2.Series

A series is an infinite ordered set of terms combined together by the addition operator. The term "infinite series" is sometimes used to emphasize the fact that series contain an infinite number of terms.

Definition 2.1:- For any sequence $< x_n >$ the associated **series** is defined as the ordered formal sum $x_1 + x_2 + x_3 + \cdots$. We use the symbol $\sum$ denote series.

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots$$

Examples:- 1) $\sum_{n=1}^{\infty} \frac{1}{n}$ 2) $\sum_{n=1}^{\infty} 2^n$

3) $\sum_{n=1}^{\infty} \frac{1}{n!}$ 4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

Definition 2.2:- Let $< x_n >$ be a sequence of real numbers and $n \in \mathbb{N}$, then we define $\boldsymbol{s_n}$ to be sum of first nth term in the series.

i.e. $s_n = x_1 + x_2 + x_3 + \cdots + x_n$

Example:- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. let $n \in \mathbb{N}$.

Notice that $s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \rightarrow (1)$

So $2s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \rightarrow (2)$

(2) – (1) ; $s_n = 1 - \frac{1}{2^n}$.

Hence $s_n = 1 - \frac{1}{2^n}$ for each $n \in \mathbb{N}$.

Geometric Series

A series for which the ratio of each two consecutive terms $\frac{x_{n+1}}{x_n}$ is a constant function of the summation index *k* is called a Geometric Series.
So we have $x_n = ar^{n-1}$ for some $a, r \in \mathbb{R} - \{0\}$ for each $n \in \mathbb{N}$. Normally we write as $\sum_{n=1}^{\infty} ar^{n-1}$ where *a* is the starting term of the sequence and *r* is the common ratio.

Example:- 1) $\sum_{n=1}^{\infty} \frac{1}{2^n}$; $\frac{x_{n+1}}{x_n} = \frac{1}{2}$

2) $\sum_{n=1}^{\infty} \frac{1}{5^n}$; $\frac{x_{n+1}}{x_n} = \frac{1}{5}$

3) $\sum_{n=1}^{\infty} 2^n$; $\frac{x_{n+1}}{x_n} = 2$

Alternative Series

An Alternative series a series where terms alternate signs.

Example:- 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$ 2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

We can write an alternating series as $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ where x_n is positive for each $n \in \mathbb{N}$.

p-series

The series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series. p- can be any real number

2.1 Convergence of a series

Definition:- A series $\sum_{n=1}^{\infty} x_n$ said to be convergence to point L iff the associated sequence of partial sums $< s_n >$ converges to L. If such a L does not exists we called the series diverges

$$\sum_{n=1}^{\infty} x_n = L \text{ iff } \lim_{n \to \infty} s_n = L.$$

Example:- i) Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

We have shown that, $s_n = 1 - \frac{1}{2^n}$ for each $n \in \mathbb{N}$.

Therefore $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 + \lim_{n\to\infty} \frac{1}{2^n} = 1.$

Thus $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

ii) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is converges.

Solution:-

Notice that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ for each $n \in \mathbb{N}$.

Then $s_n = 1 - \frac{1}{n+1}$ for each $n \in \mathbb{N}$

Therefore,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 + \lim_{n\to\infty} \frac{1}{n+1} = 1$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n+1}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n+1} = 1$

2.3 Test for Convergence

Sometime it is too difficult to calculate the convergence value of a series. But most of the time we can fine whether the series converges or diverges. There are few test to determine the convergence of a given series.

Theorem 2.3.1: - If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_n a_n = 0$. But the converse is false.

Proof:- Suppose $\sum_{n=1}^{\infty} a_n$ converges.

Then there exists $l \in \mathbb{R}$ s.t $\lim_n a_1 + a_2 + a_3 \ldots + a_n = l$.

Also we know that $\lim_n a_1 + a_2 + a_3 \ldots + a_n + a_{n+1} = l$.

Hence , $0 = l - l = \lim_n(a_1 + a_2 + a_3 \ldots + a_n + a_{n+1}) - \lim_n(a_1 + a_2 + a_3 \ldots + a_n)$

$= \lim_n(a_1 + a_2 + a_3 \ldots + a_n + a_{n+1} - (a_1 + a_2 + a_3 \ldots + a_n)$

$= \lim_n a_{n+1}$

Thus $\lim_n a_{n+1} = 0$.

(Remember the converse of this theorem is false. We will prove it later.)

This given the first test for a divergent series.

Theorem 2.3.2:- If it is not the case that $\lim_n a_n = 0$ than $\sum_{n=1}^{\infty} a_n$ diverges.

Proof:- Follows by the previous theorem.

Example:-

i) The series $\sum_{n=1}^{\infty} \frac{n^2}{n(n+1)}$ diverges because $\lim_n \frac{n^2}{n(n+1)} = 1$

ii) The series $\sum_{n=1}^{\infty}(-1)^n$ diverges because $\lim_n(-1)^n$ does not exists.

Example:- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is diverges.

Solution:- Frist we will show that $1 + 2 + 3 + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for each $n \in \mathbb{N}$.

We will use the induction to prove the result.

Let $P(n)$ be the statement $''1 + 2 + 3 + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}''$ for each $n \in \mathbb{N}$

Notice that $\frac{1}{1} + \frac{1}{2} \geq \frac{1}{1} + \frac{1}{2}$. Thus $P(1)$ is true.

Now let $n \in \mathbb{N}$ arbitrary s.t $p(n)$ is true.

Then $1 + 2 + 3 + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.

Observe that,

$$\underbrace{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\frac{1}{2^n+3}+\cdots+\frac{1}{2^n+2^n}}_{2^n\ terms} \ \geq \ \frac{1}{2^n+2^n}+\frac{1}{2^n+2^n}+\cdots+\frac{1}{2^n+2^n}$$

$$= 2^n.\frac{1}{2^n+2^n} = \frac{2^n}{2.2^n} \ = \frac{1}{2}$$

Then,

$$1+2+3+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}} \ \geq 1+\frac{n}{2}+ \ \frac{1}{2^{n+1}} \ \ \geq \ 1+\frac{n}{2}+\frac{1}{2}=1+\frac{n+1}{2}$$

Hence $P(n+1)$ is true.

Therefore for each $n\in\mathbb{N}$ if $P(n)$ is true, then $P(n+1) is\ true$.

Then by the PMI $P(n)$ is true for each $n\in\mathbb{N}$.

Thus $1+2+3+\cdots+\frac{1}{2^n} \ \geq 1+\frac{n}{2}$ for each $n\in\mathbb{N}$.

So we have $s_{2^n} = \ 1+2+3+\cdots+\frac{1}{2^n} \ \geq 1+\frac{n}{2}$ > 0 for each $n\in\mathbb{N}$.

Since $\lim_n 1+\frac{n}{2} = \ \infty$, we have $\lim_n s_{2^n} = \ \infty$.

Also since $\frac{1}{n} > 0$ for each $n\in\mathbb{N}$, $\lim_n s_n = \ \infty$.

That is $\sum_{n=1}^{\infty}\frac{1}{n}$ is diverges and $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$.

So even if $\lim\limits_{n}\frac{1}{n} = 0$, we have $\sum_{n=1}^{\infty}\frac{1}{n}$ is diverges.

<u>Convergence of a Geometric series</u>

Consider the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ where a is the starting term of the sequence and r is the common ratio.

Let $n\in\mathbb{N}$.

Then $\quad s_n = a+ar+ar^2+\cdots+ar^{n-1} \rightarrow (1)$

If we multiply (1) by r, $\quad rs_n = ar+ar^2+ar^3+\cdots+ar^n \rightarrow (2)$

From (1)−(2), $\quad (1-r)s_n = a-ar^n$

So if $r \neq 1$, then $\quad s_n = \frac{a-ar^n}{(1-r)} = \frac{a}{1-r} - \frac{a}{1-r}.r^n$

If $r = 1$, then $s_n = \underbrace{a + a + \cdots + a}_{n\ times} = na$

So we have $s_n = \begin{cases} na, & r = 1 \\ \frac{a-ar^n}{(1-r)}, & r \neq 1 \end{cases}$

Observe that if $= 1$, then $s_n = na$. Hence $\lim_n s_n = \infty$. So $< s_n >$ diverges.

Thus $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Notice that if $r = -1$ then, $s_n = a$ if n is odd and $s_n = 0$ is n is even.

So $< s_n >$ diverge and hence $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

If $r > 1$, then we have shown that $< r^n >$ is unbounded above. So $< r^n >$ cannot converges.

Also if $r < -1$, then we have shown that $< r^n >$ is neither bounded above nor bounded below. So $< r^n >$ cannot converges.

If $|r| < 1$, then we have shown that $\lim_n r^n = 0$.

Then $< \frac{a}{1-r}.r^n >$ converges to 0. (because $\frac{a}{1-r}$ is just a constant)

Therefore, $\lim_n s_n = \lim\limits_{n} \frac{a}{1-r} - \lim\limits_{n} \frac{a}{1-r}.r^n = \frac{a}{1-r}$.

Thus $\sum_{n=1}^{\infty} ar^{n-1}$ converges and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

Let us put all above work in to one theorem.

Theorem 2.3.3:- The Geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges iff $|r| < 1$.

Example:- 1) Consider the Geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Notice that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (\frac{1}{2}) \left(\frac{1}{2}\right)^{n-1}$

So in here $a = \frac{1}{2}$ and $r = (\frac{1}{2})$.

Hence the given geometric series converges because $|r| = \left|\frac{1}{2}\right| < 1$.

Also $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n} = \frac{1/2}{1-1/2} = 1.$

2) The Geometric series $\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^{n-1}$ converges because $a = 1, r = \frac{-2}{5}$ and $|r| = \left|\frac{-2}{5}\right| = \frac{2}{5} < 1$

Also $\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^{n-1} = \frac{1}{1-(-2)/5} = \frac{5}{7}.$

3) The Geometric series $\sum_{n=1}^{\infty} 4^{n} = \sum_{n=1}^{\infty} 4.4^{n-1}$ does not converges because $|r| = |4| > 1.$

Converges of a Alternative Series

Theorem 2.3.4:- Let $< a_n >$ be a sequence of positive terms satisfying following properties.

i) $< a_n >$ is monotonically decreasing. i.e. $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$

ii) $\lim_n a_n = 0.$

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

(Remember the converse of this theorem is false. Also if $< a_n >$ does not satisfy the property (i) above does not mean $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ diverges)

Proof:- Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$. Let us prove that $< s_{2n} >$ is monotonically increasing and bounded above.

Let $n \in \mathbb{N}$. Observe that, $s_{2(n+1)} - s_{2n} = \sum_{k=1}^{2(n+1)}(-1)^{k+1}a_k - \sum_{k=1}^{2n}(-1)^{k+1}a_k$

$= (-1)^{2n+1+1}a_{2n+1} + (-1)^{2n+2+1}a_{2n++}$

$= a_{2n+1} - a_{2n+2} \geq 0.$

Therefore $< s_{2n} >$ is monotonically increasing.

Also observe that for each Let $n \in \mathbb{N}$,

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}$$

$$= a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} \dots + \underbrace{(a_{2n-2} - a_{2n-1})}_{\geq 0} - a_{2n} \leq a_1 .$$

Hence $< s_{2n} >$ is bounded above.

Thus $< s_{2n} >$ is converges.

Then there exists $l \in \mathbb{R}$ s.t $\lim_n s_{2n} = l$.

Notice that $s_{2n+1} = s_{2n} + (-1)^{2n+1+1} a_{2n+1}$

Since $< s_{2n} >$ converges to l and $< (-1)^{2n+1+1} a_{2n+1} >$ converges to 0, we have that

$< s_{2n+1} >$ converges and,

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} (-1)^{2n+1+1} a_{2n+1}$$

$$= l + 0 = l.$$

Then $\lim_n s_{2n} = l$ and $\lim_n s_{2n+1} = l$.

Thus $\lim_n s_n = l$.

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example:- Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Solution:- Clearly $< \frac{1}{n} >$ is a sequence of positive terms s.t $\frac{1}{n} \geq \frac{1}{n+1}$ for each $n \in \mathbb{N}$ and also

$\lim_n \frac{1}{n} = 0$. Then by the Alternative test series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

<u>The Comparison Test</u>

This test is only valid for series of positive terms.

Theorem 2.3.5:- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series s.t $\sum_{n=1}^{\infty} b_n$ converges and for each $n \in \mathbb{N}$, $0 < a_n \leq b_n$. Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof:- Let $t_n = \sum_{k=1}^{n} a_k$ and $s_n = \sum_{k=1}^{n} b_k$ for each $n \in \mathbb{N}$. Then $t_n \leq s_n$ for each $n \in \mathbb{N}$, And since both are series of positive terms $t_n < t_{n+1}$, $s_n < s_{n+1}$ for each $n \in \mathbb{N}$ and $< s_n >$ converges. Since $< s_n >$ converges, $< s_n >$ is bounded above. Hence $< t_n >$ is bounded above.

Now $< t_n >$ is bounded above and strictly increasing. Thus $< t_n >$ converges.

Hence $\sum_{n=1}^{\infty} a_n$ converges.

Corollary:- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series s.t $\sum_{n=1}^{\infty} a_n$ diverges and for each $n \in \mathbb{N}$, $0 < a_n \leq b_n$. Then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof:- Follows from the theorem.

Example:- i) Consider the sequence $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

Notice that $2^n < 3^n$ for each $n \in \mathbb{N}$.

Hence $\frac{1}{3^n} < \frac{1}{2^n}$ for each $n \in \mathbb{N}$.

But we know that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges. Then by the comparison test $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges

ii) Show that the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges.

Notice that $\ln(n) < n$ for each $n \in \mathbb{N}$.

Hence $\frac{1}{n} < \frac{1}{\ln(n)}$ for each $n \in \mathbb{N}$.

But we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Then by the comparison test $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges.

Convergence of p-Series

Recall that the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series. p- can be any real number.

Theorem:- Let $p \in \mathbb{R}$ be s.t $p > 1$. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Proof:- Let $n > 1$. Then $2^n - 1 > n$.

Thus

$$s_n < s_{2^n-1} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^n-1)^{\,p}}$$

$$= 1+\left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots + \left(\frac{1}{2^{(n-1)p}} + \frac{1}{(2^{n-1}+1)^p} + \cdots + \frac{1}{(2^n-1)^p}\right)$$

$$\leq 1+\left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots + \left(\frac{1}{2^{(n-1)p}} + \frac{1}{2^{(n-1)p}} + \cdots + \frac{1}{2^{(n-1)p}}\right)$$

$$= 1+\left(\frac{2}{2^p}\right) + \left(\frac{4}{4^p}\right) + \cdots + \left(\frac{2^{n-1}}{2^{(n-1)p}}\right)$$

$$= 1+\left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{4^{p-1}}\right) + \cdots + \left(\frac{1}{2^{(n-1)(p-1)}}\right)$$

$$= 1+\left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{(p-1)}}\right)^{n-1}$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{1}{2^{p-1}}\right)^{n-1} = \frac{1}{1-{}^{1}/_{2^{p-1}}}.$$

Then $\quad s_n \leq \frac{1}{1-{}^{1}/_{2^{p-1}}}$ for each $n \in \mathbb{N}$.

Thus $< s_n >$ is bounded above.

Clearly $< s_n >$ is strictly increasing and since $< s_n >$ is bounded above $< s_n >$ is converges.

Thus $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Example:- $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges because $\frac{3}{2} > 1$.

Show that $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges.

Solution:- Observe that for each $n \in \mathbb{N}$,

$$0 < \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n(\sqrt{n}+\sqrt{n})} = \frac{1}{2n^{\frac{3}{2}}}$$

Clearly $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges because $\frac{3}{2} > 1$. Hence $\sum_{n=1}^{\infty}\frac{1}{2n^{\frac{3}{2}}}$ converges

Then by the comparison test $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges.

Ratio Test

Theorem: - Let $\sum_{n=1}^{\infty} a_n$ be a sequence with non-zero terms let l be a real number s.t

$$\lim_{n} \frac{a_{n+1}}{a_n} = l.$$

i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

iii) If $l = 1$, then nothing can be said about the convergence of $\sum_{n=1}^{\infty} a_n$

(Test fails when $l = 1$)

Proof:- (i) Suppose $l < 1$.

Put $\varepsilon = \frac{1-l}{2}$. Then $\varepsilon > 0$.

Then there exists $N \in \mathbb{N}$ s.t for each $> N$, $\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon$.

Hence $\frac{a_{n+1}}{a_n} < l + \varepsilon = \frac{1+l}{2}$ for each $n > N$

Therefore $0 < \frac{a_{N+1}}{a_N} < \frac{1+l}{2}$

$$0 < \frac{a_{N+2}}{a_{N+1}} < \frac{1+l}{2}$$

$$0 < \frac{a_{N+3}}{a_{N+2}} < \frac{1+l}{2}$$

$$\vdots$$

$$0 < \frac{a_{N+n}}{a_{N+n-1}} < \frac{1+l}{2}$$

Hence $0 < \frac{a_{N+n}}{a_N} < (\frac{1+l}{2})^n$ for each $n \geq 1$.

So, $0 < a_n < a_N (\frac{1+l}{2})^{n-N}$ for each $n \geq N+1$

Then $0 < a_n < \frac{a_N}{(\frac{1+l}{2})^N} (\frac{1+l}{2})^n$ for each $n \geq N+1$

Since $0 < \frac{1+l}{2} < 1$, the geometric series $\sum_{n=1}^{\infty} \frac{a_N}{(\frac{1+l}{2})^N} (\frac{1+l}{2})^n$ converges.

Then by the comparison test,

$$\sum_{n=N}^{\infty} a_n \text{ converges.}$$

Thus $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Now suppose $l > 1$.

Put $\varepsilon = \frac{l-1}{2}$. Then $\varepsilon > 0$.

Then there exists $N \in \mathbb{N}$ s.t for each $> N$, $\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon$.

Therefore $-\frac{l-1}{2} < \frac{a_{n+1}}{a_n} - l$ for each $n > N$

Hence $1 < \frac{l+1}{2} < \frac{a_{n+1}}{a_n}$ for each $n > N$

Then we have $a_{n+1} > a_n$ for each $n > N$

Thence $0 < a_N < a_{N+1} < a_{N+2} < \cdots$

Thus this is not the case that $\lim_n a_n = 0$.

Hence $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

Notice that $\lim\limits_{n} \frac{a_{n+1}}{a_n} = \lim\limits_{n} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim\limits_{n} \frac{n}{n+1} = \lim\limits_{n} 1 + \frac{1}{n} = 1.$

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Observe that,

$$\lim_{n} \frac{a_{n+1}}{a_n} = \lim_{n} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n} (\frac{n}{n+1})^2 = \lim_{n} \left(\frac{n}{n+1}\right)\left(\frac{n}{n+1}\right) = \lim_{n} \left(\frac{n}{n+1}\right) \lim_{n} \left(\frac{n}{n+1}\right) = 1.$$

But we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Therefore when $l = 1$ we cannot say anything about the converges of $\sum_{n=1}^{\infty} a_n$.

Example:- i) Show that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Notice that,

$$l = \lim_{n} \frac{a_{n+1}}{a_n} = \lim_{n} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n} \frac{1}{n+1} = 0.$$

Since $l < 1$ by ratio test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ii) Show that $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

Observe that,

$$l = \lim_{n} \frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{2(2n+1)}{(n+1)} = 4.$$

Since $l > 1$ by the ratio test, $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges

Sometimes we won't be able to use the ratio test because the ratio may not be define.

For example consider the sequence (1,0,1,0,1,0,..). Then we cannot define $\frac{a_{2n+1}}{a_{2n}}$ for each $n \in \mathbb{N}$. So we have to use another method to test the convergence for series like this.

<u>Root Test</u>

Theorem: - Let $\sum_{n=1}^{\infty} a_n$ be a sequence with non-zero terms let l be a real number s.t

$$\lim_{n} \sqrt[n]{a_n} = l.$$

i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

iii) If $l = 1$, then nothing can be said about the convergence of $\sum_{n=1}^{\infty} a_n$

(Test fails when $l = 1$)

Example:- Show that $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

Notice that $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n}$. Hence $\lim_n \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0$.

Since $\lim_n \sqrt[n]{a_n} < 1$, the given series is converges.

Theorem:- Let $< a_n >$ and $< b_n >$ be two sequences s.t $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are converges.

Then $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Proof:- Let S_n, T_n and U_n be the partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty}(a_n \pm b_n)$ respectively. Then for each $n \in \mathbb{N}$,

$$U_n = ((a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_n \pm b_n))$$

$$= (a_1 + a_2 + \cdots + a_n) \pm (b_1 + b_2 + \cdots + b_n)$$

$$= S_n \pm T_n$$

Since $\lim_n S_n$ and $\lim_n T_n$ exists we have $\lim_n U_n$ exists and

$$\lim_n U_n = \lim_n S_n \pm \lim_n T_n$$

Thus $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Problems

1) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

2) Prove that $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} = \frac{23}{480}$

3) Find $\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)3^{n+1}}$

4) Use Ratio test to determine whether the following series are converge or diverge.

i) $\sum_{n=1}^{\infty} n^4 e^{-4}$ ii) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

iii) $\sum_{n=1}^{\infty} n^4 e^{-4}$ ii) $\sum_{n=1}^{\infty} \frac{(-10)^n}{(n+1)4^{2n+1}}$

5) Prove that $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges

6) Test for convergence

i) $\frac{2}{1} - \frac{1}{1} - \frac{1}{1} + \frac{3}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{4}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \cdots + \frac{n+1}{n} - \underbrace{\frac{1}{n} - \frac{1}{n} - \frac{1}{n} \ldots - \frac{1}{n}}_{n+1\ terms} + \cdots$

ii) $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n} + \cdots$

7) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ converges.

8) Suppose $< a_n >$ is a sequence of positive terms s.t $\sum_{n=1}^{\infty} a_n$ converges.

Show that $\sum_{n=1}^{\infty} a_n^2$ also converges.