ENGR 242: Statics

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1 Introduction

Newton's laws:
First law: $\sum \vec{F} = \vec{0}$
Second law: $\sum \vec{F} = m\vec{a}$
Third law: $\vec{F}_{A\to B} = -\vec{F}_{B\to A}$

Newton's law of gravitation:

$$\left\|\vec{F}_{A\to B}\right\| = \left\|\vec{F}_{B\to A}\right\| = G\frac{m_A m_B}{r^2}$$

F(N): attraction force
$G = 6.674 \cdot 10^{-11}\,\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$
m(kg): mass of the object
r(m): distance between the objects

2 Statics of rigid bodies

2.1 Concepts for rigid bodies

2.1.1 Principle of transmissibility

A force $\vec{F}$ can be moved anywhere on its line of action without affecting the conditions of equilibrium or motion.

2.1.2 Moments

A moment is a measurement of how much a force acting ona rigid body causes its rotation.

The moment of a force about a point is

$$\vec{M}_B = \vec{M}_A + \overrightarrow{BA} \times \vec{F}$$

2.2 Moment of a couple

Two forces $\vec{F}$ and $-\vec{F}$ having the same magnitude and opposite direction are said to form a couple. The sum of these forces is $\vec{0}$, but the sum of the moments about a given point is different from $\vec{0}$.

The value of a couple is $\left\|\vec{M}\right\| = d\left\|\vec{F}\right\|$ where d is the smallest distance between the 2 forces.

2.3 Moment of a force about an axis

2.3.1 Moment about a given axis

The value of a moment of a force $\vec{F}$ with a point of application A about an arbitrary axis BC is

$$\left\|\vec{M}_{BC}\right\| = \frac{\overrightarrow{BC}}{\left\|\overrightarrow{BC}\right\|} \bullet \left(\overrightarrow{BA} \times \vec{F}\right)$$

2.4 Reduction of a system of forces

A system of forces can be reduced to a single force $\vec{R}$ and moment $\vec{M}_O(\vec{R})$:

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O(\vec{R}) = \sum \vec{M}_O(\vec{F})$$

To find the equivalent system, the forces can be moved from their position to a point O only if their moments about O are added. A pure moment (couple) can be moved anywhere without affecting the system.

2.5 Further reduction of a system of forces

A system of forces can be reduced to a single force if the force and its moment are mutually perpendicular. This condition is satisfied if:
- Forces are coplanar (2D case)
- Forces are concurrent
- Forces are parallel to the same axis

2.6 Reduction to a wrench

A force-moment system can be reduced to a wrench, a force $\vec{R}$ and a moment $\vec{M}$ along $\vec{R}$. The ratio $p = \frac{\|\vec{M}\|}{\|\vec{R}\|}$ is called the pitch of the wrench, and we have

$$\vec{M} = \mathrm{proj}_{\vec{R}}\vec{M}_O = \frac{\vec{R} \bullet \vec{M}_O(\vec{R})}{\left\|\vec{R}\right\|^2}\vec{R}$$

2.7 Equilibrium of rigid bodies

The equations of equilibrium for a rigid body are

$$\sum \vec{F} = \vec{0} \qquad \sum \vec{M}_O(\vec{F}) = \vec{0}$$

2.8 Equilibrium in 3D

2.8.1 Reactions at supports and connections

See Pascal Klinguer paper for a comprehensive table with the reactions at supports and connections.

The basic thing is to put a 0 on the free components.

2.8.2 Steps for solving

1. Isolate the body
2. Define the basis
3. Express all forces and moments in components
4. Apply equilibrium equations $\sum \vec{F} = \vec{0}$ and $\sum \vec{M}_O(\vec{F}) = \vec{0}$

3 Centroids and centers of gravity

3.1 Planar centers of gravity and centroids

In this section, we will consider that the body is homogeneous (uniform density ρ) and have uniform thickness t.

In order to find the centroid, we need to find the area of the plate:

$$A = \iint \mathrm{d}A$$

Next, we introduce the first moments of the shape as:

$$Q_y = M_y = \iint x \, \mathrm{d}A \qquad Q_x = M_x = \iint y \, \mathrm{d}A$$

Finally, we have the centroid location $(\bar{x}, \bar{y})$ is at:

$$\bar{x} = \frac{Q_y}{A} \qquad \bar{y} = \frac{Q_x}{A}$$

The negative signs are determined using a table of sign, where the negative sign appears in the negative axis and for holes.

3.2 Wires

For a wire, a similar equation can be derived:

$$\bar{x} = \frac{Q_y}{L} = \frac{\int x \, \mathrm{d}L}{L} \qquad \bar{y} = \frac{Q_x}{L} = \frac{\int y \, \mathrm{d}L}{L}$$

where

$$L = \int \|\vec{r}'\| \, \mathrm{d}t = \int \sqrt{[x'(t)^2] + [y'(t)^2]} \, \mathrm{d}t$$

3.3 Application of centroids

3.3.1 Distributed loads on beams

One of the application of centroids is to find the concentrated load which is equivalent to the given distributed load. It is used to to get reaction forces, but not internal forces and deflections.

3.3.2 Forces on submerged surfaces

The forces on submerged surface are distributed along the length of the body. Using the the formula of pressure, we have

$$\vec{W} = t\rho h \vec{g}$$

$\vec{W}$(N): weight of the body
t(m): thickness of the body
$\rho(\mathrm{kg} \cdot \mathrm{m}^{-3})$: density of the body
h(m): vertical distance from the surface to the body
$\vec{g} = (0, 0, -9.81)\mathrm{m} \cdot \mathrm{s}^{-2}$ (on Earth)

For forces exerted by a liquid on a curved body:

1. Isolate the volume of liquid above the body
2. Determine the forces exerted on it (including weight of the liquid)
3. The force that the body exerts on the water has the same magnitude and line of action but an opposite direction.

3.4 Center of gravity and centroids of volumes

3.4.1 3D center of gravity and centroids

The mass a solid is given by

$$m = \iiint \rho(x, y, z) \, \mathrm{d}V$$

The first moments of the solid about the coordinate planes indicated by the subscripts are given by

$$M_{xy} = \iiint z\rho(x, y, z) \, \mathrm{d}V$$
$$M_{xz} = \iiint y\rho(x, y, z) \, \mathrm{d}V$$
$$M_{yz} = \iiint x\rho(x, y, z) \, \mathrm{d}V$$

The coordinates of the center of mass G of the solid are given by

$$x_G = \frac{M_{yz}}{m} \qquad y_G = \frac{M_{xz}}{m} \qquad z_G = \frac{M_{xy}}{m}$$

4 Analysis of structures

4.1 Equilibrium of a two-force body

A rigid body in equilibrium subjected to two forces is called a "two-force body". The two forces have the same magnitude and line of action, but an opposite direction.

Therefore, if a member is subjected to two forces, the internal forces are at the joints and must have the same line of action crossing the joints.

4.2 Categories of structures

Trusses: they are stationary and used to support loads, they are fully constrained and exclusively made of two-force members

Frames: they are stationary and used to support loads, and they always have at least one multi-force member

Mechanisms: they are designed to transmit and modify forces, they have moving parts and they always have at least one multi-force member

4.3 Analysis of trusses

Trusses are made of straight members connected at joints. Bolded or welded connections are assumed to be pinned together. It is usually assumed that members are pinned, therefore, the forces acting at each end are reduced to a force and no moment. Only two-force members are considered.

4.3.1 Zero-force members

If there are two members at a joint with no external forces and the two members are not aligned, then these two members are zero-force members.

If there are three members at a joint with no external forces and two are aligned, then the third member is a zero-force members.

4.3.2 Analysis of trusses by method of joints

The line of actions of all internal forces are known, the analysis is reduced to finding their magnitude and determining their stress: either "compression" or "tension".

1. Isolate the entire truss as a rigid body
2. Find reaction forces using equilibrium equations
3. Isolate each joint
4. Find internal reaction forces using equilibrium at each joint

4.3.3 Analysis of trusses by method of sections

If the force in on member or very few members are needed, the method of sections is more efficient.

1. Isolate the entire truss as a rigid body
2. Find reaction forces using equilibrium
3. Cut the truss through the members of interest
4. Represent the forces of the cut members
5. Equilibrium of one fo the cut truss (the one with less external forces)

4.4 Analysis of frames and mechanisms

4.4.1 Frames

Frames are stationary and used to support loads, and they always have at least one multi-force member.

1. Isolate the entire frame as a rigid body
2. Find reaction forces using equilibrium
3. Isolate each member
4. Find internal forces using equilibrium

4.4.2 Mechanisms

Mechanisms are designed to transmit and/or modify forces, they have moving parts and they always have at least one multi-force member.

1. Isolate the entire mechanism as a rigid body
2. Find reaction forces using equilibrium
3. Isolate each member
4. Find internal forces using equilibrium

4.5 Internal forces in members

For a straight two-force member, the internal force is equivalent to axial forces.

For a not straight member or a multi-force member, the internal forces are not limited to producing just compression or tension, they also produce shear force $\vec{V}$ and bending moment $\vec{M}$.

4.6 Beams

4.6.1 Types of loading and support

4.6.2 Shear and bending moment in a beam

Steps to analyze a beam:

1. Isolate the entire beam as a rigid body
2. Find reaction forces using equilibrium
3. Cut the beam at the desired location
4. Find the internal shearing force $\vec{V}$ and the bending moment $\vec{M}$ using the sign convention
5. Plot the values of the shearing force $\vec{V}$ and the bending moment $\vec{M}$ with respect to a distance x to get the shear and bending diagrams

Sign convention At first, shearing force $\vec{V}$ is assumed to be directed down on the left side of the cut, and up on the right side of the cut. The bending moment $\vec{M}$ is assumed to be directed counterclockwise on the left side of the cut, and clockwise on the right side of the cut.

4.6.3 Relation between loads, shear and bending moment

When there are continuous loads w, the following relation can be used:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -w \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V$$

5 Friction

5.1 Friction equations

Static equation:

$$\vec{F}_S = \mu_S \vec{N}$$

$\vec{F}_S$(N): static friction force
μ_S: coefficient of static friction
$\vec{N}$(N): normal force

Kinetic equation:

$$\vec{F}_K = \mu_K \vec{N}$$

$\vec{F}_K$(N): kinetic friction force
μ_K: coefficient of kinetic friction
$\vec{N}$(N): normal force

5.2 Friction on a horizontal surface

Case I: No friction $\implies$ no horizontal force $\implies$ no opposing friction force

Case II: No motion $\implies$ external force $\vec{P}$ not high enough to overcome static friction force $\vec{F}_S$

Case III: Motion impending (imminent) $\implies$ applied forces such that the body is just about to slide $\implies$ $\vec{F} = \vec{F}_S$

Case IV: Motion $\implies$ $\vec{F} = \vec{F}_K$

5.3 Angles of friction

Case I: Motion impending $\implies$ $\mu_S = \tan\phi_S = \frac{\|\vec{F}_S\|}{\|\vec{N}\|}$

Case II: Motion $\implies$ $\mu_K = \tan\phi_K = \frac{\|\vec{F}_K\|}{\|\vec{N}\|}$

5.4 Problems involving friction

- All applied forces are given and coefficients of frictions are known $\implies$ determine whether the body will remain at rest or slide
- Applied forces are known and the body is in impending motion $\implies$ determine friction force, normal force and coefficients of friction
- Coefficients of friction are known and the body is in impending motion $\implies$ determine one of the external forces that will cause impending motion

5.5 P_{min} and P_{max} to equilibrium

For a block with a weight $\vec{W}$ on an inclined surface, pushed by a force $\vec{P}$, we have:

$$P_{\text{max,min}} = \left\|\vec{W}\right\| \sin\theta \pm \mu_S \left\|\vec{N}\right\|$$

where P_{min} and P_{max} are the boundaries in which the block will not move.

6 Moment of inertia

6.1 Area moment of inertia

6.1.1 Calculation of the area moment of inertia

In the 2D case, we have:

$$I_x = \iint y^2 \,\mathrm{d}A \qquad I_y = \iint x^2 \,\mathrm{d}A$$

6.1.2 Poplar moment of inertia

When dealing with torsion, the polar moment of inertia is generally more important:

$$\begin{aligned} J_O &= \iint r^2 \,\mathrm{d}A = \iint x^2 + y^2 \,\mathrm{d}A \\ &= I_x + I_y \end{aligned}$$

6.1.3 Radius of gyration of an area

To get the same I between a blob and a strip, the strip should be placed at the radius of gyration K:

$$K_x = \sqrt{\frac{I_x}{A}} \qquad K_y = \sqrt{\frac{I_y}{A}} \qquad K_O = \sqrt{\frac{J_O}{A}}$$

6.2 Parallel-axis theorem and composite areas

Often, complex shapes can be broken into a sum of simple shapes. Since we need the moment of inertia of each simple shapes with respect to the same axis, we need to use the parallel-axis theorem.

The moment of inertia of the area A about a non-centroidal axis can be obtained using

$$I_x = I_{x_O} + A{d_x}^2 \qquad I_y = I_{y_O} + A{d_y}^2$$

where d_x is the distance between the two x-axis and d_y is the distance between the two y-axis.

Hence, the moment of inertia's of a composite chape made of several areas is

$$\begin{aligned} I_x &= \sum I_{x_i} + \sum A{d_x}^2 \\ I_y &= \sum I_{y_i} + \sum A{d_y}^2 \end{aligned}$$

The change in the radius of gyration is $K^2 = {K_O}^2 + d^2$

6.3 Mass moment of inertia

6.3.1 Moment of inertia of a simple mass

The mass moment of inertia is defined as:

$$I = \int r^2 \,\mathrm{d}m$$

> $I(\mathrm{kg \cdot m^2})$: mass moment of inertia about an axis
> $r(\mathrm{m})$: distance for the center of mass to the axis

We can express I in terms of the standard xyz-basis:

$$\begin{aligned} I_x &= \int y^2 + z^2 \,\mathrm{d}m \\ I_y &= \int x^2 + z^2 \,\mathrm{d}m \\ I_z &= \int x^2 + y^2 \,\mathrm{d}m \end{aligned}$$

6.3.2 Parallel-axis theorem for mass moments of inertia

$$\begin{aligned} I_{x_O} &= I_{x_G} + m({y_O}^2 + {z_O}^2) \\ I_{y_O} &= I_{y_G} + m({x_O}^2 + {z_O}^2) \\ I_{z_O} &= I_{z_G} + m({x_O}^2 + {y_O}^2) \end{aligned}$$

where G is the center of mass and O is the destination point.

6.3.3 Mass moment of inertia of a 3D body by integration

The general formula for the mass moment of inertia of a 3D body is:

$$\begin{aligned} I_x &= \iiint (y^2 + z^2)\rho(x,y,z) \,\mathrm{d}V \\ I_y &= \iiint (x^2 + z^2)\rho(x,y,z) \,\mathrm{d}V \\ I_z &= \iiint (x^2 + y^2)\rho(x,y,z) \,\mathrm{d}V \end{aligned}$$

6.4 Transformation of moment of inertia

If we want to find the axis that generate the maximum and minimum moment of inertia, we first need to define the product of inertia:

$$I_{xy} = \iint xy \,\mathrm{d}A \qquad I_{xy} = I_{x_G y_G} + x_G y_G A$$

Now, we can obtain Mohr's circles, which let use get the minimum and maximum moments of inertia for a body:

$$I_{\text{max,min}} = I_{\text{average}} \pm R = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + {I_{xy}}^2}$$