Basics of perturbative QCD

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Contents

1 The strong interactions: hadrons, quarks and gluons

Strong interactions

- One of the four fundamental forces (interactions) of Nature, besides the electromagnetic, weak and gravitational interactions.
- Binds nucleons (protons, neutrons) inside the atomic nucleus.
- Affects strongly interacting particles (hadrons):
  - Baryons (half-integer spin): proton, neutron, $\Lambda, \Sigma, \Delta, \Omega, \ldots$
  - Mesons (integer spin): $\pi, K, \eta, \rho, \phi, J/\psi, \Upsilon, \ldots$
- Since the Large Hadron Collider (LHC) collides protons on protons, the dynamics of the strong interactions plays a crucial role in all processes.
- Strong interactions are short ranged: range about 1 fm (1 Fermi) = 10^{-15} m. This leads to typical cross-sections of 1 fm^2 = 10 mb (mb = millibarn).
  1 fm corresponds to energies of a few hundred MeV and time-scales (life-times) of 10^{-24} s.
  Note: in the following, we will use "natural units" where $\hbar = c = 1 \rightarrow$ masses and momenta given in MeV or GeV.
  Some useful conversion factors:
  $\hbar = 6.582\ 118\ 99(16) \times 10^{-22}$ MeV s
  $\hbar c = 197.326\ 9631(49)$ MeV fm
  $(\hbar c)^2 = 0.389\ 379\ 304(19)$ GeV^2 mb

Quantum Chromodynamics (QCD)

- Underlying theory of the strong interactions: Quantum Chromodynamics [Fritzsch, Gell-Mann, Leutwyler [1], 1973]
- Non-abelian gauge theory of quarks and gluons with three color charges (gauge group $SU(3)$, $N_c = 3$).
- 6 flavors of quarks ($N_f = 6$):

Quark type	Electric charge ($e > 0$)	Quark mass	Comment
u	$\frac{2}{3}e$	$\overline{m}_u(2\text{ GeV}) = 2.49^{+0.81}_{-0.79}$ MeV	$\overline{\text{MS}}$-scheme
d	$-\frac{1}{3}e$	$\overline{m}_d(2\text{ GeV}) = 5.05^{+0.75}_{-0.95}$ MeV	$\overline{\text{MS}}$-scheme
s	$-\frac{1}{3}e$	$\overline{m}_s(2\text{ GeV}) = 101^{+29}_{-21}$ MeV	$\overline{\text{MS}}$-scheme
c	$\frac{2}{3}e$	$\overline{m}_c(\overline{m}_c) = 1.27^{+0.07}_{-0.09}$ GeV	$\overline{\text{MS}}$-scheme
b	$-\frac{1}{3}e$	$\overline{m}_b(\overline{m}_b) = 4.19^{+0.18}_{-0.06}$ GeV	$\overline{\text{MS}}$-scheme
t	$\frac{2}{3}e$	$m_t = 172.0 \pm 1.6$ GeV	Pole mass

  Quark mass values from Particle Data Group (PDG) 2010 [2]. Since quarks have not been observed as free particles (confinement), it is not straightforward to define their masses. For the top quark, which decays before it hadronizes, one usually uses the pole mass. For the other quarks, one employs running masses in the $\overline{\text{MS}}$-scheme at some scale μ_R. Note the scale of 2 GeV for the light quarks u, d, s.

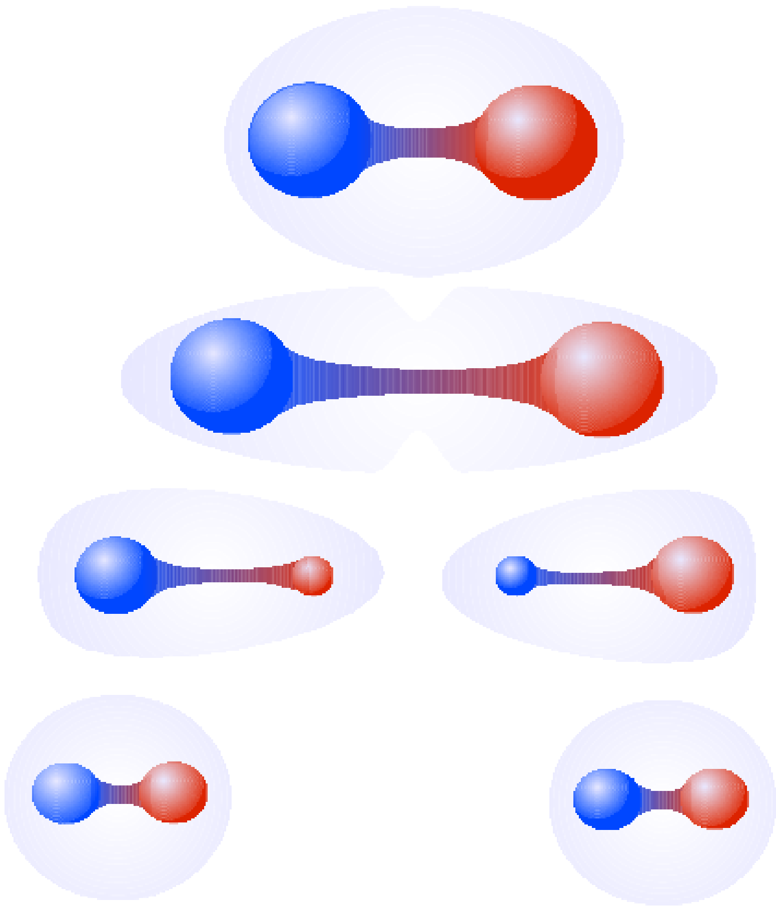
- Hadrons are composite particles, made up of quarks, antiquarks and gluons:
  - Baryons $\sim qqq$ (3 valence quarks)
  - Mesons $\sim q\bar{q}$ (1 valence quark + 1 valence anti-quark)
- Interaction strength of QCD at typical hadronic energy scale (proton mass):

$$\alpha_s(1 \text{ GeV}) \approx 0.5 \gg \alpha \approx 1/137$$

  $\alpha \equiv e^2/4\pi\epsilon_0\hbar c$: fine-structure constant, electromagnetic coupling.

- Non-perturbative phenomena of QCD at low energies:
  - Confinement: quarks and gluons are permanently confined into color-neutral hadrons (no free quarks and gluons observed in Nature).

    If we try to break up a meson into quarks, we have to put more and more energy into the system and at some point a new quark-antiquark pair is created out of the vacuum and we end up with two mesons (string-breaking, schematically):

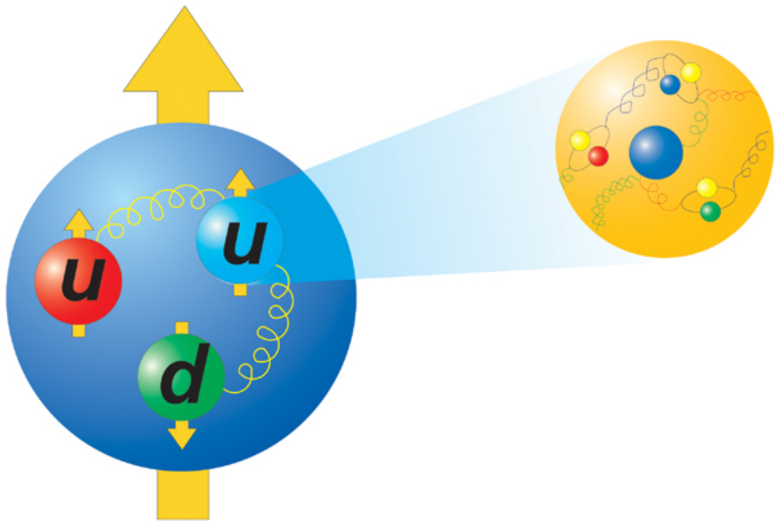


Source: NIC Jülich

    Confinement also leads to the finite range of the strong interactions, even though the gluons, which mediate the force, are massless. In fact, the strong force between the nucleons in an atomic nucleus can be described effectively by the exchange of pions (Yukawa).

– Structure of hadrons: the intrinsic structure of hadrons, described by hadronic wave functions, form factors and parton distribution functions, is governed by non-perturbative physics. Can often not be derived from theory and needs to be extracted from data.

While the proton can be thought of to consist of three valence quarks (uud), if we "look" closer (at shorter distances, i.e. probe at higher energies), the picture is much more complicated with the creation of virtual quarks, antiquarks and gluons:



Source: NIKHEF

– Spontaneous chiral symmetry breaking: pions as Goldstone bosons of spontaneously broken chiral (axial-vector) symmetry. Global symmetries for N_f massless flavors:

$$U(N_f)_L \times U(N_f)_R = SU(N_f)_V \times SU(N_f)_A \times U(1)_B \times U(1)_A \to SU(N_f)_V \times U(1)_B$$

The $U(1)_A$ symmetry is anomalous, i.e. broken by quantum effects (triangle anomaly in $\langle VVA \rangle$).

– Methods to study QCD at low energies: Lattice QCD, Effective Field Theories (Chiral Perturbation Theory), Hadronic Models (quark models, resonance models, ...)

- Asymptotic freedom [Gross + Wilczek [3]; Politzer [4], 1973]:

$$\alpha_s(Q) \to 0 \quad \text{for} \quad Q \to \infty \qquad (\alpha_s(M_Z) \approx 0.118)$$

$\Rightarrow$ can do perturbation theory (series in α_s) for large momenta Q, i.e. at the LHC, for the hard scattering of quarks and gluons.

At short distances (= at large energies), like in deep-inelastic scattering of electrons off protons, the quarks and gluons inside hadrons behave almost like free particles (parton model).

2 QCD Lagrangian and Feynman rules

Gauge Symmetry

Gauge symmetries play an important role in the construction of Lagrangians for the fundamental forces. Example: Quantum Electrodynamics (QED) = interaction of electrons and photons (electromagnetic field):

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}\left(i\gamma^\mu D_\mu - m_e\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where ψ is a 4-component Dirac field (spinor) describing the electron with mass m_e (Dirac indices suppressed in Lagrangian) and

$$\begin{aligned} D_\mu &= \partial_\mu - ieA_\mu && \text{(covariant derivative)} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu && \text{(field strength tensor of gauge field)} \end{aligned}$$

Lagrangian invariant under local $U(1)$ gauge transformations (Abelian group):

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{i\omega(x)}\psi(x) \\ A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\omega(x) \end{aligned}$$

Non-abelian gauge symmetry (Yang-Mills)

Can generalize local gauge transformation to non-abelian groups, e.g. $SU(N_c)$, where ψ transforms in fundamental representation of $SU(N_c)$ and the gauge-field A^a_μ now carries an index a of the adjoint representation:

$$\begin{aligned} \psi &= \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^{N_c} \end{pmatrix} \\ \psi(x) &\rightarrow \psi'(x) = U(x)\psi(x) \\ \hat{A}_\mu(x) \equiv A^a_\mu(x)T^a &\rightarrow \hat{A}'_\mu(x) = U(x)\hat{A}_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)(\partial_\mu U^\dagger(x)) \\ U(x) &= e^{i\omega^a(x)T^a} \in SU(N_c) \end{aligned}$$

$T^a, a = 1, \ldots N_c^2 - 1$: generators of $SU(N_c)$ in fundamental representation. Hermitean and traceless since $U^\dagger U = 1$ and $\det U = 1$. They form a Lie algebra:

$$\left[T^a, T^b\right] = if^{abc}T^c$$

with real and fully antisymmetric structure constants f^{abc}. Note that $\hat{A}_\mu(x)$ is in the Lie algebra of the group.
Normalization of generators in fundamental representation:

$$\mathrm{tr}\left(\mathrm{T^a T^b}\right) = \frac{1}{2}\delta^{\mathrm{ab}}$$

QCD Lagrangian

For $N_C = 3, N_f = 6$ (preliminary version !):

$$\mathcal{L}_{\rm QCD} = \overline{q}\left(i\gamma^\mu D_\mu - \mathcal{M}\right) q - \frac{1}{2}\mathrm{tr}\left(\hat{\mathrm{F}}_{\mu\nu}\hat{\mathrm{F}}^{\mu\nu}\right)$$

with

$$\begin{aligned}
q &= \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix} \\
\mathcal{M} &= \mathrm{diag}(m_u, m_d, m_s, m_c, m_b, m_t) \\
D_\mu &= \partial_\mu - i g_s \hat{A}_\mu \\
\hat{A}_\mu &= A_\mu^a T^a, \quad T^a = \frac{\lambda^a}{2}, a = 1, \dots, 8 \quad (\lambda^a\text{: Gell-Mann matrices}) \\
\hat{F}_{\mu\nu} &= \frac{i}{g_s}\left[D_\mu, D_\nu\right] = F_{\mu\nu}^a T^a \\
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c \quad \text{(non-Abelian field strength tensor)}
\end{aligned}$$

The covariant derivative D_μ is flavor diagonal and acting only on color indices, e.g. on

$$u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

$\mathcal{L}_{\rm QCD}$ invariant under local $SU(3)$ gauge transformations $U(x)$:

$$\begin{aligned}
\psi &\rightarrow \psi' = U\psi, \quad \psi = u, \dots, t \\
\hat{A}_\mu &\rightarrow \hat{A}'_\mu = U\hat{A}_\mu U^\dagger + \frac{i}{g_s} U(\partial_\mu U^\dagger) \\
D_\mu &\rightarrow D'_\mu = U D_\mu U^\dagger \quad (D_\mu \text{ is acting on everything that follows to the right !}) \\
\hat{F}_{\mu\nu} &\rightarrow \hat{F}'_{\mu\nu} = U\hat{F}_{\mu\nu}U^\dagger
\end{aligned}$$

Quantization

When one quantizes QCD, one usually employs the Faddeev-Popov trick [5] in the path integral to fix a gauge and to define a gluon propagator. In general, ghost fields will then also appear in the Lagrangian.

In this way we arrive at the following Lagrangian, suitable for calculations in perturbative QCD (for simplicity, we write it down for only one flavor ψ; follow largely the notations in Muta [6]):

$$\mathcal{L}_{\text{QCD}} = \overline{\psi}^i \left(i\gamma^\mu D_\mu^{ij} - m\delta^{ij} \right) \psi^j - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + (\partial^\mu \chi^{a*}) \mathcal{D}^{ab}_\mu \chi^b$$

where

$$\begin{aligned}
D_\mu^{ij} &= \delta^{ij}\partial_\mu - ig_s \left(T^c\right)^{ij} A^c_\mu \quad \text{(covariant derivative in fundamental rep.)} \\
\mathcal{D}^{ab}_\mu &= \delta^{ab}\partial_\mu - g_s f^{abc} A^c_\mu \quad \text{(covariant derivative in adjoint rep.: } (T^a_{\text{adj}})^{bc} = -if^{abc}) \\
i, j &= 1, \ldots, 3 \\
a, b, c &= 1, \ldots, 8
\end{aligned}$$

The ghost fields have been denoted by χ^a. These are complex, anti-commuting scalar fields and as for fermions, there is a minus sign for each closed ghost-loop in the Feynman rules.

To read off the Feynman rules, we split the Lagrangian as follows:

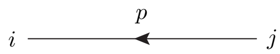
$$\begin{aligned}
\mathcal{L}_{\text{QCD}} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\
\mathcal{L}_{\text{free}} &= \overline{\psi}^i (i\gamma^\mu \partial_\mu - m) \psi^i \\
&\quad -\frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} \right) - \frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) \\
&\quad +(\partial^\mu \chi^{a*})(\partial_\mu \chi^a) \\
\mathcal{L}_{\text{int}} &= g_s \overline{\psi}^i T^a_{ij} \gamma^\mu \psi^j A^a_\mu \\
&\quad -\frac{g_s}{2} f^{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{g_s^2}{4} f^{abe} f^{cde} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu} \\
&\quad -g_s f^{abc} (\partial^\mu \chi^{a*}) \chi^b A^c_\mu
\end{aligned}$$

The 4 interactions terms in the gauged fixed Lagrangian are all related by the original gauge symmetry (BRST symmetry after gauge fixing) $\rightarrow$ only 1 independent coupling constant g_s.

In contrast to QED, we now also have vertices involving the self-interactions of 3 and 4 gluons as they carry a color charge. This has very important consequences (confinement, asymptotic freedom).

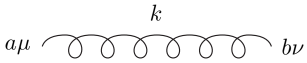
Feynman rules for QCD

Quark propagator:



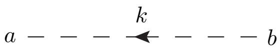
$$\delta_{ij}\frac{i}{\not{p}-m}$$

Gluon propagator:



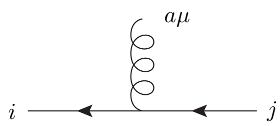
$$\delta_{ab}\frac{-i}{k^2}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Ghost propagator:



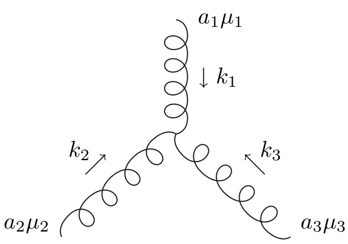
$$\delta_{ab}\frac{i}{k^2}$$

Quark-gluon vertex:



$$ig_s\gamma_\mu T^a_{ij}$$

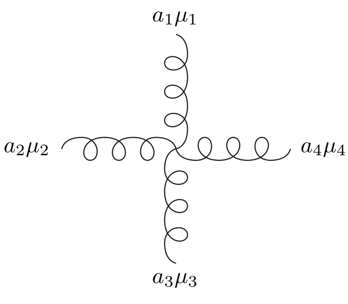
Three-gluon vertex:



$$g_s f^{a_1a_2a_3}V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$$

$$\begin{aligned}V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) &= (k_1-k_2)_{\mu_3}\,g_{\mu_1\mu_2}+(k_2-k_3)_{\mu_1}\,g_{\mu_2\mu_3}\\ &\quad +(k_3-k_1)_{\mu_2}\,g_{\mu_3\mu_1}\\ &\quad \text{(all momenta incoming)}\end{aligned}$$

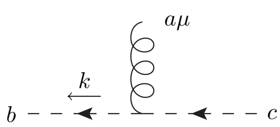
Four-gluon vertex:



$$-ig_s^2 W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}$$

$$\begin{aligned}W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4} &= \left(f^{13,24}-f^{14,32}\right)g_{\mu_1\mu_2}g_{\mu_3\mu_4}\\ &\quad +\left(f^{12,34}-f^{14,23}\right)g_{\mu_1\mu_3}g_{\mu_2\mu_4}\\ &\quad +\left(f^{13,42}-f^{12,34}\right)g_{\mu_1\mu_4}g_{\mu_3\mu_2}\\ f^{ij,kl} &= f^{a_ia_ja}f^{a_ka_la}\end{aligned}$$

Ghost-gluon vertex:

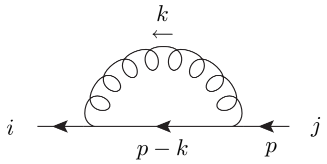


$$g_s f^{abc}k_\mu$$

3 Dimensional regularization

The need for regularization

Example: quark self-energy at one loop



Quark propagator:

$$S_{ij}(p) = \delta_{ij}\frac{i}{\not{p} - m - \Sigma(p)}, \qquad \Sigma(p) : \text{1-particle irreducible (1PI) part}$$

In Feynman gauge ($\xi = 1$):

$$\begin{aligned} -i\Sigma_{ij}(p) &= \int \frac{d^4k}{(2\pi)^4}\,(ig_s)\gamma_\mu T^a_{ik}\,\delta_{kl}\frac{i}{\not{p} - \not{k} - m + i\epsilon}\,(ig_s)\gamma_\nu T^b_{lj}\,\delta_{ab}\frac{(-i)g^{\mu\nu}}{k^2 + i\epsilon} \\ \Sigma_{ij}(p) &= \delta_{ij}\Sigma(p) \\ \Sigma(p) &= C_F\frac{g_s^2}{i}\int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{k^2((p-k)^2 - m^2)} \end{aligned}$$

with the color factor

$$(T^aT^a)_{ij} = \delta_{ij}C_F, \quad C_F = \frac{N_c^2 - 1}{2N_C} \;\text{ for }\; SU(N_c)$$

Integral linearly divergent due to high-momentum region $|k| \to \infty$:

$$\Sigma(p) \sim \int d^4k\frac{\not{k}}{k^2k^2} \sim \lim_{K\to\infty} K \qquad \text{(naive power counting)}$$

Ultraviolet (UV) divergence (= short distances)

Origin: local interaction vertex $g_s\overline{\psi}^i(x)\gamma^\mu T^a_{ij}A^a_\mu(x)\psi^j(x)$. Product of fields (operators) at same space-time point x.

UV regularization (UV regulator)

Mathematical procedure to make a quantum field theory finite (= regular) → each step in calculation is well defined, in particular loop integrals which appear in perturbation theory are finite. Physical results after renormalization (more on that later) should be unaffected, if the regulator is removed in the end. This has to be shown for each regulator.

Infrared (IR) and collinear divergences

There are other types of divergences, if there are massless particles and $k \to 0$ (IR divergence) or if two four-vectors k, p become collinear. Those will be discussed in other lectures at this School.

Dimensional regularization

The preferred choice nowadays for calculations in perturbative QCD is dimensional regularization. Based on the observation that multiple integrals are more convergent (in UV), if one reduces the number of integrals. For instance, $\Sigma(p)$ converges in $d = 2$ space-time dimensions:

$$\Sigma(p) \overset{d=4}{\sim} \int dk\, k^3 \frac{\not{k}}{k^4} \quad \overset{d=2}{\rightarrow} \quad \int dk\, k \frac{\not{k}}{k^4} \qquad \rightarrow \text{UV convergent}$$

In general

$$I = \int d^4k f(k) \quad \rightarrow \quad I_{\text{reg}}(d) = \int d^dk f(k), \qquad d < 4$$

Final result $I_{\text{reg}}(d)$ is then analytically continued to $d \in \mathbb{C}$.
$d \to 4$: single poles $\frac{1}{d-4}$ appear in $I_{\text{reg}}(d)$ (for divergent I).

Advantages of dimensional regularization:

- Preserves Lorentz invariance, translational invariance, gauge invariance, chiral invariance, unitarity.
- Can also be used to regularize IR and collinear divergences.

Disadvantages of dimensional regularization:

- Problem with consistent definition of Dirac matrix γ_5 (triangle anomaly in $\langle VVA \rangle$) or of Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$.
- Works only for loop integrals in perturbation theory, not non-perturbatively for path integral, as does lattice regularization.

For a discussion on how to precisely define an integration $\int d^dk$ for continuous or complex d from the beginning (explicit mathematical construction, consistency and uniqueness), see Chapter 4 in the book "Renormalization" by Collins [8], based on Wilson's axioms [9] for d-dimensional integration:

1. Linearity: for all $a, b \in \mathbb{C}$: $\int d^dk\, [af(k) + bg(k)] = a \int d^dk f(k) + b \int d^dk f(k)$
2. Scaling: for any number $s \in \mathbb{C}$: $\int d^dk f(s\,k) = s^{-d} \int d^dk f(k)$
3. Translation invariance: for any vector k': $\int d^dk f(k + k') = \int d^dk f(k)$

<u>d-dimensional space-time</u> (integer d for the moment !)

$$\begin{aligned}
\mu &= 0, 1, \ldots, d-1 \\
p^\mu &= (p^0, p^1, \ldots, p^{d-1}) \\
g^{\mu\nu} &= (+, -, \ldots, -) \\
g^\mu{}_\mu &= g_{\mu\nu} g^{\mu\nu} = d \qquad \text{(Attention !)} \\
\int \frac{d^4k}{(2\pi)^4} &\rightarrow \int \frac{d^dk}{(2\pi)^d} \qquad \text{(convention)}
\end{aligned}$$

<u>Dirac matrices in d-dimensions</u>

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\
(\gamma^\mu)^\dagger &= \begin{cases} \gamma^\mu, \ \mu = 0 \\ -\gamma^\mu, \mu = 1, \ldots, d-1 \end{cases} \\
\gamma^\mu \gamma_\mu &= g^\mu{}_\mu = d, \qquad \gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\gamma_\nu \\
\operatorname{tr}(\gamma_\mu \gamma_\nu) &= 4g_{\mu\nu} \quad \text{(convention)}
\end{aligned}$$

<u>Dimensionful coupling in d-dimensions</u>

Action:

$$S = \int d^dx \, \mathcal{L} \ \text{ dimensionless} \Rightarrow \dim[\mathcal{L}] = d \qquad \text{(mass dimension !)}$$

Kinetic terms:

$$\begin{aligned}
(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \ &\Rightarrow \ \dim\left[A^a_\mu\right] = \frac{d-2}{2} \\
\overline{\psi}_i \partial_\mu \psi_i \ &\Rightarrow \ \dim[\psi_i] = \frac{d-1}{2}
\end{aligned}$$

Interaction term:

$$\begin{aligned}
g_s \overline{\psi}^i \gamma^\mu A^a_\mu T^a_{ij} \psi^j \ &\Rightarrow \ \dim[g_s] + 2\dim[\psi_i] + \dim\left[A^a_\mu\right] = d \\
&\Rightarrow \ \dim[g_s] = 2 - \frac{d}{2}
\end{aligned}$$

Introduce arbitrary mass scale μ by hand:

$$g_s = g_0 \, \mu^{2-\frac{d}{2}}, \qquad g_0 \ \text{ dimensionless}$$

Some useful formulae

Feynman parametrization

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}, \qquad \frac{1}{AB^2} = 2\int_0^1 dx \frac{1-x}{(xA+(1-x)B)^3}$$

Generalization (proof by induction):

$$\prod_{i=1}^n \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\, x_i^{\alpha_i - 1} \right) \frac{\delta(1-x)}{\left(\sum_{i=1}^n x_i A_i\right)^\alpha}$$

$\alpha_i, i = 1, \ldots, n$: arbitrary complex numbers

$$\alpha = \sum_{i=1}^n \alpha_i, \quad x = \sum_{i=1}^n x_i$$

Momentum integrals in Minkowski space (if integral converges for some d)

$$\int \frac{d^dk}{(2\pi)^d} k_\mu f(k^2) = 0$$

$$\int \frac{d^dk}{(2\pi)^d} k_\mu k_\nu f(k^2) = \frac{g_{\mu\nu}}{d} \int \frac{d^dk}{(2\pi)^d} k^2 f(k^2)$$

(and similarly for more powers of k_μ)
Problem with integral that has no (external) mass or momentum scale:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^\alpha} = ?$$

If we lower d to make it convergent in UV, we get IR divergence and vice-versa.
Consistency of dimensional regularization requires:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^\alpha} = 0, \qquad \text{for all } \alpha \in \mathbb{C}$$

Momentum integrals in Euclidean space (after Wick rotation)

$$\int \frac{d^dK}{(2\pi)^d} \frac{(K^2)^a}{(K^2+\Delta)^b} = \frac{1}{(4\pi)^{\frac{d}{2}}} \Delta^{\frac{d}{2}+a-b} \frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})}$$

For $b \to 0$ we get (for generic d and a and non-vanishing Δ):

$$\int \frac{d^dK}{(2\pi)^d} \frac{1}{(K^2)^{-a}} \to 0 \quad \text{since } \frac{1}{\Gamma(b)} \to 0$$

Sketch of calculation of quark self-energy in dimensional regularization

$$\Sigma_d(p) = C_F \frac{g_0^2 \mu^{4-d}}{i} \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{k^2((p-k)^2 - m^2)}$$

Convergence in UV (large k) for $d < 3$. To simplify consider the case $m = 0$.
Perform the following steps:

- Combine denominators using Feynman parametrization.
- Interchange integrals over momentum k and Feynman parameter x. Allowed since integral convergent for $d < 3$.
- Shift momentum variable $k \to k' = k - xp, d^dk' = d^dk$.
- Since $\int d^dk'\, k'_\mu\, f(k'^2) = 0$, the integral is actually only logarithmically divergent.
- Perform Wick rotation $k'_0 = iK_0, \vec{k}' = \vec{K}$. Non-singular integral for $p^2 < 0$.
- Use scaling property of dimensional regularization.
- Some helpful identities ($d^dK = K^{d-1} dK d\Omega_d$):

$$\begin{aligned}
\int d\Omega_d &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \\
\int_0^\infty dt\, \frac{t^{p-1}}{(1+t)^{p+q}} &= B(p,q) \equiv \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \\
\int_0^1 dx\, x^{p-1}(1-x)^{q-1} &= B(p,q)
\end{aligned}$$

One obtains:

$$\Sigma_d(p) = 2C_F \frac{g_0^2}{(4\pi)^2} \not{p} \left(\frac{-p^2}{4\pi\mu^2}\right)^{\frac{d}{2}-2} (d-1) B\left(\frac{d}{2}, \frac{d}{2}\right) \Gamma\left(2 - \frac{d}{2}\right)$$

This expression can be analytically continued to the whole complex plane for $d, p^2 \in \mathbb{C}$ with the exception of the following points:

- Poles at $d = 4, 6, 8, \ldots$ from $\Gamma\left(2 - \frac{d}{2}\right)$
  $\Gamma(z)$ has simple poles at $z = -n, n = 0, 1, \ldots$; $\Gamma(z) \sim \frac{(-1)^n}{n!} \frac{1}{z+n}$ for $z \sim -n$
- Branch cut on the positive real axis in p^2-plane from $(-p^2)^{\frac{d}{2}-2}$

Performing a Laurent series expansion of $\Sigma_d(p)$ near $d = 4$, using, for $\epsilon > 0$, $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$, $\gamma = 0.57721\ldots$ (Euler's constant) and $(1-\epsilon)B(1-\epsilon, 1-\epsilon) = 1 + \epsilon + \mathcal{O}(\epsilon^2)$, one obtains, with $\Gamma(x+1) \equiv x\Gamma(x)$, the result:

$$\Sigma_d(p) = C_F \frac{g_0^2}{(4\pi)^2} \not{p} \left[\frac{2}{4-d} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right] + \mathcal{O}(4-d)$$

Logarithmic divergence (effectively) in original integral shows up as simple pole at $d = 4$. d is "close" to 4, but $d \neq 4 \to$ everything is finite (regular) !
Note: in the literature there are several conventions to define dimensions d close to (smaller than) 4 and thus some infinitesimal $\epsilon > 0$. Muta: $d = 4 - 2\epsilon$. Peskin + Schroeder: $d = 4 - \epsilon$. Sometimes $d = 4 + \epsilon, \epsilon < 0$ is used.

1-loop tensor integrals: Passarino-Veltman functions

General tensor integrals of the form

$$I^{\mu_1\ldots\mu_m}(p_1,p_2,\ldots)=\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\ldots k^{\mu_m}}{(k^2-m_1^2)((k+p_1)^2-m_2^2)((k+p_1+p_2)^2-m_3^2)\ldots}$$

can be reduced to <u>scalar</u> 1-loop integrals (transformations of variables, partial fractions are valid in dimensional regularization). Attention: various conventions are used in the literature, we follow Jegerlehner [10]. Note that everywhere $m^2\to m^2-i\epsilon$ is implied.

<u>Steps</u>

1. Covariant decomposition:

$$I^{\mu_1\ldots\mu_m}(p_1,p_2,\ldots)=p_1^{\mu_1}\ldots p_1^{\mu_m}I_1(p_1^2,p_1\cdot p_2,p_2^2,\ldots)+\ldots$$

   in terms of appropriately symmetrized tensor basis with linearly independent momenta and $g_{\mu\nu}$.

2. Contraction with $g_{\mu\nu}$:

$$\frac{k^2}{k^2-m^2}=\frac{(k^2-m^2)+m^2}{k^2-m^2}=1+\frac{m^2}{k^2-m^2}$$

3. Contraction with p_{i_μ}:

$$\begin{aligned}2k\cdot p_1 &= \left((k+p_1)^2-m_2^2\right)-\left(k^2-m_1^2\right)-\left(p_1^2-m_2^2-m_1^2\right)\\ \frac{2k\cdot p_1}{(1)(2)} &= \frac{1}{(1)}-\frac{1}{(2)}-\frac{p_1^2-m_2^2-m_1^2}{(1)(2)}\end{aligned}$$

   where $\frac{1}{(i)}$ denotes the scalar propagator with mass m_i and appropriate momenta as in $I^{\mu_1\ldots\mu_m}(p_1,p_2,\ldots)$.

Note: solving the resulting system of equations for the scalar integrals can lead to numerical instabilities for exceptional momentum configurations (on-shell momenta, massless particles), where special care needs to be taken.

<u>Conventions</u> (to conform with Passarino-Veltman [11])

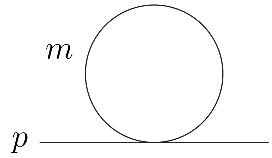
- Notation:

$$\int_k\equiv\frac{16\pi^2}{i}\int\frac{d^dk}{(2\pi)^d}$$

- Define invariant functions $I_1,\ldots$ using a factor of $(-1)^n$ where $n=$ number of propagators and factor (-1) in front of the terms with $g^{\mu_i\mu_k}$

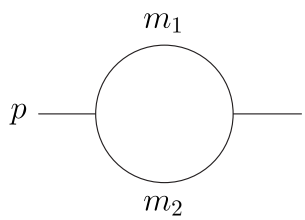
$$\int_k\frac{k^{\mu_1}\ldots k^{\mu_m}}{(k^2-m_1^2)\cdots((k+p_1+p_2+\ldots+p_{n-1})^2-m_n^2)}=(-1)^n\left(p_1^{\mu_1}\ldots p_n^{\mu_m}I_1+\ldots\right)$$

1. Tadpoles (One-point functions)



$$
\begin{aligned}
\int_k \frac{1}{k^2-m^2} &\overset{\text{shift}}{=} \int_k \frac{1}{(k+p)^2-m^2} = -A_0(m) \\
A_0(m) &= -m^2\left(\frac{2}{4-d} - \gamma + \ln(4\pi) + 1 - \ln(m^2)\right) \\
&= -\frac{2m^2}{4-d} + \text{ finite}, \qquad dA_0(m) = 4A_0(m) + 2m^2 \\
\int_k \frac{k^\mu}{(k+p)^2-m^2} &= \int_k \frac{k^\mu - p^\mu}{k^2-m^2} = \underbrace{\int_k \frac{k^\mu}{k^2-m^2}}_{=0} - p^\mu \int_k \frac{1}{k^2-m^2} = p^\mu A_0(m) \\
\int_k \frac{k^\mu k^\nu}{(k+p)^2-m^2} &= -p^\mu p^\nu A_{21} + g^{\mu\nu} A_{22} \\
&= \int_k \frac{(k-p)^\mu (k-p)^\nu}{k^2-m^2} = -p^\mu p^\nu A_0(m) + \int \frac{k^\mu k^\nu}{k^2-m^2} \\
g_{\mu\nu} \int \frac{k^\mu k^\nu}{k^2-m^2} &\overset{p=0}{=} dA_{22} = \int_k \frac{k^2}{k^2-m^2} = \underbrace{\int_k 1}_{=0 \text{ in dim. reg.}} + m^2 \int_k \frac{1}{k^2-m^2} = -m^2 A_0(m) \\
&\Rightarrow \quad A_{21} = A_0(m), \quad A_{22} = -\frac{m^2}{d} A_0(m) = -\frac{m^2}{4} A_0(m) + \frac{m^4}{8}
\end{aligned}
$$

2. Self-energies (Two-point functions)



$$
\begin{aligned}
\int_k \frac{1}{(k^2-m_1^2)\left((k+p)^2-m_2^2\right)} &= B_0(m_1, m_2, p^2) \\
B_0(m_1, m_2, p^2) &= \frac{2}{4-d} - \gamma + \ln(4\pi) \\
&\quad - \int_0^1 dx \ln\left[-p^2 x(1-x) + m_1^2(1-x) + m_2^2 x - i\epsilon\right] \\
&= \frac{2}{4-d} + \text{ finite}, \qquad dB_0 = 4B_0 - 2 \\
\int_k \frac{k^\mu}{(1)(2)} &= p^\mu B_1(m_1, m_2, p^2) \\
B_1(m_1, m_2, p^2) &= \frac{1}{2p^2}\left[-A_0(m_1) + A_0(m_2) - \left(p^2 + m_1^2 - m_2^2\right) B_0\right] \\
\int_k \frac{k^\mu k^\nu}{(1)(2)} &= p^\mu p^\nu B_{21} - g^{\mu\nu} B_{22} \\
B_{21} &= \frac{1}{3p^2}\left[-A_0(m_2) - 2\left(p^2 + m_1^2 - m_2^2\right) B_1 - m_1^2 B_0 - \frac{1}{2}\left(m_1^2 + m_2^2 - \frac{p^2}{3}\right)\right] \\
B_{22} &= \frac{1}{6}\left[A_0(m_2) - \left(p^2 + m_1^2 - m_2^2\right) B_1 - 2m_1^2 B_0 - \left(m_1^2 + m_2^2 - \frac{p^2}{3}\right)\right]
\end{aligned}
$$

Similarly for vertex functions / form-factors (three-point functions $\to C_0$) and box diagrams (four-point functions $\to D_0$).

4 Renormalization of QCD to one loop

Renormalization

In general, Green's functions which include loops are divergent, if we remove the regulator ($d \to 4$ in dim. reg.).

Important observation: the parameters in the Lagrangian (masses, couplings) and the fields are not observable quantities $\to$ can redefine masses, couplings and fields in such a way that we get finite results for physical quantities (observables), like S-matrix elements, physical masses and physical couplings. A theory is called renormalizable, if only a finite number of parameters or fields needs to be redefined.

While redefining parameters explicitly is physically intuitive, it is difficult to keep track for higher loop computations. A systematic approach uses counter-terms and renormalization constants Z. The renormalization constants Z are then adjusted in each order of perturbation theory in such a way that Green's functions are finite.

MS and $\overline{\text{MS}}$ renormalization schemes

Since there is an ambiguity to define the infinite piece of a Green's function (from the loop integral), the elimination of divergences is not unique and thus also the finite piece of the Green's function is not unique $\to$ renormalization scheme dependence.
Below will use the minimal subtraction (MS) scheme, where only the pole term $\frac{2}{4-d}$ is subtracted in the Green's function to define the Z-factors. In the modified minimal subtraction ($\overline{\text{MS}}$) scheme, all the terms $\left(\frac{2}{4-d} - \gamma + \ln(4\pi)\right)$, which always appear together, are subtracted.
These are mass-independent renormalization schemes, in contrast to mass-dependent schemes, such as the on-shell scheme (subtraction at $p^2 = m^2$) or the momentum-space subtraction (MOM) scheme (subtraction at some off-shell momentum $p^2 \neq m^2$).

Bare Lagrangian = Renormalized Lagrangian + Counter-terms

Redefine bare fields ψ^i, A^a_μ, χ^a and bare parameters g_s, m, ξ in terms of renormalized quantities:

$$\begin{aligned}
\psi^i &= Z_2^{1/2} \psi^i_r \\
A^a_\mu &= Z_3^{1/2} A^a_{r\mu} \\
\chi^a &= \tilde{Z}_3^{1/2} \chi^a_r \\
g_s &= Z_g g_{s,r} \\
m &= Z_m m_r \\
\xi &= Z_3 \xi_r
\end{aligned}$$

$Z_2, Z_3, \tilde{Z}_3$: quark field, gluon field and ghost field renormalization constants

Z_g, Z_m: coupling constant and quark mass renormalization constants

Note: to simplify the notation, we will write $g_r \equiv g_{s,r}$ in the following.

A remark about $\xi = Z_3 \xi_r$: it is nontrivial that we can choose the same Z_3 as for A^a_μ. Reason: longitudinal part of gluon propagator ($\sim k_\mu k_\nu$) is not renormalized (no radiative corrections). Follows from BRST invariance $\to$ generalized Ward-Takahashi / Slavnov-Taylor identities.

Inserting this into the Lagrangian yields

$$\mathcal{L}^{(\text{bare})}_{\text{QCD}} = \mathcal{L}_r + \mathcal{L}_{\text{CT}}$$

$\mathcal{L}_r = \mathcal{L}_{r,\text{free}} + \mathcal{L}_{r,\text{int}}$ identical to $\mathcal{L}_{\text{QCD}}$ except that all bare quantities are replaced by renormalized ones.

Counter-term Lagrangian

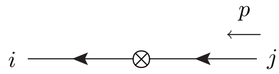
$$
\begin{aligned}
\mathcal{L}_{\rm CT} \;=\;& (Z_2-1)\overline{\psi}^i_r i\gamma^\mu\partial_\mu\psi^i_r - (Z_2 Z_m - 1)m_r\overline{\psi}^i_r\psi^i_r \\
&+(Z_3-1)\frac{1}{2}A^{a\mu}_r\delta_{ab}\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)A^{b\nu}_r + (\tilde{Z}_3-1)\left(\chi^a_r\right)^*\delta_{ab}\left(-\Box\right)\chi^b_r \\
&+(Z_{1\rm F}-1)g_r\overline{\psi}^i_r T^a_{ij}\gamma^\mu\psi^j_r A^a_{r\mu} - (Z_1-1)\frac{g_r}{2}f^{abc}(\partial_\mu A^a_{r\nu} - \partial_\nu A^a_{r\mu})A^{b\mu}_r A^{c\nu}_r \\
&-(Z_4-1)\frac{g_r^2}{4}f^{abe}f^{cde}A^a_{r\mu}A^b_{r\nu}A^{c\mu}_r A^{d\nu}_r - (\tilde{Z}_1-1)g_r f^{abc}(\partial^\mu\chi^a_r)^*\chi^b_r A^c_{r\mu}
\end{aligned}
$$

where

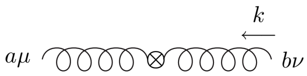
$$
Z_{1\rm F} = Z_g Z_2 Z_3^{1/2}, \quad Z_1 = Z_g Z_3^{3/2}, \quad Z_4 = Z_g^2 Z_3^2, \quad \tilde{Z}_1 = Z_g\tilde{Z}_3 Z_3^{1/2}
$$

Counter-terms treated as part of interaction Lagrangian, even if they are quadratic in the fields, since $(Z-1)$ contains at least one factor of g_r^2.

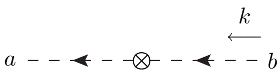
Feynman rules for counter-terms



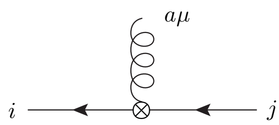
$$i\left[(Z_2-1)\not{p} - (Z_2 Z_m - 1)m_r\right]\delta_{ij}$$



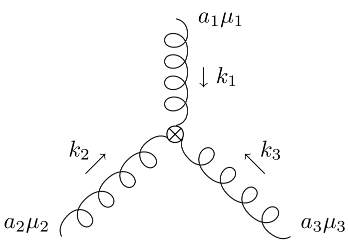
$$i(Z_3-1)\delta_{ab}\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right)$$



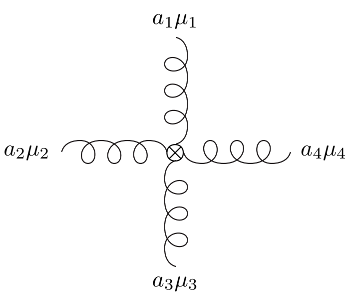
$$i(\tilde{Z}_3-1)\delta_{ab}k^2$$



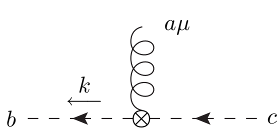
$$i(Z_{1\rm F}-1)g_r T^a_{ij}\gamma_\mu$$



$$(Z_1-1)g_r f^{a_1a_2a_3}V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$$



$$-i(Z_4-1)g_r^2 W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}$$



$$(\tilde{Z}_1-1)g_r f^{abc}k_\mu$$

Generalized Ward-Takahashi / Slavnov-Taylor identities

Important observation: the gauge coupling renormalization constant Z_g can be determined by using any of the last 4 counter-terms. One has to prove that all 4 ways lead to the same result for Z_g using the generalized Ward-Takahashi / Slavnov-Taylor identities following from BRST symmetry and a gauge-invariant regularization, such as dimensional regularization.

If the Z_g really are all the same, then one obtains the constraints (Slavov-Taylor identities in narrow sense):

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1\mathrm{F}}}{Z_2} = \frac{Z_4}{Z_1}$$

Compare with QED: $Z_{1\mathrm{F}} = Z_2$. In QCD: $\frac{Z_{1\mathrm{F}}}{Z_2} \neq 1 \Rightarrow Z_{1\mathrm{F}} \neq Z_2$.

Slavnov-Taylor identities guarantee universality of renormalized coupling g_r, but since the current $J^a_\mu = \overline{\psi}^i \gamma_\mu T^a_{ij} \psi^j$ is not gauge-invariant, its normalization is not unambiguously fixed $\Rightarrow Z_{1F} \neq Z_2$. Because of these Slavnov-Taylor identities, out of the eight renormalization constants $Z_2, Z_m, Z_3, \tilde{Z}_3, Z_{1\mathrm{F}}, Z_1, Z_4, \tilde{Z}_1$, only five are independent.

Superficially divergent Feynman amplitudes in QCD

Power counting shows that Feynman amplitudes in QCD with

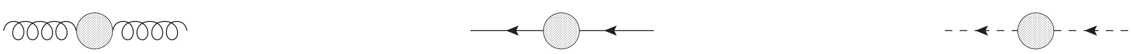
$$d_{\mathrm{div}} = 4 - N_{\mathrm{G}} - \frac{3}{2}(N_{\mathrm{F}} + N_{\mathrm{FP}}) \geq 0$$

are superficially divergent in $d = 4$ dimensions, where

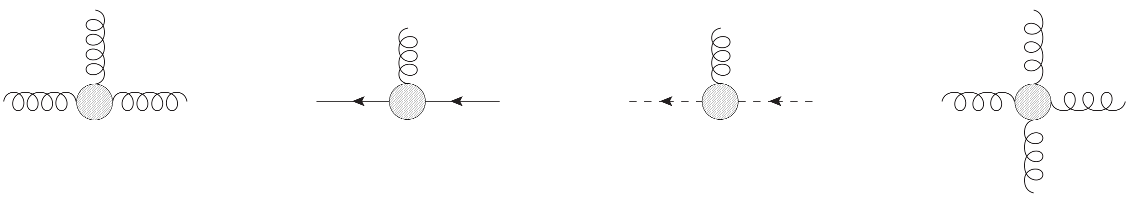
$$\begin{aligned} N_{\mathrm{G}} &= \text{number of external gluon fields} \\ N_{\mathrm{F}} &= \text{number of external quark fields} \\ N_{\mathrm{FP}} &= \text{number of external Faddeev-Popov ghost fields} \end{aligned}$$

$$\text{8 cases:} \quad (N_{\mathrm{G}}, N_{\mathrm{F}}, N_{\mathrm{FP}}) = (0,0,0), (2,0,0), (0,2,0), (0,0,2), (1,2,0), (1,0,2), (3,0,0), (4,0,0)$$

First case corresponds to vacuum diagram $\rightarrow$ normalization of generating functional $Z[J]$. Lorentz invariance forbids $(1,0,0)$.
$\rightarrow$ 7 Feynman amplitudes with overall divergences:



$d_{\mathrm{div}} = 2$ $\qquad$ $d_{\mathrm{div}} = 1$ $\qquad$ $d_{\mathrm{div}} = 1$



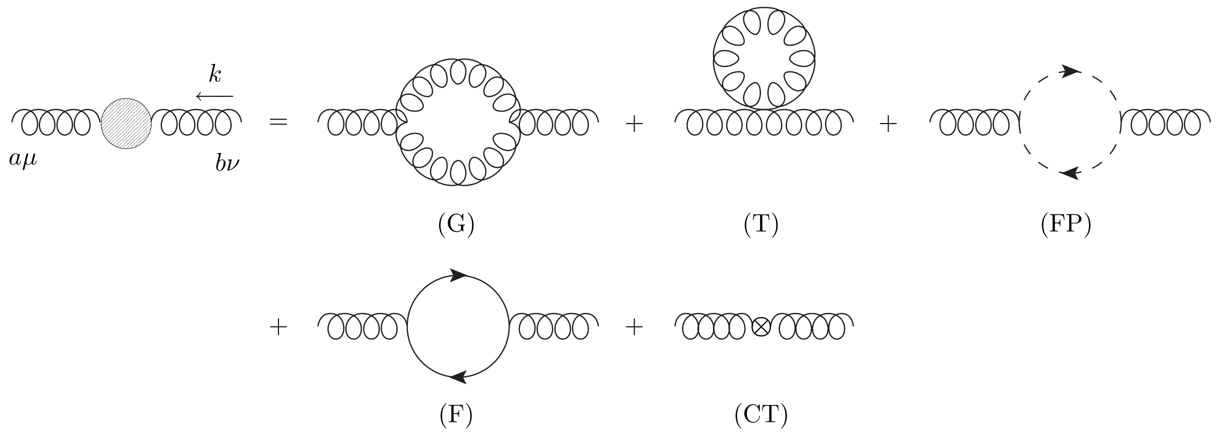
$d_{\mathrm{div}} = 1$ $\qquad$ $d_{\mathrm{div}} = 0$ $\qquad$ $d_{\mathrm{div}} = 0$ $\qquad$ $d_{\mathrm{div}} = 0$

Same structure as the 7 counter-terms written earlier ! "Suggests" renormalizability of QCD (overall divergence taken care of by the counter-terms).
Explicit calculation $\rightarrow d_{\mathrm{div}} = 0$ for all diagrams (logarithmically divergent), due to gauge and Lorentz invariance !

1-loop calculation of renormalization constants Z

Below we give the results for the renormalization constants Z of QCD at 1-loop order in the MS-scheme (see Muta [6] for details). The renormalized coupling constant in d dimensions is written as $g_r = g_R\,\mu_R^{\frac{4-d}{2}}$, where g_R is dimensionless.

<u>(i) Gluon self-energy $\Pi^{ab}_{\mu\nu}(k)$</u>



$$
\begin{aligned}
\Pi^{ab}_{\mu\nu}(k) &= \delta_{ab}\left(k_\mu k_\nu - k^2 g_{\mu\nu}\right)\Pi(k^2) \\
\Pi(k^2) &= \frac{g_R^2}{(4\pi)^2}\Bigg[\underbrace{-\frac{1}{2}C_A\left(\frac{13}{3}-\xi_r\right)}_{\text{(G)+(FP)}} + \underbrace{\frac{4}{3}T_R N_f}_{\text{(F)}, N_f \text{ flavors}}\Bigg]\frac{2}{4-d} + (Z_3 - 1) + \text{finite terms}
\end{aligned}
$$

$$
\text{(T)} \sim \int \frac{d^d q}{q^2} \equiv 0 \;\; \text{in dim. reg. (massless tadpole)}
$$

Color factors:

$$
\begin{aligned}
\operatorname{tr}\left(\mathrm{T^a T^b}\right) &= \delta_{ab}T_R, \quad f^{acd}f^{bcd} = \delta_{ab}C_A \\
\text{For } SU(N_c): \quad & T_R = \frac{1}{2}, \quad C_A = N_c
\end{aligned}
$$

Gauge invariance:

$$
k^\mu \Pi^{ab}_{\mu\nu}(k) = 0
$$

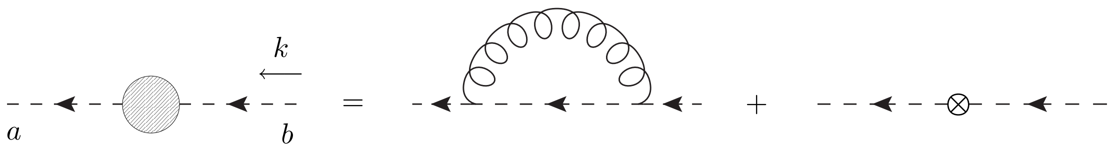
Generalized Ward-Takahashi identitity $\longrightarrow d_{\text{div}} = 0$ (only logarithmic divergence). Note: only (G) + (FP) leads to gauge invariant structure. (F) itself is gauge invariant.

$\Rightarrow$ no mass renormalization $\Rightarrow$ gluon remains massless

<u>Gluon field renormalization constant Z_3 in MS-scheme:</u>

$$
Z_3^{\text{MS}} = 1 - \frac{g_R^2}{(4\pi)^2}\left[-\frac{1}{2}C_A\left(\frac{13}{3}-\xi_r\right) + \frac{4}{3}T_R N_f\right]\frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)
$$

(ii) FP-ghost self-energy $\widetilde{\Pi}^{ab}(k)$



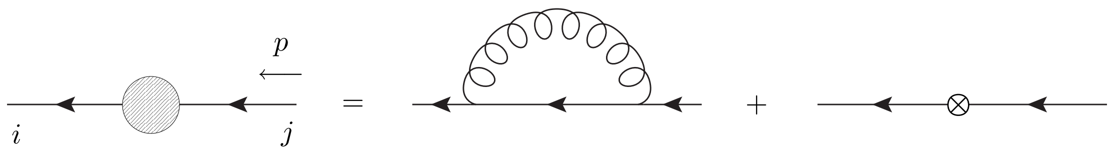
$$\widetilde{\Pi}^{ab}(k) = \delta_{ab} k^2 \left[-\frac{g_R^2}{(4\pi)^2} C_A \left(\frac{3-\xi_r}{4} \right) \frac{2}{4-d} + (\tilde{Z}_3 - 1) \right] + \text{finite terms}$$

$\sim k^2 \Rightarrow$ no mass renormalization $\Rightarrow$ FP-ghost remains massless

FP-ghost field renormalization constant $\tilde{Z}_3$ in MS-scheme:

$$\tilde{Z}_3 = 1 + \frac{g_R^2}{(4\pi)^2} C_A \left(\frac{3-\xi_r}{4} \right) \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)$$

(iii) Quark self-energy $\Sigma^{ij}(p)$



$$\begin{aligned}
\Sigma^{ij}(p) &= \delta_{ij} \left[(A\, m_r - B\, \not{p}) - (Z_2 Z_m - 1)\, m_r + (Z_2 - 1)\, \not{p} \right] + \text{finite terms} \\
A &= -\frac{g_R^2}{(4\pi)^2} C_F \left(3 + \xi_r\right) \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right) \\
B &= -\frac{g_R^2}{(4\pi)^2} C_F \xi_r \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)
\end{aligned}$$

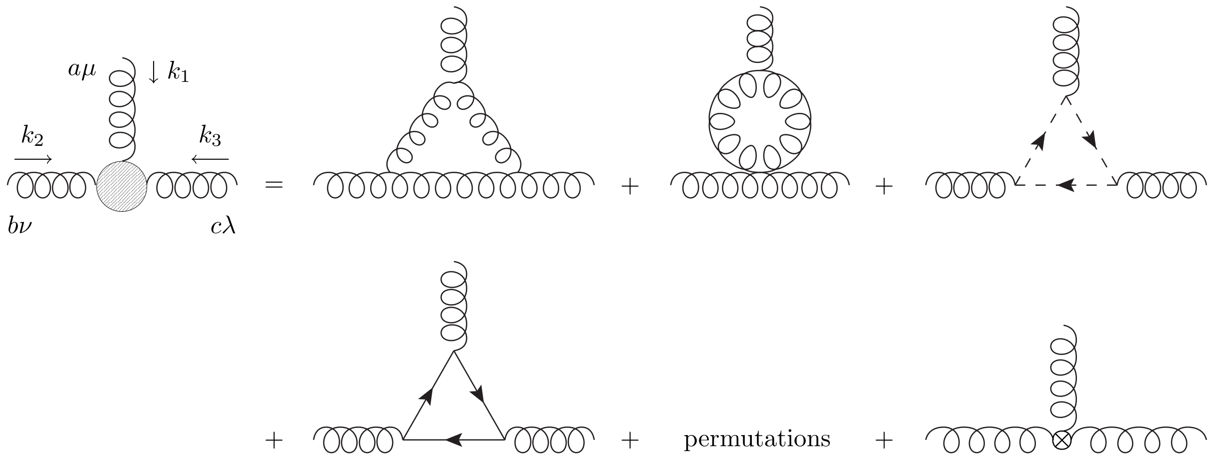
Color factor:

$$\left(T^a T^a\right)_{ij} = \delta_{ij} C_F, \quad \text{for } SU(N_c) : C_F = \frac{N_c^2 - 1}{2N_c}$$

Mass and quark field renormalization constants Z_m and Z_2 in MS-scheme:

$$\begin{aligned}
Z_m^{\rm MS} &= 1 + A - B + \mathcal{O}\left(g_R^4\right) = 1 - \frac{g_R^2}{(4\pi)^2} 3 C_F \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right) \\
Z_2^{\rm MS} &= 1 + B + \mathcal{O}\left(g_R^4\right) \qquad = 1 - \frac{g_R^2}{(4\pi)^2} C_F \xi_r \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)
\end{aligned}$$

(iv) Three-gluon vertex $\Lambda^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3)$

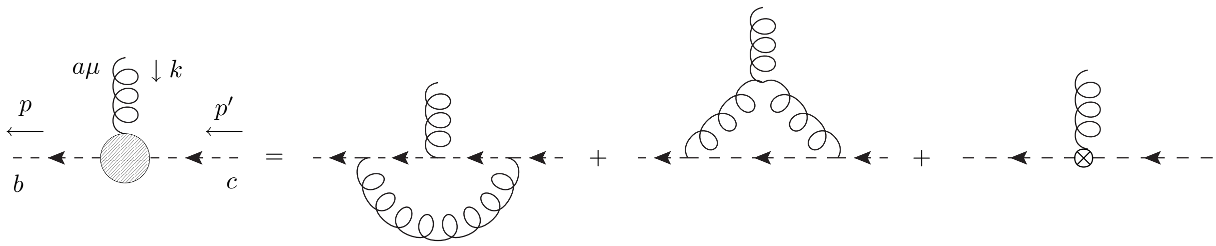


$$\Lambda^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) = -ig_R f^{abc} V_{\mu\nu\lambda}(k_1,k_2,k_3)\left[\frac{g_R^2}{(4\pi)^2}\left\{C_A\left(-\frac{17}{12}+\frac{3\xi_r}{4}\right)+\frac{4}{3}T_R N_f\right\}\frac{2}{4-d}+(Z_1-1)\right] + \text{finite terms}$$

Three-gluon vertex renormalization constant Z_1 in MS-scheme

$$Z_1^{\text{MS}} = 1-\frac{g_R^2}{(4\pi)^2}\left[C_A\left(-\frac{17}{12}+\frac{3\xi_r}{4}\right)+\frac{4}{3}T_R N_f\right]\frac{2}{4-d}+\mathcal{O}\left(g_R^4\right)$$

(v) Ghost-gluon vertex $\tilde{\Lambda}^{abc}_{\mu}(k,p,p')$

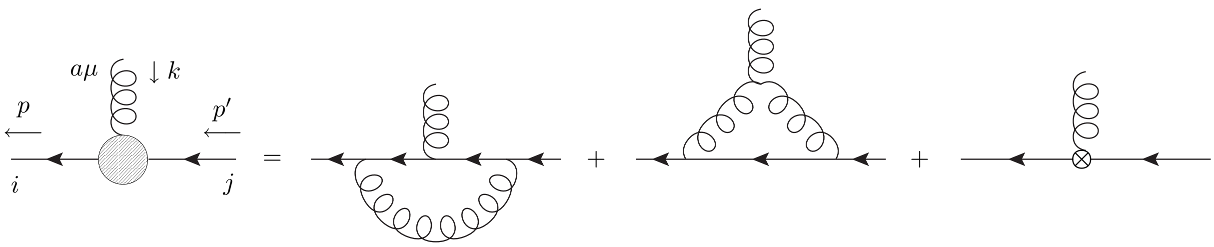


$$\tilde{\Lambda}^{abc}_{\mu}(k,p,p') = -ig_R f^{abc} p_\mu\left[\frac{g_R^2}{(4\pi)^2}C_A\frac{\xi_r}{2}\frac{2}{4-d}+\left(\tilde{Z}_1-1\right)\right]+\text{finite terms}$$

Ghost-gluon vertex renormalization constant $\tilde{Z}_1$ in MS-scheme

$$\tilde{Z}_1^{\text{MS}} = 1-\frac{g_R^2}{(4\pi)^2}C_A\frac{\xi_r}{2}\frac{2}{4-d}+\mathcal{O}\left(g_R^4\right)$$

(vi) Quark-gluon vertex $\Lambda^{aij}_{\mathrm{F}\mu}(k,p,p')$

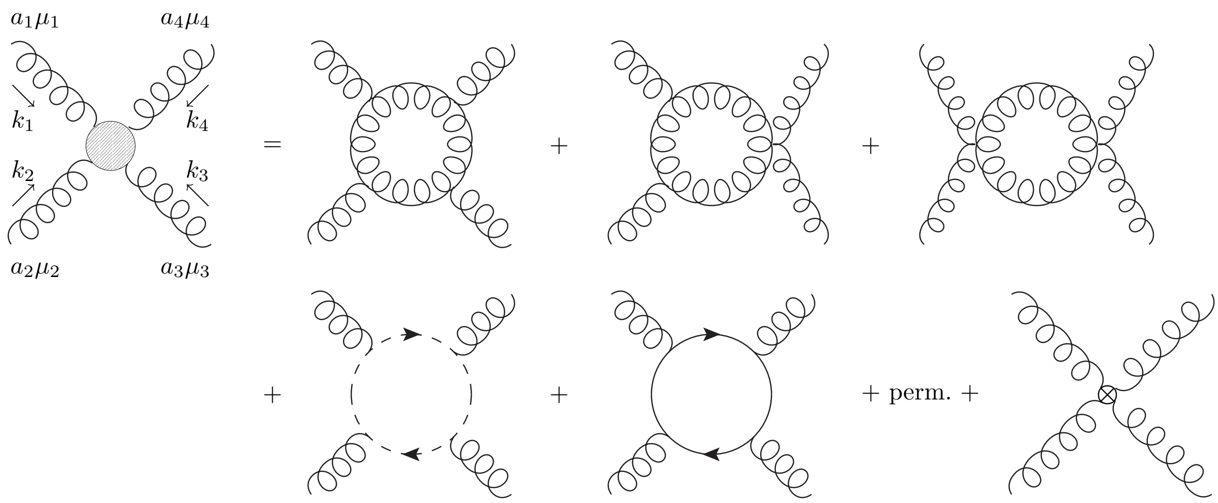


$$\Lambda^{aij}_{\mathrm{F}\mu}(k,p,p') = g_R\gamma_\mu T^a_{ij}\left[\frac{g_R^2}{(4\pi)^2}\left(\frac{3+\xi_r}{4}C_A+\xi_r C_F\right)\frac{2}{4-d}+(Z_{1\mathrm{F}}-1)\right]+\text{finite terms}$$

Quark-gluon vertex renormalization constant $Z_{1\mathrm{F}}$ in MS-scheme

$$Z^{\mathrm{MS}}_{1\mathrm{F}} = 1-\frac{g_R^2}{(4\pi)^2}\left(\frac{3+\xi_r}{4}C_A+\xi_r C_F\right)\frac{2}{4-d}+\mathcal{O}\left(g_R^4\right)$$

(vii) Four-gluon vertex $\Lambda^{a_1\ldots a_4}_{\mu_1\ldots\mu_4}(k_1\ldots k_4)$



$$\begin{aligned}\Lambda^{a_1\ldots a_4}_{\mu_1\ldots\mu_4}(k_1\ldots k_4) \quad &= \quad -g_R^2 W^{a_1\ldots a_4}_{\mu_1\ldots\mu_4}\left[\frac{g_R^2}{(4\pi)^2}\left\{\left(-\frac{2}{3}+\xi_r\right)C_A+\frac{4}{3}T_R N_f\right\}\frac{2}{4-d}+(Z_4-1)\right] \\ &\quad\; +\text{finite terms}\end{aligned}$$

Four-gluon vertex renormalization constant Z_4 in MS-scheme

$$Z^{\mathrm{MS}}_4 = 1-\frac{g_R^2}{(4\pi)^2}\left[\left(-\frac{2}{3}+\xi_r\right)C_A+\frac{4}{3}T_R N_f\right]\frac{2}{4-d}+\mathcal{O}\left(g_R^4\right)$$

Conclusions

All the 1-loop divergences in the 7 superficially divergent Feynman amplitudes can be cancelled by the counter-terms with an appropriate choice of the Z-factors.
→ Proven renormalizability of QCD at 1-loop order

Note: We did not use gauge symmetry so far (except for $\xi = Z_3\xi_r$). Can check that

$$\frac{Z_1^{\mathrm{MS}}}{Z_3^{\mathrm{MS}}} = \frac{\tilde{Z}_1^{\mathrm{MS}}}{\tilde{Z}_3^{\mathrm{MS}}} = \frac{Z_{1\mathrm{F}}^{\mathrm{MS}}}{Z_2^{\mathrm{MS}}} = \frac{Z_4^{\mathrm{MS}}}{Z_1^{\mathrm{MS}}} = 1 - \frac{g_R^2}{(4\pi)^2} C_A \left(\frac{3+\xi_r}{4}\right) \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)$$

Slavnov-Taylor identity satisfied at 1-loop order in MS-scheme (dimensional regularization preserves gauge invariance), but $Z_{1\mathrm{F}}^{\mathrm{MS}} \neq Z_2^{\mathrm{MS}}$.

Gauge coupling renormalization constant Z_g in MS-scheme

As mentioned earlier, we have four different ways to calculate Z_g, all of them are equivalent owing to the Slavnov-Taylor identities. From a practical point of view, perhaps the easiest way of calculating Z_g is to use the definition

$$Z_g = \frac{\tilde{Z}_1}{\tilde{Z}_3 Z_3^{1/2}}$$

where $\tilde{Z}_1$ appears in the ghost-gluon vertex, $\tilde{Z}_3$ in the ghost self-energy and Z_3 in the gluon self-energy. Using the relations given above, one obtains

$$Z_g^{\mathrm{MS}} = 1 - \frac{g_R^2}{(4\pi)^2} \frac{(11C_A - 4T_R N_f)}{6} \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)$$

Note: whereas $\tilde{Z}_1, \tilde{Z}_3$ and Z_3 all depend on the renormalized gauge parameter ξ_r, Z_g^{MS} is independent of ξ_r. This is also true for

$$Z_m^{\mathrm{MS}} = 1 - \frac{g_R^2}{(4\pi)^2} 3C_F \frac{2}{4-d} + \mathcal{O}\left(g_R^4\right)$$

5 Renormalization group and running α_s

Renormalization $\rightarrow$ two-fold ambiguity:

1. Arbitrariness of renormalization condition: how to define divergent and finite pieces of Green's functions (= arbitrariness of splitting bare Lagrangian in renormalized Lagrangian and counter-terms).

2. Arbitrariness in choosing renormalization scale μ_R.

$\rightarrow$ Many different expressions for physical quantities. But they all describe one unique physical reality, starting from a unique bare Lagrangian: connected by a finite renormalization.

Consider coupling and mass renormalization, i.e. relation between renormalized and bare quantities (not necessarily using dimensional regularization):

$$\begin{aligned} g_r &= Z_g^{-1} g_s, \qquad m_r = Z_m^{-1} m, \qquad && \text{at scale } \mu_R \\ g_r' &= (Z_g')^{-1} g_s, \quad m_r = (Z_m')^{-1} m, \qquad && \text{at scale } \mu_R' \end{aligned}$$

Physical quantities, such as S-matrix elements, should be the same:

$$S'(p_i; g_r', m_r', \mu_R') = S(p_i; g_r, m_r, \mu_R) \qquad (p_i : \text{fixed set of external momenta})$$

Transformation from scheme (g_r, m_r, μ_R) to (g_r', m_r', μ_R')

$$\begin{aligned} g_r' &= z_g g_r, \qquad z_g = \frac{Z_g}{Z_g'} \\ m_r' &= z_m m_r, \qquad z_m = \frac{Z_m}{Z_m'} \end{aligned}$$

is done by a finite renormalization (in ratios z_g, z_m the divergent pieces cancel out by construction of Z-factors) $\rightarrow$ physical predictions invariant under finite renormalization (transformations $\mu_R \rightarrow \mu_R'$ form an Abelian group) $\rightarrow$ <u>renormalization group (RG) equations</u>.

In practice, we can calculate S-matrix elements only in perturbation theory and truncate the series after the first few orders, e.g. to order g_r^n:

$$S'(p_i; g_r', m_r', \mu_R') - S(p_i; g_r, m_r, \mu_R) = \mathcal{O}\left(g_r^{n+1}\right)$$

$\Rightarrow$ <u>scheme dependence of physical quantities, like cross-sections at the LHC</u>

Renormalization group equations for QCD in MS ($\overline{\text{MS}}$)-scheme

For QCD and using dimensional regularization, we can rephrase the invariance of physical quantities under finite renormalization as follows. First we recall that the bare and the renormalized coupling constants acquire a mass dimension in $d \neq 4$:

$$
\begin{aligned}
g_s &= g_0\, \mu_0^{\frac{4-d}{2}} \\
g_r &= g_R\, \mu_R^{\frac{4-d}{2}}
\end{aligned}
$$

where g_0 and g_R are dimensionless coupling constants. The mass scale μ_0 for the bare coupling g_s is a fixed scale, while the mass scale μ_R for the renormalized coupling g_r is a variable parameter. μ_R is identified with the renormalization scale in the MS (or $\overline{\text{MS}}$)-scheme.
With the relation between bare and renormalized quantities, we can therefore write

$$
\begin{aligned}
g_R(\mu_R) &= \left(\frac{\mu_0}{\mu_R}\right)^{\frac{4-d}{2}} Z_g(\mu_R)^{-1}\, g_0 \\
m_R(\mu_R) &= Z_m(\mu_R)^{-1}\, m
\end{aligned}
$$

where $m_R \equiv m_r$ (to make the notation similar).
The bare parameters g and m are regarded as fixed constants and are free from any dependence on the renormalization scale μ_R. We therefore get the following set of differential equations for the running of g_R and m_R:

$$
\mu_R \frac{d}{d\mu_R} g_s = 0 \;\Rightarrow\; \mu_R \frac{d}{d\mu_R}\left(Z_g\, g_R\, \mu_R^{\frac{4-d}{2}}\right) = 0 \tag{1}
$$

$$
\mu_R \frac{d}{d\mu_R} m = 0 \;\Rightarrow\; \mu_R \frac{d}{d\mu_R}\left(Z_m\, m_R\right) = 0 \tag{2}
$$

(1) $\Rightarrow \mu_R \dfrac{d}{d\mu_R} g_R = \beta,$ $\qquad \beta = -\dfrac{4-d}{2} g_R - \dfrac{\mu_R}{Z_g}\left(\dfrac{dZ_g}{d\mu_R}\right) g_R,$ $\qquad \beta -$ function

(2) $\Rightarrow \mu_R \dfrac{d}{d\mu_R} m_R = -m_R \gamma_m,$ $\qquad \gamma_m = \dfrac{\mu_R}{Z_m}\left(\dfrac{dZ_m}{d\mu_R}\right)$

In QCD, there is a similar equation for the running of the renormalized gauge fixing parameter $\xi_R \equiv \xi_r$. Note that the divergences in the Z-factors cancel out $\Rightarrow$ β and γ_m are finite functions, if we remove the regulator (in dim. reg. $d \to 4$). Differential equations describe how $g_R(\mu_R)$, $m_R(\mu_R)$ (and $\xi_R(\mu_R)$) change with μ_R, such that physical quantities are independent of the renormalization scale μ_R.
In general renormalization scheme: $Z = Z(g_R(\mu_R), m_R(\mu_R), \xi_R(\mu_R), \mu_R) \to$ complicated, coupled set of differential equations. In MS ($\overline{\text{MS}}$) scheme a great simplification occurs:

$$
\beta^{\text{MS}} = \beta^{\text{MS}}(g_R), \qquad \gamma_m^{\text{MS}} = \gamma_m^{\text{MS}}(g_R)
$$

$\Rightarrow$ Equation (1) decouples !

QCD β-function (in MS-scheme) and asymptotic freedom

$$\begin{aligned} Z_g^{\rm MS} &= 1 - A_{11} g_R^2 \left(\frac{2}{4-d}\right) + \mathcal{O}\left(g_R^4\right), \qquad A_{11} = \frac{1}{(4\pi)^2}\frac{11C_A - 4T_R N_f}{6} \\ \beta^{\rm MS}(g_R) &= \left(\frac{4-d}{2}\right) g_R + 2A_{11} g_R^2 \left(\frac{2}{4-d}\right)\beta^{\rm MS}(g_R) + \mathcal{O}\left(g_R^5\right) \\ &= -2A_{11} g_R^3 + \mathcal{O}\left(g_R^5, 4-d\right) \end{aligned}$$

Therefore

$$\beta^{\rm MS}(g_R) = -\beta_0 g_R^3 + \mathcal{O}\left(g_R^5\right), \qquad \beta_0 = \frac{1}{(4\pi)^2}\frac{11C_A - 4T_R N_f}{3}$$

In the MS scheme, $Z_g^{\rm MS}$ is always of the form

$$Z_g^{\rm MS} = 1 + A(g_R)\left(\frac{2}{4-d}\right) + B(g_R)\left(\frac{2}{4-d}\right)^2 + \ldots$$

Then

$$\beta^{\rm MS}(g_R) = g_R^2 \frac{dA(g_R)}{dg_R}$$

Asymptotic freedom

If $\beta_0 > 0$, $g_R(\mu_R) \to 0$ with increasing $\mu_R \Rightarrow$ asymptotic freedom ($\to$ for large μ_R we can trust perturbation theory).

QCD: $\beta_0 > 0$, if $11C_A - 4T_R N_f > 0$. For real world, $N_c = 3$. $SU(3) : C_A = 3, T_R = \frac{1}{2} \Rightarrow$ asymptotically free for $N_f \leq 16$, i.e. QCD with $N_f = 6$ flavors as observed so far in Nature, is asymptotically free.
If there are no quarks in theory (pure gluodynamics): $N_f = 0 \Rightarrow$ always $\beta_0 > 0 \Rightarrow$ always asymptotically free.
Origin of asymptotic freedom: self-interactions of gluons (3-gluon and 4-gluon vertices).

Neglecting higher order terms of g_R in the β-function, the differential equation

$$\mu_R \frac{dg_R}{d\mu_R} = -\beta_0 g_R^3$$

can easily be solved

$$g_R^2(\mu_R) = \frac{g_R^2(\tilde\mu_R)}{1 + g_R^2(\tilde\mu_R)\beta_0 \ln\left(\frac{\mu_R^2}{\tilde\mu_R^2}\right)}$$

where $\tilde\mu_R$ is some fixed reference scale, e.g. $\tilde\mu_R = M_Z$, where

$$\alpha_s^{\overline{\rm MS}}(M_Z) \equiv \frac{g_R^2(M_Z)_{\overline{\rm MS}}}{4\pi} = 0.1184(7) \qquad \text{(PDG 2010 [2])}$$

Since $g_R^2(\mu_R) \to 0$ as μ_R increases, the approximate solution of the differential equation becomes better and better.

Running α_s

The β-function in the $\overline{\text{MS}}$-scheme is now known to four-loop order

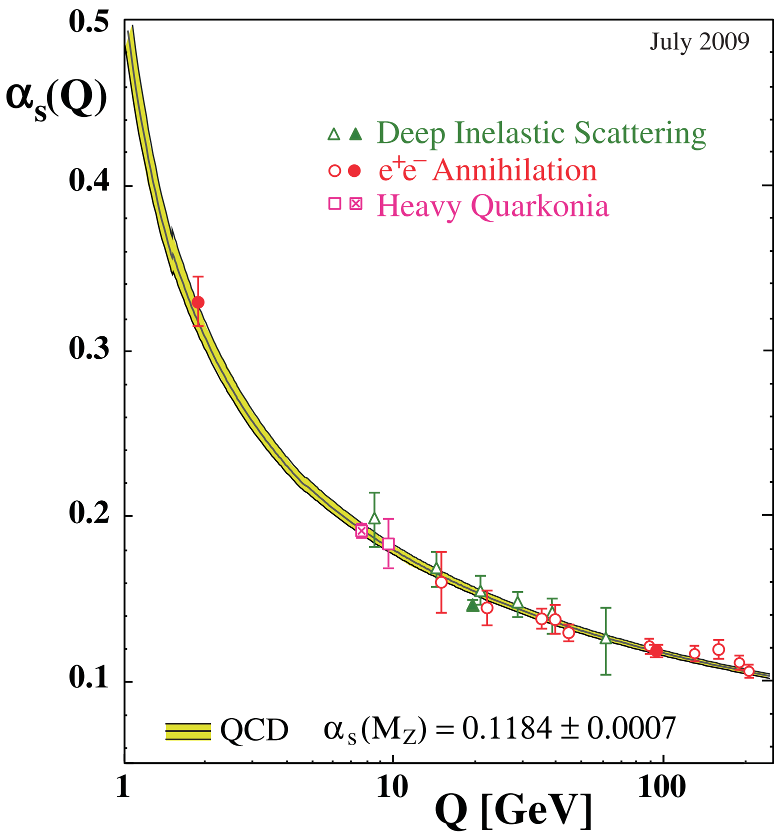
$$\begin{aligned}
\beta(g_R) &= -\beta_0 g_R^3 - \beta_1 g_R^5 - \beta_2 g_R^7 - \beta_3 g_R^9 + \mathcal{O}\left(g_R^{11}\right) \\
\beta_0 &= \frac{1}{(4\pi)^2}\frac{11C_A - 4T_R N_f}{3} \\
\beta_1 &= \frac{1}{(4\pi)^4}\left[\frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_R N_f\right] \\
\beta_2 &= \frac{1}{(4\pi)^6}\left[\frac{2857}{54}C_A^3 - \left(\frac{1415}{27}C_A^2 + \frac{205}{9}C_A C_F - 2C_F^2\right)T_R N_f + \left(\frac{158}{27}C_A + \frac{44}{9}C_F\right)T_R^2 N_f^2\right]
\end{aligned}$$

where

$$C_A = N_c \overset{N_c=3}{=} 3, \qquad C_F = \frac{N_c^2 - 1}{2N_c} \overset{N_c=3}{=} \frac{4}{3}, \qquad T_R = \frac{1}{2}$$

The results at 1-loop, 2-loop and 3-loop order were evaluated in Refs. [12], [13], [14], respectively. See Ref. [15] for the 4-loop result β_3.

Comparison of the running of α_s (4-loop running, 3-loop matching at quark mass thresholds) with experimental data shows excellent agreement and confirms the behavior of asymptotic freedom:



Source: Bethke [16]

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