MECH 361: Fluid Mechanics II

Anthony Bourboujas

March 22, 2022

1 Introduction

2 Newton's law of viscosity

3 Theoretical part

4 Velocity and acceleration

4.1 Velocity analysis

The velocity vector field $\vec{v}$ is a function of position in space as well as time:

$$\vec{v}(x,y,z,t) = v_x(x,y,z,t)\hat{\imath} + v_y(x,y,z,t)\hat{\jmath} + v_z(x,y,z,t)\hat{\mathrm{k}}$$

The magnitude of the velocity $\vec{v}$ is:

$$v = \|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

and for a 2-dimensional flow, the direction of the flow in the xy-plane:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{v_y}{v_x}$$

4.1.1 Steady flow

In a steady flow, the fluid properties may vary from point to point in the field but would remain constant at that point with respect to time t.

In a steady flow, the velocity $\vec{v}$ at a given point in space does not vary with time:

$$\frac{\partial \vec{v}}{\partial t} = \vec{0}$$

Remark. In reality, almost all flows are unsteady.

4.1.2 Unsteady flows

In unsteady flows, the velocity v does vary with time. Unsteady flows are more difficult to analyze and to investigate experimentally than are the steady flows.

Considerable simplicity often results if one can make the assumption of steady flow without compromising the usefulness of the results.

4.1.3 Uniform flow

A flow in which the velocity v is constant at any cross-section of it is called a uniform flow. Under this assumption, a 2-dimensional flow is modeled as 1-dimensional for analysis.

The density ρ and pressure P may also be assumed constant in the cross-section of a uniform flow.

4.2 Acceleration analysis

The acceleration of a fluid particle is described using partial derivative:

$$\begin{aligned}\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= \frac{\mathrm{d}v_x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\hat{\mathrm{k}} \\ &= a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{\mathrm{k}} \\ &= \frac{\partial \vec{v}}{\partial t} + v_x\frac{\partial \vec{v}}{\partial x} + v_y\frac{\partial \vec{v}}{\partial y} + v_z\frac{\partial \vec{v}}{\partial z} \\ &= \frac{\partial \vec{v}}{\partial t} + \vec{v} \bullet \vec{\nabla}\vec{v}\end{aligned}$$

where $\frac{\partial \vec{v}}{\partial t}$ is called the local acceleration (acceleration depending on time) and $\vec{v}\bullet\vec{\nabla}\vec{v}$ is the convective acceleration (acceleration depending on the position).

This means that for the x-axis, we have:

$$\begin{aligned}a_x &= \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} \\ &= \frac{\partial v_x}{\partial t} + \vec{v} \bullet \vec{\nabla} v_x\end{aligned}$$

4.3 Analysis of fluid flows

4.3.1 Stress

A state of stress is determined with 9 values at a point of a control volume using the hydrodynamic stress tensor:

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

$\sigma[\mathrm{N\,m^{-2}}]$: normal stress
$\tau[\mathrm{N\,m^{-2}}]$: shear stress

The double notation of the stresses, the first defines the surface and the second defines the direction of the component.

4.3.2 Strain

Stresses lead to strains, which are deformations either in the longitudinal direction (length or width) or in the angular (shear).

Longitudinal strains ϵ_x and ϵ_y are called normal strains are defined as:

$$\epsilon_x = \frac{\Delta l}{l} \epsilon_y = \frac{\Delta w}{w}$$

ϵ_x: longitudinal strain
Δl[m]: elongation, deformation
l[m]: original lengthϵ_y: longitudinal strain
Δw[m]: elongation, deformation
w[m]: original width

Angular strain γ is called shear strain and, for small angle, is defined as:

$$\gamma \approx \tan\gamma = \frac{\mathrm{d}x}{\mathrm{d}y}$$

4.3.3 Hooke's laws for solids

For solids, the link between strain and stress are:

$$\begin{aligned} \sigma_x &= E\epsilon_x \\ \sigma_y &= E\epsilon_y \\ \tau &= E\gamma \end{aligned}$$

where E is Young's modulus and G is the modulus of rigidity.

4.3.4 Pressure in fluids

Hydrostatic pressure Hydrostatic pressure is when the fluid is at rest pressure does not depend on orientation. It is also called the thermodynamic pressure.

Hydrostatic pressure is isotropic (equal in all directions), which implies that all shear stresses are equal to 0 when the fluid is at rest.

Hydrodynamic pressure Hydrodynamic pressure is the pressure when the fluid is in motion. It depends on orientation.

4.3.5 Compressibility

Compressibility is a measure of the change in volume of a fluid under the action of external forces.

Compressibility is studied using a similar form of Hooke's law applied to fluids:

$$\Delta P = -E\frac{\Delta V}{V}$$

ΔP[Pa]: change in pressure
E[Pa]: modulus of fluid elasticity
ΔV[m^3]: change in volume
V[m^3]: original volume

Remark. Air is 20 000 times more compressible than water.

In conclusion, in theory, liquids are incompressible fluids but gases are compressible fluids.

4.3.6 Deformation of volume of fluid particle

In general, the motion of fluid particle is composed of linear translation, rotation, linear deformation and angular deformation.

In hydrostatics, there is only normal stresses because of gravity.

In hydrodynamics, in addition to normal stresses, shear stresses exist as a result of internal micro-friction created by viscosity.

The dilation or volumetric deformation rate is a measure of compressibility:

$$\text{dilation} = \operatorname{div}\vec{v} = \vec{\nabla} \bullet \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

dilation[$\mathrm{m^{-3}\,s^{-1}}$]: dilation or volumetric deformation
$\vec{v}$[$\mathrm{m\,s^{-1}}$]: velocity vector of the flow

Remark. For an incompressible flow, the dilation is equal to $0\,\mathrm{m^{-3}\,s^{-1}}$.

5 Flow kinematic analysis

5.1 Continuity equation

The continuity equation is the conservation of mass for a liquid. The general continuity equation for an unsteady state of a compressible flow is

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_x}{\partial x} + \frac{\partial\rho v_y}{\partial y} + \frac{\partial\rho v_z}{\partial z} = \frac{\partial\rho}{\partial t} + \vec{\nabla} \bullet \rho\vec{v} = 0$$

ρ[$\mathrm{kg\,m^{-3}}$]: liquid density
t[s]: time
$\vec{v}$[$\mathrm{m\,s^{-1}}$]: flow velocity field

In cylindrical coordinates, this gives

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial r\rho v_r}{\partial r} + \frac{1}{r}\frac{\partial\rho v_\theta}{\partial\theta} + \frac{\partial\rho v_z}{\partial z} = 0$$

ρ[$\mathrm{kg\,m^{-3}}$]: liquid density
t[s]: time
r[m]: radius
θ[rad]: angle
$\vec{v}$[$\mathrm{m\,s^{-1}}$]: flow velocity field

Remark. Assuming the flow is stead and incompressible, this gives the dilation or volumetric deformation

$$\begin{aligned}\text{dilation} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \\ &= \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0\end{aligned}$$

Therefore, in this case the dilation is used as a compressibility indication and also an indication that the continuity equation is satisfied.

5.2 Linear motion and deformation

Linear translation and deformation is given by the dilation:

$$\text{dilation} = \operatorname{div}\vec{v} = \vec{\nabla} \bullet \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

5.3 Rotation analysis

Rotation of fluid particles is related to certain velocity gradients in the flow field.

The rotation vector $\vec{\omega}$ and the vorticity $\vec{\zeta}$, which is defined as twice the rotation vector, are:

$$\begin{aligned}\vec{\omega} &= \frac{1}{2}\operatorname{curl}\vec{v} = \frac{1}{2}\vec{\nabla} \times \vec{v} \\ \vec{\zeta} &= 2\vec{\omega} = \operatorname{curl}\vec{v} = \vec{\nabla} \times \vec{v}\end{aligned}$$

Remark. Vorticity is created at fluid and solid interfaces as a result of frictional forces.

5.4 Shear strain

The derivatives associated with rotation can cause teh fluid element to undergo an angular deformation, which results in a change in the shape of the element.

This change is made using the shear strain $\dot{\gamma}$, which is defined as:

$$\begin{aligned}\dot{\gamma}_{xy} &= \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \\ \dot{\gamma}_{yz} &= \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \dot{\gamma}_{zx} &= \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\end{aligned}$$

5.4.1 General solving steps

The volumetric dilation rate $\operatorname{div}\vec{v} = \vec{\nabla} \bullet \vec{v}$ answers the following questions:

- Flow is incompressible
- Flow is physically possible
- Flow satisfies the conservation of mass
- Flow satisfies the continuity equation
- the volumetric strain rate
- The divergence of the velocity vector
- is the volumetric dilation rate equal to zero?

The angular rotation is obtained using the rotation vector $\vec{\omega} = \frac{1}{2}\operatorname{curl}\vec{v}$. Alternatively, to determine if the flow is irrotational, the vorticity $\vec{\zeta} = \operatorname{curl}\vec{v} = 2\vec{\omega}$ can be used.

The shear strain (or rate of angular deformation) is found using:

$$\dot{\gamma} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}$$

and it is used in the Newton's law viscosity

$$\tau = \mu\dot{\gamma}$$

Remark. An irrotational flow is inviscid, but not any inviscid flow is irrotational since an inviscid flow cannot stop a particle from rotating if the particle was already rotating.

6 Stream function

The concept of the stream function is introduced only for two-dimensional fluid flows. Physically, it is a concept as close as it can be to the "flow rate at a point".

A stream function ψ exists if and only if the flow is a two-dimensional incompressible flow.

For a two-dimensional incompressible flow, in which the continuity equation applies, the stream function $\psi(x, y)$ is related to the velocity components of the flow with:

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

In polar coordinates, this gives

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

6.1 Advantage of the stream function

Whenever the velocity components are defined in terms of the stream function, we know that conservation will be satisfied.

For a particular problem, the stream function is unknown, but the analysis is simplified by having only one unknown function $\psi(x, y)$ rater than two velocity function $v_x(x, y)$ and $v_y(x, y)$.

6.1.1 Features of the stream function

6.1.2 Streamlines

The lines of constant stream function are streamlines, which is a line in the flow field where at each point the total fluid velocity is tangent to the line.

By considering a line of constant stream function $\mathrm{d}\psi(x, y) = 0$ implies that the slope at any point along a streamline is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v_y}{v_x}$.

6.1.3 Crossing streamlines

A fluid never crosses a streamline since by definition the velocity is tangent to the streamline.

This implies that streamlines act like a conduit for the flow.

6.1.4 Volumetric flow rate per unit width

The volumetric flow rate per unit width q passing between two streamlines is equal to the difference between the values of the stream functions. The actual numerical value associated with a particular stream line is not of particular significance, but the chang ein the value of ψ is related to the volume rate of flow.

The volumetric flow rate per unit width q is defined as

$$q = \int_{\psi_1}^{\psi_2} \mathrm{d}\psi = \psi_2 - \psi_1$$

6.1.5 Direction of flow

The direction of flow is determined by whether $\psi_2 < \psi_1$ or $\psi_2 < \psi_1$.

6.2 How to use the stream function

The stream function can be used to find the velocity components as:

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

Also, given the velocity components v_x and v_y, the stream function ψ can be found and used to develop more information about the flow. To solve those problems, either use the method of exact differential equations, or simply integrate both equations and compare the resulting stream function ψ to find the functions of integration.

7 Velocity potential

For irrotational flows, there exists a velocity potential ϕ, which explains why irrotational flows are called potential flows. The velocity potential function ϕ is defined as:

$$v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y} \qquad v_z = \frac{\partial \phi}{\partial z}$$

For an incompressible potential flow, the continuity equation becomes the Laplace equation, which is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

7.1 Potential flows

Potential flows have a potential function $\phi(x, y, z)$ and they are defined by Laplace equation, which is a linear partial differential equation. Since it is linear, various solutions can be added to obtain other solutions, meaning if ϕ_1 and ϕ_2 are two solutions to Laplace equation, then $\phi_3 = \phi_1 + \phi_2$.

7.2 Velocity potential in cylindrical coordinates

For an irrotational flow, the velocity potential ϕ in cylindrical coordinates is :

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad v_z = \frac{\partial \phi}{\partial z}$$

The Laplace equation in cylindrical coordinates is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right] + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

8 Stream function and velocity potential

A stream function exists for irrotational flows if the flow is incompressible and two-dimensional. From the definition of the stream function and the continuity equation, the stream satisfies the Laplace equation if the flow is irrotational:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

8.1 Graphical relationship

For two-dimensional incompressible irrotational flows, lines of constant velocity potential are perpendicular to lines of constant stream function:

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\mathrm{d}\psi=0} = \frac{v}{u} \qquad \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\mathrm{d}\phi=0} = -\frac{u}{v}$$

8.2 Questions relating the stream function and the velocity potential

The questions relating the stream function and the velocity potential can be:

- Given the stream function ψ, find the velocity potential ϕ;
- Given the velocity potential ϕ, find the stream function ψ;
- Plot any of the obtained stream function ψ or velocity potential ϕ.

9 Navier-Stokes, Euler, Bernoulli and continuity equations

Navier-Stokes equations are the basic differential equations describing the flow of incompressible viscous Newtonian fluids.

9.1 Bernoulli equation

The Bernoulli equation is a statement of the conservation of energy principle appropriate for flowing ideal fluids:

$$\begin{aligned}\text{Energy: } & \frac{P}{\rho} + gz + \frac{v^2}{2} = \text{constant}\\ \text{Pressure: } & P + \rho g z + \frac{\rho v^2}{2} = \text{constant}\\ \text{Head: } & \frac{P}{\rho g} + z + \frac{v^2}{2g} = \text{constant}\end{aligned}$$

P[Pa]: pressure
ρ[$\mathrm{kg\,m^{-3}}$]: density
v[$\mathrm{m\,s^{-1}}$]: velocity
$g = 9.81\,\mathrm{m\,s^{-2}}$: gravitational acceleration on Earth
z[m]: height

The assumptions for the Bernoulli equation states the flow must be inviscid, steady, incompressible and along a streamline.

The Bernoulli equations applies along a streamline for inviscid flows and ideal fluids. It can be applied everywhere between any two points in an irrotational flow field.

9.2 Euler equation

9.3 Navier-Stokes equations

The Navier-Stokes equations are the momentum equations of the differential analysis of fluid flow.

There are two methods of describing a fluid flow:
Lagrangian:
Eulerian:

Newton second law of motion for fluid mechanics in Euler's view is

$$\frac{\mathrm{d}m\vec{v}}{\mathrm{d}t} = m_{\text{in}}\vec{v}_{\text{in}} - m_{\text{out}}\vec{v}_{\text{out}} + \sum \vec{F}_{\text{external}}$$

which lead to the Navier-Stokes equations in the xy-plane:

$$\begin{aligned}x\text{-axis: } & \frac{\partial \rho v_x}{\partial t} + \frac{\partial \rho {v_x}^2}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g_x\\ y\text{-axis: } & \frac{\partial \rho v_y}{\partial t} + \frac{\partial \rho {v_y}^2}{\partial y} + \frac{\partial \rho v_x v_y}{\partial x} = \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g_y\end{aligned}$$

Those equations are global equations which applies to newtonian and non-newtonian fluids, to laminar and turbulent flows, to compressible and incompressible flows. However, there are too many unknowns, meaning they are not very useful if no assumptions are used.

Introducing the stress-strain relationship for newtonian fluid:

$$\begin{aligned}\sigma_{xx} &= -p + 2\mu\tfrac{\partial v_x}{\partial x} & \tau_{xy} &= \mu\left(\tfrac{\partial v_x}{\partial y} + \tfrac{\partial v_y}{\partial x}\right) & \tau_{xz} &= \mu\left(\tfrac{\partial v_x}{\partial z} + \tfrac{\partial v_z}{\partial x}\right)\\ \sigma_{yy} &= -p + 2\mu\tfrac{\partial v_y}{\partial y} & \tau_{yz} &= \mu\left(\tfrac{\partial v_y}{\partial z} + \tfrac{\partial v_z}{\partial y}\right) & \tau_{yx} &= \mu\left(\tfrac{\partial v_y}{\partial x} + \tfrac{\partial v_x}{\partial y}\right)\\ \sigma_{zz} &= -p + 2\mu\tfrac{\partial v_z}{\partial z} & \tau_{zx} &= \mu\left(\tfrac{\partial v_z}{\partial x} + \tfrac{\partial v_x}{\partial z}\right) & \tau_{zy} &= \mu\left(\tfrac{\partial v_z}{\partial x} + \tfrac{\partial v_x}{\partial z}\right)\end{aligned}$$

By introducing Newton's law of viscosity $\tau = \mu\frac{\mathrm{d}v_x}{\mathrm{d}y}$ and the stress-strain relationship for newtonian fluid, the newtonian Navier-Stokes equations can be found (only valid for newtonian fluids):

$$x\text{-axis: } \frac{\partial \rho v_x}{\partial t} + \frac{\partial \rho {v_x}^2}{\partial x} + \frac{\partial \rho v_x v_y}{\partial y} = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

$$y\text{-axis: } \frac{\partial \rho v_y}{\partial t} + \frac{\partial \rho {v_y}^2}{\partial y} + \frac{\partial \rho v_x v_y}{\partial x} = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

The Navier-Stokes equations for an incompressible newtonian 3-dimensional flow are

$$\begin{aligned}x\text{-axis: } & \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)\\ &= -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial x^2}\right)\end{aligned}$$

$$\begin{aligned}y\text{-axis: } & \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right)\\ &= -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial x^2}\right)\end{aligned}$$

$$\begin{aligned}z\text{-axis: } & \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)\\ &= -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial x^2}\right)\end{aligned}$$

9.4 Use of the equations

Along with the continuity equation, Navier-Stokes equations are used when viscous flows are at hand, Euler equations will be use for inviscid flow fields and Bernoulli equation for inviscid irrotational flow field along a streamline.

9.5 Solving the Navier-Stokes equations

Navier-Stokes equations will be used to solve different problems:
- Steady laminar flow between fixed parallel plates.
- Steady laminar flow between two parallel plates, one fixed and the other moving (couette flow).
- Steady laminar flow in circular tubes (Poiseuille's law)
- Steady laminar flow in an annulus.
- Steady laminar flow of a falling liquid film.

9.5.1 General solving steps

1. Analyze which velocity components describe the flow;
2. Apply the continuity equation and simplify;
3. Apply the Navier-Stokes equations with the results;
4. Double check for the body forces ρg and monitor carefully the direction of the gravitational acceleration;
5. Prove that the pressure varies hydrostatically using the two equations containing pressure differentials only;
6. Obtain the velocity profile using the third equation;
7. Apply boundary conditions to solve for the constants;
8. Obtain the equation for the volume flow rate using the velocity profile;
9. Obtain the equation for the mean velocity using the volume flow rate equation;
10. Find the maximum velocity by either using the derivative or the geometry of the problem;
11. Find the relation between the velocity profile and the maximum velocity.

9.5.2 Steady laminar flow between fixed parallel plates

For this flow, the fluid particles move in the x direction parallel to the plates, there is no velocity in the y or z directions, the gravity is in the $-y$ direction, and the distance between the plates is $2h$.

The velocity $v_y = v_z = 0$ and only v_x describe the flow. The continuity equation gives $\frac{\partial v_x}{\partial x} = 0 \implies v_x = v_x(y)$.

Navier-Stokes equations in the x, y and z directions are:

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$
$$\frac{\partial P}{\partial y} = -\rho g$$
$$\frac{\partial P}{\partial z} = 0$$

The variations in pressure equation is:

$$P = -\rho g y + f_1(x)$$

meaning it varies hydrostatically in the y direction. The velocity profile can be obtained with a double integral and using $\frac{\partial v_x}{\partial y} = \frac{\mathrm{d} v_x}{\mathrm{d} y}$ since v_x is a function of y only:

$$v_x = \frac{1}{2\mu}\frac{\partial P}{\partial x} y^2 + c_1 y + c_2$$

The no slip boundary condition can be applied in order to find the values of the constants and get the complete profile equation: at $y = \pm h$, we have $v_x = 0$, and hence

$$v_x = -\frac{1}{2\mu}\frac{\partial P}{\partial x}\left(y^2 - h^2\right)$$

The volume flow rate per unit width q between the plates is:

$$q = \int_{-h}^{h} v_x \,\mathrm{d}y = -\frac{2h^3}{3\mu}\frac{\partial P}{\partial x} = \frac{2h^3 \Delta P}{3\mu l}$$

The average velocity is found using the flow rate equation:

$$v_{x,\text{avg}} = \frac{q}{2h} = -\frac{h^2}{3\mu}\frac{\partial P}{\partial x} = \frac{h^2 \Delta P}{3\mu l}$$

The maximum velocity $v_{x,\max}$ occurs at the center between the two plates:

$$v_{x,\max} = -\frac{h^2}{2\mu}\frac{\partial P}{\partial x} = \frac{h^2 \Delta P}{2\mu l}$$

The relations between the mean velocity $v_{x,\text{avg}}$ and the maximum velocity $v_{x,\max}$ is:

$$v_{x,\max} = \frac{3}{2} v_{x,\text{avg}}$$

9.5.3 Steady laminar flow between two parallel plates, one fixed and the other moving

For this flow, the fluid particles move in the x direction parallel to the plates, there is no velocity in the y or z directions, the gravity is in the $-y$ direction, and the distance between the plates is b.

The Navier-Stokes equations are the same as the previous case since only the boundary conditions changes. Hence, Navier-Stokes equations in the x, y and z directions are:

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$
$$\frac{\partial P}{\partial y} = -\rho g$$
$$\frac{\partial P}{\partial z} = 0$$

The variations in pressure equation is

$$P = -\rho g y + f_1(x)$$

meaning it varies hydrostatically in the y direction. The velocity profile can be obtained with a double integral and using $\frac{\partial v_x}{\partial y} = \frac{\mathrm{d} v_x}{\mathrm{d} y}$ since v_x is a function of y only:

$$v_x = \frac{1}{2\mu}\frac{\partial P}{\partial x} y^2 + c_1 y + c_2$$

Now the analysis changes from the last case: the boundary conditions are $v_x = 0$ at $y = 0$ and $v_x = v_{\text{plate}}$ at $y = b$. Therefore, the velocity profile v_x is:

$$v_x = v_{\text{plate}}\frac{y}{b} + \frac{1}{2\mu}\frac{\partial P}{\partial x}\left(y^2 - by\right)$$

In the special case in which there is no pressure drop in the x direction, it means the moving plate is driving the motion of the fluid and the velocity profile reduces $v_x = v_{\text{plate}}\frac{y}{b}$, which is the proof of Newton's law of viscosity for newtonian fluids.

9.5.4 Steady laminar flow in circular tubes (Poiseuille's law)

For this flow, the polar coordinate system is used. The fluid particles move in the z direction parallel to the tube, there is no velocity in the r or θ directions, the gravity is in the $-y$ direction, the radius of the tube is R.

The velocity $v_r = v_\theta = 0$ and only v_z describe the flow. The continuity equation gives $\frac{\partial v_z}{\partial z} = 0 \implies v_z = v_z(r, \theta)$.

For this flow, the assumption of an axisymmetric flow is made: the streamlines are symmetrically located around an axis. Accordingly, the pressure P and the cylindrical velocity components v_r, v_θ and v_z are independent of the angle θ. Hence, the continuity equation gives $v_z = v_z(r)$

Navier-Stokes equations in the r, θ and z directions are:

$$\frac{\partial P}{\partial r} = -\rho g_r = -\rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\cancelto{0}{\frac{\partial P}{\partial \theta}}$$

$$\frac{\partial P}{\partial z} = \mu\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]$$

The first two equations proves that:

$$P = -\rho g r\sin\theta + f_1(z) = -\rho g y + f_1(z)$$

meaning the pressure is hydrostatically distributed at any cross-section. The velocity profile can be obtained with a double integral and using $\frac{\partial v_z}{\partial r} = \frac{\mathrm{d}v_z}{\mathrm{d}r}$ since v_z is a function of r only:

$$v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}r^2 + c_1\ln r + c_2$$

Using the existence of $r = 0$ implies $c_1 = 0$ since $\ln(0) = -\infty$ (the study of the derivative at $r = 0$ would give the same result), and using the boundary condition $v_z = 0$ at $r = R$, this implies $c_2 = -\frac{1}{4\mu}\frac{\partial P}{\partial z}R^2$, giving the following velocity profile:

$$v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left(r^2 - R^2\right)$$

The volume flow rate Q in the tube is:

$$Q = \int v_z\,\mathrm{d}A = 2\pi\int_0^R v_z r\,\mathrm{d}r = -\frac{\pi R^4}{8\mu}\frac{\partial P}{\partial z}$$

This equations is called Poiseuille's law and it relates the pressure drop and flow rate for steady, laminar flow in circular tubes. For a given pressure drop per unit length, the volume flor rate of flow is inversely proportional to the viscosity and proportional to the tube radius to the fourth power (ie. a doubling of the tube radius produces a sixteen volume flow rate increase).

The average velocity is found using the flow rate equation:

$$v_{z,\text{avg}} = \frac{Q}{A} = -\frac{R^2}{8\mu}\frac{\partial P}{\partial z} = \frac{R^2\Delta P}{8\mu l}$$

The maximum velocity $v_{z,\max}$ occurs at the center of the tube:

$$v_{z,\max} = -\frac{R^2}{2\mu}\frac{\partial P}{\partial z} = \frac{R^2\Delta P}{4\mu l}$$

The relations between the mean velocity $v_{z,\text{avg}}$ and the maximum velocity $v_{z,\max}$ is:

$$v_{z,\max} = 2v_{z,\text{avg}}$$

9.5.5 Steady laminar flow in an annulus

For this flow, the polar coordinate system is used. The fluid particles move in the z direction parallel to the tubes, there is no velocity in the r or θ directions, the gravity is in the $-y$ direction, the radius of the outer tube is r_o and the radius of the inner tube is r_i.

The velocity $v_r = v_\theta = 0$ and only v_z describe the flow. The continuity equation gives $\frac{\partial v_z}{\partial z} = 0 \implies v_z = v_z(r, \theta)$.

For this flow, the assumption of an axisymmetric flow is made: the streamlines are symmetrically located around an axis. Accordingly, the pressure P and the cylindrical velocity components v_r, v_θ and v_z are independent of the angle θ. Hence, the continuity equation gives $v_z = v_z(r)$

Navier-Stokes equations in the r, θ and z directions are:

$$\frac{\partial P}{\partial r} = -\rho g_r = -\rho g \sin\theta$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\cancelto{0}{\frac{\partial P}{\partial \theta}}$$

$$\frac{\partial P}{\partial z} = \mu\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right]$$

The first two equations proves that:

$$P = -\rho g r\sin\theta + f_1(z) = -\rho g y + f_1(z)$$

meaning the pressure is hydrostatically distributed at any cross-section. The velocity profile can be obtained with a double integral and using $\frac{\partial v_z}{\partial r} = \frac{\mathrm{d}v_z}{\mathrm{d}r}$ since v_z is a function of r only:

$$v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}r^2 + c_1\ln r + c_2$$

The analysis changes from the last case: the boundary conditions are $v_z = 0$ at $r = r_i$ and $r = r_o$, this gives the following velocity profile:

$$v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left[r^2 - {r_o}^2 + \frac{{r_i}^2 - {r_o}^2}{\ln\frac{r_o}{r_i}}\ln\frac{r}{r_o}\right]$$

The volume flow rate Q in the tube is:

$$Q = \int v_z\,\mathrm{d}A = -\frac{\pi}{8\mu}\frac{\partial P}{\partial z}\left[{r_o}^4 - {r_i}^4 - \frac{\left({r_i}^2 - {r_o}^2\right)^2}{\ln\frac{r_o}{r_i}}\right]$$
$$= \frac{\pi\Delta P}{8\mu l}\left[{r_o}^4 - {r_i}^4 - \frac{\left({r_i}^2 - {r_o}^2\right)^2}{\ln\frac{r_o}{r_i}}\right]$$

The average velocity is found using the flow rate equation:

$$v_{z,\text{avg}} = \frac{Q}{A} = -\frac{R^2}{8\mu}\frac{\partial P}{\partial z} = \frac{R^2\Delta P}{8\mu l}$$

The maximum velocity $v_{z,\max}$ occurs at r_m:

$$r_m = \sqrt{\frac{{r_o}^2 - {r_i}^2}{2\ln\frac{r_o}{r_i}}}$$

9.5.6 Steady laminar flow of a falling liquid film

The flow is on an inclined plane of angle $\alpha$and is a parallel flow in the x direction.

The velocity $v_y = v_z = 0$ and only v_x describe the flow. The continuity equation gives $\frac{\partial v_x}{\partial x} = 0 \implies v_x = v_x(y)$.

Navier-Stokes equations in the x, y and z directions are:

$$\frac{\partial P}{\partial x} = \rho g\sin\alpha + \mu\frac{\partial^2 v_x}{\partial y^2}$$
$$\frac{\partial P}{\partial y} = -\rho g\cos\alpha$$
$$\frac{\partial P}{\partial z} = 0$$

Due to the flow being a free stream, $\frac{\partial P}{\partial x} = 0$, meaning the variations in pressure equation is:

$$P = -\rho g y + c$$

meaning it varies hydrostatically in the y direction. Applying the boundary condition $P = P_{\text{atmosphere}}$ at the top of the fluid film $y = a_0$. Considering the thickness y is very small, it can be approximated that $\frac{\partial P}{\partial y} \approx 0$ and $P = P_{\text{atmosphere}}$.

This leads so the following Navier-Stokes in the x direction:

$$0 = \rho g\sin\alpha + \mu\frac{\partial^2 v_x}{\partial y^2}$$

The velocity profile can be obtained with a double integral and using $\frac{\partial v_x}{\partial y} = \frac{\mathrm{d}v_x}{\mathrm{d}y}$ since v_x is a function of y only:

$$v_x = -\frac{\rho g\sin\alpha}{2\mu}y^2 + c_1 y + c_2$$

The no slip boundary condition can be applied in order to find the values of the constants and get the complete profile equation: $v_x = 0$ at $y = 0$, and $\frac{\mathrm{d}v_x}{\mathrm{d}y} = 0$ at the interface $y = a_0$ (since the shear stress is the friction between the air and the fluid, meaning it is minimal), hence

$$v_x = \frac{\rho {a_0}^2 g\sin\alpha}{\mu}\left[\frac{y}{a_0} - \frac{1}{2}\left(\frac{y}{a_0}\right)^2\right]$$
$$= \frac{\rho g\sin\alpha}{\mu}\left[a_0 y - \frac{y^2}{2}\right]$$

Using the velocity profile, the volume flow rate Q is:

$$Q = \int_0^{a_0} v_x z\,\mathrm{d}y = \frac{\rho g {a_0}^3 W\sin\alpha}{3\mu}$$

where W is the width of the fluid film.

The average velocity is found using the flow rate equation:

$$v_{x,\text{avg}} = \frac{Q}{A} = \frac{\rho g {a_0}^2\sin\alpha}{3\mu}$$

This implies that the thickness of the film a_0 as functions of $v_{x,\text{avg}}$ and as a function of Q:

$$a_o = \sqrt{\frac{3\mu v_{x,\text{avg}}}{\rho g\sin\alpha}} \tag{9.1}$$
$$a_o = \sqrt[3]{\frac{3\mu Q}{\rho g W\sin\alpha}} \tag{9.2}$$

The maximum velocity $v_{x,\max}$ occurs at the top of the fluid film $y = a_o$:

$$v_{x,\max} = \frac{\rho g {a_0}^2\sin\alpha}{2\mu}$$

The relations between the mean velocity $v_{x,\text{avg}}$ and the maximum velocity $v_{x,\max}$ is:

$$v_{x,\max} = \frac{3}{2}v_{x,\text{avg}}$$

9.6 Dimensionless analysis of the Navier-Stokes equations

Dimensionless Navier-Stokes equations uses the characteristic length L, the velocity V, the time $T = \frac{L}{V}$ and the pressure $P = \rho V^2$.

The dimensionless parameters of the equations are:

$$\hat{x} = \frac{x}{L} \qquad \hat{y} = \frac{y}{L} \qquad \hat{v}_x = \frac{v_x}{V}$$

9.6.1 Special case: high speed, negligible friction

In this case, $\mathrm{Re} \to \infty$, which implies that the dimensionless Navier-Stokes equations becomes:

9.6.2 Special case: creeping flows, lubrication theory

In this case, $\mathrm{Re} \to 0$, which implies that the dimensionless Navier-Stokes equations becomes:

10 Potential flows

Potential flows have a potential function $\phi(x, y, z)$ and they are defined by Laplace equation, which is a linear partial differential equation. Since it is linear, various solutions can be added to obtain other solutions, meaning if ϕ_1 and ϕ_2 are two solutions to Laplace equation, then $\phi_3 = \phi_1 + \phi_2$.

Potential flows are inviscid, incompressible and irrotational. The five simple potential flows are uniform flow, source and sink, vortex flows and doublet flows. Their solution can be added in order to get a more complex flow.

10.1 Basic plane potential flows

Table 1 on the following page summarizes all the equations of the stream function ψ, the potential function ϕ and the velocity components for each basic flows.

10.1.1 Uniform flow

The uniform flow is the simplest place flow for which the streamlines are all straight and parallel, and the magnitude of the velocity is constant. For the general case in which the uniform flow has a velocity U and an angle α with the x-axis, the stream and potential functions are

$$\psi = v(y\cos\alpha - x\sin\alpha) \qquad \phi = v(x\cos\alpha + y\sin\alpha)$$

and the velocity components are

$$v_x = U\cos\alpha$$
$$v_y = U\sin\alpha$$

10.1.2 Source and sink

Let m be the volume rate of flow emanating per unit length. The flow rate m is the strength of the source or sink

$$2\pi r v_r = m \iff v_r = \frac{m}{2\pi r}$$

and $v_\theta = 0$ since the flow is purely radial.

For the a source, the stream and potential functions are

$$\psi = \frac{m}{2\pi}\theta \qquad \phi = \frac{m}{2\pi}\ln r \tag{10.1}$$

If $m > 0$ the flow is a source and if $m < 0$ the flow is a sink.

Remark. The streamlines are radial lines and the equipotential lines are concentric circles centered at the origin.

10.1.3 Vortex

A vortex represents a flow in which the streamlines are concentric circles. The velocity $\vec{v}$ is described as For a free vortex, the stream and potential functions are

$$\psi = -\frac{\Gamma}{2\pi}\ln r \qquad \phi = \frac{\Gamma}{2\pi}\theta \tag{10.2}$$

where Γ is the circulation defined as $\Gamma = 2\pi K$ where K is a constant.

If $\Gamma > 0$, the motion is counterclockwise and if $\Gamma < 0$, the motion is clockwise

The irrotational vortex is called a free vortex, while the rotational vortex is called forced vortex.

Remark. The free vortex is an irrotational flow as the rotation refers to the orientation of a fluid element and not the path followed by the element.

Remark. The streamlines are concentric circles centered at the origin and the equipotential lines are radial lines.

10.1.4 Doublet

A doublet is formed by an appropriate source-sink pair: by letting the source and sink approach one another so that the distance $2a$ between them is 0 while increasing their strength m to infinity so that the product $\frac{ma}{\pi}$ remains constant. For a doublet, the stream and potential functions are

$$\psi = -\frac{K\sin\theta}{r} \qquad \phi = \frac{K\cos\theta}{r} \tag{10.3}$$

where $K = \frac{ma}{r}$ is a constant and is the strength of the doublet.

Remark. The streamlines are circles passing through the origin and tangent to the x-axis.

10.2 Superposition of basic plane potential flows

The flow around a half body can be modeled as the superposition of a uniform flow and a source.

The flow around a closed full body can be modeled as the superposition of a uniform flow, a source and a sink.

The flow around a cylinder can be modeled as the superposition of a uniform flow and a doublet.

The flow around a rotating cylinder can be modeled as the superposition of a uniform flow, a doublet and a free vortex.

Table 1: Summary flow fields table

Flow field	Stream function	Velocity potential	Velocity components
Uniform flow	$\psi = v(y\cos\alpha - x\sin\alpha)$	$\phi = v(x\cos\alpha + y\sin\alpha)$	$v_x = U\cos\alpha$ $v_y = U\sin\alpha$
Source or sink	$\psi = \frac{m}{2\pi}\theta$	$\phi = \frac{m}{2\pi}\ln r$	$v_r = \frac{m}{2\pi r}$ $v_\theta = 0$
Free vortex	$\psi = -\frac{\Gamma}{2\pi}\ln r$	$\phi = \frac{\Gamma}{2\pi}\theta$	$v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi r}$
Doublet	$\psi = -\frac{K\sin\theta}{r}$	$\phi = \frac{K\cos\theta}{r}$	$v_r = -\frac{K\cos\theta}{r^2}$ $v_\theta = -\frac{K\sin\theta}{r^2}$

Velocity components are related to the stream function ψ and velocity potential ϕ through the relationships:

$$v_x = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} \qquad v_y = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} \qquad v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} \qquad v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

10.2.1 Solving steps

The general solving steps are:
1. Find equivalent stream ψ or potential ϕ functions by adding the flow equations and unifying the coordinate axis;
2. Find the velocity components;
3. Use Bernoulli equation to find the pressure;
4. Find the location of stagnation points.

10.3 Potential flows not at the origin

If the potential flow is not at the origin, trigonometric must be used in order to express the stream and potential functions as functions of r and θ.

For a sink and a source lying at a distance $2a$ from each each other and a distance a from the origin, the stream function ψ is

$$\psi = -\frac{m}{2\pi}\arctan\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

which for small values of a approaches:

$$\psi = -\frac{mar\sin\theta}{\pi\left(r^2 - a^2\right)}$$

11 Dimensionless groups

11.1 Reynolds number Re

Reynolds number is a measure of the ration between the inertia force and the viscous force:

$$\mathrm{Re} = \frac{\text{inertia}}{\text{viscosity}} = \frac{\rho v_{\text{avg}} L}{\mu} = \frac{v_{\text{avg}} L}{\nu}$$

The Reynolds number determines the type of flow: laminar, transitional and turbulent flow.

11.2 Froude number Fr

11.3 Weber Number We

Weber

$$\mathrm{We} = \frac{\text{inertia}}{\text{surface tension}} = \frac{\rho {v_{\text{avg}}}^2 L}{\sigma}$$

12 Viscous flows in pipes

Poiseuille's law is the best known exact solution to the Navier-Stokes equations which applies for a steady, incompressible, axisymmetric [1] and laminar through a straight circular tube of constant cross-section.

Flows in pipes are commonly called Hagen-Poiseuille flows or simply Poiseuille flows.

[1]a flow in which the streamlines are symmetrically located around an axis.

Recall the velocity profile v_z:

$$v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left(r^2 - R^2\right)$$

The volume flow rate Q in the tube is:

$$Q = \frac{\pi R^4 P}{8\mu l} = \frac{\pi D^4 P}{128\mu l}$$

Remark. Increasing the diameter has a more dramatic effect on the flow rate than increasing the pressure drop

The average velocity is:

$$v_{z,\text{avg}} = \frac{Q}{A} = -\frac{R^2}{8\mu}\frac{\partial P}{\partial z} = \frac{R^2 \Delta P}{8\mu l}$$

The maximum velocity $v_{z,\max}$ occurs at the center of the tube:

$$v_{z,\max} = \frac{R^2 \Delta P}{4\mu l}$$

The relations between the mean velocity $v_{z,\text{avg}}$ and the maximum velocity $v_{z,\max}$ is:

$$v_{z,\max} = 2 v_{z,\text{avg}}$$

12.1 Dimensionless analysis

Using the average velocity $v_{z,\text{avg}}$ as the characteristic velocity and the diameter $D = 2R$ as the characteristic length, the dimensionless velocity profile becomes:

$$\hat{v}_z = 2(1 - 4\hat{r}^2)$$

12.2 Pipe orientation and Navier-Stokes equations

12.3 Fully developed flow

Any fluid in a pipe have to enter the pipe at some location. The region of flow near where the fluid enters the pipe is called the entrance region. A fluid typically enters the pipe with a nearly uniform velocity profile, but as the fluid moves through the pipe, viscous effects cause it to stick to the pipe wall (no-slip boundary condition).

Hence, a boundary layer in which viscous effects are important is produced along the pipe wall such that the velocity profile changes with distance x along the pipe, until the fluid reaches the end of the entrance length l_e. Beyond the entrance length l_e, the velocity profile does not vary with x.

Inside the entrance region, the viscous effect are not negligible in the boundary layer, but are negligible outside of the boundary layer, that is inside the inviscid core surrounding the center line.

The flow stays fully developed unless there are bends, Ts, change in diameter, pumps, turbine, valves…

12.3.1 Entrance length

The entrance length l_e is dependent on the Reynolds number:

$$\text{Laminar flow: } \frac{l_e}{D} = 0.058\text{Re}$$
$$\text{Turbulent flow: } \frac{l_e}{D} = 4.4\sqrt[6]{\text{Re}}$$

12.4 General characteristics fo pipe flow

For all flows studied, it was assumed the pipe is completely fill with the fluid being transported. Hence, open-channel flow will not be considered.

Table 2: Closed vs. open-channel flow

12.5 Shear stress in fully developed flow

Applying momentum analysis to a fluid element, the relationship between the shear wall stress and the velocity v_z is

$$v_z(r) = \frac{\tau_{\text{wall}} D}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right] = v_{z,\max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

since $\Delta P = \frac{4l\tau_{\text{wall}}}{D}$

12.6 Laminar pipe flow equations for inclined pipes

.

12.7 Laminar and turbulent flows

Laminar flow means the flow is made of layers, which implies it is an orderly flow. On the other hand, a turbulent flow is said to have to some degree of chaotic and unpredictable motion.

12.7.1 Mechnical stability in fluids

A fluid is stable if a disturbance is not damped by the fluid motion, while if the disturbance grows and alters the motion, the fluid is said to be unstable.

12.7.2 Order and chaos

Laminar flow: The flow is an orderly flow with no chaos and when chaos (disturbances) appears and it is damped.

Turbulent flow: The flow has some degree of chaos and main orderly motion

Therefore, laminar flow is represented by smooth lines and layers. There is no exchange between layers, and the velocity profile is a smooth curve.
However, turbulent flow has sideway motion (chaos), exchange between layers, exchange of momentum and mass, which results in a flatness in the velocity profile. Turbulent flow in a pipeline is not totally chaotic as there is still an order such as the average direction of the flow.
Transitional flow is an unstable flow, which is why it is very difficult to study.

12.8 Navier-Stokes equations

It is impossible to expect a solution that exactly match the turbulent flow, but using a time average $\bar{v}_x$ and a fluctuation components v'_x, the turbulent flow can be approximated.

Turbulence are a loss of energy, but they can be useful for heat exchange and mixing as it is quicker with a turbulent flow.

The dimensionless velocity profile of a turbulent flow can be approximated as:

$$\bar{v}_z = \bar{v}_{z,\max}\left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

where n change the from laminar profile to fully developed turbulent profile. In order to know n, there is a relation ship between n and the Reynolds number Re. For most industrial applications in pipes, the Reynolds number Re is $\mathrm{Re} \approx 10^5$, which leads to $n = 7$.

The shear-stress τ has a laminar component $\tau_{\text{laminar}} = \mu\frac{\mathrm{d}\bar{v}_x}{\mathrm{d}y}$ and a turbulent component $\tau_{\text{turbulent}} = -\rho v'_x v'_y$ where v'_x and v'_y are the fluctuations.

12.8.1 Flow disturbances

In turbulent flows a viscous laminar sublayer exists close the wall. For rough walls, the disturbances are increased due to the holes and bumps of the wall.

Disturbances are minimal in laminar flow, but they are not negligible in turbulent flow as they are responsible for destabilization of the flow. Disturbances comes from the roughness of the walls and this explains why there are vanes and nets in wind tunnels in order to fight against disturbances.

12.9 Friction in pipes

Using the dimensional analysis, the friction factor for laminar flow in pipes is

$$f = \frac{64}{\mathrm{Re}} = \frac{8\tau_{\text{wall}}}{\rho v^2} \implies \Delta P = f\frac{L}{D}\frac{\rho v^2}{2}$$

However, for turbulent flows in pipes, the Moody chart is used and valid for the entire non-laminar range and is based on the Colebrook equation:

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{1}{3.7}\frac{\epsilon}{D} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

Hence the friction factor is now also a function of the dimensionless roughness of the pipe wall.

In order to get the roughness ϵ, Table 3 is used.

Also, the volumetric flow rate Q of turbulent flow is:

$$Q = 2\pi R^2 v_{\max}\frac{n^2}{(n+1)(2n+1)}$$

and

$$\frac{v_{\text{avg}}}{v_{\max}} = \frac{2n^2}{(n+1)(2n+1)}$$

12.10 Energy

The energy equation is a form of the Bernoulli equation accounting for other parameters:

$$\frac{P_1}{\rho g} + z_1 + \alpha_1\frac{{v_1}^2}{2g} + h_p = \frac{P_2}{\rho g} + z_2 + \alpha_2\frac{{v_2}^2}{2g} + h_L + h_t$$

P[Pa]: fluid pressure
$\rho[\mathrm{kg\,m^{-3}}]$: fluid density
$g = 9.81\,\mathrm{m\,s^{-2}}$: gravitational acceleration
z[m]: fluid height
α: kinetic energy coefficients
$v[\mathrm{m\,s^{-1}}]$: fluid velocity
h_p[m]: pump head
h_L[m]: head loss
h_t[m]: turbine head

12.10.1 Type of losses

12.10.2 External device (pump, turbine)

In the case of a pump, a pump head component h_p is added to the equation, where the relation between the pump power input and the head pump is

$$\dot{W}_p = \rho g Q h_p$$

$\dot{W}_p$[W]: pump power input
$\rho[\mathrm{kg\,m^{-3}}]$: fluid density
$g = 9.81\,\mathrm{m\,s^{-2}}$: gravitational acceleration
$Q[\mathrm{m^3\,s^{-1}}]$: fluid volumetric flow rate
h_p[m]: pump head

For a turbine, the concept is very similar: a turbine head component h_t is added to the equation, where the relation between the turbine power output and the head turbine is

$$\dot{W}_t = \rho g Q h_t$$

$\dot{W}_t$[W]: turbine power output
$\rho[\mathrm{kg\,m^{-3}}]$: fluid density
$g = 9.81\,\mathrm{m\,s^{-2}}$: gravitational acceleration
$Q[\mathrm{m^3\,s^{-1}}]$: fluid volumetric flow rate
h_p[m]: turbine head

Table 3: Equivalent roughness for new pipes

Pipe	ϵ [ft]	ϵ [m]
Riveted steel	$[3 \cdot 10^{-3}, 30 \cdot 10^{-3}]$	$[900 \cdot 10^{-6}, 9 \cdot 10^{-3}]$
Concrete	$[1 \cdot 10^{-3}, 10 \cdot 10^{-3}]$	$[300 \cdot 10^{-6}, 3 \cdot 10^{-3}]$
Wood stave	$[600 \cdot 10^{-6}, 3 \cdot 10^{-3}]$	$[180 \cdot 10^{-6}, 900 \cdot 10^{-6}]$
Cast iron	$850 \cdot 10^{-6}$	$260 \cdot 10^{-6}$
Galvanized iron	$500 \cdot 10^{-6}$	$150 \cdot 10^{-6}$
Commercial steel	$150 \cdot 10^{-6}$	$45 \cdot 10^{-6}$
Drawn tubing	$5 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
Plastic, glass (smooth)	0	0

12.11 Non-circular conduit

The hydraulic diameter D_h is used in case of non-circular pipes in order to use the equations of the circular pipes.

$$D_h = \frac{4A}{p_{\text{wet}}}$$

D_h[m]: hydraulic diameter
A[m^2]: cross-sectional area
$fore$

12.12 Multiple pipe flow

12.12.1 Series connection

For series pipe connection, the volumetric flow rate Q is constant. For $n \in \mathbb{R}$ pipe section, the volumetric flow rate Q is

$$Q = Q_n$$

On the other hand, the losses h_L are

$$h_L = \sum h_{L,n}$$

12.12.2 Parallel connection

For series pipe connection, the volumetric flow rate Q is constant. For $n \in \mathbb{R}$ pipe section, the volumetric flow rate Q is

$$Q = \sum Q_n$$

On the other hand, the losses h_L are

$$h_L = h_{L,n}$$