ENGR 233: Applied advanced calculus

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7 Vectors

7.1 Vectors in 2-space

A vector is described by a magnitude, a line of action and a direction.

Vector properties:

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
- $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
- $\vec{a}+\vec{0}=\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}$
- $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
- $(k_1+k_2)\vec{a}=k_1\vec{a}+k_2\vec{a}$
- $k_1(k_2\vec{a})=(k_1k_2)\vec{a}$
- $1\vec{a}=\vec{a}$
- $0\vec{a}=\vec{0}$

A 2-space vector $\vec{a}$ have two components: $\vec{a}=(a_1,a_2)$.
Vector operation with components:

Equality $\vec{a}=\vec{b} \iff a_1=b_1$ and $a_2=b_2$
Addition $\vec{a}+\vec{b}=(a_1+b_1,a_2+b_2)$
Scalar multiplication $k\vec{a}=(ka_1,ka_2)$
Magnitude $\|\vec{a}\|=\sqrt{a_1^2+a_2^2}$
Unit vector $\hat{a}=\frac{\vec{a}}{\|\vec{a}\|}$
Linear combination $\vec{u}=c_1\vec{a}+c_2\vec{b}$

The elementary vectors in $\mathbb{R}^2$ are $\hat{\imath}=(1,0)$ and $\hat{\jmath}=(0,1)$. Every vectors $\vec{a}$ in $\mathbb{R}^2$ can be represented as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$

$$\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}$$

7.2 Vectors in 3-space

A 3-space vector $\vec{a}$ have three components: $\vec{a}=(a_1,a_2,a_3)$.

The distance between two points $A=(x_A,y_A,z_A)$ and $B=(x_B,y_B,z_B)$ in $\mathbb{R}^3$ is

$$AB=\left\|\overrightarrow{AB}\right\|=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}$$

The midpoint between two points $A=(x_A,y_A,z_A)$ and $B=(x_B,y_B,z_B)$ in $\mathbb{R}^3$ is

$$I=\left(\frac{x_A+x_B}{2},\frac{y_A+y_B}{2},\frac{z_A+z_B}{2}\right)$$

The elementary vectors in $\mathbb{R}^3$ are $\hat{\imath}=(1,0,0)$, $\hat{\jmath}=(0,1,0)$ and $\hat{k}=(0,0,1)$. Every vectors $\vec{a}$ in $\mathbb{R}^3$ can be represented as a linear combination of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$

$$\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$$

7.3 Dot product

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is a scalar and is defined by:

$$\begin{aligned}\vec{a}\bullet\vec{b}&=a_1b_1+a_2b_2+a_3b_3\\&=\|\vec{a}\|\left\|\vec{b}\right\|\cos\theta\end{aligned}$$

Properties of the dot product:

- $\vec{a}\bullet\vec{b}=0 \iff \vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\theta=\frac{\pi}{2}$ $[\pi]$
- $\vec{a}\bullet\vec{b}=\vec{b}\bullet\vec{a}$
- $\vec{a}\bullet(\vec{b}+\vec{c})=\vec{a}\bullet\vec{b}+\vec{a}\bullet\vec{c}$
- $\vec{a}\bullet(k\vec{b})=(k\vec{a})\bullet\vec{b}=k(\vec{a}\bullet\vec{b})$
- $\vec{a}\bullet\vec{a}=\|\vec{a}\|^2\geqslant 0$

We can use these properties to determine the perpendicularity between two vectors: two non zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a}\bullet\vec{b}=0$.

Direction angles

The dot product is useful to determine the direction angles of a vector $\vec{a}$, which is the angle between the vector and the axis:

$$\begin{aligned}x\text{-axis: }\cos\alpha&=\frac{a_1}{\|\vec{a}\|}\\y\text{-axis: }\cos\beta&=\frac{a_2}{\|\vec{a}\|}\\z\text{-axis: }\cos\gamma&=\frac{a_3}{\|\vec{a}\|}\end{aligned}$$

These formulas implies that

$$\hat{a}=(\cos\alpha,\cos\beta,\cos\gamma)$$

Component and projection

The component of a vector $\vec{a}$ along a vector $\vec{b}$ is $\text{comp}_{\vec{b}}\vec{a}=\|\vec{a}\|\cos\theta=\frac{\vec{a}\bullet\vec{b}}{\|\vec{b}\|}$. The projection of a vector $\vec{a}$ along a vector $\vec{b}$ is $\text{proj}_{\vec{b}}\vec{a}=(\text{comp}_{\vec{b}}\vec{a})\hat{b}=\frac{\vec{a}\bullet\vec{b}}{\|\vec{b}\|^2}\vec{b}$.

7.4 Cross product

The cross product between two vectors $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - b_2 a_3)\hat{\imath} + (a_3 b_1 - b_3 a_1)\hat{\jmath} + (a_1 b_2 - b_1 a_2)\hat{k} \\ &= \left(\|\vec{a}\| \left\|\vec{b}\right\| \sin\theta\right) \hat{n}\end{aligned}$$

where $\hat{n}$ is a unit vector orthogonal to the plane of $\vec{a}$ and $\vec{b}$.

Properties of the dot product:

- $\vec{a} \times \vec{b} = 0 \iff \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\theta = 0 \ [\pi]$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- $\vec{a} \times (k\vec{b}) = (k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$
- $\vec{a} \times \vec{a} = 0$
- $\vec{a} \bullet (\vec{a} \times \vec{b}) = 0$
- $\vec{b} \bullet (\vec{a} \times \vec{b}) = 0$

We can use these properties to determine the parallelism between two vectors: two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Special products

Scalar triple product:

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \bullet \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

Applications

Area of a parallelogram: $A = \left\|\vec{a} \times \vec{b}\right\|$

Area of a triangle: $A = \frac{1}{2}\left\|\vec{a} \times \vec{b}\right\|$

Volume of a parallelepiped: $V = \left|\vec{a} \bullet \left(\vec{b} \times \vec{c}\right)\right|$

Coplanar vectors: $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if $\vec{a} \bullet (\vec{b} \times \vec{c}) = 0$.

7.5 Lines and planes in space

Vector equations of a line

The parametric equation of a line is of the form

$$\vec{r} = \vec{a}t + \vec{r}_0$$

where $\vec{r} = (x, y, z)$ are the coordinates in $\mathbb{R}^3$, $\vec{a} = (a_1, a_2, a_3)$ is the direction vector of the line, $\vec{r}_0 = (x_0, y_0, z_0)$ is any point on the line and t is a scalar variable. This equation can also be written as a system:

$$\begin{cases} x = a_1 t + x_0 \\ y = a_2 t + y_0 \\ z = a_3 t + z_0 \end{cases}$$

From those equation, the symmetric equation can be found:

$$t = \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

If any components of $\vec{a}$ is equal to 0, then, the dependent variable is set to its corresponding component of $\vec{r}_0$.

Example: If $\vec{a} = (0, a_2, a_3)$, then we have $x = x_0$ and $\frac{y-y_0}{a_2} = \frac{z-z_0}{a_3}$

Vector equations of a plane

The cartesian equation of a plane is of the form

$$\vec{n} \bullet (\vec{r} - \vec{r}_0) = 0$$

where $\vec{n} = (a, b, c)$ is the normal vector of the plane, $\vec{r} = (x, y, z)$ are the coordinates in $\mathbb{R}^3$ and $\vec{r}_0 = (x_0, y_0, z_0)$ is any point on the plane. This equation can be expanded in the form

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$

Method to find the equation of the plane containing three points A, B and C:

1. Build 3 vectors $\overrightarrow{AB}$, $\overrightarrow{CB}$ and $\overrightarrow{CX}$, where $X = (x, y, z)$
2. Compute the scalar triple product $(\overrightarrow{AB} \times \overrightarrow{CB}) \bullet \overrightarrow{CX} = 0$

Method to find the line of intersection between to planes:

1. Let either x, y or z equal to t
2. Build the system where the other 2 variables depends on t
3. Solve to get x, y, and z dependent on t

9 Vector calculus

9.1 Vector functions

A vector function is a function such that at least one component of a vector $\vec{r}$ is dependent on another variable:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Example: The vector function of a circular helix is of the form:

$$\vec{r}(t) = (\alpha\cos\beta t, \alpha\sin\beta t, kt)$$

Limit of a vector function

If $\lim_{a\to t} x(t)$, $\lim_{a\to t} y(t)$ and $\lim_{a\to t} z(t)$ exist, then

$$\lim_{a\to t} \vec{r}(t) = \left(\lim_{a\to t} x(t), \lim_{a\to t} y(t), \lim_{a\to t} z(t)\right)$$

Properties of limits If $\lim_{a\to t} \vec{r}_1(t) = \vec{L}_1$ and $\lim_{a\to t} \vec{r}_2(t) = \vec{L}_2$, then:

$$\lim_{a\to t} k\vec{r}_1(t) = k\vec{L}_1$$
$$\lim_{a\to t} \vec{r}_1(t) + \vec{r}_2(t) = \vec{L}_1 + \vec{L}_2$$
$$\lim_{a\to t} \vec{r}_1(t) \bullet \vec{r}_2(t) = \vec{L}_1 \bullet \vec{L}_2$$

A vector function $\vec{r}$ is said to be continuous at $t = a$ if:

- $\vec{r}(a)$ is defined
- $\lim_{a\to t} \vec{r}(t)$ exists
- $\lim_{a\to t} \vec{r}(t) = \vec{r}(a)$

Derivative of a vector function

If $\vec{r}(t) = (x(t), y(t), z(t))$, where x, y and z are differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}(t)\right] = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$$

Smooth curves When the components functions of a vector function $\vec{r}$ have continuous first derivatives and $\vec{r}'(t) \neq 0$ for all t in the open interval (a, b), then $\vec{r}$ is said to be a smooth function and the curve $\mathscr{C}$ traced by $\vec{r}$ is called a smooth curve.

Method to find parametric equation of the tangent line to a curve for $t = k$:

1. If not given, find the vector function $\vec{r}$
2. Find the point of tangency $\vec{r}(k)$
3. Compute $\vec{r}'$
4. Find the direction vector at the point of tangency $\vec{r}'(k)$
5. The parametric equation is $\vec{T} = \vec{r}'(k)t + \vec{r}(k)$

Properties of derivatives Let $\vec{r}_1(t)$ and $\vec{r}_2(t)$ be differentiable vector functions and $u(t)$ a differentiable scalar function:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}_1(t) + \vec{r}_2(t)\right] = \vec{r}_1'(t) + \vec{r}_2'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)\vec{r}_1(t)\right] = u(t)\vec{r}_1'(t) + u'(t)\vec{r}_1(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}_1(t) \bullet \vec{r}_2(t)\right] = \vec{r}_1'(t) \bullet \vec{r}_2(t) + \vec{r}_1(t) \bullet \vec{r}_2'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{r}_1(t) \times \vec{r}_2(t)\right] = \vec{r}_1'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2'(t)$$

Integrals of a vector function

If $\vec{r}(t) = (x(t), y(t), z(t))$, where x, y and z are integrable, then

$$\int \vec{r}(t)\,\mathrm{d}t = \left(\int x(t)\,\mathrm{d}t, \int y(t)\,\mathrm{d}t, \int z(t)\,\mathrm{d}t\right)$$
$$\int_a^b \vec{r}(t)\,\mathrm{d}t = \left(\int_a^b x(t)\,\mathrm{d}t, \int_a^b y(t)\,\mathrm{d}t, \int_a^b z(t)\,\mathrm{d}t\right)$$

Length of a space curve

If $\vec{r}(t) = (x(t), y(t), z(t))$ is a smooth function, then it can be shown that the length of the smooth curve traced by $\vec{r}$ is given by

$$s = \int_a^b \|\vec{r}'(t)\|\,\mathrm{d}t = \int_a^b \sqrt{[x'(t)^2] + [y'(t)^2] + [z'(t)^2]}\,\mathrm{d}t$$

9.2 Motion on a curve

Suppose a body or a particle moves along a curve $\mathscr{C}$ so that its position at time t is given by the vector function

$$\vec{r}(t) = (x(t), y(t), z(t))$$

The velocity and acceleration of the particle are

$$\vec{v}(t) = \vec{r}'(t) = (x'(t), y'(t), z'(t))$$
$$\vec{a}(t) = \vec{r}''(t) = (x''(t), y''(t), z''(t))$$

The speed of the particle is the magnitude of the velocity:

$$v(t) = \|\vec{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Centripetal motion

If $\vec{r}(t) = (r_0 \cos\omega t, r_0 \sin\omega t)$, where r_0 and ω are constants, then the acceleration is $\vec{r}'' = -\omega^2\vec{r}$ and the position and acceleration vectors are in opposite direction (this is the case for a circular motion). Also, $\|\vec{a}\| = \frac{\|\vec{v}\|^2}{\|\vec{r}_0\|}$.

Trajectory of a projectile

Due to the gravity, we have $\vec{a} = (0, -g)$. Let the initial velocity be $\vec{v}_0 = (v_0\cos\theta, v_0\sin\theta)$ and the initial height be $\vec{s}_0 = (0, s_0)$. Then the velocity vector $\vec{v}$ is $\vec{v} = (v_0\cos\theta, -gt + v_0\sin\theta)$, and the position vector $\vec{r}$ is $\vec{r} = (v_0 t\cos\theta, -\frac{1}{2}gt^2 + v_0 t\sin\theta + s_0)$.

9.3 Curvature and components of acceleration

Unit tangent vector

Let $\vec{r}(t)$ be a vector function defining a smooth curve $\mathscr{C}$ and let the unit tangent vector $\hat{T}(t)$ be

$$\hat{T}(t) = \hat{r}'(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

If s is the arc length parameter and is the unit tangent vector, then the curvature of $\mathscr{C}$ at a point is

$$\kappa = \left\| \frac{\mathrm{d}\hat{T}}{\mathrm{d}s} \right\| \text{ and } \kappa(t) = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|}$$

Method to find the curvature of a curve defined by $\vec{r}(t)$:

1. Compute $\vec{r}'(t)$ and $\|\vec{r}'(t)\|$
2. Compute $\hat{T}(t) = \hat{r}'(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$, $\vec{T}'(t)$ and $\left\|\vec{T}'(t)\right\|$
3. Compute $\kappa(t) = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{r}'(t)\|}$

Principal normal

Then, from the unit tangent vector $\hat{T}(t)$, we can find the principal normal which is defined as

$$\hat{N}(t) = \hat{T}'(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

In the case of motion, the curvature $\kappa(t)$ can be re-written as $\kappa(t) = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{v}(t)\|}$ and velocity $\vec{v}(t)$ will be $\vec{v}(t) = \|\vec{v}\|\,\hat{T}(t)$ Finally, using the unit tangent vector $\hat{T}(t)$, the curvature $\kappa(t)$ and the principal normal $\hat{N}(t)$, the acceleration vector can be re-written as

$$\begin{aligned} \vec{a}(t) &= a_N\hat{N} + a_T\hat{T} \\ \text{where } a_N &= \kappa\,\|\vec{v}\|^2 \text{ is the normal acceleration} \\ a_T &= \frac{\mathrm{d}\,\|\vec{v}\|}{\mathrm{d}t} \text{ is the tangential acceleration} \end{aligned}$$

Binormal vector

The binormal vector is defined as $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$.

Now, three planes can be defined:
Plane TN: osculating plane
Plane NB: normal plane
Plane TB: rectifying plane

The vectors $\hat{T}(t)$, $\hat{N}(t)$ and $\hat{B}(t)$ form a right-handed system referred to as trihedral system.

Using all those formulas, we can find several formulas

$$a_T = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\vec{v} \bullet \vec{a}}{\|\vec{v}\|}$$

$$a_N = \kappa\,\|\vec{v}\|^2 = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$$

$$\kappa(t) = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3}$$

Finally, the reciprocal ρ of curvature is called radius of curvature and is defined as

$$\rho = \frac{1}{\kappa}$$

9.4 Partial derivatives

Let $z = f(x, y)$ denote a surface. The level curves are $f(x, y) = c$ (same thing as lines of altitude on elevation maps).

For a function of three variables, we have $w = f(x, y, z)$ The level surfaces are $f(x, y, z) = c$.

For $y = f(x)$, we had

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, in case of partial derivatives, for $z = f(x, y)$ we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \\ \frac{\partial z}{\partial y} &= \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \end{aligned}$$

To compute $\frac{\partial z}{\partial x}$, we use the laws of ordinary differentiation while treating y as a constant. The same applies for computing $\frac{\partial z}{\partial y}$.

Notation of partial derivatives

Partial derivative representation are:

- First-order:
$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x$$
- Second-order:
$$\frac{\partial^2 z}{\partial x^2} = f_{xx}$$
- Mixed second-order:
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yx}$$

  If a function f has continuous second partial derivatives, then the order in which a mixed second partial derivative is done is irrelevant, i.e. $f_{yx} = f_{xy}$

Chain rule

If $z = f(u, v)$ is differentiable and $u = g(x, y)$ and $v = h(x, y)$ have continuous first-order partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

In order to simplify the calculations, we can use the tree method (can be expanded with more functions and more independent variables):

1. Place the dependent variable z at the top
2. Place u and v under z
3. Place x and y under each u and v
4. The "roots" of the tree, are partial derivative of the top with respect of the bottom
5. To get $\frac{\partial z}{\partial x}$ multiply the partial derivatives from z until x and sum them up for each root.

9.5 Directional derivatives

Gradient

The gradient of a function is a vector which points in the direction of most increase and is defined as

$$\vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$
$$\vec{\nabla} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Directional derivative

The directional derivative $D_{\vec{u}}\, z$ of $z = f(x,y)$ in the direction of a unit vector $\vec{u} = (\cos\theta, \sin\theta)$ is the generalization of partial differentiation of the function f, and it is defined as:

$$D_{\vec{u}}\, f(x,y) = \vec{\nabla} f(x,y) \bullet \vec{u} = \left\|\vec{\nabla} f\right\| \cos\phi$$

Therefore, we have $-\left\|\vec{\nabla} f\right\| \leqslant D_{\vec{u}}\, f \leqslant \left\|\vec{\nabla} f\right\|$. This can also be interpreted as: the gradient vector $\vec{\nabla} f$ points in the direction in which f increases most rapidly, whereas $-\vec{\nabla} f$ points in the direction in which f decreases most rapidly.

9.6 Tangent planes and normal lines

Tangent planes

Suppose $f(x,y) = c$ is the level curve of the differentiable function $z = f(x,y)$ that passe through a specified point $P(x_0, y_0)$, i.e. $f(x_0, y_0) = c$. Then, the following property can be shown: $\vec{\nabla} f(x_0, y_0) \bullet \vec{r}'(t_0) = 0$ which means that $\vec{\nabla} f$ is orthogonal to the level curve at P. For a function of three variables, we have $\vec{\nabla} f(x_0, y_0, z_0) \bullet \vec{r}'(t_0) = 0$, i.e. $\vec{\nabla} f$ is normal to the level surface at P.

The tangent plane at P is the plane normal to $\vec{\nabla} f$ evaluated at P. If $P(x,y,z)$ and $P(x_0,y_0,z_0)$ are points on the tangent plane and $\vec{r}$ and $\vec{r}_0$ are their respective position vectors, then a vector equation of the tangent plane is

$$\vec{\nabla} f(x_0, y_0, z_0) \bullet (\vec{r} - \vec{r}_0) = 0$$

with the following expanded form:

$$\begin{aligned} f_x(x_0,y_0,z_0)(x-x_0) & \\ + f_y(x_0,y_0,z_0)(y-y_0) & \\ + f_z(x_0,y_0,z_0)(z-z_0) &= 0 \end{aligned}$$

Normal line

The normal line to the surface at a point $P(x_0,y_0,z_0)$ is the line normal to the tangent plane of the surface at P. The parametric equation of the normal line is

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$$

9.7 Curl and divergence

Vector field

A vector field is made of a set of vector. It can be seen as a vector function in which the components can depend on several variables:

$$\vec{v}(x,y) = (P(x,y), Q(x,y))$$
$$\vec{v}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

Curl

The curl of a vector field $\vec{v}$ is another vector field such that

$$\operatorname{curl} \vec{v} = \vec{\nabla} \times \vec{v}$$

Flux

The flux is the volume of the fluid flowing through an element of surface area ΔS per unit of time. The flux can be obtained using

$$\text{flux} \;= (\text{comp}_{\vec{n}} \vec{v})\, \Delta S_{\text{base}} = (\vec{v} \bullet \vec{n})\, \Delta S_{\text{base}}$$

The net flux of $\vec{v}$ is defined as

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \Delta x \Delta y \Delta z$$

and the outward flux of $\vec{v}$ per unit volume is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Divergence

The divergence of a vector field $\vec{F} = (P, Q, R)$ is the scalar function

$$\operatorname{div} \vec{v} = \vec{\nabla} \bullet \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Physical interpretation

Curl The curl of the velocity field $\vec{v}$ is a measure of the tendency of the fluid to turn a device about its vertical axis.

Divergence The divergence is a measure of the fluid's compressibility
- If $\operatorname{div} \vec{v}(P) > 0$, then P is said to be a source for $\vec{v}$.
- If $\operatorname{div} \vec{v}(P) = 0$, then the fluid is said to be incompressible.
- If $\operatorname{div} \vec{v}(P) < 0$, then P is said to be a sink for $\vec{v}$.

If F is a vector field having continuous second partial derivatives, then $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$. If f is a scalar function with continuous second partial derivatives, then $\operatorname{curl}(\vec{\nabla} f) = \vec{0}$.

9.8 Line integrals

Terminology

Suppose $\mathscr{C}$ is a curve parameterized by $\vec{F} = (f(t), g(t))$, $a \leqslant t \leqslant b$, and $A = (f(a), g(a))$ and $B = (f(b), g(b))$

Smooth curve: if f' and g' are continuous on the closed interval $[a, b]$ and not simultaneously 0 on the open interval (a, b).

Piecewise smooth: if it consists of a finite number of smooth curves joined en to end.

Closed curve: is $A = B$

Simple closed curve: if $A = B$ and the curve does not cross itself.

Line integrals definition

Let $\vec{F} = (P(x, y), Q(x, y))$ be a function of two variables x and y defined on a region of the place containing a smooth curve $\mathscr{C}$. The line integral of $\vec{F}$ along $\mathscr{C}$ from A to B is

$$\int_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = \int_{\mathscr{C}} P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$$

A line integral along a piecewise-smooth curve $\mathscr{C}$ is defined as the sum of the integrals over the various smooth curves whose $\mathscr{C}$ is made of. This also applies to contour integrals which might be made of piecewise-smooth curve.

Physical interpretation

Work If we have a force $\vec{F} = (P(x, y, z), Q(x, y, z), R(x, y, z))$ along a curve $\mathscr{C}$, then the work done by this force is

$$W_{\mathscr{C}} = \int_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = \int_{\mathscr{C}} \mathrm{comp}_{\vec{T}} \vec{F}\,\mathrm{d}s$$

where $\hat{T}$ is the unit tangent vector. This means that the work done by a force $\vec{F}$ along a curve $\mathscr{C}$ is due entirely to the tangential component of $\vec{F}$.

Mass If a wire corresponding to a curve $\mathscr{C}$ has a variable density $\rho(x, y)$ in mass per unit length, then the mass of the wire along the curve $\mathscr{C}$ defined by $\vec{r}(t)$ is

$$m = \int_{\mathscr{C}} \rho\,\mathrm{d}s = \int_{\mathscr{C}} \rho\,\|\vec{r}'\|\,\mathrm{d}t$$

Circulation The circulation is defined as

$$\text{circulation} = \oint_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = \oint_{\mathscr{C}} \mathrm{comp}_{\vec{T}} \vec{F}\,\mathrm{d}s$$

9.9 Independent of the path

An integral is independent of the path if no matter what is the curve, the integral remains the same.

A vector function $\vec{F}$ is said to be conservative if $\vec{F}$ can be written as the gradient of a scalar function ϕ. The function ϕ is called a potential function of $\vec{F}$. In other words, $\vec{F}$ is conservative if there exists function ϕ such that $\vec{F} = \vec{\nabla}\phi$. A conservative vector field is also called a gradient vector field.

Fundamental theorem of the line integral

$$\int_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = \int_{\mathscr{C}} \vec{\nabla}\phi \bullet \mathrm{d}\vec{r} = \phi(B) - \phi(A)$$

where A is the starting point and B is the end point.

In an open connected region R, the integral is independent of the path $\mathscr{C}$ if and only if the vector field $\vec{F}$ is conservative in R or if $\oint_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = 0$ for every closed path $\mathscr{C}$ in R.

Therefore, we have:

$$\begin{aligned} \vec{F} \text{ conservative} &\iff \text{path independence} \\ &\iff \oint_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = 0 \end{aligned}$$

- In $\mathbb{R}^2$, $\vec{F} = (P(x, y), Q(x, y))$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- In $\mathbb{R}^3$, $\vec{F} = (P(x, y, z), Q(x, y, z), R(x, y, z))$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ and } \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \text{ and } \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

  This is equivalent to $\mathrm{curl}\,\vec{F} = \vec{0}$.

Method for finding $\phi(x, y)$ in $\mathbb{R}^2$ (very similar to solving an exact differential equation):

1. Check for $\vec{F} = (P(x, y), Q(x, y))$ being conservative
2. If $\vec{F} = (P(x, y), Q(x, y))$ is conservative, there exists a function $\phi(x, y)$ such that:

$$\frac{\partial \phi(x, y)}{\partial x} = P(x, y) \text{ and } \frac{\partial \phi(x, y)}{\partial y} = Q(x, y)$$

3. Find $\phi(x, y)$ by integrating $P(x, y)$ with respect to x, while holding y constant. This gives:

$$\phi(x, y) = \int P(x, y)\,\mathrm{d}x + g(y)$$

   where an arbitrary function $g(y)$ is the "constant" of integration
4. Differentiate $\phi(x, y)$ with respect to y and set it equals to $Q(x, y)$:

$$\frac{\partial \phi(x, y)}{\partial y} = \frac{\partial}{\partial y}\left[\int P(x, y)\,\mathrm{d}x\right] + g'(y) = Q(x, y)$$

5. This gives:

$$g'(y) = Q(x, y) - \frac{\partial}{\partial y}\left[\int P(x, y)\,\mathrm{d}x\right]$$

6. Integrate $g'(y)$ with respect to y
7. Substitute the result in $\phi(x,y) = \int P(x,y)\,\mathrm{d}x + g(y)$

Therefore, if $\vec{F} = (P(x,y), Q(x,y))$ is conservative, the full solution becomes:

$$\phi(x,y) = \int P(x,y)\,\mathrm{d}x + \int Q(x,y) - \frac{\partial}{\partial y}\left[\int P(x,y)\,\mathrm{d}x\right]\mathrm{d}y$$

The method for finding $\phi(x,y,z)$ in $\mathbb{R}^3$ is very similar, but $g(y)$ becomes $g(y,z)$ and therefore, we will need to do another integral in order to find a third function $h(z)$.

9.10 Double integrals

Let f be a function of two variables defined on a closed region R of $\mathbb{R}^2$. Then the double integral of f over R is given by

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Properties

Let f and g be functions of two variables that are integrable over a region R, then:

- $\iint_R kf(x,y)\,\mathrm{d}A = k\iint_R f(x,y)\,\mathrm{d}A$, where $k \in \mathbb{R}$.
- $\iint_R [f(x,y) \pm g(x,y)]\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A$
- $\iint_R f(x,y)\,\mathrm{d}A = \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_1} f(x,y)\,\mathrm{d}A$, where R_1 and R_2 are subregions of R that not overlap and $R = R_1 \cup R_2$

Computation

For region of Type I:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

For region of Type II:

$$\int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{h_1(x)}^{h_2(x)} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y$$

Fubini's theorem Let f be continuous on a region R. If R is of Type I, then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

If R is of Type II, then

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Mass

If a lamina corresponding to a region R has a variable density $\rho(x,y)$ continuous on R, then

$$m = \iint_R \rho(x,y)\,\mathrm{d}A$$

The coordinates of the center of mass of the lamina are:

$$x = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\,\mathrm{d}A$$
$$y = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)\,\mathrm{d}A$$

The moments of inertia of the lamina are:

$$I_x = \iint_R y^2\rho(x,y)\,\mathrm{d}A$$
$$I_y = \iint_R x^2\rho(x,y)\,\mathrm{d}A$$

The radius of gyration K of a lamina of mass m is defined by $K = \sqrt{\frac{I}{m}}$.

9.11 Double integrals in polar coordinates

The double integral of a function $f(r,\theta)$ with respect to an area in polar coordinates is

$$\iint_R f(r,\theta)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta) r\,\mathrm{d}\theta\,\mathrm{d}r$$

A standard double integral function of x and y can be written as a double integral using polar coordinates:

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

which is particularly useful when f contains the expression x^2+y^2 since $x^2+y^2 = r^2$.

9.12 Green's theorem

For Green's theorem, we introduce the concept of direction in the contour integrals: $\oint$ is in the positive direction and $\oint$ is in the negative direction.

Suppose that $\mathscr{C}$ is a piecewise-smooth simple closed curve bounding a simply connected region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then

$$\oint_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} = \oint_{\mathscr{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the contour integral can be taken on a curve that is more convenient, as long as this curve is fully enclosed in the region bounded by the previous curve.

9.13 Surface integrals

Surface area

Let f be a function for which the first partial derivatives f_x and f_y are continuous on a closed region R. Then the area of the surface over R is given by

$$S = \iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}A$$

The surface area of a parameterized vector function $\vec{F}(x,y)$ over R is

$$S = \iint_R \left\| \frac{\partial \vec{F}}{\partial x} \times \frac{\partial \vec{F}}{\partial y} \right\| \mathrm{d}x\,\mathrm{d}y$$

The differential of the surface area is the function function

$$\mathrm{d}S = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}A$$

Surface integral

Let G be a function of three variables defined over a region of $\mathbb{R}^3$ containing the surface S. Then the surface integral of G over S is given by

$$\iint_S G(x,y,z)\,\mathrm{d}S$$

In order to evaluate this surface integral, we project it along a planes:
xy-plane:

$$\iint_S G(x,y,z)\,\mathrm{d}S = \iint_R G(x,y,f(x,y))\sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}A$$

xz-plane:

$$\iint_S G(x,y,z)\,\mathrm{d}S = \iint_R G(x,g(x,z),z)\sqrt{1 + g_x(x,z)^2 + g_z(x,z)^2}\,\mathrm{d}A$$

yz-plane:

$$\iint_S G(x,y,z)\,\mathrm{d}S = \iint_R G(h(y,z),y,z)\sqrt{1 + h_y(y,z)^2 + h_z(y,z)^2}\,\mathrm{d}A$$

The mass m of a surface represented by $\rho(x,y,z)$ as the density of this shape at any point is given by

$$m = \iint_S \rho(x,y,z)\,\mathrm{d}S$$

Orientable surface

A surface S defined as $g(x,y,z) = 0$ can be an oriented surface. The orientation of S can be found using the normal vector function $\hat{n}(x,y,z)$, where

$$\hat{n} = \frac{\vec{\nabla} g}{\left\|\vec{\nabla} g\right\|}$$

If S is defined by $z = f(x,y)$, then we define $g(x,y,z) = z - f(z,y) = 0$ or $g(x,y,z) = f(z,y) - z = 0$ depending on the orientation of S.

A two-sided surface S defined by $z = f(x,y)$ has an upward orientation when the unit normals are directed upward (positive $\hat{\mathrm{k}}$ components), and it has a downward orientation when the unit normals are direct downward (negative $\hat{\mathrm{k}}$ components).

Integrals of vector fields

If $\vec{v} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is the velocity field of a fluid, then the total of a fluid passing through S per unit of time is called the flux of $\vec{v}$ through S and is given by

$$\text{flux} = \iint_S \vec{v} \bullet \hat{n}\,\mathrm{d}S$$

In the case of a closed surface S:
- If $\hat{n}$ is the outer normal, the surface integral gives the volume of fluid flowing out through S per unit of time
- If $\hat{n}$ is the inner normal, the surface integral gives the volume of fluid flowing in through S per unit of time

9.14 Stokes' theorem

Let S be a piecewise-smooth orientable surface bounded by a piecewise-smooth simple closed curve $\mathscr{C}$. Let $\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field for which P, Q and R are continuous and have continuous first partial derivatives in a region of $\mathbb{R}^3$ containing S. If $\mathscr{C}$ is traversed in the positive direction, then

$$\begin{aligned}\oint_{\mathscr{C}} \vec{F} \bullet \mathrm{d}\vec{r} &= \oint_{\mathscr{C}} \vec{F} \bullet \hat{T}\,\mathrm{d}S \\ &= \iint_S \left(\operatorname{curl} \vec{F}\right) \bullet \hat{n}\,\mathrm{d}S\end{aligned}$$

where $\hat{n}$ is a unit normal to S in the direction of the orientation of S.

Curl interpretation

The curl of $\vec{F}$ is the circulation of $\vec{F}$ per unit of area:

$$\left(\operatorname{curl} \vec{F} \bullet \hat{n}\right) = \lim_{r \to 0} \frac{1}{A_r} \oint_{\mathscr{C}_r} \vec{F} \bullet \mathrm{d}\vec{r}$$

9.15 Triple integral

Let F be a function of three variables defined on a closed region R of $\mathbb{R}^3$. Then the triple integral of F over R is given by

$$\iiint_D F(x,y,z)\,\mathrm{d}V = \iiint_D F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Computation

If the region D is bounded above by the graph of $z = f_1(x,y)$ and bounded below by the graph of $z = f_2(x,y)$, then it can be shown that the triple integral can be expressed as a double integral of a partial integral:

$$\begin{aligned}\iiint_D F(x,y,z)\,\mathrm{d}V &= \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

Applications

Volume If $F(x,y,z) = 1$, then the volume of the solid D is

$$V = \iiint_D \mathrm{d}V$$

Mass If $\rho(x,y,z)$ is density, then the mass of the solid D is given by

$$m = \iiint_D \rho(x,y,z)\,\mathrm{d}V$$

First moments The first moments of the solid about the coordinate planes indicated by the subscripts are given by

$$\begin{aligned}M_{xy} &= \iiint_D z\rho(x,y,z)\,\mathrm{d}V \\ M_{xz} &= \iiint_D y\rho(x,y,z)\,\mathrm{d}V \\ M_{yz} &= \iiint_D x\rho(x,y,z)\,\mathrm{d}V\end{aligned}$$

Center of mass The coordinates of the center of mass of D are given by

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

Moments of inertia The moments of inertia of D about the coordinate axes indicated by the subscripts are given by

$$\begin{aligned}I_x &= \iiint_D (y^2+z^2)\rho(x,y,z)\,\mathrm{d}V \\ I_y &= \iiint_D (x^2+z^2)\rho(x,y,z)\,\mathrm{d}V \\ I_z &= \iiint_D (x^2+y^2)\rho(x,y,z)\,\mathrm{d}V\end{aligned}$$

Radius of gyration If I is a moment of inertia of the solid about a given axis, then the radius of gyration is

$$R_g = \sqrt{\frac{I}{m}}$$

Cylindrical coordinates

The cylindrical coordinate system combines the polar description of a point in the plane with cartesian description of the z-component of a point in space. To transform the point $P = (x,y,z)$ in cartesian coordinates from its cylindrical coordinates $P = (r,\theta,z)$, we use

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Conversely, to get the cartesian coordinates from cylindrical coordinates, we use

$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\frac{y}{x} \qquad z = z$$

In order to obtain the triple integral using cylindrical coordinates, we use $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$:

$$\begin{aligned}\iiint_D F(r,\theta,z)\,\mathrm{d}V &= \iint_R \left[\int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)\,\mathrm{d}z\right]\mathrm{d}A \\ &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

Spherical coordinates

The spherical coordinates of a point P are given by the ordered triple (ρ,ϕ,θ), where ρ is the distance from the origin to P, ϕ is the angle between the positive z-axis and the vector $\overrightarrow{OP}$, and θ is the angle measured from the positive x-axis to the vector $\overrightarrow{OP}$ projected onto the xy-plane. To transform $P = (x,y,z)$ in cartesian coordinates from spherical coordinates $P = (\rho,\phi,\theta)$ are

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

Conversely, to get the cartesian coordinates from spherical coordinates, we use

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{y}{x} \qquad \phi = \arccos\frac{z}{\rho}$$

Finally, to transform from spherical coordinates (ρ,ϕ,θ) to cylindrical coordinates (r,θ,z), we use:

$$r = \rho\sin\phi \qquad \theta = \theta \qquad z = \rho\cos\phi$$

In order to obtain the triple integral using spherical coordinates, we use $\mathrm{d}V = \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$:

$$\iiint_D F(\rho,\phi,\theta)\,\mathrm{d}V =$$
$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(\phi,\theta)}^{f_2(\phi,\theta)} F(\rho,\phi,\theta)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

9.16 Divergence theorem

Let $\vec{F}(x,y) = (P(x,y), Q(x,y))$ be a $\mathbb{R}^2$ vector field. Then, another vector form of Green's theorem is

$$\oint_{\mathscr{C}} \vec{F} \bullet \hat{n}\,\mathrm{d}s = \iint_R \operatorname{div}\vec{F}\,\mathrm{d}A$$

The generalization of this to $\mathbb{R}^3$, where $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field, then

$$\iint_S \vec{F} \bullet \hat{n}\,\mathrm{d}S = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}A$$

For the surface between two spheres, we have

$$\iint_{S_a} \vec{F} \bullet \hat{n}\,\mathrm{d}S + \iint_{S_b} \vec{F} \bullet \hat{n}\,\mathrm{d}S = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}A$$

where $\hat{n}$ points outward from D, i.e. $\hat{n}$ points away from the origin on S_a and $\hat{n}$ points toward the origin on S_b.

Physical interpretation of divergence

The divergence can be interpreted as the flux per unit of volume at a specific point:

$$\operatorname{div}\vec{F}(P_0) = \lim_{r\to 0} \frac{1}{V_r} \iint_{S_r} \vec{F} \bullet \hat{n}\,\mathrm{d}S$$

Continuity equation

Using the divergence theorem, we can find the equation of continuity for fluid flow:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho\vec{F}\right) = 0$$

9.17 Change of variables in multiple integrals

For simple integrals, the change of variable is as follow: let $x = g(u)$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(g(u))g'(u)\,\mathrm{d}u$$

where $a = g(c)$ and $b = g(d)$. In multivariable calculus, we use the Jacobian $J(u)$ instead of $g'(u)$.

Let $x = f(u,v)$ and $y = g(u,v)$, then the change of variable in a double integral is of the form

$$\iint_R F(x,y)\,\mathrm{d}A = \iint_S F(f(u,v), g(u,v))J(u,v)\,\mathrm{d}A$$

Method for change of variable from (x,y) to $(u,v) = (f(x,y), g(x,y))$:

1. Find the region image S in terms of (u,v) based on the region R, which include functions and boundary points
2. Compute the non-zero Jacobian $J(u,v)$, which is defined as
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$
3. If F is continuous on R, then
$$\iint_R F(x,y)\,\mathrm{d}A = \iint_S F(f(u,v), g(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}A$$

For triple integrals, let $x = f(u,v,w)$, $y = g(u,v,w)$ and $z = h(u,v,w)$, then

$$\iiint_D F(x,y,z)\,\mathrm{d}V =$$
$$\iiint_E F(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}V$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$