Optomechanical cooling with time-dependent parameters

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We model the laser cooling of a parametrically driven optomechanical cavity using a dissipation model that accounts for the modification of the quasi-energy spectrum caused by the driving. We construct a master equation for the mechanical object using Floquet operators. When the natural frequency of the mechanical object oscillates periodically around its mean value, we derive, using an adiabatic approximation, an analytical expression for its temperature. This expression depends both on the oscillator's mean frequency and that of the frequency's oscillations around its mean value. We find that the temperature can be lower than in the non-time dependent case. Our results raise the possibility of achieving lower temperatures for the mechanical object if its natural frequency can be controlled as a function of time.

I. INTRODUCTION

Quantum cavity optomechanics studies systems composed of macroscopic mechanical objects, such as mirrors, and an optical cavity's quantized light field usually coupled via radiation pressure. In a common scheme an end-mirror of a Fabry-Perot cavity is suspended while being able to freely oscillate. When photons are reflected by the mirror, there is a momentum transfer between the light field and the mirror and hence a coupling force. As the cavity's resonance depends on its length, the mechanical displacement affects the light field inside the cavity. Some of the first theoretical work predicting this sort of coupling was performed by [1]. This interaction between the macroscopic mechanical object and the light field leads to several interesting effects such as optomechanically induced transparency [2], the optical spring effect [3] or, most relevant to this study, optomechanical cooling [4–7].

Optomechanical cooling was first proposed by Mancini, et al in [8]. It is the damping of the end mirror's mechanical motion due to the radiative coupling to the cavity field. Sideband cooling takes place when the cavity's resonance is much narrower than the mechanical frequency. It can be understood as Raman scattering [9] of incident photons, red-detuned from the cavity resonance, when the parameters are chosen to favor phonon absorption from the mechanical oscillator in order to scatter the photon into the cavity's resonance mode, resulting in cooling. For coherent quantum control over a mechanical object, it must be close to a pure quantum mechanical state [10] so effective methods of cooling macroscopic objects to low temperatures is highly desirable.

One possible avenue for improved cooling lies in controlling the mechanical resonator's frequency as a function of time [11]. The effect of this dependence on the mechanical object's final temperature was studied in [12]. In that study it was found that the final temperature of the parametrically driven harmonic oscillator was larger than the nondriven case. The master equation from which the cooling rates were derived, accounted for the natural frequency of the mechanical resonator via ad-hoc time-dependent coefficients. These were introduced after performing the Markov approximation. This approach could lead to an incomplete master equation and thus an incomplete description of the system's dynamics, as the system is being treated as essentially not time-dependent during the derivation of the master equation. In this study, we employ a different method that accounts for the time dependence throughout the entire derivation.

The formalism which we apply (section II) is based on Floquet theory and was demonstrated to be a more accurate treatment in [13]. For the case where the drive consist of a small periodic oscillation with respect to the central frequency of the mechanical oscillator, the Floquet operators can be given explicitly (section III). Under the adiabatic approximation, we derive an approximated expression for the mean phonon occupation number in the final stages of optomechanical cooling (section IV). We perform numerical calculations in order to compare this expression to the one for the non-time-dependent case (section V). We find, using the formalism presented here, that lower temperatures can be obtained if the mechanical object is parametrically driven. Our results suggest that the theoretical model of dissipation of the mechanical osscilator can have a significant influence on the resulting temperature. (section VI).

II. OPTOMECHANICAL HAMILTONIAN

A. Hamiltonian with Floquet Operators

We employed the standard Hamiltonian for optomechanical cooling [7] in a reference system that rotates with the same frequency as a laser that continuously pumps photons into the cavity.

$$H(t) = H_{cav} + H_{mec}(t) + H_{int} + H_{pump}, \qquad (1)$$

where

$$
\begin{aligned}
H_{cav} =& -\hbar\delta a^\dagger a, && (2)\\
H_{mec}(t) =& \frac{p^2}{2M}+\frac{1}{2}M\nu^2(t)x^2, && (3)\\
H_{int} =& -\hbar g a^\dagger a x, && (4)\\
H_{pump} =& \hbar\frac{\Omega}{2}(a^\dagger + a), && (5)
\end{aligned}
$$

$\delta = \omega_{laser} - \omega_{cav}$ is the frequency difference between the laser and the cavity. M is the mechanical oscillator's mass and $\nu(t)$ is its frequency. In order to employ the Floquet formalism $\nu(t)$ must be a periodic function of time. The H_{int} term models the interaction between photons and the mirror, and g sets the strength of the coupling. Finally, Ω describes the strength of the cavity pump. Readers interested in a derivation of the interaction term should consult [10]. In our case, the only term with an explicit time dependence is the term for the mechanical oscillator.

The Floquet operators are analogous to the usual creation and annihilation operators for the standard harmonic oscillator and can be expressed in terms of the mechanical oscillator's position and momentum operators [13]. These operators are

$$
\Gamma(t) = \frac{1}{2i}\left[\hat{x}\sqrt{\frac{2M}{\hbar}}\dot{f}(t) - \hat{p}\sqrt{\frac{2}{M\hbar}}f(t)\right], \qquad (6)
$$

as well as its Hermitian conjugate. $f(t)$ is the solution to the classical time-dependent harmonic oscillator equation of motion in one dimension and is generally a complex function [14]

$$
\ddot{f} + \nu(t)^2 f = 0. \qquad (7)
$$

This equation has two solutions [13] of the form

$$
f(t) = e^{i\mu t}\phi(t), \qquad (8)
$$

and its complex conjugate. $\phi(t)$ is a periodic function of time with the same period as $\nu(t)$. μ is, in general, a complex number [15]. These operators follow the usual commutation relations for creation and annihilation operators

$$
[\Gamma(t)^\dagger, \Gamma(t)] = 1. \qquad (9)
$$

Using these operators $H_{mec}(t)$ can be written in the same form as the non time-dependent harmonic oscillator with the Floquet operators taking the place of the annihilation operators, with the exception of a global time-dependent scalar coefficient [14]

$$
H_{mec}(t) = \hbar\frac{W}{|f(t)|^2}\left[\Gamma^\dagger(t)\Gamma(t) + \frac{1}{2}\right], \qquad (10)
$$

with the Wronskian W for (7). Equation (6) is then inverted and solved for the harmonic oscillator's position operator [16]

$$
\hat{x} = \frac{b^*\Gamma - b\Gamma^\dagger}{(b^*a - ba^*)} \qquad (11)
$$

with

$$
\begin{aligned}
a =& \frac{1}{2i}\sqrt{\frac{2M}{\hbar}}\dot{f}\,, && (12)\\
b =& \frac{1}{2i}\sqrt{\frac{2}{M\hbar}}f\,. && (13)
\end{aligned}
$$

These are then substituted back into the interaction Hamiltonian resulting in

$$
H_{int}(t) = g\sqrt{\frac{\hbar}{2M}}a^\dagger a[\gamma_+(t)\Gamma(t) + \gamma_-(t)\Gamma^\dagger(t)], \qquad (14)
$$

with new coefficients

$$
\begin{aligned}
\gamma_+(t) =& \frac{f^*}{(f^*\dot{f} - f\dot{f}^*)},\\
\gamma_-(t) =& \frac{f}{(f\dot{f}^* - f^*\dot{f})}\,.
\end{aligned}
$$

The Hamiltonian contains two separate harmonic oscillator-like terms that commute with each other, H_{cav} and H_{mec}, so the standard harmonic oscillator master equation structure can be employed [16][13]. The usual derivation of the master equation involves a Markov approximation, under the formalism we employ, frequency's time dependence is accounted for during this approximation [13] via the Floquet operators. This differs from previous attempts to study this type of system, where this dependence was included after the Markov approximation had been performed via time-dependent ad-hoc coefficients for the damping [12]. As demonstrated in [13], the method employed here is a more complete, and thus accurate, treatment.

The corresponding master equation is

$$
\dot{\rho} = \frac{1}{i\hbar}[H,\rho] + L_a\rho + L_\Gamma\rho, \qquad (15)
$$

where

$$
\begin{aligned}
L_a\rho =& -\frac{\kappa}{2}(n_p+1)[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] && (16)\\
& -\frac{\kappa}{2}(n_p)[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a]\,,
\end{aligned}
$$

$$
\begin{aligned}
L_\Gamma\rho =& -\frac{\gamma}{2}(n_m+1)[\Gamma^\dagger\Gamma\rho + \rho\Gamma^\dagger\Gamma - 2\Gamma\rho\Gamma^\dagger] && (17)\\
& -\frac{\gamma}{2}(n_m)[\Gamma\Gamma^\dagger\rho + \rho\Gamma\Gamma^\dagger - 2\Gamma^\dagger\rho\Gamma]\,,
\end{aligned}
$$

κ is the energy decay rate for the cavity and γ is the decay rate for the mechanical oscillator. The number of thermal excitations for the cavity and the oscillator are given by n_p and n_m respectively [17]. The superoperators L_Γ and L_a model the energy exchanges between the environment and the cavity and the mechanical resonator respectively. Equation (15) represents a master equation for a parametric optomechanical system with an improved dissipation model which accounts for the mechanical oscillator's time dependent frequency. The formalism developed in [13] was used to derive the mechanical dissipation term (17).

B. Displaced Frame

In order to eliminate the pump term and to find useful approximations, we employ a unitary transformation to shift (15) into a displaced reference frame. This transformation depends on two time-dependent coefficients, $\alpha(t)$ and $\beta(t)$, which are chosen in a convenient manner to simplify the Hamiltonian. The transformation is given by the operator

$$U_{a,\Gamma} = e^{(\alpha(t)a^\dagger - \alpha^*(t)a)} e^{(\beta(t)\Gamma^\dagger - \beta^*(t)\Gamma)}, \tag{18}$$

and results in a displaced Hamiltonian and in turn a displaced master equation

$$\dot{\rho}' = \frac{1}{i\hbar}[H', \rho'] + L_a\rho' + L_\Gamma\rho' + C(t)\rho' \,, \tag{19}$$

for the time evolution of the density operator $\rho'(t)$ which represents the coupled cavity-mechanical resonator system. The primes indicate that the transformation has been applied. The displaced Hamiltonian, which includes a pump-like term that appears when making the transformation on the L operators, is given by

$$\begin{aligned} H' =& -\hbar\delta' a^\dagger a + \hbar\frac{W}{|f(t)|^2}\Gamma\Gamma^\dagger \\ & -\hbar g\sqrt{\frac{\hbar}{2M}}[(a^\dagger a + \alpha a^\dagger + \alpha^* a)(\gamma_-(t)\Gamma^\dagger + \gamma_+(t)\Gamma)] \\ & + i\hbar(\beta^*\dot{\Gamma} - \beta\dot{\Gamma}^\dagger), \end{aligned} \tag{20}$$

with $\delta' = \delta + g\sqrt{\frac{\hbar}{2M}}(\beta+\beta^*)$. This Hamiltonian is valid as long as the coefficients $\alpha(t)$ and $\beta(t)$ fulfill the differential equations

$$\dot{\alpha} =\alpha(-\frac{\kappa}{2} + i(\delta + g\sqrt{\frac{\hbar}{2M}}(\gamma_-(t)\beta^* + \gamma_+(t)\beta)) - i\frac{\Omega}{2}, \tag{21}$$

$$\dot{\beta} =\beta(-\frac{\gamma}{2} - i\frac{W}{|f(t)|^2}) + ig\sqrt{\frac{\hbar}{2M}}|\alpha|^2\gamma_+(t). \tag{22}$$

The $C(t)$ term

$$C(t)\rho = |\beta|^2(C(t)_{+-} - C(t)_{-+})\rho,$$

where

$$\begin{aligned} C(t)_{+-} =& [\dot{\Gamma}^\dagger, \Gamma], \\ C(t)_{-+} =& [\dot{\Gamma}, \Gamma^\dagger], \end{aligned}$$

appears because the Floquet operators do not necessarily commute with their own time derivatives. In the case of the Hamiltonian for the cavity's light field, the operators contain no explicit time dependence. The Floquet operators, however, do include such a dependence, which introduces additional terms into the master equation. These terms involve the commutators between the Floquet operators and their time derivatives and contain no operator dependence whatsoever and as such are not considered part of the Hamiltonian. These commutation relations are not, in general, zero [16], they depend on the specific form of the solutions for (7), f and f^*. For example

$$C(t)_{-+} = -(C(t)_{+-})^* = \frac{i}{2}(\dot{f}^*\dot{f} - \ddot{f}^* f^*). \tag{23}$$

Proceeding further requires an explicit solution for (7). The primes in the operators will be omitted from now on as all calculations will be done in the displaced frame.

III. SOLUTION FOR SMALL OSCILLATIONS

In order to obtain an explicit form of the Floquet operators we focus on the case of small oscillations around a central frequency, specifically

$$\nu(t) = \nu_0 + \epsilon' cos(2\omega t) \tag{24}$$

with $\epsilon' \ll \nu_0$, where ν_0 is the mean frequency. This leads to the time-dependent harmonic oscillator equation

$$\ddot{f} + (\nu_0^2 + 2\epsilon'\nu_0 cos(2\omega t))f = 0, \tag{25}$$

which is a particular case of the Mathieu equation [18]. In order to guarantee stable solutions we use the scattering relation

$$\frac{\nu_0^2}{\omega^2} = n^2, \tag{26}$$

with $n \in \mathbb{Z}^+$[15]. We will focus on the case $n \gg 1$, which will be used in the next sections. The solutions for (25) are, to first order in $\epsilon = \frac{2\epsilon'\nu_0}{\omega^2}$,

$$f(t) = e^{in\omega t} + \epsilon\frac{1}{8(n+1)}e^{i(n+2)\omega t} - \epsilon\frac{1}{8(n-1)}e^{i(n-2)\omega t} \tag{27}$$

and its complex conjugate.

With an explicit solution for (25) in hand, we can solve Eq. (21). Assuming that the time scale of the dissipatives processes are fast compared with the other time scales we can focus on the stationary case ($\dot{\alpha}(t) = \dot{\beta}(t) = 0$); we also assume a weak coupling regime (coefficients of first order in g or higher are neglected) obtaining

$$\alpha_0 = \frac{\Omega}{2\delta - i\kappa}, \qquad \beta_0 = 0. \tag{28}$$

The 0 sub-index denotes that the solutions are valid only to zeroth order in the coupling parameter. In obtaining the solution we used

$$\gamma_{\pm} = \frac{\pm 1}{2i\nu_0} e^{\mp i\nu_0 t}, \tag{29}$$

and $\frac{W}{|f|^2} = \nu_0 + \frac{\epsilon\omega}{2n} cos(2\omega t)$. These expressions neglect terms of order $\frac{\epsilon}{n^2}$ and use the approximation $n + 1 \approx n - 1 \approx n$ since we work in the regime where $n >> 1$. Using (28) in (20) the Hamiltonian can be written as

$$\begin{aligned} H = & -\hbar\delta a^\dagger a + \hbar(\nu_0 + \frac{\epsilon\omega}{2n} cos(2\omega t))\Gamma^\dagger\Gamma \\ & - \hbar g\sqrt{\frac{\hbar}{2M}}(a^\dagger a + \alpha_0 a^\dagger + \alpha_0^* a)(\gamma_-(t)\Gamma^\dagger + \gamma_+(t)\Gamma). \end{aligned} \tag{30}$$

We focus on the case where $|\alpha_0| \gg 1$ [12], so that the $a^\dagger a$ term can be neglected as it is small when compared to the other two terms in the interaction. This leads to a further simplified Hamiltonian

$$\begin{aligned} H(t) = & -\hbar\delta a^\dagger a + \hbar(\nu_0 + \frac{\epsilon\omega}{2n} cos(2\omega t))\Gamma^\dagger\Gamma \\ & + \hbar g\sqrt{\frac{\hbar}{2M}}(\alpha_0 a^\dagger + \alpha_0^* a)(\frac{-e^{i\nu_0 t}}{2i\nu_0}\Gamma - \frac{e^{i\nu_0 t}}{2i\nu_0}\Gamma^\dagger). \end{aligned} \tag{31}$$

Finally, using that $C(t)$ is proportional to β_0, the master equation (19) simplifies to

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + L_a\rho + L_\Gamma\rho. \tag{32}$$

This equation looks similar to the standard optomechanical master equation but it has Floquet operators instead of creation and annihilation operator for the mechanical oscillator and an explicit time dependence in the interaction Hamiltonian

IV. LASER COOLING

Our focus is on the parameter regime where the mechanical resonator's temperature evolves much more slowly than the cavity's losses and than the mechanical frequency. This requires $(g\sqrt{\frac{\hbar}{2M}}\alpha_o)^2 \ll (\frac{\kappa}{\nu_0})$. Following the derivation found in [7], we arrive at a master equation for the density operator, after projecting into the subspace corresponding to its stationary state and tracing over the cavity degrees of freedom. The technical details of the derivation is reported in Appendix A. It leads to the master equation

$$\dot{\mu} \approx -\frac{g'^2}{2}(G(\nu_m, n_c)[\Gamma^\dagger, \Gamma\mu] - G^*(-\nu_m, n_c)[\Gamma^\dagger, \mu\Gamma]), \tag{33}$$

where we have set $g' = g\sqrt{\frac{\hbar}{2M}}\alpha_0$ for convenience and use μ as the density operator of the mechanical degree of freedom. Note that this master equation is obtained via the usual derivation, since the particular form of (27) makes the explicit time dependencies cancel out in the operator products of the form $\Gamma^\dagger\Gamma$ and $\Gamma\Gamma^\dagger$.

The cavity-quadratures correlation $G(\nu_m, n_p)$ is used to calculate the heating and cooling rates for the mirror and is given by

$$G(\nu_m, n_p) = \int_0^\infty dt e^{i\nu(t)t} Tr_a[X_a e^{L_a t} X_a \rho_{st}], \tag{34}$$

with

$$X_a = \frac{a + a^\dagger}{\sqrt{2}\alpha_0}, \tag{35}$$

the cooling and heating rates are given by

$$A_\pm(\nu_m, n_p) = g'^2 Real(G(\mp\nu_m, n_p)), \tag{36}$$

where A_+ represents heating and A_- represents cooling. The final number of phonon excitations is given by

$$\langle m \rangle = \langle \Gamma^\dagger\Gamma \rangle = \frac{A_+}{A_- - A_+}. \tag{37}$$

Thus if (34) can be calculated, the heating and cooling rates and the final temperature (represented by the average number of phonon excitations) can be obtained in a straightforward manner. This procedure is done numerically in the following section.

V. CALCULATION OF MEAN VIBRATIONAL OCCUPATION NUMBER

Assuming that the cavity is at zero temperature ($n_p = 0$) the trace inside (34) gives

$$Tr_a[X_a e^{L_a t} X_a \rho_{st}] = \frac{1}{2} e^{-(\frac{\kappa}{2} - i\delta)t}, \tag{38}$$

up to first order in ϵ (34) simplifies to

$$\begin{aligned} G(\nu, 0) = & \int_0^\infty dt e^{(i\nu_0 + \epsilon' \cos(2\omega t))t} Tr[...] \\ & \approx \frac{1}{2}\int_0^\infty e^{i(\nu_0+\delta)t} e^{-\frac{\kappa t}{2}} dt \\ & + \frac{i\epsilon\omega}{4n}\int_0^\infty cos(2\omega t) t e^{i(\nu_0+\delta)t} e^{-\frac{\kappa t}{2}} dt\,, \end{aligned}$$

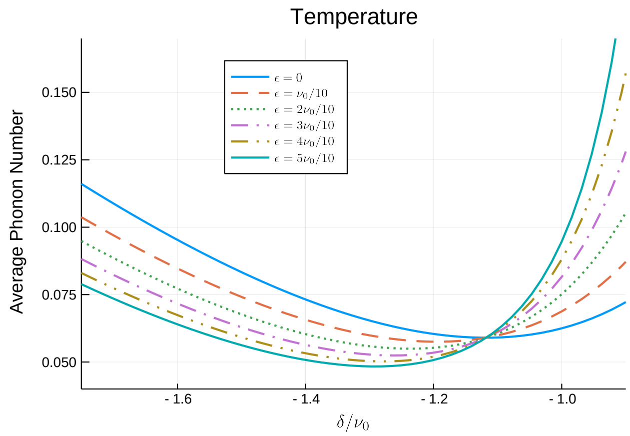

FIG. 1. Mechanical oscillator phonon number as a function of the perturbation. The correction results in a shift of the location of the minimum temperature, as well as a lower minimum. All curves cross the same point. We used the parameters in the text plus $\kappa \ll \nu_0$

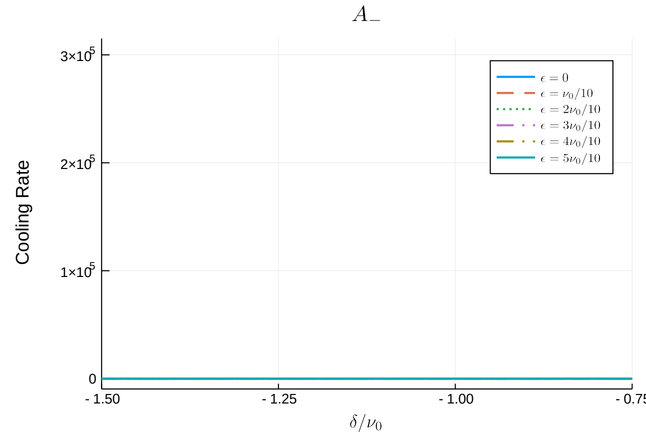

FIG. 2. Cooling coefficient A_- as a function of the perturbation. As ϵ increases the A_- presents a narrower distribution and a higher peak. The peak shifts away from the cavity resonance corresponding to ν_0. A_+ is not shown as, in this region, it is around 4 orders of magnitude smaller than A_-. We used the parameters in the text plus $\kappa \ll \nu_0$.

where the $e^{-\frac{\kappa}{2}t}$ factor acts as a cut-off for the integral, which allows for the expansion in ϵ to be performed.See Appendix B for more on the states used to evaluate the trace and Appendix B 1 for a more detailed evaluation of the trace term. This leads to an expression for (34) which consist of two parts: one corresponding to the nondriven harmonic oscillator case plus a correction term proportional to ϵ

$$\frac{G(\nu,0)}{g'^2} = \frac{1}{-\kappa + 2i(\delta+\nu_0)} + i\epsilon\frac{\omega}{4n}\frac{(-\kappa+i(\nu_0+\delta))^2 - 4\omega^2}{((-\kappa+i(\nu_0+\delta))^2+4\omega^2)^2} \tag{39}$$

We evaluate the real part of this expression numerically. In order for the adiabatic approximation to be valid, the variation in the mechanical oscillator's frequency must be slow when compared to the frequency itself ($\omega \ll \nu_0$) which translates to classical solutions with $1 \ll n$ due to (26). We assume that $\epsilon = \frac{\nu_0}{10}$. Physically, we employ a classical solution which describes a situation where the oscillator's average ν_0 frequency is the fastest one, while the secular frequency ω is the slowest one. In figure 1 it is shown the number of mechanical oscillations when $\frac{\delta}{\nu_0}$ is in the range of $[-2, 2]$. Compared with the non-driven case, the parametrically driven oscillator has a frequency shift of where to expect the minimum number of excitations, and a region where the predicted number of excitations is smaller. The reason of this behavior is that, as ϵ increases, the parametric modulation causes the cooling and heating coefficients to shift away from each other, and its distribution becomes narrower and presents a higher peak. This can be observed in figure 2. As long as ν_0 and ω fulfill (26), altering them does not change the final temperature. The parameters used are consistent with the reported controlled frequency modulation [19][20].

VI. CONCLUSIONS

Using an improved theoretical description of dissipation of a parametrically driven mechanical object, in an optomechanical setup, we found that the lowest temperature reached can be lower than in the non-driven case. This implies that periodically changing the natural frequency of the mechanical object in optomechanics can be used for reaching lower temperatures. This also opens the path for exploring what happens if the movement of a leaky cavity is taken into consideration when theoretically treating the dissipation of the cavity field, because, similarly to the setup study in this paper, the natural frequency of the cavity field changes periodically with time. We plan to explore this case in future work.

Appendix A: Laser Cooling and Projection Operators

In order to find the master equation (33) we employed projection operators like those in [21]. We also separate the L terms into

$$L_0 = (\frac{1}{i\hbar}[H_{cav}, \bullet] + L_a) + (\frac{1}{i\hbar}[H_{mec}(t), \bullet]), \tag{A1}$$

and

$$L_1 = \frac{1}{i\hbar}[H_{int}, \bullet]. \tag{A2}$$

These projection operators fulfill a completeness relation

$$1 = P + Q, \tag{A3}$$

and have the properties

1. $PL_0 = L_0P = 0$ as the corresponding eigenvalues are 0
2. $PL_1P = 0$ as the interaction does not couple states in P
3. $P^2 = P \quad Q^2 = Q$ as P and Q are projectors.

In this case $P = P_m^{\lambda_m^0} P_c^{\lambda_c^0}$. We project the master equation $\dot{\rho} = L\rho$ into both P and Q spaces and insert the completeness relation to obtain two equations. Working in the decay picture

$$\begin{aligned}\rho' =& e^{L_0 t}\rho,\\ L' =& e^{-L_0 t} L e^{L_0 t} = L_1'.\end{aligned}$$

and omitting primes, the equations are

$$\begin{aligned}P\dot{\rho} =& PL_1Q\rho,\\ Q\dot{\rho} =& QLQ\rho + QLP\rho.\end{aligned}$$

The equation for Q can be formally integrated [16]

$$\begin{aligned}Q\rho =& Q\rho(t_0) + \int_{t_0}^{t} dt' QL(t')P\rho(t')\\ &+ \int_{t_0}^{t} dt' QL(t')Q\rho(t'),\\ \simeq& Q\rho(t_0) + \int_{t_0}^{t} dt' QL_1(t')P\rho(t_0)\\ &+ \int_{t_0}^{t} dt' QL_1(t')Q\rho(t_0) + O(\eta^2),\end{aligned}$$

and this is substituted into the P equation

$$\begin{aligned}P\dot{\rho}(t) =& PL_1Q\rho(t_0) \\ &+ PL_1 \int_{t_0}^{t} dt' QL_1(t')P\rho(t_0)\\ &+ PL_1 \int_{t_0}^{t} dt' QL_1(t')Q\rho(t_0),\end{aligned} \tag{A4}$$

where only the second term is non zero. Transforming back from the decay picture

$$Pe^{-L_0 t} L_1 e^{L_0 t} \int_{t_0}^{t} dt' Q e^{-L_0 t'} L_1 e^{L_0 t'} P e^{-L_0(t_0)}\rho(t_0). \tag{A5}$$

Noting that the factors that multiply the Floquet operators cancel out their time dependence and decomposing the projectors into the forms $P = \sum_\lambda \hat{\rho}_\lambda \otimes \check{\rho}_\lambda$ and $Q = \sum_{\lambda'} \hat{\rho}_{\lambda'} \otimes \check{\rho}_{\lambda'}$ we may evaluate the L_0 terms by applying them to the nearest states

$$\begin{aligned}Pe^{-L_0 t} L_1 [\sum_{\lambda',\lambda} \int_{t_0}^{t} dt' e^{\lambda' t} \hat{\rho}_{\lambda'} \otimes \check{\rho}_{\lambda'} e^{-\lambda' t'} L_1 \\ e^{\lambda t'} \hat{\rho}_\lambda \otimes \check{\rho} e^{-\lambda(t_0)}\rho(t_0)].\end{aligned} \tag{A6}$$

We focus only on the exponential terms, which are now numbers and can be shifted accordingly

$$[\int_{t_0}^{t} dt' e^{(\lambda'-\lambda)t'}] e^{\lambda t - \lambda' t_0}. \tag{A7}$$

Here we notice that $\lambda' = \lambda_{mec}(t)$ as the cavity is taken to be at zero temperature. Since $\lambda_{mec}(t) = i\nu_0 + \epsilon\cos(2\omega t)$ we must take care of the integration. We employ the approximation

$$e^{(\lambda'-\lambda)t'} \approx e^{(i\nu_0-\lambda)t'} + i\epsilon\cos(2\omega t')t' e^{(i\nu_0-\lambda)t'}. \tag{A8}$$

Each terms is then integrated separately. The adiabatic approximation consists of taking the limit $(t-t_0)\to\infty$ or $t\to\infty, \quad t_0 \to 0$. In this limit

$$e^{\lambda t} \to 0 \qquad e^{\lambda' t} \to 1. \tag{A9}$$

This is due to λ representing the fast time scale which decays quickly, while λ' represents the slow time scale which contains stationary states. Under these conditions (A7) becomes, up to first order in ϵ

$$[\int_{t_0}^{t} dt' e^{(\lambda'-\lambda)t'}] e^{\lambda t - \lambda' t_0} = \frac{1}{a} + i\epsilon \frac{a^2-\omega^2}{(4\omega^2+a^2)^2} = I_{\lambda',\lambda}, \tag{A10}$$

with $a = i\nu_0 - \lambda$. Returning to (A6), we may now write it as

$$Pe^{-L_0 t} \sum_{\lambda',\lambda} L_1 I_{\lambda',\lambda} \rho_{\lambda'} L_1 \rho_\lambda \rho(t_0). \tag{A11}$$

We may return the projectors to their Original notation to remove the sums now sum over the possible values of the mechanical eigenvalues

$$Pe^{-L_0 t} \sum_m L_1 I_m Q L_1 P\rho(t_0). \tag{A12}$$

The full equation is multiplied by $Pe^{L_0 t}$

$$P\dot{\rho} = P[L_0]P\rho(t) + \sum_m PL_1 I_m Q L_1 P\rho(t_0). \tag{A13}$$

We then trace over all the cavity degrees of freedom and set $\mu = Tr_c[P\rho] \otimes \rho_{st}$, where we assume no correlation between the cavity and mirror states at time t_0. This leads to the equation

$$\dot{\mu}_m = L_{mec}\mu_m + Tr_c[PL_1 I(\nu_m)QL_1 P\rho(t_0)_m], \quad \text{(A14)}$$

for any given value of m. Remembering that $L_1 = \frac{1}{i\hbar}[X_a X_\Gamma, \bullet]$ and that the mirror operators Γ may leave the trace alongside μ, equation (A14) can be written as (33) where the second term has been written as

$$\int_0^t dt' e^{\imath\nu(t')t'} Tr_c[PX_a e^{L_a t'} QX_a P\rho_s t], \quad \text{(A15)}$$

for convenience.

Appendix B: The Damping Basis

Master equations of the type

$$\dot{\rho} = \frac{i}{\hbar}[H,\rho] + L\rho, \quad \text{(B1)}$$

with

$$\begin{aligned} L_a\rho = &-\frac{\kappa}{2}(n_p+1)[a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger] \\ &-\frac{\kappa}{2}(n_p)[aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a], \end{aligned} \quad \text{(B2)}$$

model the behavior of a bosonic field inside a cavity with just one mode and with ν thermal photons in contact with a thermal bath. A is a constant related to the damping and $A, n_p \geq 0$ [22]. The density operator can be expressed in the Lindblad superoperator's eigenbasis to simplify the calculations. This is known as the damping basis [22].

$$a^{\dagger l}\frac{(-1)^n}{(n_p+1)^{l+1}} : L_n^l[\frac{a^\dagger a}{n_p+1}]e^{-[\frac{a^\dagger a}{n_p+1}]} : \quad l \geq 0, \quad \text{(B3)}$$

$$\frac{(-1)^n}{(n_p+1)^{|l|+1}} : L_n^{|l|}[\frac{a^\dagger a}{n_p+1}]e^{-[\frac{a^\dagger a}{n_p+1}]} : a^{|l|} \quad l \leq 0, \quad \text{(B4)}$$

with eigenvalues

$$\lambda_n^l = -\kappa[n + \frac{|l|}{2}], \quad \text{(B5)}$$

which fulfill

$$n = 0, 1, 2..., \qquad l = 0, \pm1, \pm2, ... \quad \text{(B6)}$$

These are the right eigenstates, which correspond to $L\hat{\rho_n^l} = \lambda_n^l \hat{\rho_n^l}$, however the left eigenstates must also be considered, these correspond to $\check{\rho_n^l} L = \lambda_n^l \check{\rho_n^l}$ and have the same eigenvalues. These are

$$(\frac{-n_p}{n_p+1})^n \frac{n!}{(n+l)!} : L_n^l[\frac{a^\dagger a}{n_p}] : a^l \quad l \geq 0, \quad \text{(B7)}$$

$$(\frac{-n_p}{n_p+1})^n \frac{n!}{(n+|l|)!} a^{\dagger |l|} : L_n^{|l|}[\frac{a^\dagger a}{n_p}] : \quad l \leq 0. \quad \text{(B8)}$$

These states are orthogonal under the product

$$Tr[\hat{\rho}_{n,l}\check{\rho}_{n',l'}] = \delta_{n,n'}\delta_{l,l'}, \quad \text{(B9)}$$

and also fulfill

$$\sum_\lambda \hat{\rho}_\lambda \otimes \check{\rho}_\lambda = \not{1}, \quad \text{(B10)}$$

where the sum is over all possible eigenvalues. An important approximation can be obtained for a cavity at zero temperature, in these case the states are [22]

$$a^{\dagger l}(-1)^{a^\dagger a + n}\binom{n+l}{a^\dagger a + l} \quad l \geq 0, \quad \text{(B11)}$$

$$(-1)^{a^\dagger a + n}\binom{n+|l|}{a^\dagger a + |l|} a^{|l|} \quad l < 0, \quad \text{(B12)}$$

and the dual states are

$$\frac{n!}{(n+l)!}\binom{a^\dagger a}{n} a^l \quad l \geq 0, \quad \text{(B13)}$$

$$a^{\dagger |l|}\frac{n!}{(n+|l|)!}\binom{a^\dagger a}{n} \quad l < 0. \quad \text{(B14)}$$

1. Evaluation of Trace Term

In order to calculate the $G(\nu_m, n_c)$ terms in (33), the term

$$T = Tr_a[X_a e^{L_a t} X_a \rho_{st}], \quad \text{(B15)}$$

must be explicitly calculated first. The calculation hinges on the Damping Basis states detailed above. The cavity's stationary state is taken to be the base zero temperature state

$$\rho_{st} = |0\rangle\langle 0|, \quad \text{(B16)}$$

and we recall that

$$X_a = \frac{(\alpha_0 a^\dagger + \alpha_0^* a)}{\sqrt{2}}. \quad \text{(B17)}$$

The first step is to separate the term (B15) using (B10)

$$Tr_a[X_a e^{L_a t} X_a \rho_s t] = \sum_\lambda Tr_a[X_a e^{L_a t} \hat{\rho}_\lambda] \otimes Tr_a[\check{\rho}_\lambda X_a |0\rangle\langle 0|]. \tag{B18}$$

In this context λ refers exclusively to the cavity eigenvalues. The a term of X_a is zero when applied to ρ_{st} and so

$$T = \alpha_0 \sum_\lambda Tr_a[X_a e^{L_a t} \hat{\rho}_\lambda] \otimes Tr_a[\check{\rho}_\lambda |1\rangle\langle 0|], \tag{B19}$$

$$= \frac{\alpha_0}{\sqrt{2}} \sum_\lambda Tr_a[X_a e^{L_a t} \hat{\rho}_\lambda] \delta_{N,0}\delta_{l,-1}, \tag{B20}$$

$$= \frac{\alpha_0}{\sqrt{2}} Tr_a[X_a e^{L_a t} \hat{\rho}_{0,-1}], \tag{B21}$$

$$= \frac{\alpha_0}{\sqrt{2}} e^{-\frac{\kappa}{2}} Tr_a[X_a \hat{\rho}_{0,-1}], \tag{B22}$$

$$= \frac{\alpha_0}{2} e^{-\frac{\kappa}{2}} \alpha_0^* Tr_a[|0\rangle\langle 0|], \tag{B23}$$

$$= \frac{|\alpha_0|^2}{2} e^{-\frac{\kappa}{2}}. \tag{B24}$$

Here we used the fact that $F\rho_{0,-1}\hat{} = |0\rangle\langle 0|$.

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