

# ENGR 213: Applied Ordinary Differential Equations

Anthony Bourboulas

March 19, 2022

## 1 Introduction to differential equations

### 1.1 Definitions and terminology

A differential equation (differential equation) is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.

Differential equations can be classified by:

- **Type:**
  - Ordinary differential equations (ODE): derivatives are with respect to a single independent variable.

*Example.*

$$\frac{dy}{dx} + 5y = e^x \text{ or } \frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y$$

- Partial differential equations (PDE): derivatives are with respect to two or more independent variables.

*Example.*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2 \frac{\partial u}{\partial t} \text{ or } \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

- **Order:** the order of an ordinary differential equation or a partial differential equation is the order of the highest derivative in the equation

*Example.*

$$\begin{aligned} \frac{dy}{dx} + 5y &= e^x \rightarrow \text{order 1} \\ 2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} &= 0 \rightarrow \text{order 4} \end{aligned}$$

- **Linearity:**
  - Form of a linear ordinary differential equation:

$$\begin{aligned} a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots \\ + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \end{aligned}$$

$$\begin{aligned} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots \\ + a_1(x)y' + a_0(x)y = g(x) \end{aligned}$$

- Non-linear ordinary differential equation:
  - \* The coefficient  $a_0(x), a_1(x), \dots, a_{n-1}(x), a_n(x)$  contains the dependent variable  $y$  or its derivative
  - \* Powers of  $y, y', \dots, y^{(n-1)}, y^{(n)}$  appears in the equation
  - \* Non-linear functions of the dependent variable or its derivatives, such as  $\sin y$  or  $e^y$ , appear in the equation

- **Homogeneity:**
  - Homogeneous:  $g(x) = 0$  in a linear ordinary differential equation or  $\frac{dy}{dx}$  can be represented as  $f\left(\frac{y}{x}\right)$
  - Nonhomogeneous:  $g(x)$  is a constant or a function of  $x$  in a linear ordinary differential equation or  $\frac{dy}{dx}$  cannot be represented as  $f\left(\frac{y}{x}\right)$

Solutions of a ordinary differential equation:

- **Solution curve:** the solution is a function.
  - Families of solutions: solution contains one or more arbitrary constant  $c, c_1, \dots, c_n$
  - Particular solution: solution does not contain an arbitrary constant
- Singular solution: solution which is not member of a family of solutions
- Trivial solution:  $y(x) = 0$

**Definition.** If a linear homogeneous ordinary differential equation has solutions  $y = f(x)$  and  $y = g(x)$ , then  $y = af(x) + bg(x)$  (where  $a$  and  $b$  are constants) is also a solution.

### 1.2 Initial-value problem (IVP)

In an IVP, we seek a solution  $y(x)$  of a differential equation so that  $y(x)$  satisfies one or more initial conditions at  $x_0$  in order to get a particular solution.

**Existence of a unique solution** If  $\frac{dy}{dx} = f(x, y)$  and  $\frac{\partial f}{\partial y}$  are continuous a region  $R$ , then there exists a unique function  $y(x)$  that is a solution of the IVP.

### 1.3 Differential equations as mathematical models

A mathematical model is a description of a system or a phenomenon.

Steps of the modeling process:

1. Make assumptions and hypotheses about the system or phenomenon
2. Express these assumptions in terms of differential equations
3. Get a mathematical formulation
4. Solve the differential equations
5. Obtain solutions
6. Display predictions of the model (e.g. graphically)
7. Check the model's predictions with known facts
8. If necessary alter assumptions or increase resolution of the model

*Example.* • Population dynamics

- Radioactive decay
- Newton's law of cooling and warming
- Spread of a disease
- Chemical reactions
- Mixtures
- Draining a tank
- Series circuits
- Falling bodies
- Falling bodies and air resistance
- Suspended cables
- ... and many more

## 2 First-order differential equations

### 2.1 Separable equations

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables.

A separable first-order differential equation can be rewritten in the form  $h(y)dy = g(x)dx$ , which can be solved by integrating both sides.

### 2.2 Linear equations

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation in the dependent variable  $y$ .

The standard form of a linear differential equation is obtained by dividing both sides by the lead coefficient  $a_1(x)$ :

$$\frac{dy}{dx} + P(x)y = f(x)$$

Method of solution:

1. Rewrite the equation in its standard form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

2. Determine the integrating factor  $e^{\int P(x) dx}$
3. Write

$$\frac{d}{dx} \left[ e^{\int P(x) dx} y \right] = e^{\int P(x) dx} f(x)$$

4. Multiply both sides by  $dx$
5. Integrate both sides
6. Divide both sides by the integrating factor  $e^{\int P(x) dx}$  to isolate for  $y$

Using these steps, the solution is:

$$y = e^{-\int P(x) dx} \int e^{\int P(x) dx} f(x) dx$$

### Piecewise-linear differential equation

1. Solve the initial-value problem in several parts corresponding to the intervals over which  $f(x)$  is defined
2. Piece the two solutions together so that the solution  $y(x)$  is continuous on a larger interval

### Error functions

Error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Complementary error function:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

Property of the error functions:

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$$

### 2.3 Exact equations

A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an exact equation if the expression on the left side is an exact differential.

The condition of exact differentials is:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

Method of solution:

1. Check for exact differentials
2. If the equation is exact, there exists a function  $f(x, y)$  such that:

$$\frac{\partial f(x, y)}{\partial x} = M(x, y) \text{ and } \frac{\partial f(x, y)}{\partial y} = N(x, y)$$

3. Find  $f(x, y)$  by integrating  $M(x, y)$  with respect to  $x$ , while holding  $y$  constant. This gives:

$$f(x, y) = \int M(x, y) dx + g(y)$$

where an arbitrary function  $g(y)$  is the "constant" of integration

4. Differentiate  $f(x, y)$  with respect to  $y$  and set it equals to  $N(x, y)$ :

$$\frac{\partial f(x, y)}{\partial y} = \frac{\partial}{\partial y} \left[ \int M(x, y) dx \right] + g'(y) = N(x, y)$$

5. This gives:

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \left[ \int M(x, y) dx \right]$$

6. Integrate  $g'(y)$  with respect to  $y$
7. Substitute the result in  $f(x, y) = \int M(x, y) dx + g(y)$
8. The implicit solution of  $M(x, y)dx + N(x, y)dy = 0$  is  $f(x, y) = c$

Therefore, if the equation is exact, the full solution be-

comes:

$$f(x, y) = \int M(x, y) dx + \int N(x, y) - \frac{\partial}{\partial y} \left[ \int M(x, y) dx \right] dy = c$$

### Integrating factor for exact equations

An equivalent exact equation can be created from a non-exact equation using an integrating factor  $\mu(x, y)$ :

1. Check for integrating factor:
  - If  $\frac{N_x - M_y}{M}$  is a function of  $x$  alone, then an integrating factor for  $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$  is:

$$\mu(x) = e^{\int \frac{M_y - N_x}{M} dx}$$

- If  $\frac{N_x - M_y}{N}$  is a function of  $y$  alone, then an integrating factor for  $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$  is:

$$\mu(y) = e^{\int \frac{N_x - M_y}{N} dy}$$

2. If there exists an integrating factor  $\mu(x, y)$ , then the new exact equation is

$$[\mu(x, y)M(x, y)] dx + [\mu(x, y)N(x, y)] dy = 0$$

3. Apply the method of solution of an exact equation taking  $M(x, y) = \mu(x, y)M(x, y)$  and  $N(x, y) = \mu(x, y)N(x, y)$

## 2.4 Solution by substitution (Bernoulli, homogeneous and linear substitution)

Substitution is used to get a differential equation in a form such that a known procedure can be used to find a solution. The chain rule is often used as a tool to get the differential form:

1. Let a function  $u(x, y)$ , then  $y = g(x, u)$  and
 
$$\frac{dy}{dx} = \frac{\partial g(x, u)}{\partial x} + \frac{\partial g(x, u)}{\partial u} \frac{du}{dx}$$
2. Solve for  $u$  using  $\frac{du}{dx}$  and another method (separable, linear or exact equations)
3. Replace back  $u$  in  $y = g(x, u)$  to solve for  $y$

### Homogeneous equations

In  $M(x, y)dx + N(x, y)dy = 0$ , if  $M(tx, ty) = t^\alpha M(x, y)$  and  $N(tx, ty) = t^\alpha N(x, y)$ , then the equation is homogeneous.

Method of solution:

1. Let  $y = ux \Rightarrow dy = udx + xdu$
2. Replace  $y$  and  $dy$  in the equation by  $ux$  and  $udx + xdu$  respectively
3. Solve for  $u$  using another method (separable, linear or exact equations)
4. Replace back  $u$  by  $\frac{y}{x}$
5. Solve to get an implicit or an explicit solution

### Bernoulli's equation

A Bernoulli's equation is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \in \mathbb{R}$$

Method of solution:

1. Let  $u = y^{1-n}$ . The equation becomes:

$$\frac{du}{dx} + (1-n)P(x)u = (n-1)f(x)$$

2. Solve for  $u$  using the linear equation method
3. Solve for  $y$  by using  $y = u^{\frac{1}{1-n}}$

### Reduction to separation of variables

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substitution  $u = Ax + By + C$  if  $B \neq 0$ .

Method of solution:

1. Let  $u = Ax + By + C \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$ . The equation becomes:

$$\frac{du}{dx} = A + B \cdot f(u)$$

2. Solve for  $u$  using another method (separable, linear or exact equations)
3. Solve for  $y$  using  $y = \frac{u - Ax - C}{B}$

## 2.5 Linear models

### Growth and decay

For growth and decay, the general initial value problem is

$$\frac{dx}{dt} = kx, x(t_0) = x_0$$

where  $k$  is the constant of proportionality.

The general solution of this differential equation is  $x(t) = Ce^{kt}$ .

When  $t_0 = 0$ ,  $C = x_0$  and the solution of the differential equation becomes  $x(t) = x_0 e^{kt}$ . In order to get the value of  $k$ , another value of  $x(t)$  needs to be known.

**Half-life** The half-life is a measure of the stability of a radioactive substance. It is the time it takes for one-half of the atoms in an initial amount  $A_0$  to disintegrate.

In half-life value problems, the solution of the differential equation is  $A(t) = A_0 e^{kt}$  with  $k = -\frac{\ln 2}{t_{\text{half-life}}}$ .

One of the special value of half-life to know is the half-life of carbon - 14 which is  $t_{C-14} = 5730$  years.

### Newton's law of cooling and warming

Newton's empirical law of cooling and warming states that the rate  $\frac{dT}{dt}$  at which the temperature of a body changes is proportional to the difference between the temperature  $T$  of the body and the temperature  $T_m$  of the surroundings, i.e. the ambient temperature:

$$\frac{dT}{dt} = k(T - T_m), T(t_0) = T_0$$

where  $k$  is a constant of proportionality.

The general solution of this differential equation is  $T(t) = T_m + Ce^{kt}$ .

When  $t_0 = 0$ ,  $C = T_0 - T_m$  and the solution of the differential equation becomes  $T(t) = T_m + (T_0 - T_m)e^{kt}$ . In order to get the value of  $k$ , another value of  $T(t)$  needs to be known.

### Series circuits

**RL circuits** The instantaneous voltage  $v$  in a RL circuit is described by the following differential equation:

$$L \frac{dI}{dt} + RI = V$$

**RC circuits** The instantaneous voltage  $v$  in a RC circuit is described by the following differential equation:

$$R \frac{dQ}{dt} + \frac{1}{C}Q = V$$

**RLC circuits** The instantaneous voltage  $v$  in a RLC circuit is described by the following differential equation:

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = V$$

## 2.6 Non-linear models

### Logistic equation

Let an environment be capable of sustaining no more than a fixed number of  $P_{\max}$  individuals in its population. The quantity  $P_{\max}$  is called the carrying capacity of the environment. Let  $P_0$  be the initial population. Then, using a density-dependent model of population dynamics, we get the logistic equation as

$$\frac{dP}{dt} = P(a - bP)$$

where  $a = kP_{\max}$ ,  $b = k$  and  $k$  a constant of proportionality.

From this equation, it follows that

$$\begin{aligned} P(t) &= \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}} \\ &= \frac{P_0P_{\max}}{P_0 + (P_{\max} - P_0)e^{-kP_{\max}t}} \end{aligned}$$

To get the graph of  $P(t)$ , the second derivative of  $P$  can

be used:

$$\begin{aligned} \frac{d^2P}{dt^2} &= 2b^2P \left(P - \frac{a}{b}\right) \left(P - \frac{a}{2b}\right) \\ &= 2 \left(\frac{P_0}{P_{\max}}\right)^2 P(P - P_{\max}) \left(P - \frac{P_{\max}}{2}\right) \end{aligned}$$

### Modification of the logistic equation

There exist many variations of the logistic equation such as

$$\frac{dP}{dt} = P(a - bP) \pm h$$

where  $h$  is a rate of increasing or decreasing in the population.

Another variation of the logistic equation, known as the Gompertz differential equation, is stated as:

$$\frac{dP}{dt} = P(a - b \ln P)$$

This differential equation is used as a model in the study of the growth or decline of population, growth of solid tumors...

### Chemical reactions

Let a quantity  $a$  of chemical  $A$  and  $b$  of chemical  $B$ . If a reaction needs  $M$  part of  $A$  and  $N$  parts of  $B$  to form a quantity  $X$  of a chemical  $C$ , then the quantity of  $A$  and  $B$  remaining at any time are, respectively,

$$a - \frac{M}{M+N}X \text{ and } b - \frac{N}{M+N}X$$

The rate at which chemical  $C$  is produced can be governed by a second-order differential equation of the following

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

with  $\alpha = a \frac{M+N}{M}$  and  $\beta = b \frac{M+N}{N}$ .

Method of solution:

1. Compute  $\alpha = a \frac{M+N}{M}$  and  $\beta = b \frac{M+N}{N}$
2. Solve the differential equation using separable equation and integration of partial fractions
3. Solve the constant  $C$  of integration using  $X_0 = 0$
4. Solve for the constant  $k$  of proportionality of the reaction using a known value of  $X(t)$
5. The maximum quantity  $X_{\max}$  of chemical  $C$  is equal to the smallest value between  $\alpha$  and  $\beta$

**Leaking tank****3 Higher-order differential equations****3.1 Theory of linear equations**

A linear differential equation is an equation of the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

**Initial-value problem**

For a linear equation, an  $n$ th-order initial-value problem (IVP) is a problem where the linear equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

is solved using the initial conditions

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

**Existence of a unique solution** Let  $a_n(x)$ ,  $a_{n-1}(x)$ ,  $\dots$ ,  $a_1(x)$ ,  $a_0(x)$  and  $g(x)$  be continuous on an interval  $I$ , and let  $a_n(x) \neq 0$  for every  $x$  in this interval. If  $x = x_0$  is any point in this interval, then a solution  $y(x)$  of the initial-value problem exists on the interval and is unique.

**Boundary-value problem**

Another type of problem, called a boundary value problem (BVP) consists of solving a linear differential equation in which the dependent variable  $y$  or its derivatives are specified at different points. The general boundary conditions for a two-point boundary-value problem are

$$\begin{aligned} A_1 y(a) + B_1 y'(a) &= C_1 \\ A_2 y(b) + B_2 y'(b) &= C_2 \end{aligned}$$

**Homogeneous equations**

A linear  $n$ th order differential equation of the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

is said to be homogeneous, whereas an equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

with  $g(x) \neq 0$  is said to be nonhomogeneous.

In order to solve a nonhomogeneous linear equation, the associated homogeneous equation must first be solvable and solved.

**Differential operator**

Differential operator  $D$  transforms differentiable functions into other functions:

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = D(Dy) = D^2 y$$

and in a general manner

$$\frac{d^n y}{dx^n} = D^n y$$

Any linear differential equation can be expressed in terms of  $D$ .

*Example.*

$$y'' + 5y' + 6y = 5x - 3 \implies (D^2 + 5D + 6)y = 5x - 3$$

**Superposition principle**

**Linear dependence and independence** A set of functions  $f_1(x)$ ,  $f_2(x)$ ,  $\dots$ ,  $f_n(x)$  is said to be linearly dependent on an interval  $I$  if there exist constants  $c_1$ ,  $c_2$ ,  $\dots$ ,  $c_n$  not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every  $x$  in the interval. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent. Any set of  $n$  linearly independent solutions of the homogeneous linear differential equation on interval  $I$  is a fundamental set of solutions.

**Superposition principle for homogeneous equations** Let  $y_1$ ,  $y_2$ ,  $\dots$ ,  $y_k$  be solutions of a homogeneous  $n$ th-order differential equation on an interval  $I$ . Then, the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

where  $c_1$ ,  $c_2$ ,  $\dots$ ,  $c_k$  are arbitrary constants, is also a solution on the interval.

**Solutions of differential equations**

**Wronskian** Suppose each of the functions  $f_1(x)$ ,  $f_2(x)$ ,  $\dots$ ,  $f_n(x)$  possesses at least  $n - 1$  derivatives. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

where the primes denote derivatives, is called the Wronskian of the functions.

**Linearly independent solutions** Let  $y_1$ ,  $y_2$ ,  $\dots$ ,  $y_n$  be  $n$  solutions of a homogeneous  $n$ th-order differential

equation on an interval  $I$ . Then, the set of solutions is linearly independent on  $I$  if and only if  $W(y_1, y_2, \dots, y_n) \neq 0$  for every  $x$  in the interval.

The general solution of a homogeneous differential equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

where  $y_1, y_2, \dots, y_n$  is a fundamental set of solutions and  $c_1, c_2, \dots, c_k$  are arbitrary constants.

The general solution of a nonhomogeneous differential equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p$$

where  $y_1, y_2, \dots, y_n$  is a fundamental set of solutions,  $c_1, c_2, \dots, c_k$  are arbitrary constants and  $y_p$  is a particular solution of the differential equation.

### Another superposition principle

**Superposition principle for nonhomogeneous equations** Let  $y_{p1}, y_{p2}, \dots, y_{pk}$  be  $k$  particular solutions of a nonhomogeneous linear  $n$ th-order differential equation on an interval  $I$  corresponding, in turn, to  $k$  distinct functions  $g_1, g_2, \dots, g_n$ . That is, suppose that  $y_{p_i}$  denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g_i(x)$$

where  $i = 1, 2, \dots, k$ . Then

$$y_p(x) = y_{p1} + y_{p2} + \dots + y_{pk}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x)$$

## 3.2 Homogeneous linear equations with constant coefficients

An homogeneous linear equation with constant coefficients has the following form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where the coefficients  $a_0, a_1, \dots, a_{n-1}, a_n$  are real constants and  $a_n \neq 0$ .

The auxiliary equation of a homogeneous linear differential equation is used to solve the differential equation it is associated with:

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0$$

where  $m$  is used to find a solution for the homogeneous linear differential equation.

The three possible cases for the value of  $m$  are:

**Case I: distinct real roots** For each real root, there is a solution  $y = ce^{mx}$

**Case II: repeated real roots** For each repeated root, there is a solution  $y = c_1 e^{mx} + c_2 x e^{mx} + \dots + c_k x^{k-1} e^{mx}$ , where  $m$  is a root of multiplicity  $k$  (i.e.  $k$  roots are equal to  $m$ ).

**Case III: conjugate complex roots** For each conjugate complex root  $m = \alpha \pm i\beta$ , there is a solution  $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Method of solution:

1. Build the auxiliary equation associated with the differential equation
2. Solve for all values of  $m$
3. Follow the rule for the three possible cases of  $m$
4. Sum all the solutions in order to get the full solution  $y_c$

The most difficult step to solve the auxiliary equation is to find some roots in order to reduce the equation to a factorized form with only 2nd order terms.

Method of solution:

1. Find all the multiple (positive and negative) of the last coefficient  $a_0$  as  $p$
2. Find all the multiple (positive and negative) of the first coefficient  $a_n$  as  $q$
3. Build the values  $\frac{p}{q}$  which are possible solutions of the auxiliary equation
4. Test the values  $\frac{p}{q}$  until finding one that solves the auxiliary equation
5. Factor  $m - \frac{p}{q}$  of the equation using long division
6. Repeat the process until having only 2nd order terms

### Two equations worth knowing

Two differential equations

$$y'' + k^2 y = 0 \text{ and } y'' - k^2 y = 0, \quad k \in \mathbb{R}$$

The general solution of  $y'' + k^2 y = 0$  is

$$y_c = c_1 \cos kx + c_2 \sin kx$$

and the general solution of  $y'' - k^2 y = 0$  is

$$y_c = c_1 e^{kx} + c_2 e^{-kx} = c_3 \cosh kx + c_4 \sinh kx$$

## 3.3 Undetermined coefficients

To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

two things must be done:

1. Find the complementary function  $y_c$
2. Find any particular solution  $y_p$  of the nonhomogeneous equation
3. Sum  $y_c$  and  $y_p$  to get the general solution  $y = y_c + y_p$

The method of undetermined coefficients is limited to nonhomogeneous linear differential equations where

- The coefficients  $a_0, a_1, \dots, a_{n-1}, a_n$  are real constants
- $g(x)$  is a linear combination of functions of the type

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$P(x)e^{\alpha x}$$

$$P(x)e^{\alpha x} \cos \beta x$$

$$P(x)e^{\alpha x} \sin \beta x$$

where  $n \in \mathbb{R}^+$  and  $(\alpha, \beta) \in \mathbb{R}$

There are several cases to find  $y_p$ :

**Case I** No function in  $y_p$  is found in  $y_c$

**Case II** A function in  $y_p$  is found in  $y_c$

The form of  $y_p$  can be found using the following table.

Table 1: Trial particular solutions

$g(x)$	Form of $y_p$
Constant	$A$
$a_n x^n + \dots + a_1 x + a_0$	$Ax^n + \dots + Yx + Z$
$a \cos bx$	$A \cos bx + B \sin bx$
$a \sin bx$	$A \cos bx + B \sin bx$
$ae^{bx}$	$Ae^{bx}$
$(a_n x^n + \dots + a_0)e^{bx}$	$(Ax^n + \dots + Z)e^{bx}$
$ae^{bx} \cos cx$	$e^{bx}(A \cos cx + B \sin cx)$

**Multiplication rule** If function of  $y_p$  contains terms that duplicate terms in  $y_c$ , then that function must be multiplied by  $x^n$ , where  $n$  is the smallest positive integer that eliminates that duplication.

Method of solution

- Find the form of  $y_p$ 
  - If no function in  $y_p$  is found in  $y_c$ , then no more work is needed
  - If a function in  $y_p$  is found in  $y_c$ , then the multiplication rule must be applied
- Solve for the unknowns of  $y_p$  using the differential equation and the derivatives of  $y_p$

### 3.4 Variation of parameters

Unlike the method of undetermined coefficients, the variation of parameters method is not limited to special function of  $g(x)$ , not is it limited to differential equations with constant coefficients.

Method of solution:

- Rewrite the differential equation in its standard form (with a coefficient of 1 in front of the biggest derivative)
$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = f(x)$$
- Solve for  $y_c$ , with

$$y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

where  $y_1, y_2, \dots, y_n$  constitute a linearly independent set of solutions of the associated homogeneous differential equation. The particular solution  $y_p$  will be of the form

$$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$$

where  $u$  is a function of  $x$

- Solve for  $u_1, u_2, \dots, u_n$  using their derivative and the following formula

$$u'_k = \frac{W_k}{W}, \quad k = 1, 2, \dots, n$$

where  $W$  is the Wronskian of  $y_1, y_2, \dots, y_n$  and  $W_k$  is the determinant obtained by replacing the  $k$ th column

of the Wronskian by the column  $(0, 0, \dots, f(x))$

### 3.5 Cauchy-Euler equations

Any linear differential equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$$

where the coefficients  $a_n, a_{n-1}, \dots, a_0$  are constants, is known as a Cauchy-Euler equation or an equidimensional equation.

The auxiliary equation of a Cauchy-Euler equation is used to solve the differential equation it is associated with. In order to get the auxiliary equation, the substitution  $y = x^m$  is done, and the differential equation is set equal to 0.

*Example.* For a second-order Cauchy-Euler equation, we would get

$$am(m-1) + bm + c = 0$$

$$\iff am^2 + (b-a)m + c = 0$$

where  $m$  is used to find a solution for the Cauchy-Euler equation.

The three possible cases for the value of  $m$  are:

**Case I: distinct real roots** For each real root, there is a solution  $y = cx^m$

**Case II: repeated real roots** For each repeated root, there is a solution  $y = c_1 x^m + c_2 x^m \ln x + \dots + c_k x^m (\ln x)^{k-1}$ , where  $m$  is a root of multiplicity  $k$  (i.e.  $k$  roots are equal to  $m$ ).

**Case III: conjugate complex roots** For each conjugate complex root  $m = \alpha \pm i\beta$ , there is a solution  $y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$

Method of solution:

- Build the auxiliary equation associated with the Cauchy-Euler equation
- Solve for all values of  $m$
- Follow the rule for the three possible cases of  $m$
- Sum all the solutions in order to get the full solution  $y_c$
- Get  $y_p$  using the method of variation of parameters

A generalization of this method can be made by replacing  $x$  by  $x - x_0$ .

### 3.6 Nonlinear equations

In order to solve nonlinear equations, several method exists. One of them known as the reduction of order may be used to find solutions of special kinds of nonlinear second-order differential equations:

- When the dependent variable  $y$  is missing and we have  $F(x, y', y'') = 0$
- When the independent variable  $x$  is missing and we have  $F(y, y', y'') = 0$

In each case, the differential equation can be reduced to a first-order equation by the substitution  $u = \frac{dy}{dx}$ . In the first case, the differential equation becomes  $F(x, u, u') =$

0, where  $\frac{dy}{dx} = u$  and  $\frac{d^2y}{dx^2} = \frac{du}{dx}$ . In the second case, the differential equation becomes  $F(y, u, u') = 0$ , where  $y$  is the new independent variable and  $\frac{dy}{dx} = u$  and  $\frac{d^2y}{dx^2} = u \frac{du}{dy}$ .

### 3.7 Linear Models: initial-value problems (IVP)

#### Spring-mass systems: free undamped motion

For a free undamped motion of a spring-mass system of spring constant  $k$  and mass  $m$ , using Hooke's law ( $F = -ks$ , where  $k \in \mathbb{R}_+^*$ ) and Newton's Second law ( $F = ma$ ), the following second order differential equation can be obtained

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{k}{m}$$

This equation describes a simple harmonic motion, also called free undamped motion.

The solution of this differential equation is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

The period of free vibrations is

$$T = \frac{2\pi}{\omega}$$

and the frequency is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

An alternative form of  $x(t)$  is used to find the amplitude of free vibrations  $A$  and the phase angle  $\phi$  of the system:

$$y = A \sin(\omega t + \phi) = A \cos(\omega t + \phi - \frac{\pi}{2})$$

$$\text{where } A = \sqrt{c_1^2 + c_2^2} \text{ and } \phi = \arctan\left(\frac{c_1}{c_2}\right)$$

**Multiple spring systems** If the springs are in parallel, we have  $k_{\text{equivalent}} = k_1 + k_2 + \dots$ . If the springs are in series, we have  $\frac{1}{k_{\text{equivalent}}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots$ .

**Systems with variable spring constants** The above solution does not apply for systems with variable spring constants, e.g. aging spring or environments with rapid temperature change.

#### Spring and mass systems: free damped motion

For a free damped motion of a spring-mass system of spring constant  $k$ , mass  $m$  and damping factor  $\beta$ , the following second order differential equation can be obtained

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

$$\text{where } 2\lambda = \frac{\beta}{m} \text{ and } \omega^2 = \frac{k}{m}$$

This equation describes a free damped motion.

There are three possible cases:

**Case I:**  $\lambda^2 - \omega^2 > 0$  The system is said to be overdamped, it results in a smooth and nonoscillatory motion and the solution is

$$x(t) = e^{-\lambda t} (c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}})$$

**Case II:**  $\lambda^2 - \omega^2 = 0$  The system is said to be critically damped and the solution is

$$x(t) = e^{-\lambda t} (c_1 + c_2 t)$$

**Case III:**  $\lambda^2 - \omega^2 < 0$  The system is said to be underdamped, it results in an oscillatory motion fading over time and the solution is

$$x(t) = e^{-\lambda t} (c_1 \cos t\sqrt{\omega^2 - \lambda^2} + c_2 \sin t\sqrt{\omega^2 - \lambda^2})$$

In case III ( $\lambda^2 - \omega^2 < 0$ ), an alternative form of  $x(t)$  is used to find the amplitude of free vibrations  $A$  and the phase angle  $\phi$  of the system:

$$y = A \sin(t\sqrt{\omega^2 - \lambda^2} + \phi)$$

$$\text{where } A = \sqrt{c_1^2 + c_2^2} \text{ and } \phi = \arctan\left(\frac{c_1}{c_2}\right)$$

#### Driven motion

The inclusion of another force  $f(t)$  in the formulation of Newton's second law gives the differential equation of driven or forced motion:

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

$$\text{where } 2\lambda = \frac{\beta}{m}, \omega^2 = \frac{k}{m} \text{ and } F(t) = \frac{f(t)}{m}$$

Method of solution:

1. Find  $x_c$  of the homogeneous equation using the solutions found for the free damped motion
2. Find  $x_p$  using the method of undetermined coefficients or variation of parameters

Method of solution for driven motion without damping (pure resonance):

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \gamma t$$

$$\text{where } \omega^2 = \frac{k}{m}$$

1. Find  $x_c$  of the homogeneous equation using the solutions found for the free undamped motion
2. Find  $x_p$  using the method of undetermined coefficients or variation of parameters
3. Determine the condition of pure resonance of the system by letting  $\gamma = \omega$

If the initial conditions are  $x_0 = 0$  and  $x'_0 = 0$ , then for  $\gamma \neq \omega$ , the solution is

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)} (-\gamma \sin \omega t + \omega \sin \gamma t)$$

and for  $\gamma = \omega$ , the solution is

$$x(t) = \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t$$



## RLC-series circuits

Using Kirchhoff's loop rule, the following linear second-order differential equation is found:

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V(t)$$

The equation can be solved using previous solution method such as auxiliary equation, undetermined coefficient or variation of parameters.

## 4 Series solutions of linear equations

### 4.1 Solutions about ordinary points

#### Review of power series

A power series centered at  $a$  is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Power series characteristics:

- **Convergence:** the convergence of a power series can be checked by the ratio test
 
$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = L$$

$$L < 1$$
 The series converges absolutely  

$$L = 1$$
 The test is inconclusive  

$$L > 1$$
 The series diverges
- **Interval of convergence:** the set of all real numbers for which the series converges
- **Radius of convergence  $R$ :** the set of all real numbers for which the series converges, centered at a constant  $a$ . The interval is then inside  $a - R$  and  $a + R$ .
- **Analytic:** a function is analytic at a point  $a$  if it can be represented by a power series centered at  $a$  with  $R \in \mathbb{R}^+$

*Example.*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Power series can be added if the power of  $x$  in each sum are "in phase" and if the summation indices start with the same number. Method to add power series:

1. Identify and check the current power of  $x$  for all the series
2. Pull terms out of the series with the lowest power of  $x$  until it matches the power of the highest
3. Introduce a new counter  $k = f(n)$  equal to the power of  $x$  in the series

4. Replace the old counter  $n$  with the new one  $k$
5. Combine the series

#### Power series solutions

**Ordinary and singular points** A point  $x_0$  is said to be an ordinary point of the differential equation if all the coefficients in the standard form are analytic at  $x_0$ . A point that is not an ordinary point is said to be a singular point of the equation.

**Existence of power series solutions** If  $x = x_0$  is an ordinary point of the differential equation, we can always find linearly independent solutions in the form of a power series centered at  $x_0$ . A series solution converges at least on some interval defined by  $|x - x_0| < R$ , where  $R$  is the distance from  $x_0$  to the closest singular point.

Method of solution (method of undetermined series coefficients at  $x = 0$ ):

1. Check for the existence of a solution at  $x = 0$
2. Set the solution as

$$y = \sum_{n=0}^{\infty} c_n x^n \implies y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$\implies y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

3. Replace  $y$ ,  $y'$  and  $y''$  in the DE
4. If a coefficient is non-polynomial, replace it with its polynomial form (for  $\sin x$ ,  $\cos x$  and  $e^x$ )
5. Combine the series into one sum
6. Apply the identity property (set all terms equal to 0)
7. Get a recursion formula for the  $c$ 's
8. Solve for the  $c$ 's with different values of  $k$
9. Combine the  $c$ 's into a unique  $y$  solution
10. Factor out the common  $c$ 's
11. Find a power series formula for each  $c$
12. The distinct power series are the independent solutions expanded about the ordinary point  $x = 0$

## 5 Systems of linear DEs

### 5.1 Theory of linear systems

The normal form of a first-order system of linear equations is:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2(t)$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n(t)$$

This system can be represented by a matrix equation  $\vec{X}' = \mathbf{A}\vec{X} + \vec{F}$ . If the system is homogeneous, its matrix form is then  $\vec{X}' = \mathbf{A}\vec{X}$ .

**Existence of a unique solution** Let the entries of the matrices  $\mathbf{A}(t)$  and  $\vec{F}(t)$  be functions continuous on a common interval  $I$  that contains the point  $t_0$ . Then there exists a unique solution of the initial-value problem on the interval.

### Homogeneous systems

**Superposition principle** Let  $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_k$  be a set of solution vectors of a homogeneous system on an interval  $I$ . Then the linear combination

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + \dots + c_k\vec{X}_k$$

where  $c_1, c_2, \dots, c_k$  are arbitrary constants, is also a solution of the system on the interval.

In order to check for linearly independent solutions, the Wronskian of  $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_k$  is computed:

- If  $W \neq 0$ , the solutions vectors are independent
- If  $W = 0$ , the solutions vectors are dependent

**General solution of a homogeneous system** Let  $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$  be a fundamental set of solutions, i.e. a set independent solution vectors, of a homogeneous system on an interval  $I$ . Then the general solution of the system on the interval is

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + \dots + c_n\vec{X}_n$$

where  $c_1, c_2, \dots, c_n$  are arbitrary constants.

### Nonhomogeneous systems

#### General solution of a nonhomogeneous system

The general solution  $\vec{X}$  of a nonhomogeneous system is the sum of the general solution  $\vec{X}_c$  of the associated homogeneous system and a particular solution  $\vec{X}_p$ :

$$\vec{X} = \vec{X}_c + \vec{X}_p$$

## 5.2 Homogeneous linear systems

The matrix form of a homogeneous linear system is  $\vec{X}' = \mathbf{A}\vec{X}$  and the general solution is of the form

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + \dots$$

$$\text{where } \vec{X}_i = \vec{K}_i e^{\lambda_i t}$$

This leads to solve this system as an eigenvalue and eigenvector problem, where the eigenvalues  $\lambda_i$  of  $\mathbf{A}$  are determined using the following characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

From those eigenvalues, the corresponding eigenvectors  $\vec{K}_i$  of  $\mathbf{A}$  can be determined using

$$(\mathbf{A} - \lambda \mathbf{I})\vec{K} = \vec{0}$$

Method of solution:

- Determine the eigenvalues  $\lambda_i$  using  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$
- For each eigenvalue, determine an eigenvector using  $(\mathbf{A} - \lambda \mathbf{I})\vec{K} = \vec{0}$
- Apply the rule for one of the four possible cases:

**Case I** Distinct  $\lambda_i$ 's: the final answer is of the form

$$\vec{X} = c_1\vec{K}_1 e^{\lambda_1 t} + c_2\vec{K}_2 e^{\lambda_2 t} + \dots + c_n\vec{K}_n e^{\lambda_n t}$$

**Case II** Repeated  $\lambda_i$ 's and distinct  $\vec{K}_i$ 's: the final answer is of the form

$$\vec{X} = c_1\vec{K}_1 e^{\lambda t} + c_2\vec{K}_2 e^{\lambda t} + \dots + c_n\vec{K}_n e^{\lambda t}$$

**Case III** Repeated  $\lambda_i$ 's and  $\vec{K}_i$ 's: the final answer is of the form

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + \dots$$

where  $\vec{X}_1 = e^{\lambda t}$ ,  $\vec{X}_2 = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$ , until

$$\vec{X}_n = \vec{K} \frac{t^{n-1}}{(n-1)!} e^{\lambda t} + \vec{P} \frac{t^{n-2}}{(n-2)!} e^{\lambda t} + \dots + e^{\lambda t}$$

**Case IV** Complex  $\lambda_i$ 's: the final answer is of the form

$$\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2$$

where  $\vec{X}_1 = e^{\alpha t} \left[ \Re(\vec{K}_1) \cos \beta t - \Im(\vec{K}_1) \sin \beta t \right]$

$$\vec{X}_2 = e^{\alpha t} \left[ \Im(\vec{K}_1) \cos \beta t + \Re(\vec{K}_1) \sin \beta t \right]$$

In case III, to determine  $\vec{K}, \vec{P}, \dots, \vec{Q}$ , the following characteristic equations need to be used:

$$(\mathbf{A} - \lambda \mathbf{I})\vec{K} = \vec{0}$$

$$(\mathbf{A} - \lambda \mathbf{I})\vec{P} = \vec{K}$$

$$(\mathbf{A} - \lambda \mathbf{I})\vec{Q} = \vec{P}$$

## 6 Functions of a complex variable

### 6.1 Complex numbers

#### Terminology

A complex number is any number of the form where  $a$  and  $b$  are real numbers and  $i$  is the imaginary number such that  $i^2 = -1$ .

**Equality** Complex numbers  $z_1 = a_1 + ib_1$  and  $z_2 = a_2 + ib_2$  are equal if

$$\Re(z_1) = \Re(z_2) \text{ and } \Im(z_1) = \Im(z_2)$$

#### Arithmetic operations

**Addition**  $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

**Subtraction**  $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

**Multiplication**  $z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + b_2 a_1)$

**Division**  $\frac{z_1}{z_2} = \left( \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left( \frac{b_1 a_2 - b_2 a_1}{a_2^2 + b_2^2} \right)$

**Modulus**  $|z| = \sqrt{a^2 + b^2}$

#### Conjugate

If  $z = a + ib$ , then its conjugate is  $\bar{z} = a - ib$ .

The arithmetic operations are the same and it can be

shown that

$$\begin{aligned}\overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 \cdot z_2} &= \bar{z}_1 \cdot \bar{z}_2 \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2}\end{aligned}$$

Then using a complex number  $z$  and its conjugate  $\bar{z}$ , the following can be shown

$$\begin{aligned}z + \bar{z} &= 2a \\ z - \bar{z} &= 2bi \\ z \cdot \bar{z} &= a^2 + b^2\end{aligned}$$

## 6.2 Powers and roots

### Polar form of complex numbers

A complex number  $z$  can be expressed in polar coordinates  $(r, \theta)$  as

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

where  $r$  is the modulus of  $z$  and  $\theta$  is its argument, written as  $r = |z|$  and  $\theta = \arg z$ .

Multiplication and division with polar form:

$$\begin{aligned}z_1 \cdot z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] = r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\end{aligned}$$

### Powers of $z$

**Integer powers of  $z$**  The polar form of a complex number is useful to manage powers of  $z$  more easily, as the formula is

$$z^n = r^n (\cos n\theta + i \sin n\theta), \quad n \in \mathbb{Z}$$

**Roots of  $z$**  A number  $w$  is said to be a  $n$ th root of a nonzero complex number  $z$  if  $w^n = z$ . The  $n$ th roots of a nonzero complex number  $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$  are given by

$$\begin{aligned}w_k &= \sqrt[n]{r} \left[ \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right] \\ &= \sqrt[n]{r} e^{i \frac{\theta + 2k\pi}{n}}\end{aligned}$$

where  $k = 0, 1, 2, \dots, n-1$