Math 1A Notes

Tikhon Jelvis

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Contents

1 Functions

In the first section, we are going to cover the idea of a *function*. A function is an idea that we have seen in previous courses and other contexts; however, it is going to be disproportionately important in this course.

A function in math is a rule that maps the elements of one set, called the *domain*, to one element apiece of another set, called the *range*. Note that here we are not necessarily talking about *numbers*; rather, we used the word *element*. This is important because functions can map things which are not numbers. In fact, nothing stops a function from mapping a set of functions to a different set of function! That said, in this course, we will be primarily talking about normal functions that just map sets of numbers.

1.1 The Exponential Function (a^x)

The first function we will talk about is called the *exponential function*. This is a function like any other and can be expressed in the same notation (as $f(x)$, for example). That said, it is usually expressed as $f(x) = a^x$ where a is some number.

If a is a positive real number and x is any real number, then what a^x is depends on what x is:

- If x is an integer, then a^x is a product of x factors of a.
- If x is 0, then $a^x = a^0 = 1$.
- If x is a rational number $\frac{p}{q}$ then $a^x = \sqrt[q]{a^p}$
- If x is negative, then $a^x = \frac{1}{a^{-x}}$.
- If x is a real number, then a^x is the number b so that for all $n < x$ $a^n < b$ and for all $n > x$ $a^n > b$.

Laws of Exponents

There are several laws that make working with exponents much easier; these are called the "Laws of Exponents". They make arithmetic with exponents much easier.

- $a^{x+y} = a^x \cdot a^y$
- $(ab)^x = a^x \cdot b^x$
- $a^{xy} = (a^x)^y$
- $a^0 = 1$
- $a^1 = a$
- $a^{-1} = \frac{1}{a}$

Special Numbers (a)

In the exponential function $f(x) = a^x$, the number a is called the *base*. Here are some special bases to be aware of:

- $a = 2$ Computers and doubling. This is binary (base 2).
- $a = 10$ Base ten is traditional and a convenient number to work with.
- e Euler's number. One of the five important numbers $(0, 1, e, \pi, i)$. $e \approx 2.71828\ldots$. It's definition is subtle; we'll cover it later on.

Out of these bases, e is probably the most important for mathematics; the rest are important in the "real world", wherever that may be.

1.2 Functions Defined by Cases

So far, we have seen that a function can be defined as follows:

$$f(x) = x^3 + x^2 + x1$$

This function maps x to the result of an expression, like so: $x \rightarrow x^3 + x^2 + x + 1$. However, a function does not have to be the result of an *equation*; it just has to map one set to another.

Another way to define a function is to define it *by cases*. This means that you define what happens to x over various intervals. The classic example of a function defined by cases is the absolute value function $|x|$. It is defined as follows:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

What this means is that when x is negative, the function returns $-x$ and when x is non-negative, it returns x. In practice, this means that the returned value is never negative—the range of this function is the set $[0, \infty)$. Coincidentally, this function can also be defined as $f(x) = \sqrt{x^2}$ thanks to the way the square root function works.

1.3 New Functions

Mathematics builds on previous discoveries and developments; this is both based on history and the organization of various concepts. This can be extended to *functions*; new functions can be created simply by combining old ones.

Here are some ways to combine functions:

- If f is a function and c a real number, you can multiply: $g(x) = c \cdot f(x)$.
- You can add functions: $h(x) = f(x) + g(x)$. The domain of the new function is all numbers in the domains of *both* f and g. This is the intersection of the two domains.
- You can also multiply functions; this works much like addition.
- You can also divide functions ($h(x) = \frac{f(x)}{g(x)}$); here the domain is the intersection between the two functions' domains *except* for the set of numbers x where $g(x) = 0$.
- You can also modify the independent variable ($g(x) = f(x+c), h(x) = f(c \cdot x) \dots$)
- Another option: composition. You can take a function of a function. E.g.: $f \circ g(x) = f(g(x))$. The domain is the domain of x where x is in the domain of g and $g(x)$ is also in the domain of f.

1.4 Square Root ($\sqrt{}$)

- Nonnegative = ≥ 0.
- All nonnegative numbers have a nonnegative square root.
- $\sqrt{x} = x^{\frac{1}{2}}$
- If $x < 0$, there is no real root.
- The domain is $[0, \infty)$.

1.5 One-to-one Functions

Definition : A function f is a *one-to-one* function where all outputs ($f(x)$) are unique.

Examples:

- $f(x) = \frac{1}{y}$ If $\frac{1}{x} = \frac{1}{y}$ then $x = y$.
- $f(x) = x^2$ is not; $f(3) = f(-3) = 9$.
- For different domains, it can be different: $g(x) = x^2$ with the domain $[0, \infty)$ *is* one-to-one.
- The function $\sin x$ with the domain of $\Re$, is *not* one-to-one; however, for the domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$ it *is* one-to-one.
- All *strictly* increasing functions are one-to-one.

Horizontal Line Test

If a horizontal line crosses the graph more than once, it is *not* one-to-one. If no horizontal lines cross the graph more than once, it is one-to-one. A function can be a one-to-one function even if some horizontal lines do not cross the graph *at all*.

This is like the vertical line test that tests for functions.

1.6 Inverse Functions

The function g is the **inverse** of f if $(g \circ f)(x) = x$ for all of x. The domain of g has to be equal to the range of f and vice versa.

The inverse of $f(x)$ is written as $f^{-1}(x)$. Do not confuse this with a function raised to negative one! $f^{-1}(x) \neq f(x)^{-1}$.

Which Functions Have Inverses?

The function f can only have an inverse if it is a one-to-one function. The function f will only have an inverse if it passes the horizontal line test.

Note: Any function with an interval domain that is either increasing or decreasing is one-to-one and, naturally, has and inverse. This is the main way we will identify functions with inverses in this course.

This is yet another way to make new functions: any one-to-one function can yield an inverse.

Some examples of inverse functions:

- $f(x) = x^2$ with the domain $[0, \infty)$ has the inverse $f^{-1}(x) = \sqrt{x}$.
- $f(x) = \sqrt{x-2}$ has the domain $[2, \infty)$. Find the inverse by solving $y = \sqrt{x-2}$ for x. This gives you $f^{-1}(y) = y^2 + 2$. In practice, we change the ys back to xs ($f^{-1}(x) = x^2 + 2$). This has the same meaning. The domain of this function is the range of $f(x)$, which is *not* $[2, \infty)$; it is $[0, \infty)$.
- Some functions are their own inverses ($f(x) = f^{-1}(x)$). Examples include:

  1. $f(x) = \frac{1}{x}$
  2. $f(x) = x$
  3. $f(x) = -x$

  You can produce more of these using the various methods for creating new functions outlined in section 1.3 on page 3.

A Subtlety

There are sometimes functions that *have* inverses, and it is easy to find this out, but calculating the inverse is difficult.

A good example is $f(x) = 2^x + x^3$ with the domain $\Re$. This function is increasing, but using algebra to find the actual inverse function is nontrivial.

Thus, you can find out that there *is* an inverse without finding out what the inverse actually is. Even if you can't find the inverse as a formula, it still exists!

General Properties

Here are some general properties of *all* inverse functions:

- $f(x) = y$ is the same as $f^{-1}(y) = x$.
- The ordered pair (x, y) is in the graph of f iff (y, x) is in the graph of f^{-1}.
- The graph of f is the reflection of f^{-1} about the line $y = x$.
- The functions cancel out: $f(f^{-1}(y)) = y$ in the domain of f^{-1}.

Think of the inverse as a function that undoes what the actual function does—they are opposites.

Logarithms

The inverse of the exponential function ($f(x) = a^x$) is called the logarithm. Logarithms can have any base that an exponential function can have. The most common bases are $2, e$ and 10.

Notation: $\log_a y = x$ means $a^x = y$. By default, in math classes, a log function without a specified base has base 10.

1.7 The Magic Number e

Take the graph of $y = a^x$ and the graph of $y = b^x$ where $b > a$. The latter graph would be more steep. At $x = 0$, $y = 1$ for all exponential functions as $a^0 = 1$ in every case.

At this point, there is a tangent line. This is a line that intersects the curve at only *one* point. The tangent line has a different slope for different graphs. The slope for the graph of base b is greater than the slope for the graph of base a if $b > a$.

There is *one* special number a such that the slope of the tangent line at $(0, 1)$ for $y = x^x$ is 1. That number is e.

e is an irrational number ($e \approx 2.71828\ldots$). This is a brand-new number, mostly: there is no easy relationship between e and π, at least nothing obvious and algebraic like $\frac{\pi}{2}$ or something.

The Natural Logarithm

The function $f(x) = e^x$ has an inverse. This inverse, like the inverse of any exponential function, is a logarithm. However, this is a very special logarithm, so it has a special name: It is called the "natural logarithm". So $\log_e x = \ln(x)$.

The graph of the natural logarithm is the mirror image of the graph of $y = e^x$ over the line $y = x$, just like any other inverse function! This can be used to draw the graph of $y = \ln x$ very easily.

1.8 Logarithms

The graph of any logarithm has several properties. There is a vertical asymptote along the line $x = 0$. There are no points at all in the interval $(-\infty, 0]$. The graph approaches ∞ as x approaches ∞, but increasingly slowly.

Thus, the domain of any logarithm is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Laws of Logarithms

When we work with the exponential function, we usually use the laws of exponentiation rather than relying on the definition of the function. The same is true for logarithms.

Here are some laws of logarithms:

- $\log_a x^a = x$
- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a b^x = x \cdot \log_a b$
- $\log_b c = \frac{\log_a c}{\log_a b}$

Here a can be any positive number except for 1. This is impossible because 1 to any power is still 1. $1^y = x$ is only valid if $x = 1$. Additionally, the graph of $f(x) = 1^x$ fails the horizontal line test and cannot have an inverse.

2 Limits

2.1 Intro to Limits

Secant Lines

A *secant line* is a line that passes through a curve at exactly two points. Take the function $f(x) = 100 - 5x^2$. A line passing through the points $(1, f(1))$ and $(c, f(c))$ would be a secant line. We can get the slope of this line, which is -5 for a particular value of c.

We can draw a secant line for any value of c. As c comes closer and closer to 1, the closer and closer the slope of the secant line is to that of the *tangent line*, which crosses the curve at only one point. Using this, we can guess the slope of the tangent line at $(1, f(1))$. We get -10 by plugging in a c that is very close to 1.

This means that the slope of a tangent line is the *limit* of slopes of secant lines.

Informal Definition

Basically (informally), a (finite) limit can be thought of as:

$$\lim_{x \to a} f(x) = L$$

Here L is finite. This effectively means that one can make $f(x)$ as close to L as desired by taking $f(x)$ where x is very close to a; however; x *cannot* actually be a.

Of course, this is a very vague definition that is not actually practical for use in mathematics. This is just supposed to convey the essence of limits, not actually define them in any practical terms.

Some Examples

Take this as an example:

$$\lim_{x \to 3} x^2 = 9$$

This means that as x gets closer and closer to 3, x^2 gets closer and closer to 9. Not much of a revelation, but still useful.

Another example:

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Take $\lim_{x \to 0} f(x)$ This limit is impossible to find; it does not exist.

A similar example:

$$f(x) = \frac{1}{|x|}$$

Take $\lim_{x \to 0} f(x)x$. In this case, it also does not exist; however, this is a special condition. Here, as x approaches 0, $f(x)$ gets bigger and bigger. Thus

$$\lim_{x \to 0} \frac{1}{|x|} = \infty$$

However, this limit still *does not exist*. Here you can also have negative infinity ($-\infty$), which just means that $f(x)$ would get more and more negative as you approach the limit.

Note: The notation ∞ and $-\infty$ does not mean the infinity is a number; it is not. It cannot be used as a number, even though it is sometimes written where a number would otherwise be.

Another common example: $f(x) = \frac{1}{x}$. Here, from one side, the limit $lim_{x \to 0} f(x)$ does not exist and is *neither* ∞ nor $-\infty$.

Take a weird function like $f(x) = \sin \frac{1}{x}$ which goes between 1 and -1 more and more quickly as it approaches 0. Depending on which xs one looks at, the limit $\lim_{x \to 0} f(x)$ could be take as 0, 1 or -1. In this case, the limit *does not exist*.

A last weird example: take a function like the previous except that instead of going from 1 to -1, the amplitude decreases as x gets close to 0 ($f(x) = x - \sin \frac{1}{x}$). Thus, the two values it jumps between get closer and closer to zero as x does. In this case, the limit as x approaches 0 *does* exist; the limit here is 0. Note that $f(0)$ can be anything, or even undefined; it is not pertinent to finding the limit.

One-Sided Limits

Limits can be taken from just one side, either the right (positive) or the left (negative). If a and L are real numbers, $\lim_{x \to a^+} f(x) = L$ means the limit of $f(x)$ as x approaches a from the right.

This can be very useful in certain cases. Take the function $f(x) = \ln x$. This function has nothing left of the y-axis. Thus, $lim_{x \to 0} f(x)$ cannot be found *at all*; however, $lim_{x \to a^+} f(x)$ *can* be found (it is $-\infty$).

Vertical Asymptotes

A vertical asymptote of $f(x)$ at a means that $lim_{x \to a^+} f(x) = \infty$ or $-\infty$ or $\lim_{x \to a^-} f(x) = \infty$ or $-\infty$.

2.2 Working with Limits

When working with limits, it is often necessary to find the limit of a function created from one or more simpler functions. The methods for creating functions this way are described in section 1.3 on page 3.

It is possible to find the limit of a complex function knowing the limits of the functions that make it up. This requires using a set of rules called the "Limit Laws" which describe how limits of complex functions made up of other functions behave.

Limit Laws

Assume that $f(x)$ and $g(x)$ both have a real limit as x approaches a. Here are the various rules for combining limits:

- Sums of limits:

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

- Differences of limits:

$$\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

- A limit of a function times a constant:

$$\lim_{x\to a} c \cdot f(x) = c \lim_{x\to a} f(x)$$

- Products of limits:

$$\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

- Ratios of limits:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \lim_{x\to a} g(x) \neq 0$$

- Powers:

$$\lim_{x\to a} f(x)^n = (\lim_{x\to a} f(x))^n \text{ if } n \text{ is a positive integer.}$$

- Constant function (identity):

$$\lim_{x\to a} c = c$$

2.3 Direct Substitution Property

The Limit Laws (section 2.2) help find the limits of functions based on the limits of simpler functions; however, they aren't much help for finding the limit of a function if we do not know any limits at all.

In order to find $\lim_{x\to a} f(x)$, it is often enough to simply plug a into the function f. This is called the *direct substitution property*. This property is limited however—it can only be used if the function f is a polynomial or rational function and a is in the domain of f. Functions that fit this criteria are called *continuous* at a.

While direct substitution is limited to functions continuous at a, there are ways of finding limits of functions *not* continuous at a. If you have two functions f and g where $f(x) = g(x)$ except when $x = a$, then the limit of the two functions at a is the same. This is very convenient if one function is continuous at a and the other isn't.

An example: take a function f that is defined as:

$$f(x) = \begin{cases} x & \text{if } x \neq 3 \\ 42 & \text{if } x = 3 \end{cases}$$

We cannot use the substitution property to find $\lim_{x \to 3} f(x)$ as f is not continuous at 3—$f(x) = 42$, which is blatantly wrong. However, lets take the function g such that $g(x) = x$. We know that $g(x) = f(x)$ everywhere except 3, and that's where we're trying to find the limit. Now we know that $lim_{x \to 3} f(x) = lim_{x \to a} g(x)$. Using the fact that g is continuous at 3, we can employ the substitution property to find that $lim_{x \to 3} f(x) = 3$. Now we know that $\lim_{x \to 3} f(x) = 3$.

2.4 Limits with Inequalities

Take two functions f and g such that $f(x) \leq g(x)$ for some interval. Take a number a such that a is within and not on the edge of this interval. If this is true, then:

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

If the limit of g here is finite, the same is true for g.

A special case to note is when $g(x) = C$ for all x in the interval. Here, this holds:

$$\lim_{x \to a} f(x) \leq C$$

Of course, this is only true if $\lim_{x \to a} f(x)$ exists.

These two properties can, naturally, be used in reverse. You can conclude about both $\leq$ and $\geq$ relationships.

Squeeze Theorem

The *Squeeze Theorem* is a way of getting an equality from multiple inequalities. Take three functions such that $f(x) \leq g(x) \leq h(x)$ in some open interval containing a. This basically means that the graph of g is between f and h. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. Note that this also applies to one-sided limits. This works with both finite and infinte limits.

Examples

Take the function g such that

$$g(x) = x \sin \frac{1}{x} \quad \text{for } x > 0.$$

This graph "bounces" up and down less and less as it approaches 0. Our goal is to find $\lim_{x \to a} g(x)$. To do this, we will use the Squeeze Theorem. We define two functions h and f where $h(x) = x$ and $f(x) = -x$; both of these funcitons have the same domain as the original function. Here $f(x) \leq g(x) \leq h(x)$; we have a sandwich! The latter two functions are polynomial; we can use the substitution property to find that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = 0$. As both of these functions have the same limit, the Squeeze Theorem tells us that $\lim_{x \to a} g(x) = 0$. This is a great example of how inequalities can lead to equalities.

2.5 Definition of Limits

Quick primer: mathematicians like to use Greek letters. ϵ is epsilon; δ is delta.

Another reminder: $|y - b| < \delta$. All this means is that $b - \delta < y < b + \delta$. Another one: $0 < |x - a| < \delta$. This means that $a - \delta < x < a + \delta$ and $x + a$. Knowing this, we can continue.

Definition

$\lim_{x \to a} f(x) = L$ where L and a are real numbers and f is a function defined in some open interval about a (note that $f(a)$ need not be defined). For any $\epsilon > 0$ there is some $\delta > 0$ such that whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$.

Examples

We know that

$$\lim_{x \to 2} 5x + 3 = 13$$

Now we are going to prove this using the definition of a limit. Here $L = 13$, $a = 2$ and $f(x) = 5x + 3$. First, we find $|f(x) - L|$. This is a measure of how close $f(x)$ is to L, the limit. We substitute: $|5x+3-13|$ and then $5|x-2|$. Note that $a=2$, so we now have $5|x-a|$. Now we let $\epsilon > 0$ and let δ be defined later. Now we assume $0 < |x-2| < \delta$. Now we substitute and simplify: $|5x+3-13| = 5|x-2| < 5\delta = \epsilon$. Now we know that if $\delta = \frac{\epsilon}{5}$ then $|5x+3-13| < \epsilon$. Now we just define *delta* as $\frac{\epsilon}{5}$ and we have the solution!

Another example:

$$\lim_{x \to 3} x^2 = 9$$

Our job is to show that this is true. Here $a=3$, $L=9$ and $f(x)=x^2$. In this problem, we cannot use the Limit Laws; we have to use the definition of a limit. We start by trying to find $|x-a|$:

$$f(x) - L = x^2 - 9 = (x+3)(x-3)$$

Here $x-a$ is $x-3$; the $x+3$ term is extra. Now we let $\epsilon > 0$, as we do in all such proofs. We now need to define δ. We know that if $0 < |x-3| < \delta$ then $|(x-3)(x+3)| = |x-3| \cdot |x+3| \leq \delta|x+3|$. Here people are tempted to set δ to $\frac{\epsilon}{|x+3|}$. This is invalid, because we do not know what x is; δ is a number while x is a variable. You cannot have an x in the value of δ. Instead, you should try to make $|x+3|$ disappear.

Infinite Limits

The definition of infinite limits is different from that of normal limits. They are written as follows:

$$\lim_{x \to a} f(x) = \infty$$

This means that for any finite $N < \infty$ thee is some $\delta > 0$ such that $f(x) > N$ whenever $0 < |x-a| < \delta$. Basically, for any number, $f(x)$ can be larger. As always, f has to be defined over some open interval containing a, although it doesn't have to be defined at a.

2.6 One-Sided Limits

A one-sided limit is written as follows:

$$\lim_{x \to a^+} f(x) = L$$

This means that there is some number $c > a$ such that $f(x)$ is defined from all $x \in (a, c)$ and for all $\epsilon > 0$ there is a $\delta > 0$ such that

Basically, this means that as x approaches a from the right, $f(x)$ gets closer and closer to L.

Comparison to Two-Sided Limits

It is useful to compare one-sided limits to two-sided limits. One thing that is interesting is that you can have different limits for a^+ and a^-. This means that the two sided limit as $x \to a$ does not exist. Thus, we can say that if $lim_{x \to a} f(x) = L$ then both $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$. The opposite is also true: If the limit as $x \to a^+$ is them same as the limit as $x \to a^-$, then the limit as $x \to a$ exists and is the same as the two one-sided limits.

Example:

$$f(x) = \begin{cases} x+1 & \text{if} x > 0 \\ x-1 & \text{if} x < 0 \end{cases}$$

Here $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$ but $\lim_{x \to 0} f(x)$ *does not exist*.

Infinite Limits

You can also get the limit of $x \to \pm\infty$. The following:

$$\lim_{x \to \infty} f(x) = L$$

means that for any $\epsilon > 0$, there is some M so that whenever $x > M$, $|f(x) - L| < \epsilon$. In less mathematical terms, this means that the difference between $f(x)$ and L decreases as x gets bigger and bigger. For any arbitrarily small number (ϵ), there exists a number such that for all x greater than that number, the difference between $f(x)$ and L is smaller than the arbitrarily small number.

You can use this definition to prove limits, much like for normal limits. Let's take

$$\lim_{x \to \infty} 3 + \frac{1}{x^2 + 1} = 3$$

We begin by getting $f(x) - L$, which is:

$$3 + \frac{1}{x^2+1} - 3$$

or just

$$\frac{1}{x^2+1}$$

Now we let $\epsilon > 0$ and let $M > \frac{1}{\sqrt{\epsilon}}$. Note how M increases whenever ϵ decreases, which makes sense.

You can also define limits as x approaches $-\infty$. Additionally, you can always define limits where $L = \pm\infty$. However, you can also have hybrid limits as so:

$$\lim_{x\to\infty} f(x) = \infty$$

Lines, polynomials, logarithms, exponential functions and others all have limits like this. The simplest case is that $\lim_{x\to\infty} x = \infty$. All functions larger than this also have this limit.

It is convenient to note that:

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(\frac{1}{x})p$$

Horizontal Asymptotes

There is a connection between limits of $x \to \infty$ and horizontal asymptotes. Take the graph of $f(x)$ that has the line $y = L$ as a horizontal asymptote. What this means in terms of limits is simple:

$$\lim_{x\to\infty} f(x) = L$$

This is very similar to vertical asymptotes, which are defined as limits where L is infinite and a is finite.

2.7 Why Limits?

Why do we spend all this time on limits? Learning the formal definition, playing around with $\epsilon - \delta$ proofs..etc?

Well, the fundamental concept of this course is the *derivative*. We will learn about derivatives from multiple points of view. We will look at them through graphs, methods of calculation, application and what it actually means. We cannot get the latter without referring to limits. We will learn about derivatives starting from the simpler concept of a limit.

2.8 Continuity

This helps us find limits using the substitution property. It will also be useful in the future. Here is the formal definition of continuity:

f is continuous at a if the domain is an open interval about a and $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$.

Intuitively, this means that $f(x)$ has to go through and include a with no breaks.

A broader definition concerns open intervals. We can say that $f(x)$ is continuous on the open interval (b, c) if $f(x)$ is continuous at every point in the interval.

If the limit is not open, like $[b, c)$, then it is continuous on the interval if it is continuous on (b, c) and $\lim_{x\to b^+} f(x) = f(b)$. Basically, we do not care about anything out of the interval, like all $x < b$.

Basic Functions

Life is convenient: all of the basic functions we know (polynomials, rational functions, exponential functions $f(x) = a^x$ with $a > 0$, n^{th} roots, logarithms, trig functions, inverse trig functions, absolute value function...) are continuous on their domains.

Building Up Functions

Here are some basic ideas about continuous functions:

If f, g are continuous functions on the same interval and c is a real number, then:

- $f + g$
- $f \cdot g$

- cf

are all continuous functions.

Basically we can build up more complex functions and still know that they are continuous.

Intermediate Value Theorem

Assume that:

- f is a continuous function on a finite, closed interval $[a, b]$.
- $f(a) \neq f(b)$.

Given this, if z is between $f(a)$ and $f(b)$ there is at least one value of x satisfying $a < x < b$ which is a solution of the equation $f(x) = z$. Basically, a continuous function must cross all of the horizontal lines between the two ends of the interval. The function could also cross any given line *multiple* times; all you know is that there is *at least* one solution.

Of course, if the function is not continuous, this will not hold. Additionally, this theorem does not say anything about lines where z is not between $f(a)$ and $f(b)$.

Multiple Answers

The glaring weakness of the intermediate value theorem is that it only tells you *whether* a solution exists; it does not give you the actual solution, or even if more than one solution exists. The theorem only tells us that there is an arbitrary positive number of solutions.

We can use certain facts about the function we're looking at to figure out how many answers there are. Take $\frac{1}{x} = \ln x$. When changed to $\frac{1}{x} - \ln x$, this is a strictly decreasing function, so it cannot, by definition, cross the same horizontal line twice.

3 Derivatives

This is the main part of the course! This is a very useful idea based on the concept of a limit. In very general terms, the derivative of a function f at

a is the slope of the tangent line to f at a. Derivatives do not always exist.

3.1 Definition of a Derivative

For reference, $f'(a)$ = "the derivative of f at a". If $a \in$ domain of f, then:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

As long as the limit exists and is *finite*. In other words, the derivative is the limit of the slopes of secant lines formed from both a and points successively closer to a.

Another way to think of it is as a limit of differences. Take Δx to be the change in x and Δf as the corresponding change in $f(x)$. This lets us restate the formula as:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

This means the exact same thing as the previous definition; it is merely stated in a different way.

This also leads us to another way of writing derivatives. Instead of writing $f'(x)$, we can write $\frac{df}{dx}$. Note that here df *does not have a meaning*, at least for us at this point in the course. This notation reminds us that derivatives are limits of ratios.

Tangent Lines

Previously, derivatives were referenced as the slopes of "tangent lines". Tangent lines are hard to define without derivatives, but not that the derivative *is* definable without tangent lines, so we can define the latter in terms of the former.

A tangent line is the line that goes through the point $(a, f(a))$ and has the slope equal to $f'(a)$.

Calculating Derivatives Using the Definition

We can calculate derivatives using the definition found in 3.1. Let's take a simple example:

$$f(x) = \frac{1}{x}$$

We will get the derivative of this function for all values a where $a \neq 0$. That is, we are solving:

$$\lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}$$

Now all we need to do is simplify:

$$\lim_{x \to a} \frac{\frac{a-x}{ax}}{x - a} = \lim_{x \to a} \frac{-1}{ax} = \frac{-1}{a \cdot a}$$

What we have here is the answer: $f'(a) = \frac{-1}{a^2}$ for all $a \neq 0$. This makes sense given the graph of $f(x)$.

Let's take another example:

$$f(x) = |x|$$

We want $f'(0)$. The limit we need to solve is:

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

We cannot use the rules here, but we can look at one sided limits. We now need to evaluate:

$$\lim_{x \to 0^+} \frac{|x|}{x} \text{and } \lim_{x \to 0^-} \frac{|x|}{x}$$

Here one side is 1 and the other is -1. When one-sided limits to the same value are not equal, that means that the limit at the value does not exists. If the limit does not exists, the derivative does not exist either.

This is an important point to note: not all derivatives exist. A function can easily have a derivative on all points save one, for example.

Limit $h \to 0$

Sometimes, we can greatly simplify the algebra of finding the derivative using the definition. We do this by defining a new variable h which is equal to $x - a$. This then turns $\lim_{x \to a} f(x)$ into:

$$\lim_{h \to 0} f(a + h)$$

And we finally get the formula that:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is only valid as long as a is in some open interval in the domain of f.

Basically, this approach is never *necessary*; it is merely sometimes much easier to solve problems like this rather than just using the normal definition of a derivative.

Real-World Example : Position, Velocity, Acceleration

The canonical example of derivatives in the real world comes from physics: velocity and position. Let's take a function $s(t)$ that represents the position of a particle at a given time. We can interpret

$$\frac{s(t) - s(a)}{t - a}$$

as being the net distance over the net time, which gives us the average velocity. If we take $\lim_{x \to a} \frac{s(t)-s(a)}{t-a}$, we will get the *instantaneous* velocity of the particle at time t.

This basically means that the derivative of position at a time t is the velocity at that time. In other words, $s'(t) = v(t)$. Note that the velocity of a particle is the change in position over time.

We are also generally interested in the change of velocity over time: this is called acceleration. Knowing the acceleration is very useful for understanding the behavior of a moving particle. The nice thing is that we can approach getting the acceleration from the velocity the exact same way we approached getting velocity from position: ultimately, we would find that acceleration is simply the derivative of velocity.

To summerize, we now know that acceleration is the derivative of velocity which, in turn, is the derivative of position. We can also say that the acceleration is the double derivative of position. So, we now have: $a(x) = v'(x) = s''(x)$.

3.2 Continuity

Derivatives also have other applications to functions. One important application is in finding continuity. In short, if $f(x)$ is differentiable at a, then we know that it is continuous at a. This also works backwards: if the function is not continuous at a, then the derivative of f at a does not exist.

It is very important to note that this theorem only states what it states; more pertinently, it does *not* say that if a function is continuous at a it is differentiable at a—this statement is, in fact, trivially disprovable. Let's take the absolute value function. This function does not have a derivative at 0; this was shown in section 3.1 on page 19. However, as $\lim_{x\to 0} |x|$ is 0, we know that the function is continuous at 0. Basically, make sure to never take this theorem to mean that all continuous functions are differentiable.

3.3 Derivatives as Functions

In a large amount of cases, it is convenient to know the derivative of a function at all of its points rather than just knowing it from *one* point. To do this, we need to get the derivative at x of the function $f(x)$ as a function of x. That is, we are finding $f'(x)$ as a function.

Doing this basically involves taking the appropriate limit of the function at all of the points along it. Of course, this is not as difficult as it seems.

We can also take derivatives repeatedly. Taking a derivative twice, for example, results in a "second derivative" which is usually rendered as $f''(x)$.

Notation

When you take the derivative of a function, it may not exist at all of the elements of that function's domain. There are many ways to say this:

- f is differentiable at a
- $f'(a)$ exists
- $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
- $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

- $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ where $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$, exists.

All of these statements mean the same thing, namely that the derivative of f at a can be found. This is a very good example of all of the different notations.

All of the different things exist for a reason: in certain cases, one particular notation is preferable over the other. Additionally, some things just have two names.

3.4 Calculating Derivatives

Derivatives are very useful; using limits to calculate them is cumbersome. In order to make life much easier, we will go over some rules for more quickly finding derivatives. There will be, broadly speaking, two types of rules: basic rules that let you get the derivative of basic functions and other rules that let you get the derivatives of complex functions made up of simpler ones.

The following sections contain a summary of all of the various rules we have for quickly calculating the derivative of various functions ranging in complexity from the simple to the extremely complex.

3.5 Simple Functions

Here are some rules for very simple functions which will serve as the building blocks for calculating more complex derivatives:

- If $f(x) = C$, then $f'(x) = 0$ for all x.
- If n is an integer $\neq 0$ and $f(x) = x^n$ then $f'(x) = nx^{n-1}$ for all x, unless $n < 0$ in which case the derivative does not exist at 0.
- If $f(x) = r^x$ and $r > 0$, then $f(x) = f'(0) * f(x)$ for all x. Note that if $r = e$ then $f'(x) = f(x) = e^x$.

Combinations of Functions

This is the second type of rule, where we will find the derivative of combinations of known functions. This is very useful for differentiating more complex expressions.

- **Constants:** If $g(x) = C \cdot f(x)$ then $g'(x) = C \cdot f'(x)$ for all x if $f'(x)$ exists.
- **Sums:** If $h(x) = f(x) + g(x)$ and both $g'(x)$ and $f'(x)$ exist, then $h'(x) = f'(x) + g'(x)$.
- **Products:** If $f'(x)$ and $g'(x)$ exist, and $h(x) = (f \cdot g)(x)$, then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ for all x.
- **Quotients:** Given the same conditions as for the product rule, as well as $g(x) \neq 0$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) + f(a) \cdot g'(a)}{g(a)^2}$$

Trig Functions

Apart from simple funcitons, it is also useful to be able to differentiate trigonometric functions. The two simplest ones are $\sim x$ and $\cos x$, which are all differentiable on all x in their domains. Basically, $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$. This now all that is needed to figure out the more complex functions like, say, $\tan x$.

We know from trigonometry that $tanx = \frac{\sin x}{\cos x}$. This is a very simple application of the quotient rule:

$$\frac{d}{dx} \tan x = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

This then simplifies to:

$$\frac{\cos^2 x - \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec x$$

This is true as long as $\cos x \neq 0$. If $\cos x = 0$ then the derivative is undefined since we can't divide by 0.

In summary, the derivatives of various trig functions are:

Function	Derivative	Conditions
$\sin x$	$\cos x$	All $x \in \Re$
$\cos x$	-$\sin x$	All $x \in \Re$
$\tan x$	$\sec^2 x$	All $x \in \Re$ such that $\cos x \neq 0$

3.6 Implicit Derivation

Function are usually defined explicitly, like so:

$$f(x) = x^2 + x + 2$$

Here the function is on one side; the expression for calculating the function in terms of x is on the other side. In general, this means that the function is defined in the form of:

$$f(x) = \text{somthing } x$$

This is a very convenient form for working with functions; however, sometimes functions are not written this way.

Another way to define a function is implicitly. That is, the function is defined on the same side of the equation as some other things. The general form of this is:

$$f(x,y) = 0\text{or } f(x,y) = h(x,y)$$

A more realistic example of implicit definition is the equation of a circle:

$$y^2 + x^2 = 1$$

Note that there is no immediately obvious $f(x)$ here. Also not that, in this case, y is not a function of x; rather, it is two functions.

Calculating Implicit Derivatives

Let's take an example to show how to calculate the derivative of implicitly defined functions. Here is the equation:

$$4x^2 + xy + y^2 = 24$$

This is the equation of an ellipse. We can get the derivative of this function even though it is not actually a function and it is defined implicitly. To do so, we would take the derivative in terms of x of both sides:

$$\frac{d}{dx}(4x^2 + xy + y^2) = \frac{d}{dx}(24)$$

Now, given this somewhat complicated equation, it is time to simplify. We can turn this equation into:

$$8x + (x\frac{dy}{dx} + 1 * y) + 2y\frac{dy}{dx} = 0$$

Now we have a formula with three interesting bits: x, y and $\frac{dy}{dx}$, which is what we are looking for.

Now all we have to do is simplify this and get the answer. We do this by solving for $\frac{dy}{dx}$.

$$(x + 2y)\frac{dy}{dx} + (8x + y) = 0$$

And then:

$$\frac{dy}{dx} = -\frac{8x + y}{x + 2y}$$

Now we have the solution. Note that both x and y are present in the solution; this is fine. Now, to find the slope of a tangent line at a particular point, you just have to plug the point into the newly solved equation.

A More Complicated Example

The previous example was not very complicated; it might have even been easier to simply solve the equation to get y as a set of functions of x and then just differentiate normally. However, there are plenty of implicit equations out there that are too difficult to plausibly solve this way. This is an example:

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

Solving this equation for y would be very difficult; it is much easier to just differentiate implicitly. We start by getting $\frac{d}{dx}$ of each side:

$$\frac{d}{dx}(y^2(y^2 - 4)) = \frac{d}{dx}(x^2(x^2 - 5))$$

Next we derive all the parts:

$$2y\frac{dy}{dx}\cdot(y^2-4)+y^2\cdot 2y\frac{dy}{dx}=2x(x^2-5)+2x^3$$

Next we simplify:

$$\frac{dy}{dx}(2y(y^2-4)+2y^3)=2x^3+2x(x^2-5)$$

And we finally get:

$$\frac{dy}{dx}=\frac{2x^3+2x(x^2-5)}{2y(y^2-4)+2y^3}$$

Now that we have the slope as a function, we can once again plug in any point to get the slope of the tangent at that point. However, note that this will make us sometimes divide by 0. When we get an undefined answer like that, we are most likely dealing with a vertical tangent line which has no slope.

A final note: we will not talk about when it is valid to use this method; we will just assume any time it works it is valid.

3.7 Inverse Trig Functions

We will now cover how to differentiate inverse trigonometric functions. These are functions like arcsin and arctan—given a degree measure, they return the length of a side of a triangle.

Arcsin

Lets take the arcsin function for example. The $\arcsin(x)$ function is also sometimes called $\text{Arcsine}(x)$ or $\sin^{-} 1x$.

We can get the derivative of $\arcsin x$ using implicit differentiation. We do this by taking the function:

$$\sin \arcsin x = x$$

and getting its derivative with respect to x. That is, we take:

$$\frac{d}{dx}(\sin \arcsin x)=\frac{d}{dx}(x)$$

Then we get:

$$\cos \arcsin x \cdot \frac{d}{dx} \arcsin x = 1$$

Which means that:

$$\frac{d}{dx} \arcsin x = \frac{1}{\cos \arcsin x}$$

We can further simplify this by getting the value of $\cos \arcsin x$. This is relatively simple; it can be done geometrically. First, imagine a right triangle...

Arctan

Arctan is the opposite of tangent much the same way arcsin is the opposite of sin. We can take the same approach to getting its derivative as we did for arcsin:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\arctan tan x)$$

We can now use the chain rule to get this derivative:

$$1 = \sec^2 x \cdot \arctan x$$

We can solve this for arctan:

$$\frac{d}{dx} \arctan x = \frac{1}{\sec^2 \arctan x}$$

Now we just need to solve $\frac{1}{\sec^2 \arctan x}$. First we use the identity that $\sec^2 x = 1 + \tan^2 x$:

$$\frac{1}{1 + \tan^2 \arctan x}$$

This simplifies into our final answer:

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

We can also approach simplifying $\sec^2 \arctan x$ using geometry. For this, we start with a right triangle where one side is x and the other leg is 1. This gives us the length of the which is $\sqrt{1 + x^2}$. Given the angle θ, we can solve for $\cos \theta$:

$$\cos \theta = \frac{1}{\sqrt{1 + x^2}}$$

This naturally leads us to $\sec x$:

$$\sec\theta = \sqrt{1+x^2}$$

3.8 An Important Function

Here is a function that is going to be very important in the future: $f(x) = \ln(|x|)$. This may seem like an odd function, but it will be really important in the next course.

How would we go about finding the derivative of this odd function? Unsurprisingly, the absolute value makes life less fun; the absolute value function, defined by cases, is always tricky to deal with.

We can solve this problem using a trick: $\ln|x|^2 = \ln x^2$. This is also the same as $2\ln|x|$. This then means that $\frac{1}{2}\ln x^2 = \ln|x|$. We can now solve this problem using the chain rule.

We now get:

$$\frac{d}{dx}\ln|x| = \frac{1}{2}\frac{d}{dx}\ln x^2$$

We can now proceed with the chain rule:

$$\frac{1}{2}\cdot\frac{1}{x^2}\cdot\frac{d}{dx}(x^2)$$

We can now simplify this a bit, giving us the answer:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

This is a nice result as we took a rather complex, ugly fraction and got a very simple, fundamental function as the derivative.

3.9 Differential Equations

Differential equations are, unsurprisingly, equations that involve derivatives. For example, take this equation:

$$\frac{dy}{dt} = 3y(t)$$

Here we have a derivative ($\frac{d}{dx}$) a variable t and an unknown y. One of the important notes here is that the unknown, y, is *not* a variable—it is a *function*. This means that this equation is not solvable through mere algebra.

More generally, these equations can come in the form:

$$\frac{dy}{dy}(\text{or } y'(t)) = k \cdot y(t)$$

Where k is some constant. Here $\frac{dy}{dt}$ should be read as the change in y with regards to t.

These equations are useful in many different fields, like biology. A population reproduces proportionally to its size; radioactive elements decay at a rate proportional to the amount of the element. This particular equation is the most common differential equation and is seen in many different fields particularly because it is a very simple differential equation.

Solving Simple Differential Equations

A solution to this equation would be a function y that makes both sides equal. For example, given the equation

$$\frac{dy}{dt} = 3y$$

We can get the solution

$$y(t) = e^{3t}$$

because

$$\frac{d}{dt}(e^{3t}) = 3e^{3t} = 3y$$

However, this equation is not as simple as other equations; there are, in fact, very many answers to this equation. In fact, there is an infinite amount of possible solutions to this problem. This is because we can make y a different function like:

$$y(t) = 10 \cdot e^{3t}$$

and it would still be a valid solution! Having infinitely many solutions is not a good place to be; for this reason, the solution to equations in the form $y' = ky$ is $y(t) = Ce^{kt}$ where C is any constant. This lets us describe all of the possible solutions as one.

The sort of problem we may encounter in terms of differential equations is along the lines of:

$$y' = ky \text{ and } k = 3 \text{ and } y(2) = 100$$

To solve this, we first need to use the general solution of the equation in the form of $y' = ky$, which is Ce^{kt}. Now we just need to solve for C, so $Ce^{kt} = 100$ when $t = 2$.

Unknown k

Now let's look at a different example, where the constant is not known. The following problem is an example of this:

$$\frac{dy}{dt} = ky$$

Where $y(0) = 10$ and $y(2) = 100$. We need to find $y(t)$. The special bit here is that k is not given. However, we have two pieces of additional information to use, namely the value of $y(t)$ at two separate values of t. We can plug one of these know values into the equation:

$$y(0) = 10 = \mathrm{C} \cdot e^{kt}$$

As $e^0 = 1$, we can easily solve this for C, getting $C = 10$. Next we take the other value we have and plug it into the equation, now knowing C:

$$y(2) = 100 = 10e^{2k}$$

This is easily solvable with logarithms:

$$e^{2k} = \frac{100}{\mathrm{C}} = 10, \text{ so } e^{2k} = 10.$$

This naturally means that:

$$k = \frac{\ln 10}{2}$$

Now we have all the information we need for the solution!

Another Example

Here is another sort of problem we may encounter. The difference here is that while we are given two values of $y(t)$, we are *not* given $y(0)$. This sort of problem is very common in applications such as carbon dating. Let's say that we are given:

$$\frac{dy}{dt} = ky$$

as well as:

$$y(t_1) = \frac{1}{2}y(t_2) \text{ where } t_2 \neq 0$$

We can proceed by substituting the function in for $y(t)$:

$$\frac{1}{2} = \frac{y(t_1)}{y(t_0)} = \frac{Ce^{kt_1}}{Ce^{kt_0}}$$

This can be easily simplified to:

$$\frac{1}{2} = e^{k(t_1 - t_0)}$$

We can now use logarithms to solve this, ultimately getting:

$$k = \frac{\ln \frac{1}{2}}{t_1 - t_0} = \frac{-\ln 2}{t_1 - t_0}$$

Can we tell whether this is positive or negative? Well, no. However, we can say that it is positive if $t_1 > t_0$. This equation explains two situations: if $k < 0$, then this is exponential decay; if $k > 0$, then this is called exponential growth.

Carbon Dating

As mentioned before, an important application of this sort of equation is in carbon dating. Carbon dating is a method for establishing the age of something by comparing the amount of carbon 14 it contains to the amount of carbon 14 in the atmosphere. Carbon 14 decays exponentially over time, so the difference between the ambient carbon 14 content and the content in a particular thing is enough information to give us how long the carbon has been decaying.

Let's have a function y that represents the amount of carbon 14 at a time t. Given all we know about carbon 14, we know that the following relation is true:

$$\frac{dy}{dt} = ky(t)$$

This is, happily, exactly the differential equation we have been studying just now! What an unexpected coincidence. What we would like to know is the value of k in this equation. We can do this experimentally; it has been established that carbon 14 has a half-life of about 6000 years. This is enough information to get k.

A half-life is how long it takes half of the carbon 14 to decay. In this case, a given sample will be halved in 5730 years. This is useful information as this is just like the last example; here $t_1 - t_0 = 5730$. Now, using what we found in the previous example, we know that:

$$k = \frac{-\ln 2}{5730}$$

This is all good, but we cannot measure the amount of carbon 14 the creature had originally. However, this is not really a problem, since we know the ratio between carbon-12 and carbon-14 in the atmosphere.

Here is a more concrete example of this sort of problem: let's say that t_1 is now, the elapsed time and that $t_0 = 0$, which was the time of the organism's death. We want to find t_1. We know that:

$$\frac{y(t_1)}{y(t_0)} = e^{k(t_1 - t_0)} = e^{kt_1}$$

Now all we have to do is solve. We get by knowing that the organism only has 1% of the carbon-14 we expect:

$$kt_1 = \ln 10^{-2}$$

Followed by:

$$t_1 = \frac{\ln 10^{-2}}{k} = \frac{\ln 10^{-2} \cdot 5730}{-\ln 2}$$

Just doing the arithmetic gives us the ultimate solution.

Newton's Law of Cooling

Another use of this sort of differential equation is Newton's law of cooling. This law governs change of temperature between two bodies as their temperatures approach each other. The law states that the change in the temperature is proportional to the difference in temperatures. This can be restated as:

$$\frac{d}{dt}T(t) = k(T(t) - T_s)$$

Where $T(t)$ is a function of temperature over time and T_s is the temperature of the environment. Note that this is *not* the same equation as we have been looking at lately; what we did earlier does not apply directly.

We can overcome this problem by defining a new function y such that $y(t) = T(t) - T_s$. Given this, we also know that:

$$\frac{dy}{dt} = \frac{dT}{dt} - \frac{dT_s}{dt} = k(T(t) - T_s) = ky$$

This means that we *can* get a simple differential equation for y:

$$\frac{dy}{dt} = ky$$

This naturally leads to:

$$y(t) = Ce^{kt}$$

And, substituting back, we get:

$$T(t) = T_s + Ce^{kt}$$

Now that we have this equation, the rest can be solved using the same methods we used above.

3.10 Related Rates

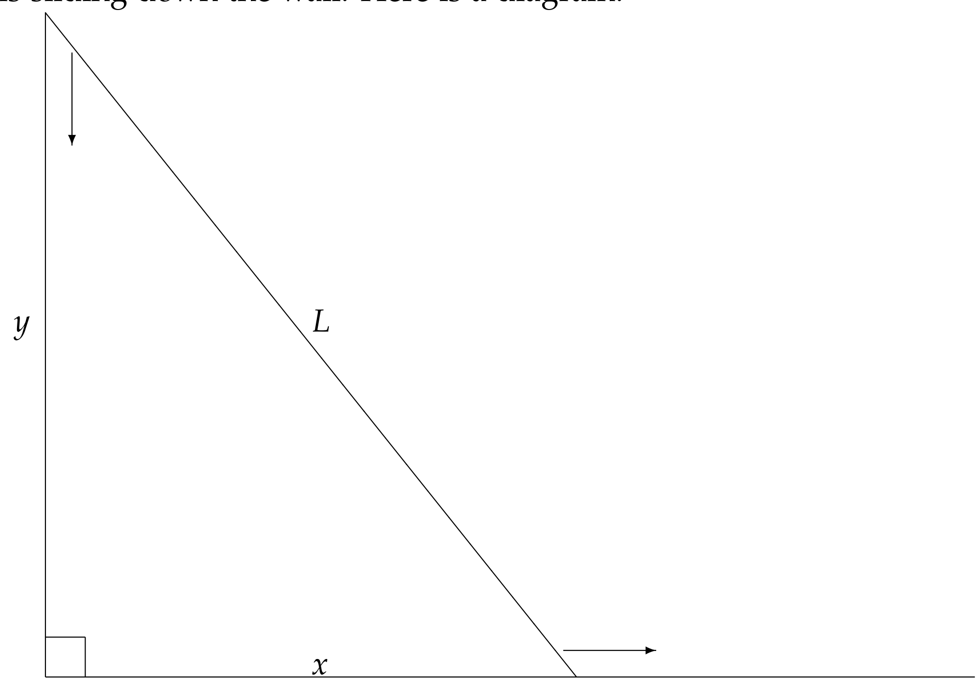
Now we are going to look at a different application of derivatives to real life. Math wouldn't be particularly useful without applications, would it?

We are now going to be applying the derivative. The derivative can be generally thought of as the *instantaneous* change in one quantity with respect to something else. The latter part can be clearly seen when the derivative is written out as $\frac{dy}{dx}$—this means "the change in y with respect to x". The other notations do not always make this very clear, but it always holds.

Now to go on about related rates problems. These problems are going to be looking at the derivative as the rate of change in something; they generally have more than one dependant variable, with all of the variables being related in some way.

Example: Ladder Problem

Here is the canonical example of a related rates problem. Imagine a ladder leaning against a wall that is perpendicular to the floor. The ladder is not anchored very well, so the bottom is sliding out and the top of the ladder is sliding down the wall. Here is a diagram:



We are given the length of the ladder as well as the distance from the bottom of the latter to the wall and the top of the ladder to the floor. Naturally, we may be given only two of the three previous values but completing that set is trivial. We are also generally given the rate of change of the position of one of the ends.

We have the following quantities in the problem:

- Time t
- Position of foot: $x(t)$
- Position of top: $y(t)$
- The speed of the ladder going down at a time t_0, $\frac{dy}{dt} = -4$

This is all the information we get. The question is to find the speed of the bottom of the ladder at time t_0 (we do not have enough information to find the speed at other times).

The first part is to find how the quantities in the problem are related. Here, this is easy: the ladder and wall make a right triangle; we can simply use the Pythagorean theorem. Thus we know that:

$$x^2(t) + y^2(t) = 13^2$$

Now, we could solve this by solving for the unknown ($x(t)$); however, we can use a better approach: implicit differentiation. We do this by taking the derivative of both sides:

$$(\frac{d}{dt})(x^2(t) + y^2(t)) = (\frac{d}{dt})(13^2)$$

This is equal to:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Now we have the relation between the two rates of change; this is better written solved for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dx}$$

Knowing this, we can simply plug in known values and solve:

$$\frac{dx}{dt} = \frac{-y(t_0)}{5}(-4)$$

Note how we do not know $y(t_0)$. This is easy to find as we have $x(t_0)$; we just plug it into the equation we got with the Pythagorean theorem, getting $y(t_0) = 12$. This gives us the final equation:

$$\frac{dx}{dt} = \frac{-12}{5} \cdot (-4) = \frac{48}{5}$$

This is the solution to the problem!

3.11 Linear Approximations

We are now going to cover a more heuristic way of solving problems. We are not going to get the *correct* answer; rather, we will get one that is more or less correct.

We are doing this because there are a lot of function which are very complicated; these are always difficult to work with. On the other hand, we have the linear equation $y = mx + b$ which is basically the easiest function there is to use. The basic idea behind linear approximations is modeling more complex functions using lines.

We will be using the tangent line of a curve to approximate it. The tangent line of the function f at the point a is

$$y = f'(a) \cdot (x - a) + f(a)$$

This can sometimes be a very good approximation of a function; at other times it can be very bad.

Basically, given the equation of a tangent line, it is a sufficiently close approximation of the function if x is sufficiently close to a. Of course, as ever, all of the nebulous terms are all relative; they depend on context and the problem at hand.

An Example: $x^{\frac{1}{3}}$

Here is a simple problem: find a good approximation for $x^{\frac{1}{3}}$ where x is close to 1. The natural answer here is $x^{\frac{1}{3}} \approx 1$; however, we can do much better than that. We know that if $f(x) = x^{\frac{1}{3}}$ then $f(1) = 1$ and $f'(a) = \frac{1}{3}x^{frac-23}$; this leads to $f'(1) = \frac{1}{3}$. Now we plug all of this into the formula for a tangent line we got above, getting:

$$x^{\frac{1}{3}} \approx 1 + \frac{1}{3}(x - 1)$$

Note how this approximation has a term, 1, and an additional term that helps correct for x when x is far away from 1. This second term approaches 0 as x approaches 1.

This approximation is actually pretty accurate; if $x = 1$, then it is exact. However, if x is far away from 1, the approximation is very bad: for

$x = 1000$, the actual value is 10 while the approximation gives us 334. But let's take a look at values of x close to 1. If x is 1.1, the approximation is close to three decimal places; if x is 1.001, you can get the answer close to seven decimal places.

Another Example:

Now we are going to look at the natural log function. To make this more interesting, we are going to take two numbers that are relatively far from each other: 1000 and 1010. We want to know how close $\ln 1010$ is to $\ln 1000$.

We will use a linear approximation here, with $a = 1000$ and $x = 1010$. We don't know what $\ln 1000$ is, but that is fine because we just need to know how close that is to $\ln 1010$. We also know that the derivative of the natural log function is $\frac{1}{x}$. Plugging this in, we get:

$$\ln 1010 \approx \frac{1}{1000}(10) + \ln 1000$$

We can simplify this to get:

$$\ln 1010 - \ln 1000 \approx 0.01$$

The actual difference is 0.00995, so we were *very* close. This really illustrates how "close" is a very relative idea: for logs, a difference of 10 between x and a for a different function could make the approximation really off, but for logarithms it is nothing.

3.12 Maxima and Minima

Now we are going to cover how to find the largest and the smallest values of a function over an interval. In such a problem, we are generally given a function f and a closed, bounded interval $[a, b]$. Given this, our goal is to find either the maximum (the largest $f(x)$ if $x \in [a, b]$) or the minimum (the smallest $f(x)$ if $x \in [a, b]$.

This sort of problem has a ton of applications; they are very useful in the real world. For example, many laws of physics tell us that something behaves in a way as to minimize another quantity. An example here is thinking of various problems involving forces as problems that try to minimize work. This is also the sort of problems that are often approached using genetic algorithms; however, that is a very different method of solving these. We are only going to approach these problems using the magic of calculus.

Given the function f with domain D, the *absolute maximum* of the function is the number $c \in D$ such that c is greater than or equal to all other values $f(x)$ where $x \in D$. The *absolute minimum* is the same except that instead of being greater than or equal to it is *less* than or equal to all other values of f in D. Functions can have *multiple* absolute minima and maxima; for example, the function $f(x) = \cos x$ has *infinite* absolute maxima. Additionally, a function need not have either; for example, $f(x) = x$ has no absolute maxima or minima.

3.13 Rolle's and the Mean Value Theorem

Now we are going to cover two related theorems. The first was discovered first; the second is very similar but more broad. Both concern the behaviour of the derivative of functions.

Rolle's Theorem

Rolle's theorem concerns functions that have a domain that is a closed interval $[a, b]$, are continuous on that interval and are diffrentiable on the interval (a, b). Given such a function, Rolle's Theorem states that if $f(a) = f(b)$ then there exists some number c such that $f'(c) = 0$.

Of course, a function that does not fulfill the criteria for Rolle's Theorem will not necessarily have a derivative that equals 0 anywhere. Take the function $f(x) = |x|$ on the domain $[-1, 1]$: there is no number c so that $f'(c) = 0$, but since the function is not differentiable at $x = 0$, it does not fulfill the requirements of Rolle's Theorem, so the theorem does not apply.

An Example

Here is a simple example of Rolle's theorem in use. Let's take the function:

$$f(x) = x^3 + 6x - 20$$

We want to show that this equation has at most one solution. Note that this function is differentiable on all real numbers. Now we need to take two numbers x_1 and x_2, assuming that $f(x_1) = f(x_2) = 0$. We can now use Rolle's theorem on the interval $[x_1, x_2]$. This tells us that, if both x_1 and x_2 exist, $f'(x)$ where $x \in [x_1, x_2]$ must equal 0 at some point. However, we know that:

$$f'(x) = 3x^2 + 6$$

This means that the derivative of the function is *never* 0! This means that there cannot be two numbers where $f(x) = 0$.

This seems to be a solution particular to this function; however, this can be easily generalized to all functions: if a function is differentiable on (a, b) and if f' is never 0, the function is one-to-one on $[a, b]$ and has an inverse on $[a, b]$.

Mean Value Theorem

Rolle's theorem is, evidently, a somewhat narrow theorem: it only deals with intervals $[a, b]$ where $f(a) = f(b)$; this is a rather rare situation. This can be abstracted to situations where $a \neq b$; this leads to the Mean Value Theorem.

The mean value theorem states that, given a function differentiable on the interval (a, b) and continuous on $[a, b]$ (just like Rolle's theorem), then there is some number $c \in (a, b)$ such that the following holds:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, the theorem stats that, given two points, there will be a point where the slope of the tangent line is that same as the slope of the secant line containing both of those points, as long as the function fulfills all of the requirements for the theorem.

To put this in more grounded terms, we turn to physics. Let's take the function $s(t)$ that represents the position of a particle. We know that the average velocity of the particle from t_1 to t_2 is equal to:

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Additionally, we know that $s'(t) = v(t)$ and represents the *instantaneous* velocity of the particle at time t. What the Mean Value Theorem says in this case is that over some interval of time, there is a time t such that the instantaneous velocity at time t is equal to the average velocity over the given interval.

Consequences

Now let's look at some of the immediate ramifications of this theorem. Let's say that for a function f, $f'(x) = 0$ for ever $x \in (a, b)$. This tells us that our function is a constant on the given interval. This can be easily shows using the Mean Value Theorem. We get the following equation:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

This can only be true if $f(x_1) = f(x_2)$. Here x_1 and x_2 can be any two numbers in the given interval.

Additionally, we get that if $f'(x) > 0$ for all $x \in (a, b)$ and $a < x_1 < x_2 < b$, then $f(x_1) < f(x_2)$. In short, this means that if all of the tangent lines slope upwards, the function is always increasing.

Yet another interesting fact is that if you have $f'(x) = g'(x)$ for all x in (a, b), then the two functions are the same over (a, b) as long as $f(a) = g(a)$. All of these finding are really rather obvious, but they are important.

Example

Now let's look at how this theorem can be applied. We want to show that $\ln 1 + x < x$ for all $x > 0$. Now, if x is very large, it is patently obvious that $\ln 1 + x$ is smaller; however, this is much less obvious for small values of x.

Both Rolle's theorem and the Mean Value Theorem concer single functions, so it is expedient to combine both of these functions in some way. We can do this by subtracting, getting:

$$f(x) = x - \ln 1 + x$$

Now we need to get the derivative of this function. We get:

$$f'(x) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x} > 0$$

Given this, we know that $f'(x) > 0$ for all $x > 0$, which means that f is increasing on all $x > 0$. This means that $f(x_2) > f(x_1)$ if $x_2 > x_1 \geq 0$. Now we can take $x_1 = 0$. We get that $f(x) > f(0) = 0 - \ln x + 1 = 0$ for all $x > 0$. This means that $\frac{f(x)-f(0)}{x-0} > 0$. This is the same as $\frac{f(x)}{x} > 0$. We can multiply by x since $x > 0$ and we get $f(x) > 0$.

3.14 Graphing

We are now going to look at how to quickly draw a graph of a function using the first and second derivatives of the function. This is a very useful trick; it lets us graph functions with relatively little information rather quickly.

Increasing/Decreasing Test (I/D)

The basis of graphing functions quickly is called the "Increasing/Decreasing Test". Given an interval (a, b), if a function f is differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$ then $f(x_z) < f(x_2)$ if $a < x_1 < x_2 < b$. This basically means that the function is increasing along an interval if its derivative on that interval is always positive.

This theorem is actually more broad than this; it can be generalized to negative derivatives. If $f'(x) < 0$ when $x \in (a, b)$, then the function is decreasing.

Additionally, this also means that if the derivative is non-negative then the function is not decreasing and if the derivative is not positive the function is not increasing. Finally, if f is continuous on a closed interval $[a, b]$, then all of this applies to the closed interval; here x_1 can be equal to a and x_2 can be equal to b.

This can also be taken to another level: if $f'(x) \geq 0$ for all x, and $f(x) = 0$ for only **one** x, the function is still increasing.

First Derivative Test

Lets assume that a function f is differentiable on (a, b). Let c be a number such that $c \in (a, b)$. If $f'(x)$ changes sign at c (that is, left of c the function is negative and right of c it is positive, or vice-versa), then the function must have a local maximum or minimum at c. If you go from - to +, then the function will have a local minimum; if it goes from + to -, then the function will have a local maximum. A natural corollary to this is that if the derivative does not change sign, the function has neither a local maximum nor a local minimum—this is also evident intuitively.

Now lets look at a nice example. We need to find the local maxima and minima for the function f where f is:

$$f(x) = 3x^4 - 8x^3 + 6x^2$$

The first step here is to find the derivative of the function, which is easy:

$$f'(x) = 12x^3 - 24x^2 + 12x$$

Now we need to find where the derivative is negative. The natural next step is to factor the derivative, getting:

$$f'(x) = 12x(x-1)^2$$

The next step is to draw a number line:

Slant Asymptotes

To graph various curves, we need to know about asymptotes. We have already covered horizontal and vertical asymptotes; now we need to look at *slant* asymptotes. A slant asymptotes is defined as a line in the form:

$$y = mx + b$$

A given function f has a slant asymptote along the line $y = mx + b$ if the following is true:

$$\lim_{x \to \infty} f(x) - (mx + b) = 0$$

This means that as increases, $f(x)$ gets closer and closer to the given line. This also naturally holds for limits as $x \to -\infty$. In practice, slant asymptotes are actually just a more general form of horizontal asymptotes; the main difference is that the line the function approaches is not horizontal.

Putting Everything Together

We can use L'Hôpital's rule to help use quickly graph functions. Let's take the function

$$f(x) = \frac{x^3}{x^2 - 3}$$

We want to find all of the interesting points of this function.

The first thing we should do is find the domain of the function. Here, we need to make sure we are not dividing by 0. We get interesting points at $x = \pm\sqrt{3}$. At these points, the function will do something interesting.

Now we want to get the first and second derivatives of the function—we know we will need them at some point in the future. The first derivative is

$$f'(x) = \frac{x^4 - 9x^2}{(x^2 - 3)^2}$$

and the second derivative is:

$$f''(x) = \frac{x(6x^2 + 54)}{x^2 - 3)^3}$$

Now we want to know if the function is even or odd. This function is odd because it is in the form $\frac{\text{odd}}{\text{even}}$, which means it is odd. This means that $f(-x) = -f(x)$.

The next step is to look for asymptotes. We know three types of asymptotes, so we need to check for all of them. We know that there are vertical asymptotes when we divide by 0. We already have the x values: $\pm\sqrt{3}$. Now we need to look for horizontal and slant asymptotes. We know that

$$\lim_{x\to\infty} f(x) = \infty$$

This means that there is no horizontal asymptote. Since we know that our function is odd, we do not need to worry about $x \to -\infty$ as the function is just a mirror image of itself in the negative direction.

Now we have to look for slant asymptotes. These are harder to find as there are two variables we need to keep track of: slope *and* y-intercept. We want to rewrite this function, so we do the following:

$$f(x) = \frac{(x^3 - 3x) + 3x}{x^2 - 3} = x + \frac{3x}{x^2 - 3}$$

This helps us find the slant asymptote. Using the rewritten formula, we find that there is a slant asymptote along the line $y = x$ as x approaches ∞. Since this function is odd, we also know that it has the same slant asymptote as x goes to $-\infty$.

Now we need to find the intervals where the function is increasing or decreasing. To do this, we will use the first derivative to find this. To do so, we need to solve:

$$f'(x) = \frac{x^4 - 9x^2}{(x^2-3)^2} = 0$$

To make life easier, we should factor:

$$f'(x) = x^2(x^2-9)(x^2-3)^{-2}$$

The only term that can change the sign is $(x^2 - 9)$. This term is negative on two intervals: $(-3, 0)$ and $(0, 3)$. Therefore, this function is increasing on $(-\infty, -3)$ and $(3, \infty)$. Additionally, since $x = 0$ at only one point on $(-3, 3)$, we know that the function is increasing throughout the *whole* interval $(-3, 3)$. However, this does not take into account that the derivative is not defined at $\pm\sqrt{3}$. Taking this into account, we actually get that the function increases on *three* separate intervals: $(-3, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, $(\sqrt{3}, 3)$.

Now that we have the intervals where the function is increasing and decreasing, we can find the local maxima and minima. We know that wherever $f'(x) = 0$ or doesn't exist there *can* be a maxima or minima. Thus, we have five points to check: -3, $-\sqrt{3}$, 0, $\sqrt{3}$, 3. We now need to look at which points the derivative changes signs. The points where this happens are -3 and 3. We know that $\pm\sqrt{3}$ are vertical asymptotes, so they can't be maxima or minima. Thus, in the end, we find that there is a local maximum at -3 and a local minimum at 3.

And, finally, for the last step, we look at concavity. We do this by looking at $f''(x)$. We factor this:

$$f''(x) = x(6x^2 - 54)(x^2-3)^{-3}$$

We know that the second derivative changes signs at $\pm\sqrt{3}$ and 0. The concavity changes at $\pm\sqrt{3}$ and 0. Since $f(x)$ is undefined at $\pm\sqrt{3}$, the sole inflection point is at $x = 0$. Thus the function is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and it is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

Now we just need to draw all of the interesting features we have found, which will give us a very good approximation of the graph.

3.15 Limits of Ratios: L'Hôpital's Rule

We are now going to cover a method of finding the limit of a ratio, called L'Hôpital's rule. This can be stated as follows: if $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists and:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

This is only true as long as $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} = \pm\infty$. In the latter case, the two infinities need not match up: one can be negative and one can be positive. Also note that you cannot use this rule when one limit is 0 and the other is $\pm\infty$.

Example

Let's find the following limit:

$$\lim_{x\to 0}\frac{1+2x}{1+3x} = \lim_{x\to 0}\frac{2}{3} = \frac{2}{3}$$

But wait! This is not right. The reason this did not work is that L'Hôpital's rule does not work on limits like this; it only works on "indeterminate" limits—in practice, it only works on limits that you could not get otherwise and fails on limits that are easy to get.

Now let's take a more useful example:

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2}$$

This is a valid limit for the rule since the limit of the top is 0 and so is the limit of the bottom. Now we get the derivative of the top and the bottom:

$$\lim_{x\to 0} - \sin x 2x = -\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} = -\frac{1}{2}$$

Note that the resulting limit from applying the rule was also indeterminate; we merely knew its value. However, if we did not, we could simply apply the rule again. Sometimes you'll need to apply the rule more than once to get the value of a limit.

Variants

This rule can be applied in several more cases:

- it works for one-sided limits (e.g. $x \to a^{\pm}$)
- it works for infinite limits (e.g. $x \to \pm\infty$)
- it also works if $\lim_{xtoa} f'(x)g'(x) = \pm\infty$; even though that limit doesn't exist, we still know that the limit $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty$, the same as the limit of the derivatives.
- it also works on products as long as they can be rewritten as a ratio. That is, we could find the value of $\lim_{x \to 0^+} x \cdot \ln x$ by using the rule on $\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$.

More Examples

Here is another example: let's find

$$\lim_{x \to \infty} \frac{\ln x}{x}$$

We know that this is equal to

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

This limit shows that $\ln x$ goes to ∞ *really* slowly: it is the opposite of the exponential function.

And yet another: let's find

$$\lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x}$$

This is a limit of the type "$\frac{\infty}{\infty}$". Now we naturally take the limit of the top and bottom:

$$\lim_{x \to 0} \frac{x - \sin x}{x \cdot \sin x}$$

This is also an indeterminate limit! It is of the type "$\frac{0}{0}$". Now we take the derivatives again:

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cdot \cos x}$$

This is again of the type "$\frac{0}{0}$". We use the rule again:

$$\lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \cdot \sin x}$$

This limit is determinate! We can find that it equals 0. Now, using L'Hôpital's rule, we can find that the first limit is also 0. This is interesting because it was a limit of the form $\frac{\infty}{\infty}$ but in the end was equal to 0!

3.16 Newton's Method

Newton's Method is used when there is an equation of the form $f(x) = 0$. You can use them method to find a solution to this equation in cases when it is not possible to find the solution algebraically. Newton' Method is a heuristic: it will not give you an *exact* solution; rather, it will get you an arbitrarily accurate approximation of the solution.

Let's start by finding the value of $\sqrt{10}$. We can do this by writing a nice function like:

$$f(x) = x^3 - 10$$

This equation, of course, is solvable using basic arithmetic and a calculator; Newton's method is actually a viable method for the calculator to solve problems like this.

The basic idea of Newton's Method is iterated improvement—you find an approximation and you make it better by applying the method repeatedly. It uses tangent line approximations.

To use the method, get a tangent line of the curve at some point; find that tangent line's x-intercept, repeat using the x-intercept as the seed value.

To make life simpler, instead of doing the algebra each time, here is the equation solved for x:

$$x = a - \frac{f(x)}{f'(x)}$$

This formula should make life easier.

Summary

Here is a quick executive summary of how to use Newton's Method:

1. Make an initial guess (x_1)—make a reasonable one if possible.
2. Define x_2 to be
$$x_1 - \frac{f(x_1)}{f'(x_1)}$$
3. Now that we have x_2, we can get the next guess (x_3) by plugging in x_2 into the equation.
4. We can repeat this process as many times as we want, get a better approximation each time.

Example

Now let's go back to finding an approximation for $\sqrt{10}$. We start out by choosing an initial guess, let's say 2 since $2^3 = 8$. Next, we get the derivative $f'(x) = 3x^2$. Now we get the next value of x:

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{12} = 2\frac{1}{6}$$

After cubing, we are now off the answer only be about $\frac{1}{6}$. To make this error smaller, we can keep going. This would be exactly the same as it was before.

4 Integrals

The last fundamental concept that we will cover is the integral. However, before we can work with integrals, we need to cover a couple of related concepts.

4.1 Area

One of the main concepts we need to understand in detail is *area*. We know what the area of a figure with straight sides is and how to find it; however, we need to understand the area of shapes with *curved* sides.

The area of a region needs to have several basic properties:

- The area should be ≥ 0.
- Area $R_1 \leq$ area R_2 if $R_1 \subset R_2$.
- Area $(R_1 \cup R_2)$ = area R_1 + area R_2 if R_1 and R_2 do not intersect.
- The area of a rectangle is its base times its width. Using this property, we can find the areas of very complicated shapes with straight sides, but it is useless for shapes with *curved* sides.

To address the area of curved shapes, we will start by looking at the special case of the area under a curve. If we know how to find the area under a curve, we can find the area of a circle by doubling the

$$f(x) = \sqrt{1 - x^2}$$

This is the top half of the unit circle. Ultimately, this just illustrates the utility of finding the area under a curve.

We can start by doing this using a bunch of rectangles. To do this, we need to divide the x-axis into equal parts and then draw rectangles that have equal widths and a height corresponding to the graph. Now, to get a good estimate of the area under the curve, we just sum up the area of the drawn rectangles.

Lets say that the starting x-coordinate is a and the ending x-coordinate is b. The location of each of the divisions of the x-axis is:

$$x_i = a + i\frac{b - a}{n}$$

The width is simply equal to:

$$\frac{b-a}{n}$$

The height of each of the rectangles is naturally equal to $f(x)$. This gives us the following to be the area under the curve:

$$f(x_o)\frac{b-a}{n} + f(x_1)\frac{b-a}{n} + \cdots + f(x_{n-1})\frac{b-a}{n}$$

This is the left-hand approximation for the area under $f(x)$; we can get the right-hand version by starting with x_1 and ending with x_n.

Both of these methods are imperfect: they are just approximations of a curve using straight lines, so they generally miss the mark, sometimes significantly.

Both of these methods have a simple property: the larger the n, the more accurate the area. Naturally, the best result would be found using $n = \infty$. However, we cannot work with ∞ as a number, so we really have to find the limit of the sum as $n \to \infty$.

Example

Let's find the area under $y = x^2$ from 0 to 1 using this method. We start by saying that:

$$R_n = \frac{1-0}{n}(x_1^2 + x_2^2 + \cdots + x_n^2)$$

Then we let

$$x_i = a + \frac{b-a}{n} \cdot i = 0 + \frac{i}{n} = \frac{i}{n}$$

This lets us get a nice expression for the area under $y = x^2$:

$$\lim_{n\to\infty} \frac{1 + 2^2 + 3^2 + \cdots + n^2}{n^3}$$

However, we are not equipped to solve this limit. Happily, there is a formula for the sum of the first n squares:

$$\frac{n(n+1)(2n+1)}{6}$$

This formula lets us restate the limit:

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$$

This is a relatively easy limit to solve; we don't even have to use L'Hôpital's rules. We ultimately get the answer: $\frac{1}{3}$.

Sums

So, we managed to find the area under x^2 using a neat forumla for adding up squares; however, we have no such formula for other functions. This makes it more difficult to find sums like that for other functions.

We know that there exists some sum in the form:

$$L_n = \frac{b-a}{n}(f(x_0) + f(x_1) + \cdots + f(x_{n-1})) \tag{1}$$

This is the aforementioned "left-hand" approximation. We can also get the right-hand approximation:

$$R_n = \frac{b-a}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)) \tag{2}$$

Both of these approximations get more accurate as n increases. There is a theorem that states that for all differentiable functions, the following is true:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n \tag{3}$$

Basically, both approximations' limits exist and are equal, and they are the *exact* area under the curve!

Now lets look over some notation: writing out sums using "..." is not ver elegant; instead, we have special "Sigma" notation for this. With this notation, we can write out R_n as:

$$R_n = \frac{b-a}{n}\left(\sum_{i=0}^{n-1} f(x_i)\right) \tag{4}$$

Riemann Sums

Riemann Sums are the more general form of the sums we have been working with. To understand how they work, we first need to understand what a sample point is. In short, the point x_n^* is any point in the interval from x_{n-1} to x_n. These points can be the endpoints of their interval, or they can be any point between the endpoints.

The Riemann Sum can then be written out as:

$$R_n = \frac{b-a}{n}\left(\sum_{i=0}^{n} f(x_i^*)\right) \tag{5}$$

This is for any sample point x_i^* in the interval $[x_{i-1}, x_i]$. Basically, a Riemann Sum is an approximation of the area under a curve using a bunch of rectangles calculated using different points in the interval.

4.2 Informal Definition

Now that we know what a Riemann Sum is, we can get a good *informal* definition of a definite integral. The following integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n \tag{6}$$

where R_n is the Riemann sum with n terms. This is not a normal limit: R_n does not only depend on n; it also depends on the choice of sample points. This changes the definition of this limit from the normal definition: it means that for any $\epsilon > 0$ there is a number N so $|R_n - I| < \epsilon$ for *all* choices of sample points, for every $n > N$.

Now that we know what a definite integral is, we can get a more general version of the previous theorem involving R_n and L_n. We can now say that if a function f is continuous except for a finite (possibly 0) number of *jump discontinuities*, then $\lim_{n\to\infty} R_n$ exists.

To understand this, we of course have to define *jump discontinuities*. A function f has a jump discontinuity at c as long as $f(c)$ exists, $\lim_{x\to c^-} f(x)$ exists and $\lim_{x\to c^+} f(x)$ exists but two of the three aren't equal.

Functions with jump discontinuities are integrable; however, functions with more complex discontinuities are not.

Notation'

Here are some quick notes about notation. The following means "definite integral of $f(x)$ on the interval $[a, b]$ for x":

$$\int_a^b f(x)dx \tag{7}$$

The values a and b are the endpoints of the interval that we care about; x is just a dummy variable that we use to show exactly what we are summing up. The x could be any other undefined variable without changing the meaning of the integral.

An Example

Let's calculate a simple integral using the method we have just described. Our goal is to find:

$$\int_0^1 e^x dx \tag{8}$$

To do this, we need to rewrite the integral as a limit:

$$\int_0^1 e^x dx = \lim_{n\to\infty} \frac{1-0}{n}\left(\sum_{i=1}^{n} e^{a+i\frac{b-a}{n}}\right) \tag{9}$$

Now all we have to do is solve this limit! We can start by simplifying:

$$R_n = n^{-1}\left(\sum_{i=1}^{n} e^{\frac{i}{n}}\right) \tag{10}$$

This is a nice geometric sequence; we know how to get the sum of geometric sequences. The formula is:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1} \text{ if } r \neq 1 \tag{11}$$

We can use this formula in the limit:

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{e^{\frac{n+1}{n}}-1}{e^{\frac{1}{n}}-1}\right) \tag{12}$$

4.3 Fundamental Theorem of Calculus

Differentiation and integration are, ultimately, opposite actions; one undoes the other. That is the gist of the fundamental theorem of calculus.