AI1003 – Mathematical Foundation of AI

Homework 2

Subspaces, Span & Projections

Part A — Span & Subspaces

1. **Topic Space in NLP (5 pts)**
   Suppose we have two "topic vectors" in a document embedding space:

   $$t_1 = (1,2,1), t_2 = (0,1,1).$$

   Do $\{t_1, t_2\}$ span all of $\mathbb{R}^3$?

   o   What is the dimension of their span?
   o   Interpret: If document embeddings live in this span, what does it mean about the diversity of topics?

2. **Subspace of Neutral Sentiment (5 pts)**
   Define a subspace of $\mathbb{R}^3$:

   $$W = \{(x, y, z): x + y + z = 0\}.$$

   o   Show W is a subspace.
   o   Give a basis for W.
   o   AI interpretation: If (x, y, z) = scores for (positive, neutral, negative), explain why this subspace represents a **"balanced sentiment" space**.

Part B — Orthogonal Projections

3. **Semantic Similarity (5 pts)**
   Consider word embeddings:

$$q = (2,1), d = (1,1).$$

- o Compute the projection of q onto the line spanned by d.
- o Interpret: Why does cosine similarity relate to this projection?
- o If q is a query and d a document, what does projection tell us about relevance?

4. **Regression as Projection (10 pts)**
Suppose we fit a linear regression model with feature matrix

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

- o Compute the projection matrix $P = X(X^T X)^{-1} X^T$.
- o Find the projected vector $\hat{y} = Py$.
- o Verify that residuals $y - \hat{y}$ are orthogonal to the columns of X.
- o AI interpretation: Explain why this "projection property" guarantees the regression line is the **best fit**.

Part C — AI Application Question

5. **Recommender System Subspaces (5 pts)**
In a recommender system, user preferences are stored as vectors in a high-dimensional space (movies × genres). Low-rank methods project users into a lower-dimensional subspace of "latent tastes."
- o Explain why using the **span of a few eigenvectors** (latent factors) gives a compact representation of users.
- o What does projecting a new user vector onto this subspace achieve?

Grading (30 pts total)

- Part A: 10 pts
- Part B: 15 pts
- Part C: 5 pts