^{\mu_1} + m\gamma^{\mu_1}\right) + \cdots = \bar{u}(p')\ 2{p'}_{\mu_1} + \ldots \\
&= \bar{u}(p')\ 2p'^{\mu_1}\ 2p'^{\mu_2}\ldots
\end{aligned}
\tag{6.3}
$$

and equation (6.2) becomes

$$
\bar{u}(p')\left(e\,\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(e\,\frac{p'^{\mu_1}}{p'\cdot(k_1+k_2)}\right)\ldots(i\mathcal{M}_{\text{hard}})\ldots \tag{6.4}
$$

The particular structure of the subsequent emissions in the previous formula can be derived even thanks to eikonal approximation, which we will study later for QCD soft emissions.

We have now to sum over all the possible different ordering of photons insertions. For the moment we do not care about possible overcounting, such the case of two photon attached togheter to form a single virtual photon, since we'll take care of this later.
There are $n!$ different possible ordering of the external photons momenta. This means that we have to sum $n!$ diagrams. To do this sum we start by defining a permutation Π. Let's take an integer i, in the range $[1, n]$, and define $\Pi(i)$ as the result of the permutation. Notice that$\Pi(i)$ must be in the same range of i. For example if we have three integer $1, 2, 3$ and we have a permutation such that $1 \to 3$, $2 \to 1$, $3 \to 2$, we have $\Pi(1) = 3$, $\Pi(2) = 1$ and $\Pi(3) = 2$. In order to sum over all the different permutations we have to use the following formula

$$
\sum_{\text{perm}} \frac{1}{p\cdot k_{\Pi(1)}}\frac{1}{p\cdot(k_{\Pi(1)}+k_{\Pi(2)})}\cdots\frac{1}{p\cdot(k_{\Pi(1)}+\cdots+k_{\Pi(n)})} = \frac{1}{p\cdot k_1}\frac{1}{p\cdot k_2}\cdots \tag{6.5}
$$

<u>proof</u>

For $n = 2$ (6.5) can be easily verified

$$
\frac{1}{p_1}\ \frac{1}{p_1+p_2} + \frac{1}{p_2}\ \frac{1}{p_1+p_2} = \left(\frac{1}{p_1}+\frac{1}{p_2}\right)\frac{1}{p_1+p_2} = \frac{1}{p_1}\ \frac{1}{p_2} \tag{6.6}
$$

Suppose now that (6.5) holds for $n-1$, we can now demonstrate it for n.

$$
\begin{aligned}
&\sum_{\text{perm}} \frac{1}{p\cdot k_{\Pi(1)}}\frac{1}{p\cdot(k_{\Pi(1)}+k_{\Pi(2)})}\cdots\frac{1}{p\cdot(k_{\Pi(1)}+\cdots+k_{\Pi(n)})} \\
&= \frac{1}{p\cdot\sum_i k_i}\sum_{\text{perm}} \frac{1}{p\cdot k_{\Pi(1)}}\frac{1}{p\cdot(k_{\Pi(1)}+k_{\Pi(2)})}\cdots\frac{1}{p\cdot(k_{\Pi(1)}+\cdots+k_{\Pi(n-1)})}
\end{aligned}
\tag{6.7}
$$

The last formula is independent from $k_{\Pi(n)}$. If now we call $i = \Pi(n)$ we can formally write the sum over the permutations as

$$
\sum_{\Pi} = \sum_{i=1}^{n}\sum_{\Pi'(i)} \tag{6.8}
$$

where the second term in the right hand side of (6.8) holds for the other $n-1$ contributions. In this way the sum over the permutations (6.7) becomes

$$\frac{1}{p\cdot\sum k}\sum_{i=1}^{n}\frac{1}{p\cdot k_1}\cdots\frac{1}{p\cdot k_{i-1}}\,\frac{1}{p\cdot k_{i+1}}\cdots\frac{1}{p\cdot k_n} \tag{6.9}$$

We have now to multiply each term in the last sum with a factor $\frac{p\cdot k_i}{p\cdot k_i}$, from which we obtain (6.5). Q.E.D.

Returning to our problem and applying (6.5) we have

$$\bar{u}(p')\left(e\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(e\frac{p'^{\mu_2}}{p'\cdot k_2}\right)\cdots\left(e\frac{p'^{\mu_n}}{p'\cdot k_n}\right) \tag{6.10}$$

In the case of a second lepton leg, with ingoing momentum, it's easy to show that the various contribution to the propagator are the same except for a minus sign for each photon, since

$$(p-\sum k_i)^2-m^2=-2p\cdot\sum k \tag{6.11}$$

If we now consider diagrams containing n soft photons connected in all the possible orders to two external leptonic legs, one with incoming and the other with outgoing momentum, we have for the amplitude the result

$$\begin{aligned}\bar{u}(p')\mathcal{M}_{\text{hard}}u(p)\;e\left(\frac{p'^{\mu_1}}{p'\cdot k_1}-\frac{p^{\mu_1}}{p\cdot k_1}\right)\;e\left(\frac{p'^{\mu_2}}{p'\cdot k_2}-\frac{p^{\mu_2}}{p\cdot k_2}\right)\cdots\\ \cdots e\left(\frac{p'^{\mu_n}}{p'\cdot k_n}-\frac{p^{\mu_n}}{p\cdot k_n}\right)\end{aligned} \tag{6.12}$$

After that we have to separate the contribution coming from the virtual and the the real photons, deciding what of these photons are reals and what are virtuals. A virtual photon actually it's nothing but a couple of photons with momenta k_i and k_j such that $k_i=-k_j=k$. After that we have to multiply for the propagator of this virtual photon and integrate over k, obtaining

$$X\equiv\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\left(\frac{p'}{p'\cdot k}-\frac{p}{p\cdot k}\right)\left(\frac{p'}{-p'\cdot k}-\frac{p}{-p\cdot k}\right) \tag{6.13}$$

The $1/2$ factor in the previous formula is due to simmetry of the the exchange of k_i and k_j. If we have m virtual photons we have to follow the same procedure for each single photon, the resulting simmetry factor being $1/m!$. Summing all the contributions due to the virtual photons we have

$$\bar{u}(p')(i\mathcal{M}_{\text{hard}})u(p)\sum_{m=0}^{\infty}\frac{X^m}{m!}=\bar{u}(p')(i\mathcal{M}_{\text{hard}})u(p)\;e^X \tag{6.14}$$

For the emission of a real photon we instead have to multiply by its polarization vector, sum over the polarizations and integrate the squared amplitude over the real photon phase space obtaining

$$Y \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}e^2(-g_{\mu\nu})\left(\frac{p'^{\mu}}{p'\cdot k}-\frac{p^{\mu}}{p\cdot k}\right)\left(\frac{p'^{\nu}}{p'\cdot k}-\frac{p^{\nu}}{p\cdot k}\right) \tag{6.15}$$

In the case of n different real emissions we still have the factor $1/n!$ (n identical bosons in the final state). This implies that the cross section for an emission of any number of real photons is

$$\sum_{n=0}^{\infty}\frac{d\sigma}{d\Omega}(p\to p'+n\gamma) = \frac{d\sigma}{d\Omega}(p\to p')\sum_{n=0}^{\infty}\frac{Y^n}{n!} = \frac{d\sigma}{d\Omega}(p\to p')e^{Y} \tag{6.16}$$

If we now combine the two previous results, in order to consider the cross section due to the emission of both real and virtual photons we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \left(\frac{d\sigma}{d\Omega}\right)_0 e^{2X}e^{Y} \tag{6.17}$$

where in the left hand side of the previous equation it appears the differential cross section as we can experimentally measure it, while in the right hand side there is the *bare* one, multiplied by the two Sudakov exponential factor due to the emission or to the exchange of infinitely many soft photons. If we try to evaluate these exponential, for example, for the n-jet cross section, we can use the calculation already done for QCD. The only subtlety is to put, at the end of the day $C_{\text{F}} = 1$ and $C_{\text{A}} = 0$ to recover correct QED results. The exponentiation of the two jet cross section then follows straightly from (6.36) and gives

$$\begin{aligned}\sigma_{2j} &= \sigma_0\left(1-\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2+\frac{1}{2!}\left(\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right)^2\ldots\right)\\ &= \sigma_0\,\exp\left\{-\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right\}\end{aligned} \tag{6.18}$$

The three jet cross section instead becomes

$$\begin{aligned}\sigma_{3j} &= \sigma_0\left(\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2-\left(\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right)^2\ldots\right)\\ &= \sigma_0\,\left(\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right)\exp\left\{-\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right\}\end{aligned} \tag{6.19}$$

and so on. The generic term n emissions is thus

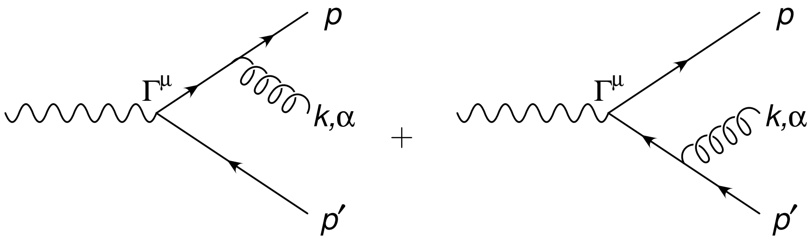
$$\begin{aligned}&\sigma_{nj} = \sigma_0\,\frac{1}{(n-2)!}\left(\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right)^{(n-2)}\exp\left\{-\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right\}\\ &= \sigma_0\frac{1}{(n-2)!}\lambda^{(n-2)}e^{-\lambda}\end{aligned} \tag{6.20}$$

The last expression is interesting because shows that summing over n all the contributions the exponential that appears cancels out the one already present, giving us back σ_0. Furthemore we have found that the generic n-th term obeys a Poisson distribution function, this implies that the average number of jets is

$$\lambda = \langle n_{\text{jet}}\rangle = 2+\frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2 \tag{6.21}$$

6.4 Soft emission in QCD

Soft emissions are very important both in QCD and QED because of their large number. In fact, we will see that since the emission spectrum behaves as $\frac{dk^0}{k^0}$, the probability to emit a soft particle is very large. Let's concentrate to the QCD case studying the emission of a soft gluon from an off-shell photon decaying in a $q\bar{q}$ couple:



$$
\begin{aligned}
M^\mu &= \bar{u}(p)\ \varepsilon_\alpha(k)\ (-ig\gamma^\alpha)t^a\ \frac{i}{\not p + \not k}\ \Gamma^\mu\ v(p') \quad + \\
&\qquad\qquad\qquad\qquad \bar{u}(p)\ \Gamma^\mu\ \frac{i}{-\not p' - \not k}\ \varepsilon_\alpha(k)\ (-ig\gamma^\alpha)\ t^a\ v(p') \\
&= \bar{u}(p)\ g\ t^a\ \left\{ \not\varepsilon\ \frac{\not p + \not k}{2p\cdot k}\ \Gamma^\mu\ -\ \Gamma^\mu\ \frac{\not p' + \not k}{2p'\cdot k}\ \not\varepsilon \right\} v(p')
\end{aligned}
$$

In previous formulae we considered a generic vertex form factor Γ^μ. Being in the soft approximation ($k << p, p'$) we can neglect now the $\not k$ factors in the numerator, so using Dirac equations one gets

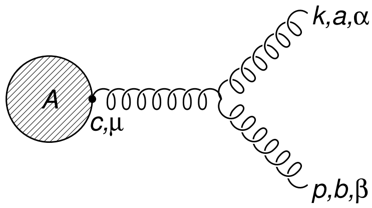
$$
M^\mu_{soft}\ \approx\ \bar{u}(p)\ g\ t^a\ \left\{ \frac{p\cdot\varepsilon}{p\cdot k}\ \Gamma^\mu\ -\ \Gamma^\mu\ \frac{p'\cdot\varepsilon}{p'\cdot k} \right\} v(p')
$$

which can be cast in the form

$$
\begin{aligned}
M^\mu_{soft} &= g\ t^a\ \bar{u}(p)\ \Gamma^\mu\ v(p')\ \varepsilon_\alpha(k)\ \left\{ \frac{p^\alpha}{p\cdot k}\ -\ \frac{p'^\alpha}{p'\cdot k} \right\} \\
&= g\ t^a\ \bar{u}(p)\ \Gamma^\mu\ v(p')\ \varepsilon_\alpha(k)\ J^\alpha
\end{aligned}
\tag{6.22}
$$

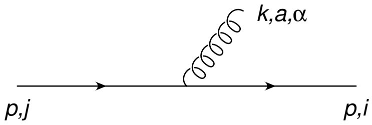
with J^α called the eikonal current. We have just showed how soft gluon emission from a quark line factorize into the product of an emission factor times the underlying amplitude. This is exactly what one would have expected since long wavelength radiation doesn't know anything about the spin: the soft limit is a classical limit while the spin is a pure quantistic object. Furthermore it worths noticing how the soft approximation doesn't spoil the current conservation equation $k_\alpha J^\alpha = 0$.

In the same way one can evaluate the soft emission amplitude from a gluon line . Starting from an amplitude $A_{c,\mu}$ and neglecting k with respect to p one has

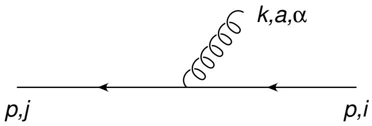


$$M_{soft} \approx -A_{c,\mu}\, ig\, f^{abc} \frac{p^\alpha}{p\cdot k}\, \varepsilon_\alpha(k)\, \varepsilon'^\mu(p) \tag{6.23}$$

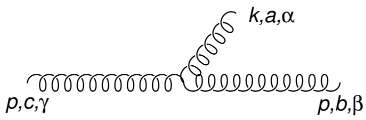
From the factorization properties seen in previous calculations one can easily extract the "Feynman rules" for soft gluons emissions. They read



$$= \quad g\, t^a_{ij} \frac{p^\alpha}{p\cdot k} \tag{6.24}$$



$$= \quad -\, g\, t^a_{ij} \frac{p^\alpha}{p\cdot k} \tag{6.25}$$



$$= \quad -ig\, f^{abc}\, g^{\beta\gamma} \frac{p^\alpha}{p\cdot k} \tag{6.26}$$

Armed with these rules let's return to the $q\bar{q}$ pair production from the decay of an off-shell photon. The amplitude for the emission of an extra soft gluon was calculated in (6.22) so the differential cross section, obtained after squaring, summing over polarizations and colors, reads

$$d\sigma_g = \sum_{spin} |A_0|^2 g^2\, \varepsilon_\alpha \varepsilon^*_\beta\, C_{\rm F} \left(\frac{-2p^\alpha p'^\beta}{(p\cdot k)(p'\cdot k)} \right) \frac{d^3k}{(2\pi)^3\, 2k^0}\, d\phi_2 \tag{6.27}$$

where $|A_0|^2$ is simply $\bar{u}(p)\, \Gamma^\mu\, v(p')$ of (6.22) times its hermitian conjugate. The last formula can be simplified using $d\sigma_0$ to indicate the differential cross section without gluon emission and considering that if the emitted gluon is soft p and p' are almost anti parallel. Defining θ the angle between the gluon momentum k and p, θ' that between k and p' and $\theta_{pp'}$ that between p and p' one has in this limit $\theta_{pp'} \approx \pi$ and $\theta' \approx \pi - \theta$. So one gets

$$\begin{aligned} d\sigma_g &= d\sigma_0\, g^2\, C_{\rm F} \left(\frac{p\cdot p'}{(p\cdot k)(p'\cdot k)} \right) \frac{k^0 dk^0 d\cos\theta}{(2\pi)^2} \frac{d\varphi}{2\pi} \\ &= d\sigma_0 \frac{\alpha_{\rm S} C_{\rm F}}{\pi} \frac{d\varphi}{2\pi} \frac{dk^0}{k^0} \frac{(1-\cos\theta_{pp'})\, d\cos\theta}{(1-\cos\theta)(1-\cos\theta')} && (6.28) \\ &= d\sigma_0 \frac{\alpha_{\rm S} C_{\rm F}}{\pi} \frac{d\varphi}{2\pi} \frac{dk^0}{k^0} \frac{2\, d\cos\theta}{(1-\cos\theta)(1+\cos\theta)} && (6.29) \end{aligned}$$

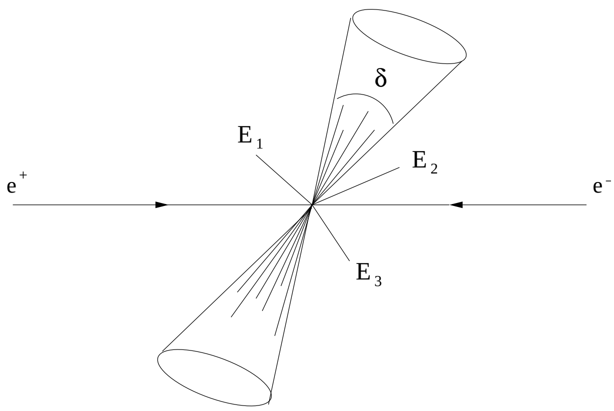
At this point if one evaluates the integral over the phase space to get the real emission cross section one encounters singularities in the soft $k^0 \to 0$ and collinear $\theta \to 0, \pi$ limits. But since the total cross section is an infrared safe observable, we know, from the KLN theorem, that the sum of real and virtual cross section must be finite. So one can write down the virtual differential cross section at hand

$$\frac{d\sigma_v}{dk^0 d\cos\theta} = -\, d\sigma_0 \frac{2\alpha_{\rm S} C_{\rm F}}{\pi} \int_0^{\sqrt{s/2}} \frac{dk'^0}{k'^0} \int_{-1}^{1} \frac{d\cos\theta'}{(1-\cos\theta')(1+\cos\theta')} \cdot \frac{\delta(k^0)}{2}\left[\delta(1-\cos\theta)+\delta(1+\cos\theta)\right] \tag{6.30}$$

Beware that the integrals appearing in the previous equation aren't phase space integrals: equation (6.30) actually represents a differential cross section. Those integrals are only a clever parametrization of the loop integral of virtual graphs, in order to get two formulae, the real one and the virtual one, with the same variables. One can now easily verify that doing the integral over the phase space of the sum of (6.29) and (6.30) the singular part of the cross section completely disappears, leaving only a finite result.

6.5 Sterman–Weinberg jets

Jet rates are an example of infrared safe observables calculable in field theory. Loosely speaking a jet is intended to be a bunch of particles *near* in the phase space. Because of this lack of rigour in the definition of what a jet is, several jet constructing algorithm can be defined. The first that has appeared in the literature and one of the simplest is the Sterman–Weinberg jet algorithm. We say that an event contributes to a jet *à la* Sterman-Weinberg, that depends on the two parameter $\delta$and ε, if there exist two cones of width δ that contain all of the energy of the event except for an ε factor. For example in the case of the event represented in the figure below



where E_1,E_2 and E_3 are the only particles outside the cones, the event is accepted as a Sterman–Weinberg jet only if

$$E_1 + E_2 + E_3 < \varepsilon E_{\rm TOT}. \tag{6.31}$$

We now present an explicit calculation for the event $e^+e^- \to q\bar{q}$. We already know that the NLO real emission correction, i.e. the emission of an extra gluon, gives an infrared divergent contribution. However, because of the infrared safety of jet definition we used, we can evade the questions about the divergent probability of the event $g\bar{q}q$.

First of all we have to extimate what are the contributions to the jet event. The diagrams we have to take in account are the Born amplitude, the virtual and the real corrections. In the case of Born and virtual amplitude we're left with a back to back kinematic even in the final state, so there always exist a cone, with width δ, that contains all but an ε part of the total energy. Hence these diagrams always contibute to the jet event. Their contribution can be easily evaluated

$$\text{Born} + \text{Virtual} = \sigma_0 - \sigma_0\ C_{\text{F}}\,\frac{2\alpha_{\text{S}}}{\pi}\int_0^E \frac{dk_0}{k_0}\int_{-1}^{1} d\cos\theta\ \frac{1}{1-\cos^2\theta} \tag{6.32}$$

As far as we concern the contribution of the real gluons we have to distinguish two cases

- If the energy of the gluon, i.e. the only particle outside the jet ,is $k_0 < \epsilon E$, the event contributes to the 2–jet cross section and we have to calculate the integral

$$R_1 = \sigma_0 C_{\text{F}}\frac{2\alpha_{\text{S}}}{\pi}\int_0^{\varepsilon E}\frac{dk_0}{k_0}\int_{-1}^{1}\frac{d\cos\theta}{1-\cos^2\theta} \tag{6.33}$$

- If $k_0 > \epsilon E$ we have to take in account only the case where θ is inside the cone defined by δ. The integral to calculate is thus

$$R_2 = \sigma_0 C_{\text{F}}\frac{2\alpha_{\text{S}}}{\pi}\int_0^{\varepsilon E}\frac{dk_0}{k_0}\left[\int_0^{\delta}\frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi}\frac{d\cos\theta}{1-\cos^2\theta}\right] \tag{6.34}$$

The 2–jet cross section is now given by summing all the contributions. Doing this sum it is immediate to observe that the singularities are absent in the integral, which means that we have an infrared safe quantity. The final integral results

$$\text{Born+Virtual+Real} = \sigma_0 - \sigma_0 C_{\text{F}}\frac{2\alpha_{\text{S}}}{\pi}\int_{\varepsilon E}^{E}\frac{dk_0}{k_0}\int_{\delta}^{\pi-\delta}\frac{d\cos\theta}{1-\cos^2\theta} \tag{6.35}$$

The final result for the two jet cross section at the first order is thus

$$\sigma_{2j} = \sigma_0\left(1 - \frac{2\alpha_{\text{S}}}{\pi}C_{\text{F}}\log\varepsilon\log\delta^2\right). \tag{6.36}$$

Notice that the previous result remains finite as long ε and δ are taken finite. At this order the calculation of σ_{3j} is straightforward. Knowing σ_{2j} and conidering that we are studing the process $e^+e^- \to q\bar{q}g$ we know that $\sigma = \sigma_{2j} + \sigma_{3j} = \sigma_0$. This implies that σ_{3j}

$$\sigma_{3j} = \frac{2\alpha_{\text{S}}}{\pi}\ C_{\text{F}}\ \log\varepsilon\ \log\delta^2 \tag{6.37}$$

Thus the more σ_{2j} becomes small, the more σ_{3j} gets large. This can be pratically obtained lowering ε and δ in the jet definition. However, some care is required since lowering the parameters one encounters values of them for whom σ_{2j} gets negative. This result has the physical interpretation that the infrared safety of SW jet definition is vanishing since the more the integrals approach singularities the more they feels the soft and collinear divergences and the perturbative series loses sense. If we want a correct physical result we have to be able to resum the series. This resummation can be explicitly done in QED, while in QCD this can't be done to all order since the more the gluon emitted becomes soft, the more the coupling $\alpha_{\rm S}$ associated with its emission becomes large. That coupling does indeed depend on the transverse momentum of the emitted gluon[1]

$$\alpha_{\rm S} = \alpha_{\rm S}(k_\perp) \tag{6.38}$$

and thus the resummation in the limit $k_\perp \to 0$ does not exponentiate as in QED, because of the IR slavery of QCD.

6.6 Angular ordering

A more interesting property of soft gluons is the angular ordering of subsequent emissions. This can be proved returning to the previous example of a virtual photon decaying in a $q\bar{q}$ pair with an extra soft gluon. The real differential cross section is exactly the same until (6.28), but now we can't assume p and p' anti parallel. Retaining the right angular dependence we can rewrite (6.28) redefining

$$\begin{aligned} \theta &= \theta_i \\ \theta' &= \theta_j \\ \theta_{pp'} &= \theta_{ij} = \theta_i - \theta_j \end{aligned}$$

and using the identity

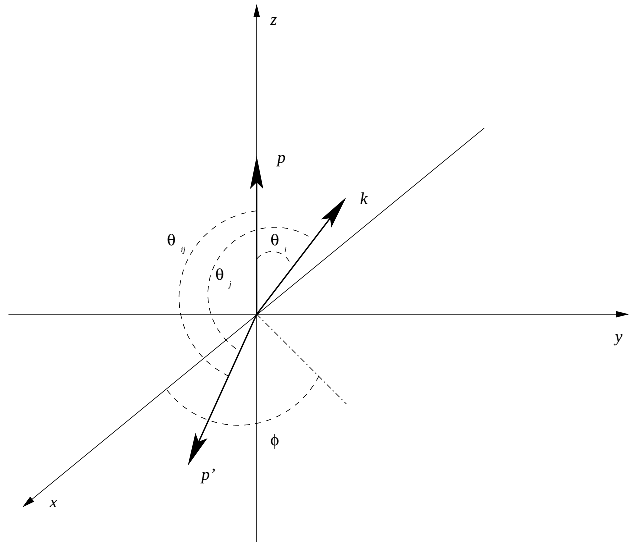
$$\begin{aligned} \frac{1-\cos\theta_{ij}}{(1-\cos\theta_i)(1-\cos\theta_j)} &= \frac{1}{2}\left[\frac{\cos\theta_i - \cos\theta_{ij}}{(1-\cos\theta_i)(1-\cos\theta_j)} + \frac{1}{1-\cos\theta_i}\right] + \\ &\quad + \frac{1}{2}\left[i \leftrightarrow j\right] \\ &\equiv W_{(i)} + W_{(j)} \end{aligned} \tag{6.39}$$

The last definition is useful since each of the W can be viewed as a sort of radiation probability from the corresponding (anti-)quark line. Mind that they aren't actually true probabilities because the $W's$ are not positive defined. Nevertheless they are important since each of them is singular only in the limit of gluon emission parallel to the corresponding (anti-)quark, while they are finite in the limit of gluon emission parallel to the other one:

$$\begin{aligned} W_{(i)} \to \infty \;\text{ if }\; \cos\theta_i \to 0 \qquad W_{(i)} \to \text{finite} \;\text{ if }\; \cos\theta_j \to 0 \\ W_{(j)} \to \infty \;\text{ if }\; \cos\theta_j \to 0 \qquad W_{(j)} \to \text{finite} \;\text{ if }\; \cos\theta_i \to 0 \end{aligned}$$

[1]Strictly speaking the natural argument for $\alpha_{\rm S}$ would be of the order of $k_\perp$, but using $k_\perp$ is a good approximation.

To prove the angular order property of $W_{(i)}$ we consider, without loosing generality, the following three-momenta configuration



Having chosen p in the z direction and p' in the (x, z) plane we are left with

$$
\begin{aligned}
p &= p^0 \ (1, 0, 0, 1) \\
p' &= p'^0 \ (1, \sin\theta_{ij}, 0, \cos\theta_{ij}) \\
k &= k^0 \ (1, \sin\theta_i \cos\varphi, \sin\theta_i \sin\varphi, \cos\theta_i).
\end{aligned}
\tag{6.40}
$$

At this point, comparing $k \cdot p' = k^0 p'^0 (1 - \cos\theta_j)$ with what one would obtain from (6.40), one can extrapolate a relation between the angles:

$$
\begin{aligned}
1 - \cos\theta_j &= 1 - \cos\theta_{ij}\cos\theta_i - \sin\theta_{ij}\sin\theta_i\cos\varphi \\
&\equiv a - b\cos\varphi
\end{aligned}
\tag{6.41}
$$

What we want is to evaluate the integral of W_i over the azimuthal angle , so we start to consider only the integral of the φ-dependent part

$$
I_{(i)} \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{a - b\cos\varphi} \tag{6.42}
$$

Defining $z = e^{i\varphi}$ we can use the residue theorem to simplify the evaluation of the integral:

$$
\begin{aligned}
I_{(i)} &= \frac{1}{2\pi} \int_{|z|=1} \frac{dz}{iz} \frac{1}{a - \frac{b}{2}\left(z + \frac{1}{z}\right)} \\
&= \frac{i}{\pi b} \int_{|z|=1} \frac{dz}{(z - z_-)(z - z_+)}
\end{aligned}
\tag{6.43}
$$

where

$$
z_{\pm} = \frac{a \pm \sqrt{a^2 - b^2}}{b} \tag{6.44}
$$

Since only z_- can lie inside the integration contour we have

$$
\begin{aligned}
I_{(i)} &= -\frac{2}{b}\,\mathrm{Res}\{\frac{1}{(z-z_-)(z-z_+)}; z=z_-\} \\
&= \frac{2}{b}\left(\frac{1}{z_+-z_-}\right) \\
&= \frac{1}{\sqrt{a^2-b^2}} \;=\; \frac{1}{\sqrt{\cos^2\theta_i+\cos^2\theta_{ij}-2\cos\theta_i\cos\theta_{ij}}} \\
&= \frac{1}{|\cos\theta_i-\cos\theta_{ij}|}.
\end{aligned}
\tag{6.45}
$$

Hence from (6.39) one has

$$
\begin{aligned}
\int_0^{2\pi}\frac{d\varphi}{2\pi}\,W_{(i)} &= \frac{1}{2}\,\frac{1}{1-\cos\theta_i}\left[1+(\cos\theta_i-\cos\theta_{ij})I_{(i)}\right] \\
&= \frac{1}{2}\,\frac{1}{1-\cos\theta_i}\left[1+\frac{\cos\theta_i-\cos\theta_{ij}}{|\cos\theta_i-\cos\theta_{ij}|}\right]
\end{aligned}
\tag{6.46}
$$

so

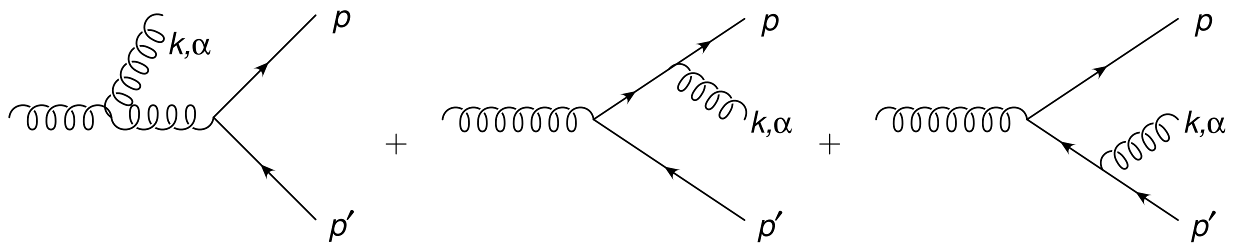
$$
\int_0^{2\pi}\frac{d\varphi}{2\pi}\,W_{(i)} = \begin{cases} \dfrac{1}{1-\cos\theta_i} & \text{if } \theta_i<\theta_{ij} \\[2ex] 0 & \text{if } \theta_i>\theta_{ij} \end{cases}
\tag{6.47}
$$

The same machinery holds for $W_{(j)}$ which results

$$
\int_0^{2\pi}\frac{d\varphi}{2\pi}\,W_{(j)} = \begin{cases} \dfrac{1}{1-\cos\theta_j} & \text{if } \theta_j<\theta_{ij} \\[2ex] 0 & \text{if } \theta_j>\theta_{ij} \end{cases}
\tag{6.48}
$$

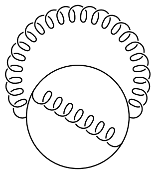
The resulting picture is that soft gluon radiation from the (anti)quark leg is admitted only in a cone of angle smaller than that between quark and antiquark momenta. Moreover one can consider the radiation inside the two cones as being uncorrelated. Thus the graphs in which the emission is from quark or antiquark can be summed incoherently, being their interference completely taken into account for by constraining the emission to be within those cones.

Repeating the calculation for the emission of an extra soft gluon, one finds the same relations between subsequent angles, but this time applied to each of the previous dipole. To see that these two contributions can be summed incoherently let's analyze more in detail the contributions to $q\bar{q}g$ final state from the decay of a gluon of momentum q. Using the soft "Feynman rules" (6.24-6.26) one has



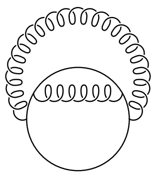
$$
\begin{aligned}
&= \left(-igf^{abc}\right) \frac{q\cdot\varepsilon}{q\cdot k} t^c M_{born} + t^b M_{born}(gt^a)\,\frac{p\cdot\varepsilon}{p\cdot k} + (-gt^a)\,\frac{p'\cdot\varepsilon}{p'\cdot k} t^b M_{born} \\
&= \left[g\, t^a\, t^b \left(\frac{q\cdot\varepsilon}{q\cdot k} - \frac{p'\cdot\varepsilon}{p'\cdot k} \right) + g\, t^b\, t^a \left(\frac{p\cdot\varepsilon}{p\cdot k} - \frac{q\cdot\varepsilon}{q\cdot k} \right) \right] M_{born}
\end{aligned}
\tag{6.49}
$$

The two factors correspond to the two possible ways in which color can flow in previous graphs: the initial gluon can be color connected to the quark or to the antiquark. The soft gluon can thus be seen as emitted independently from the quark or the antiquark. Actually this description is not entirely correct since when one tries to evaluate the probability, squaring the amplitude and summing over spin and color, one has the interference term that spoils this simplistic result. Nevertheless one finds that the interference term is suppressed by a factor $1/n^2$ since its color factor is



$$
= \left(C_{\rm F} - \frac{C_{\rm A}}{2}\right) C_{\rm F}\; n \;=\; -\frac{C_{\rm F}}{2} \;=\; -\frac{n^2-1}{4n} \;=\; O(n)
$$

compared to

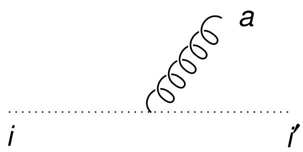


$$
= \; C_{\rm F}^2\; n \;=\; \frac{(n^4 - 2n + 1)}{4n} \;=\; O(n^3).
$$

As a result, the emission of a soft gluon can be described, at the leading order in $\frac{1}{n^2}$ expansion, as the incoherent sum of the emission from the two color currents.

Returning to our analysis of an additional gluon emission one finds that one can consider the two color contributions to $q\bar{q}g$ as independent emitters: one where the color connection is between quark and initial gluon and the other where it is between antiquark and initial gluon. The emission of the additional gluon will thus be constrained to be either in the cone formed by the gluon and the quark or within the cone made by the gluon and the antiquark. In both cases the emission angle will be smaller than the angle of the first gluon. This leads straightly to angular ordering with successive emissions taking places within cones with angles smaller and smaller.

We can summarize previous results concerning soft gluon emission from the parton i using the generator of its colour group representation $\boldsymbol{T}_i^a$, where a is the color index of the emitted gluon. Schematically



$$
\boldsymbol{T}_i^a \;=\; \begin{cases} t^a_{i'i} & \text{for outgoing } q \text{ or incoming } \bar{q} \\[2ex] -(t^a_{i'i})^T & \text{for outgoing } q \text{ or incoming } \bar{q} \\[2ex] if^{ai'i} & \text{for gluon} \end{cases}
$$

Being generators of color representations, $\boldsymbol{T}$'s satisfy

$$\boldsymbol{T}_i^2 \;=\; \begin{cases} C_{\rm F} & \text{if } i \text{ is a quark or a antiquark} \\ \\ C_{\rm A} & \text{if } i \text{ is a gluon} \end{cases}$$

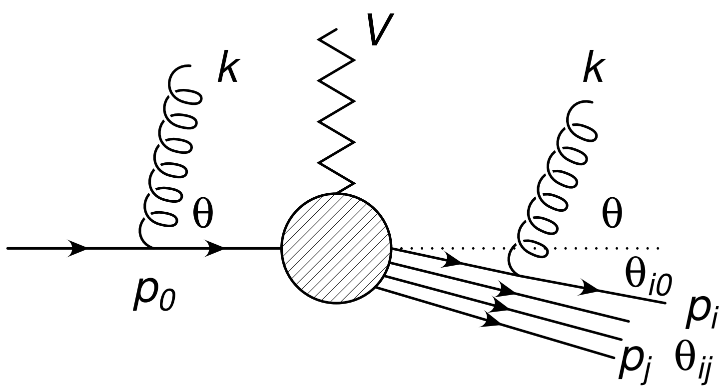
In this way, given a generic process, the emission of the soft gluon of momentum k and color a from any leg of momentum p_i can be described by

$$J^{\mu a} \;=\; \sum_{incoming} \boldsymbol{T}_i^a \frac{p_i^\mu}{p \cdot k} \;-\; \sum_{outgoing} \boldsymbol{T}_i^a \frac{p_i^\mu}{p \cdot k} \tag{6.50}$$

where the sum is split between incoming and outgoing emission leg. One can easily verify that the current conservation $k_\mu J^{\mu a} = 0$ still holds since color conservation implies $\sum_i \boldsymbol{T}_i^a = 0$.

6.7 Large-angle soft-gluon emission

Let's now concentrate on large angle soft gluon radiation, in particular let's consider the most general case of an incoming parton p_0 splitting, because of the interaction with some external field V, in a bunch of partons with small relative angles and emitting a large angle soft gluon either from one initial or finale state leg. The process can be graphically represented by



where the smallness of relative angles between partons with respect the angle of gluon emission is accounted for assuming

$$\theta \gg \theta_{i0}$$
$$\theta \gg \theta_{ij} \quad \forall j$$

Here θ_{i0} is the angle between the emitting particle and the initial parton direction while θ_{ij} is that between the emitting particle and another generic final state parton p_j. With these assumptions all radiation factors are almost the same

$$\frac{p_i^\mu}{p_i \cdot k} \;\approx\; \frac{p_0^\mu}{p_0 \cdot k} \;\; \forall i \tag{6.51}$$

thus the eikonal current

$$J^{\mu a} \;=\; \boldsymbol{T}_0^a \,\frac{p_0^\mu}{p_0 \cdot k} \;-\; \sum_i \boldsymbol{T}_i^a \,\frac{p_i^\mu}{p_i \cdot k}$$

can be simplified in

$$\begin{aligned} J^{\mu a} \;&=\; \frac{p_0^\mu}{p_0 \cdot k}\,\left[\boldsymbol{T}_0^a \;-\; \sum_i \boldsymbol{T}_i^a\right] \\ &=\; \frac{p_0^\mu}{p_0 \cdot k}\,\boldsymbol{T}_V \end{aligned} \tag{6.52}$$

because of color conservation, where $\boldsymbol{T}_V$ is the generator of the color representation of the scattering field V. Thus the probability of emitting a large angle soft gluon is proportional to the color charge of the interacting field, irrespectively of the parton that actually emits the gluon. This mechanism is called color coherence and a typical example where it is important can be found in Higgs production. Here two competitive process are the most importants: weak boson fusion and gluon-gluon fusion via top quark loop. In the former case the scattering field is a W or a Z, which are colorless, so one would expect a suppression of large angle soft gluons. In the latter instead there is the interaction with gluon fields, so one finds many large angle soft gluons, being thus able to distinguish experimentally the two process.

Chapter 7

Initial-state singularities

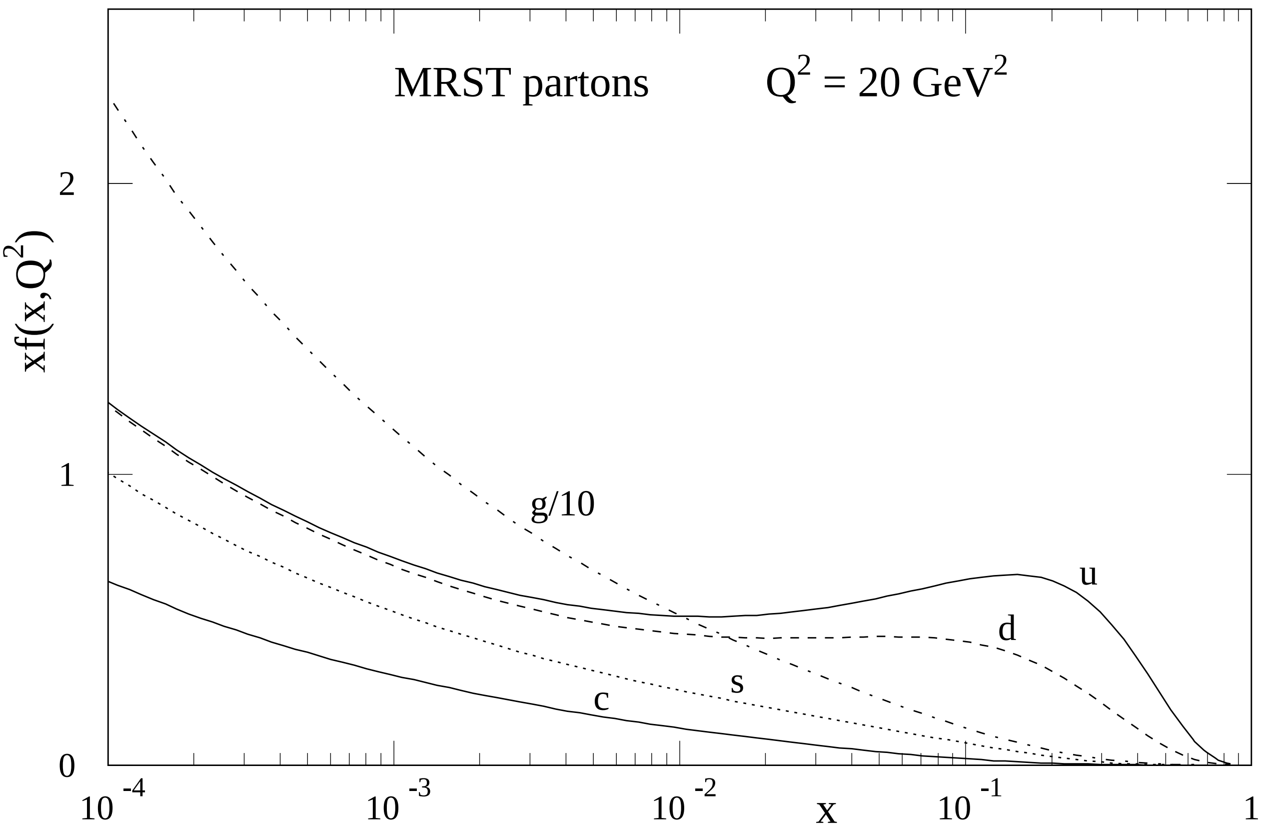
7.1 The naïve parton model

7.2 The “improved” parton model

7.2.1 The Sudakov decomposition

7.2.2 The DGLAP equations

7.2.3 Leading-logarithmic resummation, Mellin moments


MRST partons
$Q^2 = 20\ \mathrm{GeV}^2$
$xf(x,Q^2)$
x
g/10
u
d
s
c
0
1
2
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1

Appendix A

Useful mathematical functions

A.1 The Γ and B functions

The Gamma function is defined by

$$\Gamma(z) \equiv \begin{cases} \displaystyle\int_0^\infty dx\, \mathrm{e}^{-x} x^{z-1} & \mathrm{Re}\, z > 0 \\ \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^\infty dx\, \mathrm{e}^{-x} x^{z-1} & \mathrm{Re}\, z < 0,\ z \neq -n,\ n \in N_0 \end{cases} \tag{A.1}$$

With a simple change of variables $x \to x^2$

$$\Gamma(z) = \int_0^\infty dx\, \mathrm{e}^{-x} x^{z-1} = 2 \int_0^\infty dx\, \mathrm{e}^{-x^2} x^{2z-1} \tag{A.2}$$

It can be easily shown that

$$\Gamma(1) = 1 \tag{A.3}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{A.4}$$

$$\Gamma(z+1) = z\,\Gamma(z) \tag{A.5}$$

Using eq. (A.2) and changing to polar coordinates ($x = r\cos\theta, y = r\sin\theta$) we can write

$$\begin{aligned} \Gamma(a)\,\Gamma(b) &= 4 \int_0^\infty dx\, dy\, \mathrm{e}^{-x^2-y^2} y^{2a-1} x^{2b-1} \\ &= 4 \int_0^\infty dr\, r \int_0^{\pi/2} d\theta\, \mathrm{e}^{-r^2} r^{2a+2b-2} (\sin\theta)^{2a-1} (\cos\theta)^{2b-1} \\ &= 2 \int_0^{\pi/2} d\theta\, (\sin\theta)^{2a-1} (\cos\theta)^{2b-1}\, 2 \int_0^\infty dr\, \mathrm{e}^{-r^2}\, r^{2(a+b)-1} \\ &= 2 \int_0^{\pi/2} d\theta\, (\sin\theta)^{2a-1} (\cos\theta)^{2b-1}\, \Gamma(a+b) \end{aligned} \tag{A.6}$$

Since $d\sin^2\theta = 2\sin\theta\cos\theta\, d\theta$ we can write

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = 2\int_0^{\pi/2} d\theta\, \left(\sin\theta\right)^{2a-1} \left(\cos\theta\right)^{2b-1} = \int_0^1 d\sin^2\theta \left(\sin^2\theta\right)^{a-1} \left(\cos^2\theta\right)^{b-1} \tag{A.7}$$

Calling $x = \sin^2\theta$ we have

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} \tag{A.8}$$

We can define the "beta" function as

$$B(a,b) \equiv \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = 2\int_0^{\frac{\pi}{2}} d\theta\, \left(\sin\theta\right)^{2a-1} \left(\cos\theta\right)^{2b-1} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,. \tag{A.9}$$

A useful expansion is given by

$$\Gamma(1+\epsilon) = 1 - \gamma_E\,\epsilon + \frac{6\,\gamma_E^2 + \pi^2}{12}\,\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)\ , \tag{A.10}$$

where $\gamma_E = 0.5772157\ldots$ is the Euler-Mascheroni constant.

A.2 The angular volume Ω_d in d dimensions

In order to compute the total angular volume in d dimensions we proceed as follows. We consider the integral I

$$I \equiv \left(\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-x^2}\right)^d = \left(\sqrt{\pi}\right)^d \tag{A.11}$$

and we rewite the lhs of the equation as

$$I = \int_{-\infty}^{\infty} dx_1\, dx_2 \ldots dx_d\, \mathrm{e}^{-(x_1^2+x_2^2+\ldots+x_d^2)} = \int d\Omega_d \int_0^{\infty} dr\, r^{d-1}\mathrm{e}^{-r^2} \tag{A.12}$$

The r integration can be performed using eq. (A.2)

$$I = \Omega_d\, \frac{\Gamma\left(\frac{d}{2}\right)}{2} \tag{A.13}$$

so that

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{A.14}$$

A.3 The δ distribution

La δ di Dirac è una distribuzione (o funzione generalizzata) definita dal seguente integrale ($a < b$)

$$\int_a^b dx\, f(x)\, \delta(x - x_0) = \begin{cases} f(x_0) & a < x_0 < b \\ 0 & \text{altrove} \end{cases} \tag{A.15}$$

dove $f(x)$ è una funzione sufficientemente regolare nell'intorno di x_0. Inoltre, x_0 deve appartenere all'intervallo di integrazione. Altrimenti l'integrale è zero. Dalla (A.15) segue immediatamente che

$$\int dx\, \delta(x - x_0) = 1 \tag{A.16}$$

e, con un semplice cambio di variabili nell'integrale, che

$$\delta(a\,(x - x_0)) = \frac{1}{|a|}\, \delta(x - x_0)\,. \tag{A.17}$$

A partire dall'eq. (A.15), possiamo anche dare un significato alle derivate della delta. Per esempio, integrando per parti ($a < x_0 < b$)

$$\begin{aligned} \int_a^b dx\, f(x)\, \frac{d}{dx}\delta(x - x_0) &= \int_a^b dx\, \frac{d}{dx}\left[f(x)\, \delta(x - x_0)\right] - \int_a^b dx\, \frac{df(x)}{dx}\, \delta(x - x_0) \\ &= f(x)\, \delta(x - x_0)\Big|_a^b - \int_a^b dx\, \frac{df(x)}{dx}\, \delta(x - x_0) \\ &= -\int_a^b dx\, \frac{df(x)}{dx}\, \delta(x - x_0) = -\left(\frac{df}{dx}\right)_{x = x_0}\,. \end{aligned} \tag{A.18}$$

Quindi possiamo scrivere in modo formale

$$\frac{d}{dx}\delta(x - x_0) = -\delta(x - x_0)\frac{d}{dx}\,. \tag{A.19}$$

Le derivate di ordine superiore si calcolano integrando ripetutamente per parti, scaricando una alla volta le derivate dalla δ sulla funzione.

La funziona a gradino

Dalla definizione (A.15) si può scrivere (a è un parametro dato)

$$\int_{-\infty}^x dy\, \delta(y - a) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases} \;\equiv\; \theta(x - a) \tag{A.20}$$

chiamata anche "funzione θ". Derivando l'equazione si può dare significato preciso alla derivata di un gradino

$$\frac{d}{dx}\theta(x - a) = \delta(x - a) \tag{A.21}$$

Importante identità
Una relazione molto importante è la seguente

$$\delta\left[g(x)\right] = \sum_i \frac{\delta(x - x_i)}{|dg/dx(x_i)|}\,, \tag{A.22}$$

dove gli x_i sono le radici (semplici) di $g(x)$ nell'intervallo di integrazione.

"Rappresentazioni" della $\boldsymbol{\delta}$
In generale, ogni funzione "molto piccata", normalizzata a 1 e con la larghezza che va a zero, può essere usata come "rappresentazione" della distribuzione δ.

1.
$$\delta(x) = \lim_{\epsilon \to 0} R(x, \epsilon) \tag{A.23}$$

dove

$$R(x,\epsilon) = \begin{cases} 0 & x < -\epsilon \\ \dfrac{1}{2\epsilon} & -\epsilon < x < \epsilon \\ 0 & x > \epsilon \end{cases} \tag{A.24}$$

2.
$$\delta(x) = \lim_{\alpha \to \infty} \frac{\sin \alpha x}{\pi x} \tag{A.25}$$

3.
$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \tag{A.26}$$

4.
$$\delta(x) = \lim_{\alpha \to \infty} \frac{\alpha}{\sqrt{\pi}} \exp(-\alpha^2 x^2) \tag{A.27}$$

5.
$$\delta(x) = \lim_{\alpha \to \infty} \frac{\sin^2 \alpha x}{\pi \alpha x^2} \tag{A.28}$$

6.
$$\lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} = \mathrm{VP}\,\frac{1}{x} - i\pi\delta(x) \tag{A.29}$$

dove "VP" indica l'integrazione fatta in Valor Principale, ovvero, se $a < 0 < b$ e $\epsilon > 0$

$$\mathrm{VP}\int_a^b dx\,\frac{1}{x}\,f(x) = \lim_{\epsilon \to 0}\left\{\int_a^{-\epsilon} dx\,\frac{1}{x}\,f(x) + \int_\epsilon^b dx\,\frac{1}{x}\,f(x)\right\} \tag{A.30}$$

Esercizio: Dimostrare l'eq. (A.22).

Esercizio: Verificare che le funzioni dall'eq. (A.23) all'eq. (A.30) sono normalizzate a 1.

Esercizio
Dimostare che

$$\int_0^1 dx \int_0^1 dy \, \frac{\delta(1-x-y)}{(ax+by)^2} = \frac{1}{ab} \tag{A.31}$$

Esercizio
Dimostare che

$$2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{\delta(1-x-y-z)}{(ax+by+cz)^3} = \frac{1}{abc} \tag{A.32}$$

Esercizio
Dimostare che

$$6 \int_0^1 dx \int_0^1 dy \int_0^1 dz \, y \, \frac{\delta(1-x-y-z)}{(ax+by+cz)^4} = \frac{1}{ab^2c} \tag{A.33}$$

A.4 The dilogarithm function

The dilogarithm function is defined by

$$\mathrm{Li}_2\,(x) = -\int_0^x dz \, \frac{\log(1-z)}{z} \qquad x \le 1 \,, \tag{A.34}$$

and an immediate consequence of this definition is the following expansion in powers of ϵ

$$\begin{aligned} \int_0^1 dx \, x^{-1-\gamma\,\epsilon} \, (1-\alpha\, x)^{\beta\,\epsilon} &= \int_0^1 dx \, x^{-1-\gamma\,\epsilon} \left[1 + \beta\,\epsilon \log(1-\alpha\, x) + \mathcal{O}\left(\epsilon^2\right)\right] \\ &= -\frac{1}{\gamma\,\epsilon} - \beta\,\epsilon\, \mathrm{Li}_2\,(\alpha) + \mathcal{O}\left(\epsilon^2\right) \,. \end{aligned} \tag{A.35}$$

It can easiky be shown that

$$\int_0^\alpha dz \, \frac{\log(1+\beta z)}{z} = -\,\mathrm{Li}_2\,(-\alpha\beta) \tag{A.36}$$

One of the most used properties is the analytic continuation of the dilogarithm function

$$\mathrm{Li}_2\,(x \pm i\eta) = -\,\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{1}{2}\log^2 x + \frac{\pi^2}{3} \pm i\pi \log x \qquad x > 1 \,, \tag{A.37}$$

that can be demonstrated with the help of

$$\log(-x \pm i\eta) = \log x \pm i\pi \qquad x > 0 \,. \tag{A.38}$$

In addition

$$\int_0^1 dx\, x^{-1+\beta\epsilon} \frac{1}{(1+\alpha\, x)^{1+\gamma\epsilon}} = (1+\alpha)^{-\beta\epsilon} \int_0^1 dx\, x^{-1+\beta\epsilon} \left(1 - \frac{\alpha}{1+\alpha} x\right)^{(\gamma-\beta)\epsilon} \qquad \alpha > -1 \, , \tag{A.39}$$

where we have used the projective transformation

$$x \to \frac{x}{1+\alpha\,(1-x)} \, . \tag{A.40}$$

Appendix B

A few radiative corrections

B.1 The vertex correction

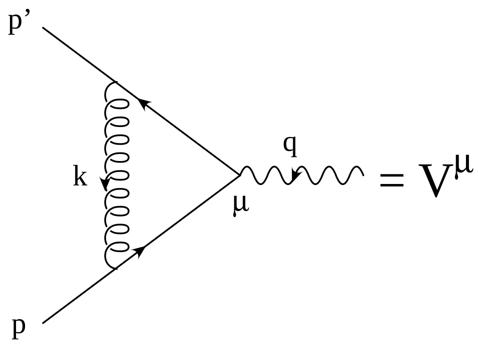


Figure B.1: Vertex correction.

We consider now the vertex correction depicted in Fig. B.1. All momenta are incoming, satisfying $p'^{\mu} + p^{\mu} + q^{\mu} = 0$, $p^2 = p'^2 = 0$, $q^2 = 2\, p \cdot p' = s$. We define the vertex correction as (trivial color factors and coupling constants not included)

$$V^{\mu} = \int \frac{d^d k}{(2\pi)^d}\, \bar{v}(p')\gamma^{\alpha} \left(\not{k} - \not{p}'\right) \gamma^{\mu} \left(\not{k} + \not{p}\right) \gamma_{\alpha} u(p)\; \frac{1}{k^2 (k+p)^2 (k+p+q)^2}. \tag{B.1}$$

We deal first with the numerator, and we contract the index α in $\bar{d}$ dimensions. We obtain

$$\begin{aligned} N \equiv \bar{v}(p')\gamma^{\alpha} \left(\not{k} - \not{p}'\right) \gamma^{\mu} \left(\not{k} + \not{p}\right) \gamma_{\alpha} u(p) \;&=\; \left(4 - \bar{d}\right) \bar{v}(p') \left(\not{k} - \not{p}'\right) \gamma^{\mu} \left(\not{k} + \not{p}\right) u(p) \\ &\quad - 2\, \bar{v}(p') \left(\not{k} + \not{p}\right) \gamma^{\mu} \left(\not{k} - \not{p}'\right) u(p), \end{aligned} \tag{B.2}$$

that can be further simplified using

$$\not{p} u(p) = 0, \qquad\qquad \bar{v}(p')\not{p}' = 0, \tag{B.3}$$

to

$$N = \bar{v}(p')\left\{2\left(2-\bar{d}\right)k^{\mu}\not{k} + \left[\left(\bar{d}-2\right)k^2 + 4\left(p\cdot k - p'\cdot k - p\cdot p'\right)\right]\gamma^{\mu}\right\}u(p). \tag{B.4}$$

The loop integration is performed in d dimensions, and we get

$$V^{\mu} = \left[\underbrace{-\frac{2-\bar{d}}{2-d}B_0(q)}_{\not{k}k^{\mu}}\underbrace{+(\bar{d}-2)B_0(q)}_{k^2\gamma^{\mu}}\underbrace{-4B_0(q)}_{(p\cdot k-p'\cdot k)\gamma^{\mu}}\underbrace{-2q^2\,C_0(p,q)}_{p\cdot p'\gamma^{\mu}}\right]\bar{v}(p')\gamma^{\mu}u(p), \tag{B.5}$$

where B_0 and C_0 are the two- and three-point scalar functions (see Passarino-Veltman). We can finally write

$$V^{\mu} = \left[\frac{\bar{d}(d-3)+14-6d}{d-2}B_0(q) - 2q^2\,C_0(p,q)\right]\bar{v}(p')\gamma^{\mu}u(p). \tag{B.6}$$

This is a completely general expression that can be further specified in **CDR** and **DR**:

- **CDR**: $\bar{d} = d$
$$V^{\mu} = \left[(d-7)B_0(q) - 2q^2\,C_0(p,q)\right]\bar{v}(p')\gamma^{\mu}u(p), \tag{B.7}$$

- **DR**: $\bar{d} = 4$
$$V^{\mu} = \left[\frac{2(d-1)}{2-d}B_0(q) - 2q^2\,C_0(p,q)\right]\bar{v}(p')\gamma^{\mu}u(p). \tag{B.8}$$

The expressions for the scalar integrals in $d = 4-2\epsilon$ dimensions are given by ($q^2 = s$)

$$B_0(p) \equiv B_0(s) \tag{B.9}$$
$$C_0(p,q) \equiv C_0(s) \tag{B.10}$$

where

$$B_0(s) = \frac{i}{(4\pi)^2}c_{\Gamma}s^{-\epsilon}e^{i\pi\epsilon}\frac{1}{\epsilon(1-2\epsilon)} \tag{B.11}$$
$$C_0(s) = \frac{i}{(4\pi)^2}c_{\Gamma}s^{-1-\epsilon}e^{i\pi\epsilon}\frac{1}{\epsilon^2} \tag{B.12}$$

with

$$c_{\Gamma} = (4\pi)^{\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}. \tag{B.13}$$

We can then write the QCD vertex correction for $q\bar{q}$ (color indexes i and j) with a vector particle, as

- **CDR**:
$$V^{\mu} = \delta_{ij}\,C_{\mathrm{F}}\left(\frac{\mu^2}{q^2}\right)^{\epsilon}\frac{\alpha_{\mathrm{S}}}{4\pi}c_{\Gamma}\left\{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \pi^2 - 8 + i\pi\left[-\frac{2}{\epsilon} - 3\right] + \mathcal{O}(\epsilon)\right\}\bar{v}(p')\gamma^{\mu}u(p) \tag{B.14}$$

- **DR**:

$$V^\mu = \delta_{ij}\, C_{\rm F} \left(\frac{\mu^2}{q^2}\right)^\epsilon \frac{\alpha_{\rm S}}{4\pi}\, c_\Gamma \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \pi^2 - 8 + 1 + i\pi \left[-\frac{2}{\epsilon} - 3 \right] + \mathcal{O}(\epsilon) \right\} \bar{v}(p')\gamma^\mu u(p) \tag{B.15}$$

Please note the difference of one unit (here written in red) between the two expressions in curly braces.

The same results can be obtained if we first perform the loop integration and then we contract the indexes in the corresponding space.

B.2 Radiative corrections to external heavy-quark lines

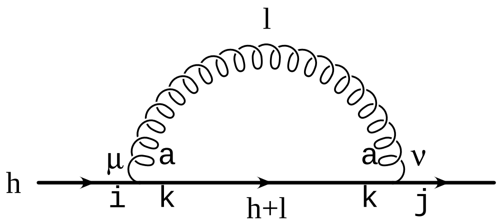


Figure B.2: Quark self-energy $\Sigma_{ij}(h)$.

In this section we describe how to treat loop corrections to **external** heavy-quark lines and how to compute the mass counterterm. We follow the same steps described in Ref. [12].

The one loop correction to a quark propagator is defined as

$$\Sigma_{ij}(h) = \int \frac{d^d l}{(2\pi)^d} \left(-i g_{\rm S} \mu^\epsilon \gamma^\nu t^a_{jk}\right) \frac{i}{\not h + \not l - m} \left(-i g_{\rm S} \mu^\epsilon \gamma^\mu t^a_{ki}\right) \frac{-i g_{\mu\nu}}{l^2}, \tag{B.16}$$

where the sum over repeated indexes is meant and μ is the mass parameter of the dimensional regularization, introduced in order to keep $g_{\rm S}$ dimensionless. With a little algebra we have

$$\begin{aligned}
\Sigma_{ij}(h) &= -g_{\rm S}^2 \mu^{2\epsilon - 2\epsilon} C_{\rm F}\, \delta_{ij} \int \frac{d^d l}{(2\pi)^d} \frac{\gamma^\mu \left(\not l + \not h + m\right) \gamma_\mu}{l^2 \left[(h+l)^2 - m^2\right]} \\
&= -g_{\rm S}^2 \mu^{2\epsilon - 2\epsilon} C_{\rm F}\, \delta_{ij} \int \frac{d^d l}{(2\pi)^d} \frac{\left(2 - \bar{d}\right)\left(\not h + \not l\right) + \bar{d}\, m}{l^2 \left[(h+l)^2 - m^2\right]} \\
&= -g_{\rm S}^2 \mu^{2\epsilon - 2\epsilon} C_{\rm F}\, \delta_{ij} \left\{ \left[\left(2 - \bar{d}\right) \not h + \bar{d}\, m\right] \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 \left[(h+l)^2 - m^2\right]} \right. \\
&\quad \left. + \left(2 - \bar{d}\right) \int \frac{d^d l}{(2\pi)^d} \frac{\not l}{l^2 \left[(h+l)^2 - m^2\right]} \right\} .
\end{aligned} \tag{B.17}$$

In the last integral we can replace

$$l_\mu \to l \cdot h \, \frac{h_\mu}{h^2} = \frac{1}{2} \left\{ \left[(h+l)^2 - m^2 \right] - h^2 - l^2 + m^2 \right\} \frac{h_\mu}{h^2} \,, \tag{B.18}$$

and we obtain

$$\begin{aligned} \Sigma_{ij}(h) &= -g_{\mathrm{S}}^2 \mu^{2\epsilon - 2\epsilon} C_{\mathrm{F}} \, \delta_{ij} \Bigg\{ \left[\left(2 - \bar{d}\right) \not{h} + \bar{d}\, m \right] B_0(h^2) \\ &\qquad + \frac{\not{h}}{2h^2} \left(2 - \bar{d}\right) \left[\left(m^2 - h^2\right) \, B_0(h^2) - A_0(m^2) \right] \Bigg\} \\ &= -g_{\mathrm{S}}^2 \mu^{2\epsilon - 2\epsilon} C_{\mathrm{F}} \, \delta_{ij} \left\{ \left[\left(-1 + \epsilon - \epsilon - (1 - \epsilon + \epsilon) \frac{m^2}{h^2} \right) \not{h} + (4 - 2\epsilon + 2\epsilon)\, m \right] B_0(h^2) \right. \\ &\qquad \left. + \frac{\not{h}}{h^2} (1 - \epsilon + \epsilon) \, A_0(m^2) \right\} \,, \end{aligned} \tag{B.19}$$

where

$$A_0(m^2) \equiv \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{l^2 - m^2} \,, \tag{B.20}$$

$$B_0(h^2) \equiv \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{l^2} \, \frac{1}{(h+l)^2 - m^2} \,, \tag{B.21}$$

and we have used $\bar{d} = 4 - 2\epsilon + 2\epsilon$.

We are interested in the expansion of $\Sigma_{ij}(h)$ around $\not{h} = m$

$$\Sigma_{ij}(h) = \Sigma_{ij}(h)\big|_{\not{h}=m} + (\not{h} - m) \, \frac{\partial \Sigma_{ij}(h)}{\partial \not{h}} \bigg|_{\not{h}=m} + \mathcal{O}\left((\not{h} - m)^2 \right) \,. \tag{B.22}$$

Using the identity

$$\not{h}\,\not{h} = h^2 \;\Rightarrow\; 2\not{h}\,\partial\not{h} = \partial h^2 \;\Rightarrow\; \frac{\partial}{\partial \not{h}} = 2\not{h} \frac{\partial}{\partial h^2} \,, \tag{B.23}$$

so that

$$\frac{\partial \Sigma_{ij}(h)}{\partial \not{h}} \bigg|_{\not{h}=m} = 2\, m \, \frac{\partial \Sigma_{ij}(h)}{\partial h^2} \bigg|_{\not{h}=m} \,, \tag{B.24}$$

and

$$A_0(m^2) = \frac{i}{(4\pi)^2} \, c_\Gamma \, \left(m^2\right)^{-\epsilon} m^2 \left(\frac{1}{\epsilon} + 1 + \mathcal{O}(\epsilon) \right) \,, \tag{B.25}$$

$$B_0(m^2) = \frac{i}{(4\pi)^2} \, c_\Gamma \, \left(m^2\right)^{-\epsilon} \left(\frac{1}{\epsilon} + 2 + \mathcal{O}(\epsilon) \right) \,, \tag{B.26}$$

$$\frac{\partial B_0(h^2)}{\partial h^2} \bigg|_{h^2=m^2} = \frac{i}{(4\pi)^2} \, c_\Gamma \, \left(m^2\right)^{-\epsilon} \frac{1}{m^2} \left(-\frac{1}{2\epsilon} - 1 + \mathcal{O}(\epsilon) \right) \,. \tag{B.27}$$

we can write

$$\begin{aligned}
\Sigma_{ij}(h) &= -g_S^2 C_F \frac{i}{(4\pi)^2} c_\Gamma \left(\frac{\mu^2}{m^2}\right)^\epsilon \delta_{ij} \left\{ \left[\frac{3}{\epsilon} + 4 + 1\right] m + (\not{h} - m) \left[-\frac{3}{\epsilon} - 4 - 1\right]\right\} \\
&\quad + \mathcal{O}\left((\not{h} - m)^2\right) \\
&\equiv \delta_{ij} \left[-i\,\delta m - i\,(\not{h} - m)\, z_Q\right] + \mathcal{O}\left((\not{h} - m)^2\right) ,
\end{aligned} \tag{B.28}$$

where

$$\begin{aligned}
\delta m &= \frac{g_S^2}{(4\pi)^2} c_\Gamma\, C_F \left(\frac{\mu^2}{m^2}\right)^\epsilon \left[\frac{3}{\epsilon} + 4 + 1\right] m \\
z_Q &= -\frac{g_S^2}{(4\pi)^2} c_\Gamma\, C_F \left(\frac{\mu^2}{m^2}\right)^\epsilon \left[\frac{3}{\epsilon} + 4 + 1\right] .
\end{aligned} \tag{B.29}$$

The full quark propagator at first order reads

$$\begin{aligned}
G_Q(h) &= \frac{i\,\delta_{ij}}{\not{h} - m} + \frac{i\,\delta_{ik}}{\not{h} - m}\, \Sigma_{kl}(h)\, \frac{i\,\delta_{lj}}{\not{h} - m} + \mathcal{O}\left(\alpha_S^2\right) \\
&= \frac{i\,\delta_{ij}}{\not{h} - m} (1 + z_Q) + \frac{i\,\delta_{ik}}{\not{h} - m} (-i\,\delta m) \frac{i\,\delta_{kj}}{\not{h} - m} + \frac{\mathcal{O}\left(h^2 - m^2\right)}{\not{h} - m} .
\end{aligned} \tag{B.30}$$

If we want that the pole of the propagator is not displaced by radiative corrections, so that m corresponds to the pole mass definition, we have to add a mass counterterm to cancel the second term of the above expression. For this reason, we define the Feynman rule for the mass counterterm as the insertion, in the fermion propagator, of the vertex $-i\,m_c$, where

$$m_c \equiv -\delta m = -\frac{g_S^2}{(4\pi)^2} c_\Gamma\, C_F \left(\frac{\mu^2}{m^2}\right)^\epsilon \left[\frac{3}{\epsilon} + 4 + 1\right] m . \tag{B.31}$$

In this way, **slightly off-shell**, the quark propagator behaves like

$$G_Q(h) = \frac{i\,\delta_{ij}}{\not{h} - m} Z_Q , \tag{B.32}$$

with

$$Z_Q = 1 + z_Q = 1 - \frac{g_S^2}{(4\pi)^2} c_\Gamma\, C_F \left(\frac{\mu^2}{m^2}\right)^\epsilon \left[\frac{3}{\epsilon} + 4 + 1\right] . \tag{B.33}$$

Appendix C

Problems

1. Compute the self-energy corrections to a quark propagator of mass m at first order in α_S. Derive the LSZ Z_Q and the mass counterterm.
2. Compute the self-energy corrections to a gluon propagator, due to a massive-quark loop (quark mass m) at first order in α_S. Derive the LSZ Z_g.
3. Compute the collinear limit for $g \to gg$ splitting (see fig. C.1) i.e. derive the singular part of the square of the invariant amplitude when two collinear gluons are produced.

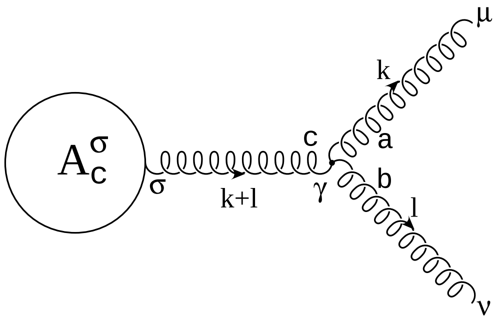


Figure C.1: Gluon splitting into a gg pair.

Check your answer against the following

$$\sum_{\text{col,spin}} \mathcal{A}^{ab}\mathcal{A}^{\dagger}_{ab} \simeq \frac{g_s^2}{2\, l\cdot k}\, 4\, C_A \left\{ -\left[-2 + \frac{1}{z} + \frac{1}{1-z} + z\,(1-z) \right] g_{\sigma\sigma'} \right.$$
$$\left. - 2\, z\,(1-z)(1-\epsilon) \left[\frac{k_{\perp\sigma} k_{\perp\sigma'}}{k_\perp^2} - \frac{g_{\perp\sigma\sigma'}}{2-2\,\epsilon} \right] \right\} A_c^{\sigma}(t) A_c^{\dagger\sigma'}(t)$$

Suggestion: please note that, in the collinear limit, the amplitude for the emission of two gluons in the final state can be decomposed into two parts: the first one contains the graph where the two gluons are emitted by a single virtual one (see fig. C.1), and the other one contains all the other graphs

$$\mathcal{A}^{ab} = \left\{ \mathcal{A}_c^\sigma(l+k)\, \frac{iP_{\sigma\gamma}(k+l)}{(k+l)^2}\, (-g_s)\, f^{abc}\, \Gamma^{\mu\nu\gamma}(-k,-l,k+l) + \mathcal{R}_{ab}^{\mu\nu} \right\} \epsilon_\mu(k)\, \tilde{\epsilon}_\nu(l)\ , \tag{C.1}$$

where a and b are the colour indexes of the final gluons, P is the spin projector of the gluon propagator, g_s is the strong coupling constant, f^{abc} are the structure constants of the SU(3) gauge group, ϵ and $\tilde{\epsilon}$ are the polarization vectors of the final gluons, $\Gamma^{\mu\nu\gamma}$ is the Lorentz part of the three-gluon vertex

$$\Gamma^{\mu\nu\gamma}(-k,-l,k+l) = (-k+l)^\gamma g^{\mu\nu} + (-2l-k)^\mu g^{\nu\gamma} + (2k+l)^\nu g^{\mu\gamma}\,, \tag{C.2}$$

and the gluon spin projector is given by

$$P^{\sigma\gamma}(p) = -g^{\sigma\gamma} + \frac{\eta^\sigma p^\gamma + \eta^\gamma p^\sigma}{\eta\cdot p} \equiv -g_\perp^{\sigma\gamma}\,, \qquad \text{with} \qquad \eta^2 = 0\,. \tag{C.3}$$

In addition, use the following Sudakov decomposition

$$k^\mu = z\, t^\mu + \xi'\eta^\mu + k_\perp^\mu \tag{C.4}$$
$$l^\mu = (1-z)\, t^\mu + \xi''\eta^\mu - k_\perp^\mu\ , \tag{C.5}$$

with $k_\perp$ such that

$$t\cdot k_\perp = 0\ , \qquad \eta\cdot k_\perp = 0\ . \tag{C.6}$$

4. Compute the collinear limit for the $g \to q\bar{q}$ splitting (see fig. C.2), i.e. the singular part of the square of

$$\mathcal{A}_{ij} = \mathcal{A}_c^\sigma(k+l)\, \frac{iP_{\rho\sigma}(k+l)}{(k+l)^2}\, \bar{u}(k)\left(-ig_s\gamma^\rho t_{ij}^c\right) v(l)\ , \tag{C.7}$$

where P is given by eq. (C.3) and t^c are the generators of SU(3) gauge symmetry.

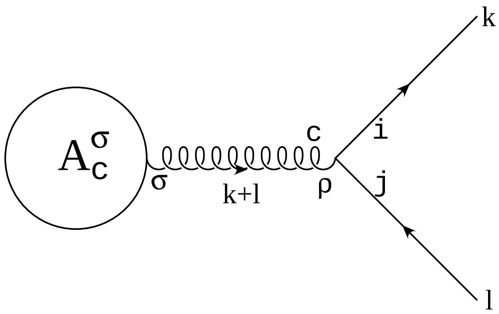


Figure C.2: Gluon splitting into a $q\bar{q}$ pair.

5. Following what has been done for the calculation of P_{qq}, compute P_{gq} and P_{gg}.

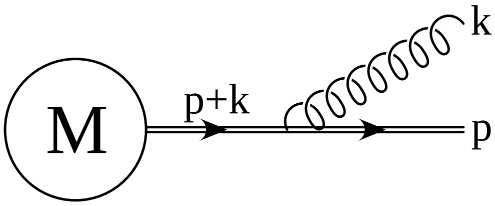


Figure C.3: Heavy quark splitting.

6. Compute the Altarelli-Parisi splitting function P_{QQ} for a final-state quark Q of mass m (see fig. C.3).

   Suggestions: Start from the following

$$A_0(p) \;=\; \mathcal{M}_0^\dagger(p)\,\mathcal{M}_0(p) = M^\dagger(p)\,(\not{p}+m)\,M(p) \tag{C.8}$$

$$A(p,k) \;=\; \mathcal{M}^\dagger(p,k)\mathcal{M}(p,k) = g_s^2\,C_F\,\frac{1}{(2\,p\cdot k)^2}\,M_0^\dagger(p+k)\;N\;M_0(p+k) \tag{C.9}$$

$$N \;=\; \sum_{\text{pol}}\epsilon^{*\mu}\epsilon^{\nu}\,[\not{p}+\not{k}+m]\,\gamma_\mu\,(\not{p}+m)\,\gamma_\nu\,[\not{p}+\not{k}+m] \tag{C.10}$$

   and derive

$$A(p,k) \;\simeq\; 8\pi\alpha_{\rm s}\,\frac{z(1-z)}{(1-z)^2m^2-k_\perp^2}\,P_{QQ}\,A_0\left(\frac{p}{z}\right) \tag{C.11}$$

$$P_{QQ} \;=\; C_F\left[\frac{1+z^2}{1-z}-\frac{2z(1-z)m^2}{(1-z)^2m^2-k_\perp^2}\right] \tag{C.12}$$

7. Cosider the scattering of two protons at center-of-mass equal to $\sqrt{S}$. After suggesting a reasonably-simple and physically-sound form for the parton distribution function for a u and d quark in a proton, compute the contribution from the ud scattering to the total cross section for $pp \to ud + X$ at order $\alpha_{\rm s}^2$ in QCD.

8. Compute the first-order QCD corrections to the deep-inelastic process $e^-q \to e^-q$ in dimensional regularization.

   Do the calculation both in the Lorentz and in the axial gauge. What can you learn from this?

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