Optomechanical cooling with time-dependent parameters

Pablo Yanes-Thomas, Marc Bienert, and Pablo Barberis-Blostein
Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas,
Universidad Nacional Autonoma de Mexico, Circuito Escolar s/n Ciudad Universitaria Mexico, D.F.
(Dated: December 9, 2018)

We model the laser cooling of a parametrically driven optomechanical cavity using a dissipation model that accounts for the modification of the quasi-energy spectrum caused by the driving. We construct a master equation for the mechanical object using Floquet operators. When the natural frequency of the mechanical object oscillates periodically around its mean value, we derive, using an adiabatic approximation, an analytical expression for its temperature. This expression depends both on the oscillator's mean frequency and that of the frequency's oscillations around its mean value. We find that the temperature can be lower than in the non-time dependent case. Our results raise the possibility of achieving lower temperatures for the mechanical object if its natural frequency can be controlled as a function of time.

I. INTRODUCTION

Quantum cavity optomechanics studies systems composed of macroscopic mechanical objects, such as mirrors, and an optical cavity's quantized light field usually coupled via radiation pressure. In a common scheme an end-mirror of a Fabry-Perot cavity is suspended while being able to freely oscillate. When photons are reflected by the mirror, there is a momentum transfer between the light field and the mirror and hence a coupling force. As the cavity's resonance depends on its length, the mechanical displacement affects the light field inside the cavity. Some of the first theoretical work predicting this sort of coupling was performed by [1]. This interaction between the macroscopic mechanical object and the light field leads to several interesting effects such as optomechanically induced transparency [2], the optical spring effect [3] or, most relevant to this study, optomechanical cooling [4–7].

Optomechanical cooling was first proposed by Mancini, et al in [8]. It is the damping of the end mirror's mechanical motion due to the radiative coupling to the cavity field. Sideband cooling takes place when the cavity's resonance is much narrower than the mechanical frequency. It can be understood as Raman scattering [9] of incident photons, red-detuned from the cavity resonance, when the parameters are chosen to favor phonon absorption from the mechanical oscillator in order to scatter the photon into the cavity's resonance mode, resulting in cooling. For coherent quantum control over a mechanical object, it must be close to a pure quantum mechanical state [10] so effective methods of cooling macroscopic objects to low temperatures is highly desirable.

One possible avenue for improved cooling lies in controlling the mechanical resonator's frequency as a function of time [11]. The effect of this dependence on the mechanical object's final temperature was studied in [12]. In that study it was found that the final temperature of the parametrically driven harmonic oscillator was larger than the nondriven case; The master equation from which the cooling rates were derived, accounted for the natural frequency of the mechanical resonator via ad-hoc time-dependent coefficients. These were introduced after performing the Markov approximation. This approach could lead to an incomplete master equation and thus an incomplete description of the system's dynamics, as the system is being treated as essentially not time-dependent during the derivation of the master equation. In this study, we employ a different method that accounts for the time dependence throughout the entire derivation.

The formalism which we apply (section II) is based on Floquet theory and was demonstrated to be a more accurate treatment in [13]. For the case where the drive consist of a small periodic oscillation with respect to the central frequency of the mechanical oscillator, the Floquet operators can be given explicitly (section III). Under the adiabatic approximation, we derive an approximated expression for the mean phonon occupation number in the final stages of optomechanical cooling (section IV). We perform numerical calculations in order to compare this expression to the one for the non-time-dependent case (section V). We find, using the formalism presented here, that lower temperatures can be obtained if the mechanical object is parametrically driven. Our results suggest that the theoretical model of dissipation of the mechanical osscilator can have a significant influence on the resulting temperature. (section VI).

II. OPTOMECHANICAL HAMILTONIAN

A. Hamiltonian with Floquet Operators

We employed the standard Hamiltonian for optomechanical cooling [7] in a reference system that rotates with the same frequency as a laser that continuously pumps photons into the cavity.

$$H(t) = H_{mec}(t) + H_{cav} + H_{int} + H_{pump}, \tag{1}$$

where

$$H_{cav} = -\hbar\delta a^\dagger a, \tag{2}$$
$$H_{mec}(t) = \frac{p^2}{2M} + \frac{1}{2}M\nu^2(t)x^2, \tag{3}$$
$$H_{int} = -\hbar g a^\dagger a x, \tag{4}$$
$$H_{pump} = \hbar\frac{\Omega}{2}(a^\dagger + a), \tag{5}$$

$\delta = \omega_{laser} - \omega_{cav}$ is the frequency difference between the laser and the cavity. M is the mechanical oscillator's mass and $\nu(t)$ is its frequency. In order to employ the Floquet formalism $\nu(t)$ must be a periodic function of time. The H_{int} term models the interaction between photons and the mirror, and g sets the strength of the coupling. Finally, Ω describes the strength of the cavity pump. Readers interested in a derivation of the interaction term should consult [10]. In our case, the only term with an explicit time dependence is the term for the mechanical oscillator.

The Floquet operators are analogous to the usual creation and annihilation operators for the standard harmonic oscillator and can be expressed in terms of the mechanical oscillator's position and momentum operators [13]. These operators are

$$\Gamma(t) = \frac{1}{2i}\left[\hat{x}\sqrt{\frac{2M}{\hbar}}\dot{f}(t) - \hat{p}\sqrt{\frac{2}{M\hbar}}f(t)\right], \tag{6}$$

as well as its Hermitian conjugate. $f(t)$ is the solution to the classical time-dependent harmonic oscillator equation of motion in one dimension and is generally a complex function [14]

$$\ddot{f} + \nu(t)^2 f = 0. \tag{7}$$

This equation has two solutions [13] of the form

$$f(t) = e^{i\mu t}\phi(t), \tag{8}$$

and its complex conjugate. $\phi(t)$ is a periodic function of time with the same period as $\nu(t)$. μ is, in general, a complex number [15]. These operators follow the usual commutation relations for creation and annihilation operators

$$[\Gamma(t)^\dagger, \Gamma(t)] = 1. \tag{9}$$

Using these operators $H_{mec}(t)$ can be written in the same form as the non time-dependent harmonic oscillator with the Floquet operators taking the place of the annihilation operators, with the exception of a global time-dependent scalar coefficient [14]

$$H_{mec}(t) = \hbar\frac{W}{|f(t)|^2}\left[\Gamma^\dagger(t)\Gamma(t) + \frac{1}{2}\right], \tag{10}$$

with the Wronskian W for (7). Equation (6) is then inverted and solved for the harmonic oscillator's position operator [16]

$$x = \frac{b^*\Gamma - b\Gamma^\dagger}{(b^*a - ba^*)} \tag{11}$$

with

$$a = \frac{1}{2i}\sqrt{\frac{2M}{\hbar}}\dot{f}, \tag{12}$$
$$b = \frac{1}{2i}\sqrt{\frac{2}{M\hbar}}f. \tag{13}$$

These are then substituted back into the interaction Hamiltonian resulting in

$$H_{int}(t) = g\sqrt{\frac{\hbar}{2M}}a^\dagger a[\gamma_+(t)\Gamma(t) + \gamma_-(t)\Gamma^\dagger(t)], \tag{14}$$

with new coefficients

$$\gamma_+(t) = \frac{f^*}{(f^*\dot{f} - f\dot{f}^*)},$$
$$\gamma_-(t) = \frac{f}{(f^*\dot{f} - f\dot{f}^*)}.$$

The Hamiltonian contains two separate harmonic oscillator-like terms that commute with each other, H_{cav} and H_{mec}, so the standard harmonic oscillator master equation structure can be employed [16][13]. The usual derivation of the master equation involves a Markov approximation, under the formalism we employ, frequency's time dependence is accounted for during this approximation [13] via the Floquet operators. This differs from previous attempts to study this type of system, where this dependence was included after the Markov approximation had been performed via time-dependent ad-hoc coefficients for the damping [12]. As demonstrated in [13], the method employed here is a more complete, and thus accurate, treatment.

The corresponding master equation is

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + L_a\rho + L_\Gamma\rho, \tag{15}$$

where

$$\begin{aligned} L_a\rho = &-\frac{\kappa}{2}(n_p + 1)[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] \\ &-\frac{\kappa}{2}(n_p)[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a], \end{aligned} \tag{16}$$

$$\begin{aligned} L_\Gamma\rho = &-\frac{\gamma}{2}(n_m + 1)[\Gamma^\dagger\Gamma\rho + \rho\Gamma^\dagger\Gamma - 2\Gamma\rho\Gamma^\dagger] \\ &-\frac{\gamma}{2}(n_m)[\Gamma\Gamma^\dagger\rho + \rho\Gamma\Gamma^\dagger - 2\Gamma^\dagger\rho\Gamma], \end{aligned} \tag{17}$$

κ is the energy decay rate for the cavity and γ is the decay rate for the mechanical oscillator. The number of thermal excitations for the cavity and the oscillator are given by n_p and n_m respectively [17]. The superoperators L_Γ and L_a model the energy exchanges between the environment and the cavity and the mechanical resonator respectively. The formalism developed in [13] was used to derive the mechanical dissipation term (17).

B. Displaced Frame

In order to eliminate the pump term and to find useful approximations, we employ a unitary transformation to shift (15) into a displaced reference frame. This transformation depends on two time-dependent coefficients, $\alpha(t)$ and $\beta(t)$, which are chosen in a convenient manner to simplify the Hamiltonian. The transformation is given by the operator

$$U_{a,\Gamma} = e^{(\alpha(t)a^\dagger - \alpha^*(t)a)}e^{(\beta(t)\Gamma^\dagger - \beta^*(t)\Gamma)}, \qquad (18)$$

and results in a displaced Hamiltonian and in turn a displaced master equation

$$\dot{\rho}' = \frac{1}{i\hbar}[H', \rho'] + L_a\rho' + L_\Gamma\rho' + C(t)\rho' , \qquad (19)$$

for the time evolution of the density operator $\rho'(t)$ which represents the coupled cavity-mechanical resonator system. The primes indicate that the transformation has been applied. The displaced Hamiltonian, which includes a pump-like term that appears when making the transformation on the L operators, is given by

$$\begin{aligned} H' =& -\hbar\delta' a^\dagger a + \hbar\frac{W}{|f(t)|^2}\Gamma\Gamma^\dagger \\ & - \hbar g\sqrt{\frac{\hbar}{2M}}[(a^\dagger a + \alpha a^\dagger + \alpha^* a)(\gamma_-(t)\Gamma^\dagger + \gamma_+(t)\Gamma)] \\ & + i\hbar(\beta^*\dot{\Gamma} - \beta\dot{\Gamma}^\dagger), \end{aligned}$$

with $\delta' = \delta + g\sqrt{\frac{\hbar}{2M}}(\beta+\beta^*)$. This Hamiltonian is valid as long as the coefficients $\alpha(t)$ and $\beta(t)$ fulfill the differential equations

$$\dot{\alpha} =\alpha(-\frac{A}{2} + i(\delta + g\sqrt{\frac{\hbar}{2M}}(\gamma_-(t)\beta^* + \gamma_+(t)\beta)) - i\frac{\Omega}{2}, \qquad (20)$$

$$\dot{\beta} =\beta(-\frac{\gamma}{2} - i\frac{W}{|f(t)|^2}) + ig\sqrt{\frac{\hbar}{2M}}|\alpha|^2\gamma_+(t). \qquad (21)$$

The $C(t)$ term

$$C(t)\rho = |\beta|^2(C(t)_{+-} - C(t)_{-+})\rho,$$

where

$$\begin{aligned} C(t)_{+-} =& [\dot{\Gamma}^\dagger, \Gamma], \\ C(t)_{-+} =& [\dot{\Gamma}, \Gamma^\dagger], \end{aligned}$$

appears because the Floquet operators do not necessarily commute with their own time derivatives. In the case of the Hamiltonian for the cavity's light field, the operators contain no explicit time dependence. The Floquet operators, however, do include such a dependence, which introduces additional terms into the master equation. These terms involve the commutators between the Floquet operators and their time derivatives and contain no operator dependence whatsoever and as such are not considered part of the Hamiltonian. These commutation relations are not, in general, zero [16], they depend on the specific form of the solutions for (7), f and f^*. For example

$$C(t)_{-+} = -(C(t)_{+-})^* = \frac{i}{2}(\dot{f}^*\dot{f} - \ddot{f}^*f^*). \qquad (22)$$

Proceeding further requires an explicit solution for (7). The primes in the operators will be omitted from now on as all calculations will be done in the displaced frame.

III. SOLUTION FOR SMALL OSCILLATIONS

In order to obtain an explicit solution for (7) we focus on the case of small oscillations around a central frequency, specifically

$$\nu(t) = \nu_0 + \epsilon' cos(2\omega t), \qquad (23)$$

with $\epsilon \ll \nu_0$ where ν_0 is the mean frequency. This leads to the time-dependent harmonic oscillator equation

$$\ddot{x} + (\nu_0^2 + 2\epsilon'\nu_0 cos(2\omega t))x = 0, \qquad (24)$$

which is a particular case of the Mathieu equation [18]. In order to put the equation into the standard form we employ the change of variables $t' = \omega t$ and set $\epsilon = \frac{2\epsilon'\nu_0}{\omega^2}$ and use the scattering relation

$$\frac{\nu_0^2}{\omega^2} = n^2 \qquad (25)$$

with $n \in \mathbb{Z}^+$ in order to guarantee stable solutions [15]. Under these restrictions and returning to the original variable t, the solutions for (23) are, to first order in ϵ and for the case of $n = 1$

$$f(t) = e^{i\omega t} + \frac{\epsilon}{16}e^{3i\omega t}, \qquad (26)$$

and its complex conjugate, which is equal to $f(-t)$.

A. Explicit Expressions for Small Oscillations

With an explicit solution for (23) in hand, we obtain explicit solutions for the Floquet operator commutator terms by direct substitution of (26) into (6) [16]

$$C(t)_{+-} = i[1 - \frac{\epsilon}{16}e^{2i\omega t} - \frac{6\epsilon}{16}e^{-2i\omega t}], \tag{27}$$
$$C(t)_{-+} = i[1 - \frac{\epsilon}{16}e^{-2i\omega t} - \frac{6\epsilon}{16}e^{2i\omega t}]. \tag{28}$$

The $\gamma_{\pm}$ coefficients can be calculated as well

$$\gamma_{\pm} = \frac{1}{\omega}e^{\mp i\omega t} \tag{29}$$

as can be the overall time-dependent factor for the time-dependent harmonic oscillator expressed in terms of Floquet operators,

$$\frac{W}{|f|^2} = \omega. \tag{30}$$

With these expressions we may solve for the $\alpha(t)$ and $\beta(t)$ coefficients. The equations are

$$\dot{\alpha} = \alpha(-\frac{A}{2} + i(\delta + g\sqrt{\frac{\hbar}{2M}}(e^{i\omega t}\beta^* + e^{-i\omega t}\beta)) - i\frac{\Omega}{2}, \tag{31}$$
$$\dot{\beta} = \beta(-\frac{\gamma}{2} - i2\omega) + ig\sqrt{\frac{\hbar}{2M}}|\alpha|^2 e^{i\omega t}, \tag{32}$$

however our focus is on the stationary case ($\dot{\alpha}(t) = \dot{\beta}(t) = 0$) and in a weak coupling regime, so coefficients of first order in g or higher are neglected, which results in two simplified equations

$$0 = \alpha(-\frac{A}{2} + i\delta) - i\frac{\Omega}{2}, \tag{33}$$
$$0 = \beta(-\frac{\gamma}{2} - i2\omega), \tag{34}$$

which are trivially solved

$$\alpha_0 = \frac{\Omega}{2\delta - iA}, \tag{35}$$
$$\beta_0 = 0. \tag{36}$$

The 0 sub-index denotes that the solutions are valid only to zeroth order in the coupling parameter.

B. Laser Cooling Hamiltonian

Under these parameters we can neglect the $C(t)$ terms as they figure into the master equation as terms of first order in ϵ and second order in β. The term involving the time derivatives of the Floquet operators can also be neglected for this reason. The Hamiltonian can then be written as

$$\begin{aligned} H = & -\hbar\delta a^\dagger a + \hbar\omega\Gamma^\dagger\Gamma \\ & - \hbar g\sqrt{\frac{\hbar}{2M}}(a^\dagger a + \alpha_0 a^\dagger + \alpha_0^* a)(\gamma_-(t)\Gamma^\dagger + \gamma_+(t)\Gamma). \end{aligned} \tag{37}$$

We focus on the case where $|\alpha| \gg 1$ [12], so that the $a^\dagger a$ term can be neglected as it is small when compared to the other two terms in the interaction. This leads to a further simplified Hamiltonian

$$\begin{aligned} H(t) = & -\hbar\delta a^\dagger a + \hbar\omega\Gamma^\dagger\Gamma \\ & + \frac{\hbar g\sqrt{\frac{\hbar}{2M}}}{\omega}(\alpha_0 a^\dagger + \alpha_0^* a)(e^{i\omega t}\Gamma^\dagger + e^{-i\omega t}\Gamma) \end{aligned} \tag{38}$$

This Hamiltonian corresponds to a standard optomechanical master equation with Floquet operators for the mechanical oscillator

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + L_a\rho + L_\Gamma\rho, \tag{39}$$

where L_a corresponds to the cavity's light field [19] and L_Γ corresponds to the time dependent harmonic oscillator [13].

Equation (39) represents a master equation for a parametric optomechanical system with an improved dissipation model which accounts for the mechanical oscillator's time dependent frequency.

IV. LASER COOLING

Our focus is on the parameter regime where the mechanical resonator's temperature evolves much more slowly than the cavity's losses and than the mechanical frequency. This requires $(g\sqrt{\frac{\hbar}{2M}}\alpha_o)^2 \ll (\frac{\kappa}{\omega_m})$. Following the derivation found in [7], we arrive at a master equation for the density operator, after projecting into the subspace corresponding to its stationary state and tracing over the cavity degrees of freedom. The technical details of the derivation is reported in App. B. It leads to the master equation

$$\dot{\mu} \approx -\frac{g'^2}{2}G(\nu_m, n_c)[\Gamma^\dagger, \Gamma\mu] - G^*(-\nu_m, n_c)[\Gamma^\dagger, \mu\Gamma], \tag{40}$$

where we have set $g' = g\sqrt{\frac{\hbar}{2M}}\alpha_0$ for convenience and use μ as the density operator of the mechanical degree of freedom. The cavity-quadratures correlation $G(\nu_m, n_p)$ can be used to calculate the heating and cooling rates for the mirror and is given by

$$G(\nu_m, n_p) = \int_0^\infty dt e^{i\nu(t)t} Tr_c[X_c e^{L_c t} X_c \rho_{st}], \tag{41}$$

with

$$X = \frac{a + a^\dagger}{\sqrt{2}\alpha_0}, \tag{42}$$

the cooling and heating rates are given by

$$A_{\pm\nu}(n_p) = g'^2 Real(G(\mp\nu_m, n_p)). \tag{43}$$

where A_+ represents heating and A_- represents cooling. The final number of phonon excitations is given by

$$\langle m \rangle = \langle \Gamma^\dagger \Gamma \rangle = \frac{A_+}{A_- - A_+}. \tag{44}$$

Thus if (41) can be calculated, the heating and cooling rates and the final temperature (represented by the average number of phonon excitations) can be obtained in a straightforward manner. This procedure is done numerically in the following section.

V. CALCULATION OF MEAN VIBRATIONAL OCCUPATION NUMBER

In order to perform numerical computations we calculate (41) up to first order in ϵ. We assume that the cavity is at zero temperature ($n_p = 0$). λ_c represents the eigenvalues for the cavity's states. The trace inside (41) can be easily calculated, which leaves the integration

$$\begin{aligned}
G(\nu, 0) &= \int_0^\infty dt e^{(i\nu(t)+\lambda_c)t} Tr[...] \\
&= \int_0^\infty e^{i\nu=\nu_0+\epsilon cos(2\omega t)t+\lambda t} Tr[...]dt, \\
&= \int_0^\infty e^{i\nu_0 t+\lambda t} e^{i\epsilon cos(2\omega t)t} Tr[...]dt, \\
&\approx \int_0^\infty e^{i\nu_0 t+\lambda t}(1 + i\epsilon cos(2\omega t)t) Tr[...]dt, \\
&= \int_0^\infty e^{i\nu_0 t+\lambda t} Tr[...]dt \\
&+ i\epsilon \int_0^\infty cos(2\omega t) t e^{i\nu_0 t+\lambda t} Tr[...]dt.
\end{aligned}$$

This leads to an expression for (41) which consist of two parts: one corresponding to the nondriven harmonic oscillator case plus a correction term proportional to ϵ

$$\frac{G(\nu, 0)}{g'^2} = \frac{1}{-k + 2i(\delta + \nu_0)} + i\epsilon \frac{(-k + i(\nu_0 + \delta))^2 - 4\omega^2}{((-k + i(\nu_0 + \delta))^2 + 4\omega^2)^2}, \tag{45}$$

We evaluate the real part of this expression numerically. In order for the adiabatic approximation to make sense, the variation in the mechanical oscillator's frequency must be slow when compared to the frequency itself ($\omega \ll \nu_0$) which translates to classical solutions with $1 \ll n$ due to (25). Assuming that, originally $\epsilon = \frac{\nu_0}{10}$ we now have the condition $\epsilon = \frac{n^2}{5}$. We wish for n to be sufficiently large so as to fulfill the condition relating ν_0 and ω but small enough so that $\epsilon \ll \nu_0$. To this end we chose $n = \sqrt{\frac{\nu_0}{2}}$ (rounded to the nearest integer) which maintains the previous ratio between the two frequencies. Physically, we employ a classical solution which describes a situation where the oscillator's average ν_0 frequency is clearly the fastest one, while the secular frequency ω is of the order of the perturbation frequency ϵ. As a calculation variable, we employ the ratio between the detuning δ and ν_0 covering the range $\frac{\delta}{\nu_0} \in [-2, 2]$

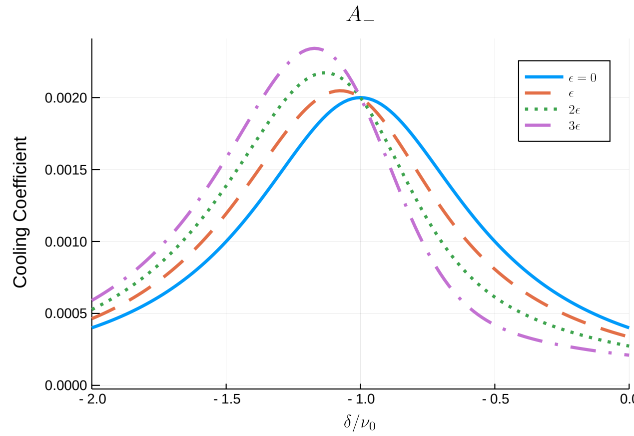

Figure 1. Cooling sideband for different perturbation levels. Besides all the scattering relations already stated, we work under $\kappa \ll \nu_0$

The results of this prediction for the number of mechanical excitations can be seen in figure 2. The prediction presents a shift of where to expect the minimum number of excitations and also, most interestingly, a region where the predicted number of excitations is smaller than in the non-driven case.

The temperature shift depends on ϵ. As long as ν_0 and ω fulfill (25), altering them does not seem to alter the behavior of the final temperature. This indicates that the shift is related to the time dependent nature of the correction. One possible explanation for this shift is that the ϵ term causes the cooling and heating sidebands to shift away from each other. The common crossing point indicates the point where the resonance is placed at a symmetry point for the sideband's position and thus the effect cancels out. After the shift, the resonance point with each of the sidebands is further away from the optimal point for the other sideband. This effect can be observed in figure 1. The parameters used appear reasonable, as controlled frequency modulation has been shown [20][21].

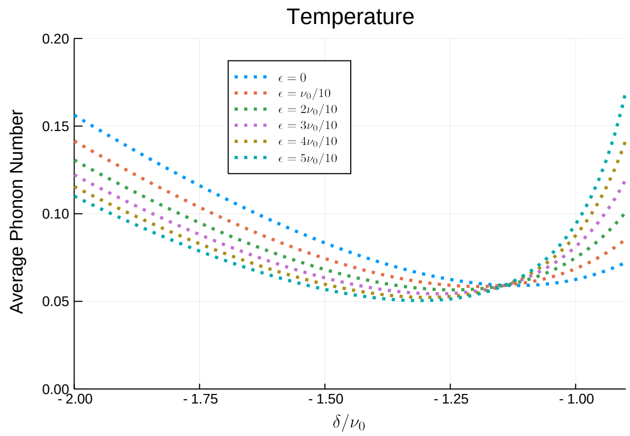

Figure 2. *Comparison between the results for temperature with varying levels of perturbation.* The correction results in a shift of the location of the minimum temperature, as well as a lower minimum. All temperature curves cross the same point. Besides all the scattering relations already stated, we work under $\kappa \ll \nu_0$

VI. CONCLUSIONS

Using an improved theoretical description of dissipation of a parametrically driven mechanical object, in an optomechanical setup, we found that the lowest temperature reached can be lower than in the non-driven case. This implies that periodically changing the natural frequency of the mechanical object in optomechanics can be used for reaching lower temperatures. This also opens the path for exploring what happens if the movement of a leaky cavity is taken into consideration when theoretically treating the dissipation of the cavity field, because, similarly to the setup study in this paper, the natural frequency of the cavity field changes periodically with time. We plan to explore this case in future work.

Appendix A: The Damping Basis

Master equations of the type

$$\dot{\rho} = \frac{i}{\hbar}[H,\rho] + L\rho, \tag{A1}$$

with

$$\begin{aligned} L_a\rho =& -\frac{A}{2}(\nu+1)[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] \\ & -\frac{A}{2}(\nu)[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a], \end{aligned} \tag{A2}$$

model the behavior of a bosonic field inside a cavity with just one mode and with ν thermal photons in contact with a thermal bath. A is a constant related to the damping and $A, \nu \geq 0$ [22]. The density operator can be expressed in the Lindblad superoperator's eigenbasis to simplify the calculations. This is known as the damping basis [22].

$$a^{\dagger l}\frac{(-1)^n}{(\nu+1)^{l+1}} : L_n^l[\frac{a^\dagger a}{\nu+1}]e^{-[\frac{a^\dagger a}{\nu+1}]} : \quad l \geq 0, \tag{A3}$$

$$\frac{(-1)^n}{(\nu+1)^{|l|+1}} : L_n^{|l|}[\frac{a^\dagger a}{\nu+1}]e^{-[\frac{a^\dagger a}{\nu+1}]} : a^{|l|} \quad l \leq 0, \tag{A4}$$

with eigenvalues

$$\lambda_n^l = -A[n + \frac{|l|}{2}], \tag{A5}$$

which fulfill

$$n = 0, 1, 2..., \qquad l = 0, \pm 1, \pm 2, ... \tag{A6}$$

These are the right eigenstates, which correspond to $L\hat{\rho_n^l} = \lambda_n^l \hat{\rho_n^l}$, however the left eigenstates must also be considered, these correspond to $\check{\rho_n^l} L = \lambda_n^l \check{\rho_n^l}$ and have the same eigenvalues. These are

$$(\frac{-\nu}{\nu+1})^n \frac{n!}{(n+l)!} : L_n^l[\frac{a^\dagger a}{\nu}] : a^l \quad l \geq 0, \tag{A7}$$

$$(\frac{-\nu}{\nu+1})^n \frac{n!}{(n+|l|)!} a^{\dagger |l|} : L_n^{|l|}[\frac{a^\dagger a}{\nu}] : \quad l \leq 0. \tag{A8}$$

These states are orthogonal under the trace

$$Tr[\hat{\rho}_{n,l}\check{\rho}_{n',l'}] = \delta_{n,n'}\delta_{l,l'}. \tag{A9}$$

An important approximation can be obtained for a cavity at zero temperature, in these case the states are [22]

$$a^{\dagger l}(-1)^{a^\dagger a + n}\binom{n+l}{a^\dagger a + l} \quad l \geq 0, \tag{A10}$$

$$(-1)^{a^\dagger a + n}\binom{n+|l|}{a^\dagger a + |l|} a^{|l|} \quad l \geq 0, \tag{A11}$$

and the dual states are

$$\frac{n!}{(n+l)!}\binom{a^\dagger a}{n} a^l \quad l \geq 0, \tag{A12}$$

$$a^{\dagger |l|}\frac{n!}{(n+|l|)!}\binom{a^\dagger a}{n} \quad l \geq 0. \tag{A13}$$

Appendix B: Laser Cooling and Projection Operators

In order to find the master equation (40) we employed projection operators like those in [23], these fulfill a completeness relation

$$1 = P + Q, \tag{B1}$$

and have the properties

1. $PL_0 = L_0P = 0$ as the corresponding eigenvalues are 0
2. $PL_1P = 0$ as the interaction does not couple states in P
3. $P^2 = P \quad Q^2 = Q$ as P and Q are projectors.

In this case $P = P_m^{\lambda_m^0} P_c^{\lambda_c^0}$. We project the master equation $\dot{\rho} = L\rho$ into both P and Q spaces and insert the completeness relation to obtain two equations. Working in the decay picture

$$\begin{aligned}
\rho' =& e^{L_0 t}\rho, \\
L' =& e^{-L_0 t} L e^{L_0 t} = L_1'.
\end{aligned}$$

and omitting primes, the equations are

$$\begin{aligned}
P\dot{\rho} =& PL_1Q\rho, \\
Q\dot{\rho} =& QLQ\rho + QLP\rho.
\end{aligned}$$

The equation for Q can be formally integrated [16]

$$\begin{aligned}
Q\rho =& Q\rho(t_0) + \int_{t_0}^{t} dt' QL(t')P\rho(t') \\
&+ \int_{t_0}^{t} dt' QL(t')Q\rho(t'), \\
\simeq& Q\rho(t_0) + \int_{t_0}^{t} dt' QL_1(t')P\rho(t_0) \\
&+ \int_{t_0}^{t} dt' QL_1(t')Q\rho(t_0) + O(\eta^2),
\end{aligned}$$

and this is substituted into the P equation

$$\begin{aligned}
P\dot{\rho}(t) =& PL_1Q\rho(t-\Delta t) \\
&+ PL_1 \int_{t_0}^{t} dt' QL_1(t')P\rho(t-\Delta t) \\
&+ PL_1 \int_{t_0}^{t} dt' QL_1(t')Q\rho(t-\Delta t),
\end{aligned} \tag{B2}$$

where only the last term is non zero. Transforming back from the decay picture

$$Pe^{-L_0 t} L_1 e^{L_0 t} \int_{t-\Delta t}^{t} dt' Q e^{-L_0 t} L_1 e^{L_0 t} P e^{-L_0 t} \rho(t-\Delta t). \tag{B3}$$

Noting that the factors that multiply the Floquet operators cancel out their time dependence and decomposing the projectors into the forms $P = \sum_\lambda = \hat{\rho}_\lambda \otimes \check{\rho}_\lambda$ and $Q = \sum_{\lambda'} = \hat{\rho}_{\lambda'} \otimes \check{\rho}_{\lambda'}$ the time integral can be evaluated to obtain

$$P\dot{\rho} = \sum_{\lambda,\lambda'} Pe^{-L_0 t} L_1 \rho_\lambda L_1 \rho_{\lambda'} \rho(t-\Delta t) \frac{1}{\lambda' - \lambda}. \tag{B4}$$

We multiply by $Pe^{L_0 t}$ and return the projectors to their previous notation to obtain a projected master equation.

[1] V. B. BRAGINSKII, A. B. MANUKIN, and M. Yu. TIKHONOV. Investigation of dissipative ponderomotive effects of electromagnetic radiation. *Soviet Physics JETP*, 1970.
[2] S Weiss, et al. Optomechanically induced transparency. *Science*, 2010C.
[3] M. Vogel, C. Mooser, and K. Karraia. Optically tunable mechanics of microlevers. *Applied Physics Letters*, 2003.
[4] M. Pinard P. F. Cohadon, A. Heidmann. Cooling of a mirror by radiation pressure. *Physical Review Letters*, 1999.
[5] Thomas Corbitt, Christopher Wipf, et al. Optical dilution and feedback cooling of a gram-scale oscillator to 6.9 mk. *Physical Review Letters*, 2007.
[6] A. Schliesser, P. Del Haye, et al. Radiation pressure cooling of a micromechanical oscillator using dynamical backaction. *Physical Review Letters*, 2006.
[7] I Wilson-Rae, N. Nooshi, et al. Cavity-assisted back action cooling of mechanical resonators. *New Journal of Physics*, 2008.
[8] Stefano Mancini, David Vitali, and Paolo Tombesi. Optomechanical cooling of a macroscopic oscillator by homodyne feedback. *Physical Review Letters*, 1998.
[9] et al Florian Marquardt. Quantum theory of optomechanical cooling. *arXiv:0803.1164v1 [quant-ph]*, 2008.
[10] Thomas J. Kippenberg, Thomas Marquardt, Markus Aspelmeyer. Cavity optomechanics. *Review of Modern Physics*, 2013.

[11] A. Jockel, M. T. Rakher. Realization of an optomechanical interface between ultracold atoms and a membrane. *Physical Review Letters*, 107, November 2011.

[12] Marc Bienert, Pablo Barberis-Blostein. Optomechanical laser cooling with mechanical modulation. *Physical Review Letters A*, 2015.

[13] Sigmund Kohler, Thomas Dittrich† , Peter Hänggi. Floquet-markov description of the parametrically driven, dissipative harmonic quantum oscillator. *Phys.Rev. E55*, 30 Sep 1998.

[14] Lowell S. Brown. Quantum motion in a paul trap. *Physical Review Letters*, 1991.

[15] M. Ward. *Supplement: Floquet Theory and the Mathieu's Equation.*

[16] Pablo Enrique Yanes Thomas. Enfriamiento optomecanico con parametros dependientes del tiempo. Master's thesis, Universidad Nacional Autonoma de Mexico, 2016.

[17] *Five Lectures on Dissipative Master Equations (in Coherent Evolution in Noisy Environments).* Springer, 2002.

[18] Artur R. Pietrykowski Marcin Piatek. Classical irregular blocks, hill's equation and pt–symmetric periodic complex potentials. *arXiv:1604.03574v1 [hep-th]*, 2016.

[19] Peter Zoller. *Quantum Noise:A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics.* Springer Science and Business Media, 2004.

[20] G. J. Milburn M. J. Woolley, A. C. Doherty and K. C. Schwab. Nanomechanical squeezing with detection via microwave cavity. *Physical Review A*, 2008.

[21] et al Andreas Jöckel. Spectroscopy of mechanical dissipation in micro-mechanical membranes. *Applied Physics Letters*, 2011.

[22] Hans-Jurgen Briegel, Berthold-Georg Englert. Quantum optical master equations: The use of damping bases. *PHYSICAL REVIEW A*, 1993.

[23] Howard Carmichael. *Statistical Methods in Quantum Optics, Vol 1.* Springer, 1999.