Задача 1

Из ЦПТ: $EX := EX_1 = EX_2 = \dots = EX_n$

$$\sqrt{n}\,(\overline{X} - EX) \to N(0, DX) \quad n \to \infty$$

$$E_\theta X = \frac{1}{\theta}$$

Дельта-метод:

$h(x) = \frac{1}{x}$ $x > 0$

$h'(x) = -\frac{1}{x^2} \neq 0$ $\forall x > 0$ $\Rightarrow$ условие на h' выполнено

Из ЦПТ для дельта-метода: $EX = \frac{1}{\theta}$ $h(EX) = \theta$

$$\sqrt{n}\left(h(\overline{X}) - \theta\right) \sim N(0, DX \cdot (h'(EX))^2)$$

$$DX = EX^2 - (EX)^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$h'(EX) = -\frac{1}{(EX)^2} = -\theta^2$$

$$\sigma^2(\theta) = DX\,(h'(EX))^2 = \frac{1}{\theta^2} \cdot \theta^4 = \theta^2$$

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \theta\right) \sim N(0, \theta^2)$$

② $X_1^2, \dots X_n^2$

Из ЦПТ

$$\sqrt{n}\left(\overline{X^2} - EX^2\right) \longrightarrow N(0, DX^2)$$

$$EX^2 = \frac{2}{\theta^2} \qquad DX^2 = EX^4 - (EX^2)^2 = \frac{24}{\theta^4} - \frac{4}{\theta^4} = \frac{20}{\theta^4}$$

$$h(x) = \sqrt{\frac{2}{x}} \qquad h(EX^2) = \theta$$

$$h'(x) = \frac{1}{\sqrt{2x^3}} \neq 0 \text{ при } x > 0$$

$$\sigma^2(\theta) = DX^2 \left(h'(EX^2)\right)^2 \qquad \frac{2}{\theta^2}$$

$$\sigma^2(\theta) = \frac{20}{\theta^4} \frac{1}{2(EX^2)^3} = \frac{20}{\theta^4} \frac{1}{16\frac{1}{\theta^6}} = \frac{20}{16}\theta^2 = \frac{5}{4}\theta^2$$

$$h\left(\overline{X^2}\right) = \sqrt{\frac{2}{\overline{X^2}}}$$

$$\sqrt{n}\left(\sqrt{\frac{2}{\overline{X^2}}} - \theta\right) \rightarrow N\left(0, \frac{5}{4}\theta^2\right)$$

$$\hat{\theta} = \sqrt{\frac{2}{\overline{X^2}}}$$

③ $$P\left(L(X_1, \dots X_n) \leq \theta \leq U(X_1, \dots X_n)\right) = \alpha$$

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \longrightarrow N\left(0, \theta^2\right)$$

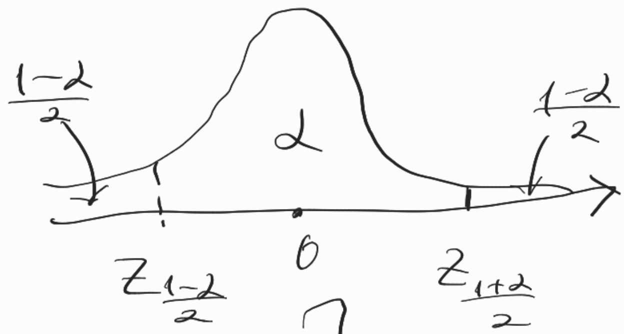
$$P\left(\left|\frac{\sqrt{n}\left(\hat{\theta} - \theta\right)}{\sigma(\theta)}\right| \leq z_{\frac{1-\alpha}{2}}\right) = \alpha$$

$$P\left(z_{\frac{1-\alpha}{2}} < \frac{\sqrt{n}\left(\hat{\theta} - \theta\right)}{\sigma(\theta)} < z_{\frac{1+\alpha}{2}}\right) = \alpha$$

$$P\left(\frac{z_{\frac{1-\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}} < (\hat{\theta}-\theta) < \frac{z_{\frac{1+\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}}\right)=\alpha$$

$$P\left(\hat{\theta}-\frac{z_{\frac{1+\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}} < \theta < \hat{\theta}-\frac{z_{\frac{1-\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}}\right)=\alpha$$

Доверительный интервал:



$$\left[\hat{\theta}-\frac{z_{\frac{1+\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}},\ \hat{\theta}-\frac{z_{\frac{1-\alpha}{2}}\,\sigma(\hat{\theta})}{\sqrt{n}}\right]$$

1) $\sigma(\hat{\theta})=\theta=\frac{1}{\overline{X}}$

$$z_{\frac{1-\alpha}{2}}=-z_{\frac{1+\alpha}{2}}$$

$$\left[\frac{1}{\overline{X}}-\frac{z_{\frac{1+\alpha}{2}}}{\overline{X}\sqrt{n}},\ \frac{1}{\overline{X}}+\frac{z_{\frac{1+\alpha}{2}}}{\overline{X}\sqrt{n}}\right]$$

2) $\sigma(\hat{\theta}) = \ldots$

2) $\sigma(\theta) = \sqrt{\frac{5}{4}\theta} = \sqrt{\frac{5}{2X^2}}$

$$\left[\sqrt{\frac{2}{\bar{X}^2}} - \sqrt{\frac{5}{2n\bar{X}^2}}\, z_{\frac{1+\alpha}{2}},\ \sqrt{\frac{2}{\bar{X}^2}} + \sqrt{\frac{5}{2n\bar{X}^2}}\, z_{\frac{1+\alpha}{2}}\right]$$