Karlsruher Institut für Technologie

Thesis

by

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Emission kernel of parton shower

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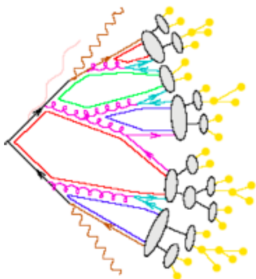

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statement of originality

I hereby confirm that I have written the accompanying thesis by myself, without contributions from any sources other than those cited in the text and acknowledgements. This applies also to all graphics, drawings, maps and images included in the thesis.

Karlsruhe, June 12, 2019

Tigran Saidnia

Abstract

Emission kernel of parton shower

For the results of particle physics experiments in connection with the strong interaction the perturbation theory is used. In order to make useful and more accurate predictions, the calculations must be carried out at least in next-to-leading order, in order to avoid large uncertainties arising from the unphysical scale dependencies. In this work, the dipole subtraction method will be used for calculating next-to-leading order corrections in QCD. The algorithm is straightforwardly implementable ingeneral purpose Monte Carlo programs.

n this paper we outline a new parton shower algorithm based on the Catani-Seymour dipole factorization. Our motivation is to have an algorithm which cannaturally cooperate with the NLO calculations.

In this paper we aim to outline some new ideas for a parton showeralgorithm along our criteria. In the next we will discuss a parton showeralgorithm based on the Catani-Seymour dipole factorization formulas [3]and give a general prescription for the matching of parton showerand fixorder computation at leading and next-to-leading order level. It was aobvious choice to use the dipole factorization because it fulfills our criteriaand it makes the NLO level matching easier because lots of modern NLOfix order computations are based on the dipole subtraction method, forexample NLOJET++[5] and MCFM[6].

The whole parton shower idea is based on the factorization properties ofthe matrix elements in the soft and collinear regions. In this region anm+1 parton matrix element can be written as product anm-parton matrixelement and an universial singular factor. Using this approximtion, onecan start from a very simple configuration and generate large multiplicitymultiparton states.

https://arxiv.org/pdf/hep-ph/0601021.pdf

The combination of Parton shower simulations with corrections obtained by matrix elements of leading order and the addition of NLO to the production process has been investigated in the past. We present a new general algorithm for calculating arbitrary jet cross sectionsin arbitrary scattering processes to next-to-leading accuracy in perturbative-QCD. The algorithm is based on the subtraction method. The key ingredi-ents are new factorization formulae, called dipole formulae, which implementin a Lorentz covariant way both the usual soft and collinear approximations,smoothly interpolating the two. The corresponding dipole phase space obeysexact factorization, so that the dipole contributions to the cross section can beexactly integrated analytically over the whole of phase space. We obtain explicitanalytic results for any jet observable in any scattering orfragmentation processin lepton, lepton-hadron or hadron-hadron collisions. Allthe analytical formu-lae necessary to construct a numerical program for next-to-leading order QCDcalculations

are provided. The algorithm is straightforwardly implementable ingeneral purpose Monte Carlo programs.
Unfortunately, standard NLO programs have significant flaws. One flawis that the final states consist just of a few partons, while in nature finalstates consist of many hadrons. A worse flaw is that the weights are oftenvery large positive numbers or very large negative numbers.There is another class of calculational tools, the shower Monte Carloevent generators, such as HERWIG[1] and PYTHIA[2]. These have thesignificant advantage that the objects in the final state consist of hadrons
We begin in Sect. 2 by giving a brief overview of the general method, describing thesubtraction procedure and how our dipole formulae are used to implement it. In Sect. 3 weestablish the notation used throughout the paper. In Sect. 4we review the factorizationproperties of QCD matrix elements in the soft and collinear limits before presenting, inSect. 5, our dipole factorization formulae, which smoothlyinterpolate these two limitingregions. After briefly recalling, in Sect. 6, the precise definitions of QCD cross sectionsat NLO, we go on to describe in detail our subtraction method for evaluating these crosssections, in Sects. 7–11. In Sect. 12 we summarize and discuss our results. Appendix Agives more details, and some examples, of the necessary colour algebra. In Appendix B weexplicitly perform the only difficult integral we encounter.In Appendix C we collect to-gether the main formulae needed to implement our method in specific calculations. Finallyin Appendix D we work through a few simple examples of applying our method to specificcross sections

Contents

Introduction

Introduction

Chapter 1

Theoretical Basics

1.1 Brief history of particle physics

Knowledge is a human need. For thousands of years we have been trying to understand the secrets of the universe. Such riddles fascinated even Johann Wolfgang von Goethe, as he wrote in his book Faust [14]; eine Tragedie, "What holds the world together in its innermost." Almost 400 years before Christ, an ancient Greek philosopher, Democritus, and his teacher Leukipp claimed that matter cannot be divided at will. Rather, there must be an Atomos (Greek: indivisible) that could no longer be subdivided. Democritus was of the opinion that there were infinitely many atoms with different geometric forms that were in contact in a certain way. He pointed out that a thing has a color, taste or even soul, based on the apparent effect of the composition of these small grains. [3]
This statement of Democritus was first laughed at by the renowned philosopher Aristotiles. It took about 2000 years for a chemist named John Dalton to deal with the subject. Based on various test series, he summarized his conclusion in his book A New System of Chemical Philosophy, that all substances consist of spherical indivisible atoms. The atoms of different elements have different masses and volumes. This was exactly the most striking difference to Democritus's atomic world.[6]
The discovery of the periodic system by **D. Mendeleev** and **P. Meyer** enabled us to arrange the atoms according to their mass in such a way that their properties occur in a certain order.[17]
In 1897 Joseph Thompson was able to obtain a stream of particles by heating metals and deflecting them by a magnetic field. This electron beam was 200 times lighter than the lightest atom, hydrogen. His conclusion was that atoms cannot be indivisible. He suggested that each atom consists of an electrically positively charged sphere in which electrically negatively charged electrons are stored - like raisins in a cake.
furthermore, renowned scientists as well as Marie and Pierre Curie have contributed much to the development of atomic theory by discovering radioactivity, Boltzmann by kinetic gas theory and Plank, the founder of quantum physics. However, one of the most important steps in the atomic model was taken by the British physicist Rutherford. He bombarded a thin aluminium foil with a radioactive sample. If Thompson's cake model were correct, only a few alpha particles would be detected behind the aluminium foil.

Surprisingly, many particles were visible, which could only be explained by the assumption that the majority of atoms consisted of empty spaces. Another miracle was that some particles could be seen above or below the target sample. Since we knew that the alpha particles were positively charged, we could assume the electric repulsive force of two positive charges. From the ideas of Planck and Rutherford, the Danish physicist Bohr (1885-1962) developed a planetary atomic model. The electrons then move around the nucleus in certain orbits, like planets orbit the sun. The orbits are also called shells. The special thing about it was that the distances of the electron orbits follow strict mathematical laws.

At first, however, it remained unclear what this core should consist of. [7, 17] In 1912, the Austrian physicist Victor Hess discovered during his balloon flights that the ionization rate of the Earth's atmosphere increases with altitude. This result was not expected because until then the Earth's radioactivity was known as the only source of air ionization. Therefore, he postulated this new type of radiation as cosmic radiation, which must originate outside the Earth's atmosphere [10].

Further investigations two years later confirmed the thesis of a cosmic background of such radiation. After this new discovery, it was discovered that the radiation consists of charged particles. In 1932, the American physicist Carl David Anderson was able to prove the postulated particle of Dirac, the positron, as a component of an air shower through his cloud chamber. For a long time, cosmic rays were the only way to analyse such exotic particles.[1] This changed when particle accelerators were able to generate particles in collisions. But even today, cosmic rays are the only way to study particles of the highest energies, since these energies cannot be reached by today's particle accelerators, such as the LHC. The LHC, the world's largest accelerator at CERN, produces particles with centre-of-mass energy equivalent to a cosmic particle of nearly $10^{17}eV$, with the energy spectrum of cosmic particles reaching up to $10^{20}eV$. However, we can only analyse such exotic particles in detail by increasing the luminosity and procession of the particle accelerators at the nucleus. The discovery of the neutron by **Chadwick** (1932) showed that atomic nuclei are made up of protons and neutrons. It was also clear that, in addition to gravitation and the electromagnetic force,there should exist two short-range forces in nature; a strong force which binds the nucleons together and a weak force which is responsible for radioactive. In the meantime it was agreed that a new theory was needed for the classification and grouping of this particle zoo. This is how the current standard model came into being.

The SLAC experiments indicate that the electrons can be scattered as quasi-free point-like constituents within the proton structure, which actually meant that the protons or neutrons are not point-like and must consist of other constituents. Through the bubble chamber a huge number of previously invisible particles (Gell-Mann's eightfold path) could suddenly be made visible, which represented contradictions to the previous physics. To explain this, the physicist Gell-Mann found basic building blocks from which all previously known atomic particles should be built. The components are later identified with quarks.

1.2 Standard model

The Standard Model in particle physics encompasses all of the Elementary particles and their interactions. It is a gauge theory spontaneously broken by the Higgs mechanism with the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.
From a theoretical point of view, the Standard Model is a quantum field theory that is based on local gauge invariance and consists of two rough parts. The electroweak sector $SU(2)_L \otimes U(1)_Y$ is called GWS (**Glashow-Weinberg-Salam**) theory and describes the gauge bosons $W^\pm, Z^0, \gamma$, the Higgs sector and its interaction with the leptons and quarks. In contrast to the other gauge bosons, the exchange particles of the weak interaction carry mass, which also affects the properties of the interactions. The color-charged sector $SU(3)_C$, the chromodynamics, deals with quarks and contains the eight massless, electrically neutral gluons as gauge bosons. The gauge groups $SU(3)_C$ and $SU(2)_L$ are non-abel gauge theories, more precisely **Yang Mills theories**. The massive particles, fermions, will be divided into two groups, leptons and quarks. Each group is arranged in 3 generations. Within the leptons there are three electrically neutral neutrinos. The mass of the particles increases from generation to generation. Neutrinos only interact weakly, whereas the charged leptons interact both weakly and electromagnetically. Quarks are characterized by the fact that they can also interact strongly [8].

1.3 Quantum chromo dynamics

Nowadays, we know there are four types of interactions.

Interaction	Energy scale	Range [m]	Mediators
Strong	~ 1	10^{-15}	g
Electromagnetic	$\sim 10^{-2}$	∞	γ
Weak	$\sim 10^{-6}$	10^{-18}	$W^\pm, Z$
Gravity	$\sim 10^{-38}$	∞	maybe graviton

Otherwise, it's clear that nucleons are made up of quark and gluons. Whereby, the gluons are the exchange bosons for this short ranged interaction. To explain the short range of the strong interaction Yukawa (1934) postulated mesons as a mediator for this force by the exchange of this massive field quanta. Three years later a candidate (π meson) was found in cosmic rays. Later on it was shown that massive gauge field quanta break the gauge symmetry so that the mediator must be massless. If it is based on the SU(3) gauge symmetry of the QCD[1] massless Lagrangian how can the strong sector be short range? Another question came from a series of experiments at SLAC. Through high-energy electron-proton scattering there was a search for evidence of the existence of quarks and their behaviour like free particles despite the energetically bound inner proton. The solution to these question was explained by Gross, Politzer and Wilczek through asymptotic freedom. This effect can be proved by the running coupling and anti screen-

[1]The quantum field theory which describes this area is called Quantum chromodynamics short QCD.

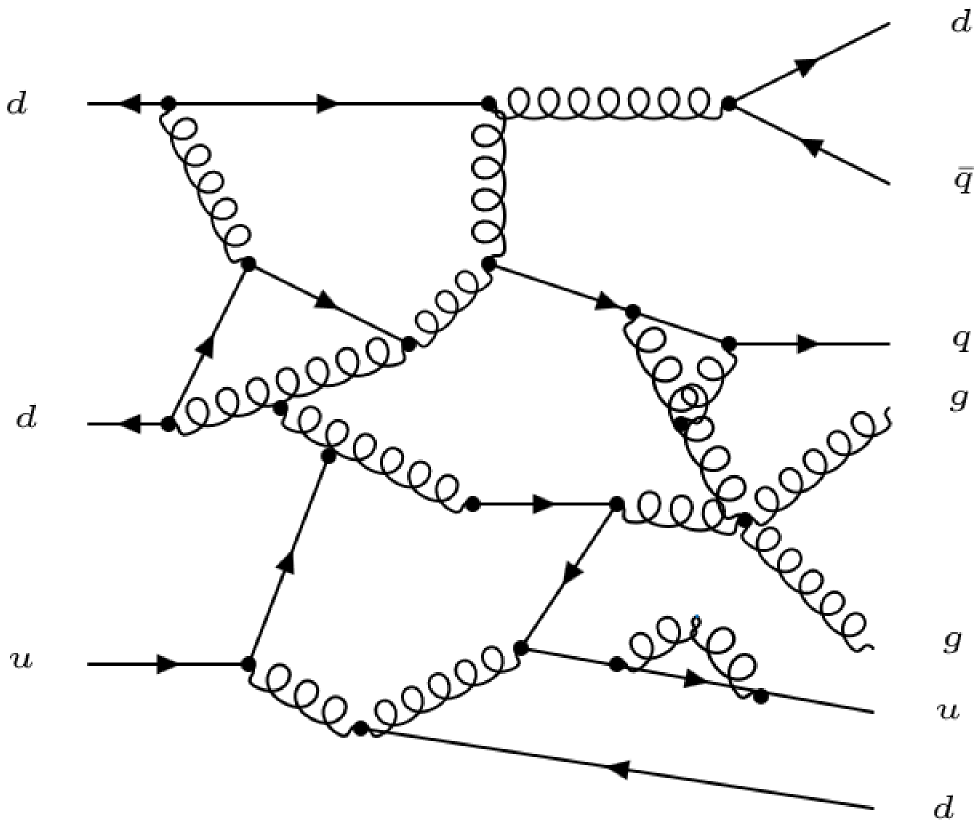


Figure 1.1: A schematic picture of neutron structure. at the left side of the resolution is too low to see. The 3 quarks picture allows us to interpretate the quantum numbers of the neutron in the valence band. We also obtain a high-resolution picture for a large Q^2. Here we have a lot of gluons (gluon sea) and quarks pairs. [2]
The interesting thing is, it doesn't matter in which energy scale we observe the quantum number of a neutron, because it is always the same.

ing in QCD. For the calculation of the propagator loop correction in QCD we have to consider both quark loops (negative contribution → screening) and gluon loops (positive contribution → anti screening).

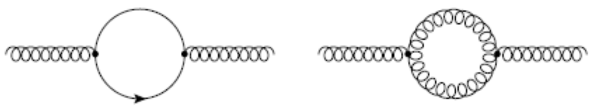


Figure 1.2: Running coupling compared for QED, with a positive and QCD with a negative beta function. The quark loop vacuum polarization diagram gives a negative contribution to $\beta_0 \sim n_f$ and the gluon loop gives a positive contribution to $\beta_0 \sim N_c$. The second contribution is bigger than the first, so that $\beta_0 > 0$ in QCD. In contrast to this, the function in QED is negative because the second contribution does not exist $N_C = 0$.

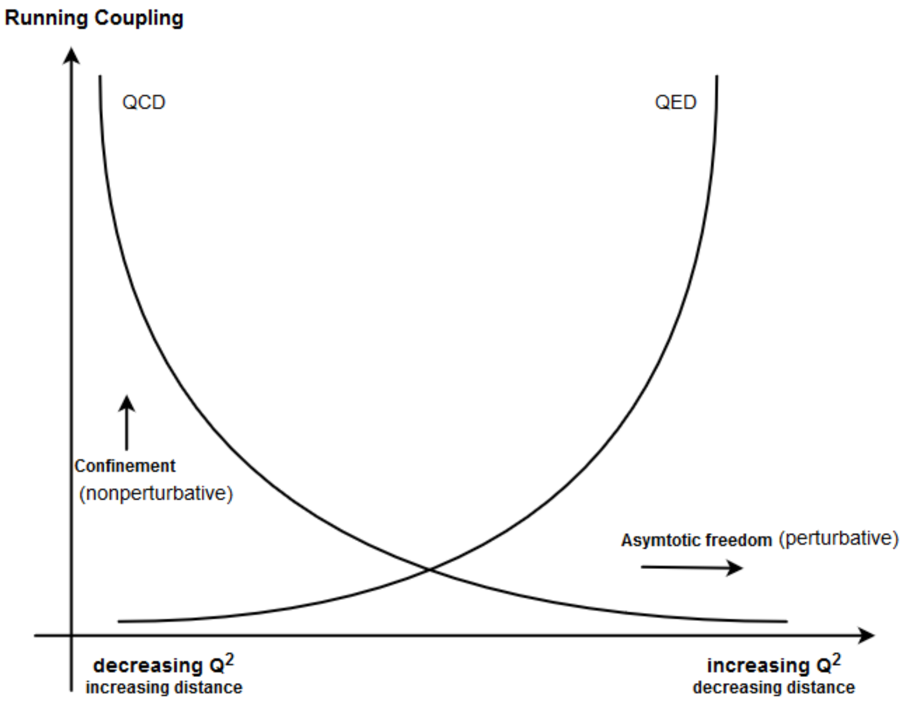
The one loop running coupling in QCD is:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) ln(\frac{Q^2}{\mu^2})} \tag{1.1}$$

Where $\beta_0 = \dfrac{11N_c - 2n_f}{12\pi}$, n_f comes from the first diagram and causes screening. n_f is the number of quarks and N_c the number of colours and comes from the second diagram (anti screening).
Obviously, with $n_f = 6$ and $N_c = 3$ in standard model we will get $\beta_0 > 0$. The Beta function is defined as:

$$\beta(\alpha) = -(\beta_0 \alpha^2 + \beta_1 \alpha^3 + \beta_2 \alpha^4 + ...) = \frac{d\,\alpha(Q^2)}{d\,ln(Q^2)} \tag{1.2}$$

e.g. $-(\beta_0 \alpha^2) < 0$ will be negative, which is actually the opposite of QED with $\beta_0 = -\dfrac{\pi}{3} \rightarrow -(\beta_0 \alpha^2) > 0$! That means the coupling constant in QCD will increase with decreasing Q^2 (increasing distance), In QED vice versa.



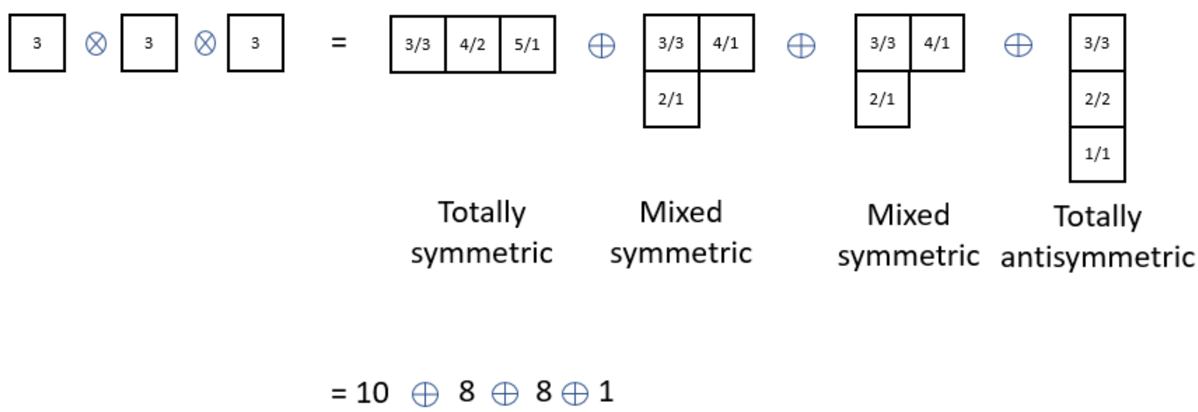
Asymptotic freedom allows us to use perturbation theory [2]. Quarks have not yet been observed as free particles. With increasing separation it will be easier to produce a quark-antiquark pair than to isolate a quark because the coupling between them is too strong.

[2] Actually there is need of two more things, if we want to make the connection between theory and experiment: either infrared safety or factorisation. That becomes discussed in the next chapter

This mechanism is called confinement. Confinement It has been confirmed in Lattice QCD, but not yet mathematically. And it belongs to nonperturbative theory. Quarks prefer to bind into hadrons which can be classified into baryons with three quarks state and mesons with a quark-antiquark state. As we know, the wave function of fermions must be antisymmetric according to the Pauli exclusion principle under the exchange of two quarks. Interestingly, there are resonance states with spin $\frac{3}{2}$ like Δ^{++}. The spins of the three up quarks are parallel to each other, have the same flavour and orbital angular momentum L=0. This means that an exchange of flavour, spin and space (orbital angular momentum) does not lead to any change. This problem is solved with the additional degree of freedom, the so-called color charge. Each quark comes in one of three colours red, green or blue and also anticolour $\bar{r}, \bar{b}, \bar{g}$ for antiquarks. The hadrons are colour singlets with regard to the hypothesis and so are invariant under rotations in colour space. The colour hypothesis describes the existence of mesons with $q\bar{q}$ and baryons with qqq. because if the wave function is odd in color, we have solved the spin statistical problem. The total wave function for each particle can be expressed in terms of:

$$\begin{aligned} \Psi_{3q} = \psi_{space} \times \chi_{spin} &\times \theta_{colour} \times \phi_{flavour} \\ O(3) \quad SU(2) &\quad SU(3) \quad SU(6) \end{aligned} \tag{1.3}$$

Now we can compute all possible colour states with Young Tableaux [15]. One uses group theory methods, for instance the Young Tableaux technique, to decompose products of irreducible representations into sums.



After using The same procedure for SU(2) and SU(6) for spins and flavours of the three quarks we will get:

$$\begin{aligned} 2 \otimes 2 \otimes 2 &= 4 \quad \oplus 2 \quad \oplus 2 \quad \oplus 0 \\ 6 \otimes 6 \otimes 6 &= 56 \oplus 70 \oplus 70 \oplus 20 \end{aligned} \tag{1.4}$$

As we can see, the total wave function is most complicated in the QCD region. That is the reason why the Lagrangian of QCD is always given in the shortened form. Before

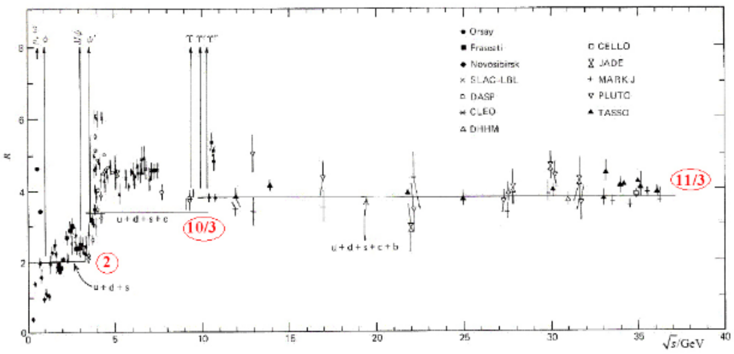
QCD is formulated as a gauge theory, an experiment should be pointed out which makes it clear why there is an additional degree of freedom in the QCD and why there is no U(1)-symmetry. Looking at the electron-positron scattering again, it is important to realize that not only $\mu^+\mu^-$, but also e^+e^-,$\tau^+\tau^-$ and also $q\bar{q}$ can arise, when the quark pairs fragment into hadrons. For the ratio:

$$R = \frac{\sigma(e^+e^- \to Hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{1.5}$$

one would expect, due to the fact that the coupling takes place between two charged particles, that only the sum over the square of the quark charges (because ${e^2}_\mu = 1$) contributes. However, there is an additional factor N_C that can be determined experimentally

$$R = N_C \sum_q e_q^2 \tag{1.6}$$

Without this factor one would expect for $u, d, s,$ u, d, s, c and u, d, s, c, b Respectively $\frac{2}{3}, \frac{10}{9}, \frac{11}{10}$ The experiment showed a third of the expected results (i.e. $N_C = 3$):



[25]

1.4 QCD Lagrangian

QCD like QED and the weak interaction theory is described by representations of a symmetry group. From the condition that the Lagrangian must be invariant under arbitrary global and local symmetry transformations (Noether's theorem) follows the interaction terms. The Lagrangian of QCD is invariant under $U(3) = U(1) \times SU(3)$ global transformation. Here only $SU(3)$ is discussed. The three Pauli matrices from $SU(2)$ can be replaced by the eight Gell-Mann λ^a with using the following relation:

$$
\begin{aligned}
&T^a = \frac{1}{2}\lambda^a \\
&[T^a, T^b] = i f^{abc} T^c \qquad \text{fundamental representation} \\
&({T^a}_{adj})_b c = -i f^{abc} \qquad \text{adjoint representation}
\end{aligned}
\tag{1.7}
$$

To quantize QCD theory, the Faddeev-Popov method [11] is usually used in the path integral to fix a gauge and define a gluon propagator. The Lagrangian is given:

$$
\begin{aligned}
&\mathscr{L} = \mathscr{L}_{free} + \mathscr{L}_{int} \\
&\mathscr{L} = \sum_f \bar{\psi}_{if}(i\gamma^\mu \partial_\mu - m_f)\psi^{if} - \frac{1}{4}{F_a}^{\mu\nu}{F^a}_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu {A^a}_\mu)(\partial^\nu {A^a}_\nu) + (\partial^\mu \chi^{a*})(\partial_\mu \chi^a) \\
&-g_s \bar{\psi}_i {T^a}_{ij} \psi_j \gamma^\mu {A^a}_\mu - \frac{g_s}{2} f^{abc}(\partial_\mu {A^a}_\nu - \partial_\nu {A^a}_\mu){A_b}^\mu {A_c}^\nu) - \frac{{g_s}^2}{4} f^{abc}({A_b}^\mu {A_c}^\nu) f^{ade}({A^d}_\mu {A^e}_\nu) \\
&-g_s f^{abc}(\partial^\mu \chi^{a*})\chi^b {A^c}_\mu
\end{aligned}
\tag{1.8}
$$

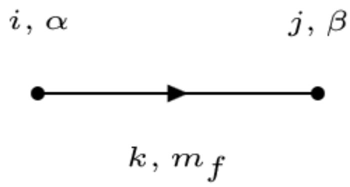
Here i,j are color indices in the fundamental representation, a color index in the adjoint representation of SU(3). f labels the six flavours of the quarks. g_s describes the strong coupling constant and ${A^a}_\mu$ is the gluon field and it corresponds to a non-abelian gauge theory with structure constants f^{abc}. χ^a is a scalar field under Lorenz group, but anti-commuting with the field strength tensor of QCD by [21, 26]

$$
{F^a}_{\mu\nu} = \partial_\mu {A^a}_\nu - \partial_\nu {A^a}_\mu - g_s f_{abc} {A^b}_\mu {A^c}_\nu \tag{1.9}
$$

It can be shown that the above Lagrangian is invariant under the following $SU(3)$ gauge transformations:

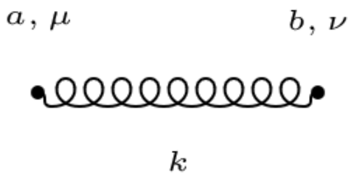
$$\begin{aligned}
&\psi'(x) \rightarrow exp(i\,\eta_a(x)\,T^a)\,\psi(x)\\
&D' \rightarrow \partial_\mu + ig_s T_a\, A'^a_\mu\\
&A'^a_\mu \rightarrow A^a{}_\mu - \frac{1}{g_s}\partial_\mu \eta^a(x) + f^{abc}\eta_b(x) A_{c\nu}(x)
\end{aligned} \tag{1.10}$$

$$L_{q,free} = \sum_f \bar{\psi}_{if}(i\gamma^\mu \partial_\mu - m_f)\delta_{ij}\psi^{jf}$$



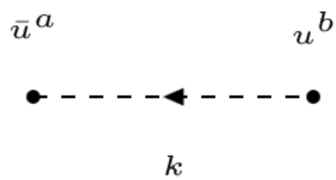
$$= \left(\frac{i}{\not{k} - m_f}\right)_{\alpha\beta}\delta_{ij}$$

$$L_{g,free} = -\frac{1}{4}F_a{}^{\mu\nu}F^a{}_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a{}_\mu)(\partial^\nu A^a{}_\nu)$$



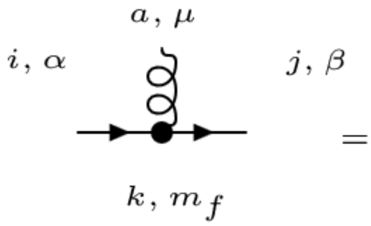
$$= \frac{-i}{k^2}\left(-g_{\mu\nu} - (1-\zeta)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}$$

$$L_{ghost,free} = (\partial^\mu \chi^{a*})(\partial_\mu \chi^a)$$



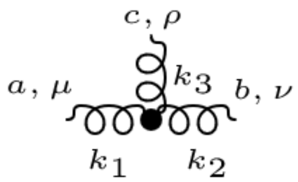
$$= \frac{i}{k^2}\delta^{ab}$$

$$L_{qg\bar{q},int} = g_s \bar{\psi}_i T^a{}_{ij} \psi_j \gamma^\mu A^a{}_\mu$$



$$= i g_s \gamma^\mu \otimes T^a{}_{ij}$$

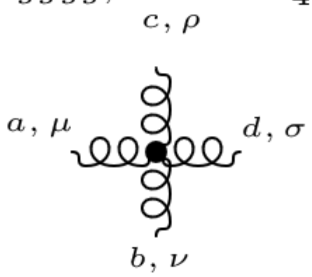
$$L_{ggg,int} = -\frac{g_s}{2} f^{abc} (\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu) A_b{}^\mu A_c{}^\nu$$



$$= -g_s f^{bca} [(k_1 - k_2)^\rho g^{\mu\nu} + (k_2 - k_3)^\mu g^{\nu\rho} + (k_3 - k_1)^\nu g^{\rho\mu}]$$

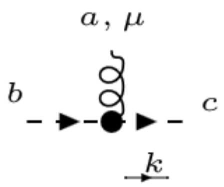
$(all\ Momenta\ are\ incoming!)$

$$L_{gggg,int} = -\frac{{g_s}^2}{4} f^{abc} (A_b{}^\mu A_c{}^\nu) f^{ade} (A^d{}_\mu A^e{}_\nu)$$



$$= -i {g_s}^2 [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]$$

$$L_{\chi g \bar{\chi}} = -g_s f^{abc} (\partial^\mu \chi^{a*}) \chi^b A^c{}_\mu$$



$$= g_s f^{abc} k^\mu$$

1.5 Colour factor calculation

In this section we will calculate the Casimir operators of the respective diagrams which will be used later. The fundamental representation in $SU(3)$ are given by[24, 26]

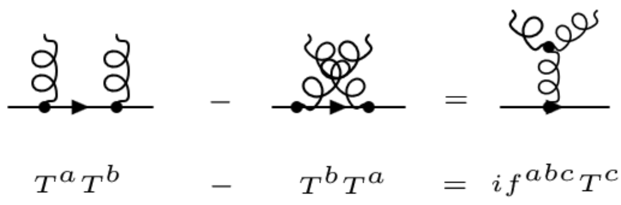
$$T^a = \vartheta^a \equiv \frac{\lambda^2}{2} \qquad with\ Gell - Mann\ matrices\ \lambda^a \tag{1.11}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}$$
$$\lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{1.12}$$

As we can see, λ^3 and λ^8 are diagonal. These generators satisfy:

Or in the adjoint representation:

$[T^a, T^b] = i f^{abc} T^c \Rightarrow$



$[F^a, F^b] = i f^{abc} F^c \Rightarrow$

The most common convention for the normalization of the generators in physics is:

$$\sum_{c,d} f^{acd} f^{bcd} = N \delta^{ab} \tag{1.13}$$

One of the most important equations for the colour factor calculation is the Jaccobi-Identity:

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0 \tag{1.14}$$

In terms of the structure constant, it turns out:

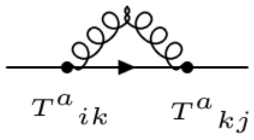
$$f^{axd} f^{bcx} + f^{cxd} f^{abx} + f^{bxd} f^{cax} = 0 \tag{1.15}$$

It follows:

$$f^{abc} = -2i\, tr(T^a [T^b, T^c]) \tag{1.16}$$

Which generalises to:

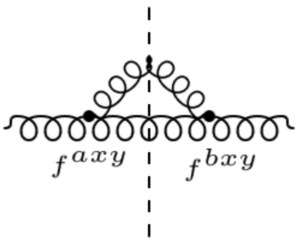
$$f^{abc} f^{xcd} = 4i\, tr(T^a [T^b, [T^c, T^d]]) \tag{1.17}$$



$$\sum_a (T^a T^a)_{ij} = C_F \delta_{ij} \quad \Rightarrow C_F$$

$$C_F = \frac{N_c^2 - 1}{2N_c} = C_F \sim \frac{N_c}{2}$$

With these relations all Casimir operators can be calculated:
Fundamental representation 3:
Adjoint representation 8:



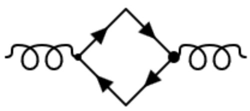
$$\sum_{xy} f^{axy} f^{bxy} = C_A \delta^{ab} \quad \Rightarrow C_A$$

$$C_A = N_C$$

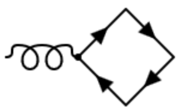
Which means the charge of the gluon is twice that of the quark because:

$$C_A = N_c = 2C_F \sim 2(\frac{N_C}{2}) \tag{1.18}$$

Trace identities:



$T_f \, \delta^{ab}$

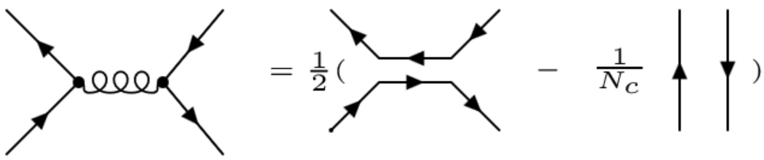


$tr(T^a) = 0$

One of the most important relation in this case is the Fierz identity. It shows the difference between QED and QCD!

$$\sum_a T_{ij}{}^a T_{kl}{}^a = \frac{1}{2}(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}) \tag{1.19}$$

Graphically it means: The charge transfer in QED takes place along the Fermion line



because photons cannot transport charges. On the other hand, the gluons can transfer color charges because they have color charges themselves.
The main relation we will use later for SU(N):

$$tr(T^a T^b) = T_{ij}{}^a T_{ji}{}^b = T_F \delta^{ab} \tag{1.20}$$

$$\sum_a (T^a T^a) = C_F \delta^{ij} \tag{1.21}$$

$$f^{acd} f^{bcd} = C_A \delta^{ab} \tag{1.22}$$

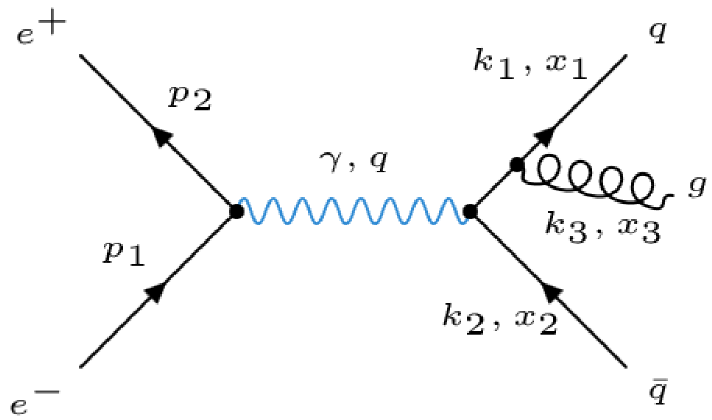
With $T_F = \frac{1}{2}$, $C_A = N$ and $C_F = \frac{N^2-1}{2N}$.

Chapter 2

Introduction to splitting function

2.1 IR and Collinar Divergences

Beyond the LO (Leading order) diagrams singularities can occur. Consider first the process $e^{-}e^{+} \rightarrow q\bar{q}g$



In order to calculate the cross section of this diagram, contemplate the gluon emission from the (anti)-quark. Since the calculation is quite long, concentrate on the final result:

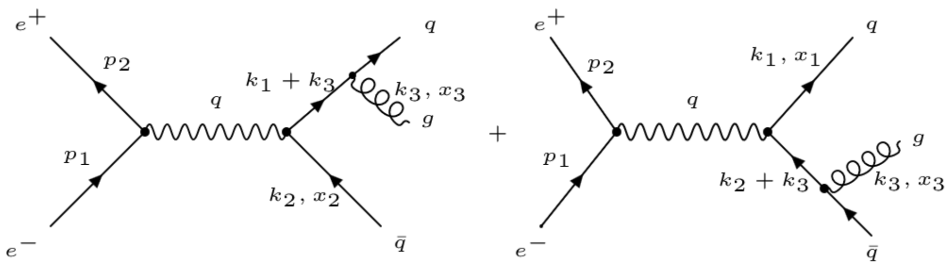


Figure 2.1: Left diagram $e^-e^+ \to qg\bar{q}$ and right $e^-e^+ \to q\bar{q}g$

$$\begin{aligned} A =& \frac{\bar{u}(k_1)(-ig_s\gamma^\nu \times T^a)[-i(\not{k}_1 + \not{k}_3)](-iee_q\gamma^\mu)v(k_2)\epsilon_\mu{}^{\lambda_1}\epsilon_\nu{}^{\lambda_2 *}}{(k_1+k_3)^2} \\ -& \frac{\bar{u}(k_1)(-iee_q\gamma^\mu)[i(\not{k}_2 + \not{k}_3)](-ig_s\gamma^\nu \times T^a)v(k_2)\epsilon_\mu{}^{\lambda_1}\epsilon_\nu{}^{\lambda_2 *}}{(k_1+k_3)^2} \\ \Rightarrow A =& -g_sT^a[\frac{\bar{u}\ \not{\epsilon}\,(\not{k}_1 + \not{k}_3)\,\Gamma\, v}{(k_1+k_3)^2} - \frac{\bar{u}\,\Gamma\,(\not{k}_2 + \not{k}_3)\ \not{\epsilon}\, v}{(k_2+k_3)^2}] \quad \text{with } \Gamma = (-iee_q\gamma^\mu)\epsilon_\mu{}^{\lambda_1} \end{aligned} \tag{2.1}$$

Under considering that the partons are on-shell, we get:

$$A = -g_sT^a[\frac{\bar{u}\ \not{\epsilon}\,(\not{k}_1 + \not{k}_3)\,\Gamma\, v}{2k_1 \cdot k_3} - \frac{\bar{u}\,\Gamma\,(\not{k}_2 + \not{k}_3)\ \not{\epsilon}\, v}{2k_2 \cdot k_3}] \tag{2.2}$$

In the soft limit with $k_0 \to 0$ we can factorize A_{soft} the amplitude in two parts:

$$A = -g_sT^a[\frac{k_1\ \epsilon}{k_1 \cdot k_3} - \frac{k_2\ \epsilon}{k_2 \cdot k_3}]A_{born} \qquad \text{with } A_{born} = \bar{u}\,\Gamma\, v \tag{2.3}$$

Where one part contains all information about colour and momenta and the other, A_{born} involves all spin information. If one calculates the cross section for it, one gets:

$$\begin{aligned} A =& -C_F g_s^2 \sigma^{born} \int \frac{d^3k}{2k_0(2\pi)^3} 2(\frac{k_1 \cdot k_2}{(k_1 \cdot k_3)(k_2 \cdot k_3)}) \\ & -C_F g_s^2 \sigma^{born} \int dcos\,\theta\, \frac{dk_0}{k_0} \frac{4}{(1-cos\,\theta)(1+cos\,\theta)} \end{aligned} \tag{2.4}$$

The energy fraction is defined as:

$$x_i = \frac{2E_i}{\sqrt{s}} = \frac{2q\ \cdot\ k_i}{s} \tag{2.5}$$

It can be shown that $\sum x_i = 2$ and thus, that only two of them are independent.
The final result is:

$$\frac{d^2\sigma}{dx_1 dx_2} = (\frac{4\pi\alpha}{s}) \sum e_i{}^2 \frac{2\alpha_s}{3\pi} \frac{x_1{}^2 + x_2{}^2}{(1-x_1)(1-x_2)} \tag{2.6}$$

There are three singularities within the final result. If the emitted photon is collinear to the outgoing quark or anti-quark ($x_1 \to 1$ or $x_2 \to 1$) and when the emitted gluon is soft ($x_1 \to 1$ and $x_2 \to 1$). The singularities come from the quark propagator in each diagram. The denominators contain terms proportional to $\frac{1}{(k_i+k_j)^2}$. We can eliminate the quark mass under on-shell condition so that:

$$\frac{1}{(k_i+k_j)^2} = \frac{1}{2k_i \cdot k_j} = \frac{1}{2E_iE_j(1-cos\theta_{ij})} = \frac{1}{s(1-x_k)} \tag{2.7}$$

can show all possibilities for three partons through a triangle:

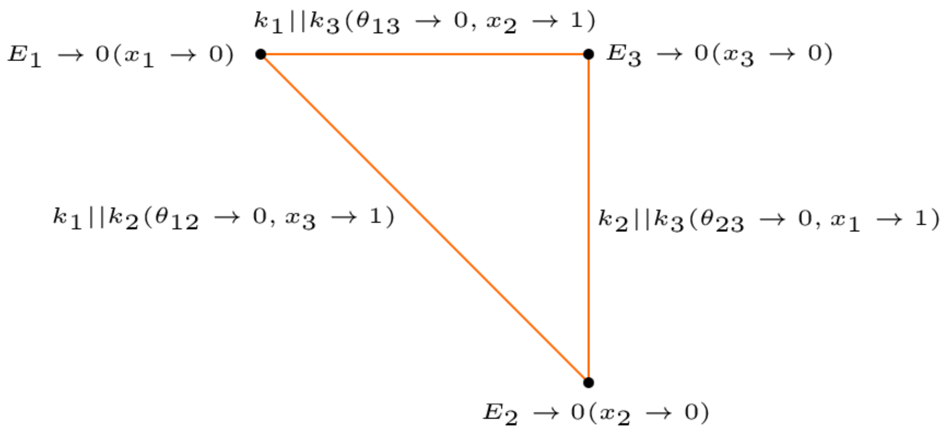


Figure 2.2: three-parton configurations at the boundaries of phase space

According to KLN-Theorem, IR singularities must cancel when summing the transition rate over all degenerate (initial and final) states. The sum of the integrals $\int_R$ and $\int_V$ over the phase space is finite. However, this is not true for the individual contributions. We will use deep inelastic scattering (DIS) to show how the infra-red singularities are

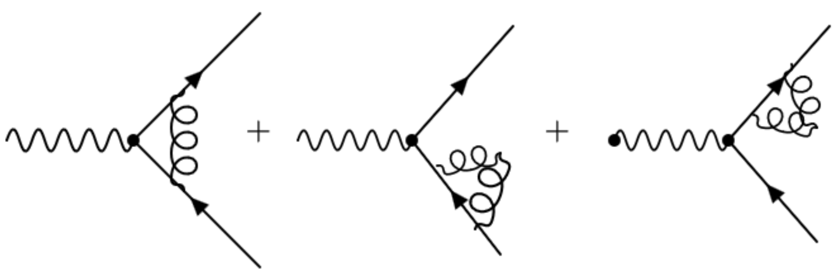


Figure 2.3: Virtual corrections: one-loop corrections to $e^-e^+ \to q\bar{q}$

absorbed in the parton distributions [2].

2.2 Subtraction method

$$|A|^2 = |A^{(0)}{}_m|^2 + |A^{(0)}{}_{m+1}|^2 + 2Re(A^{(0)*}{}_n A^{(1)}{}_m) \tag{2.8}$$

Where $|A^{(0)}{}_m|^2$ is the tree level contribution (Born sector) from LO and has no divergences, $|A^{(0)}{}_{m+1}|^2+2Re(A^{(0)*}{}_m A^{(1)}{}_m)$ comes from NLO and they are each divergent. The problem in this case is the Integrals cannot be combined due to different phase space dimensions:

$$\sigma^{NLO} = \int_{m+1} \partial\sigma^R + \int_m \partial\sigma^V \tag{2.9}$$

The real and virtual contributions are both IR divergent and need to be regularised in $d = 4 - 2\epsilon$ dim. To tackle this problem one can use the subtraction method in that way one add and subtract a local counter term $\partial\sigma^A$ with same singularity structure as terms $\partial\sigma^R$ to the integral. $\partial\sigma^A$ approximates the soft and collinear singularities of $\partial\sigma^R$.

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R - \partial\sigma^A] + \int_m [\partial\sigma^V + \int_1 \partial\sigma^A] \tag{2.10}$$

In this case, we can safely set $\epsilon \to 0$ for $\partial\sigma^R|_{\epsilon\to 0} - \partial\sigma^A|_{\epsilon\to 0}$ and calculate the integral numerically in 4-dimensions. On the other side, integrate over the one-parton phase space is integrated over analytically to explicitly cancel poles and then ϵ is set to 0.

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R|_{\epsilon\to 0} - \partial\sigma^A|_{\epsilon\to 0}] + \int_m [\partial\sigma^V + \int_1 \partial\sigma^A]_{\epsilon\to 0} \tag{2.11}$$

The virtual contribution must be UV-finite:

$$\int_m \partial\sigma^V = \int_m [\int_{loop} \partial\sigma^V{}_{bare} + \sigma^V{}_{Counter\,term}] \tag{2.12}$$

The addition of $\int_1 \partial\sigma^A$ to the $\int_m \partial\sigma^V$ ensures that IR poles are cancelled. The bare and counter contribution are separately divergent and have also different integral dimensions. One can use the same idea with the subtraction method to solve this problem [4, 5]

$$\int_m \partial\sigma^V + \int_{loop} \partial\sigma^L - \int_{loop} \partial\sigma^L = \int_m \int_{loop} [\partial\sigma^V{}_{bare} - \partial\sigma^L] + \int_m [\sigma^V{}_{Counter\,term} + \int_{loop} \partial\sigma^L] \tag{2.13}$$

$$\sigma^{NLO} = \int_{m+1} [\partial\sigma^R - \partial\sigma^A] + \int_m \int_{loop} [\partial\sigma^V{}_{bare} - \partial\sigma^L] + \int_m [\sigma^V{}_{Counter\,term} + \int_{loop} \partial\sigma^L + \partial\sigma^A] \tag{2.14}$$

Determination of emission kernels

Now we want to introduce the properties of the counter term $\partial\sigma_A$. This term needs to have the same behaviour as $\partial\sigma_R$ in d dimensions. This process and specific observable independent term has to be obtained in a way that is independent of the particular jet observable considered. It must be exactly integrable analytically over one-parton phase space in d and $\partial\sigma_R - \partial\sigma_A$ has to be integrable via Monte Carlo methods. $\partial\sigma_A$ acts as a local counter-term for $\partial\sigma_B$ At this point, one should derive improved factorization formulae which are called dipole formulae. Note that the notation below is symbolic:

$$\partial\sigma_A = \sum_{dipoles} \partial\sigma_B \otimes \partial V_{dipoles} \tag{2.15}$$

Where the sum is over dipoles for all $m+1$ configurations with consideration to a given m-parton state. $\partial\sigma_B$ describes the color/spin projection of the Born-level exclusive cross section. The symbol $\otimes$ describes phase space convolutions and sums over colour and spin indices. $\partial V_{dipoles}$ will be computed and its singular properties matched to the real part. The Dipoles are universal in the sense that they do not depend on the hard scattering. This allows the use of a factorisable mapping from the m + 1-parton phase space to an m-parton subspace. That will be clearer when the parametrisation is used in the next chapter. If we integrate over all $m+1$, we can write:

$$\int_{m+1} \partial\sigma_A = \int_m \partial\sigma_B \otimes \sum_{dipoles} \int_1 \partial V_{dipoles} \tag{2.16}$$

This is the important result because this can now be written in terms of the known m-sector from LO and the other term is a universal factor which contains all ϵ-poles.

Singularity Structure

Before we begin with the collinear limit or soft limit respectively we are going to pull up the matrix element from LO which has this below general form:

$$\mathcal{M}_m{}^{c_1,...,c_m;s_1,...,s_m}(p_1,...,p_m) \tag{2.17}$$

c_i, s_i and p_i denote respectively the colour, spin indices and the momenta for each m-parton in the tree level matrix element in the final state. A common method used in this case is to define a basis in colour+helicity space.

$$\mathcal{M}_m{}^{c_1,...,c_m;s_1,...,s_m}(p_1,...,p_m) \equiv (< c_i,..,c_m|\otimes < s_1,...,s_m|)|1,..,m >_m \tag{2.18}$$

With $< c_i,..,c_m|\otimes < s_1,...,s_m|)|$ as the basis and $|1,..,m >_m$ as a vector in this space. Thus, for the matrix element squared:

$$\begin{aligned} |\mathcal{M}_m|^2 =& (< c_i,..,c_m|\otimes < s_1,...,s_m|)(|c_i,..,c_m > \otimes|s_1,...,s_m >) < 1,..,m|1,..,m > \\ &= \delta_{c_1c_1},...\delta_{c_mc_m} \otimes \delta_{s_1s_1},...\delta_{s_ms_m}\ {}_m < 1,..,m|1,..,m >\ {}_m \end{aligned} \tag{2.19}$$

Define a colour-charge operator T_i with the emission of a gluon from each parton i:

$$T_i = T_i^c|c> \tag{2.20}$$

Its action onto the colour space is defined by:

$$< c_1, .., c_i, ..c_m, c|T_i|b_1, .., b_i, ..b_m >= \delta_{c_1 b_1} ... T^c{}_{c_i b_i} ... \delta_{c_m b_m} \tag{2.21}$$

Where $T^c{}_{c_i b_i}$ is the colour-charge matrix in the adjoint representation in the case of gluon emission or colour-charge matrix in the fundamental representation for the quark/anti-quark emission case. The following properties must be taken into account:

$$\begin{aligned}
&T_i \cdot T_j = T_j \cdot T_i && \text{if } i \neq j, \text{ commutative property} \\
&T_i^2 = C_i \quad C_i = C_A && \text{for gluon and } C_i = C_F \text{ for (anti)quark} \\
&\sum_{i=1}^{m} T_i|1, ..., m >_m = 0 && \text{for single state}
\end{aligned} \tag{2.22}$$

Thus, the square of colour-correlated tree-amplitudes for the indices I, J referring either to final-state or initial-state partons will be [4, 5]

$$|\mathcal{M}^{I,J}{}_{m,a...}|^2 = {}_{m,a...} < 1, ..., m; a, ...|T_I \cdot T_J|1, ..., m; a, ... > {}_{m}, a... \tag{2.23}$$

Dipole factorisation

Consider $(m + 1)$-partons with the general matrix element [5, 27]

$$|\mathcal{M}_{m+1}(Q; p_1, ..., p_i, ..., p_j, ..., p_{m+1})|^2 \tag{2.24}$$

We need to take collinear and soft limits which allow factorisation. In the soft region the momentum p_j can be parametrised with $p_j \to \lambda q$, $\lambda \to 0$, where q is a arbitrary four vector and λ a scale parameter. The matrix element squared is characterised by $|\mathcal{M}|^2 \sim \frac{1}{\lambda^2}$. and if p_i and p_j become collinear, we parametrise $p_j = \frac{z}{1-z} p_i$. So the matrix element will be $|\mathcal{M}|^2 \sim \frac{1}{p_i \cdot p_j}$. This will be covered in more detail in the next chapter. Here is given a summery of the behaviour of the matrix elements in different regions. Based on the Catani-Seymour method for (m + 1) parton matrix element, it's possible to factorise out parton k to give $|\mathcal{M}_m|^2$:

$$|\mathcal{M}_{m+1}|^2 \to \sum |\mathcal{M}_m|^2 \otimes V_{ij,k} \Rightarrow |\mathcal{M}_{m+1}|^2 \to \sum |\mathcal{M}_m|^2 \otimes V_{ij,k} \tag{2.25}$$

$V_{ij,k}$ a singular factor including parton k and its interaction with partons i and j from the m parton amplitude. This situation can be represented by the diagram 2.4.
Here i and k are the emitters and k plays the role of a spectator. The blobs denote the tree-level matrix elements and their complex conjugate. The dots on the right-hand side stand for non-singular terms both in the soft and collinear limits. When the partons i and j become soft and/or collinear, the singularities are factorized into the term $V_{ij,k}$ (the dashed box on the right-hand side) which embodies correlations with a single additional parton k.

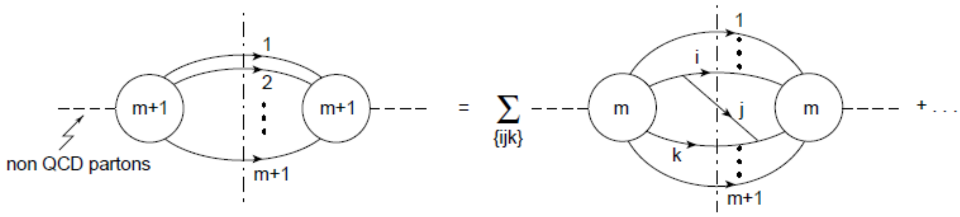


Figure 2.4: Factorisation in dipole formalism
[4]

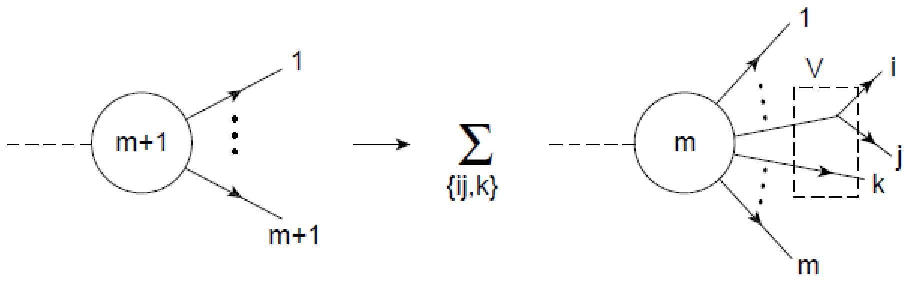
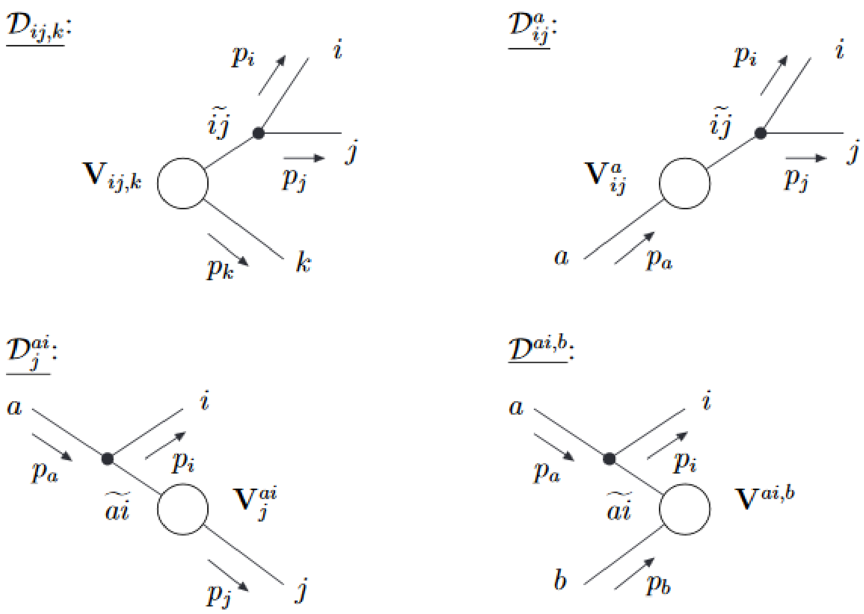


Figure 2.5: Effective diagrams for the different emitter–spectator cases. There are four different configurations depending on whether the emitter and the spectator are part of the initial or the final state.These configurations are called final-final (FF, upper left panel), final-initial(FI, upper right panel), initial-final (IF, lower left panel) and initial-initial (II,lower right panel)
[5]



In this context the different dipole factorisation for both initial states and final states shall be presented. All these different possibilities can be seen in the diagram 2.5.
The circle in the center of each sub diagram presents the m-partons matrix element and the tilde labels the collinear splitting process for the initial or final states. For this work, the first upper diagram with final-state singularities without initial-state partons, is completely sufficient and is discussed here in detail with its formula. The matrix element for this is written:

$$|\mathcal{M}_{m+1}|^2 =< 1, ..., m+1||1, ..., m+1 >= \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) + \text{finite terms} \tag{2.26}$$

The first term with the sum over dipoles is divergent as $p_i \cdot p_j \to 0$. These dipole terms are explicitly given by:

$$\mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) = \frac{-1}{2p_i \cdot p_j}{}_m < 1, ..., \tilde{ij}, ..., k, ..., m{+}1| \frac{T_k \cdot T_{ij}}{T_{ij}{}^2} V_{ij,k}|1, ..., \tilde{ij}, ..., k, ..., m{+}1 >_m \tag{2.27}$$

Where $T_k \cdot T_{ij}$ are the color charges of spectator and emitter
$V_{ij,k}$ splitting kernel in helicity space of emitter explicit form depends on parton type become proportional to Altarelli-Parisi splitting functions 2.35 and eikonal factors in collinear and soft limits.
Using the kinematics:

$$\begin{aligned}
\tilde{p}^{\mu}_{ij} &= p_i{}^{\mu} + p_j{}^{\mu} - \frac{y_{ij,k}}{1-y_{ij,k}} p_k{}^{\mu} \\
\tilde{p}^{\mu}_{k} &= \frac{1}{1-y_{ij,k}} p_k{}^{\mu} \\
\text{with} \quad y_{ij,k} &= \frac{p_i \cdot p_j}{p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i}
\end{aligned} \tag{2.28}$$

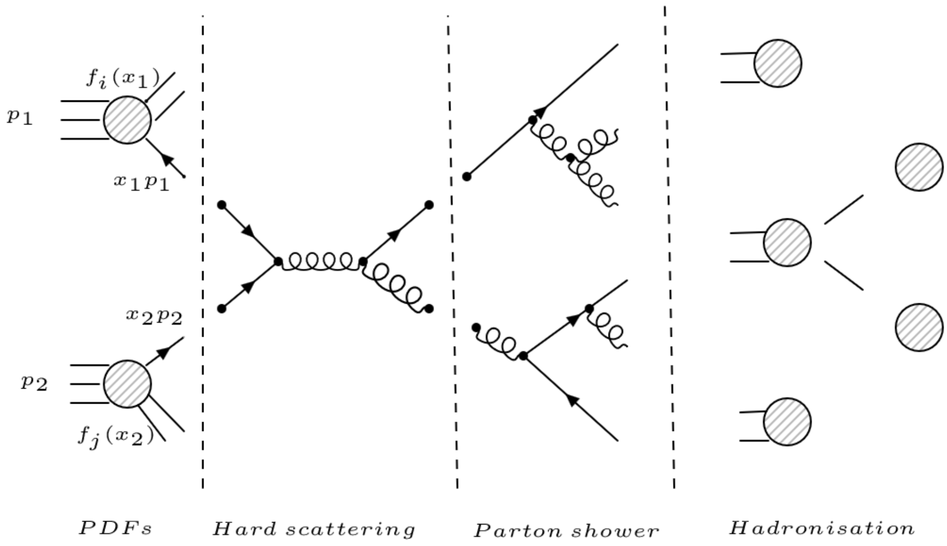
Note, that momenta are on-shell. Due to momentum conservation $p_i{}^{\mu} + p_j{}^{\mu} + p_k{}^{\mu} = \tilde{p}^{\mu}_k + \tilde{p}^{\mu}_{ij}$.

2.3 Hard scattering

The hadron hadron scattering can be written as:

$$\sigma = \sum_{ij} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \sigma_{ij}(x_1, x_2, Q^2/\mu^2 ...) \tag{2.29}$$

Here the (arbitrary) factorisation scale μ can be thought of as the scale which separates



the long and short-distance physics. Roughly speaking, a parton with a transverse momentum less than μ is then considered to be part of the hadron structure and is absorbed into the parton distribution. Partons with larger transverse momenta participate in the hard scattering process with a short-distance partonic cross-section. The factorisation theorem also applies to deep inelastic scattering. It can be shown how the IR singularities are absorbed into the PDFs. The DIS cross section can be written as [9]

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4}[(1-y)F_2(x, Q^2) + xy^2 F_1(x, Q^2)] \tag{2.30}$$

In this case we need to introduce the structure function, is defined as the charge weighted sum of the parton momentum densities which describe the probability that the parton carries a momentum fraction between x and ∂x of the proton momentum. The index i denotes the quark flavour. Parton distributions are nonpertubative and have to be obtained from experiment.

$$F_2{}^{exp}(x) = \sum_i e_i{}^2 x f_i(x) \tag{2.31}$$

The evolution of a parton distribution due to splitting can be described by the DGLAP evolution equation [28].

$$\frac{\partial f(x, \mu^2)}{\partial\, ln\, \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} f_j(y, \mu^2) P_{ij}(\frac{x}{y}) + O(\alpha_s{}^2) \tag{2.32}$$

This is a system of coupled integral or differential equations. $P_{ij}(\frac{x}{y})$ represents the probability, a daughter parton i with momentum fraction $\frac{x}{y}$ is splits from a parent parton j. The above convolution in compact notation (Mellin Convolution) in the general case:

$$\frac{\partial f_i(x,\mu^2)}{\partial\, ln\, \mu^2} = \sum_{j=-n_f}^{n_f} P_{ij} \otimes f_j(\mu^2) \tag{2.33}$$

The four splitting probabilities are illustrated in the diagram 2.6 and the transitions lead to a set of $2n_f + 1$ coupled evolution equations.

$$\frac{\partial}{\partial\, ln\, \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{bmatrix} P_{qq} & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{bmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \tag{2.34}$$

A parton distribution changes when a different parton splits or the parton itself splits. Parton densities can not be analytically determined, but it is possible to predict how they evolve from one scale to another. PDFs are measured in one process and use them as an input for another process.
he parton distribution function or the fragmentation function is acon-volution of a perturbative and a non-perturbative part. The perturbativepart is the evolution kernel which is solution of the DGLAP equation. Itis based on the collinear factorization property of QCD. Fixing the accu-racy of the calculation and the factorization scheme the evolution kernel iswell defined. The conventional factorization scheme is theMS. The non-perturbative part cannot be calculated, fortunately it is universal and wecan obtain it from fit to the data and use them for other processes. Of courseif we change the factorization scheme we have to refit the non-perturbativepart.What is the parton shower and the hadronization? We can think aboutthat is a multi-hadron fragmentation function and the parton shower is itsevolution kernel. The hadronization model is the non-perturbativepartthat we fit to the data. The perturbative part is based on the factorizationtheorem of the QCD and it is well defined up to the finite pieces. In otherwords, a different shower algorithm represent differentshower schemebutupto the leading (LL) and next-to-leading (NLL) logarithmic accuracy theymust be equivalent. If we have some free parameters other than the trivialscales in the parton shower algorithm then the different values of theseparameters correspond to different shower schemes and the tunning of thehadronization model must be redone.How to define aconventionalshower scheme? These are our criteria: i)It must be Lorentz covariant and Lorentz invariant. ii) It must be correctat LL level. It must be correct at NLL level at least in leading color appriximation. iii) The soft gluon effect must be correct at least in leading colorapproximation. iv) In every step of the shower the phase space configu-ration must be the exactm-body phase space with the exact phase spaceweight. The infrared cutoff parameter should be the only cutoff parameterin the algorithm. v) It should “smoothly” work together with the NLOand November 7, 2018 13:39 Proceedings Trim Size: 9in x 6inRingberg053matrix element matching schemes

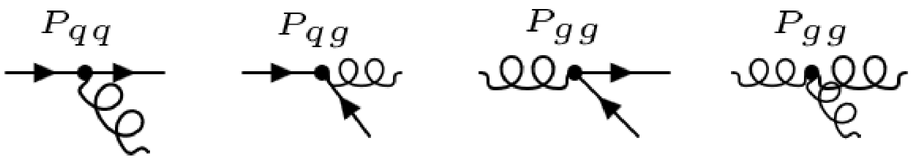


Figure 2.6: The four splitting probabilities

$$
\left.\begin{aligned}
\langle\, \hat{P}_{qq}\rangle &= C_F[\frac{1+z^2}{1-z} - \varepsilon(1-z)] \\
\langle\, \hat{P}_{gq}\rangle &= T_R[1 - \frac{2z(1-z)}{1-\varepsilon}] \\
\langle\, \hat{P}_{qg}\rangle &= C_F[\frac{1+(1-z)^2}{z} - \varepsilon z] \\
\langle\, \hat{P}_{gg}\rangle &= 2C_A[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)]
\end{aligned}\right\} \text{Alterali-Parisi} \tag{2.35}
$$

2.4 Mapping 3 partons to 2

As we have already seen in the last section, the dipole factorization is obtained from the square matrix element. Now we want to take a closer look at the four possible parton showers from the previous section. For this goal we first calculate the respective matrix elements and the complex-conjugated one of them in the known 3 parton evaluation. This results in the dipole-term, which contains the soft- and collinear singularities. Then this is parametrized with a certain kinematics in order to separate the finite terms from infinities. First of all, the Emmiter and the spectator are defined. After the substitution of the new momenta we get splitting kernel in helicity space and color charges of spectator and emitter. Then we ignore the finite terms because we are looking for the singular terms. It should be noted that at the beginning of this thesis a parametrisation was used, which unfortunately only works for LO. This was recognized later and therefore a new kinematics was used. This theoretical description will become clearer as we begin to implement the mappings. First of all the kinematics have to be introduced [13, 22, 23, 24].

2.5 Old mapping

First we start with the old mapping, as promised.

$$\left.\begin{aligned} {q_i}^\mu &= z{p_i}^\mu + y(1-z){p_j}^\mu + \sqrt{zy(1-z)}{m^\mu}_\perp \\ q^\mu &= (1-z){p_i}^\mu + yz{p_j}^\mu - \sqrt{zy(1-z)}{m^\mu}_\perp \\ {q_j}^\mu &= (1-y){p_j}^\mu \end{aligned}\right\} \quad \text{parametrisation} \tag{2.36}$$

Where q is the radiated soft momentum, q_i the momenta of the emitter and q_j the momentum of the spectator is. the dimensionless variable is given by $y = \frac{q_i \cdot q}{p_i \cdot p_j}$. Note that both the emitter and the spectator are on-shell. momentum conservation is implemented exactly:

$${q_i}^\mu + q^\mu + {q_k}^\mu = {p_i}^\mu + {p_j}^\mu + {m^\mu}_\perp \tag{2.37}$$

For this mapping it is useful to calculate some common relation:

$$\begin{aligned} &{q_i}^\mu + q^\mu = {p_i}^\mu + y{p_j}^\mu \\ &{q_j}^\mu + q^\mu = (1-z){p_i}^\mu + (1+yz-y){p_j}^\mu - \sqrt{zy(1-z)}{m^\mu}_\perp \\ &q_i \cdot q = y(1-2z+2z^2)(p_i \cdot p_j) \\ &q_i \cdot q_j = z(1-y)(p_i \cdot p_j) \\ &q_j \cdot q = (1-z)(1-y)(p_i \cdot p_j) \end{aligned} \tag{2.38}$$

2.6 new kinematic

For the general m emission case it must be defined a new mapping. The parametrisation of the splitting momenta is formalized as:

$$\begin{aligned}
{k_l}^{\mu} &= \alpha_l \alpha \Lambda^{\mu}{}_{\nu} {p_i}^{\nu} + y\beta n^{\mu} + \sqrt{y\alpha_l\beta_l} n^{\mu}{}_{\perp,l} \qquad\qquad l = 1, ..., m \\
{q_i}^{\mu} &= (1 - \sum_{l=1}^{m} \alpha_l) \alpha \Lambda^{\mu}{}_{\nu} {p_i}^{\nu} + y(1 - \sum_{l=1}^{m} \beta_l) n^{\mu} - \sqrt{y\alpha_l\beta_l} n^{\mu}{}_{\perp,l} \\
{q_k}^{\mu} &= \alpha \Lambda^{\mu}{}_{\nu} {p_k}^{\nu} \qquad\quad k = 1, ..., n \qquad k \neq i
\end{aligned} \tag{2.39}$$

$k = 1, ..., n$ labels the emission momenta and is taken to be massless ${k_l}^2 = 0$. Where the label l denotes the count of emissions. In this work we just want to considerate the one-emission kernels. The other important issue here is that all hard momenta are on-shell, ${p_k}^2 = {q_k}^2 = 0$.
n^{μ} is an auxiliary light-like vector which is necessary to specify the transverse component of $n^{\mu}{}_{\perp,l}$. To absorb the recoil we define n^{μ} as:

$$n^{\mu} = Q^{\mu} - \frac{Q^2}{2p_i \cdot Q} {p_i}^{\mu} \tag{2.40}$$

Whereby Q is the total momentum with:

$$Q^{\mu} = {q_i}^{\mu} + \sum_{l=1}^{m} k_l^{\mu} + \sum_{k=1}^{m} q_k^{\mu} = {p_i}^{\mu} + \sum_{k=1}^{m} p_k^{\mu} \tag{2.41}$$

To fulfil the condition that the emission momenta are massless, we need the following condition:

$$\begin{aligned}
n^{\mu}{}_{\perp,l} \Lambda^{\mu}{}_{\nu} {p_i}^{\nu} &= n_{\perp,l} \cdot n = n_{\perp,l} \cdot Q = 0 \\
n^{\mu}{}_{\perp,l} \cdot p_k &\neq 0
\end{aligned} \tag{2.42}$$

$n^2_{\perp,l} = -2\alpha \Lambda^{\mu}{}_{\nu} {p_i}^{\nu} n_{\mu}$ is not on-shell and in terms of single emission case we get $n^2_{\perp,1} = -2p_i \cdot Q$. The parameter y is related to the virtuality of the splitting parton:

$${q_i}^{\mu} + \sum_{l=1}^{m} k_l^{\mu} = \alpha \Lambda^{\mu}{}_{\nu} {p_i}^{\nu} + y n^{\mu} \tag{2.43}$$

With $\alpha = \sqrt{1 - y}$.

Lorenz transformation of momenta

In order to be able to work with the parametrisation, we have to do the Lorenz transformation of the Emitters, Spectator and total momentum first.

$$\begin{aligned}\alpha\Lambda^{\mu}{}_{\nu} &= p_i{}^{\mu} p_{i\nu} \frac{-y^2Q^2}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + p_i{}^{\mu} Q_{\nu} \frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} \\ &+ Q^{\mu} p_{i\nu} \frac{(y^2-y-y\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}\eta^{\mu}{}_{\nu}\end{aligned} \tag{2.44}$$

In the collinear limit of $y \to 0, \alpha \to 1$ this transformation reduces to trivial $\eta^{\mu}{}_{\nu}$. Finally we are going to compute the Lorenz transformation of the Momenta. The detailed calculation of them can be found in Appendix A.

$$\boxed{\hat{p_i}^{\mu} = \alpha\Lambda^{\mu}{}_{\nu} p_i{}^{\nu} = p_i{}^{\mu}} \tag{2.45}$$

$$\boxed{\hat{Q}^{\mu} = \tfrac{Q^2}{2p_i\cdot Q} y\, p_i{}^{\mu} + (1-y)\, Q^{\mu}} \tag{2.46}$$

$$\boxed{\hat{p_k}^{\mu} = A_1\, p_i{}^{\mu} + A_2\, Q^{\mu} + \sqrt{1-y} p_k{}^{\mu}} \tag{2.47}$$

with

$$\begin{aligned}A_1 &\equiv \frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} \\ A_2 &\equiv \frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}\end{aligned}$$

2.7 Single emission part

In terms of one emission where $l=1$ the mapping will be simplified as:

$$\begin{aligned}k_1{}^{\mu} &= (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\mu} + y\beta_1 Q^{\mu} + \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\ q_i{}^{\mu} &= (\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\mu} + y\alpha_1 Q^{\mu} - \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\ q_k{}^{\mu} &= \alpha\Lambda^{\mu}{}_{\nu} p_k{}^{\nu} \qquad k=1,...,n \qquad k\neq i\end{aligned} \tag{2.48}$$

$$\begin{aligned}k_1{}^{\mu} &= \zeta_1 p_i{}^{\mu} + \lambda_1 Q^{\mu} + \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,1} \\ q_i{}^{\mu} &= \zeta_q p_i{}^{\mu} + \lambda_q Q^{\mu} - \sqrt{y\alpha_1\beta_1} n^{\mu}{}_{\perp,l} \\ q_k{}^{\mu} &= A_1 p_i{}^{\mu} + A_2 Q^{\mu} + \sqrt{1-y} p_k^{\mu}\end{aligned}$$

2.8 Common scalar products

To investigate the mapping it is useful to determine the dot products between these four vectors. To understand the often occurring pre-factor products one should look them up in the appendix A.

$$\boxed{k_1 \cdot q_i = y(\alpha_1 + \beta_1)^2\, p_i \cdot Q = y\, p_i \cdot Q} \tag{2.49}$$

$$\boxed{k_1 \cdot q_k = [\alpha_1(1-y) + y\beta_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\beta_1\, Q \cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{2.50}$$

$$\boxed{q_i \cdot q_k = [\beta_1(1-y) + y\alpha_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\alpha_1\, Q \cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{2.51}$$

2.9 Recipe for the use of the new parametrisation

In the previous chapter we have discussed that the singularities come from the propagators in each diagram since the denominators contain according Feynmann rules terms with $\sim \frac{1}{2q_a \cdot q_b}$. Whereby a and b here place holder the respective momenta. Since the calculations are sometimes very complicated and confusing, the procedure for eliminating the finite terms is as follows:
In the calculating of the square matrix elements always appear products in the form of $p_a \cdot p_b$ both in the numerator and denominator. The denominator shows which pre-factor causes the singularity. As we know, if , we get zero in the denominator. These terms from the numerator with the same prefix can be omitted from the beginning because they appear in both the denominator and the numerator and are therefore finite. This is explicitly shown below for two common denominators.

2.9.1 Parametrization in terms of $(k_1 \cdot q_i)(k_1 \cdot q_k)$

$$\boxed{(k_1 \cdot q_i)(k_1 \cdot q_k) \approx y(1-\beta_1)(1-y)\, (p_i \cdot p_k)(p_i \cdot Q)} \tag{2.52}$$

Here you can quickly see that this term converges for $y \to 0$ and $\beta_1 \to 1$ towards zero. That means, you could ignore all terms with $y(1-\beta_1)$. However, since the equation becomes rather large quickly if we first use all the momenta products and then drop the terms with the pre-factor out of the denominator, this is already done for the scalar products. And this is exactly the biggest simplification in the calculation. The result looks like this:

$$
\begin{aligned}
{k_1}^{\eta}{k_1}^{\eta'} &= [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]{p_i}^{\eta}{p_i}^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}){p_i}^{\eta}Q^{\eta'} - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})Q^{\eta}{p_i}^{\eta'}\\
{k_1}^{\eta}{q_i}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2Q^{\eta}{p_i}^{\eta'}\\
{q_i}^{\eta}{k_1}^{\eta'} &= [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'}\\
{q_i}^{\eta}{q_i}^{\eta'} &= {\beta_1}^2{p_i}^{\eta}{p_i}^{\eta'}\\
{k_1}^{\eta}{q_k}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_i}^{\eta}{p_k}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\,{p_i}^{\eta}{p_i}^{\eta'}\\
&\quad - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\,{p_i}^{\eta}Q^{\eta'} + y\beta_1A_1\,Q^{\eta}{p_i}^{\eta'} + y\beta_1A_2\,Q^{\eta}Q^{\eta'} + y\beta_1\sqrt{1-y}Q^{\eta}{p_k}^{\eta'}\\
{q_i}^{\eta}{q_k}^{\eta'} &= A_1\beta_1{p_i}^{\eta}{p_i}^{\eta'} + A_2\beta_1{p_i}^{\eta}Q^{\eta'} + \beta_1\sqrt{1-y}{p_i}^{\eta}{p_k}^{\eta'}\\
{q_k}^{\eta}{k_1}^{\eta'} &= [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'} - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\,{p_i}^{\eta}{p_i}^{\eta'}\\
&\quad - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\,Q^{\eta}{p_i}^{\eta'} + y\beta_1A_1\,{p_i}^{\eta}Q^{\eta'} + y\beta_1A_2\,Q^{\eta}Q^{\eta'} + y\beta_1\sqrt{1-y}{p_k}^{\eta}Q^{\eta'}\\
{q_k}^{\eta}{q_i}^{\eta'} &= A_1\beta_1{p_i}^{\eta}{p_i}^{\eta'} + A_2\beta_1Q^{\eta}{p_i}^{\eta'} + \beta_1\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'}
\end{aligned}
\tag{2.53}
$$

2.9.2 Parametrization in terms of $(k_1 \cdot q_i)(k_1 \cdot q_i)$

$$\boxed{(k_1\cdot q_i)(k_1\cdot q_i) = y^2(p_i\cdot Q)(p_i\cdot Q)} \tag{2.54}$$

With the same interpretation from above one could say that this term converges just for

$y \to 0$ towards zero. That's why we will remove all product terms with y^2.

$$
\begin{aligned}
{k_1}^\eta {k_1}^{\eta'} &= [(1-\beta_1)^2 - 2y\beta_1(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y\beta_1(1-\beta_1)(\frac{Q^2}{2p_i \cdot Q}){p_i}^\eta Q^{\eta'} \\
&+ y\beta_1(1-\beta_1)(\frac{Q^2}{2p_i \cdot Q})Q^\eta {p_i}^{\eta'} \\
{k_1}^\eta {q_i}^{\eta'} &= [\beta_1(1-\beta_1) - y(1-\beta_1)^2(\frac{Q^2}{2p_i \cdot Q}) - y{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y(1-\beta_1)^2 Q^\eta {p_i}^{\eta'} \\
{q_i}^\eta {k_1}^{\eta'} &= [\beta_1(1-\beta_1) - y(1-\beta_1)^2(\frac{Q^2}{2p_i \cdot Q}) - y{\beta_1}^2(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y(1-\beta_1)^2 {p_i}^\eta Q^{\eta'} \\
{q_i}^\eta {q_i}^{\eta'} &= [{\beta_1}^2 - 2y\beta_1(1-\beta_1)(\frac{Q^2}{2p_i \cdot Q})]{p_i}^\eta {p_i}^{\eta'} + y\beta_1(1-\beta_1)(\frac{Q^2}{2p_i \cdot Q}){p_i}^\eta Q^{\eta'} \\
&+ y\beta_1(1-\beta_1)(\frac{Q^2}{2p_i \cdot Q})Q^\eta {p_i}^{\eta'} \\
{k_1}^\eta {q_k}^{\eta'} &= (1-\beta_1)A_1 {p_i}^\eta {p_i}^{\eta'} + (1-\beta_1)A_2 {p_i}^\eta Q^{\eta'} + (1-\beta_1)\sqrt{1-y}{p_i}^\eta {p_k}^{\eta'} \\
{q_i}^\eta {q_k}^{\eta'} &= A_1\beta_1 {p_i}^\eta {p_i}^{\eta'} + A_2\beta_1 {p_i}^\eta Q^{\eta'} + \beta_1\sqrt{1-y}{p_i}^\eta {p_k}^{\eta'} \\
{q_k}^\eta {k_1}^{\eta'} &= (1-\beta_1)A_1 {p_i}^\eta {p_i}^{\eta'} + (1-\beta_1)A_2 Q^\eta {p_i}^{\eta'} + (1-\beta_1)\sqrt{1-y}{p_k}^\eta {p_i}^{\eta'} \\
{q_k}^\eta {q_i}^{\eta'} &= A_1\beta_1 {p_i}^\eta {p_i}^{\eta'} + A_2\beta_1 Q^\eta {p_i}^{\eta'} + \beta_1\sqrt{1-y}{p_k}^\eta {p_i}^{\eta'}
\end{aligned}
\tag{2.55}
$$

Concept

Before the procedure is explained, at this point it should be mentioned that the steps are gradually explained in more detail in the next steps. This only provides a rough overview and can be used as a reference for the other sections.

i) First of all, look at a possible spliting. For this one has to make sure that all possible meaningful diagrams have been considered. All M_1, M_2, ${M_1}^\dagger M_2$ and $M_1 {M_2}^\dagger$ diagrams need to be indexed independently of each other. To determine the matrix elements, the Feynmann rules will be used which is explained in detail in section 1.4. Before the kinematics is used, the obtained matrix element should be simplified by matrix algebra, which is completely explained in the appendix Mathematical Tool, otherwise the calculation of the parametrisation becomes clearly more complicated.

ii) Each diagram consists of an emitter and a spactator part. The emitter part itself contains an emitter parent with the momentum $q_i + q$ in the old kinematic, of which a patron is split with q. A daughter-patron with q_i remains of the parent patron. One should select the spectator q_j skilfully, so that the diagrams are meaningful and calculable in the case of the interference terms, otherwise one must manipulate with the final results because of the unanimity of the indices. Thus a structure is achieved and the diagrams can be replaced from $M_1, {M_1}^\dagger, {M_2}^\dagger$ and M_2 side by side and even use their probability amplitude for the interference terms without having to recalculate them every time.

iii) Before starting to calculate, it will be firs tried to predict the expected result based on the contracting indices. Usually the non-contracting indices that remain form the final result as one or more tensors. This is relatively helpful when a calculation for a certain limit is performed, because it can be quickly seen from the square matrix element which terms must be calculated for the final result.

iv) When using parametrisation, it is recommended to use the concepts from the previous section. This is paractic, because when evaluating matrix elements, the multiplication of two tensors often occurs. To see which case to use for which matrix element, first look at the scalar products in the denominator that come from the propagators. Basically, there are four common scalar products listed here for this thesis:

- $(q \cdot q_i)(q \cdot q_i)$
- $(q \cdot q_j)(k_1 \cdot q_j)$
- $(q \cdot q_i)(q \cdot q_j)$
- $(q \cdot q_i)(q_i \cdot q_i)$

With the concept from the last section it was recognized exactly, which terms are finite, so that they can be omitted with the multiplication of the tensors in the apron. In other words, it is first recognized which pre-factors in the denominator cause singularities and then those terms with the same pre-factors are eliminated as finite terms. This considerably reduces the evaluation. This will become clearer later if the splitting functions are determined in the collinear limit from the respective diagram. For the new parametrisation, the following substitution is used:

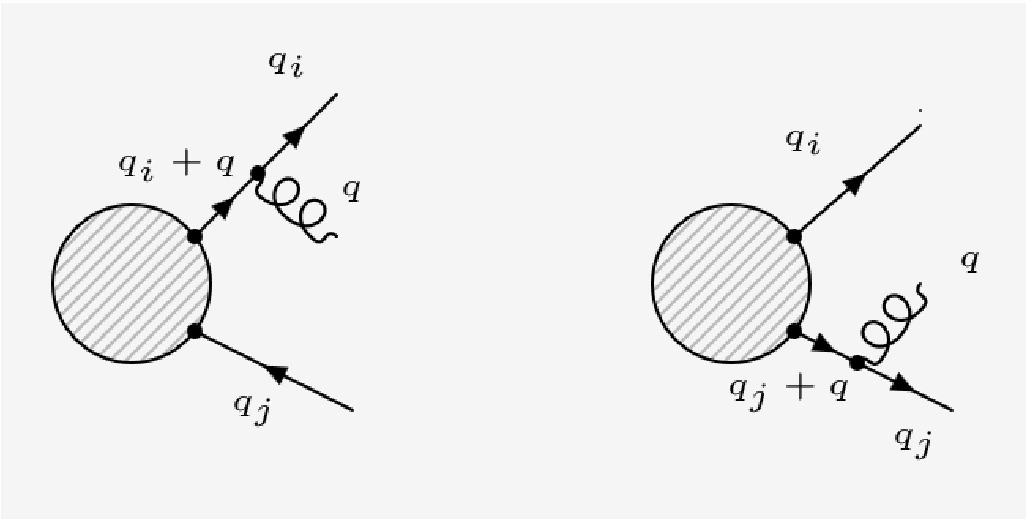
$$\begin{aligned} q_i &\to q_i \\ q &\to k_1 \\ q_j &\to q_k \end{aligned} \tag{2.56}$$

v) finally, to find out whether everything was calculated correctly, the collinear limits will be used because it is known that in this case the well known Alterali-Parisi 2.35 splitting function have to be output.

vi) In the case of indistinguishable particles in relation to the calculation of the interferometer, the momentum of the particles for the same diagram must be exchanged once in order to obtain the full result.

Chapter 3

anti-quark-gluon splitting from a parent quark

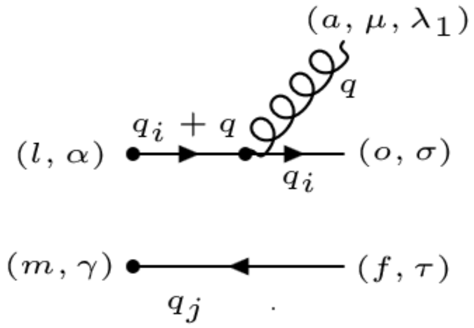


First we are going to consider a daughter quark from the splitting of a parent quark into a quark and a gluon with an arbitrary spectator like an anti-quark, see the left picture above. where $q_j + q$ is the momentum of the quark before splitting, q the momentum of the gluon and q_j of daughter quark respectively. The momentum of the spectator is q_j. The distinction between daughter and parent vanishes, when the gluon becomes soft, and a singularity develops. The other possibility to get a singularity is surely if the gluon will be collinear to quark. The splitting functions are flavour independent since the strong interaction is flavour independent. Furthermore, leading order splitting cannot change the flavour of a quark, thus we can write for the splitting functions In leading order QCD:

$$P_{\bar{q}_i\bar{q}_j} = P_{q_iq_j} = P_{qq}\delta_{ij} \tag{3.1}$$

For this aim we have to take any diagram to the account which can have the same splitting. Since there is no distinction between quark and anti-quark, one can imagine exact the same splitting variation for anti-quark with a quark as a spectator, see the right picture above.

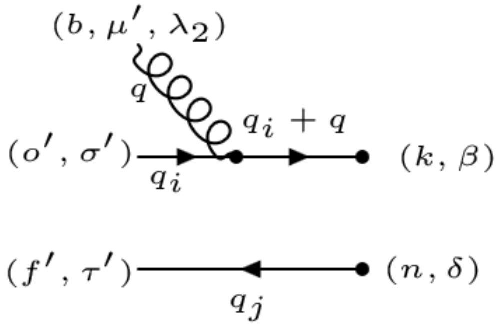
3.1 Matrix element of a quark with a gluon radiation $|M_1|^2$



If one simply calculates the amplitude of this diagram, one gets:

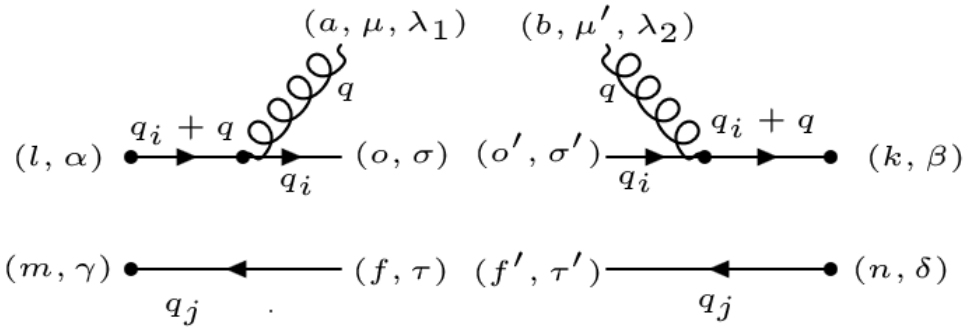
$$M_1 = [\bar{u}_\sigma(q_i)(-ig_s\gamma^\mu \times [T^a]_o{}^l)\frac{i(\not{q}_i + \not{q})}{(q_i+q)^2}\varepsilon^{\lambda_1}{}_\mu(q)]\,[v_\tau(q_j)] \tag{3.2}$$

For the quadratic matrix element we need the dagger of M_1 as well.



$$M_1{}^\dagger = [\frac{-i(\not{q}_i + \not{q})}{(q_i+q)^2}\,(ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\,u_{\sigma'}(q_i)\,\varepsilon^{\lambda_2}{}_{\mu'}(q)][\bar{v}_{\tau'}(q_j)] \tag{3.3}$$

After multiplying $M_1{}^\dagger$ and M_1 we get the desired result.



$$|M_1|^2 = M_1\, M_1{}^\dagger = [\bar{u}_\sigma(q_i)\,(-ig_s\gamma^\mu \times [T^a]_o{}^l)\,\frac{i(\not{q}_i+\not{q})}{(q_i+q)^2}\,\varepsilon^{\lambda_1}{}_\mu(q)][v_\tau(q_j)]$$
$$[\frac{-i(\not{q}_i+\not{q})}{(q_i+q)^2}\,(ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\,u_{\sigma'}(q_i)\,\varepsilon^{\lambda_2}{}_{\mu'}{}^*(q)][\bar{v}_{\tau'}(q_j)] \tag{3.4}$$

Now it's time to connect those terms which are related to each other.

$$|M_1|^2 = [\frac{-i(\not{q}_i+\not{q})}{(q_i+q)^2}\,(ig_s\gamma^{\mu'} \times [T^b]_{o'}{}^k)\,\bar{u}_\sigma(q_i)\,u_{\sigma'}(q_i)\,\varepsilon^{\lambda_2}{}^*_{\mu'}(q)\varepsilon^{\lambda_1}{}_\mu(q)$$
$$\times(-ig_s\gamma^\mu \times [T^a]_o{}^l)\,\frac{i(\not{q}_i+\not{q})}{(q_i+q)^2}][\bar{v}_{\tau'}(q_j)v_\tau(q_j)] \tag{3.5}$$

Sum over the lorenz index (σ,σ') and (τ,τ') and spin addition relation leads to:

$$\sum_{\sigma,\sigma'}\bar{u}_\sigma(q_i)\,u_{\sigma'}(q_i) = \not{q}_i\delta^{oo'},$$
$$\sum_{\tau,\tau'}\bar{v}_\tau(q_j)\,v_{\tau'}(q_j) = \not{q}_j\delta^{ff'} \tag{3.6}$$

Sum over polarization index (λ_1,λ_2) :

$$\sum_{\mu,\mu'}\varepsilon^{\lambda_2}{}^*_{\mu'}(q)\varepsilon^{\lambda_1}{}_\mu(q) = -g_{\mu\mu'}\delta^{ab} \tag{3.7}$$

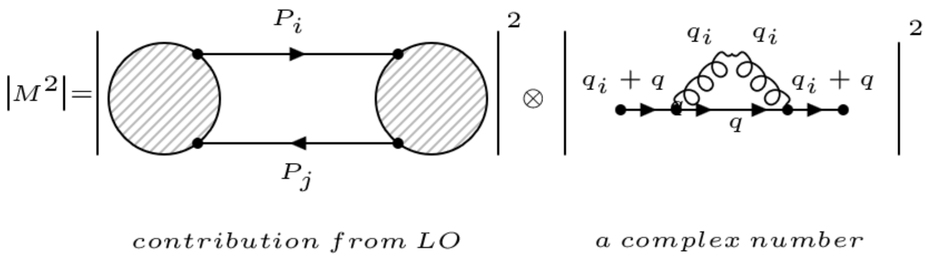
The matrix element will be simplified with:

$$|M_1|^2 = \frac{-g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[(\not{q}_i+\not{q})\;\gamma^{\mu'}\;\;\not{q}_i\,g_{\mu'\mu}\gamma^\mu\,(\not{q}_i+q)][\not{q}_j] \tag{3.8}$$

When we contract all related indices we will be actually able to make a statements about the last result.

$$|M_1|^2 = \frac{-g_s^2[T^a]_o{}^k\,[T^a]_o{}^l}{(q_i+q)^2(q_i+q)^2}[(\not{q}_i+\not{q})\;\gamma^{\mu'}\;\;\not{q}_i\,\gamma_{\mu'}\,(\not{q}_i+q)][\not{q}_j] \tag{3.9}$$

In other words we expect the tree level diagram from LO and a number:
Which graphically means:



$$|M_1|^2 = \frac{-g_s^2 [T^a]_o{}^k \, [T^a]_o{}^l}{(q_i+q)^2 (q_i+q)^2} [\not{P}_i][\not{P}_j] \; \otimes \; (a\ complex\ number) \tag{3.10}$$

Let's calculate the contribution and compare the final result with this expectation:

$$\begin{aligned} N =: \gamma^{\mu'} \; \not{q}_i \, \gamma_{\mu'} &= q_{i\sigma} \, \gamma^{\mu'} \gamma^{\sigma} \;\; \gamma_{\mu'} \\ &= q_{i\sigma} \, (\{\gamma^{\mu'}, \gamma^{\sigma}\} \; - \gamma^{\sigma} \gamma^{\mu'}) \gamma_{\mu'} \\ &= q_{i\sigma} \, 2 g^{\mu' \sigma} \, \gamma_{\mu'} \; - \; d \, \gamma^{\sigma} \\ &= (2-d) \;\; \not{q}_i \end{aligned} \tag{3.11}$$

Simplification of the bracket:

$$|M_1|^2 = -(2-d) \; \frac{g_s^2 [T^a]_o{}^k \, [T^a]_o{}^l}{(q_i+q)^2 (q_i+q)^2} [(\not{q}_i + \not{q}) \;\;\; \not{q}_i \;\; (\not{q}_i + q)][\not{q}_j] \tag{3.12}$$

$$|M_1|^2 = -(2-d) \; \frac{g_s^2 [T^a]_o{}^k \, [T^a]_o{}^l}{(q_i+q)^2 (q_i+q)^2} [\not{q}_i \;\, \not{q}_i \;\, \not{q}_i + \;\, \not{q}_i \;\, \not{q}_i \;\, \not{q} + \;\, \not{q} \;\, \not{q}_i \;\, \not{q}_i + \;\, \not{q} \;\, \not{q}_i \;\, \not{q}][\not{q}_j] \tag{3.13}$$

Momenta are on-shell, so:

$$\begin{aligned} \not{q}_i \;\; \not{q}_i &= {q_i}^2 = {m_i}^2 \\ \not{q} \;\; \not{q} &= q^2 = m^2 \\ \not{q}_j \;\; \not{q}_j &= {q_j}^2 = {m_j}^2 \end{aligned} \tag{3.14}$$

we can first neglect the mass of patrons:

$$|M_1|^2 = -(2-d) \; \frac{g_s^2 [T^a]_o{}^k \, [T^a]_o{}^l}{(2 q_i q)(2 q_i q)} [\not{q} \;\, \not{q}_i \;\, \not{q}][\not{q}_j] \tag{3.15}$$

Here we need to make the terms in the brackets simpler and:

$$\begin{aligned} L = \not{q}\ \not{q}_i\ \not{q} &= \not{q}[q_{i\sigma}q_\mu\left(\{\gamma^\mu,\gamma^\sigma\} - \gamma^\sigma\gamma^\mu\right)] \\ &= \not{q}[2{q_i}^\mu q_\mu - q_{i\sigma}q_\mu\gamma^\mu\gamma^\sigma \\ &= \not{q}(2q_iq) - q_\mu q_{i\sigma}q_\mu[\gamma^\mu\gamma^\mu\gamma^\sigma] \\ &= \not{q}(2q_iq) - q_\mu q_{i\sigma}q_\mu[\frac{\gamma^\mu\gamma^\mu}{2} + \frac{\gamma^\mu\gamma^\mu}{2}]\gamma^\sigma \\ &= \not{q}(2q_iq) - q_\mu q_{i\sigma}q_\mu[g^{\mu\mu}]\gamma^\sigma \\ &= \not{q}(2q_iq) - q_\mu q_{i\sigma}q^\mu\gamma^\sigma = \not{q}(2q_iq) - q^2\ \not{q}_i \\ &= \not{q}(2q_iq) \end{aligned} \tag{3.16}$$

After inserting the last result of L and simplify the term $(2q_iq)$ from the denominator and nominator, we get:

$$|M_1|^2 = -(2-d)\ \frac{g_s^2{[T^a]_o}^k\ {[T^a]_o}^l}{2y(1-2z+2z^2)(p_i\cdot p_j)}[\not{q}][\not{q}_j] \tag{3.17}$$

Now we are going to use the parametrisation from equation (1) to reduce the 3-member matrix element to 2-member and take out the singularity term from the amplitude.

$$|M_1|^2 = (d-2)\ \frac{g_s^2{[T^a]_o}^k\ {[T^a]_o}^l}{2y(1-2z+2z^2)(p_i\cdot p_j)}[(1-z)\ \not{p}_i + zy\ \not{p}_j - \sqrt{zy(1-z)}\ \not{m}_\perp][(1-y)\ \not{p}_j] \tag{3.18}$$

Multiplying the both sides

$$\begin{aligned} |M_1|^2 = (d-2)\ \frac{g_s^2{[T^a]_o}^k\ {[T^a]_o}^l}{2y(1-2z+2z^2)(p_i\cdot p_j)}&[(1-z)(1-y)\ \not{p}_i\ \not{p}_j \\ &+zy(1-y)\ \not{p}_j\ \not{p}_j + (1-y)\sqrt{zy(1-z)}\ \not{m}_\perp\ \not{p}_j] \end{aligned} \tag{3.19}$$

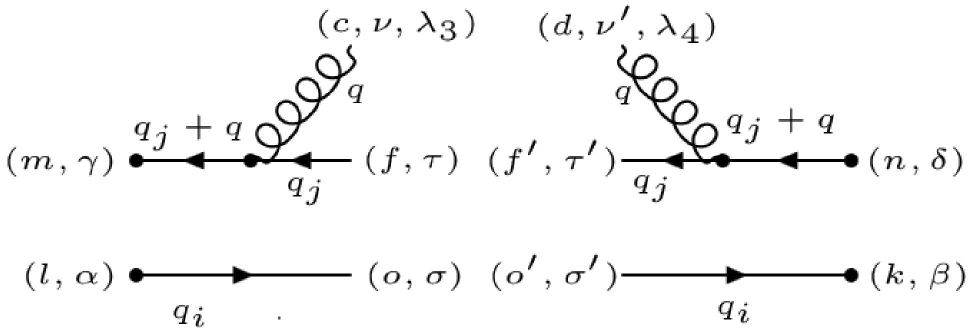
Under consideration of the fact that p_i and p_j are the on-shell momenta of the emitter and spectator partons, we can ignore the terms with $\not{p}_i\ \not{p}_i$ and $\not{p}_j\ \not{p}_j$. The $p_i \cdot m_\perp$ and $p_j \cdot m_\perp$ are always 0 because the p_i and p_j are lightlike, i.e. zero transverse component. So those terms can be neglected.

$$|M_1|^2 = \frac{g_s^2{[T^a]_o}^k\ {[T^a]_o}^l}{(p_i\cdot p_j)}[\not{p}_i][\not{p}_j] \otimes \frac{(d-2)(1-z)(1-y)}{2y(1-2z+2z^2)} \tag{3.20}$$

As discussed, we get a contribution from the LO a complex number. As you can see, the number is just for $y \to 0$ singular and not for $z \to 1$.

3.2 Matrix element of an anti-quark with a gluon radiation $|M_2|^2$

the same procedure is used to obtain the matrix element for an anti-quark with a single gluon emission.



$$|M_2|^2 = M_2\, M_2{}^\dagger = [\frac{i(\not q_j + \not q)}{(q_j+q)^2}(-ig_s\gamma^\nu \times [T^c]_f{}^m)\, v_\tau(q_j)\, \varepsilon^{\lambda_3}{}_\nu(q)]\, [u_\sigma(q_i)]$$
$$[\bar v_{\tau'}(q_j)\, (ig_s\gamma^{\nu'} \times [T^d]_{f'}{}^n)\, \frac{-i(\not q_j + \not q)}{(q_j+q)^2}\, \varepsilon^{\lambda_4}{}_{\nu'}(q)]\, [\bar u_{\sigma'}(q_i)] \tag{3.21}$$

$$|M_2|^2 = \frac{g_s^2\, [T^c]_f{}^m\, [T^d]_{f'}{}^n}{(q_j+q)^2(q_j+q)^2}[(\not q_j + \not q)\gamma^\nu\, v_\tau(q_j)\bar v_{\tau'}(q_j)\, \varepsilon^{\lambda_3}{}_\nu(q)\varepsilon^{\lambda_4}{}_{\nu'}(q)\gamma^{\nu'}(\not q_j + \not q)]$$
$$[u_\sigma(q_i)]\, [\bar u_{\sigma'}(q_i)] \tag{3.22}$$

and after sum over the lorenz and polarization indexes like (σ,σ'), (τ,τ') and (λ_3,λ_4) as well and using the spin addition relation:

$$|M_2|^2 = \frac{g_s^2\, [T^c]_f{}^m\, [T^c]_f{}^n}{(q_j+q)^2(q_j+q)^2}[(\not q_j + \not q)\gamma^\nu\ \ \not q_j\, (-g_{\nu\nu'})\gamma^{\nu'}(\not q_j + \not q)]\, [\not q_i] \tag{3.23}$$

Analogous to the last calculation from the previous section:

$$|M_2|^2 = (d-2)\frac{g_s^2\, [T^c]_f{}^m\, [T^c]_f{}^n}{(2qq_j)}[\not q]\, [\not q_i] \tag{3.24}$$

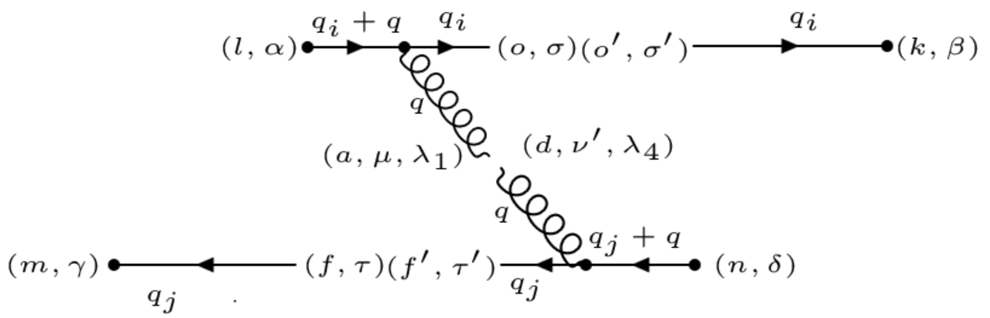
finally, we achieve:

$$|M_2|^2 = \frac{g_s^2\, [T^c]_f{}^m\, [T^c]_f{}^n}{(p_i\cdot p_j)}[\not p_i][\not p_j] \otimes \frac{-(d-2)yz^2}{2(1-z)(1-y)} \tag{3.25}$$

Interestingly, here is a term with y concerning the gluon radiation from an anti-quark. This means that this result cannot contribute to the collinear limit for soft gluon $y \to 0$.

3.3 Interference contribution

So far most of the work is done and we just have to put the results of M_1 and ${M_2}^\dagger$ next to each other, as we can see in the diagram. So we still get the interference contribution. Using the results from the previous section and we received:



$$\begin{aligned} M_1\, {M_2}^\dagger = & [\bar{u}_\sigma(q_i)\, (-ig_s\gamma^\mu \times [T^a]_o{}^l)\, \frac{i(\not{q}_i + \not{q})}{(q_i+q)^2}\, \varepsilon^{\lambda_1}{}_\mu(q)][v_\tau(q_j)] \\ & [\bar{v}_{\tau'}(q_j)\, (ig_s\gamma^{\nu'} \times [T^d]_{f'}{}^n)\, \frac{-i(\not{q}_j + \not{q})}{(q_j+q)^2}\, \varepsilon^{\lambda_4}{}_{\nu'}(q)]\, [u_{\sigma'}(q_i)] \end{aligned} \tag{3.26}$$

$$M_1\, {M_2}^\dagger = \frac{g_s^2 [T^a]_o{}^l\, [T^d]_{f'}{}^n}{(2q_i q)(2q_j q)} [\not{q}_i\, \gamma^\mu\, (\not{q}_i + \not{q})\,]\varepsilon^{\lambda_1}{}_\mu(q)\, \varepsilon^{\lambda_4}{}_{\nu'}(q)[\not{q}_j\, \gamma^{\nu'}\, (\not{q}_j + \not{q})] \tag{3.27}$$

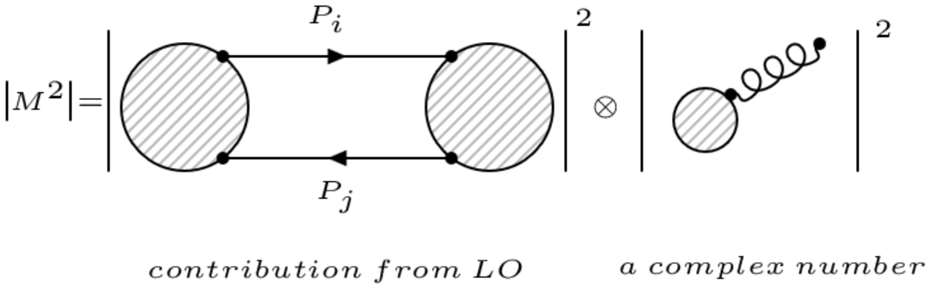
$$M_1\, {M_2}^\dagger = \frac{g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{(2q_i q)(2q_j q)} [\not{q}_i\, \gamma^\mu\, (\not{q}_i + \not{q})\,](-g_{\mu\nu'})[\not{q}_j\, \gamma^{\nu'}\, (\not{q}_j + \not{q})] \tag{3.28}$$

$$M_1\, {M_2}^\dagger = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{(2q_i q)(2q_j q)} [\not{q}_i\, \gamma^\mu\, (\not{q}_i + \not{q})\,]\, [\not{q}_j\, \gamma_\mu\, (\not{q}_j + \not{q})] \tag{3.29}$$

Expectation:

$$\begin{aligned} M_1\, {M_2}^\dagger = & \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{(2q_i q)(2q_j q)} [(z\, \not{p}_i + y(1-z)\, \not{p}_j + \sqrt{zy(1-z)}\, \not{m}_\perp)\, \gamma^\mu\, (\not{p}_i + y\, \not{p}_j)] \\ & [(1-y)\, \not{p}_j\, \gamma_\mu\, ((1-z)\, \not{p}_i + (1+yz-y)\, \not{p}_j - \sqrt{zy(1-z)}\, \not{m}_\perp)] \end{aligned} \tag{3.30}$$

$$\begin{aligned} M_1\, {M_2}^\dagger = & \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{4(1-z)(1-y)y(1-2z+2z^2)(p_i \cdot p_j)(p_i \cdot p_j)} \\ & [z\, \not{p}_i\, \gamma^\mu\, \not{p}_i + zy\, \not{p}_i\, \gamma^\mu\, \not{p}_j + y(1-z)\, \not{p}_j\, \gamma^\mu\, \not{p}_i + y^2(1-z)\, \not{p}_j\, \gamma^\mu\, \not{p}_j \\ & + \sqrt{zy(1-z)}\, \not{m}_\perp\, \gamma^\mu\, \not{p}_i + y\sqrt{zy(1-z)}\, \not{m}_\perp\, \gamma^\mu\, \not{p}_j][(1-y)(1-z)\, \not{p}_j\, \gamma_\mu\, \not{p}_i \\ & + (1-y)(1+yz-y)\, \not{p}_j\, \gamma_\mu\, \not{p}_j - (1-y)\sqrt{zy(1-z)}\, \not{p}_j\, \gamma_\mu\, \not{m}_\perp)] \end{aligned} \tag{3.31}$$



$$M_1\, M_2{}^{\dagger} = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{4(1-z)(1-y)y(1-2z+2z^2)(p_i\cdot p_j)(p_i\cdot p_j)} \\ [z\, \not{p}_i\, \gamma^{\mu}\, \not{p}_i + zy\, \not{p}_i\, \gamma^{\mu}\, \not{p}_j][(1-z)\, \not{p}_j\, \gamma_{\mu}\, \not{p}_i + (1-y)\, \not{p}_j\, \gamma_{\mu}\, \not{p}_j] \tag{3.32}$$

$$M_1\, M_2{}^{\dagger} = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{4(1-z)(1-y)y(1-2z+2z^2)(p_i\cdot p_j)(p_i\cdot p_j)} [2z\, p_i{}^{\mu}\, \not{p}_i + 2zyp_j{}^{\mu}\, \not{p}_i - zy\, \not{p}_i\, \not{p}_j\, \gamma^{\mu}] \\ [2(1-z)p_{i_{\mu}}\, \not{p}_j - (1-z)\, \not{p}_j\, \not{p}_i\, \gamma_{\mu} + 2(1-y)\, p_{j_{\mu}}\, \not{p}_j] \tag{3.33}$$

$$M_1\, M_2{}^{\dagger} = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{4(1-z)(1-y)y(1-2z+2z^2)(p_i\cdot p_j)(p_i\cdot p_j)} \\ [4z(1-z)p_i{}^2\, \not{p}_i\, \not{p}_j - 2z(1-z)\, \not{p}_i\, \not{p}_j\, \not{p}_i\, \not{p}_i + 4z(1-y)(p_i\cdot p_j)\, \not{p}_i\, \not{p}_j + \\ 4zy(1-z)(p_j\cdot p_i)\, \not{p}_i\, \not{p}_j - 2zy(1-z)\, \not{p}_i\, \not{p}_j\, \not{p}_i\, \not{p}_j + 4zy(1-y)p_j{}^2\, \not{p}_i\, \not{p}_j + \\ -2zy(1-z)\, \not{p}_i\, \not{p}_j\, \not{p}_i\, \not{p}_j + zy(1-z)\, \not{p}_i\, \not{p}_j \gamma^{\mu}\, \not{p}_j\, \not{p}_i\, \gamma_{\mu} - 2zy(1-y)\, \not{p}_i\, \not{p}_j\, \not{p}_j\, \not{p}_i] \tag{3.34}$$

$$M_1\, M_2{}^{\dagger} = \frac{-g_s^2 [T^a]_o{}^l\, [T^a]_{f'}{}^n}{y(1-2z+2z^2)(p_i\cdot p_j)} [\not{p}_i\, \not{p}_j] \otimes \frac{z}{1-z} \tag{3.35}$$

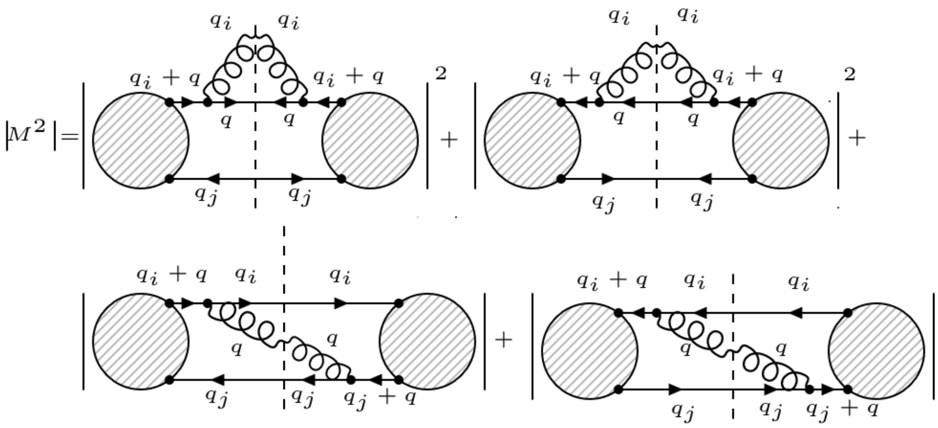
Now we use the old parametrization to collect the singularities.
Here we can use the singular term in the denominator $y(1-z)$ to drop the term with the same pre-factor and thus obtain:

3.4 Final result

One could assume that for a complete result the contribution ${M_1}^{\dagger} M_2$ is still missing.

$$| M|^2 = | M_1|^2 + | M_2|^2 + M_1 \, {M_2}^{\dagger} + {M_1}^{\dagger} M_2 \tag{3.36}$$

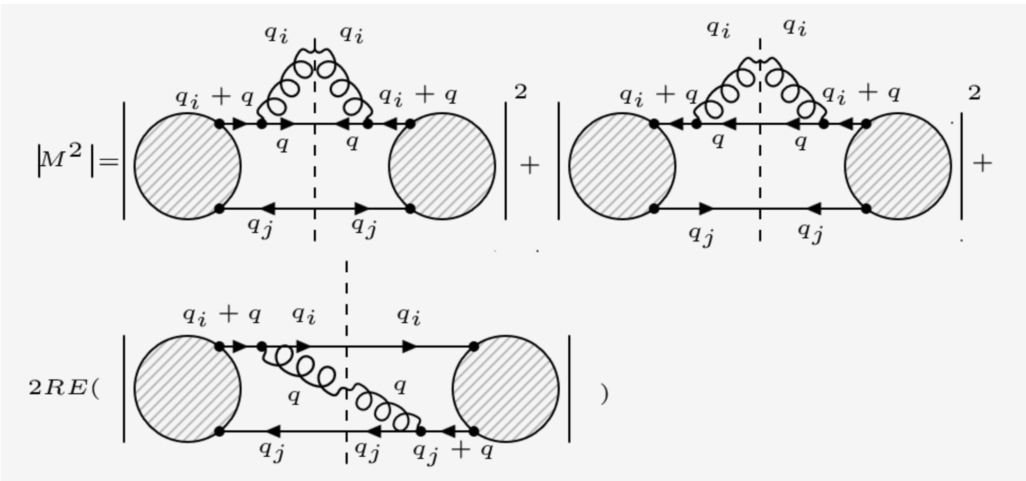
It should be noted that it is completely sufficient to calculate $M_1 \, {M_2}^{\dagger}$, because we know



it from the quadratic amount of the complex numbers, we can calculate double of real part of $2RE(M_1 \, {M_2}^{\dagger})$ instead of $M_1 \, {M_2}^{\dagger} + {M_1}^{\dagger} M_2$ and that is exactly what is preferred here.

$$| M|^2 = | M_1|^2 + | M_2|^2 + 2RE(M_1 \, {M_2}^{\dagger}) \tag{3.37}$$

Let's just add up the results from the previous sections and get:



$$\begin{aligned}
| M|^2 = & (d-2)(1-z)(1-y) \, \frac{g_s^2 {[T^a]_o}^k \, {[T^a]_o}^l}{2y(1-2z+2z^2)(p_i \cdot p_j)} [\not{p}_i][\not{p}_j] \\
& - (d-2) y z^2 \, \frac{g_s^2 \, {[T^c]_f}^m \, {[T^c]_f}^n}{2(1-z)(1-y)(p_i \cdot p_j)} [\not{p}_i][\not{p}_j] \\
& + 2RE((\frac{-2z}{z-1}) \frac{g_s^2 \, {[T^a]_o}^l \, {[T^a]_f}^n}{2y(1-2z+2z^2)(p_i \cdot p_j)} [\not{p}_i][\not{p}_j])
\end{aligned} \tag{3.38}$$

Now we use the knowledge from the introduction about the calculation of the Colour factor. With Fritz equation:

$$T^a{}_{o\,k}\,T^a{}_{l\,o} = \frac{1}{2}(\delta_{oo}\delta_{lk} - \frac{1}{N}\delta_{ok}\delta_{lo}) = \frac{1}{2}(N\delta_{lk} - \frac{1}{N}\delta_{lk}) = C_F\delta_{lk} \tag{3.39}$$

After summation over the final colour states and averaging over initial colour states we get:

$$T^a{}_{o\,k}\,T^a{}_{l\,o} = C_F\delta_{lk} = \frac{1}{N}\sum_{l=1}^{N}\delta_{lk}C_F = C_F \tag{3.40}$$

The same calculation for $T^c{}_{m\,f}\,T^c{}_{f\,n}$ and $T^a{}_{o\,l}\,T^a{}_{f\,n}$ turns C_F out as the colour factor. Now we are going to compute the splitting function in the case of the colinearity, wich means, if:

$$y \longrightarrow 0 \tag{3.41}$$

$$\mid M|^2 = \frac{g_s^2 C_F}{2y(1-2z+2z^2)(p_i\cdot p_j)}[\not{p}_i][\not{p}_j] \otimes ((d-2)(1-z) - \frac{4z}{z-1}) \tag{3.42}$$

for $d = 4 - 2\epsilon$

$$\begin{aligned}\mid M|^2 \ &= \frac{g_s^2}{y(1-2z+2z^2)(p_i\cdot p_j)}[\not{p}_i][\not{p}_j] \otimes C_F(\frac{(1+z^2)}{1-z} - \epsilon(1-z)) \\ &= \frac{g_s^2}{q_i\cdot q}[\not{p}_i][\not{p}_j] \otimes \langle\, \hat{P_{qq}}\rangle\end{aligned} \tag{3.43}$$

With Alterali-Parisi splitting function $\langle\, \hat{P_{qq}}\rangle$ in the collinear limes, which was mentioned in the previous chapter. This is exactly the confirmation of our calculation that our calculation was actually performed correctly, otherwise we would not have received the same splitting function for soft gluons.

3.5 Double-check the results with the new kinematic

One could do exactly the same calculation for the new kinematics to see if you get the same result in the collinear limit. From the next chapter we will explicitly work with the new parametrisation, because we found that the old kinematics only work in NLO and one-single emission.

$|M_1|^2$

$$|M_1|^2 = (d-2)\,\frac{g_s^2 C_F}{(2k_1\cdot q_i)}[\not{k}_1][\not{q}_k] \tag{3.44}$$

$$\begin{aligned}&|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\beta_1\ \not{Q} + \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}]\\&[A_1\ \not{p}_i + A_2\ \not{Q} + \sqrt{1-y}\ \not{p}_k]\end{aligned} \tag{3.45}$$

$$\begin{aligned}&|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)p_i\cdot Q\\&+(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}p_i\cdot p_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}n_{\perp,1}\cdot p_k]\end{aligned} \tag{3.46}$$

For the collinearity $y\to 0$ we'll get:

$$\begin{aligned}&|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)\ \not{p}_i\ \not{Q}\\&+(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}\ \not{p}_i\ \not{p}_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}\ \not{p}_k]\end{aligned} \tag{3.47}$$

$$|M_1|^2 = (d-2)(1-\beta_1)\sqrt{1-y}\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[\not{p}_i\ \not{p}_k] \tag{3.48}$$

$|M_2|^2$

$$|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}[\not{k}_1]\,[\not{q}_i] \tag{3.49}$$

$$\begin{aligned}&|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\beta_1\ \not{Q} + \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}]\\&[(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\alpha_1\ \not{Q} - \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,l}]\end{aligned} \tag{3.50}$$

Which means:

$$\begin{aligned}|M_2|^2 &\sim (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}y[...]\\|M_2|^2 &\to 0 \quad \text{for} \quad y\to 0\end{aligned} \tag{3.51}$$

$M_1\, M_2{}^\dagger$

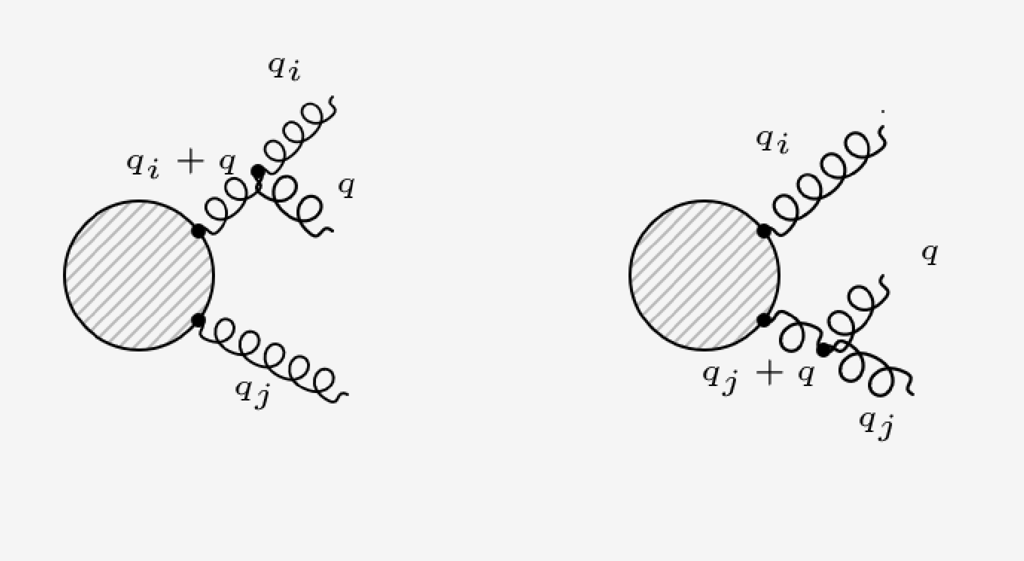
$$M_1\, M_2{}^\dagger = \frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} \\ 4(\beta_1\sqrt{1-y}p_i \cdot p_k)[\beta_1\sqrt{1-y}\ \not{p}_i\ \not{p}_k + (1-\beta_1)\sqrt{1-y}\ \not{p}_i\ \not{p}_k] \tag{3.52}$$

$$M_1\, M_2{}^\dagger = \frac{-g_s^2\, C_F}{y(1-\beta_1)\,(p_i \cdot p_k)(p_i \cdot Q)}\beta_1(p_i \cdot p_k)[\beta_1\ \not{p}_i\ \not{p}_k + (1-\beta_1)\ \not{p}_i\ \not{p}_k] \tag{3.53}$$

$$M_1\, M_2{}^\dagger = \frac{\beta_1}{(1-\beta_1)}\,\frac{-g_s^2\, C_F}{y\,(p_i \cdot Q)}[\ \not{p}_i\ \not{p}_k] \tag{3.54}$$

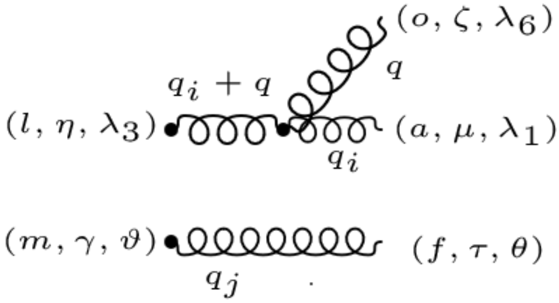
Chapter 4

Gluon radiation from a parent quark



Now consider a daughter gluon from the splitting of a parent gluon with radiation an another gluon. When the gluon becomes soft, the distinction between daughter and parent vanishes, and a singularity develops. In this chapter we are going to keep the same procedure with a difference that we won't use the old parametrisation since it only works in LO. One of the mainly challenges about this emission kernel is that the calculations are long and complicated. Before we get started, it must be mentioned that we will work here with the recipe from section 3.6, in which the respective mathematical tools were calculated in detail.

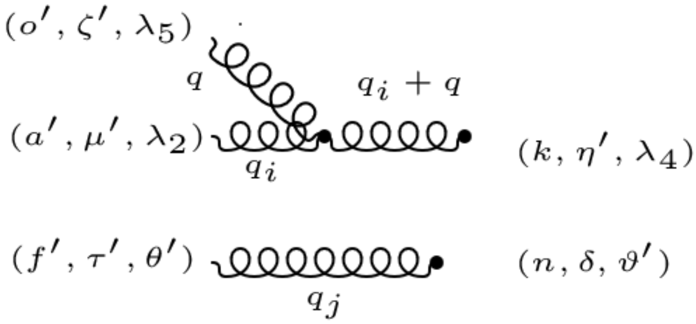
4.1 Gluon-Emitter Bubble



If we start with the first diagram and use the necessary Feynmann rules, we get:

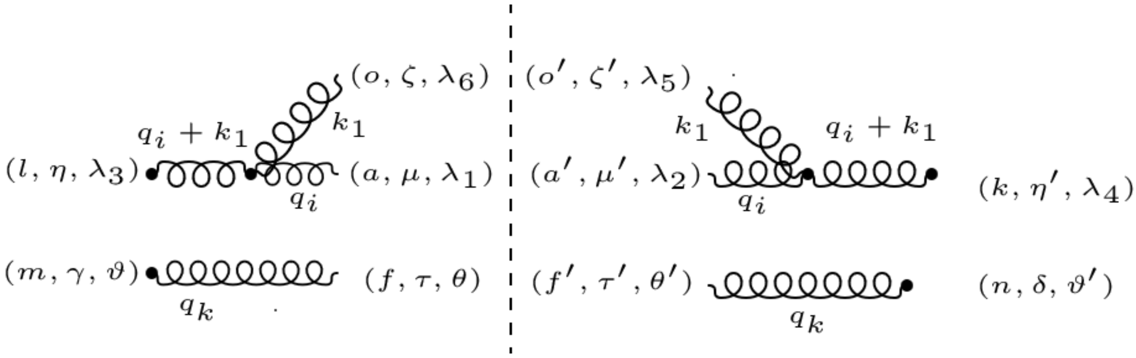
$$M_1 = [\frac{-i}{(q_i+q)^2}(-g_s f^{\,a\,o\,l}(g^{\mu\zeta}(q-q_i)^\eta - g^{\zeta\eta}(2q+q_i)^\mu + g^{\eta\mu}(2q_i+q)^\zeta) \\ \varepsilon^{\lambda_1}{}_\mu(q_i)\varepsilon^{\lambda_6}{}_\zeta(q)][\varepsilon^{\theta}{}_{\tau'}(q_j)] \tag{4.1}$$

It has to be emphasized that here only the impulse from the parent gluon is incoming and the two further impulses from the vertex are pointing out. This is why the minus signs appear in the equation. If the upper equation is daggered, you get the following diagram and the corresponding following amplitude:



$$M_1{}^\dagger = [\frac{i}{(q_i+q)^2}(-g_s f^{\,a'\,k\,o'}(-g^{\mu'\eta'}(2q_i+q)^{\zeta'} + g^{\eta'\zeta'}(2q+q_i)^{\mu'} + g^{\zeta'\mu'}(q_i-q)^{\eta'}) \\ \varepsilon^{\lambda_2}{}_{\mu'}{}^*(q_i)\varepsilon^{\lambda_5}{}_{\zeta'}{}^*(q)][\varepsilon^{\theta'}{}_{\tau'}{}^*(q_j)] \tag{4.2}$$

Let us now calculate the matrix element, in which we place the results from above next to each other, it turns out:



$$
\begin{aligned}
|M_1|^2 = [&\frac{-i}{(q_i+q)^2}(-g_s f^{\,a\,o\,l}(g^{\mu\zeta}(q-q_i)^\eta - g^{\zeta\eta}(2q+q_i)^\mu + g^{\eta\mu}(2q_i+q)^\zeta) \\
&\varepsilon^{\lambda_1}{}_\mu(q_i)\,\varepsilon^{\lambda_2}{}_{\mu'}{}^*(q_i)\varepsilon^{\lambda_6}{}_\zeta(q)\,\varepsilon^{\lambda_5}{}_{\zeta'}{}^*(q)(-g_s f^{\,a'\,k\,o'}(-g^{\mu'\eta'}(2q_i+q)^{\zeta'} + g^{\eta'\zeta'}(2q+q_i)^{\mu'} \\
&+ g^{\zeta'\mu'}(q_i-q)^{\eta'})\frac{i}{(q_i+q)^2}][g^{\gamma\delta}]
\end{aligned}
\tag{4.3}
$$

After the summation over the spin- just as well polarization indices can be obtained:

$$
|M_1|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[N^{\eta\eta'}][g^{\gamma\delta}] \tag{4.4}
$$

The tensor $N^{\eta\eta'}$ appears exactly when you contract across all possible indices.
The goal of the next step is to evaluate this tensor, which is avoided here and instead of that, it only presents the final result. The more detailed calculation can be found in Appendix A.

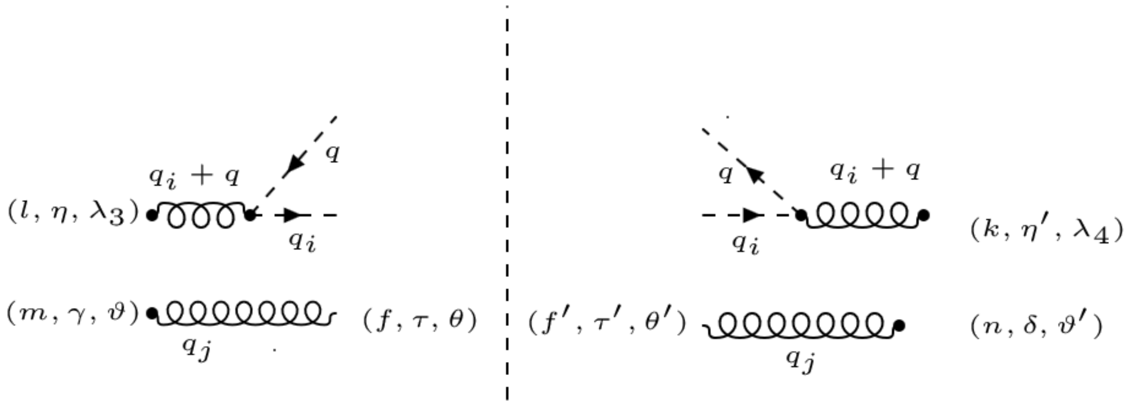
$$
\begin{aligned}
N^{\eta\eta'} \equiv [(6-d)q^\eta q^{\eta'} + (d+3)q^\eta {q_i}^{\eta'} + (d+3){q_i}^\eta q^{\eta'} + (6-d){q_i}^\eta {q_i}^{\eta'} \\
-g^{\eta\eta'}(5q^2 + 5{q_i}^2 + 8qq_i)][g^{\gamma\delta}]
\end{aligned}
\tag{4.5}
$$

Replace this result in the equation

$$
\begin{aligned}
|M_1|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^\eta q^{\eta'} + (d+3)q^\eta {q_i}^{\eta'} + (d+3){q_i}^\eta q^{\eta'} + (6-d){q_i}^\eta {q_i}^{\eta'} \\
-g^{\eta\eta'}(5q^2 + 5{q_i}^2 + 8qq_i)][g^{\gamma\delta}]
\end{aligned}
\tag{4.6}
$$

This Gluon self-energy diagram has to be corrected by Ghost Loop to get the complete result. That is exactly what we are going to do in the next section.

4.1.1 One-loop corrections to the gluon self-energy diagram(Gluon-Emitter Bubble)



In order to get a meaningful result and correct the gluon loop, the same indices must be used. For this the same diagram with a fine difference is used, in which you replace the cut off gluon propagators with ghost propagators and the rest remain as in the previous diagram.

$$|M_1|^2_{Ghost\,loop} = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[-q_i{}^{\eta}q^{\eta'} - q^{\eta}q_i{}^{\eta'}][g^{\gamma\delta}] \tag{4.7}$$

$$\begin{aligned}|M_1'|^2 = |M_1|^2 + |M_1|^2_{Ghost\,loop} = &\frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^{\eta}q^{\eta'} + (d+3)q^{\eta}q_i{}^{\eta'} \\ &+ (d+3)q_i{}^{\eta}q^{\eta'} + (6-d)q_i{}^{\eta}q_i{}^{\eta'} - g^{\eta\eta'}(5q^2 + 5q_i{}^2 + 8qq_i) - q_i{}^{\eta}q^{\eta'} - q^{\eta}q_i{}^{\eta'}][g^{\gamma\delta}]\end{aligned} \tag{4.8}$$

After simplification and addition over the same terms:

$$\begin{aligned}|M_1'|^2 = &\frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{(q_i+q)^2(q_i+q)^2}[(6-d)q^{\eta}q^{\eta'} + (d+2)q^{\eta}q_i{}^{\eta'} + (d+2)q_i{}^{\eta}q^{\eta'} + (6-d)q_i{}^{\eta}q_i{}^{\eta'} \\ &- g^{\eta\eta'}(8qq_i)][g^{\gamma\delta}]\end{aligned} \tag{4.9}$$

Implementation the new parametrisation:

$$\begin{aligned}|M_1'|^2 = &\frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y^2(\alpha_1+\beta_1)^2\,(p_i\cdot Q)\,(p_i\cdot Q)} \\ &[(6-d)(\zeta_1 p_i{}^{\eta} + \lambda_1 Q^{\eta} + \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_1 p_i{}^{\eta'} + \lambda_1 Q^{\eta'} + \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ &+ (d+2)(\zeta_1 p_i{}^{\eta} + \lambda_1 Q^{\eta} + \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_q p_i{}^{\eta'} + \lambda_q Q^{\eta'} - \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ &+ (d+2)(\zeta_q p_i{}^{\eta} + \lambda_q Q^{\eta} - \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_1 p_i{}^{\eta'} + \lambda_1 Q^{\eta'} + \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ &+ (6-d)(\zeta_q p_i{}^{\eta} + \lambda_q Q^{\eta} - \sqrt{y\alpha_1\beta_1}n^{\eta}{}_{\perp,1})(\zeta_q p_i{}^{\eta'} + \lambda_q Q^{\eta'} - \sqrt{y\alpha_1\beta_1}n^{\eta'}{}_{\perp,1}) \\ &- 8g^{\eta\eta'}[(\alpha_1{}^2 + \beta_1{}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}]\end{aligned} \tag{4.10}$$

Note that here the short version of the kinematics was used to increase the overview. Now you have to multiply the terms in brackets and simplify the matrix element.

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y^2\,(p_i\cdot Q)\,(p_i\cdot Q)}[(6-d)[\zeta_1\zeta_1 {p_i}^{\eta}{p_i}^{\eta'} + \zeta_1\lambda_1 {p_i}^{\eta} Q^{\eta'} + \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_1\zeta_1 Q^{\eta}{p_i}^{\eta'} + \lambda_1\lambda_1 Q^{\eta}Q^{\eta'} + \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} + \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& + \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(d+2)[\zeta_1\zeta_q {p_i}^{\eta}{p_i}^{\eta'} + \zeta_1\lambda_q {p_i}^{\eta}Q^{\eta'} - \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_1\zeta_q Q^{\eta}{p_i}^{\eta'} + \lambda_1\lambda_q Q^{\eta}Q^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} + \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& - \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(d+2)[\zeta_q\zeta_1 {p_i}^{\eta}{p_i}^{\eta'} + \zeta_q\lambda_1 {p_i}^{\eta}Q^{\eta'} + \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_q\zeta_1 Q^{\eta}{p_i}^{\eta'} + \lambda_q\lambda_1 Q^{\eta}Q^{\eta'} + \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& - \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}][(6-d)[\zeta_q\zeta_q {p_i}^{\eta}{p_i}^{\eta'} + \zeta_q\lambda_q {p_i}^{\eta}Q^{\eta'} - \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + \lambda_q\zeta_q Q^{\eta}{p_i}^{\eta'} + \lambda_q\lambda_q Q^{\eta}Q^{\eta'} - \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} - \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& + \sqrt{y\alpha_1\beta_1}\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}]
\end{aligned}
\tag{4.11}
$$

One replaces now the relations for the often occurring pre-factor products from appendix A and get:

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y^2\,(p_i\cdot Q)\,(p_i\cdot Q)}[(6-d)\{({\alpha_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 {p_i}^{\eta}Q^{\eta'} \\
& + \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} + y\beta_1\alpha_1 Q^{\eta}{p_i}^{\eta'} + \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} \\
& + \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} + y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} + (d+2)\{(\alpha_1\beta_1 - y({\alpha_1}^2+{\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} \\
& + y{\alpha_1}^2{p_i}^{\eta}Q^{\eta'} - \zeta_1\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} + y{\beta_1}^2Q^{\eta}{p_i}^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} + \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} \\
& + \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} - y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} \\
& + (d+2)\{(\beta_1\alpha_1 - y({\beta_1}^2+{\alpha_1}^2)(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2{p_i}^{\eta}Q^{\eta'} + \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} \\
& + y{\alpha_1}^2Q^{\eta}{p_i}^{\eta'} + \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} - \lambda_1\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} \\
& - y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} + (6-d)\{({\beta_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\beta_1\alpha_1{p_i}^{\eta}Q^{\eta'} \\
& - \zeta_q\sqrt{y\alpha_1\beta_1}{p_i}^{\eta}{n^{\eta'}}_{\perp,1} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} - \lambda_q\sqrt{y\alpha_1\beta_1}Q^{\eta}{n^{\eta'}}_{\perp,1} - \zeta_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}{p_i}^{\eta'} \\
& - \lambda_q\sqrt{y\alpha_1\beta_1}{n^{\eta}}_{\perp,1}Q^{\eta'} + y\alpha_1\beta_1{n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1}\} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}]
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y^2\,(p_i\cdot Q)\,(p_i\cdot Q)}[(6-d)\{({\alpha_1}^2-2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\eta}p_i{}^{\eta'} \\
& + y\alpha_1\beta_1 p_i{}^{\eta}Q^{\eta'} + y\beta_1\alpha_1 Q^{\eta}p_i{}^{\eta'} + y\alpha_1\beta_1 n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1}\} \\
& + (d+2)\{(\alpha_1\beta_1 - y({\alpha_1}^2+{\beta_1}^2)(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\eta}p_i{}^{\eta'} + y{\alpha_1}^2 p_i{}^{\eta}Q^{\eta'} + y{\beta_1}^2 Q^{\eta}p_i{}^{\eta'} \\
& - y\alpha_1\beta_1 n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1}\} + (d+2)\{(\beta_1\alpha_1 - y({\beta_1}^2+{\alpha_1}^2)(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\eta}p_i{}^{\eta'} \\
& + y{\beta_1}^2 p_i{}^{\eta}Q^{\eta'} + y{\alpha_1}^2 Q^{\eta}p_i{}^{\eta'} - y\alpha_1\beta_1 n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1}\} \\
& + (6-d)\{({\beta_1}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}))p_i{}^{\eta}p_i{}^{\eta'} + y\beta_1\alpha_1 p_i{}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}p_i{}^{\eta'} \\
& + y\alpha_1\beta_1 n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1}\} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}]][g^{\gamma\delta}]
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
|M_1'|^2 = {} & \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{4y\,(p_i\cdot Q)\,(p_i\cdot Q)}[[8-4d]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} - 8g^{\eta\eta'}[({\alpha_1}^2+{\beta_1}^2)p_i\cdot Q \\
& - (\beta_1(1-\beta_1))n_{\perp,1}\cdot n_{\perp,1}][g^{\gamma\delta}]]
\end{aligned}
\tag{4.14}
$$

In this step the equation for $d = 4 - 2\epsilon$ was calculated and the value of $(\alpha_1 + \beta_1)^2 p_i \cdot Q$ from equation (2.49) was replaced. What here noticeable is at this point that one y from the denominator is abbreviated with one from the nominator. The final result looks like this:

$$
|M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y\,(p_i\cdot Q)}[2[\epsilon-1]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}] \tag{4.15}
$$

4.1.2 Another way (within the concept 2.9.2)

During the analysis of the evaluation it turned out that the particularly complicated and extensive calculation can be handled with the substitutions conceived below:
The first consideration is to look in the denominators for the pre-factors that cause a singularity.
In the second step, care must be taken to ensure that the numerator consists of the addition of several terms, which mostly consist of scalar products with two four-vectors. This is the reason why these scalar products have to be considered separately, provided that the terms with the same pre-factor are eliminated from the nominator beforehand, since they are finite anyway. In the last chapter, this has already been deduced with regard to the factors in the denominator. For illustration this is calculated once for $|M_1'|^2$ with the denominator $4y^2\,(p_i \cdot Q)\,(p_i \cdot Q)$.
If one looks at this corrected matrix element, with the pre-factor y^2 in the denominator, it can be seen that in the counter the respective terms with y^2 by the multiplication of two four-vectors can be neglected instead of calculating these, since exactly these form the finite terms. This simplifies the results of the scalar products. Under this assumption, the result looks as follows:

$$
\begin{aligned}
{k_1}^{\eta}{k_1}^{\eta'} &= ({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 {p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{k_1}^{\eta}{q_i}^{\eta'} &= (\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\alpha_1}^2 {p_i}^{\eta}Q^{\eta'} + y{\beta_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{q_i}^{\eta}{k_1}^{\eta'} &= (\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2 {p_i}^{\eta}Q^{\eta'} + y{\alpha_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
{q_i}^{\eta}{q_i}^{\eta'} &= ({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 {p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1}
\end{aligned}
\tag{4.16}
$$

Now we insert these results into N which was an element of the square matrix element.

$$
\begin{aligned}
N \equiv\ & (6-d)({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 {p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\alpha_1}^2 {p_i}^{\eta}Q^{\eta'} + y{\beta_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y{\beta_1}^2 {p_i}^{\eta}Q^{\eta'} + y{\alpha_1}^2 Q^{\eta}{p_i}^{\eta'} - y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& + (6-d)({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})){p_i}^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 {p_i}^{\eta}Q^{\eta'} + y\alpha_1\beta_1 Q^{\eta}{p_i}^{\eta'} + y\alpha_1\beta_1 n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& - 8g^{\eta\eta'}[({\alpha_1}^2 + {\beta_1}^2)p_i \cdot Q - (\beta_1(1-\beta_1))n_{\perp,1} \cdot n_{\perp,1}]
\end{aligned}
\tag{4.17}
$$

Summary of the equation provides:

$$
\begin{aligned}
N \equiv\ & [(6-d)({\alpha_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q})) + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})) \\
& + (d+2)(\alpha_1\beta_1 - y({\alpha_1}^2 + {\beta_1}^2)(\frac{Q^2}{2p_i \cdot Q})) + (6-d)({\beta_1}^2 - 2\alpha_1\beta_1 y(\frac{Q^2}{2p_i \cdot Q}))]{p_i}^{\eta}{p_i}^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 + (d+2)y{\alpha_1}^2 + (d+2)y{\beta_1}^2 + (6-d)y\alpha_1\beta_1]{p_i}^{\eta}Q^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 + (d+2)y{\beta_1}^2 + (d+2)y{\alpha_1}^2 + (6-d)y\alpha_1\beta_1]Q^{\eta}{p_i}^{\eta'} \\
& + [(6-d)y\alpha_1\beta_1 - (d+2)y\alpha_1\beta_1 - (d+2)y\alpha_1\beta_1 + (6-d)y\alpha_1\beta_1]n^{\eta}_{\perp,1}n^{\eta'}_{\perp,1} \\
& - 8g^{\eta\eta'}[({\alpha_1}^2 + {\beta_1}^2)p_i \cdot Q - (\beta_1(1-\beta_1))n_{\perp,1} \cdot n_{\perp,1}]
\end{aligned}
\tag{4.18}
$$

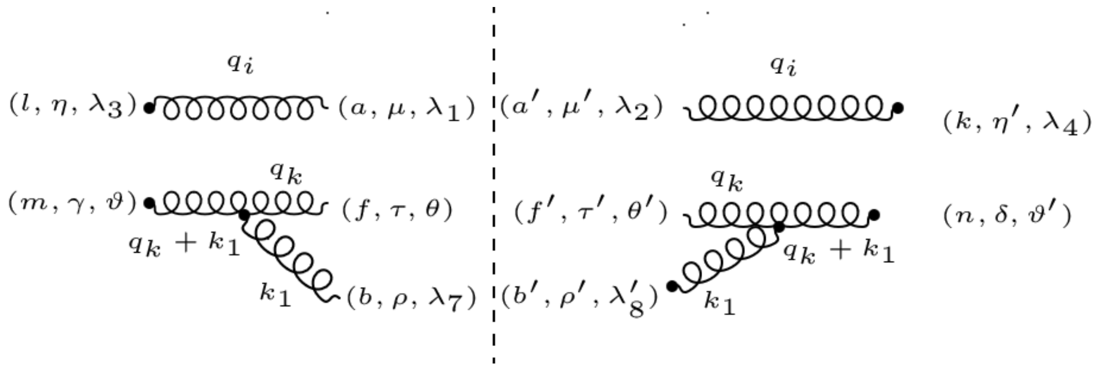
If you now insert the result into the matrix element and simplify it:

$$
|M_1'|^2 = \frac{g_s^2\, f^{\,a\,o\,l}\, f^{\,a\,k\,o}}{y\,(p_i \cdot Q)}[2[\epsilon - 1]\beta_1(1-\beta_1){n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}] \tag{4.19}
$$

conclusion

This concept allows to achieve the same result in just a few steps. This can also be done for further calculations. In the following it is imperative to orient oneself to the denominator of the matrix element.

4.2 Gluon-Spectator Bubble



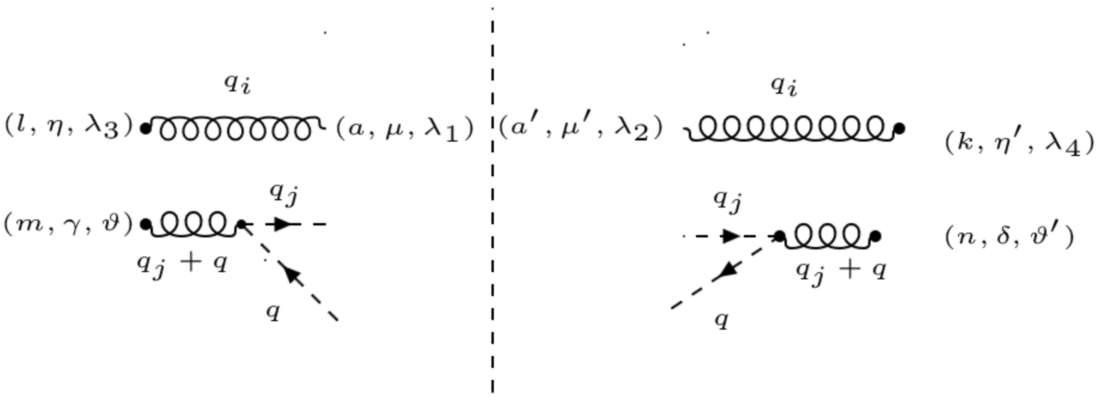
This concept can also be applied to the gluon spectator. The only difference is that a gluon is emitted by a spectator and other indices are used here.
Using the Feynmann rules, the matrix element is evaluated:

$$\begin{aligned}|M_2|^2 = &\frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[(2q+q_j)^\gamma(2q_j+q)^\delta - g^{\delta\gamma}(2q_j+q)^\rho(2q_j+q)_\rho \\ &-(2q_j+q)^\gamma(q-q_j)^\delta - g^{\delta\gamma}(2q+q_j)^\tau(2q+q_j)_\tau + (2q_j+q)^\gamma(2q+q_j)^\delta \\ &+(2q+q_j)^\gamma(q-q_j)^\delta - (q_j-q)^\gamma(2q+q_j)^\delta + (q_j-q)^\gamma(2q_j+q)^\delta \\ &+d(q_j-q)^\gamma(q-q_j)^\delta][g^{\eta\eta'}]\end{aligned} \tag{4.20}$$

It follows:

$$\begin{aligned}|M_2|^2 = &\frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[(3+d)q^\gamma {q_j}^\delta + (6-d)q^\gamma q^\delta + (6-d){q_j}^\gamma {q_j}^\delta \\ &+(3+d){q_j}^\gamma q^\delta - g^{\delta\gamma}(5{q_j}^2 + 5q^2 + 8qq_j)][g^{\eta\eta'}]\end{aligned} \tag{4.21}$$

4.2.1 One-loop corrections to the gluon self-energy diagram (Gluon-Spectator Bubble)



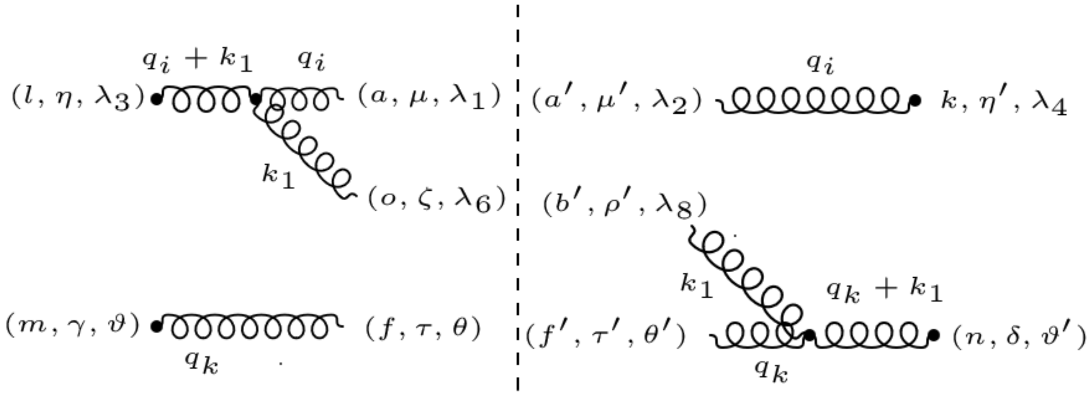
Here not all steps are shown in detail, but only the final result is presented. The reason for this is that all steps can be followed analogously to the last section.

$$|M_2|^2_{Ghost\,loop} = \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[-q_j{}^{\gamma} q^{\delta} - q^{\delta} q_j{}^{\gamma}][g^{\eta\eta'}] \tag{4.22}$$

$$\begin{aligned}|M_2'|^2 = &\frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(q_j+q)^2(q_j+q)^2}[(2+d)q^{\gamma} q_j{}^{\delta} + (6-d)q^{\gamma} q^{\delta} \\ &+(6-d)q_j{}^{\gamma} q_j{}^{\delta} + (2+d)q_j{}^{\gamma} q^{\delta} - g^{\delta\gamma}(8qq_j)][g^{\eta\eta'}]\end{aligned} \tag{4.23}$$

$$|M_2'|^2 = \frac{g_s^2\, f^{\,b\,f\,m} f^{\,b\,n\,f}}{(1-\beta_1)(1-y)\,(p_i \cdot p_k)}[-2g^{\delta\gamma}][g^{\eta\eta'}] \tag{4.24}$$

4.3 Interference term $M_1 M_2{}^\dagger$



Analogous to the last two sections, we calculate the quadratic matrix element in the case of the interference term.

$$M_1 M_2{}^\dagger = \frac{g_s^2 f^{\,l\,a\,o} f^{\,f'\,b'\,n} \delta^{aa'} \delta^{ob'} \delta^{ff'}}{(q_i+q)^2 (q_j+q)^2} [g_\mu{}^{\eta'} g_{\tau\tau'} (g^{\eta\mu}(2q_i+q)^\zeta + g^{\mu\zeta}(q-q_i)^\eta - g^{\zeta\eta}(2q+q_i)^\mu)$$
$$g_{\zeta\rho'}(g^{\tau'\rho'}(q_j-q)^\delta + g^{\rho'\delta}(2q+q_j)^{\tau'} - g^{\delta\tau'}(2q_j+q)^{\rho'}] \tag{4.25}$$

Multiply the available tensors and summarize all results:

$$\begin{aligned}
M_1 M_2{}^\dagger = \frac{g_s^2 f^{\,l\,a\,o} f^{\,f\,o\,n}}{4(q \cdot q_i)(q \cdot q_j)} \{ & g^{\eta\eta'}[2{q_i}^\gamma {q_j}^\delta + 2{q_i}^\gamma q^\delta + q^\gamma {q_j}^\delta + q^\gamma q^\delta + 4 q^\gamma {q_i}^\delta \\
& + 2q^\gamma q^\delta + 2{q_j}^\gamma {q_i}^\delta + {q_j}^\gamma q^\delta] - g^{\eta\eta'} g^{\gamma\delta}(2q \cdot q_j + q \cdot q + 4q_i \cdot q_j + 2q_i \cdot q) \\
& + g^{\gamma\eta'}[q^\eta {q_j}^\delta - q^\eta q^\delta - {q_i}^\eta {q_j}^\delta + {q_i}^\eta q^\delta] + g^{\eta'\delta}[2q^\eta q^\gamma + q^\eta {q_j}^\gamma + {q_i}^\eta q^\gamma + {q_i}^\eta {q_j}^\gamma] \\
& - g^{\gamma\delta}[2q^\eta {q_j}^{\eta'} + q^\eta q^{\eta'} - 2{q_i}^\eta {q_j}^{\eta'} - {q_i}^\eta q^{\eta'}] - g^{\gamma\eta}[2q^{\eta'} {q_j}^\delta - 2q^{\eta'} q^\delta + {q_i}^{\eta'} {q_j}^\delta - {q_i}^{\eta'} q^\delta] \\
& - g^{\eta\delta}[4q^{\eta'} q^\gamma + 2q^{\eta'} {q_j}^\gamma + 2{q_i}^{\eta'} q^\gamma + {q_i}^{\eta'} {q_j}^\gamma] + g^{\gamma\delta}[4{q_j}^\eta q^{\eta'} + 2{q_j}^\eta {q_i}^{\eta'} + q^\eta q^{\eta'} + q^\eta {q_i}^{\eta'}]\}
\end{aligned} \tag{4.26}$$

As can be seen here from the term in the denominator, this is the parametrisation in the case of (2.52)

$$
\begin{aligned}
{k_1}^{\eta} {k_1}^{\eta'} &= [(1-\beta_1)^2 - y^2 {\beta_1}^2 (\frac{Q^2}{2p_i \cdot Q})^2] {p_i}^{\eta} {p_i}^{\eta'} - y^2 {\beta_1}^2 (\frac{Q^2}{2p_i \cdot Q}) {p_i}^{\eta} Q^{\eta'} - y^2 {\beta_1}^2 (\frac{Q^2}{2p_i \cdot Q}) Q^{\eta} {p_i}^{\eta'} \\
{k_1}^{\eta} {q_i}^{\eta'} &= [\beta_1(1-\beta_1) - y {\beta_1}^2 (\frac{Q^2}{2p_i \cdot Q})] {p_i}^{\eta} {p_i}^{\eta'} + y {\beta_1}^2 Q^{\eta} {p_i}^{\eta'} \\
{q_i}^{\eta} {k_1}^{\eta'} &= [\beta_1(1-\beta_1) - y {\beta_1}^2 (\frac{Q^2}{2p_i \cdot Q})] {p_i}^{\eta} {p_i}^{\eta'} + y {\beta_1}^2 {p_i}^{\eta} Q^{\eta'} \\
{q_i}^{\eta} {q_i}^{\eta'} &= {\beta_1}^2 {p_i}^{\eta} {p_i}^{\eta'} \\
{k_1}^{\eta} {q_k}^{\eta'} &= [(1-\beta_1) - y\beta_1 (\frac{Q^2}{2p_i \cdot Q})] \sqrt{1-y} {p_i}^{\eta} {p_k}^{\eta'} - y\beta_1 (\frac{Q^2}{2p_i \cdot Q}) A_1\, {p_i}^{\eta} {p_i}^{\eta'} - y\beta_1 (\frac{Q^2}{2p_i \cdot Q}) A_2\, {p_i}^{\eta} Q^{\eta'} \\
&\quad + y\beta_1 A_1\, Q^{\eta} {p_i}^{\eta'} + y\beta_1 A_2\, Q^{\eta} Q^{\eta'} + y\beta_1 \sqrt{1-y} Q^{\eta} {p_k}^{\eta'} \\
{q_i}^{\eta} {q_k}^{\eta'} &= A_1 \beta_1 {p_i}^{\eta} {p_i}^{\eta'} + A_2 \beta_1 {p_i}^{\eta} Q^{\eta'} + \beta_1 \sqrt{1-y} {p_i}^{\eta} {p_k}^{\eta'} \\
{q_k}^{\eta} {k_1}^{\eta'} &= [(1-\beta_1) - y\beta_1 (\frac{Q^2}{2p_i \cdot Q})] \sqrt{1-y} {p_k}^{\eta} {p_i}^{\eta'} - y\beta_1 (\frac{Q^2}{2p_i \cdot Q}) A_1\, {p_i}^{\eta} {p_i}^{\eta'} - y\beta_1 (\frac{Q^2}{2p_i \cdot Q}) A_2\, Q^{\eta} {p_i}^{\eta'} \\
&\quad + y\beta_1 A_1\, {p_i}^{\eta} Q^{\eta'} + y\beta_1 A_2\, Q^{\eta} Q^{\eta'} + y\beta_1 \sqrt{1-y} {p_k}^{\eta} Q^{\eta'} \\
{q_k}^{\eta} {q_i}^{\eta'} &= A_1 \beta_1 {p_i}^{\eta} {p_i}^{\eta'} + A_2 \beta_1 Q^{\eta} {p_i}^{\eta'} + \beta_1 \sqrt{1-y} {p_k}^{\eta} {p_i}^{\eta'}
\end{aligned}
\tag{4.27}
$$

It should be noted that we have not opted for the standard way with the replacement of parametrisation and elimination of limited terms, because otherwise the calculation becomes rather confusing and long. It should be mentioned here that the standard way was still used as a comparison to determine whether the two approaches would lead to the same goal. Let's start with the evaluation of the respective terms. a summarized calculation can be found in the appendix 6.1. If you are interested in the complete solution which contains both cases, i.e. the collinear and soft region, you simply have to add the terms. Since the calculation becomes quite long and confusing, we focus on the second term, which is important for the collinear case and presents the full result, which is used for the calculation of the splitting function.
The starting point is the second term of the quadratic matrix element:

$$
-g^{\eta\eta'} g^{\gamma\delta} (2q \cdot q_j + q \cdot q + 4q_i \cdot q_j + 2q_i \cdot q) \tag{4.28}
$$

After the replacement of the corresponding Skalar products from 4.27 it results in:

$$
\begin{aligned}
M_1 {M_2}^{\dagger} &= g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta} [\frac{1}{2y(p_i \cdot Q)} + \frac{\beta_1 (\frac{Q^2}{2p_i \cdot Q})}{2y(1-\beta_1)(1-y)\,(p_i \cdot Q)} \\
&+ \frac{\beta_1\, Q \cdot p_k}{2y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} + \frac{\beta_1}{y(1-\beta_1)(p_i \cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i \cdot p_k)}]
\end{aligned}
\tag{4.29}
$$

4.4 Interference term of inverse $M_1 M_2{}^{\dagger\prime}$

If now the results from the previous sections are added together, it is noticeable that a part of the final solution falls, which is not demonstrated here. The problem is that the gluons are indistinguishable with respect to the interference term. In fact, it could not be determined which gluon has which momentum. Here exactly the sixth point from the procedure 2.9.2 appears. This problem is solved by exchanging the impulses for the same diagram once and repeating the steps. Here only the final result is presented and the detailed steps can be looked up in the appendix 6.2.

$$\begin{aligned} M_1 M_2{}^{\dagger\prime} &= g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)} + \frac{1}{2y(p_i\cdot Q)} + \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)} \\ &+ \frac{(1-\beta_1)\, Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}] \end{aligned} \tag{4.30}$$

4.5 $|M^2|$

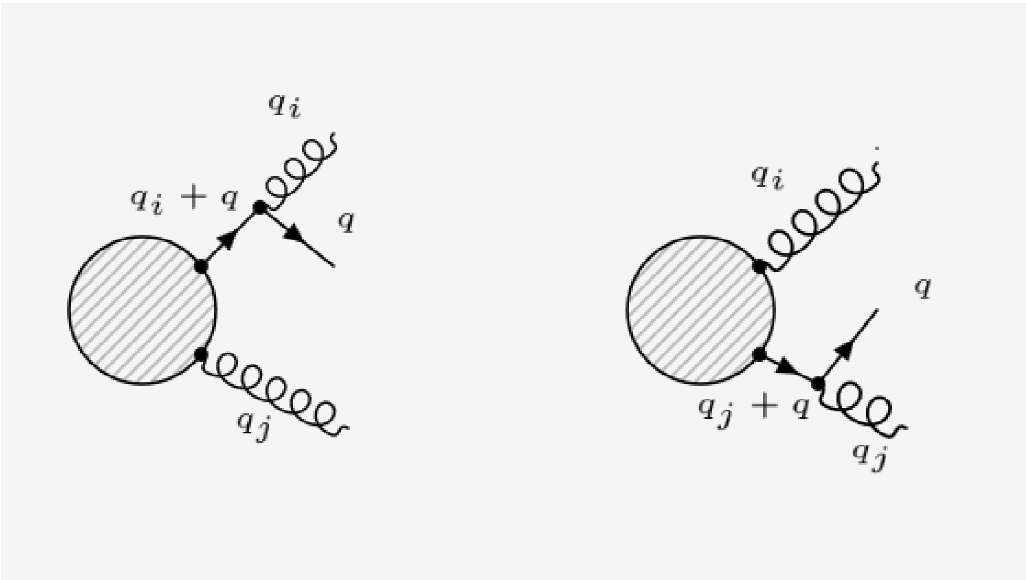
$$\begin{aligned} |M|^2 &= |M'_2|^2 + |M'_1|^2 + 2RE(M_1 M_2{}^{\dagger} + M_1 M_2{}^{\dagger\prime}) \\ |M|^2 &= \frac{g_s^2\, C_A}{y\,(p_i\cdot Q)}[2[\epsilon-1]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} - 2g^{\eta\eta'}][g^{\gamma\delta}] \\ &+ \frac{g_s^2\, C_A}{(1-\beta_1)(1-y)\,(p_i\cdot p_k)}[-2g^{\delta\gamma}][g^{\eta\eta'}] \\ &+ 2Re(g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[\frac{1}{2y(p_i\cdot Q)} + \frac{\beta_1(\frac{Q^2}{2p_i\cdot Q})}{2y(1-\beta_1)(1-y)\,(p_i\cdot Q)} \\ &+ \frac{\beta_1\, Q\cdot p_k}{2y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{\beta_1}{y(1-\beta_1)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}] \\ &+ g_s^2\, C_A\, g^{\eta\eta'} g^{\gamma\delta}[\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)} + \frac{1}{2y(p_i\cdot Q)} + \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)} \\ &+ \frac{(1-\beta_1)\, Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} + \frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}]) \end{aligned} \tag{4.31}$$

$$\begin{aligned} |M|^2 &= g_s^2\, C_A\, g^{\gamma\delta}[-2[1-\epsilon]\beta_1(1-\beta_1)n^{\eta}{}_{\perp,1}n^{\eta'}{}_{\perp,1} + \frac{\beta_1(\frac{Q^2}{2p_i\cdot Q})}{y(1-\beta_1)(1-y)\,(p_i\cdot Q)}g^{\eta\eta'} \\ &+ \frac{\beta_1\, Q\cdot p_k}{y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}g^{\eta\eta'} + \frac{2\beta_1}{y(1-\beta_1)(p_i\cdot Q)}g^{\eta\eta'} + \frac{2(1-\beta_1)}{y\beta_1(p_i\cdot Q)}g^{\eta\eta'} \\ &+ \frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{y\beta_1(1-y)\,(p_i\cdot Q)}g^{\eta\eta'} + \frac{(1-\beta_1)\, Q\cdot p_k}{y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}g^{\eta\eta'}] \end{aligned} \tag{4.32}$$

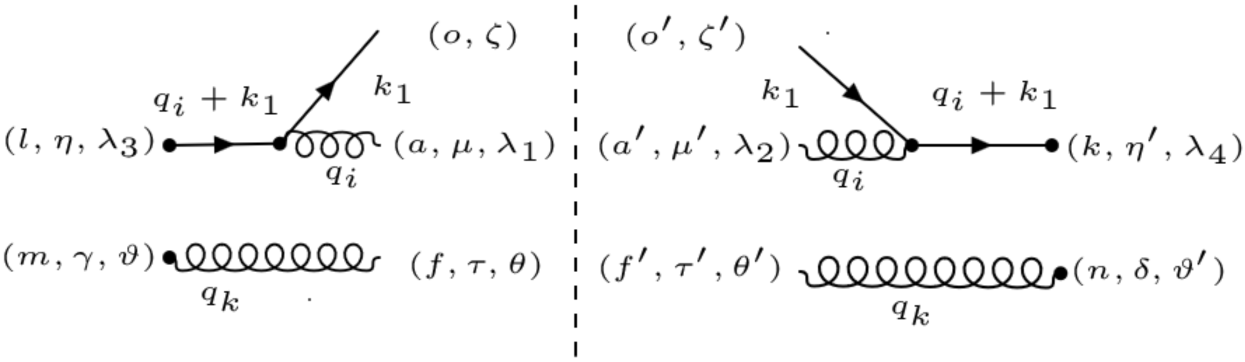
$$|M|^2 = g_s^2 \, C_A \, g^{\gamma\delta}[-2(1-\epsilon)\beta_1(1-\beta_1){n^{\eta}}_{\perp,1}{n^{\eta'}}_{\perp,1} + \frac{2\beta_1}{y(1-\beta_1)(p_i \cdot Q)} g^{\eta\eta'} + \frac{2(1-\beta_1)}{y\beta_1(p_i \cdot Q)} g^{\eta\eta'} + \frac{Q^2}{2y\beta_1(1-y)\,(p_i \cdot Q)(p_i \cdot Q)} g^{\eta\eta'} + \frac{Q \cdot p_k}{y\beta_1(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} g^{\eta\eta'}] \tag{4.33}$$

Chapter 5

A daughter gluon from a parent quark



5.1 M_1



$$|M_1|^2 = -\frac{{g_s}^2 [T^{a'}]_k{}^{o'} [T^a]_o{}^l}{4(k_1 \cdot q_i)(k_1 \cdot q_i)} [(\not{q}_i + \not{k}_1)\gamma_\mu \not{k}_1 \gamma^\mu (\not{q}_i + \not{k}_1)][-g^\delta{}_\gamma] \tag{5.1}$$

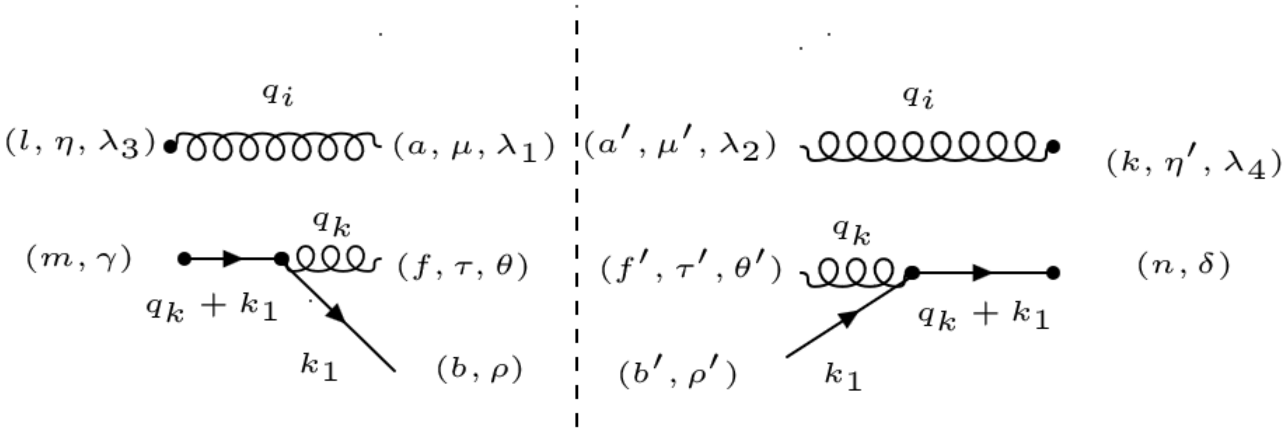
$$|M_1|^2 = -(2-d)\frac{{g_s}^2 [T^{a'}]_k{}^{o'} [T^a]_o{}^l}{4(k_1 \cdot q_i)(k_1 \cdot q_i)} [(\not{q}_i + \not{k}_1) \not{k}_1 (\not{q}_i + \not{k}_1)][-g^\delta{}_\gamma] \tag{5.2}$$

$$|M_1|^2 = -(2-d)\frac{{g_s}^2 [T^{a'}]_k{}^{o'} [T^a]_o{}^l}{2(k_1 \cdot q_i)} [\not{q}_i][-g^\delta{}_\gamma] \tag{5.3}$$

$$|M_1|^2 = -(2-d)\frac{{g_s}^2 C_F}{2y\, p_i \cdot Q}[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q})) \not{p}_i + y\beta_1 \not{Q} + \sqrt{y\alpha_1\beta_1} \not{n}_{\perp,1}][-g^\delta{}_\gamma] \tag{5.4}$$

$$|M_1|^2 = (d-2)(1-\beta_1)\frac{{g_s}^2 C_F}{2y\, p_i \cdot Q}[\not{p}_i][-g^\delta{}_\gamma] \tag{5.5}$$

5.2 M_2

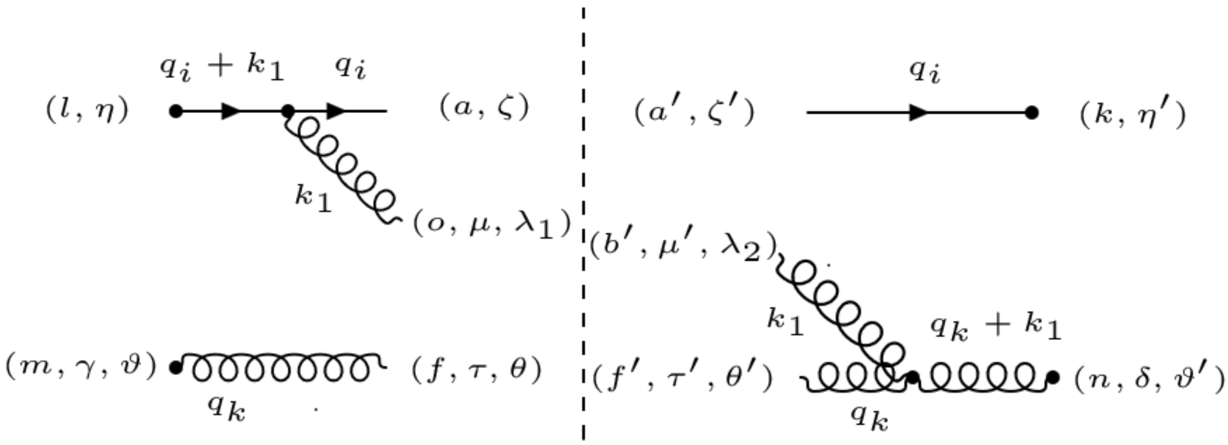


$$|M_2|^2 = -\frac{{g_s}^2 C_F}{4(k_1 \cdot q_k)(k_1 \cdot q_k)}[(\not{q}_k + \not{k}_1)\gamma_{\tau'} \not{k}_1 \gamma^{\tau}(\not{q}_k + \not{k}_1)][-g^{\eta\eta'}] \tag{5.6}$$

$$|M_2|^2 = (d-2)\frac{{g_s}^2 C_F}{4(k_1 \cdot q_k)}[\not{q}_k][-g^{\eta\eta'}] \tag{5.7}$$

It doesn't contribute to the final result!!

5.3 $M1{M_2}^\dagger$



$$M1{M_2}^\dagger = \frac{-{g_s}^2 [T^o]_a{}^l f^{f'\,b'\,n}}{4(k_1\cdot q_i)(k_1\cdot q_k)} [\not{q}_i \gamma_\mu (\not{k}_1 + \not{q}_i)] \\ [(g^{\gamma\mu}(q_k - k_1)^\delta + g^{\mu\delta}(2k_1 + q_k)^\gamma - g^{\delta\gamma}(2q_k + k_1)^\mu)] \tag{5.8}$$

$$M1{M_2}^\dagger = \frac{-{g_s}^2 [T^o]_a{}^l f^{f'\,b'\,n}}{4(k_1\cdot q_i)(k_1\cdot q_k)} [-\gamma_\mu \not{q}_i \not{k}_1 + 2(\not{k}_1 + \not{q}_i) q_{i_\mu}] \\ [g^{\gamma\mu}(q_k - k_1)^\delta + g^{\mu\delta}(2k_1 + q_k)^\gamma - g^{\delta\gamma}(2q_k + k_1)^\mu] \tag{5.9}$$

$$M1{M_2}^\dagger = \frac{-{g_s}^2 [T^o]_a{}^l f^{f'\,b'\,n}}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\ [-\gamma_\mu((\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))\not{p}_i + y\alpha_1 \not{Q})((\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\not{p}_i + y\beta_1 \not{Q}) \\ + (2((\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\not{p}_i + 2y\beta_1 \not{Q} + 2(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))\not{p}_i + 2y\alpha_1 \not{Q})(\beta_1 q_{i_\mu})] \\ [g^{\gamma\mu}(-\alpha_1 p_i)^\delta + g^{\mu\delta}(2\alpha_1 p_i)^\gamma - g^{\delta\gamma}(\alpha_1 p_i + (2-y)Q)^\mu] \tag{5.10}$$

$$M1{M_2}^\dagger = \frac{-{g_s}^2 C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\ [-\gamma_\mu(y\beta_1^2)\not{p}_i \not{Q} + 2(\not{p}_i + y\not{Q})(\beta_1 p_{i_\mu})] \\ [g^{\gamma\mu}(-\alpha_1 p_i + \sqrt{1-y}p_k)^\delta + g^{\mu\delta}(2\alpha_1 p_i + \sqrt{1-y}p_k)^\gamma - g^{\delta\gamma}(\alpha_1 p_i + 2\sqrt{1-y}p_k)^\mu] \tag{5.11}$$

$$M1{M_2}^\dagger = \frac{-{g_s}^2 C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\ [-\gamma_\mu(y\beta_1^2)\not{p}_i \not{Q}][g^{\gamma\mu}(-\alpha_1 p_i + \sqrt{1-y}p_k)^\delta + g^{\mu\delta}(2\alpha_1 p_i + \sqrt{1-y}p_k)^\gamma - g^{\delta\gamma}(\alpha_1 p_i + 2\sqrt{1-y}p_k)^\mu] \\ + [2(\not{p}_i + y\not{Q})(\beta_1 p_{i_\mu})][g^{\gamma\mu}(-\alpha_1 p_i + \sqrt{1-y}p_k)^\delta + g^{\mu\delta}(2\alpha_1 p_i + \sqrt{1-y}p_k)^\gamma - g^{\delta\gamma}(\alpha_1 p_i + 2\sqrt{1-y}p_k \tag{5.12}$$

$$\begin{aligned} M1M_2{}^{\dagger} = & \frac{-g_s{}^2 C_F}{4y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} \\ & [-\gamma_\mu(y\beta_1^2)\ \not{p}_i\ \not{Q}][g^{\gamma\mu}(-\alpha_1 p_i)^\delta + g^{\mu\delta}(\alpha_1 p_i)^\gamma - g^{\delta\gamma}((2-y)Q)^\mu][g^\delta{}_\gamma] \\ & + [2\beta_1(\ \not{p}_i + y\ \not{Q})\ ][p_i{}^\gamma(-\alpha_1 p_i)^\delta + p_i{}^\delta(2\alpha_1 p_i)^\gamma - g^{\delta\gamma}(\alpha_1 p_i + (2-y))Q \cdot p_i] \end{aligned} \tag{5.13}$$

$$\begin{aligned} M1M_2{}^{\dagger} = & \frac{-g_s{}^2 C_F}{4y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} \\ & [-\gamma_\mu(y\beta_1^2)\ \not{p}_i\ \not{Q}][-g^{\delta\gamma}(\alpha_1 p_i + 2\sqrt{1-y}p_k)^\mu] \\ & + [2\beta_1(\ \not{p}_i + y\ \not{Q})\ ][-g^{\delta\gamma}(\alpha_1 p_i + 2\sqrt{1-y})p_i \cdot p_k] \end{aligned} \tag{5.14}$$

$$\begin{aligned} M1M_2{}^{\dagger} = & \frac{-g_s{}^2 C_F}{4y(1-\beta_1)(1-y)\,(p_i \cdot p_k)(p_i \cdot Q)} \\ & [-2y\beta_1^2\sqrt{1-y}\ \not{p}_k\ \not{p}_i\ \not{Q} + 4\sqrt{1-y}\beta_1(\ \not{p}_i + y\ \not{Q})p_i \cdot p_k][-g^{\delta\gamma}] \end{aligned} \tag{5.15}$$

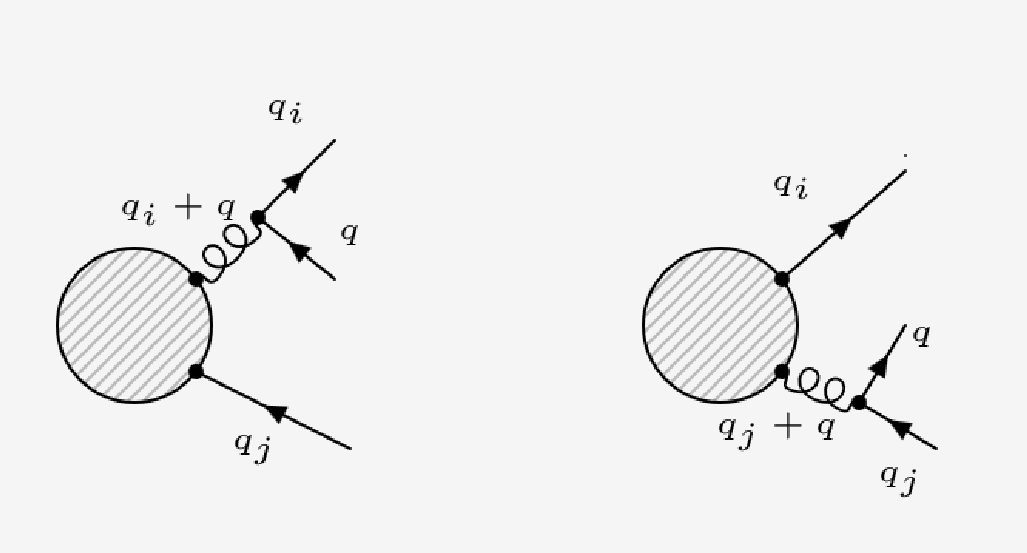
$$M1M_2{}^{\dagger} = \frac{-g_s{}^2 C_F}{y(1-\beta_1)(1-y)\,(p_i \cdot Q)}\sqrt{1-y}\beta_1[\not{p}_i][-g^{\delta\gamma}] \tag{5.16}$$

5.4 $|M|^2$

$$|M|^2 = \frac{-g_s{}^2 C_F}{2y(1-y)\,(p_i \cdot Q)}[\not{p}_i][-g^{\delta\gamma}] \otimes [2RE(\frac{2\beta_1}{1-\beta_1}) + (d-2)(1-\beta_1)] \tag{5.17}$$

Chapter 6

quark-anti-quark splitting from a parent gluon



This case concerns a daughter quark from a parent gluon which splits into a quark-anti-quark pair. Here no singularity develops since daughter and parent can always be distinguished.
This is the reason why the calculation is not mentioned here, because the evaluation is analogous to the other parts considered so far.

Appendix A

MATHEMATICAL TOOLS

Lorenz transformation of momenta $\hat{p_i}^\mu, \hat{p_k}^\mu$ and $\hat{Q}^\mu$

$$\hat{p_i}^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu = p_i{}^\mu p_{i\nu} p_i{}^\nu \frac{-y^2Q^2}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + p_i{}^\mu Q_\nu p_i{}^\nu \frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$
$$+ Q^\mu p_{i\nu} p_i{}^\nu \frac{(y^2-y-y\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}\eta^\mu{}_\nu p_i{}^\nu$$

$$\hat{p_i}^\mu = p_i{}^\mu (Q\cdot p_i)\frac{y(1+\sqrt{1-y})}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}p_i{}^\mu$$
$$= p_i{}^\mu[\frac{y(1+\sqrt{1-y})}{(2+2\sqrt{1-y}-y)} + \sqrt{1-y}] = p_i{}^\mu$$

$$\boxed{\hat{p_i}^\mu = \alpha\Lambda^\mu{}_\nu p_i{}^\nu = p_i{}^\mu} \tag{6.1}$$

$$\hat{p_k}^\mu = \alpha\Lambda^\mu{}_\nu p_k{}^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}p_k{}^\mu$$

$$\hat{p_k}^\mu = \alpha\Lambda^\mu{}_\nu p_k{}^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+ Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}p_k{}^\mu$$

with

$$A_1 \equiv \frac{-y^2Q^2(p_i\cdot p_k)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$
$$A_2 \equiv \frac{(y^2-y-y\sqrt{1-y})(p_i\cdot p_k)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}$$

$$\boxed{\hat{p_k}^\mu = A_1\, p_i{}^\mu + A_2\, Q^\mu + \sqrt{1-y} p_k{}^\mu} \tag{6.2}$$

$$\hat{Q}^\mu = \alpha\Lambda^\mu{}_\nu Q^\nu = p_i{}^\mu[\frac{-y^2Q^2(p_i\cdot Q)}{4(p_i\cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})Q^2}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]$$
$$+ Q^\mu[\frac{(y^2-y-y\sqrt{1-y})(p_i\cdot Q)}{2(p_i\cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] + \sqrt{1-y}Q^\mu$$

with

$$S_1 \equiv \frac{Q^2}{2p_i\cdot Q}[\frac{-y^2}{2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})}{(1+\sqrt{1-y}-\frac{y}{2})}] = \frac{Q^2}{2p_i\cdot Q}y$$
$$S_2 \equiv \frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y} = 1-y$$

$$\boxed{\hat{Q}^\mu = \tfrac{Q^2}{2p_i\cdot Q}y\, p_i{}^\mu + (1-y)\, Q^\mu} \tag{6.3}$$

The often occurring pre-factor products

$$\begin{aligned}
\zeta_1\zeta_1 &= (\alpha_1{}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}) + y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2)\\
\zeta_1\lambda_1 &= (y\alpha_1\beta_1 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q}))\\
\zeta_1\zeta_q &= (\alpha_1\beta_1 - y(\alpha_1{}^2+\beta_1{}^2)(\frac{Q^2}{2p_i\cdot Q}) + y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q})^2)\\
\zeta_1\lambda_q &= (y\alpha_1{}^2 - y^2\beta_1\alpha_1(\frac{Q^2}{2p_i\cdot Q}))\\
\zeta_q\zeta_q &= (\beta_1{}^2 - 2y\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}) + y^2\alpha_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2)\\
\zeta_q\lambda_1 &= (y\beta_1{}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i\cdot Q}))\\
\zeta_q\lambda_q &= (y\beta_1\alpha_1 - y^2\alpha_1{}^2(\frac{Q^2}{2p_i\cdot Q}))\\
\lambda_1\lambda_1 &= y^2\beta_1{}^2 \qquad \lambda_1\lambda_q = y^2\beta_1\alpha_1 \qquad \lambda_q\lambda_q = y^2\alpha_1{}^2
\end{aligned} \tag{6.4}$$

Common scalar products

$$\begin{aligned}
k_1 \cdot q_i &= (\zeta_1\lambda_q + \lambda_1\zeta_q)p_i \cdot Q + \lambda_1\lambda_q Q^2 - y\alpha_1\beta_1 n^2{}_{\perp,1} \\
&= [(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))y\alpha_1 + y\beta_1(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i \cdot Q}))]\, p_i \cdot Q \\
&\quad y^2\beta_1\alpha_1\, Q^2 + 2y\alpha_1\beta_1\, p_i Q \\
\Rightarrow k_1 \cdot q_i &= [y\alpha_1{}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i \cdot Q}) + y\beta_1{}^2 - y^2\alpha_1\beta_1(\frac{Q^2}{2p_i \cdot Q})]\, p_i \cdot Q \\
&\quad y^2\beta_1\alpha_1\, Q^2 + 2y\alpha_1\beta_1\, p_i Q
\end{aligned} \tag{6.5}$$

$$\boxed{k_1 \cdot q_i = y(\alpha_1 + \beta_1)^2\, p_i \cdot Q = y\, p_i \cdot Q} \tag{6.6}$$

$$\begin{aligned}
k_1 \cdot q_k &= (\zeta_1 A_2 + \lambda_1 A_1)p_i \cdot Q + \zeta_1\sqrt{1-y}\, p_i \cdot p_k + \lambda_1 A_2\, Q^2 + \lambda_1\sqrt{1-y}\, Q \cdot p_k \\
&+ \sqrt{\alpha_1\beta_1 y(1-y)}p_k \cdot n_{\perp,1} \\
&= \{[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))\frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1 + \sqrt{1-y} - \frac{y}{2})}] \\
&+ y\beta_1[\frac{-y^2Q^2(p_i \cdot p_k)}{4(p_i \cdot Q)^2(1 + \sqrt{1-y} - \frac{y}{2})} + \frac{y(1 + \sqrt{1-y})(Q \cdot p_k)}{2(p_i \cdot Q)(1 + \sqrt{1-y} - \frac{y}{2})}]\}\, p_i \cdot Q \\
&+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i \cdot Q}))\sqrt{1-y}\, p_i \cdot p_k + y\beta_1\frac{(y^2 - y - y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1 + \sqrt{1-y} - \frac{y}{2})}Q^2 \\
&+ y\beta_1\sqrt{1-y}Q \cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)}p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
k_1 \cdot q_k &= \alpha_1\frac{(y^2 - y - y\sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})}(p_i \cdot p_k) - y\beta_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})}(p_i \cdot p_k) \\
&+ y\beta_1\frac{-y^2Q^2}{4(p_i \cdot Q)(1 + \sqrt{1-y} - \frac{y}{2})}(p_i \cdot p_k) + y\beta_1\frac{y(1 + \sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})}\, Q \cdot p_k \\
&+ \alpha_1\sqrt{1-y}\, p_i \cdot p_k - y\beta_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}\, p_i \cdot p_k \\
&+ y\beta_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2 - y - y\sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})}(p_i \cdot p_k) + y\beta_1\sqrt{1-y}(Q \cdot p_k) \\
&+ \sqrt{\alpha_1\beta_1 y(1-y)}p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
k_1 \cdot q_k &= [\alpha_1\frac{(y^2 - y - y\sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})} + y\beta_1\frac{-y^2Q^2}{4(p_i \cdot Q)(1 + \sqrt{1-y} - \frac{y}{2})} + \alpha_1\sqrt{1-y} \\
&- y\beta_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}]\, p_i \cdot p_k + [y\beta_1\frac{y(1 + \sqrt{1-y})}{2(1 + \sqrt{1-y} - \frac{y}{2})} + y\beta_1\sqrt{1-y}](Q \cdot p_k) \\
&+ \sqrt{\alpha_1\beta_1 y(1-y)}p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.9}$$

$$\begin{aligned} k_1 \cdot q_k = \{&\alpha_1[\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}] \\ &+ y\beta_1(\frac{Q^2}{p_i \cdot Q})[\frac{-y^2}{4(1+\sqrt{1-y}-\frac{y}{2})} - \sqrt{1-y}]\}\, p_i \cdot p_k \\ &+ y\beta_1[\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}](Q \cdot p_k) \\ &+ \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1} \end{aligned} \tag{6.10}$$

$$\boxed{k_1 \cdot q_k = [\alpha_1(1-y) + y\beta_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\beta_1\, Q \cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{6.11}$$

$$\begin{aligned} q_i \cdot q_k =& (\zeta_q A_2 + \lambda_q A_1) p_i \cdot Q + \zeta_q\sqrt{1-y}\, p_i \cdot p_k + \lambda_q A_2\, Q^2 + \lambda_q\sqrt{1-y}\, Q \cdot p_k \\ &- \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1} \\ =& \{[(\beta_1 - y\alpha_1(\frac{Q^2}{2p_i \cdot Q}))\frac{(y^2-y-y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}] \\ &+ y\alpha_1[\frac{-y^2 Q^2 (p_i \cdot p_k)}{4(p_i \cdot Q)^2(1+\sqrt{1-y}-\frac{y}{2})} + \frac{y(1+\sqrt{1-y})(Q \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}]\}\, p_i \cdot Q \\ &+ (\beta_1 - y\alpha_1(\frac{Q^2}{2p_i \cdot Q}))\sqrt{1-y}\, p_i \cdot p_k + y\alpha_1\frac{(y^2-y-y\sqrt{1-y})(p_i \cdot p_k)}{2(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} Q^2 \\ &+ y\alpha_1\sqrt{1-y} Q \cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1} \end{aligned} \tag{6.12}$$

$$\begin{aligned} q_i \cdot q_k =& \beta_1\frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) - y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) \\ &+ y\alpha_1\frac{-y^2Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\alpha_1\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}\, Q \cdot p_k \\ &+ \beta_1\sqrt{1-y}\, p_i \cdot p_k - y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}\, p_i \cdot p_k \\ &+ y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})}(p_i \cdot p_k) + y\alpha_1\sqrt{1-y}(Q \cdot p_k) \\ &- \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1} \end{aligned} \tag{6.13}$$

$$\begin{aligned} q_i \cdot q_k = [&\beta_1\frac{(y^2-y-y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\alpha_1\frac{-y^2Q^2}{4(p_i \cdot Q)(1+\sqrt{1-y}-\frac{y}{2})} + \beta_1\sqrt{1-y} \\ &- y\alpha_1(\frac{Q^2}{2p_i \cdot Q})\sqrt{1-y}]\, p_i \cdot p_k + [y\alpha_1\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + y\alpha_1\sqrt{1-y}](Q \cdot p_k) \\ &- \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1} \end{aligned} \tag{6.14}$$

$$\begin{aligned}
k_1 \cdot q_k = \{&\beta_1[\frac{(y^2 - y - y\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}] \\
&+ y\alpha_1(\frac{Q^2}{p_i \cdot Q})[\frac{-y^2}{4(1+\sqrt{1-y}-\frac{y}{2})} - \sqrt{1-y}]\}\, p_i \cdot p_k \\
&+ y\alpha_1[\frac{y(1+\sqrt{1-y})}{2(1+\sqrt{1-y}-\frac{y}{2})} + \sqrt{1-y}](Q \cdot p_k) \\
&- \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}
\end{aligned} \tag{6.15}$$

$$\boxed{q_i \cdot q_k = [\beta_1(1-y) + y\alpha_1(\tfrac{Q^2}{2p_i \cdot Q})]\, p_i \cdot p_k + y\alpha_1\, Q \cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)} p_k \cdot n_{\perp,1}} \tag{6.16}$$

Detailed calculation of the gluon radiation of a quark

$|M_1|^2$

$$|M_1|^2 = (d-2)\,\frac{g_s^2 C_F}{(2k_1\cdot q_i)}[\not{k}_1][\not{q}_k] \tag{6.17}$$

$$\begin{aligned}|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\beta_1\ \not{Q} + \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}]\\ &[A_1\ \not{p}_i + A_2\ \not{Q} + \sqrt{1-y}\ \not{p}_k]\end{aligned} \tag{6.18}$$

$$\begin{aligned}|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)p_i\cdot Q\\ &+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}p_i\cdot p_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}n_{\perp,1}\cdot p_k]\end{aligned} \tag{6.19}$$

For the collinearity $y\to 0$ we'll get:

$$\begin{aligned}|M_1|^2 = (d-2)\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}&[(A_2(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q})) + A_1 y\beta_1)\ \not{p}_i\ \not{Q}\\ &+ (\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\sqrt{1-y}\ \not{p}_i\ \not{p}_k + A_2 y\beta_1 Q^2 + \sqrt{1-y}\sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}\ \not{p}_k]\end{aligned} \tag{6.20}$$

$$|M_1|^2 = (d-2)(1-\beta_1)\sqrt{1-y}\,\frac{g_s^2\,C_F}{2y\,p_i\cdot Q}[\not{p}_i\ \not{p}_k] \tag{6.21}$$

$|M_2|^2$

$$|M_2|^2 = (d-2)\frac{g_s^2\,[T^c]_f{}^m\,[T^c]_f{}^n}{2k_1\cdot q_k}[\not{k}_1]\,[\not{q}_i] \tag{6.22}$$

$$\begin{aligned}|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}&[(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\beta_1\ \not{Q} + \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}]\\ &[(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i + y\alpha_1\ \not{Q} - \sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,l}]\end{aligned} \tag{6.23}$$

$$\begin{aligned}|M_2|^2 = (d-2)\frac{g_s^2\,C_F}{2k_1\cdot q_k}&[y\alpha_1(\alpha_1 - y\beta_1(\frac{Q^2}{2p_i\cdot Q}))\ \not{p}_i\ \not{Q} + y\beta_1(\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q}))]\ \not{Q}\ \not{p}_i\\ &+ y^2\alpha_1\beta_1 Q^2 - y\beta_1\sqrt{y\alpha_1\beta_1}\ \not{Q}\ \not{n}_{\perp,1} + y\beta_1\sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}\ \not{Q} - y\alpha_1\beta_1\,{n_{\perp,l}}^2\\ &+ (\beta_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q})\sqrt{y\alpha_1\beta_1}\ \not{n}_{\perp,1}\ \not{p}_i - (\alpha_1 - \alpha_1 y(\frac{Q^2}{2p_i\cdot Q})\sqrt{y\alpha_1\beta_1}\ \not{p}_i\ \not{n}_{\perp,1}]\end{aligned} \tag{6.24}$$

Which means:

$$
\begin{aligned}
|M_2|^2 &\sim (d-2)\frac{g_s^2\, C_F}{2k_1\cdot q_k}y[...] \\
|M_2|^2 &\to 0 \quad \text{for} \quad y \to 0
\end{aligned}
\tag{6.25}
$$

$M_1\, M_2{}^\dagger$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2[T^a]_o{}^l\,[T^a]_{f'}{}^n}{(2q_ik_1)(2q_kk_1)}[(\not{q}_i + \not{k}_1)\ \not{q}_i\,\gamma^\mu]\,[(\not{q}_k + \not{k}_1)\ \not{q}_k\gamma_\mu] \\
&+ 4[(\not{q}_i + \not{k}_1)\, q_i{}^\mu][(\not{q}_k + \not{k}_1)q_{k\mu}]
\end{aligned}
\tag{6.26}
$$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\
&[(\not{q}_i\ \not{q}_i + \not{k}_1\ \not{q}_i)\,\gamma^\mu][(\not{q}_k\ \not{q}_k + \not{k}_1\ \not{q}_k)\gamma_\mu] + 4(q_i{}^\mu q_{k\mu})[\not{q}_i + \not{k}_1\,][\not{q}_k + \not{k}_1]
\end{aligned}
\tag{6.27}
$$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\
&[\not{k}_1\ \not{q}_i\,\gamma^\mu][\not{k}_1\ \not{q}_k\gamma_\mu] + 4(q_i\cdot q_k)[\not{q}_i\ \not{q}_k + \not{k}_1\ \not{q}_k + \not{q}_i\ \not{k}_1]
\end{aligned}
\tag{6.28}
$$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\
&4(A_1\beta_1 p_i\cdot p_i + A_2\beta_1 p_i\cdot Q + \beta_1\sqrt{1-y}p_i\cdot p_k) \\
&[A_1\beta_1\ \not{p}_i\ \not{p}_i + A_2\beta_1\ \not{p}_i\ \not{Q} + \beta_1\sqrt{1-y}\ \not{p}_i\ \not{p}_k \\
&+ [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}\ \not{p}_i\ \not{p}_k - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\ \ \not{p}_i\ \not{p}_i \\
&- y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\ \ \not{p}_i\ \not{Q} + y\beta_1 A_1\ \ \not{Q}\ \not{p}_i + y\beta_1 A_2\ \ \not{Q}\ \not{Q} + y\beta_1\sqrt{1-y}\ \not{Q}\ \not{p}_k \\
&+ [\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]\ \not{p}_i\ \not{p}_i + y\beta_1{}^2\ \not{p}_i\ \not{Q}]
\end{aligned}
\tag{6.29}
$$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\
&4(A_2\beta_1 p_i\cdot Q + \beta_1\sqrt{1-y}p_i\cdot p_k)[A_2\beta_1\ \not{p}_i\ \not{Q} + \beta_1\sqrt{1-y}\ \not{p}_i\ \not{p}_k \\
&+ [(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}\ \not{p}_i\ \not{p}_k - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\ \ \not{p}_i\ \not{Q} \\
&+ y\beta_1 A_1\ \ \not{Q}\ \not{p}_i + y\beta_1 A_2\ \ \not{Q}\ \not{Q} + y\beta_1\sqrt{1-y}\ \not{Q}\ \not{p}_k + y\beta_1{}^2\ \not{p}_i\ \not{Q}]
\end{aligned}
\tag{6.30}
$$

$$
\begin{aligned}
M_1\, M_2{}^\dagger = &\frac{-g_s^2\, C_F}{4y(1-\beta_1)(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} \\
&4(\beta_1\sqrt{1-y}p_i\cdot p_k)[\beta_1\sqrt{1-y}\ \not{p}_i\ \not{p}_k + (1-\beta_1)\sqrt{1-y}\ \not{p}_i\ \not{p}_k]
\end{aligned}
\tag{6.31}
$$

$$M_1\, M_2{}^\dagger = \frac{-g_s^2\, C_F}{y(1-\beta_1)\,(p_i\cdot p_k)(p_i\cdot Q)}\beta_1(p_i\cdot p_k)[\beta_1\ \not{p}_i\ \not{p}_k + (1-\beta_1)\ \not{p}_i\ \not{p}_k] \tag{6.32}$$

$$M_1\, M_2{}^\dagger = \frac{\beta_1}{(1-\beta_1)}\,\frac{-g_s^2\, C_F}{y\,(p_i\cdot Q)}[\not{p}_i\ \not{p}_k] \tag{6.33}$$

Evaluation of the tensor $N^{\eta\eta'}$

$$\begin{aligned} N^{\eta\eta'} \equiv g_{\mu\mu'}g_{\zeta\zeta'}[&-g^{\mu\zeta}g^{\mu'\eta'}(q-q_i)^\eta(2q_i+q)^{\zeta'} + g^{\mu\zeta}g^{\eta'\zeta'}(q-q_i)^\eta(2q+q_i)^{\mu'} \\ &+g^{\mu\zeta}g^{\zeta'\mu'}(q-q_i)^\eta(q_i-q)^{\eta'} + g^{\zeta\eta}g^{\mu'\zeta'}(2q+q_i)^\mu(2q_i+q)^{\zeta'} \\ &-g^{\zeta\eta}g^{\eta'\zeta'}(2q+q_i)^\mu(2q+q_i)^{\mu'} - g^{\zeta\eta}g^{\zeta'\mu'}(2q+q_i)^\mu(q_i-q)^{\eta'} \\ &-g^{\eta\mu}g^{\mu'\eta'}(2q_i+q)^\zeta(2q_i+q)^{\zeta'} + g^{\eta\mu}g^{\eta'\zeta'}(2q_i+q)^\zeta(2q+q_i)^{\mu'} \\ &+g^{\eta\mu}g^{\zeta'\mu'}(2q_i+q)^\zeta(q_i-q)^{\eta'}][g^{\gamma\delta}] \end{aligned} \tag{6.34}$$

$$\begin{aligned} N^{\eta\eta'} \equiv [&-(q-q_i)^\eta(2q_i+q)^{\eta'} + (q-q_i)^\eta(2q+q_i)^{\eta'} + d(q-q_i)^\eta(q_i-q)^{\eta'} \\ &+(2q+q_i)^{\eta'}(2q_i+q)^\eta - g^{\eta\eta'}(2q+q_i)^\mu(2q+q_i)_\mu - (2q+q_i)^\eta(q_i-q)^{\eta'} \\ &-g^{\eta\eta'}(2q_i+q)^\zeta(2q_i+q)_\zeta + (2q_i+q)^{\eta'}(2q+q_i)^\eta + (2q_i+q)^\eta(q_i-q)^{\eta'}][g^{\gamma\delta}] \end{aligned} \tag{6.35}$$

$$\begin{aligned} N^{\eta\eta'} \equiv [&-(q^\eta q^{\eta'} + 2q^\eta q_i{}^{\eta'} - q_i{}^\eta q^{\eta'} - 2q_i{}^\eta q_i{}^{\eta'}) + (2q^\eta q^{\eta'} + q^\eta q_i{}^{\eta'} - 2q_i{}^\eta q^{\eta'} - q_i{}^\eta q_i{}^{\eta'}) \\ &+(dq^\eta q_i{}^{\eta'} - dq^\eta q^{\eta'} - dq_i{}^\eta q_i{}^{\eta'} + dq_i{}^\eta q^{\eta'}) + (4q^{\eta'}q_i{}^\eta + 2q^{\eta'}q^\eta + 2q_i{}^{\eta'}q_i{}^\eta + q_i{}^{\eta'}q^\eta) \\ &-(-2q^\eta q^{\eta'} + 2q^\eta q_i{}^{\eta'} - q_i{}^\eta q^{\eta'} + q_i{}^\eta q_i{}^{\eta'}) + (2q^{\eta'}q^\eta + q^{\eta'}q_i{}^\eta + 4q_i{}^{\eta'}q^\eta + 2q_i{}^{\eta'}q_i{}^\eta) \\ &+(-q^\eta q^{\eta'} + q^\eta q_i{}^{\eta'} - 2q_i{}^\eta q^{\eta'} + 2q_i{}^\eta q_i{}^{\eta'}) - g^{\eta\eta'}(5q^2 + 5q_i{}^2 + 8qq_i)][g^{\gamma\delta}] \end{aligned} \tag{6.36}$$

6.1 Evaluation of the interference term $M_1 {M_2}^\dagger$

Calculation of the first Term

$$
\begin{aligned}
&g^{\eta\eta'}[2\{A_1\beta_1 {p_i}^\gamma {p_i}^\delta + A_2\beta_1 {p_i}^\gamma Q^\delta + \beta_1\sqrt{1-y}{p_i}^\gamma {p_k}^\delta\} \\
&+2\{[\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^\gamma {p_i}^\delta + y{\beta_1}^2 {p_i}^\gamma Q^\delta\} \\
&+\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_i}^\gamma {p_k}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, {p_i}^\gamma {p_i}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, {p_i}^\gamma Q^\delta \\
&+y\beta_1 A_1\, Q^\gamma {p_i}^\delta + y\beta_1 A_2\, Q^\gamma Q^\delta + y\beta_1\sqrt{1-y}Q^\gamma {p_k}^\delta\} \\
&+3\{[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]{p_i}^\gamma {p_i}^\delta - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}){p_i}^\gamma Q^\delta - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})Q^\gamma {p_i}^\delta\} \\
&+4\{[\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]{p_i}^\gamma {p_i}^\delta + y{\beta_1}^2 Q^\gamma {p_i}^\delta\} \\
&+2\{A_1\beta_1 {p_i}^\gamma {p_i}^\delta + A_2\beta_1 Q^\gamma {p_i}^\delta + \beta_1\sqrt{1-y}{p_k}^\gamma {p_i}^\delta\} \\
&+\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}{p_k}^\gamma {p_i}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, {p_i}^\gamma {p_i}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, Q^\gamma {p_i}^\delta \\
&+y\beta_1 A_1\, {p_i}^\gamma Q^\delta + y\beta_1 A_2\, Q^\gamma Q^\delta + y\beta_1\sqrt{1-y}{p_k}^\gamma Q^\delta\}]
\end{aligned}
\tag{6.37}
$$

Calculation of the second term

$$
-g^{\eta\eta'} g^{\gamma\delta}(2q\cdot q_j + q\cdot q + 4q_i\cdot q_j + 2q_i\cdot q) \tag{6.38}
$$

$$
\begin{aligned}
&-g^{\eta\eta'} g^{\gamma\delta}[2([\alpha_1(1-y) + y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\, p_i\cdot p_k + y\beta_1\, Q\cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \\
&4([\beta_1(1-y) + y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\, p_i\cdot p_k + y\alpha_1\, Q\cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1}) \\
&+2(y\, p_i\cdot Q)]
\end{aligned}
\tag{6.39}
$$

Calculation of the third term

$$
\begin{aligned}
&+ g^{\gamma\eta'}\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\sqrt{1-y}p_i{}^\eta p_k{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1\, p_i{}^\eta p_i{}^\delta - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\, p_i{}^\eta Q^{\eta'}\\
&+ y\beta_1 A_1\, Q^\eta p_i{}^\delta + y\beta_1 A_2\, Q^\eta Q^\delta + y\beta_1\sqrt{1-y}Q^\eta p_k{}^\delta\\
&- [[(1-\beta_1)^2 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2]p_i{}^\eta p_i{}^\delta - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})p_i{}^\eta Q^\delta - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})Q^\eta p_i{}^\delta]\\
&- [A_1\beta_1 p_i{}^\eta p_i{}^\delta + A_2\beta_1 p_i{}^\eta Q^\delta + \beta_1\sqrt{1-y}p_i{}^\eta p_k{}^\delta]\\
&+ [\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]p_i{}^\eta p_i{}^{\eta'} + y\beta_1{}^2 p_i{}^\eta Q^{\eta'}\}
\end{aligned}
\tag{6.40}
$$

Calculation of the fourth term

$$
\begin{aligned}
&+ g^{\eta'\delta}\{[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q}) - \beta_1]\sqrt{1-y}p_i{}^\eta p_k{}^\gamma\\
&+ [2[(1-\beta_1)^2 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2] - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + A_1\beta_1 +\\
&[\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]]p_i{}^\eta p_i{}^\gamma\\
&+ [-2y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q}) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\; + A_2\beta_1 + y\beta_1{}^2]p_i{}^\eta Q^\gamma\\
&+ [y\beta_1 A_1\; + 2y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]Q^\eta p_i{}^\gamma + y\beta_1 A_2\, Q^\eta Q^\gamma + y\beta_1\sqrt{1-y}Q^\eta p_k{}^\gamma\}
\end{aligned}
\tag{6.41}
$$

Calculation of the fifth term

$$
\begin{aligned}
&- g^{\gamma\delta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] - 2\beta_1]\sqrt{1-y}p_i{}^\eta p_k{}^{\eta'}\\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + [(1-\beta_1)^2 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})^2] - 2A_1\beta_1\\
&- [\beta_1(1-\beta_1) - y\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]]\; p_i{}^\eta p_i{}^{\eta'}\\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2\; - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q}) - y\beta_1{}^2 - 2A_2\beta_1]p_i{}^\eta Q^{\eta'}\\
&+ [2y\beta_1 A_1 - y^2\beta_1{}^2(\frac{Q^2}{2p_i\cdot Q})]\; Q^\eta p_i{}^{\eta'} + 2y\beta_1 A_2\, Q^\eta Q^{\eta'} + 2y\beta_1\sqrt{1-y}Q^\eta p_k{}^{\eta'}\}
\end{aligned}
\tag{6.42}
$$

Calculation of the sixth term

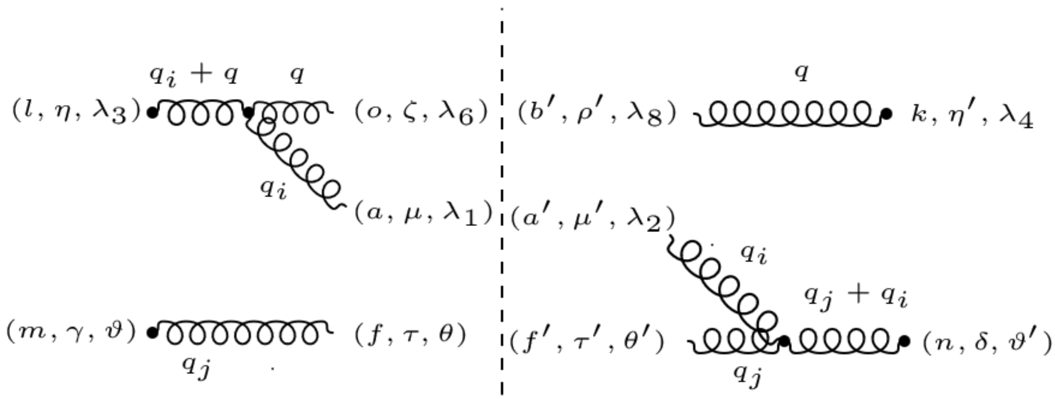
$$
\begin{aligned}
&- g^{\gamma\eta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + \beta_1]\sqrt{1-y}{p_i}^{\eta'}{p_k}^{\delta} \\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 - 2[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2] \\
&- [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})] + A_1\beta_1]\, {p_i}^{\eta'}{p_i}^{\delta} \\
&[-2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ + 2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + A_2\beta_1 - y{\beta_1}^2]{p_i}^{\eta'}Q^{\delta} \\
&+ [2y\beta_1 A_1 + 2y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\, Q^{\eta'}{p_i}^{\delta} + 2y\beta_1 A_2\, Q^{\eta'}Q^{\delta} + 2y\beta_1\sqrt{1-y}Q^{\eta'}{p_k}^{\delta}\}
\end{aligned} \tag{6.43}
$$

Calculation of the seventh term

$$
\begin{aligned}
&- g^{\eta\delta}\{[2[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + \beta_1]\sqrt{1-y}{p_i}^{\eta'}{p_k}^{\gamma} \\
&[4[(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2] - 2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + A_1\beta_1 \\
&+ 2[\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]]{p_i}^{\eta'}{p_i}^{\gamma} \\
&+ [-4y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) - 2y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ + 2y{\beta_1}^2 + A_2\beta_1]{p_i}^{\eta'}Q^{\gamma} \\
&+ [-4y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + 2y\beta_1 A_1]Q^{\eta}{p_i}^{\eta'} + 2y\beta_1 A_2\, Q^{\eta}Q^{\eta'} + 2y\beta_1\sqrt{1-y}Q^{\eta'}{p_k}^{\gamma}\}
\end{aligned} \tag{6.44}
$$

Calculation of the eighth term

$$
\begin{aligned}
&+ g^{\gamma\delta}\{[4[(1-\beta_1) - y\beta_1(\frac{Q^2}{2p_i\cdot Q})] + 2\beta_1]\sqrt{1-y}{p_k}^{\eta}{p_i}^{\eta'} \\
&+ [-4y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_1 + 2A_1\beta_1 + [\beta_1(1-\beta_1) - y{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})] \\
&+ [(1-\beta_1)^2 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})^2]]\, {p_i}^{\eta}{p_i}^{\eta'} \\
&+ [4y\beta_1 A_1 - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q})]\, {p_i}^{\eta}Q^{\eta'} + 4y\beta_1 A_2\, Q^{\eta}Q^{\eta'} + 4y\beta_1\sqrt{1-y}{p_k}^{\eta}Q^{\eta'} \\
&+ [2A_2\beta_1 - 4y\beta_1(\frac{Q^2}{2p_i\cdot Q})A_2 \ - y^2{\beta_1}^2(\frac{Q^2}{2p_i\cdot Q}) + y{\beta_1}^2]Q^{\eta}{p_i}^{\eta'}\}
\end{aligned} \tag{6.45}
$$



6.2 Evaluation of the interference term of inverse $M_1 {M_2}^{\dagger'}$

$$M_1 {M_2}^\dagger = \frac{g_s^2 f^{\,l\,o\,a} f^{\,f'\,a'\,n} \delta^{aa'} \delta^{ob'} \delta^{ff'}}{(q_i+q)^2 (q_j+q_i)^2} [g_{\zeta}{}^{\eta'} g^{\gamma}{}_{\tau'} (g^{\eta\zeta}(2q+q_i)^{\mu} + g^{\zeta\mu}(q_i-q)^{\eta} - g^{\mu\eta}(2q_i+q)^{\zeta})$$
$$g_{\mu\mu'}(g^{\tau'\mu'}(q_j-q_i)^{\delta} + g^{\mu'\delta}(2q_i+q_j)^{\tau'} - g^{\delta\tau'}(2q_j+q_i)^{\mu'}] \tag{6.46}$$

$$\begin{aligned} M_1 {M_2}^\dagger = & \frac{g_s^2 f^{\,l\,o\,a} f^{\,f\,a\,n}}{4(q\cdot q_i)(q_i \cdot q_j)} \\ & [g^{\eta\eta'}(2q+q_i)^{\gamma}(q_j-q_i)^{\delta} + g^{\eta\eta'}(2q_i+q_j)^{\gamma}(2q+q_i)^{\delta} - g^{\eta\eta'} g^{\gamma\delta}(2q+q_i)\cdot(2q_j+q_i) \\ & + g^{\gamma\eta'}(q_i-q)^{\eta}(q_j+q_i)^{\delta} + g^{\eta'\delta}(q_i-q)^{\eta}(2q_i+q_j)^{\gamma} - g^{\gamma\delta}(q_i-q)^{\eta}(2q_j+q_i)^{\eta'} \\ & - g^{\gamma\eta}(2q_i+q)^{\eta'}(q_j-q_i)^{\delta} - g^{\eta\delta}(2q_i+q)^{\eta'}(2q_i+q_j)^{\gamma} + g^{\gamma\delta}(2q_j+q_i)^{\eta}(2q_i+q)^{\eta'}] \end{aligned} \tag{6.47}$$

6.3 Parametrization in terms of $(k_1 \cdot q_i)(q_i \cdot q_k)$

$$\boxed{(k_1 \cdot q_i)(q_i \cdot q_k) \approx y\beta_1(1-y)\,(p_i \cdot Q)(p_i \cdot p_k)} \tag{6.48}$$

Calculation of the third term

$$-g^{\eta\eta'} g^{\gamma\delta}\{4k_1 \cdot q_j + 2k_1 \cdot q_i + 2q_i \cdot q_k\} \tag{6.49}$$

$$\begin{aligned} M_1 {M_2}^\dagger = & \frac{g_s^2 C_A}{4y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)} g^{\eta\eta'} g^{\gamma\delta} \\ & [4([\alpha_1(1-y) + y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\beta_1\,Q\cdot p_k + \sqrt{\alpha_1\beta_1 y(1-y)} p_k\cdot n_{\perp,1}) \\ & 2([\beta_1(1-y) + y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k + y\alpha_1\,Q\cdot p_k - \sqrt{\alpha_1\beta_1 y(1-y)} p_k\cdot n_{\perp,1}) \\ & + 2(y\,p_i\cdot Q)] \end{aligned} \tag{6.50}$$

$$-g^{\eta\eta'}g^{\gamma\delta}[4([\alpha_1(1-y)+y\beta_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k+y\beta_1\,Q\cdot p_k+\sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1})$$
$$2([\beta_1(1-y)+y\alpha_1(\frac{Q^2}{2p_i\cdot Q})]\,p_i\cdot p_k+y\alpha_1\,Q\cdot p_k-\sqrt{\alpha_1\beta_1 y(1-y)}p_k\cdot n_{\perp,1})$$
$$+2(y\,p_i\cdot Q)] \tag{6.51}$$

$$M_1 M_2{}^\dagger = g_s^2\,C_A\,g^{\eta\eta'}g^{\gamma\delta}[\frac{1-\beta_1}{y\beta_1(p_i\cdot Q)}+\frac{1}{2y(p_i\cdot Q)}+\frac{(1-\beta_1)(\frac{Q^2}{2p_i\cdot Q})}{2y\beta_1(1-y)\,(p_i\cdot Q)}$$
$$+\frac{(1-\beta_1)\,Q\cdot p_k}{2y\beta_1(1-y)\,(p_i\cdot p_k)(p_i\cdot Q)}+\frac{1}{2(1-\beta_1)(1-y)(p_i\cdot p_k)}] \tag{6.52}$$

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