ENGR 311: Transform Calculus and Partial Differential Equations

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1 Laplace transform

1.1 Definition of the Laplace transform

If $f(t)$ is defined for $t \geqslant 0$, then the improper integral $\int_0^\infty K(s,t)f(t)\,\mathrm{d}t$, is defined as a limit:

$$\int_0^\infty K(s,t)f(t)\,\mathrm{d}t = \lim_{b\to\infty}\int_0^b K(s,t)f(t)\,\mathrm{d}t$$

If the limit exists, the integral is said to exist or to be convergent; if the limit does not exist the integral does not exist and is said to be divergent. The limit will generally exist for only certain values of the variable s. The choice $K(s,t) = e^{-st}$ gives us an especially important integral transform called Laplace transform and written

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t$$

where t is in the time domain defined for $\mathbb{R}_+$ and s in the frequency domain. The s-plane is not a real plane, but a complex plane.

The formula for integration by parts used in this class is:

$$\int f_1(t)f_2(t)\,\mathrm{d}t = f_1(t)\int f_2(t)\,\mathrm{d}t - \int\left(\frac{\mathrm{d}}{\mathrm{d}t}[f_1(t)]\int f_2(t)\,\mathrm{d}t\right)\mathrm{d}t$$

1.1.1 Laplace transforms of some basic functions

$$\begin{aligned}
\mathcal{L}\left\{1\right\} &= \frac{1}{s}\\
\mathcal{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}}, n \in \mathbb{N}_+^*\\
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a}, a \in \mathbb{R}\\
\mathcal{L}\left\{\sin kt\right\} &= \frac{k}{s^2+k^2}\\
\mathcal{L}\left\{\cos kt\right\} &= \frac{s}{s^2+k^2}\\
\mathcal{L}\left\{\sinh kt\right\} &= \frac{k}{s^2-k^2}\\
\mathcal{L}\left\{\cosh kt\right\} &= \frac{s}{s^2-k^2}
\end{aligned}$$

1.1.2 Sufficient conditions for existence

A function f is said to be of exponential order if there exist constants $c > 0$, $M > 0$, and $T > 0$ such that $|f(t)| \leqslant Me^{ct}$ for all $t > T$ (meaning for t big enough, the graph of $f(t)$ does not grow faster than the graph of Me^{ct}). If $f(t)$ is piecewise continuous on the interval $\mathbb{R}_+$ and of exponential order, then $\mathcal{L}\left\{f(t)\right\}$ exists for $s > c$.

1.1.3 Linearity of Laplace transform

Linear transform:

$$\mathcal{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha\mathcal{L}\left\{f(t)\right\} + \beta\mathcal{L}\left\{g(t)\right\}$$

1.2 The inverse transform and transforms of derivatives

If $F(s)$ represents the Laplace transform of a function $f(t)$, then $f(t)$ is the inverse Laplace transform of $F(s)$:

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$$

Inverse linear transform:

$$\mathcal{L}^{-1}\left\{\alpha F(s) + \beta G(s)\right\} = \alpha\mathcal{L}^{-1}\left\{F(s)\right\} + \beta\mathcal{L}^{-1}\left\{G(s)\right\}$$

1.2.1 Inverse Laplace transforms of some basic functions

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1\\
\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} &= t^n, n \in \mathbb{N}_+^*\\
\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at}, a \in \mathbb{R}\\
\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} &= \sin kt\\
\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} &= \cos kt\\
\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} &= \sinh kt\\
\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} &= \cosh kt
\end{aligned}$$

1.2.2 Transforms of derivatives

If f, f',…,$f^{(n-1)}$ are continuous on $\mathbb{R}_+$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $\mathbb{R}_+$, then

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Remark. The Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$, which will be used to solve differential equations.

1.2.3 The unit step function

The unit step function $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leqslant t < a \\ 1, & t \geqslant a \end{cases}$$

The unit step function is used to write a piecewise-defined function in a compact form:

$$f(t) = \begin{cases} g(t), & 0 \leqslant t < a \\ h(t), & a \leqslant t < b \\ i(t), & t \geqslant b \end{cases}$$

$$\begin{aligned} \iff f(t) &= g(t)\left[\mathcal{U}(t-0) - \mathcal{U}(t-a)\right] \\ &\quad + h(t)\left[\mathcal{U}(t-a) - \mathcal{U}(t-b)\right] \\ &\quad + i(t)\left[\mathcal{U}(t-b) - \mathcal{U}(t-\infty)\right] \\ &\quad + i(t)\left[\mathcal{U}(t-b)\right] \\ \iff f(t) &= g(t) + \mathcal{U}(t-a)\left[h(t) - g(t)\right] \\ &\quad + \mathcal{U}(t-b)\left[i(t) - h(t)\right] \end{aligned}$$

1.3 Translation theorems

1.3.1 Frequency shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$, then

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

1.3.2 Time shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as} F(s)$$

This leads to the Laplace transform of the unit step function being:

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

An alternative form of the time shift theorem is

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

1.4 Additional operational properties

1.4.1 Derivatives of transforms

If the function f is piecewise continuous on $\mathbb{R}_+$, of exponential order, and $\mathcal{L}\{f(t)\} = F(s)$, then for $n \in \mathbb{N}^*$ and $s > c$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$$

1.4.2 Convolution

If functions f and g are piecewise continuous on $\mathbb{R}_+$, then the convolution of f and g, denoted by the symbol $f * g$, is a function defined by the integral

$$f * g = \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau$$

Convolution theorem If the functions f and g are piecewise continuous on $\mathbb{R}_+$, of exponential order and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then for $s > c$:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

1.4.3 Transform of an integral

Using the convolution theorem with $g(t) = 1$, the Laplace transform of the integral of f is

$$\mathcal{L}\left\{\int_0^\tau f(\tau)\,\mathrm{d}\tau\right\} = \frac{F(s)}{s}$$

1.4.4 Periodic function

If a periodic function f has a period $T \in \mathbb{R}_+^*$, then $f(t+T) = f(t)$.

If $f(t)$ is piecewise continuous on $\mathbb{R}_+$, of exponential order, and periodic with period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^- st f(t)\,\mathrm{d}t$$

1.5 The dirac delta function

1.5.1 The unit impulse function

The unit impulse function is defined as:

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \leqslant t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \leqslant t < t_0 + a \\ 0, & t \geqslant t_0 + a \end{cases}$$

1.5.2 The dirac delta function

The dirac delta function is defined as:

$$\delta\left(t-t_0\right)=\lim_{a\to 0}\delta_a\left(t-t_0\right)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

Remark. The integral of the dirac delta function is:

$$\int_0^{\infty}\delta\left(t-t_0\right)\,\mathrm{d}t=1$$

The Laplace transform of the dirac delta function is:

$$\mathcal{L}\left\{\,\delta\left(t-t_0\right)\right\}=e^{-st_0}$$

1.6 Systems of linear differential equations

When initial conditions are specified, the Laplace transform reduces a system of linear differential equations to a set of simultaneous algebraic equations in the transformed functions.

This is applicable to coupled springs, electrical networks, double pendulums...

2 Orthogonal functions and fourier series

2.1 Orthogonal functions

2.1.1 The inner product

The inner product, also called dot product, of vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ is a scalar defined as

$$(\vec{u},\vec{v})=u_1v_1+u_2v_2+u_3v_3=\vec{u}\bullet\vec{v}$$

Properties of the inner product:

$$(\vec{u},\vec{v})=(\vec{v},\vec{u}) \tag{2.1}$$
$$(k\vec{u},\vec{v})=k\left(\vec{v},\vec{u}\right),k\in\mathbb{R} \tag{2.2}$$
$$(\vec{u},\vec{u})=0\iff\vec{u}=\vec{0} \tag{2.3}$$
$$(\vec{u}+\vec{v},\vec{w})=(\vec{u},\vec{w})+(\vec{v},\vec{w}) \tag{2.4}$$

The inner product of functions on the interval $[a,b]$ is defined as

$$(f_1,f_2)=\int_a^b f_1(x)f_2(x)\,\mathrm{d}x$$

2.1.2 Orthogonality

Two vectors or functions are said to be orthogonal if their inner product is zero. A set of real-valued functions $\{\phi_0(x),\phi_1(x),\phi_2(x),\dots\}$ is said to be orthogonal on $[a,b]$ if

$$(\phi_m,\phi_n)=\int_a^b\phi_m(x)\phi_n(x)\,\mathrm{d}x=0,m\neq n,\{m,n\}\in\mathbb{N}$$

The norm $\|\vec{u}\|$ of a vector u can be expressed using the inner product:

$$\|\vec{u}\|=\sqrt{(\vec{u},\vec{u})}$$

Similarly, the norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$, $n\in\mathbb{N}$, is

$$\|\phi_n(x)\|=\sqrt{\int_a^b\phi_n(x)^2\,\mathrm{d}x}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a,b]$ with the property that $\|\phi_n(x)\|=1$ for all $n\in\mathbb{N}$, then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval.

An orthogonal set is said to be a complete set if the only continuous function orthogonal to each member of the set is the zero function $f(x)=0$.

A set of real-valued functions $\{\phi_n(x)\}$, $n\in\mathbb{N}$ is said to be orthogonal with respect to a weight function $w(x)$ on $[a,b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\,\mathrm{d}x=0,m\neq n,\{m,n\}\in\mathbb{N}$$

A real valued function $f(x)$ is said to be periodic with $T\neq 0$ if $f(x+T)=f(x)$ for all x in the domain of x. The smallest value of T which satisfies this property is the fundamental period.

2.1.3 Gram-Schmidt orthogonalization process

For a linearly independent set of real-valued functions that are continuous on an interval $[a,b]$, an orthogonal can be made from this set using the Gram-Schmidt process. The Gram-Schmidt process for turning the set of linearly independent real-valued functions $f_n(x)$ $n\in\mathbb{N}$ into the orthogonal set $\phi_n(x)$ $n\in\mathbb{N}$ is

$$\begin{aligned}
\phi_0(x)&=f_0(x)\\
\phi_1(x)&=f_1(x)-\frac{(f_1,\phi_0)}{\|\phi_0(x)\|^2}\phi_0(x)\\
\phi_2(x)&=f_2(x)-\frac{(f_2,\phi_0)}{\|\phi_0(x)\|^2}\phi_0(x)-\frac{(f_2,\phi_1)}{\|\phi_1(x)\|^2}\phi_1(x)\\
\phi_n(x)&=f_n(x)-\sum_{k=0}^{n-1}\frac{(f_n,\phi_k)}{\|\phi_k(x)\|^2}\phi_k(x)
\end{aligned}$$

2.1.4 Orthogonal series expansion

An orthogonal series expansion of f or a generalized Fourier series is

$$f(x) = \sum_{n=0}^{\infty} \frac{\int_a^b f(x)\phi_n(x)\,\mathrm{d}x}{\|\phi_n(x)\|^2}\phi_n(x) = \sum_{n=0}^{\infty} \frac{(f,\phi_n)}{\|\phi_n(x)\|^2}\phi_n(x)$$

2.2 Fourier series

The Fourier series of a function f defined on the interval $(-p,p)$is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,\mathrm{d}x \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,\mathrm{d}x \end{aligned}$$

Conditions of convergence where $f(x)$ and $f'(x)$are piecewise continuous on the interval $(-p,p)$:

- At a point of continuity: the Fourier series converges to $f(x)$.
- At a point of discontinuity: the Fourier series converges to $\frac{f(x^-)+f(x^+)}{2}$, where $f(x^-) = \lim_{a\to x^-} f(a)$ and $f(x^+) = \lim_{a\to x^+} f(a)$.

2.3 Fourier cosine and sine series

2.3.1 Odd and even functions

Odd: a function is said to be an odd function of x if $f(-x) = -f(x)$ (symmetry about the origin);
Even: a function is said to be an even function of x if $f(-x) = f(x)$ (symmetry about the y-axis).

Properties of even and odd functions:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.
- The sum (difference) of two even functions is even.
- The sum (difference) of two odd functions is odd.
- If f is even, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

- If f is odd, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Special Fourier series:

Even function: The Fourier series of an even function $(-p,p)$ is the cosine series ($b_n = 0$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,\mathrm{d}x \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,\mathrm{d}x$$

Odd function: The Fourier series of an odd function $(-p,p)$ is the sine series ($a_0 = a_n = 0$):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,\mathrm{d}x$$

2.3.2 Half-range expansion

If the range of a function is given on $(0,L)$ rather than $(-p,p)$, an arbitrary definition of f on $(-L,0)$ can be made:

1. Reflect the graph of the function about the y-axis onto $(-L,0)$ so the function is even on $(-L,L)$. This results in doing the half-range cosine expansion of f by taking $p = L$ and using the cosine series.
2. Reflect the graph of the function about the origin onto $(-L,L)$ so the function is odd on $(-L,L)$. This results in doing the half-range sine expansion of f by taking $p = L$ and using the sine series.
3. Define f on $(-L,L)$ by identity reflection (translation), $f(x) = f(x+L)$. By taking $p = \frac{L}{2}$ and using the classic Fourier series, this results in doing the periodic extension of f .

3 Boundary-value problems in rectangular coordinates

3.1 Separable partial differential equations (PDEs)

The general form of a linear second-order partial differential equation is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A, B, C, D, E, F and G are constants in $\mathbb{R}$ or functions of x and y

Remark. If $G = 0$, the equation is called a homogeneous equation; otherwise, it is nonhomogeneous .

A solution of a linear PDE is a function of the two independent variables possessing all partial derivatives occurring in the equation and satisfying the equation in some region of the xy-plane.

3.1.1 Separation of variables

The method of separation of variables can be used to find a particular solution u of the form of a product of function of x and y, where

$$u(x,y) = X(x)Y(y)$$
$$\implies \frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$
$$\implies \frac{\partial^2 u}{\partial x^2} = X''Y \qquad \frac{\partial^2 u}{\partial y^2} = XY'' \qquad \frac{\partial^2 u}{\partial x \partial y} = X'Y'$$

With this assumption, is is sometimes possible to reduce a linear PDE in two variables to two ODEs.

Separation of variables is not a general method for finding particular solutions as some linear partial differential equations are simply not separable.

3.1.2 Superposition principle

If u_1, u_2, ..., u_k are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k = \sum_{i=1}^{k} c_i u_i$$

where c_i, $i = 1, 2, \ldots, k$ are constants in $\mathbb{R}$, is also a solution.

3.1.3 Classification of equations

The general linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A, B, C, D, E, F and G are constants in $\mathbb{R}$, is said to be

hyperbolic if $B^2 - 4AC > 0$
parabolic if $B^2 - 4AC = 0$
elliptic if $B^2 - 4AC < 0$

3.2 Classical PDEs and boundary-value problems

The classical second-order partial differential equations which have important applications and mathematical physics are:
The one-dimensional heat equation:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k \in \mathbb{R}_+^*$$

The one-dimensional wave equation:

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The two-dimensional Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since the heat and wave equations depend on t, there exists initial conditions at $t = 0$.
Moreover, one of the following boundary conditions can be used to define the behavior at boundaries:
Dirichlet condition: u.
Neumann condition: $\frac{\partial u}{\partial x}$.
Robin condition: $\frac{\partial u}{\partial x} + hu$.

3.2.1 Solving the classical PDEs and boundary-value problems

Sturm-Liouville problem types Sturm-Liouville I:

$$u''(x) + \lambda u(x) = 0, \quad u(0) = 0 \text{ and } u(L) = 0$$

Non-trivial solution exists only for $\lambda = \alpha^2 > 0$: the non-trivial solution is $u_n(x) = \sin \alpha x = \sin \frac{n\pi}{L} x$.

Sturm-Liouville II:

$$u''(x) + \lambda u(x) = 0, \quad u'(0) = 0 \text{ and } u'(L) = 0$$

Non-trivial solutions exist for
$\lambda = 0$**:** the non-trivial solution is $u_0(x) = 1$.
$\lambda = \alpha^2 > 0$**:** the non-trivial solution is $u_n(x) = \cos \frac{n\pi}{L} x$.

Sturm-Liouville III:

$$u''(x) + \lambda u(x) = 0, \quad u(0) = 0 \text{ and } u'(L) = 0$$

Non-trivial solution exists only for $\lambda = \alpha^2 > 0$: the non-trivial solution is $u_n(x) = \sin \alpha x = \sin \frac{(2n-1)\pi}{2L} x$.

Sturm-Liouville IV:

$$u''(x) + \lambda u(x) = 0, \quad u'(0) = 0 \text{ and } u(L) = 0$$

Non-trivial solution exists only for $\lambda = \alpha^2 > 0$: the non-trivial solution is $u_n(x) = \cos \alpha x = \cos \frac{(2n-1)\pi}{2L} x$.

General method steps

1. Apply the change of variable $u(x,y) = X(x)Y(y)$.
2. Substitute the partial derivatives by the product of derivatives of $X(x)$ and $Y(y)$.
3. Write the equation in fraction form with all $X(x)$ and its derivatives on one side and all $Y(y)$ and its derivatives one the other side.
4. Equal those fractions to the separation variable $-\lambda$.
5. Write the ordinary differential equations using $-\lambda$.
6. Solve for $X(x)$ and $Y(y)$ using their ordinary differential equation and $-\lambda$
   - In the case of a first-order linear ordinary differential equation of the form $X'(x) + P(x)X = f(x)$, the solution is:

$$X(x) = e^{-\int P(x)\,\mathrm{d}x} \int e^{\int P(x)\,\mathrm{d}x} f(x)\,\mathrm{d}x$$
$$f(x) = 0 \implies X(x) = Ce^{-\int P(x)\,\mathrm{d}x}, \quad C \in \mathbb{R}^*$$

- In the case of a second-order linear ordinary differential equation of the form $X'' + P(\lambda)X = 0$, it must be solved for non-trivial solutions ($X(x) \neq 0$) with 3 cases in which the constants C_i are solved the homogeneous boundary conditions (or use Sturm-Liouville problem types):
  $P(\lambda) = 0$: $X(x) = C_1 x + C_2, \quad \{C_1, C_2\} \in \mathbb{R}$
  $P(\lambda) = -\alpha^2 < 0$: $X(x) = C_3 \cosh \alpha x + C_4 \sinh \alpha x, \quad \{C_3, C_4\} \in \mathbb{R}$
  $P(\lambda) = \alpha^2 < 0$: $X(x) = C_5 \cos \alpha x + C_6 \sin \alpha x, \quad \{C_5, C_6\} \in \mathbb{R}$ (Do not forget that $\sin n\pi = 0$ and $\cos(2n+1)\frac{\pi}{2} = 0$, $n \in \mathbb{N}$)
- In the case of another form of second-order ordinary differential equation, use other methods such as the auxiliary equation method.

7. Form the product solution $u_n(x,t) = X(x)Y(y)$.
8. Combine the constants C_i as A_0, A_n and B_n (functions of n).
9. Apply the superposition principle $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$.
10. Use initial and boundary conditions and Fourier series expansion to solve for A_0, A_n and B_n.
11. Obtain the complete solution $u(x,t) = X(x)Y(y)$ by substituting A_0, A_n and B_n into the solution.

3.3 Heat equation

Suppose a rod of length L has a circular cross-sectional area A and coincides with the x-acis on the interval $[0, L]$. The following assumption are made:

- The flow of heat within the rod takes place only in the x-direction.
- The lateral, or curved, surface of the rod is insulated, meaning no heat escapes from this surface.
- Not heat is being generated withing the rod.
- The rod is homogeneous, meaning its density ρ is constant.
- The specific heat γ and thermal conductivity K of the material of the rod are constants.

The governing equation for this problem is:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0, 0 \leqslant x \leqslant L, t \geqslant 0$$

which is subjected to the following conditions:
Boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$, $t \geqslant 0$.
Initial conditions: $u(x,0) = f(x)$, $0 \leqslant x \leqslant L$.

Using the general method steps for this problem, the complete solution of the one-dimensional heat equation is:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \,\mathrm{d}x \right) e^{-kt\left(\frac{n\pi}{L}\right)^2} \sin \frac{n\pi}{L} x$$

3.4 Wave equation

Consider a string of length L stretched taut between two point on the x-axis at $x = 0$ and $x = L$. The following assumptions are made:

- The string is perfectly flexible.
- The string is homogeneous, meaning its density ρ is constant.
- The displacement u are small compared to the length of the string.
- The slope of the curve is small at all points.
- The tension $\vec{T}$ acts tangent to the string, and its magnitude T is the same at all points.
- The tension is large compared with the force of gravity.
- No other external forces act on the string.

The governing equation for this problem is:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 \leqslant x \leqslant L, t \geqslant 0$$

which is subjected to the following conditions:
Boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$, $t \geqslant 0$.
Initial conditions: $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}\big|_{t=0} = g(x)$, $0 \leqslant x \leqslant L$

Using the general method steps for this problem, the complete solution of the one-dimensional wave equation is:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(\frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \,\mathrm{d}x \right) \cos \frac{n\pi a}{L} t + \left(\frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x \,\mathrm{d}x \right) \sin \frac{n\pi a}{L} t \right] \sin \frac{n\pi}{L} x$$

3.5 Laplace's equation

Laplace's equation in two and three dimensions occurs in time-independent problems involving potentials such as electrostatic, gravitational and velocity in fluid mechanics. A solution of Laplace's equation can also be interpreted as a steady-state temperature distribution.

Suppose we wish to find the steady-state temperature $u(x,y)$ in a rectangular plate whose vertical edges are insulated and whose lower and upper edges are maintained at temperatures 0 and $f(x)$

The governing equation for this problem is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$$

which is subjected to the following conditions:
Boundary conditions: $\frac{\partial u}{\partial x}\big|_{x=0} = 0$, $\frac{\partial u}{\partial x}\big|_{x=a} = 0$, $0 \leqslant y \leqslant b$.
Initial conditions: $u(x,0) = 0$, $u(x,b) = f(X)$, $0 \leqslant x \leqslant a$

Using the general method steps for this problem, the complete solution of the two-dimensional Laplace's equation is:

$$u(x,y) = \frac{y}{ab} \int_0^a f(x) \,\mathrm{d}x + \frac{2}{a} \sum_{n=1}^{\infty} \frac{1}{\sinh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi}{a} \,\mathrm{d}x \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x$$

3.5.1 Dirichlet problem

A Dirichlet problem is a boundary-value problem in which a solution to an elliptic partial differential equation within a region R is seek, such that all boundary conditions are Dirichlet conditions.

A Dirichlet problem for a rectangle can be solved by separation of variables when homogeneous boundary conditions are specified on two parallel boundaries. If the boundary conditions are nonhomogeneous on at least two perpendicular sides, the boundary-value problem is split into two problems, each of which has a homogeneous boundary conditions on parallel boundaries.

3.6 Nonhomogeneous boundary-value problems

A boundary-value problem is said to be nonhomogeneous if either the partial differential equation or the boundary conditions are nonhomogeneous.

The method of separation of variable does not always work.

3.6.1 Time-independent

A change of the dependent variable is employed

$$u(x,t) = v(x,t) + \psi(x)$$

This transforms a nonhomogeneous partial differential equation boundary value problem into one involving an ordinary differential equation and the another involving a homogeneous partial differential equation, solvable by separation of variables.

General method steps

1. Obtain the governing equation, and the boundary and initial conditions from the problem statement.
2. Apply the change of dependent variable $u(x,t) = v(x,t) + \psi(x)$.
3. Substitute the change of dependent variable into the governing equation, and the boundary and initial conditions.
4. Separate the problem into 2 sub-problems.
5. Solve the ordinary differential equation for $\psi(x)$ using the boundary conditions.
6. Solve the partial differential equation for $v(x,t)$ using the general method (Section 3.2.1 on page 5).
7. Obtain the complete solution using $u(x,t) = v(x,t) + \psi(x)$.

3.6.2 Time-dependent

A change of the dependent variable is employed

$$u(x,t) = v(x,t) + \psi(x,t)$$

$$\text{where: } \psi(x,t) = u(0,t) + \frac{x}{a}\left[u(a,t) - u(0,t)\right]$$

This transforms a nonhomogeneous partial differential equation boundary value problem into a nonhomogeneous partial differential equation with homogeneous boundary and initial conditions.

General method steps

1. Obtain the governing equation, and the boundary and initial conditions from the problem statement.
2. Apply the change of variable $u(x,t) = v(x,t) + \psi(x,t)$.
3. Find $\psi(x,t) = u(0,t) + \frac{x}{a}\left[u(a,t) - u(0,t)\right]$.
4. Substitute the change of dependent variable into the governing equation, and the boundary and initial conditions.
5. Write v terms on the left and ψ terms on the right.
6. Take the ψ terms on the right side as $-G(x,t)$.
7. Solve for $v(x,t)$ (since $\psi(x,t)$ is already known) by assuming $v(x,t)$ and $G(x,t)$ can be written as Fourier sine or cosine series for a fixed t (the functions are in terms of $t$and n only inside the infinite sums as x appears in the sine/cosine term).
8. Solve for the coefficient $G_n(t)$.
9. Find the partial derivatives of v in their sine/cosine expansion form.
10. Substitute those sine/cosine expansion into the governing equation as well as the one of $G(x,t)$.
11. Equate the coefficients $v_n(t)$ and $G_n(t)$, providing that the sine/cosine term are the same.
12. Solve for $v_n(t)$
13. Obtain the complete solution using $u(x,t) = v(x,t) + \psi(x,t)$.