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Take-Home Exam

Administrative Items

This is your take-home exam that will count towards 20% of your final grade. Marks add up to 100 (I hope). You can work in groups of up to **four** people (that's 4 or less). One submission per group, you need to specify group membership on your submissions. All members of the same group will receive the same grade. The deadline for this assignment and the precise submission guidelines are available from the department.

Please do check the Moodle page for amendments or additional information that may be posted there to clarify the questions or amend them in case unexpected difficulties appear.

Statement of your problem

You are running a fund investing in variability and you decided to call it **LSE**, short for **L**everaged **S**peculative **E**xcitement. You have access to a good data set of options premia of a rather wide set of moneyness and maturity and you decide to extract as much information from it as possible and contrast them with your own views. This will allow you to take a view and go long or short variability in many different ways. But before you can do that, you need to see what beliefs are already impounded into markets.

Outline. European options prices embed the beliefs of the market, and in particular since Breeden-Litzenberger we know that we can in fact extract the risk-neutral density of the return on the underlying between times t and $T > t$ from the prices of calls and puts expiring at time T.

In this take-home exam you will extract the density. As in real-life, complications will arise. For instance, in order to get the entire density function for time T one would need to observe options for the entire continuum of strikes from 0 to $+\infty$. In practice, one only observes a few strikes around the money, so extensive intrapolation and extrapolation will need to be done in order to construct the density function. This intra- and extrapolations can be done in the price or in the implied vol domain. Furthermore, only short-dated options prices are observed most of the time.

Once you have the risk-neutral densities of excess log-returns you can start asking the questions about trading opportunities, and in particular about the risk-term-structures. What are the expected risk-neutral anualised excess mean log-returns over a variety of maturities? What is the variance term-structure of annualised excess log-returns? What is the term-structure of risk-neutral Value-at-Risks (VaR), or quantiles? What is the market-price of variance risk? How did these entities change over time, in particular around times of market stress? Did they correctly forecast episodes of stress? How many factors drive the term-structures of risk, as determined by a principal components analysis? Can we try to build a model that captures the essential features so extracted? And if we do so, how can we use it to make money?

Data. You are given vanilla S&P500 calls premia over a number of years for a range of maturities and strikes, as well as the values of future and index prices. The calls are on the futures on the S&P500, whose price process is denoted by $F_t(T)$, and the maturities of the options coincide with the expiries of the futures contracts. Throughout we shall assume that futures prices and forward prices agree. By convention, $F_T(T) = S_T$, the value for the S&P500 *price* (not total return) index at T. The time series goes from 31.01.2006 to 31.12.2009, with a variety of strikes and maturities from 1 month to 10 years. For each call price, you are also given the forward price, the relevant annualised discount rate r and the annualised dividend yield δ, both assumed to be deterministic over the relevant maturity horizon. You do not have the S&P index, but you can reconstruct it from the data you have.

SVI. In order to intra- and extra-polate, we use what is called the SVI procedure (Stochastic Volatility Inspired) proposed by Gatheral. There are quite a few versions by now, we choose the basic one here. Total implied variance is defined as below:

$$w(k,\tau) = \sigma_{BS}^2(k,\tau)\tau \tag{1}$$

where σ_{BS} is the Black-Scholes implied volatility; τ is the maturity in years; and k is log-moneyness defined as $\ln(\frac{K}{F})$.

Raw SVI parameterisation. For a given time and maturity (t,T) and a parameter set $\chi_R = \{a, b, \rho, m, \bar{\sigma}\}$, all implicitly depending on (t,T), total implied variance can be parameterised as:

$$w(k,\tau;\chi_R) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \bar{\sigma}^2}\right\} \tag{2}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\bar{\sigma} > 0$ and $a + b\bar{\sigma}\sqrt{1-\rho^2} \geq 0$ to ensure that $w(k;\chi_r) \geq 0$ for all $k \in \mathbb{R}$.

Model M. Model M is

$$d\ln S_t = \left(r_t - \delta_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\,d\hat{W}_t^1 \tag{3}$$

$$dv_t = \kappa(\theta_t - v_t)dt + \bar{\sigma}_1\sqrt{v_t}\left(\rho\,d\hat{W}_t^1 + \sqrt{1-\rho^2}\,d\hat{W}_t^2\right) \tag{4}$$

$$d\theta_t = \eta(\bar{\theta} - \theta_t)dt + \bar{\sigma}_2\sqrt{\theta_t}\,d\hat{W}_t^3 \tag{5}$$

with $\kappa, \bar{\theta}, \nu, \bar{\sigma}_1, \bar{\sigma}_2$ all strictly positive constants and $\hat{W}$ are independent $\mathbb{Q}$ Brownians. We also write $\sigma_t^2 := v_t$. Both r and δ are deterministic, but may be time-varying.

Further Notation. Net holding period returns are $\frac{S_T}{F_t}$. Call the log holding period excess-return $R_{t,T}$,

$$R_{t,T} := \ln\left(\frac{S_T}{F_t}\right)$$

Questions

1. [4 marks] Show that $\ln\left(\frac{S_T}{F_t}\right)$ is indeed the net log holding period return.

2. [24 marks] Prove that in order to exclude static arbitrage opportunities, we need a bunch of restrictions on option prices, and thus on implied vol surfaces. Throughout, assume NA, and in particular that the value functions C, P are smooth in the underlying (this can be shown by NA).

   (a) [2 marks] Prove that the absence of arbitrage for a vertical bull spread is equivalent to the condition that $-e^{-r(T-t)} \leq \frac{\partial C}{\partial K} \leq 0$.

   (b) [2 marks] Prove that the absence of arbitrage for a vertical bear spread is equivalent to the condition that $0 \leq \frac{\partial P}{\partial K} \leq e^{-r(T-t)}$.

   (c) [4 marks] Prove that the absence of arbitrage for a butterfly is equivalent to the condition that $\frac{\partial^2 C}{\partial K^2}(S_t, t, K) \geq 0$ and $\frac{\partial^2 P}{\partial K^2}(S_t, t, K) \geq 0$.

   (d) [3 marks] Prove that the absence of arbitrage for calendar spreads on calls is equivalent to $\frac{\partial C}{\partial T} \geq 0$.

   (e) [2 marks] Prove that the absence of arbitrage for calendar spreads on puts is equivalent to $\frac{\partial P}{\partial T} \geq -re^{-r(T-t)}K$.

   (f) [2 marks] Show that no bull-spread arbitrage implies that

   $$L(K;\cdot) := -\frac{\partial p_{BS}/\partial K}{\partial p_{BS}/\partial \sigma_{IV}} \leq \frac{\partial \sigma_{IV}}{\partial K} \leq R(K;\cdot) := -\frac{\partial c_{BS}/\partial K}{\partial c_{BS}/\partial \sigma_{IV}}$$

   with $R(K;\cdot) \geq 0$ and $L(K;\cdot) \leq 0$.

   (g) [3 marks] Assume that there are real numbers $m_1 > 0$, $m_2 > 0$ and $m_2' > 0$ with m_1 arbitrarily small and m_2 and m_2' arbitrarily large, for which $m_1 \leq \sigma_{IV}(K) \leq m_2$ and $\left|\frac{\partial \sigma_{IV}(K)}{\partial K}\right| \leq m_2'$. Then

   $$\lim_{K\to 0} L(K;\cdot) = -\infty \quad \text{and} \quad \lim_{K\to\infty} L(K;\cdot) = -\infty$$

   and

   $$\lim_{K\to 0} R(K;\cdot) = +\infty \quad \text{and} \quad \lim_{K\to\infty} R(K;\cdot) = 0$$

   Interpret.

   (h) [3 marks] Show that

   $$\ell(T;\cdot) := -\frac{\partial c_{BS}/\partial T}{\partial c_{BS}/\partial \sigma_{IV}} \leq \frac{\partial \sigma_{IV}}{\partial T}$$

   with $\ell(T;\cdot) \leq 0$ and no respective upper bound on the IV slope.

   (i) [3 marks] Show that the absence of an butterfly arbitrage implies

   $$\frac{\partial^2 C^{BS}}{\partial K^2} + \frac{\partial C^{BS}}{\partial \sigma}\frac{\partial^2 \sigma_{IV}}{\partial K^2} + 2\frac{\partial \sigma_{IV}}{\partial K}\frac{\partial^2 C^{BS}}{\partial K \partial \sigma_{IV}} \geq 0$$

3. [4 marks] For each calendar time t, extract the implied volatilities for different strikes and maturities. Plot a few to test your intuition and to make sure the implied vols appear logical.

4. [4 marks] Given t and T, fit the best SVI curve to the implied variances.

   In order to do that, you can code your own routines, or make use of the R code file that will be posted to the Moodle page.

   Depict, for a few selected calendar times, the fitted implied vol surface in terms of moneyness K/F and $\ln(K/F)$.

5. [4 marks] Check that the surface does not violate any of the NA restrictions you proved above, and if it does, adjust it accordingly, describing how you chose to do so.

6. [4 marks] Using Breeden-Litzenberger, construct one family of risk-neutral densities of log excess returns returns $R_{t,t+\tau}$ (for the maturities you have data for) per calendar time. Check whether they make sense.

7. [4 marks] Depict, for a few selected calendar times, the fitted risk neutral densities.

8. [4 marks] We now construct moments directly from the risk-neutral densities. Define $\mu(t,\tau) := E_t^{\mathbb{Q}}[R_{t,t+\tau}]$.

   Only for the next two subquestions, assume Model M holds.

   (a) [2 marks] Show that within Model M, $\mu(t,\tau) = -\frac{1}{2}\int_t^T E_t^{\mathbb{Q}}[\sigma_u^2]du$.

   (b) [2 marks] Within Model M, what is the relationship between $E_t^{\mathbb{Q}}[R_{t,T}]$ and the variance-swap-strike $K(t,t,T)$, and the VIX?

9. [4 marks] We also derive the second moments directly from the risk-neutral densities. Define $V(t,T) := \mathrm{Var}_t^{\mathbb{Q}}(R_{t,T})$. The object of interest is the behaviour – the term-structure – of the mapping of annualised variances $\tau \mapsto \nu(t,\tau) := \frac{1}{\tau}V(t,t+\tau)$.

   Now fix a time t and depict on a graph, with $T-t$ on the x-axis, the term structures of μ and ν. Repeat for various days and map them, possibly using different colours for different days, onto the same plot.

10. [4 marks] Stare at the plot with the μ and the plot with the ν curves. Describe what you can see. Do the plots offer any insights as to what sort of model may be able to describe them?

11. [4 marks] Now pick $\tau = \frac{1}{12}$ and plot over the sample period three curves: (1) the actual realised VIX over the period, (2) $\mu(t,1/12)$ (or a transformation to make the plots more comparable) and (3) $\nu(t,1/12)$.

12. [4 marks] Describe and explain what you see.

13. [4 marks] Consider the term-structure curves in $\{\nu(t,\cdot)\}_{t\geq 0}$, one for each t. They move over time. PCA is a method to analyse the latent drivers of the changes (variance-covariance) in the curves over time. Using PCA routines [hints will be put on Moodle],

perform a PCA, describe the most important components. How many significant ones are there?

14. [4 marks] How does this PCA analysis square with the premises underlying Model M?

15. [4 marks] You don't need to show the following theorem.
Theorem.

$$E_t^{\mathbb{Q}}[R_{t,T}] = (\sigma_t^2 - \bar{\theta})\tilde{T}_{\sigma^2}(t,T) + \bar{\theta}\tilde{T}_{\bar{\theta}}(t,T) + (\theta_t - \bar{\theta})\tilde{T}_{\theta}(t,T)$$

$$Var_t^{\mathbb{Q}}(R_{t,T}) = (\sigma_t^2 - \bar{\theta})T_{\sigma^2}(t,T) + \bar{\theta}T_{\bar{\theta}}(t,T) + (\theta_t - \bar{\theta})T_{\theta}(t,T)$$

where

$$\begin{aligned} T_{\sigma^2}(t,T) &:= T_2(t,T) + \frac{1}{2}\tilde{T}_2(t,T) \\ T_{\bar{\theta}}(t,T) &:= T_3(t,T) + \frac{1}{2}\tilde{T}_3(t,T) \\ T_{\theta}(t,T) &:= T_4(t,T) + \frac{1}{2}\tilde{T}_4(t,T) \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_{\sigma^2}(t,T) &:= -\frac{1}{2}T_2(t,T;\rho=0) \\ \tilde{T}_{\bar{\theta}}(t,T) &:= -\frac{1}{2}T_3(t,T;\rho=0) \\ \tilde{T}_{\theta}(t,T) &:= -\frac{1}{2}T_4(t,T;\rho=0) \end{aligned}$$

where $T_i(t,T;\rho=0) = T_i(t,T)$ after having set $\rho = 0$. As to the coefficients,

$$T_2(t,T) = \left(\frac{\kappa - \rho\bar{\sigma}_1}{\kappa^2}\right) + e^{-\kappa(T-t)}\left[\frac{\rho\bar{\sigma}_1 - \kappa + \kappa\rho\bar{\sigma}_1(T-t)}{\kappa^2}\right]$$

$$T_3(t,T) = \frac{1 - e^{-\kappa(T-t)} + (T-t)\kappa(\rho\,\bar{\sigma}_1 - \kappa)}{\kappa^2}$$

$$\begin{aligned} T_4(t,T) = \left(\frac{\kappa - \rho\bar{\sigma}_1}{\eta\kappa}\right) + e^{-\eta(T-t)}\left[\frac{\kappa(\eta - \kappa + \rho\bar{\sigma}_1)}{\eta(\eta-\kappa)^2}\right] + \\ e^{-\kappa(T-t)}\left[\frac{\kappa(\kappa-\eta) + \rho\bar{\sigma}_1(\eta - 2\kappa) + \rho\bar{\sigma}_1\kappa(\eta-\kappa)(T-t)}{\kappa(\eta-\kappa)^2}\right] \end{aligned}$$

and

$$\tilde{T}_2(t,T) = \left(\frac{\bar{\sigma}_1^2}{2\kappa^3}\right) - (T-t)\,e^{-\kappa(T-t)}\left(\frac{\bar{\sigma}_1^2}{\kappa^2}\right) - e^{-2\kappa(T-t)}\left(\frac{\bar{\sigma}_1^2}{2\kappa^3}\right)$$

$$\tilde{T}_3(t,T) = -\left[\frac{\frac{\kappa^2\bar{\sigma}_2^2\left(3\eta^2+5\eta\kappa+3\kappa^2\right)}{\eta^3(\eta+\kappa)} + 3\bar{\sigma}_1^2}{4\kappa^3}\right] + \frac{1}{2}\left(\frac{\bar{\sigma}_2^2}{\eta^2} + \frac{\bar{\sigma}_1^2}{\kappa^2}\right)(T-t) +$$

$$e^{-\kappa(T-t)}\left(\frac{\frac{\kappa^2\bar{\sigma}_2^2}{\eta^2-\eta\kappa} + \bar{\sigma}_1^2}{\kappa^3}\right) + e^{-\eta(T-t)}\left[\frac{\kappa\bar{\sigma}_2^2}{\eta^3(\kappa-\eta)}\right] - e^{-2\kappa(T-t)}\left[\frac{\frac{\kappa^2\bar{\sigma}_2^2}{(\eta-\kappa)^2} + \bar{\sigma}_1^2}{4\kappa^3}\right]$$

$$- e^{-2\eta(T-t)}\left[\frac{\kappa^2\bar{\sigma}_2^2}{4\eta^3(\eta-\kappa)^2}\right] + e^{-(\eta+\kappa)(T-t)}\left[\frac{\kappa\bar{\sigma}_2^2}{\eta(\eta-\kappa)^2(\eta+\kappa)}\right]$$

$$\begin{aligned}\tilde{T}_4(t,T) = \frac{1}{2\eta^3\kappa^2(\eta-2\kappa)(\eta-\kappa)^2}&\{(\eta-2\kappa)(\eta-\kappa)^2(\eta^2\bar{\sigma}_1^2+\kappa^2\bar{\sigma}_2^2)\\ &+e^{-\kappa(T-t)}2\eta^2(\eta-2\kappa)\kappa\left[\eta\bar{\sigma}_1^2-\kappa\bar{\sigma}_2^2-\eta\bar{\sigma}_1^2(\eta-\kappa)(T-t)\right]\\ &-e^{-\eta(T-t)}2\eta\kappa^2\left[(\eta-\kappa)^2+\kappa(1-\eta\bar{\sigma}_1^2)-\kappa(\eta^2+3\eta\kappa+2\kappa^2)(T-t)\right]\\ &-e^{2\kappa(T-t)}\eta^3\left[(\eta-\kappa)^2\bar{\sigma}_1^2-\kappa^2\bar{\sigma}_2^2\right]-e^{-2\eta(T-t)}(\eta-2\kappa)\kappa^4\bar{\sigma}_2^2\\ &+e^{-(\eta+\kappa)(T-t)}2\eta^2(\eta-2\kappa)\kappa^2\bar{\sigma}_2^2\}\end{aligned}$$

You don't need to type the coefficients in, they can be found on the Moodle page in file `coefficients3FactorModel_V2.txt`

Play with the parameters and see how they affect the shapes. Explain what you see.

16. [4 marks] Choose the parameters of Model M judiciously using optimisation routines [hints to be published on Moodle].

17. [4 marks] Having chosen the parameters judiciously, generate many Monte-Carlo paths and compute at each time on each path the term structures μ and ν. If you put them onto the same graph, as you did for the empirical data earlier on, how similar do they look to the real data ones? Do they capture the relevant features? If not, what would be missing?

18. [4 marks] Construct the term-structures of quantiles in the same manner you have computed the term-structures for μ and ν. More precisely, at any moment in calendar time, pick the 0.01 and 0.05 quantiles in the left tail for each maturity τ, call them $z_{.01}(t,\tau)$ and $z_{.05}(t,\tau)$ respecively.

19. [4 marks] On a graph, plot the .01 quantile with τ on the $x-axis$, so you have as many curves (combine the points to a curve) as you have calendar dates. On a separate graph, plot the .05 quantile. Describe the salient features and discuss.

20. [4 marks] On a graph with calendar time on the x-axis, plot the time series of the quantiles for the various maturities. Add the VIX to the plot. Describe the salient features and discuss.

Good luck!

Notes on Implementation

As questions or issues arise over the weeks to come, they will be posted on the FM408 Moodle page.